Edexcel A Level Further Maths: Core Pure

Your notes

2.1 Properties of Matrices

Contents

2.1.1 Introduction to Matrices

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Introduction to Matrices

Matrices are a useful way to represent and manipulate data in order to model situations. The elements in a matrix can represent data, equations or systems and have many real-life applications.

What are matrices?

- A matrix is a **rectangular array** of **elements** (numerical or algebraic) that are arranged in **rows** and **columns**
- The **order** of a matrix is defined by the **number** of rows and columns that it has
  - The order of a matrix with m rows and n columns is $m \times n$
- A matrix $\boldsymbol{A}$ can be defined by $\boldsymbol{A} = (a_{ij})$ where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$ and a_{ij} refers to the element in row i, column j

$$\boldsymbol{A} = (a_{i,j}) = \overbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}}^{\text{Number of columns, } n = 3} \left.\vphantom{\begin{pmatrix} a \\ a \end{pmatrix}}\right\} \text{Number of rows, } m = 2$$

What type of matrices are there?

- A **column matrix** (or column vector) is a matrix with a **single column**, $n = 1$
- A **row matrix** is a matrix with a **single row**, $m = 1$
- A **square matrix** is one in which the number of rows is **equal** to the number of columns, $m = n$
- Two matrices are **equal** when they are of the **same order** and their **corresponding elements** are **equal**, i.e. $a_{ij} = b_{ij}$ for all elements
- A **zero matrix**, $\boldsymbol{O}$, is a matrix in which all the elements are 0, e.g. $\boldsymbol{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- The identity matrix, $\boldsymbol{I}$, is a **square** matrix in which all elements along the **leading diagonal** are 1 and the rest are 0, e.g. $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

What is the transpose of a matrix?

- The **transpose** of matrix **A** is denoted as $\mathbf{A}^{\mathrm{T}}$
- The **transpose** matrix is formed by interchanging the rows and columns

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$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

Examiner Tip

- Make sure that you know how to enter and store a matrix on your calculator

Worked example

Let the matrix $\boldsymbol{A} = \begin{pmatrix} 5 & -3 & 7 \\ -1 & 2 & 4 \end{pmatrix}$

a) Write down the order of $\boldsymbol{A}$.

$\boxed{A \text{ is a } 2 \times 3 \text{ Matrix}}$

b) State the value of $a_{2,3}$.

$\boxed{a_{23} = 4}$

Basic Operations with Matrices

Your notes

Just as with ordinary numbers, **matrices** can be **added** together and **subtracted** from one another, provided that they meet certain conditions.

How is addition and subtraction performed with matrices?

- Two matrices of the **same order** can be added or subtracted
- Only **corresponding elements** of the two matrices are added or subtracted
  - $\boldsymbol{A} \pm \boldsymbol{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$
- The **resultant** matrix is of the **same order** as the original matrices being added or subtracted

What are the properties of matrix addition and subtraction?

- $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ (commutative)
- $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$ (associative)
- $\boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$
- $\boldsymbol{O} - \boldsymbol{A} = -\boldsymbol{A}$
- $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$

How do I multiply a matrix by a scalar?

- Multiply **each element** in the matrix by the **scalar** value
  - $k\boldsymbol{A} = (ka_{ij})$
- The **resultant** matrix is of the **same order** as the original matrix
- Multiplication by a **negative** scalar changes the **sign** of each element in the matrix

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Worked example

Consider the matrices $\boldsymbol{A} = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix}$.

a) Find $\boldsymbol{A} + \boldsymbol{B}$.

$$A + B = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix} = \boxed{\begin{pmatrix} -2 & 8 \\ 12 & -6 \\ -1 & -8 \end{pmatrix}}$$

b) Find $\boldsymbol{A} - \boldsymbol{B}$.

$$A - B = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix} = \boxed{\begin{pmatrix} -6 & -4 \\ 2 & 12 \\ 3 & -2 \end{pmatrix}}$$

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Multiplying Matrices

Can I always multiply a matrix by another matrix?

- Not always - only if the dimensions of the matrices allow it
- If ***A*** has order $m \times n$ and ***B*** has order $q \times p$ then you the matrix ***AB*** exists only if $n = q$
- The order of the matrix ***AB*** will be $m \times p$
- It is possible for ***AB*** to exist but ***BA*** not exist and vice versa
- ***AB*** and ***BA*** both will exist if they are both **square matrices** of the same order
  - This means the dimensions are the same $n \times n$

How do I multiply a matrix by another matrix?

- To multiply a matrix by another matrix, the **number of columns** in the **first** matrix must be **equal** to the **number of rows** in the **second** matrix
- If the order of the **first** matrix is $m \times n$ and the order of the **second** matrix is $n \times p$, then the order of the **resultant** matrix will be $m \times p$
- The product of two matrices is found by multiplying the corresponding elements in the **row** of the **first** matrix with the corresponding elements in the **column** of the **second** matrix and finding the **sum** to place in the resultant matrix
  - E.g. If $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}$
    - then $\boldsymbol{AB} = \begin{bmatrix} (ag + bi + ck) & (ah + bj + cl) \\ (dg + ei + fk) & (dh + ej + fl) \end{bmatrix}$
    - then $\boldsymbol{BA} = \begin{bmatrix} (ga + hd) & (gb + he) & (gc + hf) \\ (ia + jd) & (ib + je) & (ic + jf) \\ (ka + ld) & (kb + le) & (kc + lf) \end{bmatrix}$

How do I square an expression involving matrices?

- If an expression involving matrices is squared then you are multiplying the expression by itself, so write it out in bracket form first, e.g. $(A + B)^2 = (A + B)(A + B)$
  - remember, the regular rules of algebra do not apply here and you cannot expand these brackets, instead, add together the matrices inside the brackets and then multiply the matrices together

What are the properties of matrix multiplication?

- $\boldsymbol{AB} \neq \boldsymbol{BA}$ (non-commutative)
- $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$ (associative)
- $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ (distributive)
- $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$ (distributive)

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- $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$ (identity law)
- $\boldsymbol{AO} = \boldsymbol{OA} = \boldsymbol{O}$, where $\boldsymbol{O}$ is a zero matrix
- Powers of **square** matrices: $\boldsymbol{A}^2 = \boldsymbol{AA},\ \boldsymbol{A}^3 = \boldsymbol{AAA}$ etc.

Worked example

Consider the matrices $\boldsymbol{A}=\begin{bmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix} 5 & 1 \\ -2 & 5 \\ 9 & 7 \end{bmatrix}$.

a) Find $\boldsymbol{AB}$.

$$AB = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix} \times \begin{pmatrix} 5 & 1 \\ -2 & 5 \\ 9 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (4\times5 + 2\times-2 + -5\times9) & (4\times1 + 2\times5 + -5\times7) \\ (-3\times5 + 8\times-2 + 1\times9) & (-3\times1 + 8\times5 + 1\times7) \\ (-1\times5 + -2\times-2 + 2\times9) & (-1\times1 + -2\times5 + 2\times7) \end{pmatrix}$$

$$= \begin{pmatrix} (20-4-45) & (4+10-35) \\ (-15-16+9) & (-3+40+7) \\ (-5+4+18) & (-1-10+14) \end{pmatrix}$$

$$\boxed{AB = \begin{pmatrix} -29 & -21 \\ -22 & 44 \\ 17 & 3 \end{pmatrix}}$$

b) Explain why you cannot find $\boldsymbol{BA}$.

BA cannot be found because the number of columns in B is different to the number of rows in A

c) Find $\boldsymbol{A}^2$.

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$$A^2 = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (4\times4 + 2\times-3 + -5\times-1) & (4\times2 + 2\times8 + -5\times-2) & (4\times-5 + 2\times1 + -5\times2) \\ (-3\times4 + 8\times-3 + 1\times-1) & (-3\times2 + 8\times8 + 1\times-2) & (-3\times-5 + 8\times1 + 1\times2) \\ (-1\times4 + -2\times-3 + 2\times-1) & (-1\times2 + -2\times8 + 2\times-2) & (-1\times-5 + -2\times1 + 2\times2) \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 15 & 34 & -28 \\ -37 & 56 & 25 \\ 0 & -22 & 7 \end{pmatrix}}$$

2.1.2 Determinants of Matrices

Your notes

Determinant of a 2×2 Matrix

What is a determinant?

- The **determinant** is a **numerical value** (positive or negative) calculated from the elements in a matrix and is used to find the **inverse** of a matrix
- You can only find the determinant of a **square** matrix
- The method for finding the determinant of a 2×2 matrix is given by:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$$

Worked example

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 3 & -6 \\ p & 7 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant. Given that $\det \boldsymbol{A} = -3$, find the value of p.

$\det A = 3 \times 7 - -6 \times p = 21 + 6p$

So, $-3 = 21 + 6p$

$-24 = 6p$

$\boxed{p = -4}$

Determinant of a 3×3 Matrix

What is the minor of an element in a 3×3 matrix?

- For any **element** in a 3×3 matrix, the **minor** is the determinant of the 2×2 matrix created by crossing out the row and column containing that **element**
- For the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$
  - The **minor** of the **element** *a* would be found by:
    - crossing out the first row and first column $\begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{pmatrix}$
    - finding the determinant of the remaining 2×2 matrix $\begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh$
  - The **minor** of the **element** *f* would be found by:
    - crossing out the second row and third column $\begin{pmatrix} a & b & \cancel{c} \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & h & \cancel{i} \end{pmatrix}$
    - finding the determinant of the remaining 2×2 matrix $\begin{vmatrix} a & b \\ g & h \end{vmatrix} = ah - bg$

How do I find the determinant of a 3×3 matrix?

- Finding the determinant of a 3×3 matrix is best explained using an example
- **STEP 1**
  Select **any row** or **column** in the matrix
  - e.g. Selecting Row 2 of $\boldsymbol{M} = \begin{pmatrix} 4 & -2 & 3 \\ 5 & 6 & -7 \\ -8 & 2 & -1 \end{pmatrix}$

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- **STEP 2**
  Use the **matrix of signs** $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ to find the row or column that corresponds to the row or column selected in Step 1
  - e.g. $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ Row 2 was selected so "- + -" will be needed
- **STEP 3**
  Multiply each **element** in the selected row or column by its **minor** and use the corresponding signs form the matrix of signs to determine whether to **add or subtract** each product
  - e.g. $\det \boldsymbol{M} = -(5)\begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix} + (6)\begin{vmatrix} 4 & 3 \\ -8 & -1 \end{vmatrix} - (-7)\begin{vmatrix} 4 & -2 \\ -8 & 2 \end{vmatrix}$
- **STEP 4**
  Evaluate the **minors** and calculate the **determinant**
  - e.g. $\det \boldsymbol{M} = -(5)(-4) + (6)(20) - (-7)(-8) = 84$
  - Check the answer using a calculator with a matrix feature

Examiner Tip

- In general, you can use the **top row** to find the determinant
  - this will save having to recall the matrix of signs, it will always be "**+ - +**"
  - and allow you to quickly jump from Step 1 to Step 4 in the method above
  - the matrix of signs is still needed for finding the inverse of a 3×3 matrix
- Using a row or column that **contains a 0** can speed up the process
- Do use the matrix mode on your calculator where possible
  - Look for any notes on questions about using/not using "calculator technology"
  - Consider the number of marks a question or part is worth

Your notes

Worked example

Let $\boldsymbol{M} = \begin{pmatrix} 2 & -4 & 5 \\ 2 & 2k^2 & -k \\ k & k+1 & k^2 \end{pmatrix}$ where k is a positive integer.

a) Find, in terms of k, the determinant of **M**.

Select row 1, corresponding signs are + − +

$$M = \begin{pmatrix} \boxed{2 \quad -4 \quad 5} \\ 2 \quad 2k^2 \quad -k \\ k \quad k+1 \quad k^2 \end{pmatrix}$$

Multiply each element in the row by its minor

$$\det M = +(2)\begin{vmatrix} 2k^2 & -k \\ k+1 & k^2 \end{vmatrix} - (-4)\begin{vmatrix} 2 & -k \\ k & k^2 \end{vmatrix} + (5)\begin{vmatrix} 2 & 2k^2 \\ k & k+1 \end{vmatrix}$$

Evaluate

$$\det M = (2)(2k^4 + k(k+1)) + (4)(2k^2 + k^2) + (5)(2(k+1) - 2k^3)$$

$$\det M = 4k^4 + 2k^2 + 2k + 12k^2 + 10k + 10 - 10k^3$$

$$\boxed{\det M = 4k^4 - 10k^3 + 14k^2 + 12k + 10}$$

b) Given that $\det \boldsymbol{M} = 226$, find the value of k.

$$226 = 4k^4 - 10k^3 + 14k^2 + 12k + 10$$

$$0 = 4k^4 - 10k^3 + 14k^2 + 12k - 216$$

Solve the quartic using a polynomial solver on your calculator

$$k = 3, \ -2.12..., \ 0.81... + 2.79...i, \ 0.81... - 2.79...i$$

$$\boxed{k = 3}$$

Properties of Determinants

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What are the properties of determinants of matrices?

- The determinant of an **identity matrix** is $\det(\boldsymbol{I}) = 1$
- The determinant of a **zero matrix** is $\det(\boldsymbol{O}) = 0$
- When finding the determinant of a **multiple** of a matrix or the **product** of two matrices:
  - $\det(k\boldsymbol{A}) = k^n \det(\boldsymbol{A})$ (for a $n \times n$ matrix)
  - $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B})$
  - $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$
- If $\det(\boldsymbol{A}) = 0$ then $\boldsymbol{A}$ is singular
- If $\det(\boldsymbol{A}) \neq 0$ then $\boldsymbol{A}$ is non-singular

Worked example

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 3 & -6 \\ p & 7 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant. Given that $\det \boldsymbol{A} = -3$, find the determinant of $4\boldsymbol{A}$.

$$\det(4A) = 4^2 \times -3 = \boxed{-48}$$

Your notes

2.1.3 Inverses of Matrices

Inverse of a Matrix

What is an inverse of a matrix?

- The determinant can be used to find out if a matrix is invertible or not:
  - If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is **invertible**
  - If $\det \boldsymbol{A} = 0$, then $\boldsymbol{A}$ is singular and does **not** have an inverse
- The inverse of a square matrix $\boldsymbol{A}$ is denoted as the matrix $\boldsymbol{A}^{-1}$
- The product of these matrices is an **identity** matrix, $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$
- You can use your calculator to find the inverse of matrices
  - You need to know how to find the inverse of 2×2 and 3×3 matrices by hand
- Inverses can be used to rearrange equations with matrices:
  - $\boldsymbol{AB} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C}$ (pre-multiplying by $\boldsymbol{A}^{-1}$)
  - $\boldsymbol{BA} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{C}\boldsymbol{A}^{-1}$ (post-multiplying by $\boldsymbol{A}^{-1}$)
- The inverse of a product of matrices is the product of the inverse of the matrices in reverse order:
  - $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Examiner Tip

- Many past exam questions exploit the property $\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$
  - these typically start with two, seemingly, unconnected matrices
    - **M** and **N**, say, possibly with some unknown elements
  - the result of **MN** is often a **scalar multiple** of **I**, *k***I** say
  - so **M** and **N** are (almost) inverses of each other
    - You are expected to deduce $\boldsymbol{M}^{-1} = \dfrac{1}{k}\boldsymbol{N}$
  - Look out for and practise this style of question, they are very common

Your notes

Worked example

Consider the matrices $\boldsymbol{M}=\begin{pmatrix} p & -2 & -3 \\ -3 & 0 & p \\ 1 & 2p & 1 \end{pmatrix}$ and $\boldsymbol{N}=\begin{pmatrix} -4p & -10 & -2p \\ 5 & 5 & 5 \\ -12 & -5p & -6 \end{pmatrix}$, where p is a constant.

a) Find $\boldsymbol{MN}$, writing the elements in terms of p where necessary.

$$MN = \begin{pmatrix} p & -2 & -3 \\ -3 & 0 & p \\ 1 & 2p & 1 \end{pmatrix}\begin{pmatrix} -4p & -10 & -2p \\ 5 & 5 & 5 \\ -12 & -5p & -6 \end{pmatrix}$$

$$\boxed{MN = \begin{pmatrix} -4p^2+26 & 5p-10 & 8-2p^2 \\ 0 & 30-5p^2 & 0 \\ 6p-12 & 5p-10 & 8p-6 \end{pmatrix}}$$

b) In the case $p=2$, deduce the matrix $\boldsymbol{M}^{-1}$.

Substitute $p=2$ into MN

$$MN = \begin{pmatrix} -4(2)^2+26 & 5(2)-10 & 8-2(2)^2 \\ 0 & 30-5(2)^2 & 0 \\ 6(2)-12 & 5(2)-10 & 8(2)-6 \end{pmatrix}$$

$$MN = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

Note that $MN = 10I$, $\therefore$ $M^{-1} = \frac{1}{10}N$

$$M^{-1} = \frac{1}{10}\begin{pmatrix} -4(2) & -10 & -2(2) \\ 5 & 5 & 5 \\ -12 & -5(2) & -6 \end{pmatrix}$$

$$\boxed{M^{-1} = \begin{pmatrix} -\frac{4}{5} & -1 & -\frac{2}{5} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{6}{5} & -1 & -\frac{3}{5} \end{pmatrix}}$$

Your notes

Finding the Inverse of a 2×2 Matrix

How do I find the inverse of a 2×2 matrix?

- The method for finding the inverse of a 2×2 matrix is:
  - Switch the two entries on leading diagonal
  - Change the signs of the other two entries
  - Divide by the determinant

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \; ad \neq bc$$

Your notes

Worked example

Consider the matrices $\boldsymbol{P}=\begin{pmatrix}4 & -2\\ 8 & 2\end{pmatrix}$, $\boldsymbol{Q}=\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix}$ and $\boldsymbol{R}=\begin{pmatrix}18 & 18\\ 6 & 54\end{pmatrix}$, where k is a constant.

a) Find $\boldsymbol{P}^{-1}$.

$$P^{-1} = \frac{1}{4\times 2-(-2)\times 8}\begin{pmatrix}2 & 2\\ -8 & 4\end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix}2 & 2\\ -8 & 4\end{pmatrix}$$

$$\boxed{P^{-1} = \begin{pmatrix}\frac{1}{12} & \frac{1}{12}\\ -\frac{1}{3} & \frac{1}{6}\end{pmatrix}}$$

b) Given that $\boldsymbol{PQ}=\boldsymbol{R}$ find the value of k.

$$PQ = R \Rightarrow Q = P^{-1}R$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{12} & \frac{1}{12}\\ -\frac{1}{3} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}18 & 18\\ 6 & 54\end{pmatrix}$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix} = \begin{pmatrix}\left(\frac{1}{12}\times 18+\frac{1}{12}\times 6\right) & \left(\frac{1}{12}\times 18+\frac{1}{12}\times 54\right)\\ \left(-\frac{1}{3}\times 18+\frac{1}{6}\times 6\right) & \left(-\frac{1}{3}\times 18+\frac{1}{6}\times 54\right)\end{pmatrix}$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix} = \begin{pmatrix}2 & 6\\ -5 & 3\end{pmatrix}$$

$$\boxed{k = 2}$$

Your notes

Finding the Inverse of a 3×3 Matrix

How do I find the inverse of a 3×3 matrix?

- This is easiest to see with an example
  - Use the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 3 \\ -3 & 1 & -1 \\ 4 & 2 & -2 \end{pmatrix}$
- **STEP 1**
  Find the **determinant** of a 3×3 matrix
  - The inverse **only exists** if the determinant is **non-zero**
    - e.g. $\det \boldsymbol{A} = 2\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} - 0\begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} + 3\begin{vmatrix} -3 & 1 \\ 4 & 2 \end{vmatrix} = 2(0) + 3(-10) = -30$
- **STEP 2**
  Find the **minor** for **every element** in the matrix.
  - You will sometimes see this written as a huge matrix – like below
    This is called the **matrix** of **minors** and is often denoted by ***M***
    With pen and paper, this can get quite large and cumbersome to work with so you may prefer to lay the minors out separately and form ***M*** at the end
    - e.g. $\boldsymbol{M} = \begin{pmatrix} \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} -3 & 1 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -3 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 10 & -10 \\ -6 & -16 & 4 \\ -3 & 7 & 2 \end{pmatrix}$
- **STEP 3**
  Find the matrix of **cofactors**, often denoted by **C**, by combining the **matrix of signs,** with the matrix of **minors**
  - The matrix of signs is $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$
    - e.g. $\boldsymbol{C} = \begin{pmatrix} +(0) & -(10) & +(-10) \\ -(-6) & +(-16) & -(4) \\ +(-3) & -(7) & +(2) \end{pmatrix} = \begin{pmatrix} 0 & -10 & -10 \\ 6 & -16 & -4 \\ -3 & -7 & 2 \end{pmatrix}$
- **STEP 4**
  Transpose the matrix of **cofactors** to form $\mathbf{C}^T$

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- This is sometimes called the adjugate of ***A***
  - e.g. $\boldsymbol{C}^{\mathrm{T}} = \begin{pmatrix} 0 & 6 & -3 \\ -10 & -16 & -7 \\ -10 & -4 & 2 \end{pmatrix}$
- **STEP 5**
  Find the **inverse** of ***A*** by dividing **C**$^{\mathrm{T}}$ by the determinant of ***A***
  - $\boldsymbol{A}^{-1} = \dfrac{1}{\det \boldsymbol{A}} \boldsymbol{C}^{\mathrm{T}}$
  - e.g. $\boldsymbol{A}^{-1} = \dfrac{1}{-30}\begin{pmatrix} 0 & 6 & -3 \\ -10 & -16 & -7 \\ -10 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{2}{15} & \frac{1}{10} \\ \frac{1}{3} & \frac{8}{15} & \frac{7}{30} \\ \frac{1}{3} & \frac{2}{15} & -\frac{1}{15} \end{pmatrix}$
- It is often convenient to leave ***A***$^{-1}$ as a (positive) scalar multiple of **C**$^{\mathrm{T}}$, rather than have a matrix full of fractions that can be awkward to read and follow
  - e.g. $\boldsymbol{A}^{-1} = \dfrac{1}{30}\begin{pmatrix} 0 & -6 & 3 \\ 10 & 16 & 7 \\ 10 & 4 & -2 \end{pmatrix}$

Can I use my calculator to get the inverse of a matrix?

- Yes, of course, but only where possible!
- Questions with unknown elements will generally not be solvable directly on a calculator
  - If by the end of the questions, the unknowns have been found, you can then check your answers using the calculator
- Some questions with purely numerical matrices may still ask you to show your full working without relying on calculator technology - but you can still use it at the end to check!
- Two things to be very careful with when using your calculator
  - When entering values into a matrix, check and be clear as to where the cursor moves to after each element – does it move across or down?
  - When displaying a matrix many calculators will display values as (rounded/truncated) decimals; highlighting a particular one will show the value as an exact fraction

Examiner Tip

- Do not worry too much about the various terms and language used in finding the inverse of a 3×3 matrix, learning and following the process (without a calculator) is more important
- If a question says not to rely on "calculator technology" in your answer, you must show full working throughout
  - However, you can still use your calculator to check your work at the end
  - Consider the number of marks a question is worth for a clue as to how much working may be necessary

Your notes

Your notes

Worked example

Given that $\boldsymbol{A}=\begin{pmatrix} 2 & -4 & 5 \\ 2 & 18 & -k \\ 3 & 4 & 9 \end{pmatrix}$, find $\boldsymbol{A}^{-1}$ in terms of k.

Find the determinant of A

$$\det A = +(2)\begin{vmatrix} 18 & -k \\ 4 & 9 \end{vmatrix} - (-4)\begin{vmatrix} 2 & -k \\ 3 & 9 \end{vmatrix} + (5)\begin{vmatrix} 2 & 18 \\ 3 & 4 \end{vmatrix}$$

$$\det A = (2)(162+4k) + (4)(18+3k) + (5)(-46)$$

$$\det A = 324 + 8k + 72 + 12k - 230$$

$$\det A = 166 + 20k$$

Find the matrix of minors, label it M

$$M = \begin{pmatrix} \begin{vmatrix} 18 & -k \\ 4 & 9 \end{vmatrix} & \begin{vmatrix} 2 & -k \\ 3 & 9 \end{vmatrix} & \begin{vmatrix} 2 & 18 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} -4 & 5 \\ 4 & 9 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} & \begin{vmatrix} 2 & -4 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} -4 & 5 \\ 18 & -k \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & -k \end{vmatrix} & \begin{vmatrix} 2 & -4 \\ 2 & 18 \end{vmatrix} \end{pmatrix}$$

$$M = \begin{pmatrix} 162+4k & 18+3k & -46 \\ -56 & 3 & 20 \\ 4k-90 & -2k-10 & 44 \end{pmatrix}$$

Your notes

Find the matrix of cofactors, label it C

$$C = \begin{pmatrix} 162 + 4k & -(18 + 3k) & -46 \\ -(-56) & 3 & -(20) \\ 4k - 90 & -(-2k - 10) & 44 \end{pmatrix}$$

$$C = \begin{pmatrix} 162 + 4k & -18 - 3k & -46 \\ 56 & 3 & -20 \\ 4k - 90 & 2k + 10 & 44 \end{pmatrix}$$

Transpose C

$$C^T = \begin{pmatrix} 162 + 4k & 56 & 4k - 90 \\ -18 - 3k & 3 & 2k + 10 \\ -46 & -20 & 44 \end{pmatrix}$$

Divide C^T by $\det A$

$$\boxed{A^{-1} = \frac{1}{166 + 20k}\begin{pmatrix} 162 + 4k & 56 & 4k - 90 \\ -18 - 3k & 3 & 2k + 10 \\ -46 & -20 & 44 \end{pmatrix}}$$