Your notes

Edexcel A Level Further Maths: Core Pure

6.1 Vector Lines

Contents

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6.1.1 Equations of Lines in 3D

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Equation of a Line in Vector Form

How do I find the vector equation of a line?

- You need to know:
  - The **position vector** of one point on the line
  - A **direction vector** of the line (or the position vector of another point)
- There are two formulas for getting a **vector equation** of a line:
  - **r** = **a** + *t* (**b** - **a**)
    - use this formula when you know the position vectors **a** and **b** of two points on the line
  - **r** = **a** + *t* **d**
    - use this formula when you know the position vector **a** of a point on the line and a direction vector **d**
  - Both forms could be compared to the Cartesian equation of a 2D line
    - $y = mx + c$
    - The point on the line **a** is similar to the "+c" part
    - The direction vector **d** or **b** – **a** is similar to the "m" part
- The vector equation of a line shown above can be applied equally well to vectors in **2 dimensions** and to vectors in **3 dimensions**
- Recall that vectors may be written using $\mathbf{i}, \mathbf{j}, \mathbf{k}$ **reference unit vectors** or as **column vectors**
- It follows that in a vector equation of a line either form can be employed – for example,

$$\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j}) \text{ and } \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

show the same equation written using the two different forms

How do I determine if a point is on a line?

- Each **different point** on the line corresponds to a **different value of** *t*
  - For example: if an equation for a line is **r** = 3**i** + 2**j** - **k** + t (**i** + 2**j**)
    - the point with coordinates (2, 0, -1) is on the line and corresponds to *t* = *-1*
  - However we know that the point with coordinates (-7, 5, 0) is not on this line
    - No value of *t* could make the **k** component 0

Can two different equations represent the same line?

- Why do we say *a* direction vector and not *the* direction vector? Because the magnitude of the vector doesn't matter; **only the direction is important**
  - we can multiply any direction vector by a (non-zero) constant and this wouldn't change the direction

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- Therefore there are an infinite number of options for **a** (a point on the line) and an infinite number of options for the direction vector
- For Cartesian equations – two equations will represent the same line only if they are multiples of each other
  - $x - 2y = 5$ and $3x - 6y = 15$
- For vector equations this is not true – two equations might look different but still represent the same line:
  - $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

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Examiner Tip

- Remember that the vector equation of a line can take many different forms. This means that the answer you derive might look different from the answer in a mark scheme.
- You can choose whether to write your vector equations of lines using reference unit vectors or as column vectors – use the form that you prefer!
- If, for example, an exam question uses column vectors, then it is usual to leave the answer in column vectors, but it isn't essential to do so - you'll still get the marks!

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Worked example

a) Find a vector equation of a straight line through the points with position vectors **a** = 4**i** – 5**k** and **b** = 3**i** - 3**k**

Use the position vectors to find the displacement vector between them.

$$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \Rightarrow \overrightarrow{AB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$\mathbf{r} = \underbrace{\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}}_{\text{position vector of point a}} + t\underbrace{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}_{\text{direction vector}} \quad \text{or} \quad \mathbf{r} = \underbrace{\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}}_{\text{position vector of point b}} + t\underbrace{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}_{\text{direction vector}}$$

$$\boxed{\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}$$

b) Determine whether the point C with coordinate (2, 0, -1) lies on this line.

Let $c = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, then check to see if there exists a value of t such that

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

From the 'i' component : $4 - t = 2$ ①

From the 'j' component : $0 + 0t = 0$ ② (✓) Works for all t

From the 'k' component : $-5 + 2t = -1$ ③

① $\Rightarrow t = 2$ sub into ③ $\Rightarrow -5 + (2 \times 2) = -5 + 4 = -1$ ✓

$\boxed{\text{Point C Lies on the line}}$

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Equation of a Line in Parametric Form

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How do I find the vector equation of a line in parametric form?

- By considering the three separate components of a vector in the *x*, *y* and *z* directions it is possible to write the **vector equation** of a line as **three separate equations**
  - Letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ becomes
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
    - Where $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is a position vector and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is a direction vector
  - This vector equation can then be split into its three separate component forms:
    - $x = a_1 + \lambda b_1$
    - $y = a_2 + \lambda b_2$
    - $z = a_3 + \lambda b_3$

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Worked example

Write the parametric form of the equation of the line which passes through the point (-2, 1, 0) with direction vector $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$.

Use $r = a + \lambda b$ to write the equation in vector form first:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

position vector of a point

direction vector

Separate the components into their 3 separate equations

$$\boxed{\begin{aligned} x &= -2 + 3\lambda \\ y &= 1 + \lambda \\ z &= -4\lambda \end{aligned}}$$

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Equation of a Line in Cartesian Form

- The **Cartesian** equation of a line can be found from the **vector equation of a line** by
  - Finding the vector equation of the line in parametric form
  - Eliminating λ from the parametric equations
    - λ can be eliminated by **making it the subject** of each of the parametric equations
    - For example: $x = x_0 + \lambda l$ gives $\lambda = \dfrac{x - x_0}{l}$
- In **2D** the **cartesian equation of a line** is a regular equation of a straight line simply given in the form
  - $y = mx + c$
  - $ax + by + d = 0$
  - $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ by rearranging $y - y_1 = m(x - x_1)$
- In **3D** the **cartesian equation of a line** also includes *z* and is given in the form
  - $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3} (= \lambda)$
  - where $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- If one of your variables **does not depend on** λ then this part can be written as a separate equation
  - For example: $b_2 = 0 \Rightarrow y = a_2$ gives $\dfrac{x - a_1}{b_1} = \dfrac{z - a_3}{b_3}, y = a_2$

How do I find the vector equation of a line given the Cartesian form?

- If you are given the **Cartesian** equation of a line in the form
  - $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3} (= \lambda)$
- A vector equation of the line can be found by
  - STEP 1: Set each part of the equation equal to λ individually
  - STEP 2: Rearrange each of these three equations (or two if working in 2D) to make *x*, *y*, and *z* the subjects
    - This will give you the three **parametric equations**
    - $x = a_1 + \lambda b_1$
    - $y = a_2 + \lambda b_2$

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- $z = a_3 + \lambda b_3$
- STEP 3: Write this in the vector form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
- STEP 4: Set r to equal $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- If one part of the cartesian equation is given separately and is not in terms of λ then the corresponding component in the direction vector is equal to zero

Worked example

A line has the vector equation $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$. Find the Cartesian equation of the line.

Begin by writing the equation of the line in parametric form:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{ll} x = 1 + 4\lambda & ① \\ y = -2\lambda & ② \\ z = 2 + \lambda & ③ \end{array}$$

Rearrange each equation to make λ the subject:

① $\lambda = \dfrac{x-1}{4}$

② $\lambda = \dfrac{y}{-2}$

③ $\lambda = z - 2$

Set each expression for λ equal to each other:

$$\boxed{\frac{x-1}{4} = \frac{y}{-2} = z - 2}$$

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6.1.2 Pairs of Lines in 3D

Coincident & Parallel Lines

How do I tell if two lines are parallel?

- Two lines are parallel if, and only if, their **direction vectors** are **parallel**
  - This means the direction vectors will be **scalar multiples** of each other
  - For example, the lines whose equations are $\mathbf{r}=\begin{pmatrix}2\\1\\-7\end{pmatrix}+\lambda_1\begin{pmatrix}2\\0\\-8\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\-1\\5\end{pmatrix}+\lambda_2\begin{pmatrix}-1\\0\\4\end{pmatrix}$ are parallel
    - This is because $\begin{pmatrix}2\\0\\-8\end{pmatrix}=-2\begin{pmatrix}-1\\0\\4\end{pmatrix}$

How do I tell if two lines are coincident?

- **Coincident lines** are two lines that lie directly on top of each other
  - They are indistinguishable from each other
- Two parallel lines will either **never intersect** or they are **coincident (identical)**
  - Sometimes the vector equations of the lines may look different
    - for example, the lines represented by the equations $\mathbf{r}=\begin{pmatrix}1\\-8\end{pmatrix}+s\begin{pmatrix}-4\\8\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}-3\\0\end{pmatrix}+t\begin{pmatrix}1\\-2\end{pmatrix}$ are **coincident**,
  - To check whether two lines are **coincident**:
    - First check that they are **parallel**
      - They are because $\begin{pmatrix}-4\\8\end{pmatrix}=-4\begin{pmatrix}1\\-2\end{pmatrix}$ and so their direction vectors are parallel
    - Next, determine whether **any point** on one of the lines also lies on the other
      - $\begin{pmatrix}1\\-8\end{pmatrix}$ is the position vector of a point on the first line and $\begin{pmatrix}1\\-8\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}+4\begin{pmatrix}1\\-2\end{pmatrix}$ so it also lies on the second line
    - If two parallel lines share **any point**, then they share **all points** and are **coincident**

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Intersecting Lines

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How do I determine whether two lines in 3 dimensions intersect?

- If the lines are **not parallel**, check whether they **intersect**:
  - STEP 1: Set the vector equations of the two lines equal to each other with **different variables**
    - e.g. λ and μ, for the parameters
  - STEP 2: Write the three separate equations for the **i**, **j**, and **k** components in terms of λ and μ
  - STEP 3: **Solve** two of the equations to find a value for λ and μ
  - STEP 4: **Check** whether the values of λ and μ you have found satisfy the third equation
    - If **all three** equations are satisfied, then the lines **intersect**

How do I find the point of intersection of two lines?

- If a pair of lines are **not parallel** and **do intersect**, a unique point of intersection can be found
  - If the two lines intersect, there will be a single point that will lie on both lines
- Follow the steps above to find the values of λ and μ that satisfy **all three equations**
  - STEP 5: Substitute either the value of λ or the value of μ into one of the vector equations to find the position vector of the point where the lines intersect
  - It is always a good idea to **check** in the other equations as well, you should get the same point for each line

Examiner Tip

- Make sure that you use different letters, e.g. λ and μ, to represent the parameters in vector equations of different lines
  - Check that the variable you are using has not already been used in the question

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Worked example

Line L_1 has vector equation $\mathbf{r} = \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

Line L_2 has vector equation $\mathbf{r} = \begin{pmatrix} -3 \\ 11 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

a) Show that the lines L_1 and L_2 intersect.

Set L_1 and L_2 equal to each other:

$$\begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Write the three separate equations for the $\underline{i}$, $\underline{j}$ and $\underline{k}$ components:

$$8 + 3\lambda = -3 - \mu \quad (1)$$
$$-1 - \lambda = 11 + 2\mu \quad (2)$$
$$-1 + \lambda = 2 + \mu \quad (3)$$

Solve equations (1) and (2) to find a value for λ and μ

$$8 + 3\lambda = -3 - \mu$$
$$+\,3(-1 - \lambda = 11 + 2\mu)$$
$$5 = 30 + 5\mu$$
$$\mu = -5 \quad \Rightarrow \quad 8 + 3\lambda = -3 - (-5)$$
$$\lambda = -2$$

Check these values of λ and μ in equations (3)

$$-1 + \lambda = 2 + \mu \quad \Rightarrow \quad -1 - 2 = 2 - 5$$
$$-3 = -3$$

$\boxed{\text{All three equations are satisfied, so the lines intersect}}$

b) Find the position vector of the point of intersection.

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Substitute the values of λ and μ found in part (a) into one of the vector equations:

$$\lambda = -2 \Rightarrow \begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

OR

$$\mu = -5 \Rightarrow \begin{pmatrix} -3 \\ 11 \\ 2 \end{pmatrix} - 5\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\boxed{\text{Position vector} = 2\underline{i} + \underline{j} - 3\underline{k}}$$

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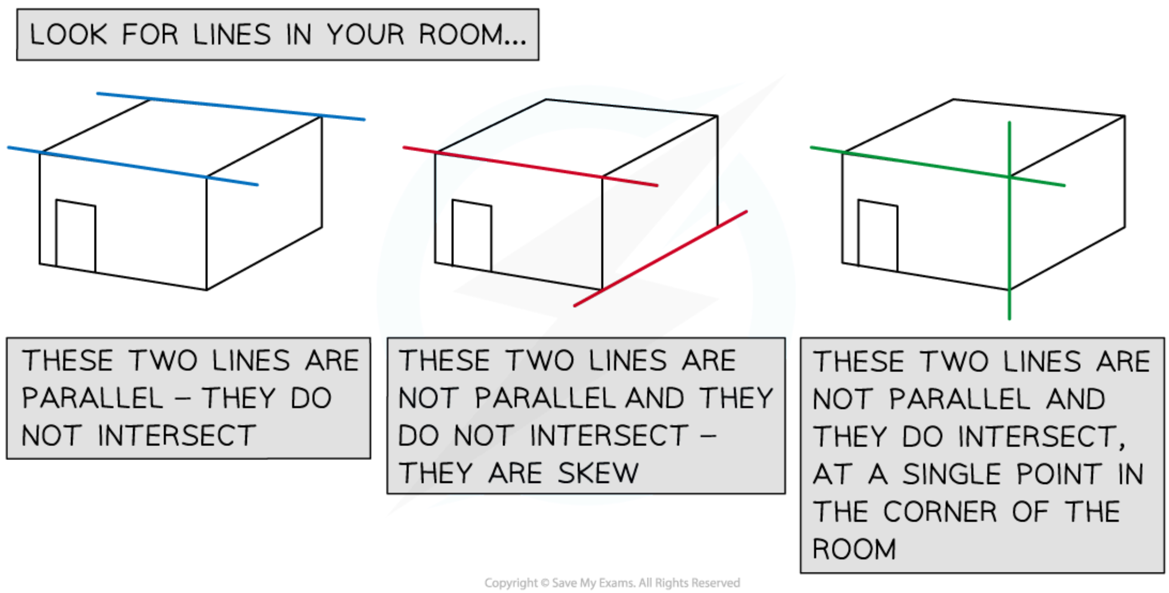
Skew Lines

What are skew lines?

- Lines that are **not parallel** and which **do not intersect** are called **skew lines**
  - This is only possible in **3-dimensions**
- If two lines are skew then there is not a plane in 3D than contains both of the lines



How do I determine whether lines in 3 dimensions are parallel, skew, or intersecting?

- First, look to see if the direction vectors are parallel:
  - if the **direction vectors are parallel**, then the **lines are parallel**
  - if the **direction vectors are not parallel**, the **lines are not parallel**
- If the lines are **parallel**, check to see if the lines are **coincident**:
  - If they **share any point**, then they are **coincident**
  - If **any point** on one line is **not on the other line**, then the lines are **not coincident**
- If the lines are **not parallel**, check whether they **intersect**:
  - STEP 1: Set the vector equations of the two lines equal to each other with **different variables**
    - e.g. λ and μ, for the parameters
  - STEP 2: Write the three separate equations for the **i**, **j**, and **k** components in terms of λ and μ
  - STEP 3: **Solve** two of the equations to find a value for λ and μ
  - STEP 4: **Check** whether the values of λ and μ you have found satisfy the third equation
    - If **all three** equations are satisfied, then the lines **intersect**
    - If **not all three** equations are satisfied, then the lines are **skew**

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Worked example

Determine whether the following pair of lines are parallel, intersect, or are skew.

$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + s(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $\boldsymbol{r} = -5\mathbf{i} + 4\mathbf{j} + \mathbf{k} + t(2\mathbf{i} - \mathbf{j})$.

STEP 1: Check to see if the lines are parallel:

$$r_1 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \qquad r = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

direction vectors

The lines are not parallel because there is no value of k such that $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = k \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

STEP 2: Check to see if the lines intersect:

$4 + 5\lambda = -5 + 2\mu$ ①

$3 + 2\lambda = 4 - \mu$ ②

$3\lambda = 1$ ③

Set up three equations for each of the $\underline{i}$, $\underline{j}$ and $\underline{k}$ components.

Equation ③: $\lambda = \frac{1}{3}$ Sub into ②: $3 + 2\left(\frac{1}{3}\right) = 4 - \mu$

$$\frac{11}{3} = 4 - \mu$$

$$\mu = \frac{1}{3}$$

Sub into ①: $4 + 5\left(\frac{1}{3}\right) = -5 + 2\left(\frac{1}{3}\right)$

$\frac{17}{3} \neq -\frac{13}{3}$ contradiction

There is no point of intersection.

$\boxed{\text{The lines are skew}}$

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6.1.3 Angle between Lines

Scalar Product

The **scalar product** is an important link between the algebra of vectors and the trigonometry of vectors. We shall see that the scalar product is somewhat comparable to the operation of multiplication on real numbers.

What is the scalar (dot) product?

- The scalar product between two vectors **a** and **b** is represented by $\mathbf{a} \cdot \mathbf{b}$
  - This is also called the dot product because of the symbol used
- The scalar product between two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- The result of taking the scalar product of two vectors is a **real number**
  - i.e. a **scalar**
- For example,

$$(3\mathbf{i} - \mathbf{k}) \cdot (2\mathbf{i} + 9\mathbf{j} + \mathbf{k}) = 3 \times 2 + 0 \times 9 + (-1) \times 1 = 6 + 0 - 1 = 5$$

and

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 2 \end{pmatrix} = 2 \times (-8) + 7 \times 2 = -16 + 14 = -2$$

- The scalar product has some important properties:
  - The order of the vectors doesn't affect the result:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- In effect we can 'multiply out' brackets:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

- This means that we can do many of the same things with vectors as we can do when operating on real numbers – for example,

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$$

- The scalar product between a vector and itself is equal to the square of its **magnitude:**

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

For example,

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 7 \end{pmatrix} = 2^2 + 7^2 = 53 \text{ and } \left|\begin{pmatrix} 2 \\ 7 \end{pmatrix}\right|^2 = 2^2 + 7^2 = 53$$

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What is the connection between the scalar product and trigonometry?

- There is another important method for finding $\mathbf{a} \cdot \mathbf{b}$ involving the angle between the two vectors θ:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

  - Here θ is the angle between the vectors when they are placed **'base to base'**
    - when the vectors are placed so that they begin at the same point
  - This formula can be derived using the cosine rule and expanding $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$
- The scalar product of two vectors gives information about the angle between the two vectors
  - If the scalar product is **positive** then the angle between the two vectors is **acute** (less than 90°)
  - If the scalar product is **negative** then the angle between the two vectors is **obtuse** (between 90° and 180°)
  - If the scalar product is **zero** then the angle between the two vectors is **90°** (the two vectors are **perpendicular**)

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How do I tell if vectors or lines are perpendicular?

- Two (non-zero) vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** if, and only if, $\mathbf{a} \cdot \mathbf{b} = 0$
  - If the **a** and **b** are perpendicular then:
    - $\theta = 90° \Rightarrow \cos\theta = 0 \Rightarrow |\mathbf{a}||\mathbf{b}|\cos\theta = 0 \Rightarrow \mathbf{a} \cdot \mathbf{b} = 0$
  - If $\mathbf{a} \cdot \mathbf{b} = 0$ then:
    - $|\mathbf{a}||\mathbf{b}|\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90° \Rightarrow$ **a** and **b** are perpendicular
  - For example, the vectors $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $-4\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular since

$$(2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) \cdot (-4\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2 \times (-4) + (-3) \times (-1) + 5 \times 1 = -8 + 3 + 5 = 0$$

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Examiner Tip

- When writing a scalar product, it's important to write a distinctive **dot** between the vectors – otherwise your meaning will not be clear.

Worked example

Find the value of *t* such that the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = (t-1)\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular to each other.

The two vectors $\underline{v}$ and $\underline{w}$ are perpendicular if $\underline{v} \cdot \underline{w} = 0$.

$\underline{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} t-1 \\ -1 \\ 1 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = 2(t-1) + t(-1) + 5(1)$

$= 2t - 2 - t + 5$

Therefore $\underline{v}$ and $\underline{w}$ are perpendicular if

$t + 3 = 0$

$\boxed{t = -3}$

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Angle between Lines

How do I find the angle between two vectors?

- Recall that a formula for the scalar (or 'dot') between vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

  - where θ is the angle between the vectors when they are placed **'base to base'**
    - that is, when the vectors are positioned so that they start at the same point
  - We arrange this formula to make $\cos\,\theta$ the subject:
  - To find the angle between two vectors
    - Calculate the scalar product between them
    - Calculate the magnitude of each vector
    - Use the formula to find $\cos\,\theta$
    - Use inverse trig to find θ

How do I find the angle between two lines?

- To find the angle between two lines, find the angle between their **direction vectors**
  - For example, if the lines have equations $\mathbf{r} = \mathbf{a}_1 + s\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + t\mathbf{d}_2$, then the angle θ between the lines is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}\right)$$

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Your notes

Worked example

Calculate the angle formed by the two vectors $\boldsymbol{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$\underline{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\underline{w} = 3\underline{i} + 4\underline{j} - \underline{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Start by finding the scalar product:

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$$

$$= (-1 \times 3) + (3 \times 4) + (2 \times -1) = 7$$

Find the magnitude of both vectors:

$$|\underline{v}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$$

$$|\underline{w}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$$

$$\cos\theta = \frac{7}{\sqrt{14} \times \sqrt{26}} = 0.3668...$$

$$\theta = \cos^{-1}(0.3668...)$$

$$\boxed{\theta = 68.5^\circ \ (3sf)}$$

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6.1.4 Shortest Distances – Lines

Your notes

Shortest Distance between a Point & a Line

How do I find the shortest distance from a point to a line?

- The shortest distance from any point to a line will always be the **perpendicular** distance
  - Given a line *l* with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a point *P* not on *l*
  - The **scalar product** of the direction vector, **b**, and the vector in the direction of the **shortest distance** will be zero
- The shortest distance can be found using the following steps:
  - STEP 1: Let the vector equation of the line be ***r*** and the point not on the line be *P*, then the point on the line closest to *P* will be the point *F*
    - The point *F* is sometimes called the foot of the perpendicular
  - STEP 2: Sketch a diagram showing the line *l* and the points *P* and *F*
    - The vector $\overrightarrow{FP}$ will be **perpendicular** to the line *l*
  - STEP 3: Use the equation of the line to find the position vector of the point *F* in terms of λ
  - STEP 4: Use this to find the displacement vector $\overrightarrow{FP}$ in terms of λ
  - STEP 5: The scalar product of the direction vector of the line *l* and the displacement vector $\overrightarrow{FP}$ will be zero
    - Form an equation $\overrightarrow{FP} \cdot \mathbf{b} = 0$ and solve to find λ
  - STEP 6: Substitute λ into $\overrightarrow{FP}$ and find the magnitude $\left|\overrightarrow{FP}\right|$
    - The shortest distance from the point to the line will be the magnitude of $\overrightarrow{FP}$
- Note that the shortest distance between the point and the line is sometimes referred to as the **length of the perpendicular**

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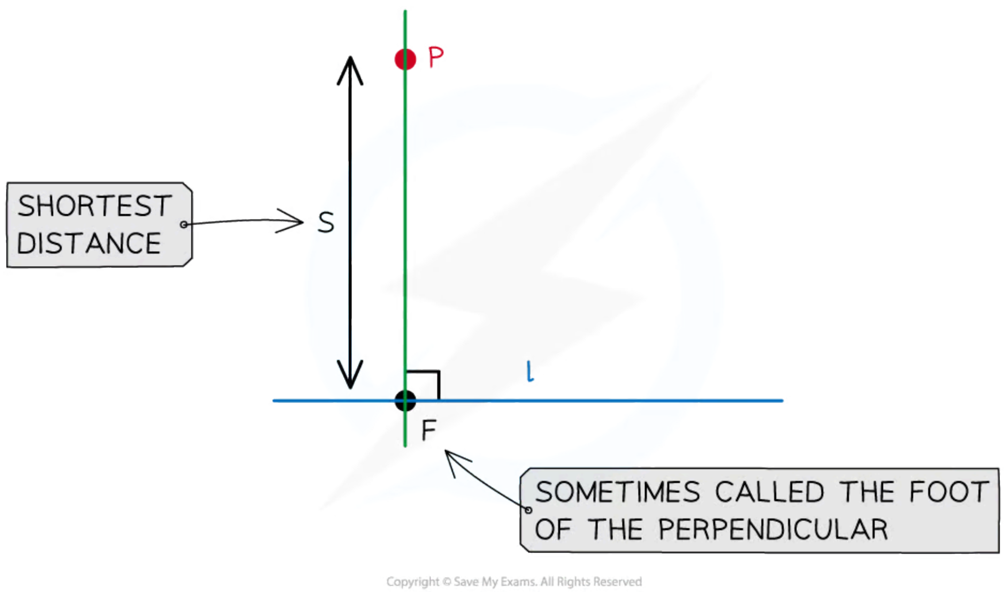

Your notes

Examiner Tip

- Column vectors can be easier and clearer to work with when dealing with scalar products.

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Your notes

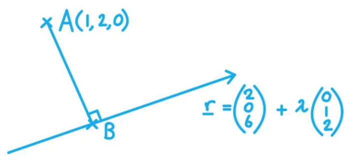
Worked example

Point A has coordinates (1, 2, 0) and the line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Find the shortest distance from A to the line l.

B is on l so can be written in terms of λ:

$$\overrightarrow{OB} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix}$$



Find $\overrightarrow{AB}$ using $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda-2 \\ 6+2\lambda \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l: $\overrightarrow{AB} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 \\ \lambda-2 \\ 6+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\lambda - 2 + 2(6+2\lambda) = 0$$

$$5\lambda + 10 = 0$$

$$\lambda = -2$$

Substitute back into $\overrightarrow{AB}$ and find the magnitude:

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2-2 \\ 6+2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$$

$$\boxed{\text{Shortest distance } = \sqrt{21} \text{ units}}$$

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Your notes

Shortest Distance between two Lines

How do we find the shortest distance between two parallel lines?

- Two **parallel** lines will never intersect
- The shortest distance between two **parallel lines** will be the **perpendicular distance** between them
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and a line l_2 with equation $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ then the shortest distance between them can be found using the following steps:
  - Remember the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are scalar multiples of each other and so either can be used here
  - STEP 1: Find the vector between $\mathbf{a}_1$ and a general coordinate from l_2 in terms of μ
  - STEP 2: Set the scalar product of the vector found in STEP 1 and the direction vector $\mathbf{d}_1$ equal to zero
  - STEP 3: Form and solve an equation to find the value of μ
  - STEP 4: Substitute the value of μ back into the equation for l_2 to find the coordinate on l_2 closest to l_1
  - STEP 5: Find the distance between $\mathbf{a}_1$ and the coordinate found in STEP 4

How do we find the shortest distance from a given point on a line to another line?

- The shortest distance from any point on a line to another line will be the **perpendicular** distance from the point to the line
- If the angle between the two lines is known or can be found then right-angled trigonometry can be used to find the perpendicular distance
- Alternatively, the equation of the line can be used to find a general coordinate and the steps above can be followed to find the shortest distance

How do we find the shortest distance between two skew lines?

- Two **skew** lines are not parallel but will never intersect
- The shortest distance between two **skew lines** will be perpendicular to **both** of the lines
- To find the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$,
  - STEP 1: Find position vectors for the points on each line that form the shortest distance
    - Point P has position vector $\mathbf{p} = \mathbf{a}_1 + \lambda\mathbf{d}_1$
    - Point Q has position vector $\mathbf{q} = \mathbf{a}_2 + \mu\mathbf{d}_2$
  - STEP 2: Find the displacement vector between P and Q
    - $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$
  - STEP 3: Form two equations by using the fact that the scalar product of the displacement vector and the direction vector of each line should equal zero
    - $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{d}_1 = 0$

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Your notes

- $(\mathbf{q}-\mathbf{p}) \cdot \mathbf{d}_2 = 0$
- STEP 4: Solve the two equations simultaneously to find the values of λ and μ
- STEP 5: Substitute the values of λ and μ into the displacement vector and take the magnitude
    - Shortest distance = $|\mathbf{q}-\mathbf{p}|$

Examiner Tip

- Exam questions will often ask for the shortest, or minimum, distance within vector questions
- If you're unsure start by sketching a quick diagram
- Sometimes calculus can be used, however usually vector methods are required

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Your notes

Worked example

Consider the skew lines l_1 and l_2 as defined by:

$$l_1 : \boldsymbol{r} = \begin{pmatrix} 6 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$l_2 : \boldsymbol{r} = \begin{pmatrix} -5 \\ 4 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Find the minimum distance between the two lines.

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Your notes

Find position vectors for general points, P and Q, on both lines:

$$\overrightarrow{OP} = \begin{pmatrix} 6+2\lambda \\ -4-3\lambda \\ 3+4\lambda \end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix} -5-\mu \\ 4+2\mu \\ -8+\mu \end{pmatrix}$$

Find the displacement vector between P and Q

$$\overrightarrow{PQ} = \begin{pmatrix} -5-\mu \\ 4+2\mu \\ -8+\mu \end{pmatrix} - \begin{pmatrix} 6+2\lambda \\ -4-3\lambda \\ 3+4\lambda \end{pmatrix} = \begin{pmatrix} -11-\mu-2\lambda \\ 8+2\mu+3\lambda \\ -11+\mu-4\lambda \end{pmatrix}$$

The scalar product of $\overrightarrow{PQ}$ and the direction vector of each line is zero:

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -11-\mu-2\lambda \\ 8+2\mu+3\lambda \\ -11+\mu-4\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 0$$

$$\Rightarrow 2(-11-\mu-2\lambda) - 3(8+2\mu+3\lambda) + 4(-11+\mu-4\lambda) = 0$$

$$-90 - 4\mu - 29\lambda = 0 \quad (1)$$

$$\overrightarrow{PQ} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -11-\mu-2\lambda \\ 8+2\mu+3\lambda \\ -11+\mu-4\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow -(-11-\mu-2\lambda) + 2(8+2\mu+3\lambda) + (-11+\mu-4\lambda) = 0$$

$$16 + 6\mu + 4\lambda = 0 \quad (2)$$

Solve the simultaneous equations to find λ and μ:

$$\lambda = -\frac{238}{79} \qquad \mu = -\frac{52}{79}$$

Substitute back into $\overrightarrow{PQ}$ and find the magnitude:

$$\left| \begin{pmatrix} -11-\left(-\frac{52}{79}\right) - 2\left(-\frac{238}{79}\right) \\ 8+2\left(-\frac{52}{79}\right) + 3\left(-\frac{238}{79}\right) \\ -11+\left(-\frac{52}{79}\right) - 4\left(-\frac{238}{79}\right) \end{pmatrix} \right| = \left| \begin{pmatrix} -\frac{341}{79} \\ -\frac{186}{79} \\ \frac{31}{79} \end{pmatrix} \right| = \sqrt{\left(-\frac{341}{79}\right)^2 + \left(-\frac{186}{79}\right)^2 + \left(\frac{31}{79}\right)^2}$$

$$\boxed{\text{Shortest distance} = 4.93 \text{ units (3 s.f.)}}$$

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