The position of a point A, relative to a fixed origin is $4\mathbf{i} + 2\mathbf{j}$.

The position of a point B, relative to a fixed origin is $-2\mathbf{i} + 7\mathbf{j}$.

a) i) Find the vector $\overrightarrow{AB}$ ii) Find the vector $\overrightarrow{BA}$

b) Find the distance AB

c) Write down any 2 vectors parallel to $\overrightarrow{AB}$

The point M is the midpoint of AB.

d) i) Write down the vector $\overrightarrow{AM}$ ii) Write down the vector $\overrightarrow{BM}$

Vectors - Core Pure

1:: Equations of straight lines in 3D

"Find an equation of the line that passes through the points $A(1,2,3)$ and $B(4,0,-2)$, giving your answer in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$"

2:: Equations of planes

"The plane Π is perpendicular to the normal vector $a = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and passes through the point P with position vector $8\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$. Find the Cartesian equation of Π."

3:: Scalar product and angles between line + line or plane + line or plane + plane.

"If the line l has equation $r = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ and point $A(3,-1,4)$ is a point on the line and B has coordinates $(5,6,6)$, find the angle between l and AB."

4:: Scalar product form of equation of plane.

$$r \cdot n = a \cdot n$$

5:: Point of intersection of two planes.

"Show that the line with equations $3i + j + k + \lambda(i - 2j - k)$ and $r = -2j + 3k + \mu(-5i + j + 4k)$ meet and find "the point of intersection."

6:: Perpendicular distance between line + line or point + line or point + plane.

"Find the shortest distance between the line l with equation $\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+3}{-1}$ and the point A has coordinates $(1,2,-1)$."

Vector Basics

$x \to \underline{i}$

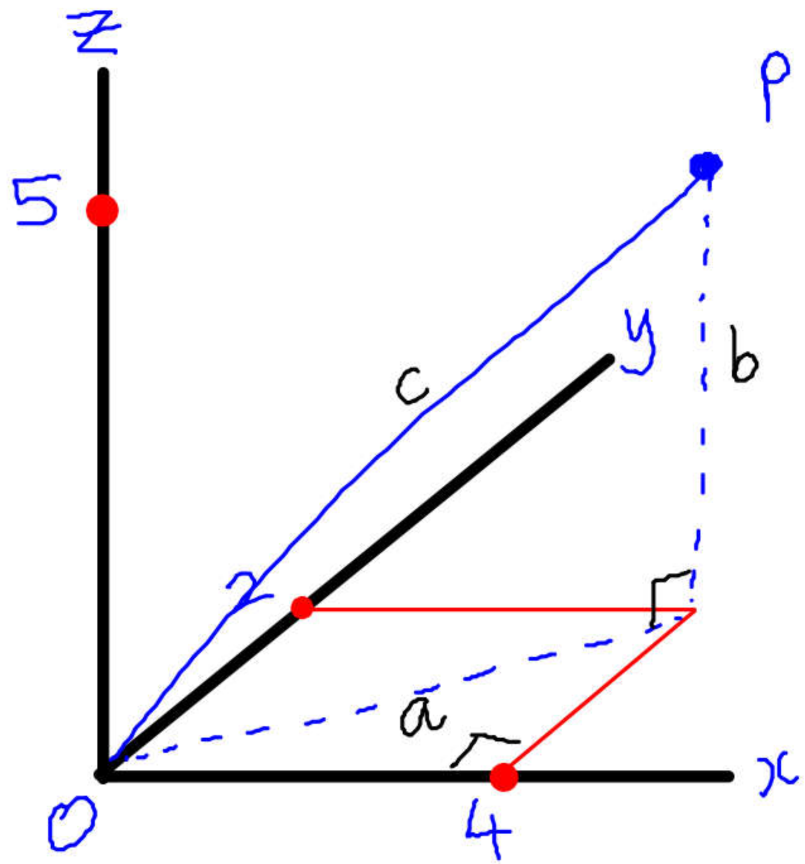
$y \to \underline{j}$

$z \to \underline{k}$

$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix}$

Distances

Find the distance from the origin to the point P(4, 2, 5)



$a = \sqrt{4^2 + 2^2}$

$b = 5$

$c^2 = a^2 + b^2$

$c = \sqrt{4^2 + 2^2 + 5^2}$

$c = \sqrt{45}$

To find the dist. to origin of (x, y, z)

$d = \sqrt{x^2 + y^2 + z^2}$

Find the distance AB

A(3, 6, -2)
B(1, 0, 5)

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$$

$$AB = |\overrightarrow{AB}| = \sqrt{2^2 + 6^2 + 7^2}$$
$$= \underline{\underline{\sqrt{89}}}$$

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

Find a unit vector in the direction $\overrightarrow{AB}$.

$$\frac{1}{\sqrt{89}} \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$$

Vector equation of a straight line in 3D

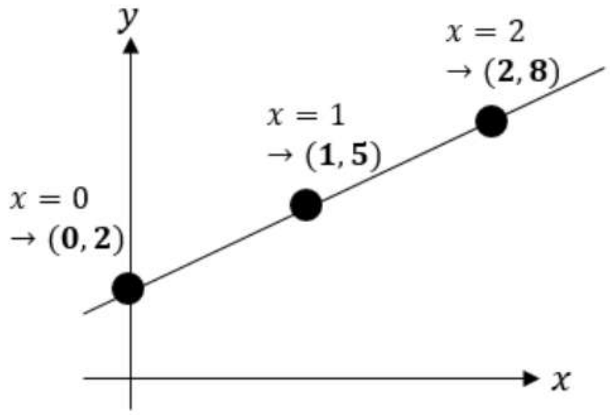
Consider the equation of a straight line in 2D:

vector slide.pptx

Vector equation of a straight line in 3D

Consider the equation of a straight line in 2D:

$$y = 3x + 2$$

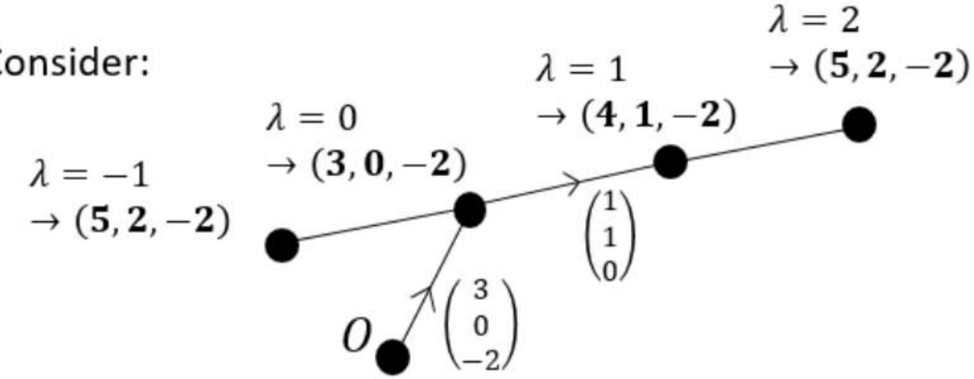


x is obviously a variable (i.e. it can vary!). As we consider different values of x, we get different points on the line.

It's worth noting that in $y = mx + c$, while x and y are variables, m and c are **constants**: after these are set for a particular line, they don't change.

Can we do something similar with vectors? Consider:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$



What happens as we vary λ?

Therefore what was the role of:

$\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$: Position vector of some arbitrary point on the line (it doesn't matter which).

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$: The **direction** of the line.

Vector equation $\boldsymbol{r}$ of a straight line:

$$\underline{r} = \underline{a} + \lambda \underline{b}$$

where $\boldsymbol{a}$ is (the position vector of) some point on the line, $\boldsymbol{b}$ is the direction vector.

λ - parameter

Important understanding points:

- $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants (i.e. fixed for a given line) while λ is a variable.

- It is often helpful to write as a single position vector, e.g:

$$\underline{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3+\lambda \\ \lambda \\ -2 \end{pmatrix}$$

- It is highly important the you can distinguish between the **position vector $\boldsymbol{r}$** of a **point on the line**, and the **direction $\boldsymbol{b}$** of the line:

Example Problem

The equation of line l_1 is $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find the vector equation of a line parallel to l_1 which passes through the point $(2,5,1)$.

$$\underline{r} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Find a vector equation of the straight line which passes through the point A, with position vector $3\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}$ and is parallel to the vector $7\boldsymbol{i} - 3\boldsymbol{k}$.

$$\underline{r} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 + 7\lambda \\ -5 \\ 4 - 3\lambda \end{pmatrix}$$

Find a vector equation of the straight line which passes through the points A and B, with coordinates $(4,5,-1)$ and $(6,3,2)$ respectively.

Find direction $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

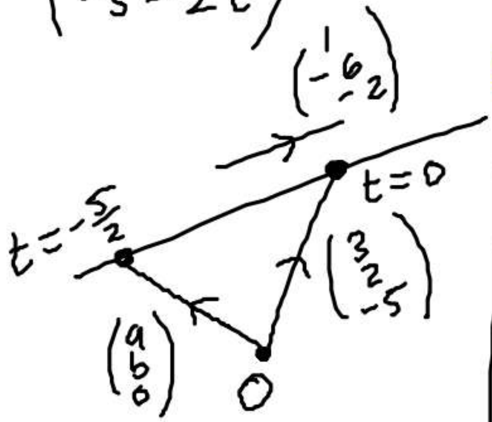
$$\underline{r} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$\begin{pmatrix} 200 \\ -200 \\ 300 \end{pmatrix}$ simplify

The straight line has vector equation
$\boldsymbol{r} = (3\boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}) + t(\boldsymbol{i} - 6\boldsymbol{j} - 2\boldsymbol{k})$.
Given that the point $(a, b, 0)$ lines on l, find the value of a and the value of b.

$$\underline{r} = \begin{pmatrix}3\\2\\-5\end{pmatrix} + t\begin{pmatrix}1\\-6\\-2\end{pmatrix} = \begin{pmatrix}3+t\\2-6t\\-5-2t\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\0\end{pmatrix} = \begin{pmatrix}3+t\\2-6t\\-5-2t\end{pmatrix}$$



$\underline{k}$ comp.

$$0 = -5-2t$$
$$t = -\frac{5}{2}$$

$$a = 3+t = 3-\frac{5}{2} = \underline{\underline{\frac{1}{2}}}$$

$$b = 2-6t = 2-6\times\left(-\frac{5}{2}\right) = \underline{\underline{17}}$$

The straight line l has vector equation
$\boldsymbol{r} = (2\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) + \lambda(6\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k})$.
Show that another vector equation of l is
$\boldsymbol{r} = (8\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) + \mu(3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

1. If same, directions must be parallel / multiples
2. If same, the "$\underline{a}$" of second eqn, must also be on the first eqn's line.

1. $\begin{pmatrix}6\\-2\\4\end{pmatrix} = 2\begin{pmatrix}3\\-1\\2\end{pmatrix}$ So they are the same direction

2. when $\lambda = 1$, $\underline{r} = \begin{pmatrix}8\\3\\1\end{pmatrix}$

So $\begin{pmatrix}8\\3\\1\end{pmatrix}$ is on the line, hence

$\underline{r} = \begin{pmatrix}8\\3\\1\end{pmatrix} + \mu\begin{pmatrix}3\\-1\\2\end{pmatrix}$ is the same line.

$$\underline{r} = \begin{pmatrix} -1+\lambda \\ \lambda \\ 3 \end{pmatrix}$$

Example Problem

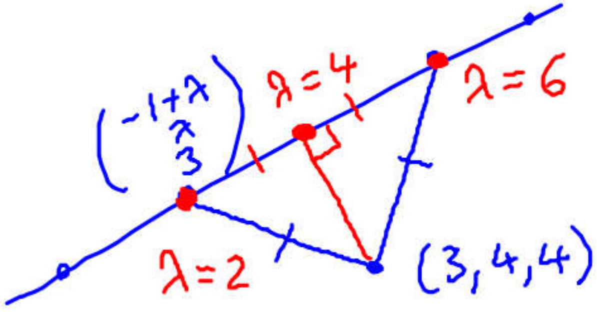
The equation of line l_1 is $\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find the coordinates of the points on l_1 which are a distance of 3 away from (3,4,4).

Ex 9A Questions:
1ace
2ace
3
4ai, 4aiii
5ac
7
8
10
11
13

Ex 9A Homework:
1bd
2bd
5b
9
12

$$\begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} \to \sqrt{4+4+1}$$



- subtract the vectors
- Pythagorise and $= 3$

$$\begin{pmatrix} -1+\lambda & -3 \\ \lambda & -4 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} -4+\lambda \\ \lambda-4 \\ -1 \end{pmatrix}$$

$$\sqrt{(\lambda-4)^2 + (\lambda-4)^2 + 1^2} = 3$$

$$\lambda^2 - 8\lambda + 16 + \lambda^2 - 8\lambda + 16 + 1 = 9$$

$$2\lambda^2 - 16\lambda + 24 = 0$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$(\lambda-6)(\lambda-2) = 0$$

$$\lambda = 2, \ \lambda = 6$$

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

(1, 2, 3) and (5, 6, 3)

If $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ is equation of straight line, then its Cartesian form is

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = (\lambda)$$

$$\underline{r} = \underline{a} + \lambda \underline{b}$$

$$\underline{r} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \\ a_3 + \lambda b_3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \\ a_3 + \lambda b_3 \end{pmatrix}$$

$\underline{i}$ comp

$x = a_1 + \lambda b_1$

$$\frac{x - a_1}{b_1} = \lambda$$

$\underline{j}$ comp

$y = a_2 + \lambda b_2$

$$\frac{y - a_2}{b_2} = \lambda$$

$\underline{k}$ comp

$z = a_3 + \lambda b_3$

$$\frac{z - a_3}{b_3} = \lambda$$

Find the Cartesian equation of the line with equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

$$\frac{x-4}{-1} = \frac{y-3}{2} = \frac{z+2}{5} = \lambda$$

$$4 - x = \frac{y-3}{2} = \frac{z+2}{5}$$

Find the Cartesian equation of the line with equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$.

$$\frac{x-2}{1} = \frac{y-5}{3} = \frac{z}{-2}$$

$$\frac{y-2}{3} = \frac{x}{1}$$

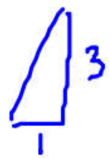
$$\underline{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The Cartesian equation of a line is $y = 3x + 2$. Find the vector form of the equation of the line.

$$y = 3x + 2$$

$$\frac{y}{3} = \frac{x + \frac{2}{3}}{1}$$

$$\underline{r} = \begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



$\left(-\frac{2}{3}, 0\right)$

The Cartesian equation of a line is $\frac{x-2}{3} = \frac{y+5}{1} = \frac{z}{4}$. Find the vector form of the equation of the line.

$$\underline{r} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 + 3\lambda \\ -5 + \lambda \\ 4\lambda \end{pmatrix}$$

Ex 9A
Q4bi, biii
Q6