

# Chapter 8 - Proof by Induction

We will use Proof of Induction for 4 different types of proof:

## 1 Summation Proofs

“Show that  $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$  for all  $n \in \mathbb{N}$ .”

## 2 Divisibility Proofs

“Prove that  $n^3 - 7n + 9$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .”

## 3 Matrix Proofs

“Prove that  $\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$  for all  $n \in \mathbb{Z}^+$ .”

## 4 *Recurrence Relation Proofs*

$$u_{n+2} = 5u_{n+1} - 6u_n \quad n \geq 1$$

Prove by induction that, for  $n \in \mathbb{Z}^+$

$$u_n = 3^n - 2^n$$

**Note:** Recall that  $\mathbb{Z}$  is the set of all integers, and  $\mathbb{Z}^+$  is the set of all positive integers. Thus  $\mathbb{N} = \mathbb{Z}^+$  (where  $\mathbb{N}$  is the set of ‘natural’ numbers).

We can often use **proof by induction** whenever we want to show some property holds for all integers (usually positive) up to infinity.

Show that  $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$  for all  $n \in \mathbb{N}$ .

We could show it's true for certain examples:

$$n = 3 \quad \longrightarrow \quad \begin{aligned} LHS &= 1 + 2 + 3 = 6 \\ RHS &= \frac{1}{2} \times 3 \times 4 = 6 \end{aligned}$$

But what is the problem with just trying some examples?

## How Proof by Induction works...

Case number

$$n = 1$$

$$n = 2$$

$$n = 3$$

$$n = 4$$

...

$$n = k$$

$$n = k + 1$$

$$n = k + 2$$

Step 1: **Base Case**  
Step 2: **Assumption**  
Step 3: **Inductive Case**  
Step 4: **Conclusion**

# Type 1: Summation Proofs

Step 1: <b>Basis:</b>	Prove the general statement is true for $n = 1$ .
Step 2: <b>Assumption:</b>	Assume the general statement is true for $n = k$ .
Step 3: <b>Inductive:</b>	Show that the general statement is then true for $n = k + 1$ .
Step 4: <b>Conclusion:</b>	The general statement is then true for all positive integers $n$ .

Show that  $\sum_{r=1}^n (2r - 1) = n^2$  for all  $n \in \mathbb{N}$ .

## More on the ‘Conclusion Step’

“Since true for  $n = 1$  and if true for  $n = k$ , true for  $n = k + 1$ ,  
 $\therefore$  true for all  $n$ .”

I lifted this straight from a mark scheme, hence use this exact wording!  
The mark scheme specifically says:

(For method mark)

**Any 3 of these seen anywhere in the proof:**

- “true for  $n = 1$ ”
- “assume true for  $n = k$ ”
- “true for  $n = k + 1$ ”
- “true for all  $n$ /positive integers”

Prove by induction that for all positive integers  $n$ ,

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

**Tip:** Write out what you are aiming for

Prove by induction that for all positive integers  $n$ ,

$$\sum_{r=1}^n r2^r = 2(1 + (n-1)2^n)$$

## Your Turn

8. (a) Prove **by induction** that, for any positive integer  $n$ ,

$$\sum_{r=1}^n r^3 = \frac{1}{4} n^2(n+1)^2.$$

(5)

## Type 2a: Divisibility Proofs

Step 1: <b>Basis:</b>	Prove the general statement is true for $n = 1$ .
Step 2: <b>Assumption:</b>	Assume the general statement is true for $n = k$ .
Step 3: <b>Inductive:</b>	Show that the general statement is then true for $n = k + 1$ .
Step 4: <b>Conclusion:</b>	The general statement is then true for all positive integers $n$ .

Prove by induction that  $3^{2n} + 11$  is divisible by 4 for all positive integers  $n$ .

Tip: Use  $f(k + 1) - f(k)$

Prove by induction that  $n^3 - 7n + 9$  is divisible by 3 for all positive integers  $n$ .



## Type 2b: Divisibility Proofs - with a 'twist'

Prove by induction that  $8^n - 3^n$  is divisible by 5.

Prove by induction that for all positive integers  $n$ ,  $11^{n+1} + 12^{2n-1}$  is divisible by 133.

Ex 8B

### Type 3: Matrix Proofs

Prove by induction that  $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 - 2^n \\ 0 & 2^n \end{pmatrix}$  for all  $n \in \mathbb{Z}^+$ .

Prove by induction that  $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$  for all  $n \in \mathbb{Z}^+$ .

## Type 4a: Recurrence Relation Proofs - 1 assumption

Given that  $u_{n+1} = 3u_n + 4$

and that  $u_1 = 1$ , prove by induction that  $u_n = 3^n - 2$

## Type 4b: Recurrence Relation Proofs - 2 assumptions

A sequence of numbers is defined by

$$\begin{aligned} u_1 &= 1 & u_2 &= 5 \\ u_{n+2} &= 5u_{n+1} - 6u_n & n &\geq 1 \end{aligned}$$

Prove by induction that, for  $n \in \mathbb{Z}^+$

$$u_n = 3^n - 2^n$$

(6)

- 1 Given that  $u_{n+1} = 5u_n + 4$ ,  $u_1 = 4$ , prove by induction that  $u_n = 5^n - 1$ .
- 2 Given that  $u_{n+1} = 2u_n + 5$ ,  $u_1 = 3$ , prove by induction that  $u_n = 2^{n+2} - 5$ .
- 3 Given that  $u_{n+1} = 5u_n - 8$ ,  $u_1 = 3$ , prove by induction that  $u_n = 5^{n-1} + 2$ .
- 4 Given that  $u_{n+1} = 3u_n + 1$ ,  $u_1 = 1$ , prove by induction that  $u_n = \frac{3^n - 1}{2}$ .
- 5 Given that  $u_{n+2} = 5u_{n+1} - 6u_n$ ,  $u_1 = 1$ ,  $u_2 = 5$  prove by induction that  $u_n = 3^n - 2^n$ .
- 6 Given that  $u_{n+2} = 6u_{n+1} - 9u_n$ ,  $u_1 = -1$ ,  $u_2 = 0$ , prove by induction that  $u_n = (n-2)3^{n-1}$ .
- 7 Given that  $u_{n+2} = 7u_{n+1} - 10u_n$ ,  $u_1 = 1$ ,  $u_2 = 8$ , prove by induction that  $u_n = 2(5^{n-1}) - 2^{n-1}$ .
- 8 Given that  $u_{n+2} = 6u_{n+1} - 9u_n$ ,  $u_1 = 3$ ,  $u_2 = 36$ , prove by induction that  $u_n = (3n-2)3^n$ .

# Exam Questions



6. (a) Prove by induction that, for all  $n \in \mathbb{Z}^+$

$$f(n) = n^5 + 4n$$

is divisible by 5

(6)

- (b) Show that  $f(-x) = -f(x)$  for all  $x \in \mathbb{R}$

(1)

- (c) Hence prove that  $f(n)$  is divisible by 5 for all  $n \in \mathbb{Z}$

(2)



8. (i) Prove by induction that for  $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix} \quad (6)$$

- (ii) Prove by induction that for  $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)



6. Prove by induction, that for all positive integers  $n$ ,

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

(6)

6. Prove by induction that for all positive integers  $n$

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

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