Chapter 14: Exponentials and Logarithms

You have encountered exponential expressions like $y = 2^x$ before, but probably not 'the' exponential function $y = e^x$. Similarly, you will learn that the inverse of $y = 2^x$ is $y = \log_2 x$.

1:: Sketch exponential graphs.

Sketch $y = 2^x$ and $y = e^x$ on the same axes.

2:: Use an interpret models that use exponential functions.

The population P of Totnes after t years is modelled using $P = Ae^{kt}$, where A, k are constants...

3:: Be able to differentiate e^{kx}.

If $y = 5e^{3x}$, determine $\frac{dy}{dx}$.

4:: Understand the log function and use laws of logs.

Solve the equation:

$$\log_2(2x) = \log_2(5x + 4) - 3$$

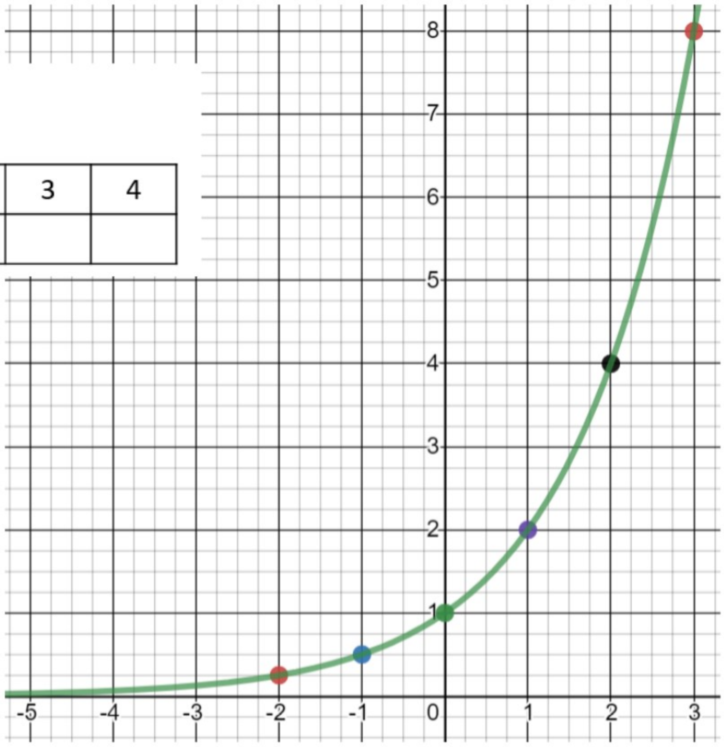
5:: Use logarithms to estimate values of constants in non-linear models.

(This is a continuation of (2))

Exponential graphs

Plot the function $y = 2^x$

x	-2	-1	0	1	2	3	4
y							



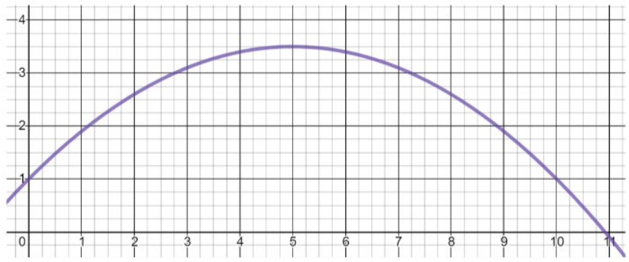
Ensure that you can distinguish between a x^a (e.g. polynomial) term and an a^x exponential term.

In the former the variable is in the **base**, and in the latter the variable is in the **power**.
2^x behaves very differently to x^2, both in its rate of growth (i.e. exponential terms grow much faster!) and how it differentiates.

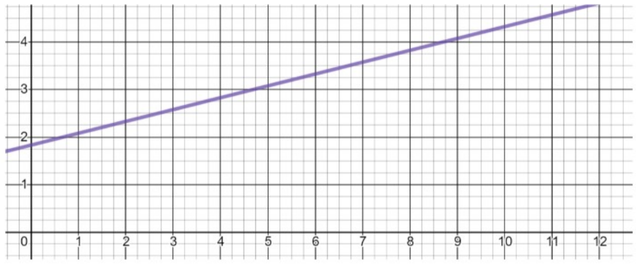
Why are exponential functions important?

Different functions can be used to model different real life scenarios:

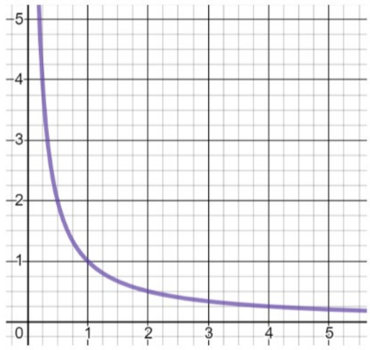
- **Quadratic functions**, for example, are useful modelling how an arrow fired by an archer flies



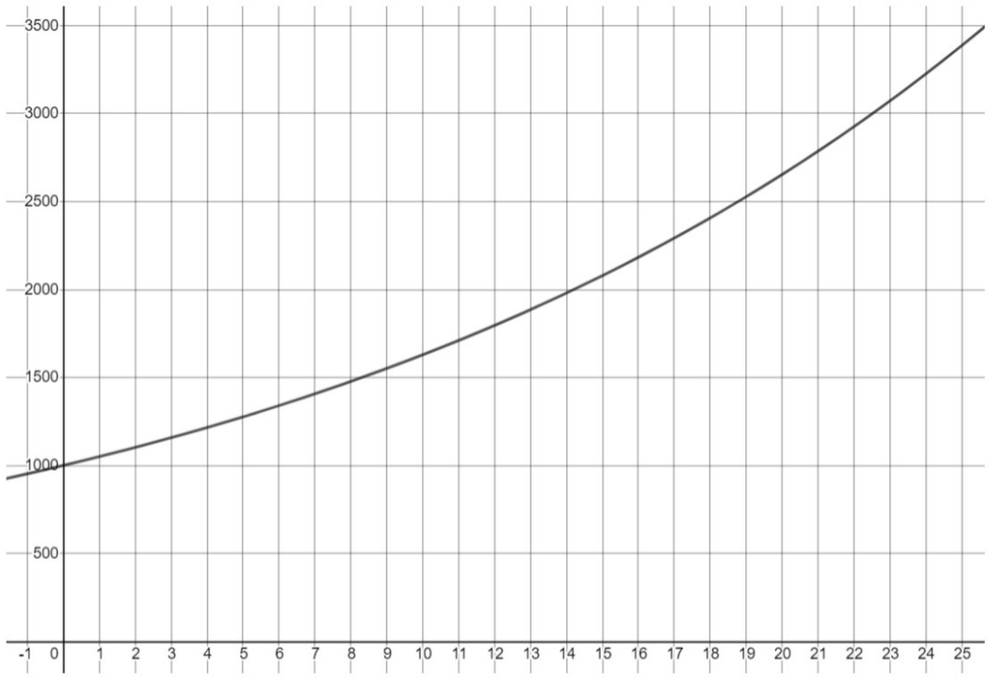
- **Linear functions** can model things that have constant growth, adding the same amount each time



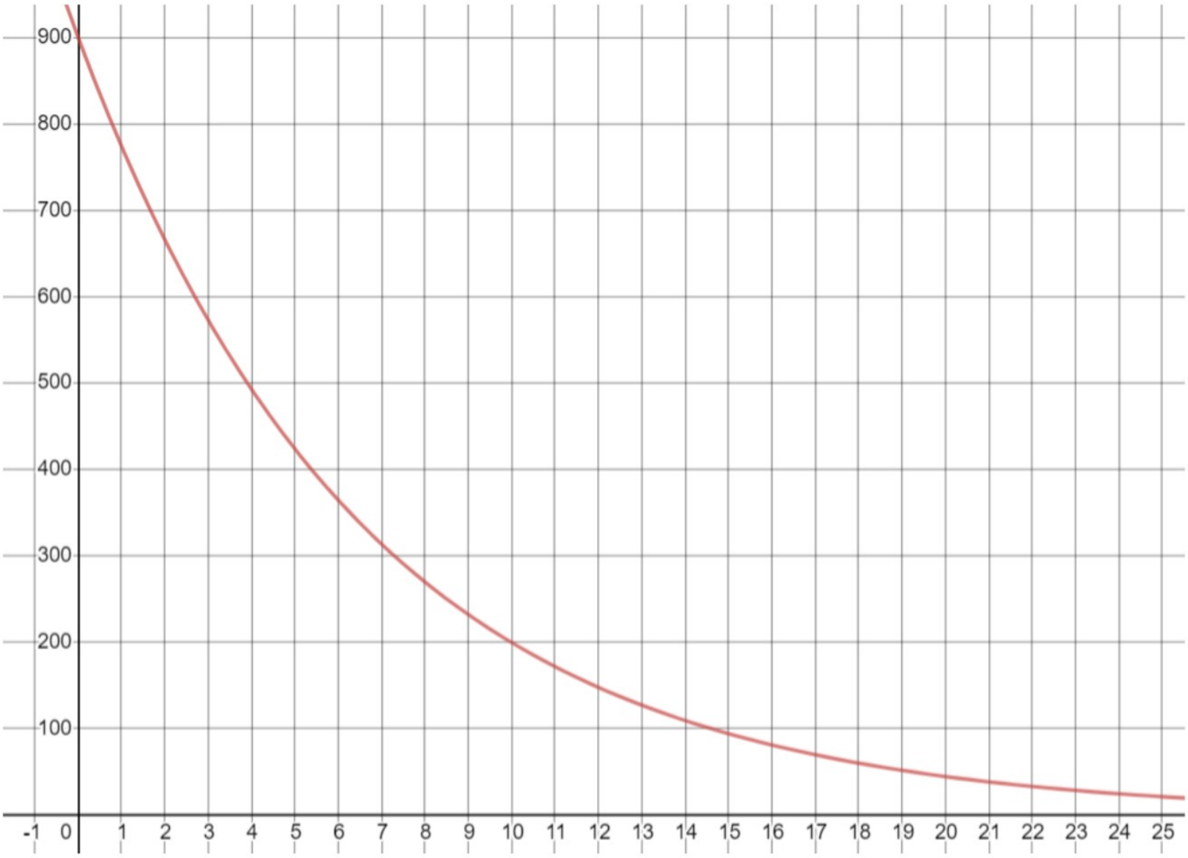
- **Reciprocal functions** can model two things that are inversely proportional



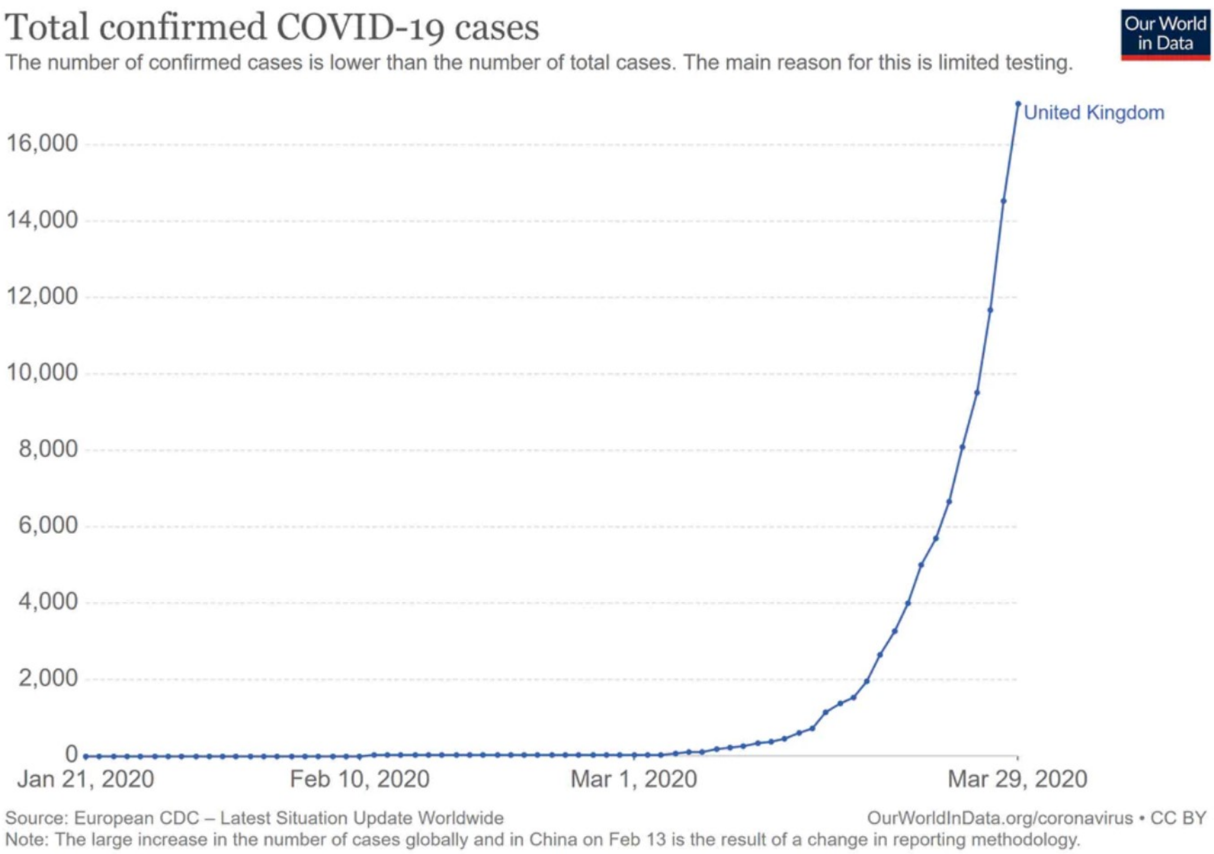
- **Exponential functions** can be used to model scenarios when we're **multiplying by the same amount** each time. For example:
  - We could use $S = 1000(1.05^t)$ to model our savings with interest, where each year we have 1.05 times as much, i.e. with 5% added interest.



- We could use $P = 900(0.86^{t})$ to model a population of a species that each year is being multiplied by 0.86, i.e. is decreasing by 14% per year.

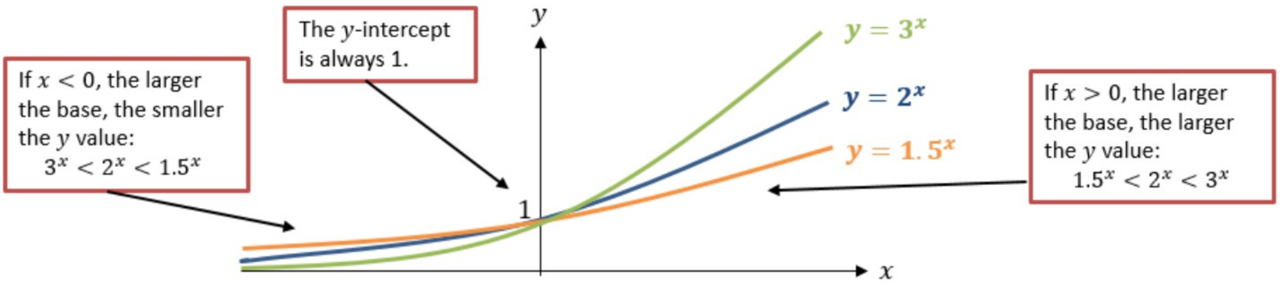


- ... and can also use exponential functions in modelling the spread of Covid-19

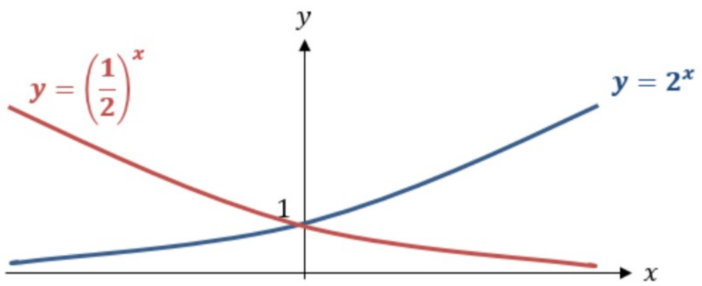


Contrasting exponential graphs

Here are sketches of $y = 3^x, y = 2^x, y = 1.5^x$



On the same axes sketch $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$

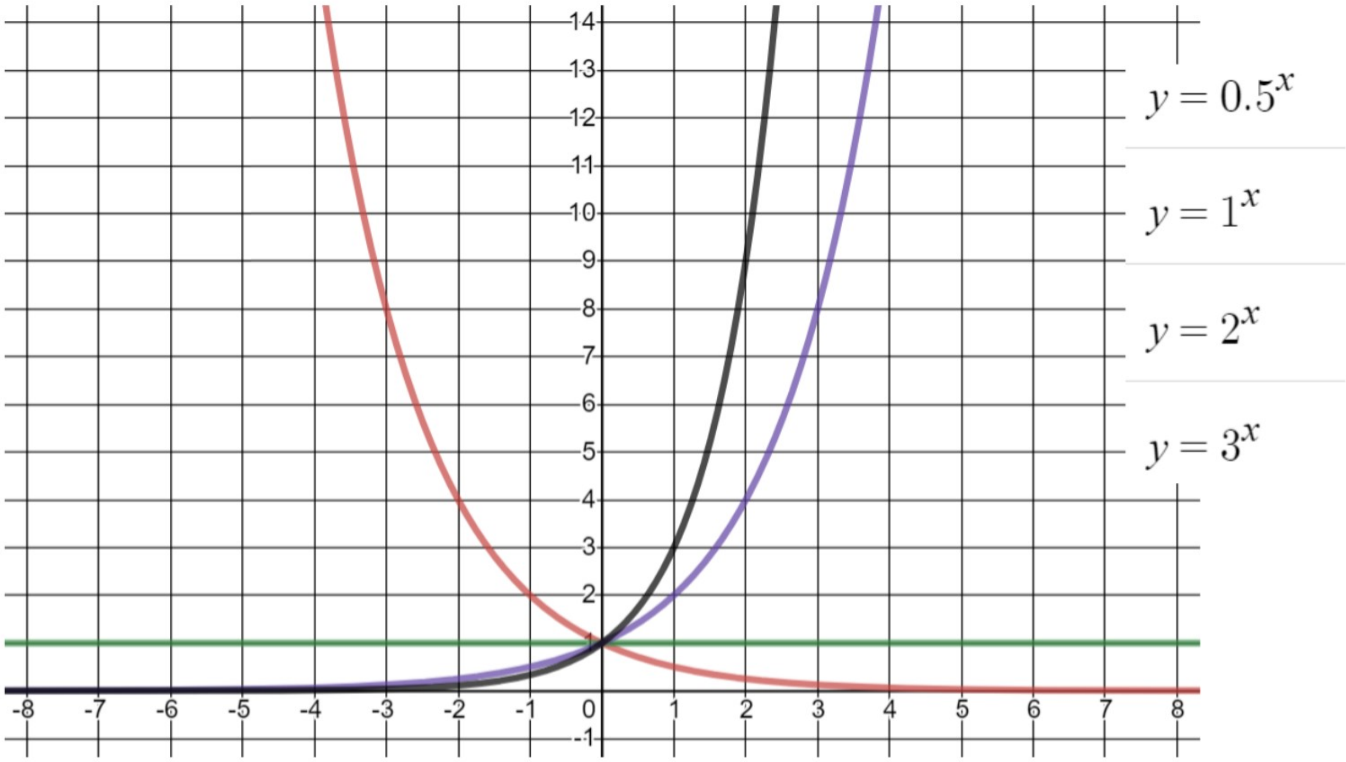


Three important notes:

- $y = 2^x$ is said to be "**exponential growing**" whereas $y = \left(\frac{1}{2}\right)^x$ is said to be "**exponentially decaying**", because it's getting smaller (halving) each time x increases by 1.
- $y = \left(\frac{1}{2}\right)^x$ is a reflection of $y = 2^x$ in the line $x = 0$.
  Proof:
  If $f(x) = 2^x, f(-x) = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$
- $\left(\frac{1}{2}\right)^x$ would usually be written 2^{-x}.
  <u>You should therefore in general be able to recognise and sketch the graph $y = a^{-x}$.</u>

https://www.desmos.com/calculator/rnfnf0yxgj

Which graph matches which function?



Graph Transformations

Sketch $y = 2^{x+3}$

The population of squirrels in a forest is modelled by the equation $\mathrm{P} = 300(1.05)^t$, where t is in years. Sketch a graph of the population over time

For $f(x) = 2^x$ sketch:

- $y = 4f(x)$
- $y = f(x) + 2$
- $y = f\left(\frac{1}{4}x\right)$

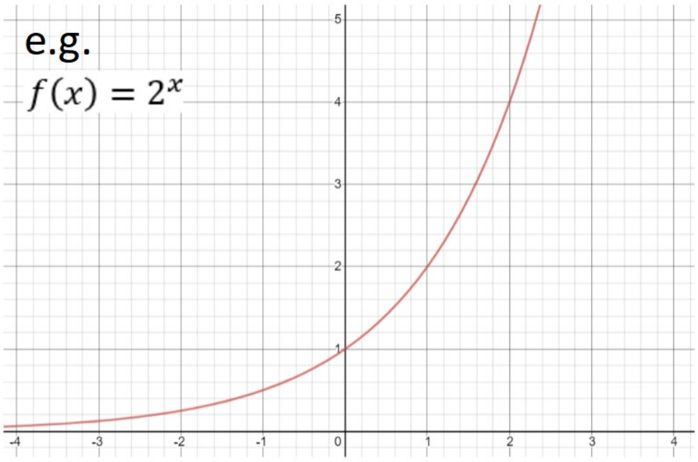
Euler's Number, e

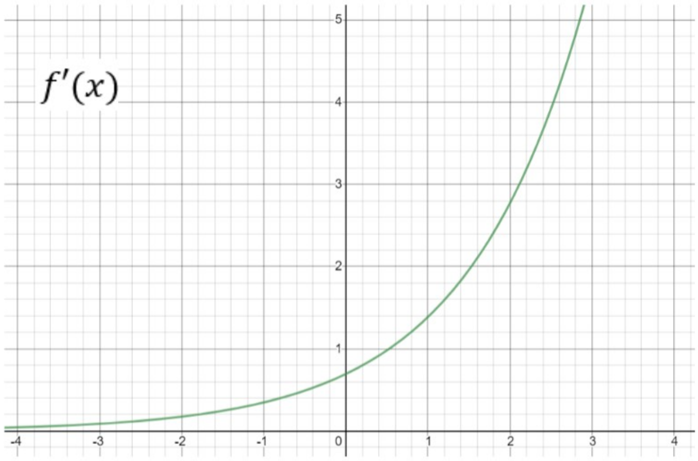
Exponential functions of the form $f(x) = a^x$ have a special property.

The graphs of their **gradient functions** $\boldsymbol{f'(x)}$ are a similar shape to the graphs of the functions themselves.

This means that $\boldsymbol{f'(x) = ka^x}$ where k is a constant.

Let's investigate.





https://www.desmos.com/calculator/pkpqpapaue

I highly recommend watching Numberphile's video on Euler's number

$e = 2.71828 \ldots$ is known as **Euler's Number.**

It is one of the five most fundamental constants in mathematics (0, 1, i, e, π).

It has the property that:

$$\boxed{y = e^x \qquad \rightarrow \qquad \frac{dy}{dx} = e^x}$$

Although any function of the form $y = a^x$ is known as **an** exponential function, e^x is known as ***the*** exponential function.

You can find the exponential function on your calculator, above the "ln" key

e.g. Find the value of e^4

Differentiating $y = ae^{kx}$

✎ If $y = e^{kx}$, where k is a constant, then $\frac{dy}{dx} = ke^{kx}$

Note: This is not a standalone rule but an application of something called the '*chain rule*', which you will encounter in Year 2.

Different e^{5x} with respect to x.

Different e^{-x} with respect to x.

Different $4e^{3x}$ with respect to x.

Note: In general, when you scale the function, you scale the derivative/integral.

More Graph Transformations

Sketch $y = e^{3x}$

Sketch $y = 5e^{-x}$

Sketch $y = 2 + e^{\frac{1}{3}x}$

Sketch $y = 50e^{\frac{1}{2}x} - 20$

Ex 14B

Exponential Modelling

There are two key features of exponential functions which make them suitable for **population growth**:

1. **a^x gets a times bigger each time x increases by 1. (Because $a^{x+1} = a \times a^x$)**
   With population growth, we typically have a fixed percentage increase each year. So suppose the growth was 10% a year, and we used the equivalent decimal multiplier, 1.1, as a. Then 1.1^t, where t is the number of years, would get 1.1 times bigger each year.
2. **The rate of increase is proportional to the size of the population at a given moment.**
   This makes sense: The 10% increase of a population will be twice as large if the population itself is twice as large.

Suppose the population P in *The Republic of Maths* is modelled by $P = 100e^{3t}$ where t is the numbers years since *The Republic* was established.
What is the initial population?
What is the initial rate of population growth?

The density of a pesticide in a given section of field, P mg/m^2, can be modelled by the equation $P = 160e^{-0.006t}$
where t is the time in days since the pesticide was first applied.

a. Use this model to estimate the density of pesticide after 15 days.

b. Interpret the meaning of the value 160 in this model.

c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.

d. Interpret the significance of the sign of your answer in part (c).

e. Sketch the graph of P against t.

Ex 14C

Logarithms

You know the inverse of many mathematical operations; we can undo an addition by 2 for example by subtracting 2. But is there an inverse function for an exponential function?

	Function		Inverse	
4	$x \times 3$	12		4
4	$x + 3$	7		4
4	x^2	16		4
4	x^5	1024		4
4	3^x	81		4

Interchanging between exponential and log form

Logarithms (really should just be called 'powers') exist in order to provide an inverse to exponential functions.

$\log_a n$ ("said log base a of n") is equivalent to $a^x = n$

The log function outputs the **missing power**

$$3^2 = 9 \quad \Longleftrightarrow \quad \log_3 9 = 2$$

Here are two methods of interchanging between these forms.
Pick your favourite!

Method 1: 'Missing Power'

$$\log_2 8 = 3$$

- Note first the base of the log must match the base of the exponential.
- $\log_2 8$ for example asks the question "2 to **what power** gives 8?"

Method 2: Do same operation to each side of equation.

$$\log_2 8 = 3$$

Since KS3 you're used to the idea of doing the same thing to each side of the equation that 'undoes' whatever you want to get rid of.

$$3x + 2 = 11$$
$$(-2) \qquad (-2)$$
$$3x = 9$$

"log base a" undoes "a to the power of" and vice versa, as they are inverse functions.

Rewrite using a logarithm

a) $6^2 = 36$

b) $2^7 = 128$

c) $64^{\frac{1}{2}} = 8$

Rewrite using a power

a) $\log_3 81 = 4$

b) $\log_2\left(\frac{1}{8}\right) = -3$

c) $\log 100 = 2$

Without using a calculator, find out the value of the following:

a) $\log_5 25$

b) $\log_3 81$

c) $\log_2 32$

d) $\log_{10} 1000$

e) $\log_4 1$

f) $\log_4 4$

g) $\log_2 \left(\frac{1}{2}\right)$

h) $\log_3 \left(\frac{1}{27}\right)$

i) $\log_2 \left(\frac{1}{16}\right)$

j) $\log_a(a^3)$

k) $\log_4(-1)$

While a log can output a negative number, we **can't log negative numbers**.

$$\log_a a =$$

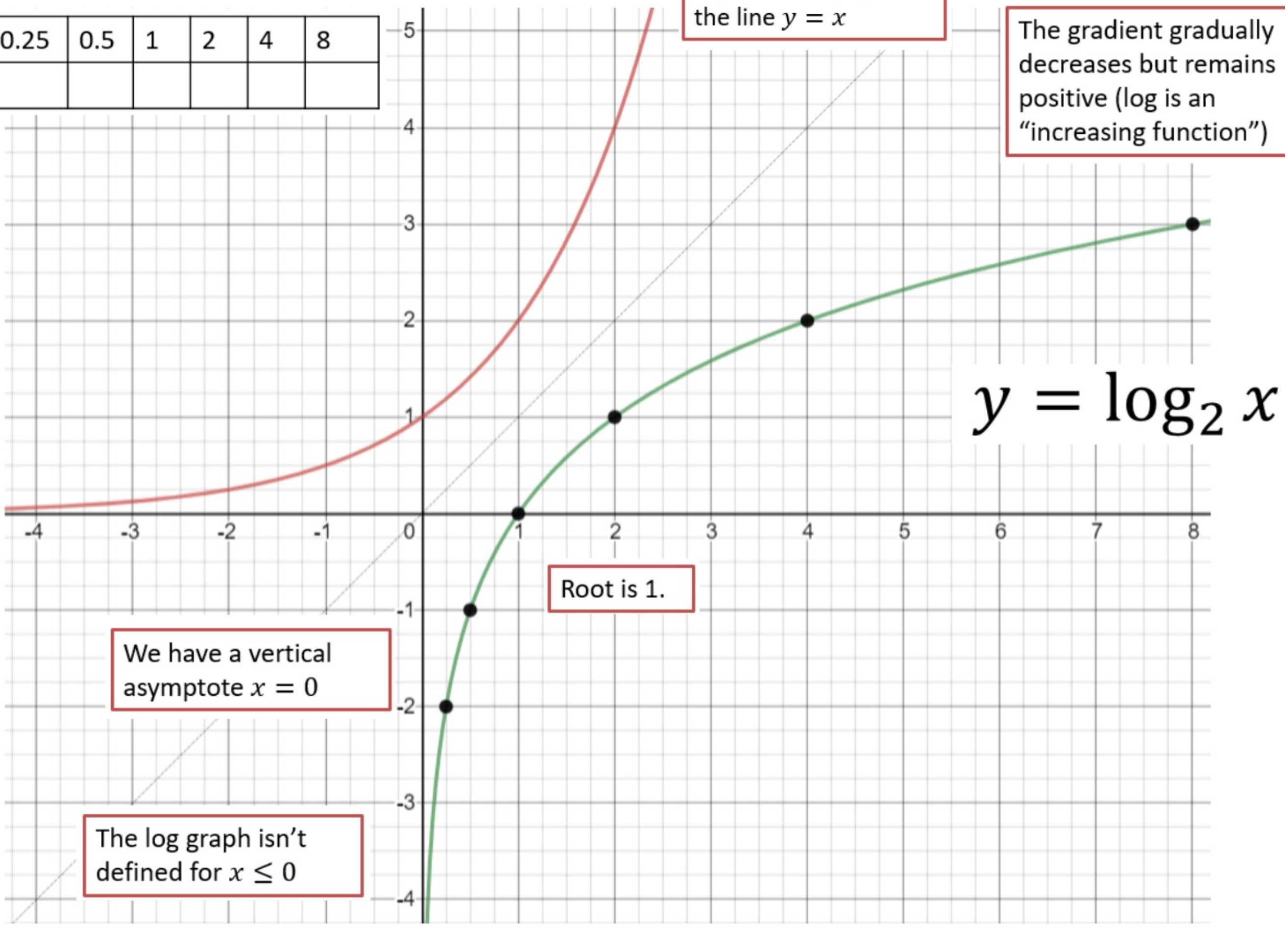
$$\log_a 1 =$$

$$\log_a(a^x) =$$

$$a^{\log_a(x)} =$$

Logarithmic graphs

x	0.25	0.5	1	2	4	8
y						



With Your Calculator

There are three buttons on your calculator for computing logs:

$\log_{\square} \square$

$\log_3 7 =$

$\log_5 0.3 =$

ln

$\ln 10 =$

$\ln e =$

ln is the "**natural log of x**", meaning "log to the base e", i.e. it the inverse of e^x.

$\ln(x) = \log_e(x)$

log

$\log 100 =$

log without a base is **base 10** by default

Ex 14D

Laws of Logs

Three main laws:

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$$

$$\log_a\left(x^k\right) = k\log_a x$$

Special cases:

$$\log_a a = 1 \quad (a > 0, a \neq 1)$$

$$\log_a 1 = 0 \quad (a > 0, a \neq 1)$$

$$\log\left(\frac{1}{x}\right) = \log(x^{-1}) = -\log(x)$$

The logs must have a consistent base.

i.e. You can move the power to the front.

We often try to avoid leaving fractions inside logs. So if the answer was:

$$\log_2\left(\frac{1}{3}\right)$$

You should write your answer as: $-\log_2 3$

Reciprocating the input negates the output.

Laws of Logs: Proof

$$\boxed{\log_a x + \log_a y = \log_a xy}$$

Let $\log_a x = b$
and $\log_a y = c$

Then

$$xy =$$

$$\boxed{\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)}$$

Let $\log_a x = b$
and $\log_a y = c$

Then

$$\frac{x}{y} =$$

$$\boxed{\log_a(x^k) = k \log_a x}$$

$$\log_a(x^k) = \log_a($$

IMPORTANT

$$\log_a(bx^k) \neq k \log_a(bx)$$

$$\log_a(bx^k) =$$

Write as a single logarithm:

a. $\log_3 6 + \log_3 7$

b. $\log 15 - \log 3$

c. $2\log_5 3 + 3\log_5 2$

d. $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

e. $3\ln 4 + 3\ln 2$

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$

a. $\log_a(x^2yz^3)$

b. $\log_a\left(\frac{x}{y^3}\right)$

c. $\log_a\left(\frac{x\sqrt{y}}{z}\right)$

d. $\log_a\left(\frac{x}{a^4}\right)$

Mistakes to avoid

These are **NOT LAWS OF LOGS**, but are mistakes students often make:

$$\log_a(b + c) \neq \log_a b + \log_a c$$

There is **no method** to simplify the **log of a sum**, only the sum of two logs!

$$(\log_2 x)^3 \neq 3\log_2 x$$

The power must be on the input (here the x), but here the power is around the entire log.

Simplifying log expressions

Many questions, particularly in Year 2 Integration, will ask for your answer in simplified, exact form. They usually ask for $a\ln 2$ or $a + b\ln c$ form

Simplify to the form $a + b\ln 2$:

a. $\ln 2e$

b. $\ln 2 - \ln 8 + \ln 16$

c. $\ln \frac{1}{2} + \ln 8e^2$

d. $\ln 2\sqrt{2} - \ln \frac{16}{\sqrt{2}} + 2\ln \sqrt{2}e$

e. $3\ln 8e + 3\ln 2$

Convert all logarithms so they are in powers of 2

Separate multiplications/divisions using log laws if necessary

Simplify by adding and subtracting

Your Turn

Simplify to the form $a + b\ln 2$:

a. $\ln 8e$

b. $\ln 4 - \ln 2 + \ln 32$

c. $\ln 2e^{\frac{1}{2}}$

d. $\ln 2\sqrt{e} - \frac{1}{3}\ln 4 + 2\ln e$

e. $5\ln 4e + 3\ln 2e^{-1}$

Answers

a. $1 + 3\ln 2$

b. $6\ln 2$

c. $\frac{1}{2} + \ln 2$

d. $\frac{5}{2} + \frac{1}{3}\ln 2$

e. $2 + 13\ln 2$

Solving Equations with Logs

Solve the equation $\log_{10} 4 + 2\log_{10} x = 2$

The strategy is to **<u>combine the logs into one</u>** and isolate on one side.

Solve the equation $2\log x = 3 - \log 6$

Careful!

$2\log_a x + \log_a y \neq 2\log_a xy$

Solve the equation $2\log_3(x+11) = 4 + 2\log_3(x-5)$

Given that $2\log_2(x+15) - \log_2 x = 6$,

(*a*) show that $x^2 - 34x + 225 = 0$.

(5)

(*b*) Hence, or otherwise, solve the equation $2\log_2(x+15) - \log_2 x = 6$.

(2)

Solving Equations with Exponentials

Solve $3^x = 20$

This is often said
"Taking logs of both sides..."

Side note: you can take logs of *any* base of both sides... what happens if we do it with ln?
I *sometimes* prefer this as it can be quicker to type (and write!)...

Solve $5^{4x-1} = 61$

Taking ln of both sides...

Solve $3^x = 2^{x+1}$

Solve $2^x 3^{x+1} = 5$, giving your answer in exact form.

Your Turn

Solve $3^{2x-1} = 5$, giving your answer to 3dp.

Solve $3^{x+1} = 4^{x-1}$, giving your answer to 3dp.

Ex 14F
NOT Q2 or Q4

Pseudo-Quadratics revisited

Solve $4^{2x} - 6(4^x) + 5 = 0$, giving your answer to 3dp.

Solve $8^{2x} - 12 = 8^x$, giving your answer to 3dp.

Solve $3^{x+1} = 3^{2x} - 4$, giving your answer to 3dp.

Solve $5^{x+1} = 5^{2x-4}$, giving your answer to 3dp.

Ex 14F
Q2, 4

Natural Logarithms

✎ The inverse of $y = e^x$ is $y = \ln x$

$$\ln e^x =$$

$$e^{\ln x} =$$

Solve $e^x = 5$

Solve $2 \ln x + 1 = 5$

Solve $\ln(3x+1) = 2$

Solve $2^x e^{x+1} = 3$ giving your answer as an exact value.

Pseudo-Quadratics in e^x

Solve $e^{2x} + 2e^x - 15 = 0$

Solve $e^{2x} + 5e^x = 6$

Solve $e^x - 2e^{-x} = 1$

Exponential Modelling

Note: there is no specific exercise on this, but many exam questions are on this topic. I have included them here

You will need to use your skills of solving equations from Chapter 14 in modelling scenarios

5. The mass, m grams, of a radioactive substance, t years after first being observed, is modelled by the equation

$$m = 25e^{-0.05t}$$

According to the model,

(a) find the mass of the radioactive substance six months after it was first observed,

(2)

(b) show that $\frac{dm}{dt} = km$, where k is a constant to be found.

(2)

13. The growth of pond weed on the surface of a pond is being investigated.

The surface area of the pond covered by the weed, A m^2, can be modelled by the equation

$$A = 0.2e^{0.3t},$$

where t is the number of days after the start of the investigation.

(*a*) State the surface area of the pond covered by the weed at the start of the investigation.

(1)

(*b*) Find the rate of increase of the surface area of the pond covered by the weed, in m^2/day, exactly 5 days after the start of the investigation.

(2)

Given that the pond has a surface area of 100 m^2,

(*c*) find, to the nearest hour, the time taken, according to the model, for the surface of the pond to be fully covered by the weed.

(4)

The pond was observed for one month. By the end of the month 90% of the surface area of the pond was covered by the weed.

(*d*) Evaluate the model in light of this information, giving a reason for your answer.

(1)

(Total for Question 13 is 8 marks)

14. The value of a car, £V, can be modelled by the equation

$$V = 15\,700\mathrm{e}^{-0.25t} + 2300 \qquad t \in \mathbb{R},\ t \geqslant 0$$

where the age of the car is t years.

Using the model,

(a) find the initial value of the car.

(1)

Given the model predicts that the value of the car is decreasing at a rate of £500 per year at the instant when $t = T$,

(b) (i) show that

$$3925\mathrm{e}^{-0.25T} = 500$$

(ii) Hence find the age of the car at this instant, giving your answer in years and months to the nearest month.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

The model predicts that the value of the car approaches, but does not fall below, £A.

(c) State the value of A.

(1)

(d) State a limitation of this model.

(1)

3. A cup of hot tea was placed on a table. At time t minutes after the cup was placed on the table, the temperature of the tea in the cup, θ °C, is modelled by the equation

$$\theta = 25 + Ae^{-0.03t}$$

where A is a constant.

The temperature of the tea was 75 °C when the cup was placed on the table.

(a) Find a complete equation for the model.

(1)

(b) Use the model to find the time taken for the tea to cool from 75 °C to 60 °C, giving your answer in minutes to one decimal place.

(2)

Two hours after the cup was placed on the table, the temperature of the tea was measured as 20.3 °C.

Using this information,

(c) evaluate the model, explaining your reasoning.

(1)

12. The value, £V, of a vintage car t years after it was first valued on 1st January 2001, is modelled by the equation

$$V = Ap^t \qquad \text{where } A \text{ and } p \text{ are constants}$$

Given that the value of the car was £32 000 on 1st January 2005 and £50 000 on 1st January 2012

(a) (i) find p to 4 decimal places,

(ii) show that A is approximately 24 800

(4)

(b) With reference to the model, interpret

(i) the value of the constant A,

(ii) the value of the constant p.

(2)

Using the model,

(c) find the year during which the value of the car first exceeds £100 000

(4)

Coming up with your own exponential model

Sometimes, the question will ask you to find an exponential model, but doesn't give you the starting form.

A general exponential model will be $y = Ae^{kt}$
If $k > 0$ it will be exponential growth
If $k < 0$ if will be exponential decay

7. In a simple model, the value, £V, of a car depends on its age, t, in years.

The following information is available for car A

- its value when new is £20 000
- its value after one year is £16 000

(a) Use an exponential model to form, for car A, a possible equation linking V with t.

(4)

The value of car A is monitored over a 10-year period.
Its value after 10 years is £2 000

(b) Evaluate the reliability of your model in light of this information.

(2)

The following information is available for car B

- it has the same value, when new, as car A
- its value depreciates more slowly than that of car A

(c) Explain how you would adapt the equation found in (a) so that it could be used to model the value of car B.

(1)

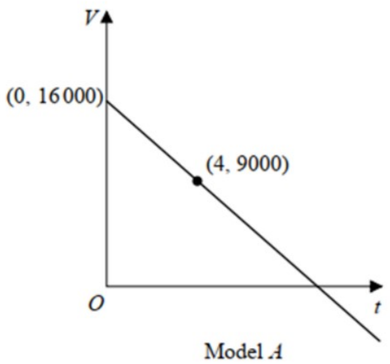
6. A company plans to extract oil from an oil field.

The daily volume of oil V, measured in barrels that the company will extract from this oil field depends upon the time, t years, after the start of drilling.

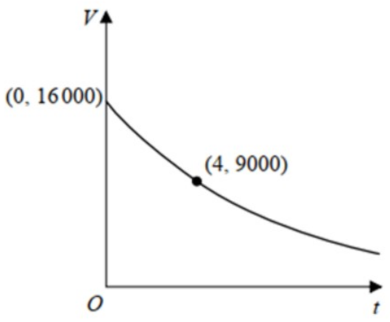
The company decides to use a model to estimate the daily volume of oil that will be extracted. The model includes the following assumptions:

- The initial daily volume of oil extracted from the oil field will be 16 000 barrels.
- The daily volume of oil that will be extracted exactly 4 years after the start of drilling will be 9000 barrels.
- The daily volume of oil extracted will decrease over time.

The diagram below shows the graphs of two possible models.



Model A



Model B

(a) (i) Use model A to estimate the daily volume of oil that will be extracted exactly 3 years after the start of drilling.

(ii) Write down a limitation of using model A.

(2)

(b) (i) Using an exponential model and the information given in the question, find a possible equation for model B.

(ii) Using your answer to (b)(i) estimate the daily volume of oil that will be extracted exactly 3 years after the start of drilling.

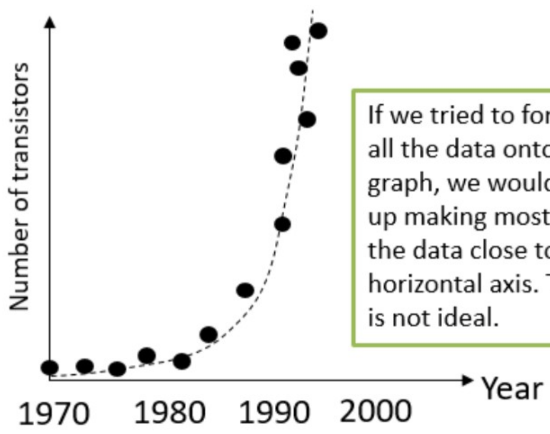
(5)

Graphs for Exponential Data

In Science and Economics, **experimental data often has exponential growth**, e.g. bacteria in a sample, rabbit populations, energy produced by earthquakes, my Twitter followers over time, etc.

Because exponential functions increase rapidly, it tends to look a bit rubbish if we tried to draw a suitable graph:

Take for example "Moore's Law", which hypothesised that the processing power of computers would double every 2 years. Suppose we tried to plot this for computers we sampled over time:

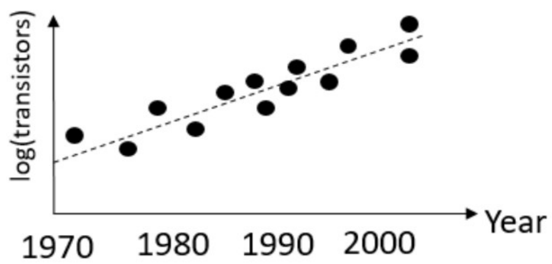


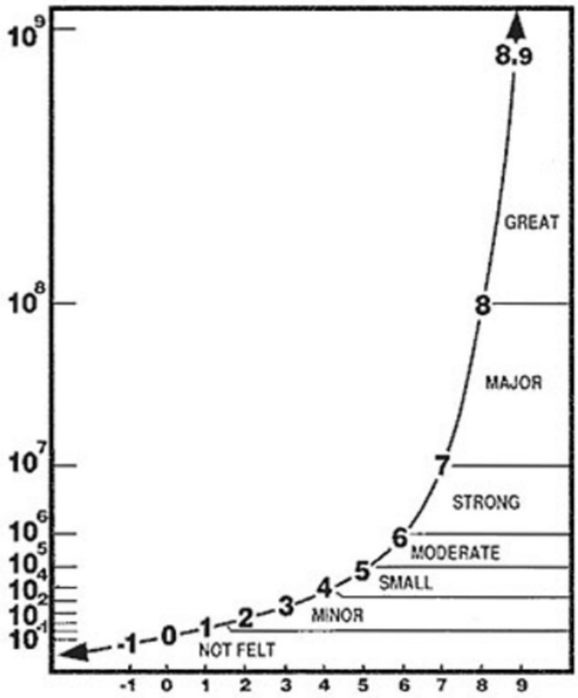
If we tried to force all the data onto the graph, we would end up making most of the data close to the horizontal axis. This is not ideal.

But suppose we **took the log** of the number of transistors for each computer. Suppose the number of transistors one year was y, then doubled 2 years later to get $2y$.
When we log (base 2) these:

$$
\begin{aligned}
y &\rightarrow \log_2 y \\
2y &\rightarrow \log_2(2y) = \log_2 2 + \log_2 y \\
&\qquad\qquad\quad = 1 + \log_2 y
\end{aligned}
$$

The logged value only increased by 1! Thus **taking the log of the values turns <u>exponential growth</u> into <u>linear growth</u> (because each time we would have doubled, we're now just adding 1), and the resulting graph is a straight line.**





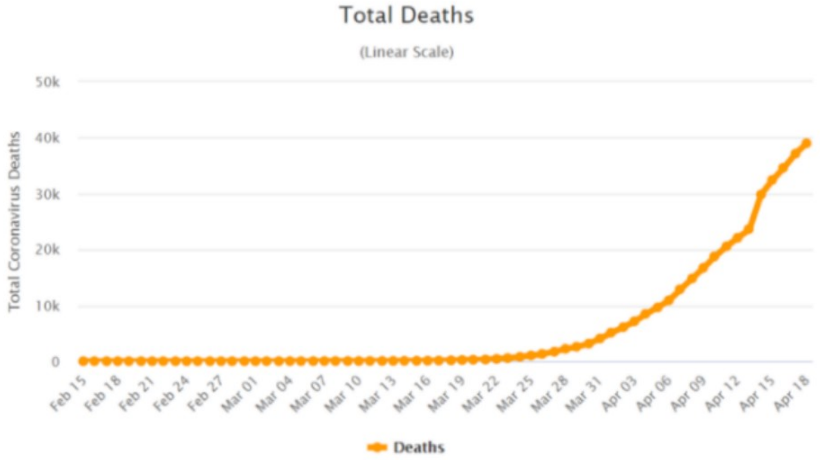
Richter Scale

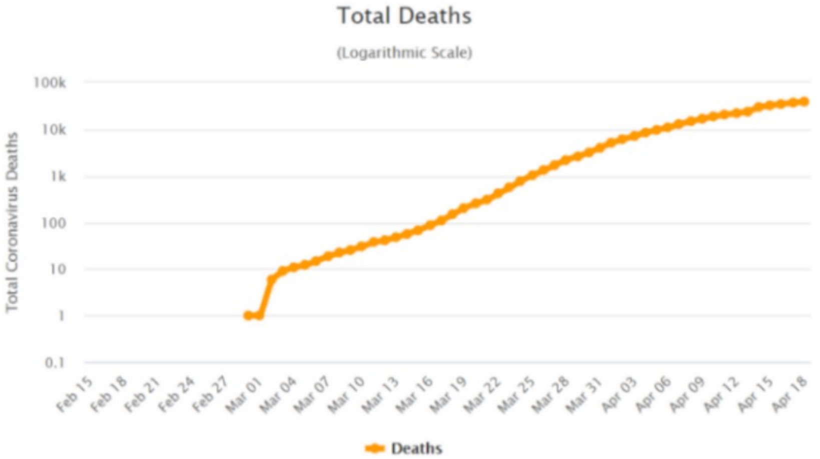
Because the energy involved in **earthquakes** decreases exponentially from the epicentre of the earthquake, such energy values recorded from different earthquakes would **vary wildly**.

The **Richter Scale** is a **<u>logarithmic scale</u>**, and takes the log (base 10) of the amplitude of the waves, giving a more even spread of values in a more sensible range.

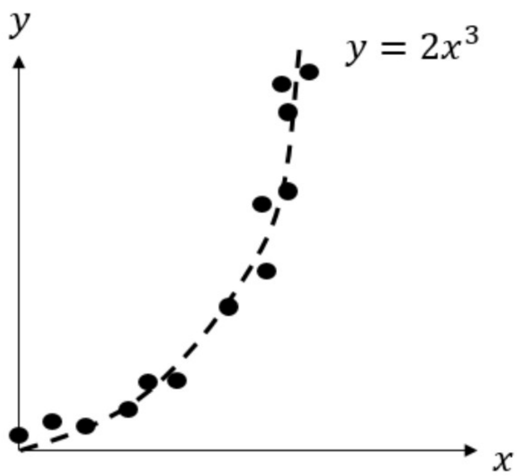
The result is that an earthquake just 1 greater on the Richter scale would in fact be 10 times as powerful.

Figure 1: COVID-19 Related Deaths in United States Between February 15th and April 18th in a linear scale (left panel) and in a log scale (right panel). Source: www.worldometers.info





Other Non-Linear Growth



We would also have similar graphing problems if we tried to plot data that followed some **polynomial function** such as a quadratic or cubic.

We will therefore look at the process to convert a **polynomial graph into a linear one**, as well as a **exponential graph into a linear one…**

Turning non-linear graphs into linear ones

This exact topic comes up again in Stats Year 2!

Case 1: Polynomial → Linear

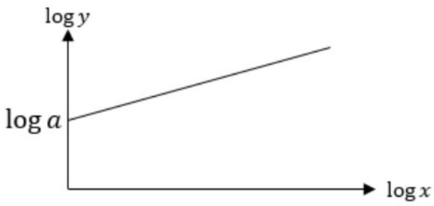
Suppose our original model was a polynomial one*:

$$y = ax^n$$

Then taking logs of both sides:

* We could also allow non-integer n; the term would then not strictly be polynomial, but we'd still say the function had "polynomial growth".

If $y = ax^n$, then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.



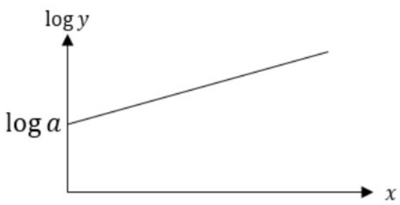
Case 2: Exponential → Linear

Suppose our original model was an exponential one:

$$y = ab^x$$

Then taking logs of both sides:

If $y = ab^x$, then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.



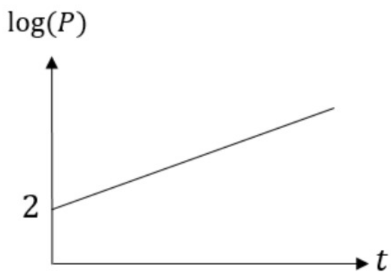
The key difference compared to Case 1 is that we're **only logging the y values** (e.g. number of transistors), not the x values (e.g. years elapsed). **Note that you <u>do not need to memorise</u> the contents of these boxes and we will work out from scratch each time…**

The graph represents the growth of a population of bacteria, P, over t hours. The graph has a gradient of 0.6 and meets the vertical axis at $(0,2)$ as shown.
A scientist suggest that this growth can be modelled by the equation $P = ab^t$, where a and b are constants to be found.

a. Write down an equation for the line.

b. Using your answer to part (a) or otherwise, find the values of a and b, giving them to 3 sf where necessary.

c. Interpret the meaning of the constant a in this model.



12. In a controlled experiment, the number of microbes, N, present in a culture T days after the start of the experiment were counted.

N and T are expected to satisfy a relationship of the form

$$N = aT^b, \qquad \text{where } a \text{ and } b \text{ are constants}$$

(a) Show that this relationship can be expressed in the form

$$\log_{10} N = m \log_{10} T + c$$

giving m and c in terms of the constants a and/or b.

(2)

Figure 3 shows the line of best fit for values of $\log_{10} N$ plotted against values of $\log_{10} T$

(b) Use the information provided to estimate the number of microbes present in the culture 3 days after the start of the experiment.

(4)

(c) Explain why the information provided could not reliably be used to estimate the day when the number of microbes in the culture first exceeds 1 000 000.

(2)

(d) With reference to the model, interpret the value of the constant a.

(1)

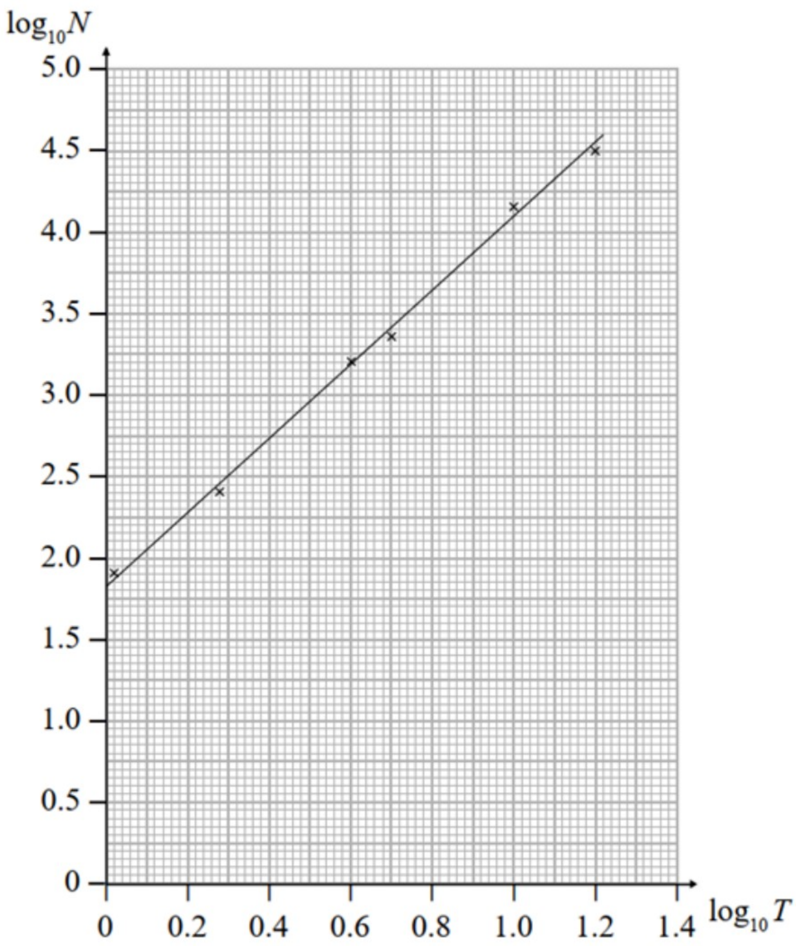


Figure 3

9. The amount of antibiotic, y milligrams, in a patient's bloodstream, t hours after the antibiotic was first given, is modelled by the equation

$$y = ab^t$$

where a and b are constants.

(a) Show that this equation can be written in the form

$$\log_{10} y = t \log_{10} b + c$$

expressing the constant c in terms of a.

(2)

A doctor measures the amount of antibiotic in the patient's bloodstream at regular intervals for the first 5 hours after the antibiotic was first given.

She plots a graph of $\log_{10} y$ against t and finds that the points on the graph lie close to a straight line passing through the point (0, 2.23) with gradient -0.076

(b) Estimate, to 2 significant figures, the value of a and the value of b.

(2)

With reference to this model,

(c) (i) give a practical interpretation of the value of the constant a,

(ii) give a practical interpretation of the value of the constant b.

(2)

(d) Use the model to estimate the time taken, after the antibiotic was first given, for the amount of antibiotic in the patient's bloodstream to fall to 30 milligrams. Give your answer, in hours, correct to one decimal place.

(2)

(e) Comment on the reliability of your estimate in part (d).

(1)

13.

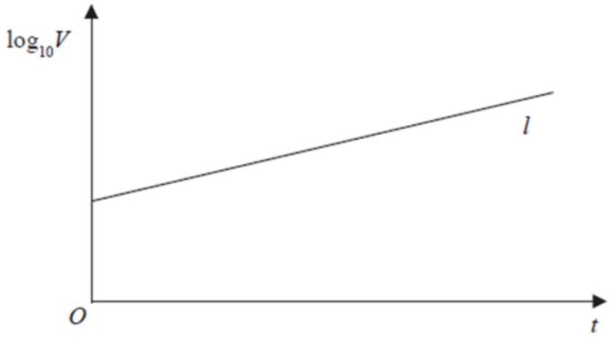


Figure 3

The value of a rare painting, £V, is modelled by the equation $V = pq^t$, where p and q are constants and t is the number of years since the value of the painting was first recorded on 1st January 1980.

The line l shown in Figure 3 illustrates the linear relationship between t and $\log_{10}V$ since 1st January 1980.

The equation of line l is $\log_{10}V = 0.05t + 4.8$

(a) Find, to 4 significant figures, the value of p and the value of q.

(4)

(b) With reference to the model interpret

(i) the value of the constant p,

(ii) the value of the constant q.

(2)

(c) Find the value of the painting, as predicted by the model, on 1st January 2010, giving your answer to the nearest hundred thousand pounds.

(2)

Exam Questions

Note: there are *many* more in the Google Drive

5. $$\mathrm{f}(x) = x^3 + ax^2 - ax + 48, \quad \text{where } a \text{ is a constant}$$

Given that $\mathrm{f}(-6) = 0$

(a) (i) show that $a = 4$

(ii) express $\mathrm{f}(x)$ as a product of two algebraic factors.

(4)

Given that $2\log_2(x+2) + \log_2 x - \log_2(x-6) = 3$

(b) show that $x^3 + 4x^2 - 4x + 48 = 0$

(4)

(c) hence explain why

$$2\log_2(x+2) + \log_2 x - \log_2(x-6) = 3$$

has no real roots.

(2)

5. A student's attempt to solve the equation $2\log_2 x - \log_2 \sqrt{x} = 3$ is shown below.

$$2\log_2 x - \log_2 \sqrt{x} = 3$$

$2\log_2 \left(\dfrac{x}{\sqrt{x}}\right) = 3$ using the subtraction law for logs

$2\log_2 \left(\sqrt{x}\right) = 3$ simplifying

$\log_2 x = 3$ using the power law for logs

$x = 3^2 = 9$ using the definition of a log

(a) Identify two errors made by this student, giving a brief explanation of each.

(2)

(b) Write out the correct solution.

(3)