Integration (Year 1)

This chapter is roughly divided into two parts: the first, **indefinite integration**, is the **opposite of differentiation**. The second, **definite integration**, allows us to **find areas under graphs** (as well as surface areas and volumes) or areas between two graphs.

1:: Find y given $\frac{dy}{dx}$

A curve has the gradient function

$$\frac{dy}{dx} = 3x + 1$$

If the curve goes through the point $(2,3)$, determine y.

2:: Evaluate definite integrals, and hence the area under a curve.

Find the area bounded between the curve with equation $y = x^2 - 2x$ and the x-axis.

3:: Find areas bound between two different lines.

Find the points of intersection of $y = x^2 - 4x + 3$ and $y = x + 9$, and hence find the area bound between the two lines.

Integrating x^n terms

Integration is the **opposite of differentiation**.
(For this reason it is also called '*antidifferentiation*')

Function $5x^3$ → multiply by power → reduce power by 1 → **Gradient Function** $15x^2$

Gradient Function $15x^2$ → increase power by 1 → divide by <u>new</u> power → **Function** $5x^3$

However, there's one added complication...

Find y when $\frac{dy}{dx} = 3x^2$

Adding 1 to the power and dividing by this power give us:

$$y = x^3$$

However, other functions would also have differentiated to $3x^2$:

$$y = x^3 + 1, \qquad y = x^3 - 4, \qquad \ldots$$

Clearly we could have had any constant, as it disappears upon differentiation.

$$\therefore \boldsymbol{y = x^3 + c}$$

c is known as a **constant of integration**

Find y when:

Exam Note: You should **always** include it for indefinite integration

$$\frac{dy}{dx} = 4x^3$$

$$\frac{dy}{dx} = x^5$$

You could also write as $\frac{x^6}{6}$. It's a matter of personal preference.

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}$$

Tip: Many students are taught to write $\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}$ (as does textbook!). This is ugly and students then often struggle to simplify it. Instead remember back to GCSE: **When you divide by a fraction, you multiply by the reciprocal.**

Find y when:

When we divide by $\frac{1}{2}$ we multiply by the reciprocal, i.e. 2.
Tip: I recommend eventually doing this in your head when the simplification would be simple.

$$\frac{dy}{dx} = \frac{4}{\sqrt{x}}$$

$$\frac{dy}{dx} = 5x^{-2}$$

$$\frac{dy}{dx} = 4x^{\frac{2}{3}}$$

$$\frac{dy}{dx} = 10x^{-\frac{2}{7}}$$

Your Turn

Find $f(x)$ when:

$$f'(x) = 2x + 7$$

$$f'(x) = x^2 - 1$$

$$f'(x) = \frac{2}{x^7}$$

$$f'(x) = \sqrt[3]{x}$$

$$f'(x) = 33x^{\frac{5}{6}}$$

Note: In case you're wondering what happens if $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$, the problem is that after adding 1 to the power, we'd be dividing by 0. You will learn how to integrate $\frac{1}{x}$ in Year 2.

Integration notation

The following notation could be used to differentiate an expression:

The dx here means differentiating "with respect to x".

$$\frac{d}{dx}(5x^2) = 10x$$

There is similarly notation for integrating an expression:

$$\int 10x \ dx = 5x^2 + c$$

"Integrate..."

"...this expression"

"...with respect to x"
(the dx is needed just as it was needed in the differentiation notation at the top of this slide)

This is known as **indefinite integration**, in contrast to definite integration, which we'll see later in the chapter.

It is called 'indefinite' because the exact expression is unknown (due to the $+c$).

Note: The brackets are required if there's multiple terms.

Find $\int (x^{-\frac{3}{2}} + 2)\, dx$

Find $\int (6t^2 - 1)\, dt$

Note the dt instead of dx.

Find $\int (px^3 + q)\, dx$ where p and q are constants.

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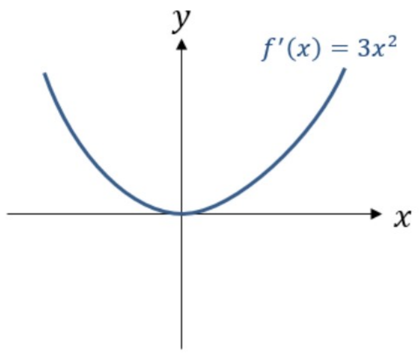
Given that $y = 2x^5 + \frac{6}{\sqrt{x}}$, $x > 0$, find in their simplest form

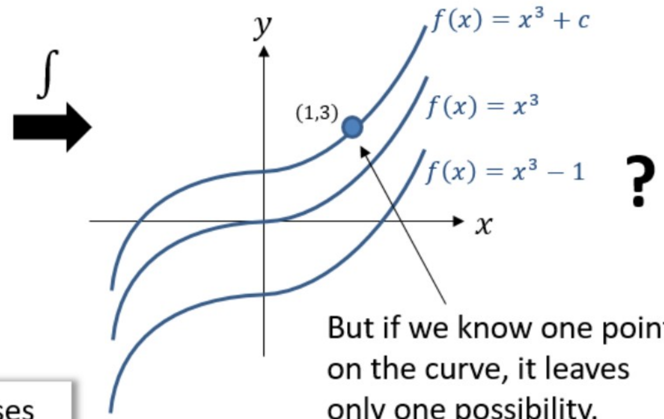
(b) $\int y\, dx$ **(3)**

Ex 13B

Finding constant of integration

Recall that when we integrate, we get a constant of integration, which could be any real value. This means **we don't know what the exact original function was**.





But if we know one point on the curve, it leaves only one possibility.

The curve with equation $y = f(x)$ passes through $(1,3)$. Given that $f'(x) = 3x^2$, find the equation of the curve.

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A curve with equation $y = \mathrm{f}(x)$ passes through the point (4, 25).

Given that

$$\mathrm{f}'(x) = \frac{3}{8}x^2 - 10x^{-\frac{1}{2}} + 1, \qquad x > 0$$

(a) find f(x), simplifying each term.

(5)

(b) Find an equation of the normal to the curve at the point (4, 25).

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(5)

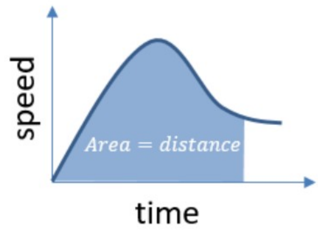
To keep you occupied if you finish (a) quickly!

Ex 13C

Definite Integration

So far we've seen integration as '**the opposite of differentiation**', allowing us to find $y = f(x)$ when we know the gradient function $y = f'(x)$.

In practical settings however the most useful use of integration is that **it finds the area under a graph**. Remember at GCSE for example when you estimated the area under a speed-time graph, using trapeziums, to get the distance?
If you knew the equation of the curve, you could get the exact area!

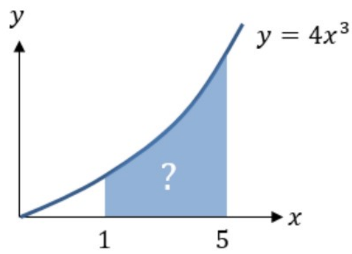


Before we do this, we need to understand how to find a **definite integral**:

These are known as **limits**, which give the values of x we're finding the area between.

We integrate as normal, but put expression in **square brackets**, meaning **we still need to evaluate the integrated expression using the limits**.

$$\int_1^5 4x^3 \, dx =$$



Write $(\ldots) - (\ldots)$ and evaluate the expression for each of the limits, top one first.

$$\int_{-3}^3 x^2 + 1 \, dx =$$

We **DON'T have a constant of integration** when doing definite integration.

Write out you working EXACTLY as seen here. The $(\ldots) - (\ldots)$ brackets are particularly crucial as you'll otherwise likely make a sign error.

You can use the $\left[\!\left[\int_b^a \square\right]\!\right]$ button on your calculator to evaluate definite integrals.
<u>But only use it to check your answer.</u>

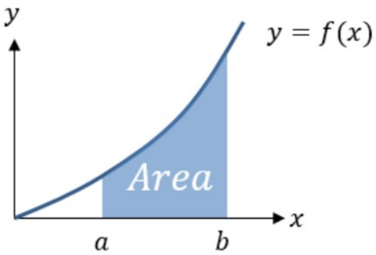
Problem Solving

Given that P is a constant and $\int_1^5 (2Px + 7)\, dx = 4P^2$, show that there are two possible values for P and find these values.

Ex 13D

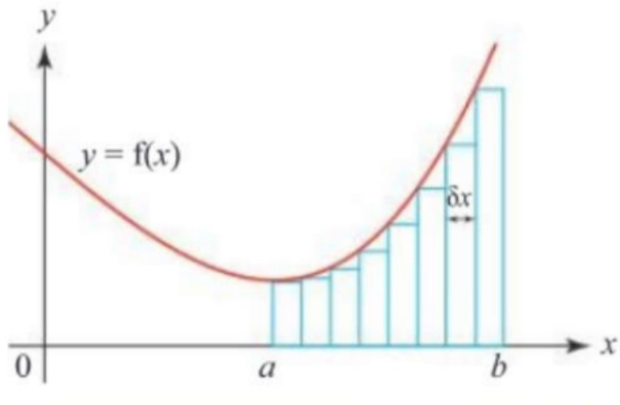
Areas under curves

Earlier we saw that the definite integral $\int_b^a \boldsymbol{f(x)}\ \boldsymbol{dx}$ gives the **area** between a positive curve $y = f(x)$, the $\boldsymbol{x}$**-axis**, and the lines $x = a$ and $x = b$.
(We'll see why this works in a sec)



Find the area of the finite region between the curve with equation $y = 20 - x - x^2$ and the x-axis.

Why does integration give the area under the curve?



$$A \approx y_1\delta x + y_2\delta x + \ldots + y_n\delta x$$

$$A \approx \sum_{i=1}^{n} y_i \delta x$$

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Ex 13E

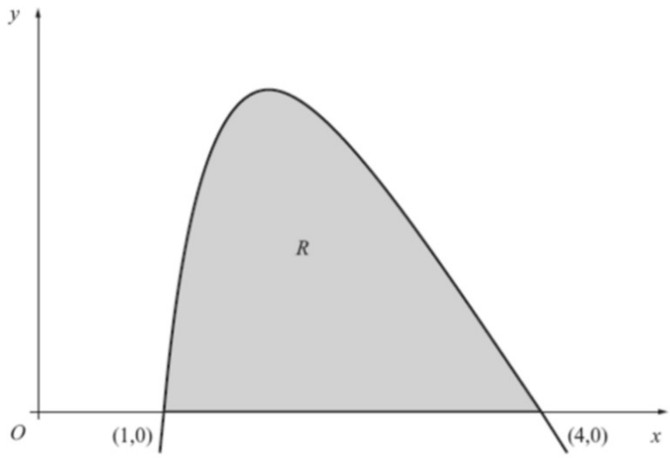


Figure 2

The finite region R, as shown in Figure 2, is bounded by the x-axis and the curve with equation

$$y = 27 - 2x - 9\sqrt{x} - \frac{16}{x^2}, \quad x > 0.$$

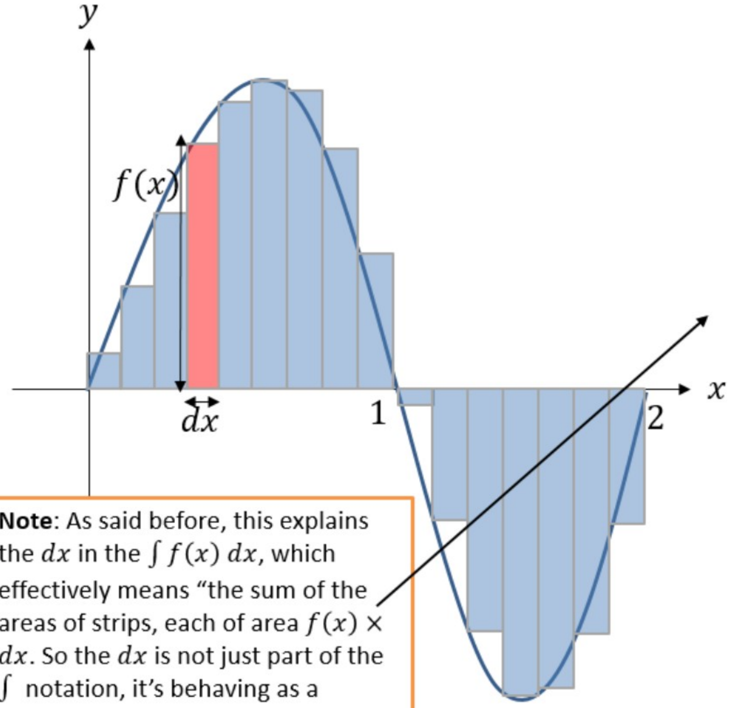
The curve crosses the x-axis at the points $(1, 0)$ and $(4, 0)$.

(c) Use integration to find the exact value for the area of R.

(6)

‘Negative Areas’

Sketch the curve $y = x(x-1)(x-2)$ (which expands to give $y = x^3 - 3x^2 + 2x$).
Now calculate $\int_0^2 x(x-1)(x-2)\ dx$. Why is this result surprising?

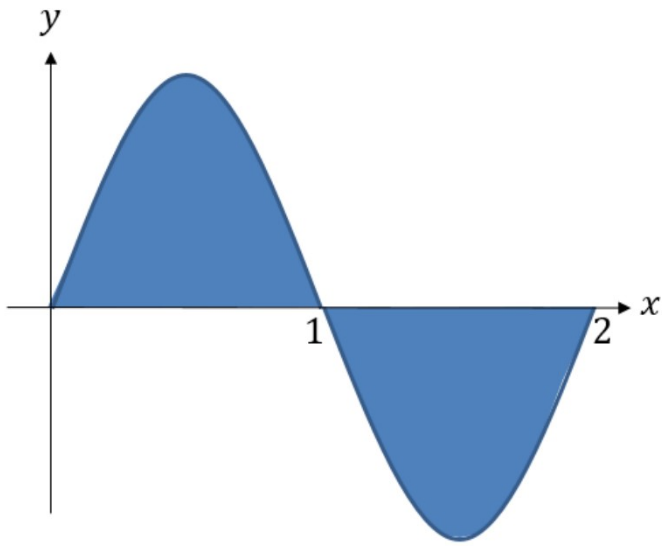


Note: As said before, this explains the dx in the $\int f(x)\,dx$, which effectively means “the sum of the areas of strips, each of area $f(x) \times dx$. So the dx is not just part of the $\int$ notation, it’s behaving as a physical quantity! (i.e. length)

Integration $\int f(x)\,dx$ is just the sum of areas of infinitely thin rectangles, where the current y value (i.e. $f(x)$) is each height, and the widths are dx.
i.e. The area of each is $f(x) \times dx$

The problem is, when $f(x)$ is negative, then $f(x) \times dx$ is negative, i.e. a negative area!
The result is that the ‘positive area’ from 0 to 1 is cancelled out by the ‘negative area’ from 1 to 2, giving an overall ‘area’ of 0.
So how do we resolve this?

Find the total area bound between the curve $y = x(x-1)(x-2)$ and the x-axis.



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Ex 13F

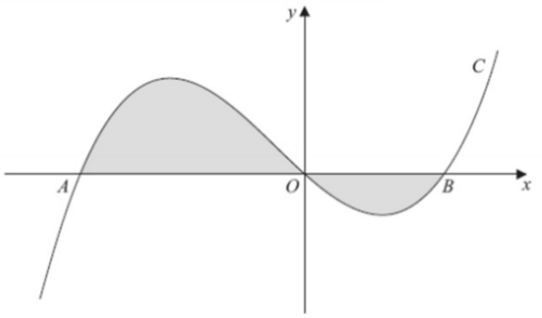


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = x(x+4)(x-2).$$

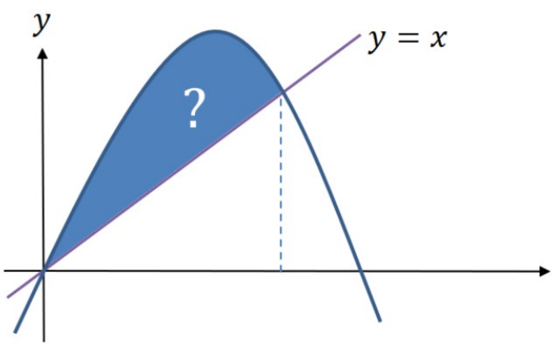
The curve C crosses the x-axis at the origin O and at the points A and B.

(*a*) Write down the x-coordinates of the points A and B. **(1)**

The finite region, shown shaded in Figure 3, is bounded by the curve C and the x-axis.

(*b*) Use integration to find the total area of the finite region shown shaded in Figure 3. **(7)**

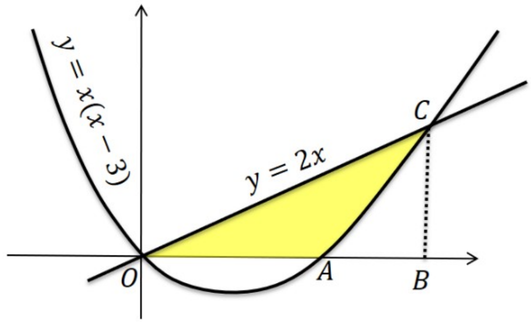
Areas between curves and lines



How could we find the area between the line and the curve?

Determine the area between the lines with equations $y = x(4-x)$ and $y = x$

A Harder One



The diagram shows a sketch of the curve with equation $y = x(x-3)$ and the line with equation $y = 2x$.
Find the area of the shaded region OAC.

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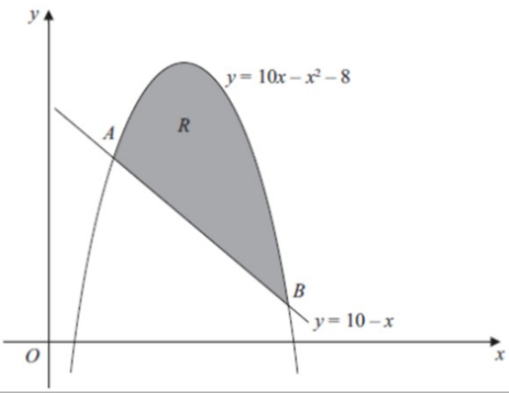


Figure 2 shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$.

The line and the curve intersect at the points A and B, and O is the origin.

(*a*) Calculate the coordinates of A and the coordinates of B. **(5)**

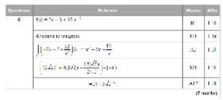
The shaded area R is bounded by the line and the curve, as shown in Figure 2.

(*b*) Calculate the exact area of R. **(7)**

Preferred Method?
If the top curve has equation $y = f(x)$ and the bottom curve $y = g(x)$, the area between them is:

$$\int_b^a \big(f(x) - g(x)\big)\, dx$$

This means you can integrate a single expression to get the final area, without any adjustment required after.



Question	Scheme	Marks	AOs
6	[illegible]	B1	1.1b
	Attempts to integrate	M1	1.1a
	[illegible]	A1	1.1b
	[illegible]	M1	1.1b
	[illegible]	A1*	1.1b
		(5 marks)	

5. Given that

$$f(x) = 2x + 3 + \frac{12}{x^2}, \quad x > 0$$

show that $\displaystyle\int_1^{2\sqrt{2}} f(x)\,dx = 16 + 3\sqrt{2}$

(5)

8.

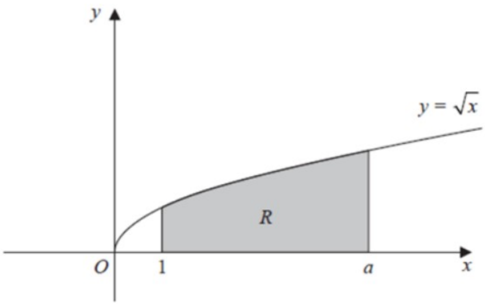


Figure 2 shows a sketch of the curve with equation $y = \sqrt{x}$, $x \geq 0$.

The region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 1$, the x-axis and the line with equation $x = a$, where a is a constant.

Given that the area of R is 10,

(a) find, in simplest form, the value of

(i) $\displaystyle\int_1^a \sqrt{8x}\ \mathrm{d}x$,

(ii) $\displaystyle\int_0^a \sqrt{x}\ \mathrm{d}x$,

(4)

(b) show that $a = 2^k$, where k is a rational constant to be found.

(4)

(Total for Question 8 is 8 marks)

14.

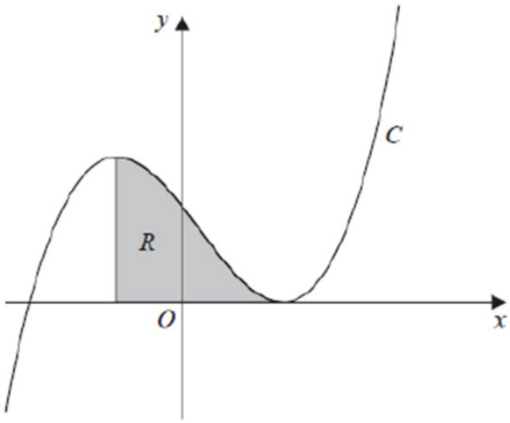


Figure 5

Figure 5 shows a sketch of the curve C with equation $y = (x - 2)^2 (x + 3)$.

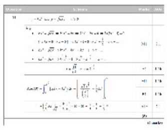
The region R, shown shaded in Figure 5, is bounded by C, the vertical line passing through the maximum turning point of C and the x-axis.

Find the exact area of R.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(Total for Question 14 is 9 marks)

14.



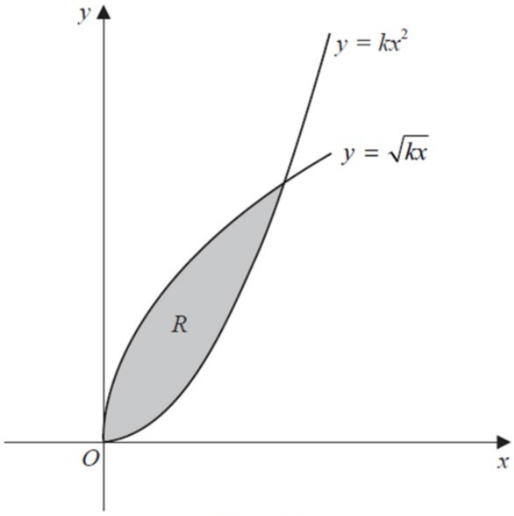


Figure 7

Figure 7 shows the curves with equations

$$y = kx^2 \qquad x \geqslant 0$$

$$y = \sqrt{kx} \qquad x \geqslant 0$$

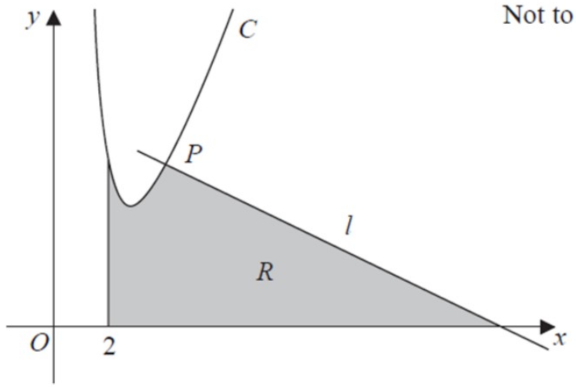
where k is a positive constant.

The finite region R, shown shaded in Figure 7, is bounded by the two curves.

Show that, for all values of k, the area of R is $\dfrac{1}{3}$

(5)

15.



Not to scale

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{32}{x^2} + 3x - 8, \qquad x > 0$$

The point $P\,(4, 6)$ lies on C.
The line l is the normal to C at the point P.

The region R, shown shaded in Figure 4, is bounded by the line l, the curve C, the line with equation $x = 2$ and the x-axis.

Show that the area of R is 46
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(10)