Vectors - Year 12

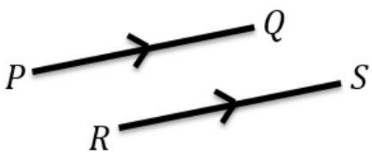
A Whereas a **coordinate** represents a **position** in space, a **vector** represents a **displacement** in space.

A vector has 2 properties:
- Direction
- Magnitude (i.e. length)

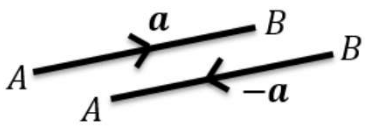
If P and Q are points then $\overrightarrow{PQ}$ is the vector between them.

B If two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, **they're the same vector** and are **parallel**.

This might seem obvious, but students sometimes think the vector is different because the movement occurred at a different point in space. Nope!

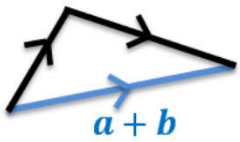


C $\overrightarrow{AB} = -\overrightarrow{BA}$ and the two vectors are parallel, equal in magnitude but in **opposite directions**.



D Triangle Law for vector addition:

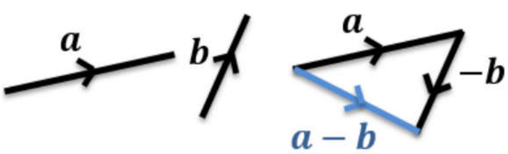
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



The vector of multiple vectors is known as the **resultant vector.**
(you will encounter this term in Mechanics)

E Vector **subtraction** is defined using vector addition and negation:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$

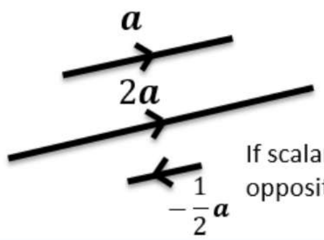


F The zero vector **0** (a bold 0), represents no movement.

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$$

In 2D: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

G A **scalar** is a normal number, which can be used to 'scale' a vector.
- The **direction** will be the **same**.
- But the **magnitude** will be **different** (unless the scalar is 1).



If scalar is negative, opposite direction.

H Any vector parallel to the vector $\boldsymbol{a}$ can be written as $\lambda\boldsymbol{a}$, where λ is a scalar.

The implication is that if we can write one vector **as a multiple of** another, then we can show they are parallel.

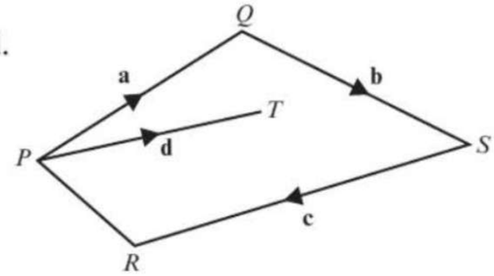
"Show $2\boldsymbol{a} + 4\boldsymbol{b}$ and $3\boldsymbol{a} + 6\boldsymbol{b}$ are parallel".

$3\boldsymbol{a} + 6\boldsymbol{b} = \frac{3}{2}(\boldsymbol{a} + 2\boldsymbol{b})$ ∴ parallel

In the diagram, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QS} = \mathbf{b}$, $\overrightarrow{SR} = \mathbf{c}$ and $\overrightarrow{PT} = \mathbf{d}$.
Find in terms of **a**, **b**, **c** and **d**:

a $\overrightarrow{QT}$ **b** $\overrightarrow{PR}$

c $\overrightarrow{TS}$ **d** $\overrightarrow{TR}$



$ABCD$ is a trapezium with AB parallel to DC and $DC = 3AB$.
M divides DC such that $DM : MC = 2 : 1$. $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$.
Find, in terms of **a** and **b**:

a $\overrightarrow{AM}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{MB}$ **d** $\overrightarrow{DA}$

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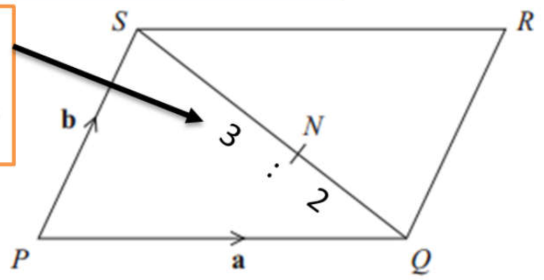
Tip: This ratio wasn't in the original diagram. I like to add the ratio as a visual aid.



$PQRS$ is a parallelogram.
N is the point on SQ such that $SN : NQ = 3 : 2$
$\overrightarrow{PQ} = \mathbf{a}$ $\overrightarrow{PS} = \mathbf{b}$

(a) Write down, in terms of **a** and **b**, an expression for $\overrightarrow{SQ}$.

(b) Express $\overrightarrow{NR}$ in terms of **a** and **b**.

Your Turn

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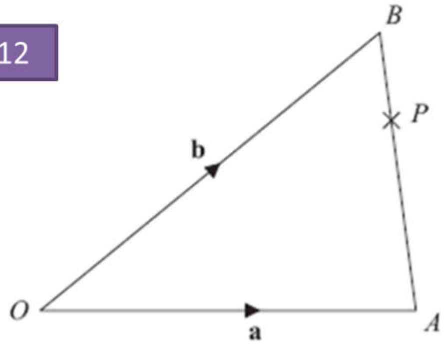


Diagram **NOT** accurately drawn

OAB is a triangle.

$\overrightarrow{OA} = \mathbf{a}$
$\overrightarrow{OB} = \mathbf{b}$

(a) Find $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

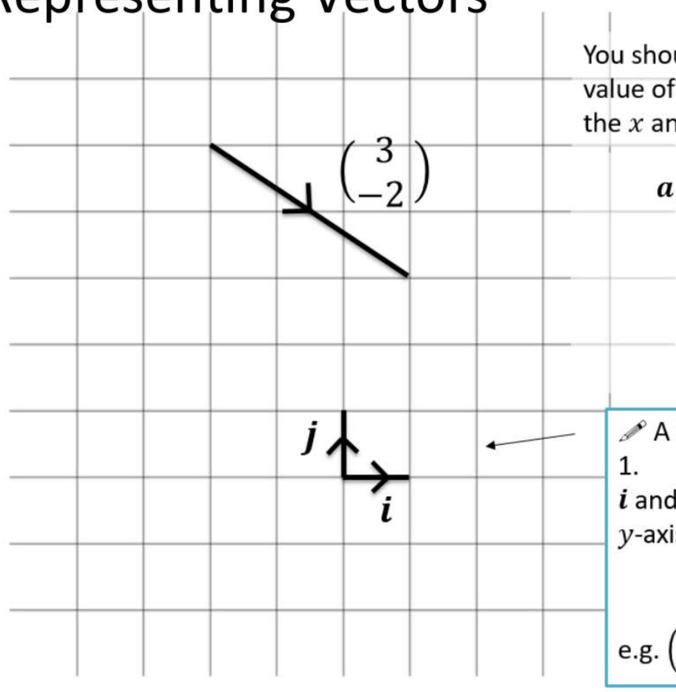
....
(1)

P is the point on *AB* such that $AP : PB = 3 : 1$

(b) Find $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Give your answer in its simplest form.

Ex 11A
Q7-11

Representing Vectors



You should already be familiar that the value of a vector is the **displacement** in the x and y direction (if in 2D).

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\boldsymbol{a} + \boldsymbol{b} =$$

$$2\boldsymbol{a} =$$

✏ A **unit vector** is a vector of magnitude 1.
$\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the x-axis and y-axis respectively.

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e.g. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4\boldsymbol{i} + 3\boldsymbol{j}$

If $\boldsymbol{a} = 3\boldsymbol{i}, \quad \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$ then:

1) Write $\boldsymbol{a}$ in vector form.
2) Find $\boldsymbol{b} + 2\boldsymbol{c}$ in $\boldsymbol{i}, \boldsymbol{j}$ form.

Given that $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j}$, find:

Your Turn

a λ if $\mathbf{c} + \lambda\mathbf{d}$ is parallel to $\mathbf{i} + \mathbf{j}$

b μ if $\mu\mathbf{c} + \mathbf{d}$ is parallel to $\mathbf{i} + 3\mathbf{j}$

c s if $\mathbf{c} - s\mathbf{d}$ is parallel to $2\mathbf{i} + \mathbf{j}$

d t if $\mathbf{d} - t\mathbf{c}$ is parallel to $-2\mathbf{i} + 3\mathbf{j}$

The resultant of the vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = p\mathbf{i} - 2p\mathbf{j}$ is parallel to the vector $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$.
Find:

a the value of p **(4 marks)**

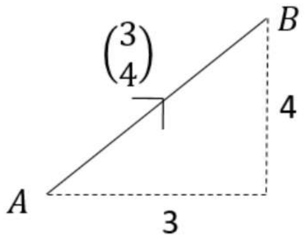
b the resultant of vectors **a** and **b**. **(1 mark)**

Skip Ex 11B

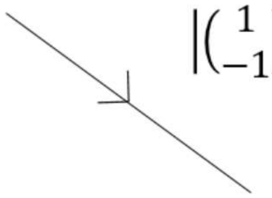
Magnitude of a Vector

The magnitude $|a|$ of a vector a is its length.

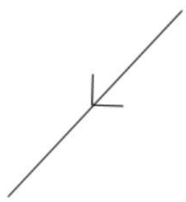
If $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$



$|\overrightarrow{AB}| =$



$\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| =$



$\left|\begin{pmatrix} -5 \\ -12 \end{pmatrix}\right| =$

$\boldsymbol{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $|\boldsymbol{a}| =$

$\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $|\boldsymbol{b}| =$

Unit Vectors

A unit vector is a vector whose magnitude is 1

There's certain operations on vectors that require the vectors to be 'unit' vectors. We just scale the vector so that its magnitude is now 1.

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

If $\boldsymbol{a}$ is a vector, then the unit vector $\hat{\boldsymbol{a}}$ in the same direction is

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Test Your Understanding: Convert the following vectors to unit vectors.

$$\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Direction of Vectors

Find the angle between $\boldsymbol{a}$ and the positive $x-$axis

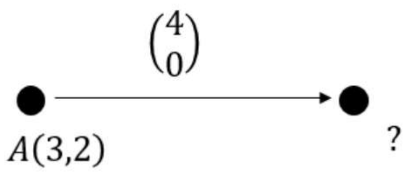
$$\boldsymbol{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Vector $\boldsymbol{a}$ has magnitude 10 and makes an angle of 30° with $\boldsymbol{j}$
Find $\boldsymbol{a}$ in $\boldsymbol{i}, \boldsymbol{j}$ and column vector format.

Ex 11C
2ab
3ab
4, 5
6ab
10, 11

Position Vectors

Suppose we started at a point $(3,2)$ and translated by the vector $\begin{pmatrix}4\\0\end{pmatrix}$:

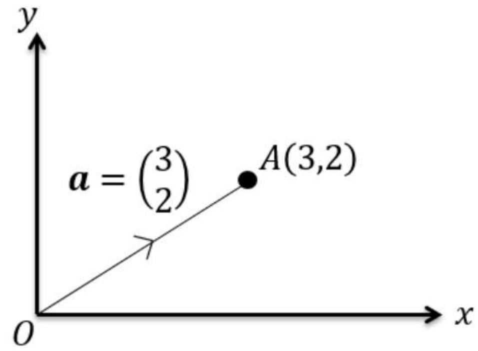


You might think we can do something like:

$$(3,2)+\begin{pmatrix}4\\0\end{pmatrix}=(7,2)$$

But only vectors can be added to other vectors. **If we treated the point $(3,2)$ as a vector**, then this solves the problem:

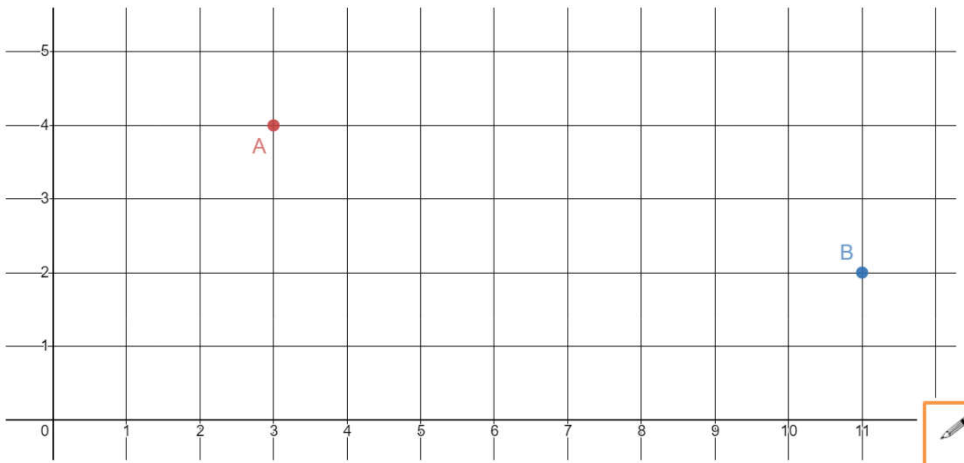
$$\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}4\\0\end{pmatrix}=\begin{pmatrix}7\\2\end{pmatrix}$$



A vector used to represent a position is unsurprisingly known as a **position vector**.
A position can be thought of as a translation from the origin, as per above. It enables us to use positions in all sorts of vector (and matrix!) calculations.

✎ The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as $\boldsymbol{a}$.

The points A and B have coordinates $(3,4)$ and $(11,2)$ respectively. Find, in terms of i and j:
a) The position vector of A
b) The position vector of B
c) The vector $\overrightarrow{AB}$



✎ For position vectors a and b:

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$
$$\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$$

The points A, B and C have coordinates $(3, -1)$, $(4, 5)$ and $(-2, 6)$ respectively, and O is the origin. Find, in terms of $\mathbf{i}$ and $\mathbf{j}$:

a **i** the position vectors of A, B and C **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{AC}$

b Find, in surd form: **i** $|\overrightarrow{OC}|$ **ii** $|\overrightarrow{AB}|$ **iii** $|\overrightarrow{AC}|$

$\overrightarrow{OA} = 5i - 2j$ and $\overrightarrow{AB} = 3i + 4j$. Find:

a) The position vector of B.

b) The exact value of $|\overrightarrow{OB}|$ in simplified surd form.

2 $\overrightarrow{OP} = 4\mathbf{i} - 3\mathbf{j}$, $\overrightarrow{OQ} = 3\mathbf{i} + 2\mathbf{j}$

a Find $\overrightarrow{PQ}$

b Find, in surd form: **i** $|\overrightarrow{OP}|$ **ii** $|\overrightarrow{OQ}|$ **iii** $|\overrightarrow{PQ}|$

3 $\overrightarrow{OQ} = 4\mathbf{i} - 3\mathbf{j}$, $\overrightarrow{PQ} = 5\mathbf{i} + 6\mathbf{j}$

a Find $\overrightarrow{OP}$

b Find, in surd form: **i** $|\overrightarrow{OP}|$ **ii** $|\overrightarrow{OQ}|$ **iii** $|\overrightarrow{PQ}|$

5 The position vectors of 3 vertices of a parallelogram are $\begin{pmatrix}4\\2\end{pmatrix}$, $\begin{pmatrix}3\\5\end{pmatrix}$ and $\begin{pmatrix}8\\6\end{pmatrix}$.

Find the possible position vectors of the fourth vertex.

6 Given that the point A has position vector $4\mathbf{i} - 5\mathbf{j}$ and the point B has position vector $6\mathbf{i} + 3\mathbf{j}$,

a find the vector $\overrightarrow{AB}$. **(2 marks)**

b find $|\overrightarrow{AB}|$ giving your answer as a simplified surd. **(2 marks)**

2 **a** $-\mathbf{i} + 5\mathbf{j}$ or $\begin{pmatrix}-1\\5\end{pmatrix}$

b **i** 5 **ii** $\sqrt{13}$ **iii** $\sqrt{26}$

3 **a** $-\mathbf{i} - 9\mathbf{j}$ or $\begin{pmatrix}-1\\-9\end{pmatrix}$

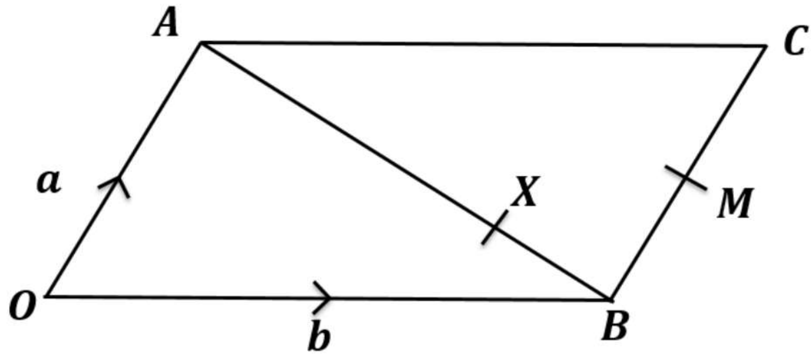
b **i** $\sqrt{82}$ **ii** 5 **iii** $\sqrt{61}$

5 $\begin{pmatrix}7\\9\end{pmatrix}$ or $\begin{pmatrix}9\\3\end{pmatrix}$

6 **a** $2\mathbf{i} + 8\mathbf{j}$ **b** $2\sqrt{17}$

Skip
Ex 11D

Solving Geometric Problems



X is a point on AB such that $AX: XB = 3: 1$. M is the midpoint of BC.
Show that $\overrightarrow{XM}$ is parallel to $\overrightarrow{OC}$.

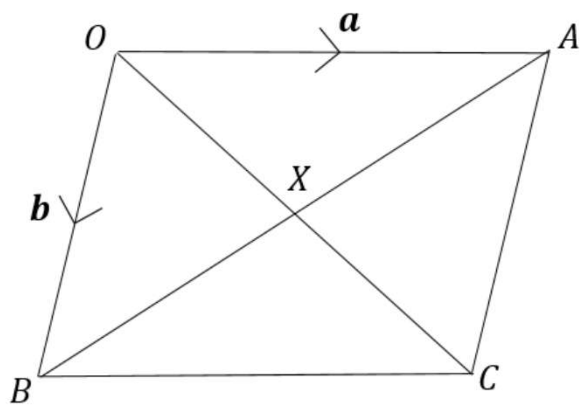
Introducing Scalars and Comparing Coefficients

Remember when we had **identities** like:

$$ax^2 + 3x \equiv 2x^2 + bx$$

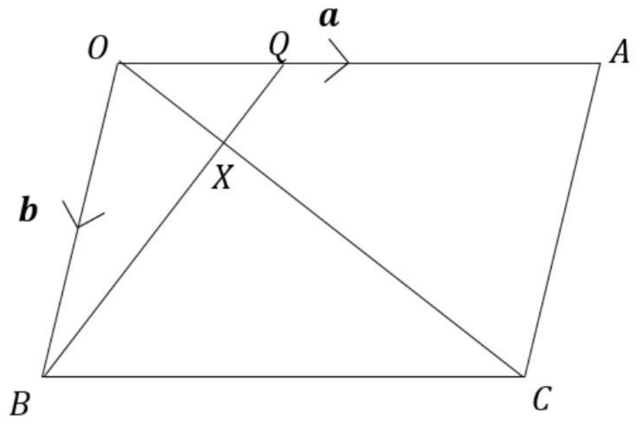
we could **compare coefficients**, so that $a = 2$ and $3 = b$.
We can do the same with (non-parallel) vectors!

$OACB$ is a parallelogram, where $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$.
The diagonals OC and AB intersect at a point X.
Prove that the diagonals bisect each other.
(Hint: Perhaps find $\overrightarrow{OX}$ in two different ways?)



'lambda'
'mu'

Your Turn



In the above diagram, $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{3}\boldsymbol{a}$. We wish to find the ratio $OX:XC$.

a) If $\overrightarrow{OX} = \lambda\, \overrightarrow{OC}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and λ.

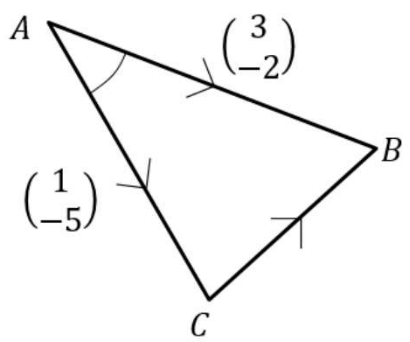
b) If $\overrightarrow{BX} = \mu\, \overrightarrow{BQ}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and μ.

c) By comparing coefficients or otherwise, determine the value of λ, and hence the ratio $OX:XC$.

Area of a Triangle

$\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{AC} = \boldsymbol{i} - 5\boldsymbol{j}$. Determine $\angle BAC$.

Strategy: Find 3 lengths of triangle then use cosine rule to find angle.



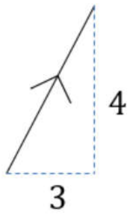
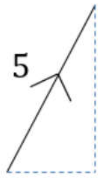
Cosine Rule:
$a^2 = b^2 + c^2 - 2bc\cos A$

Ex 11E
1, 3, 4, 5, (6)

Modelling

In Mechanics, you will see certain things can be represented as a simple number (without direction), or as a vector (with direction):

Remember a 'scalar' just means a normal number (in the context of vectors). It can be obtained using the **magnitude** of the vector.

Vector Quantity	Equivalent Scalar Quantity
Velocity e.g. $\begin{pmatrix}3\\4\end{pmatrix}\ km/h$  	Speed $=\ 5\ km/h$ 
Displacement e.g. $\begin{pmatrix}-5\\12\end{pmatrix}\ km$	Distance $= 13\ km$

Find the distance moved by a particle which travels for:

a 5 hours at velocity $(8\mathbf{i} + 6\mathbf{j})$ km h^{-1}

b 10 seconds at velocity $(5\mathbf{i} - \mathbf{j})$ m s^{-1}

c 45 minutes at velocity $(6\mathbf{i} + 2\mathbf{j})$ km h^{-1}

d 2 minutes at velocity $(-4\mathbf{i} - 7\mathbf{j})$ cm s^{-1}.