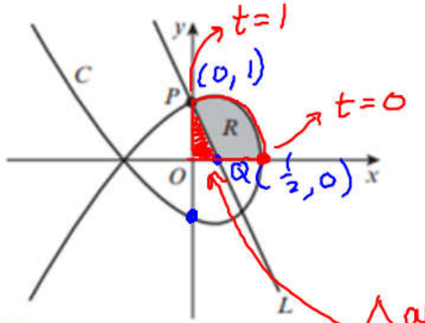
The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



Δ area

$= \frac{1}{2} \times \frac{1}{2} \times 1$

$= \frac{1}{4}$

$x = 1 - t^2$ $\quad \boxed{y = 2t - t^3}$

$0 = 1 - t^2$

$t^2 = 1$

$t = \pm 1$

$t = 1 \quad y = 2 \times 1 - 1^3 = 1$

$P(0,1)$

$0 = 2t - t^3$

$0 = t(2 - t^2)$

$t = 0 \quad t = \pm\sqrt{2}$

Equation of L

$\frac{dx}{dt} = -2t \quad \frac{dy}{dt} = 2 - 3t^2$

$\frac{dy}{dx} = \frac{2 - 3t^2}{-2t}$

$t = 1,\ \frac{dy}{dx} = \frac{2-3}{-2} = \frac{1}{2}$

Normal grad $= -2$

$y - y_1 = m(x - x_1)$

$y - 1 = -2(x - 0)$

$y = -2x + 1$

At Q, $y = 0 \quad 0 = -2x + 1$

$x = \frac{1}{2} \quad Q\left(\frac{1}{2}, 0\right)$

$$\int y\,dx = \int_1^0 y \frac{dx}{dt}\,dt = \int_1^0 (2t - t^3)(-2t)\,dt$$

$$= \int_1^0 (-4t^2 + 2t^4)\,dt$$

$$= \left[-\frac{4}{3}t^3 + \frac{2}{5}t^3\right]_1^0 = (0) - \left(-\frac{4}{3} + \frac{2}{5}\right)$$

$$= \frac{14}{15}$$

$$R = \frac{14}{15} - \frac{1}{4} = \frac{41}{60}$$

Paper C

Q2) $4^x = 2xy$. $\frac{dy}{dx}$ at $(2, 4)$.

x y

$\ln 4 \times 4^x = 2x\frac{dy}{dx} + 2y$

$u = 2x \quad v = y$

$u' = 2 \quad v' = \frac{dy}{dx}$

$\ln 4 \times 4^2 = 4\frac{dy}{dx} + 8$

$\frac{16\ln 4 - 8}{4} = \frac{dy}{dx}$

$\underline{4\ln 4 - 2 = \frac{dy}{dx}}$

$y = a^x$

$\frac{dy}{dx} = \ln a \times a^x$

$y = a^x$

$\ln y = x \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y\ln a = \ln a \times a^x$

Your Turn

Edexcel C4 Jan 2013 Q5

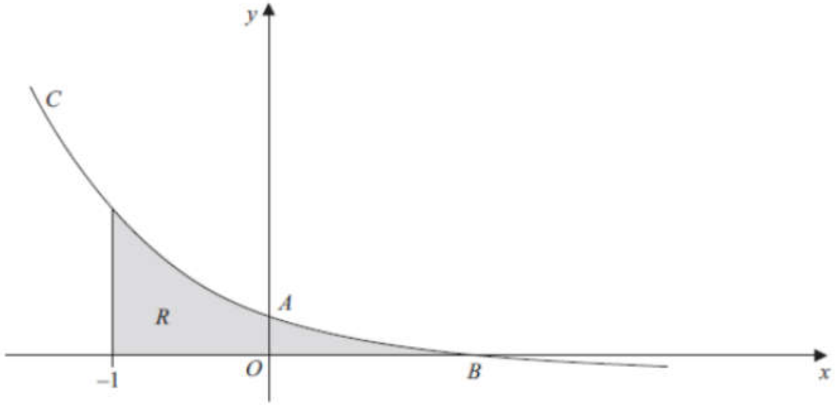


Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(*a*) Show that A has coordinates (0, 3). **(2)**

(*b*) Find the x-coordinate of the point B. **(2)**

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(*d*) Use integration to find the exact area of R.

Helping Hand:

$$\frac{d}{dx}(a^x) = a^x(\ln a)$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + c$$

	Working parametrically:
	$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1$ or $y = e^{t\ln 2} - 1$
(a)	$\{x = 0 \Rightarrow\}\ 0 = 1 - \frac{1}{2}t \Rightarrow t = 2$
	When $t = 2$, $y = 2^2 - 1 = 3$
(b)	$\{y = 0 \Rightarrow\}\ 0 = 2^t - 1 \Rightarrow t = 0$
	When $t = 0$, $x = 1 - \frac{1}{2}(0) = 1$

(d)	$\text{Area}(R) = \int (2^t - 1).\left(-\frac{1}{2}\right)dt$	Complete substitution for both y and dx	M1
	$x = -1 \to t = 4$ and $x = 1 \to t = 0$		B1
	$= \left\{-\frac{1}{2}\right\}\left(\frac{2^t}{\ln 2} - t\right)$	Either $2^t \to \frac{2^t}{\ln 2}$ or $(2^t - 1) \to \frac{(2^t)}{\pm\alpha(\ln 2)} - t$ or $(2^t - 1) \to \pm\alpha(\ln 2)(2^t) - t$	M1*
		$(2^t - 1) \to \frac{2^t}{\ln 2} - t$	A1
	$\left\{-\frac{1}{2}\left[\frac{2^t}{\ln 2} - t\right]_4^0\right\} = -\frac{1}{2}\left(\left(\frac{1}{\ln 2}\right) - \left(\frac{16}{\ln 2} - 4\right)\right)$	**Depends on the previous method mark.** Substitutes their changed limits in t and subtracts either way round.	dM1*
	$= \frac{15}{2\ln 2} - 2$	$\frac{15}{2\ln 2} - 2$ or equivalent.	A1

SKILL #11: Differential Equations

Differential equations are equations involving a mix of variables and derivatives, e.g. y, x and $\frac{dy}{dx}$.

$\cancel{\frac{d^2y}{dx^2}}$

'Solving' these equations means to get y in terms of x (with no $\frac{dy}{dx}$).

Q Find the general solution to $\frac{dy}{dx} = xy + y$

$y \longleftarrow \quad x \longrightarrow$

$$\frac{dy}{dx} = xy + y$$

$$y = Ae^{\frac{1}{2}x^2 + x}$$

$$\frac{dy}{dx} = y(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = x + 1$$

$$\int \frac{1}{y}\, dy = \int (x+1)\, dx$$

$$\ln y = \frac{1}{2}x^2 + x + c$$

$$y = e^{\frac{1}{2}x^2 + x + c} = e^{\frac{1}{2}x^2 + x} e^c = Ae^{\frac{1}{2}x^2 + x}$$

where $A = e^c$

STEP 1: Get y to the side of $\frac{dy}{dx}$ by dividing and x to the other side.
(you may need to factorise to separate out y first)

STEP 2: Integrate both sides with respect to x. $\frac{dy}{dx}\,dx$ simplifies to dy (recall that (implicitly) differentiating an expression in terms of y with respect to x introduces a $\frac{dy}{dx}$, so integrating similarly would get rid of it)

STEP 3: Make y the subject, if the question asks.

Q Find the general solution to $(1+x^2)\frac{dy}{dx} = x\tan y$

$$(1+x^2)\frac{dy}{dx} = x\tan y$$

$y \Leftarrow$ $x \longrightarrow$

$$\frac{1}{\tan y}\frac{dy}{dx} = \frac{x}{1+x^2}$$

$$\int \frac{1}{\tan y}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\int \cot y\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\ln|\sin y| = \frac{1}{2}\ln|1+x^2| + \ln k$$

If everything is "ln"

$$\ln|\sin y| = \ln k\sqrt{1+x^2}$$

$$\sin y = k\sqrt{1+x^2}$$

$$\underline{\underline{y = \arcsin\left(k\sqrt{1+x^2}\right)}}$$

$y = \arcsin\left(k\sqrt{1+x^2}\right)$

Differential Equations with Boundary Conditions

Q Find the general solution to $\frac{dy}{dx} = -\frac{3(y-2)}{(2x+1)(x+2)}$

Given that $x = 1$ when $y = 4$. Leave your answer in the form $\boxed{y = f(x)}$

Particular solutions.

$$\frac{dy}{dx} = -\frac{3(y-2)}{(2x+1)(x+2)}$$

$$\int \frac{1}{y-2}\,dy = \int \frac{-3}{(2x+1)(x+2)}\,dx$$

$$\int \frac{1}{y-2}\,dy = \int \left(-\frac{2}{2x+1} + \frac{1}{x+2}\right) dx$$

$$\ln|y-2| = -\ln|2x+1| + \ln|x+2| + \ln k$$

$x=1,\ y=4$

$$\ln 2 = -\ln 3 + \ln 3 + \ln k$$

$$\ln 2 = \ln k$$

$$k = 2$$

$$\ln|y-2| = \ln\left|\frac{k(x+2)}{2x+1}\right|$$

$$y - 2 = \frac{2(x+2)}{2x+1}$$

$$y = \frac{2x+4}{2x+1} + 2 = \frac{2x+1+3}{2x+1} + 2$$

$$= 1 + \frac{3}{2x+1} + 2$$

$$y = \frac{3}{2x+1} + 3$$

$$\frac{-3}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

$$-3 = A(x+2) + B(2x+1)$$

$x = -2$

$$-3 = -3B \Rightarrow B = 1$$

$x = -\frac{1}{2}$

$$-3 = \frac{3}{2}A \Rightarrow A = -2$$

Ex 11J

1ac

Then odd questions.

Key Tips on Differential Equations

- Get y on to LHS by dividing (possibly factorising first).

- If after integrating you have ln on the RHS, make your constant of integration $\ln k$ or $\ln A$

- Be sure to combine all your ln's together just as you did in Year 12.
  e.g.:
$$2\ln|x+1| - \ln|x| \quad \rightarrow \quad \ln\left|\frac{(x+1)^2}{x}\right|$$

- Sub in boundary conditions to work out your constant – better to do sooner rather than later.

- Exam questions ♥ partial fractions combined with differential equations.