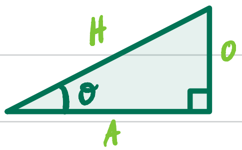
Trigonometry : Year 1

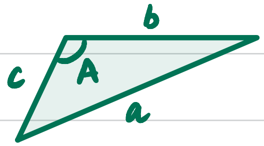
Definitions



$\sin\theta = \frac{O}{H}$ $\tan\theta = \frac{O}{A}$

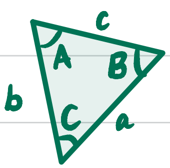
$\cos\theta = \frac{A}{H}$

Cosine Rule



$a^2 = b^2 + c^2 - 2bc\cos A$

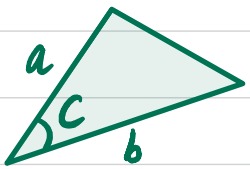
Sine Rule



$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ ↕

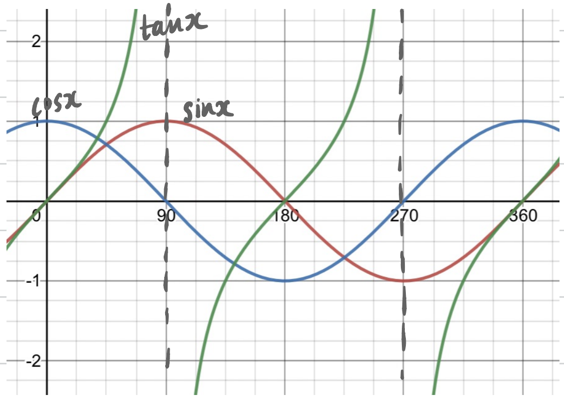
Ambiguous case: $\sin\theta = \sin(180-\theta)$

Area of a Triangle:

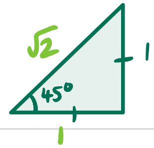


$\frac{1}{2}ab\sin C$

Graphs



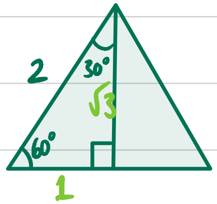
Exact Values



$\sin 0° = 0$ $\sin 90° = 1$ $\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 0° = 1$ $\cos 90° = 0$ $\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 0° = 0$ $\tan 90° = \text{undefined}$ $\tan 45° = 1$

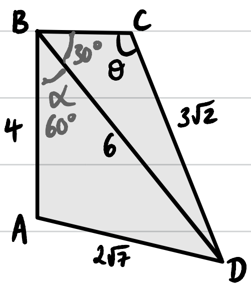


$\sin 30° = \frac{1}{2}$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\tan 60° = \sqrt{3}$

Example Problem



Given that $\angle ABC$ is a right angle and θ is obtuse, find θ

$a^2 = b^2 + c^2 - 2bc\cos A$

$(2\sqrt{7})^2 = 4^2 + 6^2 - 2\times4\times6\cos\alpha$

$28 = 16 + 36 - 48\cos\alpha$

$48\cos\alpha = 24$

$\cos\alpha = \frac{24}{48} = \frac{1}{2}$

$\alpha = 60°$

$\frac{\sin 30}{3\sqrt{2}} = \frac{\sin\theta}{6}$

$\frac{\frac{1}{2}}{3\sqrt{2}} = \frac{\sin\theta}{6}$

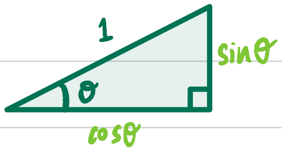
$\frac{\cancel{3}\,1}{\cancel{3}\sqrt{2}} = \sin\theta$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \cancel{45}, \; 180-45$

$135°$

More Definitions



$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cos\theta = \frac{A}{1}$

$\sin\theta = \frac{O}{1}$

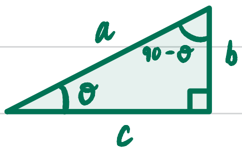
Pythagoras

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

Co-function Identity



$\sin\theta = \frac{b}{a} = \cos(90 - \theta)$

$\cos\theta = \frac{c}{a} = \sin(90 - \theta)$

Solving Equations

$\sin\theta \ldots \quad \left.\begin{matrix}\theta \\ 180 - \theta\end{matrix}\right\} \pm 360°$

$\cos\theta \ldots \quad \left.\begin{matrix}\theta \\ 360 - \theta\end{matrix}\right\} \pm 360°$

$\tan\theta \ldots \quad \theta \,\} \pm 180°$

eg. $\sin\theta = 0.6$ (Give answers to nearest degree)

$\theta = 37° \xrightarrow{+360} 397° \xrightarrow{+360}$

$= 143° \xrightarrow{+360} 503°$

Equations with Linear Input

1) Change the range
2) Solve for new range first
3) Solve these for x, θ, etc.

eg. Solve for $0° < x < 90°$, $\tan(2x - 20°) = -0.4$

$0° < 2x < 180°$

$-20° < 2x - 20 < 160°$

$2x - 20° = \cancel{-22°}, 158°$

$x = 89°$

Example Problem

a) Show that $\frac{10\sin^2\theta - 7\sin(90 - \theta) + 2}{3 + 2\cos\theta} \equiv 4 - 5\cos\theta$

b) Hence, solve for $-180° \leq x \leq 180°$

$$\frac{10\sin^2 2x - 7\sin(90 - 2x) + 2}{3 + 2\cos 2x} = 6\cos^2 2x$$

$-360° \leq 2x \leq 360°$

a)

$LHS \equiv \frac{10\sin^2\theta - 7\cos\theta + 2}{3 + 2\cos\theta} \equiv \frac{10(1 - \cos^2\theta) - 7\cos\theta + 2}{3 + 2\cos\theta}$

$\equiv \frac{10 - 10\cos^2\theta - 7\cos\theta + 2}{3 + 2\cos\theta}$

$\equiv \frac{-10\cos^2\theta - 7\cos\theta + 12}{3 + 2\cos\theta}$

$\equiv \frac{\cancel{(3 + 2\cos\theta)}(4 - 5\cos\theta)}{\cancel{3 + 2\cos\theta}}$

$\equiv 4 - 5\cos\theta$

b) $4 - 5\cos 2x = 6\cos^2 2x$

$0 = 6\cos^2 2x + 5\cos 2x - 4$

$\cos 2x = \frac{1}{2}$ or $\cos 2x = -\frac{4}{3}$

no solutions

$2x = 60°, -300°$

$300°, -60°$

$x = -150°, -30°, 30°, 150°$

Trigonometry : Year 2

Radians

$\pi \rightarrow 180°$

$2\pi \rightarrow 360°$

$\frac{\pi}{2} \rightarrow 90°$

$\frac{2\pi}{3} \rightarrow 120°$

etc.

Small Angle Approximations → radians

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

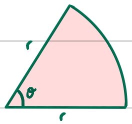
$\tan\theta \approx \theta$ $\quad \tan \underline{2\theta} \approx \underline{2\theta}$

e.g. Given that θ is small, find the approximate value of

$\frac{1-\cos\theta}{\theta\tan 2\theta} = \frac{1-(1-\frac{\theta^2}{2})}{\theta \times 2\theta} = \frac{\frac{\theta^2}{2}}{2\theta^2} = \frac{\frac{1}{2}}{2} = \underline{\underline{\frac{1}{4}}}$

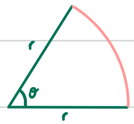
Sectors, Arcs + Segments

$\theta \rightarrow$ radians



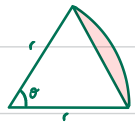
Sector Area

$\frac{1}{2}r^2\theta$



Arc Length

$r\theta$

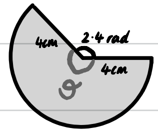


$\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

Segment Area

$\frac{1}{2}r^2(\theta - \sin\theta)$

e.g. Find the area and perimeter of :



$\theta = 2\pi - 2.4 = 3.8831...$

$\text{Area} = \frac{1}{2} \times 4^2 \times 3.883... = 31.1\text{cm}^2$ (3sf)

$\text{Perimeter} = 4 \times 3.8831... + 4 + 4$
$= 23.5\text{cm}$ (3sf)

Even more definitions

$\sec\theta = \frac{1}{\cos\theta}$ $\qquad \frac{1}{\sec\theta} = \cos\theta$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\qquad \frac{1}{\operatorname{cosec}\theta} = \sin\theta$

$\cot\theta = \frac{1}{\tan\theta}$ $\qquad \frac{1}{\cot\theta} = \tan\theta$

e.g. Simplify

$\frac{3\cancel{\sec\theta}\ \cancel{\sin\theta}\ \cancel{\tan\theta}}{2\cancel{\cos\theta}\ \cancel{\cot\theta}\ \cancel{\operatorname{cosec}\theta}} = \frac{3\sin^2\theta\tan^2\theta}{2\cos^2\theta} = \frac{3}{2}\tan^2\theta\tan^2\theta$
$= \frac{3}{2}\tan^4\theta$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\div\sin^2\theta$ $\qquad \div\cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Addition Formulae

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ $\qquad$ diff trig sc cs / same sign $\pm$ $\pm$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ $\qquad$ same trig cc ss / diff sign $\pm$ $\mp$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Harmonic Identities

max/min problems

"Express $a\sin\theta + b\cos\theta$ in the form $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$"

Double Angle Formulae

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

$\cos^2\theta$ comes first.

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

e.g. Simplify $\underline{8\sin x\cos x\cos 2x}$

$4 \times 2\sin x\cos x \times \cos 2x$

$4 \times \sin 2x\cos 2x$

$2 \times 2\sin 2x\cos 2x$

$\underline{\underline{2\sin 4x}}$

Example Problems

1a) Express $3\cos\theta + 4\sin\theta$ in the form $R\sin(\theta+\alpha)$ α in degrees $0° < \alpha < 90°$

$R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$.

$R\cos\alpha = 4 \quad \div \quad R\sin\alpha = 3$

$\tan\alpha = \frac{3}{4} \rightarrow \alpha = 36.9°$

$R = \sqrt{4^2+3^2} = 5$

$3\cos\theta + 4\sin\theta = 5\sin(\theta + 36.9)°$

2a) Prove that $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$

$LHS \equiv \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta}$

$\equiv \frac{1-\cos 2\theta}{\sin 2\theta} \equiv \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$

$\equiv \frac{\cancel{2}\sin^{\cancel{2}}\theta}{\cancel{2}\cancel{\sin\theta}\cos\theta} \equiv \tan\theta$

1b) Hence, find the minimum and maximum values of

$$\frac{4}{(3\cos\theta + 4\sin\theta)^2 + 1} = \frac{4}{(5\sin(\theta+36.9))^2+1}$$

For a fraction, it is a maximum when the denominator is as small as possible...

$\sin(\theta + 36.9) = 0 \qquad \max = \frac{4}{1} = 4$

For a min value, we want a large denominator

$\sin(\theta+36.9) = 1 \qquad \min = \frac{4}{(5)^2+1} = \frac{4}{26} = \frac{2}{13}$

2b) Hence, solve for $-\pi < y \leq \pi$

$1 + \tan^2 y = \sec^2 y$

$\tan^2 y = \sec^2 y - 1$

$(\operatorname{cosec} 2y - \cot 2y)^2 = \sec y + 1$

$(\tan y)^2 = \sec y + 1$

$\tan^2 y = \sec y + 1$

$\sec^2 y - 1 = \sec y + 1$

$\sec^2 y - \sec y - 2 = 0$

$(\sec y - 2)(\sec y + 1) = 0$

$\sec y = 2$ or $\sec y = -1$

$\cos y = \frac{1}{2}$ $\cos y = -1$

$y = \frac{\pi}{3}$ $y = \pi$

$\cancel{\frac{5\pi}{3}}$ $-\frac{\pi}{3}$

$y = -\frac{\pi}{3}, \frac{\pi}{3}, \pi$