Methods in Calculus (Chapter 3/Chapter 6)

In this chapter, we explore a variety of new techniques for integration, as well as how integration can be applied.

1:: Improper Integrals

"Evaluate $\int_1^{\infty} \frac{1}{x^2}\, dx$ or show that it is not convergent."

2:: Mean value of a function

"Find the mean value of $f(x) = \frac{4}{\sqrt{2+3x}}$ over the interval $[2,6]$."

3:: Differentiating and integrating inverse trigonometric functions

"Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$"

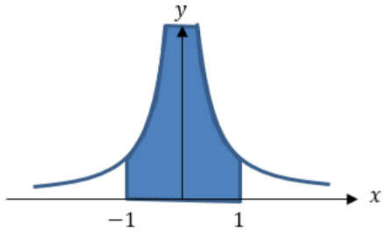
4:: Integrating using partial fractions.

"Show that $\int \frac{1+x}{x^3+9x}\, dx = A \ln\left(\frac{x^2}{x^2+9}\right) + B \arctan\left(\frac{x}{3}\right) + c$"

Improper Integrals

STARTER 1: Determine $\int_{-1}^{1} \frac{1}{x^2}\, dx$. Is there an issue?

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{1} x^{-2}\, dx = [-x^{-1}]_{-1}^{1}$$
$$= \left(-\frac{1}{1}\right) - \left(-\frac{1}{-1}\right) = -1 - \ +1 = -2$$



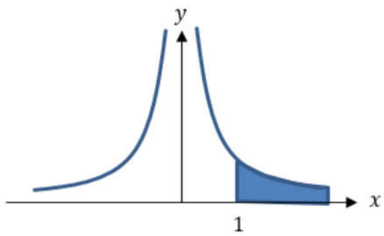
What's odd is the we ended up with a negative value. But the whole graph is above the x axis! The problem is related to integrating over a discontinuity, i.e. the function is not defined for the entire interval $[-1,1]$, notably where $x = 0$. We will see when this causes and issue and when it does not.
We say the definite integral does not exist.

STARTER 2: Determine $\int_{1}^{\infty} \frac{1}{x^2}\, dx$. Is there an issue?

(Note: **the below is seriously dodgy maths** as we're not allowed to use ∞ in calculations – we'll look at the proper way to write this in a sec)

$$[-x^{-1}]_1^{\infty} = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{1}\right) = 0 + 1 = 1$$

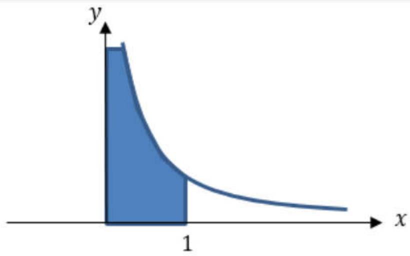
Although the graph is extending to infinity, the **area is finite** because the y values converge towards 0. The result is therefore valid this time. This is an example of an **improper integral** and because the value converged, **we say the definite integral exists**.



STARTER 3: Determine $\int_{0}^{1} \frac{1}{\sqrt{x}}\, dx$. Is there an issue?

$$\int_{0}^{1} \frac{1}{\sqrt{x}}\, dx = \int_{0}^{1} x^{-\frac{1}{2}}\, dx = [2\sqrt{x}]_0^1$$
$$= 2\sqrt{1} - 2\sqrt{0} = 2$$

This is similar to the second example. Although $y \to \infty$ as $x \to 0$ (and not defined when $x = 0$), the area is convergent and therefore finite. The result of 2 is therefore valid.



The integral $\int_a^b f(x)\, dx$ is improper if either:

- One or both of the limits is infinite
- $f(x)$ is undefined at $x = a, x = b$ are another point in the interval $[a, b]$.

✎ To find $\int_a^\infty f(x)\,dx$, determine $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx$

As mentioned, **we can't use ∞ in calculations directly**. We can make use of the lim function we saw in differentiation by first principles.

Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$ or show that it is not convergent.

Evaluate $\int_1^\infty \frac{1}{x}\,dx$ or show that it is not convergent.

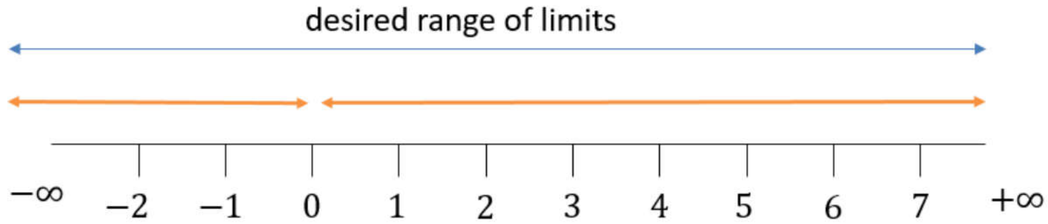
When $f(x)$ not defined for some value

We need to **avoid values** with the range $[a, b]$ **for which the expression is not defined**. But just as we avoided ∞ by considering the limit as $t \to \infty$, we can similarly find what the area converges to as x tends towards the undefined value.

Evaluate $\int_0^1 \frac{1}{x^2}\,dx$ or show that it is not convergent.

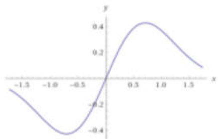
Evaluate $\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$ or show that it is not convergent.

When integrating between $-\infty$ and ∞



Suppose we want $\int_{-\infty}^{\infty} f(x)\ dx$. How could evaluate this?

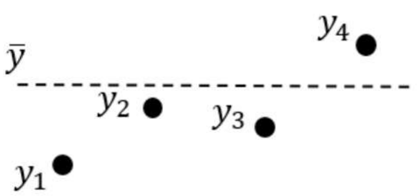
(a) Find $\int xe^{-x^2} dx$ (b) Hence show that $\int_{-\infty}^{\infty} xe^{-x^2}\ dx$ converges and find its value.



Ex 3A

The Mean Value of a Function

How would we find the mean of a set of values y values $y_1, y_2, \dots, y_n$?

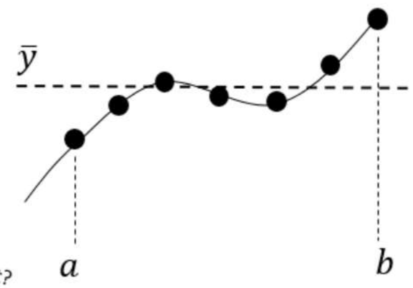


$$\bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

continuous equivalent?

continuous equivalent?

So the question then is, can we extend this to the continuous world, with a function $y = f(x)$, between $x = a$ and $x = b$?



$$\bar{y} = \frac{\int_a^b f(x)\, dx}{b - a}$$

Integration can be thought of as the continuous version of summation of the y values.

The width of the interval, $b - a$, could (sort of) be thought of as the number of points in the interval on an infinitesimally small scale.

✏ The **mean value** of the function $y = f(x)$ over the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_b^a f(x)dx$$

We write it as $\bar{y}$ or $\bar{f}$ or y_m.

Find the mean value of $f(x) = \frac{4}{\sqrt{2+3x}}$ over the interval $[2,6]$.

$f(x) = \frac{4}{1+e^x}$

(a) Show that the mean value of $f(x)$ over the interval $[\ln 2, \ln 6]$ is $\frac{4 \ln \frac{9}{7}}{\ln 3}$

(b) Use your answer to part a to find the mean value over the interval $[\ln 2, \ln 6]$ of $f(x) + 4$.

(c) Use geometric considerations to write down the mean value of $-f(x)$ over the interval $[\ln 2, \ln 6]$

Ex 3B

Differentiating inverse trigonometric functions

Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Given that $y = \arcsin x^2$ find $\frac{dy}{dx}$

Your Turn

Given that $y = \text{arcsec}\, 2x$,
show that $y = \frac{1}{x\sqrt{4x^2-1}}$

Ex 3C

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

So because of these differentiation facts, what else do we know?

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Integration (+ constant; $a > 0$ where relevant)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Integrating with inverse trigonometric functions

Use an appropriate substitution to show that
$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$

Find $\int \frac{4}{5+x^2}\,dx$

Find $\int \frac{1}{25+9x^2}\,dx$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$

Find $\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}} dx$

Find $\int \frac{x+4}{\sqrt{1-4x^2}} dx$

Ex 3D Even

$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right), \quad \ln\{x + \sqrt{x^2 - a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right), \quad \ln\{x + \sqrt{x^2 + a^2}\}$

Solving using partial fractions

Prove that $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + c$

$\frac{1}{a^2 - x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Partial Fractions involving Quadratic Factors

When you write as partial fractions, ensure you have the **most general possible non-top heavy fraction**, i.e. the 'order' (i.e. maximum power) of the numerator is **one less** than the denominator.

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Show that $\int \frac{1+x}{x^3+9x}\,dx = A\ln\left(\frac{x^2}{x^2+9}\right) + B\arctan\left(\frac{x}{3}\right) + c$, where A and B are constants to be found.

$= \frac{1}{18}\ln\left(\frac{x^2}{x^2+9}\right) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c$

If the fraction is top-heavy, you'll have a quotient. As per Pure Year 2, if the order of numerator and denominator is the same, you'll need an extra constant term. If the power is 1 greater in the numerator, you'll need a quotient of $Ax + B$, and so on.

$$\frac{4x^2 + x}{x^2 + x} = \frac{4x^2 + x}{x(x+1)} = A + \frac{B}{x} + \frac{C}{x+1}$$

(a) Express $\frac{x^4+x}{x^4+5x^2+6}$ as partial fractions.

(b) Hence find $\int \frac{x^4+x}{x^4+5x^2+6}\, dx$.

Ex 3E Even Questions

$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\{x + \sqrt{x^2 - a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\{x + \sqrt{x^2 + a^2}\}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Differentiating hyperbolic functions (Chapter 6)

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \text{sech}^2\, x$$
$$\frac{d}{dx}(\coth x) = -cosech^2 x$$

Important Memorisation Tip: They're all the same as non-hyperbolic results, other than that $cosh$ is not negated and $\text{sech}\, x$ becomes $-\text{sech}\, x \tanh x$ (i.e. **is** negated).

Prove that $\frac{d}{dx}(\sinh x) = \cosh x$

[June 2014 (R) Q3] 6. The curve C has equation

$$y = \frac{1}{2}\ln(\coth x), \qquad x > 0$$

(*a*) Show that

$$\frac{dy}{dx} = -\text{cosech}\, 2x$$ **(3)**

Hint: chain rule?

Differentiating Inverse Hyperbolic Functions

Proof?

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$$

Examples

Find $\frac{d}{dx}(artanh\ 3x)$

Given that $y = (arcosh\ x)^2$ prove that $(x^2-1)\left(\frac{dy}{dx}\right)^2 = 4y$

[June 2009 Q4] Given that $y = \operatorname{arsinh}(\sqrt{x})$, $x > 0$,

(*a*) find $\frac{dy}{dx}$, giving your answer as a simplified fraction. **(3)**

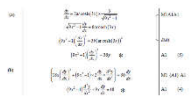
(a) $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \times \frac{1}{\sqrt{1+(\sqrt{x})^2}}$ B1, M1

$\therefore \frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{1+x}} \quad \left(= \frac{1}{2\sqrt{x(1+x)}}\right)$ A1 (3)

[June 2010 Q5] Given that $y = (\text{arcosh } 3x)^2$, where $3x > 1$, show that

(*a*) $(9x^2 - 1)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 36y,$ **(5)**

(*b*) $(9x^2 - 1)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 9x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 18.$ **(4)**



Using Maclaurin expansions for approximations

(a) Show that $\frac{d}{dx}(arsinh\ x) = \frac{1}{\sqrt{1+x^2}}$ **[We did this earlier]**
(b) Find the first two non-zero terms of the series expansion of $arsinh\ x$.
The general form for the series expansion of $arsinh\ x$ is given by

$$arsinh\ x = \sum_{r=0}^{\infty}\left(\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{2n+1}}{2n+1}$$

(c) Find, in simplest terms, the coefficient of x^5.
(d) Use your approximation up to and including the term in x^5 to find an approximate value for $arsinh\ 0.5$.
(e) Calculate the percentage error in using this approximation.

Ex 6D

Standard Integrals

Same as non-hyperbolic version?	
×	$\int \sinh x \; dx = \cosh x + C$
✓	$\int \cosh x \; dx = \sinh x + C$
✓	$\int \operatorname{sech}^2 x \; dx = \tanh x + C$
✓	$\int \operatorname{cosech}^2 x \; dx = -\coth x + C$
×	$\int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + C$
✓	$\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$

Not in this chapter but worth briefly mentioning. (sech², cosech², sech tanh, cosech coth)

Not in formula booklet. (cosech², sech tanh, cosech coth)

Was covered in Chapter 3.

$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \arcsin x + C, \qquad |x| < 1$$

$$\int \frac{1}{1+x^2} \; dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \; dx = \operatorname{arsinh} x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} \; dx = \operatorname{arcosh} x + C, \qquad x > 1$$

Recall that:

$$\boldsymbol{\int f'(ax+b)\,dx = \frac{1}{a} f(ax+b) + C}$$

e.g. $\int e^{3x+2} dx = \frac{1}{3} e^{3x+2}$

$$\int \cosh(4x-1)\,dx =$$

$$\int sinh\left(\frac{2}{3}x\right)\,dx =$$

$$\int \frac{3}{\sqrt{1+x^2}} dx =$$

$$\int \frac{4}{\sqrt{x^2-1}} dx =$$

$$\int sinh\,(3x)\;dx =$$

$$\int \frac{10}{\sqrt{x^2-1}} dx =$$

$$\int \frac{2}{\sqrt{1+x^2}} dx =$$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right), \quad \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right), \quad \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

$$\int \frac{2+5x}{\sqrt{x^2+1}}\,dx$$

$$\int \cosh^5 2x \sinh 2x \; dx$$

$$\int \tanh x \; dx$$

Using Identities

$$\int \cosh^2 3x \; dx$$

$$\int \sinh^3 x \; dx$$

Use this approach in general for small odd powers of sinh and cosh.

Other things to try...

Sometimes there are techniques which work on non-hyperbolic trig functions but doesn't work on hyperbolic ones. Just first replace any hyperbolic functions with their definition.

Find $\int e^{2x} \sinh x \; dx$

Find $\int \operatorname{sech} x \; dx$

Use the substitution $u = e^x$

Ex 6E Q1-10

Dealing with $1/\sqrt{a^2+x^2}$, $1/\sqrt{x^2-a^2}$, ...

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \sinh^2 u = \cosh^2 u$$

Sensible substitution and why?

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$x = a\sinh u$$

tan wouldn't work as well this time because the denominator would simplify to $a\sec u$, but we'd be multiplying by $a\sec^2\theta$, meaning not all the secs would cancel. With $\sinh u$ the two $\cosh u$'s obtained would fully cancel.

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$x = a\cosh u$$

Show that $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = arcosh\left(\frac{x}{a}\right) + c$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Show that $\int_5^8 \frac{1}{\sqrt{x^2-16}}\,dx = \ln\left(\frac{2+\sqrt{3}}{2}\right)$

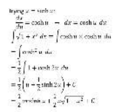
$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = arsinh\left(\frac{x}{a}\right) + c$$
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = arcosh\left(\frac{x}{a}\right) + c, \qquad x > a$$

$$arsinh\, x = \ln\left(x + \sqrt{x^2+1}\right)$$
$$arcosh\, x = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1$$

Harder Example

Show that $\int \sqrt{1+x^2}\, dx = \frac{1}{2} arsinh\, x + \frac{1}{2} x\sqrt{1+x^2} + C.$

(Hint: Use a sensible substitution)



Your Turn

Hint: You may want to factorise out $\frac{1}{\sqrt{4}}$ first, as we did in Chapter 3.

[June 2013 Q2]

(*a*) Find

$$\int \frac{1}{\sqrt{(4x^2+9)}}\, \mathrm{d}x$$

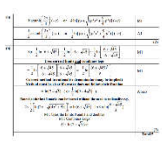
(2)

(*b*) Use your answer to part (*a*) to find the exact value of

$$\int_{-3}^{3} \frac{1}{\sqrt{(4x^2+9)}}\, \mathrm{d}x$$

giving your answer in the form $k \ln(a + b\sqrt{5})$, where a and b are integers and k is a constant.

(3)



Your Turn

Using a hyperbolic substitution, evaluate
$\int_0^6 \frac{x^3}{\sqrt{x^2+9}}\, dx$

Ex 6E Q11-20

Using $x = 3\sinh u$ yields $18(\sqrt{5} + 1)$

Integrating by Completing the Square

Determine $\int \frac{1}{x^2-8x+8} dx$

Determine $\int \frac{1}{\sqrt{12x+2x^2}}\,dx$

Ex 6E Q21-23

[June 2014(R) Q2]

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c$$

(*a*) Find the values of the constants *a*, *b* and *c*. **(3)**

Hence, or otherwise, find

(*b*) $\int \frac{1}{9x^2+6x+5}\,\mathrm{d}x$ **(2)**

(*c*) $\int \frac{1}{\sqrt{9x^2+6x+5}}\,\mathrm{d}x$ **(2)**