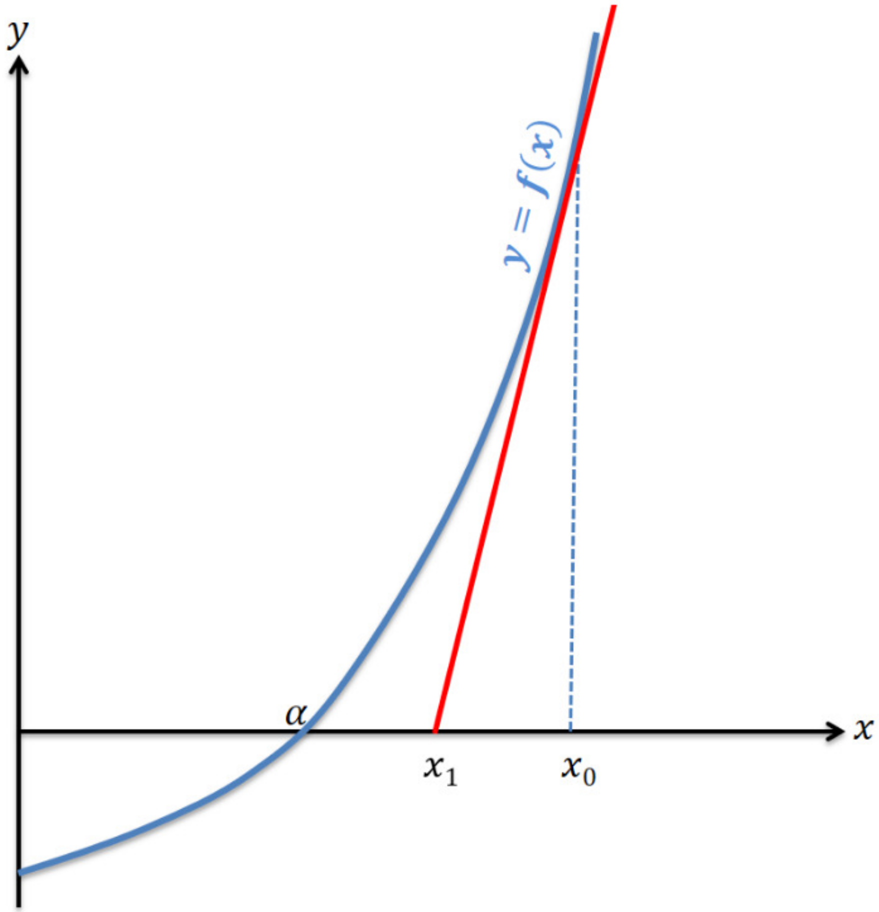
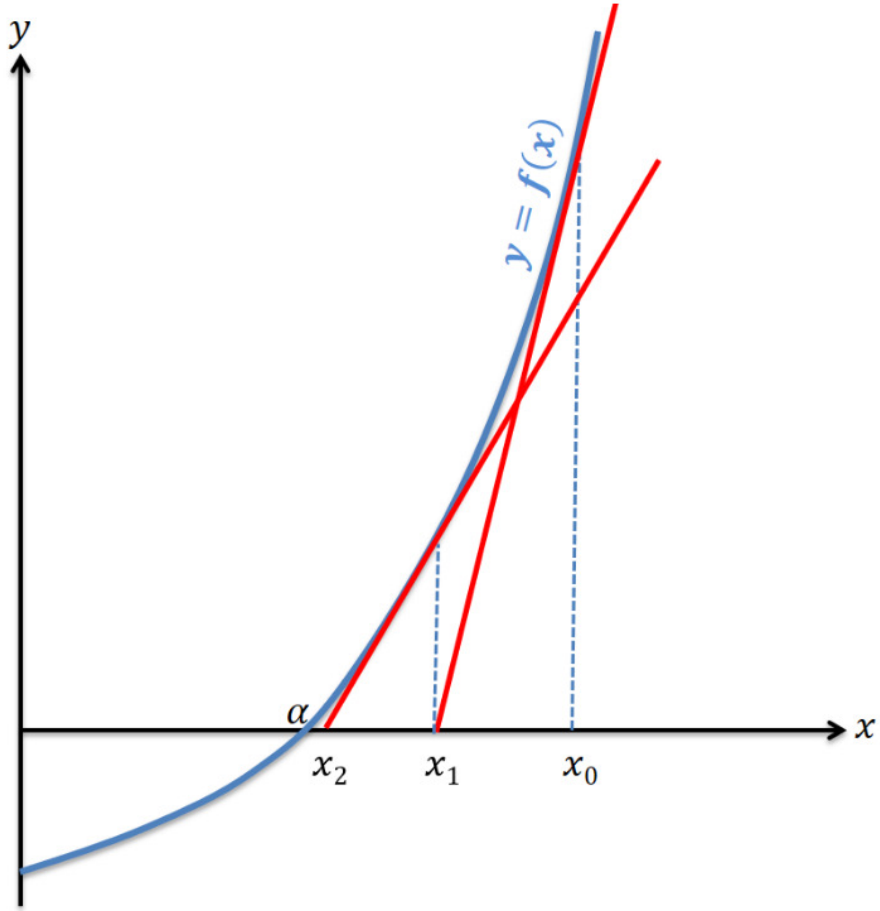
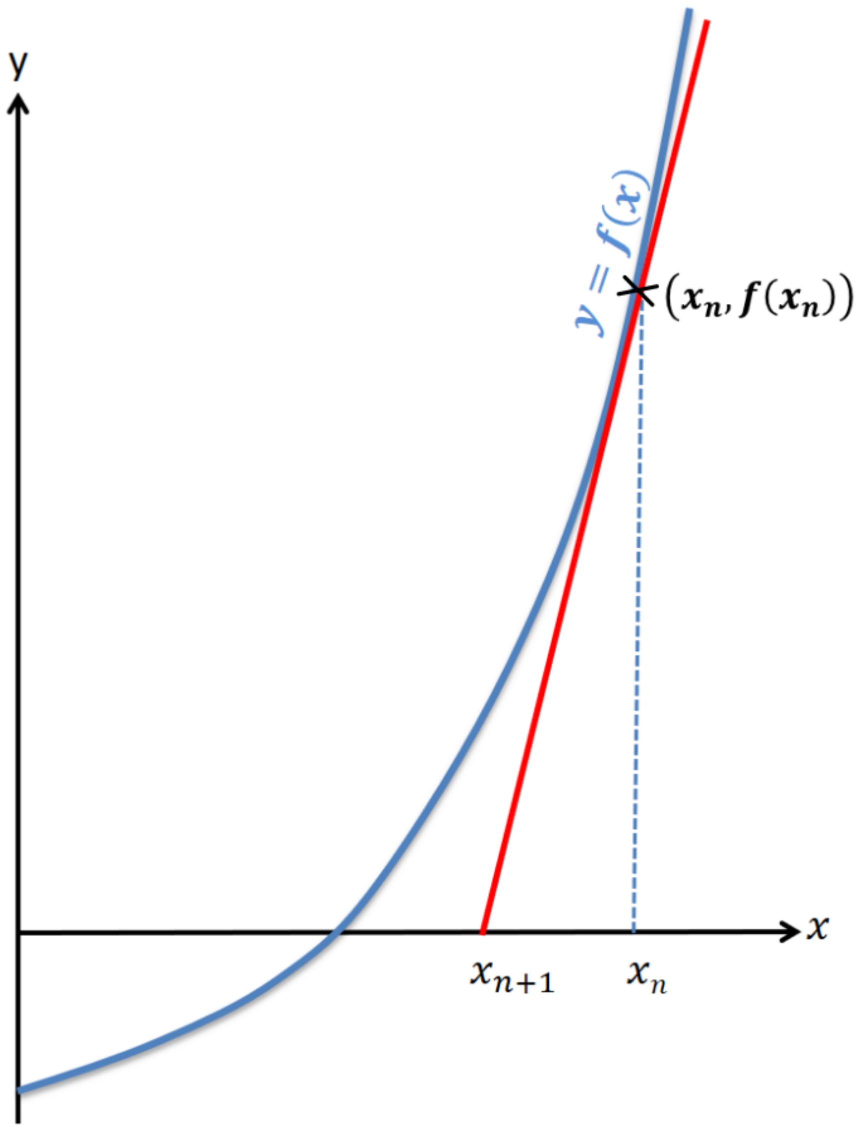
The Newton-Raphson Process/Method





A seemingly sensible thing to do is to follow the direction of the line, i.e. use the gradient of the tangent. If the line was reasonably straight, the point the tangent hits the x-axis would be close to the root.

Deriving the Formula - *not in the specification, but interesting!*



$y - y_1 = m(x - x_1)$ red line

$y - f(x_n) = f'(x_n)(x - x_n)$

When $y=0$, $x = x_{n+1}$

$-f(x_n) = f'(x_n)(x_{n+1} - x_n)$

$$\frac{-f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

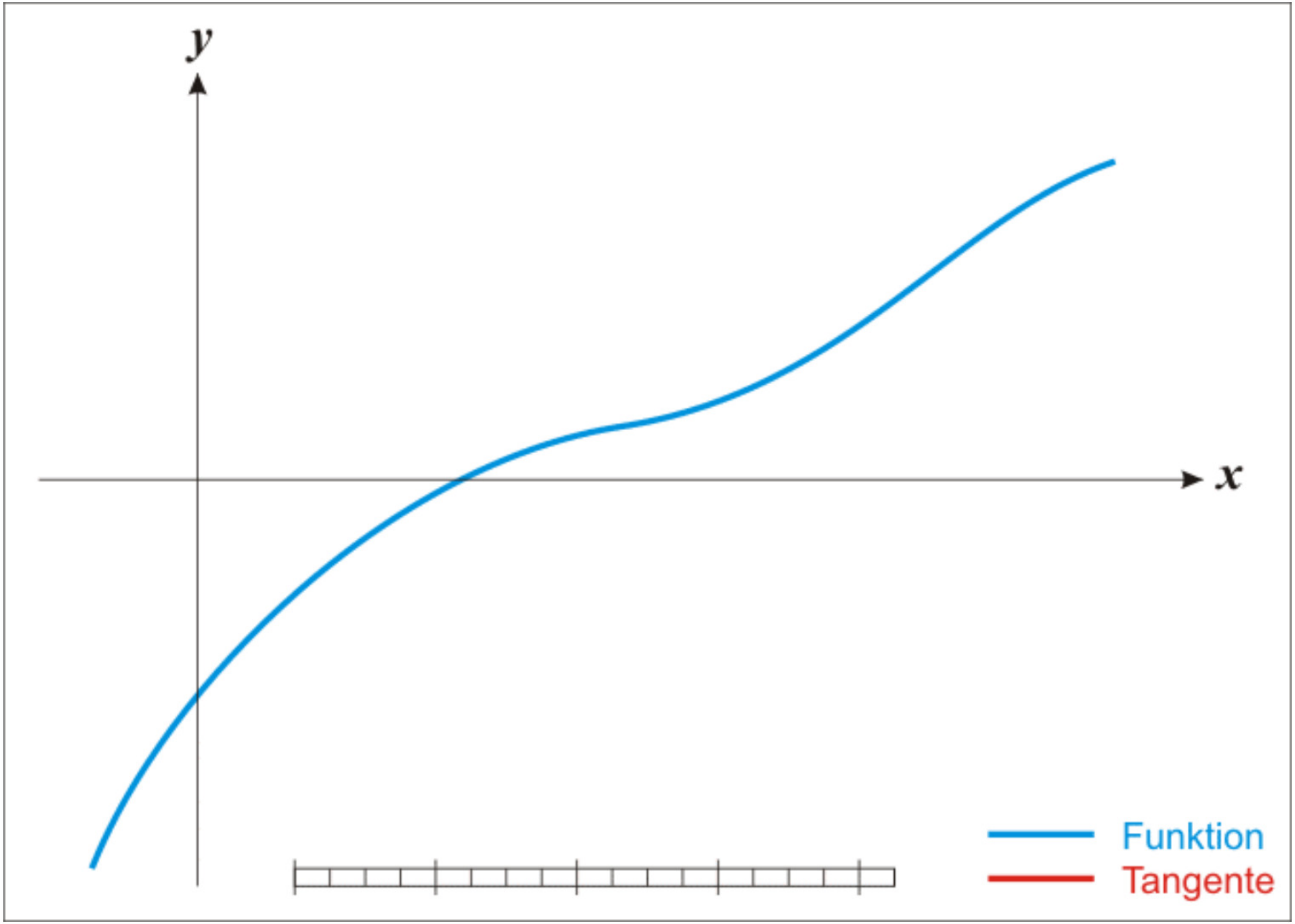
good approx.

Newton-Raphson Process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Formula Book

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$


y
x
Funktion
Tangente

Example

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Returning to our original example: $\boldsymbol{x} = \mathbf{cos}(\boldsymbol{x})$, say letting $x_0 = 0.5$

(Note: Recall that differentiation assumes radians)

$$f(x) = x - \cos x$$

$$f'(x) = 1 + \sin x$$

$$x_0 = 0.5$$

$$x_1 = 0.5 - \frac{(0.5 - \cos 0.5)}{1 + \sin 0.5}$$

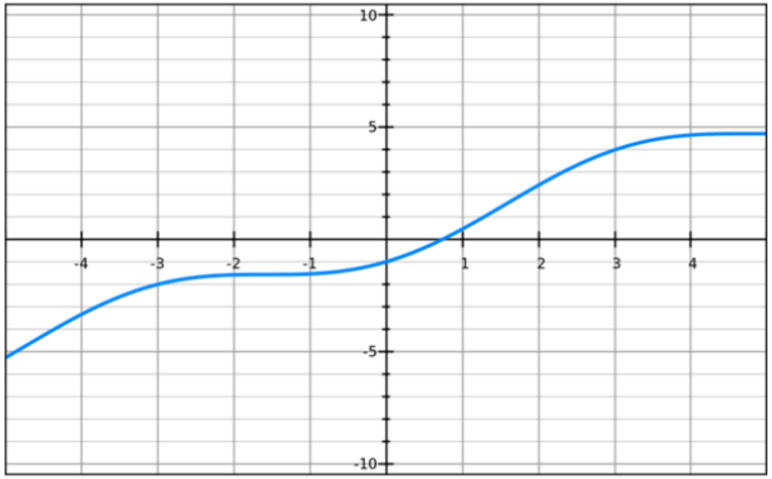
$$= 0.7552$$

$$x_2 = 0.7391$$

$$x_3 = \underline{\underline{0.7391}}$$

$$x = 0.7391$$

$$\cos x = 0.7391$$



Tip: To perform iterations quickly, do the following on your calculator:

[0.5] [=]
[ANS] – (ANS – cos(ANS))/(1 + sin(ANS))
Then hit [=].

Quick Questions

Using the Newton-Raphson process, state the recurrence relation for the following functions:

$f(x) = x^3 - 2$ ⇨ $x_{n+1} = x_n - \dfrac{x_n^3 - 2}{3x_n^2}$

$f(x) = \tan x$ ⇨ $x_{n+1} = x_n - \dfrac{\tan x_n}{\sec^2 x_n}$

$f(x) = x^2 - x - 1$ ⇨ $x_{n+1} = x_n - \dfrac{x_n^2 - x_n - 1}{2x_n - 1}$

Edexcel FP1 June 2013(R) Q3c

$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

...

The equation f(x) = 0 has a root β in the interval [–2, –1].

(c) Taking –1.5 as a first approximation to β, apply the Newton-Raphson process once to f(x) to obtain a second approximation to β.
Give your answer to 2 decimal places. **(5)**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

$$f'(x) = 2x^3 - 3x^2 + 1$$

$$x_0 = -1.5$$

$$x_1 = -1.5 - \frac{\frac{1}{2}(-1.5)^4 - (-1.5)^3 + (-1.5) - 3}{2(-1.5)^3 - 3(-1.5)^2 + 1}$$

$$= -1.3875$$

$$= \underline{-1.39}$$

$$x_2 = -1.3740$$

$$x_3 = -1.37338$$

$$x_4 = -1.37338$$

Your Turn

Edexcel FP1 Jan 2010 Q2c

$$f(x) = 3x^2 - \frac{11}{x^2}.$$

(c) Taking 1.4 as a first approximation to α, apply the Newton-Raphson procedure once to f(x) to obtain a second approximation to α, giving your answer to 3 decimal places.
(5)

$x_0 = 1.4$

$f(x) = 3x^2 - 11x^{-2}$

$f'(x) = 6x + 22x^{-3}$

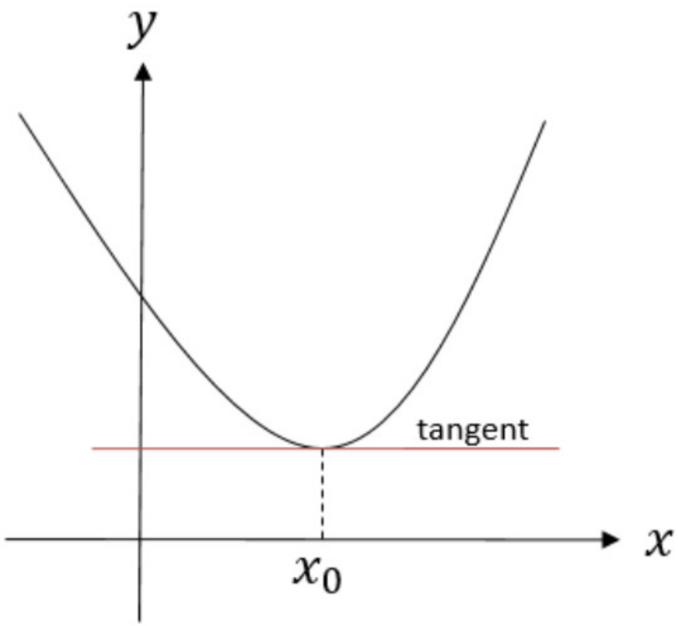
$$x_1 = 1.4 - \frac{3(1.4)^2 - 11(1.4)^{-2}}{6(1.4) + 22(1.4)^{-3}}$$

$= \underline{\underline{1.384}}$ (3dp)

Ex 10C

(c) $f'(x) = 6x + 22x^{-3}$	M1 A1
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.4 - \frac{0.268}{16.417}$, $\quad = 1.384$	M1 A1, A1
	(5)

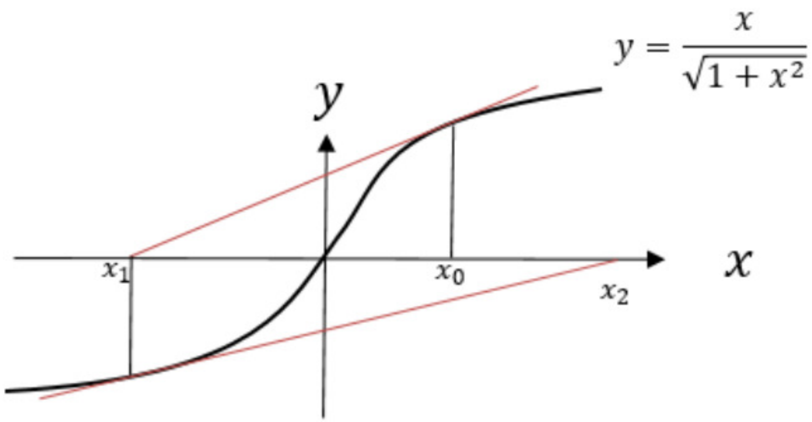
When does Newton-Raphson fail?



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the starting value x_0 was the stationary point, then $f'(x_0) = 0$, resulting in a division by 0 in the above formula.

Graphically, it is because the tangent will never reach the x-axis.



Newton-Raphson also suffers from the same drawbacks as solving by iteration, in that it's possible for the values of x_i to **diverge**.

In this example, the x_i oscillate either side of 0, but gradually getting further away from $\alpha = 0$.

Ex 10C

The value of β lies in the interval [1.5, 3]

A student takes 3 as her first approximation to β.

Given $f(3) = -1.4189$ and $f'(3) = -8.3078$ to 4 decimal places,

(c) apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation to β. Give your answer to 2 decimal places.

(2)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{\cancel{-}1.4189}{\cancel{-}8.3078}$$

$$= \underline{\underline{2.83}}$$

A different student takes a starting value of 1.5 as his first approximation to β.

(d) Use Figure 3 to explain whether or not the Newton-Raphson method with this starting value gives a good second approximation to β.

(2)

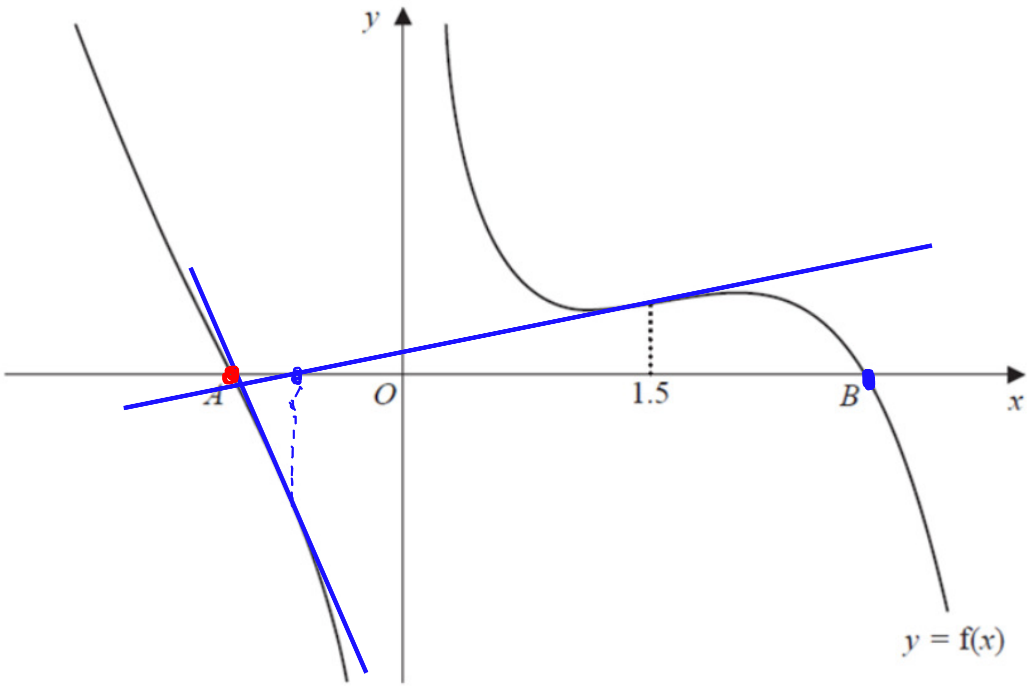


Figure 3

This is not a good starting value, as it appears to give us an approximation for the other root

Question	Scheme	Marks	AOs
7	$\mathrm{f}(x) = \dfrac{2}{x} - \mathrm{e}^x + 2x^2,\ x \in \mathbb{R}, x \neq 0$		
(a)	Evaluates both $\mathrm{f}(-1.5)$ and $\mathrm{f}(-1)$	M1	1.1b
	$\mathrm{f}(-1.5) = 2.943536507...$ and $\mathrm{f}(-1) = -0.3678794412...$ Sign change and as $\mathrm{f}(x)$ is continuous α lies between -1.5 and -1	A1	2.4
		(2)	
(b)	(i) $\{x_3 =\}\ -1.0428$	B1	1.1b
	(ii) $\{\alpha =\}\ -1.06$ (2 dp)	B1	2.2a
		(2)	
(c)	$\{x_2 =\}\ 3 - \left(\dfrac{-1.4189}{-8.3078}\right)$	M1	1.1b
	$\{= 2.829208695...\}\ = 2.83$ (2 dp)	A1	1.1b
		(2)	
(d)	• Draws a tangent to the curve at $x = 1.5$ and identifies (possibly by writing x_2) where the tangent cuts the x-axis	M1	1.1b
	and concludes either • second approximation is not good because it is not in the interval [1.5, 3] • x_2 (which is indicated on Figure 3) is nowhere near the root β	A1	2.4
		(2)	

5. The equation $2x^3 + x^2 - 1 = 0$ has exactly one real root.

(a) Show that, for this equation, the Newton-Raphson formula can be written

$$x_{n+1} = \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$$

(3)

Using the formula given in part (a) with $x_1 = 1$

(b) find the values of x_2 and x_3

(2)

(c) Explain why, for this question, the Newton-Raphson method cannot be used with $x_1 = 0$

(1)

$f(x) = 2x^3 + x^2 - 1$

$f'(x) = 6x^2 + 2x$

a) $x_{n+1} = x_n - \frac{2x_n^3 + x_n^2 - 1}{6x_n^2 + 2x_n}$

$= \frac{x_n(6x_n^2 + 2x_n) - (2x_n^3 + x_n^2 - 1)}{6x_n^2 + 2x_n}$

$= \frac{6x_n^3 + 2x_n^2 - 2x_n^3 - x_n^2 + 1}{6x_n^2 + 2x_n}$

$= \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$

b) $x_2 = \frac{4 \times 1^3 + 1^2 + 1}{6 \times 1^2 + 2 \times 1} = 0.75$

$x_3 = 0.666... = \frac{2}{3}$

c) When $x = 0$

$f'(0) = 0$ so the tangent would not meet the axis again.

Question	Scheme	Marks	AOs
5	The equation $2x^3 + x^2 - 1 = 0$ has exactly one real root		
(a)	$\left\{\mathrm{f}(x) = 2x^3 + x^2 - 1 \Rightarrow\right\}\ \mathrm{f}'(x) = 6x^2 + 2x$	B1	1.1b
	$\left\{x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)} \Rightarrow\right\}\ \{x_{n+1}\} = x_n - \dfrac{2x_n^{\ 3} + x_n^{\ 2} - 1}{6x_n^{\ 2} + 2x_n}$	M1	1.1b
	$= \dfrac{x_n(6x_n^{\ 2} + 2x_n) - (2x_n^{\ 3} + x_n^{\ 2} - 1)}{6x_n^{\ 2} + 2x_n} \Rightarrow x_{n+1} = \dfrac{4x_n^{\ 3} + x_n^{\ 2} + 1}{6x_n^{\ 2} + 2x_n}$ *	A1*	2.1
		(3)	
(b)	$\{x_1 = 1 \Rightarrow\}\ x_2 = \dfrac{4(1)^3 + (1)^2 + 1}{6(1)^2 + 2(1)}$ or $x_2 = 1 - \dfrac{2(1)^3 + (1)^2 - 1}{6(1)^2 + 2(1)}$	M1	1.1b
	$\Rightarrow x_2 = \dfrac{3}{4},\ x_3 = \dfrac{2}{3}$	A1	1.1b
		(2)	
(c)	Accept any reasons why the Newton-Raphson **method** cannot be used with $x_1 = 0$ which refer or ***allude*** to either the stationary point or the tangent. E.g. • There is a stationary point at $x = 0$ • Tangent to the curve (or $y = 2x^3 + x^2 - 1$) would not meet the x-axis • Tangent to the curve (or $y = 2x^3 + x^2 - 1$) is horizontal	B1	2.3
		(1)	
			(6 marks)

Modelling

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 15\,000\,(0.85)^x - 1000 \sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(b) Show that $f(x)$ has a root between 19 and 20.

(c) Find $f'(x)$

(d) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(e) Criticise this model with respect to the value of the car as it gets older.

a) $f(10) = 15000 \times 0.85^{10} - 1000 \sin 10$

$= 3497$

$= £3500$

b) $f(19) = 534.11\ldots$

$f(20) = -331.55\ldots$

Change in sign, $f(x)$ is continuous, so root between 19 and 20.

c) $f'(x) = 15000 \times 0.85^x \times \ln 0.85 - 1000 \cos x$

d) $x_1 = 19.5 - \dfrac{f(19.5)}{f'(19.5)}$

$= 19.528$ years

e) For some values of x, the car value becomes negative – this is not possible.

e.g. At 20 years, value is -331.