Linear Transformations with Matrices

1:: Use of matrices to represent linear transformations.

"Determine the matrix that represents the transformation $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2x+y \\ -x \end{pmatrix}$"

2:: Use matrices to represent reflections, rotations (about the origin) and enlargements.

"Describe the geometrical transformation represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$"

3:: Carry out successive transformations using matrix products.

"If $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ describe the transformation represented by the matrix $\mathbf{AB}$."

4:: Use inverse matrices to represent reverse transformations.

A matrix $\begin{pmatrix} 4 & 0 \\ -1 & 2 \end{pmatrix}$ is used to transform a point $A(x, y)$ to $B(5,5)$. Determine the point $A(x, y)$.

Linear Transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

This chapter is concerned with how we can **use matrices to represent some transformation of a point** $(\boldsymbol{x}, \boldsymbol{y})$ (written as a position vector $\begin{pmatrix} x \\ y \end{pmatrix}$).

Transforming a point $\begin{pmatrix} x \\ y \end{pmatrix}$ simply involves multiplying it by some matrix. From above we can see that multiplying by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the mapping $T\colon \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. We will see how we can use certain matrices to represent certain well-known transformations, e.g. $(x, y) \to (3x, 3y)$, i.e. an enlargement of scale factor 3 centred about the origin.

$ax + by$ is known as a **linear combination** of x and y (an algebraic form we saw in Pure Year 1 straight line equations). **<u>Each row</u>** of the matrix we're multiplying by provides an instruction of how to generate each dimension of the new coordinate system, in terms of the old dimensions x, y...

e.g. given $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x + y \end{pmatrix}$, the new x value is $2x + 3y$ and the new y value is $x + y$, i.e. linear combinations of the old x and y values.

A function $f(\boldsymbol{a})$, where $\boldsymbol{a}$ is a vector, is linear if it has the following properties:

- $f(k\boldsymbol{a}) = kf(\boldsymbol{a})$ for a constant k, i.e. scaling the original vector scales the image vector.
- $f(\boldsymbol{a} + \boldsymbol{b}) = f(\boldsymbol{a}) + f(\boldsymbol{b})$

It is possible to prove that $f\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = ax + by$ is linear, i.e. satisfies the above restrictions.

1 We can represent a translation, e.g. $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+4 \\ y \end{pmatrix}$ using a matrix.

True | False

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

2 Matrices can represent transformations which increase or decrease the number of dimensions (e.g. transform a 3D point to get a 2D point).

True | False

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

3 The origin is unaffected by any linear transformation.

True | False

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

1 We can represent a translation, e.g. $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+4 \\ y \end{pmatrix}$ using a matrix.

True False

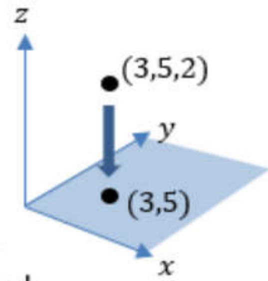
Matrices can represent any linear transformation, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$,
But $x + 4$ can't be written as $ax + by$.

2 Matrices can represent transformations which increase or decrease the number of dimensions (e.g. transform a 3D point to get a 2D point).

True False

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

This is a transformation which takes a 3D point and discards the z-value, i.e. projects a point into the x-y plane. This is relevant to 3D animation, where we need to generate a 2D image from a 3D world.



3 The origin is unaffected by any linear transformation.

True False

A linear transformation (in 2D) is $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. Thus

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0a+0b \\ 0c+0d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Find matrices to represent these linear transformations.

a) $T: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y + x \\ 3x \end{pmatrix}$

b) $V: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -2y \\ 3x + y \end{pmatrix}$

A square has coordinates $(1,1)$, $(3,1)$, $(3,3)$ and $(1,3)$. Find the vertices of the image of S under the transformation given by the matrix $\boldsymbol{M} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Sketch S and the image of S on a coordinate grid.

original → object
new → image

a) $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 0y \end{pmatrix}$

$M = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$

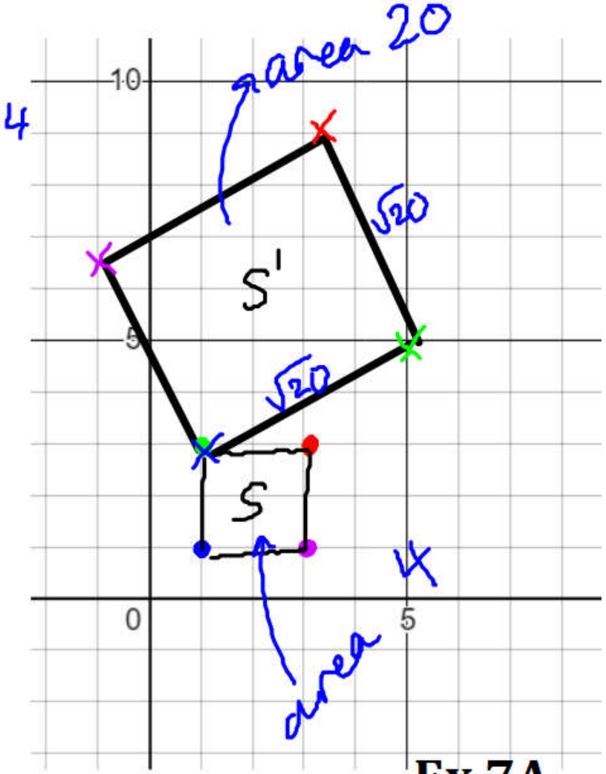
b) $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0x - 2y \\ 3x + y \end{pmatrix}$

$M = \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$

$\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 & 5 \\ 3 & 7 & 9 & 5 \end{pmatrix}$

$\left|\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\right| = -1 - 4$

$= -5$



Ex 7A

Determining a matrix for a transformation

Recall from vectors that $\boldsymbol{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix}0\\1\end{pmatrix}$ are the unit vectors representing the x and y directions. Consider what happens to each when we multiply by a matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$:

> **Just For Your Interest:** i and j are known as the *basis vectors* of the 2D coordinate space because any 2D point can be represented as a linear combination of these basis vectors, i.e. $\begin{pmatrix}x\\y\end{pmatrix} = x\boldsymbol{i} + y\boldsymbol{j}$

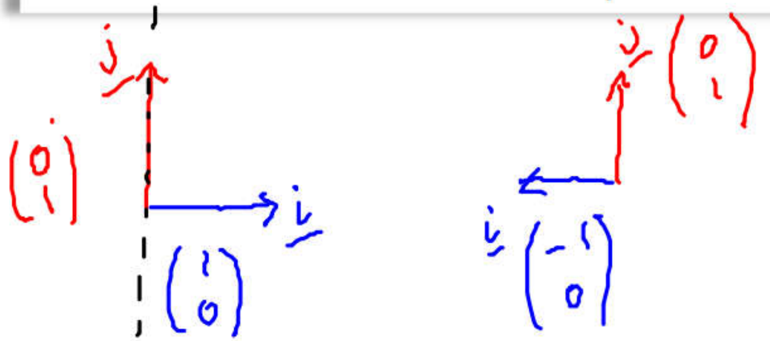
$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \qquad \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

What can we conclude about the columns of a matrix?

$\begin{pmatrix}a & b\\c & d\end{pmatrix}$ $\begin{pmatrix}a\\c\end{pmatrix}$ represents where $\underline{i}$ is transformed.

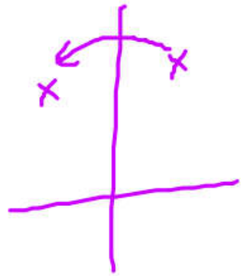
$\begin{pmatrix}b\\d\end{pmatrix}$ represents where $\underline{j}$ is transformed.

"Find a 2 × 2 matrix that represents a reflection in the y-axis."



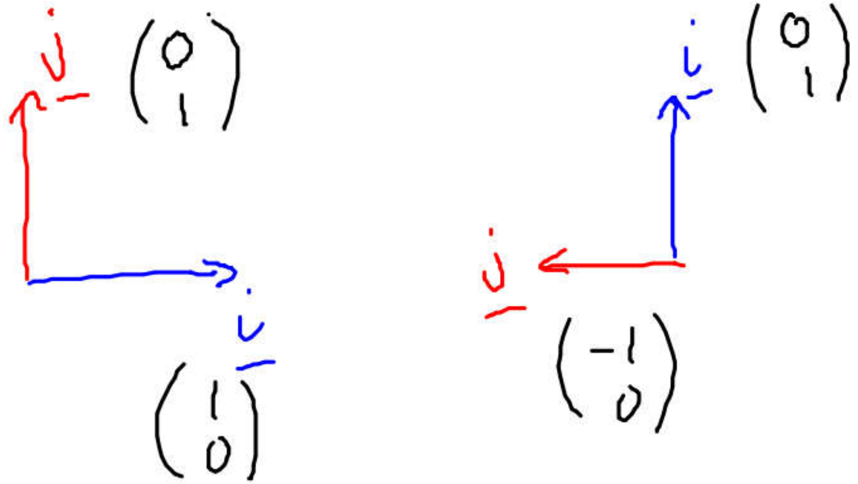
$$M = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}-3\\4\end{pmatrix}$$

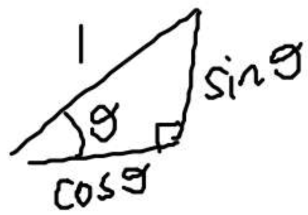


Rotation 90° about the origin.

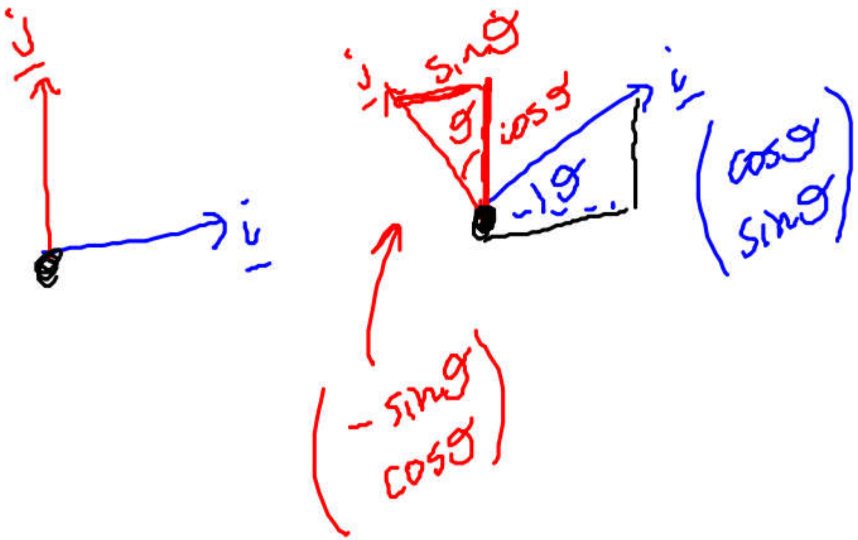
Note: Rotations by default are anticlockwise.



$$\begin{array}{cc} \text{new } \underline{i} & \text{new } \underline{j} \\ \end{array}$$
$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



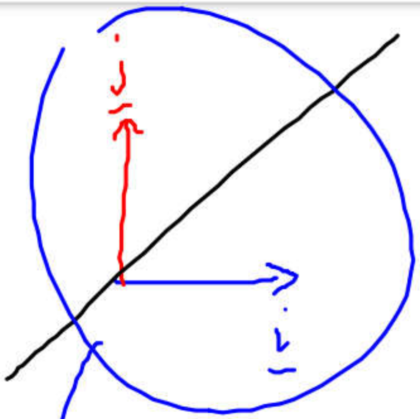
Rotation θ about the origin.

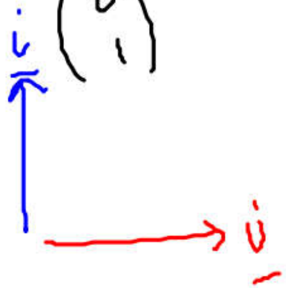


$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Your Turn

Find the matrix representing a reflection in the line $y = x$.



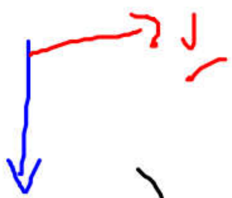


$\underline{i}$ $\begin{pmatrix}0\\1\end{pmatrix}$

$\underline{j}$ $\begin{pmatrix}1\\0\end{pmatrix}$

$M = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

Find the matrix representing a rotation by 270°.

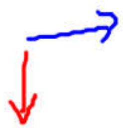


$\underline{j}$ $\begin{pmatrix}1\\0\end{pmatrix}$

$\underline{i}$ $\begin{pmatrix}0\\-1\end{pmatrix}$

$M = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$

eg $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ reflection in x-axis



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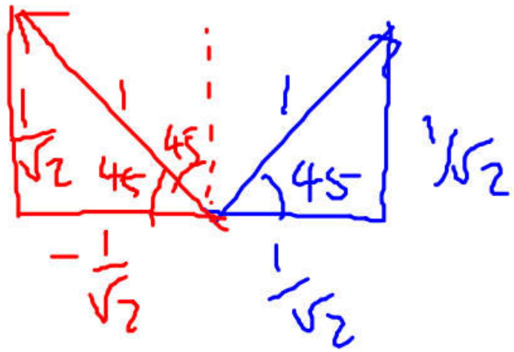
$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(*a*) Describe fully the transformations described by matrix **C**.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Rotation 45° anticlockwise about the origin

$\cos\theta = \frac{1}{\sqrt{2}}$ $\quad \sin\theta = \frac{1}{\sqrt{2}}$

$\theta = 45°$

Ex 7B