Chapter 2: Quadratics

1:: Solving quadratic equations

Solve
$(x+1)^2 - 3(x+1) + 2 = 0$

2:: Completing the square

Write $2x^2 + 8x - 5$ in the form $a(x+b)^2 + c$

3:: Quadratics as functions

If $f(x) = x^2 + 2x$, find the roots of $f(x)$.

4:: Quadratic Graphs

Sketch $y = x^2 + 4x - 5$, indicating the coordinate of the turning point and any intercepts with the axes.

5:: The Discriminant

Find the range of values of k for which $x^2 + 4x + k = 0$ has two distinct real solutions.

6:: Modelling with Quadratics

Solving Quadratic Equations

$$x^2 + 5x = 6$$

There are three ways of solving a quadratic equation.
What are they?

Solving without factorising

If the subject only appears once however, it might be easier not to expand out/factorise:

$$(x-1)^2 = 5$$

Pseudo-quadratics

When we have an expression like say $x^2 + 3x - 2$, we say it is "quadratic in x".
You can have quadratics in other variables, too!

Solve $x - 6\sqrt{x} + 8 = 0$

substitution | hardcore

1 Solve $(x+3)^2 = x + 5$ using factorisation.

2 Solve $(2x+1)^2 = 5$

3 Solve $\sqrt{x+3} = x - 3$

However, squaring both sides of an equation **can generate false solutions.**

4 Solve $2x + \sqrt{x} - 1 = 0$

'Pseudo' Quadratic Equations

$$x^2 + 3\sqrt{x} - 10 = 0$$

$$y^{\frac{4}{3}} - 5y^{\frac{2}{3}} - 14 = 0$$

$a^4 - a^2 - 12 = 0$

$2^{2x} - 9(2^x) + 8 = 0$

$b^{\frac{2}{3}} + 2b^{\frac{1}{3}} - 8 = 0$

Make up your own!

Ex2A/B

Completing the Square

"Completing the square" means putting a quadratic in the form

$$(x + a)^2 + b$$

or

$$a(x + b)^2 + c$$

Expand:

$(x + \mathbf{9})^2 =$

$(x - \mathbf{5})^2 =$

Therefore if we had $x^2 + 12x$, how could we write it in the form $(x + a)^2 + b$?

$x^2 + 12x =$

$x^2 + 8x =$

$x^2 - 2x =$

$x^2 - 6x + 7 =$

Express $2x^2 + 12x + 7$ in the form $a(x + b)^2 + c$

Express $5 - 3x^2 + 6x$ in the form $a - b(x + c)^2$

Your Turn

Express $3x^2 - 18x + 4$ in the form $a(x+b)^2 + c$

Express $20x - 5x^2 + 3$ in the form $a - b(x+c)^2$

Ex2C

Solving Equations by Completing the Square

Solve the equation:

$$3x^2 - 18x + 4 = 0$$

Ex2D

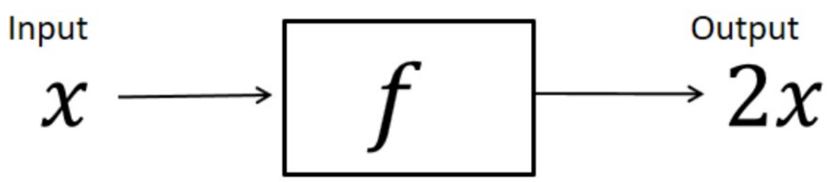
Proving the Quadratic Formula

If $ax^2 + bx + c = 0$, prove that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Functions

A function is something which **provides a rule on how to map inputs to outputs.**

We saw at GCSE that functions were a formal way of describing a 'number machine':



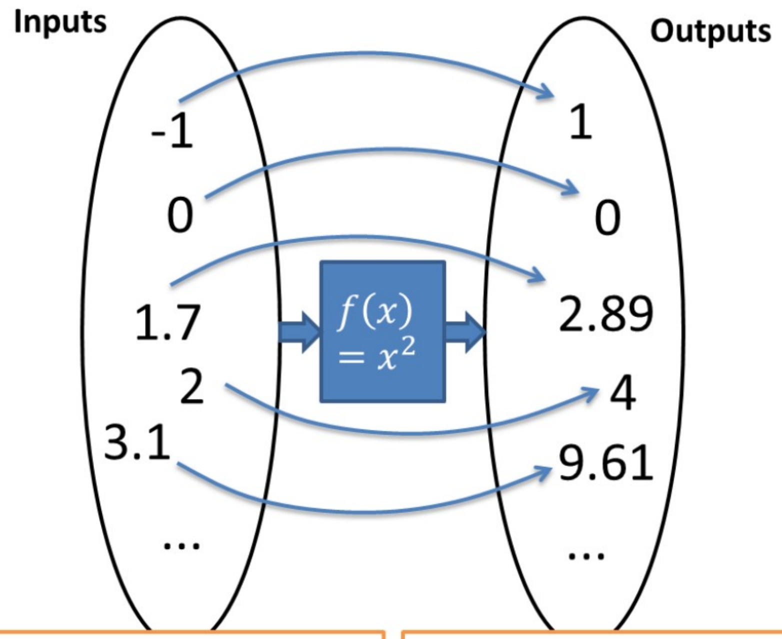
Name of the function (often f or g)

Input

Output

$$f(x) = 2x$$

You'll cover functions extensively in future chapters, but for now, you need to understand the following concepts:



The domain of a function could potentially be **any** real number. If so, we'd write:

$$x \in \mathbb{R}$$

The input x... is a member of... the set of real numbers

✏ The **domain** of a function is the set of possible inputs.

✏ The **range** of a function is the set of possible outputs.

We might be interested in what inputs x give an output of 0. These are known as the **roots** of the function.

✏ The **roots/zeroes** of a function are the values of x for which $f(x) = 0$.

If $f(x) = x^2 - 3x$ and $g(x) = x + 5, \quad x \in \mathbb{R}$

a) Find $f(-4)$

b) Find the values of x for which $f(x) = g(x)$

c) Find the roots of $f(x)$.

d) Find the roots of $g(x)$.

Maxima/Minima of Quadratics

Determine the minimum value of the function $f(x) = x^2 - 6x + 2$, and state the value of x for which this minimum occurs.

This means we want to minimise the **output** of the function.

You might try a (bad) approach of trying a few values of x and try to see what makes the output as small as possible...

$$f(1) = 1 - 6 + 2 = -3$$
$$f(2) = 4 - 12 + 2 = -6$$
$$f(3) = 9 - 18 + 2 = -7$$
$$f(4) = 16 - 24 + 2 = -6$$

This looks like the minimum as the value starts going up after.

But the best way to find the minimum/maximum value of a quadratic is to **complete the square**:

$$f(x) = (x - 3)^2 - 7$$

Since anything squared is at least 0, the smallest we can make the bracket is 0, which occurs when $x = 3$.

$$f(1) = (-2)^2 - 7 = -3$$
$$f(2) = (-1)^2 - 7 = -6$$
$$f(3) = 0^2 - 7 = -7$$
$$f(4) = 1^2 - 7 = -6$$

✏ If $f(x) = (x + a)^2 + b$, the minimum value of $f(x)$ is b, which occurs when $x = -a$.

$\boldsymbol{f(x)}$	Completed square	Min/max value of $\boldsymbol{f(x)}$	$\boldsymbol{x}$ for which this min/max occurs
$x^2 + 4x + 9$			
$x^2 - 10x + 21$			
$10 - x^2$			
$8 - x^2 + 6x$			

Your Turn

1. Find the minimum value of $f(x) = 2x^2 + 12x - 5$ and state the value of x for which this occurs.

2. Find the roots of the function $f(x) = 2x^2 + 3x + 1$

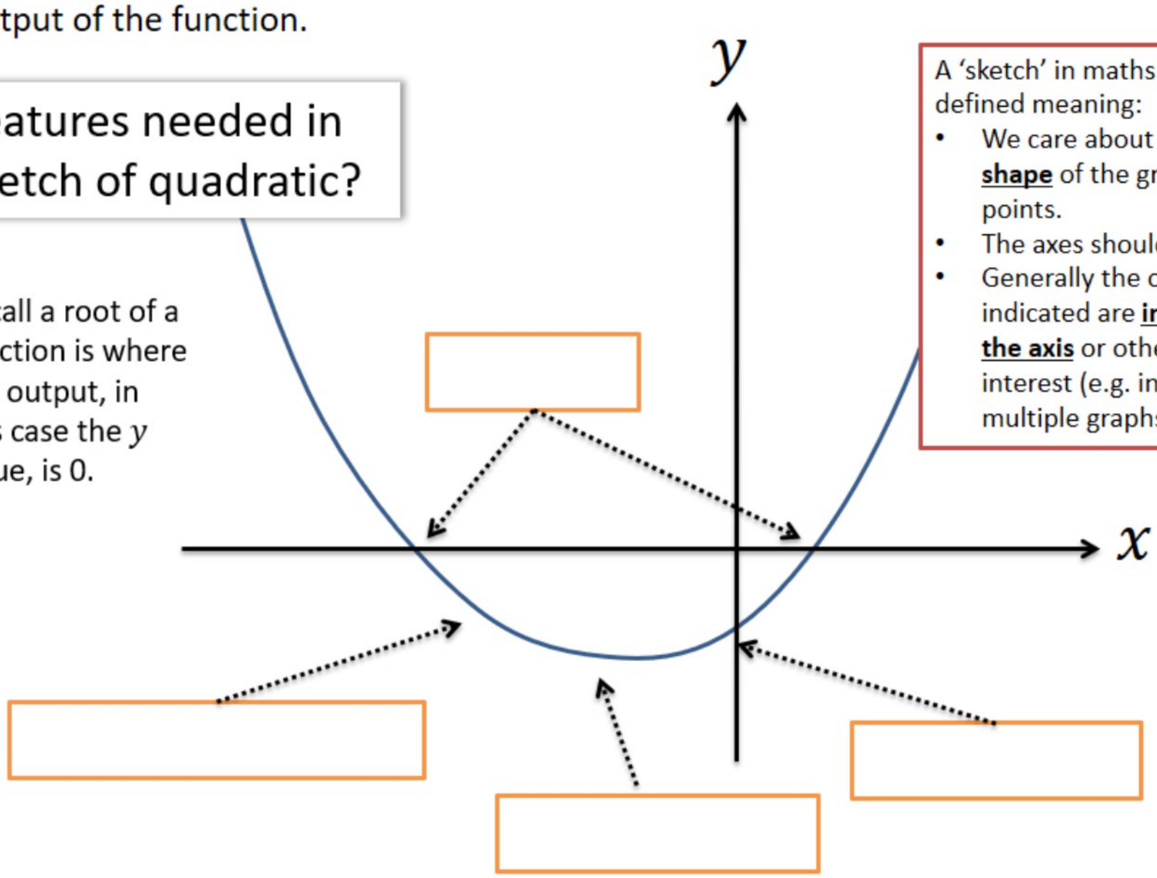
3. Find the roots of the function $f(x) = x^4 - x^2 - 6$

Quadratic Graphs

Recall that x refers to the input of a function, and the expression $f(x)$ refers to the output. For graph sketches, we often write $y = f(x)$, i.e. we set the y values to be the output of the function.

Features needed in sketch of quadratic?

Recall a root of a function is where the output, in this case the y value, is 0.



A 'sketch' in maths has a clearly defined meaning:

- We care about the **general shape** of the graph, not exact points.
- The axes should have **no scale**.
- Generally the only coordinates indicated are **intercepts with the axis** or other points of interest (e.g. intersections of multiple graphs)

Sketch the graph of $y = x^2 + 3x - 4$ and find the coordinates of the turning point.

Sketch the graph of $y = 4x - 2x^2 - 3$ and find the coordinates of the turning point. Write down the equation of the line of symmetry.

Your Turn

Sketch the following, indicating any intercepts with the axis, the turning point and the equation of the line of symmetry.

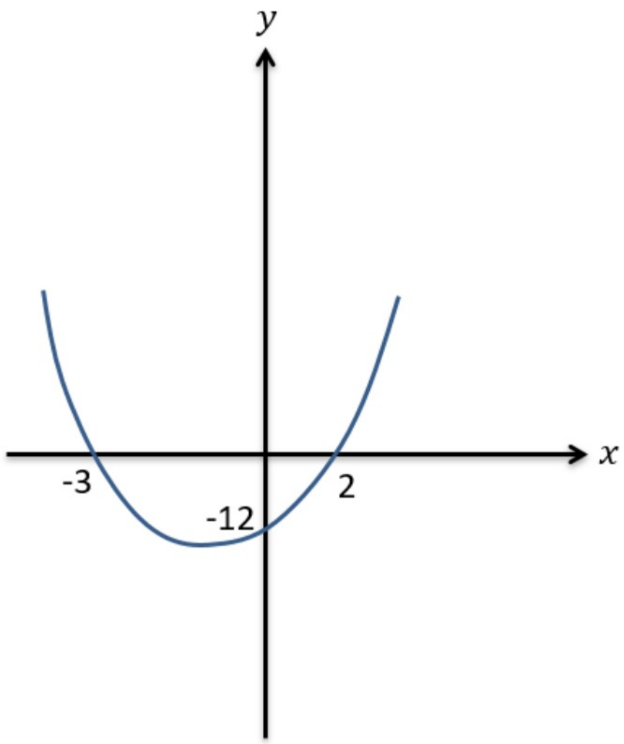
a $y = x^2 + 4$

b $y = x^2 - 7x + 10$

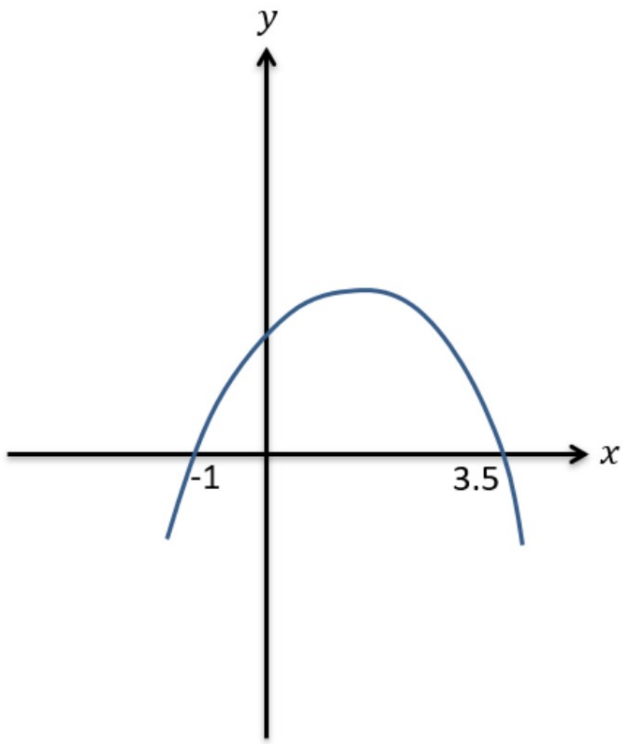
c $y = 5x + 3 - 2x^2$

d $y = x^2 + 4x + 11$

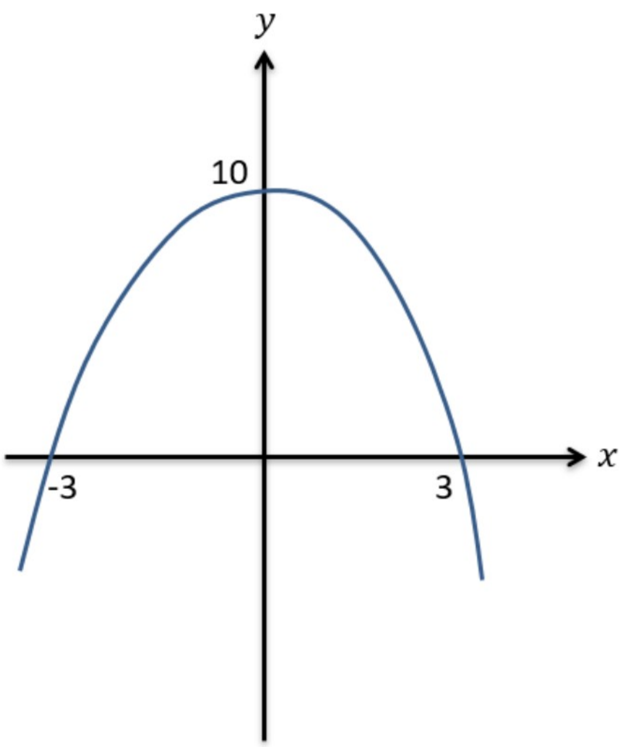
Determining the Equation using a Graph



Determine the equation of this quadratic graph, in the form $y = ax^2 + bx + c$.



Determine the equation of this quadratic graph, in the form $y = ax^2 + bx + c$, where a, b, c are integers.



Determine an equation of this quadratic graph.

Ex2F

How many **distinct** real solutions do each of the following have?

$$x^2 - 12x + 36 = 0$$

$$x^2 + x + 3 = 0$$

$$x^2 - 2x - 1 = 0$$

The Discriminant

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

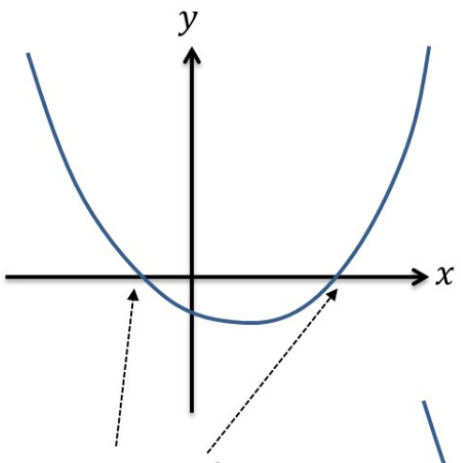
$b^2 - 4ac$ is known as the discriminant.

Looking at this formula, when in general do you think we have:

- No real roots?

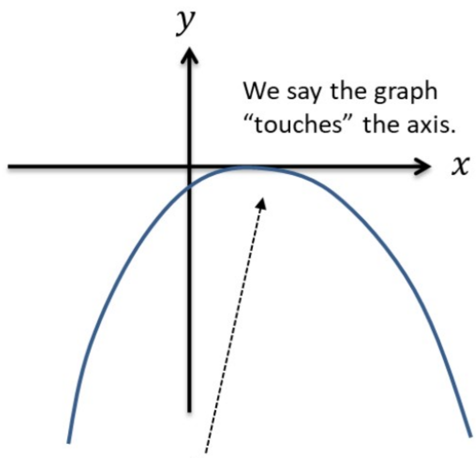
- Equal roots?

- Two distinct roots?



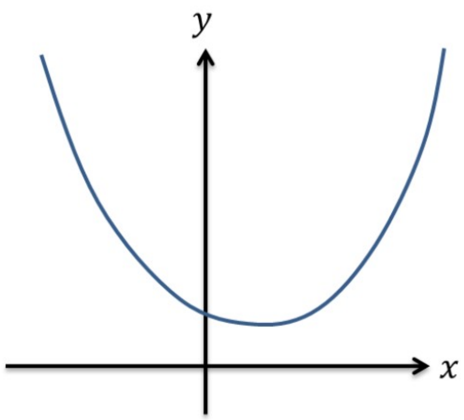
Distinct real roots

$$\boldsymbol{b^2 - 4ac > 0}$$



Equal roots

$$\boldsymbol{b^2 - 4ac = 0}$$



No real roots

$$\boldsymbol{b^2 - 4ac < 0}$$

Equation	Discriminant	Number of Distinct Real Roots
$x^2 + 3x + 4 = 0$		
$x^2 - 4x + 1 = 0$		
$x^2 - 4x + 4 = 0$		
$2x^2 - 6x - 3 = 0$		
$x - 4 - 3x^2 = 0$		
$1 - x^2 = 0$		

Problems involving the Discriminant

8. The equation $x^2 + 2px + (3p + 4) = 0$, where p is a positive constant, has equal roots.

(*a*) Find the value of p. **(4)**

(*b*) For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$. **(2)**

$$x^2 + 5kx + (10k + 5) = 0$$

where k is a constant.
Given that this equation has equal roots, determine the value of k.

Find the range of values of k for which $x^2 + 6x + k = 0$ has two distinct real solutions.

Modelling with Quadratics

A spear is thrown over level ground from the top of a tower.
The height, in metres, of the spear above the ground after t seconds is modelled by the function: $h(t) = 12.25 + 14.7t - 4.9t^2, \quad t \geq 0$

a) Interpret the meaning of the constant term 12.25 in the model.
b) After how many seconds does the spear hit the ground?
c) Write $h(t)$ in the form $A - B(t - C)^2$, where A, B and C are constants to be found.
d) Using your answer to part c or otherwise, find the maximum height of the spear above the ground, and the time at which this maximum height is reached?

11. An archer shoots an arrow.

The height, H metres, of the arrow above the ground is modelled by the formula

$$H = 1.8 + 0.4d - 0.002d^2, \qquad d \geqslant 0$$

where d is the horizontal distance of the arrow from the archer, measured in metres.

Given that the arrow travels in a vertical plane until it hits the ground,

(a) find the horizontal distance travelled by the arrow, as given by this model.

(3)

(b) With reference to the model, interpret the significance of the constant 1.8 in the formula.

(1)

(c) Write $1.8 + 0.4d - 0.002d^2$ in the form

$$A - B(d - C)^2$$

where A, B and C are constants to be found.

(3)

It is decided that the model should be adapted for a different archer.

The adapted formula for this archer is

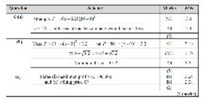
$$H = 2.1 + 0.4d - 0.002d^2, \qquad d \geqslant 0$$

Hence or otherwise, find, for the adapted model

(d) (i) the maximum height of the arrow above the ground.

(ii) the horizontal distance, from the archer, of the arrow when it is at its maximum height.

(2) Ex2H



A company makes a particular type of children's toy.

The annual profit made by the company is modelled by the equation

$$P = 100 - 6.25(x - 9)^2$$

where P is the profit measured in thousands of pounds and x is the selling price of the toy in pounds.

A sketch of P against x is shown in Figure 1.

Using the model,

(a) explain why £15 is not a sensible selling price for the toy.

(2)

Given that the company made an annual profit of more than £80 000

(b) find, according to the model, the least possible selling price for the toy.

(3)

The company wishes to maximise its annual profit.

State, according to the model,

(c) (i) the maximum possible annual profit,

(ii) the selling price of the toy that maximises the annual profit.

(2)

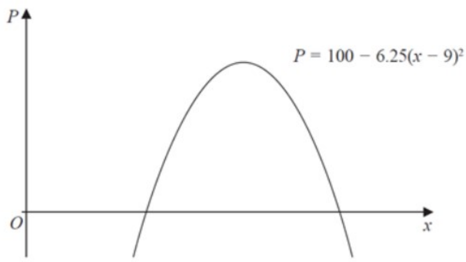


Figure 1