Geometric Sequences

common ratio r

$$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24, 48, \ldots$$

Geometric Sequence

✏ A geometric sequence is one in which there is a **common ratio** between terms.

Identify the common ratio r:

1. $1, 2, 4, 8, 16, 32, \ldots$ $r = 2$

2. $27, 18, 12, 8, \ldots \frac{16}{3}$ $r = \frac{18}{27} = \frac{2}{3}$

3. $10, 5, 2.5, 1.25, \ldots$ $r = \frac{1}{2}$ or $r = 0.5$

4. $5, -5, 5, -5, 5, -5, \ldots$ $r = -1$

An **alternating sequence** is one which oscillates between positive and negative.

5. $x, -2x^2, 4x^3$ $r = -2x$

$-\frac{2x^{\cancel{2}}}{\cancel{x}}$ $\frac{4x^{\cancel{3}}}{-2\cancel{x^2}}$

6. $1, p, p^2, p^3, \ldots$ $r = p$

7. $4, -1, 0.25, -0.0625, \ldots$ $r = -\frac{1}{4}$

n^{th} term of an geometric sequence

1st Term	2nd Term	3rd Term	$\cdots$	n^{th} term
a	ar	ar^2	$\cdots$	ar^{n-1}

n^{th} term of geometric sequence:

$$u_n = ar^{n-1}$$

The second term of a geometric sequence is 4 and the 4th term is 8. The common ratio is positive. Find the exact values of:

a) The common ratio.

b) The first term.

c) The 10th term.

a) $u_2 = 4$ $u_4 = 8$

$4 = ar^{2-1}$ $8 = ar^{4-1}$

$\boxed{4 = ar}$ ② $\boxed{8 = ar^3}$ ①

① ÷ ② $\frac{ar^3}{ar} = \frac{8}{4}$

$r^2 = 2$

$r = \sqrt{2}$

b) Using ②

$4 = a\sqrt{2}$

$a = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = \underline{\underline{2\sqrt{2}}}$.

c) $u_{10} = ar^9$

$= 2\sqrt{2} \times (\sqrt{2})^9$

$= \underline{\underline{64}}$

$2\sqrt{2}$, 4, $4\sqrt{2}$, 8 $8\sqrt{2}$, 16, $16\sqrt{2}$, The numbers $3, x$ and $x + 6$ form the first three terms of a positive geometric sequence. Find:

a) The value of x.
b) The 10th term in the sequence.

$r = \frac{6}{3} = 2$

a) $r = \frac{x}{3}$ $r = \frac{x+6}{x}$

$\frac{x}{3} = \frac{x+6}{x}$

$x^2 = 3x + 18$

$x^2 - 3x - 18 = 0$

$(x-6)(x+3) = 0$

$x = 6$ $x = -3$

3, 6, 12

But geometric seq. is +ve, so $x = 6$

3, −3, 3

b) $u_{10} = ar^9$

$= 3 \times 2^9$

$= 1536$

n^{th} term with inequalities

Sequence

What is the first term in the geometric progression 3, 6, 12, 24, ... to exceed 1 million?

$a = 3$

$r = 2$

$u_n = 3 \times 2^{n-1}$

$3 \times 2^{n-1} > 1{,}000{,}000$

$2^{n-1} > \frac{1000000}{3}$

$(n-1)\ln 2 > \ln\left(\frac{1000000}{3}\right)$

$(n-1) > \frac{\ln \frac{1000000}{3}}{\ln 2}$

$n-1 > 18.3\ldots$

$n > 19.3\ldots$

$u_{19} = 3 \times 2^{18}$

$= 786{,}432$

First term is when $n = 20$.

$u_{20} = 3 \times 2^{19} = 1{,}572{,}864$

Your Turn

All the terms in a geometric sequence are positive.
The third term of the sequence is 20 and the fifth term 80. What is the 20th term?

a? r?

u_{20}

$u_3 = 20$ $\quad$ $u_5 = 80$

$20 = ar^2$ ② $\quad$ $80 = ar^4$ ①

① ÷ ②

$\frac{80}{20} = \frac{ar^4}{ar^2}$

$4 = r^2$

$r = 2$

Use ②

$20 = a \times 2^2$

$a = 5$

$u_{20} = 5 \times 2^{19}$

$= 2,621,440$

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x+6, \qquad 5x-6$$

a) Determine the possible values of x.
b) Given the common ratio is positive, find the common ratio.
c) Hence determine the possible values for the first term of the sequence.

$r = 2$

a) $r = \frac{x+6}{x} = \frac{5x-6}{x+6}$

$(x+6)^2 = 5x^2 - 6x$

$x^2 + 12x + 36 = 5x^2 - 6x$

$0 = 4x^2 - 18x - 36$

$x = 6$ or $x = -1.5$

b) If $x = 6$ $\quad$ $6, 12, 24$

$x = -1.5$ $\quad$ $-1.5, 4.5, -13.5$ → $\cancel{r = -3}$

c) 3

Ex 3C

Sum of the first n terms of a **geometric series**

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

Proof:

Exam Note: This once came up in an exam.

$$\text{①} \quad S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$$

$$\text{②} \quad rS_n = ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} + ar^n$$

$$\text{①} - \text{②} \quad S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Find the sum of the first 10 terms.

$$S_{10} = \frac{3(1-2^{10})}{1-2} = \underline{\underline{3069}}$$

3, 6, 12, 24, 48, ...

$a = 3$ $n = 10$

$r = 2$

$$S_{10} = \frac{4\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} = \underline{\underline{7.9921875}}$$

$$= \frac{1023}{128}$$

$4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$a = 4$ $n = 10$

$r = \frac{1}{2}$

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \cdots$ to n terms would exceed 2 000 000.

$a = 1$

$r = 2$

$$S_n = \frac{1(1-2^n)}{1-2}$$

$$= \frac{1-2^n}{-1}$$

$$S_n = 2^n - 1$$

$$S_n > 2000000$$

$$2^n - 1 > 2000000$$

$$2^n > 2{,}000{,}001$$

$$n \ln 2 > \ln 2000001$$

$$n > \frac{\ln 2{,}000{,}001}{\ln 2}$$

$$n > 20.93..$$

n is an integer,

so $\underline{\underline{n = 21}}$

Your Turn

Edexcel C2 June 2011 Q6

256, 192, 144

The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio, ✓ **(2)**

(*b*) the first term, ✓ **(2)**

(*d*) the smallest value of *n* for which the sum of the first *n* terms of the series exceeds 1000. **(4)**

a) $r = \frac{144}{192} = 0.75$

b) $192 = ar$

$192 = 0.75a$

$a = 256$

c) $S_n = \frac{256(1 - 0.75^n)}{1 - 0.75}$

$S_n = 1024(1 - 0.75^n)$

$S_n > 1000$

$1024(1 - 0.75^n) > 1000$

$1 - 0.75^n > \frac{1000}{1024}$

$1 - \frac{1000}{1024} > 0.75^n$

$\frac{3}{128} > 0.75^n$

$\ln \frac{3}{128} > n \ln 0.75$

$\ln 0.75 < 0$

$\frac{\ln \frac{3}{128}}{\ln 0.75} < n$

$13.04 < n$

$n = 14$

Ex 3D