Homework

$$\cancel{\int \frac{1}{\sin^2 x} + \frac{1}{x}\,dx}$$

$$\frac{\cos}{\sin^2} = \frac{\cos}{\sin} \times \frac{1}{\sin}$$

Ex 11 B Q2 d

$$\int \frac{3 - 2\cos\frac{1}{2}x}{\sin^2\frac{1}{2}x}\,dx = \int \left(\frac{3}{\sin^2\frac{1}{2}x} - \frac{2\cos\frac{1}{2}x}{\sin^2\frac{1}{2}x} \right) dx$$

$$= \int \left(3\operatorname{cosec}^2\tfrac{1}{2}x - 2\cot\tfrac{1}{2}x\operatorname{cosec}\tfrac{1}{2}x \right) dx$$

$$= -6\cot\tfrac{1}{2}x + 4\operatorname{cosec}\tfrac{1}{2}x + c$$

$$-6 \times \tfrac{1}{2}x - \operatorname{cosec}^2\tfrac{1}{2}x \qquad 4 \times \tfrac{1}{2}x - \operatorname{cosec}\tfrac{1}{2}x\cot\tfrac{1}{2}x$$

SKILL #3: Integrating using Trig Identities

Some expressions, such as $\sin^2 x$ and $\sin x \cos x$ can't be integrated directly, but we can use one of our trig identities to replace it with an expression we can easily integrate.

Q Find $\int \sin^2 x \; dx$

$$\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$
$$= \frac{1}{2}\int (1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Q Find $\int \sin 3x \cos 3x \; dx$

$$\int \sin 3x \cos 3x \, dx = \int \frac{1}{2}\sin 6x \, dx$$
$$= -\frac{1}{12}\cos 6x + c$$

$\sin 2x = 2\sin x \cos x$

$\frac{1}{2}\sin 2x = \sin x \cos x$

Q Find $\int \cos^2 x \; dx$

$$\int \cos^2 x \, dx = \int \left(\frac{1}{2}\cos 2x + \frac{1}{2}\right) dx$$
$$= \frac{1}{4}\sin 2x + \frac{1}{2}x + c$$

Q Find $\int \tan^2 x \; dx$

$1 + \tan^2 x = \sec^2 x$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1)\, dx$$
$$= \tan x - x + c$$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x$

Q Find $\int (\sec x + \tan x)^2 \, dx$

$$\int (\sec x + \tan x)^2 dx$$

$$= \int (\sec^2 x + 2\sec x \tan x + \underbrace{\tan^2 x}_{\sec^2 x - 1}) dx$$

✓ ✓ ?

$$= \int (2\sec^2 x + 2\sec x \tan x - 1) dx$$

$$= 2\tan x + 2\sec x - x + c$$

Q ~~ta~~ 1 first column

2 " "

Q3

3 Show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{2+\pi}{8}$

Q Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 3x \; dx$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 3x \; dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right) dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 - \cos 6x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{3} - \underline{\frac{1}{6}\sin 2\pi}\right) - \left(\frac{\pi}{6} - \frac{1}{6}\underline{\sin \pi}\right)\right)$$

zero

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$= \underline{\underline{\frac{\pi}{12}}}$$

Ex 11C

SKILL #4: Reverse Chain Rule

There's certain more complicated expressions which look like the result of having applied the chain rule. I call this process '**consider then scale**':

1. **Consider some expression that will differentiate to something similar to it.**
2. **Differentiate, and adjust for any scale difference.**

ln blah $\Rightarrow \frac{1}{\text{blah}} \times$ blah'

$$\int x(x^2+5)^3\,dx$$

The first x looks like it arose from differentiating the x^2 inside the brackets.

consider $(x^2+5)^4$

diff. $4(x^2+5)^3 \times 2x$

$8x(x^2+5)^3$

scale we want $\frac{1}{8}$ of it.

$\int x(x^2+5)^3\,dx = \frac{1}{8}(x^2+5)^4 + c$

$\frac{1}{8} \times 2x \times 4(x^2+5)^3$

$$\int \cos x \sin^2 x\,dx$$

The $\cos x$ probably arose from differentiating the sin.

consider $(\sin x)^3$

diff. $3\cos x \sin^2 x$

scale we want $\frac{1}{3}$

$\int \cos x \sin^2 x\,dx = \frac{1}{3}\sin^3 x + c$

$\frac{1}{3} \times 3 \times \sin^2 x \times \cos x$

$$\int \frac{2x}{x^2+1}\,dx$$

The $2x$ probably arose from differentiating the x^2.

consider $\ln|x^2+1|$

diff. $\frac{2x}{x^2+1}$

No scaling required.

$\int \frac{2x}{x^2+1}\,dx = \ln|x^2+1| + c$

Integration by Inspection/Reverse Chain Rule: Use common sense to **consider some expression** that would differentiate to the expression given. Then **scale** appropriately.

Common patterns:

$$\int k\frac{f'(x)}{f(x)}dx \rightarrow \quad Try\ \ln|f(x)|$$

In words: "If the bottom of a fraction differentiates to give the top (forgetting scaling), try ln of the bottom".

$$\int kf'(x)[f(x)]^n \rightarrow \quad Try\ [f(x)]^{n+1}$$

$$\int \frac{x^2}{x^3+1}\,dx$$

consider $\ln|x^3+1|$

diff. $\frac{3x^2}{x^3+1}$

Scale $\frac{1}{3}$

$$\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\ln|x^3+1| + c$$

$$\int x\,e^{x^2+1}\,dx$$

consider e^{x^2+1}

diff. $2x\,e^{x^2+1}$

scale $\frac{1}{2}$

$$\int xe^{x^2+1}\,dx = \frac{1}{2}e^{x^2+1} + c$$

$$\int \frac{4x^3}{x^4-1}\,dx = \ln|x^4-1| + c$$

$$\int \frac{\cos x}{\sin x + 2}\,dx = \ln|\sin x + 2| + c$$

$$\int \cos x \; e^{\sin x}\,dx = e^{\sin x} + c$$

$$\int \cos x\,(\sin x - 5)^7\,dx = \frac{1}{8}(\sin x - 5)^8 + c$$

$$\int x^2(x^3+5)^7 = \frac{1}{24}(x^3+5)^8 + c$$

$$\int \frac{x}{(x^2+5)^3}\,dx = -\frac{1}{4}(x^2+5)^{-2} + c$$

$x(x^2+5)^{-3}$

Ex 11 D
Q 1a–h
Q 2acegi

2 Find the following integrals.

Homework
Ex 11D Questions
2bdfhj
3-6

b $\int \operatorname{cosec}^2 2x \cot 2x \, dx$

d $\int \cos x \, e^{\sin x} \, dx$

f $\int x(x^2 + 1)^{\frac{3}{2}} \, dx$

h $\int \frac{2x + 1}{\sqrt{x^2 + x + 5}} \, dx$

j $\int \frac{\sin x \cos x}{\cos 2x + 3} \, dx$

3 Find the exact value of each of the following:

a $\int_0^3 (3x^2 + 10x)\sqrt{x^3 + 5x^2 + 9} \, dx$

b $\int_{\frac{\pi}{9}}^{\frac{2\pi}{9}} \frac{6 \sin 3x}{1 - \cos 3x} \, dx$

c $\int_4^7 \frac{x}{x^2 - 1} \, dx$

d $\int_0^{\frac{\pi}{4}} \sec^2 x \, e^{4 \tan x} \, dx$

4 Given that $\int_0^k kx^2 e^{x^3} \, dx = \frac{2}{3}(e^8 - 1)$, find the value of k.

5 Given that $\int_0^\theta 4 \sin 2x \cos^4 2x \, dx = \frac{4}{5}$ where $0 < \theta < \pi$, find the exact value of θ.

6 **a** By writing $\cot x = \frac{\cos x}{\sin x}$, find $\int \cot x \, dx$.

b Show that $\int \tan x \, dx \equiv \ln|\sec x| + c$.