# Edexcel A Level Further Maths: Core Pure

Your notes

## 1.1 Complex Numbers & Argand Diagrams

### Contents

Your notes

# 1.1.1 Introduction to Complex Numbers

## Cartesian Form of Complex Numbers

Complex numbers are a set of numbers which contain both a real part and an imaginary part. The set of complex numbers is denoted as $\mathbb{C}$.

**What is an imaginary number?**

- Up until now, when we have encountered an equation such as $x^2 = -1$ we would have stated that there are "no real solutions" as the solutions are $x = \pm\sqrt{-1}$ which are not real numbers
- To solve this issue, mathematicians have defined one of the square roots of negative one as $\mathrm{i}$; an imaginary number
  - $\sqrt{-1} = \mathrm{i}$
  - $\mathrm{i}^2 = -1$
- We can use the rules for manipulating surds to manipulate imaginary numbers.
- We can do this by rewriting surds to be a multiple of $\sqrt{-1}$ using the fact that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

**What is a complex number?**

- Complex numbers have both a real part and an imaginary part
  - For example: $3 + 4\mathrm{i}$
  - The real part is 3 and the imaginary part is 4
    - Note that the imaginary part does not include the '$\mathrm{i}$'
- Complex numbers are often denoted by $z$ and we can refer to the real and imaginary parts respectively using $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$
- In general:
  - $z = a + b\mathrm{i}$ This is the Cartesian form of *z*
  - $\mathrm{Re}(z) = a$
  - $\mathrm{Im}(z) = b$
- It is important to note that two complex numbers are equal if, and only if, both the real and imaginary parts are identical.
  - For example, $3 + 2\mathrm{i}$ and $3 + 3\mathrm{i}$ are **not equal**

### Examiner Tip

- Be careful in your notation of complex and imaginary numbers.
- For example:
  $(3\sqrt{5})i$ could also be written as $3i\sqrt{5}$, but if you wrote $3\sqrt{5}\,i$ this could easily be confused with $3\sqrt{5i}$.

Your notes

## Worked example

a) Solve the equation $x^2 = -9$

$x^2 = -9$

$x = \pm\sqrt{-9}$

Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ $\quad x = \pm\sqrt{9}\sqrt{-1}$

$\boxed{x = \pm 3i}$

b) Solve the equation $(x+7)^2 = -16$, giving your answers in Cartesian form.

$(x+7)^2 = -16$

$x + 7 = \pm\sqrt{-16}$

$x + 7 = \pm\sqrt{16}\sqrt{-1}$

$x + 7 = \pm 4i$ Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

Rearrange answer into Cartesian form:

$\boxed{x = -7 \pm 4i}$

# Operations with Complex Numbers

Your notes

## How do I add and subtract complex numbers?

- When adding and subtracting complex numbers, simplify the real and imaginary parts separately
  - Just like you would when collecting like terms in algebra and surds, or dealing with different components in vectors
  - $(a+b\mathrm{i})+(c+d\mathrm{i})=(a+c)+(b+d)\mathrm{i}$
- Complex numbers can also be multiplied by a constant in the same way as algebraic expressions:
  - $k(a+b\mathrm{i})=ka+kb\mathrm{i}$

## How do I multiply complex numbers?

- The most important thing to bear in mind when multiplying complex numbers is that $\mathrm{i}^2=-1$
- We can still apply our usual rules for multiplying algebraic terms:
  - $a(b+c)=ab+ac$
  - $(a+b)(c+d)=ac+ad+bc+bd$
- Sometimes when a question describes multiple complex numbers, the notation $z_1,\ z_2,\ \ldots$ is used to represent each complex number

## How do I deal with higher powers of i?

- Because $\mathrm{i}^2=-1$ this can lead to some interesting results for higher powers of i
  - $\mathbf{i}^3=\mathbf{i}^2\times\mathbf{i}=\ -\mathbf{i}$
  - $\mathbf{i}^4=(\mathbf{i}^2)^2=(-1)^2=1$
  - $\mathbf{i}^5=(\mathbf{i}^2)^2\ \times\mathbf{i}=\mathbf{i}$
  - $\mathbf{i}^6=(\mathbf{i}^2)^3=(-1)^3=\ -1$
- We can use this same approach of using $\mathrm{i}^2$ to deal with much higher powers
  - $\mathbf{i}^{23}=(\mathbf{i}^2)^{11}\times\mathbf{i}=(-1)^{11}\times\mathbf{i}=\ -\mathbf{i}$
  - Just remember that -1 raised to an even power is 1 and raised to an odd power is -1

### Examiner Tip

- Most calculators used at A-Level can work with complex numbers and you can use these to check your working.
- You should still show your full working though to ensure you get all marks though.

Your notes

## Worked example

a) Simplify the expression $2(8-6\text{i})-5(3+4\text{i})$.

Expand the brackets

$2(8-6i)-5(3+4i) = 16-12i-15-20i$

Collect the real and imaginary parts

$16-15-12i-20i$

Simplify

$\boxed{1-32i}$

b) Given two complex numbers $z_1 = 3+4\text{i}$ and $z_2 = 6+7\text{i}$, find $z_1 \times z_2$.

Expand the brackets

$(3+4i)(6+7i) = 18+21i+24i+28i^2$

$= 18+21i+24i+(28)(-1)$ Using $i^2=-1$

Collect the real and imaginary parts

$18+21i+24i-28 = 18-28+(21+24)i$

Simplify

$\boxed{-10+45i}$

## Complex Conjugation & Division

Your notes

When dividing complex numbers, we can use the **complex conjugate** to make the denominator a real number, which makes carrying out the division much easier.

### What is a complex conjugate?

- For a given complex number $z = a + bi$, **the complex conjugate of $z$ is denoted as $z^*$**, where $z^* = a - bi$
- If $z = a - bi$ then $z^* = a + bi$
- You will find that:
  - $z + z^*$ is always real because $(a + bi) + (a - bi) = 2a$
    - For example: $(6 + 5i) + (6 - 5i) = 6 + 6 + 5i - 5i = 12$
  - $z - z^*$ is always imaginary because $(a + bi) - (a - bi) = 2bi$
    - For example: $(6 + 5i) - (6 - 5i) = 6 - 6 + 5i - (-5i) = 10i$
  - $z \times z^*$ is always real because $(a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$ (as $i^2 = -1$)
    - For example: $(6 + 5i)(6 - 5i) = 36 + 30i - 30i - 25i^2 = 36 - 25(-1) = 61$

### How do I divide complex numbers?

- When we divide complex numbers, we can express the calculation in the form of a fraction, and then **start by multiplying the top and bottom by the conjugate of the denominator**:
  - $\dfrac{a + bi}{c + di} = \dfrac{a + bi}{c + di} \times \dfrac{c - di}{c - di}$
- This ensures we are multiplying by 1; so not affecting the overall value
- This gives us a real number as the denominator because we have a complex number multiplied by its conjugate ($zz^*$)
- This process is very similar to "rationalising the denominator" with surds which you may have studied at GCSE

### Examiner Tip

- We can speed up the process for finding $zz^*$ by using the basic pattern of $(x + a)(x - a) = x^2 - a^2$
- We can apply this to complex numbers: $(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$ (using the fact that $i^2 = -1$)
- So $3 + 4i$ multiplied by its conjugate would be $3^2 + 4^2 = 25$

Your notes

### Worked example

Find the value of $(1+7\mathrm{i}) \div (3-\mathrm{i})$.

Rewrite as a fraction: $\dfrac{1+7i}{3-i}$

complex conjugate of $3-i$ is $3+i$

Multiply top and bottom of the fraction by the complex conjugate of the denominator.

$$\frac{1+7i}{3-i} \times \frac{3+i}{3+i} = \frac{(1+7i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{3+i+21i+7\overset{=-1}{i^2}}{9+3i-3i-i^2}$$

the imaginary parts eliminate each other

$$= \frac{3+22i+(-7)}{9-(-1)}$$

Simplify $= \dfrac{-4+22i}{10}$

Write in Cartesian form $= -\dfrac{4}{10} + \dfrac{22}{10}i$

$\boxed{-\dfrac{2}{5} + \dfrac{11}{5}i}$ Simplify final answer.

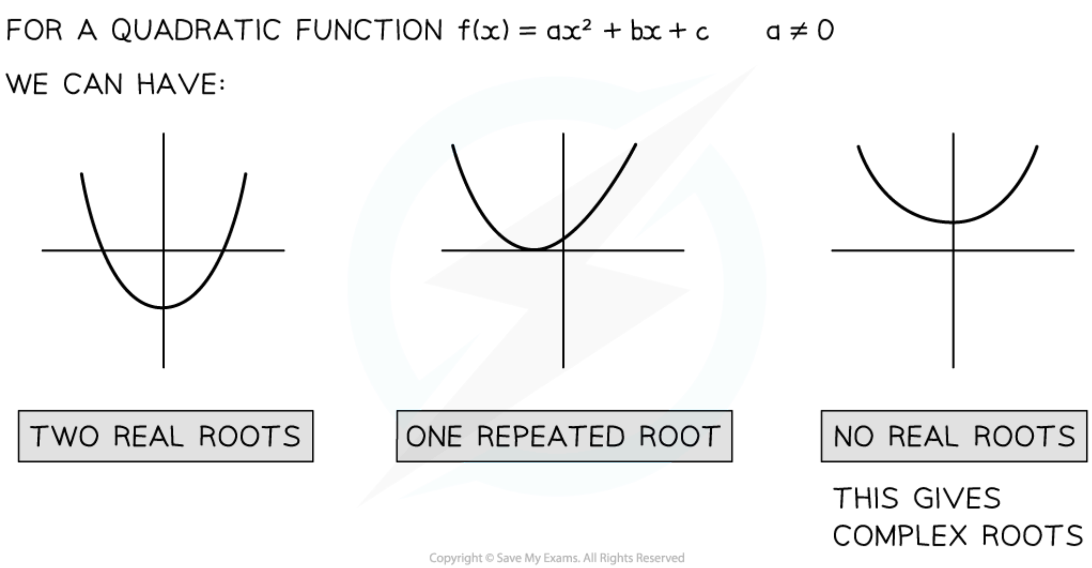
Your notes

# 1.1.2 Solving Equations with Complex Roots

## Solving Quadratic Equations with Complex Roots

### What are complex roots?

- Complex numbers provide solutions for quadratic equations which have **no real roots**




- Complex roots occur when solving a quadratic with a **negative discriminant**
  - This leads to square rooting a negative number

### How do we solve a quadratic equation with complex roots?

- We solve an equation with complex roots in the same way we solve any other quadratic equations
  - If in the form $az^2 + b = 0 \; (a \neq 0)$ we can **rearrange** to solve
  - If in the form $az^2 + bz + c = 0 \; (a \neq 0)$ we can **complete the square** or use the **quadratic formula**
- We use the property $\mathrm{i} = \sqrt{-1}$ along with a manipulation of surds
  - $\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a} \times \sqrt{-1}$
- When the **coefficients** of the quadratic equation are **real**, complex roots occur in **complex conjugate pairs**
  - If $z = m + n\mathrm{i} \; (\mathrm{n} \neq 0)$ is a root of a quadratic with real coefficients then $z^* = m - n\mathrm{i}$ is also a root
- When the coefficients of the quadratic equation are **non-real**, the solutions will **not** be complex conjugates

- To solve these use the quadratic formula

Your notes

### How do we find a quadratic equation given a complex root?

- We can find the equation of the form $\boldsymbol{z}^2 + \boldsymbol{bz} + \boldsymbol{c} = 0$ if you are given a complex root in the form $\boldsymbol{m} + \boldsymbol{n}\mathrm{i}$
  - We know that the complex conjugate $\boldsymbol{m} - \boldsymbol{n}\mathbf{i}$ is another root,
  - This means that $z - (m + n\mathrm{i})$ and $z - (m - n\mathrm{i})$ are **factors** of the quadratic equation
  - Therefore $z^2 + bz + c = [z - (m + n\mathrm{i})][z - (m - n\mathrm{i})]$
    - Writing this as $((z - m) - n\mathrm{i})((z - m) + n\mathrm{i})$ will speed up expanding
  - **Expanding** and **simplifying** gives us a quadratic equation where $b$ and $c$ are **real** numbers

## Examiner Tip

- Once you have your final answers you can check your roots are correct by substituting your solutions back into the original equation.
- You should get 0 if correct! [Note: 0 is equivalent to $0 + 0\mathbf{i}$]

Your notes

## Worked example

a) Solve the quadratic equation $z^2 - 2z + 5 = 0$ and hence, factorise $z^2 - 2z + 5$.

Use the quadratic formula or completing the square to find the solutions.

$a = 1 \quad b = -2 \quad c = 5$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$\boxed{z_1 = 1 + 2i \qquad z_2 = 1 - 2i}$$

If the solutions are $z_1 = 1 + 2i$ and $z_2 = 1 - 2i$

then the factors must be $z - (1+2i)$ and $z - (1-2i)$

$$z^2 - 2z + 5 = (z - (1+2i))(z - (1-2i))$$

$$\boxed{(z - 1 - 2i)(z - 1 + 2i)}$$

b) Given that one root of a quadratic equation is $z = 2 - 3i$, find the quadratic equation in the form $az^2 + bz + c = 0$, where $a$, $b$, and $c \in \mathbb{R}$, $a \neq 0$.

Your notes

If $2-3i$ is one root then $2+3i$ must be the other root and the two factors must be $z-(2-3i)$ and $z-(2+3i)$.

$$(z-(2-3i))(z-(2+3i))=0$$

$$((z-2)+3i)((z-2)-3i)=0$$

$$(z-2)^2-(3i)^2=0$$

$$z^2-4z+4-9i^2=0$$

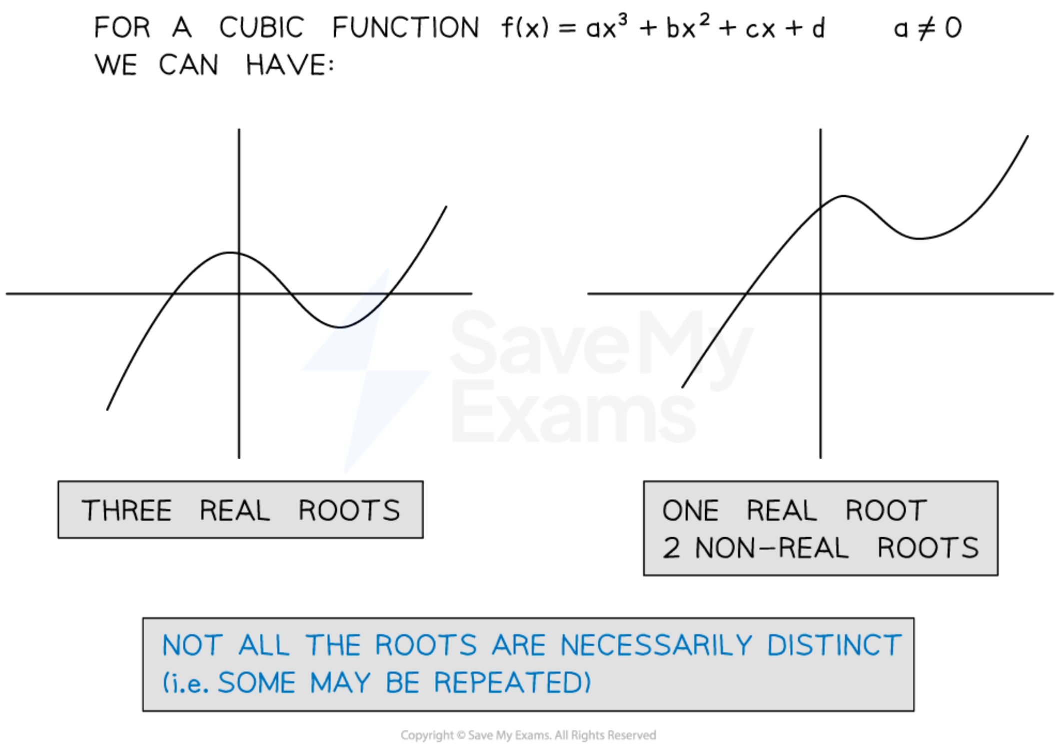
$i^2=-1 \therefore -9i^2=9$

$$\boxed{z^2-4z+13=0}$$

## Solving Polynomial Equations with Complex Roots

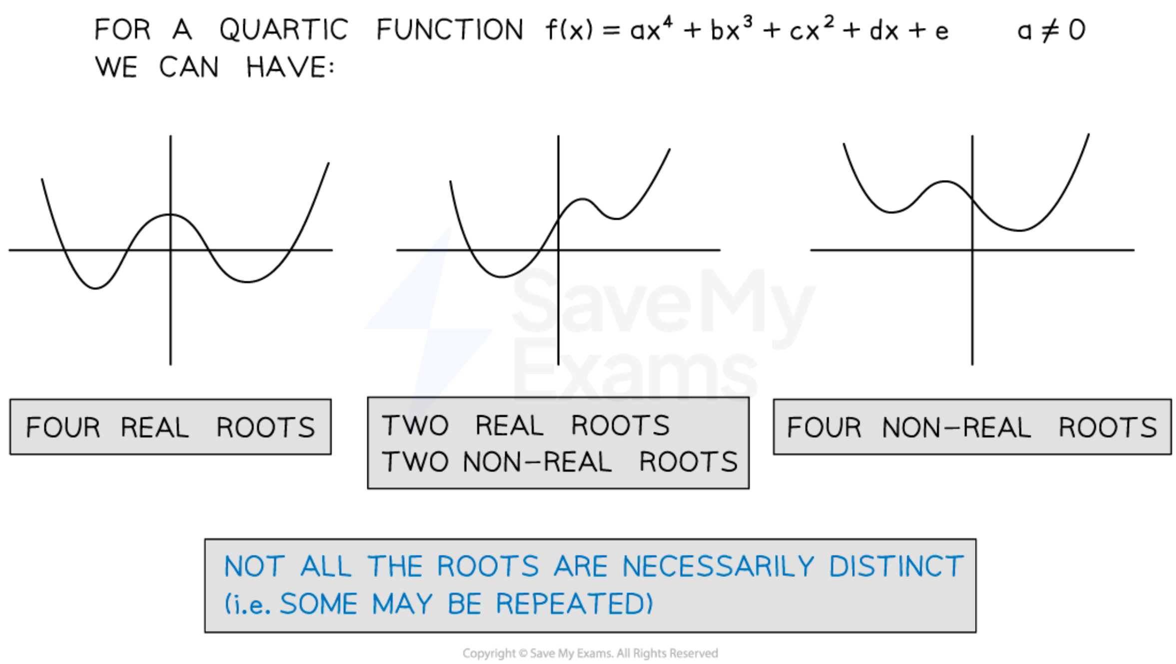
### How many roots should a polynomial have?

- We know that every **quadratic** equation has **two roots** (not necessarily distinct or real)
- This is a particular case of a more general rule:
  - Every polynomial equation, with real coefficients, of degree *n* has *n* roots
  - The *n* roots are not necessarily all **distinct** and therefore we need to count any **repeated** roots that may occur individually
- From the above rule we can state the following:
  - A **cubic** equation of the form $ax^3 + bx^2 + cx + d = 0$ can have either:
    - 3 **real** roots
    - Or 1 **real** root and a complex **conjugate** pair
  - A **quartic** equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ will have one of the following cases for roots:
    - 4 **real** roots
    - 2 **real** and 2 **nonreal** (a complex conjugate pair)
    - 4 **nonreal** (two complex conjugate pairs)
- When a real polynomial of any degree has one complex root it will always also have the complex conjugate as a root




*Number of roots of a cubic function*

Your notes




*Number of roots of a quartic function*

### How do we solve a cubic equation with complex roots?

- Steps to solve a cubic equation with complex roots
  - If we are told that $m + n\mathrm{i}$ is a root, then we know $m - n\mathrm{i}$ is also a root
  - This means that $(z-(m+n\mathrm{i}))$ and $(z-(m-n\mathrm{i}))$ are **factors** of the cubic equation
  - **Multiply** the above factors together gives us a quadratic factor of the form $(Az^2 + Bz + C)$
  - We need to find the third factor $(z-\alpha)$
  - **Multiply** the factors and **equate** to our original equation to get

$$(Az^2 + Bz + C)(z-\alpha) = ax^3 + bx^2 + cx + d$$

- From there either
  - **Expand** and **compare coefficients** to find $\alpha$
  - Or use **polynomial division** to find the factor $(z-\alpha)$
- Finally, write your **three roots** clearly

### How do we solve a polynomial of any degree with complex roots?

- When asked to find the roots of any polynomial when we are given one, we use almost the same method as for a cubic equation
  - State the initial root and its conjugate and write their factors as a quadratic factor (as above) we will have two unknown roots to find, write these as factors $(z - \alpha)$ and $(z - \beta)$
  - The unknown factors also form a quadratic factor $(z - \alpha)(z - \beta)$

Your notes

- Then continue with the steps from above, either **comparing coefficients** or using **polynomial division**
  - If using polynomial division, then solve the quadratic factor you get to find the roots $\alpha$ and $\beta$

## How do we solve polynomial equations with unknown coefficients?

- Steps to find unknown variables in a given equation when given a root:
  - **Substitute** the given root $p + q$i into the equation f(z) = 0
  - **Expand** and **group** together the **real** and **imaginary** parts (these expressions will contain our unknown values)
  - **Solve** as simultaneous equations to find the unknowns
  - **Substitute** the values into the **original** equation
  - From here continue using the previously described methods for finding other roots for the polynomial

## How do we factorise a polynomial when given a complex root?

- If we are given a root of a polynomial of any degree in the form $z = p + q\text{i}$
  - We know that the complex conjugate, $z^* = p - q\text{i}$ is another root
  - We can write $(z - (p + q\text{i}))$ and $(z - (p - q\text{i}))$ as two linear factors
    - Or rearrange into one quadratic factor
  - This can be multiplied out with another factor to find further factors of the polynomial
- For higher order polynomials more than one root may be given
  - If the further given root is complex then its complex conjugate will also be a root
  - This will allow you to find further factors

### Examiner Tip

- As with solving quadratic equations, we can substitute our solutions back into the original equation to check we get 0
- You can speed up multiplying two complex conjugate factors together by
  - rewrite $(z - (p + q\text{i}))(z - (p - q\text{i}))$ as $((z - p) - q\text{i})((z - p) + q\text{i})$
  - Then $((z - p) - q\text{i})((z - p) + q\text{i}) = (z - p)^2 - (q\text{i})^2 = (z - p)^2 + q^2$

Your notes

## Worked example

Given that one root of a polynomial p(*x*) = $z^3 + z^2 - 7z + 65$ is $2 - 3i$, find the other roots.

If $2-3i$ is one root then $2+3i$ must be the other root and two of the factors must be $z-(2-3i)$ and $z-(2+3i)$

Therefore a quadratic factor is $z^2 - 4z + 13$

There must exist a linear factor $(az + b)$

$\therefore (az+b)(z^2-4z+13) = z^3 + z^2 - 7z + 65$

Compare coefficients: $az^3 = z^3$ coefficient of $z^3$

$a = 1$

$13b = 65$ constant coefficient

$b = 5$

Therefore the factors are $z-(2-3i)$, $z-(2+3i)$ and $(z+5)$

$(z-(2-3i))(z-(2+3i))(z+5) = 0$

$$\boxed{z = (2 \pm 3i) \text{ and } z = -5}$$

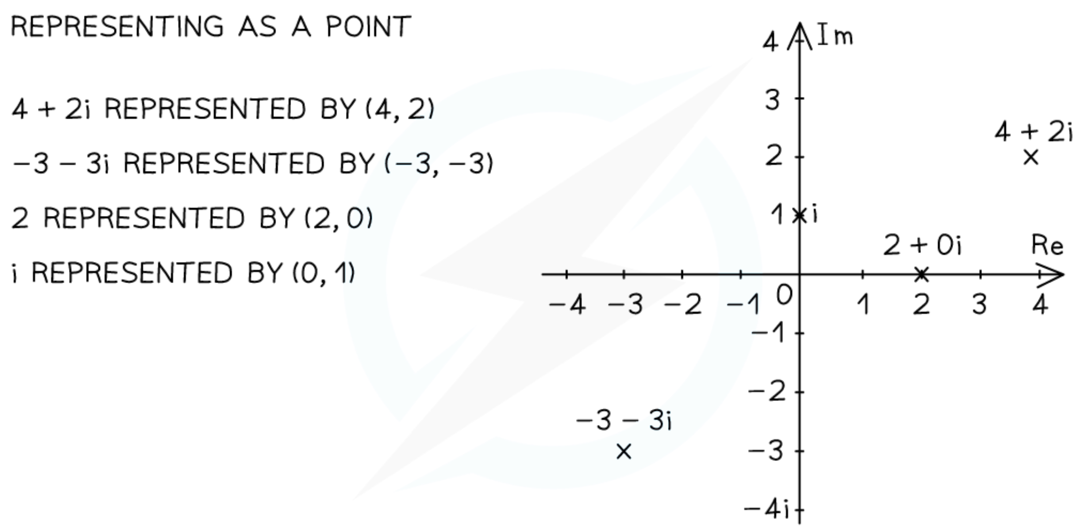
Your notes

# 1.1.3 Modulus & Argument

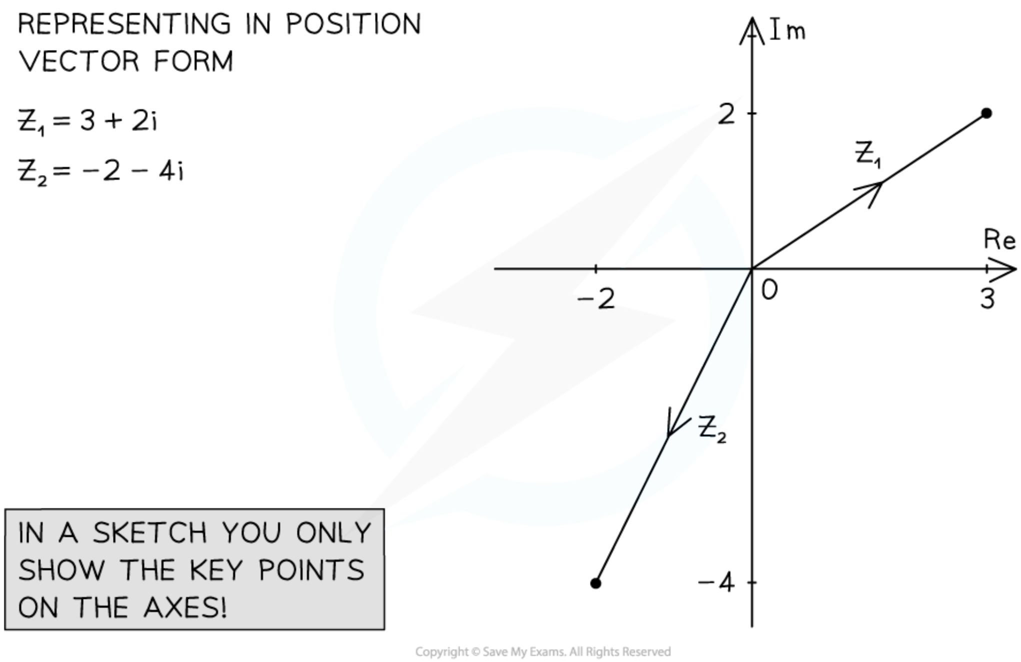
## Argand Diagrams – Basics

### What is an Argand diagram?

- An Argand diagram is a **geometrical** way to represent complex numbers as either a **point** or a **vector** in two-dimensional space
  - We can represent the complex number $x + yi$ by the **point** with **cartesian coordinate** $(x, y)$
- The **real** component is represented by points on the **x-axis**, called the **real axis**, Re
- The **imaginary** component is represented by points on the **y-axis**, called the **imaginary axis**, Im



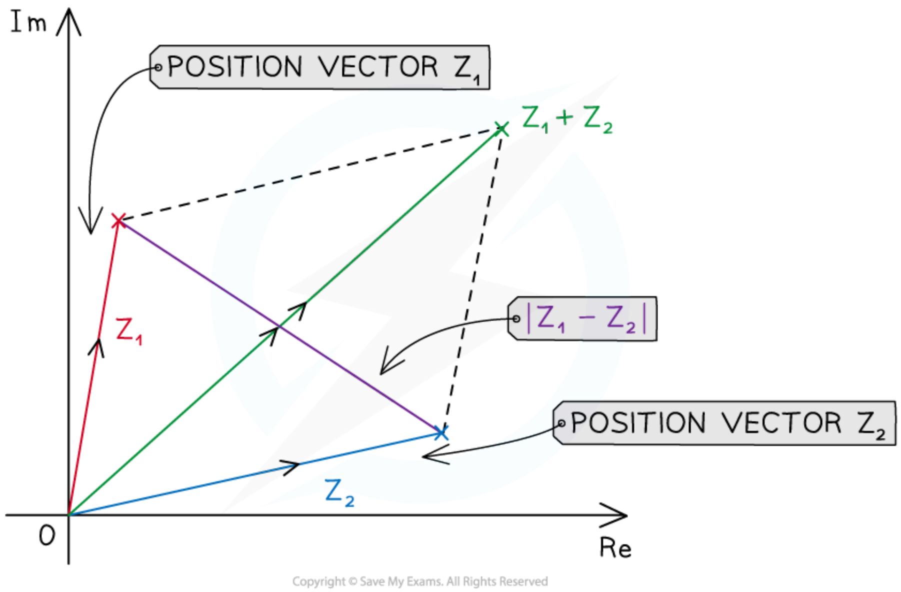
Your notes




- You may be asked to show **roots** of an equation in an Argand diagram
  - First **solve** the equation
  - Draw a quick sketch, only adding essential information to the axes
  - Plot the points and label clearly

### How can I use an Argand diagram to visualise $|z_1 + z_2|$ and $|z_1 - z_2|$?

- Plot two complex numbers $z_1$ and $z_2$
- Draw a line from the origin to each complex number
- Form a parallelogram using the two lines as two adjacent sides
- The modulus of their sum $|z_1 + z_2|$ will be the length of the diagonal of the parallelogram starting at the origin
- The modulus of their difference $|z_1 - z_2|$ will be the length of the diagonal between the two complex numbers

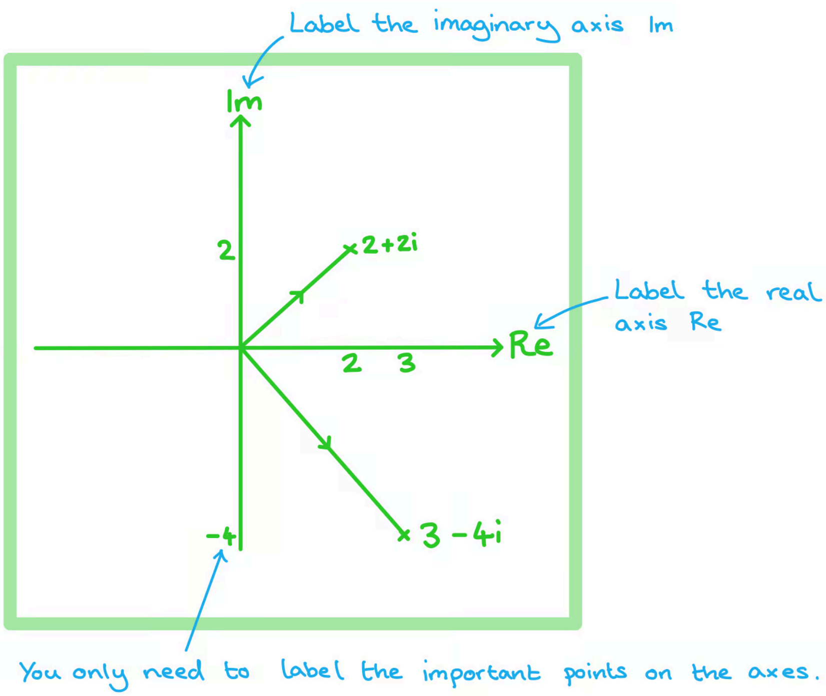

Im
POSITION VECTOR $Z_1$
$Z_1 + Z_2$
$Z_1$
$|Z_1 - Z_2|$
POSITION VECTOR $Z_2$
$Z_2$
0
Re
Copyright © Save My Exams. All Rights Reserved

Your notes

Your notes

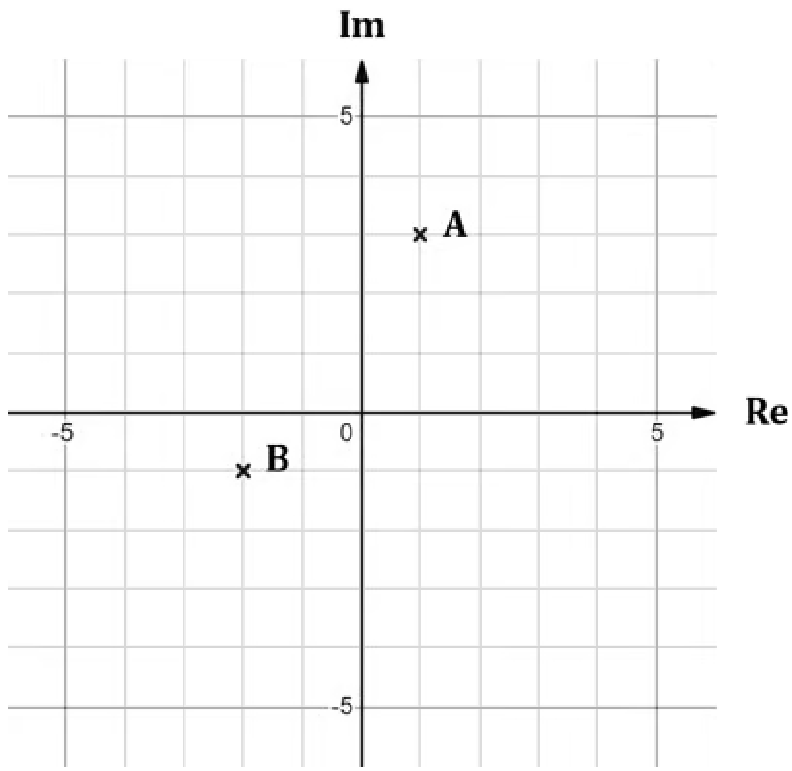
## Worked example

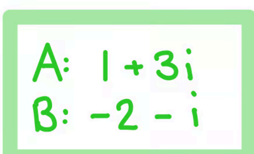
a) Plot the complex numbers $z_1 = 2 + 2i$ and $z_2 = 3 - 4i$ as points on an Argand diagram.




b) Write down the complex numbers represented by the points A and B on the Argand diagram below.

Your notes



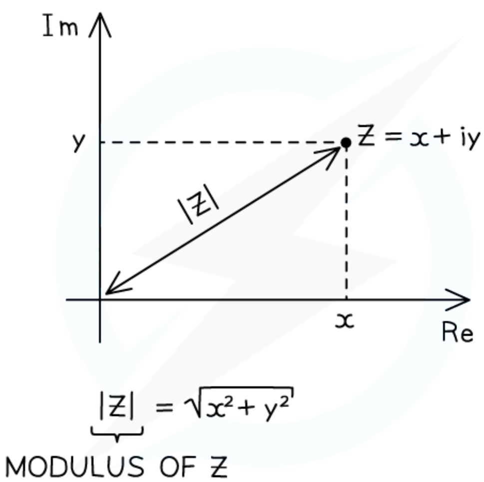

## Examiner Tip

- When setting up an Argand diagram you do not need to draw a fully scaled axes, you only need the essential information for the points you want to show, this will save a lot of time.

## Modulus & Argument

### How do I find the modulus of a complex number?

- The modulus of a complex number is its **distance** from the origin when plotted on an Argand diagram
- The modulus of $z$ is written $|z|$
- If $z = x + \mathrm{i}y$, then we can use **Pythagoras** to show...
  - $|z| = \sqrt{x^2 + y^2}$
- A modulus is always **positive**
- the modulus is related to the complex **conjugate** by...
  - $zz^* = z^*z = |z|^2$
  - This is because $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$
- In general, $|z_1 + z_2| \neq |z_1| + |z_2|$
  - e.g. both $z_1 = 3 + 4\mathrm{i}$ and $z_2 = -3 + 4\mathrm{i}$ have a modulus of 5, but $z_1 + z_2$ simplifies to $8\mathrm{i}$ which has a modulus of 8

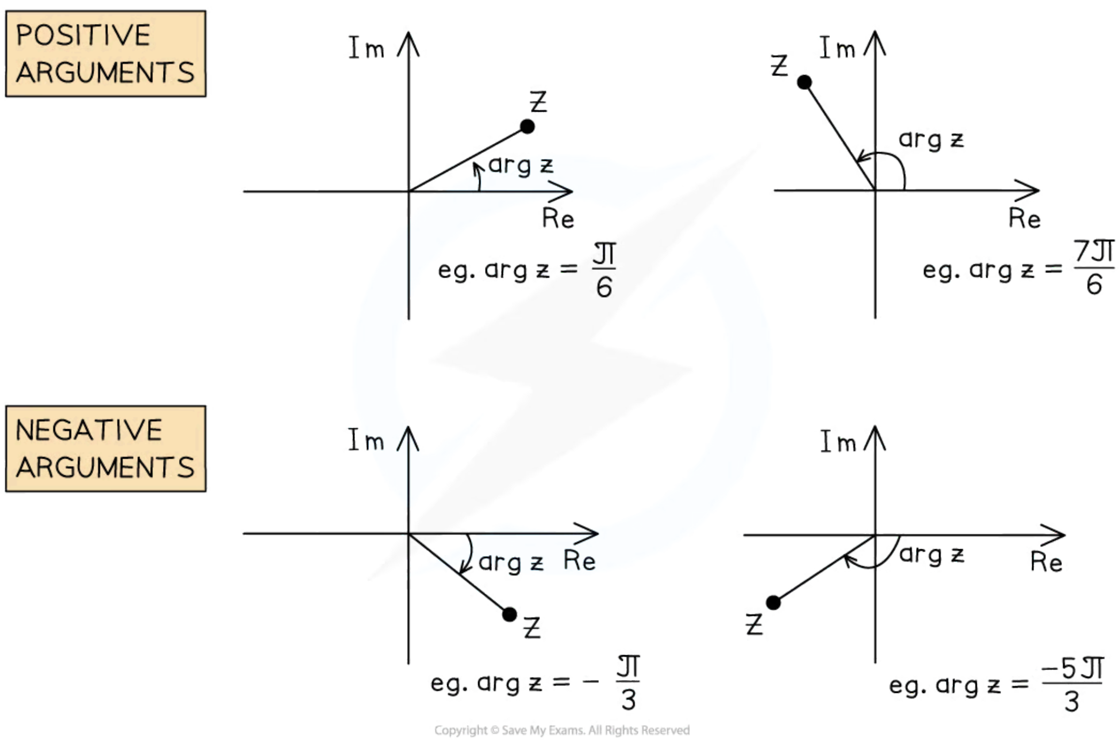





### How do I find the argument of a complex number?

- The argument of a complex number is the anti-clockwise **angle** that it makes when starting at the positive real axis on an Argand diagram
- Arguments are measured in **radians**
  - Sometimes these can be given exact in terms of $\pi$
- The argument of $z$ is written $\arg z$
- Arguments can be calculated using right-angled **trigonometry**
  - This involves using the tan ratio plus a sketch to decide whether it is positive/negative and acute/obtuse
- Arguments are usually given in the range $-\pi < \arg z \leq \pi$

- This is called the **principal argument**
- Negative arguments are for complex numbers in the third and fourth quadrants
- Occasionally you could be asked to give arguments in the range $0 < \arg z \leq 2\pi$
- The argument of zero, $\arg 0$ is undefined (no angle can be drawn)




### Examiner Tip

- Always draw a sketch to see which quadrant the complex number is in

Your notes

Your notes

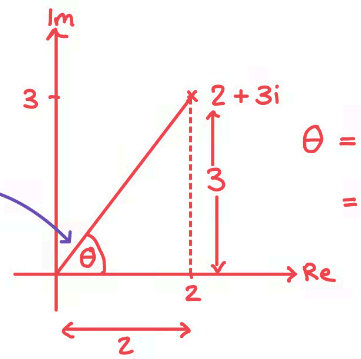
## Worked example

a) Find the modulus and argument of $z = 2 + 3\text{i}$

$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

Draw a sketch to help find the argument:

The argument is the counter-clockwise angle taken from the positive $x$-axis




$\theta = \tan^{-1}\left(\frac{3}{2}\right)$

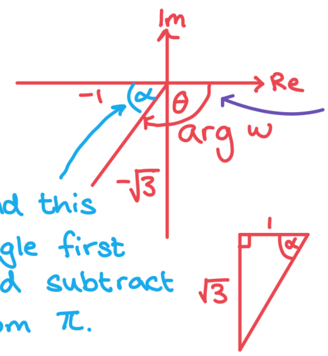
$= 0.9827...$

$$\boxed{\begin{aligned} \text{Mod } z &= |z| = \sqrt{13} \\ \arg z &= \theta = 0.983 \text{ (3sf)} \end{aligned}}$$

b) Find the modulus and argument of $w = -1 - \sqrt{3}\text{i}$

Your notes

$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4}$




If the argument is measured clockwise from the positive $x$-axis then it will be negative.

Find this angle first and subtract from $\pi$.

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$$\boxed{\begin{aligned} \text{Mod } z &= |z| = 2 \\ \arg z &= -\theta = -\frac{2\pi}{3} \end{aligned}}$$

# 1.1.4 Modulus-Argument Form

Your notes

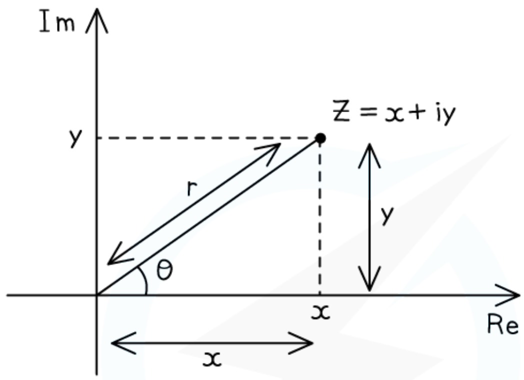
## Modulus-Argument Form

The complex number $z = x + \mathrm{i}y$ is said to be in Cartesian form. There are, however, other ways to write a complex number, such as in modulus-argument (polar) form.

### How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number, $z = x + \mathrm{i}y$, is written in terms of its real part, $x$, and its imaginary part, $y$
- If we let $r = |z|$ and $\theta = \arg z$, then it is possible to write a complex number in terms of its modulus, $r$, and its argument, $\theta$, called the **modulus-argument (polar) form**, given by...
  - $z = r(\cos\theta + \mathrm{i}\sin\theta)$
- It is usual to give arguments in the range $-\pi < \theta \leq \pi$
  - Negative arguments should be shown clearly, e.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ without simplifying $\cos(-\frac{\pi}{3})$ to either $\cos\left(\frac{\pi}{3}\right)$ or $\frac{1}{2}$
  - Occasionally you could be asked to give arguments in the range $0 \leq \theta < 2\pi$
- If a complex number is given in the form $z = r(\cos\theta - \mathrm{i}\sin\theta)$, then it is not currently in modulus-argument (polar) form due to the minus sign, but can be converted as follows...
  - By considering transformations of trigonometric functions, we see that $-\sin\theta \equiv \sin(-\theta)$ and $\cos\theta \equiv \cos(-\theta)$
  - Therefore $z = r(\cos\theta - \mathrm{i}\sin\theta)$ can be written as $z = r(\cos(-\theta) + \mathrm{i}\sin(-\theta))$, now in the correct form and indicating an argument of $-\theta$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
  - E.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ becomes $z = 2\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}\,\mathrm{i}$

Your notes




$r = |z|$
$\theta = \arg z$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$\therefore z = x + iy \Rightarrow z = r\cos\theta + ir\sin\theta$

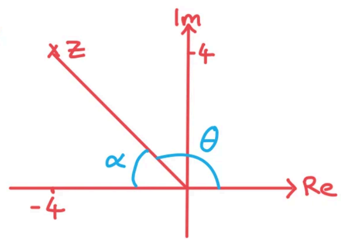
$z = r(\cos\theta + i\sin\theta)$

MODULUS – ARGUMENT (POLAR) FORM

### Worked example

Write $z = -4 + 4\mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ where $r$ and $\theta$ are exact.

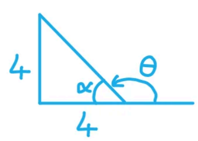
Draw a sketch to help:




Use Pythagoras to find $|z|$: $|z| = \sqrt{(-4)^2 + (4)^2} \quad (= \sqrt{32})$

$= 4\sqrt{2}$ – this is $r$.

Use trigonometry to find $\arg z$: $\tan\alpha = \frac{4}{4}$




$\alpha = \tan^{-1}\left(\frac{4}{4}\right)$

$= \frac{\pi}{4}$

$\arg z = \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Write in modulus - argument form: $r = 4\sqrt{2}$, $\theta = \frac{3\pi}{4}$

$$\boxed{z = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)}$$

Your notes

Your notes

## Operations using Modulus-Argument Form

### What are the rules for moduli and arguments under multiplication and division?

- When two complex numbers, $z_1$ and $z_2$, are **multiplied** to give $z_1 z_2$, their **moduli** are also **multiplied**
  - $|z_1 z_2| = |z_1||z_2|$
- When two complex numbers, $z_1$ and $z_2$, are **divided** to give $\frac{z_1}{z_2}$, their **moduli** are also **divided**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- When two complex numbers, $z_1$ and $z_2$, are **multiplied** to give $z_1 z_2$, their **arguments** are **added**
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- When two complex numbers, $z_1$ and $z_2$, are **divided** to give $\frac{z_1}{z_2}$, their **arguments** are **subtracted**
  - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

### How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily (often quicker than when in Cartesian form)
- To **multiply** two complex numbers, $z_1$ and $z_2$, in modulus-argument (polar) form we use the rules from above to **multiply their moduli** and **add their arguments**
  - $|z_1 z_2| = |z_1||z_2|$
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- So if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then the rules above give...
  - $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
- Sometimes the new argument, $\theta_1 + \theta_2$, does not lie in the range $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$ if this is being used)
  - An out-of-range argument can be adjusted by either **adding or subtracting** $2\pi$
  - E.g. If $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{\pi}{2}$ then $\theta_1 + \theta_2 = \frac{7\pi}{6}$

Your notes

- This is currently not in the range , but by subtracting $2\pi$ from $\frac{7\pi}{6}$ to give $-\frac{5\pi}{6}$, a new argument is formed that lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when…
  - …multiplying three complex numbers together, $z_1 z_2 z_3$, or more
  - …finding powers of a complex number (e.g. $z^2$ can be written as $zz$)
- Whilst not examinable, the rules for multiplication can be proved algebraically by multiplying $z_1 = r_1(\cos\theta_1 + \text{i}\sin\theta_1)$ by $z_2 = r_2(\cos\theta_2 + \text{i}\sin\theta_2)$, expanding the brackets and using compound angle formulae

## How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers, $z_1$ and $z_2$ in modulus-argument (polar) form, we use the rules from above to **divide their moduli** and **subtract their arguments**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- So if $z_1 = r_1(\cos\theta_1 + \text{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \text{i}\sin\theta_2)$ then the rules above give…
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \text{i}\sin(\theta_1 - \theta_2))$
- As with multiplication, sometimes the new argument, $\theta_1 - \theta_2$, can lie out of the range $-\pi < \theta \leq \pi$ (or the range $0 < \theta \leq 2\pi$ if this is being used)
  - You can **add or subtract** $2\pi$ to bring out-of-range arguments back in range
- Whilst not examinable, the rules for division can be proved algebraically by dividing $z_1 = r_1(\cos\theta_1 + \text{i}\sin\theta_1)$ by $z_2 = r_2(\cos\theta_2 + \text{i}\sin\theta_2)$, using complex division and compound angle formulae

Your notes

## Worked example

Let $\square_1 = 4\sqrt{2}\cos\frac{3\square}{4} + \text{i}\sin\frac{3\square}{4}$ and $\square_2 = \sqrt{8}\cos\frac{\square}{2} - \text{i}\sin\frac{\square}{2}$

a) Find $\square_1\square_2$, giving your answer in the form $\square\cos\square + \text{i}\sin\square$ where $0 \leq \square < 2\square$

$$z_1 = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$z_2 = \sqrt{8}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$$

$z_2$ needs to be in form $r(\cos\theta + i\sin\theta)$

So use: $\cos x = \cos(-x)$

$-\sin x = \sin(-x)$

$$z_2 = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

For $z_1 z_2$ multiply the moduli and add the arguments

$$z_1 z_2 = \left(4\sqrt{2} \times 2\sqrt{2}\right)\left(\cos\left(\frac{3\pi}{4} + -\frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{4} + -\frac{\pi}{2}\right)\right)$$

$$\boxed{z_1 z_2 = 16\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$$

b) Find $\frac{\square_1}{\square_2}$, giving your answer in the form $\square\cos\square + \text{i}\sin\square$ where $-\pi \leq \square < \square$

Your notes

For $\frac{z_1}{z_2}$ divide the moduli and subtract the arguments

$$\frac{z_1}{z_2} = \frac{4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}{2\sqrt{2}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)}$$

$$= 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$\frac{5\pi}{4}$ is not in the range $-\pi \leq \theta \leq \pi$

So subtract $2\pi$ to bring it into the range

$$\boxed{\frac{z_1}{z_2} = 2\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)}$$

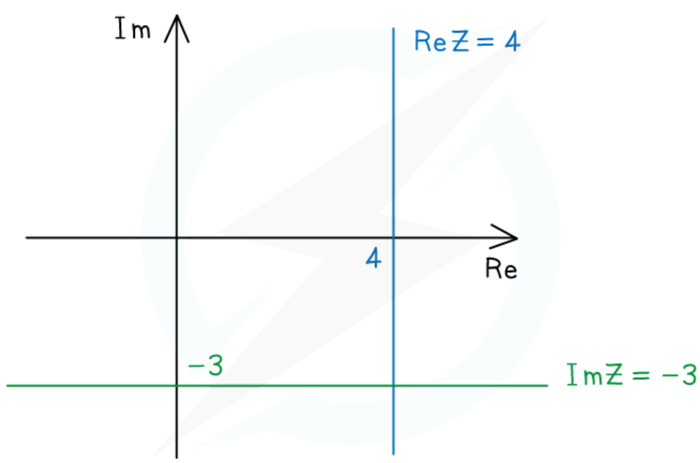
Your notes

## 1.1.5 Loci in Argand Diagrams

## Loci in Argand Diagrams

### How do I sketch the locus of $\text{Re } z = k$ or $\text{Im } z = k$ on an Argand diagram?

- All complex numbers, $z = x + \text{i}y$, that satisfy the equation $\text{Re } z = k$ lie on a **vertical line** with Cartesian equation $x = k$
  - Any complex number along this vertical line will have a **real part** of $k$
- All complex numbers, $z = x + \text{i}y$, that satisfy the equation $\text{Im } z = k$ lie on a **horizontal line** with Cartesian equation $y = k$
  - Any complex number along this horizontal line will have an **imaginary part** of $k$
- E.g. The loci $\text{Re } z = 4$ and $\text{Im } z = -3$ are represented by the vertical line $x = 4$ and the horizontal line $y = -3$





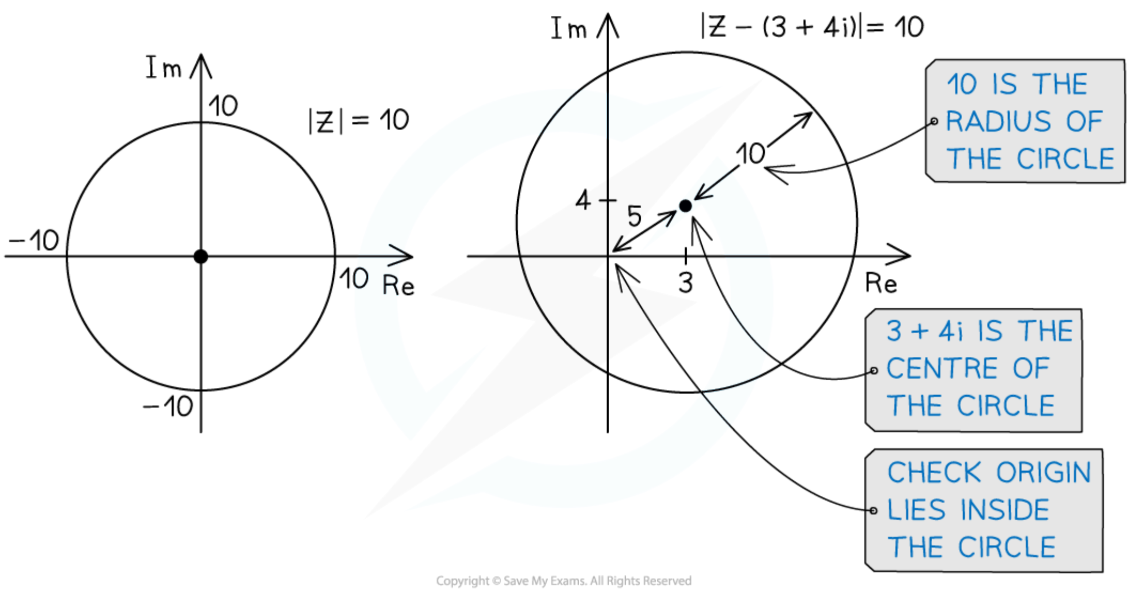

*Sketching the loci of* $\text{Re } z = 4$ *and* $\text{Im } z = -3$

### How do I sketch the locus of $|z - a| = k$ on an Argand diagram?

- All complex numbers, $z$, that satisfy the equation $|z| = k$ lie on a **circle** of **radius** $k$ about the **origin**
  - E.g. the locus of $|z| = 10$ is a circle of radius 10, centred at the origin, as every complex number on that circle has a modulus of 10
- For a given complex number, $a$, all complex numbers, $z$, that satisfy the equation $|z - a| = k$ lie on a **circle** of **radius** $k$ about the **centre** $a$
  - This is because $|z - a|$ represents the distance between complex numbers $z$ and $a$

Your notes

- E.g. the locus of $|z-(3+4i)|=10$ is a circle of radius 10 about $(3+4i)$
- Many equations need to be adjusted algebraically into the correct $|z-a|$ form
  - E.g. to find the centre of the circle $|z-8i+3|=12$, first rewrite it as $|z-(-3+8i)|=12$, giving the centre as $-3+8i$
  - E.g. to find the centre of the circle $|z+i|=2$, first rewrite it as $|z-(-i)|=2$, giving the centre as $(-i)$
  - Note that the centre of the circle $|z|=5$ is the origin (it can be thought of as $|z-0|=5$)
- In order to sketch correctly, **check** whether the **origin** lies **outside**, **on** or **inside** the circle
  - E.g. for the locus of $|z-(3+4i)|=10,$ the distance from the centre of the circle, $3+4i$, to the origin is 5 (by Pythagoras), which is **less** than the radius of 10; a sketch must therefore show the origin **inside** the circle
- By knowing the radius and centre of a circle, the Cartesian equation of the circle can be found
  - The circle $|z-(3+4i)|=10$ has a radius of 10 and centre of $(3,4)$ in coordinates, so the equation of the circle is $(x-3)^2+(y-4)^2=100$



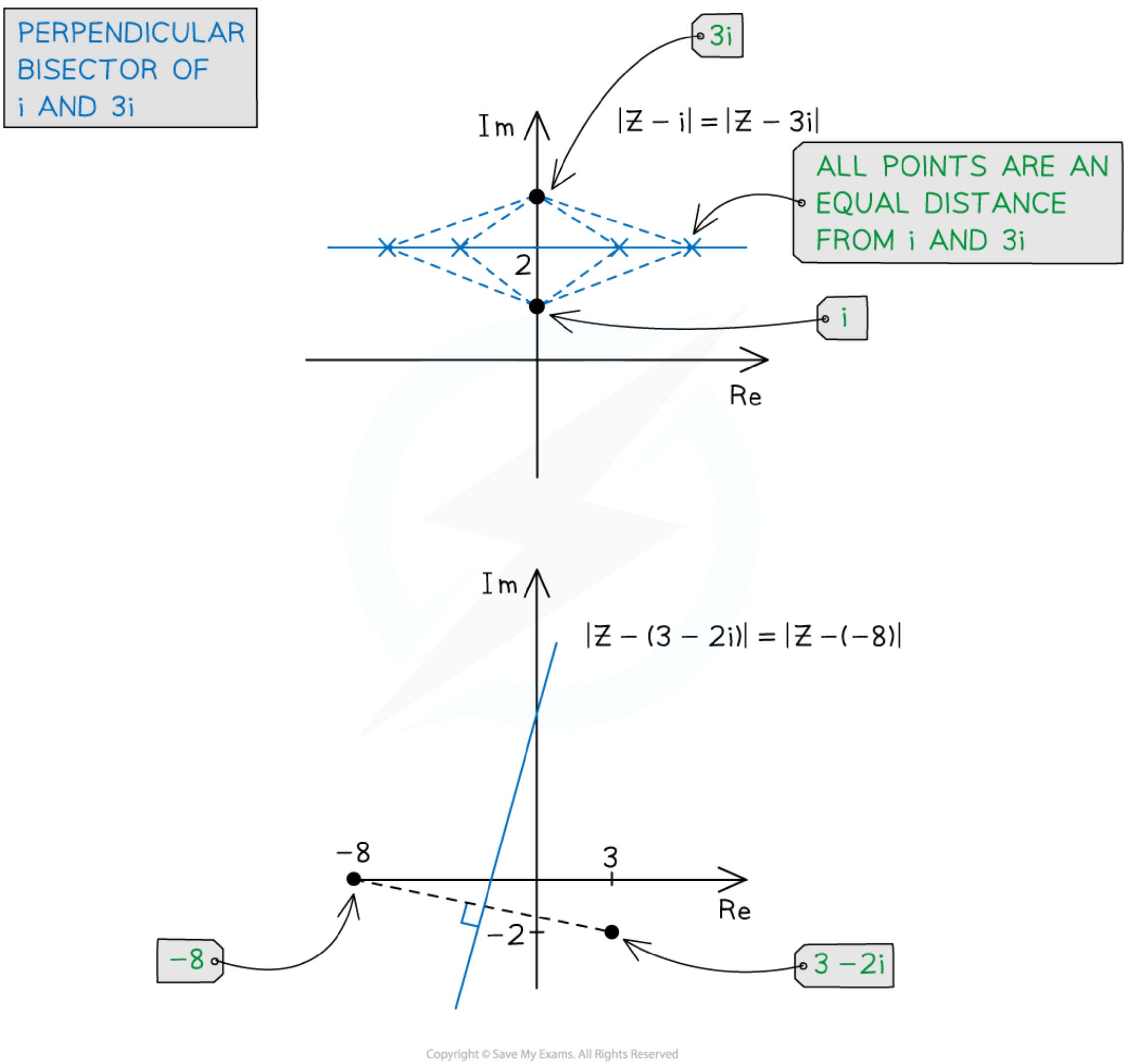

***Sketching the loci of*** $|z|=10$ ***and*** $|z-(3+4i)|=10$

### How do I sketch the locus of $|z-a|=|z-b|$ on an Argand diagram?

- For two given complex numbers, $a$ and $b$, all complex numbers, $z$, that satisfy the equation $|z-a|=|z-b|$ lie on the **perpendicular bisector** of $a$ and $b$
  - This is because the distance from $z$ to $a$ must equal the distance from $z$ to $b$
    - a condition that is satisfied by all the complex numbers, $z$, on the perpendicular bisector of $a$ and $b$

Your notes

- E.g. the locus of $|z-3+2\mathrm{i}| = |z+8|$ can be rewritten as $|z-(3-2\mathrm{i})| = |z-(-8)|$ which is the perpendicular bisector of the points $3-2\mathrm{i}$ and $-8$
- A **sketch** of the perpendicular bisector is sufficient, without finding its exact equation (though this could be found using coordinate geometry methods)




***Sketching the loci of*** $|z-\mathrm{i}| = |z-3\mathrm{i}|$ ***and*** $|z-(3-2\mathrm{i})| = |z-(-8)|$

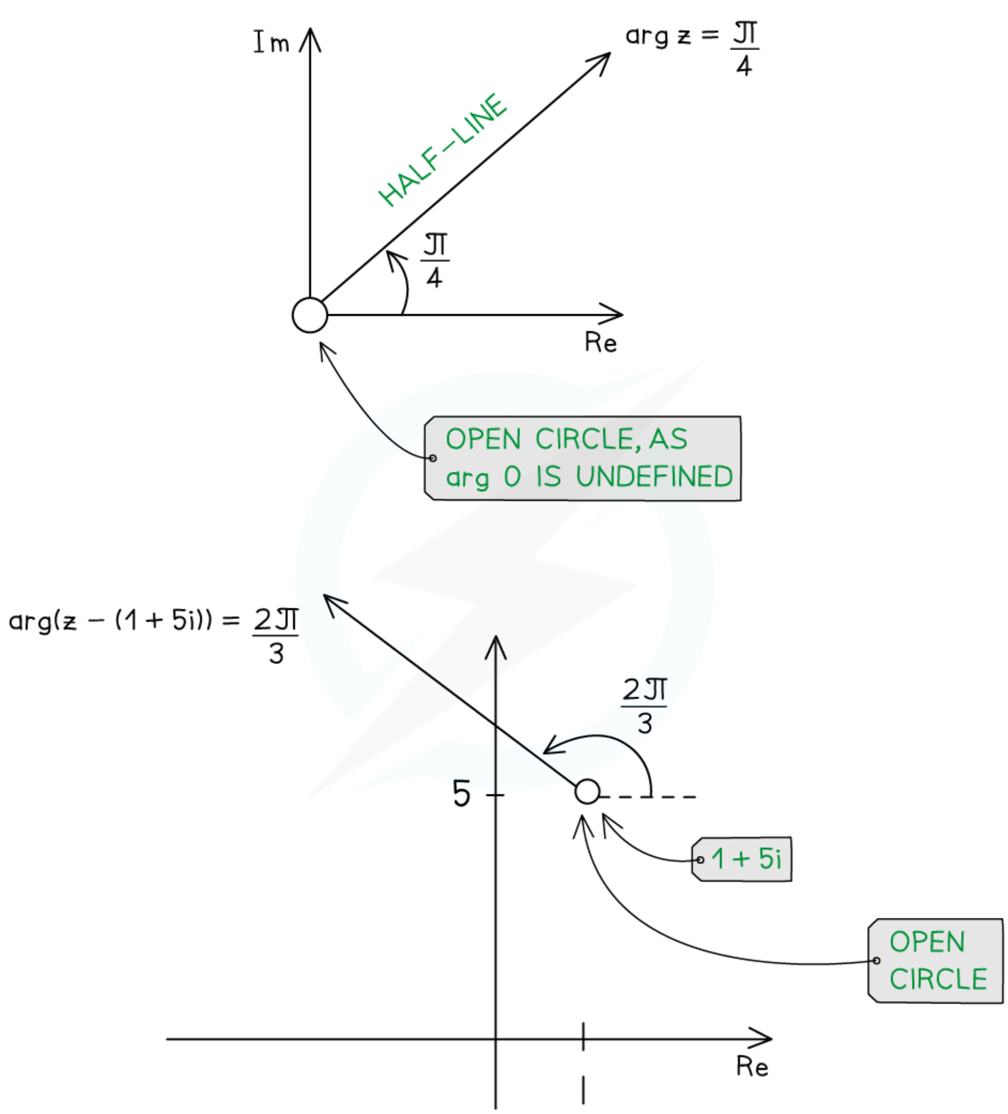
## How do I sketch the locus of $\arg\ (z-a) = \alpha$ on an Argand diagram?

- All complex numbers, $z$, that satisfy the equation $\arg\ z = \alpha$ lie on a **half-line** from the **origin** at an angle of $\alpha$ to the positive real axis

- Although the half-line starts at the origin, the origin itself ($z = 0$) does not satisfy the equation $\arg z = \alpha$ as $\arg 0$ is undefined (there is no angle at the origin)
- To show the **exclusion** of $z = 0$ from the locus of $\arg z = \alpha$, a small **open circle** at the origin is used
- E.g. the locus of $\arg z = \frac{\pi}{4}$ is a half-line of angle $\frac{\pi}{4}$ to the positive real axis, starting from the origin, with an open circle at the origin
- For a given complex number, $a$, all complex numbers, $z$, that satisfy the equation $\arg (z - a) = \alpha$ lie on a **half-line** from the **point** $a$ at an **angle** of $\alpha$ to the positive real axis, with an open circle to show the **exclusion** of $z = a$
  - E.g. the locus of $\arg(z - 1 - 5\mathrm{i}) = \frac{2\pi}{3}$ can be rewritten as $\arg(z - (1 + 5\mathrm{i})) = \frac{2\pi}{3}$, which is a half-line of angle $\frac{2\pi}{3}$ measured from the point $1 + 5\mathrm{i}$, with an open circle at $1 + 5\mathrm{i}$ to show its exclusion
- In some cases, the **equation** of the half-line can be found using a **sketch** to help
  - E.g. the locus of $\arg z = \frac{\pi}{4}$ is the half-line $y = x$ for $x > 0$
  - E.g. the locus of $\arg(z - (8 + 5\mathrm{i}) = -\frac{\pi}{4}$ can be thought of, in coordinate geometry, as the half-line through $(8, 5)$ with gradient -1, giving $y = -x + 13$ for $x > 8$
  - Whilst **not examinable**, the half-line equation for a more general angle, $\arg z = \alpha$, is $y = (\tan \alpha)x$ for $x > 0$, as the $\text{gradient} = \frac{\text{opposite}}{\text{adjacent}} = \tan \alpha$

Your notes

Your notes






*Sketching the loci of* $\arg\ z = \dfrac{\pi}{4}$ *and* $\arg(z-(1+5\mathrm{i})) = \dfrac{2\pi}{3}$

### Examiner Tip

- In the exam, do not worry about making your diagrams perfect.
- A quick sketch with all the key features is sufficient.

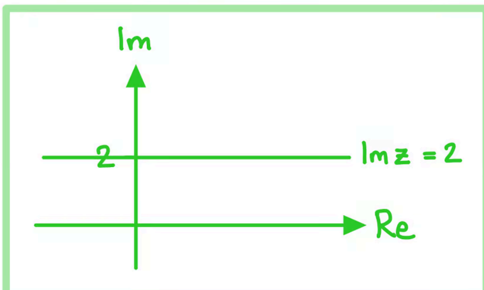
Your notes

## Worked example

On separate axes, sketch the locus of points representing complex numbers, $z$, that satisfy the following equations:

a) $\text{Im}\, z = 2$

A horizontal line through 2 on the imaginary axis.




b) $|z + 5 - 12\text{i}| = 8$

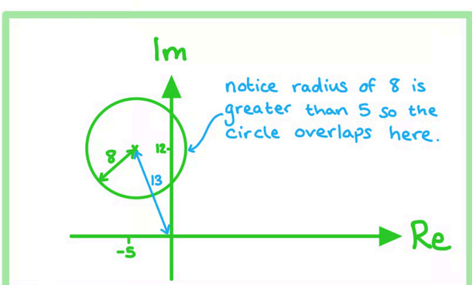
$|z + 5 - 12i| = 8$ Rearrange into the form $|z - a|$

$|z - (-5 - 12i)| = 8$ A circle of radius 8 with centre $-5 + 12i$

Distance from centre to origin is $\sqrt{(-5)^2 + (12)^2}$ $(= \sqrt{169})$

Check if origin is inside or outside of circle:

Radius is $8 < 13$ $\therefore$ origin is outside of circle.



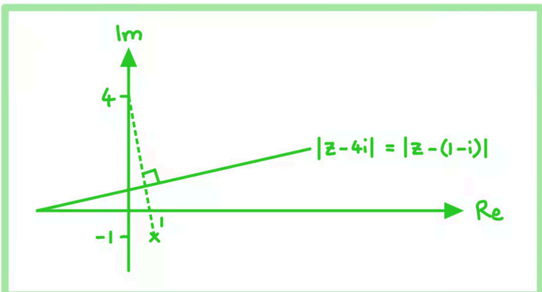

c) $|z - 4\text{i}| = |z + \text{i} - 1|$

Your notes

Rearrange $|z + i - 1|$ into the form $|z - a|$

$|z - 4i| = |z - (1 - i)|$

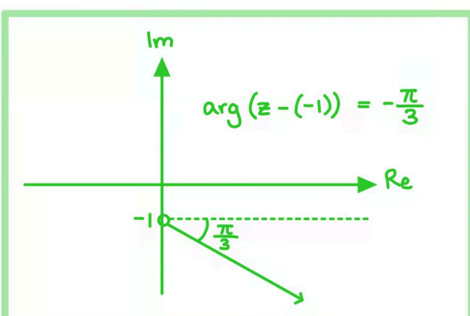
Need to draw the perpendicular bisector of $4i$ and $1 - i$




d) $$\arg(z + i) = -\frac{\pi}{3}$$

Rearrange $\arg(z+i)$ into the form $\arg(z-a)$

$\arg(z - (-i)) = -\frac{\pi}{3}$




- Include angle clearly on diagram
- Use an open circle to exclude $z = -i$ (which will give arg0 which is undefined)
- Include an arrowhead to indicate the half-line continues this way

Your notes

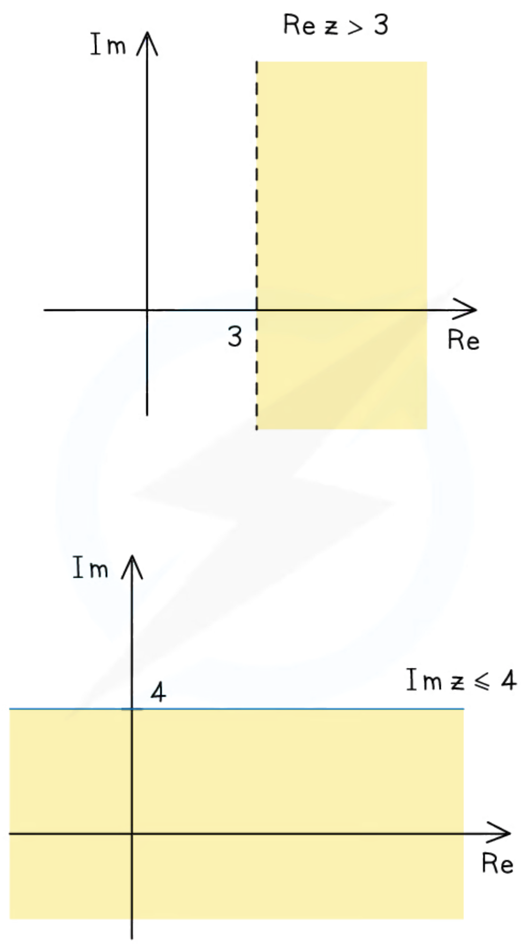
# 1.1.6 Regions in Argand Diagrams

## Inequalities & Regions in Argand Diagrams

### How do I sketch inequalities such as $\text{Re } z < k$ or $\text{Im } z < k$ on an Argand diagram?

- The inequality $\text{Re } z < 2$ is satisfied by all complex numbers, $z = x + \mathrm{i}y$, with a real part less than 2
  - On an Argand diagram, this would be the **region** to the left of the vertical line $x = 2$
  - The vertical line itself, $x = 2$, should be drawn dotted to show that points on this line are not permitted due to the strict inequality (<)
- In general, **dotted** lines are used with **strict** inequalities ( < or > ) and **solid** lines are used with inequalities that **can be equal** ( ≤ or ≥ )
- Here is how to represent the following inequalities as regions on an Argand diagram...
  - $\text{Re } z < k$: **shade** the region to the **left** of the **dotted** vertical line (or **solid** vertical line for ≤)
  - $\text{Re } z > k$: **shade** the region to the **right** of the **dotted** vertical line (or **solid** vertical line for ≥)
- Similarly, here is how to represent the following inequalities as regions on an Argand diagram...
  - $\text{Im } z < k$: **shade** the region **below** the **dotted** horizontal line (or **solid** horizontal line for ≤)
  - $\text{Im } z > k$: **shade** the region **above** the **dotted** horizontal line (or **solid** horizontal line for ≥)

Your notes



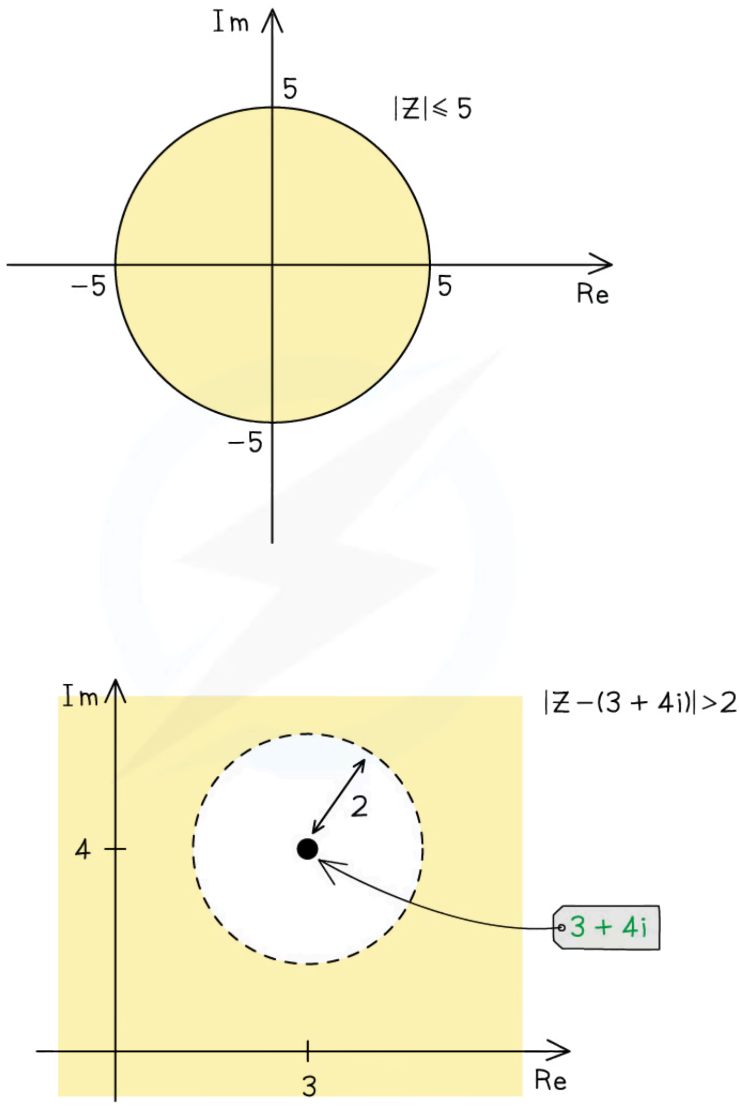



*Sketching the inequalities* $\text{Re } z > 3$ *and* $\text{Im } z \le 4$

## How do I sketch inequalities such as $|z - a| < k$ on an Argand diagram?

- For a given complex number $a$, the inequality $|z - a| < k$ represents all complex numbers, $z$, that lie inside the circle of radius $k$, centred at $a$
- Here is how to represent the following inequalities as regions on an Argand diagram...
  - $|z - a| < k$: **shade** the region **inside** the circle of radius $k$, centred at $a$
  - $|z - a| > k$: **shade** the (unbounded) region **outside** circle of radius $k$, centred at
  - Again use a **dotted** line for strict inequalities (< and >) and a **solid** line for weak inequalities ( ≤ or ≥ )

Your notes



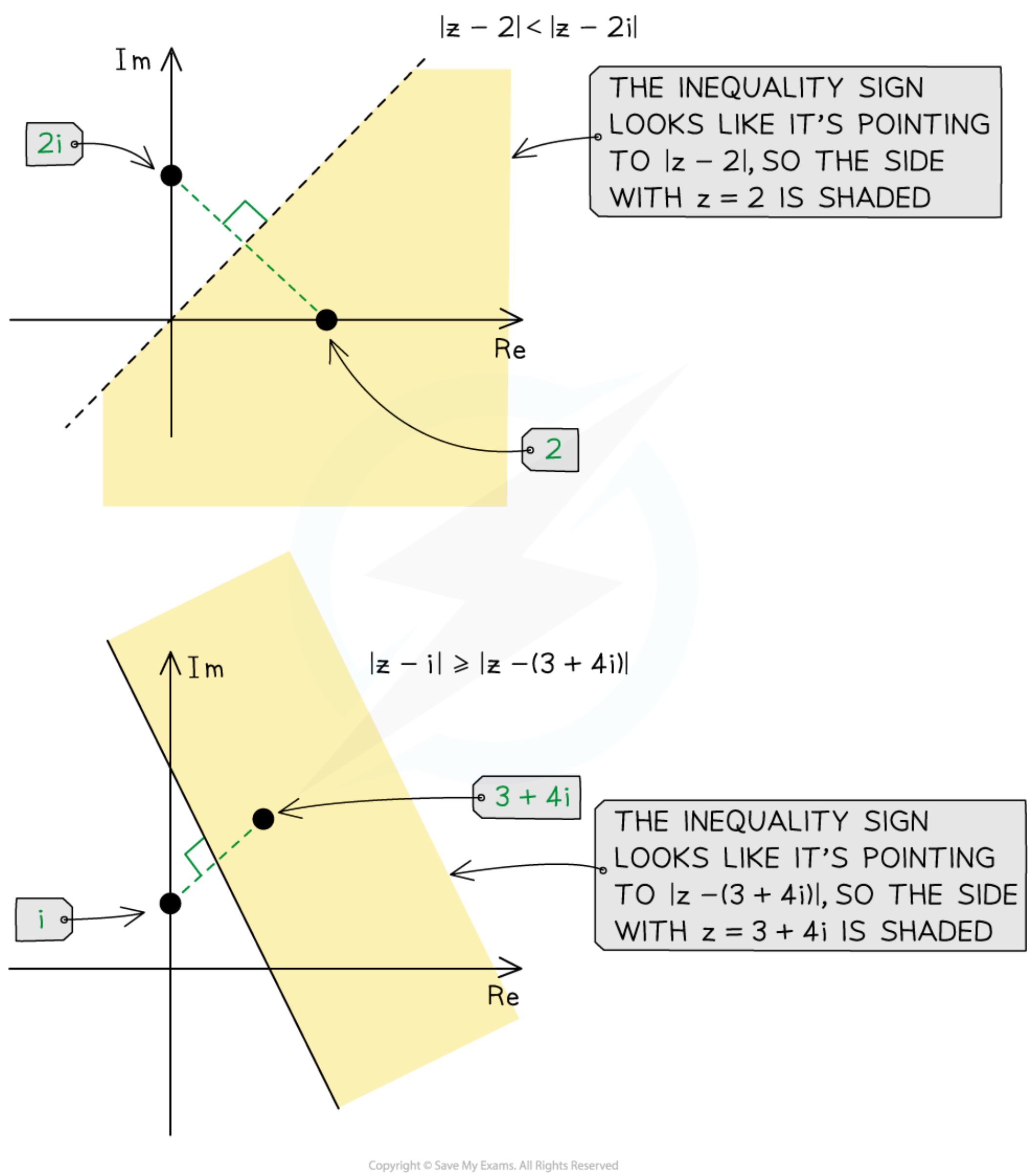



***Sketching the inequalities*** $|z| \leq 5$ ***and*** $|z-(3+4\mathrm{i})| > 2$

### How do I sketch inequalities such as $|z-a| < |z-b|$ on an Argand diagram?

- For two given complex numbers $a$ and $b$, the inequality $|z-a| < |z-b|$ represents all complex numbers, $z$, that lie closer to $a$ than to $b$
- Here is how to represent the following inequalities as regions on an Argand diagram...
  - $|z-a| < |z-b|$: **shade** the **region** on the side of the perpendicular bisector of $a$ and $b$ that **includes the point** $a$

- $|z - a| > |z - b|$: **shade** the **region** on the side of the **dotted** perpendicular bisector of $a$ and $b$ that **includes the point** $b$
- A good way to remember which side to shade is that the inequality sign points (like an arrow) to the side to be shaded



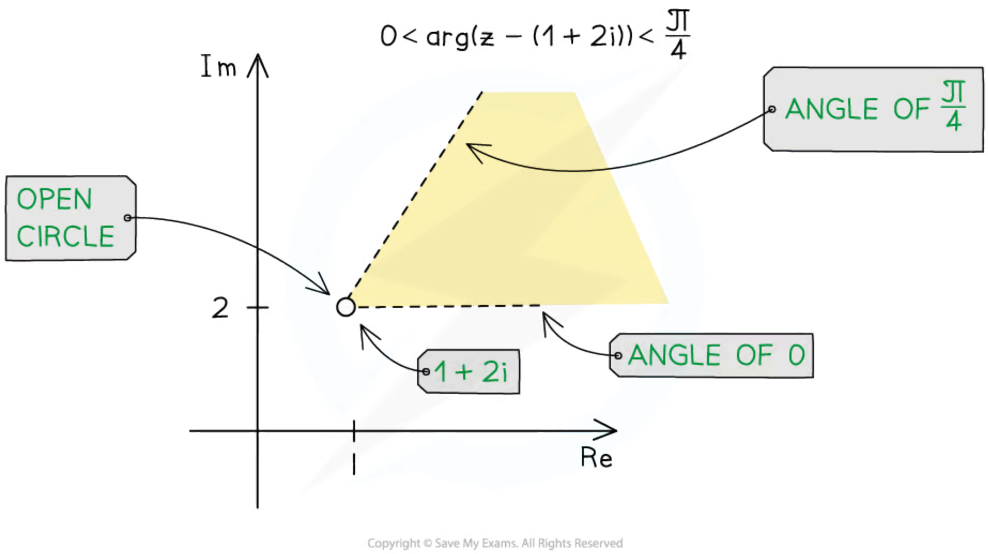

*Sketching the inequalities* $|z - 2| < |z - 2i|$ *and* $|z - i| \geq |z - (3 + 4i)|$

**How do I sketch inequalities such as $\alpha < \arg(z - a) < \beta$ on an Argand diagram?**

Your notes

Your notes

- For a given complex number, $a$, the inequality $\alpha < \arg(z-a) < \beta$ represents all complex numbers, $z$, that have an **argument** between $\alpha$ and $\beta$, as **measured from** the point $a$
- Here is how to represent the following inequalities as regions on an Argand diagram…
  - $\alpha < \arg(z-a) < \beta$: **shade** the **wedge-shaped region** between the half-line $\arg(z-a) = \alpha$ and the half-line $\arg(z-a) = \beta$, with an open circle at $z = a$ to show the **exclusion** of this point (to avoid the undefined value of $\arg 0$)



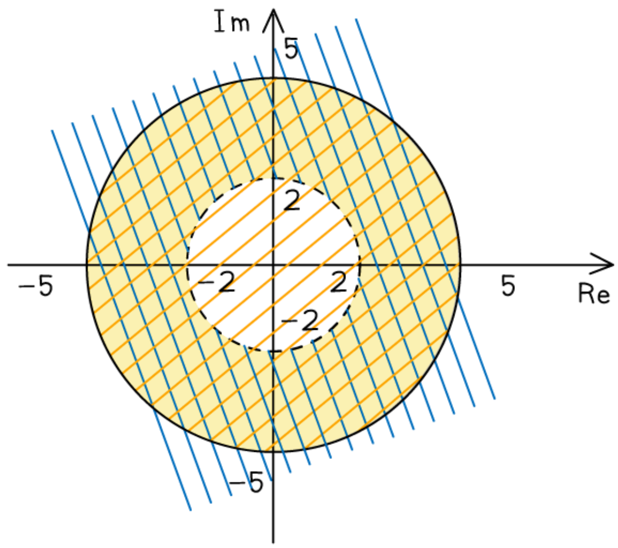

*Sketching the inequalities* $0 < \arg(z-(1+2i)) < \dfrac{\pi}{4}$

### How do I draw multiple inequalities on the same Argand diagram?

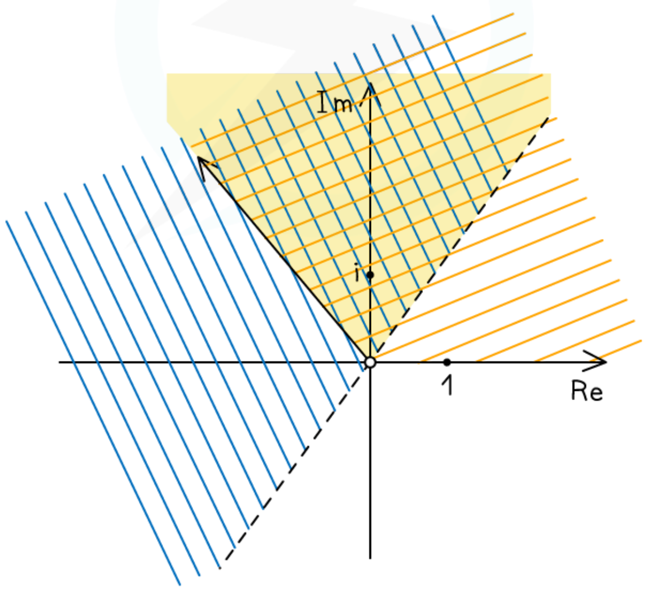
- To sketch a region that satisfies multiple inequalities, we need to find the **intersection** of all the regions (where all regions overlap)
  - E.g. the region satisfied by $2 < |z| \leq 5$ can be found as follows…
    - Using diagonal lines in the same direction, lightly shade the region $|z| > 2$ (the outside of a circle of radius 2 about the origin)
    - Using diagonal lines in a different direction, lightly shade the region $|z| \leq 5$ (the inside of a circle of radius 5 about the origin)
    - Where the diagonal lines **cross each other** highlights the region satisfying both inequalities; this should be shaded **boldly**
  - E.g. the region satisfied by $|z-i| < |z-1|$ and $0 \leq \arg z \leq \dfrac{3\pi}{4}$ can be found as follows…
    - Using diagonal lines in the same direction, lightly shade the region $|z-i| < |z-1|$ (the side including $z = i$ from the perpendicular bisector of the points $i$ and 1)

Your notes

- Using diagonal lines in a different direction, lightly shade the region $0 \leq \arg z \leq \frac{3\pi}{4}$ (the wedge-shaped region from 0 to $\frac{3\pi}{4}$ radians from the origin)
- Where the diagonal lines **cross each other** highlights the region satisfying both inequalities; this should be shaded **boldly**




$|z| > 2$:
$|z| \leqslant 5$:
OVERLAP
$(2 < |z| \leqslant 5)$:




$|z - i| < |z - 1|$:
$0 \leqslant \arg z \leqslant \dfrac{3\pi}{4}$:
OVERLAP:

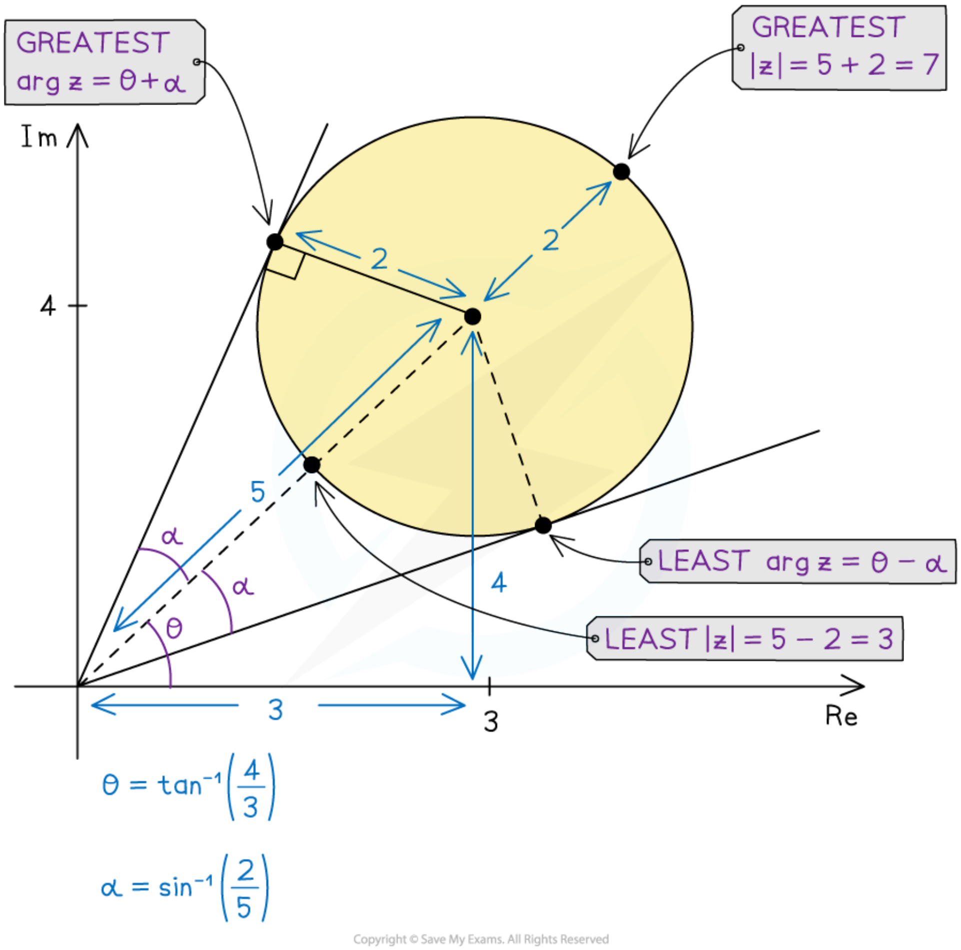
Your notes

*The inequalities* $2 < |z| \le 5$ *on the left, the inequalities* $|z - i| < |z - 1|$ *and* $0 \le \arg z < \frac{3\pi}{4}$ *on the right*

## How do I find the greatest (or least) values of $|z|$ or $\arg z$ in a region?

- Every complex number, $z$, that lies within a region on an Argand diagram will have its own modulus, $|z|$, that is measured from the origin
- For any shaped region…
  - The **least** value of $|z|$ is the **distance** from the **origin** to the **nearest point** in the region
  - The **greatest** value of $|z|$ is the **distance** from the **origin** to the **farthest point** in the region
  - Sometimes the least value is zero (if the origin is in the region) and/or the greatest value is infinite (if the region is unbounded)
- For circular regions that do not contain the origin…
  - Find the length from the origin to the centre of the circle…
  - then add a radius for the greatest value of $|z|$ or subtract a radius for the least value of $|z|$
- Every complex number, $z$, that lies within a region on an Argand diagram will have its own argument, $\arg z$, as measured from the origin
- For any shaped region…
  - The **least** value of is $\arg z$ the **smallest angle** a line through the origin can make to a point in the region
  - The **greatest** value of $\arg z$ is the **largest angle** a line through the origin can make to a point in the region
- For circular regions that do not contain the origin…
  - Find $\theta$, the argument of the centre…
  - Find $\alpha$, the angle between a tangent to the circle through the origin and the line from the origin to the centre…
    - This can be done using trigonometry, as a radius meets a tangent at right angles
  - The greatest value of $\arg z$ is $\theta + \alpha$ and the least value of $\arg z$ is $\theta - \alpha$

Your notes




***Find greatest and least values of*** $|z|$ ***and*** $\arg z$

## Examiner Tip

- When shading an unbounded region in the exam, make sure you extend your shading outwards, crossing any axes where necessary.
- When shading multiple inequalities to find a common region, lots of methods are accepted; these include lightly shading individual regions to see where they overlap, or starting to shade along the boundaries of individual regions to find the common region, or marking individual regions to find the region with multiple marks, etc.

Your notes

## Worked example

a) On separate Argand diagrams, shade the region whose points represent complex numbers $z$ satisfying

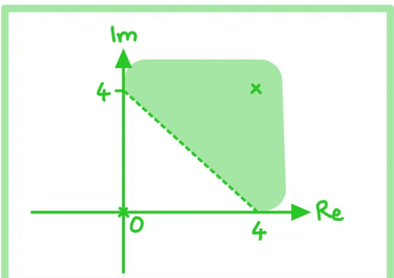
(i) the inequality $|z - 4 - 4\mathrm{i}| < |z|$

(ii) both inequalities $|z + 1 - \mathrm{i}| < 1$ and $-\dfrac{\pi}{4} \leq \arg(z + 2 - \mathrm{i}) \leq \dfrac{\pi}{4}$

(i) Rearrange $|z-4-4i|$ into the form $|z-a|$

$|z-(4+4i)| < |z|$

Need to draw the perpendicular bisector of $4+4i$ and $0$

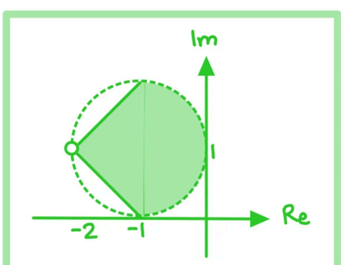



• Shade the side with the point $(4+4i)$ in

(ii) Rearrange $|z+1-i|$ and $(z+2-i)$ into the form $|z-a|$

$|z-(-1+i)| < 1$ will be a circle, radius 1, centre $-1+i$

$-\frac{\pi}{4} \leq \arg(z-(-2+i)) \leq \frac{\pi}{4}$ will be a wedge region from $-2+i$ between $\pm\frac{\pi}{4}$




• Use a dotted circumference for the $<$ inequality

• Use an open circle to exclude $z=-2+i$

• Use solid half lines at angles $\pm\frac{\pi}{4}$

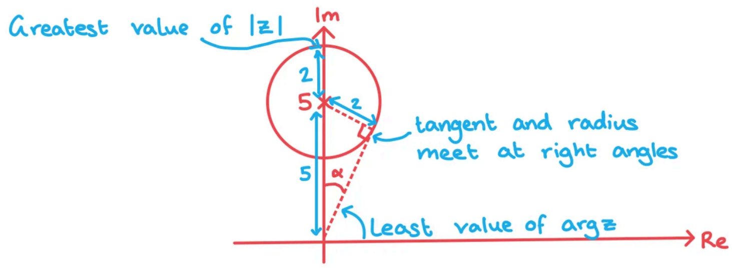
b) For complex numbers $z$ satisfying the inequality $|z - 5\mathrm{i}| \leq 2$, find

(i) the greatest value of $|z|$

(ii) the least value of $\arg(z)$

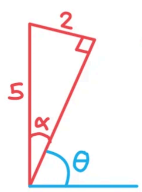
Draw a sketch of $|z-5i| \leq 2$ to help:

Circle, radius 2, centre 5i




(i) $\boxed{\text{Greatest value of } |z| \text{ is } 7}$

(ii)




$$\sin\alpha = \frac{2}{5} \Rightarrow \alpha = \sin^{-1}\left(\frac{2}{5}\right)$$

$$= 0.4115...$$

$$\theta = \frac{\pi}{2} - 0.4115... = 1.159...$$

$\boxed{\text{Least value of arg } z \text{ is } 1.16 \text{ radians } (3 \text{ s.f.})}$

Your notes