Trigonometry and Modelling (Chapter 7)

1a:: Addition Formulae

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

1b:: Double Angle Formulae

"Solve, for $0 \leq x < 2\pi$, the equation
$$2\tan 2y \tan y = 3$$
giving your solutions to 3sf."

3:: The Harmonic Identity: $a\cos x \pm b\sin x$

"Find the maximum value of $2\sin x + \cos x$ and the value of x for which this maximum occurs."

4:: Modelling

"The sea depth of the tide at a beach can be modelled by $x = Rsin\left(\frac{2\pi t}{5} + \alpha\right)$, where t is the hours after midnight..."

Addition Formulae

Addition Formulae allow us to deal with a **sum or difference of angles**.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Trigonometric identities

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$

How it looks in the formula book

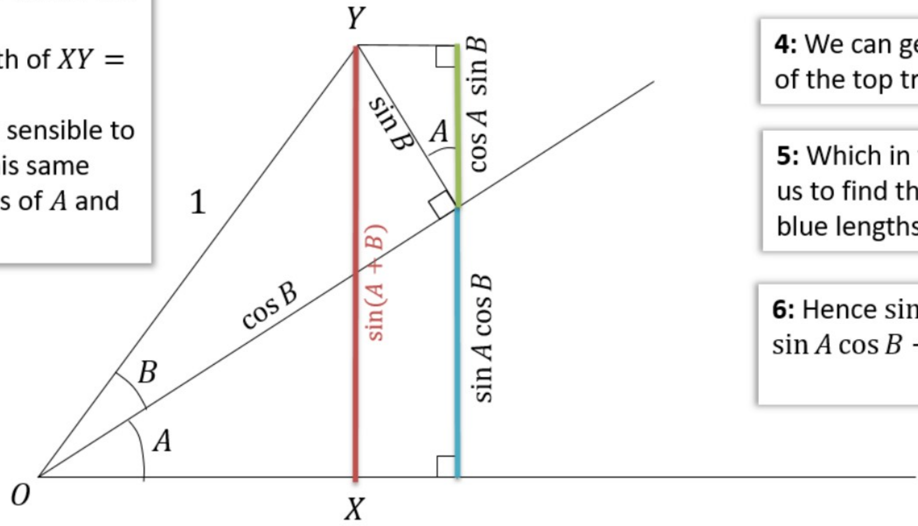
Proof of $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$

(Not needed for exam)

1: Suppose we had a line of length 1 projected an angle of $A+B$ above the horizontal.
Then the length of $XY = \sin(A+B)$
It would seem sensible to try and find this same length in terms of A and B individually.

2: We can achieve this by forming two right-angled triangles.

3: Then we're looking for the combined length of these two lines.



4: We can get the lengths of the top triangle…

5: Which in turn allows us to find the green and blue lengths.

6: Hence $\sin(A+B) = \sin A \cos B + \cos A \sin B$ □

Why is $\sin(A+B)$ not just $\sin(A) + \sin(B)$?

Because $\boldsymbol{sin}$ is a <u>function</u>, not a quantity that can be expanded out like this. It's a bit like how $(\boldsymbol{a}+\boldsymbol{b})^2 \not\equiv \boldsymbol{a}^2 + \boldsymbol{b}^2$.
We can easily disprove it with a counterexample.

Now can you reproduce them without peeking at your notes?

$\mathbf{sin}(\boldsymbol{A}+\boldsymbol{B}) \equiv$

$\mathbf{sin}(\boldsymbol{A}-\boldsymbol{B}) \equiv$

$\mathbf{cos}(\boldsymbol{A}+\boldsymbol{B}) \equiv$

$\mathbf{cos}(\boldsymbol{A}-\boldsymbol{B}) \equiv$

$\mathbf{tan}(\boldsymbol{A}+\boldsymbol{B}) \equiv$

$\mathbf{tan}(\boldsymbol{A}-\boldsymbol{B}) \equiv$

How to memorise:

First notice that for all of these the first thing on the RHS is the same as the first thing on the LHS!

- For sin, the operator in the middle is the same as on the LHS.
- For cos, it's the opposite.
- For tan, it's the same in the numerator, opposite in the denominator.

- For sin, we mix sin and cos.
- For cos, we keep the cos's and sin's together.

Express the following as a single sine, cosine or tangent:

$\cos 110°\cos 15° - \sin 15°\sin 110°$

$\sin 5x \cos 2x - \cos 5x \sin 2x$

$\sin 20°\cos 35° + \cos 20°\sin 35°$

$\sin 10°\sin 15° - \cos 10°\cos 15°$

$$\frac{\tan\frac{1}{2}\theta + \tan\frac{1}{4}\theta}{1 - \tan\frac{1}{2}\theta\tan\frac{1}{4}\theta}$$

$$\frac{\tan\frac{5\pi}{12} - \tan\frac{\pi}{6}}{1 + \tan\frac{5\pi}{12}\tan\frac{\pi}{6}}$$

Ex 7A Q5-9
Ex 7B Q2

Proof of tan identities

Edexcel C3 Jan 2012 Q8

(a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(4)

Given that $2\sin(x+y) = 3\cos(x-y)$ express $\tan x$ in terms of $\tan y$.

Calculate the exact value of $\tan x$ if

$$\tan(x+60) = 5$$

Uses of Addition Formulae

Using a suitable angle formulae, show that $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$.

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$, find the value of: (a) $\cos(A - B)$ (b) $\tan(A + B)$

Your Turn

Without using a calculator, determine the exact value of:

a) $\cos(75°)$

b) $\tan(75°)$

Challenging question

Edexcel June 2013 Q3

Given that

$$2\cos(x+50)° = \sin(x+40)°.$$

(*a*) Show, without using a calculator, that

$$\tan x° = \frac{1}{3}\tan 40°.$$

Ex 7B
Q4-10

Double Angle Formulae

Double-angle formulae allow you to halve the angle within a trig function.

$$\sin(2A) \equiv 2\sin A\cos A$$
$$\cos(2A) \equiv \cos^2 A - \sin^2 A$$
$$\equiv 2\cos^2 A - 1$$
$$\equiv 1 - 2\sin^2 A$$

Tip: The way I remember what way round these go is that the cos on the RHS is 'attracted' to the cos on the LHS, whereas the sin is pushed away.

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

These are all easily derivable by just setting $A = B$ in the compound angle formulae. e.g.

$$\sin(2A) = \sin(A+A)$$
$$= \sin A\cos A + \cos A\sin A$$
$$= 2\sin A\cos A$$

Use the double-angle formulae to write each of the following as a single trigonometric ratio.

a) $\cos^2 50° - \sin^2 50°$

b) $\frac{2\tan\left(\frac{\pi}{6}\right)}{1-\tan^2\left(\frac{\pi}{6}\right)}$

c) $\frac{4\sin 70°}{\sec 70°}$

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Use the double-angle formulae to write each of the following as a single trigonometric ratio.

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

d) $\sqrt{\cos 2x + 1}$

e) $\sin^3 x\cos^3 x$

f) $8\cos x\sin x\cos 2x$

g) $1 - 4\sin^2 x + 4\sin^4 x$

Given that $x = 3\sin\theta$ and $y = 3 - 4cos2\theta$, eliminate θ and express y in terms of x.

Note: This question is an example of turning a set of **parametric equations** into a single **Cartesian** one. You will cover this in the next chapter.

Given that $\cos x = \frac{3}{4}$ and x is acute, find the exact value of
(a) $\sin 2x$ (b) $\tan 2x$

Solving Trigonometric Equations

Using the Addition Formulae

Solve $4\cos(x - 30) = 8\sqrt{2}\sin x$ for $0 \leq x \leq 360°$.

Note: The arguments do not match, so first we must deal with this!

Using the Double Angle Formulae

Solve $3\cos 2x - \cos x + 2 = 0$ for $0 \leq x \leq 360°$.

Note: The arguments do not match, so first we must deal with this!

Solve $2\tan 4y \tan 2y = 3$ for $0 \leq y < \pi$, giving your answer to 2dp.

Much trickier example

By noting that $3A = 2A + A$, :

a) Show that $\sin(3A) = 3\sin A - 4\sin^3 A$.

b) Hence or otherwise, solve, for $0 < \theta < 2\pi$, the equation $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$

Exam Note: A question pretty much just like this came up in an exam once.

Ex 7D

Edexcel C3 Jan 2013 Q6

6. (i) Without using a calculator, find the exact value of

$$(\sin 22.5° + \cos 22.5°)^2.$$

You must show each stage of your working.

(5)

(ii) (*a*) Show that $\cos 2\theta + \sin\theta = 1$ may be written in the form

$$k\sin^2\theta - \sin\theta = 0, \text{ stating the value of } k.$$

(2)

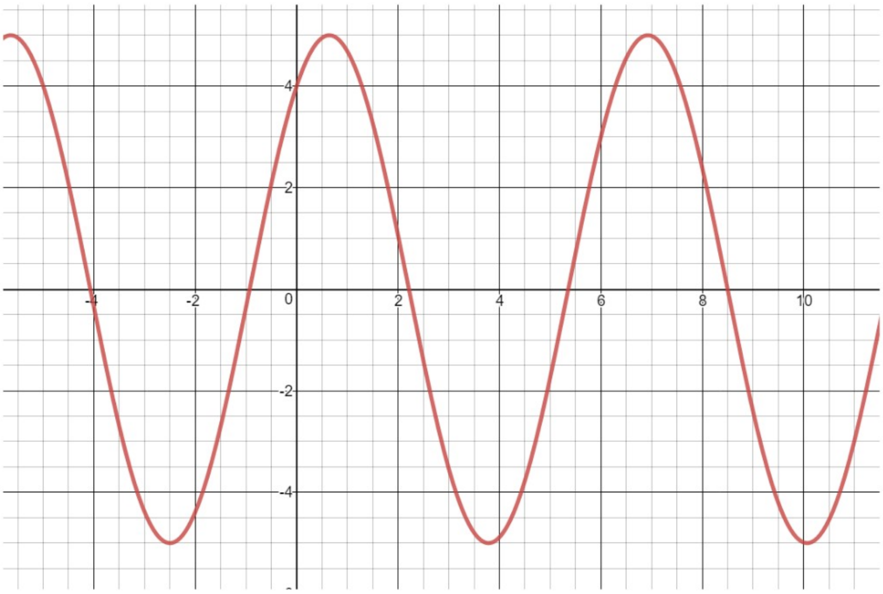
(*b*) Hence solve, for $0 \leq \theta < 360°$, the equation

$$\cos 2\theta + \sin\theta = 1.$$

(4)

The Harmonic Identity

Here is the graph of y= 3sin$x$ + 4cos$x$



What do you notice?

Q Put $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ giving α in degrees to 1dp.

STEP 1: Expanding:

STEP 2: Comparing coefficients:

STEP 3: Using the fact that $R^2\sin^2\alpha + R^2\cos^2\alpha = R^2$:

If $R\cos\alpha = 3$ and $R\sin\alpha = 4$ then $R^2\cos^2\alpha = 3^2$ and $R^2\sin^2\alpha = 4^2$.
$R^2\sin^2\alpha + R^2\cos^2\alpha = 3^2 + 4^2$
$R^2(\sin^2\alpha + \cos^2\alpha) = 3^2 + 4^2$
$R^2 = 3^2 + 4^2$
$R = \sqrt{3^2 + 4^2}$
(You can write just the last line in exams)

STEP 4: Using the fact that $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$:

STEP 5: Put values back into original expression.

Express each of the following in the given form, where $R > 0$ and $0 < \alpha < 90$.

Give the exact value of R and the value of α correct to 1 decimal place.

a $5\cos x - 12\sin x$

b $4\cos 2x + 4\sin 2x$

Express each of the following in the given form, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the exact value of R and the value of α correct to 2 decimal places where appropriate.

a $\sin x - 2\cos x$

b $8\sin 3x + 6\cos 3x$

Solomon Worksheet F Q1-3
(Ex7E)

Harmonic Identity Questions

Harmonic Identity Worksheet F.pdf
Harmonic Identity F answers.pdf

1 Express each of the following in the form $R\cos(x-\alpha)°$, where $R>0$ and $0<\alpha<90$.
Give the values of R and α correct to 1 decimal place where appropriate.

a $\cos x° + \sin x°$ **b** $3\cos x° + 4\sin x°$

c $2\sin x° + \cos x°$ **d** $\cos x° + \sqrt{3}\sin x°$

2 Express each of the following in the given form, where $R>0$ and $0<\alpha<90$.
Give the exact value of R and the value of α correct to 1 decimal place.

a $5\cos x° - 12\sin x°$, $R\cos(x+\alpha)°$ **b** $4\sin x° + 2\cos x°$, $R\sin(x+\alpha)°$

c $\sin x° - 7\cos x°$, $R\sin(x-\alpha)°$ **d** $8\cos 2x° - 15\sin 2x°$, $R\cos(2x+\alpha)°$

3 Express each of the following in the given form, where $R>0$ and $0<\alpha<\frac{\pi}{2}$.
Give the exact value of R and the value of α correct to 2 decimal places where appropriate.

a $3\sin x - 2\cos x$, $R\sin(x-\alpha)$ **b** $3\cos x + \sqrt{3}\sin x$, $R\cos(x-\alpha)$

c $8\sin 3x + 6\cos 3x$, $R\sin(3x+\alpha)$ **d** $\cos x + \frac{1}{2}\sin x$, $R\cos(x-\alpha)$

4 Find the maximum value that each expression can take and the smallest positive value of x, in degrees, for which this occurs.

a $24\sin x - 7\cos x$ **b** $4\cos 2x + 4\sin 2x$

c $3\cos x - 5\sin x$ **d** $5\sin 3x + \cos 3x$

5 **a** Express $3\sin x° - 3\cos x°$ in the form $R\sin(x-\alpha)°$, where $R>0$ and $0<\alpha<90$.

b Hence, describe two transformations that would map the graph of $y=\sin x°$ onto the graph of $y = 3\sin x° - 3\cos x°$.

6 By first expressing each curve in an appropriate form, sketch each of the following for x in the interval $0 \le x \le 360°$, showing the coordinates of any turning points.

a $y = 12\cos x + 5\sin x$ **b** $y = \sin x - 2\cos x$

c $y = 2\sqrt{3}\cos x - 6\sin x$ **d** $y = 9\sin x + 4\cos x$

7 **a** Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$.

b Solve the equation $\sqrt{3}\cos x - \sin x = 1$ for x in the interval $0 \le x \le 2\pi$, giving your answers in terms of π.

8 Solve each equation for x in the interval $0 \le x \le 2\pi$, giving your answers to 2 decimal places.

a $6\sin x + 8\cos x = 5$ **b** $2\cos x - 2\sin x = 1$

c $7\sin x - 24\cos x - 10 = 0$ **d** $3\cos x + \sin x + 1 = 0$

e $\cos 2x + 4\sin 2x = 3$ **f** $5\sin x - 8\cos x + 7 = 0$

9 Solve each equation for x in the interval $-180° \le x \le 180°$, giving your answers to 1 decimal place where appropriate.

a $\sin x + \cos x = 1$ **b** $4\cos x - \sin x + 2 = 0$

c $\cos\frac{x}{2} + 5\sin\frac{x}{2} - 4 = 0$ **d** $6\sin x = 5 - 3\cos x$

Solving Equations by using the Harmonic Identity

Q

Put $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $0 < \alpha < 90°$
Hence solve, for $0 < \theta < 360$, the equation $2\cos\theta + 5\sin\theta = 3$

Finding Maxima/Minima by using the Harmonic Identity

Q

(Without using calculus), find the maximum value of $12\cos\theta + 5\sin\theta$, and give the smallest positive value of θ at which it arises.

More Maxima/Minima

What is the maximum value of the expression and determine the **smallest positive** value of θ (in degrees) at which it occurs.

Expression	Maximum	(Smallest) $\boldsymbol{\theta}$ at max
$20\sin\theta$		
$5 - 10\sin\theta$		
$3\cos(\theta + 20°)$		
$\dfrac{2}{10 + 3\sin(\theta - 30)}$		

Hint: investigate how the expression behaves when you set sin(x) or cos(x) to 1 or -1

Ex 7E
Q 9, 14

Edexcel C3 Jan 2013 Q4

4. (*a*) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

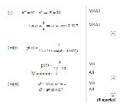
(*b*)
$$\mathrm{p}(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \quad 0 \le \theta \le 2\pi.$$

Calculate

(i) the maximum value of $\mathrm{p}(\theta)$,

(ii) the value of θ at which the maximum occurs.

(4)



(a) Express $\sin\theta - 2\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

Give the exact value of R and the value of α, in radians, to 3 decimal places.

(3)

$$\mathrm{M}(\theta) = 40 + (3\sin\theta - 6\cos\theta)^2$$

(b) Find

(i) the maximum value of $\mathrm{M}(\theta)$,

(ii) the smallest value of θ, in the range $0 < \theta \leqslant 2\pi$, at which the maximum value of $\mathrm{M}(\theta)$ occurs.

(3)

$$\mathrm{N}(\theta) = \frac{30}{5 + 2(\sin 2\theta - 2\cos 2\theta)^2}$$

(c) Find

(i) the maximum value of $\mathrm{N}(\theta)$,

(ii) the largest value of θ, in the range $0 < \theta \leqslant 2\pi$, at which the maximum value of $\mathrm{N}(\theta)$ occurs.

(3)

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

9 (a)	$R = \sqrt{5}$	B1	
	$\tan\alpha = 2 \Rightarrow \alpha =$ awrt 1.107	M1A1	[3]
(b)(i)	$40 + 9R^2 = 85$	M1A1	
(ii)	$\theta = \frac{\pi}{2} + 1.107 \Rightarrow \theta =$ awrt 2.68	B1	[3]
(c)(i)	6	B1	
(ii)	$2\theta - 1.107 = 3\pi \Rightarrow \theta =$ awrt 5.27	M1A1	[3]
		(9 marks)	

(*a*) Express $2 \sin \theta - 4 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

Give the value of α to 3 decimal places.

(3)

$$H(\theta) = 4 + 5(2\sin 3\theta - 4\cos 3\theta)^2$$

Find

(*b*) (i) the maximum value of $H(\theta)$,

(ii) the smallest value of θ, for $0 \le \theta \le \pi$, at which this maximum value occurs.

(3)

Find

(*c*) (i) the minimum value of $H(\theta)$,

(ii) the largest value of θ, for $0 \le \theta \le \pi$, at which this minimum value occurs.

(3)

9.(a)	$R = \sqrt{20}$	B1	
	$\tan\alpha = \frac{4}{2} \Rightarrow \alpha =$ awrt 1.107	M1A1	(3)
(b)(i)	$4 + 5R^2 = 104$	B1ft	
(ii)	$3\theta - 1.107 = \frac{\pi}{2} \Rightarrow \theta =$ awrt 0.89	M1A1	(3)
(c)(i)	4	B1	
(ii)	$3\theta - 1.107 = 2\pi \Rightarrow \theta =$ awrt 2.46	M1A1	(3)
		(9 marks)	

Proving Trigonometric Identities

Double Angle

$\sin 2x = 2\sin x\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 2\cos^2 x - 1$ $\cos 2x = 1 - 2\sin^2 x$
$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Prove that $\tan 2\theta \equiv \frac{2}{\cot\theta - \tan\theta}$

Prove that $\frac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

Prove that $\cot 2x + cosec\ 2x \equiv \cot x$

By writing $\cos x = \cos\left(2 \times \frac{x}{2}\right)$ or otherwise, prove the identity $\frac{1-\cos x}{1+\cos x} \equiv \tan^2\left(\frac{x}{2}\right)$

1 Prove the following identities.

a $\dfrac{\cos 2A}{\cos A + \sin A} \equiv \cos A - \sin A$

b $\dfrac{\sin B}{\sin A} - \dfrac{\cos B}{\cos A} \equiv 2\operatorname{cosec} 2A \sin(B - A)$

c $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$

d $\dfrac{\sec^2\theta}{1 - \tan^2\theta} \equiv \sec 2\theta$

e $2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \equiv \sin 2\theta$

f $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} \equiv 2$

g $\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv 2\sin\theta$

h $\dfrac{\sec\theta - 1}{\sec\theta + 1} \equiv \tan^2\dfrac{\theta}{2}$

Modelling with Trigonometry

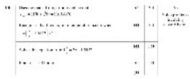
(a) Express $5\cos\theta - 8\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \pi$. Write R in surd form and give the value of α correct to 4 decimal places.

(4 marks)

The temperature of a kiln, T °C, used to make pottery can be modelled by the equation $T = 1100 + 5cos\left(\frac{x}{3}\right) - 8sin\left(\frac{x}{3}\right), 0 \leq x \leq 72$, where x is the time in hours since the pottery was placed in the kiln.

(b) Calculate the maximum value of T predicted by this model and the value of x, to 2 decimal places, when this maximum first occurs.

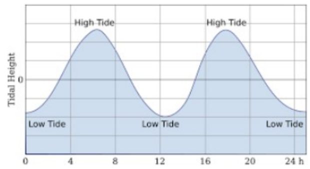
(4 marks)



(c) Calculate the times during the first 24 hours when the temperature is predicted, by this model, to be exactly 1097 °C.

(4 marks)

(*a*) Express $2 \sin \theta - 1.5 \cos \theta$ in the form $R \sin (\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.



Give the value of α to 4 decimal places.

(3)

(*b*) (i) Find the maximum value of $2 \sin \theta - 1.5 \cos \theta$.

(ii) Find the value of θ, for $0 \leq \theta < \pi$, at which this maximum occurs.

(3)

Tom models the height of sea water, H metres, on a particular day by the equation

$$H = 6 + 2 \sin \left(\frac{4\pi t}{25} \right) - 1.5 \cos \left(\frac{4\pi t}{25} \right), \quad 0 \leq t < 12,$$

where t hours is the number of hours after midday.

(*c*) Calculate the maximum value of H predicted by this model and the value of t, to 2 decimal places, when this maximum occurs.

(3)

(*d*) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres.

(6)

(*e*) for how long is the boat above 7m?

Ex 7G
Q6, 7
Mixed Ex7
Q26, 27

(a) Express $10\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$
Give the exact value of R and give the value of α, in degrees, to 2 decimal places.
(3)

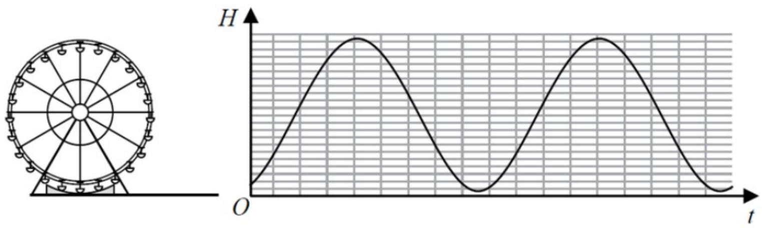


Figure 3

The height above the ground, H metres, of a passenger on a Ferris wheel t minutes after the wheel starts turning, is modelled by the equation

$$H = a - 10\cos(80t)° + 3\sin(80t)°$$

where a is a constant.

Figure 3 shows the graph of H against t for two complete cycles of the wheel.

Given that the initial height of the passenger above the ground is 1 metre,

(b) (i) find a complete equation for the model,

(ii) hence find the maximum height of the passenger above the ground.
(2)

(c) Find the time taken, to the nearest second, for the passenger to reach the maximum height on the second cycle.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)
(3)

It is decided that, to increase profits, the speed of the wheel is to be increased.

(d) How would you adapt the equation of the model to reflect this increase in speed?
(1)