Aiming for A*: ‘in between’ Pure 1 and Pure 2

[Edexcel]

Bicen Maths

Note:

- I have used questions from the international A-Level, old specification A-Level, and other exam boards – I only used ‘recent’ years of the old specification, as this is when examiners were gearing up for the new style of questions, so they’re a good fit!
- This is to ensure the questions are all ‘fresh’, as at this stage I know that exam question fatigue is real.
- I’ve used my knowledge to select ones that match Edexcel’s style as much as possible.

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OCR Pure 1 2020

Try these questions whilst we wait for everyone to arrive

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1 (a) For a small angle θ, where θ is in radians, show that $2\cos\theta + (1-\tan\theta)^2 \approx 3 - 2\theta$. **[3]**

(b) Hence determine an approximate solution to $2\cos\theta + (1-\tan\theta)^2 = 28\sin\theta$. **[2]**

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2 A sequence of transformations maps the curve $y = e^x$ to the curve $y = e^{2x+3}$.

Give details of these transformations. **[3]**

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4 Prove algebraically that $n^3 + 3n - 1$ is odd for all positive integers n. **[4]**

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1 Beth states that for all real numbers p and q, if $p^2 > q^2$ then $p > q$.

Prove that Beth is **not** correct. **[2]**

Small Angle Approximations

$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ $\tan\theta \approx \theta$

$\sin\theta \approx \theta$

a) $2\cos\theta + (1-\tan\theta)^2 \approx 2(1-\frac{1}{2}\theta^2) + (1-\theta)^2$

$= 2 - \theta^2 + 1 - 2\theta + \theta^2$

$= 3 - 2\theta$

b) $2\cos\theta + (1-\tan\theta)^2 = 28\sin\theta$

$3 - 2\theta = 28\theta$

$3 = 30\theta$

$\underline{\underline{\theta = 0.1}}$

Transformations

Let $f(x) = e^x$

then $f(2x+3) = e^{2x+3}$

This is a translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and stretch in x direction, S.F. $\frac{1}{2}$

Proof

Use exhaustion with odd/evens

Let $n = 2k$

$$
\begin{aligned}
n^3 + 3n - 1 &= (2k)^3 + 3(2k) - 1 \\
&= 8k^3 + 6k - 1 \\
&= 2(4k^3 + 3k) - 1 \quad \text{so odd}
\end{aligned}
$$

Let $n = 2k + 1$

$$
\begin{aligned}
n^3 + 3n - 1 &= (2k+1)^3 + 3(2k+1) - 1 \\
&= 8k^3 + 3 \times 4k^2 + 3 \times 2k + 1 + 6k + 3 - 1 \\
&= 8k^3 + 12k^2 + 12k + 3 \\
&= 2(4k^3 + 6k^2 + 6k + 1) + 1 \quad \text{so odd}
\end{aligned}
$$

Hence it is always odd.

<u>Disproof by counter example</u>

eg. $2 > -3$ but $2^2 < (-3)^2$

$4 < 9$

So Beth is not correct

1	(a)		$2\left(1-\frac{1}{2}\theta^2\right)+(1-\theta)^2$	B1	2.1	Correct statement
			$2-\theta^2+1-2\theta+\theta^2$	M1	2.1	Attempt to expand and simplify given expression
			$=3-2\theta$ **A.G.**	A1	2.1	Obtain given answer
				[3]		
	(b)		$3-2\theta=28\theta$	M1	1.1a	Use $28\sin\theta \approx 28\theta$ and attempt to solve
			$\theta=0.1$	A1	1.1	Obtain 0.1 oe
				[2]		

2			Refers to translation and stretch	**M1**	**1.1**	In either order; ignore details here; allow any equivalent wording (such as move or shift for translation) to describe geometrical transformations but not statements such as add –3 to x (do not accept 'enlargement' or 'shear' for stretch)
			State translation $\begin{pmatrix}-3\\0\end{pmatrix}$	**A1**	**1.1**	Or state translation in x-direction by –3 (units); accept horizontal to indicate direction or parallel to the x-axis; term 'translate' or 'translation' needed for award of A1
			State stretch by scale factor 0.5 parallel to the x-axis	**A1**	**1.1**	Or in the x direction or horizontally; term 'stretch' needed for award of A1; these two transformations must be in this order – if details correct for M1A1A1 but order wrong, award M1A1A0
				[3]		

4			If n is even then n can be written as $2m$. $n^3+3n-1 = 8m^3 + 6m - 1$	E1	2.1	Consider when n is even
			$= 2(4m^3 + 3m) - 1$ For all m, $2(4m^3 + 3m)$ is even, hence $2(4m^3 + 3m) - 1$ is odd	E1	2.4	Conclude from useable form
			If n is odd then n can be written as $2m + 1$ $n^3+3n-1 = 8m^3 + 12m^2 + 6m + 1 + 6m + 3 - 1$ $= 8m^3 + 12m^2 + 12m + 3$	E1	2.1	Consider when n is odd
			$= 2(4m^3 + 6m^2 + 6m) + 3$ For all m, $2(4m^3 + 6m^2 + 6m)$ is even, hence $2(4m^3 + 6m^2 + 6m) + 3$ is odd	E1	2.4	Conclude from useable form

1			For example $(-3)^2 = 9 > 2^2 = 4$ and $(-3) < 2$ So Beth is not correct	M1 E1 [2]

What came up in Paper 1?

- factor theorem
- binomial expansion
- numerical methods, Newton-Raphson
- differentiation first principles
- quotient rule/decreasing functions
- modulus graphs
- simple differential equation (tank)
- functions - composite, inverse, domain, range
- geometric series (included indices)
- area integration/differentiation problem
- sector/segment areas
- harmonic identity (Rcos Rsin)
- integration by substitution
- differential equation - set up and solve (balloon)
- proof, inc. proof by contradiction

Key

- Doesn't usually appear in both
- Could potentially appear again
- Could easily appear again

What *should* come up in Paper 2?

- Small angle approximations
- Graph transformations
- Proof (again!)
- Sequences + series – arithmetic and/or sigma notation
- Circles
- Vectors
- Trigonometric identities equations
- Implicit differentiation, dx/dy
- Connected rates of change + optimisation problems
- More integration - limit of a sum, by parts, partial fractions
- Parametric differentiation + integration
- Exponential + logarithmic equations
- Modelling - quadratics, linear, exp & logs
- Trapezium rule
- Inequalities? Regions?
- Numerical methods?
- Surds?
- Pseudo-quadratics?

Key

- Not covered in this session
- Covered in the 'starter' – use textbook and Exam Qs PDF for more practice questions
- Covered in the main session or Your Turn questions

After the session, make sure you:

- Review anything you didn't understand
- Complete the Your Turn questions
- Review the red and blue topics
- Look over your notes for topics which have already come up, *just in case*

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Sequences and Series

- They usually have at least 2 questions on this topic

- $S_n = \frac{n}{2}(2a + (n-1)d)$
- $u_n = a + (n-1)d$

Arithmetic series

$$S_n = \tfrac{1}{2}n(a + l) = \tfrac{1}{2}n[2a + (n-1)d]$$

- $S_n = \frac{a(1-r^n)}{1-r}$
- $u_n = ar^{n-1}$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

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$u_n = a+(n-1)d$

3 An arithmetic progression has first term 2 and common difference d, where $d \neq 0$. The first, third and thirteenth terms of this progression are also the first, second and third terms, respectively, of a geometric progression.

By determining d, show that the arithmetic progression is an increasing sequence. **[5]**

$u_1 = 2$	$u_1 = 2$
$u_3 = 2+2d$	$u_2 = 2r$
$u_{13} = 2+12d$	$u_3 = 2r^2$

$2+2d = 2r$

$1+d = r$

$2+12d = 2r^2$

$1+6d = r^2$

$1+6d = (1+d)^2$

$1+6d = 1+2d+d^2$

$0 = d^2-4d$

$d(d-4)=0$

$\underline{\underline{d=4}}$

As $d>0$, it is increasing.

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3 A particular phone battery will last 10 hours when it is first used. Every time it is recharged, it will only last 98% of its previous time.

Find the maximum total length of use for the battery. **[3]**

$10 + 10\times0.98 + 10\times0.98\times0.98 + 10\times0.98^3 + \dots$

$S_\infty = \frac{a}{1-r} = \frac{10}{1-0.98} = \frac{10}{0.02} = \underline{\underline{500}}$ hours

3			$2+2d=2r$	B1	1.1	Or for $a+2d=ar$	
			$2+12d=2r^2$	B1	1.1	Or for $a+12d=ar^2$	
			$1+6d=(1+d)^2$ or $2+12d=2(1+d)^2$	M1*	1.1	Setting up an equation in d or r only – dependent on one **B** mark	$2+12(r-1)=2r^2$
			$d^2-4d=0 \Rightarrow d=...$	M1dep*	1.1	Solving their two-term quadratic equation in d (or three-term quadratic in r)	$r^2-6r+5=0$ $(r-5)(r-1)=0$ $\Rightarrow r=...$
			$d=4$ and as the common difference is positive the progression is an increasing sequence	A1	2.4	Correct value for d and link to increasing sequence – must either say that d is positive (oe) or state at least the correct first four terms and comment that they are increasing	Condone no mention of $d \neq 0$
				[5]			

3			$10 + 10 \times 0.98 + 10 \times 0.98^2$ or $10 + 9.8 + 9.604$	M1	3.1b	Use of GP with common ratio 0.98
			$\frac{10}{1-0.98}$	M1	1.1	Sum to infinity
			500	A1	3.2a	
				[3]		

2. A sequence $u_1, u_2, u_3 \ldots$ is defined by

$$u_1 = 20$$

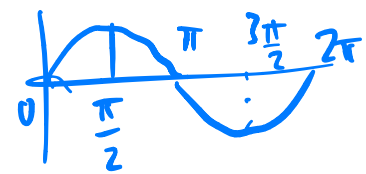
$$u_{n+1} = u_n + 5\sin\left(\frac{n\pi}{2}\right) - 3(-1)^n$$

(a) (i) Show that $u_2 = 28$

(ii) Find the value of u_3 and the value of u_4

(3)

Given that the sequence is periodic with order 4



(b) (i) write down the value of u_5

(ii) find the value of $\sum_{r=1}^{25} u_r$

(3)

i) $u_2 = u_1 + 5\sin\left(\frac{\pi}{2}\right) - 3(-1)^1$

$= 20 + 5 + 3 = 28$

ii) $u_3 = u_2 + 5\sin\pi - 3(-1)^2$

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$= 28 + 0 - 3$

$= 25$

$u_4 = u_3 + 5\sin\frac{3\pi}{2} - 3(-1)^3$

$= 25 - 5 + 3$

$= 23$

b) i) 20, 28, 25, 23, $\boxed{20}$ u_5

$u_5 = 20$

ii) $\sum_{r=1}^{25} u_r = u_1 + u_2 + u_3 + \dots + u_{25}$

$= \underbrace{6(20 + 28 + 25 + 23)}_{\text{sum of } u_1 \text{ to } u_{24}} + \underset{u_{25}}{20}$

$= 596$

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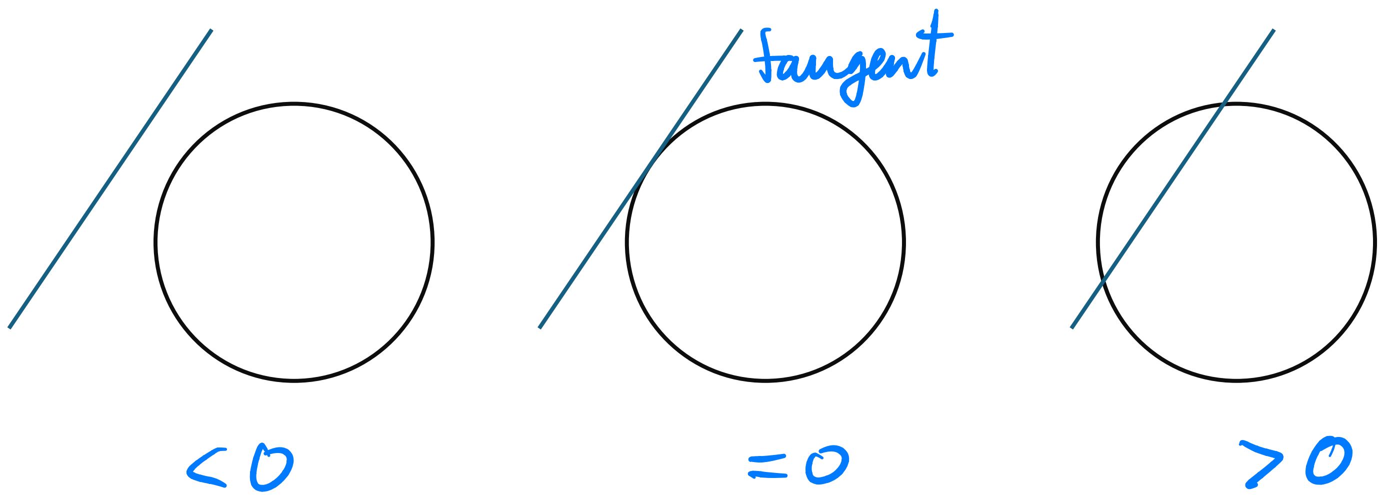
Question	Scheme	Marks	AOs
2(a)(i)	e.g. $(u_2 =)20+5\sin\left(\frac{\pi}{2}\right)-3(-1)^1=28$ *	B1*	2.1
(ii)	$u_3=28+5\sin\left(\frac{2\pi}{2}\right)-3(-1)^2\ (=25)$ **or** $u_4=\text{"25"}+5\sin\left(\frac{3\pi}{2}\right)-3(-1)^3\ (=23)$	M1	1.1b
	$u_3=25$ and $u_4=23$	A1	1.1b
		(3)	
(b)(i)	$(u_5=)20$	B1	2.2a
(ii)	e.g. $\sum_{r=1}^{25}u_r=6(20+28+\text{"25"}+\text{"23"})+20$	M1	3.1a
	$=596$	A1	1.1b
		(3)	
			(6 marks)

Circles

Tips:

- Use completing the square to get to standard form
- Use discriminant to decide how many intersections

$b^2 - 4ac$



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14. The circle C has equation

$$x^2 + y^2 + 16y + k = 0$$

where k is a constant.

(a) Find the coordinates of the centre of C.

(2)

Given that the radius of C is 10

(b) find the value of k.

(2)

The point $A(a, -16)$, where $a > 0$, lies on the circle C. The tangent to C at the point A crosses the x-axis at the point D and crosses the y-axis at the point E.

(c) Find the exact area of triangle ODE.

(7)

a) $x^2 + (y+8)^2 - 64 + k = 0$

$x^2 + (y+8)^2 = 64 - k$ r^2

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$C(0, -8) = 0$

b) $64 - k = 100$

$-k = 36$

$k = -36$

c) $y = -16$

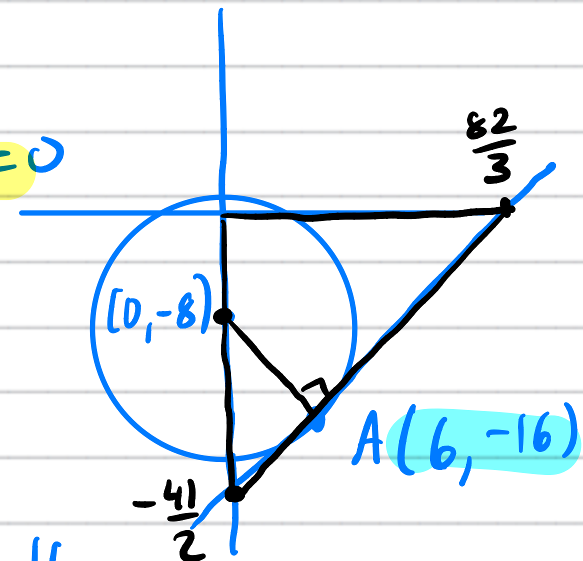
$x^2 + (-16+8)^2 - 64 - 36 = 0$

$x^2 + 64 - 64 - 36 = 0$

$x^2 = 36$

$x = \pm 6$

$x = 6, \; y = -16$



- find gradient of radius
  - $-$ve reciprocal
- find eq. of tangent
  - $x=0$, find y
  - $y=0$, find x
- Find the area.

$(0, -8)$
$(6, -16)$

$$m = \frac{-8 - -16}{0 - 6} = \frac{8}{-6} = \frac{4}{-3}$$

$$m_{\perp} = \frac{3}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y + 16 = \frac{3}{4}(x - 6)$$

$x = 0$

$$y + 16 = \frac{3}{4}(-6)$$

$$y = -\frac{41}{2}$$

$y = 0$

$$16 = \frac{3}{4}(x - 6)$$

$$x = \frac{82}{3}$$

$$\text{Area} = \frac{1}{2} \times \frac{41}{2} \times \frac{82}{3} = \frac{1681}{6}$$

14	Mark (a) and (b) together		
(a)	$(0, -8)$	$x = 0$ **or** $y = -8$ (May be seen on a sketch)	B1
		$x = 0$ **and** $y = -8$ (May be seen on a sketch)	B1
			(2)
(b)	Uses 64, 100 and k (not k^2) to obtain a value for k		M1
	$k = -36$	cao	A1
	$k = -36$ scores both marks		
			(2)
14(c)	$y = -16 \Rightarrow a = 6$	Correct x-coordinate. Allow $x = 6$ or just sight of 6. May be seen on a sketch.	B1
	$m_N = \frac{-16+8}{6-0}\left(=-\frac{4}{3}\right)$ or $m_N = \frac{-16+8}{a-0}\left(=-\frac{8}{a}\right)$	Correct attempt at gradient using the centre and their A. Allow one sign slip. If they use O for the centre, this is M0. Allow if in terms of a i.e. if they haven't found or can't find a.	M1
	$m_T = -1 \div "-\frac{4}{3}" = ...$ or $m_T = -1 \div "-\frac{8}{a}" = ...$	Correct use of perpendicular gradient rule. Allow if in terms of a.	M1
	$y + 16 = \frac{3}{4}(x - "6")$ or $y + 16 = \frac{a}{8}(x - "6")$	Correct straight line method using a gradient which is **not** the radius gradient and their A or (a, -16). Allow a gradient in terms of a.	M1
	$x = 0 \Rightarrow y = -\frac{41}{2},\ y = 0 \Rightarrow x = \frac{82}{3}$	Correct values	A1
	Area $= \frac{1}{2} \times \frac{41}{2} \times \frac{82}{3}$	Correct method for area using vertices of the form (0, 0), (X, 0) and (0, Y) where X and Y are numeric and have come from the intersections of their tangent with the axes. Allow negative lengths here. **Dependent on the previous M mark.**	**d**M1
	$= \frac{1681}{6}$ or $280\frac{1}{6}$ or $280.1\dot{6}$ (clear dot over 6)	Cao. Must be **positive** and may be recovered from sign errors on $-\frac{41}{2}$ and/or $\frac{82}{3}$ **but must be from a correct tangent equation**.	A1

Vectors

- I'm going to concentrate on **geometric** problems here – *but of course, you should learn the basics for magnitude, angles with axes, etc.*
- They've not asked one of these style of questions for a long time, so I think they're worth revisiting

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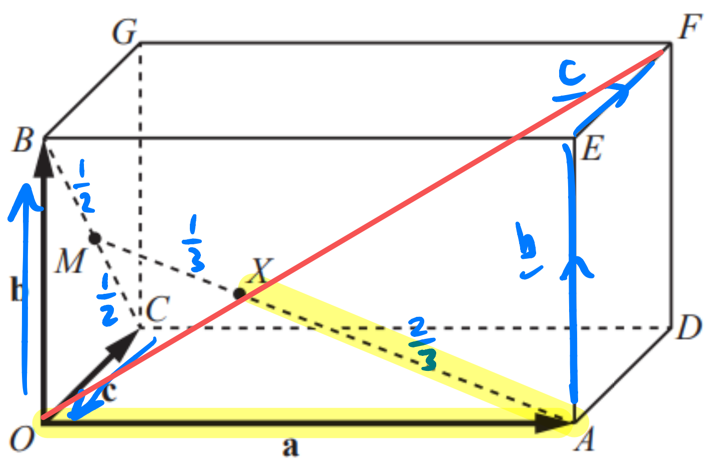
9 Points A, B and C have position vectors **a**, **b** and **c** relative to an origin O in 3-dimensional space. Rectangles $OADC$ and $BEFG$ are the base and top surface of a cuboid.



$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$
$$= \underline{a} + \frac{2}{3}\overrightarrow{AM}$$
$$= \underline{a} + \frac{2}{3}\left(-\underline{a} + \underline{c} + \frac{1}{2}\overrightarrow{CB}\right)$$
$$= \underline{a} - \frac{2}{3}\underline{a} + \frac{2}{3}\underline{c} + \frac{1}{3}\overrightarrow{CB}$$
$$= \frac{1}{3}\underline{a} + \frac{2}{3}\underline{c} + \frac{1}{3}(\underline{b} - \underline{c})$$
$$= \frac{1}{3}\underline{a} + \frac{2}{3}\underline{c} + \frac{1}{3}\underline{b} - \frac{1}{3}\underline{c}$$
$$= \frac{1}{3}\underline{a} + \frac{1}{3}\underline{b} + \frac{1}{3}\underline{c}$$
$$= \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$$

- The point M is the midpoint of BC.
- The point X lies on AM such that $AX = 2XM$.

(a) Find $\overrightarrow{OX}$ in terms of **a**, **b** and **c**, simplifying your answer. **[4]**

(b) Hence show that the lines OF and AM intersect. **[2]**

$\overrightarrow{OF} = \underline{a} + \underline{b} + \underline{c}$

$= 3\overrightarrow{OX}$

As $\overrightarrow{OF}$ is parallel to $\overrightarrow{OX}$, so X is on OF and X is on AM,

AM and OF intersect.

9	(a)		**Summary method:**		
			$\overrightarrow{OM} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ or $\mathbf{b} + \frac{1}{2}(-\mathbf{b} + \mathbf{c})$ oe	B1	Can be implied
			$\overrightarrow{AM}$ or $\overrightarrow{MA}$ attempted in terms of **a**, **b** and **c** $(= \pm(\frac{1}{2}(\mathbf{b} + \mathbf{c}) - \mathbf{a})$ oe)	M1	May be included in working, eg $\overrightarrow{AX} = \frac{2}{3}(\frac{1}{2}(\mathbf{b} + \mathbf{c}) - \mathbf{a})$ Not necessarily correct
			$\overrightarrow{OX} = \mathbf{a} + \frac{2}{3}\overrightarrow{AM}$ or $\overrightarrow{OM} + \frac{1}{3}\overrightarrow{MA}$ oe attempted in terms of **a**, **b** and **c**	M1	Not necessarily correct
			$\overrightarrow{OX} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$	A1	or equivalent simplified form

9	(b)		$\overrightarrow{OF} = \mathbf{a} + \mathbf{b} + \mathbf{c}$	B1*	
			Hence X lies on OF, so AM and OF intersect	B1dep [2]	Both statements needed. NB dep on B1

Trigonometry

- In my view, the following are essential to know by heart:

Simple:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

Pythagorean:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

Double angle:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x$$

- Use the shortcuts for finding the solutions:

$\sin x = \sin(180 - x)$ then ± 360

$\cos x = \cos(360 - x)$ then ± 360

$\tan x = \tan(x + 180)$

- Work on the ‘messy’ side, or one where can add fractions

9. (a) Prove that

$$\sin 2x - \tan x \equiv \tan x \cos 2x, \qquad x \neq (2n+1)90°, \qquad n \in \mathbb{Z}$$

(4)

(b) Given that $x \neq 90°$ and $x \neq 270°$, solve, for $0 \leqslant x < 360°$,

$$\sin 2x - \tan x = 3\tan x \sin x$$

Give your answers in degrees to one decimal place where appropriate.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

a) $\sin 2x - \tan x \equiv 2\sin x\cos x - \dfrac{\sin x}{\cos x}$

$$\equiv \frac{2\sin x\cos^2 x - \sin x}{\cos x}$$

$$\equiv \frac{\sin x(2\cos^2 x - 1)}{\cos x}$$

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$$\equiv \tan x \cos 2x$$

b) $\cancel{\tan x}\cos 2x = 3\cancel{\tan x}\sin x$

$$\boxed{\tan x = 0}$$

$$x = 0°, 180°$$

$$\cos 2x = 3\sin x$$

$$1 - 2\sin^2 x = 3\sin x$$

$$0 = 2\sin^2 x + 3\sin x - 1$$

$\sin x = \dfrac{-3+\sqrt{17}}{4}$ or $\sin x = \cancel{\dfrac{-3-\sqrt{17}}{4}}$

$\hookrightarrow -1.78$

$$x = 16.3°, 163.7°$$

180 − ANS

$$x = 0°, 180°$$

9(a)	$\sin 2x - \tan x = 2\sin x\cos x - \tan x$	M1
	$= \dfrac{2\sin x\cos^2 x}{\cos x} - \dfrac{\sin x}{\cos x}$	M1
	$= \dfrac{\sin x}{\cos x} \times (2\cos^2 x - 1)$	
	$= \tan x\cos 2x$	dM1 A1*
		(4)
(b)	$\tan x\cos 2x = 3\tan x\sin x \Rightarrow \tan x(\cos 2x - 3\sin x) = 0$	
	$\cos 2x - 3\sin x = 0$	M1
	$\Rightarrow 1 - 2\sin^2 x - 3\sin x = 0$	M1
	$\Rightarrow 2\sin^2 x + 3\sin x - 1 = 0 \Rightarrow \sin x = \dfrac{-3 \pm \sqrt{17}}{4} \Rightarrow x = \ldots$	M1
	Two of $x = 16.3°, 163.7°, 0, 180°$	A1
	All four of $x = 16.3°, 163.7°, 0, 180°$	A1
		(5)
		(9 marks)

Implicit Differentiation

- If you differentiate y with respect to x, it becomes $\frac{dy}{dx}$
- If you differentiate a function in y with respect to x, do as you expect, then multiply by $\frac{dy}{dx}$
- I recommend writing out the product rule carefully for any product expressions including a y
  - e.g. $4x^2y^3$

$u = 4x^2 \quad \times \quad v = y^3$

$u' = 8x \quad\quad v' = 3y^2 \frac{dy}{dx}$

- If you have $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, then $f(x, y) = 0$ for zero (horizontal) gradient, and $g(x, y) = 0$ for undefined (vertical) gradient

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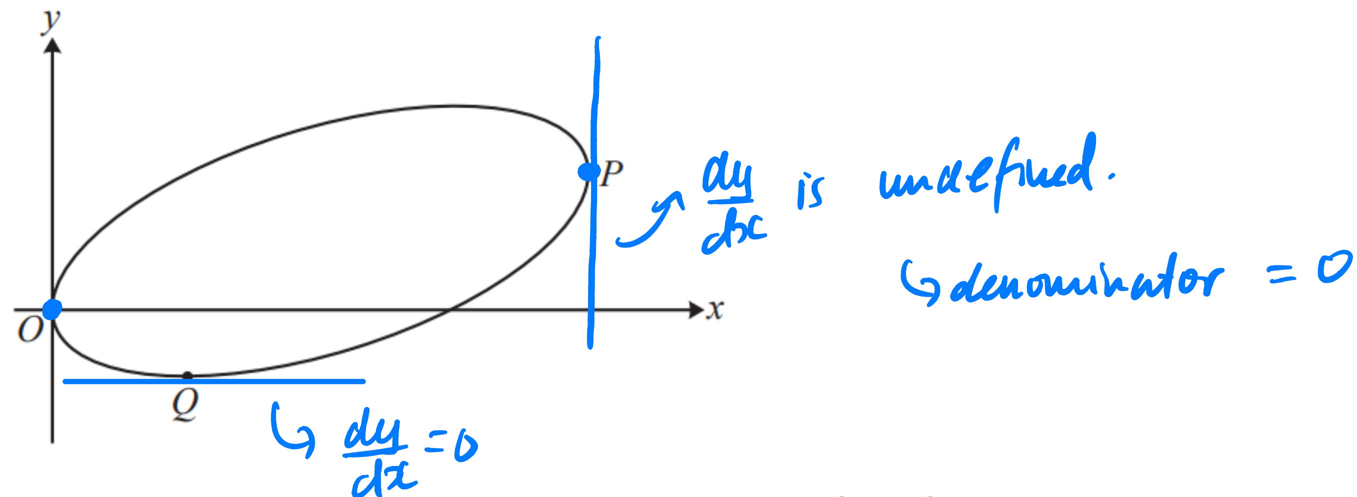
6 **In this question you must show detailed reasoning.**



The diagram shows the curve with equation $4xy = 2(x^2 + 4y^2) - 9x$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x-4y-9}{4x-16y}$. **[3]**

At the point P on the curve the tangent to the curve is parallel to the y-axis and at the point Q on the curve the tangent to the curve is parallel to the x-axis.

(b) Show that the distance PQ is $k\sqrt{5}$, where k is a rational number to be determined. **[8]**

I've pre-done some of (b) so as not to waste your time in the session

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$$4xy = 2x^2 + 8y^2 - 9x$$

$u = 4x$ $\quad$ $v = y$

$u' = 4$ $\quad$ $v' = \frac{dy}{dx}$

$$4x\frac{dy}{dx} + 4y = 4x + 16y\frac{dy}{dx} - 9$$

$$4x\frac{dy}{dx} - 16y\frac{dy}{dx} = 4x - 4y - 9$$

$$\frac{dy}{dx}(4x - 16y) = 4x - 4y - 9$$

$$\frac{dy}{dx} = \frac{4x - 4y - 9}{4x - 16y}$$

b) Find P, $\quad 4x - 16y = 0$

$$4x = 16y$$

$$x = 4y$$

$$4(4y)y = 2(4y)^2 + 8y^2 - 9(4y)$$

$$16y^2 = 32y^2 + 8y^2 - 36y$$

$$0 = 24y^2 - 36y$$

$$0 = y(24y - 36)$$

$y = 0$

or

$$24y = 36$$

$$y = \frac{3}{2}$$

$$x = 4 \times \frac{3}{2} = 6$$

$$P\left(6, \frac{3}{2}\right)$$

Q//

$\frac{dy}{dx} = 0$ $\qquad$ $4x - 4y - 9 = 0$

$4x = 4y + 9$

$x = y + 9/4$

$4xy = 2x^2 + 8y^2 - 9x$

$4(y + \frac{9}{4})y = 2(y + \frac{9}{4})^2 + 8y^2 - 9(y + \frac{9}{4})$

$4y^2 + 9y = 2(y^2 + \frac{9}{2}y + \frac{81}{16}) + 8y^2 - 9y - \frac{81}{4}$

$4y^2 + 9y = 2y^2 + 9y + \frac{81}{8} + 8y^2 - 9y - \frac{81}{4}$

$0 = 6y^2 - 9y - \frac{81}{8}$

$y = \frac{9}{4}$, $\underline{\underline{y = -\frac{3}{4}}}$

↪ as Q is below x-axis

$x = y + \frac{9}{4}$

$= -\frac{3}{4} + \frac{9}{4} = \frac{3}{2}$

$Q\left(\frac{3}{2}, -\frac{3}{4}\right)$

$P\left(6, \frac{3}{2}\right)$

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$$PQ = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{4}\right)^2} = \frac{9}{4}\sqrt{5}$$

6	(a)		**DR**			$4xy = 2(x^2 + 4y^2) - 9x$
				M1	1.1	For correct differentiation of either LHS or RHS, even if not in an equation
			$4y + 4x\dfrac{dy}{dx} = 4x + 16y\dfrac{dy}{dx} - 9$	**A1**	1.1	
			$4x\dfrac{dy}{dx} - 16y\dfrac{dy}{dx} = 4x - 4y - 9 \Rightarrow \dfrac{dy}{dx} = \dfrac{4x-4y-9}{4x-16y}$	**A1**	2.1	**AG** (at least one line of working from correct differentiation to given answer)
				[3]		
6	(b)		**DR**			
			(At P) $4x - 16y = 0$	**M1***	3.1a	
			$x = 4y \Rightarrow 16y^2 = 2(16y^2 + 4y^2) - 36y$ $24y^2 - 36y = 0$	**M1dep***	2.1	Forms two-term quadratic equation in y or x (if correct $x^2 - 6x = 0$)
			$y(2y-3) = 0 \Rightarrow y = \frac{3}{2}$ $P\left(6, \frac{3}{2}\right)$	**A1**	1.1	
			(At Q) $4x - 4y - 9 = 0$	**M1***	3.1a	
			$\Rightarrow 4x\left(x - \frac{9}{4}\right) = 2x^2 + 8\left(x - \frac{9}{4}\right)^2 - 9x$ $4x^2 - 24x + 27 = 0$	**M1dep***	2.1	Forms three-term quadratic equation in y or x (if correct $16y^2 - 24y - 27 = 0$)
			$Q\left(\frac{3}{2}, -\frac{3}{4}\right)$ only	**A1**	3.2a	
			$PQ^2 = \left(6 - \frac{3}{2}\right)^2 + \left(\frac{3}{2} - \left(-\frac{3}{4}\right)\right)^2$	**M1**	1.1	Correct implies distance formula for their P and Q
			$PQ = \frac{9}{4}\sqrt{5}$	**A1**	2.2a	www
				[8]		

(ii) Given $x = \sin^2 2y$, $0 < y < \frac{\pi}{4}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of y.

Write your answer in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p\operatorname{cosec}(qy), \qquad 0 < y < \frac{\pi}{4}$$

where p and q are constants to be determined.

(5)

$x = (\sin 2y)^2$

$\frac{dx}{dy} = 2\sin 2y \times 2\cos 2y$

$\frac{dx}{dy} = 4\sin 2y\cos 2y$

$\frac{dx}{dy} = 2 \times 2\sin 2y\cos 2y$

$\frac{dx}{dy} = 2\sin 4y$

$\frac{dy}{dx} = \frac{1}{2}\operatorname{cosec} 4y$

$2\sin A\cos A = \sin 2A$

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			(5)
(ii)	$x = \sin^2 2y \Rightarrow \dfrac{dx}{dy} = 2\sin 2y \times 2\cos 2y$	M1A1	
	Uses $\sin 4y = 2\sin 2y \cos 2y$ in their expression	M1	
	$\dfrac{dx}{dy} = 2\sin 4y \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sin 4y} = \dfrac{1}{2}\operatorname{cosec} 4y$	M1A1	
			(5)
			(10 marks)

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Modelling with differentiation

- Connected rates of change
  - Write out what you are looking for, then split the derivative up:
    - $\frac{dh}{dt} = \frac{dh}{} \times \frac{}{dt}$

      ↑ rate of change of h

      $\frac{}{dt}$ rate
- Optimisation problems
  - Setup by eliminating variables, differentiate and set to 0, then solve

4.

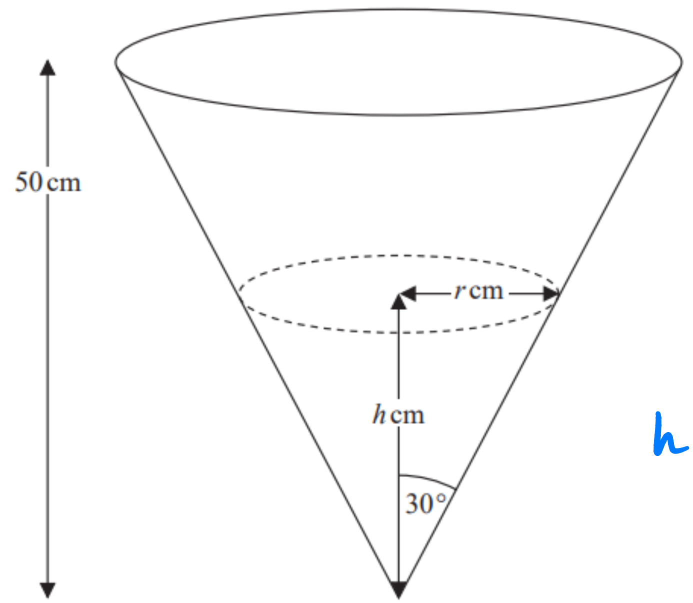
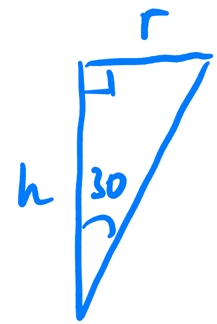


Diagram not drawn to scale



$\tan 30 = \frac{r}{h}$

$r = h \tan 30$

$r = \frac{\sqrt{3}}{3} h$

Figure 1

A water container is made in the shape of a hollow inverted right circular cone with semi-vertical angle of 30°, as shown in Figure 1. The height of the container is 50 cm.

When the depth of the water in the container is h cm, the surface of the water has radius r cm and the volume of water is V cm³.

(a) Show that $V = \frac{1}{9}\pi h^3$

[*You may assume the formula* $V = \frac{1}{3}\pi r^2 h$ *for the volume of a cone.*] → eliminate r

(2)

Given that the volume of water in the container increases at a constant rate of 200 cm³ s⁻¹,

$\frac{dV}{dt} = 200$

(b) find the rate of change of the depth of the water, in cm s⁻¹, when $h = 15$
Give your answer in its simplest form in terms of π.

(4)

a) $V = \frac{1}{3}\pi r^2 h$

$r = \frac{\sqrt{3}}{3} h$

$r^2 = \frac{1}{3} h^2$

$V = \frac{1}{3}\pi \left(\frac{1}{3} h^2\right) h = \frac{1}{9}\pi h^3$

b) $\frac{dV}{dh} = \frac{1}{3}\pi h^2 = \frac{\pi h^2}{3}$

$\frac{dV}{dt} = 200$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dh}{dt} = \frac{3}{\pi h^2} \times 200$

$= \frac{600}{\pi h^2}$

$= \frac{600}{\pi \times 15^2} = \frac{8}{3\pi}$ cm s^{-1}

4. (a)	$\frac{r}{h} = \tan 30 \Rightarrow r = h\tan 30 \left\{\Rightarrow r = \frac{h}{\sqrt{3}} \text{ or } r = \frac{\sqrt{3}}{3}h\right\}$ **or** $\frac{h}{r} = \tan 60 \Rightarrow r = \frac{h}{\tan 60} \left\{\Rightarrow r = \frac{h}{\sqrt{3}} \text{ or } r = \frac{\sqrt{3}}{3}h\right\}$ **or** $\frac{r}{\sin 30} = \frac{h}{\sin 60} \Rightarrow r = \frac{h\sin 30}{\sin 60} \left\{\Rightarrow r = \frac{h}{\sqrt{3}} \text{ or } r = \frac{\sqrt{3}}{3}h\right\}$ **or** $h^2 + r^2 = (2r)^2 \Rightarrow r^2 = \frac{1}{3}h^2$	Correct use of trigonometry to find r in terms of h **or** correct use of Pythagoras to find r^2 in terms of h^2	M1
	$\left\{V = \frac{1}{3}\pi r^2 h \Rightarrow\right\} V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h \Rightarrow V = \frac{1}{9}\pi h^3$ *	Correct proof of $V = \frac{1}{9}\pi h^3$ or $V = \frac{1}{9}h^3\pi$ Or shows $\frac{1}{9}\pi h^3$ or $\frac{1}{9}h^3\pi$ with some reference to $V =$ in their solution	A1 *
			[2]
(b) **Way 1**	$\frac{dV}{dt} = 200$		
	$\frac{dV}{dh} = \frac{1}{3}\pi h^2$	$\frac{1}{3}\pi h^2$ o.e.	B1
	Either • $\left\{\frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt} \Rightarrow\right\} \left(\frac{1}{3}\pi h^2\right)\frac{dh}{dt} = 200$ • $\left\{\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} \Rightarrow\right\} \frac{dh}{dt} = 200 \times \frac{1}{\frac{1}{3}\pi h^2}$	**either** $\left(\text{their } \frac{dV}{dh}\right) \times \frac{dh}{dt} = 200$ **or** $200 \div \left(\text{their } \frac{dV}{dh}\right)$	M1
	When $h = 15,\ \frac{dh}{dt} = 200 \times \frac{1}{\frac{1}{3}\pi(15)^2} \left\{= \frac{200}{75\pi} = \frac{600}{225\pi}\right\}$	**dependent on the previous M mark**	dM1
	$\frac{dh}{dt} = \frac{8}{3\pi}\ (\text{cm s}^{-1})$	$\frac{8}{3\pi}$	A1 **cao**
			[4]
			6

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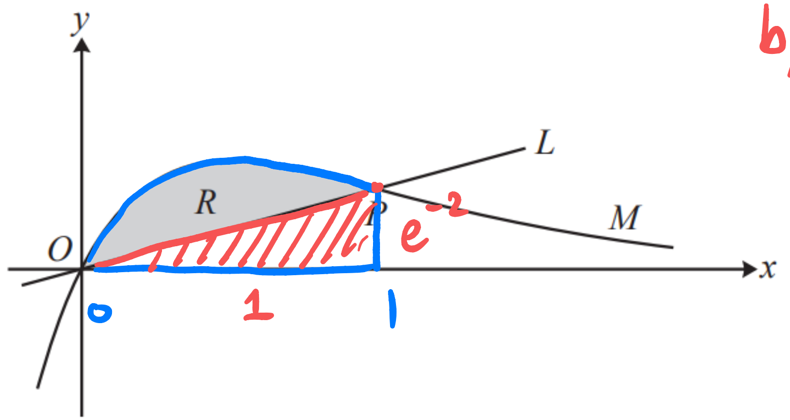
Integration

- By parts
- Partial fractions

- … more in the parametric section…

8



b)

$x = 1, \quad y = 1 \times e^{-2 \times 1}$
$= e^{-2}$

area $= 1 \times e^{-2} \times \frac{1}{2}$
$= \frac{1}{2}e^{-2}$

$\frac{d^2y}{dx^2} = 0$

The diagram shows the curve M with equation $y = xe^{-2x}$.

(a) Show that M has a point of inflection at the point P where $x = 1$. **[5]**

The line L passes through the origin O and the point P. The shaded region R is enclosed by the curve M and the line L.

(b) Show that the area of R is given by

$\frac{1}{4}(a + be^{-2})$,

where a and b are integers to be determined. **[6]**

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a) $y = xe^{-2x}$

$u = x \quad v = e^{-2x}$

$u' = 1 \quad v' = -2e^{-2x}$

$$\frac{dy}{dx} = e^{-2x} - 2xe^{-2x}$$

$$= e^{-2x}(1-2x)$$

$u = 1-2x \quad v = e^{-2x}$

$u' = -2 \quad v' = -2e^{-2x}$

$$\frac{d^2y}{dx^2} = -2e^{-2x} - 2e^{-2x}(1-2x)$$

$$= -2e^{-2x} - 2e^{-2x} + 4xe^{-2x}$$

$$= e^{-2x}(4x-4)$$

When $x=1$, $\frac{d^2y}{dx^2} = e^{-2}(4-4) = 0$

$x = 0.9 \quad \frac{d^2y}{dx^2} = e^{-2\times0.9}(4\times0.9-4) = -0.06\ldots < 0$

$x = 1.1 \quad \frac{d^2y}{dx^2} = 0.0443 > 0$

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As there is a change in sign, and $\frac{d^2y}{dx^2}=0$ at $x=1$, there is a point of inflection.

b) $$\int_0^1 xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \int_0^1 \frac{1}{2}e^{-2x}\,dx$$

$u = x \qquad v = -\frac{1}{2}e^{-2x}$

$u' = 1 \qquad v' = e^{-2x}$

$$= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1$$

$$= -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} - \left(0 - \frac{1}{4}\right)$$

$$= \underline{-\frac{3}{4}e^{-2} + \frac{1}{4}}$$

$$R = -\frac{3}{4}e^{-2} + \frac{1}{4} - \frac{1}{2}e^{-2}$$

$$= \frac{1}{4} - \frac{5}{4}e^{-2}$$

$$= \underline{\frac{1}{4}\left(1 - 5e^{-2}\right)}$$

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8	(a)			M1*	2.1	Differentiates y with respect to x – answer of the form $\pm e^{-2x} \pm \lambda x e^{-2x}$
			$y' = e^{-2x}(1-2x)$	A1	1.1	
			$y'' = e^{-2x}(-4+4x)$	A1ft	1.1	Follow through their first derivative
				M1dep*	3.1a	Solves $y''=0$ (or attempts to verify $y''=0$ by substituting $x=1$) **or** considers sign change either side of y''
			$y''=0$ at $x=1$ and $y''(0.5) = -2e^{-1} < 0$, $y''(1.5) = 2e^{-3} > 0$ (so change of sign indicates a point of inflection at $x = 1$)	A1 [5]	2.2a	Conclusion not required for this mark
8	(b)		$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx$	M1*	2.1	Integration by parts – of the form $\pm\alpha xe^{-2x} \pm \beta\int e^{-2x}dx$
			$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$	A1	1.1	
			$\int_0^1 xe^{-2x}dx = \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1$ $= \left(-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right) - \left(0 - \frac{1}{4}\right)$	M1dep*	1.1	Use of correct limits in their fully integrated expression – need not be simplified (or equivalent)
			$\frac{1}{4} - \frac{3}{4}e^{-2}$	A1	1.1	Allow unsimplified
			Area of triangle below $OP = \frac{1}{2}e^{-2}$	B1	1.1	Or by correctly evaluating $\int_0^1 e^{-2}x\,dx$
			$= \frac{1}{4}\left(1 - 5e^{-2}\right)$	A1	2.2a	$a = 1,\ b = -5$ (must be in this form)
				[6]		

Parametrics

- 'Parametric world' is easier to work in than 'Cartesian world' – i.e. only create a Cartesian equation if they ask for one

$$\frac{dy}{\cancel{dt}} \times \frac{\cancel{dt}}{dx}$$

- Differentiation: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$$\int y \frac{dx}{dt}\, dt$$

- Integration: $\int y \frac{dx}{dt}\ dt$ *Make sure the limits are in t*

- Be prepared to use a lot of different techniques in these questions – differentiation, integration, exponentials, logarithms, trigonometric identities…

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5.

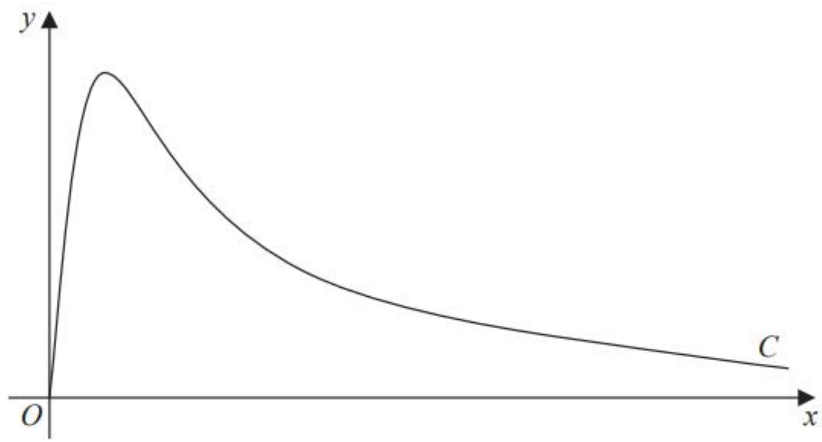

Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 4\tan t, \qquad y = 5\sqrt{3}\sin 2t, \qquad 0 \leqslant t < \frac{\pi}{2}$$

The point P lies on C and has coordinates $\left(4\sqrt{3}, \frac{15}{2}\right)$.

(a) Find the exact value of $\frac{dy}{dx}$ at the point P.

Give your answer as a simplified surd.

(4)

The point Q lies on the curve C, where $\frac{dy}{dx} = 0$

(b) Find the exact coordinates of the point Q.

(2)

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a) $x = 4\tan t$ $\qquad$ $y = 5\sqrt{3}\sin 2t$

$\frac{dx}{dt} = 4\sec^2 t$ $\qquad$ $\frac{dy}{dt} = 10\sqrt{3}\cos 2t$

$\div$

Find t at $\left(4\sqrt{3}, \frac{15}{2}\right)$

$4\sqrt{3} = 4\tan t$

$\sqrt{3} = \tan t$

$t = \frac{\pi}{3}$

$\frac{dy}{dx} = \frac{10\sqrt{3}\cos 2t}{4\sec^2 t}$

$t = \frac{\pi}{3}$

$\frac{dy}{dx} = \frac{10\sqrt{3}\cos\left(\frac{2\pi}{3}\right)}{4\sec^2\left(\frac{\pi}{3}\right)}$

$\frac{15}{2} = 5\sqrt{3}\sin 2t$

$\sin 2t = \frac{\sqrt{3}}{2}$

π - ANS

$2t = \frac{\pi}{3}, \frac{2\pi}{3}$

$t = \frac{\pi}{6}, \frac{\pi}{3}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sec\frac{\pi}{3} = 2$

$\sec^2\frac{\pi}{3} = 4$

$= -\frac{5\sqrt{3}}{16}$

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BICEN MATHS

b)

$$10\sqrt{3}\cos 2t = 0$$
$$\cos 2t = 0$$
$$2t = \frac{\pi}{2}$$
$$t = \frac{\pi}{4}$$

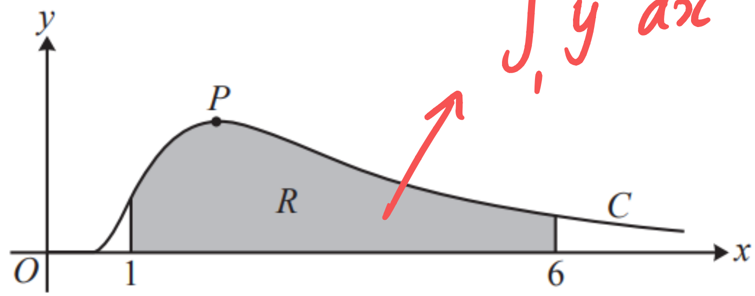
$$x = 4\tan t$$
$$= 4\tan\frac{\pi}{4}$$
$$= 4$$

$$y = 5\sqrt{3}\sin 2t$$
$$= 5\sqrt{3}\sin\left(2\times\frac{\pi}{4}\right)$$
$$= 5\sqrt{3}$$

$$Q\left(4, 5\sqrt{3}\right)$$

5.	$x = 4\tan t, \quad y = 5\sqrt{3}\sin 2t, \qquad 0 \leqslant t < \frac{\pi}{2}$		
(a) **Way 1**	$\frac{dx}{dt} = 4\sec^2 t,\ \frac{dy}{dt} = 10\sqrt{3}\cos 2t$	**Either both** x and y are differentiated correctly with respect to t **or** their $\frac{dy}{dt}$ divided by their $\frac{dx}{dt}$ **or** applies $\frac{dy}{dt}$ multiplied by their $\frac{dt}{dx}$	M1
	$\Rightarrow \frac{dy}{dx} = \frac{10\sqrt{3}\cos 2t}{4\sec^2 t}\quad \left\{= \frac{5}{2}\sqrt{3}\cos 2t\cos^2 t\right\}$	Correct $\frac{dy}{dx}$ (Can be implied)	A1 oe
	$\left\{\text{At } P\left(4\sqrt{3}, \frac{15}{2}\right),\ t = \frac{\pi}{3}\right\}$		
	$\frac{dy}{dx} = \frac{10\sqrt{3}\cos\left(\frac{2\pi}{3}\right)}{4\sec^2\left(\frac{\pi}{3}\right)}$	**dependent on the previous M mark** ***Some evidence*** of substituting $t = \frac{\pi}{3}$ or $t = 60^\circ$ into their $\frac{dy}{dx}$	dM1
	$\frac{dy}{dx} = -\frac{5}{16}\sqrt{3}$ or $-\frac{15}{16\sqrt{3}}$	$-\frac{5}{16}\sqrt{3}$ or $-\frac{15}{16\sqrt{3}}$ **from a correct solution only**	A1 **cso**
			[4]
(b)	$\left\{10\sqrt{3}\cos 2t = 0 \Rightarrow t = \frac{\pi}{4}\right\}$		
	So $x = 4\tan\left(\frac{\pi}{4}\right),\ y = 5\sqrt{3}\sin\left(2\left(\frac{\pi}{4}\right)\right)$	At least one of either $x = 4\tan\left(\frac{\pi}{4}\right)$ **or** $y = 5\sqrt{3}\sin\left(2\left(\frac{\pi}{4}\right)\right)$ **or** $x = 4$ **or** $y = 5\sqrt{3}$ **or** $y =$ awrt 8.7	M1
	Coordinates are $\left(4, 5\sqrt{3}\right)$	$\left(4, 5\sqrt{3}\right)$ **or** $x = 4, y = 5\sqrt{3}$	A1
			[2]
			6

5



$\int_1^6 y\, dx$

$\int_{\square}^{\square} y \frac{dx}{dt}\, dt$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

The diagram shows the curve C with parametric equations

$x = \frac{3}{t}$, $y = t^3 e^{-2t}$, where $t > 0$.

The maximum point on C is denoted by P. $\rightarrow \frac{dy}{dx} = 0$

(a) Determine the exact coordinates of P. **[4]**

The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 6$.

(b) Show that the area of R is given by

$$\int_a^b 3te^{-2t}\, dt,$$

where a and b are constants to be determined. **[3]**

(c) Hence determine the exact area of R. **[5]**

a) $y = t^3 e^{-2t}$ $\qquad x = 3t^{-1}$

$u = t^3 \quad v = e^{-2t}$

$u' = 3t^2 \quad v' = -2e^{-2t}$

$$\frac{dy}{dt} = -2t^3 e^{-2t} + 3t^2 e^{-2t}$$

$$\frac{dx}{dt} = -3t^{-2}$$

$$\frac{dy}{dx} = \frac{-2t^3 e^{-2t} + 3t^2 e^{-2t}}{-3t^{-2}}$$

max point, $\frac{dy}{dx} = 0$,

$-2t^3 e^{-2t} + 3t^2 e^{-2t} = 0$

$e^{-2t}(3t^2 - 2t^3) = 0$

$e^{-2t} \neq 0$

$3t^2 = 2t^3$

$3 = 2t$

$t = \frac{3}{2}$

When $t = \frac{3}{2}$

$y = \left(\frac{3}{2}\right)^3 e^{-2\times\frac{3}{2}} = \frac{27}{8} e^{-3}$ $\qquad x = 3 \times \left(\frac{3}{2}\right)^{-1} = 2$

$P\left(2, \frac{27}{8}e^{-3}\right)$

b) $\int_1^6$

$x = \frac{3}{t}$

$\int y \frac{dx}{dt}\, dt$

$x = 6 \ldots \quad t = \frac{3}{x} = \frac{3}{6} = \frac{1}{2}$

$x = 1 \ldots \quad t = \frac{3}{1} = 3$

$$\int_3^{1/2} \left(t^3 e^{-2t}\right)\left(-3t^{-2}\right) dt = \underline{\underline{\int_{1/2}^3 3te^{-2t}\, dt}}$$

remove −
and flip limits.

$$3\int_{1/2}^3 te^{-2t}\, dt$$

c) int. by parts

$u = t \qquad v = -\frac{1}{2}e^{-2t}$

$u' = 1 \qquad v' = e^{-2t}$

$$3\int_{1/2}^{3} te^{-2t}\,dt = 3\left(-\frac{1}{2}te^{-2t} + \int_{1/2}^{3}\frac{1}{2}e^{-2t}\,dt\right)$$

$$= 3\left[-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right]_{1/2}^{3}$$

$$= 3\left(-\frac{3}{2}e^{-6} - \frac{1}{4}e^{-6} - \left(-\frac{1}{4}e^{-1} - \frac{1}{4}e^{-1}\right)\right)$$

$$= 3\left(-\frac{7}{4}e^{-6} + \frac{1}{2}e^{-1}\right)$$

$$= -\frac{21}{4}e^{-6} + \frac{3}{2}e^{-1}$$

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5	(a)		$\frac{dy}{dt}=3t^2e^{-2t}+t^3\left(-2e^{-2t}\right)$	M1*	2.1	Attempts to differentiate y with respect to t using the product rule – answer of the form $\frac{dy}{dt}=\lambda t^2e^{-2t}+\mu t^3e^{-2t}$ or $y'=\alpha x^{-5}e^{-6x^{-1}}\left(\beta x+\gamma\right)$
			$\frac{dy}{dt}=0\Rightarrow t^2e^{-2t}(3-2t)=0\Rightarrow t=\ldots$	M1dep*	1.1	Sets their derivative equal to zero and solves for t
			$t=\frac{3}{2}$	A1	1.1	From correct working only (or for $x=2$)
			$P\left(2,\frac{27}{8}e^{-3}\right)$	A1	2.2a	From correct working only y-coordinate must be exact but ISW
				[4]		
5	(b)		$\frac{dx}{dt}=-3t^{-2}$ and $\int y\frac{dx}{dt}dt$	M1	2.1	Differentiates x with respect to t and attempts to set up integral for the required area
			$x=6\Rightarrow t=0.5$ and $x=1\Rightarrow t=3$	B1	1.1	Stating 0.5 and 3 is sufficient for this mark
			Area $=\int_3^{0.5}t^3e^{-2t}\left(-\frac{3}{t^2}\right)dt=\int_3^{0.5}-3te^{-2t}\,dt=\int_{0.5}^{3}3te^{-2t}\,dt$	A1	2.2a	Must be correctly shown
				[3]		
5	(c)		$u=3t$, and dv or $\frac{dv}{dt}=e^{-2t}$	M1*	1.1	Integrating by parts as far as $f(t)\pm\int g(t)dt$
			$\int 3te^{-2t}\,dt=-\frac{3}{2}te^{-2t}+\frac{3}{2}\int e^{-2t}\,dt$	A1	1.1	
			$=\ldots-\frac{3}{4}e^{-2t}(+c)$	A1	1.1	Allow correct un-simplified for both A marks
			$\left[-\frac{3}{2}te^{-2t}-\frac{3}{4}e^{-2t}\right]_{0.5}^{3}=\left(-\frac{3}{2}(3)e^{-6}-\frac{3}{4}e^{-6}\right)-\left(-\frac{3}{2}(0.5)e^{-1}-\frac{3}{4}e^{-1}\right)$	M1dep*	1.1	Use of their t-limits (so not 1 and 6) in fully integrated expression (must subtract bottom limit from top limit)
			Area $=-\frac{21}{4}e^{-6}+\frac{3}{2}e^{-1}$	A1	2.2a	ISW once correct exact answer seen
				[5]		

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Exponentials and Logarithms - modelling

- Remember – modelling can only ask you to substitute, solve, or comment – don't reinvent maths just because it is in context!

- If it asks for rate, differentiate

- For non-linear to linear, take logs of both sides – I've done loads of these before, so only covered in a Your Turn question.

9. The amount of an antibiotic in the bloodstream, from a given dose, is modelled by the formula

$$x = De^{-0.2t}$$

where x is the amount of the antibiotic in the bloodstream in milligrams, D is the dose given in milligrams and t is the time in hours after the antibiotic has been given.

A first dose of 15 mg of the antibiotic is given. $D = 15$

(a) Use the model to find the amount of the antibiotic in the bloodstream 4 hours after the dose is given. Give your answer in mg to 3 decimal places.

(2)

A second dose of 15 mg is given 5 hours after the first dose has been given. Using the same model for the second dose,

(b) show that the **total** amount of the antibiotic in the bloodstream 2 hours after the second dose is given is 13.754 mg to 3 decimal places.

(2)

No more doses of the antibiotic are given. At time T hours after the second dose is given, the total amount of the antibiotic in the bloodstream is 7.5 mg.

$\hookrightarrow x = 7.5, \quad t = T$

(c) Show that $T = a\ln\left(b + \frac{b}{e}\right)$, where a and b are integers to be determined.

(4)

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a) $x = De^{-0.2t}$ $t = 4$ $D = 15$

$x = 15e^{-0.2\times 4}$

$= \underline{6.740}$ mg (3dp)

b) 1st dose 7 hours

2nd dose 2 hours

$D = 15$

$x = \underline{15e^{-0.2\times 7}} + \underline{15e^{-0.2\times 2}}$

$= \underline{13.754}$ mg

2nd dose

c) $x = 15e^{-0.2(T+5)} + 15e^{-0.2T}$

1st

$x = 7.5$

$7.5 = 15e^{-0.2T\,\underline{-1}} + 15e^{-0.2T}$

$\frac{1}{2} = e^{-0.2T} \times e^{-1} + e^{-0.2T}$ $\div 15$

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$$\frac{1}{2} = \underline{e^{-0.2T}}(e^{-1}+1)$$

$$\frac{1}{2} \times \frac{1}{e^{-1}+1} = e^{-0.2T}$$

$$\frac{1}{2(\frac{1}{e}+1)} = e^{-0.2T}$$

Aim: $a\ln\left(b+\frac{b}{e}\right)$

$$\ln\left(\frac{1}{2(\frac{1}{e}+1)}\right) = \underline{-0.2T}$$

$\times -5$

$$-5\ln\left(\frac{1}{2(\frac{1}{e}+1)}\right) = T$$

$$5\ln\left(2\left(\frac{1}{e}+1\right)\right) = T$$

$$T = 5\ln\left(\frac{2}{e}+2\right)$$

$$\underline{T = 5\ln\left(2+\frac{2}{e}\right)}$$

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BICEN MATHS

9(a)	Subs $D=15$ and $t=4$ $\quad x=15\mathrm{e}^{-0.2\times 4}=6.740\ (mg)$	M1A1
		(2)
(b)	$15\mathrm{e}^{-0.2\times 7}+15\mathrm{e}^{-0.2\times 2}=13.754(mg)$	M1A1*
		(2)
(c)	$15\mathrm{e}^{-0.2\times T}+15\mathrm{e}^{-0.2\times(T+5)}=7.5$	M1
	$15\mathrm{e}^{-0.2\times T}+15\mathrm{e}^{-0.2\times T}\mathrm{e}^{-1}=7.5$	
	$15\mathrm{e}^{-0.2\times T}(1+\mathrm{e}^{-1})=7.5 \Rightarrow \mathrm{e}^{-0.2\times T}=\dfrac{7.5}{15(1+\mathrm{e}^{-1})}$	dM1
	$\mathrm{T}=-5\ln\left(\dfrac{7.5}{15\left(1+\mathrm{e}^{-1}\right)}\right)=5\ln\left(2+\dfrac{2}{\mathrm{e}}\right)$	A1, A1
		(4)
		(8 marks)

Exponentials and Logarithms - equations

- Follow the log laws! You should know these by now…
  - Adding logs… multiply the input
  - Subtracting logs… divide the input
  - Factor of a log… becomes input's power
    - And vice versa!

- Note: check your answers, your input cannot be negative for a logarithm

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2. Find the exact solutions, in their simplest form, to the equations

(a) $e^{3x-9} = 8$

(3)

(b) $\ln(2y + 5) = 2 + \ln(4 - y)$

(4)

a) $e^{3x-9} = 8$

$3x - 9 = \ln 8$

$x = \dfrac{\ln 8 + 9}{3}$

b) $\ln(2y+5) - \ln(4-y) = 2$

$\ln\left(\dfrac{2y+5}{4-y}\right) = 2$

$\dfrac{2y+5}{4-y} = e^2$

$2y+5 = 4e^2 - e^2 y$

$2y + e^2 y = 4e^2 - 5$

$y(2+e^2) = 4e^2 - 5$

$$y = \frac{4e^2 - 5}{2 + e^2}$$

$$-\ln a = \ln(a)^{-1}$$

$$= \ln\left(\frac{1}{a}\right)$$

$$-\ln\left(\frac{2}{5}\right) = \ln\left(\frac{5}{2}\right)$$

2.(a)		$e^{3x-9} = 8 \Rightarrow 3x - 9 = \ln 8$ $\Rightarrow x = \dfrac{\ln 8 + 9}{3}, = \ln 2 + 3$
(b)		$\ln(2y+5) = 2 + \ln(4-y)$ $\ln\left(\dfrac{2y+5}{4-y}\right) = 2$ $\left(\dfrac{2y+5}{4-y}\right) = e^2$ $2y + 5 = e^2(4-y) \Rightarrow 2y + e^2 y = 4e^2 - 5 \Rightarrow y = \dfrac{4e^2 - 5}{2 + e^2}$

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Your Turn Questions

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Your Turn

7. A sequence is defined by

$$u_1 = 3$$
$$u_{n+1} = u_n - 5, \qquad n \geqslant 1$$

Find the values of

(a) u_2, u_3 and u_4 **(2)**

(b) u_{100} **(3)**

(c) $\sum_{i=1}^{100} u_i$ **(3)**

a) $u_2 = u_1 - 5$
$= 3 - 5$
$= -2$

$u_3 = u_2 - 5$
$= -2 - 5$
$= -7$

$u_4 = u_3 - 5$
$= -7 - 5$
$= -12$

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b) $3, \ -2, \ -7, \ -12$ arithmetic

$a + (n-1)d$

$a = 3$

$d = -5$

$$u_{100} = a + (100-1)d$$
$$= 3 + 99 \times (-5)$$
$$= -492$$

c) $\sum_{i=1}^{100} u_i = u_1 + u_2 + u_3 + \dots + u_{100}$

$\frac{n}{2}(2a + (n-1)d)$

$\frac{n}{2}(a + l)$

$n = 100 \quad d = -5$

$a = 3$

$$S_{100} = \frac{100}{2}(6 + 99(-5)) = -24450$$

<u>or</u>

$$S_{100} = \frac{100}{2}(3 - 492) = -24450$$

7.		
(a)	$u_2 = -2,\ u_3 = -7$ and $u_4 = -12$	M1, A1 **[2]**
(b)	$d = -5$ and arithmetic	B1
	Uses $a + (n - 1)\,d$ with $a = 3$ and $n = 100$, to give -492	M1, A1 **[3]**
(c)	$S_{100} = \frac{n}{2}(2a + (n-1)d)$ *or* $\frac{n}{2}(a + l)$	M1
	$S_{100} = \frac{100}{2}(6 + 99 \times -5)$ *or* $\frac{100}{2}(3 + -492)$	dM1
	$= -24\,450$	A1 **[3]**
		8 marks

2 A circle with centre C has equation $x^2+y^2-6x+4y+4=0$.

(a) Find

(i) the coordinates of C, **[2]**

(ii) the radius of the circle. **[1]**

(b) Determine the set of values of k for which the line $y=kx-3$ does not intersect or touch the circle. **[5]**

a) CTS

$$x^2-6x+y^2+4y+4=0$$

$$(x-3)^2-9+(y+2)^2-\cancel{4}+\cancel{4}=0$$

$$(x-3)^2+(y+2)^2=9$$

i) $C(3,-2)$

ii) radius is $\sqrt{9}=3$

b) $y = kx - 3$ $\quad\quad (\quad)(\quad)$

$$x^2 - 6x + (kx-3)^2 + 4(kx-3) + 4 = 0$$

$$x^2 - 6x + k^2x^2 - 6kx + 9 + 4kx - 12 + 4 = 0$$

$$(1+k^2)x^2 + (-2k-6)x + 1 = 0$$

$$b^2 - 4ac < 0$$

$$(-2k-6)^2 - 4(1+k^2)1 < 0$$

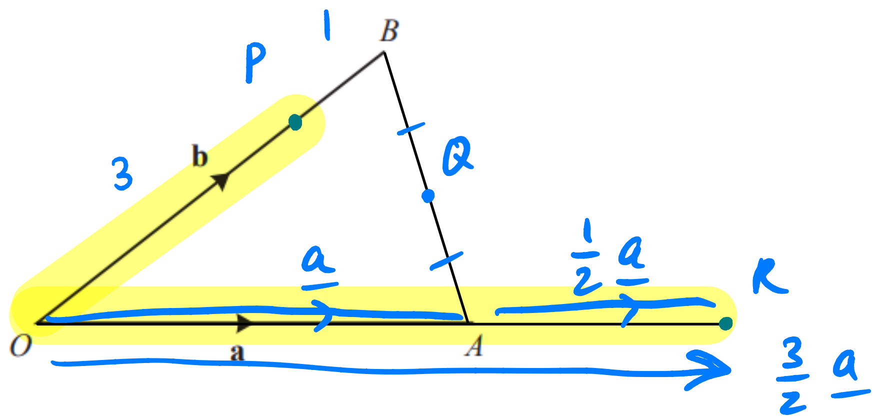
$$\cancel{4k^2} + 24k + 36 - 4 - \cancel{4k^2} < 0$$

$$24k < -32$$

$$k < -\frac{4}{3}$$

2	**(a)**	**(i)**	$(x-3)^2-9+(y+2)^2-4+4=0 \Rightarrow (x-3)^2+(y+2)^2=9$	**M1**	**1.1**	$(x\pm3)^2$ and $(y\pm2)^2$ seen (or implied by correct answer) or one correct coordinate
			$C\ (3, -2)$	**A1**	**1.1**	Accept $x = 3$ and $y = -2$
				[2]		
2	**(a)**	**(ii)**	$r = 3$	**B1**	**1.1**	Allow if stated explicitly in **(a)(i)** but not written down in **(a)(ii)** www for r
				[1]		
2	**(b)**		$(x-3)^2+(kx-3+2)^2=9$ or $x^2+(kx-3)^2-6x+4(kx-3)+4=0$	**M1***	**3.1a**	Substitutes the correct equation of the line into any form of their equation of the circle
			$(1+k^2)x^2+(-6-2k)x+1=0$	**A1**	**1.1**	oe (all terms on the same side – may not be factorised but should be simplified to 5 terms)
			$(-6-2k)^2-4(1+k^2)(1)$	**M1dep***	**3.1a**	Correct explicit use of discriminant on their 3TQ to get an expression in k only
			$36+24k+4k^2-4-4k^2<0 \Rightarrow 32+24k<0$	**M1dep***	**2.1**	Discriminant < 0 and simplify to the form $ak+b<0$ (oe)
			$k<-\frac{4}{3}$	**A1**	**2.2a**	Fully correct (no additional values)
				[5]		

5



The diagram shows points A and B, which have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O. P is the point on OB such that $OP : PB = 3:1$ and Q is the midpoint of AB.

(a) Find $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. **[2]**

The line OA is extended to a point R, so that PQR is a straight line.

(b) Explain why $\overrightarrow{PR} = k\,(2\mathbf{a} - \mathbf{b})$, where k is a constant. **[2]**

(c) Hence determine the ratio $OA : AR$. **[4]**

a) $\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ}$

$= \frac{1}{4}\underline{b} + \frac{1}{2}(\underline{a} - \underline{b})$

$\overrightarrow{PB} = \frac{1}{4}\overrightarrow{OB}$

$= \frac{1}{4}\underline{b}$

$\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BA} = \frac{1}{2}(\underline{a} - \underline{b})$

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$$= \frac{1}{4}\underline{b} + \frac{1}{2}\underline{a} - \frac{1}{2}\underline{b}$$

$$= \frac{1}{2}\underline{a} - \frac{1}{4}\underline{b}$$

b) $\overrightarrow{PR}$ is in the same direction as $\overrightarrow{PQ}$, as they forms a straight line.

$$\overrightarrow{PQ} = \frac{1}{2}\underline{a} - \frac{1}{4}\underline{b} = \frac{1}{4}(2\underline{a} - \underline{b})$$

So $\overrightarrow{PR} = k(2\underline{a} - \underline{b})$ as this is the same direction

c) $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$ $\qquad \overrightarrow{PO} = -\frac{3}{4}\underline{b}$

$\overrightarrow{PR} = -\frac{3}{4}\underline{b} + \mu\underline{a}$ $\qquad \overrightarrow{OR} = \mu\underline{a}$

$$k(2\underline{a} - \underline{b}) = \mu\underline{a} - \frac{3}{4}\underline{b}$$

$2k\underline{a} - k\underline{b} = \mu\underline{a} - \frac{3}{4}\underline{b}$ $\qquad k = \frac{3}{4}$ $\qquad 2k = \mu$

OA : AR

1 : $\frac{1}{2}$

2 : 1

$\mu = 2 \times \frac{3}{4} \times \frac{3}{2}$

5	(a)		$\overrightarrow{BQ} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$	B1	1.1a	Correct $\overrightarrow{BQ}$ or $\overrightarrow{QB}$
			$\overrightarrow{PQ} = \frac{1}{4}\mathbf{b} + \frac{1}{2}(\mathbf{a} - \mathbf{b}) = \frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b}$	B1	1.1	Correct $\overrightarrow{PQ}$
				[2]		
	(b)		$\overrightarrow{PR}$ has the same direction as $\overrightarrow{PQ}$, so vector must be a multiple of $\overrightarrow{PQ}$	B1	2.4	Explain parallel (or collinear) vectors have direction vectors that are multiples of each other
			So $\overrightarrow{PR} = \lambda(\frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b}) = \frac{1}{4}\lambda(2\mathbf{a} - \mathbf{b})$ $= k(2\mathbf{a} - \mathbf{b})$ **A.G.**	B1	2.1	Show given answer convincingly
				[2]		
	(c)		$\overrightarrow{AR} = -\mathbf{a} + \frac{3}{4}\mathbf{b} + k(2\mathbf{a} - \mathbf{b})$	B1	1.1	Correct expression for $\overrightarrow{AR}$ (or $\overrightarrow{OR}$), in terms of k
			$\overrightarrow{AR}$ multiple of $\mathbf{a}$ only, $\frac{3}{4}\mathbf{b} - k\,\mathbf{b} = 0$	M1	3.1a	Use coefficient of $\mathbf{b} = 0$
			Obtain $k = \frac{3}{4}$	A1	1.1	Obtain correct value for k
			ratio $OA : AR = 2{:}1$	A1	1.1	Correct ratio (allow $1{:}\ \frac{1}{2}$) oe
				[4]		

Your Turn

Hint: (a) is in tan and cot… keep it that way!

8. (a) Prove that

$$2\cot 2x + \tan x \equiv \cot x \qquad x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

(4)

(b) Hence, or otherwise, solve, for $-\pi \leqslant x < \pi$,

$$\underbrace{6\cot 2x + 3\tan x}_{3\cot x} = \operatorname{cosec}^2 x - 2$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Give your answers to 3 decimal places.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

a) $2 \times \frac{1}{\tan 2x} + \tan x \equiv \cancel{2} \times \frac{(1-\tan^2 x)}{\cancel{2}\tan x} + \tan x$

$$\equiv \frac{1-\tan^2 x}{\tan x} + \frac{\tan^2 x}{\tan x}$$

$$\equiv \frac{1-\cancel{\tan^2 x}+\cancel{\tan^2 x}}{\tan x}$$

$$\equiv \underline{\underline{\cot x}}$$

b) $3\cot x = \operatorname{cosec}^2 x - 2$

$3\cot x = 1+\cot^2 x - 2$

$0 = \cot^2 x - 3\cot x - 1$

Pythagorean

$1+\cot^2 x = \operatorname{cosec}^2 x$

$\left(\text{Let } y = \cot x \quad 0 = y^2 - 3y - 1\right)$

$\cot x = \dfrac{3+\sqrt{13}}{2}$ $\qquad$ $\cot x = \dfrac{3-\sqrt{13}}{2}$

$\tan x = \dfrac{2}{3+\sqrt{13}}$ $\qquad$ $\tan x = \dfrac{2}{3-\sqrt{13}}$

$x = \underline{0.294}$ $\qquad$ $x = \underline{-1.277}$

$) -\pi$ $\qquad$ $) +\pi$

$x = \underline{-2.848}$ $x = \underline{1.865}$

8 (a)	$2\cot 2x + \tan x \equiv \dfrac{2}{\tan 2x} + \tan x$	B1
	$\equiv \dfrac{(1-\tan^2 x)}{\tan x} + \dfrac{\tan^2 x}{\tan x}$	M1
	$\equiv \dfrac{1}{\tan x}$	M1
	$\equiv \cot x$	A1*
		(4)
(b)	$6\cot 2x + 3\tan x = \operatorname{cosec}^2 x - 2 \Rightarrow 3\cot x = \operatorname{cosec}^2 x - 2$	
	$\Rightarrow 3\cot x = 1 + \cot^2 x - 2$	M1
	$\Rightarrow 0 = \cot^2 x - 3\cot x - 1$	A1
	$\Rightarrow \cot x = \dfrac{3 \pm \sqrt{13}}{2}$	M1
	$\Rightarrow \tan x = \dfrac{2}{3 \pm \sqrt{13}} \Rightarrow x = ..$	M1
	$\Rightarrow x = 0.294, -2.848, -1.277, 1.865$	A2,1,0
		(6)
		(10 marks)

Your Turn

BICEN MATHS

★★★★★

Hint: Can $1 = \sin^2 x + \cos^2 x$ help us at some point?

8. (a) Prove that

$$\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A}, \qquad A \neq \frac{(2n+1)\pi}{4}, \quad n \in \mathbb{Z}$$

(5)

(b) Hence solve, for $0 \leqslant \theta < 2\pi$,

$$\sec 2\theta + \tan 2\theta = \frac{1}{2}$$

Give your answers to 3 decimal places.

(4)

a) $$\sec 2A + \tan 2A = \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$$

$$= \frac{1 + \sin 2A}{\cos 2A}$$

$$= \frac{1 + 2\sin A \cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{\cos^2 A + \sin^2 A + 2\sin A\cos A}{(\cos A - \sin A)(\cos A + \sin A)}$$

$$= \frac{(\cos A + \sin A)\cancel{(\cos A + \sin A)}}{(\cos A - \sin A)\cancel{(\cos A + \sin A)}}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A}$$

b) $$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1}{2}$$

$$\cos\theta + \sin\theta = \frac{1}{2}\cos\theta - \frac{1}{2}\sin\theta$$

$$\frac{3}{2}\sin\theta = -\frac{1}{2}\cos\theta$$

$$3\sin\theta = -\cos\theta$$

$$\downarrow \div \cos\theta$$

$$3\tan\theta = -1$$

$$\tan\theta = -1/3$$

$$\theta = \cancel{-0.3217\ldots} \xrightarrow{+\pi} \underline{2.820}, \xrightarrow{+\pi} \underline{5.961}$$

8(a)	$\sec 2A + \tan 2A = \dfrac{1}{\cos 2A} + \dfrac{\sin 2A}{\cos 2A}$	B1
	$= \dfrac{1 + \sin 2A}{\cos 2A}$	M1
	$= \dfrac{1 + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$	M1
	$= \dfrac{\cos^2 A + \sin^2 A + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$	
	$= \dfrac{(\cos A + \sin A)(\cos A + \sin A)}{(\cos A + \sin A)(\cos A - \sin A)}$	M1
	$= \dfrac{\cos A + \sin A}{\cos A - \sin A}$	A1*
		(5)
(b)	$\sec 2\theta + \tan 2\theta = \dfrac{1}{2}$ Þ $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \dfrac{1}{2}$	
	Þ $2\cos\theta + 2\sin\theta = \cos\theta - \sin\theta$	
	Þ $\tan\theta = -\dfrac{1}{3}$	M1 A1
	Þ $\theta =$ *awrt* 2.820, 5.961	dM1A1
		(4)
		(9 marks)

Your Turn

6. A curve has equation

$$4y^2 + 3x = 6ye^{-2x}$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

The curve crosses the y-axis at the origin and at the point P.

(b) Find the equation of the normal to the curve at P, writing your answer in the form $y = mx + c$ where m and c are constants to be found.

(4)

a) $8y\frac{dy}{dx} + 3 = -12ye^{-2x} + 6\frac{dy}{dx}e^{-2x}$

$u = 6y \qquad v = e^{-2x}$

$u' = 6\frac{dy}{dx} \qquad v' = -2e^{-2x}$

$3 + 12ye^{-2x} = \frac{dy}{dx}\left(6e^{-2x} - 8y\right)$

$$\frac{dy}{dx} = \frac{3 + 12ye^{-2x}}{6e^{-2x} - 8y}$$

b) $4y^2 + 3x = 6ye^{-2x}$

$x=0$

$4y^2 = 6ye^0$

$4y^2 = 6y$

$2y^2 = 3y$

$2y^2 - 3y = 0$

$y(2y-3) = 0$

$y=0 \qquad y=\frac{3}{2} \qquad P\left(0, \frac{3}{2}\right)$

$x=0$
$y=\frac{3}{2}$ $\quad \frac{dy}{dx} = \frac{3+12\times\frac{3}{2}}{6-8\times\frac{3}{2}} = \frac{21}{-6} = -\frac{7}{2}$

Normal gradient $\frac{2}{7}$

$$y = \frac{2}{7}x + \frac{3}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{3}{2} = \frac{2}{7}(x - 0)$$

$$y = \frac{2}{7}x + \frac{3}{2}$$

6(a)	$4y^2+3x=6y\mathrm{e}^{-2x}$	
	$4y^2+3x \to 8y\dfrac{\mathrm{d}y}{\mathrm{d}x}+3$	B1
	$6y\mathrm{e}^{-2x} \to -12y\mathrm{e}^{-2x}+6\mathrm{e}^{-2x}\dfrac{\mathrm{d}y}{\mathrm{d}x}$	M1 A1
	$8y\dfrac{\mathrm{d}y}{\mathrm{d}x}+3=-12y\mathrm{e}^{-2x}+6\mathrm{e}^{-2x}\dfrac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{12y\mathrm{e}^{-2x}+3}{6\mathrm{e}^{-2x}-8y}$ oe	M1 A1
		(5)
(b)	Sets $x=0$ in $4y^2+3x=6y\mathrm{e}^{-2x} \Rightarrow y=\dfrac{3}{2}$ oe	B1
	Substitutes $\left(0,\dfrac{3}{2}\right)$ in their $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{12y\mathrm{e}^{-2x}+3}{6\mathrm{e}^{-2x}-8y}=\left(\dfrac{7}{-2}\right)$	M1
	$m_N=-1\div"\dfrac{7}{-2}" \Rightarrow y="\dfrac{2}{7}"x+"\dfrac{3}{2}"$	**d**M1
	$y=\dfrac{2}{7}x+\dfrac{3}{2}$ oe e.g. $y=\dfrac{6}{21}x+\dfrac{3}{2}$	A1
		(4)

5. The point P lies on the curve with equation

$$x = (4y - \sin 2y)^2$$

Given that P has (x, y) coordinates $\left(p, \frac{\pi}{2}\right)$, where p is a constant,

(a) find the exact value of p. Find x when $y = \frac{\pi}{2}$ **(1)**

The tangent to the curve at P cuts the y-axis at the point A.

(b) Use calculus to find the coordinates of A. **(6)**

a) $x = \left(4\left(\frac{\pi}{2}\right) - \sin\left(2 \times \frac{\pi}{2}\right)\right)^2 = (2\pi - \sin\pi)^2$

$= (2\pi)^2$

$= 4\pi^2$ $p = 4\pi^2$

b) Find tangent

↳ diff. $\frac{dy}{dx}$ $y = \frac{\pi}{2}$

Find y when $x = 0$

$$x = (\underline{4y - \sin 2y})^2$$

$$\frac{dx}{dy} = 2(4y - \sin 2y)(4 - 2\cos 2y)$$

$$\frac{dx}{dy} = 2(2\pi - \sin\pi)(4 - 2\cos\pi)$$

$$= 2(2\pi)(6)$$

$$= 24\pi$$

$$\frac{dy}{dx} = \frac{1}{24\pi}$$

Tangent $\quad y - \frac{\pi}{2} = \frac{1}{24\pi}(x - 4\pi^2)$ when $x = 0$

$$y - \frac{\pi}{2} = \frac{1}{24\pi}(-4\pi^2)$$

$$y = \frac{\pi}{2} - \frac{\pi}{6} = \underline{\frac{\pi}{3}}$$

5.(a)	$p = 4\pi^2$ or $(2\pi)^2$	B1
		(1)
(b)	$x = (4y - \sin 2y)^2 \Rightarrow \dfrac{dx}{dy} = 2(4y - \sin 2y)(4 - 2\cos 2y)$	M1A1
	Sub $y = \dfrac{\pi}{2}$ into $\dfrac{dx}{dy} = 2(4y - \sin 2y)(4 - 2\cos 2y)$	
	Þ $\dfrac{dx}{dy} = 24\pi$ $(= 75.4)$ / $\dfrac{dy}{dx} = \dfrac{1}{24\pi}(= 0.013)$	M1
	Equation of tangent $y - \dfrac{\pi}{2} = \dfrac{1}{24\pi}(x - 4\pi^2)$	M1
	Using $y - \dfrac{\pi}{2} = \dfrac{1}{24\pi}(x - 4\pi^2)$ with $x = 0$ Þ $y = \dfrac{\pi}{3}$ cso	M1, A1
		(6)
		(7 marks)

8. Given that

$$y = 8\tan(2x), \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

show that

$$\frac{dx}{dy} = \frac{A}{B + y^2}$$

where A and B are integers to be found.

(4)

$$y = 8\tan 2x$$

$$\frac{dy}{dx} = 16\sec^2 2x$$

$$\frac{dx}{dy} = \frac{1}{16\sec^2 2x}$$

$$\frac{dx}{dy} = \frac{1}{16 + 16\tan^2 2x}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$16 + 16\tan^2 2x = 16\sec^2 2x$$

$$y^2 = 64\tan^2 2x \quad \div 4$$

$$\frac{y^2}{4} = 16\tan^2 2x$$

$$\frac{dx}{dy} = \frac{1}{16 + \frac{y^2}{4}} \quad \begin{matrix} \times 4 \\ \times 4 \end{matrix}$$

$$\frac{dx}{dy} = \frac{4}{64 + y^2}$$

$A = 4$

$B = 64$

8	Differentiates wrt x $\quad \dfrac{dy}{dx} = 16\sec^2(2x)$ oe	M1
	Inverts to get $\dfrac{dx}{dy} = \dfrac{1}{16\sec^2 2x}$ $= \dfrac{1}{16(1+\tan^2 2x)}$ | $\dfrac{dy}{dx} = 16\left(1+\tan^2(2x)\right)$ $= 16\left(1+\left(\dfrac{y}{8}\right)^2\right)$	dM1
	$\dfrac{dx}{dy} = \dfrac{A}{B+y^2}$	ddM1
	$= \dfrac{4}{64+y^2}$	A1
		(4)
		(4 marks)

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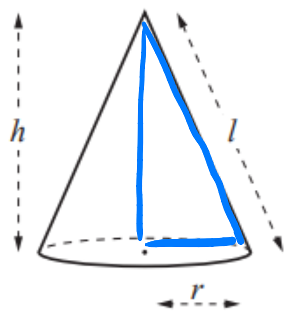
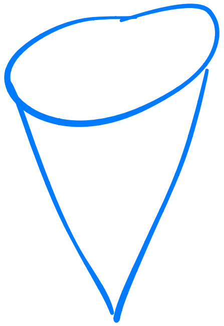
Your Turn

5





For a cone with base radius r, height h and slant height l, the following formulae are given.

Curved surface area, $S = \pi r l$

Volume, $V = \frac{1}{3}\pi r^2 h$ → r and h

A container is to be designed in the shape of an inverted cone with no lid. The base radius is r m and the volume is V m^3.

The area of the material to be used for the cone is 4π m^2.

eliminate h

(a) Show that $V = \frac{1}{3}\pi\sqrt{16r^2 - r^6}$. → only r **[4]**

(b) **In this question you must show detailed reasoning.**

It is given that V has a maximum value for a certain value of r.

Find the maximum value of V, giving your answer correct to 3 significant figures. **[5]**

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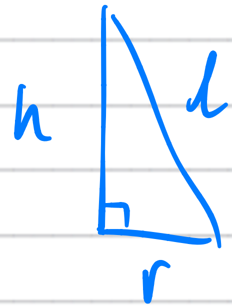
a) $S = \pi r l$

$4\pi = \pi r l$

$4 = r l$

$\frac{4}{r} = l$



Find h in terms of r and l.

$l^2 = h^2 + r^2$

$\frac{16}{r^2} = h^2 + r^2$

$\frac{16}{r^2} - r^2 = h^2$

$h = \sqrt{\frac{16}{r^2} - r^2}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi r^2 \sqrt{\frac{16}{r^2} - r^2}$

$V = \frac{1}{3}\pi \sqrt{r^4} \sqrt{\frac{16}{r^2} - r^2}$

$\sqrt{a}\sqrt{b} = \sqrt{ab}$

$$V = \frac{1}{3}\pi\sqrt{r^4\left(\frac{16}{r^2} - r^2\right)}$$

$$V = \frac{1}{3}\pi\sqrt{16r^2 - r^6} \qquad = \frac{1}{3}\pi\left(16r^2 - r^6\right)^{1/2}$$

b) $$\frac{dV}{dr} = \frac{1}{3}\pi \times \frac{1}{2}\left(16r^2 - r^6\right)^{-1/2}\left(32r - 6r^5\right)$$

$$\frac{dV}{dr} = \frac{\pi\left(32r - 6r^5\right)}{6\sqrt{16r^2 - r^6}}$$

$$\frac{dV}{dr} = 0$$

$$32r - 6r^5 = 0$$

$$32r = 6r^5$$

$$32 = 6r^4$$

$$\frac{16}{3} = r^4$$

$$r = \sqrt[4]{16/3}$$

$$r = 1.519\ldots$$

$\cancel{r \neq 0}$

$$V = \frac{1}{3}\pi\sqrt{16r^2 - r^6}$$

$$= 5.20\ m^3$$

(3sf

5	(a)		An example of a correct method:			Other correct methods may be seen eg $lr = 4$: B1, find h ito l & r: B1, Subst h & lr into V: M1, convincing: A1
			$l = \frac{4}{r}$ or $l = \frac{4\pi}{r\pi}$ exactly (not $lr = 4$)	B1	3.1a	Express l correctly in terms of r May be implied
			$(h = \sqrt{l^2 - r^2})$ $h = \sqrt{\frac{16}{r^2} - r^2}$ or $\frac{\sqrt{16-r^4}}{r}$ oe	B1	1.1	Express h (or h^2) correctly in terms of r alone
			$V = \frac{1}{3}\pi r^2 \sqrt{\frac{16}{r^2} - r^2}$ or $\frac{1}{3}\pi r^2 \frac{\sqrt{16-r^4}}{r}$ oe	M1	1.1	Sub their h (in terms of r alone) into $\frac{1}{3}\pi r^2 h$
			$(= \frac{\pi}{3}\sqrt{16r^2 - r^6}$ **AG**$)$	A1 [4]	2.1	Must see a correct previous expression in terms of r only, and the answer
5	(b)	DR	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\pi}{3}\sqrt{16r^2 - r^6}\right)$	M1	1.1a	Attempt differentiate V or $\frac{V}{\pi}$ or $3V$
			$\frac{\pi(32r - 6r^5)}{3 \times 2\sqrt{16r^2 - r^6}} = 0$ oe (Their derivative = 0)	A1	2.1	Correct derivative of one of the above Condone missing brackets.
			$r = \frac{2}{\sqrt[4]{3}}$ or $\sqrt[4]{\frac{16}{3}}$ oe or 1.52 (3 sf) Allow 1.5 or $r^2 = \frac{4}{\sqrt{3}}$	A1	1.1	Lose this mark if incorrect values of r also given, eg $r = \pm 2$ obtained from $(16r^2 - r^6)^{-\frac{1}{2}} = 0$
			$r = -\frac{2}{\sqrt[4]{3}}$ or -1.52 invalid OR $r = 0$ invalid or $r > 0$	B1f	3.2a	Comment needed, about their negative r (ft) or about $r = 0$
			$(V_{\max} = \frac{\pi}{3}\sqrt{16 \times 1.51967^2 - 1.51967^6})$ Max $V = 5.20$ (3 sf) Allow 5.2 or a.r.t. 5.2	A1 [5]	1.1	Condone $V = 5.20\ \mathrm{m}^3$

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Your Turn

This one is a little weird – so I hope by showing you, you'll be ready for it in the exam!
It's where you have an exponential and trig product integral.

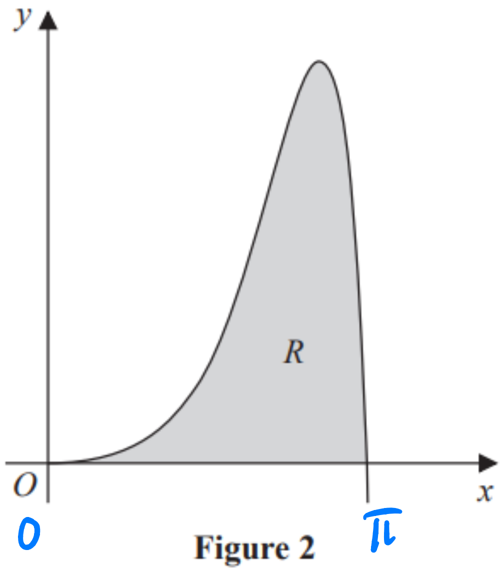

Figure 2

(a) Find $\int e^{2x} \sin x \, dx$

(5)

Figure 2 shows a sketch of part of the curve with equation

$$y = e^{2x} \sin x \qquad x \geqslant 0$$

The finite region R is bounded by the curve and the x-axis and is shown shaded in Figure 2.

(b) Show that the exact area of R is $\dfrac{e^{2\pi} + 1}{5}$

(*Solutions relying on calculator technology are not acceptable.*)

(2)

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$u = e^{2x}$, $v = -\cos x$

$u' = 2e^{2x}$, $v' = \sin x$

$$\int e^{2x} \sin x \, dx = -e^{2x} \cos x + \int 2e^{2x} \cos x \, dx$$

$u = 2e^{2x}$, $v = \sin x$

$u' = 4e^{2x}$, $v' = \cos x$

$$\int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2e^{2x} \sin x - \int 4 e^{2x} \sin dx$$

$$5 \int e^{2x} \sin x \, dx = 2e^{2x} \sin x - e^{2x} \cos x$$

$$\int e^{2x} \sin x \, dx = \frac{2}{5} e^{2x} \sin x - \frac{1}{5} e^{2x} \cos x + c$$

b) $$\left[\frac{2}{5} e^{2x} \sin x - \frac{1}{5} e^{2x} \cos x \right]_0^{\pi} = \frac{2}{5} e^{2\pi} \times 0 - \frac{1}{5} e^{2\pi}(-1) - \left(0 - \frac{1}{5} e^{0}\right)$$

$$= \frac{1}{5} e^{2\pi} + \frac{1}{5} = \frac{e^{2\pi} + 1}{5}$$

Question Number	Scheme	Marks
7(a) Way 1	$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}\cos x\,dx$	M1
	$= ... - \frac{1}{4}e^{2x}\cos x - \int \frac{1}{4}e^{2x}\sin x\,dx$	**d**M1
	$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \int \frac{1}{4}e^{2x}\sin x\,dx$	A1
	$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x \Rightarrow \int e^{2x}\sin x\,dx = ...$	**dd**M1
	$= \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + c$	A1
		(5)
7(a) Way 2	$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + \int 2e^{2x}\cos x\,dx$	M1
	$= ... + 2e^{2x}\sin x - \int 4e^{2x}\sin x\,dx$	**d**M1
	$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x - \int 4e^{2x}\sin x\,dx$	A1
	$5\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2e^{2x}\sin x \Rightarrow \int e^{2x}\sin x\,dx = ...$	**dd**M1
	$= \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + c$	A1
		(5)
(b)	$\left(\frac{2}{5}e^{2\pi}\sin\pi - \frac{1}{5}e^{2\pi}\cos\pi\right) - \left(\frac{2}{5}e^{0}\sin 0 - \frac{1}{5}e^{0}\cos 0\right) = ...$	M1
	$= \frac{1}{5}e^{2\pi} + \frac{1}{5} = \frac{e^{2\pi}+1}{5}$ *	A1*
		(2)
		(7 marks)

Your Turn

5. (a) Express $\dfrac{9(4+x)}{16-9x^2}$ in partial fractions.

(3)

Given that

$$f(x) = \frac{9(4+x)}{16-9x^2}, \qquad x \in \mathbb{R}, \qquad -\frac{4}{3} < x < \frac{4}{3}$$

(b) express $\int f(x)\,dx$ in the form $\ln(g(x))$, where $g(x)$ is a rational function.

(4)

5a) $\dfrac{36+9x}{(4+3x)(4-3x)} = \dfrac{A}{4+3x} + \dfrac{B}{4-3x}$

$36+9x = A(4-3x) + B(4+3x)$

$x = \frac{4}{3}$ $\quad 48 = 8B$ $\quad x = -\frac{4}{3}$ $\quad 24 = 8A$

$B = 6$ $\quad A = 3$

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$$\frac{3}{4+3x} + \frac{6}{4-3x}$$

b) $$\int \left(\frac{3}{4+3x} + \frac{6}{4-3x} \right) dx = \ln|4+3x| - 2\ln|4-3x| + \ln k$$

$$= \ln|4+3x| - \ln|(4-3x)^2| + \ln k$$

$$= \ln\left(\frac{k(4+3x)}{(4-3x)^2} \right)$$

5(a)	$\dfrac{9(4+x)}{16-9x^2} \equiv \dfrac{A}{(4-3x)} + \dfrac{B}{(4+3x)} \Rightarrow A \text{ or } B$	M1
	$A = 6$ or $B = 3$ obtained at any point of the solution	A1
	$\dfrac{9(4+x)}{16-9x^2} \equiv \dfrac{6}{(4-3x)} + \dfrac{3}{(4+3x)}$	A1
		(3)
(b)	$\displaystyle\int \dfrac{9(4+x)}{16-9x^2}\,\mathrm{d}x \equiv \int \dfrac{A}{(4-3x)} + \dfrac{B}{(4+3x)}\,\mathrm{d}x$	
	$= -\dfrac{A}{3}\ln(4-3x) + \dfrac{B}{3}\ln(4+3x)\ (+c)$	M1 A1ft
	$(= -2\ln(4-3x) + \ln(4+3x)\ (+c))$	
	$= \ln\dfrac{(4+3x)}{(4-3x)^2} + c\ , = \ln\dfrac{k(4+3x)}{(4-3x)^2}$ or $\ln\left\lvert\dfrac{k(4+3x)}{(4-3x)^2}\right\rvert$	M1, A1
		(4)
		(7 marks)

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Your Turn

The curve C shown in Figure 3 has parametric equations

$$x = 3\cos t, \qquad y = 9\sin 2t, \qquad 0 \leqslant t \leqslant 2\pi$$

The curve C meets the x-axis at the origin and at the points A and B, as shown in Figure 3.

(a) Write down the coordinates of A and B. **(2)**

(b) Find the values of t at which the curve passes through the origin. **(2)**

(c) Find an expression for $\frac{dy}{dx}$ in terms of t, and hence find the gradient of the curve when $t = \frac{\pi}{6}$ **(4)**

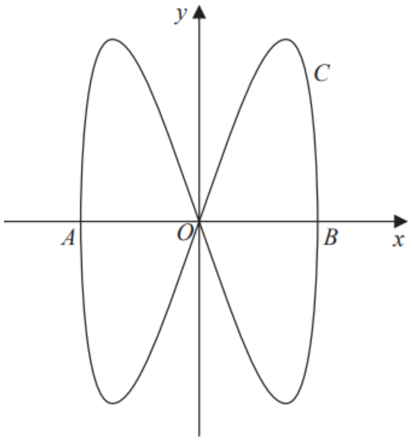

Figure 3

(d) Show that the cartesian equation for the curve C can be written in the form

$$y^2 = ax^2(b - x^2)$$

where a and b are integers to be determined. **(4)**

a) $B(3, 0)$

$A(-3, 0)$

b) $x = 0$ and $y = 0$

$3\cos t = 0$

$\cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

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$t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$

c) $x = 3\cos t$ $\qquad$ $y = 9\sin 2t$

$\frac{dx}{dt} = -3\sin t$ $\qquad$ $\frac{dy}{dt} = 18\cos 2t$

$$\frac{dy}{dx} = \frac{18\cos 2t}{-3\sin t} = -6\frac{\cos 2t}{\sin t}$$

$$t = \frac{\pi}{6}$$

$$\frac{dy}{dx} = -6\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}}$$

$$= -6$$

d) $x = 3\cos t$ $\qquad$ $y = 9\sin 2t = 18\sin t\cos t$

$\frac{x}{3} = \cos t$

$\sin^2 t = 1 - \cos^2 t$

$$y = 18\sqrt{1 - \frac{x^2}{9}}\;\frac{x}{3}$$

$\downarrow$

$$= 1 - \frac{x^2}{9}$$

$$\sin t = \sqrt{1 - \frac{x^2}{9}}$$

$$y = 6x\sqrt{1 - \frac{x^2}{9}}$$

$$y^2 = 36x^2\left(1 - \frac{x^2}{9}\right)$$

$$y^2 = 36x^2 \times \frac{1}{9}(9 - x^2)$$

$$\underline{\underline{y^2 = 4x^2(9 - x^2)}}$$

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11 (a)	(3,0) and (-3, 0)	B1, B1 **(2)**
(b)	$\frac{\pi}{2}$ and $\frac{3\pi}{2}$	M1 A1 **(2)**
(c)	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{18\cos 2t}{-3\sin t}$	M1 A1
	$= \frac{18 \times \frac{1}{2}}{-3 \times \frac{1}{2}} = -6$	dM1 A1 **(4)**
(d)	$y^2 = 81 \times 4\sin^2 t\cos^2 t$ | $y = 9 \times 2\sin t\cos t$	M1
	Attempts to replace $\cos^2 t = \frac{x^2}{9}$ *and* $\sin^2 t = 1 - \frac{x^2}{9}$ | Attempts to replace $\cos t = \frac{x}{3}$ *and* $\sin t = \sqrt{1 - \frac{x^2}{9}}$	M1
	Correct eqn $y^2 = 81 \times 4 \times \left(1 - \frac{x^2}{9}\right) \times \frac{x^2}{9}$ | Correct eqn $y = 9 \times 2 \times \sqrt{1 - \frac{x^2}{9}} \times \frac{x}{3}$	A1
	Obtain $y^2 = 4x^2(9 - x^2)$ | Obtain $y^2 = 4x^2(9 - x^2)$	A1 **(4)**
		(12 marks)

Your Turn

Figure 4 shows a sketch of part of the curve C with parametric equations

$$x = 3\theta \sin\theta, \qquad y = \sec^3\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point $P(k, 8)$ lies on C, where k is a constant.

(a) Find the exact value of k.

(2)

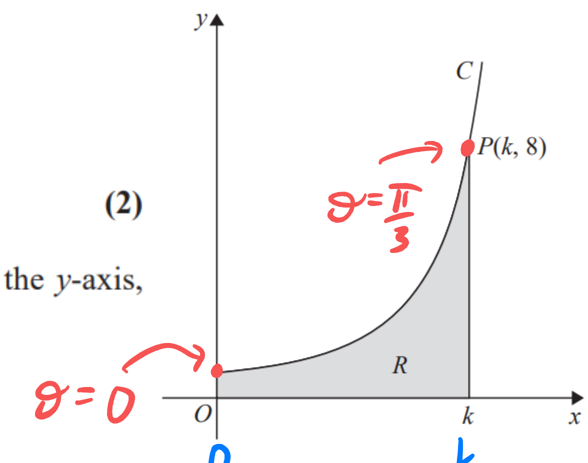

Figure 4

The finite region R, shown shaded in Figure 4, is bounded by the curve C, the y-axis, the x-axis and the line with equation $x = k$.

(b) Show that the area of R can be expressed in the form

$$\lambda \int_{\alpha}^{\beta} \left(\theta \sec^2\theta + \tan\theta \sec^2\theta\right) \mathrm{d}\theta$$

where λ, α and β are constants to be determined.

(4)

(c) Hence use integration to find the exact value of the area of R.

(6)

a) $y = 8, \quad x = k$

$8 = \sec^3\theta$

$2 = \sec\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \frac{\pi}{3}$ M

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$$k = 3\theta\sin\theta$$
$$= 3\times\frac{\pi}{3}\times\sin\frac{\pi}{3}$$
$$= \pi\times\frac{\sqrt{3}}{2}$$
$$k = \frac{\sqrt{3}}{2}\pi$$

$$x = 3\theta\sin\theta$$

$u = 3\theta \quad v = \sin\theta$

$u' = 3 \quad v' = \cos\theta$

$$\frac{dx}{d\theta} = 3\theta\cos\theta + 3\sin\theta$$

b)

$$R = \int_0^{\frac{\pi}{3}} y\,\frac{dx}{d\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \sec^3\theta\,(3\theta\cos\theta + 3\sin\theta)\,d\theta$$

$\sec^2\theta \times \cancel{\sec\theta} \times \cancel{\cos\theta}$ $\quad$ $\sec^2\theta\times\frac{\sin\theta}{\cos\theta}$

$$= \int_0^{\frac{\pi}{3}} (3\theta\sec^3\theta\cos\theta + 3\sec^3\theta\sin\theta)\,d\theta$$

$$= 3\int_0^{\frac{\pi}{3}} (\theta\sec^2\theta + \tan\theta\sec^2\theta)\,d\theta$$

$$\lambda = 3,\quad \alpha = 0,\quad \beta = \frac{\pi}{3}$$

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$$\int \theta\sec^2\theta \, d\theta = \theta\tan\theta - \int \tan\theta \, d\theta$$

$\hookrightarrow$ formula book.

IBP

$$= \theta\tan\theta - \ln|\sec\theta| \; (+c)$$

$u = \theta \quad v = \tan\theta$

$u' = 1 \quad v' = \sec^2\theta$

$$\int \tan\theta\sec^2\theta \, d\theta = \frac{1}{2}\tan^2\theta \; (+c)$$

$$3\int_0^{\frac{\pi}{3}} \left(\theta\sec^2\theta + \tan\theta\sec^2\theta\right) d\theta = \left[\theta\tan\theta - \ln|\sec\theta| + \frac{1}{2}\tan^2\theta\right]_0^{\frac{\pi}{3}}$$

$$= 3\left(\frac{\pi}{3}\times\sqrt{3} - \ln 2 + \frac{1}{2}\times 3 - \cancel{(0 - \ln 1 + 0)}\right)$$

$$= 3\left(\frac{\sqrt{3}}{3}\pi - \ln 2 + \frac{3}{2}\right)$$

$$= \sqrt{3}\pi - 3\ln 2 + \frac{9}{2}$$

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Question	Scheme	Notes	Marks
8.	$x = 3\theta\sin\theta,\ y = \sec^3\theta,\ 0 \le \theta < \frac{\pi}{2}$		
(a)	$\{\text{When } y=8,\}\ 8 = \sec^3\theta \Rightarrow \cos^3\theta = \frac{1}{8} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ $k \text{ (or } x) = 3\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$	Sets $y=8$ to find θ **and** attempts to substitute their θ into $x = 3\theta\sin\theta$	M1
	so $k \text{ (or } x) = \frac{\sqrt{3}\pi}{2}$	$\frac{\sqrt{3}\pi}{2}$ or $\frac{3\pi}{2\sqrt{3}}$	A1
	Note: Obtaining two value for k without accepting the correct value is final A0		[2]
(b)	$\frac{dx}{d\theta} = 3\sin\theta + 3\theta\cos\theta$	$3\theta\sin\theta \to 3\sin\theta + 3\theta\cos\theta$ Can be implied by later working	B1
	$\left\{\int y\frac{dx}{d\theta}\{d\theta\}\right\} = \int(\sec^3\theta)(3\sin\theta + 3\theta\cos\theta)\{d\theta\}$	Applies $(\pm K\sec^3\theta)\left(\text{their } \frac{dx}{d\theta}\right)$ Ignore integral sign and $d\theta$; $K \neq 0$	M1
	$= 3\int\theta\sec^2\theta + \tan\theta\sec^2\theta\, d\theta$	Achieves the correct result no errors in their working, e.g. bracketing or manipulation errors. **Must have** integral sign and $d\theta$ in their final answer.	A1 *
	$x=0$ and $x=k \Rightarrow \underline{\alpha=0}$ and $\underline{\beta=\frac{\pi}{3}}$	$\alpha=0$ and $\beta=\frac{\pi}{3}$ or evidence of $0\to0$ and $k\to\frac{\pi}{3}$	B1
	Note: The work for the final B1 mark must be seen in part (b) only.		[4]
(c) **Way 1**	$\left\{\int\theta\sec^2\theta\, d\theta\right\} = \theta\tan\theta - \int\tan\theta\{d\theta\}$	$\theta\sec^2\theta \to A\theta g(\theta) - B\int g(\theta),\ A>0,\ B>0,$ where $g(\theta)$ is a trigonometric function in θ **and** $g(\theta) =$ their $\int\sec^2\theta\, d\theta$. [Note: $g(\theta) \neq \sec^2\theta$]	M1
		dependent on the previous M mark **Either** $\lambda\theta\sec^2\theta \to A\theta\tan\theta - B\int\tan\theta,\ A>0,\ B>0$ **or** $\theta\sec^2\theta \to \theta\tan\theta - \int\tan\theta$	dM1
	$= \theta\tan\theta - \ln(\sec\theta)$ **or** $= \theta\tan\theta + \ln(\cos\theta)$	$\theta\sec^2\theta \to \theta\tan\theta - \ln(\sec\theta)$ or $\theta\tan\theta + \ln(\cos\theta)$ **or** $\lambda\theta\sec^2\theta \to \lambda\theta\tan\theta - \lambda\ln(\sec\theta)$ or $\lambda\theta\tan\theta + \lambda\ln(\cos\theta)$	A1
	Note: Condone $\theta\sec^2\theta \to \theta\tan\theta - \ln(\sec x)$ or $\theta\tan\theta + \ln(\cos x)$ for A1		
	$\left\{\int\tan\theta\sec^2\theta\, d\theta\right\}$	$\tan\theta\sec^2\theta$ or $\lambda\tan\theta\sec^2\theta \to \pm C\tan^2\theta$ or $\pm C\sec^2\theta$ or $\pm Cu^{-2}$, where $u=\cos\theta$	M1
	$= \frac{1}{2}\tan^2\theta$ or $\frac{1}{2}\sec^2\theta$ or $\frac{1}{2u^2}$ where $u=\cos\theta$ or $\frac{1}{2}u^2$ where $u=\tan\theta$	$\tan\theta\sec^2\theta \to \frac{1}{2}\tan^2\theta$ or $\frac{1}{2}\sec^2\theta$ or $\frac{1}{2\cos^2\theta}$ or $\tan^2\theta - \frac{1}{2}\sec^2\theta$ or $0.5u^{-2}$, where $u=\cos\theta$ or $0.5u^2$, where $u=\tan\theta$ or $\lambda\tan\theta\sec^2\theta \to \frac{\lambda}{2}\tan^2\theta$ or $\frac{\lambda}{2}\sec^2\theta$ or $\frac{\lambda}{2\cos^2\theta}$ or $0.5\lambda u^{-2}$, where $u=\cos\theta$ or $0.5\lambda u^2$, where $u=\tan\theta$	A1
	$\{\text{Area}(R)\} = \left[3\theta\tan\theta - 3\ln(\sec\theta) + \frac{3}{2}\tan^2\theta\right]_0^{\frac{\pi}{3}}$ **or** $\left[3\theta\tan\theta - 3\ln(\sec\theta) + \frac{3}{2}\sec^2\theta\right]_0^{\frac{\pi}{3}}$		
	$= \left(3\left(\frac{\pi}{3}\right)\sqrt{3} - 3\ln2 + \frac{3}{2}(3)\right) - (0)$ **or** $\left(3\left(\frac{\pi}{3}\right)\sqrt{3} - 3\ln2 + \frac{3}{2}(4)\right) - \left(\frac{3}{2}\right)$		
	$= \frac{9}{2} + \sqrt{3}\pi - 3\ln2$ **or** $\frac{9}{2} + \sqrt{3}\pi + 3\ln\left(\frac{1}{2}\right)$ **or** $\frac{9}{2} + \sqrt{3}\pi - \ln8$ **or** $\ln\left(\frac{1}{8}e^{\frac{9}{2}+\sqrt{3}\pi}\right)$		A1 o.e.
			[6]
			12

Your Turn

Sorry if another rabbit population question is a little triggering...

The number of rabbits on an island is modelled by the equation

$$P = \frac{100e^{-0.1t}}{1 + 3e^{-0.9t}} + 40, \qquad t \in \mathbb{R}, t \geqslant 0$$

where P is the number of rabbits, t years after they were introduced onto the island

A sketch of the graph of P against t is shown in Figure 3.

Figure 3

(a) Calculate the number of rabbits that were introduced onto the island.

(1)

(b) Find $\frac{dP}{dt}$

(3)

The number of rabbits initially increases, reaching a maximum value P_T when $t = T$

(c) Using your answer from part (b), calculate

(i) the value of T to 2 decimal places,

(ii) the value of P_T to the nearest integer.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(4)

For $t > T$, the number of rabbits decreases, as shown in Figure 3, but never falls below k, where k is a positive constant.

(d) Use the model to state the maximum value of k.

(1)

a) $t = 0$

$P = \frac{100}{1+3} + 40$

$P = 65$ rabbits

b) $P = \dfrac{100e^{-0.1t}}{1+3e^{-0.9t}} + 40$

$u = 100e^{-0.1t}$ $\quad v = 1+3e^{-0.9t}$

$u' = -10e^{-0.1t}$ $\quad v' = -2.7e^{-0.9t}$

$$\frac{dP}{dt} = \frac{-10e^{-0.1t}(1+3e^{-0.9t}) + 270e^{-0.1t}e^{-0.9t}}{(1+3e^{-0.9t})^2}$$

c) $\dfrac{dP}{dt} = 0$

$$-10e^{-0.1t}(1+3e^{-0.9t}) + 270e^{-t} = 0$$

$$27\cancel{0}e^{-t} = 1\cancel{0}e^{-0.1t}(1+3e^{-0.9t})$$

$$27e^{-t} = e^{-0.1t} + 3e^{-t}$$

$$\frac{24}{e^{t}} = e^{-0.1t}$$

$$24 = e^{t}e^{-0.1t}$$

$$24 = e^{0.9t}$$

$$\ln 24 = 0.9t$$

$$t = \frac{1}{0.9}\ln 24$$

$$t = 3.53117..$$

$$T = 3.53 \text{ (2dp)}$$

ii) $P_T = 102.44...$

$= 102$ rabbits.

d) $$p = \frac{100e^{-0.1t}}{1+3e^{-0.9t}} + 40$$

As $t \to \infty$

$e^{-0.1t} \to 0$

$e^{-0.9t} \to 0$

$$P \to \frac{0}{1} + 40 \to 40$$

$$\underline{\underline{k = 40}}$$

8 (a)	$P_0 = \dfrac{100}{1+3} + 40 = 65$	B1	
			(1)
	$\dfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{kt} = C\mathrm{e}^{kt}$	M1	
(b)	$\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{\left(1+3\mathrm{e}^{-0.9t}\right)\times -10\mathrm{e}^{-0.1t} - 100\mathrm{e}^{-0.1t}\times -2.7\mathrm{e}^{-0.9t}}{\left(1+3\mathrm{e}^{-0.9t}\right)^2}$	M1 A1	
			(3)
(c)(i)	At maximum $\quad -10\mathrm{e}^{-0.1t} - 30\mathrm{e}^{-0.1t}\times \mathrm{e}^{-0.9t} + 270\mathrm{e}^{-0.1t}\times \mathrm{e}^{-0.9t} = 0$		
	$\mathrm{e}^{-0.1t}\left(-10 + 240\mathrm{e}^{-0.9t}\right) = 0$		
	$\mathrm{e}^{-0.9t} = \dfrac{10}{240} \qquad$ oe $\mathrm{e}^{0.9t} = 24$	M1	
	$-0.9t = \ln\left(\dfrac{1}{24}\right) \Rightarrow t = \dfrac{10}{9}\ln(24) = 3.53$	M1, A1	
(c) (ii)	Sub $t = 3.53 \Rightarrow P_T = 102$	A1	
			(4)
(d)	40	B1	
			(1)
		9 marks	

5. A bath is filled with hot water. The temperature, θ °C, of the water in the bath, t minutes after the bath has been filled, is given by

$$\theta = 20 + Ae^{-kt}$$

where A and k are positive constants.

$t=0, \quad \theta=38$

Given that the temperature of the water in the bath is initially 38 °C,

(a) find the value of A.

(2)

The temperature of the water in the bath 16 minutes after the bath has been filled is 24.5 °C.

(b) Show that $k = \frac{1}{8}\ln 2$

$t=16 \quad \theta=24.5$

(4)

Using the values for k and A,

(c) find the temperature of the water 40 minutes after the bath has been filled, giving your answer to 3 significant figures.

(2)

(d) Explain why the temperature of the water in the bath cannot fall to 19 °C.

(1)

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$$38 = 20 + Ae^{0}$$

$$38 = 20 + A$$

$$\underline{\underline{A = 18}}$$

b) $$24.5 = 20 + 18e^{-k \times 16} \qquad \theta = 20 + 18e^{-kt}$$

$$4.5 = 18e^{-16k}$$

$$\frac{1}{4} = e^{-16k}$$

$$\ln\frac{1}{4} = -16k$$

$$k = -\frac{1}{16}\ln\frac{1}{4}$$

$$k = \frac{1}{16}\ln 4$$

$$k = \frac{1}{16}\ln 2^{2} = \frac{2}{16}\ln 2 = \underline{\underline{\frac{1}{8}\ln 2}}$$

c) $\theta = 20 + 18e^{-\frac{1}{8}\ln 2 \times 40}$

$= 20.5625$

$= \underline{20.6^\circ C}$ (3sf)

d) The model shows that 20°C is the min, so it cannot fall to 19°C.

5(a)	$t=0, \theta=38 \Rightarrow 38=20+Ae^{-k\times 0}$	For substituting $t=0$ and $\theta=38$ into $\theta=20+Ae^{-kt}$	M1
	$\Rightarrow A=18$	Correct value for A	A1
	A = 18 with no working scores both marks		
			(2)
(b)	$t=16, \theta=24.5 \Rightarrow 24.5=20+"18"e^{-k\times 16}$	For substituting $t=16$ and $\theta=24.5$ into $\theta=20+\text{their }"A"e^{-kt}$	M1
	$\Rightarrow 18e^{-k\times 16}=4.5$ or $e^{-k\times 16}=\frac{1}{4}$	This mark is for a correct equation with the **constants combined.** Allow equivalent correct equations e.g. $e^{16k}=4$	A1
	$\Rightarrow e^{16k}=4 \Rightarrow 16k=\ln 4$ or $\Rightarrow \ln 18e^{-k\times 16}=\ln 4.5 \Rightarrow \ln 18+\ln e^{-k\times 16}=\ln 4.5 \Rightarrow \ln e^{-k\times 16}=\ln\frac{1}{4}$ $\Rightarrow -16k=\ln\frac{1}{4}$ Uses correct log or exponential work to move from: $e^{\pm nk}=C$ to $\pm nk=\alpha\ln C$ **or** $pe^{\pm nk}=q$ to $\pm nk=\alpha\ln\beta$		M1
	$-16k=\ln\frac{1}{4} \Rightarrow k=-\frac{1}{16}\ln\frac{1}{4}=\frac{1}{8}\ln 2$* Shows that $k=\frac{1}{8}\ln 2$ There must be **at least one intermediate line** between their $\pm nk=\alpha\ln C$ or their $\pm nk=\alpha\ln\beta$ and the printed answer. So for example $-16k=\ln\frac{1}{4} \Rightarrow k=\frac{1}{8}\ln 2$* scores A0 as there is no intermediate line.		A1*
	Note: The marks in part (b) can be scored by using $\theta=20+Ae^{-kt}$ and substituting 2 out of: $A=18$, $\theta=24.5$, $k=\frac{1}{8}\ln 2$ to show that the 3rd variable is correct followed by a conclusion e.g. so $k=\frac{1}{8}\ln 2$		
			(4)
(c)	$t=40 \Rightarrow \theta=20+"18"e^{-\frac{1}{8}\ln 2\times 40}$	Substitutes $t=40$ into the given equation with their A and the given value of k to obtain a value for θ	M1
	$\Rightarrow \theta=\text{awrt } 20.6(°C)$	Awrt 20.6	A1
	Correct answer only scores both marks		
			(2)

5(d)	**Examples:** • The lower limit is 20 • $\theta>20$ • As t tends to infinity temperature tends to 20 • The temperature cannot go below 20 • e^{-kt} tends towards zero so the temperature tends to 20 • e^{-kt} is always positive so the temperature is always bigger than 20 • Substitutes $\theta=19$ in $\theta=20+"18"e^{-kt}$ (may be implied by e.g. $e^{-kt}=-\frac{1}{18}$) and states e.g. that you cannot find the log of a negative number or "which is not possible" Do not accept e^{-kt} cannot be negative without reference to the "20"	B1
		(1)

3. The value of a car is modelled by the formula

$$V = 16\,000e^{-kt} + A, \qquad t \geqslant 0, t \in \mathbb{R}$$

where V is the value of the car in pounds, t is the age of the car in years, and k and A are positive constants.

$t=0, V=17500$

Given that the value of the car is £17 500 when new and £13 500 two years later,

$t=2, V=13500$

(a) find the value of A,

(1)

(b) show that $k = \ln\left(\dfrac{2}{\sqrt{3}}\right)$

(4)

(c) Find the age of the car, in years, when the value of the car is £6000

Give your answer to 2 decimal places.

(4)

a) $17500 = 16000e^{0} + A$

$A = 1500$

b) $13500 = 16000e^{-k\times 2} + 1500$

$\frac{12000}{16000} = e^{-2k}$

$\ln\left(\frac{3}{4}\right) = -2k$

$k = -\frac{1}{2}\ln\left(\frac{3}{4}\right)$

$k = \frac{1}{2}\ln\left(\frac{4}{3}\right) = \underline{\ln\left(\frac{2}{\sqrt{3}}\right)}$

c) $6000 = 16000e^{\frac{1}{2}\ln\left(\frac{3}{4}\right)\times t} + 1500$

$\frac{9}{32} = e^{\frac{1}{2}\ln\left(\frac{3}{4}\right)\times t}$

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$$\ln \frac{9}{32} = \frac{1}{2} \ln\left(\frac{3}{4}\right) \times t$$

$$t = \underline{8.82 \text{ years}} \quad (2dp)$$

3(a)	$A = 1500$	B1
		(1)
(b)	Sub $t = 2, V = 13500 \Rightarrow 16000e^{-2k} = 12000$	M1
	$\Rightarrow e^{-2k} = \frac{3}{4}$ 0.75 oe	A1
	$\Rightarrow k = -\frac{1}{2}\ln\frac{3}{4}, = \ln\sqrt{\frac{4}{3}} = \ln\left(\frac{2}{\sqrt{3}}\right)$	dM1, A1*
		(4)
(c)	Sub $6000 = 16000e^{-\ln\left(\frac{2}{\sqrt{3}}\right)T} + '1500' \Rightarrow e^{-\ln\left(\frac{2}{\sqrt{3}}\right)T} = C$	M1
	$\Rightarrow e^{-\ln\left(\frac{2}{\sqrt{3}}\right)T} = \frac{45}{160} = 0.28125$	A1
	$\Rightarrow T = -\frac{\ln\left(\frac{45}{160}\right)}{\ln\left(\frac{2}{\sqrt{3}}\right)} = 8.82$	M1 A1
		(4)
		(9 marks)

Your Turn

★★★★★

11 In a science experiment a substance is decaying exponentially. Its mass, M grams, at time t minutes is given by $M = 300e^{-0.05t}$.

(i) Find the time taken for the mass to decrease to half of its original value. **[3]**

A second substance is also decaying exponentially. Initially its mass was 400 grams and, after 10 minutes, its mass was 320 grams.

(ii) Find the time at which both substances are decaying at the same rate. **[8]**

i) orig value of M when $t=0$, $M = 300$

half of 300 is 150

$$150 = 300e^{-0.05t}$$

$$0.5 = e^{-0.05t}$$

$$\ln 0.5 = -0.05t$$

$$\frac{1}{-0.05}\ln 0.5 = t$$

$t = 13.9$ minutes

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ii) $N = 400e^{-kt}$ $\quad t=10, \quad N=320$

$320 = 400e^{-10k}$

$0.8 = e^{-10k}$

$\ln 0.8 = -10k$

$k = 0.0223$ (3sf)

$N = 400e^{-0.0223t}$ $\qquad M = 300e^{-0.05t}$

rate

$\downarrow$

diff.

$\frac{dN}{dt} = -8.92e^{-0.0223t}$ $\qquad \frac{dM}{dt} = -15e^{-0.05t}$

$-8.92e^{-0.0223t} = -15e^{-0.05t}$ $\quad \overline{e^{0.05t}}$

$e^{0.05t} \times e^{-0.0223t} = \frac{15}{8.92}$

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$$e^{0.0277t} = \frac{15}{8.92}$$

$$0.0277t = \ln \frac{15}{8.92}$$

$$t = 18.76\ldots \text{ minuter}$$

$$= \underline{18.8} \text{ minutes}$$

11	(i)		When $t = 0$, $M = 300$	B1	2.2a	Identify that the initial mass is 300g
			$300e^{-0.05t} = 150$ $e^{-0.05t} = 0.5$ $-0.05t = \ln 0.5$	M1	3.1a	Equate to 150 and attempt to solve
			$t = 13.9$ (minutes)	A1	1.1	Obtain 13.86, or better
				[3]		
	(ii)		$M_2 = 400e^{kt}$	B1	2.2a	State or imply $400e^{kt}$
			$320 = 400e^{10k}$ $k = 0.1\ln 0.8$	M1	1.1a	Attempt to find k
			$M_2 = 400e^{-0.0223t}$	A1	1.1	Obtain correct expression for mass of second substance
			Substance 1: $\frac{dM_1}{dt} = -15e^{-0.05t}$ Substance 2: $\frac{dM_2}{dt} = -8.93e^{-0.0223t}$	M1	3.1a	Attempt differentiation at least once
				A1ft	1.1	Both derivatives correct
			$-15e^{-0.05t} = -8.93e^{-0.0223t}$ $e^{0.0277t} = 1.681$	M1	3.1a	Equate derivatives and rearrange as far as $e^{f(t)} = c$
			$0.0277t = 0.519$	M1	1.1	Attempt to solve equation of form $e^{f(t)} = c$
			time = 18.75 minutes	A1	3.2a	Obtain correct value for t Allow 18.7, 18.8 or 19 mins

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Your Turn

(ii) Given that

$$2\log_4(3x+5) = \log_4(3x+8) + 1, \qquad x > -\frac{5}{3}$$

(a) show that

$$9x^2 + 18x - 7 = 0$$

(4)

(b) Hence solve the equation

$$2\log_4(3x+5) = \log_4(3x+8) + 1, \qquad x > -\frac{5}{3}$$

(2)

$$\log_4(3x+5)^2 = \log_4(3x+8) + 1$$

$$\log_4 \frac{(3x+5)^2}{3x+8} = 1$$

$$\frac{(3x+5)^2}{3x+8} = 4$$

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DO NOT WRITE IN THIS AREA

a) $9x^2 + 30x + 25 = 4(3x+8)$

$9x^2 + 18x - 7 = 0$

b) $(3x-1)(3x+7) = 0$

$x = \frac{1}{3}$ $\quad$ $\cancel{x = -\frac{7}{3}}$

(ii) (a)	Ignore labels (a) and (b) in part ii and mark work as seen	
	$\log_4(3x+5)^2 =$ Applies power law of logarithms	M1
	Uses $\log_4 4 = 1$ or $4^1 = 4$	M1
	Uses quotient or product rule so e.g. $\log(3x+5)^2 = \log 4(3x+8)$ or $\log\dfrac{(3x+5)^2}{(3x+8)} = 1$	M1
	Obtains with no errors $9x^2 + 18x - 7 = 0$ *	A1* cso **(4)**
(b)	Solves given or "their" quadratic equation by any of the standard methods	M1
	Obtains $x = \frac{1}{3}$ and $-\frac{7}{3}$ and rejects $-\frac{7}{3}$ to give just $\frac{1}{3}$	A1 **(2)** **[8]**

Your Turn

7. (i) $2\log(x + a) = \log(16a^6)$, where a is a positive constant

Find x in terms of a, giving your answer in its simplest form.

(3)

(ii) $\log_3(9y + b) - \log_3(2y - b) = 2$, where b is a positive constant

Find y in terms of b, giving your answer in its simplest form.

(4)

i) $\log(x+a)^2 = \log(16a^6)$

$(x+a)^2 = 16a^6$

$x + a = 4a^3$

$x = 4a^3 - a$

ii) $\log_3\left(\frac{9y+b}{2y-b}\right) = 2$

$\frac{9y+b}{2y-b} = 9$

$9y + b = 18y - 9b$

$10b = 9y$

$y = \frac{10}{9}b$

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7. (i)	Use of power rule so $\log(x+a)^2 = \log 16a^6$ or $2\log(x+a) = 2\log 4a^3$ or $\log(x+a) = \log(16a^6)^{\frac{1}{2}}$	M1
	Removes logs and square roots, **or** halves then removes logs to give $(x+a) = 4a^3$ Or $x^2 + 2ax + a^2 - 16a^6 = 0$ followed by factorisation or formula to give $x = \sqrt{16a^6} - a$	M1
	$(x =)\ 4a^3 - a$ (depends on previous M's and must be this expression or equivalent)	A1cao **(3)**
(ii) Way 1	$\log_3 \frac{(9y+b)}{(2y-b)} = 2$ Applies quotient law of logarithms	M1
	$\frac{(9y+b)}{(2y-b)} = 3^2$ Uses $\log_3 3^2 = 2$	M1
	$(9y+b) = 9(2y-b) \Rightarrow y =$ Multiplies across and makes y the subject	M1
	$y = \frac{10}{9}b$	A1cso **(4)**
Way 2	Or : $\log_3(9y+b) = \log_3 9 + \log_3(2y-b)$ 2nd M mark	M1
	$\log_3(9y+b) = \log_3 9(2y-b)$ 1st M mark	M1
	$(9y+b) = 9(2y-b) \Rightarrow y = \frac{10}{9}b$ Multiplies across and makes y the subject	M1 A1cso **(4)** **[7]**