Chapter 10: Numerical Methods

In the GCSE9-1 syllabus you covered 'iteration', which allowed you to find successfully better approximations to the solutions of an equation. We'll revisit this, but also see a more powerful method for approximating solutions.

1:: Locating Roots

What it means to find the root of an equation and when we can be sure a root lies in a stated range.

"Show that $f(x) = x^3 - 4x^2 + 3x + 1$ has a root between $x = 1.4$ and $x = 1.5$".

2:: Using iteration to approximate roots to $f(x) = 0$

[Jan 2010] 2. $\quad \mathrm{f}(x) = x^3 + 2x^2 - 3x - 11$

(*a*) Show that $\mathrm{f}(x) = 0$ can be rearranged as **(2)**

$$x = \sqrt{\left(\frac{3x+11}{x+2}\right)}, \quad x \neq -2.$$

The equation $\mathrm{f}(x) = 0$ has one positive root α.

The iterative formula $x_{n+1} = \sqrt{\left(\frac{3x_n+11}{x_n+2}\right)}$ is used to find an approximation to α.

(*b*) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2, x_3 and x_4. **(3)**

3:: The Newton-Raphson Method

A numerical method that tends to converge to (i.e. approach) the root faster, by **following the tangent of the graph.**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Why do we need numerical methods?

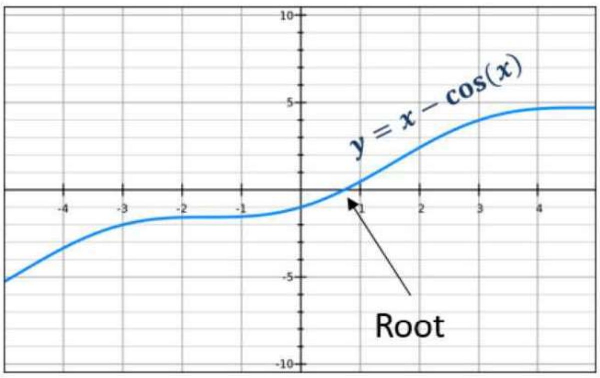
Finding the root of a function $f(x)$ is to **solve the equation $f(x) = 0$**
(i.e. the inputs such that the output of the function is 0)

However, for some functions, the 'exact' root is either complicated and difficult to calculate:

$$\boldsymbol{x^3 + 2x^2 - 3x + 4 = 0} \quad \Rightarrow \quad x = \frac{1}{3}\left(-2 - \frac{13}{\sqrt[3]{89 - 6\sqrt{159}}} - \sqrt[3]{89 - 6\sqrt{159}}\right)$$

or there's no 'algebraic' expression at all! (involving roots, logs, sin, cos, etc.)

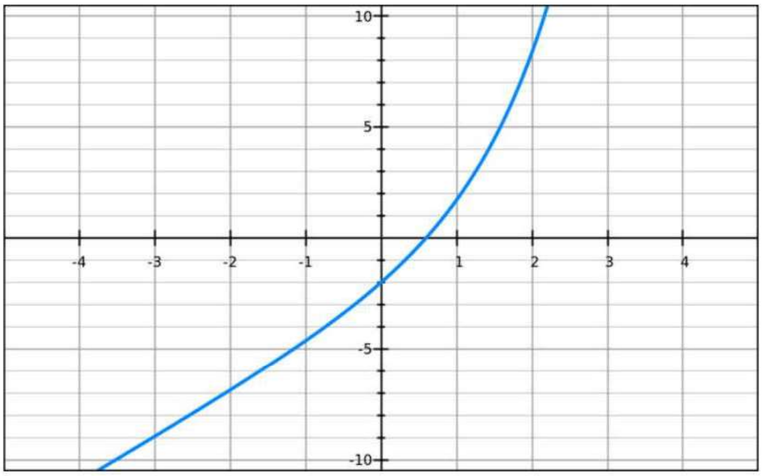
$$\boldsymbol{x - \cos(x) = 0} \quad \Rightarrow \quad \textbf{Exact solution not expressible}$$



But there are a variety of 'numerical methods' which get progressively better solutions to an equation in the form $f(x) = 0$.
You have already seen 'iteration' at GCSE as one such method.

Proving a solution lies in a range

Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$



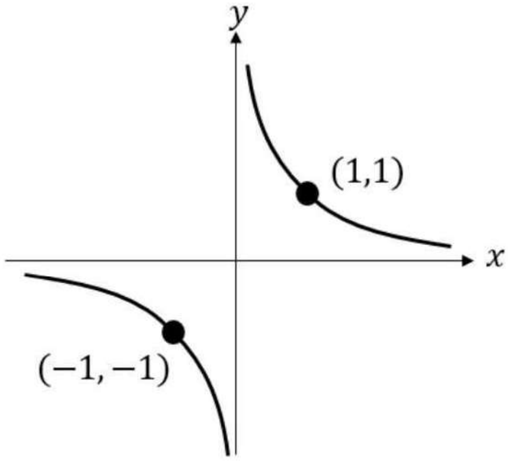
Exam Tip: In the mark scheme they're looking for:

1. Finding the function output for the two values.
2. Referring to a 'change in sign'.
3. Commenting that f(x) is continuous

…why only if the function is continuous?

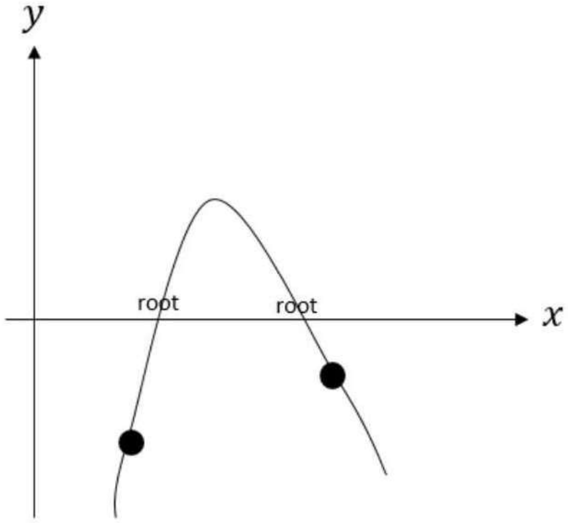
When $f(x) = \frac{1}{x}$, then $f(-1) = -1$ and $f(1) = 1$. There is a change in sign therefore $f(x)$ has a root in the range [-1, 1]

Why is this incorrect?



A function is **continuous** if the line **does not 'jump'**. A root is only guaranteed with a sign change if the function is continuous, as otherwise the line can skip past 0 (in this case due to a vertical asymptote.

No sign change doesn't *always* mean there isn't a root



Beware! Just because there isn't a sign change, doesn't mean there's no root in that interval.

The sign change method fails to detect a root if there were an **even number of roots** in that interval.

Proving a solution to a given accuracy

Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

The root of $g(x) = 0$ is α.

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

(a) Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagrams shows that the function $y = \ln(x) - \frac{1}{x}$ has only one root.

(b) Show that this root lies in the interval $1.7 < x < 1.8$

(c) Given that the root of $f(x)$ is α, show that $\alpha = 1.763$ correct to 3 decimal places.

Using iteration to approximate a root

Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

(a) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

Tip: The difficulty is that there's multiple choices of x to isolate on one side of the equation. Therefore use the target equation to give clues for how to rearrange.

✎ To solve $f(x) = 0$ by an iterative method, rearrange into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$

We'll see why it works later.

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(b) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

x_0, x_1, x_2 represent successively better approximations of the root, where x_0 is the starting value.

Calculator Tip: Initially type x_0 (i.e. 2) onto your calculator.
Now just type:

$$\ln(6 - ANS) + 1$$

And then spam your $=$ key to get successive iterations.

Exam Tip: Show the substitution for x_1 to ensure you get the method mark. But then just write the final value for x_2 and thereafter, as the remaining marks will be 'accuracy' ones.

If the x_n values get closer and closer together the iterations **converge to the root**, so iteration has failed. The iteration is **convergent**.

If the x_n values get further and further apart the iterations **diverge**, so iteration has failed. The iteration is **divergent**.

If they bounce back and forth between values, we say the iteration **oscillates** or is **periodic** or is **non-convergent**.

Your Turn

Edexcel C3 June 2012 Q2

$$f(x) = x^3 + 3x^2 + 4x - 12$$

(*a*) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{4(3-x)}{(3+x)}\right)}, \quad x \neq -3. \qquad \textbf{(3)}$$

The equation $x^3 + 3x^2 + 4x - 12 = 0$ has a single root which is between 1 and 2.

(*b*) Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{4(3-x_n)}{(3+x_n)}\right)}, \quad n \geq 0,$$

with $x_0 = 1$ to find, to 2 decimal places, the value of x_1, x_2 and x_3. **(3)**

The root of $f(x) = 0$ is α.

(*c*) By choosing a suitable interval, prove that $\alpha = 1.272$ to 3 decimal places. **(3)**

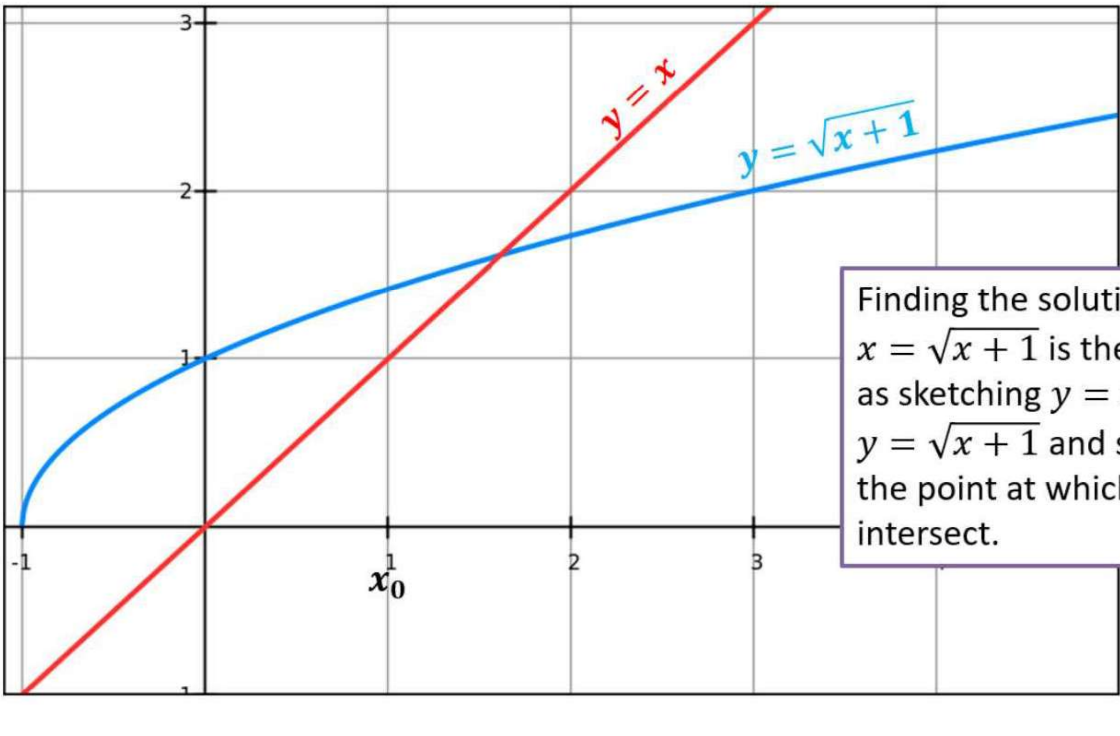
Why does this method work?

Rearrange $x^2 - x - 1 = 0$ to make 3 different iterative formulae

Staircase diagrams

Solve $x^2 - x - 1 = 0$

Recall we put in the form $x = g(x)$: in this case $x = \sqrt{x+1}$ is one possible rearrangement.
We can then use the recurrence $x_{n+1} = \sqrt{x_n + 1}$. Why does **this** recurrence work? (and not others?)



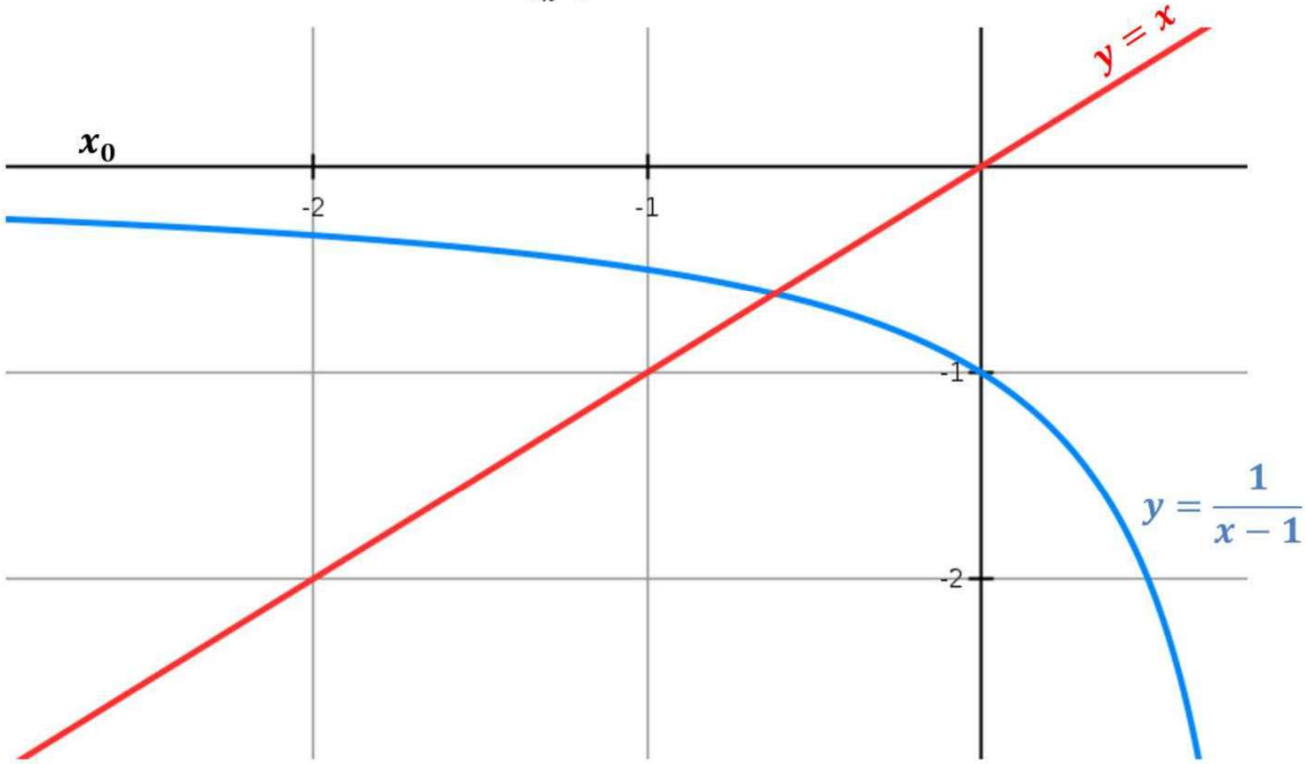
Finding the solution to $x = \sqrt{x+1}$ is the same as sketching $y = x$ and $y = \sqrt{x+1}$ and seeing the point at which they intersect.

Cobweb diagrams

Solve $x^2 - x - 1 = 0$

We could also have rearranged differently to $x = \frac{1}{x-1}$

Therefore we use the recurrence $x_{n+1} = \frac{1}{x_n - 1}$. What happens this time?

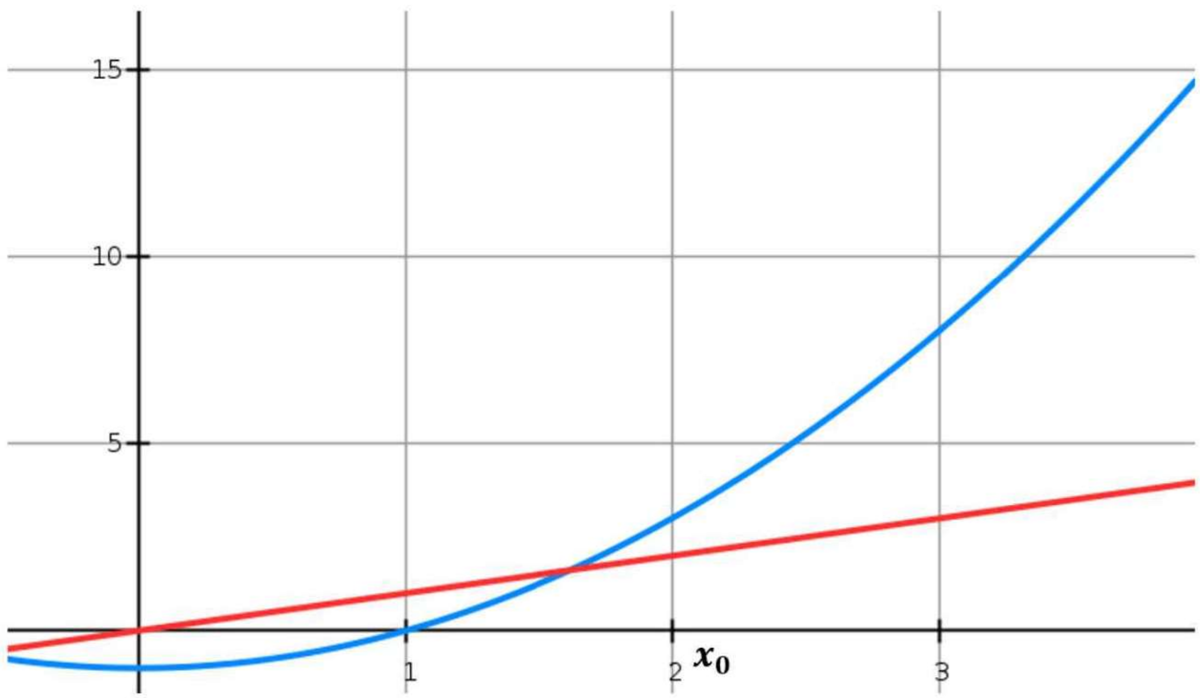


And when iteration fails…

Solve $x^2 - x - 1 = 0$

But again, we could have rearranged differently! $x = x^2 - 1$

Therefore we use the recurrence $x_{n+1} = x_n^2 - 1$. What happens this time?

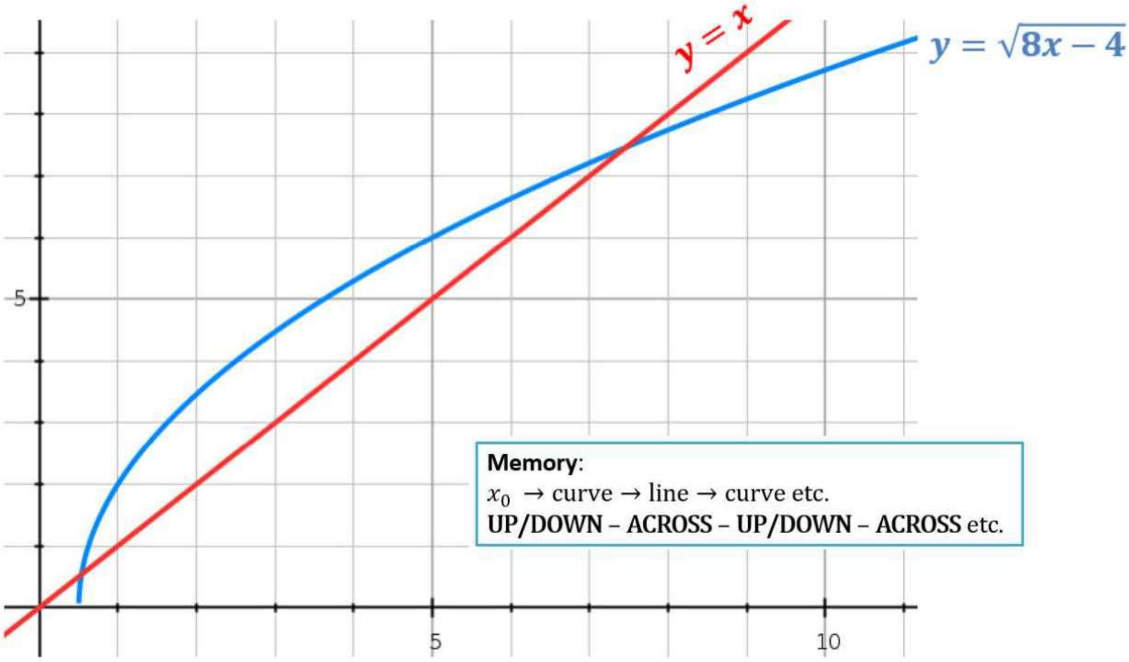


Your Turn

$$f(x) = x^2 - 8x + 4$$

(a) Show that the root of the equation $f(x) = 0$ can be written as $x = \sqrt{8x - 4}$

(b) Using the iterative formula $x_{n+1} = \sqrt{8x_n - 4}$, and starting with $x_0 = 1$, draw a staircase diagram, indicating x_0, x_1, x_2 on your x-axis, as well as the root α.



Ex 10B

4. The curve with equation $y = 2\ln(8 - x)$ meets the line $y = x$ at a single point, $x = \alpha$.

(a) Show that $3 < \alpha < 4$

(2)

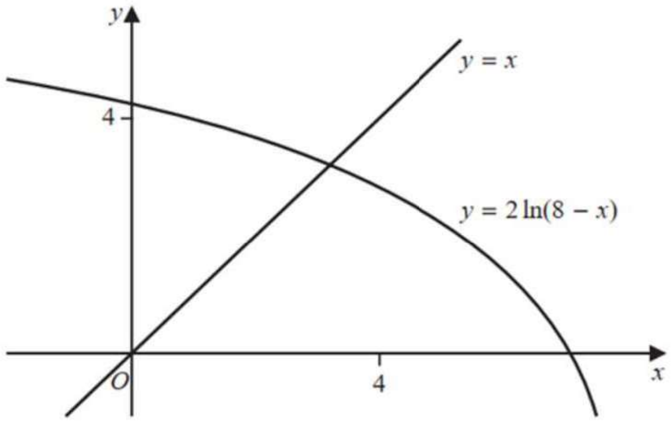


Figure 2

Figure 2 shows the graph of $y = 2\ln(8 - x)$ and the graph of $y = x$.

A student uses the iteration formula

$$x_{n+1} = 2\ln(8 - x_n), \quad n \in \mathbb{N}$$

in an attempt to find an approximation for α.

Using the graph and starting with $x_1 = 4$

(b) determine whether or not this iteration formula can be used to find an approximation for α, justifying your answer.

(2)

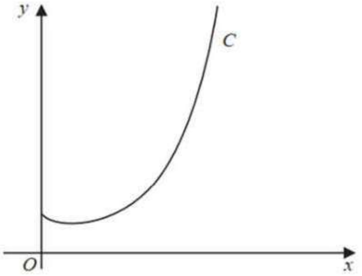


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x,\ x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{\ 1-x_n}$$

Using this formula with $x_1 = 1.5$

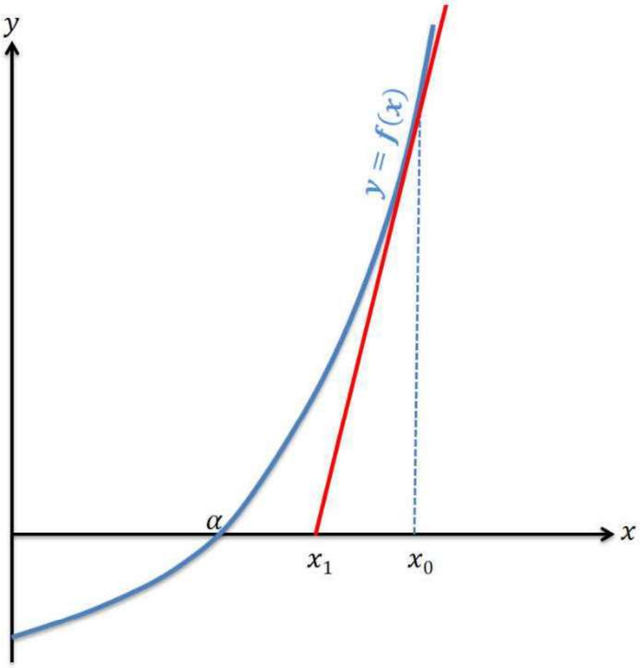
(c) find x_4 to 3 decimal places,

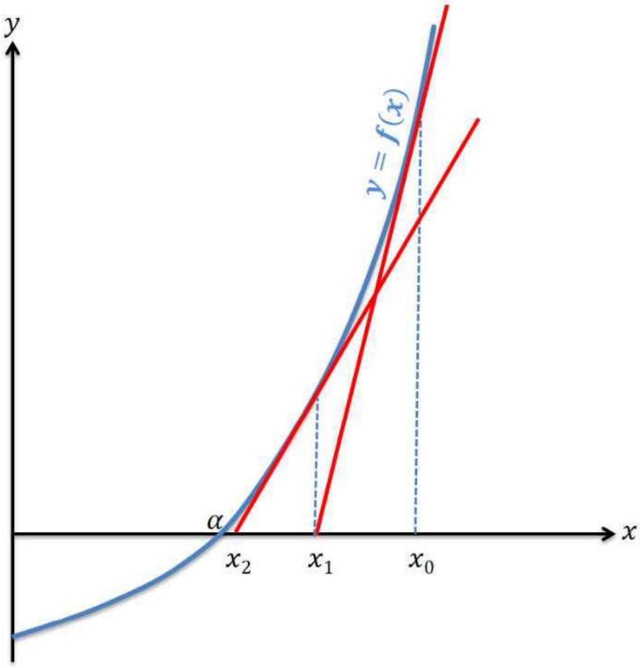
(2)

(d) describe the long-term behaviour of x_n

(2)

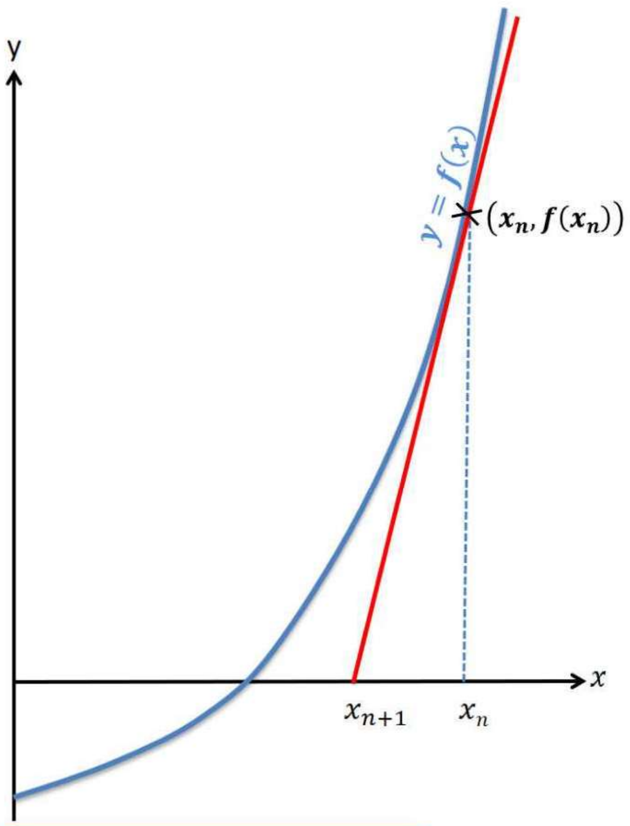
The Newton-Raphson Process/Method





A seemingly sensible thing to do is to follow the direction of the line, i.e. use the gradient of the tangent. If the line was reasonably straight, the point the tangent hits the x-axis would be close to the root.

Deriving the Formula - *not in the specification, but interesting!*

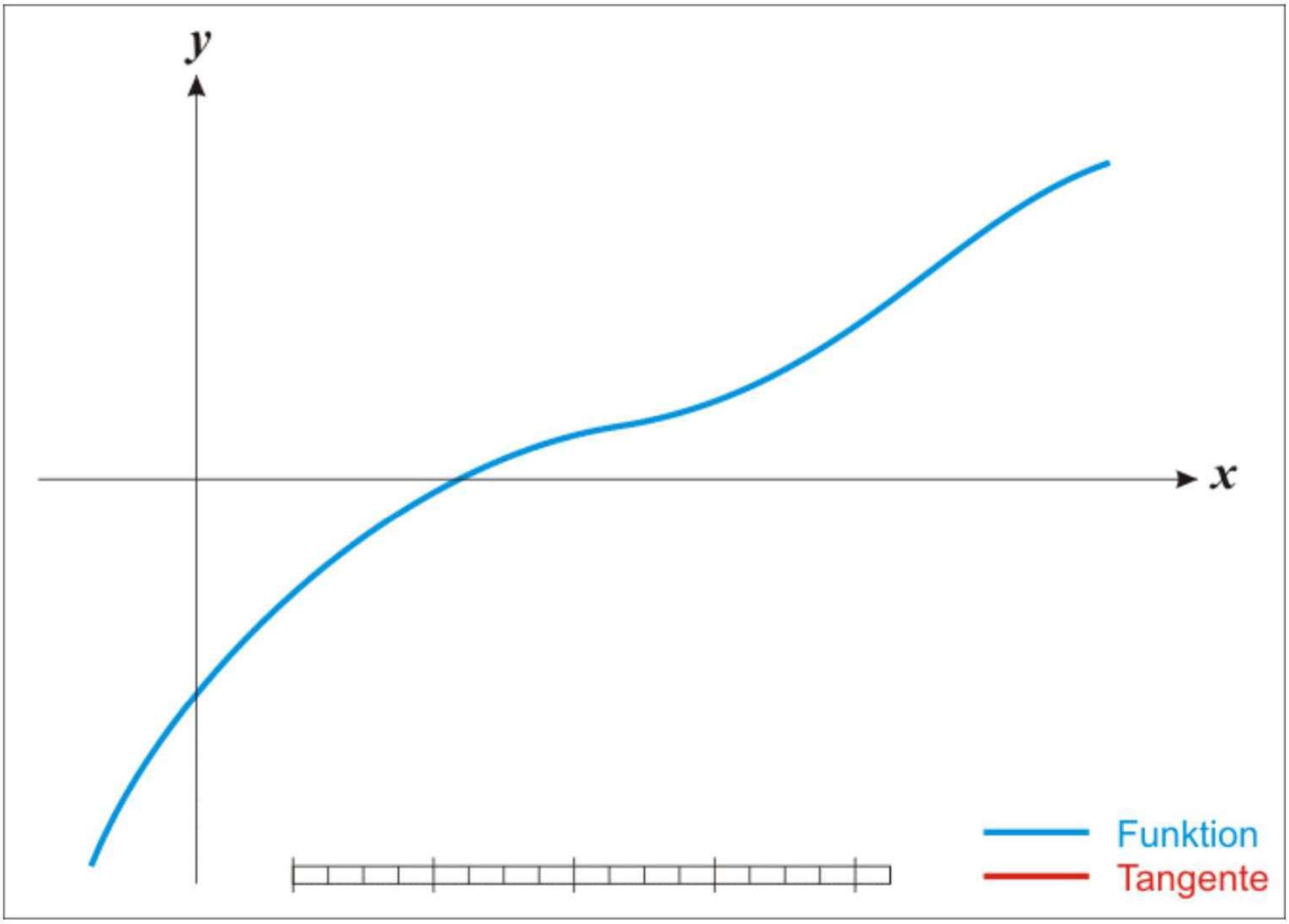


Newton-Raphson Process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Formula Book

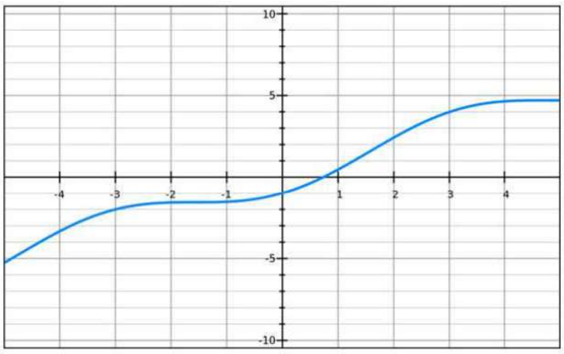
The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$



Example

Returning to our original example: $\boldsymbol{x = \cos(x)}$, say letting $x_0 = 0.5$

(Note: Recall that differentiation assumes radians)



Tip: To perform iterations quickly, do the following on your calculator:

[0.5] [=]
[ANS] – (ANS – cos(ANS))/(1 + sin(ANS))
Then hit [=].

Quick Questions

Using the Newton-Raphson process, state the recurrence relation for the following functions:

$f(x) = x^3 - 2$ $\Rightarrow$

$f(x) = \tan x$ $\Rightarrow$

$f(x) = x^2 - x - 1$ $\Rightarrow$

Edexcel FP1 June 2013(R) Q3c

$$\mathrm{f}(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

...

The equation $\mathrm{f}(x) = 0$ has a root β in the interval $[-2, -1]$.

(c) Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to obtain a second approximation to β.
Give your answer to 2 decimal places.

(5)

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Your Turn

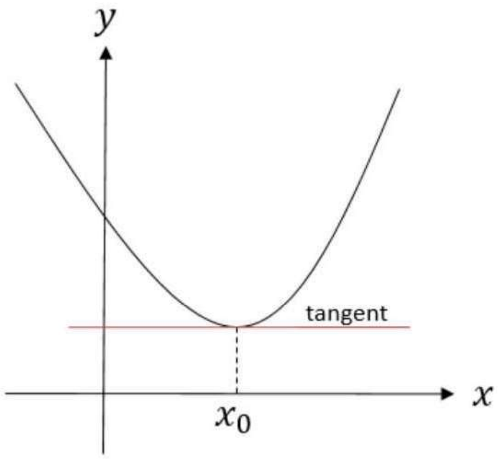
Edexcel FP1 Jan 2010 Q2c

$$f(x) = 3x^2 - \frac{11}{x^2}.$$

(*c*) Taking 1.4 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to α, giving your answer to 3 decimal places.
(5)

Ex 10C

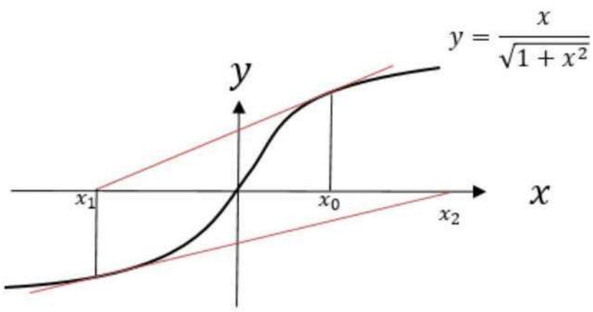
When does Newton-Raphson fail?



$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

If the starting value x_0 was the stationary point, then $f'(x_0) = 0$, resulting in a division by 0 in the above formula.

Graphically, it is because the tangent will never reach the x-axis.



Newton-Raphson also suffers from the same drawbacks as solving by iteration, in that it's possible for the values of x_i to **diverge**.
In this example, the x_i oscillate either side of 0, but gradually getting further away from $\alpha = 0$.

Ex 10C

The value of β lies in the interval [1.5, 3]

A student takes 3 as her first approximation to β.

Given $f(3) = -1.4189$ and $f'(3) = -8.3078$ to 4 decimal places,

(c) apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation to β.
Give your answer to 2 decimal places.

(2)

A different student takes a starting value of 1.5 as his first approximation to β.

(d) Use Figure 3 to explain whether or not the Newton-Raphson method with this starting value gives a good second approximation to β.

(2)

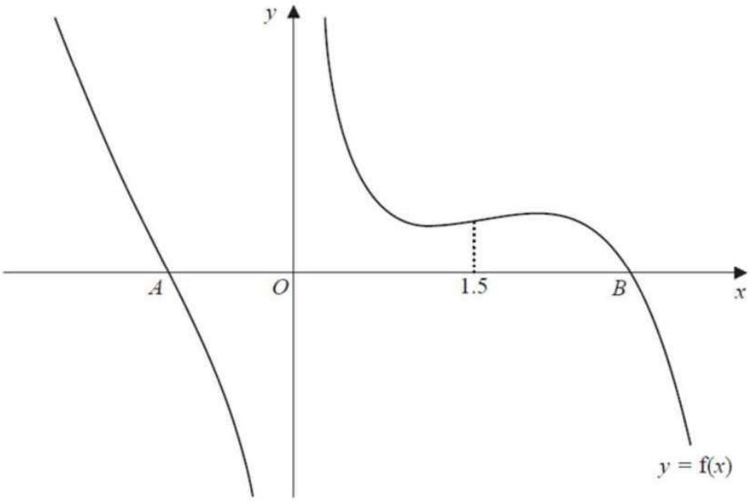
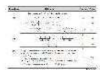


Figure 3



5. The equation $2x^3 + x^2 - 1 = 0$ has exactly one real root.

(a) Show that, for this equation, the Newton-Raphson formula can be written

$$x_{n+1} = \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$$

(3)

Using the formula given in part (a) with $x_1 = 1$

(b) find the values of x_2 and x_3

(2)

(c) Explain why, for this question, the Newton-Raphson method cannot be used with $x_1 = 0$

(1)

Modelling

The price of a car in £s, x years after purchase, is modelled by the function

$$f(x) = 15\,000\,(0.85)^x - 1000\sin x\,, \qquad x > 0$$

(a) Use the model to find the value, to the nearest hundred £s, of the car 10 years after purchase.

(b) Show that $f(x)$ has a root between 19 and 20.

(c) Find $f'(x)$

(d) Taking 19.5 as a first approximation, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation for the time when the value of the car is zero. Give your answer to 3 decimal places.

(e) Criticise this model with respect to the value of the car as it gets older.

Ex 10D