Proof

Yr 12

Direct proof (sometimes called proof by deduction)

Proof by exhaustion

Disproof using a counter-example

Yr 13

Proof by contradiction

Proof

Direct proof

Proof by exhaustion

Disproof using a counter-example

Proof by contradiction

$2n+1$ $\qquad - \qquad (\quad)^2$

Prove that every odd integer is the difference of two perfect squares

Let n be an integer.

$$2n+1 = n^2 + 2n + 1 - n^2$$

$$= (n+1)^2 - n^2$$

So, every odd integer can be expressed as the difference of 2 perfect squares

Further, the perfect squares are consecutive

↳ next to each other in order.

Odd number	$2n+1$	Where n is an integer
A *different* odd number	$2m+1$	Where m is an integer
Consecutive odd numbers	$2n-1, 2n+1, etc.$	Where n is an integer
Even number	$2n$	Where n is an integer
A *different* even number	$2m$	Where m is an integer
Consecutive even numbers	$2n, 2n+2, etc.$	Where n is an integer
a is a factor of b	$b=na$	Where n is an integer
A rational number	$\frac{a}{b}$	Where a and b are integers, and have a highest common factor of 1 (i.e. a fraction in its lowest terms)

Prove that the difference between the squares of any two consecutive integers is equal to the sum of these two integers.

Let n be any integer.

2 consecutive integers are therefore n and $n+1$

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2$$
$$= 2n + 1$$
$$= n + (n+1)$$

This result proves the statement.

Proof

Direct proof
Proof by exhaustion → try out / exhaust all the possibilities
Disproof using a counter-example
Proof by contradiction

Prove that no square number ends in an 8.

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$8^2 = 64$
$9^2 = 81$
$10^2 = 100$

If a number ends in a 1 or a 9, its square ends in a 1.
Similarly, if it ends in a 2 or a 8, its square ends in a 4
If ending in a 3 or a 7, its square ends in a 9.
If ending in a 4 or a 6, its square ends in a 6.
If ending in a 5, its square ends in a 5.
If ending in a 0, its square ends in a 0.
Hence, no square number will end in an 8.
They will only end in a 1, 4, 5, 6, 9 or 0

Proof

Direct proof
Proof by exhaustion
Disproof using a counter-example
Proof by contradiction

It is suggested that for every prime number p, 2p + 1 is also prime.

Prove that this is false.

We need just one example that doesn't work!

$p = 2$, $2p + 1 = 5$
$p = 3$, $2p + 1 = 7$
$p = 5$, $2p + 1 = 11$
$p = 7$, $2p + 1 = 15$ But 15 is not prime, so the statement is false

counter-example

Proof

Direct proof
Proof by exhaustion
Disproof using a counter-example
Proof by contradiction

To prove a statement is true by contradiction:

- **Assume** that the statement is in fact **false**.
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove that there is no greatest odd integer.

Assume, for contradiction, that there is a greatest odd integer, say n.

But, $n+2$ is also an odd integer, and clearly $n+2>n$.

So n is not the greatest odd integer.

Hence, there is no greatest odd integer.

✏ To prove a statement is true by contradiction:

- **Assume** that the statement is in fact **false**.
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove by contradiction that if n^2 is even, then n must be even.

false version

The negation of "if A then B" is "if A, then not B".

Assume for contradiction that if n^2 is even, then n must be odd.

If n is odd, then $n = 2m+1$ for some integer m.

$$n^2 = (2m+1)^2 = 4m^2 + 4m + 1$$
$$= 2(2m^2 + 2m) + 1$$

Hence, n^2 is therefore odd. But this is a contradiction to our assumption.

Therefore, if n^2 is even, then n must be even.

Use proof by contradiction to show that there exist no integers a and b for which $25a + 15b = 1$.

(4 marks)

Assume for contradiction, that integers a and b exist such that $25a + 15b = 1$

$$25a + 15b = 1 \quad (\div 5)$$

$$5a + 3b = \frac{1}{5}$$

We know that because a and b are integers, so are 5a and 3b. But the sum of two integers is an integer, and our statement says $5a + 3b = \frac{1}{5}$, which is not an integer. So we have a contradiction.

Hence, there exist no integers a and b for which $25a + 15b = 1$

Use proof by contradiction to show that, given a rational number a and an irrational number b, $a - b$ is irrational.

(4 marks)

A **rational number** is one that can be expressed in the form $\frac{a}{b}$ where a, b are integers.

An **irrational** number cannot be expressed in this form, e.g. $\pi, e, \sqrt{3}$. $(\sqrt{2} - 4)$

The set of all rational numbers is $\mathbb{Q}$ (real numbers: $\mathbb{R}$, natural numbers: $\mathbb{N}$, integers: $\mathbb{Z}$).

Assume, for contradiction, that given a rational number a, and an irrational number b, $a-b$ is rational.

Let $a = \frac{c}{d}$, $a - b = \frac{e}{f}$ where c, d, e, f are integers

$$a - \frac{e}{f} = b$$

$$\frac{c}{d} - \frac{e}{f} = b$$

$$\frac{cf - de}{df} = b$$

Because c, d, e, f are all integers, $\frac{cf-de}{df}$ is a rational number

Hence b is rational. But this is a contradiction.

So, given a rational number a and irrational number b, $a-b$ is therefore irrational.

Prove that the square root of 2 is irrational.

A **rational number** is one that can be expressed in the form $\frac{a}{b}$ where a, b are integers.

An **irrational** number cannot be expressed in this form, e.g. $\pi, e, \sqrt{3}$.

The set of all rational numbers is $\mathbb{Q}$ (real numbers: $\mathbb{R}$, natural numbers: $\mathbb{N}$, integers: $\mathbb{Z}$).

Assume, for contradiction, that $\sqrt{2}$ is rational.

i.e. $\sqrt{2} = \frac{a}{b}$ where a and b are integers with no common factors.

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

So, a is an even number.

Let $a = 2k$ where k is an integer.

$$2b^2 = (2k)^2 = 4k^2$$

$$b^2 = 2k^2$$

So, b is an even number.

Because both a and b are even, they have a common factor! This is a contradiction.

Hence $\sqrt{2}$ is irrational.

Prove by contradiction that there are infinitely many prime numbers.

(6 marks)

Assume, for contradiction, that there are finitely many primes.

Let the list of primes be, $p_1, p_2, p_3, p_4, \ldots, p_n$. 2, 3, 5, 7, 11, 13

Let $N = p_1 \times p_2 \times p_3 \times \ldots \times p_n$

$N = 2 \times 3 \times 5 \times 7 \times 11 \times 13$
$= 30,030$

$N + 1 = 30,031$

Then $N+1 = p_1 \times p_2 \times \ldots \times p_n + 1$

$$\begin{array}{c} 30031 \\ / \quad \backslash \\ \underline{59} \times \underline{509} \end{array}$$

$N+1$ is not divisible by any of $p_1, p_2, p_3, \ldots, p_n$

Hence, $N+1$ is either itself prime or its prime factors are not in our original list.

This means our list is incomplete, hence we have a contradiction, and so there are infinitely many primes.

Prove by contradiction that there are infinitely many primes.

Assume that there is a finite number of prime numbers.
Therefore we can list all the prime numbers:

$$p_1, p_2, p_3, \dots, p_n$$

Consider the number:

$$N = (p_1 \times p_2 \times \cdots \times p_n) + 1$$

When you divide N by any of $p_1, p_2, \dots, p_n$, the remainder will always be 1.
Therefore N is not divisible by any of these primes.

Therefore N must itself be prime, or its prime factorisation contains only primes not in our original list.
This contradicts the assumption that $p_1, p_2, \dots, p_n$ contained the list of all prime numbers.

Therefore, there are an infinite number of primes.

8. (i) Show that $y^2 - 4y + 7$ is positive for all real values of y.

(2)

Comp. the. sq.

(ii) Bobby claims that

$$e^{3x} \geqslant e^{2x} \qquad x \in \mathbb{R}$$

Determine whether Bobby's claim is always true, sometimes true or never true, justifying your answer.

(2)

(iii) Elsa claims that

is a member

'for $n \in \mathbb{Z}^+$, if n^2 is even, then n must be even'

positive integers

Use proof by contradiction to show that Elsa's claim is true.

(2)

(iv) Ying claims that

$\sqrt{2}$ $\quad$ $\pi + 3$

$-\sqrt{2}$ $\quad$ $3 - \pi$

'the sum of two different irrational numbers is irrational'

Determine whether Ying's claim is always true, sometimes true or never true, justifying your answer.

(2)

8 (i)	E.g. $y^2 - 4y + 7 = (y-2)^2 - 4 + 7$	M1	2.1
	$= (y-2)^2 + 3 \geqslant 3$, as $(y-2)^2 \geqslant 0$ and so $y^2 - 4y + 7$ is positive for all real values of y	A1	2.2a
		(2)	
(ii)	For an explanation or statement to show when (Bobby's) claim $e^{3x} \geqslant e^{2x}$ fails. This could be e.g. • when $x = -1$, $e^{-3} < e^{-2}$ **or** e^{-3} is not greater than or equal to e^{-2} • when $x < 0$, $e^{3x} < e^{2x}$ **or** e^{3x} is not greater than or equal to e^{2x}	M1	2.3
	Followed by an explanation or statement to show when (Bobby's) claim $e^{3x} \geqslant e^{2x}$ is true. This could be e.g. • $\boldsymbol{x = 2}$, $e^6 \geqslant e^4$ **or** e^6 is greater than or equal to e^4 • when $x \geqslant 0$, $e^{3x} \geqslant e^{2x}$ **and** a correct conclusion. E.g. • (Bobby's) claim is sometimes true	A1	2.4
		(2)	
(ii)	Assuming $e^{3x} \geqslant e^{2x}$, then $\ln(e^{3x}) \geqslant \ln(e^{2x}) \Rightarrow 3x \geqslant 2x \Rightarrow x \geqslant 0$	M1	2.3
Alt 1	Correct algebra, using logarithms, leading from $e^{3x} \geqslant e^{2x}$ to $x \geqslant 0$ **and** a correct conclusion. E.g. (Bobby's) claim is sometimes true	A1	2.4
(iii)	Assume that n^2 is even and n is odd. So $n = 2k+1$, where k is an integer.	M1	2.1
	$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ So n^2 is odd which contradicts n^2 is even. So (Elsa's) claim is true.	A1	2.4
		(2)	
(iv)	For an explanation or statement to show when (Ying's) claim "the sum of two different irrational numbers is irrational" fails This could be e.g. • $\pi, 9 - \pi$; sum $= \pi + 9 - \pi = 9$ is not irrational	M1	2.3
	Followed by an explanation or statement to show when (Ying's) claim "the sum of two different irrational numbers is irrational" is true. This could be e.g. • $\pi, 9 + \pi$; sum $= \pi + 9 + \pi = 2\pi + 9$ is irrational and a correct conclusion. E.g. • (Ying's) claim is sometimes true	A1	2.4

14. (i) Kayden claims that

$$3^x \geqslant 2^x$$

Determine whether Kayden's claim is always true, sometimes true or never true, justifying your answer.

(2)

(ii) Prove that $\sqrt{3}$ is an irrational number.

Contradiction

(6)

Question	Scheme	Marks	AOs
14 (i)	For an explanation or statement to show when the claim $3^x \ldots 2^x$ fails This could be e.g. • when $x = -1$, $\frac{1}{3} < \frac{1}{2}$ or $\frac{1}{3}$ is not greater than or equal to $\frac{1}{2}$ • when $x < 0$, $3^x < 2^x$ or 3^x is not greater than or equal to 2^x	M1	2.3
	followed by an explanation or statement to show when the claim $3^x \ldots 2^x$ is true. This could be e.g. • $x = 2$, $9 \ldots 4$ or 9 is greater than or equal to 4 • when $x \ldots 0$, $3^x \ldots 2^x$ and a correct conclusion. E.g. • so the claim $3^x \ldots 2^x$ is sometimes true	A1	2.4
		(2)	
(ii)	Assume that $\sqrt{3}$ is a rational number So $\sqrt{3} = \frac{p}{q}$, where p and q integers, $q \neq 0$, and the HCF of p and q is 1	M1	2.1
	$\Rightarrow p = \sqrt{3}q \Rightarrow p^2 = 3q^2$	M1	1.1b
	$\Rightarrow p^2$ is divisible by 3 and so p is divisible by 3	A1	2.2a
	So $p = 3c$, where c is an integer From earlier, $p^2 = 3q^2 \Rightarrow (3c)^2 = 3q^2$	M1	2.1
	$\Rightarrow q^2 = 3c^2 \Rightarrow q^2$ is divisible by 3 and so q is divisible by 3	A1	1.1b
	As both p and q are both divisible by 3 then the HCF of p and q is not 1 This contradiction implies that $\sqrt{3}$ is an irrational number	A1	2.4
		(6)	
			(8 marks)

Ex 1 A