Vectors - Year 2

1:: Distance between two points.

What's the distance between $(1,0,4)$ and $(-3,5,9)$?

2:: i, j, k notation for vectors

$$\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \rightarrow \boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$$

3:: Magnitude of a 3D vector and using it to find angle between vector and a coordinate axis.

"Find the angles that the vector $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ makes with each of the positive coordinate axis."

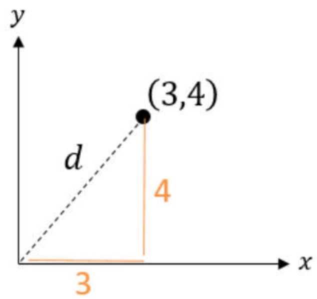
4:: Solving Geometric Problems

Same as Year 1 but with 3D vectors.

5:: Application to Mechanics

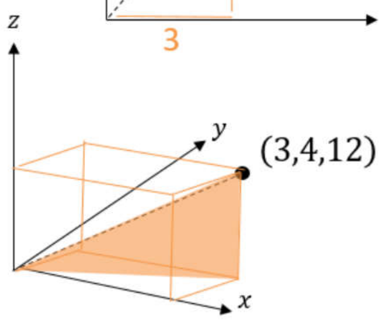
Using $F = ma$ with 3D force/acceleration vectors and understanding distance is the magnitude of the 3D displacement vector, etc.

Distance from the origin and magnitude of a vector



In 2D, how did we find the distance from a point to the origin?

How about in 3D then?



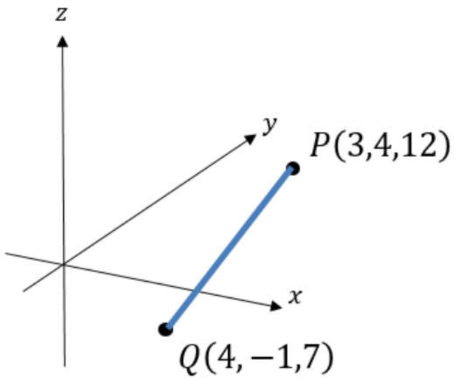
From Year 1 you will be familiar with the magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$ being its length.
We can see from above that this nicely extends to 3D:

✎ The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$|\boldsymbol{a}| = \sqrt{x^2 + y^2 + z^2}$$

And the distance of (x, y, z) from the origin is $\sqrt{x^2 + y^2 + z^2}$

Distance between two 3D points



How do we find the distance between P and Q?

✏ The distance between two points is:
$d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$

Δx means "change in x"

Quickfire Questions:

Distance of $(4,0,-2)$ from the origin:

$\left|\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}\right| =$

Distance between $(0,4,3)$ and $(5,2,3)$.

Tip: Because we're squaring, it doesn't matter whether the change is negative or positive.

Distance between $(1,1,1)$ and $(2,1,0)$.

Distance between $(-5,2,0)$ and $(-2,-3,-3)$.

Your Turn

Find the distance from the origin to the point $P(7,7,7)$.

The coordinates of A and B are $(5,3,-8)$ and $(1,k,-3)$ respectively.
Given that the distance from A to B is $3\sqrt{10}$ units, find the possible values of k.

In 2D you were previously introduced to $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as unit vectors in each of the x and y directions.

It meant for example that $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$ could be written as $8\boldsymbol{i} - 2\boldsymbol{j}$ since $8\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

Unsurprisingly, in 3D:

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Quickfire Questions

1 Put in i, j, k notation:

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$

$\begin{pmatrix} -7 \\ 3 \\ 0 \end{pmatrix} =$

$\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} =$

2 Write as a column vector:

$4\boldsymbol{j} + \boldsymbol{k} =$

$\boldsymbol{i} - \boldsymbol{j} =$

3 If $A(1,2,3), B(4,0,-1)$ then

$\overrightarrow{AB} =$

4 If $\boldsymbol{a} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$ then $3\boldsymbol{a} + 2\boldsymbol{b} =$

Reminder:

✏ For position vectors a and b:

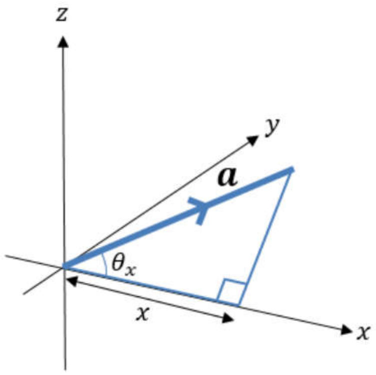
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Find the magnitude of $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$ is $2\boldsymbol{a} - 3\boldsymbol{b}$ parallel to $4\boldsymbol{i} - 5\boldsymbol{k}$?

Angles between vectors and an axis



How could you work out the angle between a vector and the x-axis?

The angle between $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the x-axis is:

$$\cos \theta_x = \frac{x}{|\boldsymbol{a}|}$$

and similarly for the y and z axes.

Find the angles that the vector $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ makes with each of the positive coordinate axis.

The points A and B have position vectors $4\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ relative to a fixed origin, O. Find $\overrightarrow{AB}$ and show that ΔOAB is isosceles.

Find the angle that the vector $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ makes with the x-axis.
By similarly considering the angle that $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ makes with the x-axis, determine the area of OAB where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. (Hint: draw a diagram)

Geometric Problems

A, B, C and D are the points $(2, -5, -8)$, $(1, -7, -3)$, $(0, 15, -10)$ and $(2, 19, -20)$ respectively.

a. Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$, giving your answers in the form $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$.

b. Show that the lines AB and DC are parallel and that $\overrightarrow{DC} = 2\overrightarrow{AB}$.

c. Hence describe the quadrilateral $ABCD$.

P, Q and R are the points $(4, -9, -3)$, $(7, -7, -7)$ and $(8, -2, 0)$ respectively. Find the coordinates of the point S so that $PQRS$ forms a parallelogram.

Introducing Scalars and Comparing Coefficients

Remember when we had **identities** like:

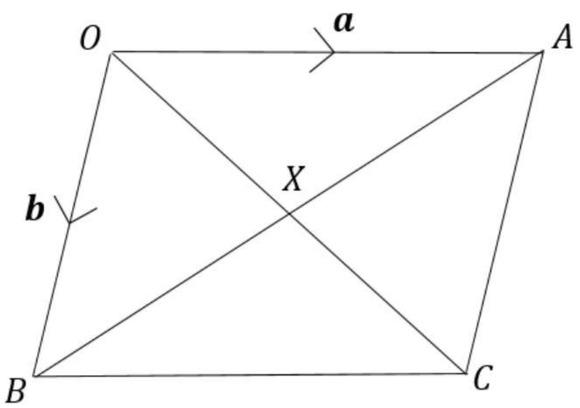
$$ax^2 + 3x \equiv 2x^2 + bx$$

we could **compare coefficients**, so that $a = 2$ and $3 = b$.

We can do the same with (non-parallel) vectors!

$OACB$ is a parallelogram, where $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. The diagonals OC and AB intersect at a point X. Prove that the diagonals bisect each other.

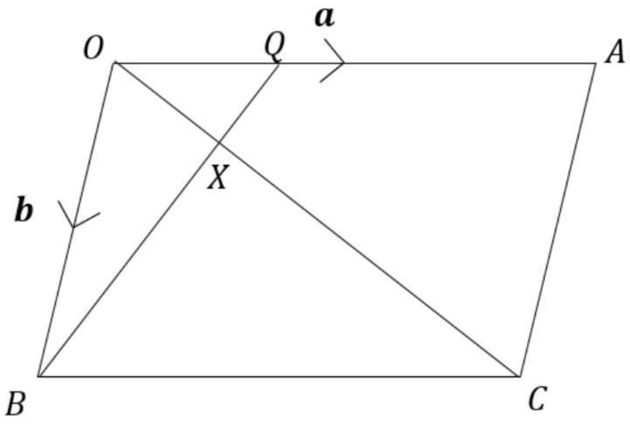
(Hint: Perhaps find $\overrightarrow{OX}$ in two different ways?)



'lambda'

'mu'

Your Turn



In the above diagram, $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{3}\boldsymbol{a}$. We wish to find the ratio $OX : XC$.

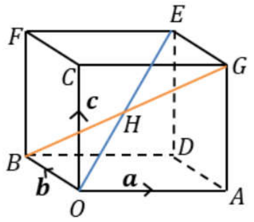
a) If $\overrightarrow{OX} = \lambda\,\overrightarrow{OC}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and λ.

b) If $\overrightarrow{BX} = \mu\,\overrightarrow{BQ}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and μ.

c) By comparing coefficients or otherwise, determine the value of λ, and hence the ratio $OX : XC$.

Given that
$3\boldsymbol{i} + (p+2)\boldsymbol{j} + 120\boldsymbol{k} = p\boldsymbol{i} - q\boldsymbol{j} + 4pqr\boldsymbol{k}$,
find the values of p, q and r.

The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G. Vectors a, b and c are the position vectors of the vertices A, B and C respectively. Prove that the diagonals OE and BG bisect each other.



The strategy behind this type of question is to find the point of intersection in 2 ways, and compare coefficients.

Application to Mechanics

Out of displacement, speed, acceleration, force, mass and time, all but mass and time are vectors. Clearly these can act in 3D space.

	Vector	Scalar
Force	$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} N$	
Acceleration	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ms^{-2}$	
Displacement	$\begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix} m$	
Velocity	$\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} ms^{-1}$	

A particle of mass 0.5 kg is acted on by three forces.

$$F_1 = (2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\ N$$
$$F_2 = (-\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k})\ N$$
$$F_3 = (4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})\ N$$

a. Find the resultant force R acting on the particle.
b. Find the acceleration of the particle, giving your answer in the form $(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k})$ ms^{-2}.
c. Find the magnitude of the acceleration.

Given that the particle starts at rest,
d. Find the distance travelled by the particle in the first 6 seconds of its motion.

Ex 12D