Edexcel A Level Further Maths: Core Pure

Your notes

9.1 Proof by Induction

Contents

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9.1.1 Intro to Proof by Induction

Intro to Proof by Induction

What is proof by induction?

- **Proof by induction** is a way of proving a **result is true for a set of integers** by showing that if it is **true for one integer then it is true for the next integer**
- It can be thought of as dominoes:
  - All dominoes will fall down if:
    - The first domino falls down
    - Each domino falling down causes the next domino to fall down

What are the steps for proof by induction?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
  - This is **normally n = 1 or 0** but it could be any integer
    - In the dominoes analogy this is showing that the first domino falls down
- **STEP 2: The assumption step**
  - **Assume** the result is true for $n = k$ for some integer k
    - In the dominoes analogy this is assuming that a random domino falls down
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for $n = k + 1$
  - The assumption from STEP 2 will be needed at some point
    - In the dominoes analogy this is showing that the random domino that we assumed falls down will cause the next one to fall down
- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}, n \geq 1$ by mathematical induction
  - The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

What type of statements might I be asked to prove by induction?

- There are **4 main applications** that you could be asked
  - Formulae for sums of series
  - Formulae for recursive sequences
  - Expression for the power of a matrix
  - Showing an expression is always divisible by a specific value
- Induction is always used to prove **de Moivre's theorem**

- It is unlikely that you will be asked unfamiliar applications in your exam but induction is used in other areas of maths
  - Proving formulae for nth derivative of functions
  - Proving formulae involving factorials

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Proving de Moivre's Theorem by Induction

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How is de Moivre's Theorem proved?

- When written in Euler's form the proof of de Moivre's theorem is easy to see:
  - Using the index law of brackets: $(re^{i\theta})^n = r^n e^{in\theta}$
- However Euler's form cannot be used to prove de Moivre's Theorem when it is in modulus-argument (polar) form
- **Proof by induction** can be used to prove de Moivre's Theorem for positive integers:
  - To prove de Moivre's Theorem for all positive integers, n
  - $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
- STEP 1: Prove it is true for $n = 1$
  - $[r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$
  - So de Moivre's Theorem is true for $n = 1$
- STEP 2: Assume it is true for $n = k$
  - $[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$
- STEP 3: Show it is true for $n = k + 1$
  - $[r(\cos\theta + i\sin\theta)]^{k+1} = ([r(\cos\theta + i\sin\theta)]^k)([r(\cos\theta + i\sin\theta)]^1)$
  - According to the assumption this is equal to
    - $(r^k(\cos k\theta + i\sin k\theta))\,(r(\cos\theta + i\sin\theta))$
  - Using laws of indices and multiplying out the brackets:
    - $= r^{k+1}[\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta]$
  - Letting $i^2 = -1$ and collecting the real and imaginary parts gives:
    - $= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$
  - Recognising that the real part is equivalent to $\cos(k\theta + \theta)$ and the imaginary part is equivalent to $\sin(k\theta + \theta)$ gives
    - $r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$
  - So de Moivre's Theorem is true for $n = k + 1$
- STEP 4: Write a conclusion to complete the proof
  - The statement is true for $n = 1$, and if it is true for $n = k$ it is also true for $n = k + 1$
  - Therefore, by the principle of mathematical induction, the result is true for all positive integers, n
- De Moivre's Theorem works for all real values of n
  - However you could only be asked to prove it is true for positive integers

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Worked example

Show, using proof by mathematical induction, that for a complex number $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and for all positive integers, *n*,

$$z^n = [r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$$

Step 1: Prove it is true for $n=1$

$$z^1 = [r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$$

Step 2: Assume it is true for $n=k$

$$z^k = [r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$$

Step 3: Show it is true for $n=k+1$

$$z^{k+1} = [r(\cos\theta + i\sin\theta)]^{k+1}$$

Addition law of indices: $a^k a^1 = a^{k+1}$

$$= \left([r(\cos\theta + i\sin\theta)]^k\right)\left([r(\cos\theta + i\sin\theta)]^1\right)$$

using the assumption

$r^k \times r^1 = r^{k+1}$

$$= [r^k(\cos k\theta + i\sin k\theta)][r(\cos\theta + i\sin\theta)]$$

$$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$i^2 = -1$

$$= r^{k+1}[\cos k\theta\cos\theta + \cos k\theta(i\sin\theta) + \cos\theta(i\sin k\theta) + i^2\sin k\theta\sin\theta]$$

$$= r^{k+1}[\cos k\theta\cos\theta + i(\cos k\theta\sin\theta + \cos\theta\sin k\theta) - \sin k\theta\sin\theta]$$

collect Re and Im parts

$$= r^{k+1}[\underbrace{\cos k\theta\cos\theta - \sin k\theta\sin\theta}_{=\cos(k\theta+\theta)} + i\underbrace{(\cos k\theta\sin\theta + \sin k\theta\cos\theta)}_{=\sin(k\theta+\theta)}]$$

$$= r^{k+1}[\cos(k\theta+\theta) + i\sin(k\theta+\theta)]$$

$$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$$

So true for $n = k+1$

Step 4: Write a conclusion:

De Moivre's theorem is true for $n=1$, and if it is true for $n=k$ it is also true for $n=k+1$.
Therefore it is true for all $n \in \mathbb{Z}^+$

9.1.2 Common Cases of Proof by Induction

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Proof by Induction – Sequences

What are the steps for proof by induction with sequences?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
    - If the recursive relation formula for the next term involves the previous **two** terms then you need to show the position-to-term formula works the **first two given terms** which will be given as part of the definition of the sequence
  - This is **normally *n* = 1 or 0** but it could be any integer
    - For example: To prove $u_n = 3^n - 2$ is the position-to-term formula for the recursive sequence $u_{n+1} = 3u_n + 4,\ u_1 = 1$:
    - $u_1 = 3^1 - 2 = 1$
- **STEP 2: The assumption step**
  - **Assume** the result is true for ***n* = *k*** for some integer *k*
    - For example: Assume $u_k = 3^k - 2$ is true
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for ***n* = *k* + 1**
  - It can be helpful to write down what you want to show
  - The assumption from STEP 2 will be needed at some point
    - For example: Want to show $u_{k+1} = 3^{k+1} - 2$
  - The trick to this step is to use the recursive relation formula:
    - $u_{k+1} = 3u_k + 4$
- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}, n \geq 1$ by mathematical induction
  - The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

Your notes

Worked example

A sequence is defined by $u_{n+1} = 5u_n - 4,\ u_1 = 2$ for $n \geq 1$.

Prove by mathematical induction that $u_n = 5^{n-1} + 1$ for $n \geq 1$.

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Want to prove $u_n = 5^{n-1} + 1$

Basic step

Show true for $n=1$ $\quad u_1 = 5^{1-1} + 1 = 2$ ✓ $\therefore$ true for $n=1$

Assumption step

Assume true for $n=k$ $\quad$ Assume $u_k = 5^{k-1} + 1$

Inductive step

Show true for $n=k+1$ $\quad$ Want to show $u_{k+1} = 5^{(k+1)-1} + 1 = 5^k + 1$

$u_{k+1} = 5u_k - 4$ $\quad$ using definition of sequence

$= 5(5^{k-1} + 1) - 4$ $\quad$ using assumption

$= 5^k + 5 - 4$

$= 5^k + 1$

$\therefore$ true for $n = k+1$

Conclusion step

Explain

If true for $n=k$ then true for $n=k+1$.
Since it is true for $n=1$, the statement
is true for all $n \in \mathbb{Z}^+$
$u_k = 5^{k-1} + 1$

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Proof by Induction – Series

What are the steps for proof by induction with series?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
  - This is **normally *n* = 1 or 0** but it could be any integer
    - For example: To prove $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ is true for all integers $n \geq 1$ you would first need to show it is true for $n = 1$:
    - $\sum_{r=1}^{1} r^2 = \frac{1}{6}(1)((1)+1)(2(1)+1)$
- **STEP 2: The assumption step**
  - **Assume** the result is true for ***n* = *k*** for some integer *k*
    - For example: Assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for ***n* = *k* + 1**
  - It can be helpful to simplify LHS & RHS separately and show they are identical
  - The assumption from STEP 2 will be needed at some point
    - For example: $LHS = \sum_{r=1}^{k+1} r^2$ and $RHS = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$
  - The trick to this step is to use:

$$\sum_{r=1}^{k+1} f(r) = f(k+1) + \sum_{r=1}^{k} f(r)$$

- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}, n \geq 1$ by mathematical induction
  - The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

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Worked example

Prove by induction that $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$ for $n \in \mathbb{Z}^+$.

Want to prove $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$

Basic step
Show true for $n=1$

$LHS = \sum_{r=1}^{1} r(r-3) = (1)(1-3) = -2$

$RHS = \frac{1}{3}(1)(1-4)(1+1) = -2$ $\quad \therefore LHS = RHS$ so true for $n=1$

Assumption step
Assume true for $n=k$

Assume $\sum_{r=1}^{k} r(r-3) = \frac{1}{3}k(k-4)(k+1)$

Inductive step
Show true for $n=k+1$

$RHS = \frac{1}{3}(k+1)((k+1)-4)((k+1)+1) = \frac{1}{3}(k+1)(k-3)(k+2)$

$LHS = \sum_{r=1}^{k+1} r(r-3) = (k+1)((k+1)-3) + \sum_{r=1}^{k} r(r-3)$

$= (k+1)(k-2) + \frac{1}{3}k(k-4)(k+1)$ Using assumption

$= \frac{1}{3}(k+1)[3(k-2) + k(k-4)]$ Factorise $\frac{1}{3}(k+1)$

$= \frac{1}{3}(k+1)[k^2 - k - 6]$

$= \frac{1}{3}(k+1)(k-3)(k+2)$

$\therefore LHS = RHS$ so true for $n = k+1$

Conclusion step
Explain

If true for $n=k$ then true for $n=k+1$.
Since it is true for $n=1$, the statement is true for all $n \in \mathbb{Z}^+$

$\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$

Your notes

Proof by Induction - Divisibility

What are the steps for proof by induction with series?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
  - This is **normally *n* = 1 or 0** but it could be any integer
    - For example: To prove $f(n) = 4^n - 1$ is divisible by 3 for all integers $n \geq 1$ you would first need to show it is true for $n = 1$:
    - $f(1) = 4 - 1 = 3 = 3(1)$
- **STEP 2: The assumption step**
  - **Assume** the result is true for ***n* = *k*** for some integer k
    - For example: Assume $f(k) = 4 - 1$ is divisible by 3
  - There is nothing to do for this step apart from writing down the assumption
  - The trick for this step is to write the expression as a multiple of the number
    - For example: $4^k - 1 = 3p$ for some integer p
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for ***n* = *k* + 1**
  - The assumption from STEP 2 will be needed at some point
    - For example: Show that $f(k+1) = 4^{k+1} - 1$ is divisible by 3
  - The trick to this step is to use:

$$a^{k+1} = a \times a^k$$

    - For example: $f(k+1) = 4(4^k) - 1 = 4(1 + 3p) - 1 = 3(p+1)$
  - Another trick is to show that $f(k+1) - f(k)$ is divisible by the number
  - Be careful with numbers like 6 as for these you can instead show it is divisible by both 3 and 2
- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}, n \geq 1$ by mathematical induction
  - The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

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Worked example

Prove by induction that $f(n) = 6^n + 13^{n+1}$ is divisible by 7 for all integers $n \geq 0$.

Want to prove $f(n) = 6^n + 13^{n+1}$ is divisible by 7 for $n \geq 0$

Basic step
Show true for $n=0$ $\quad f(0) = 6^0 + 13^1 = 14 = 7(2)$ ✓ $\therefore$ true for $n=0$

Assumption step
Assume true for $n=k$ $\quad$ Assume $f(k) = 6^k + 13^{k+1} = 7p$ for some $p \in \mathbb{Z}$

Inductive step
Show true for $n=k+1$

$$
\begin{aligned}
f(k+1) &= 6^{k+1} + 13^{k+2} \\
&= 6(6^k) + 13(13^{k+1}) && \text{Rearrange assumption} \\
&= 6(7p - 13^{k+1}) + 13(13^{k+1}) && 6^k + 13^{k+1} = 7p \\
&= 7(6p) - 6(13^{k+1}) + 13(13^{k+1}) \\
&= 7(6p) + 7(13^{k+1}) \\
&= 7(6p + 13^{k+1})
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 7

Conclusion step
Explain

If true for $n=k$ then true for $n=k+1$.
Since it is true for $n=0$, the statement is true for all $n \in \mathbb{Z}_0^+$
$6^n + 13^{n+1}$ is divisible by 7

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Proof by Induction – Matrices

What are the steps for proof by induction with matrices?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
  - This is **normally *n* = 1** but it could be any integer
    - For example: To prove $\boldsymbol{M}^n = \begin{pmatrix} 2^n & 0 \\ 1-2^n & 1 \end{pmatrix}$ where $\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ for $n \geq 1$:
    - $\begin{pmatrix} 2^1 & 0 \\ 1-2^1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$
- **STEP 2: The assumption step**
  - **Assume** the result is true for ***n* = *k*** for some integer *k*
    - For example: Assume $\boldsymbol{M}^k = \begin{pmatrix} 2^k & 0 \\ 1-2^k & 1 \end{pmatrix}$ is true
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for ***n* = *k* + 1**
  - It can be helpful to write down what you want to show
  - The assumption from STEP 2 will be needed at some point
    - For example: Want to show $\boldsymbol{M}^k = \begin{pmatrix} 2^{k+1} & 0 \\ 1-2^{k+1} & 1 \end{pmatrix}$
  - The trick to this step is to use write:
    - $\boldsymbol{M}^{k+1} = \boldsymbol{M}\boldsymbol{M}^{\mathbf{k}}$
    - The entries in the matrix can get messy so it can be clearer to write $\boldsymbol{M}^{k+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then find expressions for a, b, c, d on separate lines
- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}$, $n \geq 1$ by mathematical induction
  - The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

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Worked example

The matrix $\mathbf{M}$ is given by $M = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$.

Prove, using mathematical induction, that $\mathbf{M}^n = \begin{pmatrix} 2^n & 2(2^n - 1) \\ 0 & 1 \end{pmatrix}$.

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Want to prove $M^n = \begin{pmatrix} 2^n & 2(2^n-1) \\ 0 & 1 \end{pmatrix}$

Basic step

Show true for $n=1$

$M^1 = \begin{pmatrix} 2^1 & 2(2^1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ ✓ $\therefore$ true for $n=1$

Assumption step

Assume true for $n=k$

Assume $M^k = \begin{pmatrix} 2^k & 2(2^k-1) \\ 0 & 1 \end{pmatrix}$

Inductive step

Show true for $n=k+1$

$M^{k+1} = MM^k$

using assumption

$= \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 2(2^k-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a = 2(2^k) + 2(0) = 2^{k+1}$

$b = 2\left(2(2^k-1)\right) + 2(1) = 2(2^{k+1}) - 4 + 2 = 2(2^{k+1}-1)$

$c = 0(2^k) + 1(0) = 0$

$d = 0\left(2(2^k-1)\right) + 1(1) = 1$

$\therefore M^{k+1} = \begin{pmatrix} 2^{k+1} & 2(2^{k+1}-1) \\ 0 & 1 \end{pmatrix}$ true for $n=k+1$

Conclusion step

Explain

If true for $n=k$ then true for $n=k+1$.

Since it is true for $n=1$, the statement is true for all $n \in \mathbb{Z}^+$

$M^n = \begin{pmatrix} 2^n & 2(2^n-1) \\ 0 & 1 \end{pmatrix}$