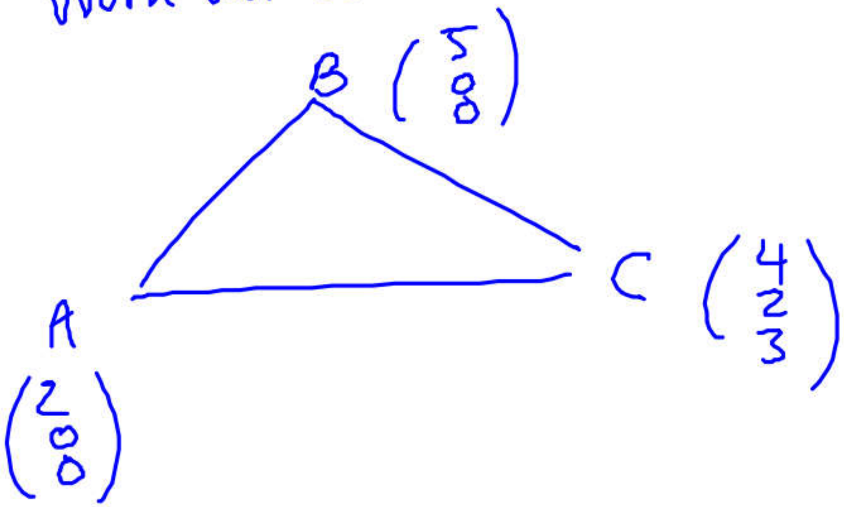
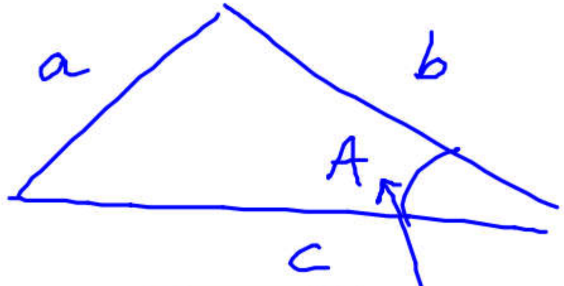
Ex 12B Q16

A scalene $\triangle$ has coords $\overset{A}{(2,0,0)}$ $\overset{B}{(5,0,0)}$ and $\overset{C}{(4,2,3)}$.

Work out area.





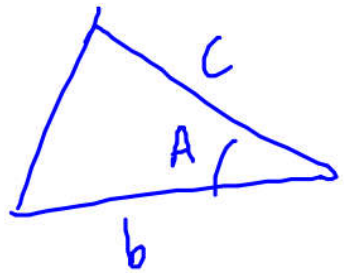
$\boxed{\text{cosine rule}}$

$$a^2 = b^2 + c^2 - 2bc\cos \underline{A}$$

Area $\triangle$

$\frac{1}{2}ab\sin C$

$\frac{1}{2}cb\sin A$



Geometric Problems

A, B, C and D are the points $(2, -5, -8)$, $(1, -7, -3)$, $(0, 15, -10)$ and $(2, 19, -20)$ respectively.

a. Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$, giving your answers in the form $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$.

b. Show that the lines AB and DC are parallel and that $\boxed{\overrightarrow{DC} = 2\overrightarrow{AB}}$.

c. Hence describe the quadrilateral $ABCD$.

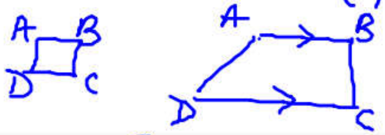
a) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 1 \\ -7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \\ -8 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} = -\underline{i} - 2\underline{j} + 5\underline{k}$

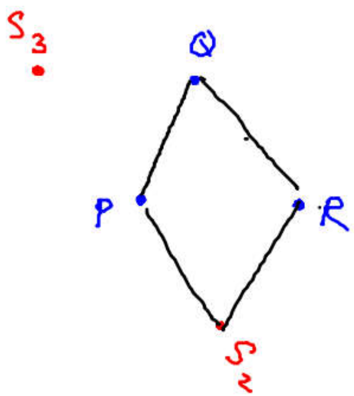
$\overrightarrow{DC} = \begin{pmatrix} 0 \\ 15 \\ -10 \end{pmatrix} - \begin{pmatrix} 2 \\ 19 \\ -20 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 10 \end{pmatrix} = -2\underline{i} - 4\underline{j} + 10\underline{k}$.

b) $\overrightarrow{DC} = 2\begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 10 \end{pmatrix} = 2\overrightarrow{AB}$

c) 

Trapezium.

P, Q and R are the points $(4, -9, -3)$, $(7, -7, -7)$ and $(8, -2, 0)$ respectively. Find the coordinates of the point S so that $PQRS$ forms a parallelogram.



$\overrightarrow{QP} = \overrightarrow{RS}$

$\overrightarrow{PQ} = \overrightarrow{SR}$

$\overrightarrow{PS} = \overrightarrow{QR}$

$\overrightarrow{QP} = \underline{s} - \underline{r}$

$\overrightarrow{QP} + \underline{r} = \underline{s}$

$\overrightarrow{QP} = \begin{pmatrix} 4 \\ -9 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -7 \\ -7 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix}$

$\underline{s} = \underline{r} + \overrightarrow{QP}$

$= \begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 4 \end{pmatrix}$

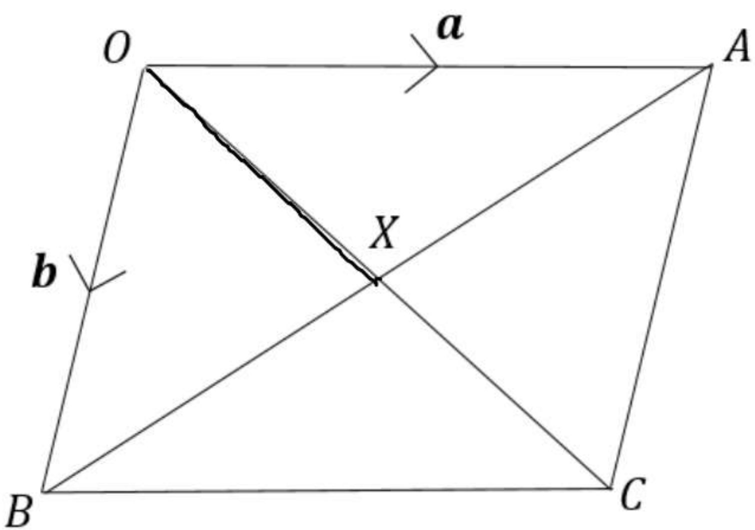
$S(5, -4, 4)$.

Introducing Scalars and Comparing Coefficients

Remember when we had **identities** like:

$$ax^2 + 3x \equiv 2x^2 + bx$$

we could **compare coefficients**, so that $a = 2$ and $3 = b$.
We can do the same with (non-parallel) vectors!



$OACB$ is a parallelogram, where $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$.
The diagonals OC and AB intersect at a point X.
Prove that the diagonals bisect each other. $\rightarrow$ in half.
(Hint: Perhaps find $\overrightarrow{OX}$ in two different ways?)

Way 1 $\quad \overrightarrow{OX} = \lambda \overrightarrow{OC}$ (λ is a fraction, $0 \le \lambda \le 1$)

$$= \lambda(\underline{a} + \underline{b})$$

$$= \lambda \underline{a} + \lambda \underline{b}$$

λ 'lambda'
μ 'mu'

Way 2 $\quad \overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}$

$$= \underline{b} + \mu \overrightarrow{BA}$$ (μ is a fraction, $0 \le \mu \le 1$)

$$= \underline{b} + \mu(-\underline{b} + \underline{a})$$

$$\overrightarrow{OX} = \mu \underline{a} + (1-\mu)\underline{b}$$

comparing $\underline{a}$ coeff. $\quad \lambda = \mu$

compare $\underline{b}$ coeff. $\quad \lambda = 1 - \mu$

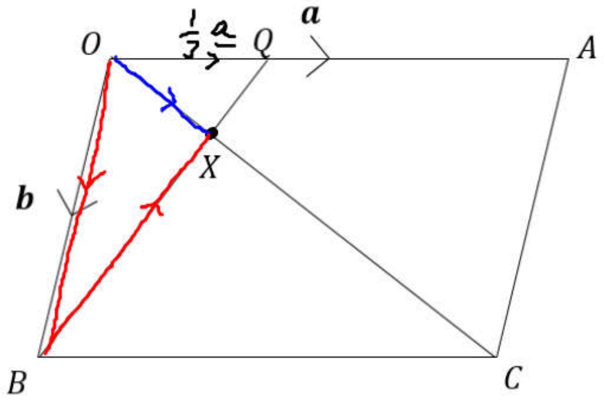
$\mu = 1 - \mu$

$2\mu = 1$

$\mu = \frac{1}{2} = \lambda$

So, the diagonals bisect.

Your Turn



In the above diagram, $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{3}\boldsymbol{a}$. We wish to find the ratio $OX: XC$.

a) If $\overrightarrow{OX} = \lambda\, \overrightarrow{OC}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and λ.

b) If $\overrightarrow{BX} = \mu\, \overrightarrow{BQ}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and μ.

c) By comparing coefficients or otherwise, determine the value of λ, and hence the ratio $OX: XC$.

a)

$\overrightarrow{OX} = \lambda \overrightarrow{OC}$

$(\overrightarrow{OC} = \underline{a} + \underline{b})$

$\overrightarrow{OX} = \lambda(\underline{a} + \underline{b})$

$= \lambda \underline{a} + \lambda \underline{b}$

b)

$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}$ $\quad (\overrightarrow{BQ} = -\underline{b} + \frac{1}{3}\underline{a})$

$= \underline{b} + \mu \overrightarrow{BQ}$

$= \underline{b} + \mu(-\underline{b} + \frac{1}{3}\underline{a})$

$= \underline{b} - \mu \underline{b} + \frac{1}{3}\mu \underline{a}$

$\overrightarrow{OX} = \frac{1}{3}\mu \underline{a} + (1-\mu)\underline{b}$

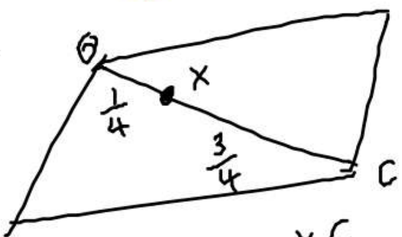
c)

$\lambda = \frac{1}{3}\mu \rightarrow 3\lambda = \mu$

$\lambda = 1 - \mu$

$\lambda = 1 - 3\lambda$

$\lambda = \frac{1}{4}$



$OX : XC$

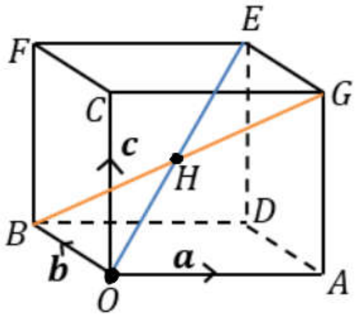
$\frac{1}{4} : \frac{3}{4}$

$1 : 3$

Given that
$3\boldsymbol{i} + (p+2)\boldsymbol{j} + 120\boldsymbol{k} = p\boldsymbol{i} - q\boldsymbol{j} + 4pqr\boldsymbol{k}$,
find the values of p, q and r.

$p = 3 \quad q = -5 \quad 4pqr = 120$
$r = 120 \div (4 \times 3 \times -5) = -2$

The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G. Vectors a, b and c are the position vectors of the vertices A, B and C respectively. Prove that the diagonals OE and BG bisect each other.



The strategy behind this type of question is to find the point of intersection in 2 ways, and compare coefficients.

WAY 1

$\overrightarrow{OH} = \lambda \overrightarrow{OE}$

$(\overrightarrow{OE} = \underline{a} + \underline{b} + \underline{c})$

$\overrightarrow{OH} = \lambda \underline{a} + \lambda \underline{b} + \lambda \underline{c}$

Way 2

$\overrightarrow{OH} = \overrightarrow{OB} + \mu \overrightarrow{BG}$

$(\overrightarrow{BG} = \underline{a} - \underline{b} + \underline{c})$

$\overrightarrow{OH} = \underline{b} + \mu \underline{a} - \mu \underline{b} + \mu \underline{c}$

$= \mu \underline{a} + (1-\mu)\underline{b} + \mu \underline{c}$

$\lambda = \mu$

$\lambda = 1 - \mu$

$\lambda = 1 - \lambda$

$\lambda = \frac{1}{2}, \mu = \frac{1}{2}$ So they bisect