Your Turn

Edexcel FP2(Old) June 2011 Q3

Find the general solution of the differential equation

$$x\frac{dy}{dx} + 5y = \frac{\ln x}{x}, \qquad x > 0$$

$$\frac{dy}{dx} + \frac{5y}{x} = \frac{\ln x}{x^2}$$

$$\int P(x)\,dx = \int \frac{5}{x}\,dx = 5\ln x = \ln x^5$$

$$e^{\int P(x)\,dx} = x^5$$

$$x^5\frac{dy}{dx} + 5x^4y = x^3\ln x$$

$$\frac{d}{dx}(x^5y) = x^3\ln x$$

$u = \ln x \quad v = \frac{1}{4}x^4$

$u' = \frac{1}{x} \quad v' = x^3$

$$x^5y = \int x^3\ln x\,dx$$

$$= \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3\,dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

$$x^5y = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

$$y = \frac{1}{4x}\ln x - \frac{1}{16x} + \frac{C}{x^5}$$

$\frac{dy}{dx} + 5\frac{y}{x} = \frac{\ln x}{x^2}$ Integrating factor $e^{\int \frac{5}{x}}$	M1
$e^{\int \frac{5}{x}} = e^{5\ln x} = x^5$	A1
$\int x^3\ln x\,dx = \frac{x^4\ln x}{4} - \int \frac{x^3}{4}\,dx$	M1 M1 A1
$= \frac{x^4\ln x}{4} - \frac{x^4}{16}$ $(+C)$	A1
$x^5y = \frac{x^4\ln x}{4} - \frac{x^4}{16} + C$ $\quad y = \frac{\ln x}{4x} - \frac{1}{16x} + \frac{C}{x^5}$	M1 A1
	(8)

Remember!

- can you separate the variables? DO THIS!
- can you use reverse product rule first? THEN DO THIS!
- put it in the form dy/dx + P(x)y = Q(x), then use IF

Q13 and Q14... may not need IF...!

Ex 7A
Q7 onwards

Classwork
Mixed Exercise 7 Q1-12

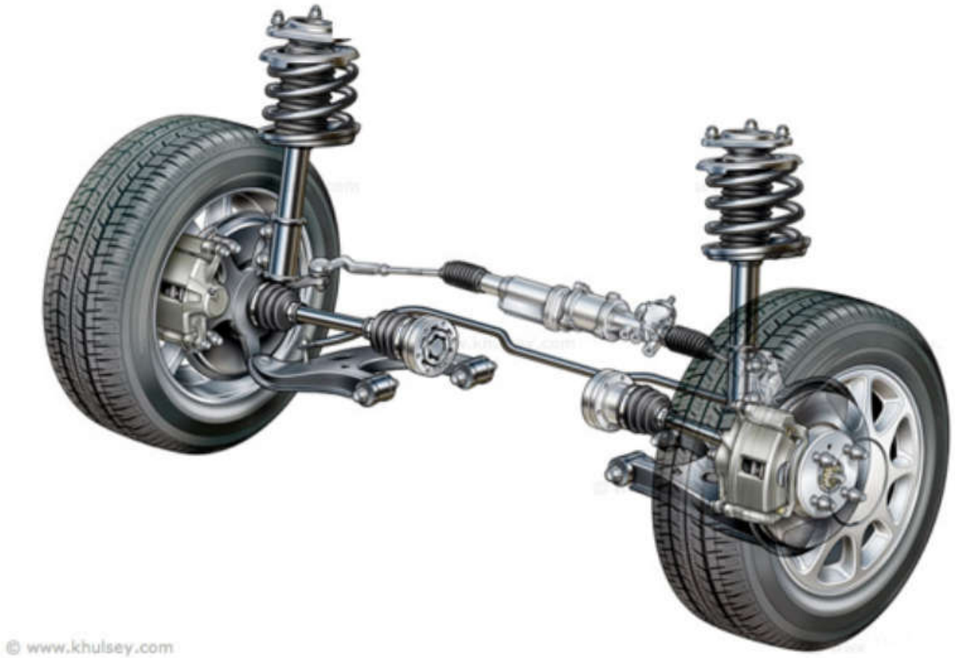
BE WARNED! Some of them are separating the variables!

Second Order Differential Equation Intro

We've already seen that differential equations are equations which relate x and y with derivatives. Unsurprisingly, second order differential equations involve the second derivative.

Shock absorbers as part of suspension of car subject to force down of car acting under acceleration, and forces up: damping force (proportional to velocity) and restoring force (proportional to extension of spring)

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - cx$$



Simple second order differential equations

We know from the previous chapter that the solution of $a\frac{dy}{dx} + by = 0$ is $y = Ae^{-\frac{b}{a}x}$.

Let's 'guess' that the solution of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is similar, and of the form Ae^{mx}

Are the any restrictions on the constants?

Let $y = Ae^{mx}$

$$\frac{dy}{dx} = Ame^{mx}$$

$$\frac{d^2y}{dx^2} = Am^2e^{mx}$$

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$$

$$Ae^{mx}(am^2 + bm + c) = 0$$

$$Ae^{mx} \neq 0$$

$$\boxed{am^2 + bm + c = 0}$$ Auxiliary Equation.

$m_1 =$

$m_2 =$

$$y = Ae^{m_1x} + Be^{m_2x}$$

A, B are constants.

- The equation $am^2 + bm + c = 0$ is called the auxiliary equation, and if m is a root of the auxiliary equation then $y = Ae^{mx}$ is a solution of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

- When the auxiliary equation has **two real distinct roots** α and β, the general solution of the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants.

Find the general solution of the equation $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$

A.E. $2m^2 + 5m + 3 = 0$

$m = -1 \quad m = -\frac{3}{2}$

$y = Ae^{-x} + Be^{-\frac{3}{2}x}$

$\frac{dy}{dx} = -Ae^{-x} - \frac{3}{2}Be^{-\frac{3}{2}x}$

$\frac{d^2y}{dx^2} = Ae^{-x} + \frac{9}{4}Be^{-\frac{3}{2}x}$

$2Ae^{-x} + \frac{9}{2}Be^{-\frac{3}{2}x} - 5Ae^{-x} - \frac{15}{2}Be^{-\frac{3}{2}x}$
$+ 3Ae^{-x} + 3Be^{-\frac{3}{2}x} = 0$

This is known as a **homogeneous** second-order differential equation because the RHS is 0. We will encountered nonhomogeneous equations later in the chapter.

Variants: $b^2 - 4ac = 0$

In the previous examples, the auxiliary equation had distinct roots, i.e. $b^2 - 4ac > 0$. What if we have equal roots?

❑ When the auxiliary equation has two equal roots α, the general solution is
$$y = (A + Bx)e^{\alpha x}$$

$Ae^{3x} + Bxe^{3x}$

$Ae^{\alpha x} + Be^{\alpha x} = (A+B)e^{\alpha x}$
$= Ce^{\alpha x}$

This is because the root of the auxiliary equation $m^2 - 6m + 9 = 0$ is 3.

Show that $(A + Bx)e^{3x}$ satisfies $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Side Note: The reason we have to use $y = (A + Bx)e^{\alpha x}$ instead of $\underline{Ae^{\alpha x} + Be^{\alpha x}}$ is similar to why in Pure Year 2 partial fractions, we have to use $\frac{A}{x} + \frac{B}{x^2}$ if we had a repeated denominator x.

$y = (A+Bx)e^{3x}$

$\frac{dy}{dx} = 3(A+Bx)e^{3x} + Be^{3x}$

$\frac{d^2y}{dx^2} = 9(A+Bx)e^{3x} + 3Be^{3x} + 3Be^{3x}$

$= 9(A+Bx)e^{3x} + 6Be^{3x}$

$9(A+Bx)e^{3x} + 6Be^{3x} - 18(A+Bx)e^{3x} - 6Be^{3x}$

$+ 9(A+Bx)e^{3x} = (9A - 18A + 9A)e^{3x}$

$+ (9Bx + 6B - 18Bx - 6B + 9Bx)e^{3x}$

$= 0 + 0 = \underline{\underline{0}}$

Variants: $b^2 - 4ac < 0$

This is actually exactly the same as when we usually have distinct real roots!

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 16y = 0$

A.E. $m^2 + 16 = 0$

$m = \pm 4i$

$y = Ae^{4ix} + Be^{-4ix}$

$= A(\cos 4x + i\sin 4x) + B(\cos 4x - i\sin 4x)$

$= (A+B)\cos 4x + (A-B)i\sin 4x$

$\cancel{P = A+B}$

$\cancel{Q = (A-B)i}$

$y = P\cos 4x + Q\sin 4x$

$\frac{d^2y}{dx^2} = -16P\cos 4x - 16Q\sin 4x$

If the auxiliary equation has two imaginary roots $\pm i\omega$, the general solution is $\boldsymbol{y = A\cos\omega x + B\sin\omega x}$ where A and B are arbitrary constants.

ω omega

Variants: $b^2 - 4ac < 0$

So what about more general complex roots $a \pm bi$?

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$

A.E. $m^2 - 6m + 34 = 0$

$m = 3 \pm 5i$

$y = Ae^{(3+5i)x} + Be^{(3-5i)x}$

$y = e^{3x}\left(Ae^{5ix} + Be^{-5ix}\right)$

$y = e^{3x}\left(P\cos 5x + Q\sin 5x\right)$

$y = e^{3x}($

If the auxiliary equation has two complex roots $p \pm iq$, the general solution is

$$\boldsymbol{y = e^{px}(A\cos qx + B\sin qx)}$$

where A and B are arbitrary constants.

Summary of Auxiliary Equation results

- When the auxiliary equation has **two real distinct roots** α and β, the general solution of the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants.

- When the auxiliary equation has two equal roots α, the general solution is
$$y = (A + Bx)e^{\alpha x}$$

If the auxiliary equation has two <u>imaginary</u> roots $\pm i\omega$, the general solution is $\boldsymbol{y = A\cos\omega x + B\sin\omega x}$ where A and B are arbitrary constants.

If the auxiliary equation has two <u>complex</u> roots $p \pm iq$, the general solution is
$$\boldsymbol{y = e^{px}(A\cos qx + B\sin qx)}$$
where A and B are arbitrary constants.

Find solutions to differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ with the following auxiliary equations (and helpfully provided roots).

Auxiliary Equation	Roots	General Solution
$m^2 + 6m + 8 = 0$	$m = -2, -4$	$y = Ae^{-2x} + Be^{-4x}$
$m^2 - 1 = 0$	$m = \pm 1$	$y = Ae^{x} + Be^{-x}$
$m^2 - 2m + 1 = 0$	$m = 1$	$y = (A + Bx)e^{x}$
$m^2 + 4 = 0$	$m = \pm 2i$	$y = P\cos 2x + Q\sin 2x$
$m^2 + 10x + 25 = 0$	$m = -5$	$y = (A + Bx)e^{-5x}$
$m^2 - 12m + 45 = 0$	$m = 6 \pm 3i$	$y = e^{6x}(P\cos 3x + Q\sin 3x)$
$m^2 + 10 = 0$	$m = \pm\sqrt{10}$	$y = Ae^{\sqrt{10}x} + Be^{-\sqrt{10}x}$
$m^2 + 2m + 5$	$m = -1 \pm 2i$	$y = e^{-x}(P\cos 2x + Q\sin 2x)$

Ex 7B - first column of questions
Be quick! These needn't be that demanding.