Edexcel A Level Further Maths: Core Pure

2.2 Transformations using Matrices

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2.2.1 Transformations using a Matrix

Transformations using a Matrix

What is a transformation matrix?

- A transformation matrix is used to determine the coordinates of an **image** from the **transformation** of an **object**
  - reflections, rotations, enlargements and stretches
  - Commonly used transformation matrices include
- (In 2D) a multiplication by any 2×2 matrix could be considered a transformation (in the 2D plane)
  - This can be done similarly in higher dimensions
- An individual point in the plane can be represented as a position vector, $\begin{pmatrix} x \\ y \end{pmatrix}$
  - Several points, that create a shape say, can be written as a position matrix $\begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ y_1 & y_2 & y_3 & \cdots \end{pmatrix}$
- A matrix transformation will be of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
  - where $\begin{pmatrix} x \\ y \end{pmatrix}$ represents any point in the 2D plane
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a given matrix

How do I find the coordinates of an image under a transformation?

- The coordinates (x', y') - the image of the point (x, y) under the transformation with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- Similarly, for a position matrix

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 & \cdots \\ y'_1 & y'_2 & y'_3 & \cdots \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ y_1 & y_2 & y_3 & \cdots \end{pmatrix}$$

- A calculator can be used for matrix multiplication

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  - If matrices involved are small, it may be as quick to do this manually

- **STEP 1**
  Determine the transformation matrix (**T**) and the position matrix (**P**)
  The transformation matrix, if uncommon, will be given in the question
  The position matrix is determined from the coordinates involved, it is best to have the coordinates in order, to avoid confusion

- **STEP 2**
  Set up and perform the matrix multiplication required to determine the image position matrix, **P'**
  P' = **TP**

- **STEP 3**
  Determine the coordinates of the image from the image position matrix, **P'**

How do I find the coordinates of the original point given the image under a transformation?

- To 'reverse' a transformation we would need the **inverse transformation matrix**
  - i.e. $\boldsymbol{T}^{-1}$
  - For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is given by $\dfrac{1}{\det \boldsymbol{T}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
    - where $\det \boldsymbol{T} = ad - bc$
  - A calculator can be used to work out inverse matrices
- You would rearrange $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
  - $\dfrac{1}{\det \boldsymbol{T}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Examiner Tip

- Read the question carefully to determine if you have the points before or after a transformation

Worked example

A quadrilateral, Q, has the four vertices A(2, 5), B(5, 9), C(11, 9) and D(8, 5).

Find the coordinates of the image of Q under the transformation $\boldsymbol{T} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$.

STEP 1: Determine the transformation and position matrices

$$\underset{\sim}{T} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \qquad \underset{\sim}{P} = \begin{pmatrix} 2 & 5 & 11 & 8 \\ 5 & 9 & 9 & 5 \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ A & B & C & D \end{array}$$

STEP 2: $\underset{\sim}{P}' = \underset{\sim}{T}\underset{\sim}{P}$

$$\underset{\sim}{P}' = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 & 11 & 8 \\ 5 & 9 & 9 & 5 \end{pmatrix}$$

$$\underset{\sim}{P}' = \begin{pmatrix} 6-5 & 15-9 & 33-9 & 24-5 \\ -2+10 & -5+18 & -11+18 & -8+10 \end{pmatrix}$$

$$\underset{\sim}{P}' = \begin{pmatrix} 1 & 6 & 24 & 19 \\ 8 & 13 & 7 & 2 \end{pmatrix}$$

Use a calculator for matrix multiplication

STEP 3: Determine the image coordinates from P'

$\boxed{A'(1, 8) \quad B'(6, 13) \quad C'(24, 7) \quad D'(19, 2)}$

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Determinant of a Transformation Matrix

What does the determinant of a transformation matrix (A) represent?

- The **absolute value** of the **determinant** of a transformation matrix is the **area scale factor** (2D) or **volume scale factor** (3D)
  - Area scale factor = $|\det \boldsymbol{A}|$ if 2×2
  - Volume scale factor = $|\det \boldsymbol{A}|$ if 3×3
- The area/volume of the **image** will be **product** of the **area/volume** of the **object** and $|\det \boldsymbol{A}|$
  - Area of image = $|\det \boldsymbol{A}|$ × Area of object (if 2×2)
  - Volume of image = $|\det \boldsymbol{A}|$ × Volume of object (if 3×3)
- Note the area will reduce if $|\det \boldsymbol{A}| < 1$
- If the determinant is **negative** then the **orientation** of the shape will be **reversed**
  - For example: the shape has been reflected

How do I solve problems involving the determinant of a transformation matrix?

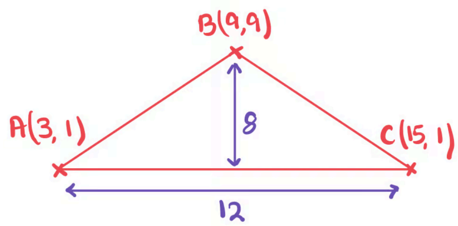
- Problems may involve comparing areas of **objects** and **images**
  - This could be as a percentage, proportion, etc
- Missing value(s) from the transformation matrix (and elsewhere) can be deduced if the determinant of the transformation matrix is known
- Remember to use the **absolute value** of the determinant
  - This can lead to multiple answers to equations
  - Use your calculator to solve these

Worked example

An isosceles triangle has vertices A(3, 1), B(15, 1) and C(9, 9).

a) Find the area of the isosceles triangle.

A sketch will help find the area



Area $= \frac{1}{2} \times 12 \times 8$ ("$A = \frac{1}{2}bh$")

$\boxed{\therefore \text{Area of } \Delta ABC = 48 \text{ square units}}$

b) Triangle $\triangle$ABC is transformed using the matrix $\boldsymbol{T} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$. Find the area of the transformed triangle.

Area scale factor is $|\det \underset{\sim}{T}|$

$|\det \underset{\sim}{T}| = 3 \times 2 - 2 \times -1 = 8$

$\therefore$ Area of image $= 48 \times 8 = 384$

$\boxed{\text{Area of transformed triangle} = 384 \text{ square units}}$

c) Triangle $\triangle$ABC is now transformed using the matrix $\boldsymbol{U} = \begin{pmatrix} a & -2 \\ 3 & a^2 \end{pmatrix}$ where $a \in \mathbb{Z}$. Given that the area of the image is twice as large as the area of the object, find the value of a.

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$\det \underset{\sim}{U} = a \times a^2 - -2 \times 3 = a^3 + 6$

$\therefore |a^3+6| = 2$

For $a^3+6=2$, $a^3=-4$, $a \notin \mathbb{Z}^-$, reject

For $a^3+6=-2$, $a^3=-8$, $a=-2$, $a \in \mathbb{Z}^-$

$\boxed{\therefore a=-2}$

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2.2.2 Geometric Transformations with Matrices

2D Transformations

What is meant by a 2D geometric transformation?

- The following transformations can be represented (in 2D) using **multiplication** of a **2×2** matrix
  - rotations (about the origin)
  - reflections
  - enlargements
  - (horizontal) stretches parallel to the x-axis
  - (vertical) stretches parallel to the y-axis

What are the matrices for geometric transformations?

- **Rotation**
  - Anticlockwise (or counter-clockwise) through angle θ about the origin
    - $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
      - This is given in the **formula booklet**
  - Clockwise through angle θ about the origin
    - $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
  - In both cases
    - θ > 0
    - θ may be measured in degrees or radians
- **Reflection**
  - In the line $y=(\tan\theta)x$
    - $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
      - This is given in the **formula booklet**
  - θ may be measured in degrees or radians
  - for a reflection in the x-axis, θ = 0° (0 radians)
  - for a reflection in the y-axis, θ = 90° (π/2 radians)
- **Enlargement**
  - Scale factor *k*, centre of enlargement at the origin (0, 0)
    - $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
- **Horizontal stretch** (or stretch parallel to the x-axis)
  - Scale factor *k*

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- - $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
- **Vertical stretch** (or stretch parallel to the y-axis)
  - Scale factor *k*
    - $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

How do I find the matrix of a 2D transformation?

- Let the transformation matrix be $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- The image of the point (*x*, *y*) after the transformation is (*x*', *y*') which can be found by:
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$
- You can find the values of *a, b, c, d* by seeing where the points (1, 0) and (0, 1) are transformed
  - (1, 0) is transformed to (*a*, c)
    - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$
  - (0, 1) is transformed to (*b*, *d*)
    - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

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Worked example

Triangle PQR has coordinates P(-1, 4), Q(5, 4) and R(2, -1).

The transformation ***T*** is a reflection in the line $y = x\sqrt{3}$.

a) Find the matrix ***T*** that represents a reflection in the line $y = x\sqrt{3}$.

From formula booklet: Reflection in line $y = (\tan\theta)x$

is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$y = x\sqrt{3}$, $\therefore \tan\theta = \sqrt{3}$, $\theta = 60°$

$\therefore \underset{\sim}{T} = \begin{pmatrix} \cos 120° & \sin 120° \\ \sin 120° & -\cos 120° \end{pmatrix}$

$$\boxed{\underset{\sim}{T} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}}$$

b) Find the position matrix, ***P'***, representing the coordinates of the images of points P, Q and R under the transformation ***T.***

Your notes

$\underset{\sim}{P}' = \underset{\sim}{T}\,\underset{\sim}{P}$ ("$\underset{\sim}{P}' = \underset{\sim}{A}\,\underset{\sim}{P}$")

$$\therefore \underset{\sim}{P}' = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{array}{c} \begin{pmatrix} -1 & 5 & 2 \\ 4 & 4 & -1 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ P & Q & R \end{matrix} \end{array}$$

position matrix P

(Use a calculator for matrix multiplication)

$$\boxed{\underset{\sim}{P}' = \begin{pmatrix} \frac{1}{2}(1+4\sqrt{3}) & -\frac{1}{2}(5-4\sqrt{3}) & -\frac{1}{2}(2+\sqrt{3}) \\ \frac{1}{2}(4-\sqrt{3}) & \frac{1}{2}(4+5\sqrt{3}) & -\frac{1}{2}(1-2\sqrt{3}) \end{pmatrix}}$$

Be careful copying a calculator display

$$-\frac{2+\sqrt{3}}{2} \neq \frac{-2+\sqrt{3}}{2}$$

Successive Transformations

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The order in which transformations occur can lead to different results – for example a reflection in the x-axis followed by clockwise rotation of 90° is different to rotation first, followed by the reflection.

Therefore, when one transformation is followed by another order is critical.

What is a composite transformation?

- A composite function is the result of applying more than one function to a point or set of points
  - e.g. a **rotation**, followed by an **enlargement**
- It is possible to find a **single** composite function **matrix** that does the same job as applying the individual transformation matrices

How do I find a single matrix representing a composite transformation?

- Multiplication of the transformation matrices
- However, the order in which the matrices is important
  - If the transformation represented by matrix **M** is applied first, and is then followed by another transformation represented by matrix **N**
    - the composite matrix is **NM**
      e. **P'** = **NMP**
      (**NM** is not necessarily equal to **MN**)
    - The matrices are **applied** right to left
    - The composite function matrix is **calculated** left to right
  - Another way to remember this is, starting from **P**, always **pre-multiply** by a transformation matrix
    - This is the same as applying **composite functions** to a value
    - The function (or matrix) furthest to the right is applied first

How do I apply the same transformation matrix more than once?

- If a transformation, represented by the matrix **T**, is applied twice we would write the composite transformation matrix as $\mathbf{T}^2$
  - $\mathbf{T}^2 = \mathbf{TT}$
- This would be the case for any number of repeated applications
  - $\mathbf{T}^5$ would be the matrix for five applications of a transformation
- A calculator can quickly calculate $\mathbf{T}^2$, $\mathbf{T}^5$, etc
- Problems may involve considering patterns and sequences formed by repeated applications of a transformation
  - The coordinates of point(s) follow a particular pattern
    - (20, 16) – (10, 8) – (5, 4) – (2.5, 2) ...
  - The area of a shape increases/decreases by a constant factor with each application

e.g. if one transformation doubles the area then three applications will increase the (original) area eight times (2^3)

Your notes

Examiner Tip

- When performing multiple transformations on a set of points, make sure you put your transformation matrices in the correct order, you can check this in an exam but sketching a diagram and checking that the transformed point ends up where it should
- You may be asked to show your workings but you can still check that you have performed you matrix multiplication correctly by putting it through your calculator

Your notes

Worked example

The matrix **E** represents an enlargement with scale factor 0.25, centred on the origin.
The matrix **R** represents a rotation, 90° anticlockwise about the origin.

a) Find the matrix, **C**, that represents a rotation, 90° anticlockwise about the origin followed by an enlargement of scale factor 0.25, centred on the origin.

$\underset{\sim}{C} = \underset{\sim}{E}\underset{\sim}{R}$

$\underset{\sim}{C} = \underbrace{\begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}}_{\substack{\text{enlargement} \\ k=0.25}} \underbrace{\begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix}}_{\substack{\text{rotation, clockwise} \\ \theta=90°}}$

Use the formula booklet

$\underset{\sim}{C} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Use a calculator for matrix multiplication

$\boxed{\therefore \underset{\sim}{C} = \begin{pmatrix} 0 & -0.25 \\ 0.25 & 0 \end{pmatrix}}$

b) A square has position matrix $T_0 = \begin{pmatrix} 0 & 0 & 256 & 256 \\ 0 & 256 & 256 & 0 \end{pmatrix}$. T_n represents the position matrix of the image square after it has been transformed n times by matrix **C**. Find T_4

$\underset{\sim}{T_4} = \underset{\sim}{C}^4 \underset{\sim}{T_0} = \begin{pmatrix} 0 & -0.25 \\ 0.25 & 0 \end{pmatrix}^4 \begin{pmatrix} 0 & 0 & 256 & 256 \\ 0 & 256 & 256 & 0 \end{pmatrix}$

Use a calculator, typing this in carefully as one calculation

$\boxed{\therefore \underset{\sim}{T_4} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}}$

c) Find the single transformation matrix that would map T_4 to T_0.

Your notes

$\underset{\sim}{T_4}$ to $\underset{\sim}{T_0}$ would be the inverse of $\underset{\sim}{C}^4$.

(Note that $[C^4]^{-1}$ does not mean $\underset{\sim}{C}^{-4}$)

Use a calculator to find $[C^4]^{-1}$ in one calculation

$$\boxed{[C^4]^{-1} = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}}$$

3D Transformations

Transforming 3D coordinates with matrices

- We can apply transformations to coordinates in 3D the same way that we apply them in 2D
- We do this by multiplying the transformation matrix by the position vector we wish to transform
- Rather than transforming 2×1 matrices (2D position vectors) we are now transforming 3×1 matrices (3D position vectors), $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
  - You can group together coordinates into a larger position matrix
    - For example, all four vertices of a rectangle in 3D can become a 3×4 position matrix, $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$
    - This is helpful as you can transform the entire shape in one matrix multiplication
- 3D transformations will be confined to
  - A **reflection** in one of x=0, y=0, or z=0
  - A **rotation** about one of the coordinate axes
- As with 2D transformations, the transformation matrix describes how the unit vectors in each direction (**i**, **j**, and **k**) are mapped
  - For a transformation matrix **T**, $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
    - The image of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} a \\ d \\ g \end{bmatrix}$
    - The images of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are the 2nd and 3rd columns of **T** respectively

Reflection matrices in 3D

- A reflection in the plane x=0 is given by the matrix $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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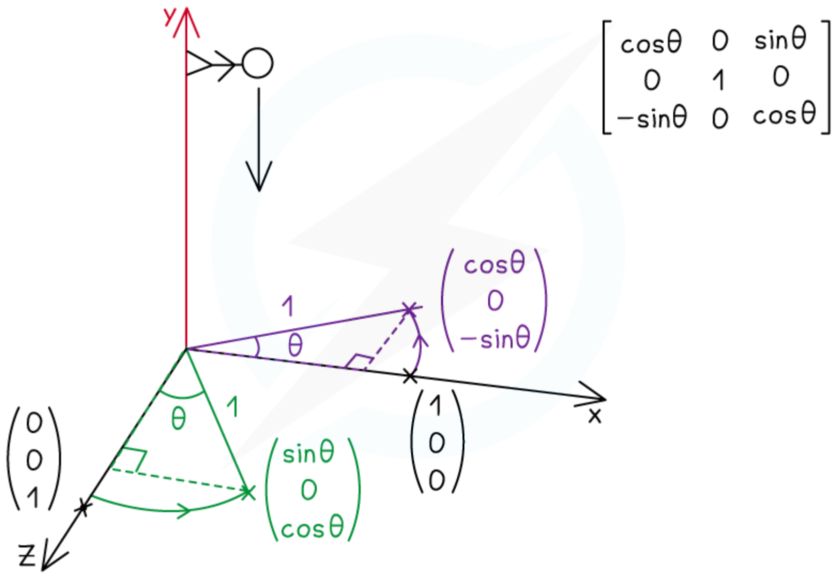
- Notice that the first column, the image of **i**, is multiplied by -1, or 'mirrored', whilst **j** and **k** stay the same
- A reflection in the plane y=0 is given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- A reflection in the plane z=0 is given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
- You are **not** given these transformation matrices in the formula book

Rotation matrices in 3D

- An anticlockwise rotation around the z-axis by angle θ is given by $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - Notice that the z coordinate of a point would be unchanged by this transformation
  - This makes it equivalent to a rotation around the origin in 2D
    - Therefore the top left corner of this matrix is the same as the 2×2 matrix for an anticlockwise rotation around the origin, given in the formula book
- An anticlockwise rotation around the x-axis by angle θ is given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
  - Notice that the x coordinate is unaffected
- An anticlockwise rotation around the y-axis by angle θ is given by $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
  - Notice that the y coordinate is unaffected
- When we describe an "anticlockwise rotation around the x/y/z-axis" this is from the perspective of standing on the positive axis in question, looking towards the origin
- You are **not** given these transformation matrices in the formula book
  - You are however given the matrix for an anticlockwise rotation about the origin in 2D, which may be useful; $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Your notes

ROTATION ANTI-CLOCKWISE AROUND y-AXIS



Examiner Tip

- When describing a rotation, remember to state the axis and direction of rotation
- Use your calculator where possible for matrix multiplication, or checking your answer to matrix multiplication if required to show working

Your notes

Worked example

The 3×3 matrix **T**, represents an anticlockwise rotation around the x-axis by 120°.

a) Find the matrix **T**.

anti-clockwise, x-axis, $120°$

$$\underset{\sim}{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 120° & -\sin 120° \\ 0 & \sin 120° & \cos 120° \end{pmatrix}$$

Only the 2x2 rotation matrix is in the formulae booklet

$$\boxed{\therefore \underset{\sim}{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}$$

b) Find the image of point A (2,4,6) under the transformation represented by **T**.

$$\underset{\sim}{A}' = \underset{\sim}{T}\,\underset{\sim}{A}$$

$$\underset{\sim}{A}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

Use a calculator for matrix multiplication

$$\underset{\sim}{A}' = \begin{pmatrix} 2 \\ -2-3\sqrt{3} \\ 2\sqrt{3}-3 \end{pmatrix}$$

$\boxed{\therefore \text{ Image of point A has coordinates } (2, -2-3\sqrt{3}, -3+2\sqrt{3})}$

c) The image of A is now reflected in the plane z=0, and labelled B. Find the coordinates of B.

A reflection in the plane $z=0$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This is NOT in the formulae booklet

$$\therefore \underset{\sim}{A}'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -2-3\sqrt{3} \\ -3+2\sqrt{3} \end{pmatrix}$$

Use a calculator

$$\underset{\sim}{A}'' = \begin{pmatrix} 2 \\ -2-3\sqrt{3} \\ 3-2\sqrt{3} \end{pmatrix}$$

$\boxed{\therefore \text{Point B has coordinates } (2, -2-3\sqrt{3}, 3-2\sqrt{3})}$

Your notes

2.2.3 Invariant Points & Lines

Your notes

Invariant Points

What is an invariant point?

- When applying transformations to a shape or collection of points, there may be some points that stay in their original position; these are known as **invariant points**

How can I find invariant points?

- If the point given by position vector $\mathbf{x}$; $\begin{pmatrix} x \\ y \end{pmatrix}$, is invariant under transformation $\mathbf{T}$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we can say that $\mathbf{Tx}=\mathbf{x}$
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$
- This will create a system of simultaneous equations which can be solved to find the invariant point
  - $\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$
    - $(a-1)x+by=0$
    - $cx+(d-1)y=0$
- The origin (0,0) is always invariant under a linear transformation

Examiner Tip

- Where the question allows, use your calculator to help solve the simultaneous equations
- Test your found invariant point by multiplying it by the transformation matrix, and making sure you still end up with the same point (invariant)

Your notes

Worked example

Find any invariant points under the transformation given by $\mathbf{T}=\begin{pmatrix}4 & 2\\ -2 & 6\end{pmatrix}$.

For an invariant point $Tx = x$

$$\begin{pmatrix}4 & 2\\ -2 & 6\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

Multiplying $\begin{pmatrix}4x+2y\\ -2x+6y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$

top row ① $3x+2y=0$

bottom row ② $-2x+5y=0$

Solve on calculator or by hand

$x=0 \quad y=0$

$\boxed{\text{(0,0) is the only invariant point}}$

Your notes

A Line of Invariant Points

What is a line of invariant points?

- If every point on a line is mapped to itself under a particular transformation, then it is **a line of invariant points**
  - For example, a line of reflection is a line of invariant points

How can I find a line of invariant points?

- Use the same strategy as for finding a single invariant point:
  - If the point given by position vector $\mathbf{x}$; $\begin{pmatrix} x \\ y \end{pmatrix}$, is invariant under transformation $\mathbf{T}$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we can say that $\mathbf{Tx} = \mathbf{x}$
    - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
  - This will create a system of simultaneous equations which can be solved to find the invariant point(s)
    - $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
    - $(a-1)x + by = 0$
    - $cx + (d-1)y = 0$
- If there is a line of invariant points, rather than solving to find a single solution (a point), the two equations will be able to simplify to the same equation
  - This means that there are infinitely many solutions, and therefore infinitely many invariant points
    - A line contains infinitely many points
  - Your solution will be the equation of the invariant line e.g. y=3x

Examiner Tip

- It may not always be obvious that the two equations reduce to the same thing (they could be an awkward multiple of each other)
- Use your calculator's simultaneous equation solver; it will tell you that there are infinitely many solutions

Your notes

Worked example

Find the equation of the line of invariant points under the transformation given by $\mathbf{T} = \begin{pmatrix} 7 & -2 \\ 6 & -1 \end{pmatrix}$

For an invariant point $T\mathbf{x} = \mathbf{x}$

$$\begin{pmatrix} 7 & -2 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Multiplying $\begin{pmatrix} 7x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

top row ① $6x - 2y = 0$

bottom row ② $6x - 2y = 0$

Both reduce to $6x = 2y \Rightarrow y = 3x$

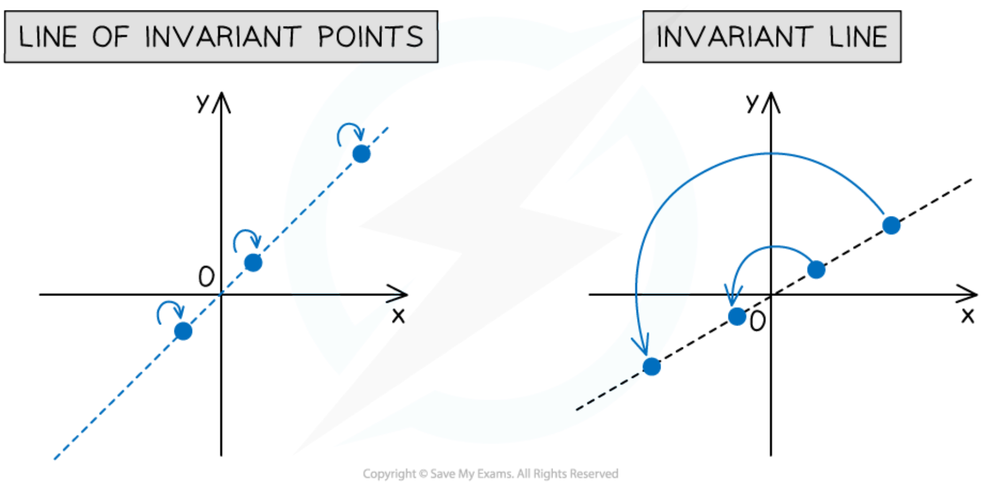
$\boxed{y = 3x \text{ is a line of invariant points}}$

Your notes

Invariant Lines

What's the difference between a line of invariant points and an invariant line?

- If every point on a line is mapped to itself under a particular transformation, then it is **a line of invariant points**
  - Every single point on the line must stay in the same place
- With an **invariant line** however, every point on the line must simply map to another point on the same line
  - We are only concerned with the overall line; not the individual points



How do I find an invariant line?

- We can use a similar strategy to finding invariant points, with two slight changes
  - Use $y = mx + c$ to write the original position vector as $\begin{pmatrix} x \\ mx + c \end{pmatrix}$
  - Write the transformed position vector as $\begin{pmatrix} x' \\ mx' + c \end{pmatrix}$ using the same idea
    - Notice that the values of m and c will be the same, but different *x* and *y* coordinates
    - This because it is a different point, on the same line
- For an invariant line under transformation $\mathbf{T}$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we can write
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$
- This will create a system of simultaneous equations which can be solved to find the invariant line(s)
  - $ax + b(mx + c) = x'$

- $cx + d(mx + c) = mx' + c$
- The first equation can be substituted into the second to give an equation in terms of the variable *x* and the constants *m* and c
- This equation can then be solved to find the values of *m* and c by equating the coefficients of *x*, and then equating the constant terms
  - There may be multiple solutions for *m* and c if there are multiple invariant lines

Your notes

Worked example

Find the equation of any invariant lines under the transformation $\mathbf{T} = \begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix}$

Your notes

Your notes

For an invariant line $T\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$$

Multiplying $\begin{pmatrix} 2x - (mx+c) \\ -3x \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$

top row ① $2x - (mx+c) = x'$

bottom row ② $-3x = mx' + c$

Substitute ① into ②

$$-3x = m(2x - mx - c) + c$$

$$-3x = 2mx - m^2x - mc + c$$

$$x(m^2 - 2m - 3) + mc - c = 0$$

Comparing coefficients of x on both sides

$$m^2 - 2m - 3 = 0 \Rightarrow m = 3 \text{ or } m = -1$$

Comparing constant terms on both sides

$$mc - c = 0$$

when $m=3$	when $m=-1$
$3c - c = 0$	$-c - c = 0$
$c = 0$	$c = 0$

$\boxed{y = 3x \text{ and } y = -x \text{ are invariant lines}}$