Integration (Year 1)

This chapter is roughly divided into two parts: the first, **indefinite integration**, is the **opposite of differentiation**. The second, **definite integration**, allows us to **find areas under graphs** (as well as surface areas and volumes) or areas between two graphs.

1:: Find y given $\frac{dy}{dx}$

A curve has the gradient function

$$\frac{dy}{dx} = 3x + 1$$

If the curve goes through the point (2,3), determine y.

2:: Evaluate definite integrals, and hence the area under a curve.

Find the area bounded between the curve with equation $y = x^2 - 2x$ and the *x*-axis.

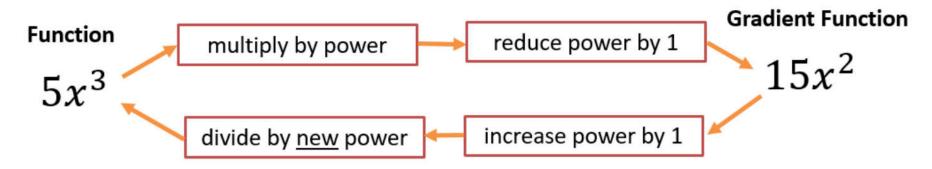
3:: Find areas bound between two different lines.

Find the points of intersection of $y = x^2 - 4x + 3$ and y = x + 9, and hence find the area bound between the two lines.

Integrating x^n terms

Integration is the opposite of differentiation.

(For this reason it is also called 'antidifferentiation')



However, there's one added complication...

Find
$$y$$
 when $\frac{dy}{dx} = 3x^2$

Adding 1 to the power and dividing by this power give us:

$$y = x^3$$

However, other functions would also have differentiated to $3x^2$:

$$y = x^3 + 1$$
, $y = x^3 - 4$, ...

Clearly we could have had any constant, as it disappears upon differentiation.

Find y when:

Exam Note: You should <u>always</u> include it for indefinite integration

$$\frac{dy}{dx} = 4x^3$$

$$y = \frac{4}{4}x^{4} = x^{4} + ce$$

$$\frac{dy}{dx} = x^5 \qquad y = \frac{1}{6}x^6 +$$

You could also write as $\frac{x^6}{6}$. It's a matter of personal preference.

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$$y = 3x \frac{2}{3}x^{3/2} + c = 2x^{3/2} + c$$

 $\left(\frac{\dot{3}}{2}\right)$

X = (multiply by

The reciporocal)

Tip: Many students are taught to write $\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}$ (as does textbook!). This is ugly and students then often struggle to simplify it. Instead remember back to GCSE: When you divide by a fraction, you multiply by the reciprocal.

Find y when:

$$\frac{dy}{dx} = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$$

$$y = 4x2x^{1/2} + 0$$

 $y = 8x^{1/2} + 0$

$$\frac{dy}{dx} = 5x^{-2}$$

$$y=-5x^{-1}+c$$

$$\frac{dy}{dx} = 4x^{\frac{2}{3}}$$

$$\frac{dy}{dx} = 10x^{-\frac{2}{7}}$$

$$\frac{dy}{dx} = \frac{4}{\sqrt{x}} = 4x^{-1/2}$$

$$y = 4x^{2}x^{2} + 0$$

$$y = 8x^{1/2} + 0$$
Tip: I recommend eventually doing this in your head when the simplification would be simple.

When we divide by $\frac{1}{2}$ we multiply by the

reciprocal, i.e. 2.

would be simple.
$$dy = 12 \times 8 \times 2^{2/3}$$

$$y = 4x3x^{5/3} + C = \frac{12}{5}x^{5/3} + C$$

$$y = 16x7 x^{5/4} + c$$

= $14x^{5/4} + c$

Your Turn

f'(x) is f(x) differentiated

Find f(x) when:

$$f'(x) = 2x + 7$$
 $f(x) = x^2 + 7x + C$
 $f'(x) = x^2 - 1$ $f(x) = \frac{1}{3}x^3 - x + C$

$$f'(x) = \frac{2}{x^7} = 2x^{-7}$$
 $f(x) = -\frac{1}{3}x^{-6} + C$

$$f'(x) = \sqrt[3]{x} = x^{1/3}$$
 $f(x) = \frac{3}{4}x^{4/3} + c$

$$f'(x) = \sqrt{3}x^{\frac{5}{6}}$$
 $f(x) = 33 \times \frac{6}{11} \times \frac{11}{6} + C = 18 \times \frac{11}{6} + C$

Note: In case you're wondering what happens if $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$, the problem is that after adding 1 to the power, we'd be dividing by 0. You will learn how to integrate $\frac{1}{x}$ in Year 2.

$$\mathbf{a} x^5$$

b
$$10x^4$$

$$c - x^{-2}$$

b
$$10x^4$$
 c $-x^{-2}$ **d** $-4x^{-3}$ **e** $x^{\frac{2}{3}}$ **f** $4x^{\frac{1}{2}}$

e
$$x^{\frac{2}{3}}$$

f
$$4x^{\frac{1}{2}}$$

$$g - 2x^6$$

h
$$x^{-\frac{1}{2}}$$

g
$$-2x^6$$
 h $x^{-\frac{1}{2}}$ **i** $5x^{-\frac{3}{2}}$ **j** $6x^{\frac{1}{3}}$ **k** $36x^{11}$ **l** $-14x^{-8}$

j
$$6x^{\frac{1}{3}}$$

$$k 36x^{11}$$

$$-14x^{-8}$$

$$\mathbf{m} - 3x^{-\frac{2}{3}}$$
 $\mathbf{n} - 5$ $\mathbf{o} 6x$ $\mathbf{p} 2x^{-0.4}$

$$p 2x^{-0.4}$$

2 Find y when $\frac{dy}{dx}$ is given by the following expressions. In each case simplify your answer.

a
$$x^3 - \frac{3}{2}x^{-\frac{1}{2}} - 6x^{-2}$$
 b $4x^3 + x^{-\frac{2}{3}} - x^{-2}$ **c** $4 - 12x^{-4} + 2x^{-\frac{1}{2}}$

b
$$4x^3 + x^{-\frac{2}{3}} - x^{-2}$$

c
$$4 - 12x^{-4} + 2x^{-\frac{1}{2}}$$

d
$$5x^{\frac{2}{3}} - 10x^4 + x^{-3}$$

$$e^{-\frac{4}{3}x^{-\frac{4}{3}}} - 3 + 8x$$

d
$$5x^{\frac{2}{3}} - 10x^4 + x^{-3}$$
 e $-\frac{4}{3}x^{-\frac{4}{3}} - 3 + 8x$ **f** $5x^4 - x^{-\frac{3}{2}} - 12x^{-5}$

3 Find f(x) when f'(x) is given by the following expressions. In each case simplify your answer.

a
$$12x + \frac{3}{2}x^{-\frac{3}{2}} + 5$$

b
$$6x^5 + 6x^{-7} - \frac{1}{6}x^{-\frac{7}{6}}$$
 c $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$$\mathbf{c} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

d
$$10x^4 + 8x^{-3}$$

e
$$2x^{-\frac{1}{3}} + 4x^{-\frac{5}{3}}$$

$$\mathbf{f} = 9x^2 + 4x^{-3} + \frac{1}{4}x^{-\frac{1}{2}}$$



(E/P) 4 Find y given that $\frac{dy}{dx} = (2x + 3)^2$.

(4 marks)

Problem-solving

Start by expanding the brackets.

Exercise 13A

1 **a**
$$y = \frac{1}{6}x^6 + c$$

$$y = x^{-1} + c$$

$$y = \frac{3}{5}x^{\frac{5}{3}} + c$$

$$y = -\frac{2}{7}x^7 + c$$

i
$$y = -10x^{-\frac{1}{2}} + c$$

$$\mathbf{k} \quad y = 3x^{12} + c$$

$$\mathbf{m} \ y = -9x^{\frac{1}{3}} + c$$

o
$$y = 3x^2 + c$$

b
$$y = 2x^5 + c$$

d
$$y = 2x^{-2} + c$$

$$\mathbf{f} \quad y = \frac{8}{3}x^{\frac{3}{2}} + c$$

h
$$y = 2x^{\frac{1}{2}} + c$$

$$y = \frac{9}{2}x^{\frac{4}{3}} + c$$

1
$$y = 2x^{-7} + c$$

$$\mathbf{n} \quad y = -5x + c$$

$$\mathbf{p} \quad y = \frac{10}{3}x^{0.6} + c$$

$$x^2 = \frac{1}{3}x^3 + C$$

$$\frac{dy}{dx} = x^2$$

$$y = \frac{1}{3}x^3 + c$$

2 **a**
$$y = \frac{1}{4}x^4 - 3x^{\frac{1}{2}} + 6x^{-1} + c$$
 b $y = x^4 + 3x^{\frac{1}{2}} + x^{-1} + c$

$$y = x^2 + 5x^3 + x^2 + c$$

c
$$y = 4x + 4x^{-3} + 4x^{\frac{1}{2}} + c$$
 d $y = 3x^{\frac{5}{3}} - 2x^{5} - \frac{1}{2}x^{-2} + c$

$$\mathbf{f} \quad y = r^5 + 2r^{-\frac{1}{2}} + 3r^{-4} + c$$

e
$$y = 4x^{-\frac{1}{3}} - 3x + 4x^2 + c$$
 f $y = x^5 + 2x^{-\frac{1}{2}} + 3x^{-4} + c$

3 **a**
$$f(x) = 6x^2 - 3x^{-\frac{1}{2}} + 5x + c$$
 b $f(x) = x^6 - x^{-6} + x^{-\frac{1}{6}} + c$

c
$$f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + c$$

c
$$f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + c$$
 d $f(x) = 2x^5 - 4x^{-2} + c$

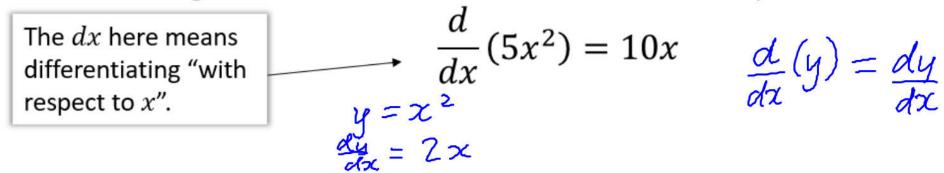
e
$$f(x) = 3x^{\frac{2}{3}} - 6x^{-\frac{2}{3}} + c$$

$$\mathbf{f} \quad \mathbf{f}(x) = 3x^3 - 2x^{-2} + \frac{1}{2}x^{\frac{1}{2}} + c$$

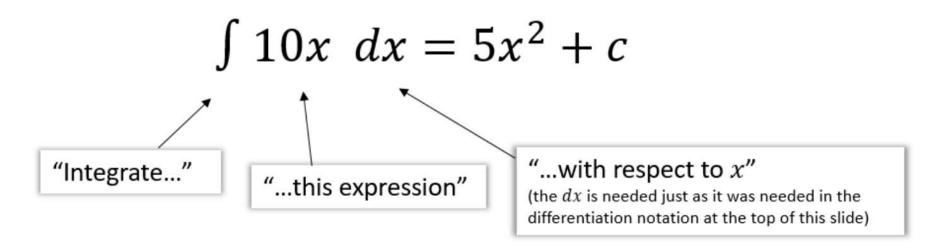
4
$$y = \frac{4x^3}{3} + 6x^2 + 9x + c$$

Integration notation

The following notation could be used to differentiate an expression:



There is similarly notation for integrating an expression:



This is known as **indefinite integration**, in contrast to definite integration, which we'll see later in the chapter.

It is called 'indefinite' because the exact expression is unknown (due to the +c).

Note: The brackets are required if there's multiple terms.

Find
$$\int (x^{-\frac{3}{2}} + 2) dx$$

$$\int (x^{-3/2} + 2) dx = -2x^{-1/2} + 2x + c$$

Find
$$\int (6t^2 - 1) dt$$

Note the dt instead of dx.

$$\int (6t^2 - 1) dt = 2t^3 - t + c$$

Find
$$\int (px^3 + q) dx$$
 where p and q are constants. In a number of $\int (px^3 + q) dx = \int (px^3 + q) dx = \int (px^4 + q) dx + C$

$$\int (px^4 + q) dx = \int (px^4 + q) dx + C$$

Edexcel C1 May 2014(R) Q4b

Given that $y = 2x^5 + \frac{6}{\sqrt{x}}$, x > 0, find in their simplest form

(b)
$$\int y \, dx$$
 (3)

$$\int y \, dx = \int (2x^5 + 6x^{-1/2}) \, dx$$
$$= \frac{1}{3}x^6 + 12x^{1/2} + C$$

Ex 13B

Q4,5,6,7

a
$$\int (4x^3 - 3x^{-4} + r) dx$$

b
$$\int (x + x^{-\frac{1}{2}} + x^{-\frac{3}{2}}) dx$$

Hint In Q4 part c you are integrating with respect to x, so treat p and t as constants.

c
$$\int (px^4 + 2t + 3x^{-2})dx$$

5 Find the following integrals:

a
$$\int (3t^2 - t^{-2}) dt$$

b
$$\int (2t^2 - 3t^{-\frac{3}{2}} + 1) dt$$

c
$$\int (pt^3 + q^2 + px^3) dt$$

6 Find the following integrals:

$$\mathbf{a} \int \frac{(2x^3 + 3)}{x^2} \, \mathrm{d}x$$

b
$$\int (2x+3)^2 dx$$

$$x^{1/2}(2x+3)$$

c $\int (2x+3)\sqrt{x} dx$
 $2x^{3/2} + 3x^{1/2}$

a
$$\int \frac{(2x^3 + 3)}{x^2} dx$$
 b $\int (2x + 3)^2 dx$

7 Find $\int f(x) dx$ when $f(x)$ is given by the following:

a $\left(x + \frac{1}{x}\right)^2$ b $(\sqrt{x} + 2)^2$

$$\left(x + \frac{1}{x}\right)^2$$

b
$$(\sqrt{x} + 2)^2$$

c
$$\left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right)$$

$$\frac{1}{2}$$
 $\frac{3}{1}$ $\frac{3}$

Each term must be in the form kxn before integrating!

Homework Questions

6 Find the following integrals:

$$\mathbf{a} \int \frac{(2x^3 + 3)}{x^2} \, \mathrm{d}x$$

b
$$\int (2x+3)^2 dx$$

c
$$\int (2x+3)\sqrt{x} \, dx$$

7 Find $\int f(x)dx$ when f(x) is given by the following:

$$\mathbf{a} \left(x + \frac{1}{x}\right)^2$$

b
$$(\sqrt{x} + 2)^2$$

c
$$\left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right)$$

8 Find the following integrals:

$$\mathbf{a} \int \left(x^{\frac{2}{3}} + \frac{4}{x^3}\right) \mathrm{d}x$$

$$\mathbf{b} \int \left(\frac{2+x}{x^3} + 3\right) \mathrm{d}x$$

c
$$\int (x^2 + 3)(x - 1) dx$$

$$\mathbf{d} \int \frac{(2x+1)^2}{\sqrt{x}} \mathrm{d}x$$

$$e^{\int \left(3 + \frac{\sqrt{x} + 6x^3}{x}\right) dx}$$

$$\mathbf{f} \quad \int \sqrt{x} (\sqrt{x} + 3)^2 \, \mathrm{d}x$$

9 Find the following integrals:

$$a \int \left(\frac{A}{x^2} - 3\right) dx$$

$$\mathbf{b} \int \left(\sqrt{Px} + \frac{2}{x^3} \right) \mathrm{d}x$$

$$\mathbf{c} \int \left(\frac{p}{x^2} + q\sqrt{x} + r\right) \mathrm{d}x$$

10 Given that $f(x) = \frac{6}{x^2} + 4\sqrt{x} - 3x + 2$, x > 0, find $\int f(x) dx$.