Chapter 1a (Year 2): Algebraic Methods, Proof

a) Proof by deduction

Prove that the product of two odd numbers is odd.

b) Proof by exhaustion

Prove that $n^2 + n$ is even for all integers n.

c) Disproof by counter example

Disprove the statement: "$n^2 - n + 41$ is prime for all integers n."

d) Proof by contradiction (A2)

Prove that the square root of 2 is irrational.

d) Proof by Contradiction

To prove a statement is true by contradiction:

- **Assume** that the statement is in fact **false.**
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove that there is no greatest odd integer.

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- **Assume** that the statement is in fact **false**.
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove by contradiction that if n^2 is even, then n must be even.

The negation of "if A then B" is "if A, then not B".

Use proof by contradiction to show that there exist no integers a and b for which $25a + 15b = 1$.

(4 marks)

Use proof by contradiction to show that, given a rational number a and an irrational number b, $a - b$ is irrational.

(4 marks)

A **rational number** is one that can be expressed in the form $\frac{a}{b}$ where a, b are integers.

An **irrational** number cannot be expressed in this form, e.g. $\pi, e, \sqrt{3}$.

The set of all rational numbers is $\mathbb{Q}$ (real numbers: $\mathbb{R}$, natural numbers: $\mathbb{N}$, integers: $\mathbb{Z}$).

Prove that the square root of 2 is irrational.

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Prove by contradiction that there are infinitely many primes.

Assume that there is a finite number of prime numbers.
Therefore we can list all the prime numbers:

$$p_1, p_2, p_3, \dots, p_n$$

Consider the number:

$$N = (p_1 \times p_2 \times \cdots \times p_n) + 1$$

When you divide N by any of $p_1, p_2, \dots, p_n$, the remainder will always be 1.
Therefore N is not divisible by any of these primes.

Therefore N must itself be prime, or its prime factorisation contains only primes not in our original list.
This contradicts the assumption that $p_1, p_2, \dots, p_n$ contained the list of all prime numbers.

Therefore, there are an infinite number of primes.

Exam Questions A2

8. (i) Show that $y^2 - 4y + 7$ is positive for all real values of y.

(2)

(ii) Bobby claims that

$$e^{3x} \geqslant e^{2x} \qquad x \in \mathbb{R}$$

Determine whether Bobby's claim is always true, sometimes true or never true, justifying your answer.

(2)

(iii) Elsa claims that

'for $n \in \mathbb{Z}^+$, if n^2 is even, then n must be even'

Use proof by contradiction to show that Elsa's claim is true.

(2)

(iv) Ying claims that

'the sum of two different irrational numbers is irrational'

Determine whether Ying's claim is always true, sometimes true or never true, justifying your answer.

(2)

14. (i) Kayden claims that

$$3^x \geqslant 2^x$$

Determine whether Kayden's claim is always true, sometimes true or never true, justifying your answer.

(2)

(ii) Prove that $\sqrt{3}$ is an irrational number.

(6)