Edexcel A Level Further Maths: Core Pure

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3.3 Maclaurin Series

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3.3.1 Maclaurin Series

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Maclaurin Series

What is a Maclaurin Series?

- A Maclaurin series is a way of representing a function as an infinite sum of increasing integer powers of x ($x^1, x^2, x^3,$ etc.)
  - If all of the infinite number of terms are included, then the Maclaurin series is exactly equal to the original function
  - If we **truncate** (i.e., shorten) the Maclaurin series by stopping at some particular power of x, then the Maclaurin series is only an approximation of the original function
- A truncated Maclaurin series will always be exactly equal to the original function for $x = 0$
- In general, the approximation from a truncated Maclaurin series becomes less accurate as the value of x moves further away from zero
- The accuracy of a truncated Maclaurin series approximation can be improved by including more terms from the complete infinite series
  - So, for example, a series truncated at the x^7 term will give a more accurate approximation than a series truncated at the x^3 term

How do I find the Maclaurin series of a function 'from first principles'?

- Use the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

- This formula is in your exam formula booklet
- STEP 1: Find the values of $f(0),\ f'(0),\ f''(0),$ etc. for the function
  - An exam question will specify how many terms of the series you need to calculate (for example, "up to and including the term in x^4")
  - You may be able to use your calculator to find these values directly without actually having to find all the necessary derivatives of the function first
- STEP 2: Put the values from Step 1 into the general Maclaurin series formula
- STEP 3: Simplify the coefficients as far as possible for each of the powers of x

Is there a connection Maclaurin series expansions and binomial theorem series expansions?

- Yes there is!
- For a function like $(1 + x)^n$ the binomial theorem series expansion is **exactly the same** as the Maclaurin series expansion for the same function
  - So unless a question specifically tells you to use the general Maclaurin series formula, you can use the binomial theorem to find the Maclaurin series for functions of that type

- Or if you've forgotten the binomial series expansion formula for $(1+x)^n$ where n is not a positive integer, you can find the binomial theorem expansion by using the general Maclaurin series formula to find the Maclaurin series expansion

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Worked example

Use the Maclaurin series formula to find the Maclaurin series for $f(x) = \sqrt{1+2x}$ up to and including the term in x^4.

$$f(x) = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$$

STEP 1: $f(0) = 1 \quad f'(0) = 1 \quad f''(0) = -1$

$f'''(0) = 3 \quad f^{(4)}(0) = -15$

STEP 2: $f(x) = 1 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(3) + \frac{x^4}{4!}(-15) + \ldots$

STEP 3: Up to the x^4 term,

$$\boxed{\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4}$$

Note: This is the same as the binomial theorem expansion of $(1+2x)^{\frac{1}{2}}$

Maclaurin Series of Standard Functions

Is there an easier way to find the Maclaurin series for standard functions?

- Yes there is!
- The following Maclaurin series expansions of standard functions are contained in your exam formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \frac{x^r}{r!} + \ldots \text{ valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \text{ valid for } -1 < x \le 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \text{ valid for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \text{ valid for all } x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \text{ valid for } -1 \le x \le 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1 \times 2 \times \ldots \times r}x^r + \ldots \text{ valid for } |x| < 1$$

- Unless a question specifically asks you to derive a Maclaurin series using the general Maclaurin series formula, you can use those standard formulae from the exam formula booklet in your working

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Maclaurin Series of Compound Functions

How can I find the Maclaurin series for a composite function?

- A **composite function** is a 'function of a function' or a 'function within a function'
  - For example sin(2*x*) is a composite function, with 2*x* as the 'inside function' which has been put into the simpler 'outside function' sin *x*
  - Similarly e^{x^2} is a composite function, with x^2 as the 'inside function' and e^x as the 'outside function'
- To find the Maclaurin series for a composite function:
  - STEP 1: Start with the Maclaurin series for the basic 'outside function'
    - Usually this will be one of the 'standard functions' whose Maclaurin series are given in the exam formula booklet
  - STEP 2: Substitute the 'inside function' every place that *x* appears in the Maclaurin series for the 'outside function'
    - So for sin(2*x*), for example, you would substitute 2*x* everywhere that *x* appears in the Maclaurin series for sin *x*
  - STEP 3: Expand the brackets and simplify the coefficients for the powers of *x* in the resultant Maclaurin series
- This method can theoretically be used for quite complicated 'inside' and 'outside' functions
  - On your exam, however, the 'inside function' will usually not be more complicated than something like *kx* (for some constant *k*) or x^n (for some constant power *n*)

How can I find the Maclaurin series for a product of two functions?

- To find the Maclaurin series for a product of two functions:
  - STEP 1: Start with the Maclaurin series of the individual functions
    - For each of these Maclaurin series you should only use terms up to an appropriately chosen power of *x* (see the worked example below to see how this is done!)
  - STEP 2: Put each of the series into brackets and multiply them together
    - Only keep terms in powers of *x* up to the power you are interested in
  - STEP 3: Collect terms and simplify coefficients for the powers of *x* in the resultant Maclaurin series

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Worked example

a) Find the Maclaurin series for the function $f(x) = \ln(1 + 3x)$, up to and including the term in x^4.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1}\frac{x^r}{r} + \dots \quad (-1 < x \leq 1)$$

[from exam formula booklet]

STEP 1 : $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

STEP 2 : $\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \dots$

STEP 3 : $\boxed{\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \dots}$

b) Find the Maclaurin series for the function $g(x) = e^x \sin x$, up to and including the term in x^4.

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$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{(for all } x\text{)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{(for all } x\text{)}$$

[from exam formula booklet]

STEP 1: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ ← Higher powers of x here will give powers higher than 4 when multiplied by the sin x series.

$\sin x = x - \frac{x^3}{6} + \ldots$ ← Don't need terms in powers of x higher than 4

STEP 2: $e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$

$$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^3}{6} - \frac{x^4}{6} \underbrace{- \cancel{\frac{x^5}{12}} - \cancel{\frac{x^6}{36}}}_{\text{Discard terms for powers higher than 4}} + \ldots$$

Note that the x^4 terms cancel out

STEP 3: $\boxed{e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots}$