Hyperbolic Functions

Definitions

$\sinh x =$

$\cosh x =$

$\tanh x =$

Plus $\operatorname{cosech} x$, $\operatorname{sech} x$ and $\coth x$

F $\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$

F $\operatorname{arcosh} x = \ln(x + \sqrt{x^2-1})$

F $\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Identities

Same as trigonometric, Osborn's Rule

" or negated from trig identity"

eg F $\cosh^2 x \quad \sinh^2 x = 1$, $1 \quad \tanh^2 x = \operatorname{sech}^2 x$

Equations

i)

ii)

Method of Differences

When to use...

or form

Maclaurin Expansion

Formula

F $f(x) =$

Lots of standard results in formula book

Compound Functions

Can sub into standard results eg. $\cos(2x^2)$

eg. $\ln\left(\frac{1-3x}{\sqrt{1+5x}}\right) =$

Volumes of Revolution

Same as CP1, Parametric same as P2

x-axis $\pi\int y^2\,dx$

y-axis $\pi\int x^2\,dy$

Polar Coordinates

Useful facts

$x =$, $y =$, $r^2 =$, $r \geq 0$

Area

♥F $A =$

$\sin^2\theta =$

$\cos^2\theta =$

Tangents

$= 0$ | $= 0$

Key: F in the formula booklet

♥F in the formula booklet but recommend learning by heart

Complex Numbers

Check CP1!

Exponential Form

$z =$ same rules for $+, -, \times, \div$

De Moivre's Theorem

F♥ $z^n =$

Application 1, expressing in terms of trig powers

e.g. $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

- Start with De Moivre's
- Binomial Expansion on LHS. Tip: Let $\cos\theta =$, $\sin\theta =$
- Compare Re for , Im for
- Use $\sin^2\theta = 1 - \cos^2\theta$ as necessary

Further Identities

$z + \frac{1}{z} =$ $\qquad z - \frac{1}{z} =$

$z^n + \frac{1}{z^n} =$ $\qquad z^n - \frac{1}{z^n} =$

Application 2, expressing trig powers

eg. $\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

- Start with $\left(z - \frac{1}{z}\right)^5 =$
- Binomial Expansion on LHS
- Reuse $z^n \pm \frac{1}{z^n}$ identities

Sums of Series Check P2 Formulae

Even further identities - "hyperbolic"

$\cos n\theta =$

$\sin n\theta =$

$e^{ni\theta} + e^{-ni\theta} =$

$e^{ni\theta} - e^{-ni\theta} =$

Simplifying complex exponential denominators 1

e.g. $\frac{}{e^{ni\theta} \pm 1}$ and $\frac{}{1 \pm e^{ni\theta}}$

Multiply by e to

e.g. $\frac{3}{e^{5i\theta}+1}$

Simplifying complex exponential denominators 2

eg. $\frac{}{ce^{ni\theta} \pm k}$ and $\frac{}{k \pm ce^{ni\theta}}$ with either $c \neq 1$, $k \neq 1$

Multiply by

e.g. $\frac{3}{e^{4i\theta}+2}$

Infinite Series

e.g. $C + iS$ where $C = 1 + \frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta \ldots$

$S = \frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta \ldots$

- Convert to exponential form
- Use infinite series $\frac{a}{1-r}$ ♥F
- Simplify using above techniques
- Compare Re and Im parts

Roots

e.g. Solve $z^n = w$

- exp form for w
- raise to power
- to power, ensuring $< \arg \leq$

Roots of Unity: $1, \omega, \omega^2, \ldots, \omega^{n-1}$

- form
- sum of roots of unity, $S_n =$
- multiply by ω,
- not centred at origin?

Further Calculus

Improper Integrals

...improper if

or... if at any point of interval

Use limits eg. $\int_0^\infty e^{-x}\,dx =$

Mean Value

$\bar{y} = \bar{f} =$

Mean of $f(x) + k =$

$kf(x) =$

Further Differentiation

Standard Results

$f(x)$	$f'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{cosech} x$	
$\operatorname{sech} x$	
$\coth x$	
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

F (arcsin to tanh); F (arsinh to artanh)

- Hyperbolics: only and change from their trigonometric versions.
- These also, of course, give integration facts.
- Use the formula book!

Further Integration

Standard Results

$f(x)$	$\int f(x)\,dx$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

F

- All in the formula book, should be relied on
- All x^2 can be replaced with $(x+k)^2$, changing x to in standard result.

'Sensible' or 'Appropriate' Substitutions

Asks for substitution? Use formula book as a guide

e.g. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ Use $x =$

e.g. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$ Use $x =$

Partial Fractions

i) improper?

ii) repeated factors?

iii) non-linear factors?

eg. $\frac{x^6+3}{x(x^2+2)^2} =$

Completing the Square

Used for quad. denominators in ax^2+bx+c form

First Order Differential Equations

i) Separating the Variables

$\frac{dy}{dx} = f(x)g(y)$

ii) Reverse Product Rule

e.g. $x^3\frac{dy}{dx} + 3x^2y = \sin x$

iii) Integrating Factor

Must be in $\frac{dy}{dx} + P(x)y = Q(x)$ form

Integrating Factor is

Multiply through by I.F.

Harmonic Motion

$a = \frac{dv}{dt} = \ddot{x} =$

For $x = a\sin(\omega t + \alpha)$

amplitude is , period is

Types

See above, top right.

Forced Harmonic Motion if $\ddot{x} + k\dot{x} + \omega^2 =$

Second Order Differential Equations

$ay'' + by' + cy = f(x)$

General Solution

$y =$

Auxiliary Equation

Complementary Functions

A.E. Roots	C.F.	Harmonic?
2 real, α, β		
1 real repeated, α		
2 imaginary, $\pm iq$		
2 complex, $p \pm iq$		

Particular Integrals

$f(x)$	P. I
a	
$ax+b$	
ax^2+bx+c	
ke^{px}	
kxe^{px}	
$m\cos qx + n\sin qx$	

Exceptions

If $f(x)$ appears anywhere in CF, multiply PI by x, x^2

e.g. $f(x) = e^{2x}$, C.F. $y = Ae^{2x} + Be^{3x}$, P.I. Try $y =$

$f(x) = e^{3x}$, C.F. $y = (A+Bx)e^{3x}$, P.I. Try $y =$

$f(x) = dx + e$, C.F. $y = A + Be^{2x}$, P.I. Try $y =$

Method

Diff. P.I. twice, sub into D.E., solve to find v, λ, μ.

Coupled First Order Differential Equations

Method

I $\frac{dx}{dt} = ax + by + f(t)$

II $\frac{dy}{dt} = cx + dy + g(t)$

1) Make y subject in I
2) Find $\dot{y}$
3) Sub into II $\rightarrow$ creates 2nd ODE in x
4) Solve for x
5) Find $\dot{x}$
6) Sub into equation from 1, find y

Useful Notation	
x	y
$\dot{x}$	$\dot{y}$
$\ddot{x}$	$\ddot{y}$

BICEN MATHS