Sequences and Series

1:: Arithmetic Series

Determine the value of $2 + 4 + 6 + \cdots + 100$

2:: Geometric Series

The first term of a geometric sequence is 3 and the second term 1. Find the sum to infinity.

3:: Sigma Notation

Determine the value of

$$\sum_{r=1}^{100}(3r+1)$$

4:: Recurrence Relations

If $a_1 = k$ and $a_{n+1} = 2a_n - 1$, determine a_3 in terms of k.

Types of sequences

common difference d

+3 +3 +3

2, 5, 8, 11, 14, … **Arithmetic Sequence**

An arithmetic sequence is one which has a common difference between terms.

common ratio r

$\times 2$ $\times 2$ $\times 2$

3, 6, 12, 24, 48, … **Geometric Sequence**

(We will explore these later in the chapter)

1, 1, 2, 3, 5, 8, …

This is the **Fibonacci Sequence**. The terms follow a **recurrence relation** because each term can be generated using the previous ones. We will encounter recurrence relations later in the chapter.

The fundamentals of sequences

u_n The n^{th} **term**. So u_3 would refer to the 3rd term.

n The **position** of the term in the sequence.

$n = 3$ $u_3 = 8$

$$2, 5, 8, 11, 14, \ldots$$

nth term of an arithmetic sequence

We use a to denote the **first term**. d is the **difference** between terms, and n is the **position** of the term we're interested in. Therefore:

1st Term	2nd Term	3rd Term	$\cdots$	n^{th} term
a	$a + d$	$a + 2d$	$\cdots$	$a + (n - 1)d$

n^{th} term of arithmetic sequence:

$$u_n = a + (n - 1)d$$

Example 1

The nth term of an arithmetic sequence is $u_n = 55 - 2n$.

a) Write down the first 3 terms of the sequence.

b) Find the first term in the sequence that is negative.

Example 2

Find the nth term of each arithmetic sequence.

a) 6, 20, 34, 48, 62

b) 101, 94, 87, 80, 73

Tip: Always write out $a =, d =, n =$ first.

A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.
Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b.

For which values of x would the expression $-8, x^2$ and $17x$ form the first three terms of an arithmetic sequence.

11. The second, third and fourth terms of an arithmetic sequence are $2k$, $5k - 10$ and $7k - 14$ respectively, where k is a constant.

Show that the sum of the first n terms of the sequence is a square number.

(5)

Edexcel C1 May 2014(R) Q10

Xin has been given a 14 day training schedule by her coach.

Xin will run for A minutes on day 1, where A is a constant.

She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.

(a) Show that on day 14, Xin will run for

$$(A + 13d + 13) \text{ minutes.}$$

(2)

Yi has also been given a 14 day training schedule by her coach.

Yi will run for $(A - 13)$ minutes on day 1.

She will then increase her running time by $(2d - 1)$ minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

(b) find the value of d.

(3)

Ex 3A

Arithmetic Series

A **series** is a **sum** of terms in a sequence.
You will encounter 'series' in many places in A Level Maths and Further Maths: Arithmetic Series, Geometric Series, Binomial Series, Taylor Series...

n^{th} term	Sum of first n terms
$u_n = a + (n-1)d$	$S_n = \frac{n}{2}(2a + (n-1)d)$

Example:

Take an arithmetic sequence $2, 5, 8, 11, 14, 17, \ldots$

$$S_5 = 2 \quad + 5 \quad + 8 + 11 + 14$$

Let's prove it!

Proving more generally:

Exam Note: The proof has been an exam question before. It's also a university interview favourite!

Alternative Formula

$$a + (a + d) + \cdots + L$$

Suppose last term was L.
We saw earlier that each opposite pair of terms (first and last, second and second last, etc.) added to the same total, in this case $a + L$.

There are $\frac{n}{2}$ pairs, therefore:

$$S_n = \frac{n}{2}(a + L)$$

Find the sum of the first 30 terms of the following arithmetic sequences...

$2 + 5 + 8 + 11 + 14 \ldots$

$100 + 98 + 96 + \cdots$

$p + 2p + 3p + \cdots$

Find the minimum number of terms for the sum of $4 + 9 + 14 + \cdots$ to exceed 2000.

Worded Arithmetic Series

Edexcel C1 Jan 2012 Q9

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P.
Salary increases by £($2T$) each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £(P + 1800).
Salary increases by £T each year, forming an arithmetic sequence.

(*a*) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$£(10P + 90T).$$

(2)

For the 10-year period, the total earned is the same for both salary schemes.

(*b*) Find the value of T.

(4)

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850.

(*c*) Find the value of P.

(3)

Ex 3B

Geometric Sequences

common ratio r

$$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24, 48, \ldots$$

Geometric Sequence

Identify the common ratio r:

A geometric sequence is one in which there is a **common ratio** between terms.

1. $1, 2, 4, 8, 16, 32, \ldots$
2. $27, 18, 12, 8, \ldots$
3. $10, 5, 2.5, 1.25, \ldots$
4. $5, -5, 5, -5, 5, -5, \ldots$
5. $x, -2x^2, 4x^3$
6. $1, p, p^2, p^3, \ldots$
7. $4, -1, 0.25, -0.0625, \ldots$

An **alternating sequence** is one which oscillates between positive and negative.

nth term of an geometric sequence

1st Term	2nd Term	3rd Term	…	n^{th} term
a	ar	ar^2	…	ar^{n-1}

n^{th} term of geometric sequence:

$$u_n = ar^{n-1}$$

The second term of a geometric sequence is 4 and the 4th term is 8. The common ratio is positive. Find the exact values of:

a) The common ratio.
b) The first term.
c) The 10th term.

The numbers $3, x$ and $x + 6$ form the first three terms of a positive geometric sequence. Find:

a) The value of x.
b) The 10^{th} term in the sequence.

n^{th} term with inequalities

What is the first term in the geometric progression $3, 6, 12, 24, \ldots$ to exceed 1 million?

Your Turn

All the terms in a geometric sequence are positive.
The third term of the sequence is 20 and the fifth term 80. What is the 20^{th} term?

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x+6, \qquad 5x-6$$

a) Determine the possible values of x.
b) Given the common ratio is positive, find the common ratio.
c) Hence determine the possible values for the first term of the sequence.

Ex 3C

Sum of the first n terms of a **geometric series**

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r}$$

Proof:

Exam Note: This once came up in an exam.

$$S_n = \frac{a(1-r^n)}{1-r}$$

Find the sum of the first 10 terms.

$3, 6, 12, 24, 48, \ldots$

$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \cdots$ to n terms would exceed 2 000 000.

Your Turn

Edexcel C2 June 2011 Q6

The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio, **(2)**

(*b*) the first term, **(2)**

(*d*) the smallest value of *n* for which the sum of the first *n* terms of the series exceeds 1000. **(4)**

Ex 3D

Sum to Infinity - Divergent vs Convergent

What can you say about the sum of each series up to infinity?

1 + 2 + 4 + 8 + 16 + ...

1 – 2 + 3 – 4 + 5 – 6 + ...

1 + 0.5 + 0.25 + 0.125 + ...

- The infinite series will converge provided that $-1 < r < 1$ (which can be written as $|r| < 1$), because the terms will get smaller.
- Provided that $|r| < 1$, what happens to r^n as $n \to \infty$?

A geometric series is convergent if $|r| < 1$.

- How therefore can we use the $S_n = \frac{a(1-r^n)}{1-r}$ formula to find the sum to infinity, i.e. S_∞?

For a <u>convergent</u> geometric series,

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{a}{1-r}$$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$	$\boldsymbol{a} =$	$\boldsymbol{r} =$	$\boldsymbol{S_\infty} =$
$27, -9, 3, -1, \ldots$			
$p, p^2, p^3, p^4, \ldots$ $where\ -1 < p < 1$			
$p, 1, \frac{1}{p}, \frac{1}{p^2}, \ldots$			

The fourth term of a geometric series is 1.08 and the seventh term is 0.23328.

a) Show that this series is convergent.

b) Find the sum to infinity of this series.

For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a) Find the possible values of r.

b) Given that all the terms in the series are positive, find the value of a.

10. In a geometric series the common ratio is r and sum to n terms is S_n

Given

$$S_\infty = \frac{8}{7} \times S_6$$

show that $r = \pm\frac{1}{\sqrt{k}}$, where k is an integer to be found.

(4)

Your Turn

Edexcel C2 May 2011 Q6

6. The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio,
(2)

(*b*) the first term,
(2)

(*c*) the sum to infinity,
(2)

(*d*) the smallest value of n for which the sum of the first n terms of the series exceeds 1000.
(4)

Ex 3E

Sigma Notation

What does each bit of this expression mean?

The Greek letter, capital sigma, means 'sum'.

$$\sum_{r=1}^{5}(2r+1)$$

The numbers top and bottom tells us what r varies between. **<u>It goes up by 1 each time.</u>**

We work out this expression for each value of r (between 1 and 5), and add them together.

$$\underbrace{=3}_{r=1}\quad\underbrace{+5}_{r=2}\quad\underbrace{+7}_{r=3}\quad\underbrace{+9}_{r=4}\quad\underbrace{+11}_{r=5}\quad=35$$

If the expression being summed (in this case $2r+1$) is **linear**, we get an **arithmetic series**. We can therefore apply our usual approach of establishing a, d and n before applying the S_n formula.

	First few terms?	Values of a, n, d or r?	Final result?
$\sum_{n=1}^{7} 3n$			
$\sum_{k=5}^{15} (10-2k)$			
$\sum_{k=1}^{12} 5\times 3^{k-1}$			

On your calculator

$$\sum_{k=5}^{12} 2 \times 3^k$$

Given that $\sum_{r=1}^{k} 2 \times 3^r = 59\,046$,

a show that $k = \dfrac{\log 19\,683}{\log 3}$

b For this value of k, calculate $\sum_{r=k+1}^{13} 2 \times 3^r$.

Exam Question - challenging!

13. Given that p is a positive constant,

(a) show that

$$\sum_{n=1}^{11} \ln(p^n) = k \ln p$$

where k is a constant to be found,

(2)

(b) show that

$$\sum_{n=1}^{11} \ln(8p^n) = 33 \ln(2p^2)$$

(2)

(c) Hence find the set of values of p for which

$$\sum_{n=1}^{11} \ln(8p^n) < 0$$

giving your answer in set notation.

(2)

Recurrence Relations - a **VERY** popular exam topic

$$u_n = 2n^2 + 3$$

This is an example of a position-to-term sequence, because each term is based on the position n.

$$u_{n+1} = 2u_n + 4$$
$$u_1 = 3$$

We need the first term because the recurrence relation alone is not enough to know what number the sequence starts at.

But a term might be defined based on previous terms.
If u_n refers to the current term, u_{n+1} refers to the next term.
So the example in words says "the next term is twice the previous term + 4"

This is known as a term-to-term sequence, or more formally as a **recurrence relation**, as the sequence 'recursively' refers to itself.

Important Note: With recurrence relation questions, the **the sequence will likely not be arithmetic nor geometric**. So your previous u_n and S_n formulae do not apply.

6. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$
$$x_{n+1} = (x_n)^2 - kx_n, \qquad n \geq 1,$$

where k is a constant.

(*a*) Find an expression for x_2 in terms of k.

(1)

(*b*) Show that $x_3 = 1 - 3k + 2k^2$.

(2)

Given also that $x_3 = 1$,

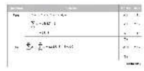
(*c*) calculate the value of k.

(3)

(*d*) Hence find the value of $\sum_{n=1}^{100} x_n$.

(3)

3. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by



$$a_1 = 3$$

$$a_{n+1} = \frac{a_n - 3}{a_n - 2}, \qquad n \in \mathbb{N}$$

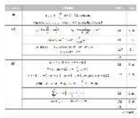
(a) Find $\displaystyle\sum_{r=1}^{100} a_r$

(3)

(b) Hence find $\displaystyle\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r$

(1)

15. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by



$$a_{n+1} = k - \frac{3k}{a_n} \qquad n \in \mathbb{Z}^+$$

where k is a constant.

The sequence is periodic of order 3

Given that $a_2 = 2$

(a) show that $k^2 + k - 12 = 0$

(3)

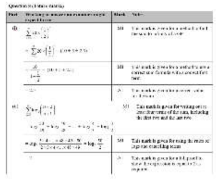
Given that $a_1 \neq a_2$

(b) find the value of $\sum_{r=1}^{121} a_r$

(4)

8. (i) Find the value of

$$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^{r}$$



(3)

(ii) Show that

$$\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) = 2$$

(3)

Edexcel C1 Jan 2012

4. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = a x_n + 5, \qquad n \geq 1,$$

where a is a constant.

(*a*) Write down an expression for x_2 in terms of a.

(1)

(*b*) Show that $x_3 = a^2 + 5a + 5$.

(2)

Given that $x_3 = 41$

(*c*) find the possible values of a.

(3)

Ex 3G

Increasing, decreasing and periodic sequences

A sequence is **strictly increasing** if the terms are always increasing, i.e. $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.
e.g. $1, 2, 4, 8, 16, \ldots$

Similarly a sequence is **strictly decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$

A sequence is **periodic** if the terms repeat in a cycle. The **order** k of a sequence is **how often it repeats**, i.e. $u_{n+k} = u_n$ for all n.
e.g. 2, 3, 0, 2, 3, 0, 2, 3, 0, 2, … is periodic and has order 3.

For each sequence:

i) State whether the sequence is increasing, decreasing or periodic.

ii) If the sequence is periodic, write down its order.

a) $u_{n+1} = u_n + 3, \; u_1 = 7$

b) $u_{n+1} = (u_n)^2, \quad u_1 = \frac{1}{2}$

c) $u_{n+1} = \sin(90n°)$

Ex 3H

Modelling

Anything involving compound changes (e.g. bank interest) will form a geometric sequence, as there is a constant ratio between terms.
We can therefore use formulae such as S_n to solve problems.

Bruce starts a new company. In year 1 his profits will be £20 000. He predicts his profits to increase by £5000 each year, so that his profits in year 2 are modelled to be £25 000, in year 3, £30 000 and so on. He predicts this will continue until he reaches annual profits of £100 000. He then models his annual profits to remain at £100 000.

a) Calculate the profits for Bruce's business in the first 20 years.
b) State one reason why this may not be a suitable model.
c) Bruce's financial advisor says the yearly profits are likely to increase by 5% per annum. Using this model, calculate the profits for Bruce's business in the first 20 years.

Ex 3I

8. There were 2100 tonnes of wheat harvested on a farm during 2017.

The mass of wheat harvested during each subsequent year is expected to increase by 1.2% per year.

(a) Find the total mass of wheat expected to be harvested from 2017 to 2030 inclusive, giving your answer to 3 significant figures.

(2)

Each year it costs

- £5.15 per tonne to harvest the first 2000 tonnes of wheat
- £6.45 per tonne to harvest wheat in excess of 2000 tonnes

(b) Use this information to find the expected cost of harvesting the wheat from 2017 to 2030 inclusive. Give your answer to the nearest £1000

(3)

12. A company extracted 4500 tonnes of a mineral from a mine during 2018.

The mass of the mineral which the company expects to extract in each subsequent year is modelled to decrease by 2% each year.

(a) Find the total mass of the mineral which the company expects to extract from 2018 to 2040 inclusive, giving your answer to 3 significant figures.

(2)

(b) Find the mass of the mineral which the company expects to extract during 2040, giving your answer to 3 significant figures.

(2)

The costs of extracting the mineral each year are assumed to be:

- £800 per tonne for the first 1500 tonnes
- £600 per tonne for any amount in excess of 1500 tonnes

The expected cost of extracting the mineral from 2018 to 2040 inclusive is £x million.

(c) Find the value of x, giving your answer to 3 significant figures.

(3)

6. A small company which makes batteries for electric cars has a 10-year plan for growth.

- In year 1 the company will make 2 600 batteries
- In year 10 the company aims to make 12 000 batteries

In order to calculate the number of batteries it will need to make each year, from year 2 to year 9, the company considers two models, Model A and Model B.

In Model A the number of batteries made will increase by the same **number** each year.

(a) Using Model A, determine the number of batteries the company will make in year 2

(3)

In Model B the number of batteries made will increase by the same **percentage** each year.

(b) Using Model B, determine the number of batteries the company will make in year 2

Give your answer to the nearest 10 batteries.

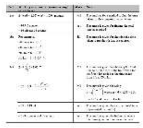
(3)

Sam calculates the total number of batteries made from year 1 to year 10 inclusive using each of the two models.

(c) Calculate the difference between the two totals, giving your answer to the nearest 100 batteries.

(3)

11. A competitor is running a 20 kilometre race.



She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre.
After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds,

(2)

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \leqslant r \leqslant 20$, is

$$6 \times 1.05^{r-4}$$

(1)

(c) estimate the total time, in minutes and seconds, that she will take to complete the race.

(4)