

AS Topics

Algebraic Expressions	Quadratics	Simultaneous Equations & Inequalities	Graphs & Transformations	Straight Line Graphs
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Circles	Algebraic Methods	Binomial Expansion	Trigonometry – Sine and Cosine Rules	Trigonometry – Identities and Equations
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Vectors	Differentiation	Integration	Exponentials and Logarithms
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Contains:
AS SAMs
AS 2018
AS 2019
AS 2020
AS 2021
AS 2022



A2 Topics

HOME

A2 Topics

Algebraic
Methods – proof,
partial fractions,
etc.

Functions and
Graphs

Sequences and
Series

Binomial
Expansion

Radians – small
angle, arc length,
etc.

Trigonometry –
sec, cosec, cot,
identities

Trigonometry –
addition, double
angle, harmonic

Parametric
Equations

Differentiation

Numerical
Methods

Integration

Vectors

Contains:
A2 SAMs
A2 2018
A2 2019
A2 2020
A2 2021
A2 2022



AS Topics

HOME

Algebraic Expressions



12. A student was asked to give the exact solution to the equation

$$2^{2x+4} - 9(2^x) = 0$$

The student's attempt is shown below:

$$2^{2x+4} - 9(2^x) = 0$$

$$2^{2x} + 2^4 - 9(2^x) = 0$$

Let $2^x = y$

$$y^2 - 9y + 8 = 0$$

$$(y - 8)(y - 1) = 0$$

$$y = 8 \text{ or } y = 1$$

$$\text{So } x = 3 \text{ or } x = 0$$

- (a) Identify the two errors made by the student.

(2)

- (b) Find the exact solution to the equation.

(2)



Question	Scheme		Marks	AOs
12(a)	$2^{2x} + 2^4$ is wrong in line 2 - it should be $2^{2x} \times 2^4$		B1	2.3
	In line 4, 2^4 has been replaced by 8 instead of by 16		B1	2.3
			(2)	
(b)	<u>Way 1:</u> $2^{2x+4} - 9(2^x) = 0$ $2^{2x} \times 2^4 - 9(2^x) = 0$ Let $2^x = y$ $16y^2 - 9y = 0$	<u>Way 2:</u> $(2x + 4)\log 2 - \log 9 - x\log 2 = 0$	M1	2.1
	$y = \frac{9}{16}$ or $y = 0$ So $x = \log_2\left(\frac{9}{16}\right)$ or $\frac{\log\left(\frac{9}{16}\right)}{\log 2}$ o.e. with no second answer	$x = \frac{\log 9}{\log 2} - 4$ o.e.	A1	1.1b
				(2)

(4 marks)



2. Find, using algebra, all real solutions to the equation

(i) $16a^2 = 2\sqrt{a}$ (4)

(ii) $b^4 + 7b^2 - 18 = 0$ (4)



Question 2 (Total 8 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(i)	$16a^2 = 2\sqrt{a} \text{ so } \frac{16a^2}{2a^{\frac{1}{2}}} = 1$ $8a^{\frac{3}{2}} = 1 \quad \text{so} \quad a^{\frac{3}{2}} = \frac{1}{8}$	M1	This mark is given for a method to find an equation to solve with the terms in a on one side
	$a = \left(\frac{1}{8}\right)^{\frac{3}{2}}$	M1	This mark is given for finding a way to deal with the indices when solving the equation
	$a = \frac{1}{4}$	A1	This mark is given for finding one correct solution to the equation
	$a = 0$ is also a solution	B1	This mark is given for deducing that $a = 0$ is also a solution
(ii)	$b^4 + 7b^2 - 18 = 0$ factorises to $(b^2 + 9)(b^2 - 2) = 0$	M1	This mark is given for factorising the equation given
	$b^2 = -9, 2$	A1	This mark is given for finding two correct solutions for b^2
	For real solutions, $b^2 = 2$ only	M1	This mark is given for recognising that $b = \sqrt{-9}$ is not a real solution
	$b = \sqrt{2}, -\sqrt{2}$	A1	This mark is given for finding the two real solutions to the equation



3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(i) Solve the equation

$$x\sqrt{2} - \sqrt{18} = x$$

writing the answer as a surd in simplest form.

(3)

(ii) Solve the equation

$$4^{3x-2} = \frac{1}{2\sqrt{2}}$$

(3)



Question	Scheme	Marks	AOs
3 (i)	$x\sqrt{2} - \sqrt{18} = x \Rightarrow x(\sqrt{2} - 1) = \sqrt{18} \Rightarrow x = \frac{\sqrt{18}}{\sqrt{2} - 1}$	M1	1.1b
	$\Rightarrow x = \frac{\sqrt{18}}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$	dM1	3.1a
	$x = \frac{\sqrt{18}(\sqrt{2} + 1)}{1} = 6 + 3\sqrt{2}$	A1	1.1b
		(3)	
(ii)	$4^{3x-2} = \frac{1}{2\sqrt{2}} \Rightarrow 2^{6x-4} = 2^{-\frac{3}{2}}$	M1	2.5
	$6x - 4 = -\frac{3}{2} \Rightarrow x = \dots$	dM1	1.1b
	$x = \frac{5}{12}$	A1	1.1b
		(3)	
(6 marks)			



2. In this question you should show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Given

$$\frac{9^{x-1}}{3^{y+2}} = 81$$

express y in terms of x , writing your answer in simplest form.

(3)



Question	Scheme	Marks	AOs
2	$\frac{9^{x-1}}{3^{y+2}} = 81 \Rightarrow \frac{3^{2x-2}}{3^{y+2}} = 3^4 \text{ or } \frac{9^{x-1}}{3^{y+2}} = 81 \Rightarrow \frac{9^{x-1}}{9^{\frac{1}{2}(y+2)}} = 9^2$	M1	1.1b
	$\Rightarrow 2x - 2 - y - 2 = 4 \Rightarrow y = \text{ or } \Rightarrow x - 1 - \frac{1}{2}y - 1 = 2 \Rightarrow y =$	dM1	1.1b
	$\Rightarrow y = 2x - 8$	A1	1.1b
		(3)	
Alt	Eg. $\log_3\left(\frac{9^{x-1}}{3^{y+2}}\right) = \log_3 81$	M1	1.1b
	$\Rightarrow (x-1)\log_3(9^{x-1}) - (y+2)\log_3(3^{y+2}) = 4$	dM1	1.1b
	$\Rightarrow 2(x-1) - y - 2 = 4 \Rightarrow y =$		
	$\Rightarrow y = 2x - 8$	A1	1.1b
(3 marks)			



6.

In this question you should show all stages of your working.

Solutions relying on calculator technology are not acceptable.

- (a) Using algebra, find all solutions of the equation

$$3x^3 - 17x^2 - 6x = 0 \quad (3)$$

- (b) Hence find all real solutions of

$$3(y - 2)^6 - 17(y - 2)^4 - 6(y - 2)^2 = 0 \quad (3)$$



Question	Scheme	Marks	AOs
6 (a)	$3x^3 - 17x^2 - 6x = 0 \Rightarrow x(3x^2 - 17x - 6) = 0$	M1	1.1a
	$\Rightarrow x(3x+1)(x-6) = 0$	dM1	1.1b
	$\Rightarrow x = 0, -\frac{1}{3}, 6$	A1	1.1b
		(3)	
(b)	Attempts to solve $(y-2)^2 = n$ where n is any solution ≥ 0 to (a)	M1	2.2a
	Two of $2, 2 \pm \sqrt{6}$	A1ft	1.1b
	All three of $2, 2 \pm \sqrt{6}$	A1	2.1
		(3)	
(6 marks)			

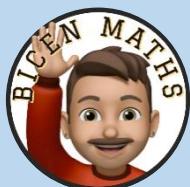


1. Given

$$2^x \times 4^y = \frac{1}{2\sqrt{2}}$$

express y as a function of x .

(3)



Question 1 (Total 3 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	$2^x \times (2^2)^y = 2^{-\frac{3}{2}} \Rightarrow 2^{x+2y} = 2^{-\frac{3}{2}}$	M1	This mark is given for writing all terms in the same base and applying an index law
	$x + 2y = -\frac{3}{2}$	M1	This mark is given for writing an equation to link x and y
	$y = -\frac{1}{2}x - \frac{3}{4}$	A1	This mark is given for rearranging to find a correct expression of y as a function of x



Simultaneous Equations & Inequalities



7. The curve C has equation

$$y = \frac{k^2}{x} + 1 \quad x \in \mathbb{R}, x \neq 0$$

where k is a constant.

- (a) Sketch C stating the equation of the horizontal asymptote.

(3)

The line l has equation $y = -2x + 5$

- (b) Show that the x coordinate of any point of intersection of l with C is given by a solution of the equation

$$2x^2 - 4x + k^2 = 0$$

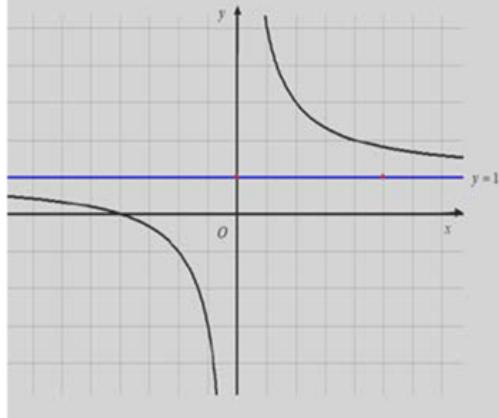
(2)

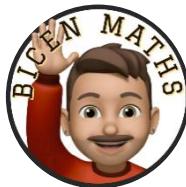
- (c) Hence find the exact values of k for which l is a tangent to C .

(3)



Question 7 (Total 8 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)		M1	This mark is given for a graph with shape $\frac{1}{x}$ in the first quadrant
		A1	This mark is given for a fully correct sketch
		B1	This mark is given for the asymptote $y = 1$ correctly shown on the sketch
(b)	$\frac{k^2}{x} + 1 = -2x + 5$	M1	This mark is given for deducing the point of intersection
	$k^2 + x = -2x^2 + 5x$ $-2x^2 + 5x - x - k^2 = 0$ $2x^2 - 4x + k^2 = 0$	A1	This mark is given for correct working to show the result required
(c)	$16 = 4 \times 2 \times k^2$ $16 = 8k^2$	M1	This mark is given for deducing that the equation has a single root and setting $b^2 - 4ac = 0$
		A1	This mark is given for correctly finding $b^2 - 4ac = 0$
	$k^2 = \pm\sqrt{2}$	A1	This mark is given for finding the two exact values of k



1.

In this question you should show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 20$$

writing your answer in set notation.

(3)



Question	Scheme	Marks	AOs
1	Finds critical values $x^2 - x > 20 \Rightarrow x^2 - x - 20 > 0 \Rightarrow x = (5, -4)$	M1	1.1b
	Chooses outside region for their values Eg. $x > 5, x < -4$	M1	1.1b
	Presents solution in set notation $\{x : x < -4\} \cup \{x : x > 5\}$ oe	A1	2.5
		(3)	
(3 marks)			

Notes

M1: Attempts to find the critical values using an algebraic method. Condone slips but an allowable method should be used and two critical values should be found

M1: Chooses the outside region for their critical values. This may appear in incorrect inequalities such as $5 < x < -4$

A1: Presents in set notation as required $\{x : x < -4\} \cup \{x : x > 5\}$. Accept $\{x < -4 \cup x > 5\}$.
Do not accept $\{x < -4, x > 5\}$

Note: If there is a contradiction of their solution on different lines of working do not penalise intermediate working and mark what appears to be their final answer.



2.

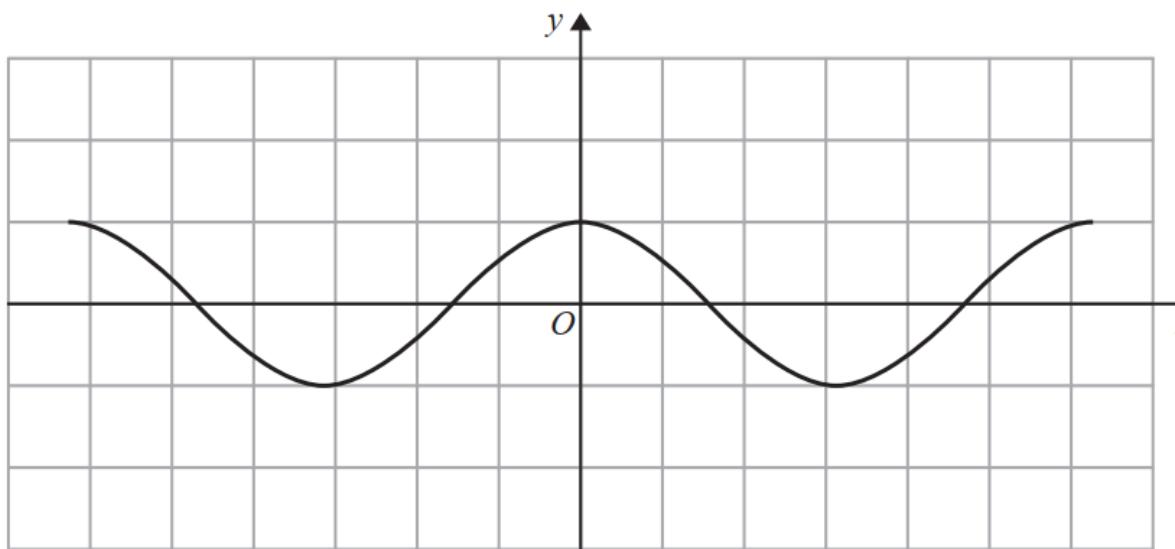


Figure 1

Figure 1 shows a plot of part of the curve with equation $y = \cos x$ where x is measured in radians. Diagram 1, on the opposite page, is a copy of Figure 1.

- (a) Use Diagram 1 to show why the equation

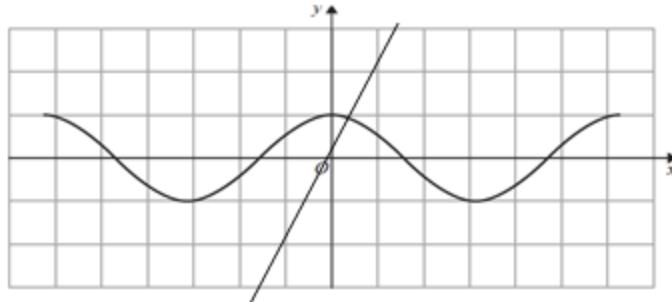
$$\cos x - 2x - \frac{1}{2} = 0$$

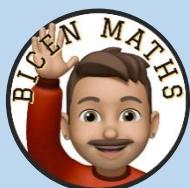
has only one real root, giving a reason for your answer.

(2)



Question 2 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)		M1	This mark is given for plotting the line $y = 2x + \frac{1}{2}$ on the diagram with a correct gradient and intercept
	Only one intersection means that there is one root	A1	This mark is given for a reason why there is only one real root



7.

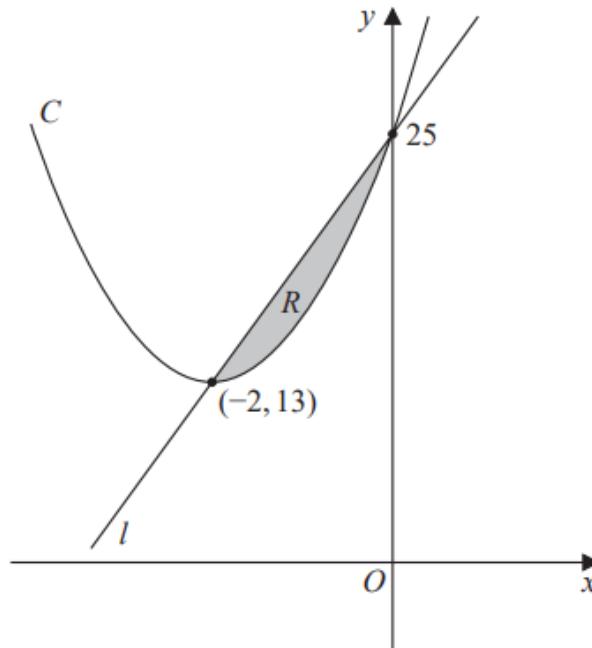


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$ and a straight line l .

The curve C meets l at the points $(-2, 13)$ and $(0, 25)$ as shown.

The shaded region R is bounded by C and l as shown in Figure 1.

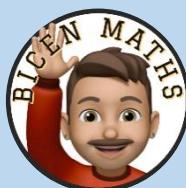
Given that

- $f(x)$ is a quadratic function in x
- $(-2, 13)$ is the minimum turning point of $y = f(x)$

use inequalities to define R .

(5)

HOME



Question	Scheme	Marks	AOs
7	Attempts equation of line Eg Substitutes $(-2, 13)$ into $y = mx + 25$ and finds m	M1	1.1b
	Equation of l is $y = 6x + 25$	A1	1.1b
	Attempts equation of C Eg Attempts to use the intercept $(0, 25)$ within the equation $y = a(x \pm 2)^2 + 13$, in order to find a	M1	3.1a
	Equation of C is $y = 3(x + 2)^2 + 13$ or $y = 3x^2 + 12x + 25$	A1	1.1b
	Region R is defined by $3(x + 2)^2 + 13 < y < 6x + 25$ o.e.	B1ft	2.5
		(5)	
(5 marks)			



Graphs & Transformations



13. (a) Factorise completely $x^3 + 10x^2 + 25x$

(2)

(b) Sketch the curve with equation

$$y = x^3 + 10x^2 + 25x$$

showing the coordinates of the points at which the curve cuts or touches the x -axis.

(2)

The point with coordinates $(-3, 0)$ lies on the curve with equation

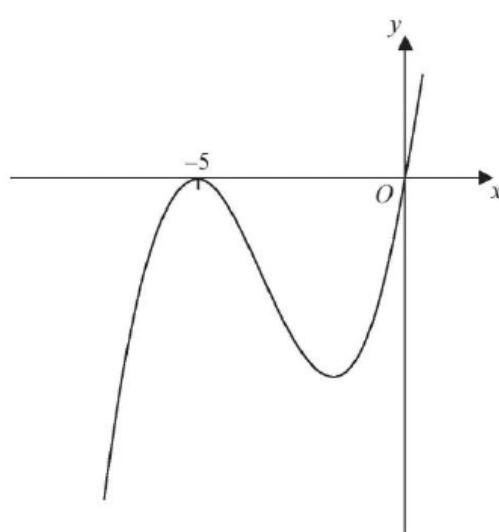
$$y = (x + a)^3 + 10(x + a)^2 + 25(x + a)$$

where a is a constant.

(c) Find the two possible values of a .

(3)



Question	Scheme	Marks	AOs
13(a)	$x^3 + 10x^2 + 25x = x(x^2 + 10x + 25)$ $= x(x+5)^2$	M1 A1	1.1b 1.1b
		(2)	
(b)	 <p>A cubic with correct orientation</p> <p>Curve passes through the origin $(0, 0)$ and touches at $(-5, 0)$ (see note below for ft)</p>	M1 A1ft	1.1b 1.1b
		(2)	
(c)	<p>Curve has been translated a to the left</p> $a = -2$ $a = 3$	M1 A1ft A1ft	3.1a 3.2a 1.1b
		(3)	



9.

$$g(x) = 4x^3 - 12x^2 - 15x + 50$$

- (a) Use the factor theorem to show that $(x + 2)$ is a factor of $g(x)$.

(2)

- (b) Hence show that $g(x)$ can be written in the form $g(x) = (x + 2)(ax + b)^2$, where a and b are integers to be found.

(4)

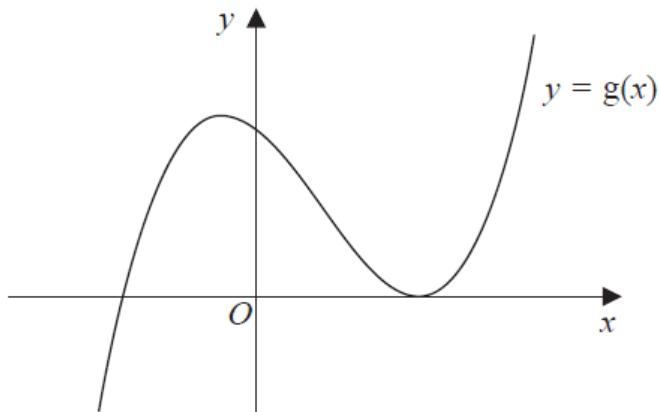


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = g(x)$

- (c) Use your answer to part (b), and the sketch, to deduce the values of x for which

- (i) $g(x) \leq 0$
- (ii) $g(2x) = 0$

(3)



Question	Scheme	Marks	AOs
9(a)	$(g(-2)) = 4 \times -8 - 12 \times 4 - 15 \times -2 + 50$	M1	1.1b
	$g(-2) = 0 \Rightarrow (x+2)$ is a factor	A1	2.4
		(2)	
(b)	$4x^3 - 12x^2 - 15x + 50 = (x+2)(4x^2 - 20x + 25)$	M1 A1	1.1b 1.1b
	$= (x+2)(2x-5)^2$	M1 A1	1.1b 1.1b
		(4)	
(c)	(i) $x \leq -2, x = 2.5$	M1 A1ft	1.1b 1.1b
	(ii) $x = -1, x = 1.25$	B1ft	2.2a
		(3)	
(9 marks)			



11.

$$f(x) = 2x^3 - 13x^2 + 8x + 48$$

- (a) Prove that
- $(x - 4)$
- is a factor of
- $f(x)$
- .

(2)

- (b) Hence, using algebra, show that the equation
- $f(x) = 0$
- has only two distinct roots.

(4)

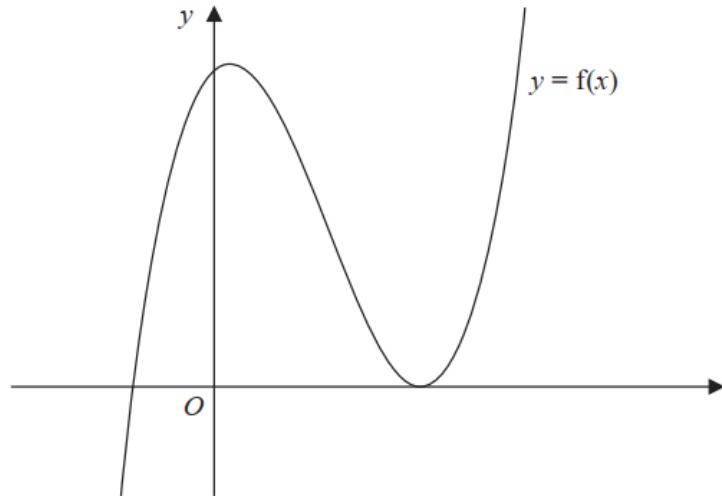
**Figure 2**

Figure 2 shows a sketch of part of the curve with equation $y = f(x)$.

- (c) Deduce, giving reasons for your answer, the number of real roots of the equation

$$2x^3 - 13x^2 + 8x + 46 = 0$$

(2)

Given that k is a constant and the curve with equation $y = f(x + k)$ passes through the origin,

- (d) find the two possible values of
- k
- .

(2)



Question 11 (Total 10 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$f(4) = (2 \times 4^3) - (13 \times 4^2) + (8 \times 4) + 48$ $= 128 - 208 + 32 + 48$	M1	This mark is given for a method to find $f(4)$
	$f(4) = 0$ so $(x - 4)$ is a factor	A1	
(b)	$2x^3 - 13x^2 + 8x + 48 =$ $(x - 4)(2x^2 - 5x - 12)$	M1	This mark is given for attempting to factorise the expression ($2x^2$ and -12 seen)
		A1	This mark is given for a fully correct factorisation
	$2x^2 - 5x - 12 = (x - 4)(2x + 3)$	M1	This mark is given for factorising the expression $2x^2 - 5x - 12$
	$f(x) = (x - 4)^2(2x + 3)$ Thus $f(x) =$ only has two roots 4 and $-\frac{3}{2}$	A1	This mark is given for a valid explanation of why the expression only has two roots
(c)	The curve will move two units down	M1	This mark is given for deducing that the curve will be translated by two units
	The curve will cross the axis at three places and so have three roots	A1	This mark is given for deducing that the curve will intersect the x-axis in three places and so the expression will have three roots
(d)	Since $f(x)$ passes through the origin, $f(0) = 0$ so	M1	This mark is given for deducing that $f(0) = 0$ when $f(x + k) = 0$
	$f(x + k) = 0$ when $k = 4, -\frac{3}{2}$	A1	This mark is given for the correct two values of k



Part (c) requires differentiation – leave this if you haven't studied it yet

7. (a) Factorise completely $9x - x^3$

(2)

The curve C has equation

$$y = 9x - x^3$$

- (b) Sketch C showing the coordinates of the points at which the curve cuts the x -axis.

(2)

The line l has equation $y = k$ where k is a constant.

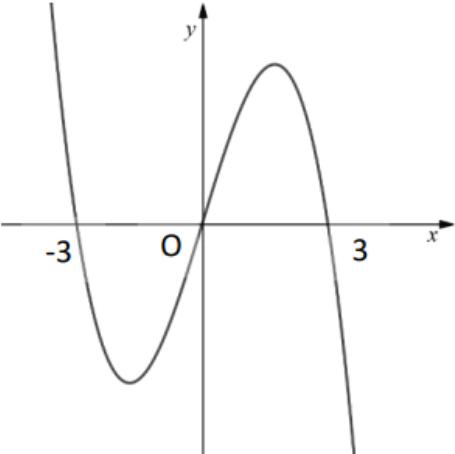
Given that C and l intersect at 3 distinct points,

- (c) find the range of values for k , writing your answer in set notation.

Solutions relying on calculator technology are not acceptable.

(3)



7(a)	$9x - x^3 = x(9 - x^2)$	M1	1.1b
	$9x - x^3 = x(3 - x)(3 + x)$ oe	A1	1.1b
		(2)	
(b)		A cubic with correct orientation Passes through origin, (3, 0) and (-3, 0)	B1 B1 1.1b 1.1b
(c)	$y = 9x - x^3 \Rightarrow \frac{dy}{dx} = 9 - 3x^2 = 0 \Rightarrow x = (\pm)\sqrt{3} \Rightarrow y = \dots$ $y = (\pm)6\sqrt{3}$ $\{k \in \mathbb{Q} : -6\sqrt{3} < k < 6\sqrt{3}\}$ oe	M1 A1 A1ft (3)	3.1a 1.1b 2.5

(7 marks)



5. $f(x) = 2x^2 + 4x + 9 \quad x \in \mathbb{R}$

(a) Write $f(x)$ in the form $a(x + b)^2 + c$, where a , b and c are integers to be found.

(3)

(b) Sketch the curve with equation $y = f(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point.

(3)

(c) (i) Describe fully the transformation that maps the curve with equation $y = f(x)$ onto the curve with equation $y = g(x)$ where

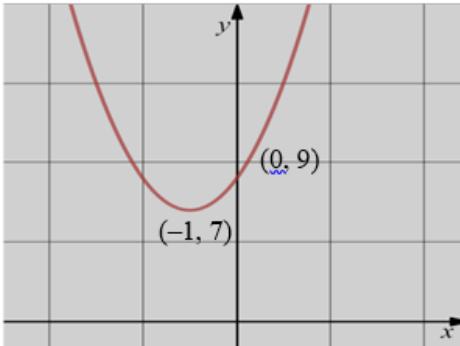
$$g(x) = 2(x - 2)^2 + 4x - 3 \quad x \in \mathbb{R}$$

(ii) Find the range of the function

$$h(x) = \frac{21}{2x^2 + 4x + 9} \quad x \in \mathbb{R}$$

(4)



Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$2x^2 + 4x + 9 = 2(x + b)^2 + c$	B1	This mark is given for writing $f(x)$ in the form $a(x + b)^2 + c$ with $a = 2$
	$2x^2 + 4x + 9 = 2(x + 1)^2 + c$	M1	This mark is given for writing $f(x)$ in the form $a(x + b)^2 + c$ with $a = 2$ and $b = 1$
	$2x^2 + 4x + 9 = 2(x + 1)^2 + 7$	A1	This mark is given for writing $f(x)$ in the form $a(x + b)^2 + c$ with $a = 2$, $b = 1$ and $c = 7$
(b)		B1	This mark is given for a U shaped curve in any position
		B1	This mark is given for a y-intercept shown at $(0, 9)$
		B1	This mark is given for a minimum shown at $(-1, 7)$
(c)(i)	$g(x) = 2(x - 2)^2 + 4(x - 2) + 5$	M1	This mark is given for writing $g(x)$ in the form $a(x + b)^2 + c$ and comparing to $f(x)$
	Translation of $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$	A1	This mark is given for deducing the translation of $y = f(x)$ to $y = g(x)$
(c)(ii)	$h(x) = \frac{21}{2(x + 1)^2 + 7}$ Maximum value = $\frac{21}{7}$ (when $x = -1$)	M1	This mark is given for writing $h(x)$ in the form $\frac{21}{a(x + b)^2 + c}$ and finding its maximum value
	$0 < h(x) \leq 3$	A1	This mark is given for finding the correct range of the function $h(x)$



Circles



17. A circle C with centre at $(-2, 6)$ passes through the point $(10, 11)$.

(a) Show that the circle C also passes through the point $(10, 1)$.

(3)

The tangent to the circle C at the point $(10, 11)$ meets the y axis at the point P and the tangent to the circle C at the point $(10, 1)$ meets the y axis at the point Q .

(b) Show that the distance PQ is 58 explaining your method clearly.

(7)



Question	Scheme		Marks	AOs
17 (a)	<u>Way 1:</u> Finds circle equation $(x \pm 2)^2 + (y \mp 6)^2 =$ $(10 \pm (-2))^2 + (11 \mp 6)^2$	<u>Way 2:</u> Finds distance between (-2, 6) and (10, 11)	M1	3.1a
	Checks whether (10, 1) satisfies their circle equation	Finds distance between (-2, 6) and (10, 1)	M1	1.1b
	Obtains $(x+2)^2 + (y-6)^2 = 13^2$ and checks that $(10+2)^2 + (1-6)^2 = 13^2$ so states that (10, 1) lies on C^*	Concludes that as distance is the same (10, 1) lies on the circle C^*	A1*	2.1
				(3)
(b)	Finds radius gradient $\frac{11-6}{10-(-2)}$ or $\frac{1-6}{10-(-2)}$ (m)			M1 3.1a
	Finds gradient perpendicular to their radius using $-\frac{1}{m}$			M1 1.1b
	Finds (equation and) y intercept of tangent (see note below)			M1 1.1b
	Obtains a correct value for y intercept of their tangent i.e. 35 or -23			A1 1.1b
	<u>Way 1:</u> Deduces gradient of second tangent	<u>Way 2:</u> Deduces midpoint of PQ from symmetry (0, 6)	M1	1.1b
	Finds (equation and) y intercept of second tangent	Uses this to find other intercept	M1	1.1b
	So obtains distance $PQ = 35 + 23 = 58^*$			A1* 1.1b
				(7)



14. The circle C has equation

$$x^2 + y^2 - 6x + 10y + 9 = 0$$

(a) Find

- (i) the coordinates of the centre of C
- (ii) the radius of C

(3)

The line with equation $y = kx$, where k is a constant, cuts C at two distinct points.

(b) Find the range of values for k .

(6)



Question	Scheme	Marks	AOs
14 (a)	Attempts to complete the square $(x \pm 3)^2 + (y \pm 5)^2 = \dots$	M1	1.1b
	(i) Centre $(3, -5)$	A1	1.1b
	(ii) Radius 5	A1	1.1b
		(3)	
(b)	Uses a sketch or otherwise to deduce $k = 0$ is a critical value	B1	2.2a
	Substitute $y = kx$ in $x^2 + y^2 - 6x + 10y + 9 = 0$	M1	3.1a
	Collects terms to form correct 3TQ $(1+k^2)x^2 + (10k-6)x + 9 = 0$	A1	1.1b
	Attempts $b^2 - 4ac < 0$ for their a , b and c leading to values for k $"(10k-6)^2 - 36(1+k^2) < 0" \rightarrow k = \dots, \dots \left(0 \text{ and } \frac{15}{8}\right)$	M1	1.1b
	Uses $b^2 - 4ac > 0$ and chooses the outside region (see note) for their critical values (Both a and b must have been expressions in k)	dM1	3.1a
	Deduces $k < 0, k > \frac{15}{8}$ oe	A1	2.2a
		(6)	
			(9 marks)



10. A circle C has equation

$$x^2 + y^2 - 4x + 8y - 8 = 0$$

(a) Find

- (i) the coordinates of the centre of C ,
- (ii) the exact radius of C .

(3)

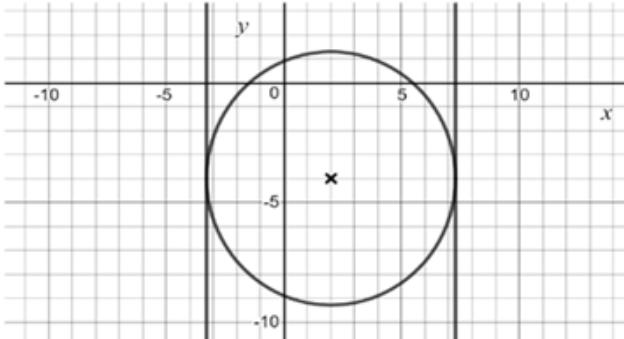
The straight line with equation $x = k$, where k is a constant, is a tangent to C .

(b) Find the possible values for k .

(2)



Question 10 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$(x - 2)^2 + (y + 4)^2 - 4 - 16 - 8 = 0$	M1	This mark is given for a method to complete the square
	Centre at $x = 2$ and $y = -4$, $(2, -4)$	A1	This mark is given for finding the correct coordinates of the centre of the circle
	$(x - 2)^2 + (y + 4)^2 - 28 = 0$ Radius = $\sqrt{28} = 2\sqrt{7}$	A1	This mark is given for finding the exact radius of the circle
(b)	 Tangent of $x = k$ touches circle at $2 + \sqrt{28}$ and $2 - \sqrt{28}$	M1	This mark is given for adding or subtracting the length of the radius of the circle from 2
		A1	This mark is given for deducing both values of k



11. (i) A circle C_1 has equation

$$x^2 + y^2 + 18x - 2y + 30 = 0$$

The line l is the tangent to C_1 at the point $P(-5, 7)$.

Find an equation of l in the form $ax + by + c = 0$, where a , b and c are integers to be found.

(5)

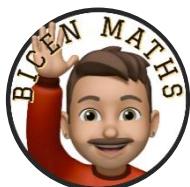
(ii) A different circle C_2 has equation

$$x^2 + y^2 - 8x + 12y + k = 0$$

where k is a constant.

Given that C_2 lies entirely in the 4th quadrant, find the range of possible values for k .

(4)



Question	Scheme	Marks	AOs
11. (i)	$x^2 + y^2 + 18x - 2y + 30 = 0 \Rightarrow (x+9)^2 + (y-1)^2 = \dots$	M1	1.1b
	Centre $(-9, 1)$	A1	1.1b
	Gradient of line from $P(-5, 7)$ to " $(-9, 1)$ " = $\frac{7-1}{-5+9} = \left(\frac{3}{2}\right)$	M1	1.1b
	Equation of tangent is $y - 7 = -\frac{2}{3}(x + 5)$	dM1	3.1a
	$3y - 21 = -2x - 10 \Rightarrow 2x + 3y - 11 = 0$	A1	1.1b
		(5)	
(ii)	$x^2 + y^2 - 8x + 12y + k = 0 \Rightarrow (x-4)^2 + (y+6)^2 = 52 - k$	M1	1.1b
	Lies in Quadrant 4 if radius $< 4 \Rightarrow "52 - k" < 4^2$	M1	3.1a
	$\Rightarrow k > 36$	A1	1.1b
	Deduces $52 - k > 0 \Rightarrow$ Full solution $36 < k < 52$	A1	3.2a
		(4)	
(9 marks)			



15.

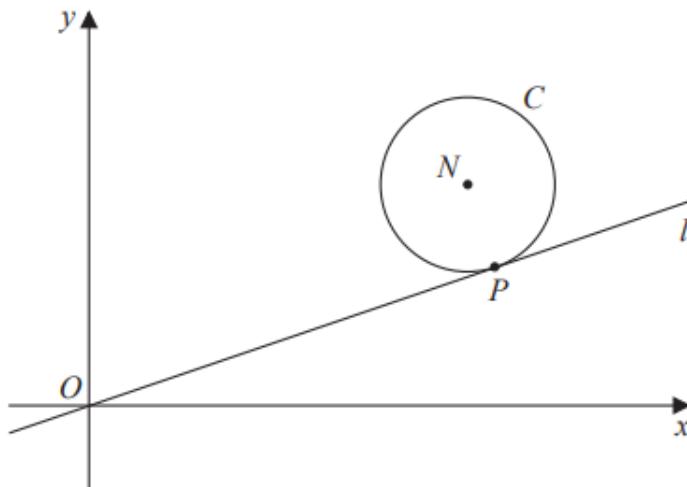


Figure 4

Figure 4 shows a sketch of a circle C with centre $N(7, 4)$

The line l with equation $y = \frac{1}{3}x$ is a tangent to C at the point P .

Find

(a) the equation of line PN in the form $y = mx + c$, where m and c are constants,

(2)

(b) an equation for C .

(4)

The line with equation $y = \frac{1}{3}x + k$, where k is a non-zero constant, is also a tangent to C .

(c) Find the value of k .

(3)

HOME



Question	Scheme	Marks	AOs
15 (a)	Deduces the line has gradient "-3" and point (7, 4) Eg $y - 4 = -3(x - 7)$ $y = -3x + 25$	M1 A1 (2)	2.2a 1.1b
(b)	Solves $y = -3x + 25$ and $y = \frac{1}{3}x$ simultaneously $P = \left(\frac{15}{2}, \frac{5}{2} \right)$ oe Length $PN = \sqrt{\left(\frac{15}{2} - 7 \right)^2 + \left(4 - \frac{5}{2} \right)^2} = \left(\sqrt{\frac{5}{2}} \right)$	M1 A1 M1	3.1a 1.1b 1.1b
	Equation of C is $(x - 7)^2 + (y - 4)^2 = \frac{5}{2}$ o.e.	A1	1.1b
		(4)	
(c)	Attempts to find where $y = \frac{1}{3}x + k$ meets C using vectors Eg: $\begin{pmatrix} 7.5 \\ 2.5 \end{pmatrix} + 2 \times \begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix}$	M1	3.1a
	Substitutes their $\left(\frac{13}{2}, \frac{11}{2} \right)$ in $y = \frac{1}{3}x + k$ to find k	M1	2.1
	$k = \frac{10}{3}$	A1	1.1b
		(3)	
		(9 marks)	



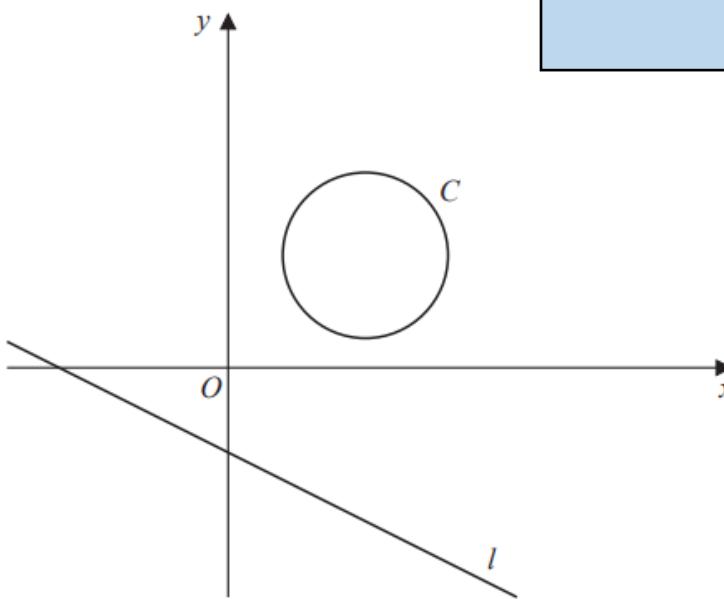


Figure 3

Figure 3 shows the circle C with equation

$$x^2 + y^2 - 10x - 8y + 32 = 0$$

and the line l with equation

$$2y + x + 6 = 0$$

(a) Find

- (i) the coordinates of the centre of C ,
- (ii) the radius of C .

(3)

(b) Find the shortest distance between C and l .

(5)



11(a)	$(x \pm 5)^2 + (y \pm 4)^2$	M1
	(i) Centre is (5, 4)	A1
	(ii) Radius is 3	A1
		(3)
(b)	$2y + x + 6 = 0 \Rightarrow y = -\frac{1}{2}x - 3 \Rightarrow -\frac{1}{2} \rightarrow 2$	B1
	$m_N = 2 \Rightarrow y - 4 = 2(x - 5)$	M1
	$y - 4 = 2(x - 5), 2y + x + 6 = 0 \Rightarrow x = ..., y = ...$	
	Intersection is at $\left(\frac{6}{5}, -\frac{18}{5}\right)$ oe	A1
	Distance from centre to intersection is $\sqrt{\left(5 - \frac{6}{5}\right)^2 + \left(4 + \frac{18}{5}\right)^2}$	dM1
	So distance required is $\sqrt{\left("5" - "6"\right)^2 + \left("4" + "18"\right)^2} - "3"$	
	$= \frac{19\sqrt{5}}{5} - 3$ (or awrt 5.50)	A1
		(5)



3. A circle C has equation

$$x^2 + y^2 - 4x + 10y = k$$

where k is a constant.

(a) Find the coordinates of the centre of C .

(2)

(b) State the range of possible values for k .

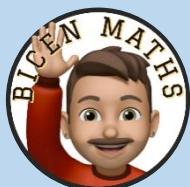
(2)



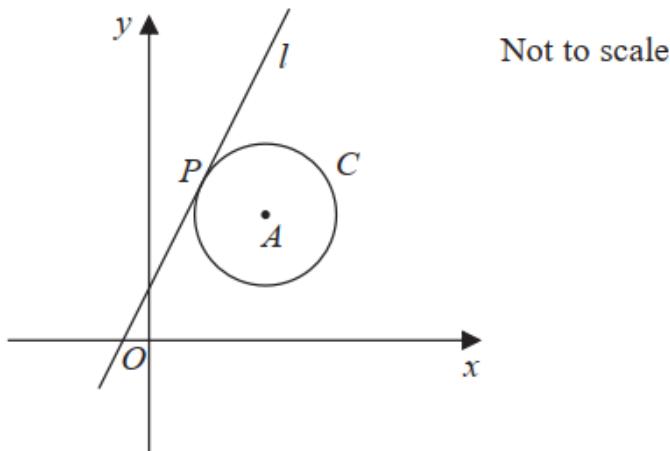
A2 SAMs Paper 1

Circles

Question	Scheme	Marks	AOs
3(a)	Attempts $(x-2)^2 + (y+5)^2 = \dots$	M1	1.1b
	Centre $(2, -5)$	A1	1.1b
		(2)	
(b)	Sets $k + 2^2 + 5^2 > 0$	M1	2.2a
	$\Rightarrow k > -29$	A1ft	1.1b
		(2)	
(4 marks)			



6.

**Figure 3**

The circle C has centre A with coordinates $(7, 5)$.

The line l , with equation $y = 2x + 1$, is the tangent to C at the point P , as shown in Figure 3.

(a) Show that an equation of the line PA is $2y + x = 17$

(3)

(b) Find an equation for C .

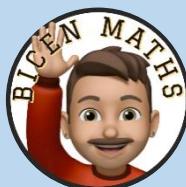
(4)

The line with equation $y = 2x + k$, $k \neq 1$ is also a tangent to C .

(c) Find the value of the constant k .

(3)

Question	Scheme	Marks	AOs
6 (a)	Deduces that gradient of PA is $-\frac{1}{2}$	M1	2.2a
	Finding the equation of a line with gradient $-\frac{1}{2}$ and point $(7, 5)$	M1	1.1b
	$y - 5 = -\frac{1}{2}(x - 7)$		
	Completes proof $2y + x = 17$ *	A1*	1.1b
(b)		(3)	
	Solves $2y + x = 17$ and $y = 2x + 1$ simultaneously	M1	2.1
	$P = (3, 7)$	A1	1.1b
	Length $PA = \sqrt{(3-7)^2 + (7-5)^2} = (\sqrt{20})$	M1	1.1b
	Equation of C is $(x-7)^2 + (y-5)^2 = 20$	A1	1.1b
(c)		(4)	
	Attempts to find where $y = 2x + k$ meets C using $\vec{OA} + \vec{PA}$	M1	3.1a
	Substitutes their $(11, 3)$ in $y = 2x + k$ to find k	M1	2.1
	$k = -19$	A1	1.1b
		(3)	
		(10 marks)	



Circle C_1 has equation $x^2 + y^2 = 100$

Circle C_2 has equation $(x - 15)^2 + y^2 = 40$

The circles meet at points A and B as shown in Figure 3.

- (a) Show that angle $AOB = 0.635$ radians to 3 significant figures, where O is the origin.

(4)

The region shown shaded in Figure 3 is bounded by C_1 and C_2

- (b) Find the perimeter of the shaded region, giving your answer to one decimal place.

(4)

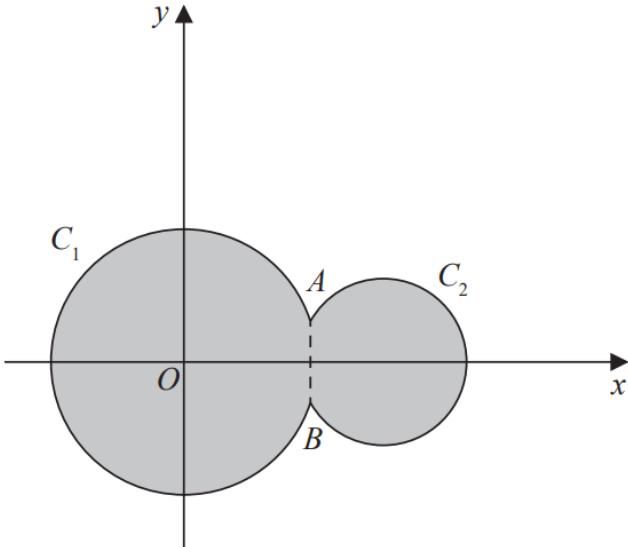


Figure 3

Question	Scheme	Marks	AOs
11 (a)	Solves $x^2 + y^2 = 100$ and $(x - 15)^2 + y^2 = 40$ simultaneously to find x or y E.g. $(x - 15)^2 + 100 - x^2 = 40 \Rightarrow x = \dots$	M1	3.1a
	Either $\Rightarrow -30x + 325 = 40 \Rightarrow x = 9.5$ Or $y = \frac{\sqrt{39}}{2} = \text{awrt } \pm 3.12$	A1	1.1b
	Attempts to find the angle AOB in circle C_1 Eg Attempts $\cos \alpha = \frac{"9.5"}{10}$ to find α then $\times 2$	M1	3.1a
	Angle $AOB = 2 \times \arccos\left(\frac{9.5}{10}\right) = 0.635 \text{ rads (3sf)} *$	A1*	2.1
		(4)	
(b)	Attempts $10 \times (2\pi - 0.635) = 56.48$	M1	1.1b
	Attempts to find angle AXB or AXO in circle C_2 (see diagram) E.g. $\cos \beta = \frac{15 - "9.5"}{\sqrt{40}} \Rightarrow \beta = \dots$ (Note $AXB = 1.03 \text{ rads}$)	M1	3.1a
	Attempts $10 \times (2\pi - 0.635) + \sqrt{40} \times (2\pi - 2\beta)$	dM1	2.1
	$= 89.7$	A1	1.1b
		(4)	
			(8 marks)



14. A circle C with radius r

- lies only in the 1st quadrant
- touches the x -axis and touches the y -axis

The line l has equation $2x + y = 12$

(a) Show that the x coordinates of the points of intersection of l with C satisfy

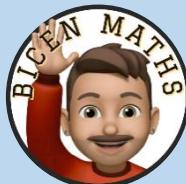
$$5x^2 + (2r - 48)x + (r^2 - 24r + 144) = 0 \quad (3)$$

Given also that l is a tangent to C ,

(b) find the two possible values of r , giving your answers as fully simplified surds. (4)



14 (a)	<p>C is</p> $(x-r)^2 + (y-r)^2 = r^2 \quad \text{or} \quad x^2 + y^2 - 2rx - 2ry + r^2 = 0$ $y = 12 - 2x, \quad x^2 + y^2 - 2rx - 2ry + r^2 = 0$ $\Rightarrow x^2 + (12 - 2x)^2 - 2rx - 2r(12 - 2x) + r^2 = 0$ <p style="text-align: center;">or</p>	B1	2.2a
	$y = 12 - 2x, \quad (x-r)^2 + (y-r)^2 = r^2$ $\Rightarrow (x-r)^2 + (12 - 2x - r)^2 = r^2$		
	$x^2 + 144 - 48x + 4x^2 - 2rx - 24r + 4rx + r^2 = 0$ $\Rightarrow 5x^2 + (2r - 48)x + (r^2 - 24r + 144) = 0 \quad *$	A1*	2.1
		(3)	
(b)	$b^2 - 4ac = 0 \Rightarrow (2r - 48)^2 - 4 \times 5 \times (r^2 - 24r + 144) = 0$ $r^2 - 18r + 36 = 0 \quad \text{or any multiple of this equation}$ $\Rightarrow (r-9)^2 - 81 + 36 = 0 \Rightarrow r = \dots$ $r = 9 \pm 3\sqrt{5}$	M1	3.1a
		A1	1.1b
		dM1	1.1b
		A1	1.1b
		(4)	
	(7 marks)		



7. The circle C has equation

$$x^2 + y^2 - 10x + 4y + 11 = 0$$

(a) Find

- (i) the coordinates of the centre of C ,
- (ii) the exact radius of C , giving your answer as a simplified surd.

(4)

The line l has equation $y = 3x + k$ where k is a constant.

Given that l is a tangent to C ,

- (b) find the possible values of k , giving your answers as simplified surds.

(5)



Question	Scheme	Marks	AOs
7(a)(i)	$(x-5)^2 + (y+2)^2 = \dots$	M1	1.1b
	$(5, -2)$	A1	1.1b
(ii)	$r = \sqrt{5^2 + (-2)^2 - 11}$	M1	1.1b
	$r = 3\sqrt{2}$	A1	1.1b
		(4)	
(b)	$y = 3x + k \Rightarrow x^2 + (3x+k)^2 - 10x + 4(3x+k) + 11 = 0$	M1	2.1
	$\Rightarrow x^2 + 9x^2 + 6kx + k^2 - 10x + 12x + 4k + 11 = 0$		
	$\Rightarrow 10x^2 + (6k+2)x + k^2 + 4k + 11 = 0$	A1	1.1b
	$b^2 - 4ac = 0 \Rightarrow (6k+2)^2 - 4 \times 10 \times (k^2 + 4k + 11) = 0$	M1	3.1a
	$\Rightarrow 4k^2 + 136k + 436 = 0 \Rightarrow k = \dots$	M1	1.1b
	$k = -17 \pm 6\sqrt{5}$	A1	2.2a
		(5)	
(9 marks)			



3. A circle has equation

$$x^2 + y^2 - 10x + 16y = 80$$

(a) Find

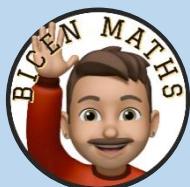
- (i) the coordinates of the centre of the circle,
- (ii) the radius of the circle.

(3)

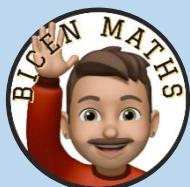
Given that P is the point on the circle that is furthest away from the origin O ,

(b) find the exact length OP

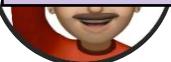
(2)



3 (a)	(i) $x^2 + y^2 - 10x + 16y = 80 \Rightarrow (x-5)^2 + (y+8)^2 = \dots$	M1
	Centre $(5, -8)$	A1
	(ii) Radius 13	A1
		(3)
(b)	Attempts $\sqrt{5^2 + 8^2} + 13$	M1
	$13 + \sqrt{89}$ but ft on their centre and radius	A1ft
		(2)



Algebraic Methods – algebraic fractions, polynomial division, factor theorem, proof



11. (a) Prove that for all positive values of x and y

$$\sqrt{xy} \leq \frac{x+y}{2} \quad (2)$$

(b) Prove by counter example that this is not true when x and y are both negative.

(1)

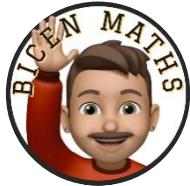


Question	Scheme	Marks	AOs
11 (a) Way 1	Since x and y are positive, their square roots are real and so $(\sqrt{x} - \sqrt{y})^2 \geq 0$ giving $x - 2\sqrt{xy} + y \geq 0$	M1	2.1
	$\therefore 2\sqrt{xy} \leq x + y$ provided x and y are positive and so $\sqrt{xy} \leq \frac{x + y}{2}$ *	A1*	2.2a
		(2)	
Way 2 Longer method	Since $(x - y)^2 \geq 0$ for real values of x and y , $x^2 - 2xy + y^2 \geq 0$ and so $4xy \leq x^2 + 2xy + y^2$ i.e. $4xy \leq (x + y)^2$	M1	2.1
	$\therefore 2\sqrt{xy} \leq x + y$ provided x and y are positive and so $\sqrt{xy} \leq \frac{x + y}{2}$ *	A1*	2.2a
		(2)	
(b)	Let $x = -3$ and $y = -5$ then LHS = $\sqrt{15}$ and RHS = -4 so as $\sqrt{15} > -4$ result does not apply	B1	2.4
		(1)	
(3 marks)			



15. Given $n \in \mathbb{N}$, prove that $n^3 + 2$ is not divisible by 8

(4)



Question 15 (Total 4 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	If n is even, $n = 2k$ and $n^3 + 2 = (2k)^3 + 2$ $= 8k^3 + 2$	M1	This mark is given for finding expressions for n and $n^3 + 2$ when n is even
	$8k^3 + 2$ is two more than a multiple of 8 and so not divisible by 8	A1	This mark is given for a correct conclusion following correct working
	If n is odd, $n = 2k + 1$ and $n^3 + 2 = (2k + 1)^3 + 2$ $= 8k^3 + 12k^2 + 6k + 3$	M1	This mark is given for finding expressions for n and $n^3 + 2$ when n is odd
	$8k^3 + 12k^2 + 6k + 3$ is an odd number and so not divisible by 8	A1	This mark is given for a correct conclusion following correct working



13. (a) Prove that for all positive values of a and b

$$\frac{4a}{b} + \frac{b}{a} \geq 4 \quad (4)$$

(b) Prove, by counter example, that this is not true for all values of a and b .

(1)



Question	Scheme	Marks	AOs
13 (a)	States $(2a - b)^2 \dots 0$	M1	2.1
	$4a^2 + b^2 \dots 4ab$	A1	1.1b
	(As $a > 0, b > 0$) $\frac{4a^2}{ab} + \frac{b^2}{ab} \dots \frac{4ab}{ab}$	M1	2.2a
	Hence $\frac{4a}{b} + \frac{b}{a} \dots 4$ *	CSO	A1* 1.1b
		(4)	
(b)	$a = 5, b = -1 \Rightarrow \frac{4a}{b} + \frac{b}{a} = -20 - \frac{1}{5}$ which is less than 4	B1	2.4
		(1)	
(5 marks)			

* .



10. A student is investigating the following statement about natural numbers.

$$\text{“}n^3 - n \text{ is a multiple of 4”}$$

(a) Prove, using algebra, that the statement is true for all odd numbers.

(4)

(b) Use a counterexample to show that the statement is not always true.

(1)



Question	Scheme	Marks	AOs
10(a)	Selects a correct strategy. E.g uses an odd number is $2k \pm 1$	B1	3.1a
	Attempts to simplify $(2k \pm 1)^3 - (2k \pm 1) = \dots$	M1	2.1
and factorise $8k^3 \pm 12k^2 \pm 4k = 4k(2k^2 \pm 3k \pm 1) =$	dM1	1.1b
	Correct work with statement $4 \times \dots$ is a multiple of 4	A1	2.4
		(4)	
(b)	Any counter example with correct statement. Eg. $2^3 - 2 = 6$ which is not a multiple of 4	B1	2.4
		(1)	
(5 marks)			
Alt (a)	Selects a correct strategy. Factorises $k^3 - k = k(k-1)(k+1)$	B1	3.1a
	States that if k is odd then both $k-1$ and $k+1$ are even	M1	2.1
	States that $k-1$ multiplied by $k+1$ is therefore a multiple of 4	dM1	1.1b
	Concludes that $k^3 - k$ is a multiple of 4 as it is odd \times multiple of 4	A1	2.4
		(4)	
Notes:			



14. (i) A student states

“if x^2 is greater than 9 then x must be greater than 3”

Determine whether or not this statement is true, giving a reason for your answer.

(1)

(ii) Prove that for all positive integers n ,

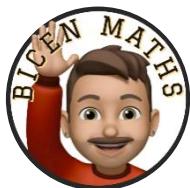
$$n^3 + 3n^2 + 2n$$

is divisible by 6

(3)



14(i)	The statement is not true because e.g. when $x = -4$, $x^2 = 16$ (which is > 9 but $x < 3$)	B1 (1)
(ii)	$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2)$ $n(n + 1)(n + 2)$ is the product of 3 consecutive integers As $n(n + 1)(n + 2)$ is a multiple of 2 and a multiple of 3 it must be a multiple of 6 and so $n^3 + 3n^2 + 2n$ is divisible by 6 for all integers n	M1 A1 A1 (3)



16. Use algebra to prove that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3

(4)



Question	Scheme	Marks	AOs
16	NB any natural number can be expressed in the form: $3k, 3k + 1, 3k + 2$ or equivalent e.g. $3k - 1, 3k, 3k + 1$		
	Attempts to square any two distinct cases of the above	M1	3.1a
	Achieves accurate results and makes a valid comment for any two of the possible three cases: E.g. $(3k)^2 = 9k^2 (= 3 \times 3k^2)$ is a multiple of 3	A1 M1 on EPEN	1.1b
	$(3k+1)^2 = 9k^2 + 6k + 1 = 3 \times (3k^2 + 2k) + 1$ is one more than a multiple of 3 $(3k+2)^2 = 9k^2 + 12k + 4 = 3 \times (3k^2 + 4k + 1) + 1$ (or $(3k-1)^2 = 9k^2 - 6k + 1 = 3 \times (3k^2 - 2k) + 1$) is one more than a multiple of 3		
	Attempts to square in all 3 distinct cases. E.g. attempts to square $3k, 3k + 1, 3k + 2$ or e.g. $3k - 1, 3k, 3k + 1$	M1 A1 on EPEN	2.1
	Achieves accurate results for all three cases and gives a minimal conclusion (allow tick, QED etc.)	A1	2.4
		(4)	
	(4 marks)		

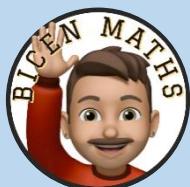


11. Prove, using algebra, that

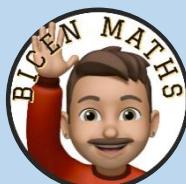
$$n(n^2 + 5)$$

is even for all $n \in \mathbb{N}$.

(4)



11	$n(n^2 + 5)$ Attempts even or odd numbers Sets $n = 2k$ or $n = 2k \pm 1$ oe and attempts $n(n^2 + 5)$	
	Achieves $2k(4k^2 + 5)$ (for $n = 2k$) and states “even” Or achieves $(2k+1)(4k^2 + 4k + 6) = 2(2k+1)(2k^2 + 2k + 3)$ (for $n = 2k + 1$) and states “even” Or e.g. achieves $(2k-1)(4k^2 - 4k + 6) = 2(2k-1)(2k^2 - 2k + 3)$ (for $n = 2k - 1$) and states “even”	M1 A1
	Attempts even and odd numbers Sets $n = 2k$ and $n = 2k \pm 1$ oe and attempts $n(n^2 + 5)$	dM1
	Achieves $2k(4k^2 + 5)$ (for $n = 2k$) and states “even” and achieves $(2k \pm 1)(4k^2 \pm 4k + 6) = 2(2k \pm 1)(2k^2 \pm 2k + 3)$ (for $n = 2k \pm 1$) and states “even” Correct work and states even for both WITH a final conclusion showing that true for all $n (\in \mathbb{N})$ or e.g. true for all even and odd numbers.	A1
		(4)



Trigonometry – Sine and Cosine Rules



8.

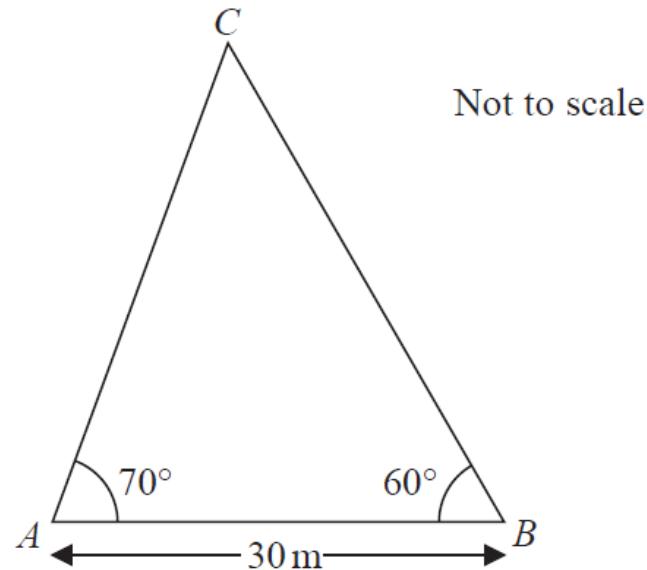


Figure 1

A triangular lawn is modelled by the triangle ABC , shown in Figure 1. The length AB is to be 30 m long.

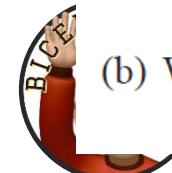
Given that angle $BAC = 70^\circ$ and angle $ABC = 60^\circ$,

(a) calculate the area of the lawn to 3 significant figures.

(4)

(b) Why is your answer unlikely to be accurate to the nearest square metre?

(1)



Question	Scheme	Marks	AOs
8(a)	Way 1 Finds third angle of triangle and uses or states $\frac{x}{\sin 60^\circ} = \frac{30}{\sin "50^\circ"}$	Way 2 Finds third angle of triangle and uses or states $\frac{y}{\sin 70^\circ} = \frac{30}{\sin "50^\circ"}$	M1 2.1
	So $x = \frac{30 \sin 60^\circ}{\sin 50^\circ}$ (= 33.9)		
	Area = $\frac{1}{2} \times 30 \times x \times \sin 70^\circ$ or $\frac{1}{2} \times 30 \times y \times \sin 60^\circ$ = 478 m ²	A1	1.1b
		M1	3.1a
		A1ft	1.1b
			(4)
(b)	Plausible reason e.g. Because the angles and the side length are not given to four significant figures Or e.g. The lawn may not be flat	B1	3.2b
			(1)
			(5 marks)



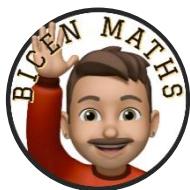
7. In a triangle ABC , side AB has length 10 cm, side AC has length 5 cm, and angle $BAC = \theta$ where θ is measured in degrees. The area of triangle ABC is 15 cm^2
- (a) Find the two possible values of $\cos \theta$ (4)

Given that BC is the longest side of the triangle,

- (b) find the exact length of BC . (2)



Question	Scheme	Marks	AOs
7 (a)	Uses $15 = \frac{1}{2} \times 5 \times 10 \times \sin \theta$	M1	1.1b
	$\sin \theta = \frac{3}{5}$ oe	A1	1.1b
	Uses $\cos^2 \theta = 1 - \sin^2 \theta$	M1	2.1
	$\cos \theta = \pm \frac{4}{5}$	A1	1.1b
		(4)	
(b)	Uses $BC^2 = 10^2 + 5^2 - 2 \times 10 \times 5 \times -\frac{4}{5}$ "	M1	3.1a
	$BC = \sqrt{205}$	A1	1.1b
		(2)	
		(6 marks)	



6.

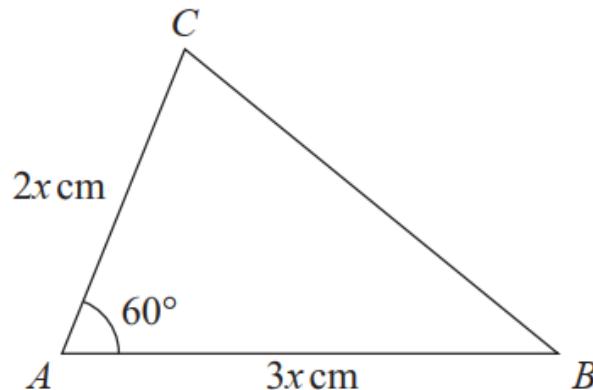


Figure 1

Figure 1 shows a sketch of a triangle ABC with $AB = 3x$ cm, $AC = 2x$ cm and angle $CAB = 60^\circ$

Given that the area of triangle ABC is $18\sqrt{3}$ cm²

- (a) show that $x = 2\sqrt{3}$ (3)

- (b) Hence find the exact length of BC , giving your answer as a simplified surd. (3)



Question 6 (Total 6 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$18\sqrt{3} = \frac{1}{2} \times 2x \times 3x \times \sin 60^\circ$	M1	This mark is given for use of the formula $A = \frac{1}{2} ab \sin C$ for the area of the triangle
	$18\sqrt{3} = 3x^2 \times \frac{\sqrt{3}}{2}$ $x^2 = 12$	M1	This mark is given for using a value of $\sin 60^\circ$ to find a value for x^2
	$x = \sqrt{12}$ $= \sqrt{(4 \times 3)}$ $= 2\sqrt{3}$	A1	This mark is given for a full solution to show that $x = 2\sqrt{3}$
(b)	$BC^2 =$ $(6\sqrt{3})^2 + (4\sqrt{3})^2 - 2 \times 6\sqrt{3} \times 4\sqrt{3} \times \cos 60^\circ$	M1	This mark is given for using the cosine rule to start to find the length BC
	$BC^2 = 84$	A1	This mark is given for finding a value for BC^2
	$BC = 2\sqrt{21}$	A1	This mark is given for a correct answer presented as a simplified surd



5.

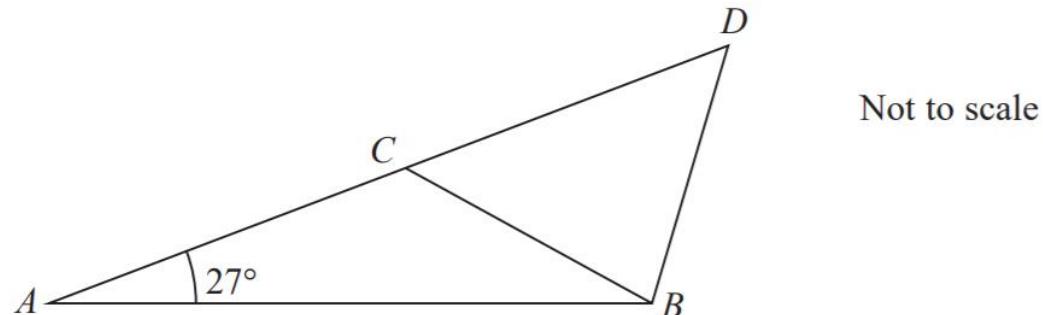
**Figure 1**

Figure 1 shows the design for a structure used to support a roof.

The structure consists of four steel beams, AB , BD , BC and AD .

Given $AB = 12 \text{ m}$, $BC = BD = 7\text{m}$ and angle $BAC = 27^\circ$

(a) find, to one decimal place, the size of angle ACB .

(3)

The steel beams can only be bought in whole metre lengths.

(b) Find the minimum length of steel that needs to be bought to make the complete structure.

(3)



Question	Scheme	Marks	AOs
5 (a)	States $\frac{\sin \theta}{12} = \frac{\sin 27}{7}$	M1	1.1b
	Finds $\theta = \text{awrt } 51^\circ$ or $\text{awrt } 129^\circ$	A1	1.1b
	$= \text{awrt } 128.9^\circ$	A1	1.1b
		(3)	
(b)	Attempts to find part or all of AD Eg $AD^2 = 7^2 + 12^2 - 2 \times 12 \times 7 \cos 101.9 = (AD = 15.09)$		
	Eg $(AC)^2 = 7^2 + 12^2 - 2 \times 12 \times 7 \cos(180 - "128.9" - 27)$	M1	1.1b
	Eg $12 \cos 27$ or $7 \cos "51"$		
	Full method for the total length = $12 + 7 + 7 + "15.09" =$	dM1	3.1a
	$= 42 \text{ m}$	A1	3.2a
		(3)	
(6 marks)			



7. A parallelogram $PQRS$ has area 50 cm^2

Given

- PQ has length 14 cm
- QR has length 7 cm
- angle SPQ is obtuse

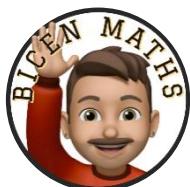
find

(a) the size of angle SPQ , in degrees, to 2 decimal places,

(3)

(b) the length of the diagonal SQ , in cm, to one decimal place.

(2)



Question	Scheme	Marks	AOs
7 (a)	Sets $50 = 7 \times 14 \sin(SPQ)$ oe	B1	1.2
	Finds $180^\circ - \arcsin\left(\frac{50}{98}\right)$	M1	1.1b
	$= 149.32^\circ$	A1	1.1b
		(3)	
(b)	Method of finding SQ $SQ^2 = 14^2 + 7^2 - 2 \times 14 \times 7 \cos 149.32^\circ$	M1	1.1b
	$= 20.3$ cm	A1	1.1b
		(2)	
		(5 marks)	
Alt(a)	States or uses $14h = 50$ or $7h_1 = 50$	B1	1.2
	Full method to find obtuse $\angle SPQ$. In this case it is $90^\circ + \arccos\left(\frac{h}{7}\right)$ or $90^\circ + \arccos\left(\frac{h_1}{14}\right)$	M1	1.1b
	awrt 149.32°	A1	1.1b



4.

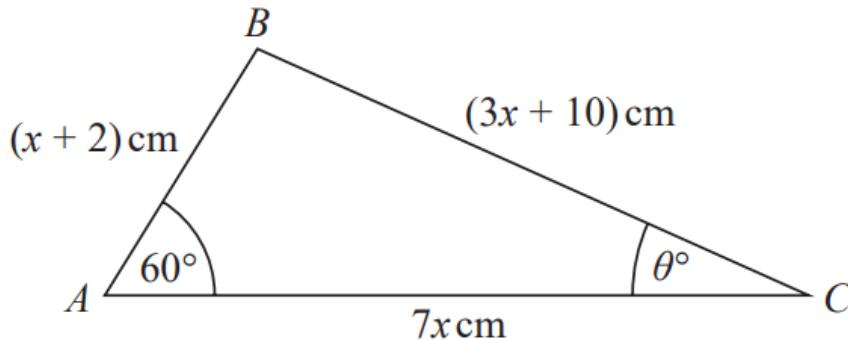
**Figure 1**

Figure 1 shows a sketch of triangle ABC with $AB = (x + 2)$ cm, $BC = (3x + 10)$ cm, $AC = 7x$ cm, angle $BAC = 60^\circ$ and angle $ACB = \theta^\circ$

(a) (i) Show that $17x^2 - 35x - 48 = 0$

(3)

(ii) Hence find the value of x .

(1)

(b) Hence find the value of θ giving your answer to one decimal place.

(2)



4(a)(i)

$$(3x+10)^2 = (x+2)^2 + (7x)^2 - 2(x+2)(7x)\cos 60^\circ \quad \text{oe}$$

M1

Uses $\cos 60^\circ = \frac{1}{2}$, expands the brackets and proceeds to a 3 term quadratic equation

dM1

$$17x^2 - 35x - 48 = 0 *$$

A1*

(3)

$$x = 3$$

B1

(1)

(ii)

$$\frac{5}{\sin ACB} = \frac{19}{\sin 60^\circ} \Rightarrow \sin ACB = \dots \left(\frac{5\sqrt{3}}{38} \right)$$

M1

or e.g.

$$5^2 = 21^2 + 19^2 - 2 \times 19 \times 21 \cos ACB \Rightarrow \cos ACB = \dots \left(\frac{37}{38} \right)$$

(b)

$$\theta = \text{awrt } 13.2$$

A1

(2)



Vectors



3. Given that the point A has position vector $3\mathbf{i} - 7\mathbf{j}$ and the point B has position vector $8\mathbf{i} + 3\mathbf{j}$,
- find the vector \overrightarrow{AB} (2)
 - Find $|\overrightarrow{AB}|$. Give your answer as a simplified surd. (2)



Question	Scheme	Marks	AOs
3(a)	Attempts $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ or similar	M1	1.1b
	$\overrightarrow{AB} = 5\mathbf{i} + 10\mathbf{j}$	A1	1.1b
		(2)	
(b)	Finds length using 'Pythagoras' $ AB = \sqrt{(5)^2 + (10)^2}$	M1	1.1b
	$ AB = 5\sqrt{5}$	A1ft	1.1b
		(2)	
(4 marks)			



3. Given that the point A has position vector $4\mathbf{i} - 5\mathbf{j}$ and the point B has position vector $-5\mathbf{i} - 2\mathbf{j}$,

(a) find the vector \overrightarrow{AB} ,

(2)

(b) find $|\overrightarrow{AB}|$.

Give your answer as a simplified surd.

(2)



Question	Scheme	Marks	AOs
3(a)	Attempts $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ or similar	M1	1.1b
	$\overrightarrow{AB} = -9\mathbf{i} + 3\mathbf{j}$	A1	1.1b
		(2)	
(b)	Finds length using 'Pythagoras' $ AB = \sqrt{(-9)^2 + (3)^2}$	M1	1.1b
	$ AB = 3\sqrt{10}$	A1ft	1.1b
		(2)	
(4 marks)			



16. (i) Two non-zero vectors, \mathbf{a} and \mathbf{b} , are such that

$$|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$$

Explain, geometrically, the significance of this statement.

(1)

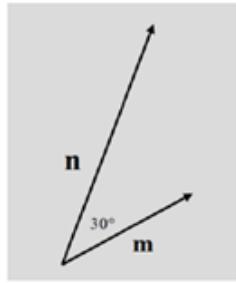
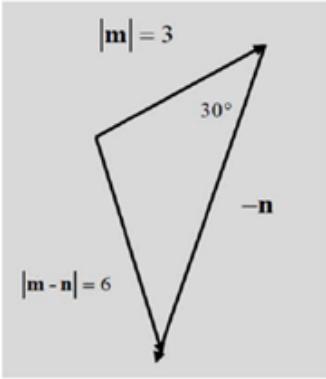
- (ii) Two different vectors, \mathbf{m} and \mathbf{n} , are such that $|\mathbf{m}| = 3$ and $|\mathbf{m} - \mathbf{n}| = 6$
The angle between vector \mathbf{m} and vector \mathbf{n} is 30°

Find the angle between vector \mathbf{m} and vector $\mathbf{m} - \mathbf{n}$, giving your answer, in degrees,
to one decimal place.

(4)



Question 16 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(i)	a and b lie in the same direction	B1	This mark is given for a valid explanation
(ii)	 	M1	This mark is given for showing the vector problem graphically (may be implied)
	$\frac{\sin 30^\circ}{6} = \frac{\sin \theta}{3}$ $\sin \theta = \frac{1.5}{6} = \frac{1}{4}$	M1	This mark is given for using the sine rule as a method to find the angle between $-\mathbf{n}$ and $\mathbf{n} - \mathbf{m}$
	$\theta = 14.5^\circ$	A1	This mark is given for finding the the angle between $-\mathbf{n}$ and $\mathbf{n} - \mathbf{m}$
	Angle between m and m - n $= (180 - 30 - 14.5) = 135.5^\circ$	A1	This mark is given for the angle between vector m and vector m - n



2. [In this question the unit vectors \mathbf{i} and \mathbf{j} are due east and due north respectively.]

A coastguard station O monitors the movements of a small boat.

At 10:00 the boat is at the point $(4\mathbf{i} - 2\mathbf{j})$ km relative to O .

At 12:45 the boat is at the point $(-3\mathbf{i} - 5\mathbf{j})$ km relative to O .

The motion of the boat is modelled as that of a particle moving in a straight line at constant speed.

- (a) Calculate the bearing on which the boat is moving, giving your answer in degrees to one decimal place.

(3)

- (b) Calculate the speed of the boat, giving your answer in km h^{-1}

(3)



Question	Scheme	Marks	AOs
2(a)			
	Attempts to find an "allowable" angle Eg $\tan \theta = \frac{7}{3}$	M1	1.1b
	A full attempt to find the bearing Eg $180^\circ + 67^\circ$	dM1	3.1b
	Bearing = awrt 246.8°	A1	1.1b
		(3)	
(b)	Attempts to find the distance travelled = $\sqrt{(4-(-3))^2 + (-2+5)^2} = (\sqrt{58})$	M1	1.1b
	Attempts to find the speed = $\frac{\sqrt{58}}{2.75}$	dM1	3.1b
	= awrt 2.77 km h^{-1}	A1	1.1b
		(3)	
(6 marks)			



4. [In this question the unit vectors \mathbf{i} and \mathbf{j} are due east and due north respectively.]

A stone slides horizontally across ice.

Initially the stone is at the point $A(-24\mathbf{i} - 10\mathbf{j})$ m relative to a fixed point O .

After 4 seconds the stone is at the point $B(12\mathbf{i} + 5\mathbf{j})$ m relative to the fixed point O .

The motion of the stone is modelled as that of a particle moving in a straight line at constant speed.

Using the model,

(a) prove that the stone passes through O ,

(2)

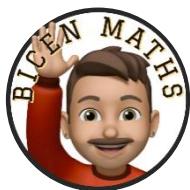
(b) calculate the speed of the stone.

(3)



Question	Scheme	Marks	AOs
4(a)	Attempts to compare the two position vectors. Allow an attempt using two of \overrightarrow{AO} , \overrightarrow{OB} or \overrightarrow{AB} E.g. $(-24\mathbf{i} - 10\mathbf{j}) = -2 \times (12\mathbf{i} + 5\mathbf{j})$	M1	1.1b
	Explains that as \overrightarrow{AO} is parallel to \overrightarrow{OB} (and the stone is travelling in a straight line) the stone passes through the point O .	A1	
			(2)
(b)	Attempts distance $AB = \sqrt{(12+24)^2 + (10+5)^2}$	M1	
	Attempts speed = $\frac{\sqrt{(12+24)^2 + (10+5)^2}}{4}$	dM1	Alternatively, allow an attempt finding the gradient using any two of AO , OB or AB Alternatively attempts to find the equation of the line through A and B proceeding as far as $y = ...x$ Condone sign slips.
	Speed = 9.75 ms^{-1}	A1	
			(3)
		(5)	
Alt(a)	Attempts to find the equation of the line which passes through A and B E.g. $y - 5 = \frac{5+10}{12+24}(x-12)$ ($y = \frac{5}{12}x$)	M1	
	Shows that when $x=0$, $y=0$ and concludes the stone passes through the point O .	A1	

Notes
(a)
M1: Attempts to compare the two position vectors. Allow an attempt using two of \overrightarrow{AO} , \overrightarrow{OB} or \overrightarrow{AB} either way around. E.g. States that $(-24\mathbf{i} - 10\mathbf{j}) = -2 \times (12\mathbf{i} + 5\mathbf{j})$
dM1: Dependent upon the previous mark. Look for an attempt at $\sqrt{a^2 + b^2}$ where a or b is correct
A1: States that as \overrightarrow{AO} is parallel to \overrightarrow{OB} or as AO is parallel to OB (and the stone is travelling in a straight line) the stone passes through the point O . Alternatively, shows that the point $(0,0)$ is on the line and concludes (the stone) passes through the point O .
(b)
M1: Attempts to find the distance AB using a correct method. Condone slips but expect to see an attempt at $\sqrt{a^2 + b^2}$ where a or b is correct
A1: 9.75 ms^{-1} Requires units



3. The triangle PQR is such that $\vec{PQ} = 3\mathbf{i} + 5\mathbf{j}$ and $\vec{PR} = 13\mathbf{i} - 15\mathbf{j}$

(a) Find \vec{QR}

(2)

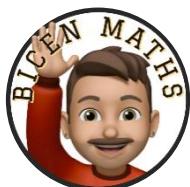
(b) Hence find $|\vec{QR}|$ giving your answer as a simplified surd.

(2)

The point S lies on the line segment QR so that $QS:SR = 3:2$

(c) Find \vec{PS}

(2)



3(a)

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{PR} - \overrightarrow{PQ} = 13\mathbf{i} - 15\mathbf{j} - (3\mathbf{i} + 5\mathbf{j}) \\ &= 10\mathbf{i} - 20\mathbf{j}\end{aligned}$$

M1

A1

(2)

(b)

$$|\overrightarrow{QR}| = \sqrt{10^2 + (-20)^2}$$

M1

$$= 10\sqrt{5}$$

A1ft

(2)

(c)

$$\overrightarrow{PS} = \overrightarrow{PQ} + \frac{3}{5} \overrightarrow{QR} = 3\mathbf{i} + 5\mathbf{j} + \frac{3}{5}(10\mathbf{i} - 20\mathbf{j}) = \dots$$

or

M1

$$\overrightarrow{PS} = \overrightarrow{PR} + \frac{2}{5} \overrightarrow{RQ} = 13\mathbf{i} - 15\mathbf{j} + \frac{2}{5}(-10\mathbf{i} + 20\mathbf{j}) = \dots$$

$$= 9\mathbf{i} - 7\mathbf{j}$$

A1

(2)



10.

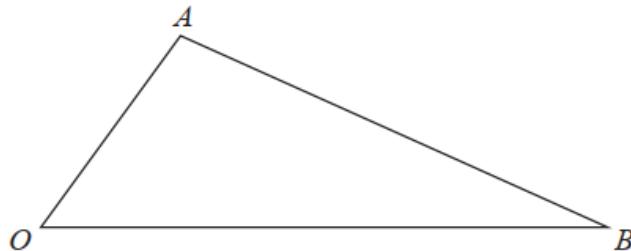


Figure 7

Figure 7 shows a sketch of triangle OAB .

The point C is such that $\overrightarrow{OC} = 2\overrightarrow{OA}$.

The point M is the midpoint of AB .

The straight line through C and M cuts OB at the point N .

Given $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$

(a) Find \overrightarrow{CM} in terms of \mathbf{a} and \mathbf{b}

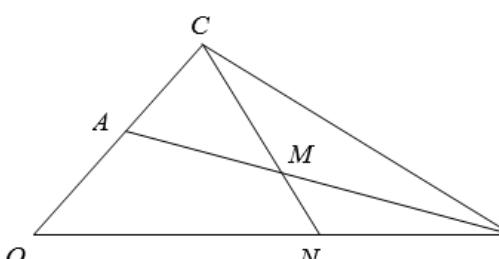
(2)

(b) Show that $\overrightarrow{ON} = \left(2 - \frac{3}{2}\lambda\right)\mathbf{a} + \frac{1}{2}\lambda\mathbf{b}$, where λ is a scalar constant.

(2)

(c) Hence prove that $ON:NB = 2:1$

(2)

Part	Working or answer an examiner might expect to see	Mark	Notes
			
(a)	$\overrightarrow{CM} = \overrightarrow{CA} + \overrightarrow{AM} = \overrightarrow{CA} + \frac{1}{2} \overrightarrow{AB}$	M1	This mark is given for a method to find an expression for \overrightarrow{CM}
	$\overrightarrow{CM} = -\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = -\frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$	A1	This mark is given for a correct expression for \overrightarrow{CM} in terms of \mathbf{a} and \mathbf{b}
(b)	$\overrightarrow{ON} = \overrightarrow{OC} + \overrightarrow{CN} = \overrightarrow{OC} + \lambda \overrightarrow{CM}$	M1	This mark is given for a method to find an expression for \overrightarrow{ON}
	$\begin{aligned}\overrightarrow{ON} &= 2\mathbf{a} + \lambda \left(-\frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} \right) \\ &= \left(2 - \frac{3}{2}\lambda \right) \mathbf{a} + \left(\frac{1}{2}\lambda \right) \mathbf{b}\end{aligned}$	A1	This mark is given for a correct expression for \overrightarrow{ON} in terms of \mathbf{a} and \mathbf{b}
(c)	$\left(2 - \frac{3}{2}\lambda \right) = 0 \text{ so } \lambda = \frac{4}{3}$	M1	This mark is given for deducing that the coefficient of $\mathbf{a} = 0$ and finding a value for λ
	$\overrightarrow{ON} = 0 \times \mathbf{a} + \frac{2}{3}\mathbf{b}$ <p>Hence $ON:NB = \frac{2}{3} : \frac{1}{3} = 2:1$</p>	A1	This mark is given for finding \overrightarrow{ON} and giving a valid conclusion

2. Relative to a fixed origin, points P , Q and R have position vectors \mathbf{p} , \mathbf{q} and \mathbf{r} respectively.

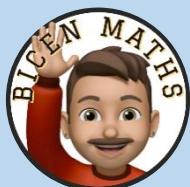
Given that

- P , Q and R lie on a straight line
- Q lies one third of the way from P to R

show that

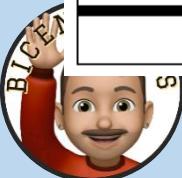
$$\mathbf{q} = \frac{1}{3}(\mathbf{r} + 2\mathbf{p})$$

(3)



Question Number	Scheme	Marks	AO's
2	<p>Attempts any one of $(\pm \overrightarrow{PQ} =) \pm (\mathbf{q} - \mathbf{p})$, $(\pm \overrightarrow{PR} =) \pm (\mathbf{r} - \mathbf{p})$, $(\pm \overrightarrow{QR} =) \pm (\mathbf{r} - \mathbf{q})$</p> <p>Or e.g. $(\pm \overrightarrow{PQ} =) \pm (\overrightarrow{OQ} - \overrightarrow{OP})$, $(\pm \overrightarrow{PR} =) \pm (\overrightarrow{OR} - \overrightarrow{OP})$, $(\pm \overrightarrow{QR} =) \pm (\overrightarrow{OR} - \overrightarrow{OQ})$</p>	M1	1.1b
	<p>Attempts e.g.</p> $\mathbf{r} - \mathbf{q} = 2(\mathbf{q} - \mathbf{p})$ $\mathbf{r} - \mathbf{p} = 3(\mathbf{q} - \mathbf{p})$ $\frac{2}{3}(\mathbf{q} - \mathbf{p}) = \frac{1}{3}(\mathbf{r} - \mathbf{q})$ $\mathbf{q} = \mathbf{p} + \frac{1}{3}(\mathbf{r} - \mathbf{p})$ $\mathbf{q} = \mathbf{r} + \frac{2}{3}(\mathbf{p} - \mathbf{r})$	dM1	3.1a
	<p>E.g.</p> $\Rightarrow \mathbf{r} - \mathbf{q} = 2\mathbf{q} - 2\mathbf{p} \Rightarrow 2\mathbf{p} + \mathbf{r} = 3\mathbf{q} \Rightarrow \mathbf{q} = \frac{1}{3}(\mathbf{r} + 2\mathbf{p})^*$	A1*	2.1
		(3)	

(3 marks)



Differentiation



16.

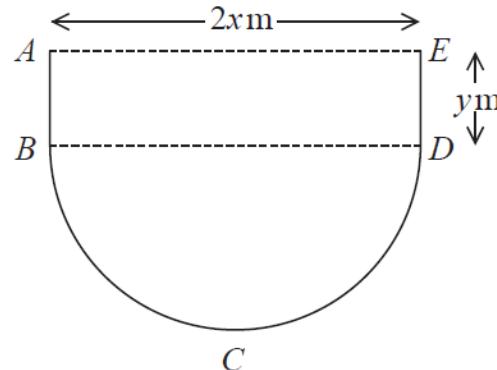
**Figure 4**

Figure 4 shows the plan view of the design for a swimming pool.

The shape of this pool $ABCDE$ consists of a rectangular section $ABDE$ joined to a semicircular section BCD as shown in Figure 4.

Given that $AE = 2x$ metres, $ED = y$ metres and the area of the pool is 250 m^2 ,

- (a) show that the perimeter, P metres, of the pool is given by

$$P = 2x + \frac{250}{x} + \frac{\pi x}{2} \quad (4)$$

- (b) Explain why $0 < x < \sqrt{\frac{500}{\pi}}$ (2)

- (c) Find the minimum perimeter of the pool, giving your answer to 3 significant figures. (4)



Question	Scheme	Marks	AOs
16(a)	Sets $2xy + \frac{\pi x^2}{2} = 250$	B1	2.1
	Obtain $y = \frac{250 - \frac{\pi x^2}{2}}{2x}$ and substitute into P	M1	1.1b
	Use $P = 2x + 2y + \pi x$ with their y substituted	M1	2.1
	$P = 2x + \frac{250}{x} - \frac{\pi x^2}{2x} + \pi x = 2x + \frac{250}{x} + \frac{\pi x}{2}$ *	A1*	1.1b
		(4)	
(b)	$x > 0$ and $y > 0$ (distance) $\Rightarrow \frac{250 - \frac{\pi x^2}{2}}{2x} > 0$ or $250 - \frac{\pi x^2}{2} > 0$ o.e.	M1	2.4
	As x and y are distances they are positive so $0 < x < \sqrt{\frac{500}{\pi}}$ *	A1*	3.2a
		(2)	
(c)	Differentiates P with negative index correct in $\frac{dP}{dx}; x^{-1} \rightarrow x^{-2}$	M1	3.4
	$\frac{dP}{dx} = 2 - \frac{250}{x^2} + \frac{\pi}{2}$	A1	1.1b
	Sets $\frac{dP}{dx} = 0$ and proceeds to $x =$	M1	1.1b
	Substitutes their x into $P = 2x + \frac{250}{x} + \frac{\pi x}{2}$ to give perimeter = 59.8 M	A1	1.1b
		(4)	
		(10 marks)	



8. A lorry is driven between London and Newcastle.

In a simple model, the cost of the journey £ C when the lorry is driven at a steady speed of v kilometres per hour is

$$C = \frac{1500}{v} + \frac{2v}{11} + 60$$

(a) Find, according to this model,

(i) the value of v that minimises the cost of the journey,

(ii) the minimum cost of the journey.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

(b) Prove by using $\frac{d^2C}{dv^2}$ that the cost is minimised at the speed found in (a)(i).

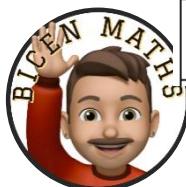
(2)

(c) State one limitation of this model.

(1)



Question	Scheme	Marks	AOs
8 (a)(i)	$C = \frac{1500}{v} + \frac{2v}{11} + 60 \Rightarrow \frac{dC}{dv} = -\frac{1500}{v^2} + \frac{2}{11}$	M1 A1	3.1b 1.1b
	Sets $\frac{dC}{dv} = 0 \Rightarrow v^2 = 8250$	M1	1.1b
	$\Rightarrow v = \sqrt{8250} \Rightarrow v = 90.8 \text{ (km h}^{-1}\text{)}$	A1	1.1b
(ii)	For substituting their $v = 90.8$ in $C = \frac{1500}{v} + \frac{2v}{11} + 60$	M1	3.4
	Minimum cost = awrt (£) 93	A1 ft	1.1b
		(6)	
(b)	Finds $\frac{d^2C}{dv^2} = +\frac{3000}{v^3}$ at $v = 90.8$	M1	1.1b
	$\frac{d^2C}{dv^2} = (+0.004) > 0$ hence minimum (cost)	A1 ft	2.4
		(2)	
(c)	It would be impossible to drive at this speed over the whole journey	B1	3.5b
		(1)	
		(9 marks)	



5. A curve has equation

$$y = 3x^2 + \frac{24}{x} + 2 \quad x > 0$$

(a) Find, in simplest form, $\frac{dy}{dx}$

(3)

(b) Hence find the exact range of values of x for which the curve is increasing.

(2)



Question 5 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)		M1	This mark is given for recognising that x^n becomes x^{n-1} when differentiating
	$\frac{dy}{dx} = 6x - \frac{24}{x^2}$	A1	This mark is given for one of the two terms $6x$ or $-\frac{24}{x^2}$ given correctly
		A1	This mark is given for $\frac{dy}{dx}$ given fully correct
(b)	$6x - \frac{24}{x^2} > 0$	M1	This mark is given for setting $\frac{dy}{dx}$ greater than 0 (allow \geq)
	$6x^3 - 24 > 0$ $x^3 - 4 > 0$ $x > \sqrt[3]{4}$	A1	This mark is given for the exact range of values of x for which the curve is increasing (allow \geq)



1. A curve has equation

$$y = 2x^3 - 4x + 5$$

Find the equation of the tangent to the curve at the point $P(2, 13)$.

Write your answer in the form $y = mx + c$, where m and c are integers to be found.

Solutions relying on calculator technology are not acceptable.

(5)



Question	Scheme	Marks	AOs
1	Attempts to differentiate $x^n \rightarrow x^{n-1}$ seen once	M1	1.1b
	$y = 2x^3 - 4x + 5 \Rightarrow \frac{dy}{dx} = 6x^2 - 4$	A1	1.1b
	For substituting $x = 2$ into their $\frac{dy}{dx} = 6x^2 - 4$	dM1	1.1b
	For a correct method of finding a tangent at $P(2,13)$. Score for $y - 13 = "20"(x - 2)$	ddM1	1.1b
	$y = 20x - 27$	A1	1.1b
		(5)	
(5 marks)			



14. A curve has equation $y = g(x)$.

Given that

- $g(x)$ is a cubic expression in which the coefficient of x^3 is equal to the coefficient of x
- the curve with equation $y = g(x)$ passes through the origin
- the curve with equation $y = g(x)$ has a stationary point at $(2, 9)$

(a) find $g(x)$,

(7)

(b) prove that the stationary point at $(2, 9)$ is a maximum.

(2)



Question	Scheme	Marks	AOs
14 (a)	Deduces $g(x) = ax^3 + bx^2 + ax$	B1	2.2a
	Uses $(2, 9) \Rightarrow 9 = 8a + 4b + 2a$ $\Rightarrow 10a + 4b = 9$	M1 A1	2.1 1.1b
	Uses $g'(2) = 0 \Rightarrow 0 = 12a + 4b + a$ $\Rightarrow 13a + 4b = 0$	M1 A1	2.1 1.1b
	Solves simultaneously $\Rightarrow a, b$	dM1	1.1b
	$g(x) = -3x^3 + \frac{39}{4}x^2 - 3x$	A1	1.1b
			(7)
(b)	Attempts $g''(x) = -18x + \frac{39}{2}$ and substitutes $x = 2$	M1	1.1b
	$g''(2) = -\frac{33}{2} < 0$ hence maximum	A1	2.4
			(2)
(9 marks)			



5.

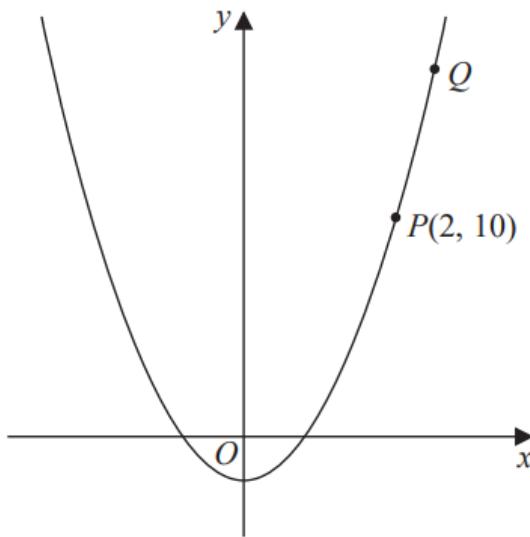
**Figure 1**

Figure 1 shows part of the curve with equation $y = 3x^2 - 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P .

(2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ , giving your answer in terms of h in simplest form.

(3)

(c) Explain briefly the relationship between part (b) and the answer to part (a).

(1)



Question	Scheme	Marks	AOs
5(a)	Attempts to find the value of $\frac{dy}{dx}$ at $x = 2$	M1	1.1b
	$\frac{dy}{dx} = 6x \Rightarrow$ gradient of tangent at P is 12	A1	1.1b
		(2)	
(b)	$\text{Gradient } PQ = \frac{3(2+h)^2 - 2 - 10}{(2+h) - 2} \text{ oe}$	B1	1.1b
	$= \frac{3(2+h)^2 - 12}{(2+h) - 2} = \frac{12h + 3h^2}{h}$	M1	1.1b
	$= 12 + 3h$	A1	2.1
		(3)	
(c)	Explains that as $h \rightarrow 0$, $12 + 3h \rightarrow 12$ and states that the gradient of the chord tends to the gradient of (the tangent to) the curve	B1	2.4
		(1)	
(6 marks)			



Note: this question tests many skills from several different chapters, not just differentiation

16. The curve C has equation $y = f(x)$ where

$$f(x) = ax^3 + 15x^2 - 39x + b$$

and a and b are constants.

Given

- the point $(2, 10)$ lies on C
- the gradient of the curve at $(2, 10)$ is -3

(a) (i) show that the value of a is -2

(ii) find the value of b .

(4)

(b) Hence show that C has no stationary points.

(3)

(c) Write $f(x)$ in the form $(x - 4)Q(x)$ where $Q(x)$ is a quadratic expression to be found.

(2)

(d) Hence deduce the coordinates of the points of intersection of the curve with equation

$$y = f(0.2x)$$

and the coordinate axes.

(2)



Question	Scheme	Marks	AOs
16 (a) (i)	Uses $\frac{dy}{dx} = -3$ at $x = 2 \Rightarrow 12a + 60 - 39 = -3$	M1	1.1b
	Solves a correct equation and shows one correct intermediate step $12a + 60 - 39 = -3 \Rightarrow 12a = -24 \Rightarrow a = -2 *$	A1*	2.1
(a) (ii)	Uses the fact that $(2, 10)$ lies on C $10 = 8a + 60 - 78 + b$	M1	3.1a
	Subs $a = -2$ into $10 = 8a + 60 - 78 + b \Rightarrow b = 44$	A1	1.1b
		(4)	
(b)	$f(x) = -2x^3 + 15x^2 - 39x + 44 \Rightarrow f'(x) = -6x^2 + 30x - 39$	B1	1.1b
	Attempts to show that $-6x^2 + 30x - 39$ has no roots Eg. calculates $b^2 - 4ac = 30^2 - 4 \times -6 \times -39 = -36$	M1	3.1a
	States that as $f'(x) \neq 0 \Rightarrow$ hence $f(x)$ has no turning points *	A1*	2.4
		(3)	
(c)	$-2x^3 + 15x^2 - 39x + 44 \equiv (x - 4)(-2x^2 + 7x - 11)$	M1	1.1b
		A1	1.1b
		(2)	
(d)	Deduces either intercept. $(0, 44)$ or $(20, 0)$	B1 ft	1.1b
	Deduces both intercepts $(0, 44)$ and $(20, 0)$	B1 ft	2.2a
		(2)	

(11 marks)



12. A company makes drinks containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is 355 cm^3

The metal used

- for the circular base and the curved side costs 0.04 pence/cm²
- for the circular top costs 0.09 pence/cm²

Both metals used are of negligible thickness.

(a) Show that the total cost, C pence, of the metal for one container is given by

$$C = 0.13\pi r^2 + \frac{28.4}{r} \quad (4)$$

(b) Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures. (4)

(c) Using $\frac{d^2C}{dr^2}$ prove that the cost is minimised for the value of r found in part (b). (2)

(d) Hence find the minimum value of C , giving your answer to the nearest integer. (2)



$$V = \pi r^2 h = 355 \Rightarrow h = \frac{355}{\pi r^2}$$

$$\left(\text{or } rh = \frac{355}{\pi r} \text{ or } \pi rh = \frac{355}{r} \right)$$

$$C = 0.04(\pi r^2 + 2\pi rh) + 0.09(\pi r^2)$$

$$C = 0.13\pi r^2 + 0.08\pi rh = 0.13\pi r^2 + 0.08\pi r\left(\frac{355}{\pi r^2}\right)$$

$$C = 0.13\pi r^2 + \frac{28.4}{r} *$$

(4)

(b)

$$\frac{dC}{dr} = 0.26\pi r - \frac{28.4}{r^2}$$

M1

A1

$$\frac{dC}{dr} = 0 \Rightarrow r^3 = \frac{28.4}{0.26\pi} \Rightarrow r = \dots$$

M1

$$r = \sqrt[3]{\frac{1420}{13\pi}} = 3.26\dots$$

A1

(4)

(c)

$$\left(\frac{d^2C}{dr^2} = \right) 0.26\pi + \frac{56.8}{r^3} = 0.26\pi + \frac{56.8}{"3.26"^3}$$

M1

$$\left(\frac{d^2C}{dr^2} = \right) (2.45..) > 0 \text{ Hence minimum (cost)}$$

A1

(2)

(d)

$$C = 0.13\pi("3.26")^2 + \frac{28.4}{"3.26"}$$

M1

$$(C =) 13$$

A1

(2)



1. The curve C has equation

$$y = 3x^4 - 8x^3 - 3$$

(a) Find (i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 2$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)



Question	Scheme	Marks	AOs
1(a)	(i) $\frac{dy}{dx} = 12x^3 - 24x^2$	M1 A1	1.1b 1.1b
	(ii) $\frac{d^2y}{dx^2} = 36x^2 - 48x$	A1ft	1.1b
		(3)	
(b)	Substitutes $x = 2$ into their $\frac{dy}{dx} = 12 \times 2^3 - 24 \times 2^2$	M1	1.1b
	Shows $\frac{dy}{dx} = 0$ and states "hence there is a stationary point"	A1	2.1
		(2)	
(c)	Substitutes $x = 2$ into their $\frac{d^2y}{dx^2} = 36 \times 2^2 - 48 \times 2$	M1	1.1b
	$\frac{d^2y}{dx^2} = 48 > 0$ and states "hence the stationary point is a minimum"	A1ft	2.2a
		(2)	
(7 marks)			



14. A company decides to manufacture a soft drinks can with a capacity of 500 ml.

The company models the can in the shape of a right circular cylinder with radius r cm and height h cm.

In the model they assume that the can is made from a metal of negligible thickness.

- (a) Prove that the total surface area, S cm², of the can is given by

$$S = 2\pi r^2 + \frac{1000}{r} \quad (3)$$

Given that r can vary,

- (b) find the dimensions of a can that has minimum surface area. (5)

- (c) With reference to the shape of the can, suggest a reason why the company may choose not to manufacture a can with minimum surface area. (1)



A2 SAMs Paper 2

Differentiation

Question	Scheme	Marks	AOs
14(a)	Sets $500 = \pi r^2 h$	B1	2.1
	Substitute $h = \frac{500}{\pi r^2}$ into $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \times \frac{500}{\pi r^2}$	M1	2.1
	Simplifies to reach given answer $S = 2\pi r^2 + \frac{1000}{r}$ *	A1*	1.1b
		(3)	
(b)	Differentiates S with both indices correct in $\frac{dS}{dr}$	M1	3.4
	$\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2}$	A1	1.1b
	Sets $\frac{dS}{dr} = 0$ and proceeds to $r^3 = k$, k is a constant	M1	2.1
	Radius = 4.30 cm	A1	1.1b
	Substitutes their $r = 4.30$ into $h = \frac{500}{\pi r^2} \Rightarrow$ Height = 8.60 cm	A1	1.1b
		(5)	
	States a valid reason such as <ul style="list-style-type: none"> The radius is too big for the size of our hands If $r = 4.3$ cm and $h = 8.6$ cm the can is square in profile. All drinks cans are taller than they are wide The radius is too big for us to drink from They have different dimensions to other drinks cans and would be difficult to stack on shelves with other drinks cans 	B1	3.2a
		(1)	
		9 marks	



2. A curve C has equation

$$y = x^2 - 2x - 24\sqrt{x}, \quad x > 0$$

(a) Find (i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 4$

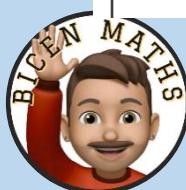
(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)



Question	Scheme	Marks	AOs
2(a)	(i) $\frac{dy}{dx} = 2x - 2 - 12x^{-\frac{1}{2}}$	M1 A1	1.1b 1.1b
	(ii) $\frac{d^2y}{dx^2} = 2 + 6x^{-\frac{3}{2}}$	B1ft	1.1b
		(3)	
(b)	Substitutes $x = 4$ into their $\frac{dy}{dx} = 2 \times 4 - 2 - 12 \times 4^{-\frac{1}{2}} = \dots$	M1	1.1b
	Shows $\frac{dy}{dx} = 0$ and states "hence there is a stationary point" oe	A1	2.1
		(2)	
(c)	Substitutes $x = 4$ into their $\frac{d^2y}{dx^2} = 2 + 6 \times 4^{-\frac{3}{2}} = (2.75)$	M1	1.1b
	$\frac{d^2y}{dx^2} = 2.75 > 0$ and states "hence minimum"	A1ft	2.2a
		(2)	
(7 marks)			



[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.

The volume of the tank is 6 m^3 .

(a) Show that, according to the model, the surface area of the tank, in m^2 , is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2 \quad (4)$$

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum. (4)

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer. (2)

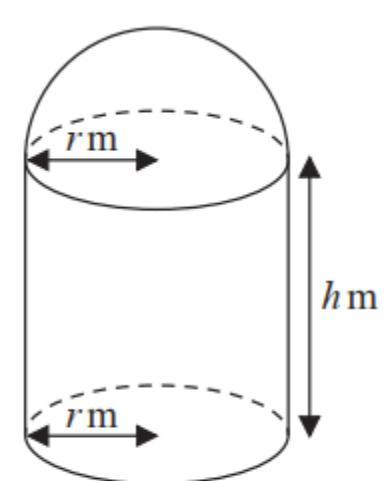


Figure 9



Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$6 = \pi r^2 h + \frac{2}{3} \pi r^3$	B1	This mark is given for a method to find the volume of the cylinder and the semi-hemisphere
	$A = 3\pi r^2 + 2\pi \left(\frac{6 - \frac{2}{3}\pi r^3}{\pi r} \right)$	M1	This mark is given for a method to find the surface area of the tank
		A1	This mark is given for finding an expression for the surface area of the tank
	$A = 3\pi r^2 + \frac{12}{r} - \frac{4\pi r^2}{3} = \frac{12}{r} + \frac{5\pi r^2}{3}$	A1	This mark is given for a fully correct proof to show the surface area of the tank as required
(b)	$A = \frac{12}{r} + \frac{5\pi r^2}{3} \Rightarrow \frac{dA}{dr} = -\frac{12}{r^2} + \frac{10\pi r}{3}$	M1	This mark is given for a method to differentiate to find r
		A1	This mark is given for accurately differentiating to find r
	When $\frac{dA}{dr} = 0, -\frac{12}{r^2} + \frac{10\pi r}{3} = 0$ $r^3 = \frac{18}{5\pi}$	M1	This mark is given for a method to set $\frac{dA}{dr} = 0$ to find a value for r
	$r = 1.046$	A1	This mark is given for finding the radius for which the surface area is a minimum
(c)	$A = \frac{12}{1.046} + \frac{5\pi(1.046)^2}{3}$	M1	This mark is given for a method to substitute a value for r
	$A = 17 \text{ m}^2$	A1	This mark is given for correctly finding the minimum surface area of the tank (to the nearest integer)



6.

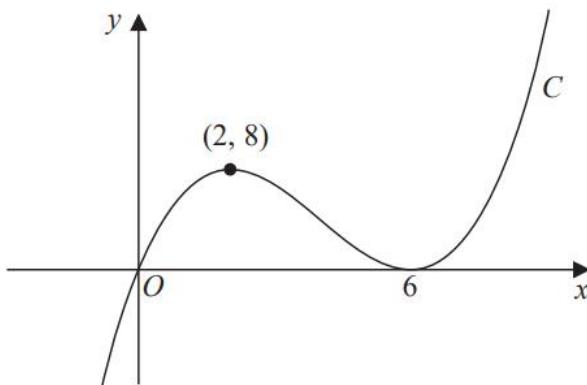


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$ where $f(x)$ is a cubic expression in x .

The curve

- passes through the origin
- has a maximum turning point at $(2, 8)$
- has a minimum turning point at $(6, 0)$

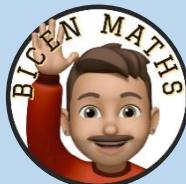
(a) Write down the set of values of x for which

$$f'(x) < 0 \quad (1)$$

The line with equation $y = k$, where k is a constant, intersects C at only one point.

(b) Find the set of values of k , giving your answer in set notation. (2)

(c) Find the equation of C . You may leave your answer in factorised form. (3)



6 (a)	$2 < x < 6$	B1
		(1)
(b)	States either $k > 8$ or $k < 0$	M1
	States e.g. $\{k : k > 8\} \cup \{k : k < 0\}$	A1
		(2)
(c)	Please see notes for alternatives	
	States $y = ax(x - 6)^2$ or $f(x) = ax(x - 6)^2$	M1
	Substitutes $(2, 8)$ into $y = ax(x - 6)^2$ and attempts to find a	dM1
	$y = \frac{1}{4}x(x - 6)^2$ or $f(x) = \frac{1}{4}x(x - 6)^2$ o.e	A1
		(3)



A company makes toys for children.

Figure 5 shows the design for a solid toy that looks like a piece of cheese.

The toy is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.8$ radians
- faces ABC and DEF are congruent
- edges AD , CF and BE are perpendicular to faces ABC and DEF
- edges AD , CF and BE have length h cm

Given that the volume of the toy is 240 cm^3

- (a) show that the surface area of the toy, $S \text{ cm}^2$, is given by

$$S = 0.8r^2 + \frac{1680}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

- (b) find the value of r for which S has a stationary point.

(4)

- (c) Prove, by further differentiation, that this value of r gives the minimum surface area of the toy.

(2)

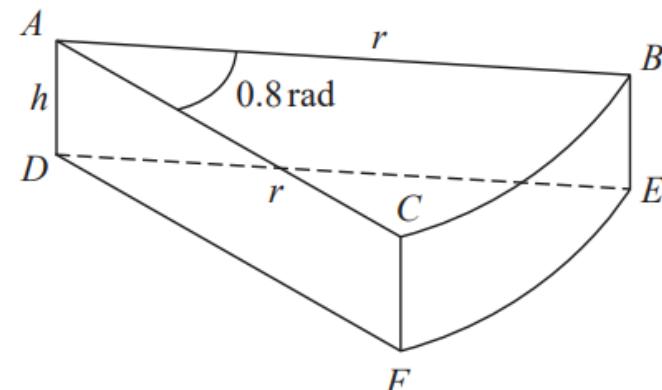


Figure 5

15 (a)	Sets up an allowable equation using volume = 240 E.g. $\frac{1}{2}r^2 \times 0.8h = 240 \Rightarrow h = \frac{600}{r^2}$ o.e.	M1 A1
	Attempts to substitute their $h = \frac{600}{r^2}$ into $(S =) \frac{1}{2}r^2 \times 0.8 + \frac{1}{2}r^2 \times 0.8 + 2rh + 0.8rh$	dM1
	$S = 0.8r^2 + 2.8rh = 0.8r^2 + 2.8 \times \frac{600}{r} = 0.8r^2 + \frac{1680}{r}$ *	A1*
		(4)
(b)	$\left(\frac{dS}{dr}\right) = 1.6r - \frac{1680}{r^2}$	M1 A1
	Sets $\frac{dS}{dr} = 0 \Rightarrow r^3 = 1050$ $r = \text{awrt } 10.2$	dM1 A1
		(4)
(c)	Attempts to substitute their positive r into $\left(\frac{d^2S}{dr^2}\right) = 1.6 + \frac{3360}{r^3}$ and considers its value or sign	M1
	E.g. Correct $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3}$ with $\frac{d^2S}{dr^2}_{r=10.2} = 5 > 0$ proving a minimum value of S	A1
		(2)



Integration

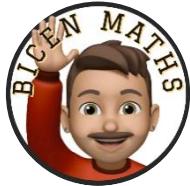


5. Given that

$$f(x) = 2x + 3 + \frac{12}{x^2}, \quad x > 0$$

show that $\int_1^{2\sqrt{2}} f(x)dx = 16 + 3\sqrt{2}$

(5)



Question	Scheme	Marks	AOs
5	$f(x) = 2x + 3 + 12x^{-2}$	B1	1.1b
	Attempts to integrate	M1	1.1a
	$\int \left(+2x + 3 + \frac{12}{x^2} \right) dx = x^2 + 3x - \frac{12}{x}$	A1	1.1b
	$\left((2\sqrt{2})^2 + 3(2\sqrt{2}) - \frac{12(\sqrt{2})}{2 \times 2} \right) - (-8)$	M1	1.1b
	$= 16 + 3\sqrt{2} *$	A1*	1.1b
(5 marks)			



1. Find

$$\int \left(\frac{2}{3}x^3 - 6\sqrt{x} + 1 \right) dx$$

giving your answer in its simplest form.

(4)

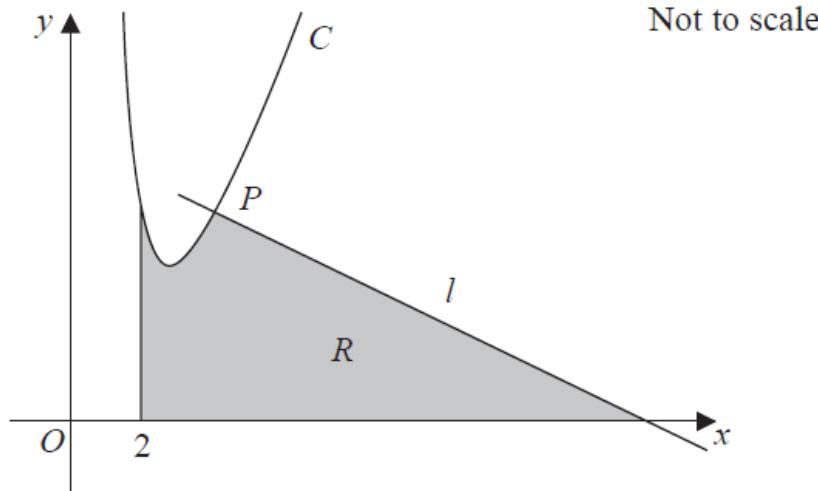


Question	Scheme	Marks	AOs
1	$\int \left(\frac{2}{3}x^3 - 6\sqrt{x} + 1 \right) dx$		
	Attempts to integrate awarded for any correct power	M1	1.1a
	$\int \left(\frac{2}{3}x^3 - 6\sqrt{x} + 1 \right) dx = \frac{2}{3} \times \frac{x^4}{4} + \dots + x$	A1	1.1b
	$= \dots - 6 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \dots$	A1	1.1b
	$= \frac{1}{6}x^4 - 4x^{\frac{3}{2}} + x + c$	A1	1.1b

(4 marks)



15.



Not to scale

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{32}{x^2} + 3x - 8, \quad x > 0$$

The point $P(4, 6)$ lies on C .

The line l is the normal to C at the point P .

The region R , shown shaded in Figure 4, is bounded by the line l , the curve C , the line with equation $x = 2$ and the x -axis.

Show that the area of R is 46

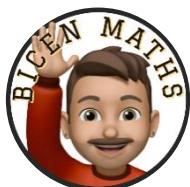
(Solutions based entirely on graphical or numerical methods are not acceptable.)

(10)

HOME



Question	Scheme	Marks	AOs
15.	For the complete strategy of finding where the normal cuts the x -axis. Key points that must be seen are <ul style="list-style-type: none">• Attempt at differentiation• Attempt at using a changed gradient to find equation of normal• Correct attempt to find where normal cuts the x - axis	M1	3.1a
	$y = \frac{32}{x^2} + 3x - 8 \Rightarrow \frac{dy}{dx} = -\frac{64}{x^3} + 3$	M1 A1	1.1b 1.1b
	For a correct method of attempting to find Either the equation of the normal: this requires substituting $x = 4$ in their $\frac{dy}{dx} = -\frac{64}{x^3} + 3 = (2)$, then using the perpendicular gradient rule to find the equation of normal $y - 6 = -\frac{1}{2}(x - 4)$ Or where the equation of the normal at $(4,6)$ cuts the x - axis. As above but may not see equation of normal. Eg $0 - 6 = -\frac{1}{2}(x - 4) \Rightarrow x = \dots$ or an attempt using just gradients $-\frac{1}{2} = \frac{6}{a - 4} \Rightarrow a = \dots$	dM1	For the complete strategy of finding the values of the two key areas. Points that must be seen are <ul style="list-style-type: none">• There must be an attempt to find the area under the curve by integrating between 2 and 4• There must be an attempt to find the area of a triangle using $\frac{1}{2} \times (16 - 4) \times 6$ or $\int_4^{16} \left(-\frac{1}{2}x + 8 \right) dx$ The "16" cannot have just been made up.
	Normal cuts the x -axis at $x = 16$	A1	$\int \frac{32}{x^2} + 3x - 8 dx = -\frac{32}{x} + \frac{3}{2}x^2 - 8x$ $\text{Area under curve} = \left[-\frac{32}{x} + \frac{3}{2}x^2 - 8x \right]_2^4 = (-16) - (-26) = (10)$ Total area = $10 + 36 = 46^*$
			(10)



3. (a) Given that k is a constant, find

$$\int \left(\frac{4}{x^3} + kx \right) dx$$

simplifying your answer.

(3)

- (b) Hence find the value of k such that

$$\int_{0.5}^2 \left(\frac{4}{x^3} + kx \right) dx = 8$$

(3)



Question 3 (Total 6 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\int 4x^{-3} + kx \, dx = -2x^{-2} + \frac{1}{2} kx$	M1	This mark is given for recognising that x^n becomes x^{n+1} when integrating
	$-\frac{2}{x^2} + \frac{kx^2}{2} + c$	A1	This mark is given for two correctly integrated terms (without c)
	$-\frac{2}{x^2} + \frac{kx^2}{2} + c$	A1	This mark is given for a full answer with a constant (in any correct form)
(b)	$\left[-\frac{2}{x^2} + \frac{k}{2} x^2 \right]_0^2 =$ $\left(-\frac{2}{2^2} + \frac{4k}{2} \right) - \left(-\frac{2}{0.5^2} + \frac{0.5^2 k}{2} \right) = 8$	M1	This mark is given for substituting the limits 2 and 0.5 and setting equal to 8
	$\left(-\frac{1}{2} + 2k \right) - \left(-8 + \frac{k}{8} \right) = 8$ $7.5 + \frac{15}{8} k = 8$	M1	This mark is given for a method to solve a linear equation in k
	$k = \frac{4}{15}$	A1	This mark is given for finding a correct value for k



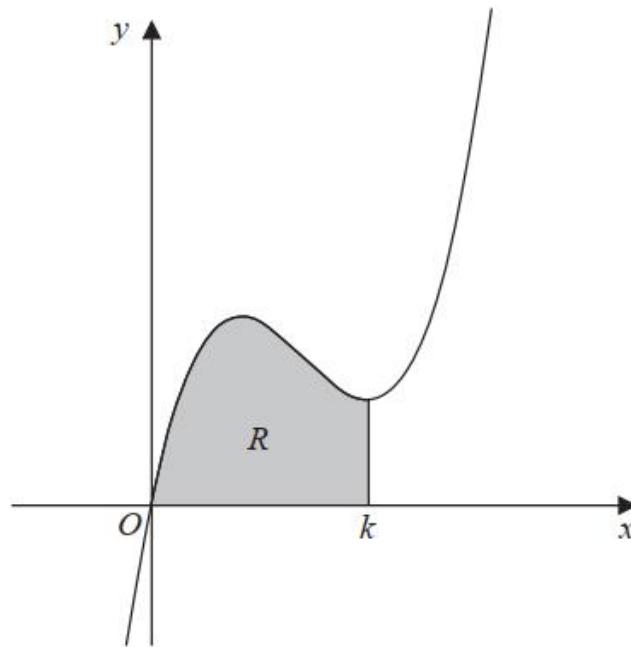
**Figure 3**

Figure 3 shows a sketch of part of the curve with equation

$$y = 2x^3 - 17x^2 + 40x$$

The curve has a minimum turning point at $x = k$.

The region R , shown shaded in Figure 3, is bounded by the curve, the x -axis and the line with equation $x = k$.

Show that the area of R is $\frac{256}{3}$

(Solutions based entirely on graphical or numerical methods are not acceptable.)

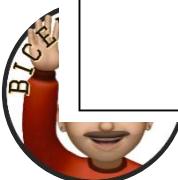
(7)

HOME



Question 13 (Total 7 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	$\frac{dy}{dx} = 6x^2 - 34x + 40$	B1	This mark is given for the equation correctly differentiated
	$\frac{dy}{dx} = 0 \text{ when } 6x^2 - 34x + 40 = 0$	M1	This mark is given for setting the equation equal to zero
	$2(3x - 5)(x - 4) = 0$	M1	This mark is given for factorising the expression
	$(x = \frac{5}{3}), x = 4,$	A1	This mark is given for finding two solutions and choosing $x = 4$ as the upper limit of the integral
	$R = \int_0^4 2x^3 - 17x^2 + 40x \, dx$ $= \left[\frac{1}{2}x^4 - \frac{17}{3}x^3 + 20x^2 \right]_0^4$	B1	This mark is given for integrating the expression from 0 to 4
	$R = (\frac{1}{2} \times 4^4) - (\frac{17}{3} \times 4^3) + (20 \times 4^2)$	M1	This mark is given for a calculation for find the area
	$R = 127 - \frac{1088}{3} + 320 = \frac{256}{3}$	A1	This mark is given for a full proof with correct notation and no errors



7. Given that k is a positive constant and $\int_1^k \left(\frac{5}{2\sqrt{x}} + 3 \right) dx = 4$

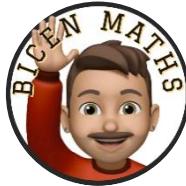
(a) show that $3k + 5\sqrt{k} - 12 = 0$

(4)

(b) Hence, using algebra, find any values of k such that

$$\int_1^k \left(\frac{5}{2\sqrt{x}} + 3 \right) dx = 4$$

(4)



Question	Scheme	Marks	AOs
7 (a)	$x^n \rightarrow x^{n+1}$	M1	1.1b
	$\int \left(\frac{5}{2\sqrt{x}} + 3 \right) dx = 5\sqrt{x} + 3x$	A1	1.1b
	$[5\sqrt{x} + 3x]^k = 4 \Rightarrow 5\sqrt{k} + 3k - 8 = 4$	dM1	1.1b
	$3k + 5\sqrt{k} - 12 = 0 *$	A1*	2.1
		(4)	
(b)	$3k + 5\sqrt{k} - 12 = 0 \Rightarrow (3\sqrt{k} - 4)(\sqrt{k} + 3) = 0$	M1	3.1a
	$\sqrt{k} = \frac{4}{3}, (-3)$	A1	1.1b
	$\sqrt{k} = \dots \Rightarrow k = \dots$ oe	dM1	1.1b
	$k = \frac{16}{9}, \cancel{(-3)}$	A1	2.3
		(4)	
(8 marks)			

--



3. Find

$$\int \frac{3x^4 - 4}{2x^3} dx$$

writing your answer in simplest form.

(4)



Question	Scheme	Marks	AOs			
3	$\int \frac{3x^4 - 4}{2x^3} dx = \int \frac{3}{2}x - 2x^{-3} dx$	M1 A1	1.1b 1.1b			
	$= \frac{3}{2} \times \frac{x^2}{2} - 2 \times \frac{x^{-2}}{-2} \quad (+ c)$	dM1	3.1a			
	$= \frac{3}{4}x^2 + \frac{1}{x^2} + c \quad \text{o.e}$	A1	1.1b			
		(4)				
	(4 marks)					
Notes:						
(i) M1: Attempts to divide to form a sum of terms. Implied by two terms with one correct index.						
A1: $\int \frac{3x^4}{2x^3} - \frac{4}{2x^3} dx$ scores this mark.						
A1: $\int \frac{3}{2}x - 2x^{-3} dx$ o.e such as $\frac{1}{2} \int (3x - 4x^{-3}) dx$. The indices must have been processed on both terms. Condone spurious notation or lack of the integral sign for this mark.						
dM1: For the full strategy to integrate the expression. It requires two terms with one correct index. Look for $= ax^p + bx^q$ where $p = 2$ or $q = -2$						
A1: Correct answer $\frac{3}{4}x^2 + \frac{1}{x^2} + c$ o.e. such as $\frac{3}{4}x^2 + x^{-2} + c$						



9. Find the value of the constant k , $0 < k < 9$, such that

$$\int_k^9 \frac{6}{\sqrt{x}} dx = 20 \quad (4)$$



Question	Scheme	Marks	AOs
9	$\int_k^9 \frac{6}{\sqrt{x}} dx = \left[ax^{\frac{1}{2}} \right]_k^9 = 20 \Rightarrow 36 - 12\sqrt{k} = 20$	M1 A1	1.1b 1.1b
	Correct method of solving Eg. $36 - 12\sqrt{k} = 20 \Rightarrow k =$	dM1	3.1a
	$\Rightarrow k = \frac{16}{9}$ oe	A1	1.1b
		(4)	
		(4 marks)	

Notes:

M1: For setting $\left[ax^{\frac{1}{2}} \right]_k^9 = 20$

A1: A correct equation involving k Eg. $36 - 12\sqrt{k} = 20$

dM1: For a whole strategy to find k . In the scheme it is awarded for setting $\left[ax^{\frac{1}{2}} \right]_k^9 = 20$, using both

limits and proceeding using correct index work to find k . It cannot be scored if $k^{\frac{1}{2}} < 0$

A1: $k = \frac{16}{9}$



14. A curve C has equation $y = f(x)$ where

$$f(x) = -3x^2 + 12x + 8$$

- (a) Write $f(x)$ in the form

$$a(x + b)^2 + c$$

where a , b and c are constants to be found.

(3)

The curve C has a maximum turning point at M .

- (b) Find the coordinates of M .

(2)

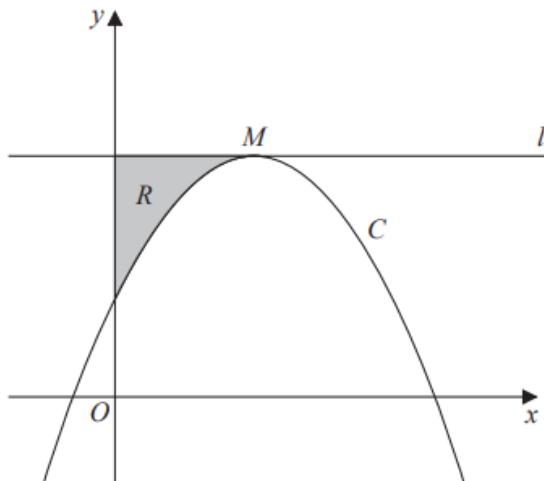


Figure 3

Figure 3 shows a sketch of the curve C .

The line l passes through M and is parallel to the x -axis.

The region R , shown shaded in Figure 3, is bounded by C , l and the y -axis.

- (c) Using algebraic integration, find the area of R .

(5)



Question	Scheme	Marks	AOs
14 (a)	$f(x) = -3x^2 + 12x + 8 = -3(x \pm 2)^2 + \dots$	M1	1.1b
	$= -3(x - 2)^2 + \dots$	A1	1.1b
	$= -3(x - 2)^2 + 20$	A1	1.1b
		(3)	
(b)	Coordinates of $M = (2, 20)$	B1ft	1.1b
		B1ft	2.2a
		(2)	
(c)	$\int -3x^2 + 12x + 8 \, dx = -x^3 + 6x^2 + 8x$	M1	1.1b
		A1	1.1b
	Method to find $R = \text{their } 2 \times 20 - \int_0^2 (-3x^2 + 12x + 8) \, dx$	M1	3.1a
	$R = 40 - [-2^3 + 24 + 16]$	dM1	1.1b
	$= 8$	A1	1.1b
		(5)	
		(10 marks)	



1. Find

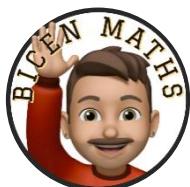
$$\int \left(8x^3 - \frac{3}{2\sqrt{x}} + 5 \right) dx$$

giving your answer in simplest form.

(4)



Question	Scheme	Marks	AOs
1	$x^n \rightarrow x^{n+1}$	M1	1.1b
	$\int \left(8x^3 - \frac{3}{2\sqrt{x}} + 5 \right) dx = \frac{8x^4}{4} \dots + 5x$	A1	1.1b
	$= \dots - 2 \times \frac{3}{2} x^{\frac{1}{2}} + \dots$	A1	1.1b
	$= 2x^4 - 3x^{\frac{1}{2}} + 5x + c$	A1	1.1b
		(4)	
		(4 marks)	



8.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The air pressure, P kg/cm², inside a car tyre, t minutes from the instant when the tyre developed a puncture is given by the equation

$$P = k + 1.4e^{-0.5t} \quad t \in \mathbb{R} \quad t \geq 0$$

where k is a constant.

Given that the initial air pressure inside the tyre was 2.2 kg/cm²

(a) state the value of k .

(1)

From the instant when the tyre developed the puncture,

(b) find the time taken for the air pressure to fall to 1 kg/cm²

Give your answer in minutes to one decimal place.

(3)

(c) Find the rate at which the air pressure in the tyre is decreasing exactly 2 minutes from the instant when the tyre developed the puncture.

Give your answer in kg/cm² per minute to 3 significant figures.

(2)



8(a)	$(k =) 0.8$	B1 (1)
(b)	$1 = 0.8 + 1.4e^{-0.5t} \Rightarrow 1.4e^{-0.5t} = 0.2$	M1
	$-0.5t = \ln\left(\frac{0.2}{1.4}\right) \Rightarrow t = \dots$	M1
	awrt 3.9 minutes	A1 (3)
(c)	$\left(\frac{dP}{dt} = \right) -0.7e^{-0.5t}$ $\left(\frac{dP}{dt} \right)_{t=2} = -0.7e^{-0.5 \times 2}$	M1
	= awrt 0.258 (kg/cm ² per minute)	A1 (2)



9. (a) Given that $p = \log_3 x$, where $x > 0$, find in simplest form in terms of p ,

(i) $\log_3\left(\frac{x}{9}\right)$

(ii) $\log_3\left(\sqrt{x}\right)$

(2)

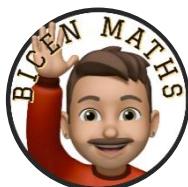
(b) Hence, or otherwise, solve

$$2\log_3\left(\frac{x}{9}\right) + 3\log_3\left(\sqrt{x}\right) = -11$$

giving your answer as a simplified fraction.

Solutions relying on calculator technology are not acceptable.

(4)



9(a)(i)

$$\log_3\left(\frac{x}{9}\right) = \log_3 x - \log_3 9 = p - 2$$

B1

(ii)

$$\log_3(\sqrt{x}) = \frac{1}{2}p$$

B1

(2)

(b)

$$2\log_3\left(\frac{x}{9}\right) + 3\log_3(\sqrt{x}) = -11 \Rightarrow 2p - 4 + \frac{3}{2}p = -11 \Rightarrow p = \dots$$

M1

$$p = -2$$

A1

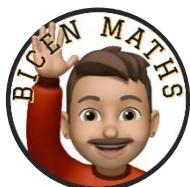
$$\log_3 x = -2 \Rightarrow x = 3^{-2}$$

M1

$$x = \frac{1}{9}$$

A1

(4)



10.

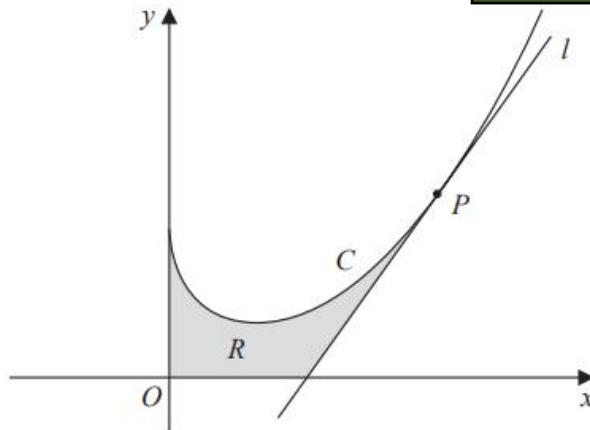


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{1}{3}x^2 - 2\sqrt{x} + 3 \quad x \geq 0$$

The point P lies on C and has x coordinate 4

The line l is the tangent to C at P .

(a) Show that l has equation

$$13x - 6y - 26 = 0 \quad (5)$$

The region R , shown shaded in Figure 2, is bounded by the y -axis, the curve C , the line l and the x -axis.

(b) Find the exact area of R . (5)



10(a)

$$y = \frac{1}{3}x^2 - 2\sqrt{x} + 3 \Rightarrow \frac{dy}{dx} = \frac{2}{3}x - x^{-\frac{1}{2}}$$

M1
A1

$$x = 4 \Rightarrow y = \frac{13}{3}$$

B1

$$\left(\frac{dy}{dx} \right)_{x=4} = \frac{2}{3} \times 4 - 4^{-\frac{1}{2}} \left(= \frac{13}{6} \right) \therefore y - \frac{13}{3} = \frac{13}{6}(x - 4)$$

M1

$$13x - 6y - 26 = 0 *$$

A1*

(5)

(b)

$$\int \left(\frac{x^2}{3} - 2\sqrt{x} + 3 \right) dx = \frac{x^3}{9} - \frac{4}{3}x^{\frac{3}{2}} + 3x (+c)$$

M1
A1

$$y = 0 \Rightarrow x = 2$$

B1

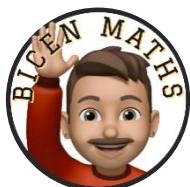
$$\text{Area of } R \text{ is } \left[\frac{x^3}{9} - \frac{4}{3}x^{\frac{3}{2}} + 3x \right]_0^4 - \frac{1}{2} \times (4 - "2") \times " \frac{13}{3} " = \frac{76}{9} - \frac{13}{3}$$

M1

$$= \frac{37}{9}$$

A1

(5)



15.

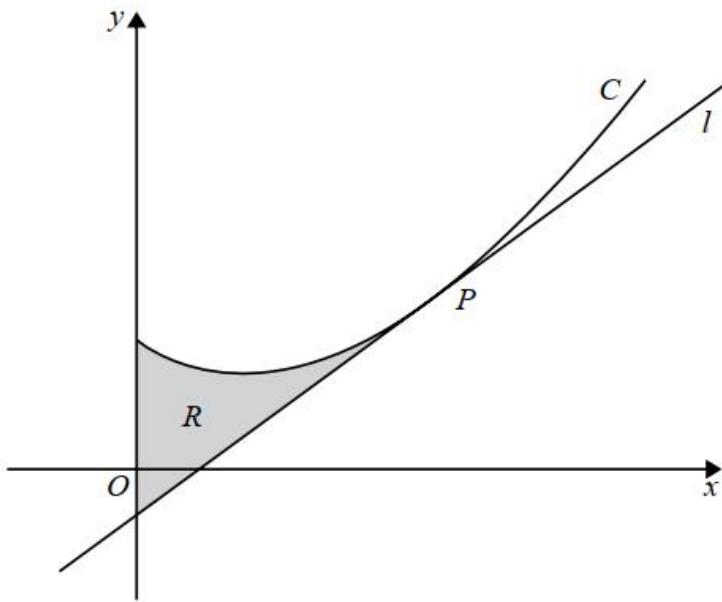
**Figure 4**

Figure 4 shows a sketch of the curve C with equation

$$y = 5x^{\frac{3}{2}} - 9x + 11, x \geq 0$$

The point P with coordinates $(4, 15)$ lies on C .

The line l is the tangent to C at the point P .

The region R , shown shaded in Figure 4, is bounded by the curve C , the line l and the y -axis.

Show that the area of R is 24, making your method clear.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(10)



Question	Scheme	Marks	AOs
15	$\frac{dy}{dx} = \frac{15}{2}x^{\frac{1}{2}} - 9$	M1 A1	3.1a 1.1b
	Substitutes $x = 4 \Rightarrow \frac{dy}{dx} = 6$	M1	2.1
	Uses (4, 15) and gradient $\Rightarrow y - 15 = 6(x - 4)$	M1	2.1
	Equation of l is $y = 6x - 9$	A1	1.1b
	Area $R = \int_0^4 \left(5x^{\frac{3}{2}} - 9x + 11 \right) - (6x - 9) dx$	M1	3.1a
	$= \left[2x^{\frac{5}{2}} - \frac{15}{2}x^2 + 20x (+c) \right]_0^4$	A1	1.1b
	Uses both limits of 4 and 0		
	$\left[2x^{\frac{5}{2}} - \frac{15}{2}x^2 + 20x \right]_0^4 = 2 \times 4^{\frac{5}{2}} - \frac{15}{2} \times 4^2 + 20 \times 4 - 0$	M1	2.1
	Area of $R = 24 *$	A1*	1.1b
	Correct notation with good explanations	A1	2.5
		(10)	
		(10 marks)	



Figure 2 shows a sketch of part of the curve with equation $y = x(x + 2)(x - 4)$.

The region R_1 shown shaded in Figure 2 is bounded by the curve and the negative x -axis.

- (a) Show that the exact area of R_1 is $\frac{20}{3}$

The region R_2 also shown shaded in Figure 2 is bounded by the curve, the positive x -axis and the line with equation $x = b$, where b is a positive constant and $0 < b < 4$

Given that the area of R_1 is equal to the area of R_2

- (b) verify that b satisfies the equation

$$(b + 2)^2 (3b^2 - 20b + 20) = 0$$

(4)

(4)

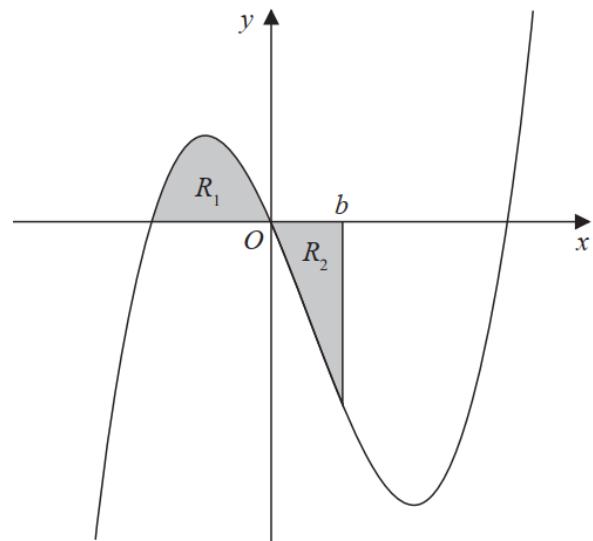


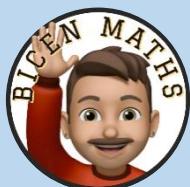
Figure 2

The roots of the equation $3b^2 - 20b + 20 = 0$ are 1.225 and 5.442 to 3 decimal places.

The value of b is therefore 1.225 to 3 decimal places.

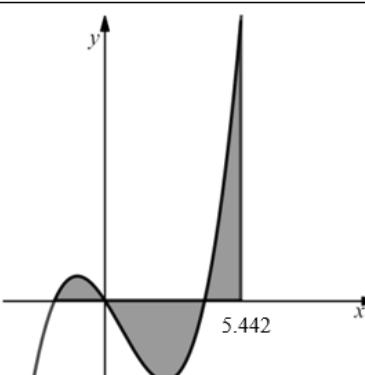
- (c) Explain, with the aid of a diagram, the significance of the root 5.442

(2)



A2 2019 Paper 1

Integration

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$y = x(x + 2)(x - 4) = x^3 - 2x^2 - 8x$	B1	This mark is given for expanding brackets as a first step to a solution
	$\int_{-2}^0 x^3 - 2x^2 - 8x \, dx$	M1	This mark is given for a method to find the exact area of R_1
	$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - 4x^2 \right]_{-2}^0$	M1	This mark is given for a method to evaluate the integral
	$= 0 - (4 - \frac{-16}{3} - 16) = \frac{20}{3}$	A1	This mark is given for a full method to show the exact value of R_1
(b)	$\frac{1}{4}b^4 - \frac{2}{3}b^3 - 4b^2 = -\frac{20}{3}$	M1	This mark is given for deducing the area of $R_2 = -\frac{20}{3}$
	$3b^4 - 8b^3 - 48b^2 + 80 = 0$	A1	This mark is given for rearranging the equation to a quartic
	$(b+2)^2(3b^2 - 20b + 20)$ $= (b^2 + 4b + 4)(3b^2 - 20b + 20)$	M1	This mark is given for expanding the equation given
	$= 3b^4 - 8b^3 - 48b^2 + 80 = 0$ The two equations are the same, so verified	A1	This mark is for showing, and stating, that the equations are the same
		B1	This mark is given for a sketch of the curve with $b = 5.442$ shown
	Between $x = -2$ and $b = 5.442$, the area above the x -axis is the same as the area below the x -axis	B1	This mark is given for a valid explanation of the significance of the root 5.442



5.

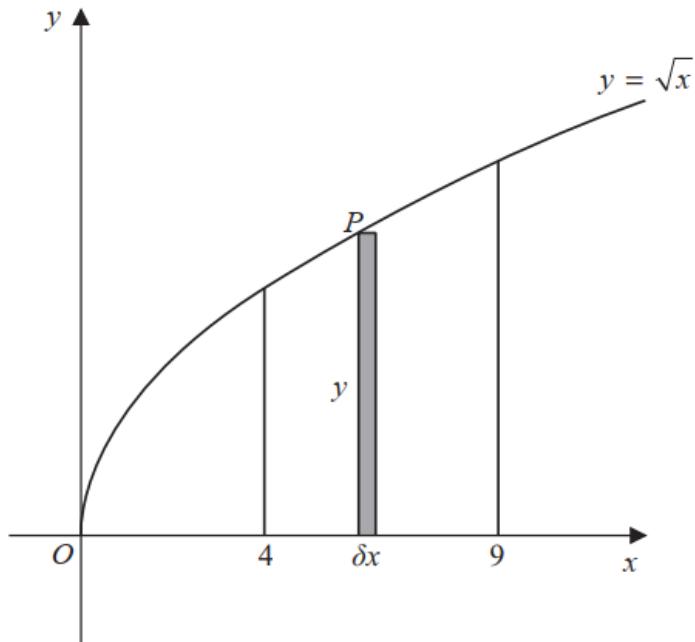


Figure 3

Figure 3 shows a sketch of the curve with equation $y = \sqrt{x}$

The point $P(x, y)$ lies on the curve.

The rectangle, shown shaded on Figure 3, has height y and width δx .

Calculate

$$\lim_{\delta x \rightarrow 0} \sum_{x=4}^9 \sqrt{x} \delta x \quad (3)$$



Question 5 (Total 3 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	$\lim_{\delta x \rightarrow 0} \sum_{x=4}^9 \sqrt{x} \delta x = \int_4^9 \sqrt{x} dx$	B1	This mark is given for writing the expression for a sum as an integral
	$\left[\frac{2}{3} x^{\frac{3}{2}} \right]_4^9 = \frac{2}{3} \times 9^{\frac{3}{2}} - \frac{2}{3} \times 4^{\frac{3}{2}}$	M1	This mark is given for a method to evaluate the integral
	$= \frac{38}{3}$	A1	This mark is given for a correct evaluation of the integral



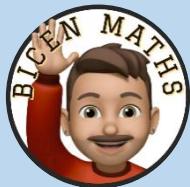
8. A curve C has equation $y = f(x)$

Given that

- $f'(x) = 6x^2 + ax - 23$ where a is a constant
- the y intercept of C is -12
- $(x + 4)$ is a factor of $f(x)$

find, in simplest form, $f(x)$

(6)



8

$$f'(x) = 6x^2 + ax - 23 \Rightarrow f(x) = 2x^3 + \frac{1}{2}ax^2 - 23x + c$$

M1
A11.1b
1.1b

$$"c" = -12$$

B1

2.2a

$$f(-4) = 0 \Rightarrow 2 \times (-4)^3 + \frac{1}{2}a(-4)^2 - 23(-4) - 12 = 0$$

dM1

3.1a

$$a = \dots (6)$$

dM1

1.1b

$$(f(x) =) 2x^3 + 3x^2 - 23x - 12$$

Or Equivalent e.g.

$$(f(x) =)(x+4)(2x^2 - 5x - 3) \quad (f(x) =)(x+4)(2x+1)(x-3)$$

A1cso

2.1

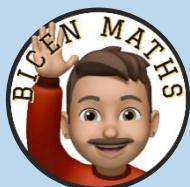
(6)**(6 marks)**

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

- (a) Use differentiation to find the equation of l , giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

- (b) Hence verify that l meets C again on the y -axis.

(1)

The finite region R , shown shaded in Figure 2, is bounded by the curve C and the line l .

- (c) Use algebraic integration to find the exact area of R .

(4)

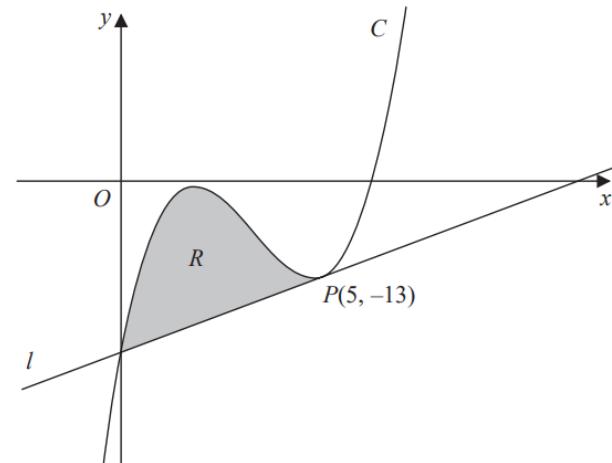
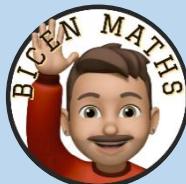


Figure 2



Question	Scheme	Marks	AOs
7(a)	$y = x^3 - 10x^2 + 27x - 23 \Rightarrow \frac{dy}{dx} = 3x^2 - 20x + 27$	B1	1.1b
	$\left(\frac{dy}{dx}\right)_{x=5} = 3 \times 5^2 - 20 \times 5 + 27 (= 2)$	M1	1.1b
	$y + 13 = 2(x - 5)$	M1	2.1
	$y = 2x - 23$	A1	1.1b
		(4)	
(b)	Both C and l pass through $(0, -23)$ and so C meets l again on the y -axis	B1	2.2a
		(1)	
(c)	$\pm \int (x^3 - 10x^2 + 27x - 23 - (2x - 23)) dx$ $= \pm \left(\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{25}{2}x^2 \right)$	M1 A1ft	1.1b 1.1b
	$\left[\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{25}{2}x^2 \right]_0^5$ $= \left(\frac{625}{4} - \frac{1250}{3} + \frac{625}{2} \right)(-0)$	dM1	2.1
	$= \frac{625}{12}$	A1	1.1b
		(4)	



8.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

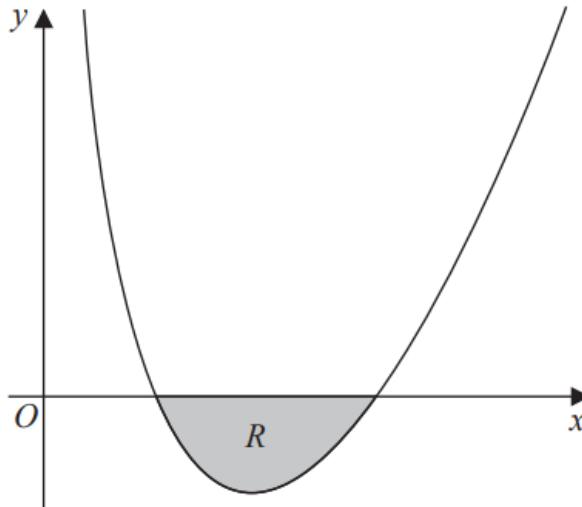


Figure 3

Figure 3 shows a sketch of part of a curve with equation

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} \quad x > 0$$

The region R , shown shaded in Figure 3, is bounded by the curve and the x -axis.

Find the exact area of R , writing your answer in the form $a\sqrt{2} + b$, where a and b are constants to be found.

(6)

HOME



8	$y = \frac{(x-2)(x-4)}{4\sqrt{x}} = \frac{x^2 - 6x + 8}{4\sqrt{x}} = \frac{1}{4}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$	M1 A1
	$\int \frac{1}{4}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} dx = \frac{1}{10}x^{\frac{5}{2}} - x^{\frac{3}{2}} + 4x^{\frac{1}{2}} (+c)$	dM1 A1
	Deduces limits of integral are 2 and 4 and applies to their	
	$\frac{1}{10}x^{\frac{5}{2}} - x^{\frac{3}{2}} + 4x^{\frac{1}{2}}$	M1
	$\left(\frac{32}{10} - 8 + 8\right) - \left(\frac{2}{5}\sqrt{2} - 2\sqrt{2} + 4\sqrt{2}\right) = \frac{16}{5} - \frac{12}{5}\sqrt{2}$ $\text{Area } R = \frac{12}{5}\sqrt{2} - \frac{16}{5} \quad \left(\text{or } \frac{16}{5} - \frac{12}{5}\sqrt{2} \right)$	A1
		(6)



Exponentials and Logarithms



14.

A town's population, P , is modelled by the equation $P = ab^t$, where a and b are constants and t is the number of years since the population was first recorded.

The line l shown in Figure 2 illustrates the linear relationship between t and $\log_{10}P$ for the population over a period of 100 years.

The line l meets the vertical axis at $(0, 5)$ as shown. The gradient of l is $\frac{1}{200}$.

(a) Write down an equation for l .

(2)

(b) Find the value of a and the value of b .

(4)

(c) With reference to the model interpret

(i) the value of the constant a ,

(ii) the value of the constant b .

(2)

(d) Find

(i) the population predicted by the model when $t = 100$, giving your answer to the nearest hundred thousand,

(ii) the number of years it takes the population to reach 200 000, according to the model.

(3)

(e) State two reasons why this may not be a realistic population model.

(2)

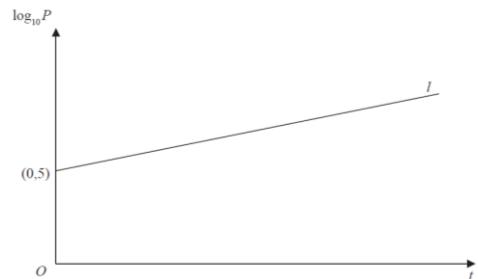
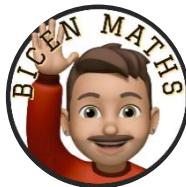


Figure 2



Question	Scheme	Marks	AOs
14(a)	$\log_{10} P = mt + c$	M1	1.1b
	$\log_{10} P = \frac{1}{200}t + 5$	A1	1.1b
		(2)	
(b)	<u>Way 1:</u> As $P = ab^t$ then $\log_{10} P = t \log_{10} b + \log_{10} a$	<u>Way 2:</u> As $\log_{10} P = \frac{t}{200} + 5$ then $P = 10^{\left(\frac{t}{200}+5\right)} = 10^5 10^{\left(\frac{t}{200}\right)}$	M1 M1
	$\log_{10} b = \frac{1}{200}$ or $\log_{10} a = 5$	$a = 10^5$ or $b = 10^{\left(\frac{1}{200}\right)}$	M1
	So $a = 100\ 000$ or $b = 1.0116$		A1
	Both $a = 100\ 000$ and $b = 1.0116$ (awrt 1.01)		A1
			(4)
	The initial population	B1	3.4
(c)(i)	The proportional increase of population each year	B1	3.4
(d)(i)	300000 to nearest hundred thousand	B1	3.4
(d)(ii)	Uses $200000 = ab^t$ with their values of a and b or $\log_{10} 200000 = \frac{1}{200}t + 5$ and rearranges to give $t =$ 60.2 years to 3sf	M1	3.4
(e)	Any two valid reasons- e.g. <ul style="list-style-type: none"> • 100 years is a long time and population may be affected by wars and disease 	A1ft	1.1b
			(3)



5. A student's attempt to solve the equation $2\log_2 x - \log_2 \sqrt{x} = 3$ is shown below.

$$2\log_2 x - \log_2 \sqrt{x} = 3$$

$$2\log_2 \left(\frac{x}{\sqrt{x}} \right) = 3$$

using the subtraction law for logs

$$2\log_2 (\sqrt{x}) = 3$$

simplifying

$$\log_2 x = 3$$

using the power law for logs

$$x = 3^2 = 9$$

using the definition of a log

- (a) Identify two errors made by this student, giving a brief explanation of each.

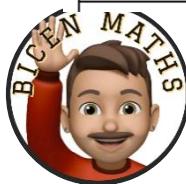
(2)

- (b) Write out the correct solution.

(3)



Question	Scheme	Marks	AOs	
5 (a)	Identifies one of the two errors "You cannot use the subtraction law without dealing with the 2 first" " They undo the logs incorrectly. It should be $x = 2^3 = 8$ "	B1	2.3	
	Identifies both errors. See above.	B1	2.3	
		(2)		
(b)	$\log_2\left(\frac{x^2}{\sqrt{x}}\right) = 3$	$\frac{3}{2}\log_2(x) = 3$	M1	1.1b
	$x^{\frac{3}{2}} = 2^3$ or $\frac{x^2}{\sqrt{x}} = 2^3$	$x = 2^2$	M1	1.1b
	$x = (2^3)^{\frac{2}{3}} = 4$	$x = 4$	A1	1.1b
		(3)		
			(5 marks)	



13.

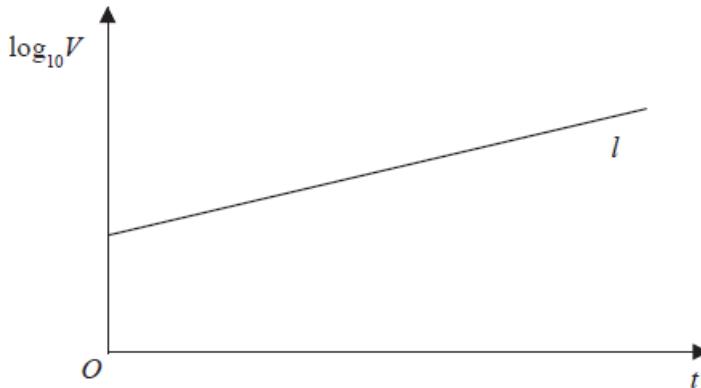


Figure 3

The value of a rare painting, £ V , is modelled by the equation $V = pq^t$, where p and q are constants and t is the number of years since the value of the painting was first recorded on 1st January 1980.

The line l shown in Figure 3 illustrates the linear relationship between t and $\log_{10}V$ since 1st January 1980.

The equation of line l is $\log_{10}V = 0.05t + 4.8$

(a) Find, to 4 significant figures, the value of p and the value of q .

(4)

(b) With reference to the model interpret

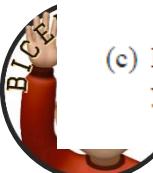
(i) the value of the constant p ,

(ii) the value of the constant q .

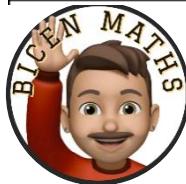
(2)

(c) Find the value of the painting, as predicted by the model, on 1st January 2010, giving your answer to the nearest hundred thousand pounds.

(2)



Question	Scheme	Marks	AOs
13(a)	For a correct equation in p or q $p = 10^{4.8}$ or $q = 10^{0.05}$	M1	1.1b
	For $p = \text{awrt } 63100$ or $q = \text{awrt } 1.122$	A1	1.1b
	For correct equations in p and q $p = 10^{4.8}$ and $q = 10^{0.05}$	dM1	3.1a
	For $p = \text{awrt } 63100$ and $q = \text{awrt } 1.122$	A1	1.1b
		(4)	
(b)	(i) The value of the painting on 1st January 1980	B1	3.4
	(ii) The proportional increase in value each year	B1	3.4
		(2)	
(c)	Uses $V = 63100 \times 1.122^{30}$ or $\log V = 0.05 \times 30 + 4.8$ leading to $V =$ $= \text{awrt } (\text{\pounds})2000000$	M1	3.4
		A1	1.1b
		(2)	
(8 marks)			



14. The value of a car, £ V , can be modelled by the equation

$$V = 15700e^{-0.25t} + 2300 \quad t \in \mathbb{R}, t \geq 0$$

where the age of the car is t years.

Using the model,

- (a) find the initial value of the car.

(1)

Given the model predicts that the value of the car is decreasing at a rate of £500 per year at the instant when $t = T$,

- (b) (i) show that

$$3925e^{-0.25T} = 500$$

- (ii) Hence find the age of the car at this instant, giving your answer in years and months to the nearest month.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

The model predicts that the value of the car approaches, but does not fall below, £ A .

- (c) State the value of A .

(1)

- (d) State a limitation of this model.

(1)



Question 14 (Total 9 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$(15\ 700 \times e^0) + 2300 = 18\ 000$	B1	This mark is given for a correct value for the initial value of the car
(b)(i)	$\frac{dV}{dt} = (-0.25 \times 15\ 700) e^{-0.25t}$ $= -3925e^{-0.25t}$	M1	This mark is given for making the link between gradient and rate of change and finding $\frac{dV}{dt} = ke^{-0.25t}$
		A1	This mark is given for a fully correct expression for $\frac{dV}{dt}$
	$-3925e^{-0.25T} = -500$ thus $3925e^{-0.25T} = 500$	A1	This mark is given fully correct working to show that $3925e^{-0.25T} = 500$
(b)(ii)	$e^{-0.25T} = \frac{500}{3925}$ $-0.25T = \ln \frac{500}{3925}$	M1	This mark is given for the start of a method to find the age of the car using logarithms
	$T = \frac{\ln 0.127 \dots}{-0.25} = \frac{-2.0605 \dots}{-0.25} = 8.24 \dots$	A1	This mark is given for rearranging and solving for T
	8 years and 3 months	A1	This mark is given for finding the answer in years and months to the nearest month
(c)	£2300	B1	This mark is given for deducing from the original equation that as $e^{-0.25t}$ tends to zero, V tends to 2300
(d)	Other factors can affect the price such as mileage or condition The price may rise as the car becomes rare	B1	This mark is given for any valid limitation to the model stated



8. The temperature, $\theta^\circ\text{C}$, of a cup of tea t minutes after it was placed on a table in a room, is modelled by the equation

$$\theta = 18 + 65e^{-\frac{t}{8}} \quad t \geq 0$$

Find, according to the model,

- (a) the temperature of the cup of tea when it was placed on the table, (1)
- (b) the value of t , to one decimal place, when the temperature of the cup of tea was 35°C . (3)
- (c) Explain why, according to this model, the temperature of the cup of tea could not fall to 15°C . (1)

The temperature, $\mu^\circ\text{C}$, of a second cup of tea t minutes after it was placed on a table in a different room, is modelled by the equation

$$\mu = A + Be^{-\frac{t}{8}} \quad t \geq 0$$

where A and B are constants.

Figure 2 shows a sketch of μ against t with two data points that lie on the curve.

The line l , also shown on Figure 2, is the asymptote to the curve.

Using the equation of this model and the information given in Figure 2

- (d) find an equation for the asymptote l . (4)

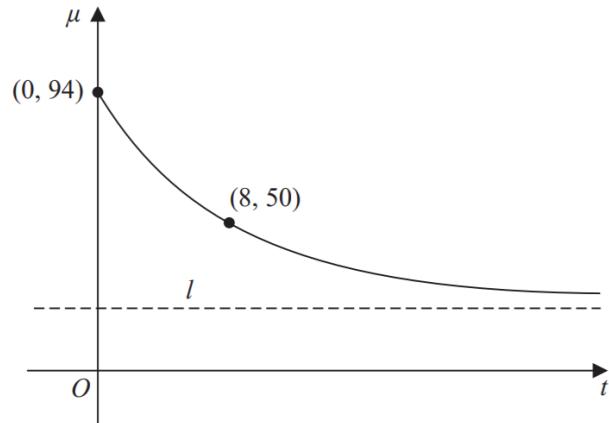


Figure 2



Question	Scheme	Marks	AOs
8 (a)	Temperature = 83°C	B1	3.4
		(1)	
(b)	$18 + 65e^{-\frac{t}{8}} = 35 \Rightarrow 65e^{-\frac{t}{8}} = 17$	M1	1.1b
	$t = -8 \ln\left(\frac{17}{65}\right)$	dM1	1.1b
	$\ln 65 - \frac{t}{8} = \ln 17 \Rightarrow t = \dots$	A1	1.1b
	$t = 10.7$	(3)	
(c)	States a suitable reason <ul style="list-style-type: none"> As $t \rightarrow \infty, \theta \rightarrow 18$ from above. The minimum temperature is 18°C 	B1	2.4
		(1)	
(d)	$A + B = 94$ or $A + Be^{-1} = 50$	M1	3.4
	$A + B = 94$ and $A + Be^{-1} = 50$	A1	1.1b
	Full method to find at least a value for A	dM1	2.1
	Deduces $\mu = \frac{50e - 94}{e - 1}$ or accept $\mu = \text{awrt } 24.4$	A1	2.2a
		(4)	
(9 marks)			

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12. An advertising agency is monitoring the number of views of an online advert.

The equation

$$\log_{10} V = 0.072t + 2.379 \quad 1 \leq t \leq 30, t \in \mathbb{N}$$

is used to model the total number of views of the advert, V , in the first t days after the advert went live.

- (a) Show that $V = ab^t$ where a and b are constants to be found.

Give the value of a to the nearest whole number and give the value of b to 3 significant figures.

(4)

- (b) Interpret, with reference to the model, the value of ab .

(1)

Using this model, calculate

- (c) the total number of views of the advert in the first 20 days after the advert went live.

Give your answer to 2 significant figures.

(2)



Question	Scheme		Marks	AOs
12 (a)	$\log_{10} V = 0.072t + 2.379$ $\Rightarrow V = 10^{0.072t+2.379}$ $\Rightarrow V = 10^{0.072t} \times 10^{2.379}$	$V = ab^t$ $\Rightarrow \log_{10} V = \log_{10} a + \log_{10} b^t$ $\Rightarrow \log_{10} V = \log_{10} a + t \log_{10} b$	B1	2.1
	States either $a = 10^{2.379}$ or $b = 10^{0.072}$	States either $\log_{10} a = 2.379$ or $\log_{10} b = 0.072$	M1	1.1b
	$a = 239$ or $b = 1.18$	$a = 239$ or $b = 1.18$	A1	1.1b
	Either $V = 239 \times 1.18^t$ or imply by $a = 239, b = 1.18$		A1	1.1b
			(4)	
(b)	The value of ab is the (total) number of views of the advert 1 day after it went live.		B1	3.4
			(1)	
(c)	Substitutes $t = 20$ in either equation and finds V Eg $V = 239 \times 1.18^{20}$		M1	3.4
	Awrt 6500 or 6600		A1	1.1b
			(2)	
(7 marks)				



11. The owners of a nature reserve decided to increase the area of the reserve covered by trees.

Tree planting started on 1st January 2005.

The area of the nature reserve covered by trees, $A \text{ km}^2$, is modelled by the equation

$$A = 80 - 45e^{ct}$$

where c is a constant and t is the number of years after 1st January 2005.

Using the model,

- (a) find the area of the nature reserve that was covered by trees just before tree planting started.

(1)

On 1st January 2019 an area of 60 km^2 of the nature reserve was covered by trees.

- (b) Use this information to find a complete equation for the model, giving your value of c to 3 significant figures.

(4)

On 1st January 2020, the owners of the nature reserve announced a long-term plan to have 100 km^2 of the nature reserve covered by trees.

- (c) State a reason why the model is not appropriate for this plan.

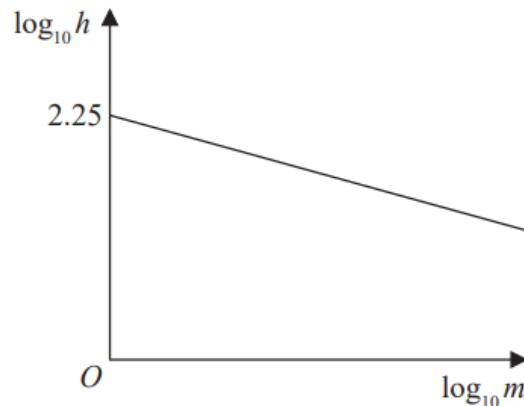
(1)



Question	Scheme	Marks	AOs
11 (a)	$35 \text{ (km}^2\text{)}$	B1	3.4
		(1)	
(b)	Sets their $60 = 80 - 45e^{14c} \Rightarrow 45e^{14c} = 20$	M1 A1	1.1b 1.1b
	$\Rightarrow c = \frac{1}{14} \ln\left(\frac{20}{45}\right) = \dots - 0.0579$	dM1	3.1b
	$A = 80 - 45e^{-0.0579t}$	A1	3.3
		(4)	
(c)	Gives a suitable answer <ul style="list-style-type: none"> The maximum area covered by trees is only 80 km^2 The "80" would need to be "100" Substitutes 100 into the equation of the model and shows that the formula fails with a reason eg. you cannot take a log of a negative number 	B1	3.5b
		(1)	
(6 marks)			



13.

**Figure 2**

The resting heart rate, h , of a mammal, measured in beats per minute, is modelled by the equation

$$h = pm^q$$

where p and q are constants and m is the mass of the mammal measured in kg.

Figure 2 illustrates the linear relationship between $\log_{10} h$ and $\log_{10} m$

The line meets the vertical $\log_{10} h$ axis at 2.25 and has a gradient of -0.235

(a) Find, to 3 significant figures, the value of p and the value of q .

(3)

A particular mammal has a mass of 5 kg and a resting heart rate of 119 beats per minute.

(b) Comment on the suitability of the model for this mammal.

(3)

(c) With reference to the model, interpret the value of the constant p .

(1)



Question	Scheme		Marks	AOs
13 (a)	$\log_{10} h = 2.25 - 0.235 \log_{10} m$ $\Rightarrow h = 10^{2.25 - 0.235 \log_{10} m}$ $\Rightarrow h = 10^{2.25} \times m^{-0.235}$	$h = pm^q$ $\Rightarrow \log_{10} h = \log_{10} p + \log_{10} m^q$ $\Rightarrow \log_{10} h = \log_{10} p + q \log_{10} m$	M1	1.1b
	Either one of $p = 10^{2.25}$ $q = -0.235$	Or either one of $\log_{10} p = 2.25$ $q = -0.235$	A1	1.1b
	$\Rightarrow p = 178$ and $q = -0.235$		A1	2.2a
				(3)
(b)	$h = "178" \times 5^{-0.235}$	$\log_{10} h = "2.25" - "0.235" \log_{10} 5$	M1	3.1b
	$h = 122$	$h = 122$	A1	1.1b
	Reasonably accurate (to 2 sf) so suitable		A1ft	3.2b
				(3)
(c)	$"p"$ would be the (resting) heart rate (in bpm) of a mammal with a mass of 1 kg		B1	3.4
				(1)
(7 marks)				



5. The mass, A kg, of algae in a small pond, is modelled by the equation

$$A = pq^t$$

where p and q are constants and t is the number of weeks after the mass of algae was first recorded.

Data recorded indicates that there is a linear relationship between t and $\log_{10} A$ given by the equation

$$\log_{10} A = 0.03t + 0.5$$

- (a) Use this relationship to find a complete equation for the model in the form

$$A = pq^t$$

giving the value of p and the value of q each to 4 significant figures.

(4)

- (b) With reference to the model, interpret

- (i) the value of the constant p ,
- (ii) the value of the constant q .

(2)

- (c) Find, according to the model,

- (i) the mass of algae in the pond when $t = 8$, giving your answer to the nearest 0.5 kg,
- (ii) the number of weeks it takes for the mass of algae in the pond to reach 4 kg.

(3)

- (d) State one reason why this may not be a realistic model in the long term.

(1)



5(a)	$p = 10^{0.5}$ (or $\log_{10} p = 0.5$) or $q = 10^{0.03}$ (or $\log_{10} q = 0.03$)	M1
	$p = \text{awrt } 3.162$ or $q = \text{awrt } 1.072$	A1
	$p = 10^{0.5}$ (or $\log_{10} p = 0.5$) and $q = 10^{0.03}$ (or $\log_{10} q = 0.03$)	dM1
	$A = 3.162 \times 1.072^t$	A1
		(4)
(b)(i)	The initial mass (in kg) of algae (in the pond).	B1
(b)(ii)	The ratio of algae from one week to the next.	B1
		(2)
(c)(i)	5.5 kg	B1
(c)(ii)	$4 = "3.162" \times "1.072"^t$ or $\log_{10} 4 = 0.03 t + 0.5$	M1
	$\text{awrt } 3.4$ (weeks)	A1
		(3)
(d)	<ul style="list-style-type: none"> The model predicts unlimited growth. The weather may affect the rate of growth 	B1
		(1)



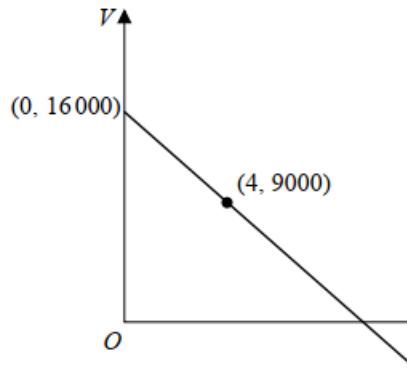
6. A company plans to extract oil from an oil field.

The daily volume of oil V , measured in barrels that the company will extract from this oil field depends upon the time, t years, after the start of drilling.

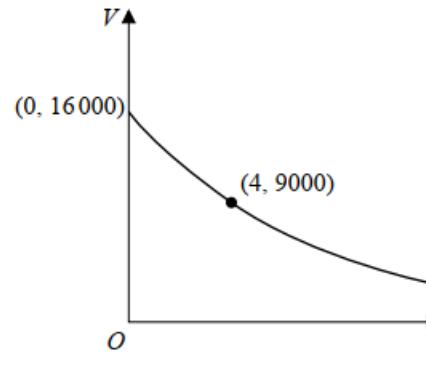
The company decides to use a model to estimate the daily volume of oil that will be extracted. The model includes the following assumptions:

- The initial daily volume of oil extracted from the oil field will be 16 000 barrels.
- The daily volume of oil that will be extracted exactly 4 years after the start of drilling will be 9000 barrels.
- The daily volume of oil extracted will decrease over time.

The diagram below shows the graphs of two possible models.

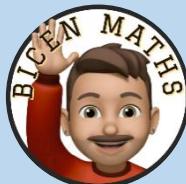


Model A



Model B

- (a) (i) Use model A to estimate the daily volume of oil that will be extracted exactly 3 years after the start of drilling.
(ii) Write down a limitation of using model A. (2)
- (b) (i) Using an exponential model and the information given in the question, find a possible equation for model B.
(ii) Using your answer to (b)(i) estimate the daily volume of oil that will be extracted exactly 3 years after the start of drilling. (5)



Question	Scheme	Marks	AOs
6 (a)(i)	10750 barrels	B1	3.4
(ii)	<p>Gives a valid limitation, for example</p> <ul style="list-style-type: none"> The model shows that the daily volume of oil extracted would become negative as t increases, which is impossible States when $t = 10, V = -1500$ which is impossible States that the model will only work for $0 \leq t \leq \frac{64}{7}$ 	B1	3.5b
		(2)	
(b)(i)	<p>Suggests a suitable exponential model, for example $V = Ae^{kt}$</p> <p>Uses $(0, 16000)$ and $(4, 9000)$ in $\Rightarrow 9000 = 16000e^{4k}$</p>	M1	3.3
	$\Rightarrow k = \frac{1}{4} \ln\left(\frac{9}{16}\right)$ awrt -0.144	dM1	3.1b
	$V = 16000e^{\frac{1}{4} \ln\left(\frac{9}{16}\right)t}$ or $V = 16000e^{-0.144t}$	A1	1.1b
(ii)	Uses their exponential model with $t = 3 \Rightarrow V = \text{awrt } 10400$ barrels	B1ft	3.4
		(5)	
		(7 marks)	



12. In a controlled experiment, the number of microbes, N , present in a culture T days after the start of the experiment were counted.

N and T are expected to satisfy a relationship of the form

$$N = aT^b, \quad \text{where } a \text{ and } b \text{ are constants}$$

- (a) Show that this relationship can be expressed in the form

$$\log_{10} N = m \log_{10} T + c$$

giving m and c in terms of the constants a and/or b .

(2)

(4)

(2)

(1)

Figure 3 shows the line of best fit for values of $\log_{10} N$ plotted against values of $\log_{10} T$

- (b) Use the information provided to estimate the number of microbes present in the culture 3 days after the start of the experiment.

- (c) Explain why the information provided could not reliably be used to estimate the day when the number of microbes in the culture first exceeds 1 000 000.

- (d) With reference to the model, interpret the value of the constant a .

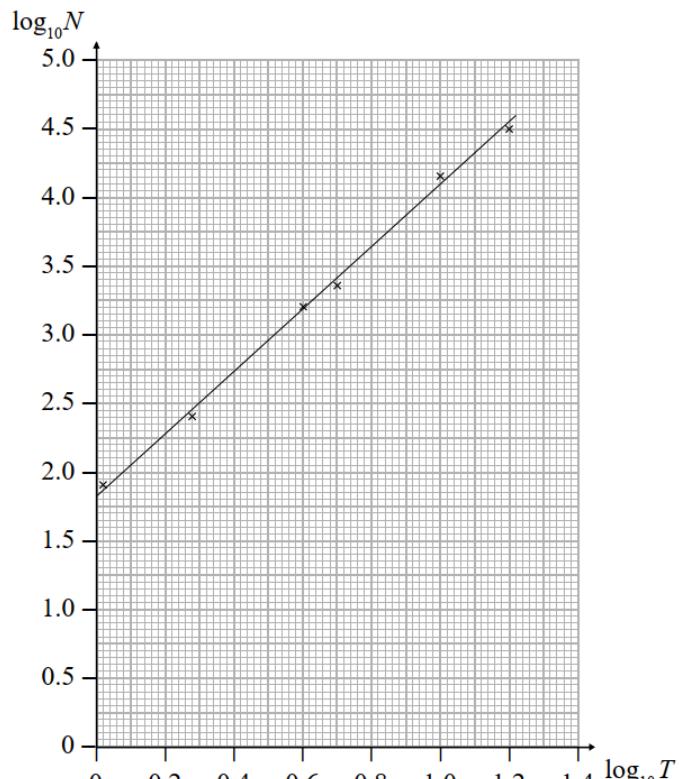
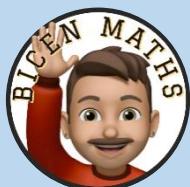


Figure 3



Question	Scheme	Marks	AOs
12 (a)	$N = aT^b \Rightarrow \log_{10} N = \log_{10} a + \log_{10} T^b$	M1	2.1
	$\Rightarrow \log_{10} N = \log_{10} a + b \log_{10} T$ so $m = b$ and $c = \log_{10} a$	A1	1.1b
		(2)	
(b)	Uses the graph to find either a or b $a = 10^{\text{intercept}}$ or $b = \text{gradient}$	M1	3.1b
	Uses the graph to find both a and b $a = 10^{\text{intercept}}$ and $b = \text{gradient}$	M1	1.1b
	Uses $T = 3$ in $N = aT^b$ with their a and b	M1	3.1b
	Number of microbes ≈ 800	A1	1.1b
		(4)	
(c)	$N = 1000000 \Rightarrow \log_{10} N = 6$	M1	3.4
	We cannot 'extrapolate' the graph and assume that the model still holds	A1	3.5b
		(2)	
(d)	States that ' a ' is the number of microbes 1 day after the start of the experiment	B1	3.2a
		(1)	
(9 marks)			



5. The mass, m grams, of a radioactive substance, t years after first being observed, is modelled by the equation

$$m = 25e^{-0.05t}$$

According to the model,

- (a) find the mass of the radioactive substance six months after it was first observed,

(2)

- (b) show that $\frac{dm}{dt} = km$, where k is a constant to be found.

(2)



Question	Scheme	Marks	AOs
5(a)	Substitutes $t = 0.5$ into $m = 25e^{-0.05t} \Rightarrow m = 25e^{-0.05 \times 0.5}$ $\Rightarrow m = 24.4\text{g}$	M1 A1	3.4 1.1b
			(2)
(b)	States or uses $\frac{d}{dt}(e^{-0.05t}) = \pm Ce^{-0.05t}$ $\frac{dm}{dt} = -0.05 \times 25e^{-0.05t} = -0.05m \Rightarrow k = -0.05$	M1 A1	2.1 1.1b
			(2)
			(4 marks)



12. The value, £ V , of a vintage car t years after it was first valued on 1st January 2001, is modelled by the equation

$$V = Ap^t \quad \text{where } A \text{ and } p \text{ are constants}$$

Given that the value of the car was £32 000 on 1st January 2005 and £50 000 on 1st January 2012

- (a) (i) find p to 4 decimal places,
(ii) show that A is approximately 24 800

(4)

- (b) With reference to the model, interpret
(i) the value of the constant A ,
(ii) the value of the constant p .

(2)

Using the model,

- (c) find the year during which the value of the car first exceeds £100 000

(4)



Question	Scheme	Marks	AOs
12 (a)	(i) Method to find p Eg. Divides $32000 = Ap^4$ by $50000 = Ap^{11}$ $p^7 = \frac{50000}{32000} \Rightarrow p = \sqrt[7]{\frac{50000}{32000}} = \dots$	M1	3.1a
	$p = 1.0658$	A1	1.1b
	(ii) Substitutes their $p = 1.0658$ into either equation and finds A $A = \frac{32000}{1.0658^4} \text{ or } A = \frac{50000}{1.0658^{11}}$	M1	1.1b
	$A = 24795 \rightarrow 24805 \approx 24800 *$	A1*	1.1b
			(4)
(b)	$A / (\text{£})24800$ is the value of the car on 1st January 2001	B1	3.4
	$p / 1.0658$ is the factor by which the value rises each year. Accept that the value rises by 6.6 % a year (ft on their p)	B1	3.4
			(2)
(c)	Attempts $100000 = 24800 \times 1.0658^t$		
	$1.0658^t = \frac{100000}{24800}$	M1	3.4
	$t = \log_{1.0658} \left(\frac{100000}{24800} \right)$	dM1	1.1b
	$t = 21.8 \text{ or } 21.9$	A1	1.1b
	cso 2022	A1	3.2a
			(4)
			(10 marks)



7. In a simple model, the value, £ V , of a car depends on its age, t , in years.

The following information is available for car A

- its value when new is £20 000
- its value after one year is £16 000

- (a) Use an exponential model to form, for car A , a possible equation linking V with t .

(4)

The value of car A is monitored over a 10-year period.

Its value after 10 years is £2 000

- (b) Evaluate the reliability of your model in light of this information.

(2)

The following information is available for car B

- it has the same value, when new, as car A
- its value depreciates more slowly than that of car A

- (c) Explain how you would adapt the equation found in (a) so that it could be used to model the value of car B .

(1)



Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$V = Ae^{-kt}$	M1	This mark is given for suggesting a suitable exponential model for V in terms of t
	When $t = 0$ and $V = 20\ 000$, $A = 20\ 000$	M1	This mark is given for using the model to show the initial value for A is £20 000
	When $t = 1$ and $V = 16\ 000$, $16\ 000 = 20\ 000e^{-1k}$ $k = \ln 0.8 = -0.223$	M1	This mark is given for using the value of the car after one year to find a value for k
	$V = 20\ 000e^{-0.223t}$	A1	This mark is given for finding a fully correct exponential model
(b)	When $t = 10$, $V = £2150$	M1	This mark is given for finding a value for V when $t = 10$
	This model is reliable since the value £2150 is close to £2000	A1	This mark is given for a valid statement comparing the two possible values of the car after 10 years
(c)	For example: The value of k should be increased (e.g. $V = 20\ 000e^{-0.1t}$) A constant should be added (e.g. $V = 20\ 000e^{-0.223t} + 2000$)	B1	This mark is given for a statement suggesting a valid adaptation



9. Given that $a > b > 0$ and that a and b satisfy the equation

$$\log a - \log b = \log(a - b)$$

- (a) show that

$$a = \frac{b^2}{b - 1} \quad (3)$$

- (b) Write down the full restriction on the value of b , explaining the reason for this restriction.

(2)



Question 9 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\log a - \log b = \log \frac{a}{b}$	B1	This mark is given for restating the log equation using $\log \frac{a}{b}$
	$a - b = \frac{a}{b}$ $ab - b^2 = a$ $ab - a = b^2$	M1	This mark is given for rearranging so that terms in a are on one side of the equation
	$a(b - 1) = b^2$ $a = \frac{b^2}{(b - 1)}$	A1	This mark is for rearranging to show the result required
(b)	$b \neq 1$	B1	This mark is given for deducing that $b \neq 1$
	Since $a > 0$, $\frac{b^2}{(b - 1)} > 0$ $b > 1$ since b^2 is positive	B1	This mark is given for stating that $b > 1$ and explaining the reason for the restriction



9. A research engineer is testing the effectiveness of the braking system of a car when it is driven in wet conditions.

The engineer measures and records the braking distance, d metres, when the brakes are applied from a speed of $V \text{ km h}^{-1}$.

Graphs of d against V and $\log_{10} d$ against $\log_{10} V$ were plotted.

The results are shown below together with a data point from each graph.

- (a) Explain how Figure 5 would lead the engineer to believe that the braking distance should be modelled by the formula

$$d = kV^n \quad \text{where } k \text{ and } n \text{ are constants}$$

with $k \approx 0.017$

Using the information given in Figure 5, with $k = 0.017$

- (b) find a complete equation for the model giving the value of n to 3 significant figures.

Sean is driving this car at 60 km h^{-1} in wet conditions when he notices a large puddle in the road 100 m ahead. It takes him 0.8 seconds to react before applying the brakes.

- (c) Use your formula to find out if Sean will be able to stop before reaching the puddle.

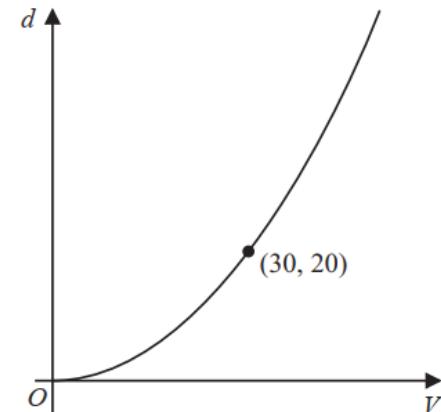


Figure 5

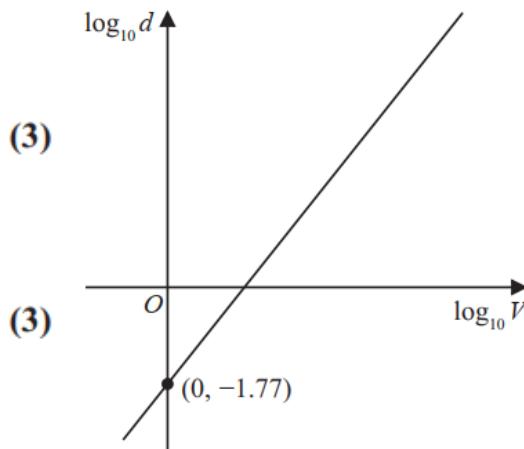


Figure 6



Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	If $d = kV^n$, then $\log_{10} d = \log_{10} k + n \log_{10} V$	M1	This mark is given for
	Plotting $\log_{10} d$ against $\log_{10} V$ will result in a straight line with gradient n and intercept $\log_{10} k$	A1	This mark is given for an explanation of why the second graph shows that $d = kV^n$
	$\log_{10} k = -1.77$ $k = 10^{-1.77} = 0.01698\dots \approx 0.017$	A1	This mark is for showing fully that $k \approx 0.017$
(b)	$d = kV^n$ When $V = 30$, $d = 20$ and $k = 0.17$ then $20 = 0.017 \times 30^n$	M1	This mark is given for substituting in the formula as a method to find the value of n
	$n \log 30 = \log \left(\frac{20}{0.017} \right)$	M1	This mark is given for a correct expression for n
	$n = 2.08$ to 3 significant figures $d = 0.017 \times V^{2.08}$	A1	This mark is given for finding a correct value of n to 3 significant figures and writing a complete equation for the model
(c)	$\frac{60}{3600} \times 0.8 \times 1000 = 13.33 \text{ m}$	M1	This mark is given for a method to find the distance, in metres, covered in the reaction time of 0.8 seconds
	$d = 0.017 \times 60^{2.08} = 84.92 \text{ m}$	M1	This mark is given for a method to use the formula to find the stopping distance
	$13.33 \text{ m} + 84.92 \text{ m} = 98.25 \text{ m}$ Sean will be able to stop before reaching the puddle	A1	This mark is given for finding a correct value of the total stopping distance and giving a valid conclusion



2. By taking logarithms of both sides, solve the equation

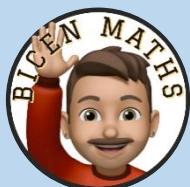
$$4^{3p-1} = 5^{210}$$

giving the value of p to one decimal place.

(3)



Question	Scheme	Marks	AOs
2	$4^{3p-1} = 5^{210} \Rightarrow (3p-1)\log 4 = 210\log 5$	M1	1.1b
	$\Rightarrow 3p = \frac{210\log 5}{\log 4} + 1 \Rightarrow p = \dots$	dM1	2.1
	$p = \text{awrt } 81.6$	A1	1.1b
		(3)	
		(3 marks)	



3. (a) Given that

$$2 \log(4 - x) = \log(x + 8)$$

show that

$$x^2 - 9x + 8 = 0$$

(3)

(b) (i) Write down the roots of the equation

$$x^2 - 9x + 8 = 0$$

(ii) State which of the roots in (b)(i) is **not** a solution of

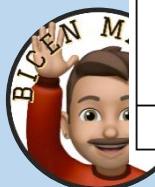
$$2 \log(4 - x) = \log(x + 8)$$

giving a reason for your answer.

(2)



3(a)	$2 \log(4-x) = \log(4-x)^2$	B1	1.2
	$2 \log(4-x) = \log(x+8) \Rightarrow \log(4-x)^2 = \log(x+8)$		
	$(4-x)^2 = (x+8)$		
	or	M1	1.1b
	$2 \log(4-x) = \log(x+8) \Rightarrow \log(4-x)^2 - \log(x+8) = 0$		
	$\frac{(4-x)^2}{(x+8)} = 1$		
	$16 - 8x + x^2 = x + 8 \Rightarrow x^2 - 9x + 8 = 0 *$	A1*	2.1
		(3)	
	(a) Alternative - working backwards:		
	$x^2 - 9x + 8 = 0 \Rightarrow (4-x)^2 - x - 8 = 0$	B1	1.2
	$\Rightarrow (4-x)^2 = x + 8$		
	$\Rightarrow \log(4-x)^2 = \log(x+8)$	M1	1.1b
	$\Rightarrow 2 \log(4-x) = \log(x+8) * \text{ Hence proved.}$	A1	2.1
(b)	(i) $(x =) 1, 8$	B1	1.1b
	(ii) 8 is not a solution as $\log(4-8)$ cannot be found	B1	2.3
		(2)	
(5 marks)			



5. The curve with equation $y = 3 \times 2^x$ meets the curve with equation $y = 15 - 2^{x+1}$ at the point P .
Find, using algebra, the exact x coordinate of P .

(4)



Question	Scheme	Marks	AOs
5	$15 - 2^{x+1} = 3 \times 2^x$	B1	1.1b
	$\Rightarrow 15 - 2 \times 2^x = 3 \times 2^x \Rightarrow 2^x = 3$ or e.g. $\Rightarrow \frac{15}{2^x} - 2 = 3 \Rightarrow 2^x = 3$	M1	1.1b
	$2^x = 3 \Rightarrow x = \dots$	dM1	1.1b
	$x = \log_2 3$	A1cs0	1.1b
		(4)	
	Alternative		
	$y = 3 \times 2^x \Rightarrow 2^x = \frac{y}{3} \Rightarrow y = 15 - 2 \times \frac{y}{3}$	B1	1.1b
	$3y + 2y = 45 \Rightarrow y = 9 \Rightarrow 3 \times 2^x = 9 \Rightarrow 2^x = 3$	M1	1.1b
	$2^x = 3 \Rightarrow x = \dots$	dM1	1.1b
	$x = \log_2 3$	A1cs0	1.1b
		(4 marks)	



9. A quantity of ethanol was heated until it reached boiling point.

The temperature of the ethanol, $\theta^{\circ}\text{C}$, at time t seconds after heating began, is modelled by the equation

$$\theta = A - Be^{-0.07t}$$

where A and B are positive constants.

Given that

- the initial temperature of the ethanol was 18°C
- after 10 seconds the temperature of the ethanol was 44°C

(a) find a complete equation for the model, giving the values of A and B to 3 significant figures.

(4)

Ethanol has a boiling point of approximately 78°C

(b) Use this information to evaluate the model.

(2)



Question	Scheme	Marks	AOs
9(a)	$t = 0, \theta = 18 \Rightarrow 18 = A - B$ or $t = 10, \theta = 44 \Rightarrow 44 = A - Be^{-0.7}$	M1	3.1b
	$t = 0, \theta = 18 \Rightarrow 18 = A - B$ and $t = 10, \theta = 44 \Rightarrow 44 = A - Be^{-0.7}$ and $\Rightarrow A = \dots, B = \dots$	M1	3.1a
	At least one of: $A = 69.6, B = 51.6$ but allow awrt 70/awrt 52	A1 M1 on EPEN	1.1b
	$\theta = 69.6 - 51.6e^{-0.07t}$	A1	3.3
		(4)	
(b)	<p>The maximum temperature is “69.6”(°C) (according to the model) (The model has an) upper limit of “69.6”(°C) (The model suggests that) the boiling point is “69.6”(°C)</p> <p>Model is not appropriate as 69.6(°C) is much lower than 78(°C)</p>	B1ft	3.4
		B1ft	3.5a
		(2)	
	(6 marks)		



8. A scientist is studying the growth of two different populations of bacteria.

The number of bacteria, N , in the **first** population is modelled by the equation

$$N = Ae^{kt} \quad t \geq 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 1000 bacteria in this population at the start of the study
- it took exactly 5 hours from the start of the study for this population to double

- (a) find a complete equation for the model.

(4)

- (b) Hence find the rate of increase in the number of bacteria in this population exactly 8 hours from the start of the study. Give your answer to 2 significant figures.

(2)

The number of bacteria, M , in the **second** population is modelled by the equation

$$M = 500e^{1.4kt} \quad t \geq 0$$

where k has the value found in part (a) and t is the time in hours from the start of the study.

Given that T hours after the start of the study, the number of bacteria in the two different populations was the same,

- (c) find the value of T .

(3)



Question	Scheme	Marks	AOs
8(a)	$A = 1000$	B1	3.4
	$2000 = 1000e^{5k}$ or $e^{5k} = 2$	M1	1.1b
	$e^{5k} = 2 \Rightarrow 5k = \ln 2 \Rightarrow k = \dots$	M1	2.1
	$N = 1000e^{\left(\frac{1}{5}\ln 2\right)t}$ or $N = 1000e^{0.139t}$	A1	3.3
		(4)	
(b)	$\frac{dN}{dt} = 1000 \times \left(\frac{1}{5} \ln 2\right) e^{\left(\frac{1}{5} \ln 2\right)t}$ or $\frac{dN}{dt} = 1000 \times 0.139 e^{0.139t}$	M1	3.1b
	$\left(\frac{dN}{dt}\right)_{t=8} = 1000 \times \left(\frac{1}{5} \ln 2\right) e^{8 \times \frac{1}{5} \ln 2}$ or $\left(\frac{dN}{dt}\right)_{t=8} = 1000 \times 0.139 e^{0.139 \times 8}$		
	$= \text{awrt } 420$	A1	1.1b
		(2)	
(c)	$500e^{1.4 \times \left(\frac{1}{5} \ln 2\right)T} = 1000e^{\left(\frac{1}{5} \ln 2\right)T}$ or $500e^{1.4 \times "0.139"t} = 1000e^{0.139t}$	M1	3.4
	Correct method of getting a linear equation in T E.g. $0.08T \ln 2 = \ln 2$ or $1.4 \times "0.339" T = \ln 2 + "0.339" t$	M1	2.1
	$T = 12.5$ hours	A1	1.1b
		(3)	
		(9 marks)	



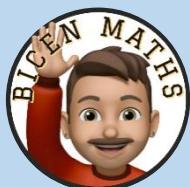
3. Using the laws of logarithms, solve the equation

$$\log_3 (12y + 5) - \log_3 (1 - 3y) = 2$$

(3)



Question	Scheme	Marks	AOs
3	$\log_3(12y+5) - \log_3(1-3y) = 2 \Rightarrow \log_3 \frac{12y+5}{1-3y} = 2$ or e.g. $2 = \log_3 9$	B1 M1 on EPEN	1.1b
	$\log_3 \frac{12y+5}{1-3y} = 2 \Rightarrow \frac{12y+5}{1-3y} = 3^2 \Rightarrow 9 - 27y = 12y + 5 \Rightarrow y = \dots$ or e.g. $\log_3(12y+5) = \log_3(3^2(1-3y)) \Rightarrow (12y+5) = 3^2(1-3y) \Rightarrow y = \dots$	M1	2.1
	$y = \frac{4}{39}$	A1	1.1b
		(3)	
	(3 marks)		



10. The time, T seconds, that a pendulum takes to complete one swing is modelled by the formula

$$T = al^b$$

where l metres is the length of the pendulum and a and b are constants.

- (a) Show that this relationship can be written in the form

$$\log_{10} T = b \log_{10} l + \log_{10} a \quad (2)$$

A student carried out an experiment to find the values of the constants a and b .

The student recorded the value of T for different values of l .

Figure 3 shows the linear relationship between $\log_{10} l$ and $\log_{10} T$ for the student's data.

The straight line passes through the points $(-0.7, 0)$ and $(0.21, 0.45)$

Using this information,

- (b) find a complete equation for the model in the form

$$T = al^b$$

giving the value of a and the value of b , each to 3 significant figures.

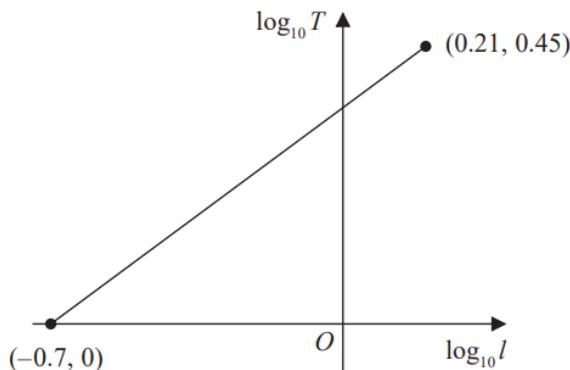
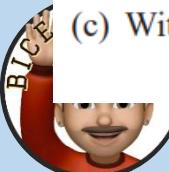


Figure 3

- (c) With reference to the model, interpret the value of the constant a .

(1)



Question	Scheme	Marks	AOs
10(a)	$T = al^b \Rightarrow \log_{10} T = \log_{10} a + \log_{10} l^b$	M1	2.1
	$\Rightarrow \log_{10} T = \log_{10} a + b \log_{10} l *$ or $\Rightarrow \log_{10} T = b \log_{10} l + \log_{10} a *$	A1*	1.1b
		(2)	
(b)	$b = 0.495$ or $b = \frac{45}{91}$	B1	2.2a
	$0 = "0.495" \times -0.7 + \log_{10} a \Rightarrow a = 10^{0.346\dots}$ or $0.45 = "0.495" \times 0.21 + \log_{10} a \Rightarrow a = 10^{0.346\dots}$	M1	3.1a
	$T = 2.22l^{0.495}$	A1	3.3
		(3)	
	The time taken for one swing of a pendulum of length 1 m	B1	3.2a
		(1)	
			(6 marks)



10. A scientist is studying the number of bees and the number of wasps on an island.

The number of bees, measured in thousands, N_b , is modelled by the equation

$$N_b = 45 + 220 e^{0.05t}$$

where t is the number of years from the start of the study.

According to the model,

(a) find the number of bees at the start of the study,

(1)

(b) show that, exactly 10 years after the start of the study, the number of bees was increasing at a **rate** of approximately 18 thousand per year.

(3)

The number of wasps, measured in thousands, N_w , is modelled by the equation

$$N_w = 10 + 800 e^{-0.05t}$$

where t is the number of years from the start of the study.

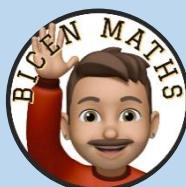
When $t = T$, according to the models, there are an equal number of bees and wasps.

(c) Find the value of T to 2 decimal places.

(4)



10 (a)	265 thousand	B1
		(1)
(b)	Attempts $\frac{dN_b}{dt} = 11e^{0.05t}$	M1
	Substitutes $t = 10$ into their $\frac{dN_b}{dt}$	M1
	$\frac{dN_b}{dt} = \text{awrt } 18.1$ which is approximately 18 thousand per year *	A1*
		(3)
(c)	Sets $45 + 220e^{0.05t} = 10 + 800e^{-0.05t} \Rightarrow 220e^{0.05t} + 35 - 800e^{-0.05t} = 0$	M1
	Correct quadratic equation $\Rightarrow 220(e^{0.05t})^2 + 35e^{0.05t} - 800 = 0$	A1
	$e^{0.05t} = 1.829, (-1.988) \Rightarrow 0.05t = \ln(1.829)$	M1
	$T = 12.08$	A1
		(4)



2. (a) Sketch the curve with equation

$$y = 4^x$$

stating any points of intersection with the coordinate axes.

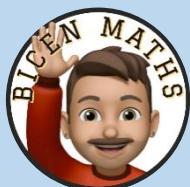
(2)

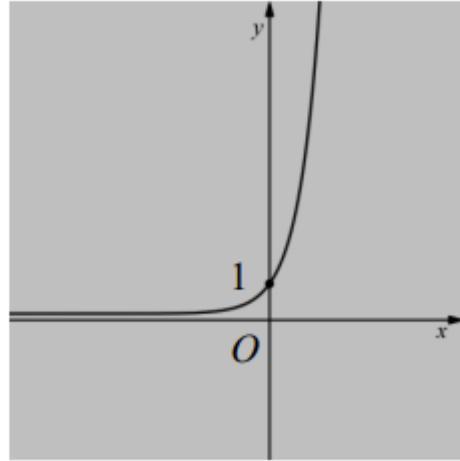
(b) Solve

$$4^x = 100$$

giving your answer to 2 decimal places.

(2)



2(a)		Correct shape or correct intercept – see notes	B1
(b)	$4^x = 100 \Rightarrow x = \log_4 100$ <p style="text-align: center;">or</p> $\text{e.g. } x \log 4 = \log 100 \Rightarrow x = \frac{\log 100}{\log 4}$ $\Rightarrow (x =) \text{awrt } 3.32$	M1	(2)

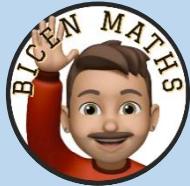


Algebraic Methods – proof, partial fractions, etc.



16. (a) Express $\frac{1}{P(11 - 2P)}$ in partial fractions.

(3)



Question	Scheme	Marks	AOs
16(a)	Sets $\frac{1}{P(11-2P)} = \frac{A}{P} + \frac{B}{(11-2P)}$	B1	1.1a
	Substitutes either $P=0$ or $P=\frac{11}{2}$ into $1=A(11-2P)+BP \Rightarrow A \text{ or } B$	M1	1.1b
	$\frac{1}{P(11-2P)} = \frac{1}{P} + \frac{2}{11-2P}$	A1	1.1b
		(3)	



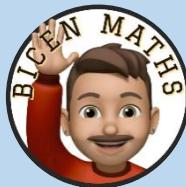
3. (a) “If m and n are irrational numbers, where $m \neq n$, then mn is also irrational.”

Disprove this statement by means of a counter example.

(2)

- (b) (i) Sketch the graph of $y = |x| + 3$
- (ii) Explain why $|x| + 3 \geq |x + 3|$ for all real values of x .

(3)



Question	Scheme	Marks	AOs	
3	Statement: "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."			
(a)	E.g. $m = \sqrt{3}$, $n = \sqrt{12}$	M1	1.1b	
	$\{mn =\} (\sqrt{3})(\sqrt{12}) = 6$ ⇒ statement untrue or 6 is not irrational or 6 is rational	A1	2.4	
		(2)		
(b)(i), (ii) Way 1	<p>$y = x + 3$</p> <p>$y = x + 3$</p> <p>{-3} O</p>	V shaped graph {reasonably} symmetrical about the y -axis with vertical intercept $(0, 3)$ or 3 stated or marked on the positive y -axis	B1	1.1b
		Superimposes the graph of $y = x + 3 $ on top of the graph of $y = x + 3$	M1	3.1a
	the graph of $y = x + 3$ is either the same or above the graph of $y = x + 3 $ {for corresponding values of x } or when $x \geq 0$, both graphs are equal (or the same) when $x < 0$, the graph of $y = x + 3$ is above the graph of $y = x + 3 $	A1	2.4	
		(3)		
(b)(ii) Way 2	<u>Reason 1</u> When $x \geq 0$, $ x + 3 = x + 3 $ <u>Reason 2</u> When $x < 0$, $ x + 3 > x + 3 $	Any one of Reason 1 or Reason 2 Both Reason 1 and Reason 2	M1 A1	3.1a 2.4
			(5 marks)	



11.

$$\frac{1 + 11x - 6x^2}{(x - 3)(1 - 2x)} \equiv A + \frac{B}{(x - 3)} + \frac{C}{(1 - 2x)}$$

(a) Find the values of the constants A , B and C .

(4)



Question	Scheme	Marks	AOs
11	$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}$		
(a) Way 1	$1+11x-6x^2 \equiv A(1-2x)(x-3) + B(1-2x) + C(x-3) \Rightarrow B = \dots, C = \dots$	M1	2.1
	$A = 3$	B1	1.1b
	Uses substitution or compares terms to find either $B = \dots$ or $C = \dots$	M1	1.1b
	$B = 4$ and $C = -2$ which have been found using a correct identity	A1	1.1b
		(4)	
(a) Way 2	{long division gives} $\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv 3 + \frac{-10x+10}{(x-3)(1-2x)}$		
	$-10x+10 \equiv B(1-2x) + C(x-3) \Rightarrow B = \dots, C = \dots$	M1	2.1
	$A = 3$	B1	1.1b
	Uses substitution or compares terms to find either $B = \dots$ or $C = \dots$	M1	1.1b
	$B = 4$ and $C = -2$ which have been found using $-10x+10 \equiv B(1-2x) + C(x-3)$	A1	1.1b
		(4)	



6. (a) Given that

$$\frac{x^2 + 8x - 3}{x + 2} \equiv Ax + B + \frac{C}{x + 2} \quad x \in \mathbb{R} \quad x \neq -2$$

find the values of the constants A , B and C

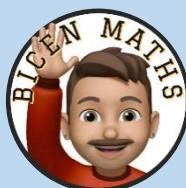
(3)

(b) Hence, using algebraic integration, find the exact value of

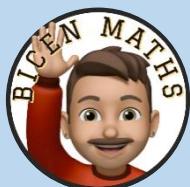
$$\int_0^6 \frac{x^2 + 8x - 3}{x + 2} dx$$

giving your answer in the form $a + b \ln 2$ where a and b are integers to be found.

(4)



6(a)	$x^2 + 8x - 3 = (Ax + B)(x + 2) + C \text{ or } Ax(x + 2) + B(x + 2) + C$ $\Rightarrow A = \dots, B = \dots, C = \dots$ <p style="text-align: center;">or</p> $ \begin{array}{r} x+6 \\ x+2 \overline{)x^2+8x-3} \\ x^2+2x \\ \hline 6x-3 \\ 6x+12 \\ \hline -15 \end{array} $		
		M1	1.1b
	Two of $A = 1, B = 6, C = -15$	A1	1.1b
	All three of $A = 1, B = 6, C = -15$	A1	1.1b
		(3)	



9.

$$f(x) = \frac{50x^2 + 38x + 9}{(5x + 2)^2(1 - 2x)} \quad x \neq -\frac{2}{5} \quad x \neq \frac{1}{2}$$

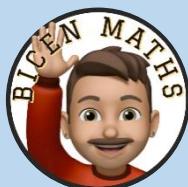
Given that $f(x)$ can be expressed in the form

$$\frac{A}{5x + 2} + \frac{B}{(5x + 2)^2} + \frac{C}{1 - 2x}$$

where A , B and C are constants

- (a) (i) find the value of B and the value of C
(ii) show that $A = 0$

(4)



Question	Scheme	Marks	AOs
9(a)(i)	$50x^2 + 38x + 9 \equiv A(5x+2)(1-2x) + B(1-2x) + C(5x+2)^2$ $\Rightarrow B = \dots \quad \text{or} \quad C = \dots$	M1	1.1b
	$B = 1$ and $C = 2$	A1	1.1b
(a)(ii)	E.g. $x = 0 \quad x = 0 \Rightarrow 9 = 2A + B + 4C$ $\Rightarrow 9 = 2A + 1 + 8 \Rightarrow A = \dots$	M1	2.1
	$A = 0^*$	A1*	1.1b
		(4)	



15. (i) Use proof by exhaustion to show that for $n \in \mathbb{N}$, $n \leq 4$

$$(n + 1)^3 > 3^n \quad (2)$$

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

(4)



Question	Scheme	Marks	AOs
15(i)	$n = 1, 2^3 = 8, 3^1 = 3, (8 > 3)$ $n = 2, 3^3 = 27, 3^2 = 9, (27 > 9)$ $n = 3, 4^3 = 64, 3^3 = 27, (64 > 27)$ $n = 4, 5^3 = 125, 3^4 = 81, (125 > 81)$ So if $n \leq 4, n \in \mathbb{N}$ then $(n + 1)^3 > 3^n$	M1	2.1
		A1	2.4
		(2)	
(ii)	Begins the proof by negating the statement. "Let m be odd" or "Assume m is not even" Set $m = (2p \pm 1)$ and attempt $m^3 + 5 = (2p \pm 1)^3 + 5 = \dots$ $= 8p^3 + 12p^2 + 6p + 6$ AND deduces even	M1	2.4
		M1	2.1
		A1	2.2a
	Completes proof which requires reason and conclusion <ul style="list-style-type: none"> reason for $8p^3 + 12p^2 + 6p + 6$ being even acceptable statement such as "this is a contradiction so if $m^3 + 5$ is odd then m must be even" 	A1	2.4
		(4)	
			(6 marks)



7. (i) Given that p and q are integers such that

$$pq \text{ is even}$$

use algebra to prove by contradiction that at least one of p or q is even.

(3)

(ii) Given that x and y are integers such that

- $x < 0$
- $(x + y)^2 < 9x^2 + y^2$

show that $y > 4x$

(2)



7 (i)	For setting up the contradiction: There exists integers p and q such that pq is even and both p and q are odd	B1
	For example, sets $p = 2m + 1$ and $q = 2n + 1$ and then attempts $pq = (2m + 1)(2n + 1) = \dots$	M1
	Obtains $pq = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1$ $= 2(2mn + m + n) + 1$	A1*
	States that this is odd, giving a contradiction so " if pq is even, then at least one of p and q is even" *	(3)
(ii)	$(x + y)^2 < 9x^2 + y^2 \Rightarrow 2xy < 8x^2$	M1
	States that as $x < 0 \Rightarrow 2y > 8x$ $\Rightarrow y > 4x$ *	A1*
		(2)



Sequences and Series



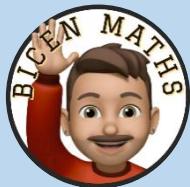
10. In a geometric series the common ratio is r and sum to n terms is S_n

Given

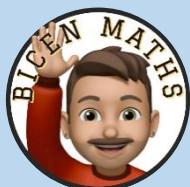
$$S_{\infty} = \frac{8}{7} \times S_6$$

show that $r = \pm \frac{1}{\sqrt{k}}$, where k is an integer to be found.

(4)



Question	Scheme	Marks	AOs
10	Attempts $S_{\infty} = \frac{8}{7} \times S_6 \Rightarrow \frac{a}{1-r} = \frac{8}{7} \times \frac{a(1-r^6)}{1-r}$	M1	2.1
	$\Rightarrow 1 = \frac{8}{7} \times (1 - r^6)$	M1	2.1
	$\Rightarrow r^6 = \frac{1}{8} \Rightarrow r = ..$	M1	1.1b
	$\Rightarrow r = \pm \frac{1}{\sqrt{2}}$ (so $k = 2$)	A1	1.1b
(4 marks)			

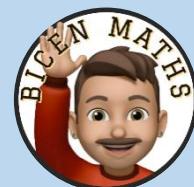


4. (i) Show that $\sum_{r=1}^{16} (3 + 5r + 2^r) = 131\,798$ (4)

(ii) A sequence u_1, u_2, u_3, \dots is defined by

$$u_{n+1} = \frac{1}{u_n}, \quad u_1 = \frac{2}{3}$$

Find the exact value of $\sum_{r=1}^{100} u_r$ (3)



Question	Scheme	Marks	AOs
4	(i) $\sum_{r=1}^{16} (3 + 5r + 2^r) = 131\,798$; (ii) $u_1, u_2, u_3, \dots, : u_{n+1} = \frac{1}{u_n}, u_1 = \frac{2}{3}$		
(i) Way 1	$\left\{ \sum_{r=1}^{16} (3 + 5r + 2^r) = \right\} \sum_{r=1}^{16} (3 + 5r) + \sum_{r=1}^{16} (2^r)$	M1	3.1a
	$= \frac{16}{2} (2(8) + 15(5)) + \frac{2(2^{16} - 1)}{2 - 1}$	M1	1.1b
		M1	1.1b
	$= 728 + 131\,070 = 131\,798 *$	A1*	2.1
		(4)	
(i) Way 2	$\left\{ \sum_{r=1}^{16} (3 + 5r + 2^r) = \right\} \sum_{r=1}^{16} 3 + \sum_{r=1}^{16} (5r) + \sum_{r=1}^{16} (2^r)$	M1	3.1a
	$= (3 \times 16) + \frac{16}{2} (2(5) + 15(5)) + \frac{2(2^{16} - 1)}{2 - 1}$	M1	1.1b
		M1	1.1b
	$= 48 + 680 + 131\,070 = 131\,798 *$	A1*	2.1
		(4)	
(i) Way 3	Sum = $10 + 17 + 26 + 39 + 60 + 97 + 166 + 299 + 560 + 1077 + 2106 + 4159 + 8260 + 16457 + 32846 + 65619 = 131\,798 *$		
		M1	3.1a
		M1	1.1b
		M1	1.1b
		A1*	2.1
		(4)	
(ii)	$\left\{ u_1 = \frac{2}{3} \right\}, u_2 = \frac{3}{2}, u_3 = \frac{2}{3}, \dots \text{(can be implied by later working)}$	M1	1.1b
	$\left\{ \sum_{r=1}^{100} u_r = \right\} 50\left(\frac{2}{3}\right) + 50\left(\frac{3}{2}\right) \text{ or } 50\left(\frac{2}{3} + \frac{3}{2}\right)$	M1	2.2a
	$= \frac{325}{3} \left(\text{or } 108\frac{1}{3} \text{ or } 108.\overline{3} \text{ or } \frac{1300}{12} \text{ or } \frac{650}{6} \right)$	A1	1.1b
		(3)	
(7 marks)			



8. (i) Find the value of

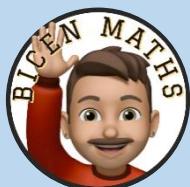
$$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$$

(3)

(ii) Show that

$$\sum_{n=1}^{48} \log_5 \left(\frac{n+2}{n+1} \right) = 2$$

(3)



Question 8 (Total 6 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(i)	$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$ $= \sum_{r=1}^{\infty} 20 \times \left(\frac{1}{2}\right)^r - (10 + 5 + 2.5)$	M1	This mark is given for a method to find the sum to infinity of a GP
	$= \frac{10}{1 - \frac{1}{2}} - (10 + 5 + 2.5)$	M1	This mark is given for a method to use a correct sum formula with a correct first term
	= 2.5	A1	This mark is given for a correct value for the sum
(ii)	$\sum_{n=1}^{48} \log_5 \left(\frac{n+2}{n+1} \right)$ $= \log_5 \frac{50}{49} + \log_5 \frac{49}{48} + \dots + \log_5 \frac{4}{3} + \log_5 \frac{3}{2}$	M1	This mark is given for writing out at least four terms of the sum, including the first two and the last two
	$= \log_5 \frac{3 \times 4 \times \dots \times 48 \times 49 \times 50}{2 \times 3 \times 4 \times \dots \times 48 \times 49} = \log_5 \frac{50}{2}$	M1	This mark is given for using the rules of logs and cancelling terms
	= 2	A1	This mark is given for a full proof to show the expression is equal to 2 as required

5. A car has six forward gears.

The fastest speed of the car

- in 1st gear is 28 km h^{-1}
- in 6th gear is 115 km h^{-1}

Given that the fastest speed of the car in successive gears is modelled by an **arithmetic sequence**,

(a) find the fastest speed of the car in 3rd gear.

(3)

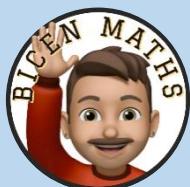
Given that the fastest speed of the car in successive gears is modelled by a **geometric sequence**,

(b) find the fastest speed of the car in 5th gear.

(3)



Question	Scheme	Marks	AOs
5 (a)	Uses $115 = 28 + 5d \Rightarrow d = (17.4)$	M1	3.1b
	Uses $28 + 2 \times "17.4" = \dots$	M1	3.4
	$= 62.8 (\text{km h}^{-1})$	A1	1.1b
		(3)	
(b)	Uses $115 = 28r^5 \Rightarrow r = (1.3265)$	M1	3.1b
	Uses $28 \times "1.3265^4" = \dots$ or $\frac{115}{"1.3265"}$	M1	3.4
	$= 86.7 (\text{km h}^{-1})$	A1	1.1b
		(3)	
(6 marks)			



13. A sequence of numbers a_1, a_2, a_3, \dots is defined by

$$a_{n+1} = \frac{k(a_n + 2)}{a_n} \quad n \in \mathbb{N}$$

where k is a constant.

Given that

- the sequence is a periodic sequence of order 3
- $a_1 = 2$

(a) show that

$$k^2 + k - 2 = 0 \quad (3)$$

(b) For this sequence explain why $k \neq 1$

(1)

(c) Find the value of

$$\sum_{r=1}^{80} a_r \quad (3)$$



Question	Scheme	Marks	AOs
13 (a)	Uses the sequence formula $a_{n+1} = \frac{k(a_n + 2)}{a_n}$ once with $a_1 = 2$	M1	1.1b
	$(a_1 = 2), a_2 = 2k, a_3 = k + 1, a_4 = \frac{k(k+3)}{k+1}$	M1	3.1a
	Finds four consecutive terms and sets a_4 equal to a_1 (oe)		
	$\frac{k(k+3)}{k+1} = 2 \Rightarrow k^2 + 3k = 2k + 2 \Rightarrow k^2 + k - 2 = 0$ *	A1*	2.1
		(3)	
(b)	States that when $k = 1$, all terms are the same and concludes that the sequence does not have a period of order 3	B1	2.3
		(1)	
(c)	Deduces the repeating terms are $a_{1/4} = 2, a_{2/5} = -4, a_{3/6} = -1,$	B1	2.2a
	$\sum_{n=1}^{80} a_k = 26 \times (2 + -4 + -1) + 2 + -4$	M1	3.1a
	$= -80$	A1	1.1b
		(3)	
(7 marks)			



3. The sequence u_1, u_2, u_3, \dots is defined by

$$u_{n+1} = k - \frac{24}{u_n} \quad u_1 = 2$$

where k is an integer.

Given that $u_1 + 2u_2 + u_3 = 0$

(a) show that

$$3k^2 - 58k + 240 = 0$$

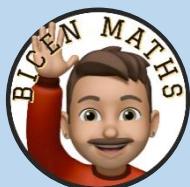
(3)

(b) Find the value of k , giving a reason for your answer.

(2)

(c) Find the value of u_3

(1)



Question	Scheme	Marks	AOs
3(a)	$u_2 = k - 12, u_3 = k - \frac{24}{k-12}$	M1	1.1b
	$u_1 + 2u_2 + u_3 = 0 \Rightarrow 2 + 2(k-12) + k - \frac{24}{k-12} = 0$	dM1	1.1b
	$\Rightarrow 3k - 22 - \frac{24}{k-12} = 0 \Rightarrow (3k-22)(k-12) - 24 = 0$	A1*	2.1
	$\Rightarrow 3k^2 - 36k - 22k + 264 - 24 = 0$		
	$\Rightarrow 3k^2 - 58k + 240 = 0 *$		
(3)			
(b)	$k = 6, \left(\frac{40}{3}\right)$	M1	1.1b
	$k = 6$ as k must be an integer	A1	2.3
		(2)	
(c)	$(u_3 =) 10$	B1	2.2a
		(1)	
(6 marks)			



5.

In this question you should show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

A company made a profit of £20 000 in its first year of trading, Year 1

A model for future trading predicts that the yearly profit will increase by 8% each year, so that the yearly profits will form a geometric sequence.

According to the model,

(a) show that the profit for Year 3 will be £23 328

(1)

(b) find the first year when the yearly profit will exceed £65 000

(3)

(c) find the total profit for the first 20 years of trading, giving your answer to the nearest £1000

(2)



Question	Scheme	Marks	AOs
5(a)	$u_3 = \text{£}20\,000 \times 1.08^2 = (\text{£})23\,328 *$	B1*	1.1b
		(1)	
(b)	$20\,000 \times 1.08^{n-1} > 65\,000$	M1	1.1b
	$1.08^{n-1} > \frac{13}{4} \Rightarrow n-1 > \frac{\ln(3.25)}{\ln(1.08)}$ or e.g. $1.08^{n-1} > \frac{13}{4} \Rightarrow n-1 > \log_{1.08}\left(\frac{13}{4}\right)$	M1	3.1b
	Year 17	A1	3.2a
		(3)	
(c)	$S_{20} = \frac{20\,000(1-1.08^{20})}{1-1.08}$	M1	3.4
	Awrt (£) 915 000	A1	1.1b
		(2)	
	(6 marks)		



1. In an arithmetic series

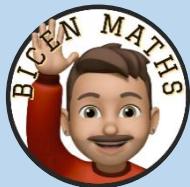
- the first term is 16
- the 21st term is 24

(a) Find the common difference of the series.

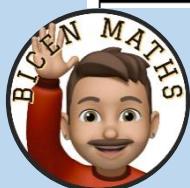
(2)

(b) Hence find the sum of the first 500 terms of the series.

(2)



Question	Scheme	Marks	AOs
1(a)	$16 + (21-1) \times d = 24 \Rightarrow d = \dots$	M1	1.1b
	$d = 0.4$	A1	1.1b
	Answer only scores both marks.		
		(2)	
(b)	$S_n = \frac{1}{2}n\{2a + (n-1)d\} \Rightarrow S_{500} = \frac{1}{2} \times 500 \{2 \times 16 + 499 \times "0.4"\}$ $= 57900$	M1	1.1b
	Answer only scores both marks	A1	1.1b
		(2)	
	(b) Alternative using $S_n = \frac{1}{2}n\{a + l\}$ $l = 16 + (500-1) \times "0.4" = 215.6 \Rightarrow S_{500} = \frac{1}{2} \times 500 \{16 + "215.6"\}$ $= 57900$	M1	1.1b
		A1	1.1b
			(4 marks)

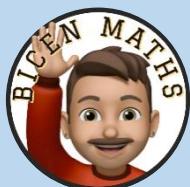


9. Show that

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{9}{28} \quad (3)$$



Question	Scheme	Marks	AOs
9	$a = \left(\frac{3}{4}\right)^2 \quad \text{or} \quad a = \frac{9}{16}$ <p style="text-align: center;">or</p> $r = -\frac{3}{4}$	B1	2.2a
	$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{\frac{9}{16}}{1 - \left(-\frac{3}{4}\right)} = \dots$	M1	3.1a
	$= \frac{9}{28} *$	A1*	1.1b
		(3)	



3. A sequence of terms a_1, a_2, a_3, \dots is defined by

$$a_1 = 3$$

$$a_{n+1} = 8 - a_n$$

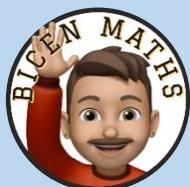
- (a) (i) Show that this sequence is periodic.
(ii) State the order of this periodic sequence.

(2)

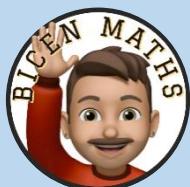
- (b) Find the value of

$$\sum_{n=1}^{85} a_n$$

(2)



3(a)(i)	$a_1 = 3, a_2 = 5, a_3 = 3 \dots$	B1
(ii)	2	B1
		(2)
(b)	$\sum_{n=1}^{85} a_n = 42 \times (3+5) + 3 \text{ o.e.}$	M1
	$= 339$	A1
		(2)



Radians – small angle, arc
length, etc.



- Given that θ is small and is measured in radians, use the small angle approximations to find an approximate value of

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta}$$

(3)



Question	Scheme	Marks	AOs
1	Attempts either $\sin 3\theta \approx 3\theta$ or $\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2}$ in $\frac{1-\cos 4\theta}{2\theta \sin 3\theta}$	M1	1.1b
	Attempts both $\sin 3\theta \approx 3\theta$ and $\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2} \rightarrow \frac{1-\left(1-\frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta}$ and attempts to simplify	M1	2.1
	$= \frac{4}{3}$ oe	A1	1.1b
		(3)	
(3 marks)			



2.

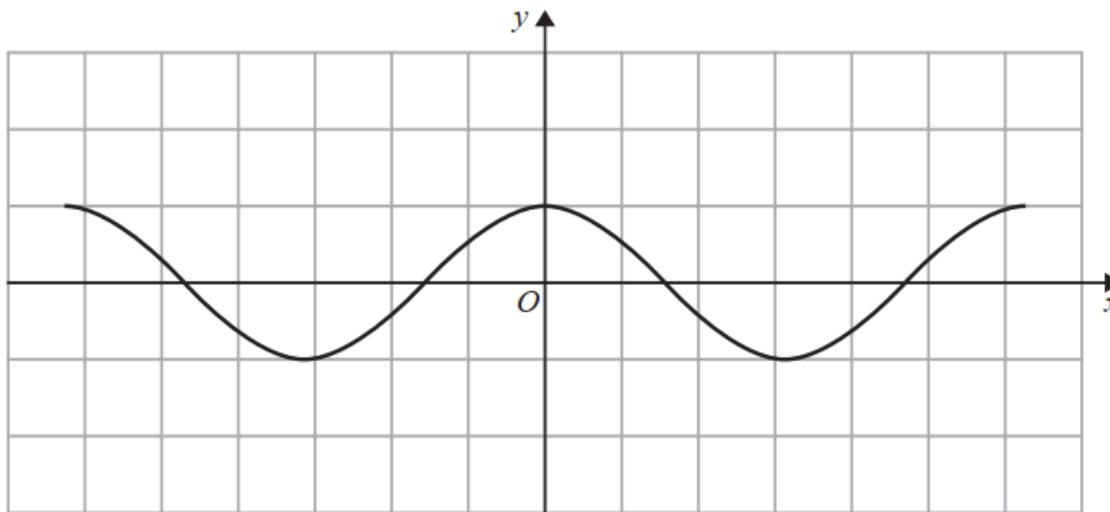


Figure 1

Figure 1 shows a plot of part of the curve with equation $y = \cos x$ where x is measured in radians. Diagram 1, on the opposite page, is a copy of Figure 1.

- (a) Use Diagram 1 to show why the equation

$$\cos x - 2x - \frac{1}{2} = 0$$

has only one real root, giving a reason for your answer.

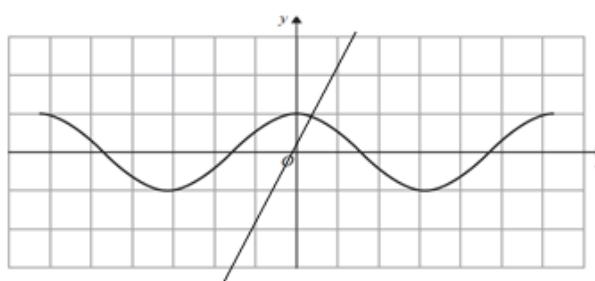
(2)

Given that the root of the equation is α , and that α is small,

- (b) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places. (3)



Question 2 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)		M1	This mark is given for plotting the line $y = 2x + \frac{1}{2}$ on the diagram with a correct gradient and intercept
	Only one intersection means that there is one root	A1	This mark is given for a reason why there is only one real root
(b)	$1 - \frac{x^2}{2} - 2x - \frac{1}{2} = 0$	M1	This mark is given for using the small angle approximation $\cos x = 1 - \frac{x^2}{2}$ in the given equation
	$x^2 + 4x - 1 = 0$	M1	This mark is given for rearranging to find a quadratic equation to solve
	0.236 or $-2 + \sqrt{5}$	A1	This mark is given for finding the correct (positive) solution for x



4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$4 \sin \frac{\theta}{2} + 3 \cos^2 \theta \approx a + b\theta + c\theta^2$$

where a , b and c are integers to be found.

(3)



Question	Scheme	Marks	AOs
4	<p>Examples:</p> $4\sin \frac{\theta}{2} \approx 4\left(\frac{\theta}{2}\right), \quad 3\cos^2 \theta \approx 3\left(1 - \frac{\theta^2}{2}\right)^2$ $3\cos^2 \theta = 3(1 - \sin^2 \theta) \approx 3(1 - \theta^2)$ $3\cos^2 \theta = 3\frac{(\cos 2\theta + 1)}{2} \approx \frac{3}{2}\left(1 - \frac{4\theta^2}{2} + 1\right)$	M1	1.1a
	<p>Examples:</p> $4\sin \frac{\theta}{2} + 3\cos^2 \theta \approx 4\left(\frac{\theta}{2}\right) + 3\left(1 - \frac{\theta^2}{2}\right)^2$ $4\sin \frac{\theta}{2} + 3\cos^2 \theta = 4\left(\frac{\theta}{2}\right) + 3(1 - \sin^2 \theta) \approx 2\theta + 3(1 - \theta^2)$ $4\sin \frac{\theta}{2} + 3\cos^2 \theta = 4\sin \frac{\theta}{2} + 3\frac{(\cos 2\theta + 1)}{2} \approx 4\left(\frac{\theta}{2}\right) + \frac{3}{2}\left(1 - \frac{4\theta^2}{2} + 1\right)$	dM1	1.1b
	$= 2\theta + 3(1 - \theta^2 + \dots) = 3 + 2\theta - 3\theta^2$	A1	2.1
		(3)	
		(3 marks)	



Trigonometry – sec, cosec, cot, identities



7. (i) Solve, for $0 \leq x < \frac{\pi}{2}$, the equation

$$4 \sin x = \sec x$$

(4)



Question	Scheme	Marks	AOs
7	(i) $4\sin x = \sec x, 0 \leq x < \frac{\pi}{2}$; (ii) $5\sin\theta - 5\cos\theta = 2, 0 \leq \theta < 360^\circ$		
(i) Way 1	For $\sec x = \frac{1}{\cos x}$	B1	1.2
	$\{4\sin x = \sec x \Rightarrow\} 4\sin x \cos x = 1 \Rightarrow 2\sin 2x = 1 \Rightarrow \sin 2x = \frac{1}{2}$	M1	3.1a
	$x = \frac{1}{2}\arcsin\left(\frac{1}{2}\right)$ or $\frac{1}{2}\left(\pi - \arcsin\left(\frac{1}{2}\right)\right) \Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$	dM1 A1	1.1b 1.1b
		(4)	



12.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\operatorname{cosec} \theta - \sin \theta \equiv \cos \theta \cot \theta \quad \theta \neq (180n)^\circ \quad n \in \mathbb{Z} \quad (3)$$

(b) Hence, or otherwise, solve for $0 < x < 180^\circ$

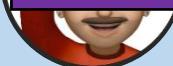
$$\operatorname{cosec} x - \sin x = \cos x \cot(3x - 50^\circ) \quad (5)$$



Question	Scheme	Marks	AOs
12 (a)	States or uses $\text{cosec } \theta = \frac{1}{\sin \theta}$	B1	1.2
	$\text{cosec } \theta - \sin \theta = \frac{1}{\sin \theta} - \sin \theta = \frac{1 - \sin^2 \theta}{\sin \theta}$	M1	2.1
	$= \frac{\cos^2 \theta}{\sin \theta} = \cos \theta \times \frac{\cos \theta}{\sin \theta} = \cos \theta \cot \theta$ *	A1*	2.1
(3)			
(b)	$\text{cosec } x - \sin x = \cos x \cot(3x - 50^\circ)$ $\Rightarrow \cos x \cot x = \cos x \cot(3x - 50^\circ)$		
	$\cot x = \cot(3x - 50^\circ) \Rightarrow x = 3x - 50^\circ$	M1	3.1a
	$x = 25^\circ$	A1	1.1b
	Also $\cot x = \cot(3x - 50^\circ) \Rightarrow x + 180^\circ = 3x - 50^\circ$	M1	2.1
	$x = 115^\circ$	A1	1.1b
	Deduces $x = 90^\circ$	B1	2.2a
	(5)		
(8 marks)			



Trigonometry – addition, double angle, harmonic



9. (a) Prove that

$$\tan \theta + \cot \theta \equiv 2\operatorname{cosec}2\theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

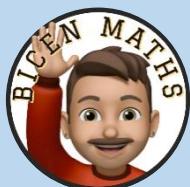
(4)

(b) Hence explain why the equation

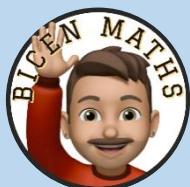
$$\tan \theta + \cot \theta = 1$$

does not have any real solutions.

(1)



Question	Scheme	Marks	AOs
9(a)	$\tan \theta + \cot \theta \equiv \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$	M1	2.1
	$\equiv \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}$	A1	1.1b
	$\equiv \frac{1}{\frac{1}{2} \sin 2\theta}$	M1	2.1
	$\equiv 2 \operatorname{cosec} 2\theta *$	A1*	1.1b
		(4)	
(b)	States $\tan \theta + \cot \theta = 1 \Rightarrow \sin 2\theta = 2$ AND no real solutions as $-1 \leq \sin 2\theta \leq 1$	B1	2.4
		(1)	
		(5 marks)	



7. (i) Solve, for $0 \leq x < \frac{\pi}{2}$, the equation

$$4 \sin x = \sec x$$

(4)

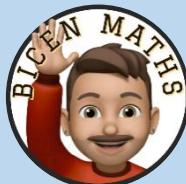
(ii) Solve, for $0 \leq \theta < 360^\circ$, the equation

$$5 \sin \theta - 5 \cos \theta = 2$$

giving your answers to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)



Question	Scheme	Marks	AOs	
7	(i) $4\sin x = \sec x, 0 \leq x < \frac{\pi}{2}$; (ii) $5\sin\theta - 5\cos\theta = 2, 0 \leq \theta < 360^\circ$			
(i) Way 1	For $\sec x = \frac{1}{\cos x}$	B1	1.2	
	$\{4\sin x = \sec x \Rightarrow\} 4\sin x \cos x = 1 \Rightarrow 2\sin 2x = 1 \Rightarrow \sin 2x = \frac{1}{2}$	M1	3.1a	
	$x = \frac{1}{2}\arcsin\left(\frac{1}{2}\right)$ or $\frac{1}{2}\left(\pi - \arcsin\left(\frac{1}{2}\right)\right) \Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$	dM1 A1	1.1b 1.1b	
		(4)		
(ii)	Complete strategy, i.e. <ul style="list-style-type: none"> Expresses $5\sin\theta - 5\cos\theta = 2$ in the form $R\sin(\theta - \alpha) = 2$, finds both R and α, and proceeds to $\sin(\theta - \alpha) = k, k < 1, k \neq 0$ Applies $(5\sin\theta - 5\cos\theta)^2 = 2^2$, followed by applying both $\cos^2\theta + \sin^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$ to proceed to $\sin 2\theta = k, k < 1, k \neq 0$ 	M1	3.1a	
	$R = \sqrt{50}$ $\tan \alpha = 1 \Rightarrow \alpha = 45^\circ$	$(5\sin\theta - 5\cos\theta)^2 = 2^2 \Rightarrow 25\sin^2\theta + 25\cos^2\theta - 50\sin\theta\cos\theta = 4 \Rightarrow 25 - 25\sin 2\theta = 4$	M1	1.1b
	$\sin(\theta - 45^\circ) = \frac{2}{\sqrt{50}}$	$\sin 2\theta = \frac{21}{25}$	A1	1.1b
	dependent on the first M mark			
	e.g. $\theta = \arcsin\left(\frac{2}{\sqrt{50}}\right) + 45^\circ$	e.g. $\theta = \frac{1}{2}\left(\arcsin\left(\frac{21}{25}\right)\right)$	dM1	1.1b
	$\theta = \text{awrt } 61.4^\circ, \text{ awrt } 208.6^\circ$		A1	2.1
	Note: Working in radians does not affect any of the first 4 marks			
		(5)		
		(9 marks)		



12. (a) Prove that

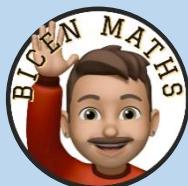
$$1 - \cos 2\theta \equiv \tan \theta \sin 2\theta, \quad \theta \neq \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z} \quad (3)$$

(b) Hence solve, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the equation

$$(\sec^2 x - 5)(1 - \cos 2x) = 3 \tan^2 x \sin 2x$$

Give any non-exact answer to 3 decimal places where appropriate.

(6)



Question	Scheme	Marks	AOs
12	$1 - \cos 2\theta \equiv \tan \theta \sin 2\theta, \theta \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$		
(a) Way 1	$\tan \theta \sin 2\theta = \left(\frac{\sin \theta}{\cos \theta}\right)(2 \sin \theta \cos \theta)$	M1	1.1b
	$= \left(\frac{\sin \theta}{\cos \theta}\right)(2 \cancel{\sin \theta} \cancel{\cos \theta}) = 2 \sin^2 \theta = 1 - \cos 2\theta *$	M1	1.1b
		A1*	2.1
		(3)	
(a) Way 2	$1 - \cos 2\theta = 1 - (1 - 2 \sin^2 \theta) = 2 \sin^2 \theta$	M1	1.1b
	$= \left(\frac{\sin \theta}{\cos \theta}\right)(2 \sin \theta \cos \theta) = \tan \theta \sin 2\theta *$	M1	1.1b
		A1*	2.1
		(3)	
	$(\sec^2 x - 5)(1 - \cos 2x) = 3 \tan^2 x \sin 2x, -\frac{\pi}{2} < x < \frac{\pi}{2}$		
(b) Way 1	$(\sec^2 x - 5) \tan x \sin 2x = 3 \tan^2 x \sin 2x$ or $(\sec^2 x - 5)(1 - \cos 2x) = 3 \tan x(1 - \cos 2x)$		
	Deduces $x = 0$	B1	2.2a
	Uses $\sec^2 x = 1 + \tan^2 x$ and cancels/factorises out $\tan x$ or $(1 - \cos 2x)$ e.g. $(1 + \tan^2 x - 3 \tan x - 5) \tan x = 0$ or $(1 + \tan^2 x - 3 \tan x - 5)(1 - \cos 2x) = 0$ or $1 + \tan^2 x - 5 = 3 \tan x$	M1	2.1
	$\tan^2 x - 3 \tan x - 4 = 0$	A1	1.1b
	$(\tan x - 4)(\tan x + 1) = 0 \Rightarrow \tan x = \dots$	M1	1.1b
	$x = -\frac{\pi}{4}, 1.326$	A1	1.1b
		A1	1.1b
		(6)	
		(9 marks)	



6. (a) Solve, for $-180^\circ \leq \theta \leq 180^\circ$, the equation

$$5 \sin 2\theta = 9 \tan \theta$$

giving your answers, where necessary, to one decimal place.

[Solutions based entirely on graphical or numerical methods are not acceptable.]

(6)

- (b) Deduce the smallest positive solution to the equation

$$5 \sin(2x - 50^\circ) = 9 \tan(x - 25^\circ)$$

(2)



Question 6 (Total 8 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$5 \sin 2\theta = 9 \tan \theta \Rightarrow$ $10 \sin \theta \cos \theta = 9 \times \frac{\sin \theta}{\cos \theta}$	M1	This mark is given for a method to substitute terms to form an equation in terms of $\cos \theta$
	$10 \cos^2 \theta = 9$	M1	This mark is given for a correct equation in terms of $\cos \theta$
	$\theta = \arccos \pm \sqrt{\frac{9}{10}}$	M1	This mark is given for finding a value for θ in terms of <u>arccos</u>
	$\theta = \pm 18.4^\circ, \pm 161.6^\circ$	A1	This mark is given for any one value of 18.4° or 161.6° found.
		A1	This mark is given for four values of θ found correctly
	$\theta = \pm 0^\circ, 180^\circ$	B1	This mark is given for the deduction of the two other solutions for θ
(b)	$10 \cos^2(x - 25) = 9$ x has smallest positive value when $x - 25^\circ = -18.4^\circ$	M1	This mark is given for finding an equation to solve for x
	$x = 6.6^\circ$	A1	This mark is given for correctly finding the smallest positive solution to the equation



12. (a) Prove

$$\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} \equiv 2 \cot 2\theta \quad \theta \neq (90n)^\circ, n \in \mathbb{Z}$$

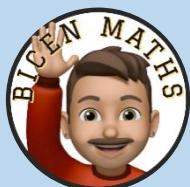
(4)

(b) Hence solve, for $90^\circ < \theta < 180^\circ$, the equation

$$\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = 4$$

(3)

giving any solutions to one decimal place.



Question 12 (Total 7 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = \frac{\cos 3\theta \cos \theta + \sin 3\theta}{\sin \theta \cos \theta}$	M1	This mark is given for a method to form a single fraction
	$= \frac{\cos(3\theta - \theta)}{\sin \theta \cos \theta}$	M1	This mark is given for a method to use a compound angle formula on the numerator
	$= \frac{\cos 2\theta}{\frac{1}{2} \sin 2\theta}$	M1	This mark is given for a method to use a compound angle formula on the denominator
	$= 2 \cot 2\theta$	A1	This mark is given for a fully correct proof to show the answer required
(b)	$\tan 2\theta = \frac{1}{2}$	M1	This mark is given for deducing that the value of $\tan 2\theta$
	$180^\circ + 26.6^\circ$	M1	This mark is given for finding the solution in the third quadrant for $\arctan \frac{1}{2}$
	$\theta = 103.3^\circ$	A1	This mark is given for finding a correct value for θ



10.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\cos 3A \equiv 4 \cos^3 A - 3 \cos A$$

(4)

(b) Hence solve, for $-90^\circ \leq x \leq 180^\circ$, the equation

$$1 - \cos 3x = \sin^2 x$$

(4)



Question	Scheme	Marks	AOs
10 (a)	$\cos 3A = \cos (2A + A) = \cos 2A\cos A - \sin 2A\sin A$	M1	3.1a
	$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$	dM1	1.1b
	$= (2\cos^2 A - 1)\cos A - 2\cos A(1 - \cos^2 A)$	ddM1	2.1
	$= 4\cos^3 A - 3\cos A *$	A1*	1.1b
		(4)	
(b)	$1 - \cos 3x = \sin^2 x \Rightarrow \cos^2 x + 3\cos x - 4\cos^3 x = 0$	M1	1.1b
	$\Rightarrow \cos x(4\cos^2 x - \cos x - 3) = 0$		
	$\Rightarrow \cos x(4\cos x + 3)(\cos x - 1) = 0$	dM1	3.1a
	$\Rightarrow \cos x = \dots$		
	Two of $-90^\circ, 0, 90^\circ$, awrt 139°	A1	1.1b
	All four of $-90^\circ, 0, 90^\circ$, awrt 139°	A1	2.1
		(4)	
			(8 marks)



10.

In this question you should show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

- (a) Given that $1 + \cos 2\theta + \sin 2\theta \neq 0$ prove that

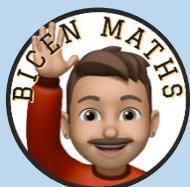
$$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} \equiv \tan \theta \quad (4)$$

- (b) Hence solve, for $0 < x < 180^\circ$

$$\frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} = 3 \sin 2x$$

giving your answers to one decimal place where appropriate.

(4)



Question	Scheme	Marks	AOs
10(a)	$\frac{1-\cos 2\theta + \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = \frac{1-(1-2\sin^2 \theta) + 2\sin \theta \cos \theta}{1+\cos 2\theta + \sin 2\theta}$ or $\frac{1-\cos 2\theta + \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = \frac{1-\cos 2\theta + \sin 2\theta}{1+(2\cos^2 \theta - 1) + 2\sin \theta \cos \theta}$	M1	2.1
	$\frac{1-\cos 2\theta + \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = \frac{1-(1-2\sin^2 \theta) + 2\sin \theta \cos \theta}{1+(2\cos^2 \theta - 1) + 2\sin \theta \cos \theta}$	A1	1.1b
	$= \frac{2\sin^2 \theta + 2\sin \theta \cos \theta}{2\cos^2 \theta + 2\sin \theta \cos \theta} = \frac{2\sin \theta(\sin \theta + \cos \theta)}{2\cos \theta(\cos \theta + \sin \theta)}$	dM1	2.1
	$= \frac{\sin \theta}{\cos \theta} = \tan \theta *$	A1*	1.1b
		(4)	
(b)	$\frac{1-\cos 4x + \sin 4x}{1+\cos 4x + \sin 4x} = 3\sin 2x \Rightarrow \tan 2x = 3\sin 2x \text{ o.e}$	M1	3.1a
	$\Rightarrow \sin 2x - 3\sin 2x \cos 2x = 0$		
	$\Rightarrow \sin 2x(1-3\cos 2x) = 0$	A1	1.1b
	$\Rightarrow (\sin 2x = 0,) \cos 2x = \frac{1}{3}$		
	$x = 90^\circ, \text{ awrt } 35.3^\circ, \text{ awrt } 144.7^\circ$	A1	1.1b
		A1	2.1
		(4)	
		(8 marks)	



14.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$2 \sin(x - 60^\circ) = \cos(x - 30^\circ)$$

show that

$$\tan x = 3\sqrt{3} \quad (4)$$

(b) Hence or otherwise solve, for $0 \leq \theta < 180^\circ$

$$2 \sin 2\theta = \cos(2\theta + 30^\circ)$$

giving your answers to one decimal place.

(4)



14(a)	Attempts to use both $\sin(x - 60^\circ) = \pm \sin x \cos 60^\circ \pm \cos x \sin 60^\circ$ $\cos(x - 30^\circ) = \pm \cos x \cos 30^\circ \pm \sin x \sin 30^\circ$	M1
	Correct equation $2\sin x \cos 60^\circ - 2\cos x \sin 60^\circ = \cos x \cos 30^\circ + \sin x \sin 30^\circ$	A1
	Either uses $\frac{\sin x}{\cos x} = \tan x$ and attempts to make $\tan x$ the subject E.g. $(2\cos 60^\circ - \sin 30^\circ) \tan x = \cos 30^\circ + 2\sin 60^\circ$	
	Or attempts $\sin 30^\circ$ etc with at least two correct and collects terms in $\sin x$ and $\cos x$ E.g. $\left(2 \times \frac{1}{2} - \frac{1}{2}\right) \sin x = \left(2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) \cos x$	M1
	Proceeds to given answer showing all key steps E.g. $\frac{1}{2} \tan x = \frac{3\sqrt{3}}{2} \Rightarrow \tan x = 3\sqrt{3}$ *	A1*
(b)		(4)
	Deduces that $x = 2\theta + 60^\circ$	B1
	$\tan(2\theta + 60^\circ) = 3\sqrt{3} \Rightarrow 2\theta + 60^\circ = 79.1^\circ, 259.1^\circ, \dots$	M1
	Correct method to find one value of θ E.g. $\theta = \frac{79.1^\circ - 60^\circ}{2}$	dM1
	$\theta = \text{awrt } 9.6^\circ, 99.6^\circ$ (See note)	A1
		(4)



Parametric Equations



5. A curve C has parametric equations

$$x = 2t - 1, \quad y = 4t - 7 + \frac{3}{t}, \quad t \neq 0$$

Show that the Cartesian equation of the curve C can be written in the form

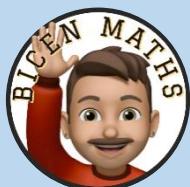
$$y = \frac{2x^2 + ax + b}{x + 1}, \quad x \neq -1$$

where a and b are integers to be found.

(3)



Question	Scheme	Marks	AOs
5	Attempts to substitute $= \frac{x+1}{2}$ into $y \Rightarrow y = 4\left(\frac{x+1}{2}\right) - 7 + \frac{6}{(x+1)}$	M1	2.1
	Attempts to write as a single fraction $y = \frac{(2x-5)(x+1)+6}{(x+1)}$	M1	2.1
	$y = \frac{2x^2 - 3x + 1}{x+1} \quad a = -3, b = 1$	A1	1.1b
(3 marks)			



14. A curve C has parametric equations

$$x = 3 + 2 \sin t, \quad y = 4 + 2 \cos 2t, \quad 0 \leq t < 2\pi$$

- (a) Show that all points on C satisfy $y = 6 - (x - 3)^2$

(2)

- (b) (i) Sketch the curve C .

- (ii) Explain briefly why C does not include all points of $y = 6 - (x - 3)^2$, $x \in \mathbb{R}$

(3)

The line with equation $x + y = k$, where k is a constant, intersects C at two distinct points.

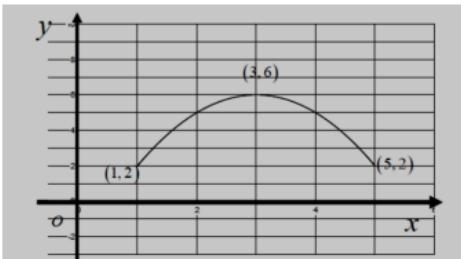
- (c) State the range of values of k , writing your answer in set notation.

(5)



A2 2018 Paper 1

Parametric Equations

Question	Scheme	Marks	AOs
14(a)	Attempts to use $\cos 2t = 1 - 2 \sin^2 t \Rightarrow \frac{y-4}{2} = 1 - 2\left(\frac{x-3}{2}\right)^2$	M1	2.1
	$\Rightarrow y-4 = 2 - 4 \times \frac{(x-3)^2}{4} \Rightarrow y = 6 - (x-3)^2 *$	A1*	1.1b
		(2)	
(b)	 Suitable reason : Eg states as $x = 3 + 2 \sin t, 1 \leqslant x \leqslant 5$	M1	1.1b
		A1	1.1b
		B1	2.4
		(3)	
(c)	Either finds the lower value for $k = 7$ or deduces that $k < \frac{37}{4}$	B1	2.2a
	Finds where $x + y = k$ meets $y = 6 - (x-3)^2$ $\Rightarrow k - x = 6 - (x-3)^2$ and proceeds to 3TQ in x or y	M1	3.1a
	Correct 3TQ in x $x^2 - 7x + (k+3) = 0$ Or y $y^2 + (7-2k)y + (k^2 - 6k + 3) = 0$	A1	1.1b
	Uses $b^2 - 4ac = 0 \Rightarrow 49 - 4 \times 1 \times (k+3) = 0 \Rightarrow k = \left(\frac{37}{4}\right)$ or $(7-2k)^2 - 4 \times 1 \times (k^2 - 6k + 3) = 0 \Rightarrow k = \left(\frac{37}{4}\right)$	M1	2.1
	Range of values for $k = \left\{k : 7 \leqslant k < \frac{37}{4}\right\}$	A1	2.5
		(5)	



4.

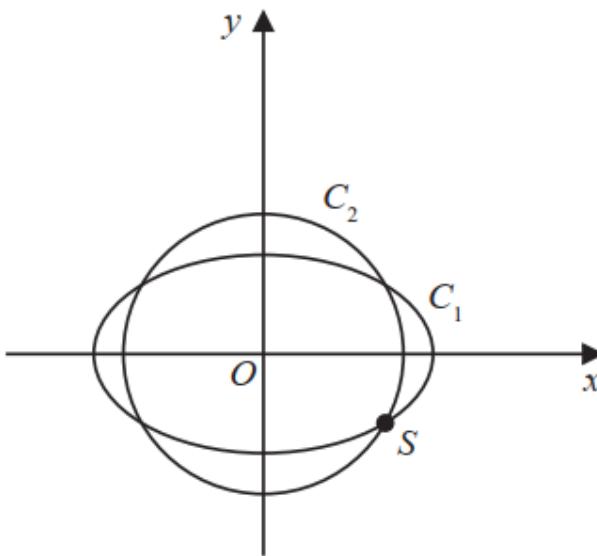


Figure 2

The curve C_1 with parametric equations

$$x = 10 \cos t, \quad y = 4\sqrt{2} \sin t, \quad 0 \leq t < 2\pi$$

meets the circle C_2 with equation

$$x^2 + y^2 = 66$$

at four distinct points as shown in Figure 2.

Given that one of these points, S , lies in the 4th quadrant, find the Cartesian coordinates of S .
(6)



Part	Working or answer an examiner might expect to see	Mark	Notes
	$(10 \cos t)^2 + (4\sqrt{2} \sin t)^2 = 66$	M1	This mark is given for combining the two equations to show where the curve and circle meet
	$100 (\cos t)^2 + 32(1 - \cos t)^2 = 66$	M1	This mark is given for forming an equation in $\cos t$ only
	$68 \cos^2 t = 34$	A1	This mark is given for simplifying to find an equation in terms of $\cos t$
	$\cos t = \pm \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}$	M1	This mark is given for finding a value for t
	$x = 10 \times \frac{1}{\sqrt{2}}$ $y = 4\sqrt{2} \times -\sin \frac{\pi}{4} = 4\sqrt{2} \times -\frac{1}{\sqrt{2}}$	M1	This mark is given for a method to substitute back into the original equations to find value for x and y
	$S = (5\sqrt{2}, -4)$	A1	This mark is given for the correct coordinates of S



13. A curve C has parametric equations

$$x = \frac{t^2 + 5}{t^2 + 1} \quad y = \frac{4t}{t^2 + 1} \quad t \in \mathbb{R}$$

Show that all points on C satisfy

$$(x - 3)^2 + y^2 = 4 \quad (3)$$

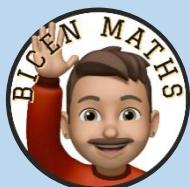


Question	Scheme	Marks	AOs
13	$(x-3)^2 + y^2 = \left(\frac{t^2+5}{t^2+1} - 3\right)^2 + \left(\frac{4t}{t^2+1}\right)^2$	M1	3.1a
	$= \frac{(2-2t^2)^2 + 16t^2}{(t^2+1)^2} = \frac{4+8t^2+4t^4}{(t^2+1)^2}$	dM1	1.1b
	$\frac{4(t^4+2t^2+1)}{(t^2+1)^2} = \frac{4(t^2+1)^2}{(t^2+1)^2} = 4*$	A1*	2.1
		(3)	

M1: Attempts to substitute the given parametric forms into the Cartesian equation or the lhs of the Cartesian equation. There may have been an (incorrect) attempt to multiply out the $(x-3)^2$ term.

dM1: Attempts to combine (at least the lhs) using correct processing into a single fraction, multiplies out and collects terms on the numerator.

A1*: Fully correct proof showing all key steps



Differentiation



13. The curve C has parametric equations

$$x = 2 \cos t, \quad y = \sqrt{3} \cos 2t, \quad 0 \leq t \leq \pi$$

- (a) Find an expression for $\frac{dy}{dx}$ in terms of t .

(2)

The point P lies on C where $t = \frac{2\pi}{3}$

The line l is the normal to C at P .

- (b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q .

- (c) Find the exact coordinates of Q .

You must show clearly how you obtained your answers.

(6)



A2 SAMs Paper 1

Differentiation

Question	Scheme	Marks	AOs
13(a)	Attempts $\frac{dy}{dx} = \frac{\cancel{dy}/dt}{\cancel{dx}/dt}$	M1	1.1b
	$\frac{dy}{dx} = \frac{\sqrt{3} \sin 2t}{\sin t} (= 2\sqrt{3} \cos t)$	A1	1.1b
		(2)	
(b)	Substitutes $t = \frac{2\pi}{3}$ in $\frac{dy}{dx} = \frac{\sqrt{3} \sin 2t}{\sin t} = (-\sqrt{3})$	M1	2.1
	Uses gradient of normal = $-\frac{1}{\cancel{dy}/dx} = \left(\frac{1}{\sqrt{3}}\right)$	M1	2.1
	Coordinates of $P = \left(-1, -\frac{\sqrt{3}}{2}\right)$	B1	1.1b
	Correct form of normal $y + \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x+1)$	M1	2.1
	Completes proof $\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ *	A1*	1.1b
		(5)	
(c)	Substitutes $x = 2\cos t$ and $y = \sqrt{3} \cos 2t$ into $2x - 2\sqrt{3}y - 1 = 0$	M1	3.1a
	Uses the identity $\cos 2t = 2\cos^2 t - 1$ to produce a quadratic in $\cos t$	M1	3.1a
	$\Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$	A1	1.1b
	Finds $\cos t = \frac{5}{6}, \cancel{\frac{-1}{2}}$	M1	2.4
	Substitutes their $\cos t = \frac{5}{6}$ into $x = 2\cos t, y = \sqrt{3} \cos 2t,$	M1	1.1b
	$Q = \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$	A1	1.1b
		(6)	
(13 marks)			



15.

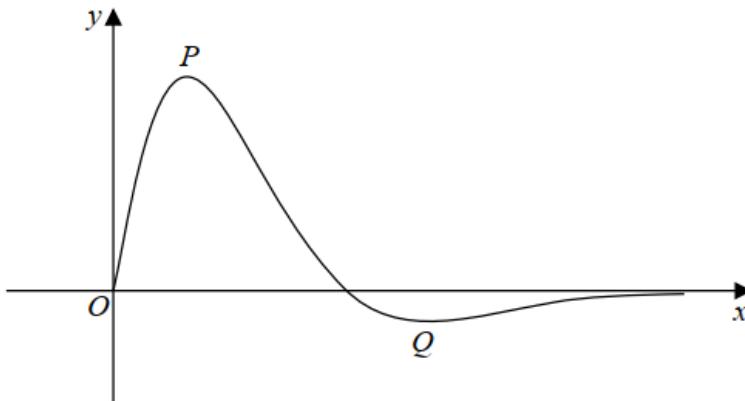


Figure 5

Figure 5 shows a sketch of the curve with equation $y = f(x)$, where

$$f(x) = \frac{4 \sin 2x}{e^{\sqrt{2}x-1}}, \quad 0 \leq x \leq \pi$$

The curve has a maximum turning point at P and a minimum turning point at Q as shown in Figure 5.

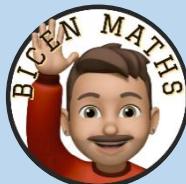
- (a) Show that the x coordinates of point P and point Q are solutions of the equation

$$\tan 2x = \sqrt{2} \quad (4)$$

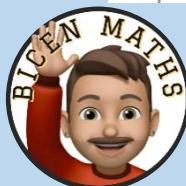
- (b) Using your answer to part (a), find the x -coordinate of the minimum turning point on the curve with equation

(i) $y = f(2x)$.

(ii) $y = 3 - 2f(x)$. (4)



Question	Scheme	Marks	AOs
15(a)	Attempts to differentiate using the quotient rule or otherwise	M1	2.1
	$f'(x) = \frac{e^{\sqrt{2}x-1} \times 8 \cos 2x - 4 \sin 2x \times \sqrt{2}e^{\sqrt{2}x-1}}{(e^{\sqrt{2}x-1})^2}$	A1	1.1b
	Sets $f'(x) = 0$ and divides/ factorises out the $e^{\sqrt{2}x-1}$ terms	M1	2.1
	Proceeds via $\frac{\sin 2x}{\cos 2x} = \frac{8}{4\sqrt{2}}$ to $\Rightarrow \tan 2x = \sqrt{2} *$	A1*	1.1b
		(4)	
(b)	(i) Solves $\tan 4x = \sqrt{2}$ and attempts to find the 2 nd solution	M1	3.1a
	$x = 1.02$	A1	1.1b
	(ii) Solves $\tan 2x = \sqrt{2}$ and attempts to find the 1 st solution	M1	3.1a
	$x = 0.478$	A1	1.1b
		(4)	
		(8 marks)	

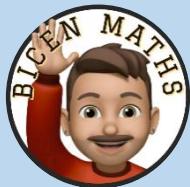


3. Given $y = x(2x + 1)^4$, show that

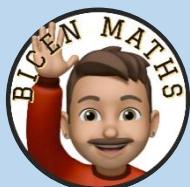
$$\frac{dy}{dx} = (2x + 1)^n(Ax + B)$$

where n , A and B are constants to be found.

(4)



Question	Scheme	Marks	AOs
3	Attempts the product and chain rule on $y = x(2x+1)^4$	M1	2.1
	$\frac{dy}{dx} = (2x+1)^4 + 8x(2x+1)^3$	A1	1.1b
	Takes out a common factor $\frac{dy}{dx} = (2x+1)^3 \{(2x+1)+8x\}$	M1	1.1b
	$\frac{dy}{dx} = (2x+1)^3(10x+1) \Rightarrow n = 3, A = 10, B = 1$	A1	1.1b
(4 marks)			



5. Given that

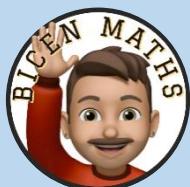
$$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta} \quad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

show that

$$\frac{dy}{d\theta} = \frac{A}{1 + \sin 2\theta} \quad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

where A is a rational constant to be found.

(5)



Question	Scheme	Marks	AOs
5	$\frac{dy}{d\theta} = \frac{(2\sin\theta + 2\cos\theta)3\cos\theta - 3\sin\theta(2\cos\theta - 2\sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$	M1 A1	1.1b 1.1b
	Expands and uses $\sin^2\theta + \cos^2\theta = 1$ at least once in the numerator or the denominator or uses $2\sin\theta\cos\theta = \sin 2\theta$ in $\Rightarrow \frac{dy}{d\theta} = \frac{\dots}{\dots \cdot C\sin\theta\cos\theta}$	M1	3.1a
	Expands and uses $\sin^2\theta + \cos^2\theta = 1$ the numerator and the denominator AND uses $2\sin\theta\cos\theta = \sin 2\theta$ in $\Rightarrow \frac{dy}{d\theta} = \frac{P}{Q + R\sin 2\theta}$	M1	2.1
	$\Rightarrow \frac{dy}{d\theta} = \frac{3}{2 + 2\sin 2\theta} = \frac{\frac{3}{2}}{1 + \sin 2\theta}$	A1	1.1b
(5 marks)			



9.

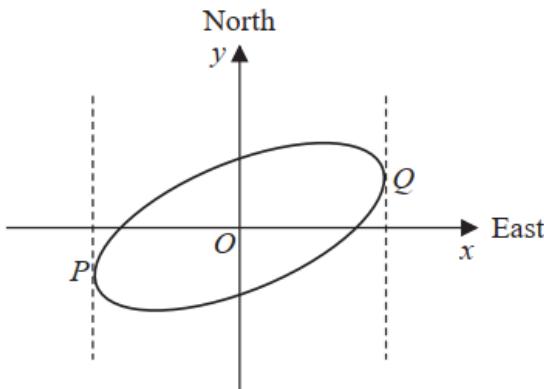


Figure 4

Figure 4 shows a sketch of the curve with equation $x^2 - 2xy + 3y^2 = 50$

(a) Show that $\frac{dy}{dx} = \frac{y-x}{3y-x}$ (4)

The curve is used to model the shape of a cycle track with both x and y measured in km.

The points P and Q represent points that are furthest west and furthest east of the origin O , as shown in Figure 4.

Using part (a),

(b) find the exact coordinates of the point P . (5)

(c) Explain briefly how to find the coordinates of the point that is furthest north of the origin O . (You **do not** need to carry out this calculation). (1)



Question	Scheme	Marks	AOs
9(a)	Either $3y^2 \rightarrow Ay \frac{dy}{dx}$ or $2xy \rightarrow 2x \frac{dy}{dx} + 2y$	M1	2.1
	$2x - 2x \frac{dy}{dx} - 2y + 6y \frac{dy}{dx} = 0$	A1	1.1b
	$(6y - 2x) \frac{dy}{dx} = 2y - 2x$	M1	2.1
	$\frac{dy}{dx} = \frac{2y - 2x}{6y - 2x} = \frac{y - x}{3y - x}$ *	A1*	1.1b
		(4)	
(b)	$\left(\text{At } P \text{ and } Q \frac{dy}{dx} \rightarrow \infty \Rightarrow \right)$ Deduces that $3y - x = 0$	M1	2.2a
	Solves $y = \frac{1}{3}x$ and $x^2 - 2xy + 3y^2 = 50$ simultaneously	M1	3.1a
	$\Rightarrow x = (\pm)5\sqrt{3}$ OR $\Rightarrow y = (\pm)\frac{5}{3}\sqrt{3}$	A1	1.1b
	Using $y = \frac{1}{3}x \Rightarrow x = ..$ AND $y = ..$	dM1	1.1b
	$P = \left(-5\sqrt{3}, -\frac{5}{3}\sqrt{3} \right)$	A1	2.2a
		(5)	
(c)	Explains that you need to solve $y = x$ and $x^2 - 2xy + 3y^2 = 50$ simultaneously and choose the positive solution	B1ft	2.4
		(1)	
(10 marks)			



11.

$$\frac{1 + 11x - 6x^2}{(x - 3)(1 - 2x)} \equiv A + \frac{B}{(x - 3)} + \frac{C}{(1 - 2x)}$$

(a) Find the values of the constants A , B and C .

(4)

$$f(x) = \frac{1 + 11x - 6x^2}{(x - 3)(1 - 2x)} \quad x > 3$$

(b) Prove that $f(x)$ is a decreasing function.

(3)



Question	Scheme	Marks	AOs
11	$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}$		
(a) Way 1	$1+11x-6x^2 \equiv A(1-2x)(x-3) + B(1-2x) + C(x-3) \Rightarrow B = \dots, C = \dots$	M1	2.1
	$A = 3$	B1	1.1b
	Uses substitution or compares terms to find either $B = \dots$ or $C = \dots$	M1	1.1b
	$B = 4$ and $C = -2$ which have been found using a correct identity	A1	1.1b
		(4)	
(a) Way 2	{long division gives} $\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv 3 + \frac{-10x+10}{(x-3)(1-2x)}$		
	$-10x+10 \equiv B(1-2x) + C(x-3) \Rightarrow B = \dots, C = \dots$	M1	2.1
	$A = 3$	B1	1.1b
	Uses substitution or compares terms to find either $B = \dots$ or $C = \dots$	M1	1.1b
	$B = 4$ and $C = -2$ which have been found using $-10x+10 \equiv B(1-2x) + C(x-3)$	A1	1.1b
		(4)	
(b)	$f(x) = 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)} \quad \{ = 3 + 4(x-3)^{-1} - 2(1-2x)^{-1} \}; \quad x > 3$		
	$f'(x) = -4(x-3)^{-2} - 4(1-2x)^{-2} \quad \left\{ = -\frac{4}{(x-3)^2} - \frac{4}{(1-2x)^2} \right\}$	M1	2.1
		A1ft	1.1b
	Correct $f'(x)$ and as $(x-3)^2 > 0$ and $(1-2x)^2 > 0$, then $f'(x) = -(+ve) - (+ve) < 0$, so $f(x)$ is a decreasing function	A1	2.4
		(3)	
		(7 marks)	



12.

$$f(x) = 10e^{-0.25x} \sin x, \quad x \geq 0$$

- (a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

- (b) Sketch the graph of H against t where

$$H(t) = |10e^{-0.25t} \sin t| \quad t \geq 0$$

showing the long-term behaviour of this curve.

(2)

The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

- (c) the maximum height of the ball above the ground between the first and second bounce.

(3)

- (d) Explain why this model should not be used to predict the time of each bounce.

(1)

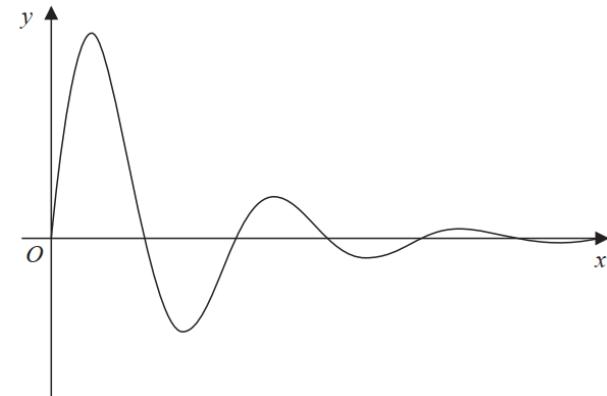
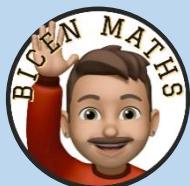
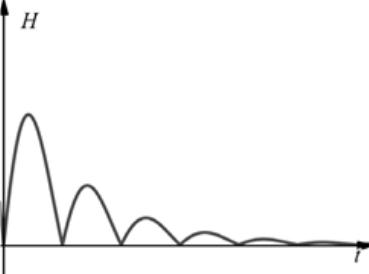


Figure 3



A2 2019 Paper 1

Differentiation

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$f'(x) = -2.5e^{-0.25x} \sin x + 10e^{-0.25x} \cos x$	M1	This mark is given for a method to differentiate to find an expression for $f'(x)$
		A1	This mark is given for correctly differentiating to find an expression for $f'(x)$
	$f'(x) = 0$ $\Rightarrow e^{-0.25x} (-2.5 \sin x + 10 \cos x) = 0$ $\Rightarrow (-2.5 \sin x + 10 \cos x) = 0$	M1	This mark is given for setting $f'(x) = 0$ and finding as method to solve for $\tan x$
	$\frac{\sin x}{\cos x} = \frac{10}{2.5}$ $\tan x = 4$	A1	This mark is given for showing that $\tan x = 4$ as required.
(b)		M1	This mark is given for a graph with a correct shape
		A1	This mark is given for a graph with heights > 0
(c)	$\tan x = 4, x = 1.326$ $t = \pi + 1.326 = 4.47$	M1	This method is given for finding a value for t between the first and second bounce
	$H(4.47) = 10e^{-0.25 \times 4.47} \sin 4.47 $	M1	This mark is given for substituting the value of $t = \pi + \arctan 4$ into $H(t)$
	$= 3.27 \times -0.97 $ $= 3.17$ metres	A1	This mark is given for finding the maximum height of the ball
(d)	The time between each bounce should not stay the same when the heights of each bounce are getting smaller	B1	This mark is given for a valid explanation of why the model should not be used to predict the time of each bounce



14. The curve C , in the standard Cartesian plane, is defined by the equation

$$x = 4 \sin 2y \quad -\frac{\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

- (a) Find the value of $\frac{dy}{dx}$ at the origin. (2)
- (b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.
(ii) Explain the relationship between the answers to (a) and (b)(i). (2)
- (c) Show that, for all points (x, y) lying on C ,

$$\frac{dy}{dx} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found. (3)



Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\frac{dx}{dy} = 8 \cos 2y \Rightarrow \frac{dy}{dx} = \frac{1}{8 \cos 2y}$	M1	This mark is given for differentiating and inverting
	At $(0, 0)$, $\frac{dy}{dx} = \frac{1}{8}$	A1	This mark is given for finding $\frac{dy}{dx}$ when $y = 0$
(b)(i)	$\sin 2y \approx 2y \Rightarrow x \approx 8y$	B1	This mark is given for finding an approximation for x
(b)(ii)	When x and y are small, $x = 4 \sin 2y$ approximates to the line $x = 8y$	B1	This mark is given for a valid explanation of the relationship between x and y when both are small
(c)	$\sin^2 2y + \cos^2 2y = 1$ $\Rightarrow \cos^2 2y = 1 - \sin^2 2y$ $x = 4 \sin 2y \Rightarrow \sin^2 2y = \left(\frac{x}{4}\right)^2$	M1	This mark is given for a method to use find an expression for $\sin^2 2y$ in terms of x
	$\frac{dy}{dx} = \frac{1}{8 \cos 2y} = \frac{1}{8 \sqrt{1 - \left(\frac{x}{4}\right)^2}}$	A1	This mark is given for an unsimplified expression for $\frac{dy}{dx}$
	$\frac{dy}{dx} = \frac{1}{2\sqrt{16 - x^2}}$	A1	This mark is given for a fully correct answer with $a = 2$ and $b = 16$



11.

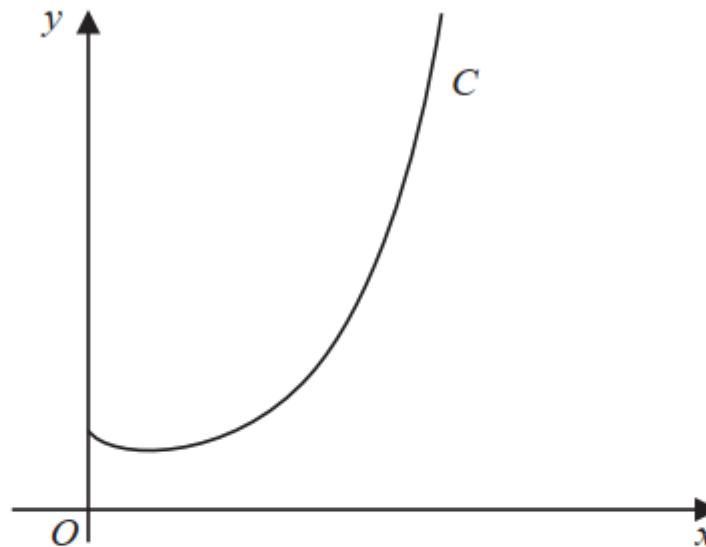
**Figure 8**

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$

- (a) Find, by firstly taking logarithms, the x coordinate of the turning point of C .

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)



Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$y = x^x \Rightarrow \ln y = x \ln x$	M1	This mark is for a method to find the x -coordinate of the turning point of C by taking logarithms
	$\ln y = x \ln x \Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln x + 1$	M1	This mark is given for a method using implicit differentiation
		A1	This mark is given for a correct expression for $\frac{1}{y} \frac{dy}{dx}$
	Setting $\frac{dy}{dx} = 0$, $\ln x + 1 = 0$	M1	This mark is given for a method for finding the turning point of C by setting $\frac{dy}{dx} = 0$
	$x = e^{-1}$	A1	This mark is given for correctly finding a value for the x -coordinate of the turning point of C



Figure 2 shows a sketch of the curve C with equation $y = f(x)$ where

$$f(x) = 4(x^2 - 2)e^{-2x} \quad x \in \mathbb{R}$$

- (a) Show that $f'(x) = 8(2 + x - x^2)e^{-2x}$
- (b) Hence find, in simplest form, the exact coordinates of the stationary points of C .

The function g and the function h are defined by

$$g(x) = 2f(x) \quad x \in \mathbb{R}$$

$$h(x) = 2f(x) - 3 \quad x \geq 0$$

- (c) Find (i) the range of g
(ii) the range of h

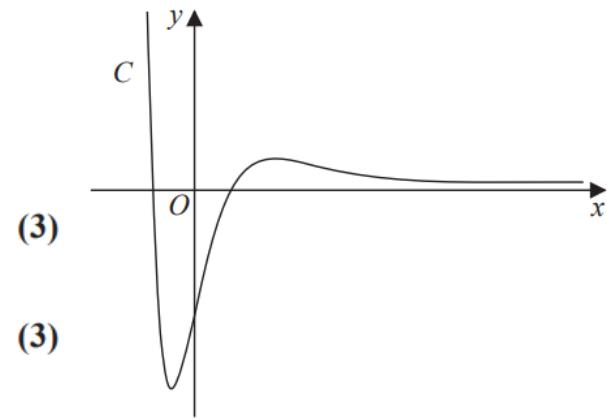
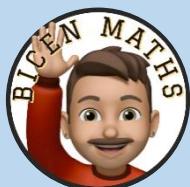


Figure 2

(3)



Question	Scheme	Marks	AOs
9(a)	$f(x) = 4(x^2 - 2)e^{-2x}$		
	Differentiates to $e^{-2x} \times 8x + 4(x^2 - 2) \times -2e^{-2x}$	M1 A1	1.1b 1.1b
	$f'(x) = 8e^{-2x} \{x - (x^2 - 2)\} = 8(2 + x - x^2)e^{-2x}$ *	A1*	2.1
			(3)
(b)	States roots of $f'(x) = 0$ $x = -1, 2$	B1	1.1b
	Substitutes one x value to find a y value	M1	1.1b
	Stationary points are $(-1, -4e^2)$ and $(2, 8e^{-4})$	A1	1.1b
			(3)
(c)	(i) Range $[-8e^2, \infty)$ o.e. such as $g(x) \geq -8e^2$	B1ft	2.5
	(ii) For <ul style="list-style-type: none"> Either attempting to find $2f(0) - 3 = 2 \times -8 - 3 = (-19)$ and identifying this as the lower bound Or attempting to find $2 \times "8e^{-4}" - 3$ and identifying this as the upper bound 	M1	3.1a
	Range $[-19, 16e^{-4} - 3]$	A1	1.1b
			(3)
			(9 marks)



15. The curve C has equation

$$x^2 \tan y = 9 \quad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81} \quad (4)$$

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27}$

(3)



A2 2020 Paper 1

Differentiation

Question	Scheme	Marks	AOs
15 (a)	$x^2 \tan y = 9 \Rightarrow 2x \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$	M1 A1	3.1a 1.1b
	Full method to get $\frac{dy}{dx}$ in terms of x using $\sec^2 y = 1 + \tan^2 y = 1 + f(x)$	M1	1.1b
	$\frac{dy}{dx} = \frac{-2x \times \frac{9}{x^2}}{x^2 \left(1 + \frac{81}{x^4}\right)} = \frac{-18x}{x^4 + 81} *$	A1*	2.1
		(4)	
(b)	$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$ $\Rightarrow \frac{d^2y}{dx^2} = \frac{-18 \times (x^4 + 81) - (-18x)(4x^3)}{(x^4 + 81)^2} = \frac{54(x^4 - 27)}{(x^4 + 81)^2} \text{ o.e.}$	M1 A1	1.1b 1.1b
	States that when $x < \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} < 0$ when $x = \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} = 0$	A1	2.4
	AND when $x > \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} > 0$ giving a point of inflection when $x = \sqrt[4]{27}$		
		(3)	

(7 marks)

HOME



14. Given that

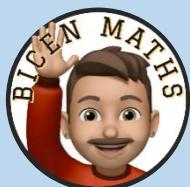
$$y = \frac{x - 4}{2 + \sqrt{x}} \quad x > 0$$

show that

$$\frac{dy}{dx} = \frac{1}{A\sqrt{x}} \quad x > 0$$

where A is a constant to be found.

(4)



Question	Scheme	Marks	AOs
14	$y = \frac{x-4}{2+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{2+\sqrt{x} - (x-4)\frac{1}{2}x^{-\frac{1}{2}}}{(2+\sqrt{x})^2}$	M1 A1	2.1 1.1b
	$= \frac{2+\sqrt{x} - (x-4)\frac{1}{2}x^{-\frac{1}{2}}}{(2+\sqrt{x})^2} = \frac{2+\sqrt{x} - \frac{1}{2}\sqrt{x} + 2x^{-\frac{1}{2}}}{(2+\sqrt{x})^2} = \frac{2\sqrt{x} + \frac{1}{2}x + 2}{\sqrt{x}(2+\sqrt{x})^2}$	M1	1.1b
	$= \frac{x+4\sqrt{x}+4}{2\sqrt{x}(2+\sqrt{x})^2} = \frac{(2+\sqrt{x})^2}{2\sqrt{x}(2+\sqrt{x})^2} = \frac{1}{2\sqrt{x}}$	A1	2.1
	(4)	(4 marks)	

Notes

M1: Attempts to use a correct rule e.g. quotient or product (& chain) rule to achieve the following forms

Quotient : $\frac{\alpha(2+\sqrt{x}) - \beta(x-4)x^{-\frac{1}{2}}}{(2+\sqrt{x})^2}$ but be tolerant of attempts where the $(2+\sqrt{x})^2$ has been

incorrectly expanded

Product: $\alpha(2+\sqrt{x})^{-1} + \beta x^{-\frac{1}{2}}(x-4)(2+\sqrt{x})^{-2}$

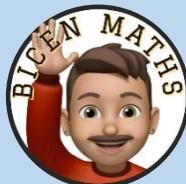
Alternatively with $t = \sqrt{x}$, $y = \frac{t^2-4}{2+t} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t(2+t) - (t^2-4)}{(2+t)^2} \times \frac{1}{2}x^{-\frac{1}{2}}$ with same rules

A1: Correct derivative in any form. Must be in terms of a single variable (which could be t)

M1: Following a correct attempt at differentiation, it is scored for multiplying both numerator and denominator by \sqrt{x} and collecting terms to form a single fraction. It can also be scored from $\frac{uv' - vu'}{v^2}$

For the $t = \sqrt{x}$, look for an attempt to simplify $\frac{t^2 + 4t + 4}{(2+t)^2} \times \frac{1}{2t}$

A1: Correct expression showing all key steps with no errors or omissions. $\frac{dy}{dx}$ must be seen at least once



5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \quad x \in \mathbb{R}$$

(a) Find

(i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)



Question	Scheme	Marks	AOs
5(a)(i)	$\frac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$	M1 A1	1.1b 1.1b
(ii)	$\frac{d^2y}{dx^2} = 60x^2 - 144x + 84$	A1ft	1.1b
			(3)
(b)(i)	$x = 1 \Rightarrow \frac{dy}{dx} = 20 - 72 + 84 - 32$	M1	1.1b
	$\frac{dy}{dx} = 0$ so there is a stationary point at $x = 1$	A1	2.1
	Alternative for (b)(i)		
	$20x^3 - 72x^2 + 84x - 32 = 4(x-1)^2(5x-8) = 0 \Rightarrow x = \dots$	M1	1.1b
	When $x = 1$, $\frac{dy}{dx} = 0$ so there is a stationary point	A1	2.1
(b)(ii)	Note that in (b)(ii) there are no marks for just evaluating $\left(\frac{d^2y}{dx^2}\right)_{x=1}$		
	E.g. $\left(\frac{d^2y}{dx^2}\right)_{x=0.8} = \dots$ $\left(\frac{d^2y}{dx^2}\right)_{x=1.2} = \dots$	M1	2.1
	$\left(\frac{d^2y}{dx^2}\right)_{x=0.8} > 0$, $\left(\frac{d^2y}{dx^2}\right)_{x=1.2} < 0$	A1	2.2a
	Hence point of inflection		(4)
	Alternative 1 for (b)(ii)		
	$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 60x^2 - 144x + 84 = 0$ (is inconclusive)		
	$\left(\frac{d^3y}{dx^3}\right) = 120x - 144 \Rightarrow \left(\frac{d^3y}{dx^3}\right)_{x=1} = \dots$	M1	2.1
	$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 0$ and $\left(\frac{d^3y}{dx^3}\right)_{x=1} \neq 0$	A1	2.2a
	Hence point of inflection		
	Alternative 2 for (b)(ii)		
	E.g. $\left(\frac{dy}{dx}\right)_{x=0.8} = \dots$ $\left(\frac{dy}{dx}\right)_{x=1.2} = \dots$	M1	2.1
	$\left(\frac{dy}{dx}\right)_{x=0.8} < 0$, $\left(\frac{dy}{dx}\right)_{x=1.2} < 0$	A1	2.2a
	Hence point of inflection		



8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{dy}{dx} = \frac{apx^2 + bqy}{qx + cy}$$

where a , b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q .

(5)



Question	Scheme	Marks	AOs
8(a)	$\frac{d}{dx}(3y^2) = 6y \frac{dy}{dx}$ or $\frac{d}{dx}(qxy) = qx \frac{dy}{dx} + qy$	M1	2.1
	$3px^2 + qx \frac{dy}{dx} + qy + 6y \frac{dy}{dx} = 0$	A1	1.1b
	$(qx + 6y) \frac{dy}{dx} = -3px^2 - qy \Rightarrow \frac{dy}{dx} = \dots$	dM1	2.1
	$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$	A1	1.1b
		(4)	
(b)	$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$	M1	1.1b
	$19x + 26y + 123 = 0 \Rightarrow m = -\frac{19}{26}$	B1	2.2a
	$\frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)} = \frac{26}{19} \quad \text{or} \quad \frac{q(-1) + 6(-4)}{3p(-1)^2 + q(-4)} = -\frac{19}{26}$	M1	3.1a
	$p - 4q = 22, \quad 57p - 102q = 624 \Rightarrow p = \dots, q = \dots$	dM1	1.1b
	$p = 2, \quad q = -5$	A1	1.1b
		(5)	
		(9 marks)	



13. The curve C has parametric equations

$$x = \sin 2\theta \quad y = \operatorname{cosec}^3 \theta \quad 0 < \theta < \frac{\pi}{2}$$

- (a) Find an expression for $\frac{dy}{dx}$ in terms of θ (3)
- (b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$ (3)



Question	Scheme	Marks	AOs
13(a)	$y = \text{cosec}^3 \theta \Rightarrow \frac{dy}{d\theta} = -3 \text{cosec}^2 \theta \text{cosec} \theta \cot \theta$	B1	1.1b
	$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$	M1	1.1b
	$\frac{dy}{dx} = \frac{-3 \text{cosec}^3 \theta \cot \theta}{2 \cos 2\theta}$	A1	1.1b
		(3)	
(b)	$y = 8 \Rightarrow \text{cosec}^3 \theta = 8 \Rightarrow \sin^3 \theta = \frac{1}{8} \Rightarrow \sin \theta = \frac{1}{2}$	M1	3.1a
	$\theta = \frac{\pi}{6} \Rightarrow \frac{dy}{dx} = \frac{-3 \text{cosec}^3 \left(\frac{\pi}{6}\right) \cot \left(\frac{\pi}{6}\right)}{2 \cos \left(\frac{2\pi}{6}\right)} = \dots$		
	or	M1	2.1
	$\sin \theta = \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{-3 \times \frac{1}{2} \times \frac{\cos \theta}{\sin \theta}}{2 \left(1 - 2 \sin^2 \theta\right)} = \frac{-3 \times 8 \times \frac{\sqrt{3}/2}{1/2}}{2 \left(1 - 2 \times \frac{1}{4}\right)}$	A1	2.2a
		(3)	
			(6 marks)



12. The function f is defined by

$$f(x) = \frac{e^{3x}}{4x^2 + k}$$

where k is a positive constant.

(a) Show that

$$f'(x) = (12x^2 - 8x + 3k)g(x)$$

where $g(x)$ is a function to be found.

(3)

Given that the curve with equation $y = f(x)$ has at least one stationary point,

(b) find the range of possible values of k .

(3)



12(a)

$$f(x) = \frac{e^{3x}}{4x^2 + k} \Rightarrow f'(x) = \frac{(4x^2 + k)3e^{3x} - 8xe^{3x}}{(4x^2 + k)^2}$$

or

M1
A1

$$f(x) = e^{3x} (4x^2 + k)^{-1} \Rightarrow f'(x) = 3e^{3x} (4x^2 + k)^{-1} - 8xe^{3x} (4x^2 + k)^{-2}$$

$$f'(x) = \frac{(12x^2 - 8x + 3k)e^{3x}}{(4x^2 + k)^2}$$

A1
(3)**(b)**If $y = f(x)$ has at least one stationary point then

B1

$$12x^2 - 8x + 3k = 0 \text{ has at least one root}$$

Applies $b^2 - 4ac \geq 0$ with $a = 12, b = -8, c = 3k$

M1

$$0 < k \leq \frac{4}{9}$$

A1

(3)

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2 \tan t + 1 \quad y = 2 \sec^2 t + 3 \quad -\frac{\pi}{4} \leq t \leq \frac{\pi}{3}$$

The line l is the normal to C at the point P where $t = \frac{\pi}{4}$

- (a) Using parametric differentiation, show that an equation for l is

$$y = -\frac{1}{2}x + \frac{17}{2}$$

- (b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x - 1)^2 + 5$$

The straight line with equation

$$y = -\frac{1}{2}x + k \quad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

- (c) Find the range of possible values for k .

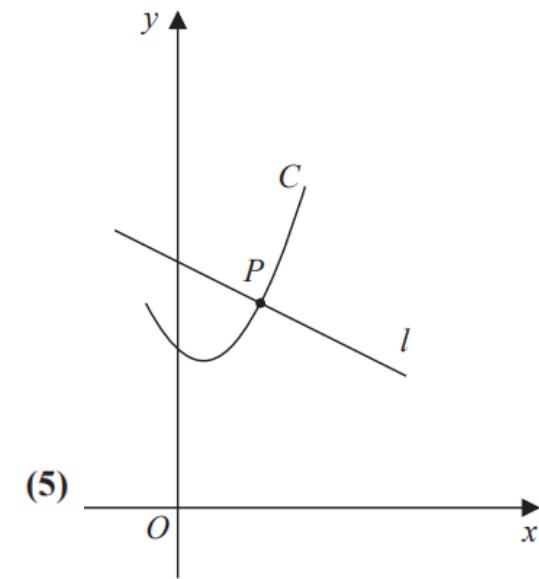
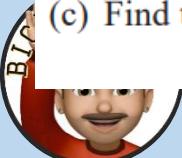


Figure 6

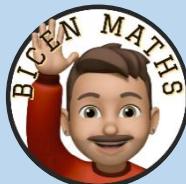
(2)

(5)

(5)



16(a)	<p>Attempts $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sec^2 t \tan t}{2\sec^2 t} (= 2 \tan t)$</p> <p>At $t = \frac{\pi}{4}$, $\frac{dy}{dx} = 2, x = 3, y = 7$</p> <p>Attempts equation of normal $y - 7 = -\frac{1}{2}(x - 3)$</p> <p>$y = -\frac{1}{2}x + \frac{17}{2}$ *</p>	M1 A1
		M1
		M1
		A1*
		(5)
(b)	<p>Attempts to use $\sec^2 t = 1 + \tan^2 t \Rightarrow \frac{y-3}{2} = 1 + \left(\frac{x-1}{2}\right)^2$</p> <p>$\Rightarrow y - 3 = 2 + \frac{(x-1)^2}{2} \Rightarrow y = \frac{1}{2}(x-1)^2 + 5$ *</p>	M1
		A1*
		(2)
(c)	<p>Attempts the lower limit for k:</p> <p>$\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k \Rightarrow x^2 - x + (11 - 2k) = 0$</p> <p>$b^2 - 4ac = 1 - 4(11 - 2k) = 0 \Rightarrow k = \dots$</p> <p>$(k =) \frac{43}{8}$</p> <p>Attempts the upper limit for k:</p> <p>$(x, y)_{t=-\frac{\pi}{4}} : t = -\frac{\pi}{4} \Rightarrow x = 2 \tan\left(-\frac{\pi}{4}\right) + 1 = -1, y = 2 \sec^2\left(-\frac{\pi}{4}\right) + 3 = 7$</p> <p>$(-1, 7), y = -\frac{1}{2}x + k \Rightarrow 7 = \frac{1}{2} + k \Rightarrow k = \dots$</p> <p>$(k =) \frac{13}{2}$</p> <p>$\frac{43}{8} < k \leqslant \frac{13}{2}$</p>	M1 A1 A1 A1 (5)



Numerical Methods



14.

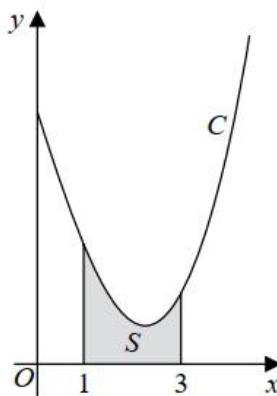


Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region S , shown shaded in Figure 4, is bounded by the curve C , the line with equation $x = 1$, the x -axis and the line with equation $x = 3$

The table below shows corresponding values of x and y with the values of y given to 4 decimal places as appropriate.

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958

- (a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of S , giving your answer to 3 decimal places.

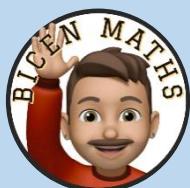
(3)

- (b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S .

(1)



Question	Scheme	Marks	AOs
14(a)	Uses or implies $h = 0.5$	B1	1.1b
	For correct form of the trapezium rule =	M1	1.1b
	$\frac{0.5}{2} \{3 + 2.2958 + 2(2.3041 + 1.9242 + 1.9089)\} = 4.393$	A1	1.1b
		(3)	
(b)	Any valid statement reason, for example <ul style="list-style-type: none"> • Increase the number of strips • Decrease the width of the strips • Use more trapezia 	B1	2.4
		(1)	



2. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in m s^{-1} .

Time (s)	0	5	10	15	20	25
Speed (m s^{-1})	2	5	10	18	28	42

Using all of this information,

- (a) estimate the length of runway used by the jet to take off.

(3)

Given that the jet accelerated smoothly in these 25 seconds,

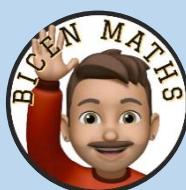
- (b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off.

(1)



Question 2 (Total 4 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\left(\frac{1}{2} \times 5\right)(42 + 2 + 2 \times (5 + 10 + 18 + 28))$	M1	This mark is given for a method to use the trapezium rule as an approximation to the area under the curve
		M1	This mark is given for a correct terms used for the trapezium rule
	415 m	A1	This mark is given for a correct estimate of the length of the runway
(b)	An overestimate since the area of the five trapezia is greater than the area under the curve	B1	This mark is given for a valid explanation



5. The table below shows corresponding values of x and y for $y = \log_3 2x$

The values of y are given to 2 decimal places as appropriate.

x	3	4.5	6	7.5	9
y	1.63	2	2.26	2.46	2.63

- (a) Using the trapezium rule with all the values of y in the table, find an estimate for

$$\int_3^9 \log_3 2x \, dx \quad (3)$$

Using your answer to part (a) and making your method clear, estimate

(b) (i) $\int_3^9 \log_3 (2x)^{10} \, dx$

(ii) $\int_3^9 \log_3 18x \, dx \quad (3)$



5(a)	States or uses $h = 1.5$	B1
	Full attempt at the trapezium rule $= \frac{1}{2} \{1.63 + 2.63 + 2 \times (2 + 2.26 + 2.46)\}$	M1
	$= \text{awrt } 13.3 \text{ or } \frac{531}{40}$	A1
		(3)
(b)(i)	$\int_3^9 \log_3 (2x)^{10} dx = 10 \times "13.3" = \text{awrt } 133 \text{ or e.g. } \frac{531}{4}$	
(ii)	$\begin{aligned} \int_3^9 \log_3 18x dx &= \int_3^9 \log_3 (9 \times 2x) dx = \int_3^9 2 + \log_3 2x dx \\ &= [2x]_3^9 + \int_3^9 \log_3 2x dx = 18 - 6 + \int_3^9 \log_3 2x dx = \dots \end{aligned}$	
	Awrt 25.3 or $\frac{1011}{40}$	



Integration



4. Given that a is a positive constant and

$$\int_a^{2a} \frac{t+1}{t} dt = \ln 7$$

show that $a = \ln k$, where k is a constant to be found.

(4)



Question	Scheme	Marks	AOs
4	Writes $\int \frac{t+1}{t} dt = \int 1 + \frac{1}{t} dt$ and attempts to integrate	M1	2.1
	$= t + \ln t (+c)$	M1	1.1b
	$(2a + \ln 2a) - (a + \ln a) = \ln 7$	M1	1.1b
	$a = \ln \frac{7}{2}$ with $k = \frac{7}{2}$	A1	1.1b
(4 marks)			



14.

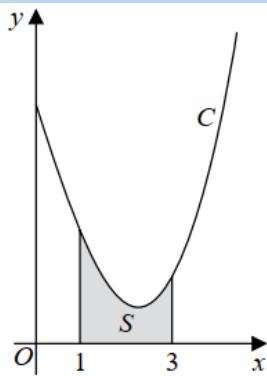


Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region S , shown shaded in Figure 4, is bounded by the curve C , the line with equation $x = 1$, the x -axis and the line with equation $x = 3$

The table below shows corresponding values of x and y with the values of y given to 4 decimal places as appropriate.

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958

- (a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of S , giving your answer to 3 decimal places.

(3)

- (b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S .

(1)

- (c) Show that the exact area of S can be written in the form $\frac{a}{b} + \ln c$, where a , b and c are integers to be found.

(6)



Question	Scheme	Marks	AOs
14(a)	Uses or implies $h = 0.5$	B1	1.1b
	For correct form of the trapezium rule =	M1	1.1b
	$\frac{0.5}{2} \{3 + 2.2958 + 2(2.3041 + 1.9242 + 1.9089)\} = 4.393$	A1	1.1b
		(3)	
(b)	Any valid statement/reason, for example <ul style="list-style-type: none"> • Increase the number of strips • Decrease the width of the strips • Use more trapezia 	B1	2.4
		(1)	
(c)	For integration by parts on $\int x^2 \ln x \, dx$	M1	2.1
	$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx$	A1	1.1b
	$\int -2x + 5 \, dx = -x^2 + 5x \quad (+c)$	B1	1.1b
	All integration attempted and limits used $\text{Area of } S = \int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \, dx = \left[\frac{x^3}{9} \ln x - \frac{x^3}{27} - x^2 + 5x \right]_{x=1}^{x=3}$	M1	2.1
	Uses correct ln laws, simplifies and writes in required form $\text{Area of } S = \frac{28}{27} + \ln 27 \quad (a = 28, b = 27, c = 27)$	A1	1.1b
		(6)	
		(10 marks)	



9. Given that A is constant and

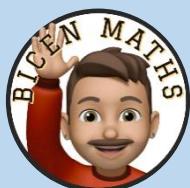
$$\int_1^4 (3\sqrt{x} + A) dx = 2A^2$$

show that there are exactly two possible values for A .

(5)



Question	Scheme	Marks	AOs	
9	$\int (3x^{0.5} + A) dx = 2x^{1.5} + Ax (+c)$	M1 A1	3.1a 1.1b	
	Uses limits and sets = $2A^2 \Rightarrow (2 \times 8 + 4A) - (2 \times 1 + A) = 2A^2$	M1	1.1b	
	Sets up quadratic and attempts to solve	Sets up quadratic and attempts $b^2 - 4ac$	M1	1.1b
	$\Rightarrow A = -2, \frac{7}{2}$ and states that there are two roots	States $b^2 - 4ac = 121 > 0$ and hence there are two roots	A1	2.4
(5 marks)				



16. (a) Express $\frac{1}{P(11 - 2P)}$ in partial fractions. (3)

A population of meerkats is being studied.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{22}P(11 - 2P), \quad t \geq 0, \quad 0 < P < 5.5$$

where P , in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that there were 1000 meerkats in the population when the study began,

- (b) determine the time taken, in years, for this population of meerkats to double, (6)

- (c) show that

$$P = \frac{A}{B + Ce^{-\frac{1}{2}t}}$$

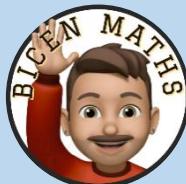
where A , B and C are integers to be found. (3)



A2 SAMs Paper 2

Integration

Question	Scheme	Marks	AOs
16(a)	Sets $\frac{1}{P(11-2P)} = \frac{A}{P} + \frac{B}{(11-2P)}$	B1	1.1a
	Substitutes either $P=0$ or $P=\frac{11}{2}$ into $1=A(11-2P)+BP \Rightarrow A \text{ or } B$	M1	1.1b
	$\frac{1}{P(11-2P)} = \frac{1/11}{P} + \frac{2/11}{(11-2P)}$	A1	1.1b
		(3)	
(b)	Separates the variables $\int \frac{22}{P(11-2P)} dP = \int 1 dt$	B1	3.1a
	Uses (a) and attempts to integrate $\int \frac{2}{P} + \frac{4}{(11-2P)} dP = t + c$	M1	1.1b
	$2\ln P - 2\ln(11-2P) = t + c$	A1	1.1b
	Substitutes $t=0, P=1 \Rightarrow t=0, P=1 \Rightarrow c=(-2\ln 9)$	M1	3.1a
	Substitutes $P=2 \Rightarrow t=2\ln 2 + 2\ln 9 - 2\ln 7$	M1	3.1a
	Time = 1.89 years	A1	3.2a
		(6)	
(c)	Uses ln laws $2\ln P - 2\ln(11-2P) = t - 2\ln 9$ $\Rightarrow \ln\left(\frac{9P}{11-2P}\right) = \frac{1}{2}t$	M1	2.1
	Makes 'P' the subject $\Rightarrow \left(\frac{9P}{11-2P}\right) = e^{\frac{1}{2}t}$ $\Rightarrow 9P = (11-2P)e^{\frac{1}{2}t}$ $\Rightarrow P = f\left(e^{\frac{1}{2}t}\right) \text{ or } \Rightarrow P = f\left(e^{-\frac{1}{2}t}\right)$	M1	2.1
	$\Rightarrow P = \frac{11}{2+9e^{-\frac{1}{2}t}} \Rightarrow A=11, B=2, C=9$	A1	1.1b
		(3)	
		(12 marks)	



7. Given that $k \in \mathbb{Z}^+$

(a) show that $\int_k^{3k} \frac{2}{(3x - k)} dx$ is independent of k , (4)

(b) show that $\int_k^{2k} \frac{2}{(2x - k)^2} dx$ is inversely proportional to k . (3)



Question	Scheme	Marks	AOs
7 (a)	$\int \frac{2}{(3x-k)} dx = \frac{2}{3} \ln(3x-k)$	M1 A1	1.1a 1.1b
	$\int_k^{3k} \frac{2}{(3x-k)} dx = \frac{2}{3} \ln(9k-k) - \frac{2}{3} \ln(3k-k)$	dM1	1.1b
	$= \frac{2}{3} \ln\left(\frac{8k}{2k}\right) = \frac{2}{3} \ln 4 \text{ oe}$	A1	2.1
		(4)	
(b)	$\int \frac{2}{(2x-k)^2} dx = -\frac{1}{(2x-k)}$	M1	1.1b
	$\int_k^{2k} \frac{2}{(2x-k)^2} dx = -\frac{1}{(4k-k)} + \frac{1}{(2k-k)}$	dM1	1.1b
	$= \frac{2}{3k} \left(\infty \frac{1}{k} \right)$	A1	2.1
		(3)	

(7 marks)

HOME

13. Show that

$$\int_0^2 2x\sqrt{x+2} \, dx = \frac{32}{15}(2 + \sqrt{2}) \quad (7)$$



Question	Scheme for Substitution		Marks	AOs
13	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2} dx$ Award for <ul style="list-style-type: none">• Using a valid substitution $u = \dots$, changing the terms to u's• integrating and using appropriate limits .		M1	3.1a
	Substitution $u = \sqrt{x+2} \Rightarrow \frac{dx}{du} = 2u \text{ oe}$	Substitution $u = x+2 \Rightarrow \frac{dx}{du} = 1 \text{ oe}$	B1	1.1b
	$\begin{aligned} & \int 2x\sqrt{x+2} dx \\ &= \int A(u^2 \pm 2)u^2 du \end{aligned}$	$\begin{aligned} & \int 2x\sqrt{x+2} dx \\ &= \int A(u \pm 2)\sqrt{u} du \end{aligned}$	M1	1.1b
	$= Pu^5 \pm Qu^3$	$= Su^{\frac{5}{2}} \pm Tu^{\frac{3}{2}}$	dM1	2.1
	$= \frac{4}{5}u^5 - \frac{8}{3}u^3$	$= \frac{4}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}$	A1	1.1b
	Uses limits 2 and $\sqrt{2}$ the correct way around	Uses limits 4 and 2 the correct way around	ddM1	1.1b
	$= \frac{32}{15}(2 + \sqrt{2}) *$		A1*	2.1
	(7)			
(7 marks)				



13.

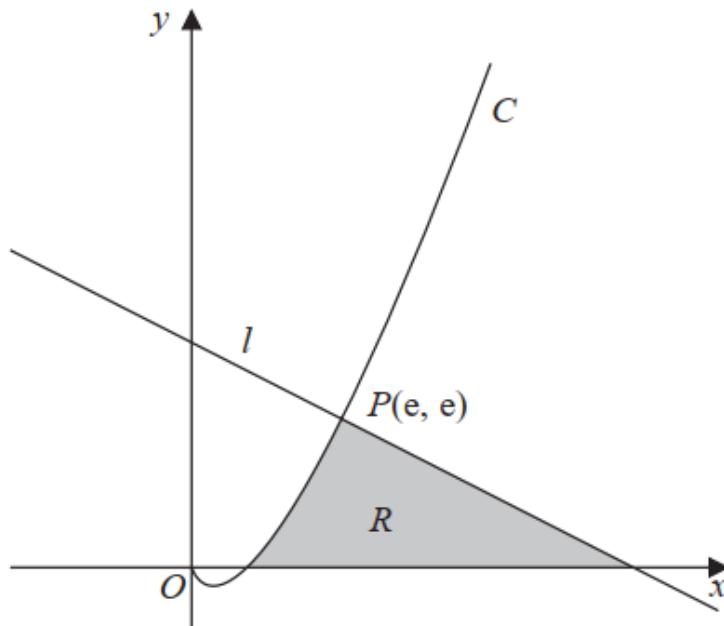


Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x \ln x$, $x > 0$

The line l is the normal to C at the point $P(e, e)$

The region R , shown shaded in Figure 2, is bounded by the curve C , the line l and the x -axis.

Show that the exact area of R is $Ae^2 + B$ where A and B are rational numbers to be found.

(10)



Question	Scheme	Marks	AOs
13	$C: y = x \ln x; l$ is a normal to C at $P(e, e)$ Let x_A be the x -coordinate of where l cuts the x -axis		
	$\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right) \quad \{= 1 + \ln x\}$	M1	2.1
		A1	1.1b
	$x = e, m_T = 2 \Rightarrow m_N = -\frac{1}{2} \Rightarrow y - e = -\frac{1}{2}(x - e)$ $y = 0 \Rightarrow -e = -\frac{1}{2}(x - e) \Rightarrow x = \dots$	M1	3.1a
	l meets x -axis at $x = 3e$ (allow $x = 2e + e \ln e$)	A1	1.1b
	{Areas:} either $\int_1^e x \ln x dx = [\dots]_1^e = \dots$ or $\frac{1}{2}((\text{their } x_A) - e)e$	M1	2.1
	$\left\{ \int x \ln x dx = \right\} \frac{1}{2}x^2 \ln x - \int \frac{1}{x} \left(\frac{x^2}{2} \right) \{dx\}$	M1	2.1
	$\left\{ = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x \{dx\} \right\} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$	dM1 A1	1.1b 1.1b
	$\text{Area}(R_1) = \int_1^e x \ln x dx = [\dots]_1^e = \dots$; $\text{Area}(R_2) = \frac{1}{2}((\text{their } x_A) - e)e$ and so, $\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2) \quad \{= \frac{1}{4}e^2 + \frac{1}{4} + e^2\}$	M1	3.1a
	$\text{Area}(R) = \frac{5}{4}e^2 + \frac{1}{4}$	A1	1.1b
		(10)	



13. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \quad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

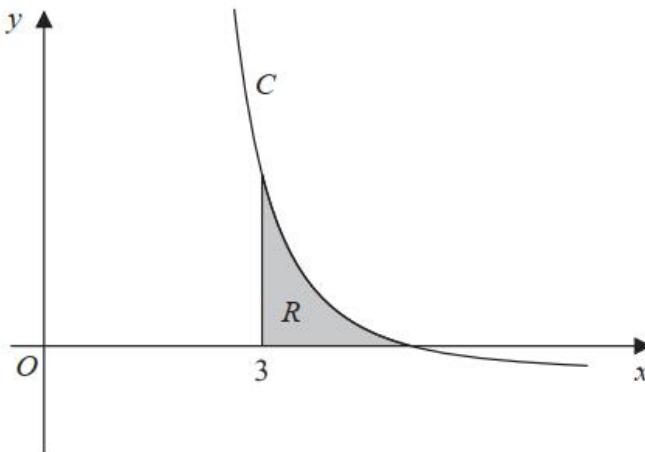


Figure 4

Figure 4 shows a sketch of part of the curve C . The region R , shown shaded in Figure 4, is bounded by the curve C , the x -axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(8)

A2 2019 Paper 1

Integration

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	The asymptote is found where $2x - q = 0$ Hence $q = 4$	B1	This mark is given for explaining that the asymptote at $x = 2$ is a solution of $2x - q = 0$
	$y = \frac{p - 3x}{(2x - 4)(x + 3)}$ $\frac{1}{2} = \frac{p - 9}{(6 - 4)(3 + 3)}$	M1	This mark is given for substituting $x = 3$, $y = \frac{1}{2}$ (and $q = 4$)
	$p - 9 = 6$ $p = 15$	A1	This mark is given for solving for p and showing that $p = 15$, as required
(b)	$\frac{15 - 3x}{(2x - 4)(x + 3)} = \frac{A}{(2x - 4)} + \frac{B}{(x + 3)}$	M1	This mark is given for a method to use partial fractions
	$= \frac{1.8}{(2x - 4)} - \frac{2.4}{(x + 3)}$	M1	This mark is given for finding values for A and B
	$= \frac{0.9}{(x - 2)} - \frac{2.4}{(x + 3)}$	A1	This mark is given for a fully simplified expression
	$I = \int \frac{15 - 3x}{(2x - 4)(x + 3)} dx$ $= m \ln(2x - 4) + n \ln(x + 3)$	M1	This mark is given for a method to integrate to find the area of R
	$= 0.9 \ln(2x - 4) + 2.4 \ln(x + 3)$	A1	This mark is given for a correct expression for the area of R
	$\text{Area } R = \left[0.9 \ln(2x - 4) - 2.4 \ln(x + 3) \right]_3^5$	M1	This mark is given for deducing an expression for the area of R ($y = 0$ when $x = 5$)
	$= [0.9 \ln 6 - 2.4 \ln 8] - [0.9 \ln 2 - 2.4 \ln 6]$ $= [0.9 \ln 6 + 2.4 \ln 6] - [7.2 \ln 2 + 0.9 \ln 2]$ $= 3.3 \ln 6 - 8.1 \ln 2$ $= 3.3 \ln 3 + 3.3 \ln 2 - 8.1 \ln 2$	M1	This mark is given for a method to find the exact area of R
	$= 3.3 \ln 3 - 4.8 \ln 2$	A1	This mark is given for a correct value of the area of R with $a = 3.3$ and $b = 4.8$



10. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3 \, dx}{(x-1)(3+2\sqrt{x-1})} = \int_p^q \frac{6 \, du}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

- (b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3 \, dx}{(x-1)(3+2\sqrt{x-1})} = \ln a$$

where a is a rational constant to be found.

(6)



Question	Scheme	Marks	AOs
10 (a)	$x = u^2 + 1 \Rightarrow dx = 2udu \text{ oe}$	B1	1.1b
	Full substitution $\int \frac{3dx}{(x-1)(3+2\sqrt{x-1})} = \int \frac{3 \times 2u \, du}{(u^2 + 1 - 1)(3 + 2u)}$	M1	1.1b
	Finds correct limits e.g. $p = 2, q = 3$	B1	1.1b
	$= \int \frac{3 \times 2u \, du}{u^2(3+2u)} = \int \frac{6 \, du}{u(3+2u)} *$	A1*	2.1
		(4)	
(b)	$\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} \Rightarrow A = \dots, B = \dots$	M1	1.1b
	Correct PF. $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$	A1	1.1b
	$\int \frac{6 \, du}{u(3+2u)} = 2 \ln u - 2 \ln(3+2u) \quad (+c)$	dM1 A1ft	3.1a 1.1b
	Uses limits $u = "3", u = "2"$ with some correct ln work leading to $k \ln b$ E.g. $(2 \ln 3 - 2 \ln 9) - (2 \ln 2 - 2 \ln 7) = 2 \ln \frac{7}{9}$	M1	1.1b
	$\ln \frac{49}{36}$	A1	2.1
		(6)	
		(10 marks)	



- 1 The table below shows corresponding values of x and y for $y = \sqrt{\frac{x}{1+x}}$

The values of y are given to 4 significant figures.

x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452

- (a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}} dx$$

giving your answer to 3 significant figures.

(3)

- (b) Using your answer to part (a), deduce an estimate for $\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}} dx$

(1)

Given that

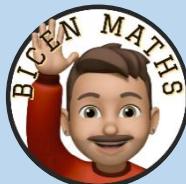
$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}} dx = 4.535 \text{ to 4 significant figures}$$

- (c) comment on the accuracy of your answer to part (b).

(1)



Question	Scheme	Marks	AOs
1(a)	$h = 0.5$	B1	1.1a
	$A \approx \frac{0.5}{2} \{0.5774 + 0.8452 + 2(0.7071 + 0.7746 + 0.8165)\}$	M1	1.1b
	= awrt 1.50	A1	1.1b
	For reference: The integration on a calculator gives 1.511549071 The full accuracy for y values gives 1.504726147 The accuracy from the table gives 1.50475		
		(3)	
(b)	$3 \times$ their (a) If (a) is correct, allow awrt 4.50 or awrt 4.51 even with no working. Only allow 4.5 if (a) is correct and working is shown e.g. 3×1.5	B1ft	2.2a
	If (a) is incorrect allow $3 \times$ their (a) given to at least 3sf but do not be too concerned about the accuracy (as they may use rounded or rounded value from (a))		
	For reference the integration on a calculator gives 4.534647213		
		(1)	
(c)	This mark depends on the B1 having been awarded in part (b) with awrt 4.5 Look for a sensible comment. Some examples: <ul style="list-style-type: none">• The answer is accurate to 2 sf or one decimal place• Answer to (b) is accurate as $4.535 \approx 4.50$• Very accurate as 4.535 to 2 sf is 4.5• $4.51425 < 4.535$ so my answer is underestimate but not too far off• It is an underestimate but quite close• It is a very good estimate• High accuracy• (Quite) accurate• It is less than 1% out• $4.535 - 4.5 = 0.035$ so not far out But not just "it is an underestimate" or Calculates percentage error correctly using awrt 4.50 or awrt 4.51 or 4.5 (No comment is necessary in these cases although one may be given) Examples: $\left \frac{4.535 - 4.50}{4.535} \right \times 100 = 0.77\% \text{ or } \left \frac{4.535 - 4.51}{4.535} \right \times 100 = 0.55\% \text{ or }$ $\left \frac{4.535 - 4.51425}{4.535} \right \times 100 = 0.46\% \text{ or } \left \frac{4.50}{4.535} \right \times 100 = 99\%$	B1	3.2b
	In these cases don't be too concerned about accuracy e.g. allow 1sf. This mark should be withheld if there are any contradictory statements		
		(1)	
		(5 marks)	



6. (a) Given that

$$\frac{x^2 + 8x - 3}{x + 2} \equiv Ax + B + \frac{C}{x + 2} \quad x \in \mathbb{R} \quad x \neq -2$$

find the values of the constants A , B and C

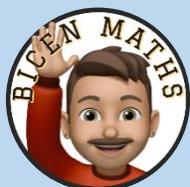
(3)

(b) Hence, using algebraic integration, find the exact value of

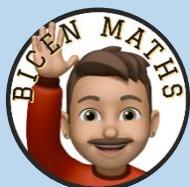
$$\int_0^6 \frac{x^2 + 8x - 3}{x + 2} \, dx$$

giving your answer in the form $a + b \ln 2$ where a and b are integers to be found.

(4)



6(a)	$x^2 + 8x - 3 = (Ax + B)(x + 2) + C$ or $Ax(x + 2) + B(x + 2) + C$ $\Rightarrow A = \dots, B = \dots, C = \dots$		
	or		
	$\begin{array}{r} x+6 \\ x+2 \overline{)x^2+8x-3} \\ \underline{x^2+2x} \\ 6x-3 \\ \underline{6x+12} \\ -15 \end{array}$	M1	1.1b
	Two of $A = 1, B = 6, C = -15$	A1	1.1b
	All three of $A = 1, B = 6, C = -15$	A1	1.1b
		(3)	
6(b)	$\int \frac{x^2 + 8x - 3}{x + 2} dx = \int x + 6 - \frac{15}{x + 2} dx = \dots - 15 \ln(x + 2)$	M1	1.1b
	$= \frac{1}{2}x^2 + 6x - 15 \ln(x + 2) \quad (+c)$	A1ft	1.1b
	$\int_0^6 \frac{x^2 + 8x - 3}{x + 2} dx = \left[\frac{1}{2}x^2 + 6x - 15 \ln(x + 2) \right]_0^6$ $= (18 + 36 - 15\ln 8) - (0 + 0 - 15\ln 2)$	M1	2.1
	$= 18 + 36 - (15 - 45)\ln 2 \text{ or e.g. } 18 + 36 + 15 \ln\left(\frac{2}{8}\right)$		
	$= 54 - 30 \ln 2$	A1	1.1b
		(4)	
			(7 marks)



The curve shown in Figure 3 has parametric equations

$$x = 6 \sin t \quad y = 5 \sin 2t \quad 0 \leq t \leq \frac{\pi}{2}$$

The region R , shown shaded in Figure 3, is bounded by the curve and the x -axis.

(a) (i) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t \, dt$

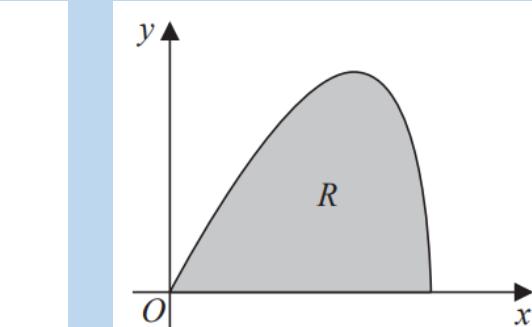


Figure 3

(ii) Hence show, by algebraic integration, that the area of R is exactly 20

(3)

(3)

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4. Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

(b) calculate the width of the walkway.

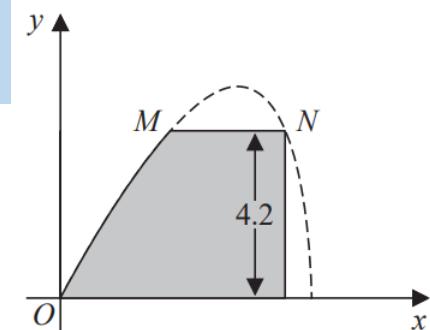
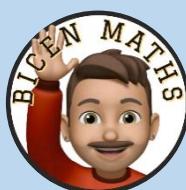


Figure 4

(5)



Question	Scheme	Marks	AOs
12(a)(i)	$y \times \frac{dx}{dt} = 5 \sin 2t \times 6 \cos t \text{ or } 5 \times 2 \sin t \cos t \times 6 \cos t$	M1	1.2
	(Area =) $\int 5 \sin 2t \times 6 \cos t dt = \int 5 \times 2 \sin t \cos t \times 6 \cos t dt$ or $\int 5 \sin 2t \times 6 \cos t dt = \int 60 \sin t \cos^2 t dt$	dM1	1.1b
	(Area =) $\int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t dt *$	A1*	2.1*
		(3)	
(a)(ii)	$\int 60 \sin t \cos^2 t dt = -20 \cos^3 t$	M1 A1	1.1b 1.1b
	Area = $[-20 \cos^3 t]_0^{\frac{\pi}{2}} = 0 - (-20) = 20 *$	A1*	2.1
		(3)	
(b)	$5 \sin 2t = 4.2 \Rightarrow \sin 2t = \frac{4.2}{5}$	M1	3.4
	$t = 0.4986..., 1.072...$	A1	1.1b
	Attempts to finds the x values at both t values	dM1	3.4
	$t = 0.4986... \Rightarrow x = 2.869...$	A1	1.1b
	$t = 1.072 \Rightarrow x = 5.269...$	A1	3.2a
	Width of path = 2.40 metres	(5)	
		(11 marks)	



Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \quad x > 0$$

The finite region R , shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x -axis and the line with equation $x = 4$

The table below shows corresponding values of x and y , with the values of y given to 4 decimal places.

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

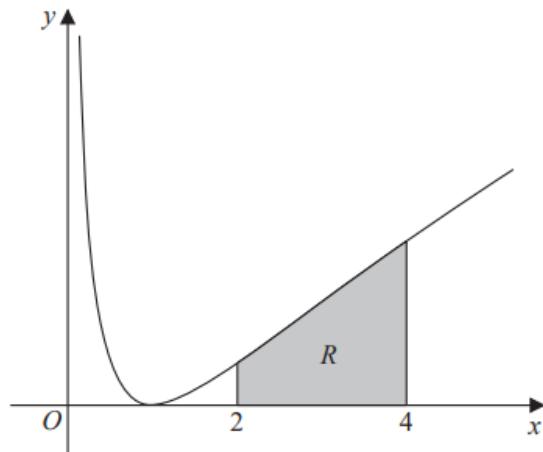


Figure 2

- (a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R , giving your answer to 3 significant figures.

(3)

- (b) Use algebraic integration to find the exact area of R , giving your answer in the form

$$y = a(\ln 2)^2 + b \ln 2 + c$$

where a , b and c are integers to be found.

(5)



Question	Scheme	Marks	AOs
11(a)	$h = 0.5$	B1	1.1b
	$A \approx \frac{1}{2} \times \frac{1}{2} \left\{ 0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694) \right\}$	M1	1.1b
	$= 2.41$	A1	1.1b
		(3)	
(b)	$\int (\ln x)^2 dx = x(\ln x)^2 - \int x \times \frac{2 \ln x}{x} dx$	M1	3.1a
	$= x(\ln x)^2 - 2 \int \ln x dx = x(\ln x)^2 - 2(x \ln x - \int dx)$	A1	1.1b
	$= x(\ln x)^2 - 2 \int \ln x dx = x(\ln x)^2 - 2x \ln x + 2x$	dM1	2.1
	$\int_2^4 (\ln x)^2 dx = \left[x(\ln x)^2 - 2x \ln x + 2x \right]_2^4$		
	$= 4(\ln 4)^2 - 2 \times 4 \ln 4 + 2 \times 4 - (2(\ln 2)^2 - 2 \times 2 \ln 2 + 2 \times 2)$	ddM1	2.1
	$= 4(2 \ln 2)^2 - 16 \ln 2 + 8 - 2(\ln 2)^2 + 4 \ln 2 - 4$		
	$= 14(\ln 2)^2 - 12 \ln 2 + 4$	A1	1.1b
		(5)	
		(8 marks)	



12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}} \, dx = \int_p^q \frac{2(u-1)^3}{u} \, du$$

where p and q are constants to be found.

(3)

(b) Hence show that

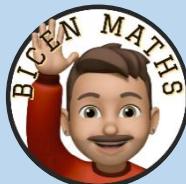
$$\int_0^{16} \frac{x}{1+\sqrt{x}} \, dx = A - B \ln 5$$

where A and B are constants to be found.

(4)



Question	Scheme	Marks	AOs
12(a)	$u = 1 + \sqrt{x} \Rightarrow x = (u-1)^2 \Rightarrow \frac{dx}{du} = 2(u-1)$ <p style="text-align: center;">or</p> $u = 1 + \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$	B1	1.1b
	$\int \frac{x}{1+\sqrt{x}} dx = \int \frac{(u-1)^2}{u} 2(u-1) du$ <p style="text-align: center;">or</p> $\int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times 2x^{\frac{1}{2}} du = \int \frac{2x^{\frac{3}{2}}}{u} du = \int \frac{2(u-1)^3}{u} du$	M1	2.1
	$\int_0^{16} \frac{x}{1+\sqrt{x}} dx = \int_1^5 \frac{2(u-1)^3}{u} du$	A1	1.1b
		(3)	
(b)	$2 \int \frac{u^3 - 3u^2 + 3u - 1}{u} du = 2 \int \left(u^2 - 3u + 3 - \frac{1}{u} \right) du = \dots$	M1	3.1a
	$= (2) \left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u \right]$	A1	1.1b
	$= 2 \left[\frac{5^3}{3} - \frac{3(5)^2}{2} + 3(5) - \ln 5 - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1 \right) \right]$	dM1	2.1
	$= \frac{104}{3} - 2 \ln 5$	A1	1.1b
		(4)	
		(7 marks)	



4. (a) Express $\lim_{\delta x \rightarrow 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x$ as an integral.

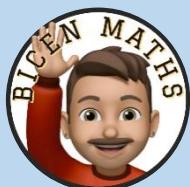
(1)

(b) Hence show that

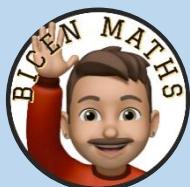
$$\lim_{\delta x \rightarrow 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \ln k$$

where k is a constant to be found.

(2)



4 (a)	$\lim_{\delta x \rightarrow 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \int_{2.1}^{6.3} \frac{2}{x} dx$	B1
		(1)
(b)	$= [2 \ln x]_{2.1}^{6.3} = 2 \ln 6.3 - 2 \ln 2.1$	M1
	$= \ln 9 \quad \text{CSO}$	A1
		(2)



12.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Show that

$$\int_1^{e^2} x^3 \ln x \, dx = ae^8 + b$$

where a and b are rational constants to be found.

(5)

12

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \times \frac{1}{x} \, dx$$

M1

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} (+c)$$

M1
A1

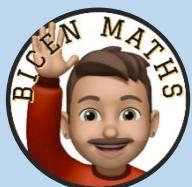
$$\int_1^{e^2} x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x - \frac{x^4}{16} \right]_1^{e^2} = \left(\frac{e^8}{4} \ln e^2 - \frac{e^8}{16} \right) - \left(-\frac{1^4}{16} \right)$$

M1

$$= \frac{7}{16} e^8 + \frac{1}{16}$$

A1

(5)



16.

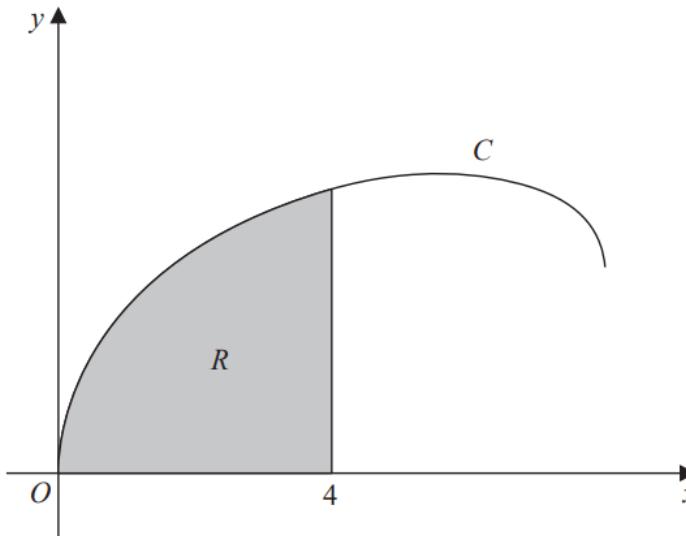


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8 \sin^2 t \quad y = 2 \sin 2t + 3 \sin t \quad 0 \leq t \leq \frac{\pi}{2}$$

The region R , shown shaded in Figure 6, is bounded by C , the x -axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a (8 - 8 \cos 4t + 48 \sin^2 t \cos t) dt$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R .

(4)



16 (a)	<p>Attempts $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times 16 \sin t \cos t$ and uses $\sin 2t = 2 \sin t \cos t$</p>	M1
	<p>Correct expanded integrand. Usually for one of</p> $(R) = \int \underline{\underline{48 \sin^2 t \cos t + 16 \sin^2 2t \, dt}}$ $(R) = \int \underline{\underline{48 \sin^2 t \cos t + 64 \sin^2 t \cos^2 t \, dt}}$ $(R) = \int \underline{\underline{24 \sin 2t \sin t + 16 \sin^2 2t \, dt}}$	A1
	Attempts to use $\cos 4t = 1 - 2 \sin^2 2t = (1 - 8 \sin^2 t \cos^2 t)$	M1
	$R = \int_0^a 8 - 8 \cos 4t + 48 \sin^2 t \cos t \, dt$ *	A1*
	Deduces $a = \frac{\pi}{4}$	B1
		(5)
(b)	$\int 8 - 8 \cos 4t + 48 \sin^2 t \cos t \, dt = 8t - 2 \sin 4t + 16 \sin^3 t$	M1 A1
	$\left[8t - 2 \sin 4t + 16 \sin^3 t \right]_0^\pi = 2\pi + 4\sqrt{2}$	M1 A1
		(4)



Vectors



7.

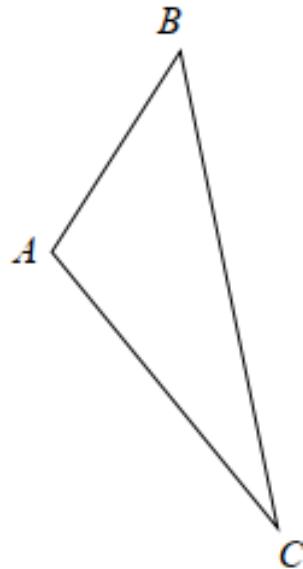
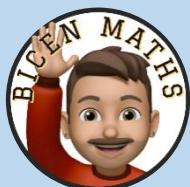
**Figure 2**

Figure 2 shows a sketch of a triangle ABC .

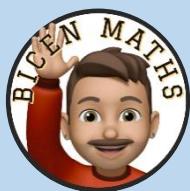
Given $\vec{AB} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\vec{BC} = \mathbf{i} - 9\mathbf{j} + 3\mathbf{k}$,

show that $\angle BAC = 105.9^\circ$ to one decimal place.

(5)



Question	Scheme	Marks	AOs
7	Attempts $\vec{AC} = \vec{AB} + \vec{BC} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \mathbf{i} - 9\mathbf{j} + 3\mathbf{k} = 3\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$	M1	3.1a
	Attempts to find any one length using 3-d Pythagoras	M1	2.1
	Finds all of $ AB = \sqrt{14}$, $ AC = \sqrt{61}$, $ BC = \sqrt{91}$	A1ft	1.1b
	$\cos BAC = \frac{14 + 61 - 91}{2\sqrt{14}\sqrt{61}}$	M1	2.1
	angle $BAC = 105.9^\circ$ *	A1*	1.1b
		(5)	
	(5 marks)		



2. Relative to a fixed origin O ,

the point A has position vector $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$,

the point B has position vector $(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$,

and the point C has position vector $(a\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$, where a is a constant and $a < 0$

D is the point such that $\overrightarrow{AB} = \overrightarrow{BD}$.

(a) Find the position vector of D .

(2)

Given $|\overrightarrow{AC}| = 4$

(b) find the value of a .

(3)



Question	Scheme	Marks	AOs
2	$\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\overrightarrow{OB} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OC} = a\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $a < 0$ $\overrightarrow{AB} = \overrightarrow{BD}$, $ \overrightarrow{AB} = 4$		
(a)	E.g. $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \overrightarrow{OB} + \overrightarrow{AB}$ or $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \overrightarrow{OB} + \overrightarrow{AB} = \overrightarrow{OB} + \overrightarrow{OB} - \overrightarrow{OA} = 2\overrightarrow{OB} - \overrightarrow{OA}$ or $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \overrightarrow{OB} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AB} = \overrightarrow{OA} + 2\overrightarrow{AB}$		
	$= \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \left\{ = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix} \right\}$ or $= \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2 \left(\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \right) \quad \left\{ = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix} \right\}$	M1	3.1a
	$= \begin{pmatrix} 6 \\ -7 \\ 10 \end{pmatrix} \quad \text{or} \quad 6\mathbf{i} - 7\mathbf{j} + 10\mathbf{k}$	A1	1.1b
			(2)
(b)	$(a-2)^2 + (5-3)^2 + (-2--4)^2$	M1	1.1b
	$\left\{ \overrightarrow{AC} = 4 \Rightarrow \right\} (a-2)^2 + (5-3)^2 + (-2--4)^2 = (4)^2$ $\Rightarrow (a-2)^2 = 8 \Rightarrow a = \dots \quad \text{or} \quad \Rightarrow a^2 - 4a - 4 = 0 \Rightarrow a = \dots$	dM1	2.1
	(as $a < 0 \Rightarrow a = 2 - 2\sqrt{2}$ (or $a = 2 - \sqrt{8}$))	A1	1.1b
			(3)
			(5 marks)



3. Relative to a fixed origin O

- point A has position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$
- point B has position vector $3\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$
- point C has position vector $2\mathbf{i} - 16\mathbf{j} + 4\mathbf{k}$

(a) Find \overrightarrow{AB}

(2)

(b) Show that quadrilateral $OABC$ is a trapezium, giving reasons for your answer.

(2)



Question	Scheme	Marks	AOs
3 (a)	$\overrightarrow{AB} = (3\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}) - (2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k})$	M1	1.1b
	$= \mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$	A1	1.1b
		(2)	
(b)	States $\overrightarrow{OC} = 2 \times \overrightarrow{AB}$	M1	1.1b
	Explains that as OC is parallel to AB , so $OABC$ is a trapezium.	A1	2.4
		(2)	
(4 marks)			



6.

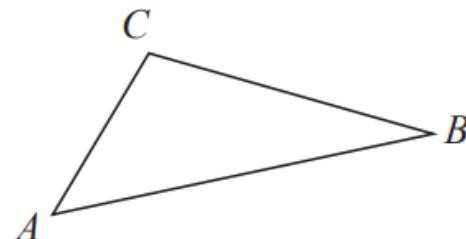


Figure 1

Figure 1 shows a sketch of triangle ABC .

Given that

- $\vec{AB} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$
- $\vec{BC} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$

(a) find \vec{AC}

(2)

(b) show that $\cos A B C = \frac{9}{10}$

(3)



6(a)	$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k} + \mathbf{i} + \mathbf{j} + 4\mathbf{k} = ...$	M1	1.1b
	$= -2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$	A1	1.1b
		(2)	
(b)	At least 2 of $(AC^2) = "2^2 + 3^2 + 1^2", (AB^2) = 3^2 + 4^2 + 5^2, (BC^2) = 1^2 + 1^2 + 4^2$	M1	1.1b
	$2^2 + 3^2 + 1^2 = 3^2 + 4^2 + 5^2 + 1^2 + 1^2 + 4^2 - 2\sqrt{3^2 + 4^2 + 5^2}\sqrt{1^2 + 1^2 + 4^2} \cos ABC$	M1	3.1a
	$14 = 50 + 18 - 2\sqrt{50}\sqrt{18} \cos ABC$	A1*	2.1
	$\Rightarrow \cos ABC = \frac{50 + 18 - 14}{2\sqrt{50}\sqrt{18}} = \frac{9}{10} *$		
		(3)	
	(b) Alternative		
	$AB^2 = 3^2 + 4^2 + 5^2, BC^2 = 1^2 + 1^2 + 4^2$	M1	1.1b
	$\overrightarrow{BA} \cdot \overrightarrow{BC} = (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = 27 = \sqrt{3^2 + 4^2 + 5^2}\sqrt{1^2 + 1^2 + 4^2} \cos ABC$	M1	3.1a
	$27 = \sqrt{50}\sqrt{18} \cos ABC \Rightarrow \cos ABC = \frac{27}{\sqrt{50}\sqrt{18}} = \frac{9}{10} *$	A1*	2.1

(5 marks)



9.

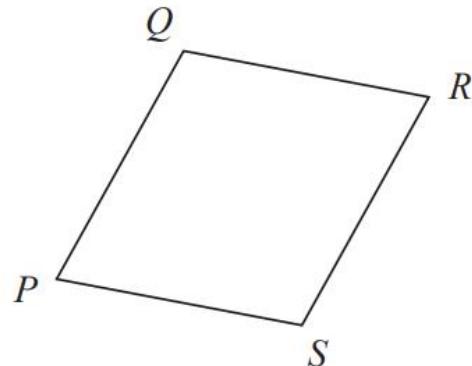
**Figure 3**

Figure 3 shows a sketch of a parallelogram $PQRS$.

Given that

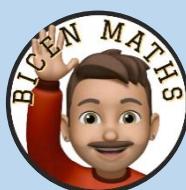
- $\overrightarrow{PQ} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$
- $\overrightarrow{QR} = 5\mathbf{i} - 2\mathbf{k}$

(a) show that parallelogram $PQRS$ is a rhombus.

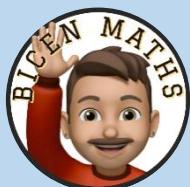
(2)

(b) Find the exact area of the rhombus $PQRS$.

(4)



9(a)	Attempts both $ \vec{PQ} = \sqrt{2^2 + 3^2 + (-4)^2}$ and $ \vec{QR} = \sqrt{5^2 + (-2)^2}$	M1
	States that $ \vec{PQ} = \vec{QR} = \sqrt{29}$ so $PQRS$ is a rhombus	A1
(b)		(2)
	Attempts BOTH $\vec{PR} = \vec{PQ} + \vec{QR} = 7\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$	M1
	AND $\vec{QS} = -\vec{PQ} + \vec{PS} = 3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$	
	Correct $\vec{PR} = 7\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and $\vec{QS} = 3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$	A1
	Correct method for area $PQRS$. E.g. $\frac{1}{2} \times \vec{PR} \times \vec{QS} $	dM1
	$= \sqrt{517}$	A1
		(4)



13. Relative to a fixed origin O

- the point A has position vector $4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$
- the point B has position vector $4\mathbf{j} + 6\mathbf{k}$
- the point C has position vector $-16\mathbf{i} + p\mathbf{j} + 10\mathbf{k}$

where p is a constant.

Given that A , B and C lie on a straight line,

(a) find the value of p .

(3)

The line segment OB is extended to a point D so that \overrightarrow{CD} is parallel to \overrightarrow{OA}

(b) Find $|\overrightarrow{OD}|$, writing your answer as a fully simplified surd.

(3)



13(a)	Attempts two of the relevant vectors $\pm \overrightarrow{AB} = \pm(-4\mathbf{i} + 7\mathbf{j} + \mathbf{k})$ $\pm \overrightarrow{AC} = \pm(-20\mathbf{i} + (p+3)\mathbf{j} + 5\mathbf{k})$ $\pm \overrightarrow{BC} = \pm(-16\mathbf{i} + (p-4)\mathbf{j} + 4\mathbf{k})$	M1
	Uses two of the three vectors in such a way as to find the value of p . E.g. $p+3=5\times 7$	dM1
	$p = 32$	A1
		(3)
(a) Alternative:		
	$r_{AB} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} + \lambda(-4\mathbf{i} + 7\mathbf{j} + \mathbf{k})$	M1
	$4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} + \lambda(-4\mathbf{i} + 7\mathbf{j} + \mathbf{k}) = -16\mathbf{i} + p\mathbf{j} + 10\mathbf{k} \Rightarrow \lambda = 5$	dM1
	$4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} + \lambda(-4\mathbf{i} + 7\mathbf{j} + \mathbf{k}) = -16\mathbf{i} + p\mathbf{j} + 10\mathbf{k} \Rightarrow p = 35 - 3$	
	$p = 32$	A1
(b)	Deduces that $\overrightarrow{OD} = \lambda \overrightarrow{OB} = 4\lambda\mathbf{j} + 6\lambda\mathbf{k}$ and attempts $\overrightarrow{CD} = 16\mathbf{i} + (4\lambda - "32")\mathbf{j} + (6\lambda - 10)\mathbf{k}$	M1
	Correct attempt at λ using the fact that \overrightarrow{CD} is parallel to \overrightarrow{OA} $\overrightarrow{CD} = 16\mathbf{i} + (4\lambda - "32")\mathbf{j} + (6\lambda - 10)\mathbf{k}$	dM1
	$\overrightarrow{OA} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ $4\lambda - 32 = -12 \Rightarrow \lambda = \dots$ OR $6\lambda - 10 = 20 \Rightarrow \lambda = \dots$	
	$ \overrightarrow{OD} = 5 \times \sqrt{4^2 + 6^2} = 10\sqrt{13}$	A1
		(3)

