Chapter 8 - Proof by Induction

We will use Proof of Induction for 4 different types of proof:

1 Summation Proofs

"Show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$."

2 Divisibility Proofs

"Prove that $n^3 - 7n + 9$ is divisible by 3 for all $n \in \mathbb{Z}^+$."

3 Matrix Proofs

"Prove that $\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$."

4 *Recurrence Relation Proofs*

$$u_{n+2} = 5u_{n+1} - 6u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3^n - 2^n$$

Note: Recall that $\mathbb{Z}$ is the set of all integers, and $\mathbb{Z}^+$ is the set of all positive integers. Thus $\mathbb{N} = \mathbb{Z}^+$ (where $\mathbb{N}$ is the set of 'natural' numbers).

We can often use **proof by induction** whenever we want to show some property holds for all integers (usually positive) up to infinity.

Show that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

We could show it's true for certain examples:

But what is the problem with just trying some examples?

$n = 3 \quad \Rightarrow \quad LHS = 1 + 2 + 3 = 6$

$$RHS = \frac{1}{2} \times 3 \times 4 = 6$$

How Proof by Induction works…

Case number

$n=1$	$n=2$	$n=3$	$n=4$	…	$n=k$	$n=k+1$	$n=k+2$

Step 1: **Base Case**
Step 2: **Assumption**
Step 3: **Inductive Case**
Step 4: **Conclusion**

Type 1: Summation Proofs

Step 1: **Basis:**	Prove the general statement is true for $n = 1$.
Step 2: **Assumption:**	Assume the general statement is true for $n = k$.
Step 3: **Inductive:**	Show that the general statement is then true for $n = k + 1$.
Step 4: **Conclusion:**	The general statement is then true for all positive integers n.

Show that $\sum_{r=1}^{n}(2r - 1) = n^2$ for all $n \in \mathbb{N}$.

More on the 'Conclusion Step'

"Since true for $n = 1$ and if true for $n = k$, true for $n = k + 1$,
$\therefore$ true for all n."

I lifted this straight from a mark scheme, hence use this exact wording!
The mark scheme specifically says:

(For method mark)
Any 3 of these seen anywhere in the proof:

- **"true for $n = 1$"**
- **"assume true for $n = k$"**
- **"true for $n = k + 1$"**
- **"true for all n/positive integers"**

Prove by induction that for all positive integers n,

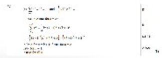
$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Tip: Write out what you are aiming for

Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r2^r = 2(1 + (n-1)2^n)$$

Your Turn



8. (*a*) Prove **by induction** that, for any positive integer n,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

(5)

Ex 8A

Type 2a: Divisibility Proofs

Step 1: **Basis:**	Prove the general statement is true for $n = 1$.
Step 2: **Assumption:**	Assume the general statement is true for $n = k$.
Step 3: **Inductive:**	Show that the general statement is then true for $n = k + 1$.
Step 4: **Conclusion:**	The general statement is then true for all positive integers n.

Prove by induction that $3^{2n} + 11$ is divisible by 4 for all positive integers n.

Tip: Use $f(k+1) - f(k)$

Prove by induction that $n^3 - 7n + 9$ is divisible by 3 for all positive integers n.

Type 2b: Divisibility Proofs - with a 'twist'

Prove by induction that $8^n - 3^n$ is divisible by 5.

Prove by induction that for all positive integers n, $11^{n+1} + 12^{2n-1}$ is divisible by 133.

Ex 8B

Type 3: Matrix Proofs

Prove by induction that $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.

Prove by induction that $\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}^n = \begin{pmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.

Type 4a: Recurrence Relation Proofs - 1 assumption

Given that $u_{n+1} = 3u_n + 4$
and that $u_1 = 1$, prove by induction that $u_n = 3^n - 2$

Type 4b: Recurrence Relation Proofs - 2 assumptions

A sequence of numbers is defined by

$$u_1 = 1 \qquad u_2 = 5$$

$$u_{n+2} = 5u_{n+1} - 6u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3^n - 2^n$$

(6)

1 Given that $u_{n+1} = 5u_n + 4$, $u_1 = 4$, prove by induction that $u_n = 5^n - 1$.

2 Given that $u_{n+1} = 2u_n + 5$, $u_1 = 3$, prove by induction that $u_n = 2^{n+2} - 5$.

3 Given that $u_{n+1} = 5u_n - 8$, $u_1 = 3$, prove by induction that $u_n = 5^{n-1} + 2$.

4 Given that $u_{n+1} = 3u_n + 1$, $u_1 = 1$, prove by induction that $u_n = \frac{3^n - 1}{2}$.

5 Given that $u_{n+2} = 5u_{n+1} - 6u_n$, $u_1 = 1$, $u_2 = 5$ prove by induction that $u_n = 3^n - 2^n$.

6 Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = -1$, $u_2 = 0$, prove by induction that $u_n = (n-2)3^{n-1}$.

7 Given that $u_{n+2} = 7u_{n+1} - 10u_n$, $u_1 = 1$, $u_2 = 8$, prove by induction that $u_n = 2(5^{n-1}) - 2^{n-1}$.

8 Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = 3$, $u_2 = 36$, prove by induction that $u_n = (3n-2)3^n$.

Exam Questions

6. (a) Prove by induction that, for all $n \in \mathbb{Z}^+$

$$f(n) = n^5 + 4n$$

is divisible by 5

(6)

(b) Show that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$

(1)

(c) Hence prove that $f(n)$ is divisible by 5 for all $n \in \mathbb{Z}$

(2)

8. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

(6)

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

2. Prove by induction that for all positive integers n,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

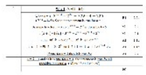
(6)

6. Prove by induction, that for all positive integers n,

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

(6)

6. Prove by induction that for all positive integers n



$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)