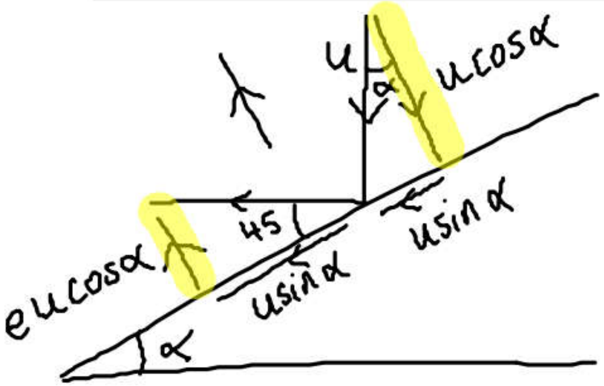
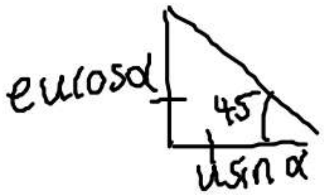
3. A small smooth ball of mass m is falling vertically when it strikes a fixed smooth plane which is inclined to the horizontal at an angle α, where $0° < \alpha < 45°$. Immediately before striking the plane the ball has speed u. Immediately after striking the plane the ball moves in a direction which makes an angle of 45° with the plane. The coefficient of restitution between the ball and the plane is e. Find, in terms of m, u and e, the magnitude of the impulse of the plane on the ball.

(Total 11 marks)





$e u\cos\alpha = u\sin\alpha$

$\boxed{e = \tan\alpha}$

$I = m(v-u)$

$I = m(eu\cos\alpha - -u\cos\alpha)$

$I = m(eu\cos\alpha + u\cos\alpha)$

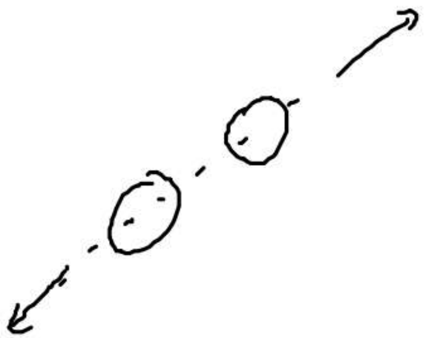
$I = mu\cos\alpha(e+1)$

$1 + e^2 = \tan^2\alpha + 1$

$1 + e^2 = \sec^2\alpha$

$\frac{1}{1+e^2} = \cos^2\alpha$

$\cos\alpha = \frac{1}{\sqrt{1+e^2}}$



$I = \frac{mu(e+1)}{\sqrt{1+e^2}}$

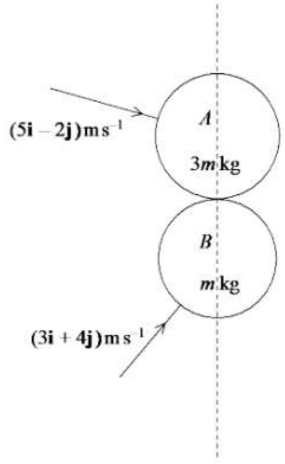


Figure 1

Two smooth uniform spheres A and B have masses $3m$ kg and m kg respectively and equal radii. The spheres are moving on a smooth horizontal surface. Initially, sphere A has velocity $(5\mathbf{i} - 2\mathbf{j})$ m s^{-1} and sphere B has velocity $(3\mathbf{i} + 4\mathbf{j})$ m s^{-1}. When the spheres collide, the line joining their centres is parallel to $\mathbf{j}$, as shown in Figure 1.

The coefficient of restitution between the two spheres is e.

The kinetic energy of sphere B immediately after the collision is 85% of its kinetic energy immediately before the collision.

Find

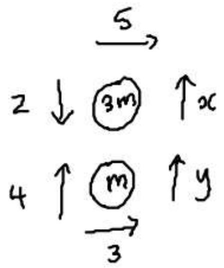
(a) the velocity of each sphere immediately after the collision,

(9)

(b) the value of e.

(3)

(Total 12 marks)



a) PCLM

$4m - 6m = 3mx + my$

$-2 = 3x + y$

NLR

$e = \frac{x-y}{6}$

$6e = x - y$

$\frac{1}{2}m(y^2+3^2) = 0.85 \times \frac{1}{2}m(4^2+3^2)$

$y^2 + 9 = 0.85 \times 5^2$

$y^2 = \frac{49}{4} \quad y = -\frac{7}{2}$

$6e - 2 = 4x$

$x = \frac{3}{2}e - \frac{1}{2}$

$y = -2 - \frac{9}{2}e + \frac{3}{2}$

$y = -\frac{1}{2} - \frac{9}{2}e$

$\frac{-2 + \frac{7}{2}}{3} = x$

$x = \frac{1}{2}$

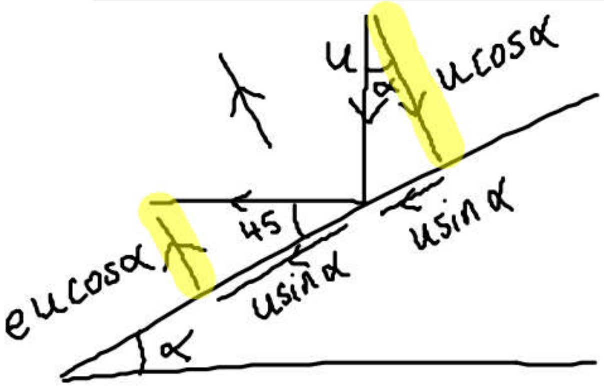
A $5\mathbf{i} + \frac{1}{2}\mathbf{j}$

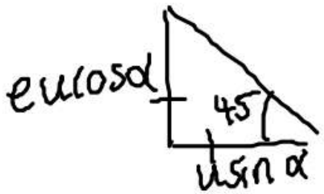
B $3\mathbf{i} - \frac{7}{2}\mathbf{j}$

b) $e = \frac{x-y}{6} = \frac{\frac{1}{2} - -\frac{7}{2}}{6} = \frac{2}{3}$

3. A small smooth ball of mass m is falling vertically when it strikes a fixed smooth plane which is inclined to the horizontal at an angle α, where $0° < \alpha < 45°$. Immediately before striking the plane the ball has speed u. Immediately after striking the plane the ball moves in a direction which makes an angle of 45° with the plane. The coefficient of restitution between the ball and the plane is e. Find, in terms of m, u and e, the magnitude of the impulse of the plane on the ball.

(Total 11 marks)





$$e\cancel{u}\cos\alpha = \cancel{u}\sin\alpha$$

$$\boxed{e = \tan\alpha}$$

$$I = m(v-u)$$

$$I = m(eu\cos\alpha - -u\cos\alpha)$$

$$I = m(eu\cos\alpha + u\cos\alpha)$$

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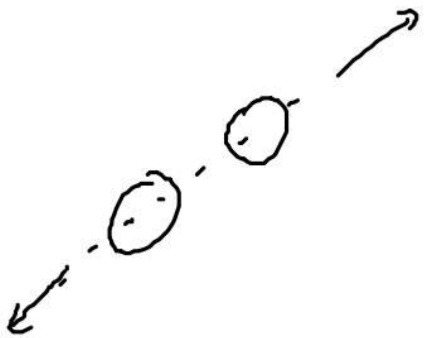

$$I = \frac{mu(e+1)}{\sqrt{1+e^2}}$$

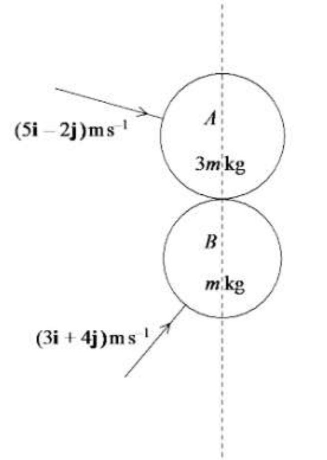

Figure 1

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Find

(a) the velocity of each sphere immediately after the collision,

(9)

(b) the value of e.

(3)

(Total 12 marks)

$\xrightarrow{5}$

$2 \downarrow$ (3m) $\uparrow x$

$4 \uparrow$ (m) $\uparrow y$

$\xrightarrow[3]{}$

a) PCLM

$4m - 6m = 3mx + my$

$-2 = 3x + y$

NLR

$e = \dfrac{x-y}{6}$

$6e = x - y$

$\frac{1}{2}m(y^2+3^2) = 0.85 \times \frac{1}{2}m(4^2+3^2)$

$y^2 + 9 = 0.85 \times 5^2$

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$\dfrac{-2 + \frac{7}{2}}{3} = x$

$x = \frac{1}{2}$

A $5\mathbf{i} + \frac{1}{2}\mathbf{j}$

B $3\mathbf{i} - \frac{7}{2}\mathbf{j}$

b) $e = \dfrac{x-y}{6} = \dfrac{\frac{1}{2} - -\frac{7}{2}}{6} = \dfrac{2}{3}$

8. A large container initially contains 3 litres of pure water. $\frac{1}{4}$ 0.25
Contaminated water starts pouring into the container at a constant rate of 250 ml per minute and you may assume the contaminant dissolves completely.

$\frac{1}{8}$ 0.125

At the same time, the container is drained at a constant rate of 125 ml per minute. The water in the container is continually mixed.

The amount of contaminant in the water pouring into the container, at time t minutes after pouring began, is modelled to be $(5 \quad e^{-0.1t})$ mg per litre.

Let m be the amount of contaminant, in milligrams, in the container at time t minutes after the contaminated water begins pouring into the container.

(a) (i) Write down an expression for the total volume of water in litres in the container at time t.

$3 + 0.125t \longrightarrow t = 30, \text{vol} = 6.75$

(ii) Hence show that the amount of contaminant in the container can be modelled by the differential equation

$$\frac{dm}{dt} = \frac{5 - e^{-0.1t}}{4} - \frac{m}{24 + t}$$

going $0.25(5 - e^{-0.1t}) = \frac{5 - e^{-0.1t}}{4}$

leaving $-\frac{m}{3 + 0.125t} \times \frac{1}{8} = -\frac{m}{24+t}$

(4)

(b) By solving the differential equation, find an expression for the amount of contaminant, in milligrams, in the container t minutes after the contaminated water begins to be poured into the container.

(8)

After 30 minutes, the concentration of contaminant in the water was measured as 3.79 mg per litre.

(c) Assess the model in light of this information, giving a reason for your answer.

(2)

$\frac{dm}{dt} = \frac{5 - e^{-0.1t}}{4} - \frac{m}{24+t}$

I.F. $\int P\,dt = \int \frac{1}{24+t}\,dt = \ln|24+t|$

I.F. is $24+t$

$\frac{dm}{dt} + \frac{1}{24+t}m = \frac{5 - e^{-0.1t}}{4}$

$u = t \quad v = -10e^{-0.1t}$

$u' = 1 \quad v' = e^{-0.1t}$

$(24+t)\frac{dm}{dt} + m = \frac{5 - e^{-0.1t}}{4}(24+t)$

$\frac{d}{dt}\left((24+t)m\right) = \frac{(5 - e^{-0.1t})}{4}(24+t)$

$(24+t)m = \frac{1}{4}\int\left(120 + 5t - 24e^{-0.1t} - te^{-0.1t}\right)dt$

$(24+t)m = \frac{1}{4}\left[120t + \frac{5}{2}t^2 + 240e^{-0.1t} - \left(-10te^{-0.1t} + \int 10e^{-0.1t}\,dt\right)\right]$

$(24+t)m = \frac{1}{4}\left[120t + \frac{5}{2}t^2 + 240e^{-0.1t} + 10te^{-0.1t} + 100e^{-0.1t}\right] + c$

$t = 0 \quad m = 0$

$0 = \frac{1}{4}(240 + 100) + c$

$c = -85$

$m = \frac{30t + \frac{5}{8}t^2 + 85e^{-0.1t} + \frac{5}{2}te^{-0.1t} - 85}{24+t}$

c) $t = 30 \quad m = 25.65677$ per litre, 6.75 litres

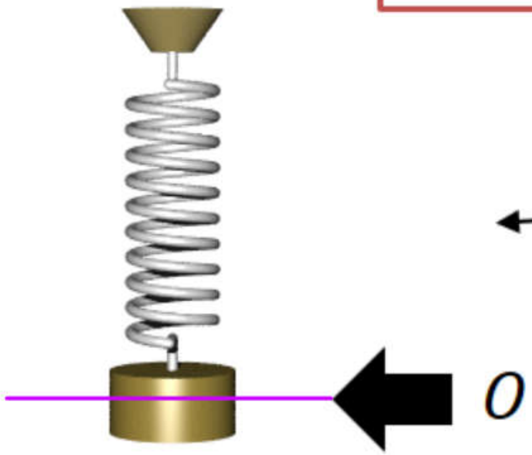
$\frac{25.656\ldots}{6.75} = 3.80$ mg per litre

So this is very close and the model is a good fit.

8(a)(i)	Container contains $3+0.25t-0.125t = 3+0.125t$ litres after t minutes	B1	3.3
(ii)	Rate of contaminant out $=0.125\times\dfrac{m}{3+0.125t}$ mg per minute	M1	3.3
	Rate of contaminant in $= 0.25\times(5-\mathrm{e}^{-0.1t})$ mg per minute	B1	2.2a
	$\dfrac{\mathrm{d}m}{\mathrm{d}t}=\dfrac{5-\mathrm{e}^{-0.1t}}{4}-\dfrac{m}{24+t}$ *	A1*	1.1b
		(4)	
(b)	Rearranges to form $\dfrac{\mathrm{d}m}{\mathrm{d}t}+\dfrac{m}{24+t}=\dfrac{5-\mathrm{e}^{-0.1t}}{4}$ and attempts integrating factor (may be by recognition).	M1	3.1a
	I.F. $=\left(\mathrm{e}^{\int\frac{1}{24+t}\,\mathrm{d}t}=\mathrm{e}^{\ln(24+t)}\right)=24+t$	A1	1.1b
	$(24+t)m=\dfrac{1}{4}\int(24+t)(5-\mathrm{e}^{-0.1t})\,\mathrm{d}t=\dfrac{1}{4}\int 120+5t-24\mathrm{e}^{-0.1t}-t\mathrm{e}^{-0.1t}\,\mathrm{d}t=..$	M1	3.1a
	$=\dfrac{1}{4}\left(120t+\dfrac{5t^2}{2}-\dfrac{24\mathrm{e}^{-0.1t}}{-0.1}+\ldots\right)$	A1	1.1b
	$\int t\mathrm{e}^{-0.1t}\,\mathrm{d}t=t\dfrac{\mathrm{e}^{-0.1t}}{-0.1}-\int 1\times\dfrac{\mathrm{e}^{-0.1t}}{-0.1}\,\mathrm{d}t=t\dfrac{\mathrm{e}^{-0.1t}}{-0.1}-\dfrac{\mathrm{e}^{-0.1t}}{(-0.1)^2}$	M1 A1	1.1b 1.1b
	So $(24+t)m=\dfrac{5}{8}t^2+30t+85\mathrm{e}^{-0.1t}+\dfrac{5}{2}t\mathrm{e}^{-0.1t}+c$		
	When $t=0$, $m=0$ as initially no contaminant in the container, so $0=0+0+85+0+c\Rightarrow c=-85$	M1	3.4
	$m=\dfrac{1}{24+t}\left(\dfrac{5}{8}t^2+30t+85\mathrm{e}^{-0.1t}+\dfrac{5}{2}t\mathrm{e}^{-0.1t}-85\right)$	A1	2.2b
		(8)	
(c)	When $t=30$ $m=25.65677...$ and $V=6.75$, hence the concentration is 3.80 mg per litre.	M1	3.4
	This resembles the measured value very closely and could easily be explained by minor inaccuracies in measurements, so the model seems to be suitable over this timeframe.	A1	3.5a
		(2)	

Simple Harmonic Motion

Simple Harmonic Motion (SHM) is motion in which the acceleration of a particle P is always towards a fixed point O on the line of motion of P. The **acceleration is proportional to the displacement x** of P from O.



We can see that when the particle is moving away from O, it is decelerating, as the acceleration is towards O.

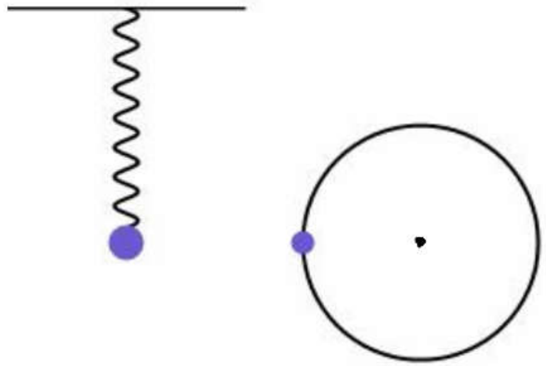
Because of the compression/extension of the spring, as we double the displacement from O, we double the acceleration towards O, i.e. the acceleration is not constant (as it would be if acting under gravity).

O is the **centre of oscillation**.

$\frac{d^2x}{dt^2}$

$$\ddot{x} = -\omega^2 x$$

$a = -kx$



Second derivative of displacement is acceleration (recall the dot notation differentiates with respect to time).

Negative because acceleration acting towards O

ω^2 is constant of proportionality. ω is the **angular velocity** of the particle.

(i.e. the number of oscillations per 2π seconds)

By the chain rule, $\ddot{x} = \frac{dv}{dt} = \frac{dv}{\cancel{dx}} \times \frac{\cancel{dx}}{dt} = v\frac{dv}{dx}$

$a = v\frac{dv}{dx} = \ddot{x}$

$$\boxed{\ddot{x} = v\frac{dv}{dx}}$$

$v\frac{dv}{dx} = -w^2 x$

Also note that $\ddot{x}$ can be written as d^2x/dt^2

A particle P moves with simple harmonic motion about a point O. Given that the maximum displacement of the particle from O is a,

(a) show that $v^2 = \omega^2(a^2 - x^2)$ where v is the velocity of the particle and ω^2 is a constant.

(b) show that $x = a\sin(\omega t + \alpha)$, where α is an arbitrary constant.

SHM $\boxed{\ddot{x} = -w^2 x}$

a)

$$v\frac{dv}{dx} = -w^2 x$$

$$\int v\,dv = \int -w^2 x\,dx$$

$$\frac{1}{2}v^2 = -\frac{1}{2}w^2x^2 + c$$

$$v^2 = -w^2x^2 + 2c$$

max disp. $x = a$, $v = 0$

$$0 = -w^2a^2 + 2c$$

$$2c = w^2a^2$$

$$v^2 = w^2a^2 - w^2x^2$$

$$v^2 = w^2(a^2 - x^2)$$

SHM

$$\boxed{\ddot{x} = -w^2 x}$$

$$\ddot{x} + w^2 x = 0$$

$$\frac{d^2x}{dt^2} + w^2 x = 0$$

A.E. $m^2 + w^2 = 0$

$$m^2 = -w^2$$

$$m = \pm wi$$

G.S.

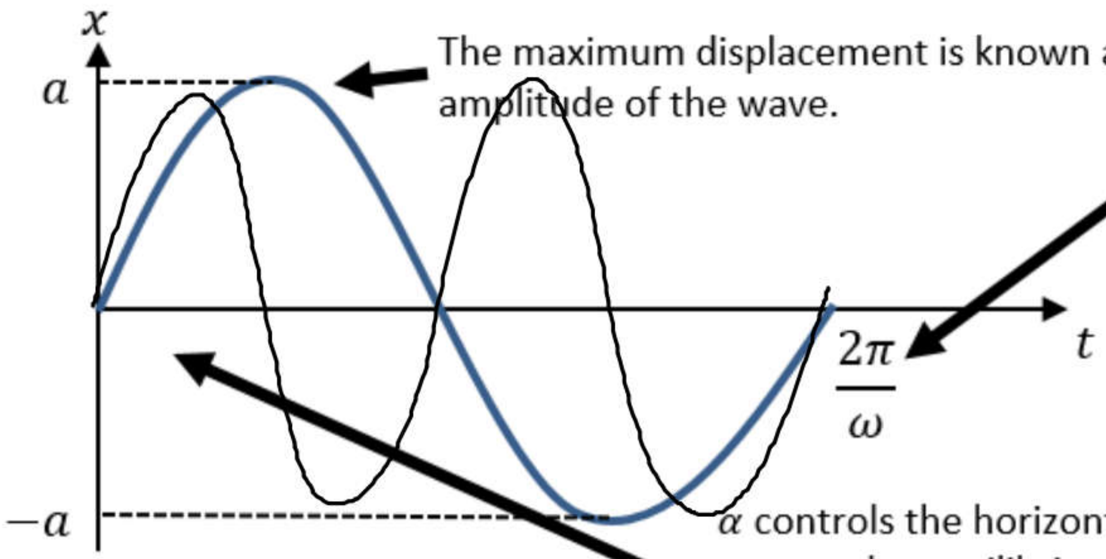
$$x = P\cos wt + Q\sin wt$$

$$x = R\sin(wt + \alpha)$$

But we know max $x = a$

so $x = a\sin(wt + \alpha)$

where is a constant



The maximum displacement is known as the amplitude of the wave.

If $\omega = 1$ then the sin wave would repeat every 2π, i.e. we'd have one oscillation per 2π. Therefore ω gives the number of oscillations per 2π. The **period** is the time for one oscillation.

α controls the horizontal translation. If $\alpha = 0$, we start at the equilibrium position.

stretch $\frac{1}{\omega}$

$$x = a\sin(\underline{\omega t} + \alpha).$$

amplitude.

$$\text{period} = \frac{2\pi}{\omega}$$

A particle is moving along a straight line. At time t seconds its displacement, x m from a fixed point O is such that

$$\frac{d^2x}{dt^2} = -4x$$

$\omega^2 = 4$
$\omega = 2$

Given that at $t = 0, x = 1$ and the particle is moving with velocity 4 ms^{-1},
(a) find an expression for the displacement of the particle after t seconds
(b) hence determine the maximum displacement of the particle from O.

a) $\frac{d^2x}{dt^2} = -4x$

A.E. $m^2 + 4 = 0$
$m = \pm 2i$.

G.S. $x = P\cos 2t + Q\sin 2t$.

$1 = P$

$\frac{dx}{dt} = -2P\sin 2t + 2Q\cos 2t$.

$4 = 2Q$

$Q = 2$

$x = \cos 2t + 2\sin 2t$.

$\frac{dx}{dt} = -2\sin 2t + 4\cos 2t$

b) $x = R\sin(2t + \alpha)$

$R = \sqrt{2^2 + 1^2} = \sqrt{5} = \underline{\underline{2.24\text{ m}}}$.

$v = 0$

$2\sin 2t = 4\cos 2t$
$\tan 2t = 2$
$2t = 1.107$
$t = 0.5535\ldots$

$x = \cos 2t + 2\sin 2t$
$= \underline{2.24\text{ m}}$ (3sf)

period of one oscillation $= \frac{2\pi}{2} = \pi$

A particle P, is attached to the ends of two identical elastic springs. The free ends of the springs are attached to two points A and B. The point C lies between A and B such that ABC is a straight line and $AC \neq BC$.
The particle is held at C and then released from rest.
At time t seconds, the displacement of the particle from C is x m and its velocity is v ms^{-1}. The subsequent motion of the particle can be described by the differential equation $\ddot{x} = -25x$.
(a) Describe the motion of the particle.
Given that $x = 0.4$ and $v = 0$ when $t = 0$,
(b) solve the differential equation to find x as a function of t
(c) state the period of the motion and calculate the maximum speed of P.

a) SHM

b) $\ddot{x} = -25x$

A.E. $m^2 + 25 = 0$

$m = \pm 5i$

G.S. $x = P\cos 5t + Q\sin 5t$.

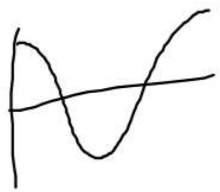
$0.4 = P$

$\frac{dx}{dt} = v = -5P\sin 5t + 5Q\cos 5t$.

$0 = 5Q$

$Q = 0$

P.S. $x = 0.4\cos 5t$



c) period $= \frac{2\pi}{w} = \frac{2}{5}\pi$

$= 1.26$ secs (3sf)

max speed is when $x = 0$

$0 = \cos 5t$

$5t = \frac{\pi}{2}$

$t = \frac{\pi}{10}$

$v = -2\sin 5t$

$v = -2\sin\frac{\pi}{2}$

$v = -2$ max speed is 2.