# Conditional Probability

**1**:: Set Notation

How sets are used to describe a sample space/event and how notation like $A \cap B$ is used to combine sets.

**2**:: Conditional Probability in Venn Diagrams

The notation $P(A|B)$ means "the probability of $A$ given that $B$ happened". How we can find such probabilities using a Venn Diagram.

**3**:: Formula for Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

**4**:: Tree Diagrams

"I have 3 red and 4 green balls in a bag. I take one ball out the bag, keep it, then take another. **Given that** the second ball was green, determine the probability the first was red."

# **RECAP** :: Using sets for sample spaces and events

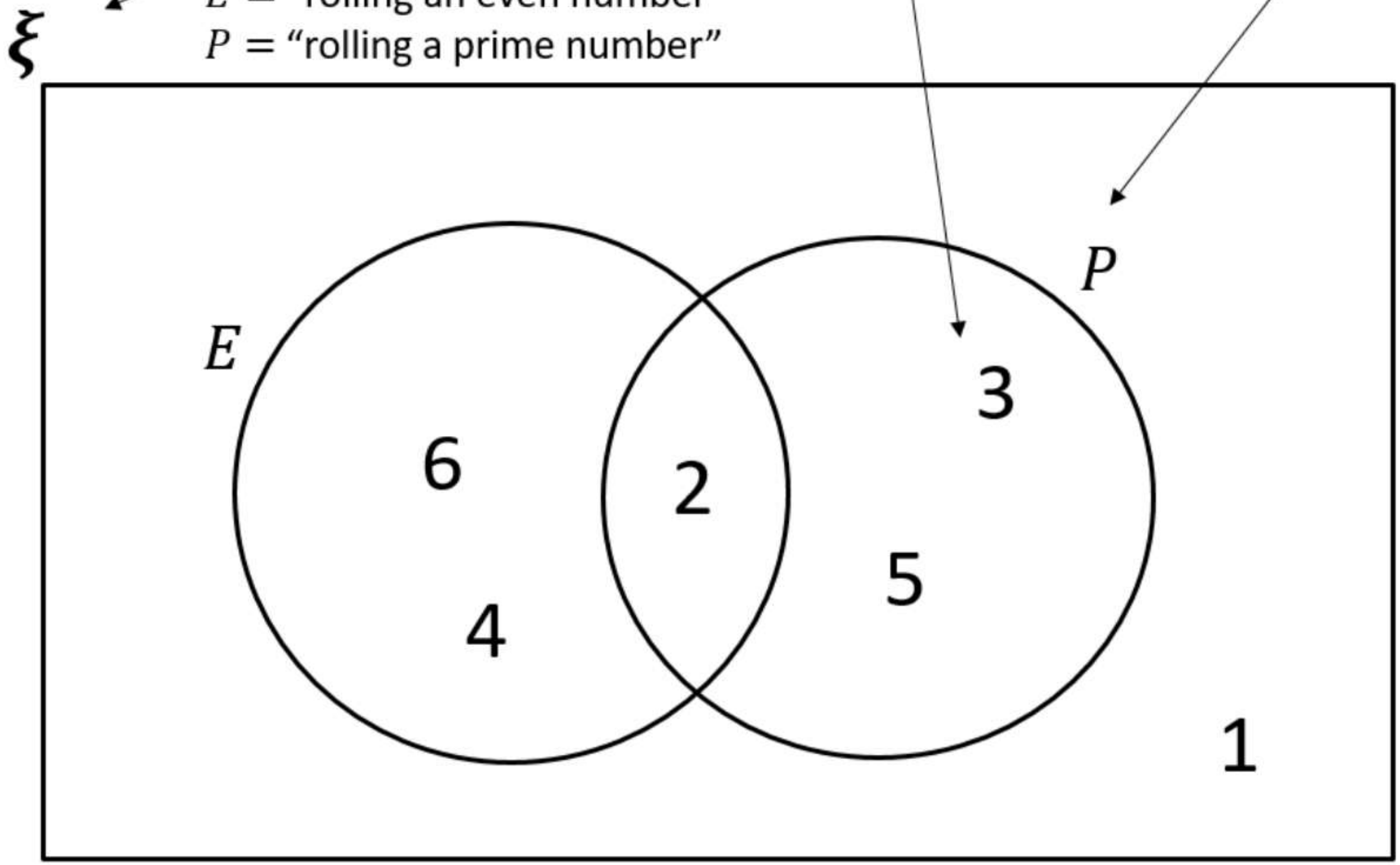
In general, sets are used to represent **collections of items**.

A **sample space** is **set of all possible outcomes**. We use $\xi$ (Greek 'Xi'), or sometimes just $S$, to represent this set. We use a rectangle in a Venn Diagram.

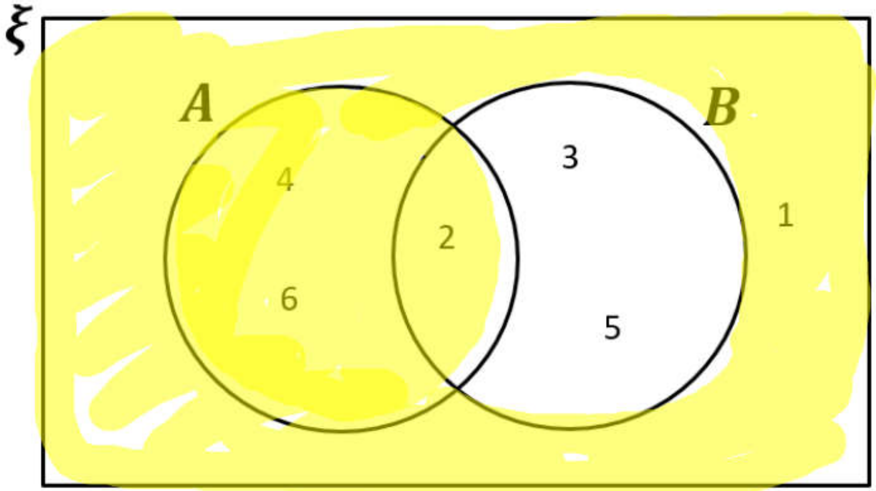
Each number represents an **outcome**.

In probability, an **event** is a **set of one or more outcomes**. These are the circles in the Venn Diagram. We use capital letters for the variables representing sets.

$E =$ "rolling an even number"
$P =$ "rolling a prime number"



# Combining events/sets




$\xi$ = the whole sample space (1 to 6)

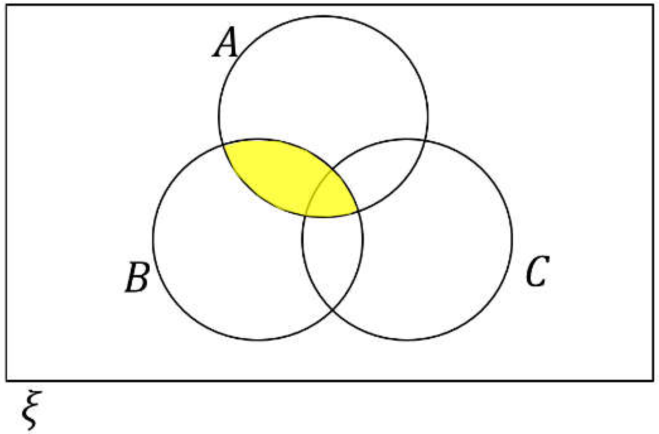
$A$ = even number on a die thrown

$B$ = prime number on a die thrown

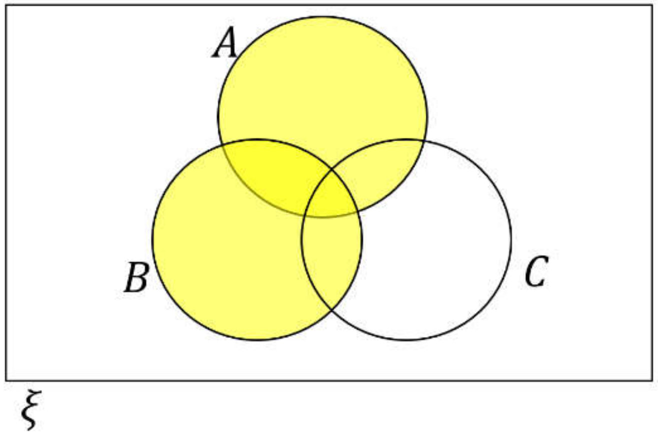
$A \cup B'$

| | What does it mean in this context? | What is the resulting set of outcomes? |
|---|---|---|
| $A'$ | **Not A (the "<u>complement</u>" of $A$).** i.e. Not rolling an even number. | $\{1, 3, 5\}$ |
| $A \cup B$ | **A or B (the "<u>union</u>" of $A$ and $B$).** i.e. Rolling an even or prime number. | $\{2,3,4,5,6\}$ |
| $A \cap B$ | **A and B (the "<u>intersection</u>" of $A$ and $B$).** i.e. Rolling a number which is even and prime. | $\{2\}$ |
| $A \cap B'$ | **"A and not B".** Rolling a number which is even and not prime. | $\{4,6\}$ |
| $(A \cup B)'$ | Rolling a number which is not [even or prime]. | $\{1\}$ |
| $(A \cap B)'$ | Rolling a number which is not [even and prime]. | $\{1,3,4,5,6\}$ |

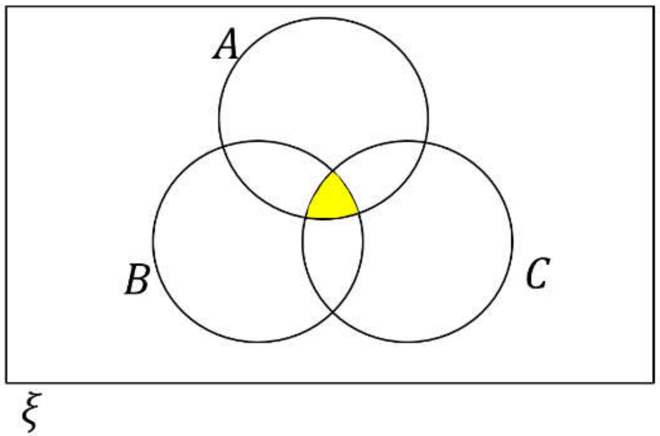
# More complex Venn diagrams



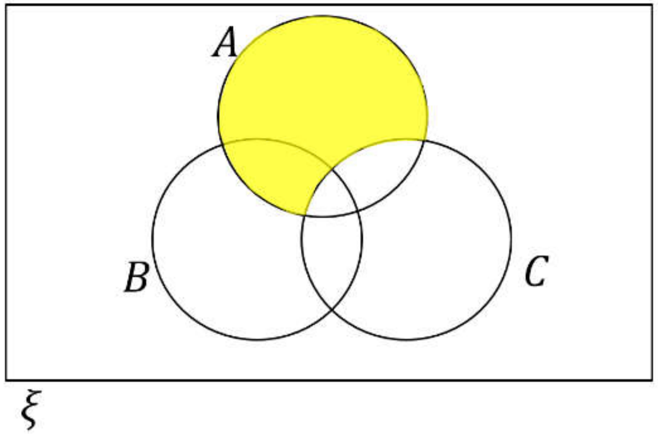

$A \cap B$



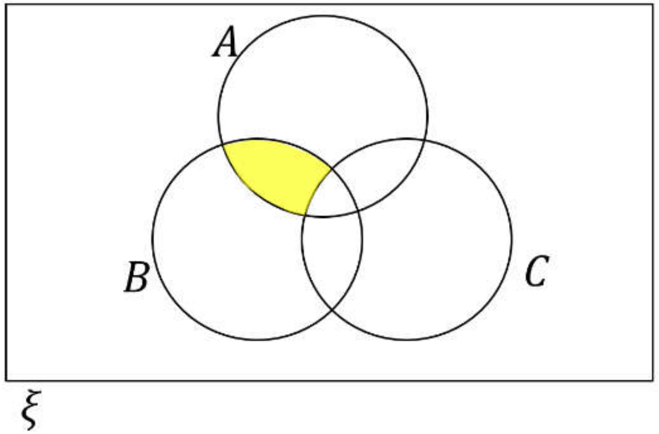

$A \cup B$



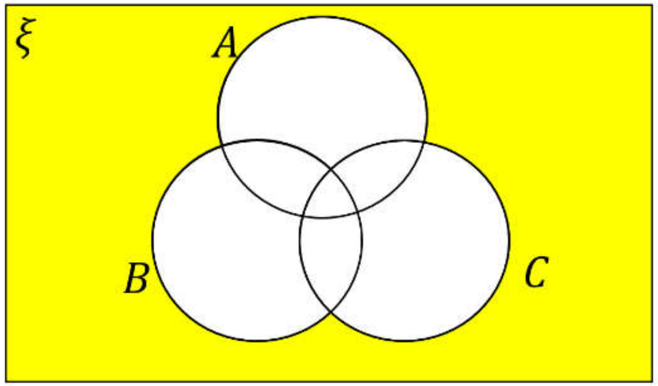

$A \cap B \cap C$



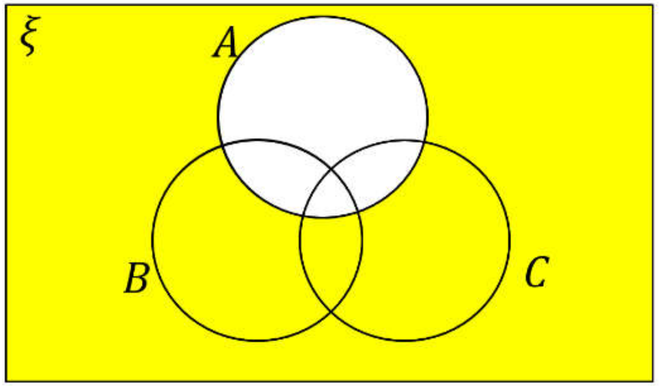

$A \cap C'$



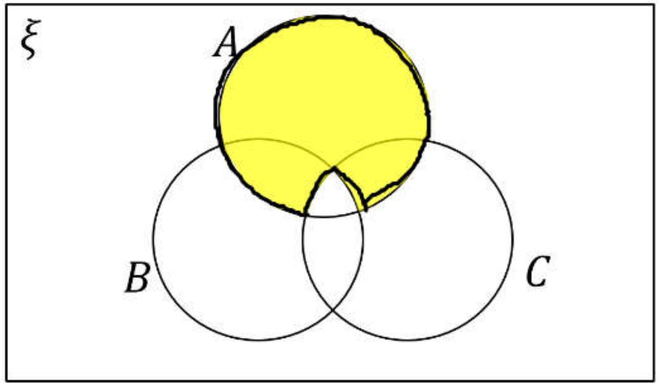

$A \cap B \cap C'$



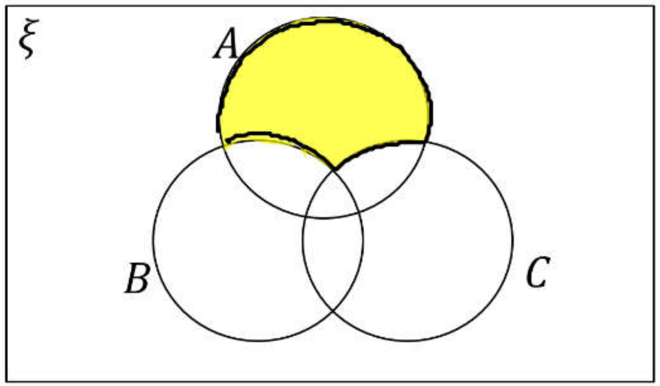

$(A \cup B \cup C)'$




$A'$




$A \cap (B \cap C)'$




$A \cap B' \cap C'$

$A \cap (B \cup C)'$

# Examples

Venn Diagram can either contain:

(a) The **specific outcomes** in each set

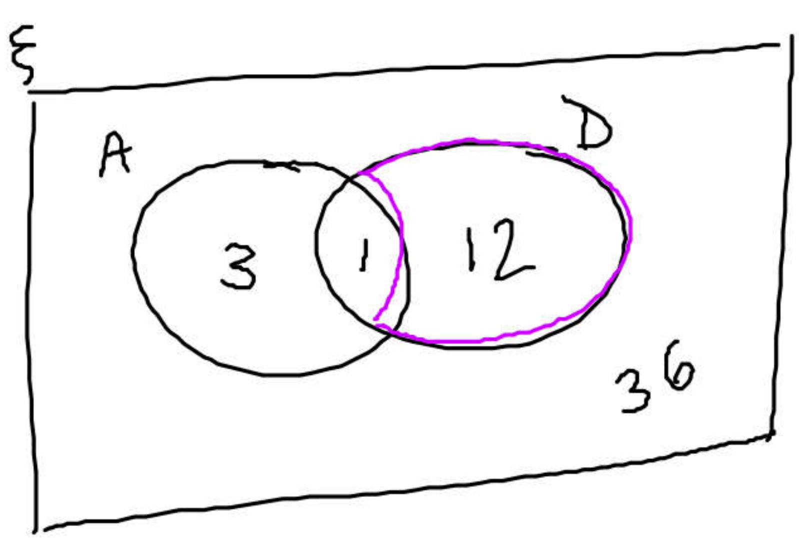
(b) The number of items in the set (i.e. **frequencies**)

(c) The **probability** of being in that set.

This will usually be stated or made obvious from the context.

A card is selected at random from a pack of 52 playing cards. Let $A$ be the event that the card is an ace and $D$ the event that the card is a diamond. Find:

a) $P(A \cap D)$ b) $P(A \cup D)$ c) $P(A')$ d) $P(A' \cap D)$




a) $\frac{1}{52}$

b) $\frac{16}{52}$

c) $\frac{48}{52}$

d) $\frac{12}{52}$

Given that $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.25$,

a. Explain why events $A$ and $B$ are not independent.

Given also that $P(C) = 0.2$, that events $A$ and $C$ are mutually exclusive and that events $B$ and $C$ are independent,

b. Draw a Venn diagram to illustrate the events $A, B$ and $C$, showing the probabilities for each region.

c. Find $P((A \cap B') \cup C)$ $= 0.25$

**If events $A$ and $B$ are independent.**

$$P(A \cap B) = P(A) \times P(B)$$

**If events $A$ and $B$ are mutually exclusive:**

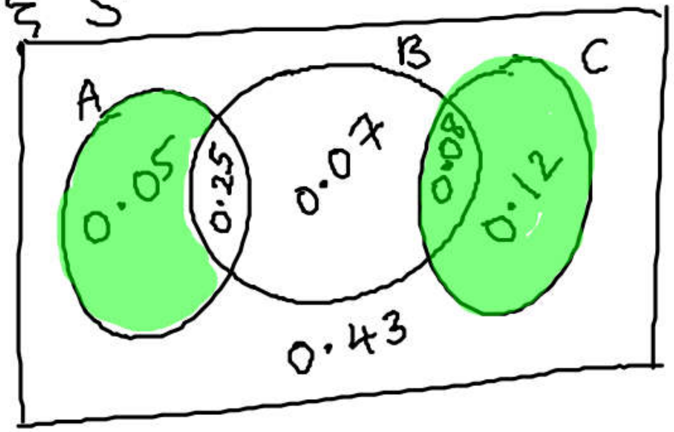
$$P(A \cap B) = 0$$
$$P(A \cup B) = P(A) + P(B)$$

a) $P(A) \times P(B) \neq P(A \cap B)$

$0.3 \times 0.4 \neq 0.25$

$0.12 \neq 0.25$

b) ξ S

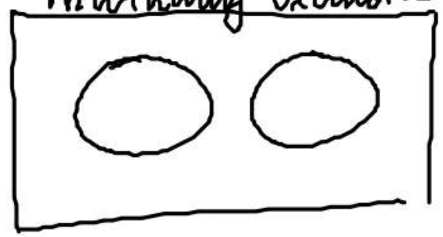



$P(B \cap C) = P(B) \times P(C)$

$= 0.4 \times 0.2$

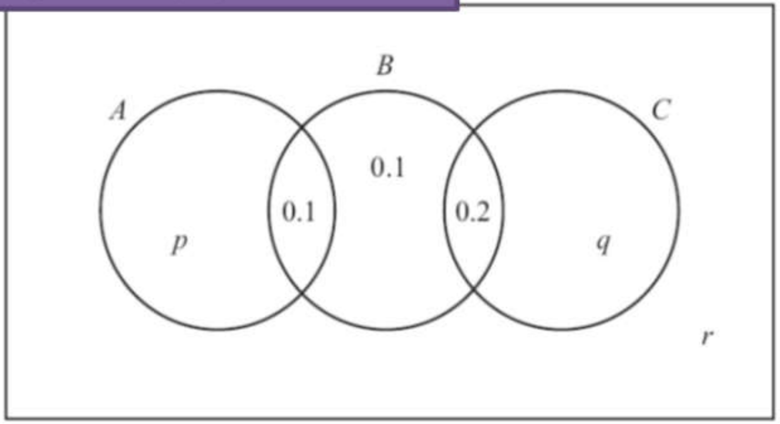
$= 0.08$

mutually exclusive



c)

May 2013 (R) Q6




The Venn diagram in Figure 1 shows three events $A$, $B$ and $C$ and the probabilities associated with each region of $B$. The constants $p$, $q$ and $r$ each represent probabilities associated with the three separate regions outside $B$.

The events $A$ and $B$ are independent.

(a) Find the value of $p$. **(3)**

$P(A \cap B) = P(A) \times P(B)$

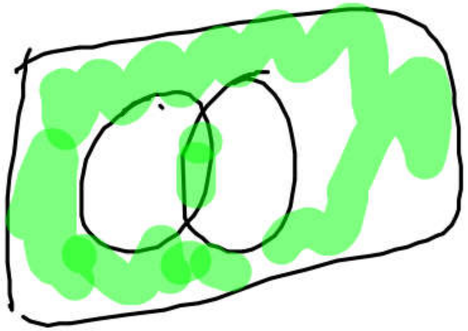
$0.1 = (0.1 + p) \times 0.4$

$\frac{0.1}{0.4} - 0.1 = p$

$\underline{\underline{p = 0.15}}$

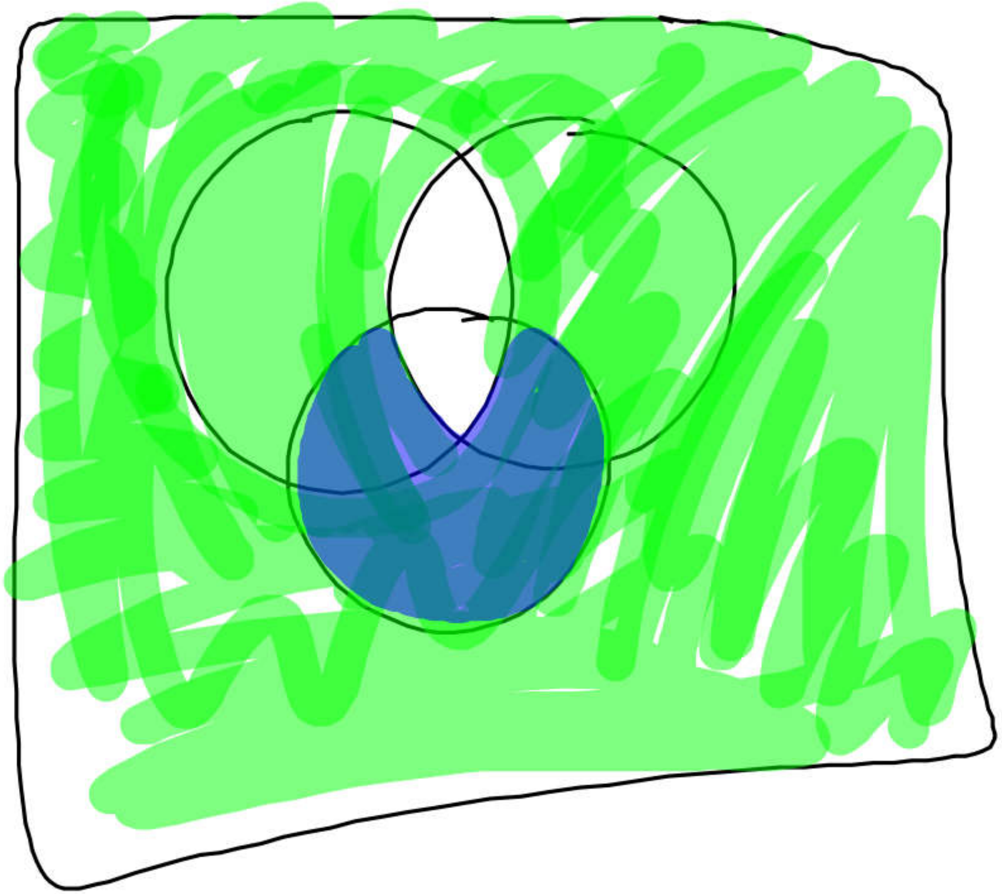
Ex 2A

Q Odd



$(A \cup B)' \cup (A \cap B)$

**Ex 2A**

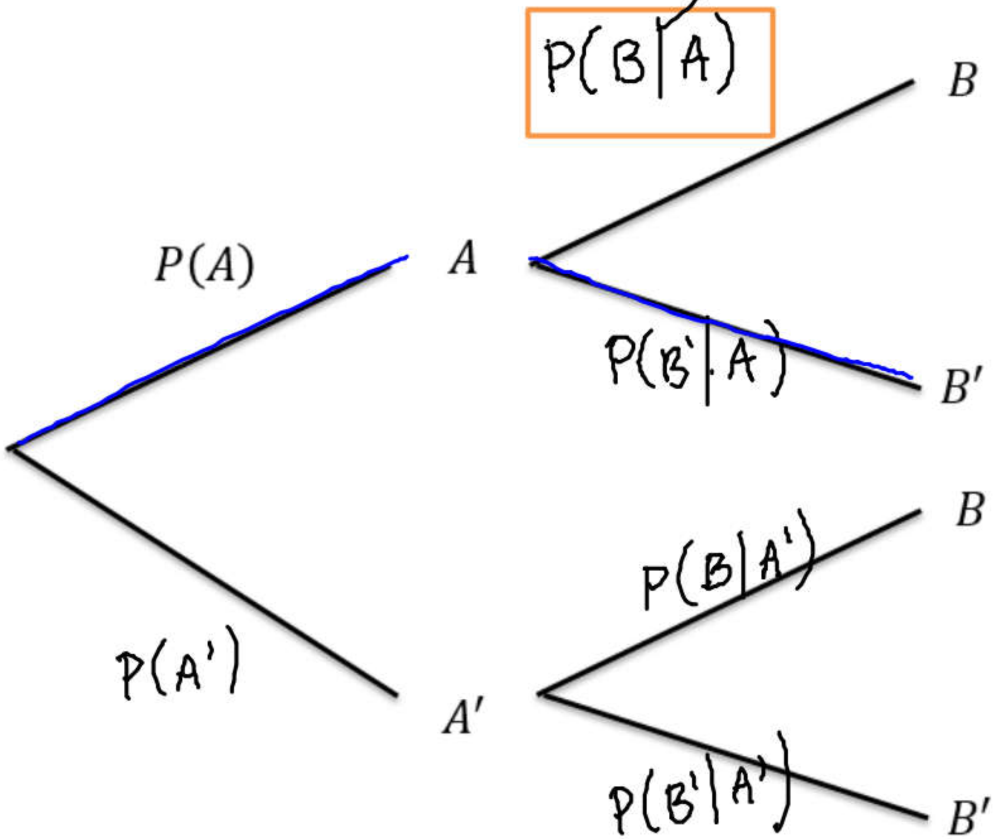


$(A' \cup B') \cap C$

# Conditional Probability

"given that"

Think about how we formed a probability tree at GCSE:




$$P(A \cap B) = P(A) \times P(B|A)$$

$$P(A \cap B') = P(A) \times P(B'|A)$$

$$P(B'|A) = \frac{P(A \cap B')}{P(A)}$$

$$P(A'|B') = \frac{P(A' \cap B')}{P(B')}$$

Alternatively (and more commonly):

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

**Memory Tip**: You're dividing by the event you're conditioning on.

**1** The following two-way table shows what foreign language students in Year 9 study.
$B$ is the event that the student is a boy. $F$ is the event they chose French as their language.

| | $\boldsymbol{B}$ | $\boldsymbol{B'}$ | Total |
|---|---|---|---|
| $\boldsymbol{F}$ | 14 | 38 | 52 |
| $\boldsymbol{F'}$ | 26 | 22 | 48 |
| Total | 40 | 60 | 100 |

**a** Determine the probability of:
$P(F|B')$ A girl is chosen. What is the prob F?

**Method 1: Using the formula:**

$$P(F|B') = \frac{P(F \cap B')}{P(B')} = \frac{38}{60}$$

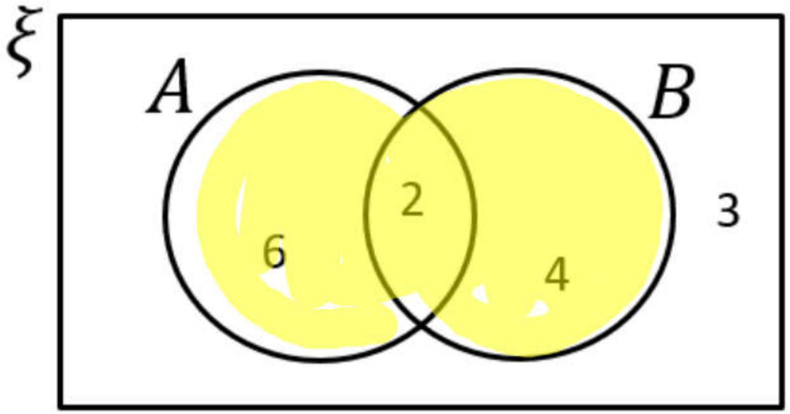
**Method 2: Restricted sample space.**

$$\frac{38}{60}$$

**b** $P(B|F') = \frac{26}{48}$

---

**2** Using the Venn Diagram, determine:




Frequencies

**a** $P(A|B)$

**Method 1: Using the formula**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/15}{6/15} = \frac{2}{6} = \frac{1}{3}$$

**Method 2: Restricted sample space**

$$\frac{2}{6} = \frac{1}{3}$$

**b** $P(A'|B') = \frac{3}{9} = \frac{1}{3}$

**c** $P(B|(A \cup B)) = \frac{6}{12} = \frac{1}{2}$

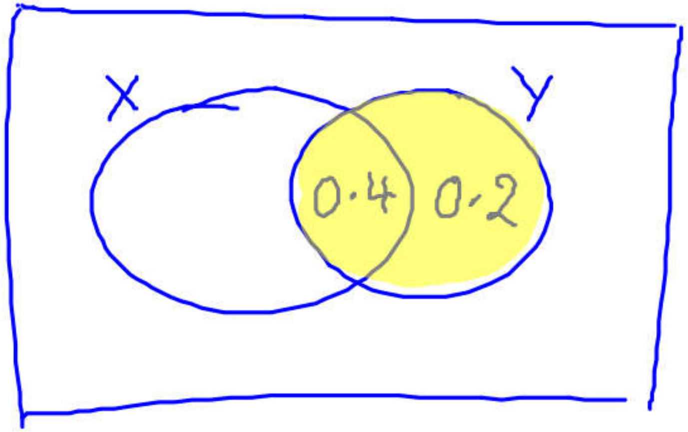
**a** Given that $P(A) = 0.5$ and $P(A \cap B) = 0.3$, what is $P(B|A)$?

**Tip**: The 'restricted sample space' method also works for Venn Diagrams with probabilities.

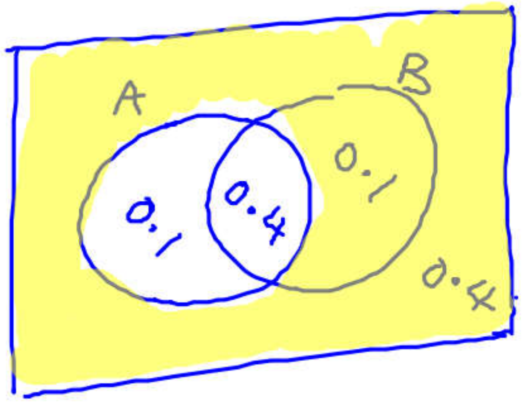
$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.5} = \underline{\underline{0.6}}.$$

**b** Given that $P(Y) = 0.6$ and $P(X \cap Y) = 0.4$, what is $P(X'|Y)$?
(Hint: Drawing a Venn Diagram will help!)

$$P(X'|Y) = \frac{0.2}{0.6} = \frac{1}{3}$$




**c** Given that $P(A) = 0.5$, $P(B) = 0.5$ and $P(A \cap B) = 0.4$, what is $P(B|A')$?




$$\frac{0.1}{0.5} = \underline{\underline{0.2}}.$$

$$\frac{P(B \cap A')}{P(A')}$$

## Your Turn

The events $E$ and $F$ are such that

$$P(E) = 0.28 \quad P(E \cup F) = 0.76 \qquad P(E \cap F') = 0.11$$

Find

*a)* $P(E \cap F) =$

*b)* $P(F) =$

*c)* $P(E'|F') =$

*(Drawing a Venn diagram is often helpful!)*