Edexcel A Level Further Maths: Further Mechanics 1

Your notes

Elastic Strings & Springs

Contents

Hooke's Law

Your notes

Introduction to Hooke's Law

What is Hooke's Law?

- Up to now, strings have been modelled as **inextensible**
  - they cannot stretch (inelastic)
  - we assume that the tension measured at any point along the string is the same **constant** value
- Things that **stretch** (or **compress**, e.g. springs) are called **elastic**
- Imagine two elastic strings held **taut** and at **rest**, but with one stretched further than the other
  - measuring the tension at different points along one string gives the same value,
  - but that "value" will be **higher** for the more stretched string than for the less stretched string
- **Hooke's Law** tells us that the value of **tension**, T N, depends on how far it's been stretched (the **extension**, x metres) beyond its **natural** (unstretched) length (l metres)
  - The law is $T=\frac{\lambda}{l}x$
  - where λ is the **modulus of elasticity**, with units of **Newtons**,
    - it measures the **stiffness** of the material the string (or spring) is made from,
    - the **higher** λ is, the **stiffer** the string / spring is
- **Springs** can be **compressed** but elastic **strings** can't (they'd go **slack**)
  - Hooke's Law works for compression of springs too
  - Instead of measuring extension, x measures the **length of compression** (from its natural length)
    - just make sure any tension arrows **reverse direction** to be compression (thrust) arrows!

Examiner Tip

- In more algebraic questions, the modulus of elasticity may be given in the form kmg Newtons, where k is a constant

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Worked example

An elastic string of natural length l metres and modulus of elasticity 20 N is stretched to a total length of $4l$ metres.

Find the tension in the string.

Hooke's law (" $T = \frac{\lambda x}{l}$ ")

$\lambda = 20$ modulus of elasticity

$x = 4l - l = 3l$ extension

" l " $= l$ natural length

$T = \frac{20}{\cancel{l}}(3\cancel{l})$

$\boxed{\therefore T = 60 \text{ N}}$

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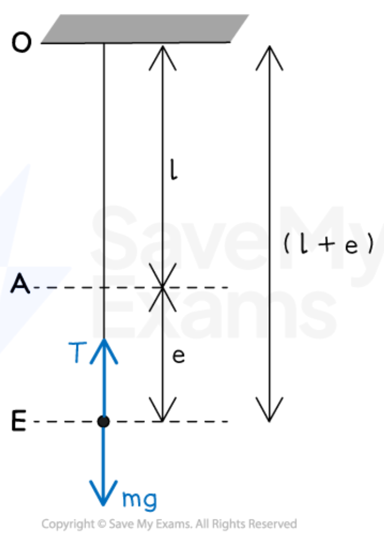
Hooke's Law – Equilibrium

How do I use Hooke's Law for particles at rest under gravity?

- Imagine a particle of mass m kg attached to the end of a **light elastic string** of natural length l metres, with modulus of elasticity λ N
- The other end of the string is attached to a ceiling at the point *O* and the particle hangs **at rest** at the point *E*, **vertically beneath** *O*
- The **total length** of the string, *OE*, will be **greater** than its natural length, *OA* (where *A* is l metres from *O*)
  - The **weight** of the **particle** has stretched the string downwards and the system is now in **equilibrium**
  - The **equilibrium extension** is often labelled as e metres
    - The total length is therefore $(l + e)$ metres



- **Two equations** can be formed and solved **simultaneously** to find e
  - Applying **Newton's 2nd Law** (*F=ma*) upwards at *E* gives the **first** equation:
    - $T - mg = 0$
    - You could also resolve downwards giving the same relationship
  - Applying **Hooke's Law** gives the **second** equation:
    - $T = \frac{\lambda}{l}e$

How do I use Hooke's Law for particles resting on a smooth inclined plane?

- Imagine a particle of mass m kg attached to the end of a **light elastic string** of natural length l metres, with modulus of elasticity λ N

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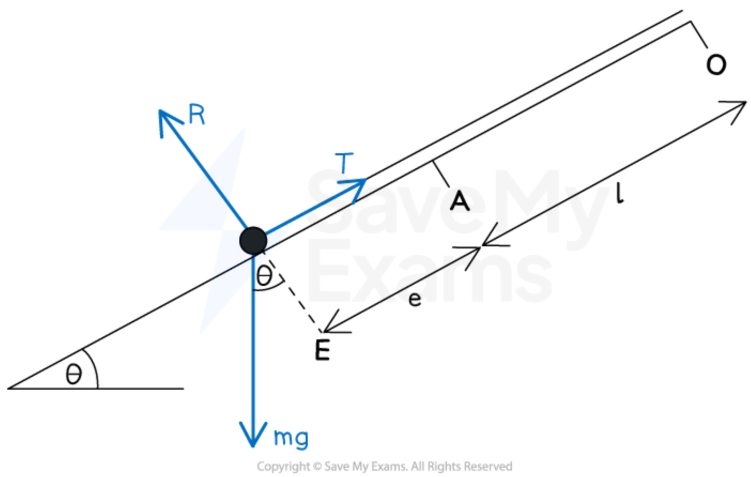
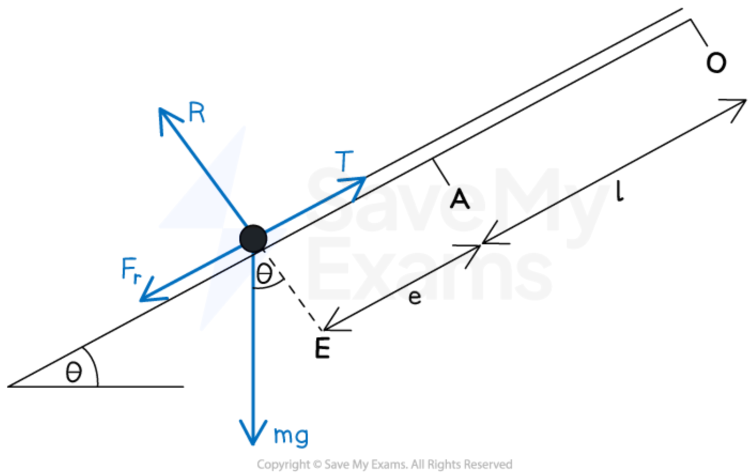
- The other end of the string is attached to the top of a **smooth inclined plane** at the point *O* and the particle lies **at rest** at the point *E* on the slope
  - The **equilibrium extension** is e metres
  - The **angle of inclination** of the plane is θ



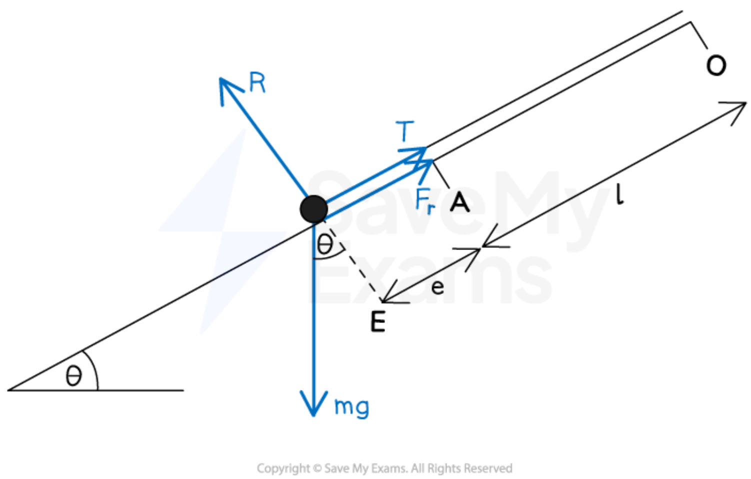
- Again, **two equations** can be formed, which can be solved **simultaneously** to find e
  - **Newton's 2nd Law** up the slope at *E* gives the **first** equation:
    - $T - mg\sin\theta = 0$
  - **Hooke's Law** gives the **second** equation:
    - $T = \frac{\lambda}{l}e$
- **Compression** questions can have a spring attached to the **bottom** of the inclined plane
  - In this case, it rests on the slope at the point *E*, **less** than its natural length l metres from the bottom
  - Use e to measure the **equilibrium compression**
  - T now goes **up** the slope (**thrust**)

What if the particle rests on a rough inclined plane?

- A **rough** slope has a **range of points** on it at which the particle would **remain in equilibrium**
- With the setup above, the **furthest** possible equilibrium position from *O* is on the slope where the string has extended so much that the particle is **on the point of sliding back up the slope**
  - This still counts as **equilibrium** (as it's not actually moved yet)
  - Friction has reached its **limiting** value, $F_r = \mu R$ (as it's on the point of moving)
  - Friction acts **down** the slope (stopping it from actually sliding up)



- Provided the string doesn't go **slack** (an **extension** of **zero**), a **shortest** possible equilibrium position from O exists on the slope where the string is extended only a small amount from its natural length and the particle is **on the point of sliding down the slope**
  - Again, this counts as **equilibrium** and friction reaches its **limiting value**
  - Friction now acts **up** the slope (stopping it from actually sliding down)



- In either case, **four equations** can be solved to find e
  - The **first** is from Newton's 2nd Law **parallel** to the slope
    - being careful to draw friction in the correct **direction**
    - $T \pm F_r - mg\sin\theta = 0$
  - The **second** is from Newton's 2nd Law **perpendicular** to the slope
    - This gives you the **reaction**
    - $R - mg\cos\theta = 0$
  - The **third** is from **Hooke's Law**

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- $T = \frac{\lambda}{l}e$
  - The **fourth** is from friction reaching it's **limiting value**
    - $F_r = \mu R$
- If the particle is in equilibrium at a point **between** the two extremes above, friction is **no longer "limiting"**
  - You **cannot** use $F_r = \mu R$ (as $F_r < \mu R$)
- If, instead, the string is replaced by a **spring**, springs can't go slack so its shortest equilibrium position from O may, in fact, be when it's under **compression**
  - i.e. when the particle is **less** than the **natural length**, l, from O

Examiner Tip

- Questions often use capital letters (*O*, *A*, *B*, *C*...) for points, with questions like "find *OB*", so add your own symbols ($l,\ e,\ x, \ldots$) to see what distance is being asked for!

Your notes

Worked example

A particle of mass m kg is attached to one end of a light elastic spring of natural length l metres and with modulus of elasticity $2mg$ N. The other end of the spring is attached to the point O at the top of a rough ramp inclined at θ degrees to the horizontal, where $\tan\,\theta = \frac{3}{4}$. The particle is on the ramp such that it is on the point of moving up the slope. The coefficient of friction between the particle and the ramp is $\frac{1}{2}$.

Show that the total length of the spring is $\frac{3}{2}l$ metres.

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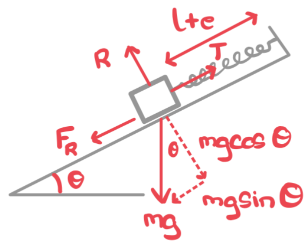
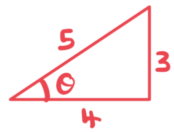
Draw a diagram



$\lambda = 2mg$
$\tan\theta = 3/4$
$\sin\theta = 0.6$
$\cos\theta = 0.8$
$\mu = 0.5$



Resolving perpendicular to the ramp (↖), equilibrium

$R = mg\cos\theta$
$\therefore R = 0.8mg$

Friction is at its maximum since the particle is on the point of moving up the slope

$F_R = \mu R = 0.5(0.8mg) = 0.4mg$

Resolving up the ramp (↗)

$T - F_R - mg\sin\theta = 0$
$T = 0.4mg + 0.6mg = mg$

Hooke's law ("$T = \frac{\lambda}{l}x$")

$\cancel{mg}^{1} = \frac{2\cancel{mg}}{l}e \qquad \therefore e = \frac{l}{2}$

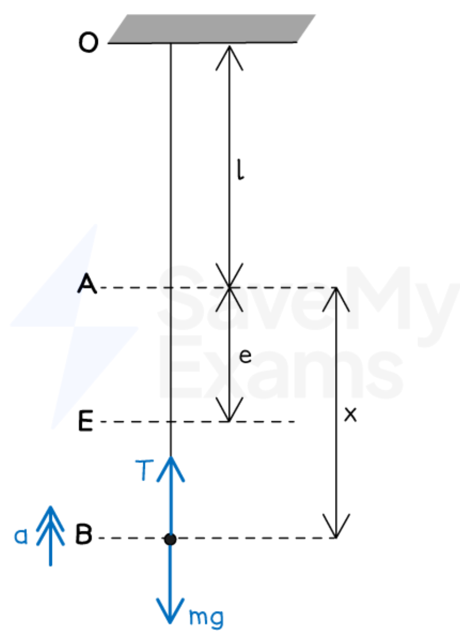
$\boxed{\therefore \text{Total length of the spring is } l + \frac{l}{2} = \frac{3l}{2} \text{ metres}}$

Your notes

Hooke's Law – Dynamics

How do I find acceleration using Hooke's Law?

- You must first understand **Hooke's Law** in **equilibrium** before looking at moving (dynamical) particles
- Imagine a particle of mass m kg attached to the end of a **light elastic string** of natural length l metres, with modulus of elasticity λ N
- The other end of the string is attached to a ceiling at the point *O* and the particle hangs **at rest** at the point *E*, **vertically beneath** *O*
  - The **total length** of the string is $(l + e)$ metres, where e is the **equilibrium extension**
  - It helps to use T_0 as the **equilibrium tension** (to avoid **confusing** it with a **different** tension, T, below)
  - e can be found by drawing a force diagram in **equilibrium** and solving two equations
    - **Newton's 2nd Law**, $T_0 - mg = 0$
    - **Hooke's Law**, $T_0 = \frac{\lambda}{l} e$
- Now imagine pulling the particle down to the point *B* with extension x where $x > e$, under tension T N, then letting go
  - the particle is **beyond** its equilibrium point, so will want to **accelerate** back towards *E*
    - draw an **acceleration arrow** to show this initial acceleration
    - the **initial acceleration** is **not zero** (but the initial speed is, as it's released from **rest**)



- The **two equations** to find the **initial acceleration** are:

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        - $T - mg = ma$ from **Newton's 2nd Law** upwards
            - here, the **order matters** (resolve upwards, as it's moving in that direction)
        - $T = \frac{\lambda}{l}x$ from **Hooke's Law**
- After that initial release, the particle **oscillates** about the **equilibrium position**, *E* (not about its natural length)
    - the **maximum speed** is when it passes through *E*
        - at this point, it's **acceleration** is **zero**
    - **Springs** lead to **full** oscillations (assuming it doesn't hit the ceiling), but **strings** may become **slack** if the initial extension is very large
        - in particular, they become slack when the **extension is zero** (at its **natural length**)
        - after this point, the particle moves freely like a **projectile** under gravity
- The same theory can be applied to strings and springs on smooth or rough **inclined planes**

Examiner Tip

- You can't use SUVAT equations to find distances and initial accelerations because oscillating particles don't have constant acceleration

Your notes

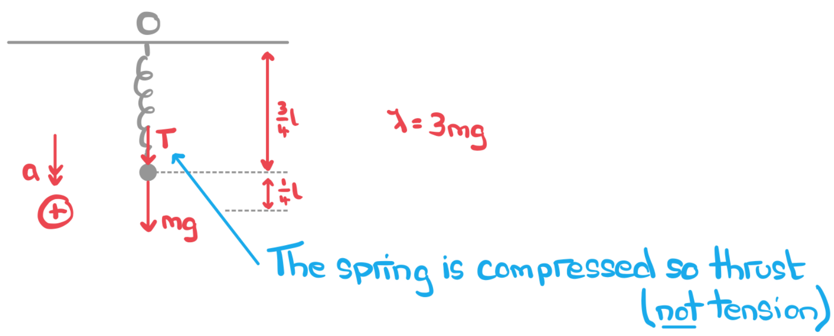
Worked example

A particle of mass m kg is attached to one end of a light elastic spring of natural length l metres, with modulus of elasticity $3mg$ N. The other end of the spring is attached to a fixed point O on a ceiling and the particle hangs vertically beneath O. The particle is initially held at a distance of $\frac{3}{4}l$ from O and released from rest.

i) Find, in terms of g, the initial acceleration of the particle.

Your notes

Draw a diagram



Newton's 2nd law (N2L, "F = ma")

$$mg + T = ma$$

Hooke's law ("$T = \frac{\lambda x}{l}$")

$$T = \frac{3mg}{l}\left(\tfrac{1}{4}l\right) = \frac{3}{4}mg$$

Solving these two equations simultaneously

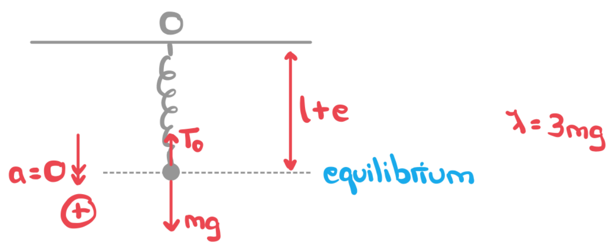
$$\cancel{m}g + \frac{3}{4}\cancel{m}g = \cancel{m}a$$

$$\boxed{\therefore a = \frac{7}{4}g \ \text{m s}^{-2}}$$

ii) Find, in terms of l, the total distance travelled by the particle when it first reaches its maximum speed.

Your notes

Maximum speed occurs when the particle is in the equilibrium position. At this point, acceleration is zero and tension is T_0.



N2L: $mg - T_0 = 0$

Hooke's law: $T_0 = \dfrac{3mg}{l}e$

Solving simultaneously (to find e):

$$\cancel{mg} - \frac{3\cancel{mg}e}{l} = 0$$

$$3e = l$$

$$e = \frac{l}{3}$$

The particle started at $\frac{3}{4}l$ m from O, so total distance travelled will be $(e + \frac{1}{4}l)$ m

$$\frac{l}{3} + \frac{1}{4}l = \frac{7}{12}l$$

$\boxed{\therefore \text{Particle has travelled } \frac{7}{12}l \text{ metres when it first reaches its maximum speed.}}$

Elastic Potential Energy

Your notes

Elastic Potential Energy

What is Elastic Potential Energy?

- It takes putting in **energy** to stretch an elastic string beyond its natural length
  - This is the **work done** in stretching it
- Once it is stretched and held in position, the energy put in is now **stored** in the string
  - This is a type of **potential** energy, because the string has the potential to contract and release it
- This stored energy is called the **elastic potential energy (EPE)**
  - Note that the "**work done** to stretch it" and the "**elastic energy** stored in it" have the same value
    - Questions may use either phrase
- It also takes energy to **compress** a spring from its natural length
  - The compressed spring has energy stored in it, ready to "spring" open
  - This is also **elastic potential energy**

How do I calculate Elastic Potential Energy?

- Let an elastic string (or spring) of natural length l metres and modulus of elasticity λ N be stretched to a new length of $(l+x)$ metres, where x metres is the extension
  - The formula for **elastic potential energy** is $\frac{\lambda}{2l}x^2$
    - Sometimes written $\frac{1}{2}\left(\frac{\lambda}{l}\right)x^2$ (in a form like kinetic energy)
    - The units are in **Joules**
    - It's also the same as "**work done** by stretching"
- This formula works for the **compression** of a spring
  - x represents the **length of compression** from its natural length
- The formula can be derived by knowing that, in general, **work done** is the **area** under a **force-distance** graph
  - The area under the graph of $T=\frac{\lambda}{l}x$ can be found by **integration** to give $\frac{\lambda}{2l}x^2$
    - Or by noting it's a **triangle** of base x and height $\frac{\lambda}{l}x$ then using $\frac{1}{2}\times$ base $\times$ height

Examiner Tip

- Be careful when selecting which formula to use, as they are very similar!
  - $T = \frac{\lambda}{l}x$ is Hooke's Law, for finding a force
  - $EPE = \frac{\lambda}{2l}x^2$ is for finding the elastic potential energy

Your notes

Worked example

An elastic string of natural length 1.5 metres and modulus of elasticity 60 N is stretched to a total length of 3.5 metres.

Find the energy stored in the string.

Elastic potential energy ("$EPE = \frac{\lambda}{l}x^2$")

$\lambda = 60$

$x = 3.5 - 1.5 = 2$

$l = 1.5$

$EPE = \frac{60}{2 \times 1.5}(2)^2$

$\boxed{\therefore EPE = 80\text{ J}}$

Work-Energy Principle with Elasticity

How do I include Elastic Potential Energy in the Work-Energy Principle?

- The **Work-Energy Principle** is an **energy balance**
  - Total **final** energy = total **initial** energy ± work done
    - or, using subscripts for final and initial, $E_f = E_i \pm WD$
- "Total energy" can now include **elastic potential energy (EPE)**
  - e.g. the total initial energy is $E_i = GPE_i + KE_i + EPE_i = mgh_i + \frac{1}{2}mv_i^2 + \frac{\lambda}{2l}x_i^2$
- The **± work done** terms now refer to any **non-gravitational** and **non-elastic** forces
  - e.g. friction (-), driving force (+), air resistance (-), etc
  - But **not weight** (mg) and **no longer tensions** that use **Hooke's Law** ($\frac{\lambda}{l}x$)
    - These are both **already** accounted for in the GPE and EPE parts

How do I know when to use the Work-Energy Principle?

- You can use it when a particle moves from one position to a **different** position, for example:
  - when it has been **pulled down** and released from rest
  - when it has been **projected** at a certain **speed**
  - when you need to find the distance it has **moved**
  - This would **not include** a particle hanging at rest (in equilibrium)
    - Here, you could use Newton's 2nd Law + Hooke's Law
- You can use it when a question involves finding **speeds**
  - Speeds can be found from **kinetic energy**
    - Not from Newton's 2nd Law or Hooke's Law
- You should **not use it** if asked to find the **initial acceleration**
  - **Acceleration** is part of Newton's 2nd Law
- It is **not required** if asked to find the **equilibrium extension**
  - The particle is not moving, so Newton's 2nd Law + Hooke's Law works
- SUVAT equations **cannot** be used for particles on elastic strings or springs as their acceleration is **not constant**
  - They can only be used if particles **detach** from the elastic string (e.g. the string breaks, or becomes slack, etc)
    - The particle then becomes a **projectile** under gravity
    - Don't forget that the **Work-Energy Principle** can also be used on **projectiles**, and in some cases (e.g. finding speeds) it's **much quicker** than SUVAT

Your notes

Examiner Tip

- The **maximum** speed for a particle on an elastic string occurs when it's **acceleration is zero**, which is when it passes through its **equilibrium** position
  - You may need to find this position using Newton's 2nd Law + Hooke's Law in equilibrium

Your notes

Worked example

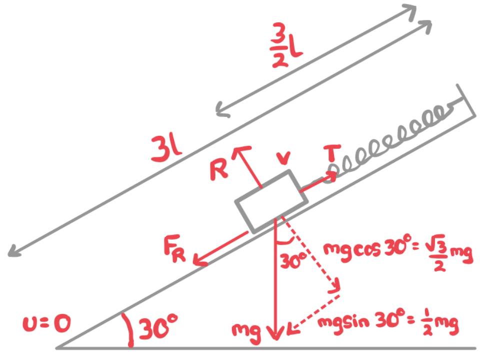
A particle of mass m kg is attached to one end of a light elastic spring of natural length l metres and with modulus of elasticity $4mg$ N. The other end of the spring is attached to the point at the top of a rough slope of length $3l$ metres at 30° to the horizontal. The particle is held on the bottom of the slope and released from rest. The coefficient of friction between the particle and the ramp is $\frac{\sqrt{3}}{4}$.

Find, in terms of g and l, the speed of the particle as it passes through the point halfway up the slope.

Your notes

Your notes

Draw a diagram and include important information



$\lambda = 4mg$

$\mu = \frac{\sqrt{3}}{4}$

Resolving perpendicular to the plane to find R / friction ($\nwarrow$)

$R = \frac{\sqrt{3}}{2}mg \qquad \therefore F_R = \frac{\sqrt{3}}{4} \times \frac{\sqrt{3}}{2}mg = \frac{3}{8}mg$

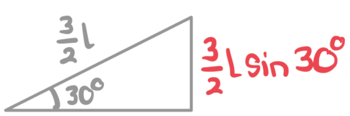
Work done is against (−) friction (this is the only non-gravitational and non-elastic force)
"WD = Fd"

$WD = \frac{3}{8}mg \times \frac{3}{2}l = \frac{9}{16}lmg$

Now consider the initial and final energies

Kinetic energy ("$\frac{1}{2}mv^2$"): $KE_i = 0$ (since $u = 0$)
$KE_f = \frac{1}{2}mv^2$

Gravitational potential energy ("mgh"):



$GPE_i = 0$ (since $h = 0$)
$GPE_f = mg\left(\frac{3}{4}l\right) = \frac{3}{4}lmg$

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Elastic potential energy ("$\frac{\lambda}{2l}x^2$"):

extension at bottom of slope

$$EPE_i = \frac{4mg}{2l}(2l)^2 = 8lmg$$

$$EPE_f = \frac{4mg}{2l}\left(\frac{1}{2}l\right)^2 = \frac{1}{2}lmg$$

extension halfway up the slope

Now use the work-energy principle to set up an equation for v
"$E_f = E_i \pm WD$"

$$\frac{1}{2}\cancel{m}v^2 + \frac{3}{4}l\cancel{m}g + \frac{1}{2}l\cancel{m}g = 8l\cancel{m}g - \frac{9}{16}l\cancel{m}g$$

$$\frac{1}{2}v^2 = \frac{99}{16}lg$$

$\therefore$ Speed, halfway up slope is $v = \frac{3}{4}\sqrt{22lg}$ m s^{-1}

Problem Solving with Strings & Springs

Your notes

Problem Solving with Strings & Springs

What if the particle is fixed between two different springs?

- There will be two **different** tensions and extensions
  - For extensions, you can use x and y
    - or, in the case of **equilibrium** extensions, e_A and e_B
- You will need an **extra equation** relating x to y, found by **summing** all the **lengths**
  - e.g. if the total distance is $5l$ then $l + x + y + l = 5l$
- Springs can go into **compression**, but if you're not told (or it's not clear), **draw both springs under tension**
  - If it turns out that x or y are **negative** later on, then you know they were actually under compression
- If a particle is attached to the **midpoint** of a spring, you can either
  - Treat the spring as a whole
    - The two tensions either side have the form $T = \frac{\lambda}{l}x$ where x is the total extension
  - Or treat the particle as being fixed between **two half-springs**, halving their lengths (**natural lengths** and **extensions**), but keeping the **same modulus of elasticity** (the material they're made from hasn't changed)
    - The two tensions either side have the form $T = \frac{\lambda}{\left(\frac{l}{2}\right)}\left(\frac{x}{2}\right)$
    - This simplifies to $T = \frac{\lambda}{l}x$ (showing that both methods give the same answer)

What if the particle is suspended at angles by two elastic strings?

- A particle of mass m kg has a **weight** of mg N acting downwards
- The particle could be held up (suspended) by **two light elastic strings** acting at **angles** to the horizontal
  - Label the **different** tensions T_1 and T_2
- If angles are not given, use the **geometry** of the situation to find $\sin\theta$ and $\cos\theta$ (by **trigonometry**)
  - This helps later for **resolving** the tensions in Newton's 2nd Law
- If a particle is attached to the **midpoint** of an elastic string and forms a triangle due to its weight, you can either
  - Treat the string as a **whole** (calculations with full extension and full natural length)
  - Or treat the setup as a particle attached to **two identical half-strings** (each with half the natural length, half the extension, but the same modulus of elasticity)

- In either case, this situation will be **symmetric**
  - If the particle is pulled vertically downwards and released, it will **accelerate vertically upwards**

Your notes

What other questions can be asked about springs and strings?

- There are so many different situations that it's impossible to know, but the tools used are the same
  - **Newton's 2nd Law**, **Hooke's Law** and the **Work-Energy Principle**
- If the situation has **uniform rods** suspended by light elastic strings, you may also need to take **moments**
- Some questions may be more **algebraic**
- Other questions could have a **change** halfway through
  - e.g. the string breaks and the particle becomes a **projectile**

Examiner Tip

- Using your own subscripts can help to avoid confusing tensions and extensions from situations with multiple strings or springs

Your notes

Worked example

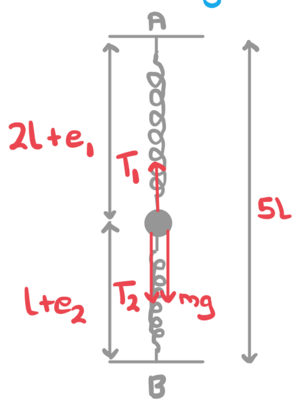
A light spring of natural length $2l$ metres and modulus of elasticity $2mg$ N has one end attached to the point *A* on a ceiling and its other end attached to a particle of mass m kg suspended vertically beneath *A*.

A different spring of natural length l metres and modulus of elasticity $4mg$ N has one end attached to the particle and its other end attached to the point *B* on the floor, where *B* is a distance of $5l$ metres vertically beneath *A*. The system is in equilibrium.

Find, in terms of l, the height of the particle above the floor.

Your notes

Draw a diagram



$\lambda_1 = 2mg$
$\lambda_2 = 4mg$

$l+e_2$ is the length required by the question
Considering the length AB

$$2l + e_1 + l + e_2 = 5l$$
$$e_1 + e_2 = 2l$$
$$e_1 = 2l - e_2$$

We need e_2 but not e_1 so write e_1 in terms of e_2 for later use

Resolving, vertically ($\uparrow$), equilibrium

$T_1 = T_2 + mg$ (Using N2L, $T_1 - T_2 - mg = m \times 0$)

Using Hooke's law ("$T = \frac{\lambda}{l}x$") for both springs

$$T_1 = \frac{2mg}{2l}e_1 = \frac{mg}{l}(2l - e_2)$$

e_1

$$T_2 = \frac{4mg}{l}e_2$$

Substitute these into $T_1 = T_2 + mg$ to set up an equation for e_2

Your notes

$$\frac{\cancel{mg}}{L}(2l - e_2) = \frac{4\cancel{mg}}{l}e_2 + \cancel{mg}$$

$$2l - e_2 = 4e_2 + L$$

$$5e_2 = l$$

$$e_2 = \frac{L}{5}$$

$$\therefore l + e_2 = L + \frac{L}{5} = \frac{6l}{5}$$

$\boxed{\therefore \text{The height of the particle above the floor is } \frac{6}{5}l \text{ metres}}$