Methods in Calculus (Chapter 3/Chapter 6)

In this chapter, we explore a variety of new techniques for integration, as well as how integration can be applied.

1:: Improper Integrals

"Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$ or show that it is not convergent."

2:: Mean value of a function

"Find the mean value of $f(x) = \frac{4}{\sqrt{2+3x}}$ over the interval $[2,6]$."

3:: Differentiating and integrating inverse trigonometric functions

"Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$"

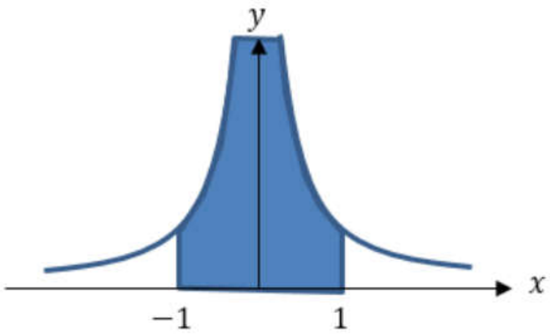
4:: Integrating using partial fractions.

"Show that $\int \frac{1+x}{x^3+9x}\,dx = A\ln\left(\frac{x^2}{x^2+9}\right) + B\arctan\left(\frac{x}{3}\right) + c$"

Improper Integrals

STARTER 1: Determine $\int_{-1}^{1} \frac{1}{x^2}\, dx$. Is there an issue?

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{1} x^{-2}\, dx = [-x^{-1}]_{-1}^{1}$$
$$= \left(-\frac{1}{1}\right) - \left(-\frac{1}{-1}\right) = -1 - +1 = -2$$



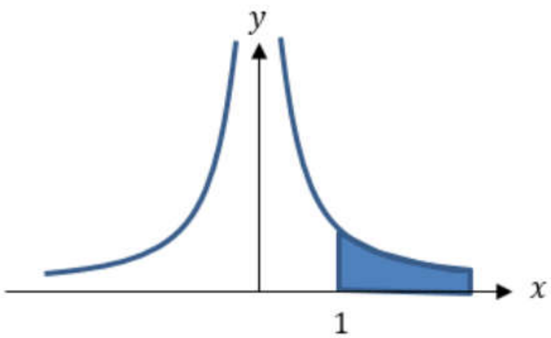
What's odd is the we ended up with a negative value. But the whole graph is above the x axis! The problem is related to integrating over a discontinuity, i.e. the function is not defined for the entire interval $[-1,1]$, notably where $x = 0$. We will see when this causes and issue and when it does not.
We say the definite integral does not exist.

STARTER 2: Determine $\int_{1}^{\infty} \frac{1}{x^2}\, dx$. Is there an issue?

(Note: **the below is seriously dodgy maths** as we're not allowed to use ∞ in calculations – we'll look at the proper way to write this in a sec)

$$[-x^{-1}]_1^{\infty} = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{1}\right) = 0 + 1 = 1$$

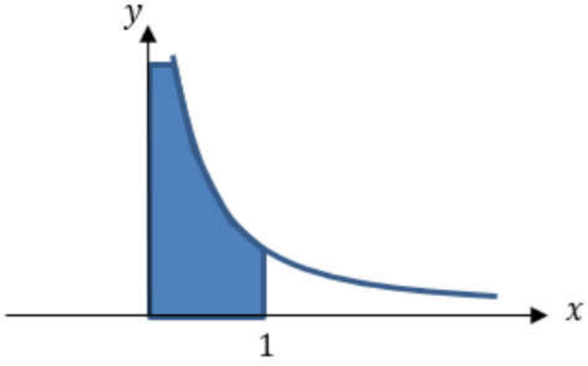
Although the graph is extending to infinity, the **area is finite** because the y values converge towards 0. The result is therefore valid this time. This is an example of an **improper integral** and because the value converged, **we say the definite integral exists**.



STARTER 3: Determine $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. Is there an issue?

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx = [2\sqrt{x}]_0^1$$
$$= 2\sqrt{1} - 2\sqrt{0} = 2$$

This is similar to the second example. Although $y \to \infty$ as $x \to 0$ (and not defined when $x = 0$), the area is convergent and therefore finite. The result of 2 is therefore valid.



✎ The integral $\int_a^b f(x)\,dx$ is improper if either:

- One or both of the limits is infinite
- $f(x)$ is undefined at $x = a, x = b$ are another point in the interval $[a, b]$.

To find $\int_a^\infty f(x)\,dx$, determine $\lim_{t\to\infty} \int_a^t f(x)\,dx$

As mentioned, **we can't use ∞ in calculations directly**. We can make use of the lim function we saw in differentiation by first principles.

Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$ or show that it is not convergent.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[-x^{-1}\right]_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} - \left(-\frac{1}{1}\right)\right)$$

$$= 1$$

Evaluate $\int_1^\infty \frac{1}{x}\,dx$ or show that it is not convergent.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \left[\ln x\right]_1^t$$

$$= \lim_{t\to\infty} \left(\ln t - \ln 1\right)$$

$$= \lim_{t\to\infty} (\ln t)$$

As $t \to \infty$, $\ln t \to \infty$

So, it is not convergent.

When $f(x)$ not defined for some value

We need to **avoid values** with the range $[a, b]$ **for which the expression is not defined**. But just as we avoided ∞ by considering the limit as $t \to \infty$, we can similarly find what the area converges to as x tends towards the undefined value.

Evaluate $\int_0^1 \frac{1}{x^2}\, dx$ or show that it is not convergent.

At zero, $\frac{1}{x^2}$ is not defined.

$$\int_0^1 \frac{1}{x^2} dx = \lim_{t \to 0} \int_t^1 \frac{1}{x^2} dx$$

$$= \lim_{t \to 0} \left[-x^{-1} \right]_t^1$$

$$= \lim_{t \to 0} \left(-\frac{1}{1} - \left(-\frac{1}{t} \right) \right)$$

As $t \to 0$, $\frac{1}{t} \to \infty$

$\therefore$ It does not converge.

Evaluate $\int_0^2 \frac{x}{\sqrt{4-x^2}}\, dx$ or show that it is not convergent.

At $x=2$, $\frac{x}{\sqrt{4-x^2}}$ is undefined.

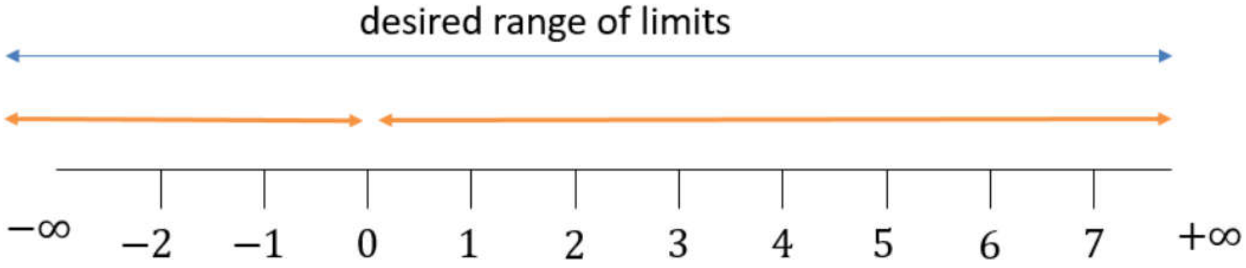
$$\int_0^2 x(4-x^2)^{-1/2} dx = \lim_{t \to 2} \int_0^t x(4-x^2)^{-1/2} dx$$

$$= \lim_{t \to 2} \left[-(4-x^2)^{1/2} \right]_0^t$$

$$= \lim_{t \to 2} \left(-(4-t^2)^{1/2} + (4)^{1/2} \right)$$

$$= \underline{\underline{2}}$$

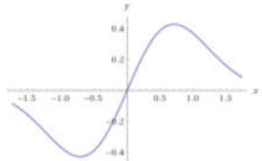
When integrating between $-\infty$ and ∞



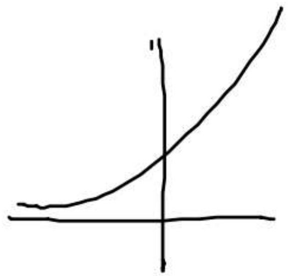
Suppose we want $\int_{-\infty}^{\infty} f(x)\ dx$. How could evaluate this?

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{0} f(x)\,dx + \lim_{t \to \infty} \int_{0}^{t} f(x)\,dx$$

(a) Find $\int xe^{-x^2}\,dx$ (b) Hence show that $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ converges and find its value.



a) $\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + c$



b) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \lim_{t \to -\infty} \int_{t}^{0} xe^{-x^2}\,dx + \lim_{t \to \infty} \int_{0}^{t} xe^{-x^2}\,dx$

$$= \lim_{t \to -\infty} \left[-\frac{1}{2}e^{-x^2}\right]_{t}^{0} + \lim_{t \to \infty} \left[-\frac{1}{2}e^{-x^2}\right]_{0}^{t}$$

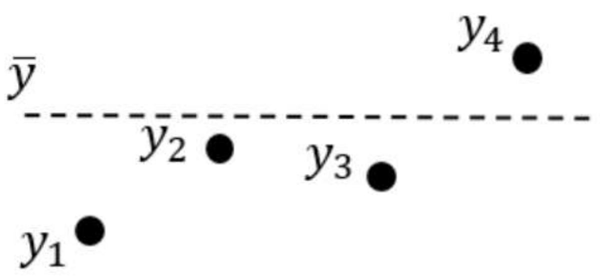
$$= \lim_{t \to -\infty} \left(-\frac{1}{2} + \frac{1}{2}e^{-t^2}\right) + \lim_{t \to \infty} \left(-\frac{1}{2}e^{-t^2} + \frac{1}{2}\right)$$

$$= -\frac{1}{2} + \frac{1}{2} = \underline{\underline{0}}$$

Q1a, 3a
Odd Question
Ex 3A

The Mean Value of a Function

How would we find the mean of a set of values y values $y_1, y_2, \dots, y_n$?

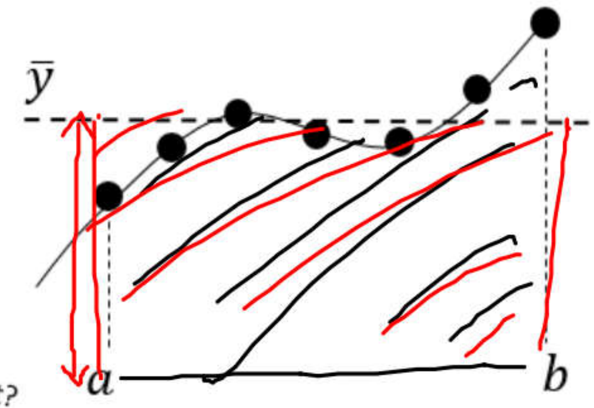


$$\bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

continuous equivalent?

continuous equivalent?

So the question then is, can we extend this to the continuous world, with a function $y = f(x)$, between $x = a$ and $x = b$?



Area $= (b-a)\bar{y}$

$$\bar{y} = \frac{\int_a^b f(x)\,dx}{b-a}$$

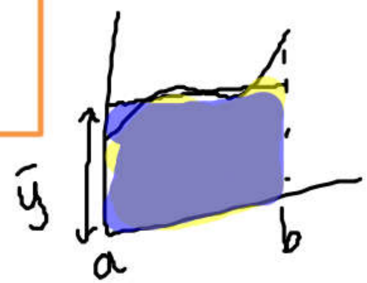
Integration can be thought of as the continuous version of summation of the y values.

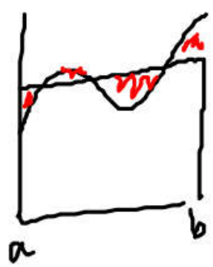
The width of the interval, $b - a$, could (sort of) be thought of as the number of points in the interval on an infinitesimally small scale.

The **mean value** of the function $y = f(x)$ over the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_b^a f(x)dx$$

We write it as $\bar{y}$ or $\bar{f}$ or y_m. $\overline{f(x)}$





$\int f(x)\,dx = (b-a)\times\bar{y}$

Find the mean value of $f(x) = \frac{4}{\sqrt{2+3x}}$ over the interval $[2,6]$.

$$\int_2^6 4(2+3x)^{-1/2}\,dx$$

$$= \left[\frac{8}{3}(2+3x)^{1/2}\right]_2^6$$

$$= \frac{8}{3}\sqrt{20} - \frac{8}{3}\sqrt{8}$$

$$= \frac{16}{3}\sqrt{5} - \frac{16}{3}\sqrt{2}$$

$$= \frac{16}{3}(\sqrt{5}-\sqrt{2})$$

$$\overline{f(x)} = \frac{1}{4} \times \frac{16}{3}(\sqrt{5}-\sqrt{2})$$

$$\frac{1}{b-a} = \frac{1}{6-2}$$

$$= \frac{4}{3}(\sqrt{5}-\sqrt{2})$$

$f(x) = \frac{4}{1+e^x}$

(a) Show that the mean value of $f(x)$ over the interval $[\ln 2, \ln 6]$ is $\frac{4\ln\frac{9}{7}}{\ln 3}$

(b) Use your answer to part a to find the mean value over the interval $[\ln 2, \ln 6]$ of $f(x) + 4$.

(c) Use geometric considerations to write down the mean value of $-f(x)$ over the interval $[\ln 2, \ln 6]$

$$\int_{\ln 2}^{\ln 6} \frac{4}{1+e^x}\,dx$$

$u = 1 + e^x$

$\frac{du}{dx} = e^x$

$\frac{du}{dx} = u - 1$

$\frac{1}{u-1}\,du = dx$

x	u
$\ln 2$	3
$\ln 6$	7

$$\int_{\ln 2}^{\ln 6} \frac{4}{1+e^x}\,dx = \int_3^7 \frac{4}{u} \times \frac{1}{u-1}\,du = \int_3^7 \frac{4}{u(u-1)}\,du$$

$$\frac{4}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$4 = A(u-1) + Bu$

$u = 1$ $u = 0$

$4 = B$ $-4 = A$

$$= \int_3^7 \left(\frac{4}{u-1} - \frac{4}{u}\right)du$$

$$= \Big[4\ln|u-1| - 4\ln|u|\Big]_3^7$$

$$= \left[4\ln\left|\frac{u-1}{u}\right|\right]_3^7 = 4\ln\frac{6}{7} - 4\ln\frac{2}{3}$$

$$= 4\ln\frac{9}{7}$$

Ex 3B

$$\overline{f(x)} = \frac{1}{\ln 6 - \ln 2} \times 4\ln\frac{9}{7} = \frac{4\ln\frac{9}{7}}{\ln 3}$$

b) $\frac{4\ln\frac{9}{7}}{\ln 3} + 4$ c) $-\frac{4\ln\frac{9}{7}}{\ln 3}$