Differential Equations - Core Pure 2

Differential equations are equations which relate x and y with derivatives. e.g.

The rate of temperature loss is proportional to the current temperature.	$\frac{dT}{dt} = -kT$
The rate of population change is proportional to $P\left(1-\frac{P}{M}\right)$ where P is the current population and M is the limiting size of the population (the Verhulst-Pearl Model)	$\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right)$
Suppose x is GDP (Gross Domestic Product). Rate of change of GDP is proportional to current GDP.	$\frac{dx}{dt} = kx$

As you might imagine, they're used a lot in physics and engineering, including modelling radioactive decay, mixing fluids, cooling materials and bodies falling under gravity against resistance.

A 'first order' differential equation means the equation contains the first derivative ($\frac{dy}{dx}$) but not the second derivative or beyond.

Separating the Variables

x and y are said to be '**separated**' because we can express the RHS as a product of two separate expressions: one in terms of just x and one in terms of just y.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Divide through by $g(y)$ and times through by dx, and slap an integral on the front!

Find general solutions to $\frac{dy}{dx} = -\frac{y}{x}$

Using reverse product rule

We will see in a bit how to solve equations of the form $\frac{dy}{dx} + Py = Q$ (where P and Q are functions of x). We'll practice a particular part of this method before going for the full thing.

Find general solutions of the equation $x^3 \frac{dy}{dx} + 3x^2 y = \sin x$

What is different about this equation?

Quickfire Questions:

$$\frac{d}{dx}(x^2 y) =$$

$$\frac{d}{dx}(y \sin(x)) =$$

So it appears whatever term ends up on front of the $\frac{dy}{dx}$ will be on the front of the y in the integral.

$$x^4 \frac{dy}{dx} + 4x^3 y \quad \rightarrow$$

$$e^x \frac{dy}{dx} + e^x y \quad \rightarrow$$

$$(\ln x)\frac{dy}{dx} + \frac{y}{x} \quad \rightarrow$$

Find general solutions of the equation $x^3 \frac{dy}{dx} + 3x^2 y = \sin x$

Find general solutions of the equation
$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$

Find general solutions of the equation
$4xy\frac{dy}{dx} + 2y^2 = x^2$

Ex 7A
Q3
Q5a
Q6

$$dy/dx + Py = Q$$

But what if we can't use the product rule backwards?

Find the general solution of $\frac{dy}{dx} - 4y = e^x$

We can multiply through by the integrating factor $e^{\int P\,dx}$. This then produces an equation where we can use the previous reverse-product-rule trick (we'll prove this in a bit).

$$I.F. =$$

Then multiplying through by the integrating factor:

Then we can solve in the usual way:

Proof that Integrating Factor works

Solve the general equation $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x.

Suppose $f(x)$ is the Integrating Factor. As usual we'd multiply by it:

$$f(x)\frac{dy}{dx} + f(x)Py = f(x)Q$$

If we can use the reverse product rule trick on the LHS, then it would be of the form:

$$f(x)\frac{dy}{dx} + f'(x)y$$

Thus comparing the coefficients of the two LHSs:

$$f'(x) = f(x)P$$

Dividing by $f(x)$ and integrating:

$$\int \frac{f'(x)}{f(x)}\, dx = \int P\, dx$$

$$\ln|f(x)| = \int P\, dx$$

$$f(x) = e^{\int P\, dx}$$

When there's something on front of the dy/dx

Find the general solution of $\cos x \frac{dy}{dx} + 2y\sin x = \cos^4 x$

What shall we do first so that we have an equation like before?

STEP 1: Divide by anything on front of dy/dx

STEP 2: Determine IF

STEP 3: Multiply through by IF and use product rule backwards.

STEP 4: Integrate and simplify.

Your Turn

Edexcel FP2(Old) June 2011 Q3

Find the general solution of the differential equation

$$x\frac{dy}{dx} + 5y = \frac{\ln x}{x}, \qquad x > 0$$

Remember!

- can you separate the variables? DO THIS!
- can you use reverse product rule first? THEN DO THIS!
- put it in the form dy/dx + P(x)y = Q(x), then use IF

Q13 and Q14... may not need IF...!

Ex 7A
Q7 onwards

Homework
Core Pure Year 2
Ex 7A Q7bdfhj, Q13, Q15
Mixed Exercise 7 Q1-12 (NOT 6, 7, 10)

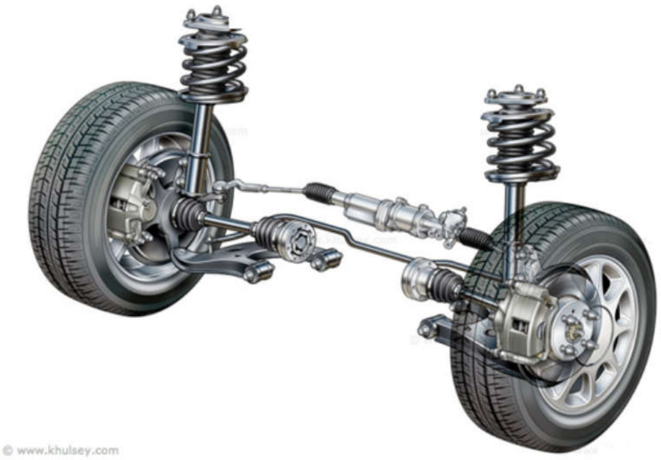
BE WARNED! Some of them are separating the variables!

Second Order Differential Equation Intro

We've already seen that differential equations are equations which relate x and y with derivatives. Unsurprisingly, second order differential equations involve the second derivative.

Shock absorbers as part of suspension of car subject to force down of car acting under acceleration, and forces up: damping force (proportional to velocity) and restoring force (proportional to extension of spring)

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - cx$$



Simple second order differential equations

We know from the previous chapter that the solution of $a\frac{dy}{dx} + by = 0$ is $y = Ae^{-\frac{b}{a}x}$.
Let's 'guess' that the solution of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is similar, and of the form Ae^{mx}
Are the any restrictions on the constants?

Let $y = Ae^{mx}$

- The equation $am^2 + bm + c = 0$ is called the auxiliary equation, and if m is a root of the auxiliary equation then $y = Ae^{mx}$ is a solution of the differential equation
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
- When the auxiliary equation has **two real distinct roots** α and β, the general solution of the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants.

Find the general solution of the equation $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$

This is known as a **homogeneous** second-order differential equation because the RHS is 0. We will encountered nonhomogeneous equations later in the chapter.

Variants: $b^2 - 4ac = 0$

In the previous examples, the auxiliary equation had distinct roots, i.e. $b^2 - 4ac > 0$. What if we have equal roots?

- When the auxiliary equation has two equal roots α, the general solution is
$$y = (A + Bx)e^{\alpha x}$$

This is because the root of the auxiliary equation $m^2 - 6m + 9 = 0$ is 3.

Show that $(A + Bx)e^{3x}$ satisfies $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Side Note: The reason we have to use $y = (A + Bx)e^{\alpha x}$ instead of $Ae^{\alpha x} + Be^{\alpha x}$ is similar to why in Pure Year 2 partial fractions, we have to use $\frac{A}{x} + \frac{B}{x^2}$ if we had a repeated denominator x.

Variants: $b^2 - 4ac < 0$

This is actually exactly the same as when we usually have distinct real roots!

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 16y = 0$

If the auxiliary equation has two imaginary roots $\pm i\omega$, the general solution is $\boldsymbol{y = A\cos\omega x + B\sin\omega x}$ where A and B are arbitrary constants.

Variants: $b^2 - 4ac < 0$

So what about more general complex roots $a \pm bi$?

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$

If the auxiliary equation has two complex roots $p \pm iq$, the general solution is

$$\boldsymbol{y = e^{px}(A\cos qx + B\sin qx)}$$

where A and B are arbitrary constants.

Summary of Auxiliary Equation results

- When the auxiliary equation has **two real distinct roots** α and β, the general solution of the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants.

- When the auxiliary equation has two equal roots α, the general solution is
$$y = (A + Bx)e^{\alpha x}$$

If the auxiliary equation has two imaginary roots $\pm i\omega$, the general solution is $\boldsymbol{y = A\cos\omega x + B\sin\omega x}$ where A and B are arbitrary constants.

If the auxiliary equation has two complex roots $p \pm iq$, the general solution is
$$\boldsymbol{y = e^{px}(A\cos qx + B\sin qx)}$$
where A and B are arbitrary constants.

Find solutions to differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ with the following auxiliary equations (and helpfully provided roots).

Auxiliary Equation	**Roots**	**General Solution**
$m^2 + 6m + 8 = 0$	$m = -2, -4$	
$m^2 - 1 = 0$	$m = \pm 1$	
$m^2 - 2m + 1 = 0$	$m = 1$	
$m^2 + 4 = 0$	$m = \pm 2i$	
$m^2 + 10x + 25 = 0$	$m = -5$	
$m^2 - 12m + 45 = 0$	$m = 6 \pm 3i$	
$m^2 + 10 = 0$	$m = \pm\sqrt{10}$	
$m^2 + 2m + 5$	$m = -1 \pm 2i$	

Ex 7B - first column of questions
Be quick! These needn't be that demanding.

Particular Integrals

So far we've always had 0 in the RHS of the differential equation.
What if we have some function in terms of x?

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \boldsymbol{f(x)}$$

When the RHS is not 0, we have a **non-homogeneous** second order differential equation.

Solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \mathbf{0}$ first to obtain what is known as the **complementary function**. (C.F.)

Then solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \boldsymbol{f(x)}$ which can be found using appropriate substitution and comparing coefficients. Solution known as particular integral. (P.I.)

$$\boldsymbol{y = C.F. + P.I.}$$

This is because $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$ for the C.F. is 0 and $f(x)$ for the P.I., which sum to $f(x)$

Forms of PI's to use

Form of f(x)	**Form of particular integral**
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

WARNING!
The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an x^2 in front of your usual PI form

Find the **particular integral** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3}$

Hence find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3}$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{2x}$

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3x^2}$

$m^2 - 5m + 6 = 0$
$(m - 2)(m - 3) = 0$
$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \boldsymbol{e^x}$

$m^2 - 5m + 6 = 0$
$(m - 2)(m - 3) = 0$
$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{13\ sin\ 3x}$

$m^2 - 5m + 6 = 0$
$(m - 2)(m - 3) = 0$
$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \boldsymbol{e^{2x}}$

$m^2 - 5m + 6 = 0$
$(m - 2)(m - 3) = 0$
$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$

It's that cheeky little x.

June 2010 Q8

(a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation **(4 marks)**

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(b) Using your answer to part (a), find the general solution of the differential equation **(3 marks)**

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

Be warned: x is being used here as y was previous used.

June 2012 Q4

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t$$

Exercise 7C First column of 1, then rest of the questions

Boundary Conditions

Sometimes you're given certain conditions, which allows us to find constants (just as we could with first order differential equations).

Find y in terms of x, given that $\frac{d^2y}{dx^2} - y = 2e^x$, and that $\frac{dy}{dx} = 0$ and $y = 0$ at $x = 0$.

General solution: $\boldsymbol{y = Ae^x + Be^{-x} + xe^x}$

June 2010 Q8 (revisited!)

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(c) Given that at $x = 0, y = 0$ and $\frac{dy}{dx} = 5$, find the particular solution to this differential equation, giving your solution in the form $y = f(x)$ **(5)**

(d) Sketch the curve with equation $y = f(x)$ for $0 \leq x \leq \pi$ **(2)**

You previously found the general solution in (b) as $y = A\cos 5x + B\sin 5x + \frac{3}{10}x\sin 5x$

Exercise 7D Odd questions