$f(x) = \operatorname{artanh} x.$

$f'(x) = \frac{1}{1-x^2} = (1-x^2)^{-1}$

$f''(x) = 2x(1-x^2)^{-2}$

$f'''(x) = 2(1-x^2)^{-2} + 8x^2(1-x^2)^{-3}$

$f''''(x) = 8x(1-x^2)^{-3} + 16x(1-x^2)^{-3} + 48x^3(1-x^2)^{-4}$

$= 24x(1-x^2)^{-3} + 48x^3(1-x^2)^{-4}$

$f'''''(x) = 24(1-x^2)^{-3} + 144x^2(1-x^2)^{-4} + 144x^2(1-x^2)^{-4}$
$+ 384x^4(1-x^2)^{-5}$

$f(0)=0$ $\quad f'''(0)=2$ $\quad f^{(5)}(0)=24.$

$f'(0)=1$ $\quad f''''(0)=0$

$f''(0)=0$

$f(x) = x + \frac{2x^3}{6} + \frac{24x^5}{5!} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5.$

$f(x) = \cosh x$

$f'(x) = \sinh x$

$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$

~~an~~

$\operatorname{artanh} x \cosh x$

$\left(x + \frac{x^3}{3} + \frac{x^5}{5}\right)\left(1 + \frac{x^2}{2} + \frac{x^4}{24}\right)$

$\approx x + \frac{x^3}{3} + \frac{x^3}{2}$

$\approx x + \frac{5}{6}x^3$

Standard Integrals

Same as non-hyperbolic version?		
✗	$\int \sinh x \ dx = \cosh x + C$	
✓	$\int \cosh x \ dx = \sinh x + C$	
✓	$\int \operatorname{sech}^2 x \ dx = \tanh x + C$	
✓	$\int \operatorname{cosech}^2 x \ dx = -\coth x + C$	Not in formula booklet.
✗	$\int \operatorname{sech} x \tanh x \ dx = -\operatorname{sech} x + C$	
✓	$\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$	

Not in this chapter but worth briefly mentioning.

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C, \qquad |x| < 1$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

Was covered in Chapter 3.

$$\int \frac{1}{\sqrt{1+x^2}} \, dx = \operatorname{arsinh} x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} \, dx = \operatorname{arcosh} x + C, \qquad x > 1$$

Recall that:

$$\int \boldsymbol{f'(ax+b)\,dx = \frac{1}{a} f(ax+b) + C}$$

e.g. $\int e^{3x+2} dx = \frac{1}{3} e^{3x+2}$

$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\dfrac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\dfrac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert = \dfrac{1}{a}\operatorname{artanh}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$

$$\int \cosh(4x-1)\,dx = \frac{1}{4}\sinh(4x-1) + c$$

$$\int sinh\left(\frac{2}{3}x\right)\,dx = \frac{3}{2}\cosh\left(\frac{2}{3}x\right) + c$$

$$\int \frac{3}{\sqrt{1+x^2}}dx = 3\operatorname{arsinh} x + c$$

$$\int \frac{4}{\sqrt{x^2-1}}dx = 4\operatorname{arcosh} x + c$$

$$\int sinh\,(3x)\ dx = \frac{1}{3}\cosh 3x + c$$

$$\int \frac{10}{\sqrt{x^2-1}}dx = 10\operatorname{arcosh} x + c$$

$$\int \frac{2}{\sqrt{1+x^2}}dx = 2\operatorname{arsinh} x + c$$

$$\int \cosh(4x-1)\,dx = \boldsymbol{\frac{1}{4}\sinh(4x-1) + C}$$

$$\int sinh\left(\frac{2}{3}x\right)\,dx = \boldsymbol{\frac{3}{2}\cosh\left(\frac{2}{3}x\right) + C}$$

$$\int \frac{3}{\sqrt{1+x^2}}dx = \boldsymbol{3\ arsinh\ x}$$

$$\int \frac{4}{\sqrt{x^2-1}}dx = \boldsymbol{4\ arcosh\ x}$$

$$\int sinh\,(3x)\ dx = \boldsymbol{\frac{1}{3}\cosh(3x) + C}$$

$$\int \frac{10}{\sqrt{x^2-1}}dx = \boldsymbol{10\ arcosh\ x}$$

$$\int \frac{2}{\sqrt{1+x^2}}dx = \boldsymbol{2\ arsinh\ x}$$

$$\int \frac{2+5x}{\sqrt{x^2+1}}\,dx$$

$$= \int \left(\frac{2}{\sqrt{x^2+1}} + \frac{5x}{\sqrt{x^2+1}} \right) dx$$

$5x(x^2+1)^{-1/2}$

$$= 2\,\mathrm{arsinh}\,x + 5(x^2+1)^{1/2} + c$$

standard result.

reverse chain rule

$$\int \cosh^5 2x \sinh 2x \; dx$$

$$\int \cosh^5 2x \sinh 2x =$$

$$\frac{1}{12} \cosh^6 2x + c$$

$$\int \tanh x \; dx$$

$$\int \tanh x \; dx = \int \frac{\sinh x}{\cosh x} \, dx$$

$$= \ln(\cosh x) + c$$

Using Identities

$$\int \cosh^2 3x \; dx$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cosh 6x\right) dx$$

$$= \frac{1}{2}x + \frac{1}{12}\sinh 6x + c$$

even powers

$$\boxed{\begin{aligned} \cosh^2 x &= \frac{1}{2} + \frac{1}{2}\cosh 2x \\ \sin^2 x &= \frac{1}{2} - \frac{1}{2}\cos 2x \\ \sinh^2 x &= \frac{1}{2}\cosh 2x - \frac{1}{2}. \end{aligned}}$$

O. R.

$$\int \sinh^3 x \; dx$$

$$= \int \sinh x \sinh^2 x \; dx$$

ODD POWERS

$$= \int \sinh x\left(\cosh^2 x - 1\right) dx$$

$$= \int \left(\sinh x \cosh^2 x - \sinh x\right) dx$$

$$= \frac{1}{3}\cosh^3 x - \cosh x + c$$

Use this approach in general for small odd powers of sinh and cosh.

Other things to try...

Sometimes there are techniques which work on non-hyperbolic trig functions but doesn't work on hyperbolic ones. Just first replace any hyperbolic functions with their definition.

Find $\int e^{2x} \sinh x \ dx$

$$\int e^{2x}\left(\frac{e^{x}-e^{-x}}{2}\right)dx$$

$$=\frac{1}{2}\int\left(e^{3x}-e^{x}\right)dx$$

$$=\frac{1}{2}\left(\frac{1}{3}e^{3x}-e^{x}\right)+c$$

$$=\frac{1}{6}e^{3x}-\frac{1}{2}e^{x}+c$$

Find $\int \operatorname{sech} x \ dx$

Use the substitution $u = e^x$

$$\int \operatorname{sech} x \, dx = \int \frac{2}{e^{x}+e^{-x}}\,dx$$

$$= \int \frac{2e^{x}}{e^{2x}+1}\,dx$$

$u = e^x$

$\frac{du}{dx} = e^x$

$\frac{du}{dx} = u$

$dx = \frac{1}{u}\,du$

$$= \int \frac{2\cancel{u}}{u^{2}+1}\times\frac{1}{\cancel{u}}\,du$$

$$= \int \frac{2}{u^{2}+1}\,du$$

$$= 2\arctan u + c$$

$$= 2\arctan(e^{x}) + c$$

Ex 6E Q1-10

Dealing with $1/\sqrt{a^2+x^2}$, $1/\sqrt{x^2-a^2}$, ...

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \sinh^2 u = \cosh^2 u$$

Sensible substitution and why?

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$x = a\sinh u$$

tan wouldn't work as well this time because the denominator would simplify to $a\sec u$, but we'd be multiplying by $a\sec^2\theta$, meaning not all the secs would cancel. With $\sinh u$ the two $\cosh u$'s obtained would fully cancel.

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$x = a\cosh u$$

Show that $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = arcosh\left(\frac{x}{a}\right) + c$

$x = a\cosh u$

$\frac{dx}{du} = a\sinh u$

$x^2 - a^2 = a^2\cosh^2 u - a^2$

$x^2 - a^2 = a^2\sinh^2 u$

$\sqrt{x^2-a^2} = a\sinh u$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \int \frac{1}{a\sinh u} \times a\sinh u\,du$

$= \int 1\,du$

$= u + c$

$= arcosh\left(\frac{x}{a}\right) + c$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Show that $\int_5^8 \frac{1}{\sqrt{x^2-16}}\,dx = \ln\left(\frac{2+\sqrt{3}}{2}\right)$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = arsinh\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = arcosh\left(\frac{x}{a}\right) + c, \qquad x > a$$

$$arsinh\, x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$arcosh\, x = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1$$

$$artanh\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad |x| < 1$$

$$\int_5^8 \frac{1}{\sqrt{x^2-16}}\,dx = \left[arcosh\left(\frac{x}{4}\right)\right]_5^8$$

$a^2 = 16$

$a = 4$

$$= \left[\ln\left(\frac{x}{4} + \sqrt{\frac{x^2}{16} - 1}\right)\right]_5^8$$

$$= \ln\left(2 + \sqrt{4-1}\right) - \ln\left(\frac{5}{4} + \sqrt{\frac{25}{16} - 1}\right)$$

$$= \ln(2+\sqrt{3}) - \ln\left(\frac{5}{4} + \frac{3}{4}\right)$$

$$= \ln(2+\sqrt{3}) - \ln 2$$

$$= \ln\left(\frac{2+\sqrt{3}}{2}\right)$$

Harder Example

Show that $\int \sqrt{1+x^2}\,dx = \frac{1}{2} arsinh\,x + \frac{1}{2}x\sqrt{1+x^2} + C.$

(Hint: Use a sensible substitution)

$x = \sinh u$

$1 + x^2 = 1 + \sinh^2 u$

$1 + x^2 = \cosh^2 u$

$\sqrt{1+x^2} = \cosh u$

$x = \sinh u$

$\frac{dx}{du} = \cosh u$

$dx = \cosh u \, du$

$u = \text{arsinh}\,x$

$\int \sqrt{1+x^2}\,dx = \int \cosh u \times \cosh u \, du$

$= \int \cosh^2 u \, du$

$= \int \left(\frac{1}{2} + \frac{1}{2}\cosh 2u\right) du$

$= \frac{1}{2}u + \frac{1}{4}\sinh 2u + C$

$= \frac{1}{2}\text{arsinh}\,x + \frac{1}{2}\sinh u \cosh u + C$

$= \frac{1}{2}\text{arsinh}\,x + \frac{1}{2}x\sqrt{1+x^2} + C$

$\sinh 2u$

$2\sinh u \cosh u$

$x = \sinh u$

$\sqrt{1+x^2} = \cosh u$

Using $x = \sinh u$:

$\frac{dx}{du} = \cosh u \rightarrow dx = \cosh u \, du$

$\int \sqrt{1+x^2}\,dx = \int \cosh u \times \cosh u \, du$

$= \int \cosh^2 u \, du$

$= \frac{1}{2}\int 1 + \cosh 2u \, du$

$= \frac{1}{2}\left(u + \frac{1}{2}\sinh 2u\right) + C$

$= \frac{1}{2}\text{arsinh}\,x + \frac{1}{2}x\sqrt{1+x^2} + C$

Your Turn

Hint: You may want to factorise out $\frac{1}{\sqrt{4}}$ first, as we did in Chapter 3.

[June 2013 Q2]

(*a*) Find

$$\int \frac{1}{\sqrt{(4x^2+9)}}\,\mathrm{d}x$$

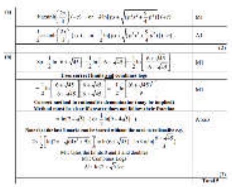
(2)

(*b*) Use your answer to part (*a*) to find the exact value of

$$\int_{-3}^{3} \frac{1}{\sqrt{(4x^2+9)}}\,\mathrm{d}x$$

giving your answer in the form $k\ln(a + b\sqrt{5})$, where a and b are integers and k is a constant.

(3)



Your Turn

Using a hyperbolic substitution, evaluate
$\int_0^6 \frac{x^3}{\sqrt{x^2+9}}\,dx$

$x = 3\sinh u$

$\sqrt{x^2+9} = \sqrt{9\cosh^2 u}$
$= 3\cosh u$

$\frac{dx}{du} = 3\cosh u$

$dx = 3\cosh u \, du$

x	u
0	0
6	arsinh2

$\cosh^2 u - \sinh^2 u = 1$

$6 = 3\sinh u \rightarrow \sinh u = 2$

arsinh2

$u = \text{arsinh}\,2$

$\sinh u = 2$
$\cosh u = \sqrt{1+\sinh^2 u} = \sqrt{5}$

$$\int_0^6 \frac{x^3}{\sqrt{x^2+9}}\,dx = \int_0^{\text{arsinh}2} \frac{27\sinh^3 u}{3\cancel{\cosh u}} \times 3\cancel{\cosh u}\,du$$

$$= 27\int_0^{\text{arsinh}2} \sinh^3 u\,du$$

$$= 27\int_0^{\text{arsinh}2} \sinh u(\cosh^2 u - 1)\,du$$

$$= 27\int_0^{\text{arsinh}2} (\sinh u\cosh^2 u - \sinh u)\,du$$

$$= 27\left[\tfrac{1}{3}\cosh^3 u - \cosh u\right]_0^{\text{arsinh}2}$$

$$= 27\left(\left(\tfrac{1}{3}(\sqrt{5})^3 - \sqrt{5}\right) - \left(\tfrac{1}{3} - 1\right)\right)$$

$$= 27\left(\frac{5\sqrt{5}}{3} - \sqrt{5} + \frac{2}{3}\right)$$

$$= 45\sqrt{5} - 27\sqrt{5} + 18$$

$$= 18\sqrt{5} + 18$$

$$= \underline{18(\sqrt{5}+1)}$$

Ex 6E Q11-20