Linear Transformations with Matrices

1:: Use of matrices to represent linear transformations.

"Determine the matrix that represents the transformation $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2x + y \\ -x \end{pmatrix}$"

2:: Use matrices to represent reflections, rotations (about the origin) and enlargements.

"Describe the geometrical transformation represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$"

3:: Carry out successive transformations using matrix products.

"If $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ describe the transformation represented by the matrix $\mathbf{AB}$."

4:: Use inverse matrices to represent reverse transformations.

A matrix $\begin{pmatrix} 4 & 0 \\ -1 & 2 \end{pmatrix}$ is used to transform a point $A(x, y)$ to $B(5,5)$. Determine the point $A(x, y)$.

Linear Transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$

This chapter is concerned with how we can **use matrices to represent some transformation of a point** $(\boldsymbol{x}, \boldsymbol{y})$ (written as a position vector $\begin{pmatrix} x \\ y \end{pmatrix}$).

Transforming a point $\begin{pmatrix} x \\ y \end{pmatrix}$ simply involves multiplying it by some matrix. From above we can see that multiplying by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the mapping $T: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. We will see how we can use certain matrices to represent certain well-known transformations, e.g. $(x, y) \rightarrow (3x, 3y)$, i.e. an enlargement of scale factor 3 centred about the origin.

$ax + by$ is known as a **linear combination** of x and y (an algebraic form we saw in Pure Year 1 straight line equations). **Each row** of the matrix we're multiplying by provides an instruction of how to generate each dimension of the new coordinate system, in terms of the old dimensions x, y...

e.g. given $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x + y \end{pmatrix}$, the new x value is $2x + 3y$ and the new y value is $x + y$, i.e. linear combinations of the old x and y values.

A function $f(\boldsymbol{a})$, where $\boldsymbol{a}$ is a vector, is linear if it has the following properties:

- $f(k\boldsymbol{a}) = kf(\boldsymbol{a})$ for a constant k, i.e. scaling the original vector scales the image vector.
- $f(\boldsymbol{a} + \boldsymbol{b}) = f(\boldsymbol{a}) + f(\boldsymbol{b})$

It is possible to prove that $f\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = ax + by$ is linear, i.e. satisfies the above restrictions.

1 We can represent a translation, e.g. $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+4 \\ y \end{pmatrix}$ using a matrix.

True | False

2 Matrices can represent transformations which increase or decrease the number of dimensions (e.g. transform a 3D point to get a 2D point).

True | False

3 The origin is unaffected by any linear transformation.

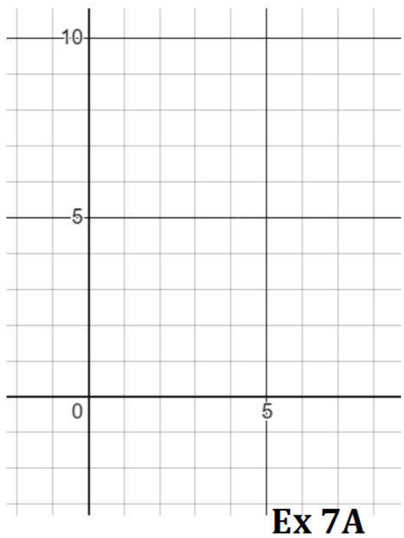
True | False

Find matrices to represent these linear transformations.

a) $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y+x \\ 3x \end{pmatrix}$

b) $V: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -2y \\ 3x+y \end{pmatrix}$

A square has coordinates $(1,1)$, $(3,1)$, $(3,3)$ and $(1,3)$. Find the vertices of the image of S under the transformation given by the matrix $\boldsymbol{M} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Sketch S and the image of S on a coordinate grid.



Ex 7A

Determining a matrix for a transformation

Recall from vectors that $\boldsymbol{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix}0\\1\end{pmatrix}$ are the unit vectors representing the x and y directions. Consider what happens to each when we multiply by a matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$:

Just For Your Interest: i and j are known as the *basis vectors* of the 2D coordinate space because any 2D point can be represented as a linear combination of these basis vectors, i.e. $\begin{pmatrix}x\\y\end{pmatrix} = x\boldsymbol{i} + y\boldsymbol{j}$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \qquad \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

What can we conclude about the columns of a matrix?

"Find a 2×2 matrix that represents a reflection in the y-axis."

Rotation $90°$ about the origin.

Note: Rotations by default are anticlockwise.

Rotation θ about the origin.

Your Turn

Find the matrix representing a reflection in the line $y = x$.

Find the matrix representing a rotation by $270°$.

Edexcel FP1(Old) Jan 2009 Q10

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Describe fully the transformations described by matrix $\mathbf{C}$.

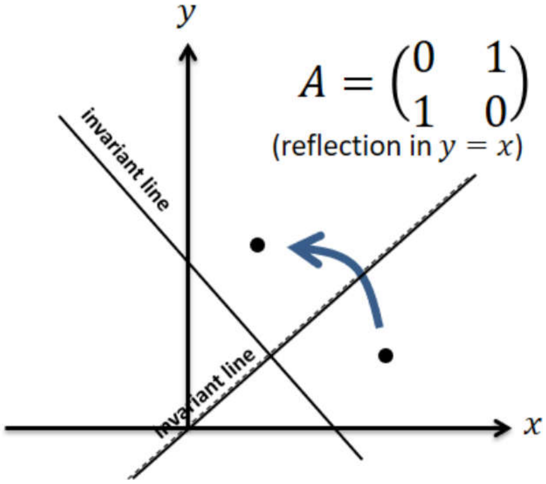
Invariant points and lines

An **invariant point** is one which is unaffected by a transformation.

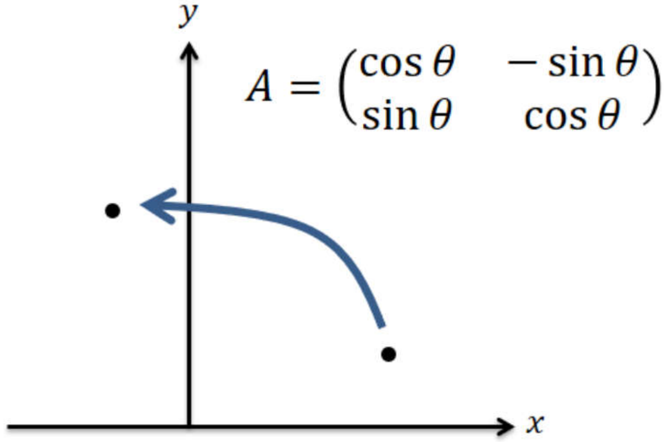
An **invariant line** is when each point on the line transformed to give another point on the same line.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(reflection in $y = x$)



$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Invariant lines: $\boldsymbol{y = x}$

$\boldsymbol{y = -x + k}$

(each line reflects to give the same)

Invariant line: **None! (unless $\theta = 180°$; any straight line through origin will be invariant))**

Invariant point: **(0, 0)**

3.

$$\mathbf{P} = \begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}$$

The matrix **P** represents a linear transformation, T, of the plane.

(a) Describe the invariant points of the transformation T.

(3)

(b) Describe the invariant lines of the transformation T.

(6)

3.

$$\mathbf{P} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

The matrix **P** represents a linear transformation, T, of the plane.

(a) Describe the invariant point of the transformation T.

(b) Describe the invariant lines of the transformation T.

(6)

Enlargements

Describe the effect of the following matrices.

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ represents a stretch scale factor a parallel to the x-axis and a stretch scale factor b parallel to the y-axis. When $a = b$ this represents an enlargement.

Using det(A)

$A(1,1), B(1,2), C(2,2)$ are points on a triangle. The transformation with matrix $\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ is applied to the triangle to produce a new triangle with vertices A', B' and C'.

(a) Determine the coordinates of A', B', C'.

(b) What is the area of triangle ABC?

(c) What is the area of triangle $A'B'C'$?

(d) Determine $\det(M)$. What do you notice?

Area scale factor

We saw in this example that:

✎ Area of image $=$ Area of object $\times\ |\det(\mathbf{M})|$

i.e. the determinant tells us how the area is scaled under the transformation with matrix $\mathbf{M}$.

(The proof of this is not covered here)

Area of Object	Transformation Matrix	Area of Image
4	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
3	$\begin{pmatrix} 2 & 0 \\ 9 & 4 \end{pmatrix}$	
9	$\begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}$	
1	$\begin{pmatrix} -5 & 2 \\ -4 & -2 \end{pmatrix}$	

Edexcel Jan 2011 Q8

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(*a*) Find det **A**.

(1)

The triangle R is transformed to the triangle S by the matrix **A**.
Given that the area of triangle S is 72 square units,

(*c*) find the area of triangle R.

(2)

Ex 7C

More invariant points

1.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

(2)

The transformation T of the plane is represented by the matrix **M**.

The triangle R is transformed to the triangle S by the transformation T.

Given that the area of S is 63 square units,

(b) find the area of R.

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation T.

(2)

Combined Transformations

We know that for a position vector $\boldsymbol{x}$ and a matrix $\mathbf{A}$ representing some transformation, then $\mathbf{A}\boldsymbol{x}$ is the transformed point.

If we wanted to apply a transformation represented by a matrix $\mathbf{A}$ followed by another represented by $\mathbf{B}$, what transformation matrix do we use to represent the combined transformation?

This is because to apply the effect of A followed by B, we have:

$$B(A\boldsymbol{x}) = (BA)\boldsymbol{x}$$

(because matrix multiplication is 'associative'*)

Tip: Ensure that you put these matrices in the right order – the first that gets applied is on the right!

$$\begin{tikzcd} \boldsymbol{x} \arrow[r, bend left, "\mathbf{A}"] \arrow[rr, bend left=60, "\mathbf{BA}"] & {} \arrow[r, bend left, "\mathbf{B}"] & \boldsymbol{y} \end{tikzcd}$$

* A binary operator $\otimes$ is **associative** if $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, i.e. when we multiply matrices, the order in which we multiply them doesn't matter.
Similarly addition on real numbers is associative, e.g. $1 + (2 + 3) = (1 + 2) + 3$.
However subtraction and division are not, e.g. $(16 \div 2) \div 8 \neq 16 \div (2 \div 8)$.

Represent as a single matrix the transformation representing a reflection in the line $y = x$ followed by a stretch on the x axis by a factor of 4.

Represent as a single matrix the transformation representing a rotation $90°$ anticlockwise about the point $(0,0)$ followed by a reflection in the line $y = x$.

What single transformation is this?

Edexcel Jan 2013 Q4

The transformation U, represented by the 2×2 matrix $\mathbf{P}$, is a rotation through 90° anticlockwise about the origin.

(a) Write down the matrix $\mathbf{P}$. **(1)**

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line $y = -x$.

(b) Write down the matrix $\mathbf{Q}$. **(1)**

Given that U followed by V is transformation T, which is represented by the matrix $\mathbf{R}$,

(c) express $\mathbf{R}$ in terms of $\mathbf{P}$ and $\mathbf{Q}$, **(1)**

(d) find the matrix $\mathbf{R}$, **(2)**

(e) give a full geometrical description of T as a single transformation. **(2)**

Ex 7D

More invariant points!

5.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **A**.
(3)

The transformation V, represented by the 2×2 matrix **B**, is a reflection in the line $y = -x$

(b) Write down the matrix **B**.
(1)

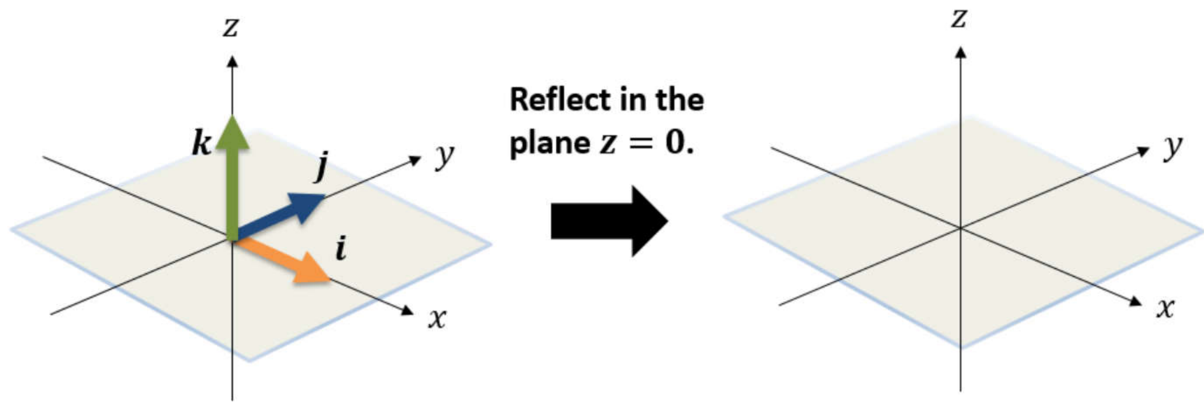
Given that U followed by V is the transformation T, which is represented by the matrix **C**,

(c) find the matrix **C**.
(2)

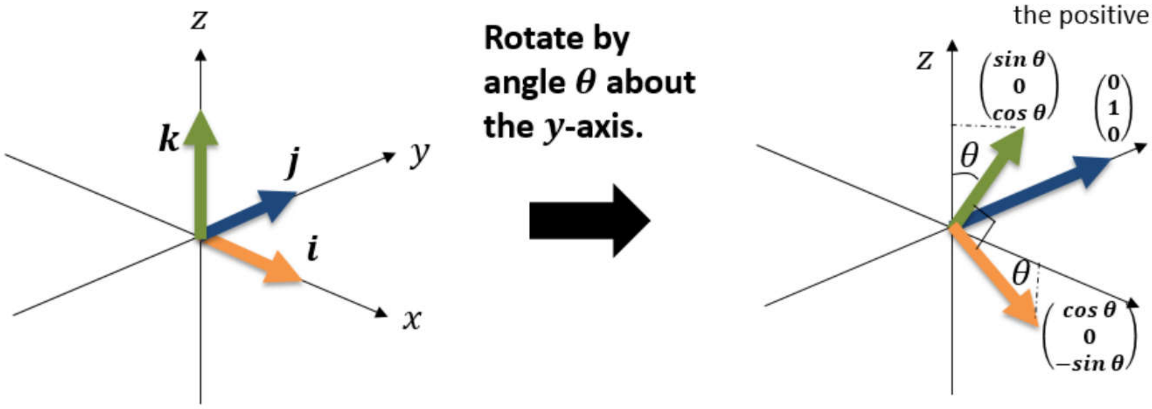
(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T.
(4)

Linear transformations in 3D

We saw earlier that we could determine the matrix corresponding to a transformation by transforming each of the unit vectors (i.e. the axes) and using these as the columns of the matrix. This works in 3D too!



Reminder: The rotation is anticlockwise relative to the positive y axis.



$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

✎ Rotation θ about x-axis: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

Rotation θ about y-axis: $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Rotation θ about z-axis: $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Tip: You can tell whether it's a rotation in the x, y or z axes by looking whether the 1 is in the 1st, 2nd of 3rd row/column.

Ex 7E

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe the transformation represented by $\mathbf{M}$.

(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by $\mathbf{M}$.

Inverse matrices for inverse transformations

✎ Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are column vectors. Then if $\boldsymbol{Ax} = \boldsymbol{y}$, then $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y}$.

The inverse matrix therefore allows us to retrieve the original point/position vector before a transformation.

The triangle T has vertices at A, B and C. The matrix $M = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$ transforms T to the triangle T' with vertices at $A'(4,3)$, $B'(4,10)$ and $C'(-4,-3)$. Determine the coordinates of A, B and C.

Edexcel June 2012 Q9

Ex 7F

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}.$$

(*a*) Find det **M**. **(1)**

Given that

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(*e*) describe fully the single geometrical transformation represented by **A**. **(2)**

The transformation represented by **A** followed by the transformation represented by **B** is equivalent to the transformation represented by **M**.

(*f*) Find **B**. **(4)**

Tip: If $\mathbf{M} = \mathbf{BA}$, make sure you multiply the <u>end</u> of each by $\mathbf{A}^{-1}$:

$$\mathbf{MA}^{-1} = \mathbf{BAA}^{-1}$$
$$\mathbf{MA}^{-1} = \mathbf{BI} = \mathbf{B}$$