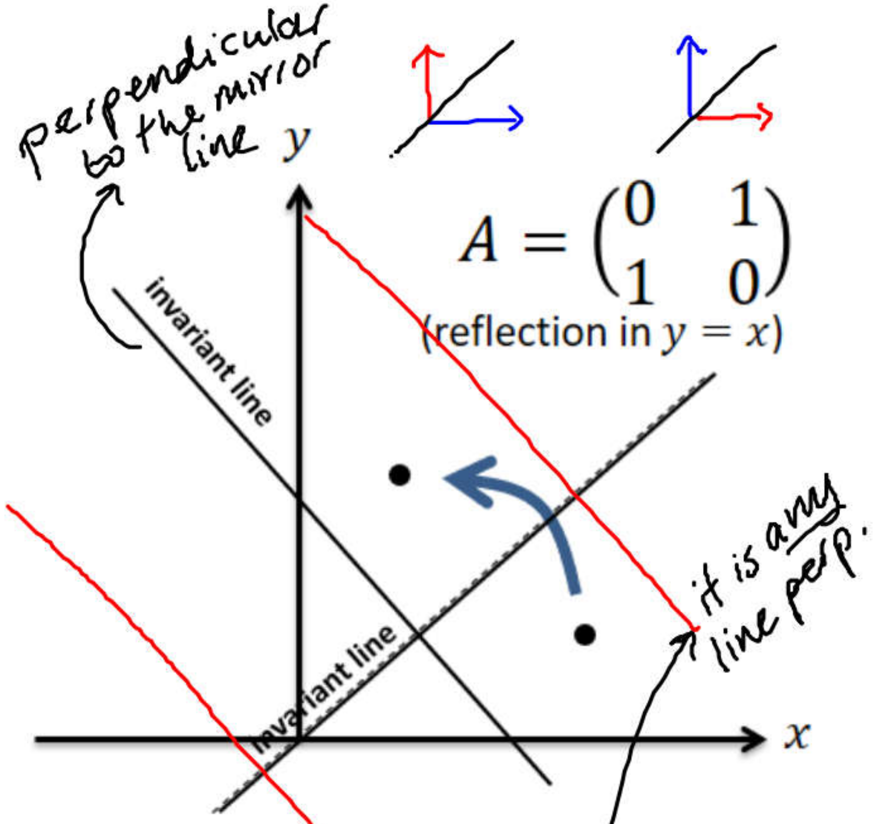
Invariant points and lines

An **invariant point** is one which is unaffected by a transformation.

An **invariant line** is when each point on the line transformed to give another point on the same line.



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

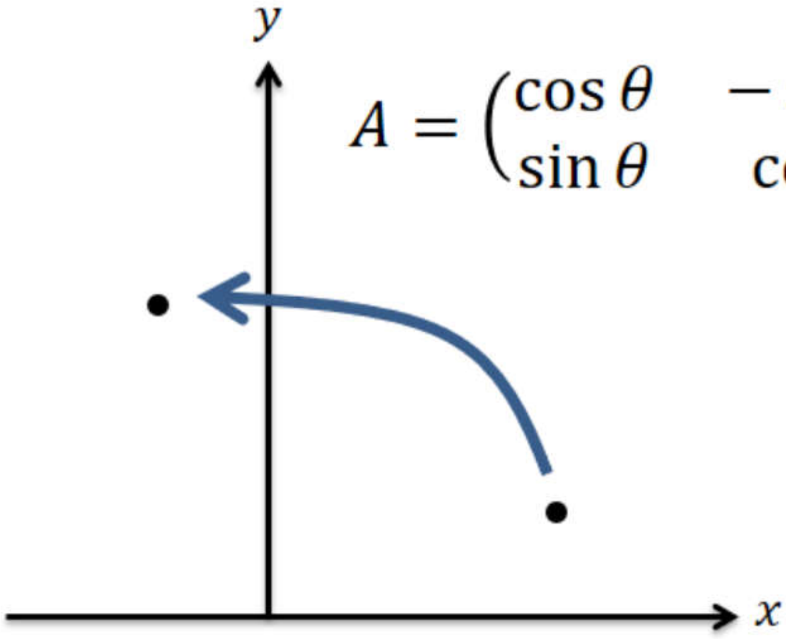
(reflection in $y = x$)

Invariant lines: $\boldsymbol{y = x}$

$\boldsymbol{y = -x + k}$

(each line reflects to give the same)

inv: points lie on $y=x$

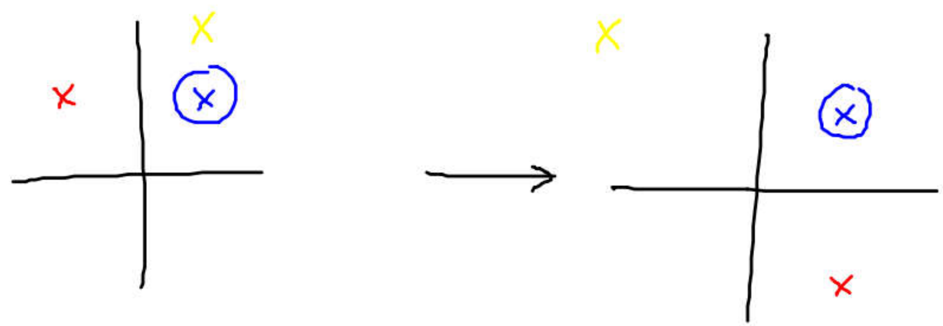


$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

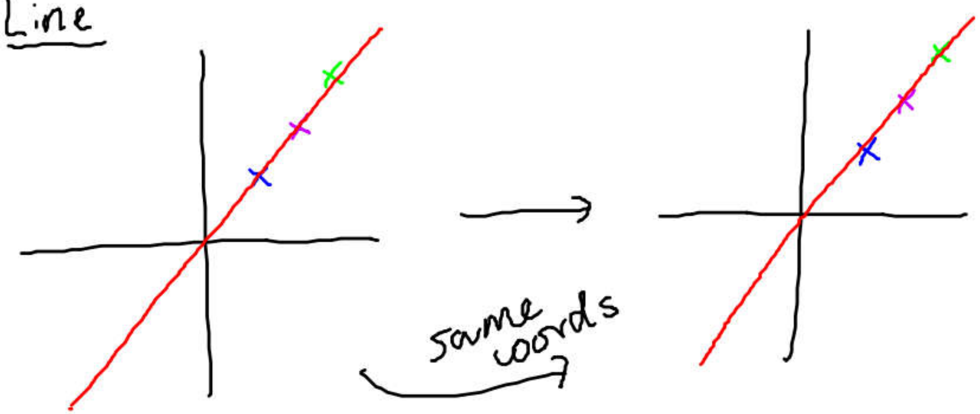
Invariant line: **None! (unless $\boldsymbol{\theta = 180°}$; any straight line through origin will be invariant))**

Invariant point: **(0, 0)**

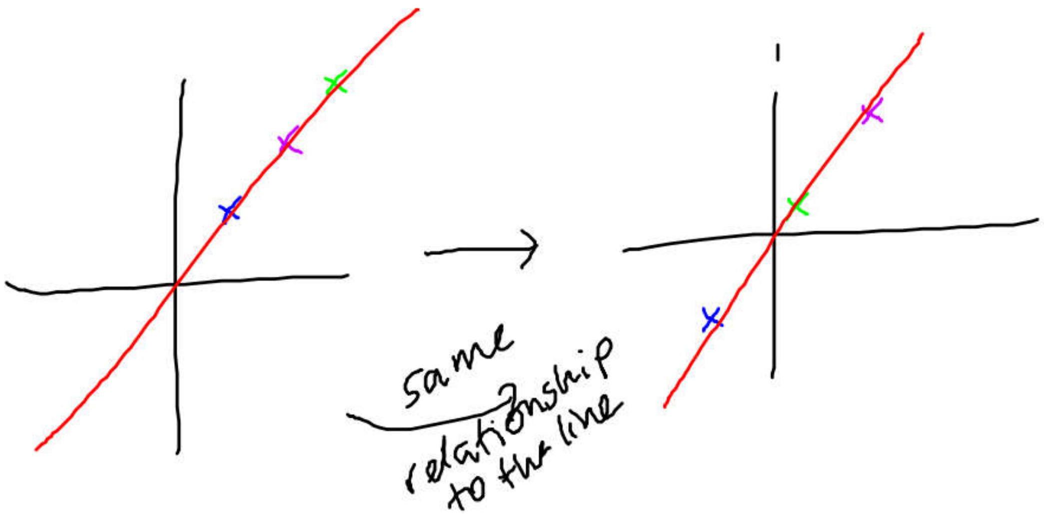
Point



Line



A line made up of invariant points



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$y = x$

$x = y$

$$(x, y) \xrightarrow{\text{invariant}} (x, y)$$

3.

$$\mathbf{P} = \begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}$$

The matrix **P** represents a linear transformation, T, of the plane.

(a) Describe the invariant points of the transformation T.

(3)

(b) Describe the invariant lines of the transformation T.

(6)

a) $(0,0)$ is invariant.

$$\begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+3y \\ 4x+7y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$3x + 3y = x$

$3y = -2x$

$y = -\frac{2}{3}x$

$4x + 7y = y$

$6y = -4x$

$y = -\frac{2}{3}x$

All points on the line $y = -\frac{2}{3}x$ are invariant.

b) $y = -\frac{2}{3}x$ is an invariant line.

Other invariant lines will be of the form $y = mx + c$

$$\begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$$

$(x, mx+c) \longrightarrow (x', mx'+c)$

New coordinate $(x \to x')$ but same relationship, same line.

$3x + 3(mx+c) = x'$

$3x + 3mx + 3c$

$4x + 7(mx+c) = mx' + c$

$4x + 7mx + 7c = m(3x + 3mx + 3c) + c$

$\underline{4x} + \underline{7mx} + \underline{7c} = \underline{3mx} + \underline{3m^2x} + \underline{3cm} + \underline{c}$

$0 = 3m^2x - 4mx + 3cm - 6c - 4x$

$0 = x(3m^2 - 4m - 4) + 3c(m-2)$

$0 = x(3m+2)(m-2) + 3c(m-2)$

If $m=2$ then equation is correct. c can take any value.

If $m = -\frac{2}{3}$ and $c=0$, equ. is correct

Our inv. lines are $y = -\frac{2}{3}x$ and $y = 2x + c$.

3.

$$\mathbf{P} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

The matrix **P** represents a linear transformation, T, of the plane.

(a) Describe the invariant point of the transformation T.

$y = -2x$

(b) Describe the invariant lines of the transformation T.

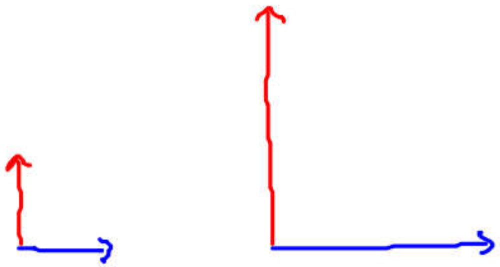
(6)

$y = -2x$ and $y = \frac{1}{2}x + c$

Enlargements

Describe the effect of the following matrices.

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$



$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

Enlargement
Centre (0,0)
S.F. 3

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Stretch in x-direction
S.F. 2

✎ $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ represents a stretch scale factor a parallel to the x-axis and a stretch scale factor b parallel to the y-axis. When $a = b$ this represents an enlargement.

Using det(A)

$A(1,1), B(1,2), C(2,2)$ are points on a triangle. The transformation with matrix $\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ is applied to the triangle to produce a new triangle with vertices A', B' and C'.

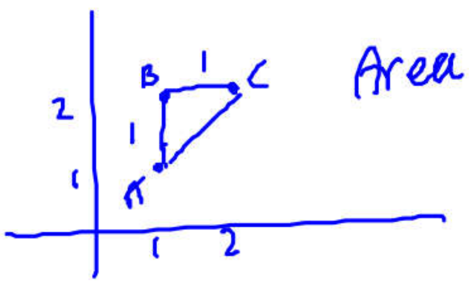
(a) Determine the coordinates of A', B', C'.

(b) What is the area of triangle ABC?

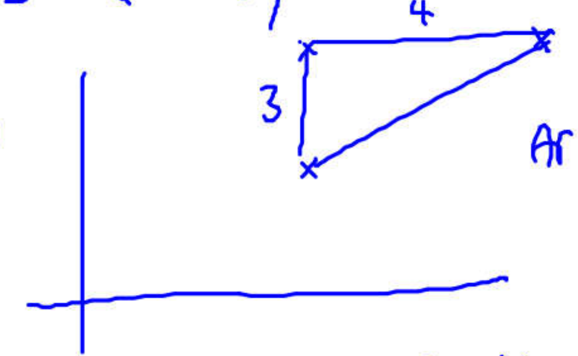
(c) What is the area of triangle $A'B'C'$?

(d) Determine $\det(M)$. What do you notice?

a) $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 8 \\ 3 & 6 & 6 \end{pmatrix}$ $A'(4,3)$ $B'(4,6)$ $C'(8,6)$

b) 

Area $= \frac{1}{2}$

c) 

Area $= 6$

d) $\det(m) = \begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} = 12$

Area of object $\times \det M =$ area of image

$\frac{1}{2} \times 12 = 6$

Area scale factor

positive value of det(M)

We saw in this example that:

✏ Area of image = Area of object × $|\det(\mathbf{M})|$

i.e. the determinant tells us how the area is scaled under the transformation with matrix $\mathbf{M}$.

(The proof of this is not covered here)

Area of Object	Transformation Matrix	Area of Image
4	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	$4 \times 2 = 8$
3	$\begin{pmatrix} 2 & 0 \\ 9 & 4 \end{pmatrix}$	$3 \times 8 = 24$
9	$\begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}$	$9 \times 1 = 9$
1	$\begin{pmatrix} -5 & 2 \\ -4 & -2 \end{pmatrix}$	$1 \times 18 = 18$

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$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(*a*) Find det **A**.

(1)

The triangle R is transformed to the triangle S by the matrix **A**.
Given that the area of triangle S is 72 square units,

(*c*) find the area of triangle R.

(2)

Ex 7C
Odd questions.

(a)	$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$ $\det \mathbf{A} = 2(3) - (-1)(-2) = 6 - 2 = \underline{4}$	$\underline{4}$	**B1** **(1)**
(c)	$\text{Area}(R) = \dfrac{72}{4} = \underline{18}\ (\text{units})^2$	$\dfrac{72}{\text{their } \det \mathbf{A}}$ or $72(\text{their } \det \mathbf{A})$ $\underline{18}$ or ft answer.	**M1** **A1**$\sqrt{\ }$ **(2)**

Ex 7C