Your notes

# Edexcel A Level Further Maths: Core Pure

## 8.2 Second Order Differential Equations

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# 8.2.1 Solving Second Order Differential Equations

## Second Order Differential Equations

### What is a second order differential equation?

- A **second order differential equation** is an equation containing second order derivatives (and possibly first order derivatives) but no higher order derivatives
  - For example $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x$ is a second order differential equation
  - And so is $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 5\frac{\mathrm{d}x}{\mathrm{d}t} + 7x = 5\sin t$
  - But $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ is not, because it contains the third order derivative $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$

### What are the types of second order differential equation?

- We divide second order differential equations into two main types
- A **homogeneous** second order differential equation is of the form $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$ where *a*, *b* and *c* are real constants
  - You may also see this written in the form $ay'' + by' + cy = 0$ where $y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$
- A **non-homogeneous** second order differential equation is of the form $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = \mathrm{f}(x)$ where *a*, *b* and *c* are real constants and where f(*x*) is a non-zero function of *x*
  - You may also see this written in the form $ay'' + by' + cy = \mathrm{f}(x)$

### How can I solve simple second order differential equations?

- If a second order differential equation is of the form $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{f}(x)$, it will often be possible to solve it simply by using repeated integration
  - A **separate integration constant** will need to be included for each of the integrations
    - This means you will end up with **two integration constants** in your final answer
    - To find the values of these constants you will need two separate initial or boundary conditions

- See the worked example below for examples of this

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## Worked example

a)

Find the general solution to the second order differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

constant of integration

$$\frac{dy}{dx} = \int (0)\,dx = A$$

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx$$

$$y = \int A\,dx = Ax + B$$

constant of integration

$$y = \int \frac{dy}{dx}\,dx$$

$$\boxed{y = Ax + B}$$

Note that this is the equation of every possible non-vertical straight line!

b)

Find the particular solution to the second order differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 24x$ that satisfies $\frac{\mathrm{d}y}{\mathrm{d}x} = -2$ and $y = 1$ when $x = 0$.

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constant of integration

$$\frac{dy}{dx} = \int 24x \, dx = 12x^2 + A \qquad \frac{dy}{dx} = \int \frac{d^2y}{dx^2} \, dx$$

$$y = \int (12x^2 + A) \, dx = 4x^3 + Ax + B \qquad y = \int \frac{dy}{dx} \, dx$$

constant of integration

Now use the boundary conditions for $x = 0$ :

$$12(0)^2 + A = -2 \implies A = -2 \qquad \frac{dy}{dx} = -2$$

$$4(0)^3 + A(0) + B = 1 \implies B = 1 \qquad y = 1$$

$$\boxed{y = 4x^3 - 2x + 1}$$

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## Auxiliary Equations & Complementary Functions

### What is a complementary function?

- For a second order **homogeneous** differential equation, the equation's **complementary function** is the general solution to the equation
  - If the differential equation contains initial or boundary conditions you may then use those to find the precise solution to the equation
- For a second order **non-homogeneous** differential equation, the equation's **complementary function** is only a part of the general solution to the equation
  - For the complete general solution you will need to include the **particular integral** as well (see the following section)
- In order to find a differential equation's complementary function we use the associated auxiliary equation

### What is an auxiliary equation?

- For a second order differential equation $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=\mathrm{f}(x)$ the associated **auxiliary equation** is $am^2+bm+c=0$
  - It is possible that f(*x*)=0, in which case the differential equation is **homogeneous**
  - The auxiliary equation is **exactly the same** whether the differential equation is homogeneous or non-homogeneous
  - The auxiliary equation is a **quadratic equation** in the variable *m*
    - The solutions to the auxiliary equation will determine the nature of the associated complementary function

### How do I use the auxiliary equation to find the associated complementary function?

- **STEP 1**: Solve the auxiliary equation $am^2+bm+c=0$ to find its roots α and β
  - It is possible that the roots will be repeated, with α = β
- **STEP 2**: The complementary function will be determined by the nature of the roots of the auxiliary equation:
  - **CASE 1**: $b^2-4ac>0$ so that α and β are **distinct real roots**
    - The complementary function is $y=A\mathrm{e}^{\alpha x}+B\mathrm{e}^{\beta x}$ where *A* and *B* are arbitrary constants
    - This holds even if one of the roots is zero – but if β = 0, say, then the complementary function will become $y=A\mathrm{e}^{\alpha x}+B$
  - **CASE 2**: $b^2-4ac=0$ so that there is only **one repeated root** α
    - The complementary function is $y=(A+Bx)\mathrm{e}^{\alpha x}$ where *A* and *B* are arbitrary constants
  - **CASE 3**: $b^2-4ac<0$ so that α and β are **complex conjugate roots** that may be written as $p\pm qi$
    - The complementary function is $y=\mathrm{e}^{px}(A\cos qx+B\sin qx)$ where *A* and *B* are arbitrary constants

- Note that if α and β are purely imaginary, then $p$ = 0 and the complementary function becomes $y = A\cos qx + B\sin qx$

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### How do I solve a second order homogeneous differential equation?

- **STEPS 1 & 2**: Use the **auxiliary equation** to find the differential equation's **complementary function** (see above)
  - The complementary function, with its arbitrary constants *A* and *B*, is the general solution to the differential equation
- **STEP 3**: If there are **initial** or **boundary conditions** associated with the differential equation, use these to find the values of the general solution's arbitrary constants
  - This gives the particular solution to the differential equation with the given initial or boundary conditions
  - Note that because there are two arbitrary constants, you will require two separate initial or boundary conditions to find both constants' values
  - If the initial or boundary conditions involve $\frac{\mathrm{d}y}{\mathrm{d}x}$ you will need to differentiate your general solution to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of the constants *A* and *B*
  - Finding the values of *A* and *B* may require solving simultaneous equations

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## Worked example

a) Find the general solution to each of the following differential equations:

(i) $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$$

(ii) $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

(iii) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

(i) STEP 1: $m^2 - m - 12 = (m-4)(m+3) = 0 \Rightarrow m = 4, -3$

STEP 2: $\boxed{y = Ae^{4x} + Be^{-3x}}$

(ii) STEP 1: $m^2 + 6m + 9 = (m+3)^2 = 0 \Rightarrow \underset{\text{repeated root}}{m = -3}$

STEP 2: $\boxed{y = (A + Bx)e^{-3x}}$

(iii) STEP 1: $m^2 - 4m + 13 = (m-2)^2 + 9 = 0 \Rightarrow m = 2 \pm 3i$

STEP 2: $\boxed{y = e^{2x}(A\cos 3x + B\sin 3x)}$

b) For each of the differential equations in part (a), find the particular solution that satisfies $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 0$.

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(i) STEP 3: $y = Ae^{4x} + Be^{-3x} \Rightarrow \frac{dy}{dx} = 4Ae^{4x} - 3Be^{-3x}$

$Ae^0 + Be^0 = A + B = 1$ — $y = 1$ when $x = 0$

$4Ae^0 - 3Be^0 = 4A - 3B = 1$ — $\frac{dy}{dx} = 1$ when $x = 0$

$\Rightarrow A = \frac{4}{7},\ B = \frac{3}{7}$ — solve simultaneous equations

$$\boxed{y = \frac{4}{7}e^{4x} + \frac{3}{7}e^{-3x}}$$

(ii) STEP 3: $y = (A + Bx)e^{-3x} \Rightarrow \frac{dy}{dx} = (B - 3A - 3Bx)e^{-3x}$

$(A + B(0))e^0 = A = 1$ — $y = 1$ when $x = 0$

$(B - 3A - 3B(0))e^0 = -3A + B = 1$ — $\frac{dy}{dx} = 1$ when $x = 0$

$\Rightarrow A = 1,\ B = 4$ — solve simultaneous equations

$$\boxed{y = (4x + 1)e^{-3x}}$$

(iii) STEP 3: $y = e^{2x}(A\cos 3x + B\sin 3x)$

$\Rightarrow \frac{dy}{dx} = e^{2x}\left((2A + 3B)\cos 3x + (2B - 3A)\sin 3x\right)$

$e^0(A\cos 0 + B\sin 0) = A = 1$ — $y = 1$ when $x = 0$

$e^0\left((2A + 3B)\cos 0 + (2B - 3A)\sin 0\right) = 2A + 3B = 1$ — $\frac{dy}{dx} = 1$ when $x = 0$

$\Rightarrow A = 1,\ B = -\frac{1}{3}$ — solve simultaneous equations

$$\boxed{y = e^{2x}\left(\cos 3x - \frac{1}{3}\sin 3x\right)}$$

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# Particular Integral

## What is a particular integral?

- A **particular integral** is part of the solution to a second order non-homogeneous differential equation of the form $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=\mathrm{f}(x)$
  - When the particular integral is substituted into the left hand side it will produce f($x$)
  - The other part of the solution is the **complementary function** associated with the differential equation

## How do I find the particular integral for a second order non-homogeneous differential equation?

- **STEP 1**: Choose the correct 'test form' of particular integral, based on the function f($x$) on the right-hand side of the non-homogeneous differential equation:

| Form of f($x$) | 'Test form' of p.i | Notes |
|---|---|---|
| $p$ | $\lambda$ | $p$ is a given constant<br>$\lambda$ is a constant to be found |
| $p+qx$ | $\lambda+\mu x$ | $p$ & $q$ are given constants<br>$\lambda$ & $\mu$ are constants to be found<br><br>Use the **full** test form of the p.i., even if there is no constant term (i.e., even if $p$ = 0) |
| $p+qx+rx^2$ | $\lambda+\mu x+\nu x^2$ | $p$, $q$ & $r$ are given constants<br>$\lambda$, $\mu$ & $\nu$ are constants to be found<br><br>Use the **full** test form of the p.i., even if there is no constant or $x$ term (i.e., even if $p$ = 0 and/or $q$ =0) |
| $p\mathrm{e}^{kx}$ | $\lambda\mathrm{e}^{kx}$ | $k$ & $p$ are given constants<br>$\lambda$ is a constant to be found |

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| | | |
|---|---|---|
| $p\cos\omega x + q\sin\omega x$ | $\lambda\cos\omega x + \mu\sin\omega x$ | $p$, $q$ & $\omega$ are given constants<br>λ & μ are constants to be found<br><br>Use the **full** test form of the p.i., even if f($x$) only contains sin or only contains cos (i.e., even if $p$ = 0 or $q$ =0) |

- If **all or part of the 'test form'** of the particular integral occurs in the differential equation's **complementary function**, you will need to **modify** the 'test form'
  - See the following section for how to do this
- **STEP 2**: Find the **first** and **second derivatives** of the 'test form' of particular integral
- **STEP 3**: **Substitute** the first and second derivatives, along with the 'test form' itself, into the **differential equation**
- **STEP 4**: By **comparing coefficients**, determine the **correct values** to use for the **constants** in the 'test form'
  - This may require solving simultaneous equations involving the various constants
  - The 'test form' with the correct values of its constants is the particular integral for the differential equation

**What if a part of the 'test form' of the particular integral already occurs in the differential equation' complementary function?**

- Because the terms of the **complementary function** are solutions to the homogeneous equation $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$, they **cannot also provide possible solutions** to the non-homogeneous equation $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = \mathrm{f}(x)$
- If the standard 'test form' for the particular integral contains terms that already occur as a part of the complementary function, then the 'test form' needs to be **modified by adding a factor of *x*** (or possibly of powers of $x$) to the terms of the 'test form'
- Some examples:
  - The equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3\mathrm{e}^x$ has complementary function $A\mathrm{e}^x + B\mathrm{e}^{-2x}$
    - The standard 'test form' of p.i. would be $\lambda\mathrm{e}^x$, but $\mathrm{e}^x$ times a constant already occurs in the complementary function
    - Therefore $\lambda x\mathrm{e}^x$ would be used as the 'test form' instead
  - The equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x} = 7$ has complementary function $A\mathrm{e}^{-x} + B$
    - The standard 'test form' of p.i. would be λ, but a constant ($B$) already occurs in the complementary function
    - Therefore $\lambda x$ would be used as the 'test form' instead

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- The equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 5e^{-x}$ has complementary function $(A + Bx)e^{-x}$
  - The standard 'test form' of p.i. would be $\lambda e^{-x}$, but $e^{-x}$ times a constant AND $xe^{-x}$ times a constant both already occur in the complementary function
  - Therefore $\lambda x^2 e^{-x}$ would be used as the 'test form' instead

## How do I solve a second order non-homogeneous differential equation?

- **STEP 1**: Use the **auxiliary equation** to find the differential equation's **complementary function** ('c.f.')
- **STEP 2**: Find the differential equation's **particular integral** ('p.i.') including the correct values of any constants
- **STEP 3**: The general solution to the differential equation is the **sum** of the **complementary function** and the **particular integral**
  - I.e., the **general solution** is $y$ = c.f. + p.i.
- **STEP 4**: If there are **initial** or **boundary conditions** associated with the differential equation, use these to find the values of the general solution's arbitrary constants
  - This gives the particular solution to the differential equation with the given initial or boundary conditions
  - Note that because there are two arbitrary constants ($A$ and $B$ from the complementary function), you will require two separate initial or boundary conditions to find both constants' values
  - If the initial or boundary conditions involve $\frac{dy}{dx}$ you will need to differentiate your general solution to find $\frac{dy}{dx}$ in terms of the constants $A$ and $B$
  - Finding the values of $A$ and $B$ may require solving simultaneous equations

### Examiner Tip

- Don't forget to include the complementary function in your solution to a second order non-homogeneous differential equation – the solution is incomplete without it!
- If your attempt to find the constants for a particular integral breaks down and doesn't appear to have a solution, make sure that you have not used a 'test form' of the p.i. that includes terms also found in the complementary function
- Finding the constants for the particular integral can be a very fiddly and algebra-heavy process – be sure to work slowly and methodically to avoid mistakes!

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## Worked example

a)

Find the general solution to the differential equation $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12x$.

STEP 1 : $m^2 + 5m + 6 = (m+2)(m+3) = 0 \Rightarrow m = -2, -3$

$y = Ae^{-2x} + Be^{-3x}$ complementary function

STEP 2 : 'Test form' of particular integral is $\lambda + \mu x$

$y = \lambda + \mu x \Rightarrow \frac{dy}{dx} = \mu \Rightarrow \frac{d^2y}{dx^2} = 0$ find derivatives of test form of p.i.

$0 + 5(\mu) + 6(\lambda + \mu x) = 6\mu x + (5\mu + 6\lambda) = 12x$ substitute into differential equation

$\Rightarrow \begin{matrix} 6\mu = 12 \\ 5\mu + 6\lambda = 0 \end{matrix} \Rightarrow \lambda = -\frac{5}{3}, \mu = 2$ compare coefficients and solve simultaneous equations

$y = 2x - \frac{5}{3}$ particular integral

STEP 3 : $\boxed{y = Ae^{-2x} + Be^{-3x} + 2x - \frac{5}{3}}$

b)

Find the general solution to the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 8e^{3x}$.

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STEP 1 : $m^2 - 4m + 3 = (m-1)(m-3) = 0 \implies m = 1, 3$

$y = Ae^x + Be^{3x}$ complementary function

STEP 2 : 'Test form' of particular integral is $\lambda x e^{3x}$

we need the $x$ here because $e^{3x}$ is already part of the complementary function

Find derivatives of test form of p.i. :

$y = \lambda x e^{3x} \implies \frac{dy}{dx} = (3\lambda x + \lambda)e^{3x} \implies \frac{d^2y}{dx^2} = (9\lambda x + 6\lambda)e^{3x}$

Substitute into differential equation:

$(9\lambda x + 6\lambda)e^{3x} - 4(3\lambda x + \lambda)e^{3x} + 3\lambda x e^{3x} = 2\lambda e^{3x} = 8e^{3x}$

$\implies 2\lambda = 8 \implies \lambda = 4$ compare coefficients and solve for $\lambda$

$y = 4xe^{3x}$ particular integral

STEP 3 : $\boxed{y = Ae^x + (B + 4x)e^{3x}}$

c)

Find the particular solution to the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 5\sin x$ that satisfies $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

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STEP 1 : $m^2 - 2m = m(m-2) = 0 \implies m = 0, 2$

$$y = A\underbrace{e^{0x}}_{= e^0 = 1} + Be^{2x} = A + Be^{2x}$$ complementary function

STEP 2 : 'Test form' of particular integral is $\lambda\cos x + \mu\sin x$

Find derivatives of test form of p.i. :

$$y = \lambda\cos x + \mu\sin x \implies \frac{dy}{dx} = \mu\cos x - \lambda\sin x \implies \frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$$

Substitute into differential equation :

$$(-\lambda\cos x - \mu\sin x) - 2(\mu\cos x - \lambda\sin x) = (-\lambda - 2\mu)\cos x + (2\lambda - \mu)\sin x = 5\sin x$$

$$\implies \begin{matrix} -\lambda - 2\mu = 0 \\ 2\lambda - \mu = 5 \end{matrix} \implies \lambda = 2,\ \mu = -1$$ compare coefficients and solve simultaneous equations

$$y = 2\cos x - \sin x$$ particular integral

STEP 3 : $y = A + Be^{2x} + 2\cos x - \sin x$ general solution

STEP 4 : $\frac{dy}{dx} = 2Be^{2x} - 2\sin x - \cos x$ differentiate general solution

$$A + Be^0 + 2\cos 0 - \sin 0 = A + B + 2 = 0$$ $y = 0$ when $x = 0$

$$2Be^0 - 2\sin 0 - \cos 0 = 2B - 1 = 1$$ $\frac{dy}{dx} = 1$ when $x = 0$

$$\implies A = -3,\ B = 1$$ solve simultaneous equations

$$\boxed{y = e^{2x} + 2\cos x - \sin x - 3}$$

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# 8.2.2 Coupled First Order Linear Equations

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## Coupled First Order Linear Equations

### What are coupled first order linear differential equations?

- **Coupled first order linear differential equations** are a pair of simultaneous differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by + \mathrm{f}(t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy + \mathrm{g}(t)$$

  - *a*, *b*, *c* and *d* are real constants
  - f(*t*) and g(*t*) are functions of *t*
    - In your exam these functions will usually be either zero or else simply equal to a constant
  - The equations are described as 'coupled' because the rate of change of each of the variables depends not only on the variable itself but also on the other variable
- Systems of coupled differential equations often occur in modelling contexts where two variables are expected to interact
  - For example *x* may refer to the size of a population of **prey** animals, and *y* to the size of a population of **predators**
    - We would expect the rate of change of the prey animal population to depend on the number of prey animals there are to reproduce, but also on the number of predator animals eating the prey animals
    - Similarly we would expect the rate of change of the predator animal population to depend on the number of predator animals there are to reproduce, but also on the number of prey animals there are for the predators to eat

### How do I solve coupled first order linear differential equations?

- You can solve **coupled systems** by turning them into an **uncoupled second order differential equation** that you know how to solve
  - For example, consider the coupled system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.6x + 2y$$

  - $$\frac{\mathrm{d}y}{\mathrm{d}t} = 3.12x + 0.4y$$

- **STEP 1**: **Rearrange one** of the equations to make the variable that is not in the derivative the subject

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- We can rearrange the first equation to get $y = 0.5\frac{\mathrm{d}x}{\mathrm{d}t} - 0.3x$
- **STEP 2**: **Differentiate both sides** of the equation from Step 1 with respect to *t*
  - Differentiating gives $\frac{\mathrm{d}y}{\mathrm{d}t} = 0.5\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 0.3\frac{\mathrm{d}x}{\mathrm{d}t}$
- **STEP 3**: **Substitute** the equations from Steps 1 and 2 into the coupled differential equation you didn't use in Step 1
  - Substituting into the second equation gives

$$0.5\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 0.3\frac{\mathrm{d}x}{\mathrm{d}t} = 3.12x + 0.4\left(0.5\frac{\mathrm{d}x}{\mathrm{d}t} - 0.3x\right)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{\mathrm{d}x}{\mathrm{d}t} - 6x = 0$$

  - The result is a **second order differential equation** in only one of the variables
- **STEP 4**: **Solve the second order differential equation** resulting from Step 3
  - Using the standard solution methods for second order differential equations gives the solution

$$x = A\mathrm{e}^{3t} + B\mathrm{e}^{-2t}$$

- **STEP 5**: To find the **solution for the other variable, substitute** the solution from Step 4 along with its derivative into the equation from Step 1
  - Differentiate $x = A\mathrm{e}^{3t} + B\mathrm{e}^{-2t}$ to get $\frac{\mathrm{d}x}{\mathrm{d}t} = 3A\mathrm{e}^{3t} - 2B\mathrm{e}^{-2t}$
  - Then substituting into the Step 1 equation gives

$$y = 0.5\left(3A\mathrm{e}^{3t} - 2B\mathrm{e}^{-2t}\right) - 0.3\left(A\mathrm{e}^{3t} + B\mathrm{e}^{-2t}\right)$$

$$y = 1.2A\mathrm{e}^{3t} - 1.3B\mathrm{e}^{-2t}$$

- **STEP 6**: If the question provides **initial** or **boundary conditions** you may use these to find the values of *A* and *B*, and then go on to interpret your solutions in the context of whatever situation the coupled system of differential equations may be modelling

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## Worked example

In the following system of coupled differential equations, the variable *x* represents the population size of a species of prey fish in a particular region of ocean, while the variable *y* represents the population size of a species of predator fish in the same region.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.4x - 0.2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.1x + 0.1y$$

a) Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 0.5\frac{\mathrm{d}x}{\mathrm{d}t} + 0.06x = 0$.

STEP 1 : $\frac{dx}{dt} = 0.4x - 0.2y \Rightarrow y = -5\frac{dx}{dt} + 2x$

STEP 2 : $\Rightarrow \frac{dy}{dt} = -5\frac{d^2x}{dt^2} + 2\frac{dx}{dt}$

STEP 3 : $-5\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 0.1x + 0.1\left(-5\frac{dx}{dt} + 2x\right)$

$\Rightarrow -5\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 0.1x - 0.5\frac{dx}{dt} + 0.2x$

$\Rightarrow 5\frac{d^2x}{dt^2} - 2.5\frac{dx}{dt} + 0.3x = 0$

$\Rightarrow \boxed{\frac{d^2x}{dt^2} - 0.5\frac{dx}{dt} + 0.06 = 0}$

b) Find the general solution for the number of prey fish in the region at time *t*.

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STEP 4: $m^2 - 0.5m - 0.06 = (m-0.2)(m-0.3) = 0$

$$\Rightarrow \quad m = 0.2,\ 0.3$$

$$\Rightarrow \quad \boxed{x = Ae^{0.2t} + Be^{0.3t}}$$

c) Find the general solution for the number of predator fish in the region at time $t$.

STEP 5: $x = Ae^{0.2t} + Be^{0.3t} \Rightarrow \frac{dx}{dt} = 0.2Ae^{0.2t} + 0.3Be^{0.3t}$

Substitute into $y = -5\frac{dx}{dt} + 2x$ from Step 1:

$$\Rightarrow \quad y = -5\left(0.2Ae^{0.2t} + 0.3Be^{0.3t}\right) + 2\left(Ae^{0.2t} + Be^{0.3t}\right)$$

$$\Rightarrow \quad \boxed{y = Ae^{0.2t} + 0.5Be^{0.3t}}$$

d) Given that there are initially 6000 prey fish and 3500 predator fish in the region, find the number of each species that the model predicts for time $t = 3$.

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STEP 6: When $t = 0$, $x = 6000$ and $y = 3500$

$$Ae^0 + Be^0 = A + B = 6000$$
$$Ae^0 + 0.5Be^0 = A + 0.5B = 3500$$

$$\Longrightarrow A = 1000, \quad B = 5000$$ solve simultaneous equations

$$x = 1000e^{0.2t} + 5000e^{0.3t}$$

$$y = 1000e^{0.2t} + 2500e^{0.3t}$$

At $t = 3$,

$x = 14120.1343...$ and $y = 7971.1265...$

Round answers to nearest whole number:

$\boxed{\text{14120 prey fish and 7971 predator fish}}$

e) Comment on the suitability of the model.

Your notes

As $t \to \infty$, $e^{0.2t} \to \infty$ and $e^{0.3t} \to \infty$.

Therefore the model predicts that both fish populations will increase at an exponential rate forever.

This is unlikely to be a realistic model of what could happen in an actual ecosystem.