

Linear Transformations with Matrices

1:: Use of matrices to represent linear transformations.

"Determine the matrix that represents the transformation $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2x + y \\ -x \end{pmatrix}$ "

2:: Use matrices to represent reflections, rotations (about the origin) and enlargements.

"Describe the geometrical transformation represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ "

3:: Carry out successive transformations using matrix products.

"If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ describe the transformation represented by the matrix AB ."

4:: Use inverse matrices to represent reverse transformations.

A matrix $\begin{pmatrix} 4 & 0 \\ -1 & 2 \end{pmatrix}$ is used to transform a point $A(x, y)$ to $B(5, 5)$. Determine the point $A(x, y)$.

Linear Transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$

This chapter is concerned with how we can **use matrices to represent some transformation of a point (x, y)** (written as a position vector $\begin{pmatrix} x \\ y \end{pmatrix}$).

Transforming a point $\begin{pmatrix} x \\ y \end{pmatrix}$ simply involves multiplying it by some matrix. From above we can see that multiplying by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the mapping $T: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. We will see how we can use certain matrices to represent certain well-known transformations, e.g. $(x, y) \rightarrow (3x, 3y)$, i.e. an enlargement of scale factor 3 centred about the origin.

$ax + by$ is known as a **linear combination** of x and y (an algebraic form we saw in Pure Year 1 straight line equations).

Each row of the matrix we're multiplying by provides an instruction of how to generate each dimension of the new coordinate system, in terms of the old dimensions x, y ...

e.g. given $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x + y \end{pmatrix}$, the new x value is $2x + 3y$ and the new y value is $x + y$, i.e. linear combinations of the old x and y values.

A function $f(\mathbf{a})$, where \mathbf{a} is a vector, is linear if it has the following properties:

- $f(k\mathbf{a}) = kf(\mathbf{a})$ for a constant k , i.e. scaling the original vector scales the image vector.
- $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$

It is possible to prove that $f\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = ax + by$ is linear, i.e. satisfies the above restrictions.

- 1 We can represent a translation, e.g.

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x+4 \\ y \end{pmatrix} \text{ using a matrix.}$$

True

False

- 2 Matrices can represent transformations which increase or decrease the number of dimensions (e.g. transform a 3D point to get a 2D point).

True

False

- 3 The origin is unaffected by any linear transformation.

True

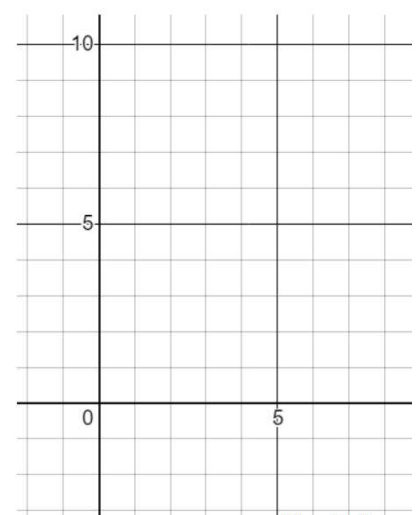
False

Find matrices to represent these linear transformations.

a) $T: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2y+x \\ 3x \end{pmatrix}$

b) $V: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -2y \\ 3x+y \end{pmatrix}$

A square has coordinates (1,1), (3,1), (3,3) and (1,3). Find the vertices of the image of S under the transformation given by the matrix $M = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Sketch S and the image of S on a coordinate grid.



Ex 7A

Determining a matrix for a transformation

Recall from vectors that $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the unit vectors representing the x and y directions. Consider what happens to each when we multiply by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Just For Your Interest: \mathbf{i} and \mathbf{j} are known as the *basis vectors* of the 2D coordinate space because any 2D point can be represented as a linear combination of these basis vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

What can we conclude about the columns of a matrix?

“Find a 2×2 matrix that represents a reflection in the y -axis.”

Rotation 90° about the origin.

Note: Rotations by default are anticlockwise.

Rotation θ about the origin.

Your Turn

Find the matrix representing a reflection in the line $y = x$.

Find the matrix representing a rotation by 270° .

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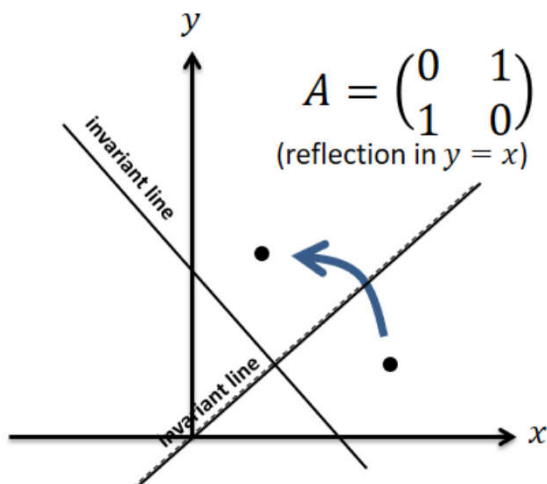
$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Describe fully the transformations described by matrix \mathbf{C} .

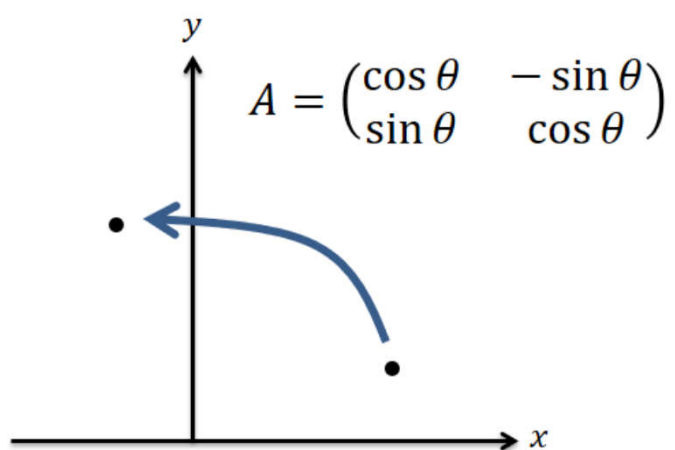
Invariant points and lines

An **invariant point** is one which is unaffected by a transformation.

An **invariant line** is when each point on the line transformed to give another point on the same line.



Invariant lines: $y = x$
 $y = -x + k$
 (each line reflects to give the same)



Invariant line: None! (unless $\theta = 180^\circ$; any straight line through origin will be invariant)

Invariant point: $(0, 0)$

3.

$$\mathbf{P} = \begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}$$

The matrix \mathbf{P} represents a linear transformation, T , of the plane.

(a) Describe the invariant points of the transformation T .

(3)

(b) Describe the invariant lines of the transformation T .

(6)

3.

$$\mathbf{P} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

The matrix \mathbf{P} represents a linear transformation, T , of the plane.

(a) Describe the invariant point of the transformation T .


(b) Describe the invariant lines of the transformation T .

(6)

Enlargements

Describe the effect of the following matrices.

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

 $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ represents a stretch scale factor a parallel to the x -axis and a stretch scale factor b parallel to the y -axis. When $a = b$ this represents an enlargement.

Using $\det(A)$

$A(1,1), B(1,2), C(2,2)$ are points on a triangle. The transformation with matrix $\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ is applied to the triangle to produce a new triangle with vertices A', B' and C' .

- (a) Determine the coordinates of A', B', C' .
- (b) What is the area of triangle ABC ?
- (c) What is the area of triangle $A'B'C'$?
- (d) Determine $\det(\mathbf{M})$. What do you notice?

Area scale factor

We saw in this example that:

 $\text{Area of image} = \text{Area of object} \times |\det(\mathbf{M})|$

i.e. the determinant tells us how the area is scaled under the transformation with matrix \mathbf{M} .

(The proof of this is not covered here)

Area of Object	Transformation Matrix	Area of Image
4	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
3	$\begin{pmatrix} 2 & 0 \\ 9 & 4 \end{pmatrix}$	
9	$\begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}$	
1	$\begin{pmatrix} -5 & 2 \\ -4 & -2 \end{pmatrix}$	

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$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(a) Find $\det \mathbf{A}$.

(1)

The triangle R is transformed to the triangle S by the matrix \mathbf{A} .
Given that the area of triangle S is 72 square units,

(c) find the area of triangle R .

(2)

More invariant points

1.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix \mathbf{M} is non-singular.

(2)

The transformation T of the plane is represented by the matrix \mathbf{M} .

The triangle R is transformed to the triangle S by the transformation T .

Given that the area of S is 63 square units,

(b) find the area of R .

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation T .

(2)

Combined Transformations

We know that for a position vector \mathbf{x} and a matrix \mathbf{A} representing some transformation, then \mathbf{Ax} is the transformed point.

If we wanted to apply a transformation represented by a matrix \mathbf{A} followed by another represented by \mathbf{B} , what transformation matrix do we use to represent the combined transformation?

This is because to apply the effect of \mathbf{A} followed by \mathbf{B} , we have:

$$\mathbf{B}(\mathbf{Ax}) = (\mathbf{BA})\mathbf{x}$$

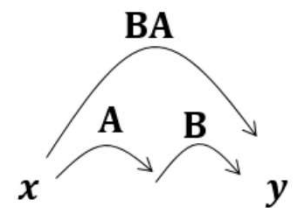
(because matrix multiplication is 'associative'*)

* A binary operator \otimes is **associative** if $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, i.e. when we multiply matrices, the order in which we multiply them doesn't matter.

Similarly addition on real numbers is associative, e.g. $1 + (2 + 3) = (1 + 2) + 3$.

However subtraction and division are not, e.g. $(16 \div 2) \div 8 \neq 16 \div (2 \div 8)$.

Tip: Ensure that you put these matrices in the right order – the first that gets applied is on the right!



Represent as a single matrix the transformation representing a reflection in the line $y = x$ followed by a stretch on the x axis by a factor of 4.

Represent as a single matrix the transformation representing a rotation 90° anticlockwise about the point $(0,0)$ followed by a reflection in the line $y = x$.

What single transformation is this?

The transformation U , represented by the 2×2 matrix \mathbf{P} , is a rotation through 90° anticlockwise about the origin.

- (a) Write down the matrix \mathbf{P} . (1)

The transformation V , represented by the 2×2 matrix \mathbf{Q} , is a reflection in the line $y = -x$.

- (b) Write down the matrix \mathbf{Q} . (1)

Given that U followed by V is transformation T , which is represented by the matrix \mathbf{R} ,

- (c) express \mathbf{R} in terms of \mathbf{P} and \mathbf{Q} , (1)
(d) find the matrix \mathbf{R} , (2)
(e) give a full geometrical description of T as a single transformation. (2)

More invariant points!

5.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

- (a) Describe fully the single geometrical transformation U represented by the matrix \mathbf{A} . (3)

The transformation V , represented by the 2×2 matrix \mathbf{B} , is a reflection in the line $y = -x$

- (b) Write down the matrix \mathbf{B} . (1)

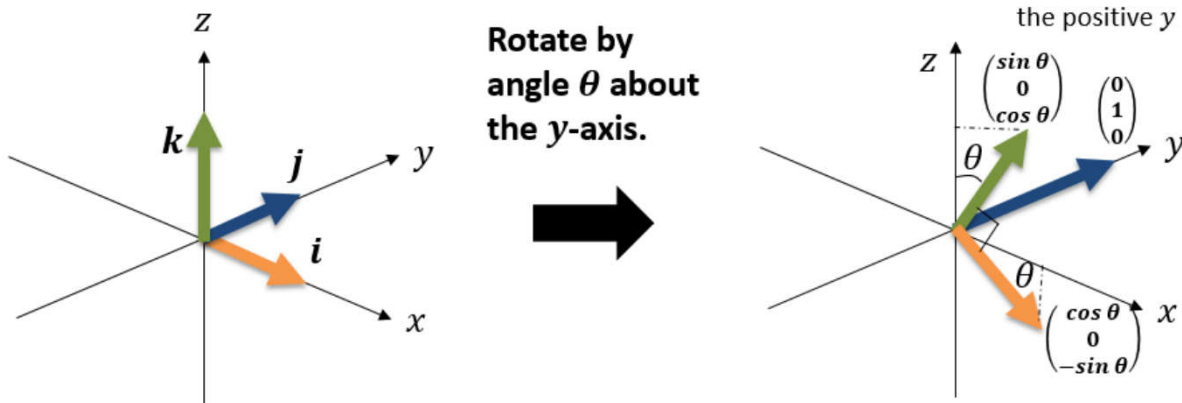
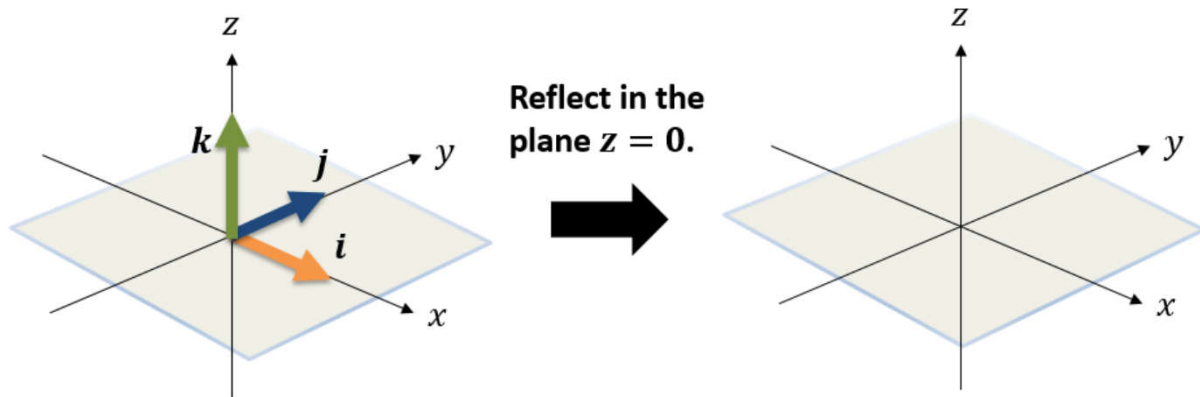
Given that U followed by V is the transformation T , which is represented by the matrix \mathbf{C} ,

- (c) find the matrix \mathbf{C} . (2)


- (d) Show that there is a real number k for which the point $(1, k)$ is invariant under T . (4)

Linear transformations in 3D

We saw earlier that we could determine the matrix corresponding to a transformation by transforming each of the unit vectors (i.e. the axes) and using these as the columns of the matrix. This works in 3D too!



Reminder: The rotation is anticlockwise relative to the positive y axis.

 Rotation θ about x-axis: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$

Rotation θ about y-axis: $\begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$

Rotation θ about z-axis: $\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

Tip: You can tell whether it's a rotation in the x, y or z axes by looking whether the 1 is in the 1st, 2nd or 3rd row/column.

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Ex 7E

- (a) Describe the transformation represented by \mathbf{M} .
(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by \mathbf{M} .

Inverse matrices for inverse transformations

 Suppose \mathbf{x} and \mathbf{y} are column vectors. Then if $\mathbf{Ax} = \mathbf{y}$, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$.

The inverse matrix therefore allows us to retrieve the original point/position vector before a transformation.

The triangle T has vertices at A , B and C . The matrix $M = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$ transforms T to the triangle T' with vertices at $A'(4,3)$, $B'(4,10)$ and $C'(-4, -3)$. Determine the coordinates of A , B and C .

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}.$$

- (a) Find $\det \mathbf{M}$. (1)

Given that

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- (e) describe fully the single geometrical transformation represented by \mathbf{A} . (2)

The transformation represented by \mathbf{A} followed by the transformation represented by \mathbf{B} is equivalent to the transformation represented by \mathbf{M} .

- (f) Find \mathbf{B} . (4)

Tip: If $\mathbf{M} = \mathbf{BA}$, make sure you multiply the end of each by \mathbf{A}^{-1} :

$$\mathbf{MA}^{-1} = \mathbf{BAA}^{-1}$$

$$\mathbf{MA}^{-1} = \mathbf{BI} = \mathbf{B}$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299	300	301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320	321	322	323	324	325	326	327	328	329	330	331	332	333	334	335	336	337	338	339	340	341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360	361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400	401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419	420	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440	441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460	461	462	463	464	465	466	467	468	469	470	471	472	473	474	475	476	477	478	479	480	481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500	501	502	503	504	505	506	507	508	509	510	511	512	513	514	515	516	517	518	519	520	521	522	523	524	525	526	527	528	529	530	531	532	533	534	535	536	537	538	539	540	541	542	543	544	545	546	547	548	549	550	551	552	553	554	555	556	557	558	559	560	561	562	563	564	565	566	567	568	569	570	571	572	573	574	575	576	577	578	579	580	581	582	583	584	585	586	587	588	589	590	591	592	593	594	595	596	597	598	599	600	601	602	603	604	605	606	607	608	609	610	611	612	613	614	615	616	617	618	619	620	621	622	623	624	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639	640	641	642	643	644	645	646	647	648	649	650	651	652	653	654	655	656	657	658	659	660	661	662	663	664	665	666	667	668	669	670	671	672	673	674	675	676	677	678	679	680	681	682	683	684	685	686	687	688	689	690	691	692	693	694	695	696	697	698	699	700	701	702	703	704	705	706	707	708	709	710	711	712	713	714	715	716	717	718	719	720	721	722	723	724	725	726	727	728	729	730	731	732	733	734	735	736	737	738	739	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756	757	758	759	760	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779	780	781	782	783	784	785	786	787	788	789	790	791	792	793	794	795	796	797	798	799	800	801	802	803	804	805	806	807	808	809	810	811	812	813	814	815	816	817	818	819	820	821	822	823	824	825	826	827	828	829	830	831	832	833	834	835	836	837	838	839	840	841	842	843	844	845	846	847	848	849	850	851	852	853	854	855	856	857	858	859	860	861	862	863	864	865	866	867	868	869	870	871	872	873	874	875	876	877	878	879	880	881	882	883	884	885	886	887	888	889	890	891	892	893	894	895	896	897	898	899	900	901	902	903	904	905	906	907	908	909	910	911	912	913	914	915	916	917	918	919	920	921	922	923	924	925	926	927	928	929	930	931	932	933	934	935	936	937	938	939	940	941	942	943	944	945	946	947	948	949	950	951	952	953	954	955	956	957	958	959	960	961	962	963	964	965	966	967	968	969	970	971	972	973	974	975	976	977	978	979	980	981	982	983	984	985	986	987	988	989	990	991	992	993	994	995	996	997	998	999	1000
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