Edexcel A Level Further Maths: Core Pure

Your notes

1.2 Exponential Form & de Moivre's Theorem

Contents

1.2.1 Exponential Form

Your notes

Exponential Form

You now know how to do lots of operations with complex numbers: add, subtract, multiply, divide, raise to a power and even square root. The last operation to learn is raising the number e to the power of an imaginary number.

How do we calculate e to the power of an imaginary number?

- Given an imaginary number (iθ) we can **define exponentiation** as
  - $e^{i\theta} = \cos\theta + i\sin\theta$
  - $e^{i\theta}$ is the **complex number** with **modulus 1** and **argument θ**
- This works with our current rules of exponents
  - $e^0 = e^{0i} = \cos 0 + i\sin 0 = 1$
    - This shows e to the power 0 would still give the answer of 1
  - $e^{i\theta_1} \times e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$
    - This is because when you **multiply complex numbers** you can **add** the **arguments**
    - This shows that when you multiply two powers you can still add the indices
  - $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1 - \theta_2)}$
    - This is because when you **divide complex numbers** you can **subtract** the **arguments**
    - This shows that when you divide two powers you can still subtract the indices

What is the exponential form of a complex number?

- Any complex number $z = a + bi$ can be written in **polar form** $z = r(\cos\theta + i\sin\theta)$
  - *r* is the modulus
  - θ is the argument
- Using the definition of $e^{i\theta}$ we can now also write z in **exponential form**
  - $z = re^{i\theta}$

Why do I need to use the exponential form of a complex number?

- It's just a **shorter** and **quicker** way of expressing complex numbers
- It makes a link between the **exponential function** and **trigonometric functions**
- It makes it easier to remember what happens with the moduli and arguments when multiplying and dividing

What are some common numbers in exponential form?

- As $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$ you can write:

- $1 = e^{2\pi i}$
- Using the same idea you can write:
  - $1 = e^0 = e^{2\pi i} = e^{4\pi i} = e^{6\pi i} = e^{2k\pi i}$ where *k* is any integer
- As $\cos(\pi) = -1$ and $\sin(\pi) = 0$ you can write:
  - $e^{\pi i} = -1$
  - Or more commonly written as $e^{i\pi} + 1 = 0$
- As $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ you can write:
  - $i = e^{\frac{\pi}{2}i}$

Examiner Tip

- The powers can be long and contain fractions so make sure you write the expression clearly.
- You don't want to lose marks because the examiner can't read your answer

Your notes

Your notes

Worked example

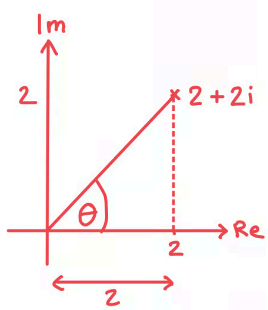
Two complex numbers are given by $z_1 = 2 + 2\text{i}$ and $z_2 = 3\text{e}^{\frac{2\pi}{3}\text{i}}$.

a) Write z_1 in the form $r\text{e}^{\text{i}\theta}$.

$z_1 = 2 + 2i$

Find the modulus: $|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Draw a sketch to help find the argument:



$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1)$

$= \frac{\pi}{4}$

$r = 2\sqrt{2}, \quad \theta = \frac{\pi}{4}$

$\boxed{z_1 = 2\sqrt{2}\,e^{\frac{\pi}{4}i}}$

b) Write z_2 in the form $a + bi$.

$z_2 = 3e^{\frac{2\pi}{3}i} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 3\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$

$\boxed{z_2 = \frac{3}{2}\left(-1 + \sqrt{3}i\right)}$

Your notes

Operations using Exponential Form

How do I multiply and divide exponential forms of complex numbers?

- If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then
  - $z_1 \times z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
    - You can clearly see that the **moduli have been multiplied** and the **arguments have been added**
  - $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$
    - You can clearly see that the **moduli have been divided** and the **arguments have been subtracted**

How do I find the complex conjugate of a complex number in exponential form?

- Simply change the sign of the argument(s)
  - If $z = re^{i\theta}$ then $z^* = re^{-i\theta}$
  - $z = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$ then $z^* = r_1 e^{-i\theta_1} + r_2 e^{-i\theta_2}$

Worked example

Consider the complex number $z = 2e^{\frac{\pi}{3}i}$. Calculate z^2 giving your answer in the form $re^{i\theta}$.

$$z^2 = \left(2e^{\frac{\pi}{3}i}\right)^2 = \left(2e^{\frac{\pi}{3}i}\right)\left(2e^{\frac{\pi}{3}i}\right) = 4e^{2\left(\frac{\pi}{3}i\right)}$$

multiply the moduli
add the arguments

$$\boxed{z^2 = 4e^{\frac{2\pi}{3}i}}$$

1.2.2 de Moivre's Theorem

Your notes

De Moivre's Theorem

What is de Moivre's Theorem?

- de Moivre's theorem can be used to find powers of complex numbers
- It states that for $z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
  - Where
    - $z \neq 0$
    - r is the modulus, $|z|$, $r \in \mathbb{R}^+$
    - θ is the argument, $\arg z$, $\theta \in \mathbb{R}$
    - $n \in \mathbb{R}$
- In Euler's form this is simply:
  - $(re^{i\theta})^n = r^n e^{in\theta}$
- In words de Moivre's theorem tells us to raise the modulus by the power of n and multiply the argument by n
- In the formula booklet de Moivre's theorem is given in both polar and Euler's form:
  - $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta}$

How do I use de Moivre's Theorem to raise a complex number to a power?

- If a complex number is in Cartesian form you will need to convert it to either modulus-argument (polar) form or exponential (Euler's) form first
  - This allows de Moivre's theorem to be used on the complex number
- You may need to convert it back to Cartesian form afterwards
- If a complex number is in the form $z = r(\cos(\theta) - i\sin(\theta))$ then you will need to rewrite it as $z = r(\cos(-\theta) + i\sin(-\theta))$ before applying de Moivre's theorem
- A useful case of de Moivre's theorem allows us to easily find the reciprocal of a complex number:
  - $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}e^{-i\theta}$
  - Using the trig identities cos(-θ) = cos(θ) and sin(-θ) = - sin(θ) gives
  - $\frac{1}{z} = z^{-1} = r^{-1}[\cos(\theta) - i\sin(\theta)] = \frac{1}{r}[\cos(\theta) - i\sin(\theta)]$
- In general
  - $z^{-n} = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)] = r^{-n}[\cos(n\theta) - i\sin(n\theta)]$

Your notes

Examiner Tip

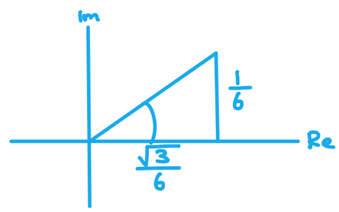
- You may be asked to find all the powers of a complex number, this means there will be a repeating pattern
  - This can happen if the modulus of the complex number is 1
  - Keep an eye on your answers and look for the point at which they begin to repeat themselves

Worked example

Find the value of $\left(\frac{\sqrt{3}}{6}+\frac{1}{6}\mathrm{i}\right)^{-3}$, giving your answer in the form $a + b$i.

Write in Polar form:

$$r = \sqrt{\left(\frac{\sqrt{3}}{6}\right)^2+\left(\frac{1}{6}\right)^2} = \frac{1}{3}$$



$$\theta = \tan^{-1}\left(\frac{\frac{1}{6}}{\frac{\sqrt{3}}{6}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\left(\frac{\sqrt{3}}{6}+\frac{1}{6}\mathrm{i}\right)^{-3} = \left(\frac{1}{3}\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right)^{-3}$$

Apply de Moivre's Theorem:

$$\left(\frac{1}{3}\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right)^{-3} = \left(\frac{1}{3}\right)^{-3}\left(\cos\left(-\frac{3\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{6}\right)\right)$$

$$= 27\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)$$

$$= 27(0-\mathrm{i})$$

$$\boxed{-27\mathrm{i}}$$

Your notes

1.2.3 Applications of de Moivre's Theorem

Multiple Angle Formulae

de Moivre's theorem can be applied to prove identities such as $\sin 5x \equiv 16\sin^5 x - 20\sin^3 x + 5\sin x$. This allows expressions involving multiple angles to be written as a polynomial function of a single trig function which makes it easier to solve equations involving different angles.

How do I write sin*k*θ or cos*k*θ in terms of powers of sinθ or cosθ?

- **STEP 1**
  Use de Moivre's theorem to write $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$
- **STEP 2**
  Use the **binomial expansion** to expand $(\cos\theta + i\sin\theta)^k = \sum_{r=0}^{k}\binom{k}{r} i^{(k-r)}\cos^r\theta\sin^{(k-r)}\theta$
- **STEP 3**
  Use $i^2 = -1$ to **simplify the expansion** and **group** the **real** terms and the **imaginary** terms separately
- **STEP 4**
  Equate the **real parts** of the expansion to cos *k*θ and equate the **imaginary parts** to sin*k*θ
- **STEP 5** (Depending on the question)
  Use $\sin^2\theta + \cos^2\theta = 1$ to write the identity in terms of sinθ only or cosθ depending on what the question asks
  - cos*k*θ can always be written as a function of just cosθ
  - sin*k*θ can be written as a function of just sinθ when *k* is odd
    - When *k* is even sin*k*θ will be a function of sinθ multiplied by a factor of cosθ

How do I write tan*k*θ in terms of powers of tanθ?

- **STEP 1**
  Find expressions for sin*k*θ and cos*k*θ using the previous method
- **STEP 2**
  Use the identity $\tan k\theta = \dfrac{\sin k\theta}{\cos k\theta}$
- **STEP 3**
  Divide each term in the fraction by the **highest power** of cosθ to write each term using powers of tanθ and secθ
- **STEP 4** (Depending on the question)
  Write everything in terms of tanθ using the identity $1 + \tan^2\theta = \sec^2\theta$

Your notes

Examiner Tip

- You can use the substitutions c = cosθ and s = sinθ to shorten your working as long as you clearly state them and change back at the end of the proof

Worked example

Prove that $\sin 5x \equiv 16\sin^5 x - 20\sin^3 x + 5\sin x$.

Use de Moivre's theorem: $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^5$

$$(\cos\theta + i\sin\theta)^5 = \sum_{r=0}^{5}\binom{5}{r} i^{(5-r)}\cos^r\theta\,\sin^{(5-r)}\theta$$

$i^4 = 1$ $\quad$ $i^3 = -i$

$$= (i)^5\cos^0\theta\sin^5\theta + 5(i)^4\cos\theta\sin^4\theta + 10(i)^3\cos^2\theta\sin^3\theta$$

$i^5 = i$

$$+ 10(i)^2\cos^3\theta\sin^2\theta + 5(i)\cos^4\theta\sin\theta + (i)^0\cos^5\theta\sin^0\theta$$

Use $i^2 = -1$ to simplify $\quad$ $i^2 = -1$ $\quad$ $i^0 = 1$

$$(\cos\theta + i\sin\theta)^5 = i\sin^5\theta + 5\cos\theta\sin^4\theta - 10i\cos^2\theta\sin^3\theta - 10\cos^3\theta\sin^2\theta + 5i\cos^4\theta\sin\theta + \cos^5\theta$$

Group the imaginary terms separately and set equal to $\sin 5x$

$$\text{Im}(\cos\theta + i\sin\theta)^5 = \sin^5\theta - 10\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta$$

$$\therefore \sin 5\theta = \sin^5\theta - 10\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta$$

Use $\sin^2\theta + \cos^2\theta \equiv 1$ to write all in terms of $\sin\theta$:

$$\sin 5\theta \equiv \sin^5\theta - 10(1-\sin^2\theta)\sin^3\theta + 5(1-\sin^2\theta)^2\sin\theta$$

$$\sin 5\theta \equiv \sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta$$

$$\equiv \sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta$$

$$\boxed{\sin 5x \equiv 16\sin^5 x - 20\sin^3 x + 5\sin x}$$

Powers of Trig Functions

Your notes

de Moivre's theorem can be applied to prove identities such as

$\sin^5\theta \equiv \frac{1}{16}(10\sin\theta - 5\sin 3\theta + \sin 5\theta)$. This allows powers of a trig function to be written in terms of multiple angles which makes them easier to integrate.

How can I write cos*k*θ and sin*k*θ in terms of $e^{i\theta}$?

- Recall $e^{i\theta} = \cos\theta + i\sin\theta$ and by de Moivre's theorem $e^{ik\theta} = \cos k\theta + i\sin k\theta$
- It follows that $e^{-ik\theta} = \cos(-k\theta) + i\sin(-k\theta) = \cos k\theta - i\sin k\theta$
- You can derive expressions for sin*k*θ and cos*k*θ using:
  - $e^{ik\theta} + e^{-ik\theta} = 2\cos k\theta$
    - $\cos k\theta = \frac{1}{2}(e^{ik\theta} + e^{-ik\theta})$
  - $e^{ik\theta} - e^{-ik\theta} = 2i\sin k\theta$
    - $\sin k\theta = \frac{1}{2i}(e^{ik\theta} - e^{-ik\theta})$

How do I write powers of sinθ or cosθ in terms of sin*k*θ or cos*k*θ?

- **STEP 1**
  Write the trig term in terms of $e^{i\theta}$
  - $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
  - $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$
- **STEP 2**
  Use the **binomial expansion** to expand $\cos^k\theta = \frac{1}{2^k}(e^{i\theta} + e^{-i\theta})^k$ or $\sin^k\theta = \frac{1}{2^k i^k}(e^{i\theta} - e^{-i\theta})^k$
  - Simplify i^k to one of i, -1, -i or 1
- **STEP 3**
  Due to symmetry you can **pair terms up** of the form $Ae^{in\theta}$ and $Ae^{-in\theta}$
  - Write as $A(e^{in\theta} + e^{-in\theta})$ or $A(e^{in\theta} - e^{-in\theta})$
  - If *k* is even then there will be a term by itself as $Be^{im\theta}e^{-im\theta} = Be^0 = B$
- **STEP 4**
  Rewrite each pair in terms of cos*n*θ or sin*n*θ

- $A\left(e^{in\theta}+e^{-in\theta}\right)=A(2\cos n\theta)$
- $A\left(e^{in\theta}-e^{-in\theta}\right)=A(2i\sin n\theta)$
- STEP 5
  Simplify the expression – remember the 2^k term!
  - $\cos^k\theta$ can always be written as an expression using only terms of the form $\cos n\theta$
  - $\sin^k\theta$ can be written as an expression using only terms of the form:
    - $\sin n\theta$ if k is odd
    - $\cos n\theta$ if k is even

How do I write powers of tanθ in terms of sin*k*θ or cos*k*θ?

- Use $\tan^k\theta=\dfrac{\sin^k\theta}{\cos^k\theta}$ and use the previous steps
- Note that the expression will be in terms of multiple angles of sin & cos and not tan

Your notes

Worked example

Prove that $\sin^5\theta \equiv \frac{1}{16}(10\sin\theta - 5\sin3\theta + \sin5\theta)$.

Write $\sin\theta$ in terms of $e^{i\theta}$: $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

Use the binomial theorem to expand $\sin^5\theta$

$$\sin^5\theta = \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^5$$

$$= \frac{1}{2^5 i^5}\left(e^{i\theta} - e^{-i\theta}\right)^5$$

$$= \frac{1}{32i}\left(e^{5i\theta} + \binom{5}{1}\left(e^{4i\theta}\right)\left(-e^{-i\theta}\right) + \binom{5}{2}\left(e^{3i\theta}\right)\left(-e^{-i\theta}\right)^2 + \binom{5}{3}\left(e^{2i\theta}\right)\left(-e^{-i\theta}\right)^3 + \binom{5}{4}\left(e^{i\theta}\right)\left(-e^{-i\theta}\right)^4 + \left(-e^{-i\theta}\right)^5\right)$$

$$= \frac{1}{32i}\left(e^{5i\theta} + 5\left(-e^{3i\theta}\right) + 10\left(e^{i\theta}\right) + 10\left(-e^{-i\theta}\right) + 5\left(e^{-3i\theta}\right) + \left(-e^{-5i\theta}\right)\right)$$

$$= \frac{1}{32i}\left(\left(e^{5i\theta} - e^{-5i\theta}\right) + 5\left(-e^{3i\theta} + e^{-3i\theta}\right) + 10\left(e^{i\theta} - e^{-i\theta}\right)\right)$$

Rearrange and pair terms up.

Use $A\left(e^{in\theta} - e^{-in\theta}\right) = A(2i\sin n\theta)$

$$\sin^5\theta = \frac{1}{32i}\left(\left(e^{5i\theta} - e^{-5i\theta}\right) - 5\left(e^{3i\theta} - e^{-3i\theta}\right) + 10\left(e^{i\theta} - e^{-i\theta}\right)\right)$$

$$= \frac{1}{32i}\left(2i\sin5\theta - 5(2i\sin5\theta) + 10(2i\sin5\theta)\right)$$

Simplify: $\frac{1}{32\cancel{i}}\left(2\cancel{i}\left(\sin5\theta - 5(\sin5\theta) + 10(\sin5\theta)\right)\right) = \frac{1}{16}\left(\sin5\theta - 5\sin5\theta + 10\sin5\theta\right)$

$$\boxed{\sin^5\theta \equiv \frac{1}{16}\left(10\sin\theta - 5\sin3\theta + \sin5\theta\right)}$$

Your notes

Your notes

Trig Series

de Moivre's theorem can be applied to find formulae for the sum of trigonometric series such as

$$\frac{1}{3}\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{27}\cos 5\theta + \ldots = \frac{\cos\theta}{5 - 3\cos 2\theta}$$

How can I find the sum of geometric series involving complex numbers?

- The **geometric series formulae** work with complex numbers
  - $S_n = \dfrac{a(1-r^n)}{1-r}$ and $S_\infty = \dfrac{a}{1-r}$ (provided $|r| < 1$)
- Suppose *w* and *z* are two complex numbers then:
  - $\sum_{k=1}^{n} wz^{k-1} = w + wz + wz^2 + \ldots + wz^{n-1} = \dfrac{w(1-z^n)}{1-z}$
  - $\sum_{k=1}^{\infty} wz^{k-1} = w + wz + wz^2 + \ldots = \dfrac{w}{1-z}$ provided $|z|<1$
    - Compare these to the geometric series formulae with $a=w$ and $r=z$

How can I find the sum of geometric series involving sinθ or cosθ?

- Using de Moivre's theorem: $\mathrm{e}^{\mathrm{i}k\theta} = \cos k\theta + \mathrm{i}\sin k\theta$
- You can find $\cos k\theta$ and $\sin k\theta$ by taking **real** and **imaginary** parts
  - $\cos k\theta = \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}k\theta}\right)$
  - $\sin k\theta = \mathrm{Im}\left(\mathrm{e}^{\mathrm{i}k\theta}\right)$
- Rewrite the series using $\mathrm{e}^{\mathrm{i}k\theta}$ to make it a **geometric series**
  - For example:
    - $\cos\theta + \cos 4\theta + \cos 7\theta + \ldots = \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{4\mathrm{i}\theta} + \mathrm{e}^{7\mathrm{i}\theta} + \ldots\right)$
    - $\sin\theta + \sin 4\theta + \sin 7\theta + \ldots = \mathrm{Im}\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{4\mathrm{i}\theta} + \mathrm{e}^{7\mathrm{i}\theta} + \ldots\right)$
- You can now use the **formulae** to find an **expression** for the sum
  - The series involving $\mathrm{e}^{\mathrm{i}k\theta}$ will be **geometric** so determine
    - whether it is **finite or infinite**
    - what is the value of *a* (the **first term**) and *r* (the **common ratio**)
- Once you have used the formula the **denominator** will be of the form $A - B\mathrm{e}^{\mathrm{i}n\theta}$
  - Multiply the numerator and denominator by $A - B\mathrm{e}^{-\mathrm{i}n\theta}$
  - The denominator will become real $A^2 + B^2 - 2AB\cos n\theta$
    - This is because $\mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{-\mathrm{i}n\theta} = 2\cos n\theta$
- If your series involved sin terms then take the **imaginary part** of the sum
- If your series involved cos terms then take the **real part** of the sum

Examiner Tip

- Exam questions normally lead you through this process
- It is common for questions to let C equal the sum of the series with cos and let S equal the sum of the series with sin
- You can then write C + iS which makes the trig terms becomes $e^{ik\theta}$

Your notes

Your notes

Worked example

Prove that $\frac{1}{3}\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{27}\cos 5\theta + \ldots = \frac{\cos\theta}{5 - 3\cos 2\theta}$.

Use de Moivre's theorem:

$$\frac{1}{3}\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{27}\cos 5\theta + \ldots = \mathrm{Re}\left(\frac{1}{3}e^{i\theta} + \frac{1}{9}e^{3i\theta} + \frac{1}{27}e^{5i\theta} + \ldots\right)$$

first term $(a) = \frac{1}{3}e^{i\theta}$ common difference $(r) = \frac{1}{3}e^{2i\theta}$

The series is infinite, so use $S_\infty = \frac{a}{1-r}$

$$\therefore \mathrm{Re}\left(\frac{1}{3}e^{i\theta} + \frac{1}{9}e^{3i\theta} + \frac{1}{27}e^{5i\theta} + \ldots\right) = \mathrm{Re}\left(\frac{\frac{1}{3}e^{i\theta}}{1 - \frac{1}{3}e^{2i\theta}}\right)$$

Multiply numerator and denominator by $\left(1 - \frac{1}{3}e^{-2i\theta}\right)$

$$\begin{aligned} e^{i\theta} - \frac{1}{3}e^{-i\theta} &= \cos\theta + i\sin\theta - \frac{1}{3}(\cos(-\theta) + i\sin(-\theta)) \\ &= \cos\theta + i\sin\theta - \frac{1}{3}(\cos\theta - i\sin\theta) \\ &= \frac{2}{3}\cos\theta + \frac{4}{3}i\sin\theta \end{aligned}$$

$$\left(\frac{\frac{1}{3}e^{i\theta}}{1 - \frac{1}{3}e^{2i\theta}}\right)\left(\frac{1 - \frac{1}{3}e^{-2i\theta}}{1 - \frac{1}{3}e^{-2i\theta}}\right) = \frac{\frac{1}{3}\left(e^{i\theta} - \frac{1}{3}e^{-i\theta}\right)}{1 - \frac{1}{3}\left(e^{-2i\theta} + e^{2i\theta}\right) + \frac{1}{9}} = \frac{\frac{1}{3}\left(\frac{2}{3}\cos\theta + \frac{4}{3}i\sin\theta\right)}{\frac{10}{9} - \frac{2}{3}\cos 2\theta}$$

$$\begin{aligned} e^{-2i\theta} + e^{2i\theta} &= \cos(-2\theta) + i\sin(-2\theta) + \cos(2\theta) + i\sin(2\theta) \\ &= \cos(2\theta) - i\sin(2\theta) + \cos(2\theta) + i\sin(2\theta) \\ &= 2\cos(2\theta) \end{aligned}$$

$$\mathrm{Re}\left(\frac{\frac{1}{3}e^{i\theta}}{1 - \frac{1}{3}e^{2i\theta}}\right) = \frac{\frac{2}{9}\cos\theta}{\frac{10}{9} - \frac{2}{3}\cos 2\theta} = \frac{\cos\theta}{5 - 3\cos 2\theta}$$

$$\boxed{\frac{1}{3}\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{27}\cos 5\theta + \ldots = \frac{\cos\theta}{5 - 3\cos 2\theta}}$$

Your notes

1.2.4 Roots of Complex Numbers

Roots of Complex Numbers

How do I find the square root of a complex number?

- The square roots of a complex number will themselves be complex:
  - i.e. if $z^2 = a + bi$ then $z = c + di$
- We can then square ($c + di$) and equate it to the original complex number ($a + bi$), as they both describe z^2:
  - $a + bi = (c + di)^2$
- Then expand and simplify:
  - $a + bi = c^2 + 2cdi + d^2i^2$
  - $a + bi = c^2 + 2cdi - d^2$
- As both sides are equal we are able to equate real and imaginary parts:
  - Equating the real components: $a = c^2 - d^2$ (1)
  - Equating the imaginary components: $b = 2cd$ (2)
- These equations can then be solved simultaneously to find the real and imaginary components of the square root
  - In general, we can rearrange (2) to make $\frac{b}{2d} = c$ and then substitute into (1)
  - This will lead to a quartic equation in terms of d; which can be solved by making a substitution to turn it into a quadratic
- The values of d can then be used to find the corresponding values of c, so we now have both components of both square roots ($c + di$)
- Note that one root will be the negative of the other root
  - g. $c + di$ and $-c - di$

How do I use de Moivre's Theorem to find roots of a complex number?

- De Moivre's Theorem states that a complex number in modulus-argument form can be raised to the power of *n* by
  - Raising the modulus to the power of *n* and multiplying the argument by *n*
- When in modulus-argument (polar) form de Moivre's Theorem can then be used to find the roots of a complex number by
  - $k = 0,\ 1,\ 2,\ \dots\ ,\ n - 1$
  - Recall that adding 2π to the argument of a complex number does not change the complex number
  - Therefore we must consider how different arguments will give the same result
  - Taking the *n*th root of the modulus and dividing the argument by *n*
  - If $z = r(\cos\theta + \mathrm{i}\sin\theta)$ then $\sqrt[n]{z} = [r(\cos(\theta + 2\pi k) + \mathrm{i}\sin(\theta + 2\pi k))]^{\frac{1}{n}}$

Your notes

- This can be rewritten as $\sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \mathrm{i}\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$
- This can be written in exponential (Euler's) form as
  - For $z^n = r\mathrm{e}^{\mathrm{i}\theta}$, $z = \sqrt[n]{r}\,\mathrm{e}^{\frac{\theta + 2\pi k}{n}\mathrm{i}}$
- The *n*th root of complex number will have *n* roots with the properties:
  - The five roots of a complex number raised to the power 5 will create a regular pentagon on an Argand diagram
  - The eight roots of a complex number raised to the power 8 will create a regular octagon on an Argand diagram
  - The *n* roots of a complex number raised to the power *n* will create a regular *n*-sided polygon on an Argand diagram
  - The modulus is $\sqrt[n]{r}$ for all roots
  - There will be *n* different arguments spaced at equal intervals on a circle centred about the origin
  - This creates some geometrically beautiful results

Examiner Tip

- de Moivre's theorem makes finding roots of complex numbers very easy, but you must be confident converting from Cartesian form into Polar and Euler's form first
  - You can use your calculator to convert between forms

Your notes

Worked example

a) Find the square roots of 5 + 12i, giving your answers in the form *a* + *b*i.

Let $z^2 = 5 + 12i$, then $z = a + bi$

$z^2 = a^2 + 2abi + b^2 i^2$ $\quad i^2 = -1$

$= a^2 + 2abi - b^2$

Therefore $5 + 12i = (a^2 - b^2) + 2abi$

Equate the real components: $a^2 - b^2 = 5$ ①

Equate the imaginary components: $2ab = 12$ ②

Solve the simultaneous equations:

$a = \frac{6}{b} \Rightarrow \left(\frac{6}{b}\right)^2 - b^2 = 5$

$b^4 + 5b^2 - 36 = 0$

$(b^2 + 9)(b^2 - 4) = 0$

$b^2 = -9$ or $b^2 = 4$

no real solutions

$b = \pm 2$

$a = \pm 3$

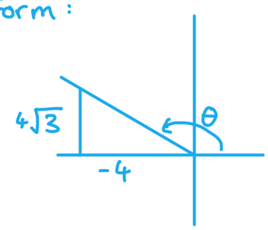
$$\boxed{z_1 = 3 + 2i \quad , \quad z_2 = -3 - 2i}$$

b) Solve the equation $z^3 = -4 + 4\sqrt{3}\,i$ giving your answers in the form $r(\cos\theta + i\sin\theta)$.

Your notes

Convert $-4+4\sqrt{3}i$ to Polar form:

$$r = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{64} = 8$$



$$\theta = \pi - \left(\tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)\right) = \frac{2\pi}{3}$$

$$-4+4\sqrt{3}i = 8\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$z^3 = -4+4\sqrt{3}i \Rightarrow z = \sqrt[3]{-4+4\sqrt{3}i}$$

$$= \left(8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right)^{\frac{1}{3}}$$

$$= 8^{\frac{1}{3}}\left(\cos\frac{1}{3}\left(\frac{2\pi}{3}+2\pi k\right) + i\sin\frac{1}{3}\left(\frac{2\pi}{3}+2\pi k\right)\right)$$

Order 3 so there are 3 roots, use $k = 0, 1, 2$:

$$\boxed{\begin{aligned} z_1 &= 2\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) \\ z_2 &= 2\left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right) \\ z_3 &= 2\left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right) \end{aligned}}$$

Your notes

Roots on an Argand Diagram

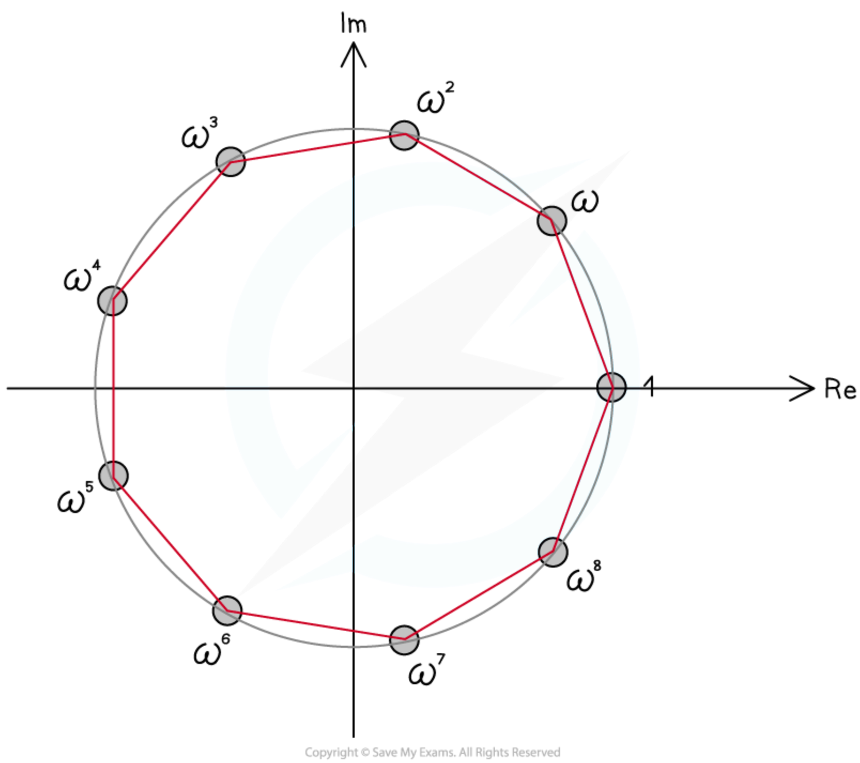
What are roots of unity?

- Roots of unity are solutions to the equation $z^n = 1$ where *n* is a positive integer
- For the equation $z^n = 1$ there are *n* roots of unity
  - $z = \mathrm{e}^{\frac{2\pi k}{n}\mathrm{i}}$ where $k = 0, 1, 2, \ldots, n-1$
    - This is given in the formula booklet
- These can be written $1, \omega, \omega^2, \ldots, \omega^{n-1}$
  - Where $\omega = \mathrm{e}^{\frac{2\pi}{n}\mathrm{i}}$
- The sum of the roots of unity is zero
  - $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$
- They can be used to find all the roots of the equation $z^n = r\mathrm{e}^{\mathrm{i}\theta}$
  - Find one root normally $\alpha = \sqrt[n]{r}\,\mathrm{e}^{\frac{\mathrm{i}\theta}{n}}$
  - Then the *n* distinct roots can be found by multiplying *α* by each root of unity
    - $\alpha, \alpha\omega, \alpha\omega^2, \ldots, \alpha\omega^{n-1}$

What are the geometric properties of roots of complex numbers?

- The *n* roots of any non-zero complex number $r\mathrm{e}^{\mathrm{i}\theta}$ lie on a circle on an Argand diagram
  - The centre will be the origin
  - The radius will be $\sqrt[n]{r}$
- The *n* roots of unity lie on the unit circle centred about the origin
- Regular polygons can be created by joining consecutive roots of a complex number with straight lines

Your notes



How can I use roots of unity to solve geometric problems?

- Roots of unity can be used to solve problems involving regular polygons centred about the origin
- Coordinates of vertices (x, y) can be considered as complex numbers $x + y\text{i}$
- If you know one vertex (x, y) you can find the others by multiplying the complex number representing the given vertex by each root of unity
  - $x + y\text{i}, (x + y\text{i})\omega, (x + y\text{i})\omega^2, \ldots, (x + y\text{i})\omega^{n-1}$
  - If you write the vertex using exponential form $re^{\text{i}\theta}$ it can make the multiplications easier
  - Then you can just add $\frac{2\pi}{n}$ to the argument to get the next vertex
- Write all vertices in Cartesian form to get the coordinates

Examiner Tip

- You can use your calculator to convert between polar and cartesian forms which may speed up your working
  - Just be aware of questions that may ask you to not use "calculator technology" where you need to show full working (but can still use calculator to check!)

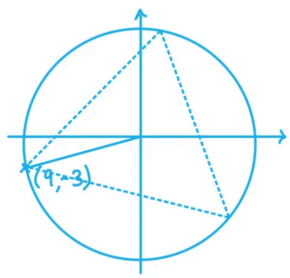
Your notes

Worked example

An equilateral triangle has its centre at the origin of a Cartesian plane. One of its vertices is at the point $(9,-3)$. Find the coordinates of the other two vertices.

Multiply the complex number representing the given vertex by each root of unity.

For $z^3 = 1$ there are 3 roots of unity ← a triangle has 3 vertices



Find the roots of unity:

$z = e^{\frac{2\pi i k}{3}}$ for $k = 1, 2, 3$

$\therefore\ z_1 = e^{\frac{2\pi i}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = e^{\frac{4\pi i}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z_3 = e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$ ← represents given vertex

Complex number representing given vertex $= 9 - 3i$:

Second coordinate: $(9-3i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{9}{2} + \frac{9\sqrt{3}}{2}i + \frac{3i}{2} - \frac{3\sqrt{3}}{2}i^2$

$= \frac{-9+3\sqrt{3}}{2} + \frac{3+9\sqrt{3}}{2}i$

Third coordinate: $(9-3i)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{9}{2} - \frac{9\sqrt{3}}{2}i + \frac{3i}{2} + \frac{3\sqrt{3}}{2}i^2$

$= \frac{-9-3\sqrt{3}}{2} + \frac{3-9\sqrt{3}}{2}i$

$\boxed{\text{Coordinates: } \left(\frac{-9+3\sqrt{3}}{2}, \frac{3+9\sqrt{3}}{2}\right) \text{ and } \left(\frac{-9-3\sqrt{3}}{2}, \frac{3-9\sqrt{3}}{2}\right)}$