Edexcel A Level Further Maths: Core Pure

Your notes

6.2 Vector Planes

Contents

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6.2.1 Equations of planes

Equation of a Plane in Vector Form

How do I find the vector equation of a plane?

- A plane is a flat surface which is two-dimensional
  - Imagine a flat piece of paper that continues on forever in both directions
- A plane in often denoted using the capital Greek letter *Π*
- The vector form of the equation of a plane can be found using **two direction vectors** on the plane
  - The direction vectors must be
    - **parallel** to the plane
    - **not parallel** to each other
    - therefore they will **intersect** at some point on the plane
- The formula for finding the **vector equation** of a plane is
  - $\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{b} + t\boldsymbol{c}$
    - Where ***r*** is the **position vector** of any point on the plane
    - ***a*** is the **position vector** of a known point on the plane
    - ***b*** and **c** are two **non-parallel direction** (displacement) **vectors** parallel to the plane
    - s and t are scalars
- The formula can also be written as
  - $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
    - Where ***r*** is the **position vector** of any point on the plane
    - ***a, b, c*** are the **position vectors** of known points on the plane
    - λ and μ are scalars
  - These formulae are **given in the formula booklet** but you must make sure you know what each part means
- As ***a*** could be the position vector of **any** point on the plane and ***b*** and **c** could be **any non-parallel** direction vectors on the plane there are infinite vector equations for a single plane

How do I determine whether a point lies on a plane?

- Given the equation of a plane $\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ then the point ***r*** with position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is on the plane if there exists a value of λ and μ such that

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- $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$$
- This means that there exists a single value of λ and μ that satisfy the three **parametric** equations:
  - $x = a_1 + \lambda b_1 + \mu c_1$
  - $y = a_2 + \lambda b_2 + \mu c_2$
  - $z = a_3 + \lambda b_3 + \mu c_2$
- Solve two of the equations first to find the values of λ and μ that satisfy the first two equation and then check that this value also satisfies the third equation
- If the values of λ and μ do not satisfy all three equations, then the point ***r*** does not lie on the plane

Examiner Tip

- The formula for the vector equation of a plane is given in the formula booklet, make sure you know what each part means
- Be careful to use different letters, e.g. λ and μ as the scalar multiples of the two direction vectors

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Worked example

The points A, B and C have position vectors $\boldsymbol{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\boldsymbol{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, and $\boldsymbol{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively, relative to the origin O.

(a) Find the vector equation of the plane.

Start by finding the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

All three points lie on the plane, so choose the position vector of one point, e.g. $\overrightarrow{OA}$, to use as 'a' in the vector equation of a plane formula.

Check that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel.

$$r = \underline{a} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$

$$\boxed{\underline{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}}$$

(This is one of many correct answers)

(b) Determine whether the point D with coordinates (-2, -3, 5) lies on the plane.

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Let D have position vector $\underline{d} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$, then the point D lies on the plane if there exists a value of λ and μ for which: $\begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

Find the parametric equations:

$-2 = 3 - 2\lambda + \mu \Rightarrow \mu - 2\lambda = -5$ ①

$-3 = 2 - 4\lambda - 3\mu \Rightarrow 3\mu + 4\lambda = 5$ ②

solve two equations for λ and μ.

$5 = -1 + 5\lambda + 4\mu \Rightarrow 4\mu + 5\lambda = 6$ ③

Find the value of λ and μ from two equations:

2①: $2\mu - 4\lambda = -10$

+②: $3\mu + 4\lambda = 5$

$5\mu = -5$

$\mu = -1$ sub into ①: $(-1) - 2\lambda = -5$

$\lambda = 2$

Check to see if λ and μ satisfy the third equation:

$4(-1) + 5(2) = -4 + 10 = 6$ ✓

$\boxed{\text{The point D lies on the plane.}}$

Your notes

Equation of a Plane in Cartesian Form

How do I find the vector equation of a plane in cartesian form?

- The **cartesian** equation of a plane is given in the form
  - $ax + by + cz = d$
  - This is **given in the formula booklet**
- A **normal vector** to the plane can be used along with a **known point on the plane** to find the cartesian equation of the plane
  - The normal vector will be a vector that is **perpendicular** to the plane
- The **scalar product** of the normal vector and any **direction vector** on the plane will be **zero**
  - The two vectors will be perpendicular to each other
  - The **direction vector** from a fixed-point A to any point on the plane, R can be written as $\mathbf{r} - \mathbf{a}$
  - Then $\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$ and it follows that $(\mathbf{n} \cdot \mathbf{r}) - (\mathbf{n} \cdot \mathbf{a}) = 0$
- This gives the **equation of a plane using the normal vector:**
  - $\mathbf{n} \cdot \mathbf{r} = \mathbf{a} \cdot \mathbf{n}$
    - Where $\mathbf{r}$ is the **position vector** of any point on the plane
    - $\mathbf{a}$ is the **position vector** of a known point on the plane
    - $\mathbf{n}$ is a vector that is **normal** to the plane
  - This is **given in the formula booklet**
- If the vector $\mathbf{r}$ is given in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{a}$ and $\mathbf{n}$ are both known vectors given in the form $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ then the Cartesian equation of the plane can be found using:
  - $\mathbf{n} \cdot \mathbf{r} = n_1 x + n_2 y + n_3 z$
  - $\mathbf{a} \cdot \mathbf{n} = a_1 n_1 + a_2 n_2 + a_3 n_3$
  - Therefore $n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$
  - This simplifies to the form $ax + by + cz = d$
    - A version of this is **given** in the **formula booklet**

How do I find the equation of a plane in Cartesian form given the vector form?

- Given the equation of the plane $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
  - Form three equations
    - $x = a_1 + \lambda b_1 + \mu c_1$

- $y = a_2 + \lambda b_2 + \mu c_2$
- $z = a_3 + \lambda b_3 + \mu c_3$

- Choose a pair of equations and use them to form an equation without μ
- Choose another pair and form another equation without μ
- Use your two expressions to form an equation without μ and λ
- Rewrite the equation in the form $ax + by + cz + d = 0$

Examiner Tip

- In an exam, using whichever form of the equation of the plane to write down a normal vector to the plane is always a good starting point

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Worked example

A plane Π has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$. Find the equation of the plane in its Cartesian form.

Use vector form of Π to form three equations:

$$x = 3 - 2\lambda + \mu \Rightarrow \mu = x - 3 + 2\lambda \quad (1)$$
$$y = 2 - 4\lambda - 3\mu \Rightarrow \mu = \frac{y - 2 + 4\lambda}{-3} \quad (2)$$
$$z = -1 + 5\lambda + 4\mu \Rightarrow \mu = \frac{z + 1 - 5\lambda}{4} \quad (3)$$

Form an equation using (1) and (2)

$$x - 3 + 2\lambda = \frac{y - 2 + 4\lambda}{-3} \Rightarrow -3x + 9 - 6\lambda = y - 2 + 4\lambda$$
$$10\lambda = -3x - y + 11 \quad (4)$$

Form an equation using (1) and (3)

$$x - 3 + 2\lambda = \frac{z + 1 - 5\lambda}{4} \Rightarrow 4x - 12 + 8\lambda = z + 1 - 5\lambda$$
$$13\lambda = -4x + z + 13 \quad (5)$$

Form an equation using (4) and (5) and rearrange

$$13(-3x - y + 11) = 10(-4x + z + 13)$$
$$-39x - 13y + 143 = -40x + 10z + 130$$

$$\boxed{x - 13y - 10z + 13 = 0}$$

6.2.2 Combinations of Lines & Planes

Your notes

Intersections of Lines & Planes

How do I tell if a line is parallel to a plane?

- A line is parallel to a plane if its **direction vector** is **perpendicular** to the plane's **normal vector**
- If you know the Cartesian equation of the plane in the form $ax + by + cz = d$ then the values of *a*, *b*, and c are the individual components of a normal vector to the plane
- The **scalar product** can be used to check in the direction vector and the normal vector are perpendicular
  - If two vectors are perpendicular their scalar product will be zero

How do I tell if the line lies inside the plane?

- If the line is parallel to the plane then it will either **never intersect** or it will lie inside the plane
  - Check to see if they have a common point
- If a line is parallel to a plane and they share **any point**, then the line lies inside the plane

How do I find the point of intersection of a line and a plane in Cartesian form?

- If a line is **not parallel** to a plane it will **intersect** it at a single point
- If both the **vector equation of the line** and the **Cartesian equation of the plane** is known then this can be found by:
- STEP 1: Set the position vector of the point you are looking for to have the individual components *x*, *y*, and *z* and substitute into the vector equation of the line
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
- STEP 2: Find the parametric equations in terms of *x*, *y*, and *z*
  - $x = x_0 + \lambda l$
  - $y = y_0 + \lambda m$
  - $z = z_0 + \lambda n$
- STEP 3: Substitute these parametric equations into the Cartesian equation of the plane and solve to find λ
  - $a(x_0 + \lambda l) + b(y_0 + \lambda m) + c(z_0 + \lambda n) = d$
- STEP 4: Substitute this value of λ back into the vector equation of the line and use it to find the position vector of the point of intersection

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- STEP 5: Check this value in the Cartesian equation of the plane to make sure you have the correct answer

How do I find the point of intersection of a line and a plane in vector form?

- Suppose you have a line with equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ and plane with equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

- Form three equations with unknowns t, λ and μ
- Solve them simultaneously on your calculator
- Substitute the values back in to get the intersection

Worked example

Find the point of intersection of the line $r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ with the plane $3x - 4y + z = 8$.

Find the parametric form of the equation of the line:

Let $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ then $x = 1 + 2\lambda$, $y = -3 - \lambda$, $z = 2 - \lambda$

Substitute into the equation of the plane:

$3(1 + 2\lambda) - 4(-3 - \lambda) + (2 - \lambda) = 8$

Solve to find λ:

$3 + 6\lambda + 12 + 4\lambda + 2 - \lambda = 8$

$\lambda = -1$

Substitute $\lambda = -1$ into the vector equation of the line:

$r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 - 2 \\ -3 + 1 \\ 2 + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

$\boxed{(-1, -2, 3)}$

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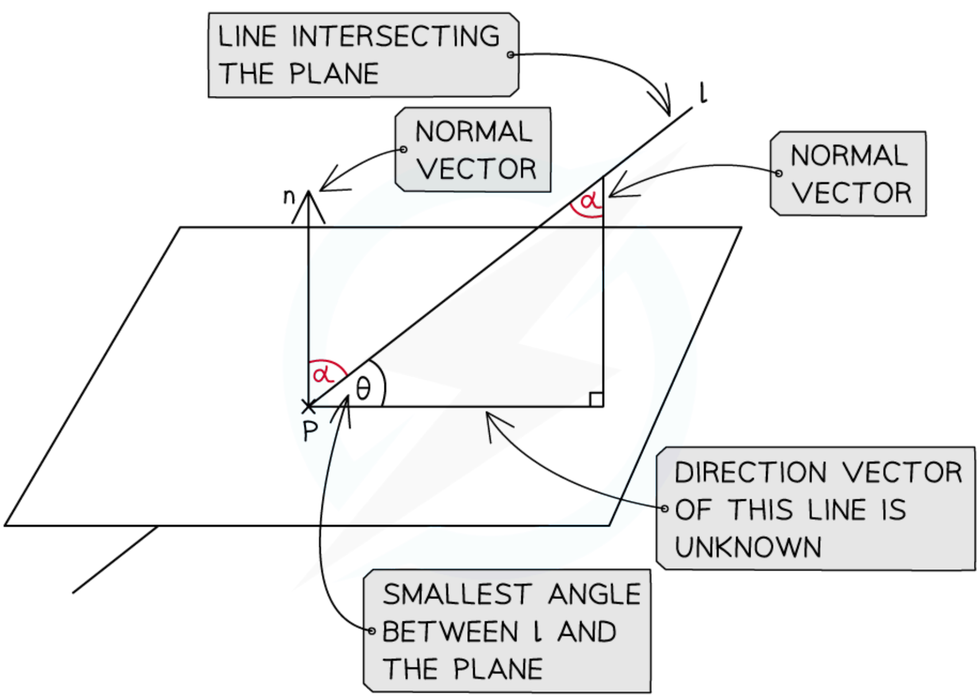
Your notes

Angle between a Line & a Plane

How do I find the angle between a line and a plane?

- When you find the angle between a line and a plane you will be finding the angle between the line itself and the line on the plane that creates the smallest angle with it
  - This means the line on the plane directly under the line as it joins the plane
- It is easiest to think of these two lines making a right-triangle with the normal vector to the plane
  - The line joining the plane will be the **hypotenuse**
  - The line on the plane will be **adjacent** to the angle
  - The normal will be **opposite** to the angle
- As you do not know the angle of the line on the plane you can instead find the angle between the **normal** and the **hypotenuse**
  - This is the angle **opposite** the angle you want to find
  - This angle **can be found** because you will know the direction vector of the line joining the plane and the normal vector to the plane
  - This angle is also equal to the angle made by the line at the point it joins the plane and the normal vector at this point
- The smallest angle between the line and the plane will be 90° minus the angle between the normal vector and the line
  - In radians this will be $\frac{\pi}{2}$ minus the angle between the normal vector and the line



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Examiner Tip

- Remember that if the scalar product is negative your answer will result in an obtuse angle
  - Taking the absolute value of the scalar product will ensure that you get the acute angle as your answer

Worked example

Find the angle in radians between the line L with vector equation $\boldsymbol{r} = (2-\lambda)\boldsymbol{i} + (\lambda+1)\boldsymbol{j} + (1-2\lambda)\boldsymbol{k}$ and the plane Π with Cartesian equation $x - 3y + 2z = 5$.

Rewrite line equation in standard vector form:

$$r = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 1-2\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \underbrace{\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}}_{\text{direction vector of the line}}$$

Find the vector normal to the plane:

$$\underbrace{x - 3y + 2z}_{\text{components of the normal vector}} = 5 \implies \text{normal vector} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

Find the angle between the direction vector and the normal vector, α:

$$\cos\alpha = \frac{\left|\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\right|}{\sqrt{(-1)^2 + (1)^2 + (-2)^2} \times \sqrt{1^2 + (-3)^2 + 2^2}} = \frac{|(-1)(1) + (1)(-3) + (-2)(2)|}{\sqrt{6}\sqrt{14}}$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\alpha$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\left(\frac{|-8|}{\sqrt{6}\sqrt{14}}\right)$$

Using the absolute value ensures we find the acute angle

$$\boxed{\theta = 1.06 \text{ radians (3s.f.)}}$$

Your notes

6.2.3 Combinations of Planes

Intersection of Planes

How do we find the line of intersection of two planes?

- Two planes will either be **parallel** or they will intersect along a **line**
  - Consider the point where a wall meets a floor or a ceiling
  - You will need to find the **equation of the line** of intersection
- If you have the Cartesian forms of the two planes then the equation of the line of intersection can be found by solving the two equations simultaneously
  - As the solution is a vector equation of a line rather than a unique point you will see below how the equation of the line can be found by part solving the equations
  - For example:
    - $2x - y + 3z = 7$ (1)
    - $x - 3y + 4z = 11$ (2)
- STEP 1: Choose one variable and substitute this variable for λ in both equations
  - For example, letting $x = \lambda$ gives:
    - $2\lambda - y + 3z = 7$ (1)
    - $\lambda - 3y + 4z = 11$ (2)
- STEP 2: Rearrange the two equations to bring λ to one side
  - Equations (1) and (2) become
    - $y - 3z = 2\lambda - 7$ (1)
    - $3y - 4z = \lambda - 11$ (2)
- STEP 3: Solve the equations simultaneously to find the two variables in terms of λ
  - 3(1) – (2) Gives
    - $z = 2 - \lambda$
  - Substituting this into (1) gives
    - $y = -1 - \lambda$
- STEP 4: Write the three parametric equations for *x*, *y*, and *z* in terms of λ and convert into the vector equation of a line in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - The parametric equations
    - $x = \lambda$
    - $y = -1 - \lambda$
    - $z = 2 - \lambda$
  - Become

Your notes

- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$
- If you have fractions in your direction vector you can change its magnitude by multiplying each one by their common denominator
  - The magnitude of the direction vector can be changed without changing the equation of a line
- An alternative method is to find two points on both planes by setting either *x*, *y*, or *z* to zero and solving the system of equations using your calculator
  - Repeat this twice to get two points on both planes
  - These two points can then be used to find the vector equation of the line between them
  - This will be the line of intersection of the planes
  - This method relies on the line of intersection having points where the chosen variables are equal to zero

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Worked example

Two planes Π_1 and Π_2 are defined by the equations:

$$\Pi_1: 3x+4y+2z=7$$

$$\Pi_2: x-2y+3z=5$$

Find the vector equation of the line of intersection of the two planes.

STEP 1: Let $z=\lambda$, then $3x+4y+2\lambda=7$ ①

You can substitute any variable here, look at the equations to see which is easiest.

$x-2y+3\lambda=5$ ②

STEP 2: ① : $3x+4y=7-2\lambda$

② : $x-2y=5-3\lambda$

Write the two equations as simultaneous equations for the two remaining constants.

STEP 3: Find x and z in terms of λ:

① − 2②:

$$\begin{aligned} &(3x+4y=7-2\lambda) \\ +\ &(2x-4y=10-6\lambda) \\ \hline &5x=17-8\lambda \\ &x=\frac{17}{5}-\frac{8\lambda}{5} \end{aligned}$$

sub into ② $\frac{17}{5}-\frac{8\lambda}{5}-2y+3\lambda=5$

$$y=\frac{7\lambda}{10}-\frac{8}{10}$$

STEP 4: $\left.\begin{aligned} x&=\frac{17}{5}-\frac{8\lambda}{5} \\ y&=\frac{7\lambda}{10}-\frac{4}{5} \\ z&=\lambda \end{aligned}\right\}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} -\frac{8}{5} \\ \frac{7}{10} \\ 1 \end{pmatrix}$

The components of the direction vector can be multiplied by a scalar without changing the direction.

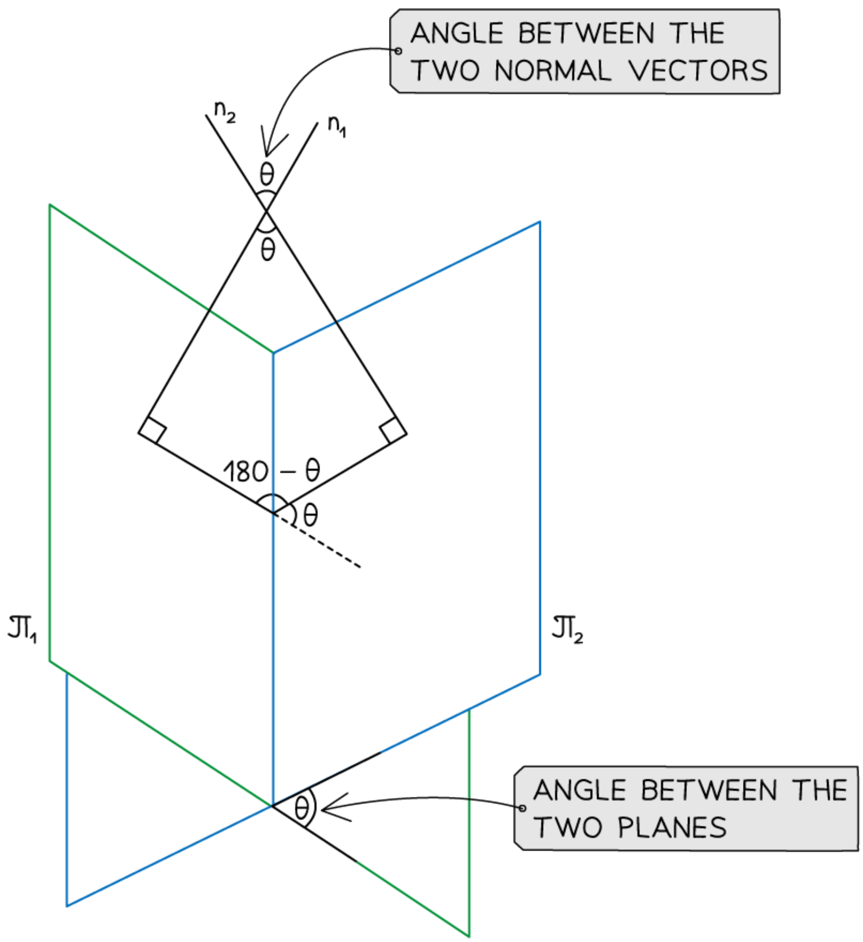
$$\boxed{r=\begin{pmatrix} 17/5 \\ -4/5 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} -16 \\ 7 \\ 10 \end{pmatrix}}$$

Your notes

Angle between two Planes

How do we find the angle between two planes?

- The angle between two planes is equal to the angle between their **normal vectors**
  - It can be found using the **scalar product** of their normal vectors
- If two planes Π_1 and Π_2 with normal vectors n_1 and n_2 meet at an angle then the two planes and the two normal vectors will form a quadrilateral
  - The angles between the planes and the normal will both be 90°
  - The angle between the two planes and the angle opposite it (between the two normal vectors) will add up to 180°



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Examiner Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Worked example

Find the acute angle between the two planes which can be defined by equations Π_1: $2x - y + 3z = 7$ and Π_2: $x + 2y - z = 20$.

Find the normal vectors of each of the planes:

$\Pi_1 : 2x - y + 3z = 7 \Rightarrow$ normal vector, $n_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\Pi_2 : x + 2y - z = 20 \Rightarrow$ normal vector, $n_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Find the angle between the two normal vectors:

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{|(2)(1) + (-1)(2) + (3)(-1)|}{\sqrt{2^2 + (-1)^2 + 3^2} \times \sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|-3|}{\sqrt{14} \times \sqrt{6}}$$

Using the absolute value ensures we find the <u>acute</u> angle.

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$$

$$\boxed{\theta = 1.24 \text{ radians } (3 \text{ s.f.})}$$

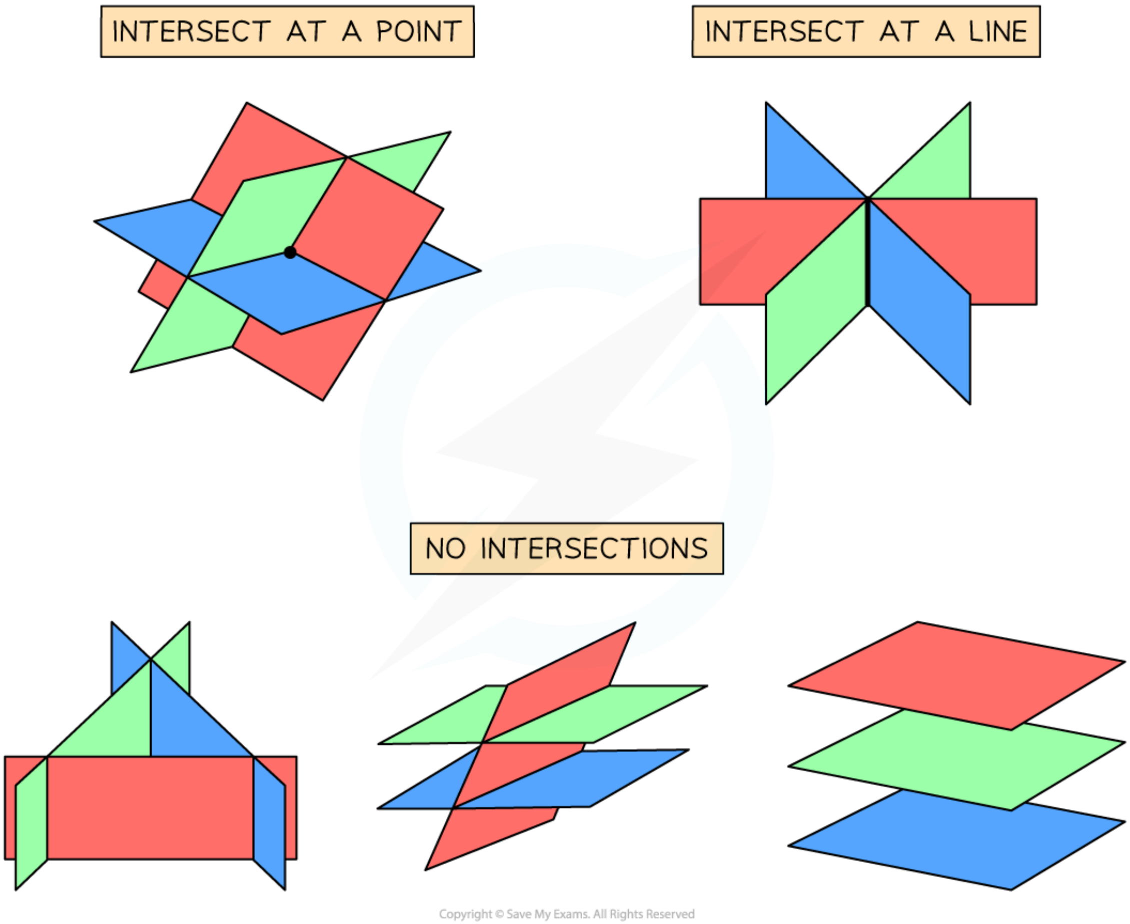
Your notes

Combinations of three Planes

What are the possible configurations of three planes?

- Form three equations using the three planes
  - $$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$
- Let the matrix M be equal to the coefficients
  - $$M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$
- If $\det M \neq 0$ then the three planes intersect at a single point
  - $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$
- If $\det M = 0$ then the three planes could
  - Be coincident or parallel
    - Check if the normal vectors are parallel
    - If they are coincident then there will be infinitely many solutions
    - If they are parallel then there will be no solutions
  - Intersect at a line
    - This is configuration is called a **sheaf**
  - Form a **triangular prism**
    - This is where pairs of planes interest at lines which are parallel to each other
  - Two could be parallel and the third could interest each plane separately

Your notes



How can I find the configuration of three planes?

- If the matrix of coefficients is **non-singular** then the planes **intersect at a single point**
- If the matrix is singular then check if any of the planes are parallel or coincident
  - $2x+3y+5z=4$ and $4x+6y+10z=8$ are coincident as they are scalar multiples
  - $2x+3y+5z=4$ and $4x+6y+10z=9$ are parallel as their normal vectors are parallel
- If the planes are **not parallel** then try to check to see if the equations are consistent
  - Consistent equations will have solutions
  - Inconsistent equations will not have any solutions
- If the planes are **not parallel** and the equations are **consistent** then they form a **sheaf**
  - They intersect at a line
  - Eliminating variables will lead to the equation of this line
  - Eliminating all variables will lead to a statement that is always true
    - Such as 0 = 0
- If the planes are **not parallel** and the equations are **inconsistent** then they form a **triangular prism**
  - They do not intersect

- Each pair of planes intersect a line and these three lines are parallel
- Eliminating all variables will lead to a statement that is never true
  - Such as 0 = 1

Your notes

Your notes

Worked example

Three planes have equations given by

$$x + 2y - z = 3$$
$$3x + 7y + z = 4$$
$$x - 9z = k$$

a) Given that the three planes intersect in a straight line, find the value of k.

Eliminate y between equations ① and ②

$$\left.\begin{aligned} ①:\ 2y &= 3 + z - x \\ ②:\ 7y &= 4 - z - 3x \end{aligned}\right\} \quad \begin{aligned} 7(3 + z - x) &= 2(4 - z - 3x) \\ 21 + 7z - 7x &= 8 - 2z - 6x \\ x &= 9z + 13 \end{aligned}$$

Substitute into equation ③

$$③:\ x - 9z = k \Rightarrow 9z + 13 - 9z = k$$

$$\boxed{k = 13}$$

b) Find a vector equation for the line of intersection.

Let $z = \lambda$ and find expressions for x and y in terms of λ

$$z = \lambda$$
$$x = 9\lambda + 13 \quad (\text{from (a)})$$

From ① $2y = 3 + z - x = 3 + \lambda - (9\lambda + 13) = -10 - 8\lambda$

$$\therefore\ y = -5 - 4\lambda$$

Write as a vector equation:

$$\left.\begin{aligned} x &= 13 + 9\lambda \\ y &= -5 - 4\lambda \\ z &= \lambda \end{aligned}\right\} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -4 \\ 1 \end{pmatrix}$$

$$\boxed{\mathbf{r} = \begin{pmatrix} 13 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -4 \\ 1 \end{pmatrix}}$$

Your notes

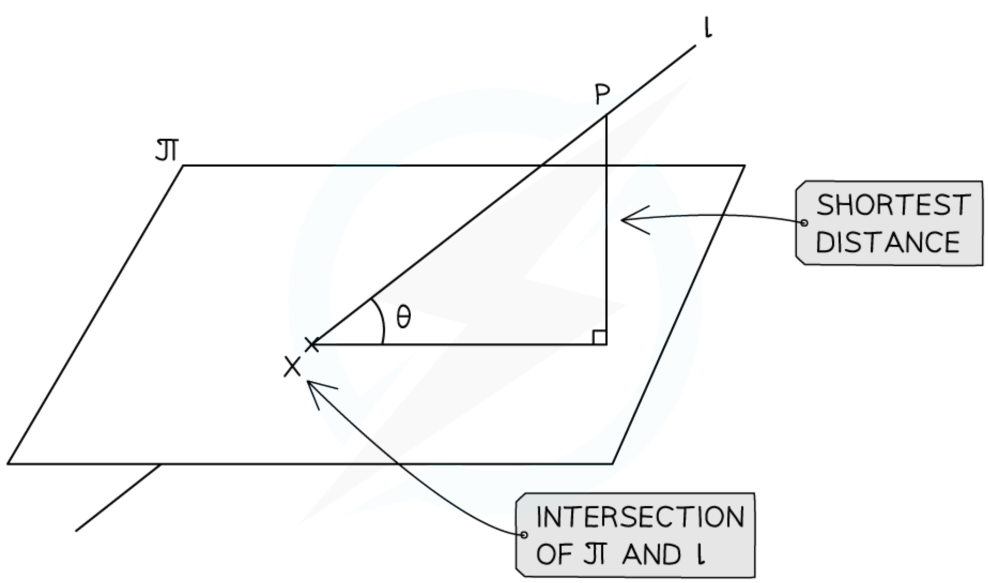
Your notes

6.2.4 Shortest Distances – Planes

Shortest Distance between a Point & a Plane

How do I find the shortest distance between a given point on a line and a plane?

- The shortest distance from any point on a line to a plane will always be the **perpendicular** distance from the point to the plane
- Given a point, *P*, on the line l with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane Π with equation $\mathbf{r} \cdot \mathbf{n} = d$
  - STEP 1: Find the vector equation of the line perpendicular to the plane that goes through the point, *P*, on l
    - This will have the position vector of the point, *P*, and the direction vector **n**
  - STEP 2: Find the coordinates of the point of intersection of this new line with Π by substituting the equation of the line into the equation of the plane
  - STEP 3: Find the distance between the given point on the line and the point of intersection
    - This will be the shortest distance from the plane to the point
- A question may provide the acute angle between the line and the plane
  - Use right-angled trigonometry to find the perpendicular distance between the point on the line and the plane
    - Drawing a clear diagram will help



Your notes

Worked example

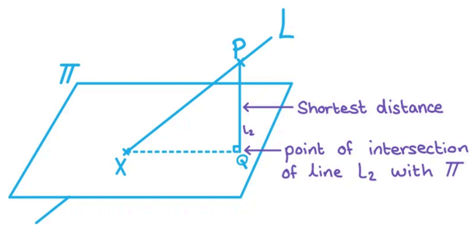
The plane Π has equation $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=6$.

The line L has equation $\mathbf{r}=\begin{pmatrix}2\\3\\1\end{pmatrix}+s\begin{pmatrix}1\\-2\\4\end{pmatrix}$.

The point $P\ (-2,\ 11,\ -15)$ lies on the line L.

Find the shortest distance between the point P and the plane Π.

Your notes



STEP 1: Use the given point, P and the known normal to the plane, $\underline{n}$ to write an equation for the line perpendicular to π, L_2.

$$\mathbf{r} = \underbrace{\begin{pmatrix}-2\\11\\-15\end{pmatrix}}_{P} + \lambda\underbrace{\begin{pmatrix}2\\-1\\1\end{pmatrix}}_{\underline{n}}$$

STEP 2: Find the point of intersection, Q, of the new line, L_2, with π.

$$\left(\begin{pmatrix}-2\\11\\-15\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix} = 6$$

$$2(-2+2\lambda)-(11-\lambda)+(\lambda-15) = 6$$

$$-4+4\lambda-11+\lambda+\lambda-15 = 6$$

$$6\lambda-30 = 6$$

$$\lambda = 6 \Rightarrow \overrightarrow{OQ} = \begin{pmatrix}-2\\11\\-15\end{pmatrix} + 6\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}10\\5\\-9\end{pmatrix}$$

STEP 3: Find the distance between P and Q.

$$|\overrightarrow{PQ}| = \sqrt{(10--2)^2+(5-11)^2+(-9--15)^2} = 6\sqrt{6} \text{ units}$$

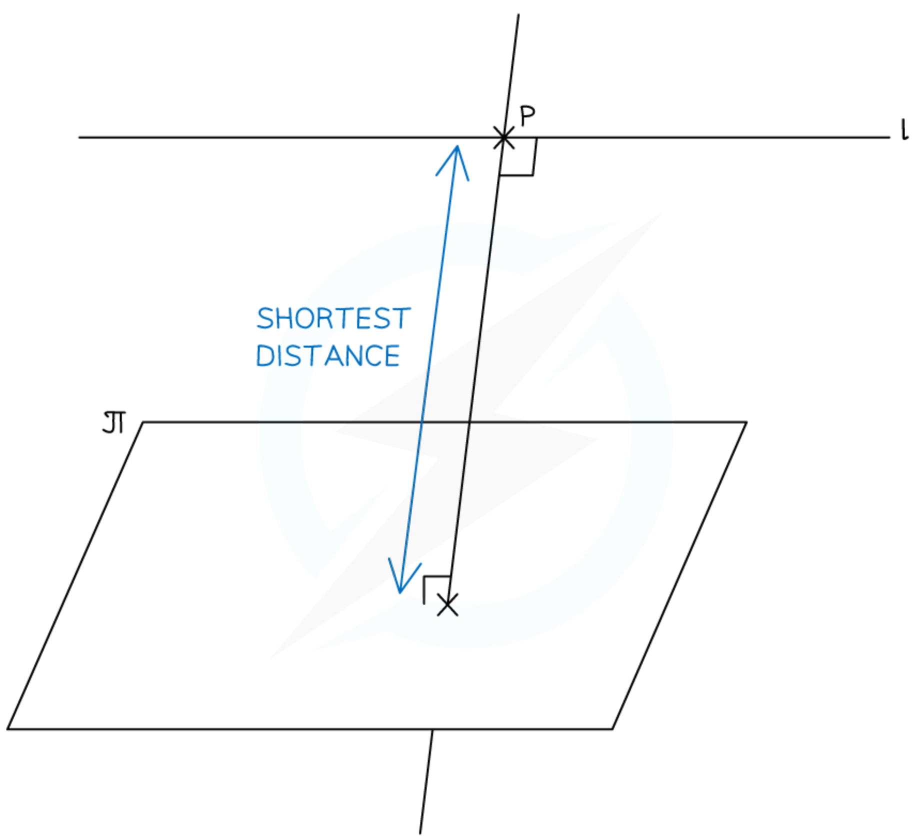
$$\boxed{\text{Shortest distance} = 6\sqrt{6} \text{ units}}$$

Your notes

Shortest Distance between a Line & a Plane

How do I find the shortest distance between a plane and a line parallel to the plane?

- The shortest distance between a line and a plane that are parallel to each other will be the **perpendicular** distance from the line to the plane
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane Π parallel to l_1 with equation $\mathbf{r} \cdot \mathbf{n} = d$
  - Where **n** is the **normal vector** to the plane
  - STEP 1: Find the equation of the line l_2 perpendicular to l_1 and Π going through the point **a** in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{n}$
  - STEP 2: Find the point of intersection of the line l_2 and Π
  - STEP 3: Find the distance between the point of intersection and the point,



Your notes

Shortest Distance between two Planes

How do I find the shortest distance between two parallel planes?

- Two **parallel** planes will never intersect
- The shortest distance between two **parallel planes** will be the **perpendicular distance** between them
- Given a plane Π_1 with equation $\mathbf{r} \cdot \mathbf{n} = d$ and a plane Π_2 with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ then the shortest distance between them can be found
  - STEP 1: The equation of the line perpendicular to both planes and through the point **a** can be written in the form **r** = **a** + s**n**
  - STEP 2: Substitute the equation of the line into $\mathbf{r} \cdot \mathbf{n} = d$ to find the coordinates of the point where the line meets Π_1
  - STEP 3: Find the distance between the two points of intersection of the line with the two planes

Consider the parallel planes defined by the equations:

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44,$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

a) Show that the two planes Π_1 and Π_2 are parallel.

Your notes

The two planes are parallel if the normal vector of one plane is perpendicular to the direction vectors on the other plane.

Parallel if $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = 0$ and $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = (3)(2) + (-5)(0) + (2)(-3)$$
$$= 6 + 0 - 6 = 0$$

$$\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (3)(1) + (-5)(1) + (2)(1)$$
$$= 3 - 5 + 2 = 0$$

$\therefore$ Π_1 and Π_2 are parallel

b) Find the shortest distance between the two planes Π_1 and Π_2.

Your notes

Find the equation of the line perpendicular to the planes through the point $(0,0,3)$

$$L: \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

Normal vector of Π_1

position vector of Π_2

Substitute the equation of L into the equation of Π_1:

$$\begin{pmatrix} 3s \\ -5s \\ 3+2s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44$$

$$3(3s) - 5(-5s) + 2(3+2s) = 44$$

$$38s + 6 = 44$$

$$s = 1$$

Substitute $s = 1$ back into the equation of L:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$$

Find the distance between $(0,0,3)$ and $(3,-5,5)$

$$d = \sqrt{3^2 + (-5)^2 + (5-3)^2} = \sqrt{38}$$

$$\boxed{\text{Shortest distance} = \sqrt{38} \text{ units}}$$