Ex 9G

Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = \sqrt{3}\sin 2t, \qquad y = 4\cos^2 t, \qquad 0 \le t \le \pi.$$

(*a*) Show that $\frac{dy}{dx} = k\sqrt{3}\tan 2t$, where k is a constant to be determined.

(*b*) Find an equation of the tangent to C at the point where $t = \frac{\pi}{3}$.

Give your answer in the form $y = ax + b$, where a and b are constants.

(a)	$\frac{dx}{dt} = 2\sqrt{3}\cos 2t$	B1
	$\frac{dy}{dt} = -8\cos t\sin t$	M1 A1
	$\frac{dy}{dx} = \frac{-8\cos t\sin t}{2\sqrt{3}\cos 2t}$	M1
	$= -\frac{4\sin 2t}{2\sqrt{3}\cos 2t}$	
	$\frac{dy}{dx} = -\frac{2}{3}\sqrt{3}\tan 2t \qquad \left(k = -\frac{2}{3}\right)$	A1
(b)	When $t = \frac{\pi}{3}$ $\qquad x = \frac{3}{2},\ y = 1$ $\qquad$ can be implied	B1
	$m = -\frac{2}{3}\sqrt{3}\tan\left(\frac{2\pi}{3}\right) \quad (=2)$	M1
	$y - 1 = 2\left(x - \frac{3}{2}\right)$	M1
	$y = 2x - 2$	A1

Differentiate with respect to x

$f(x) = \ln(\sin x)$ $\quad f'(x) = \cot x$ $\quad \int \cot x \, dx = \ln|\sin x| + c$

$g(x) = \sec(x^3)$ $\quad g'(x) = 3x^2 \sec(x^3)\tan(x^3)$

$h(x) = 2xe^x$ $\quad h'(x) = 2e^x + 2xe^x$

$u = 2x \quad v = e^x$

$u' = 2 \quad v' = e^x$

$x = 3\cos\left(\frac{y}{2}\right)$ Find $\frac{dy}{dx}$

$m(x) = (x^2 + 1)\cot x$ $\quad m'(x) = 2x\cot x - (x^2+1)\operatorname{cosec}^2 x.$

$u = x^2+1 \quad v = \cot x$

$u' = 2x \quad v' = -\operatorname{cosec}^2 x$

$n(x) = e^{(2x)}\ln x$ $\quad n'(x) = 2(\ln x)e^{2x} + \frac{e^{2x}}{x}$

$u = e^{2x} \quad v = \ln x$

$u' = 2e^{2x} \quad v' = \frac{1}{x}$

Implicit Differentiation

Implicit Functions

$x^2 + y^2 = 16$

$2y + 15x - xy = 0$

$-3\sin(xy) + 15x = y$

our topic today

There is a mixture of x and y terms.

y is not the subject.

Explicit Functions

$y = 2x + 5$

$y = \sin^2(x+3)$

$y = 13 - \ln x$

we can already differentiate these.

y is only in terms of x.

y is the subject

$$y = x^2$$

diff. w.r.t. x $\quad \frac{d}{dx} \qquad \frac{d}{dx}$

$$\frac{dy}{dx} = 2x$$

$\frac{d}{dx} y \to \frac{dy}{dx}$

To differentiate implicitly you only need to know 2 things:

- Differentiate each side of the equation (using chain rule if necessary).
- Remember that y differentiated with respect to x is, by definition, $\frac{dy}{dx}$

Try

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx}$$

and then multiply by $\frac{dy}{dx}$

$$= 2y\frac{dy}{dx}$$

$x = y^2$

$1 = 2y\frac{dy}{dx}$

$\frac{1}{2y} = \frac{dy}{dx}$

$x = y^2$

$x^{1/2} = y$

$\frac{1}{2}x^{-1/2} = \frac{dy}{dx}$

$\frac{1}{2} \times \frac{1}{y} = \frac{dy}{dx} = \frac{1}{2y}$

$x^{1/2} = y$

$x^{-1/2} = \frac{1}{y}$

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx}$$
$$= 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(x^2 + \cos y) = 2x - \sin y \times \frac{dy}{dx}$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y)\frac{dy}{dx}$$
$$= \cos y\frac{dy}{dx}$$

$$\frac{d}{dx}(e^y) = e^y\frac{dy}{dx}$$

derivative of blah

$$\frac{d}{dx}\left(e^{x^2 y}\right) = e^{x^2 y} \times \frac{d}{dx}(x^2 y)$$
$$= e^{x^2 y}\left(2xy + x^2\frac{dy}{dx}\right)$$

$u = x^2 \quad v = y$

$u' = 2x \quad v' = \frac{dy}{dx}$

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

$u = x \quad v = y$

$u' = 1 \quad v' = \frac{dy}{dx}$

$$\frac{d}{dx}(\tan(x+y)) = \sec^2(x+y) \times \left(1 + \frac{dy}{dx}\right)$$
$$= \left(1 + \frac{dy}{dx}\right)\sec^2(x+y)$$

In general, when differentiating a function of y, but with respect to x, slap a $\frac{dy}{dx}$ on the end. i.e.
$$\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$$

Find $\frac{dy}{dx}$ in terms of x and y where $x^3 + x + y^3 + 3y = 6$

$$x^3 + x + y^3 + 3y = 6$$

$$3x^2 + 1 + 3y^2\frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

rearranged so $\frac{dy}{dx}$ terms were on one side.

$$\frac{dy}{dx}(3y^2+3) = -3x^2 - 1$$

factorised $\frac{dy}{dx}$ terms

$$\frac{dy}{dx} = \frac{-3x^2-1}{3y^2+3}$$

Note: to find the gradient, you require an x AND y value.

Find the value of $\frac{dy}{dx}$ at the point $(1,1)$, where $e^{2x}\ln y = x + y - 2$

$x=1 \quad y=1$

$$e^{2x}\ln y = x + y - 2$$

$u = e^{2x} \quad v = \ln y$

$u' = 2e^{2x} \quad v' = \frac{1}{y}\frac{dy}{dx}$

$$2e^{2x}\ln y + e^{2x}\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

collect $\frac{dy}{dx}$ terms, then factorise.

$$2e^{2x}\ln y - 1 = \frac{dy}{dx} - e^{2x}\frac{1}{y}\frac{dy}{dx}$$

$$2e^{2x}\ln y - 1 = \frac{dy}{dx}\left(1 - e^{2x}\frac{1}{y}\right)$$

$$\frac{2e^{2x}\ln y - 1}{1 - e^{2x}\frac{1}{y}} = \frac{dy}{dx}$$

$x=1,\ y=1 \quad \frac{dy}{dx} = \frac{\cancel{2e^{2}\ln 1} - 1}{1 - e^{2} \times \frac{1}{1}} = \frac{-1}{1-e^{2}}$

$$= \frac{1}{e^{2}-1}$$

$x=1,\ y=1$

$$e^{2}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$e^{2}\frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$\frac{dy}{dx}(e^{2}-1) = 1 \rightarrow \frac{dy}{dx} = \frac{1}{e^{2}-1}$$

C4 Jan 2008 Q5

A curve is described by the equation

$$x^3 - 4y^2 = 12xy.$$

(*a*) Find the coordinates of the two points on the curve where $\underline{x = -8}$. **(3)**

Then solve

(*b*) Find the gradient of the curve at each of these points. **(6)**

a) $(-8)^3 - 4y^2 = 12(-8)y$

$-512 - 4y^2 = -96y$

$0 = 4y^2 - 96y + 512$

$y = 16 \quad y = 8$

$(-8, 16) \quad (-8, 8)$

b) $x^3 - 4y^2 = \underline{12xy}$

$u = 12x \quad v = y$

$u' = 12 \quad v' = \frac{dy}{dx}$

$3x^2 - 8y\frac{dy}{dx} = 12y + 12x\frac{dy}{dx}$

$3x^2 - 12y = 12x\frac{dy}{dx} + 8y\frac{dy}{dx}$

$3x^2 - 12y = \frac{dy}{dx}(12x + 8y)$

$\frac{dy}{dx} = \frac{3x^2 - 12y}{12x + 8y}$

$x = -8, y = 16$

$\frac{dy}{dx} = -3$

$x = -8 \; y = 8$

$\frac{dy}{dx} = 0.$

Ex 9H Q3, 5, 7, 9, 11.

C4 June 2014(R) Q3

$$x^2 + y^2 + 10x + 2y - 4xy = 10$$

(*a*) Find $\frac{dy}{dx}$ in terms of x and y, fully simplifying your answer. **(5)**

(*b*) Find the values of y for which $\frac{dy}{dx} = 0$. **(5)**

Hint for (b): Solve simultaneously with original equation.

	$x^2 + y^2 + 10x + 2y - 4xy = 10$		
(a)	$2x + 2y\frac{dy}{dx} + 10 + 2\frac{dy}{dx} - \left(4y + 4x\frac{dy}{dx}\right) = 0$	See notes	M1 A1 M1
	$2x + 10 - 4y + (2y + 2 - 4x)\frac{dy}{dx} = 0$	Dependent on the first M1 mark.	dM1
	$\frac{dy}{dx} = \frac{2x + 10 - 4y}{4x - 2y - 2}$		
	Simplifying gives $\frac{dy}{dx} = \frac{x + 5 - 2y}{2x - y - 1}$ $\left(= \frac{-x - 5 + 2y}{-2x + y + 1}\right)$		A1 cso oe
			[5]
(b)	$\left\{\frac{dy}{dx} = 0 \Rightarrow\right\}$ $x + 5 - 2y = 0$		M1
	So $x = 2y - 5$.		
	$(2y - 5)^2 + y^2 + 10(2y - 5) + 2y - 4(2y - 5)y = 10$		M1
	$4y^2 - 20y + 25 + y^2 + 20y - 50 + 2y - 8y^2 + 20y = 10$		
	gives $-3y^2 + 22y - 35 = 0$ or $3y^2 - 22y + 35 = 0$	$3y^2 - 22y + 35 \{= 0\}$ see notes	A1 oe
	$(3y - 7)(y - 5) = 0$ and $y = ...$	Method mark for solving a quadratic equation	dM1
	$y = \frac{7}{3}, 5$	$\{y =\} \frac{7}{3}, 5$	A1 cso
			[5]