Modelling with Differential Equations

This chapter looks at applied settings in which 1st and 2nd order differential equations might emerge.

1:: Modelling with 1st order differential equations.

"A particle P is moving along a straight line. At time t seconds, the acceleration of the particle is given by $a = t + \frac{3}{t}v,\ t \geq 0$ Given that $v = 0$ when $t = 2$, show that the velocity of the particle at time t is given by the equation $v = ct^3 - t^2$ where c is a constant to be found."

2:: Simple Harmonic Motion

Modelling the motion of a particle which has a acceleration towards a central point proportional to its displacement from this centre.

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

3:: Damped and Force Harmonic Motion

How to model the damping force on a spring (damped harmonic motion) and how to model additional forces (forced harmonic motion).

4:: Coupled First-Order Differential Equations

The differential-equation version of "simultaneous equations". e.g. Prey-predator models found in Biology.

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

At the end of Pure Year 2 (Integration), we saw how we could form and solve 1st order differential equations from context. In Core Pure Year 2 this differential equation may be of the form $\frac{dy}{dx} + Py = Q$. One common example of first order differential equations is with displacement, velocity and acceleration, e.g. $a = \frac{dv}{dt}$

A particle P starts from rest at a point O and moves along a straight line. At time t seconds the acceleration, a ms^{-2}, of P is given by

$$a = \frac{6}{(t+2)^2}, \qquad t \geq 0$$

(a) Find the velocity of P at time t seconds.
(b) Show that the displacement of P from O when $t = 6$ is $(18 - 12\ln 2)$m

A particle P is moving along a straight line. At time t seconds, the acceleration of the particle is given by $a = t + \frac{3}{t}v,\ t \geq 0$
Given that $v = 0$ when $t = 2$, show that the velocity of the particle at time t is given by the equation $v = ct^3 - t^2$ where c is a constant to be found.

Classic 'Filling a Container' Example

A storage tank initially containers 1000 litres of pure water. Liquid is removed from the tank at a constant rate of 30 litres per hour and a chemical solution is added to the tank at a constant rate of 40 litres per hour. The chemical solution contains 4 grams of copper sulphate per litre of water. Given that there are x grams of copper sulphate in the tank after t hours and that the copper sulphate immediately disperses throughout the tank on entry,

(a) Show that the situation can be modelled by the differential equation

$$\frac{dx}{dt} = 160 - \frac{3x}{100 + t}$$

(b) Hence find the number of grams of copper sulphate in the tank after 6 hours.

(c) Explain how the model could be refined.

Litres of liquid in tank after t hours:

Concentration of copper sulphate after t hours:

Rate copper sulphate **in**:

Rate copper sulphate **out**:

8. A large container initially contains 3 litres of pure water.
Contaminated water starts pouring into the container at a constant rate of 250 ml per minute and you may assume the contaminant dissolves completely.

At the same time, the container is drained at a constant rate of 125 ml per minute.
The water in the container is continually mixed.

The amount of contaminant in the water pouring into the container, at time t minutes after pouring began, is modelled to be $(5 - e^{-0.1t})$ mg per litre.

Let m be the amount of contaminant, in milligrams, in the container at time t minutes after the contaminated water begins pouring into the container.

(a) (i) Write down an expression for the total volume of water in litres in the container at time t.

(ii) Hence show that the amount of contaminant in the container can be modelled by the differential equation

$$\frac{dm}{dt} = \frac{5 - e^{-0.1t}}{4} - \frac{m}{24 + t}$$

(4)

(b) By solving the differential equation, find an expression for the amount of contaminant, in milligrams, in the container t minutes after the contaminated water begins to be poured into the container.

(8)

After 30 minutes, the concentration of contaminant in the water was measured as 3.79 mg per litre.

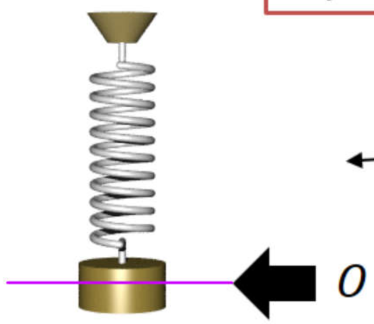
(c) Assess the model in light of this information, giving a reason for your answer.

(2)

Ex 8A Q7
Mixed Exercise 8, Q1-7

Simple Harmonic Motion

✎ Simple Harmonic Motion (SHM) is motion in which the acceleration of a particle P is always towards a fixed point O on the line of motion of P. The **acceleration is proportional to the displacement x** of P from O.



We can see that when the particle is moving away from O, it is decelerating, as the acceleration is towards O.

Because of the compression/extension of the spring, as we double the displacement from O, we double the acceleration towards O, i.e. the acceleration is not constant (as it would be if acting under gravity).

O is the **centre of oscillation.**

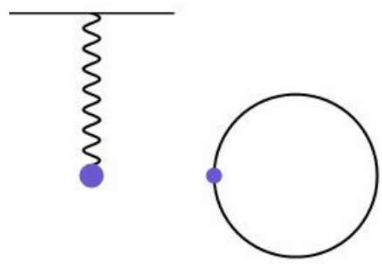
$$\ddot{x} = -\omega^2 x$$

Second derivative of displacement is acceleration (recall the dot notation differentiates with respect to time).

Negative because acceleration acting towards O

ω^2 is constant of proportionality. ω is the **angular velocity** of the particle.

(i.e. the number of oscillations per 2π seconds)



By the chain rule, $\ddot{x} = \dfrac{dv}{dt} = \dfrac{dv}{dx} \times \dfrac{dx}{dt} = v\dfrac{dv}{dx}$

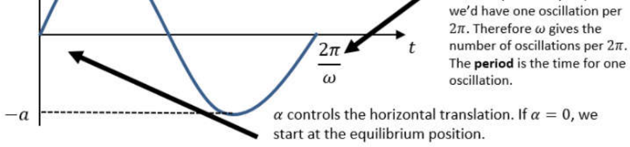
✎ $\ddot{x} = v\dfrac{dv}{dx}$

Also note that $\ddot{x}$ can be written as d^2x/dt^2

A particle P moves with simple harmonic motion about a point O. Given that the maximum displacement of the particle from O is a,

(a) show that $v^2 = \omega^2(a^2 - x^2)$ where v is the velocity of the particle and ω^2 is a constant.

(b) show that $x = a\sin(\omega t + \alpha)$, where α is an arbitrary constant.



A particle is moving along a straight line. At time t seconds its displacement, x m from a fixed point O is such that

$$\frac{d^2x}{dt^2} = -4x$$

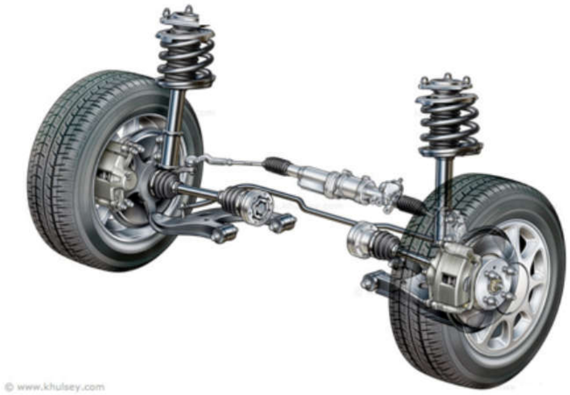
Given that at $t = 0, x = 1$ and the particle is moving with velocity 4 ms^{-1},

(a) find an expression for the displacement of the particle after t seconds

(b) hence determine the maximum displacement of the particle from O.

A particle P, is attached to the ends of two identical elastic springs. The free ends of the springs are attached to two points A and B. The point C lies between A and B such that ABC is a straight line and $AC \neq BC$.
The particle is held at C and then released from rest.
At time t seconds, the displacement of the particle from C is x m and its velocity is v ms^{-1}. The subsequent motion of the particle can be described by the differential equation $\ddot{x} = -25x$.
(a) Describe the motion of the particle.
Given that $x = 0.4$ and $v = 0$ when $t = 0$,
(b) solve the differential equation to find x as a function of t
(c) state the period of the motion and calculate the maximum speed of P.

Damped Harmonic Motion



We have seen so far that the extension/compression of the spring leads to a force, and hence an **acceleration, which is proportional to the displacement from some central point**, i.e. the more you stretch the spring, the greater the force, and hence the greater the acceleration. With this force alone, we saw this resulted in the displacement that follows a sine curve as time increases.

But we also know from practice that the amplitude gradually decreases over time (until perhaps the spring no longer oscillates), so there must be some other force at work.

This is known as the **damping force**, and is **proportional to velocity** (k is positive).
In contrast the force we previously saw, caused by the elasticity of the spring, is known as the **restoring force**.

Such motion is known as **damped harmonic motion**.

Note that k is positive

✎ For particle moving with damped harmonic motion:

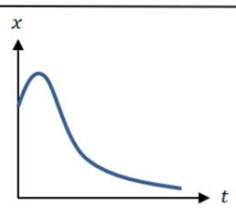
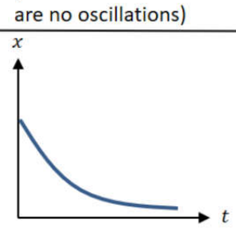
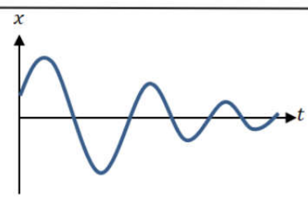
$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - \omega^2 x$$

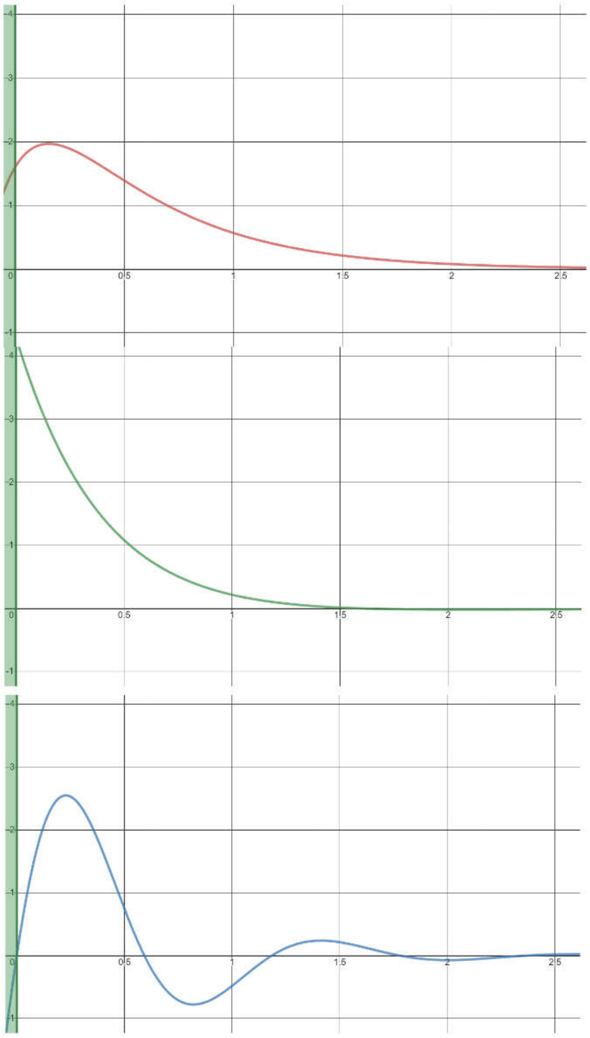
$$\Rightarrow \frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

The type of motion seen will depend on the roots to the auxiliary equation...

$$\boxed{\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0}$$

Auxiliary equation: $\boldsymbol{m^2 + km + \omega^2 = 0}$
The solution to the second-order differential equation depends on the number of roots (and hence the discriminant) of this auxiliary equation:

Roots of auxiliary:	**Distinct roots**: $k^2 - 4\omega^2 > 0$	**Equal roots**: $k^2 - 4\omega^2 = 0$	**No roots**: $k^2 - 4\omega^2 < 0$
Form of resulting solution to differential equation:	$x = Ae^{-\alpha t} + Be^{-\beta t}$	$x = (A + Bt)e^{-\alpha t}$	$x = Ae^{-at}\sin bt$
Type of damping:	**Heavy damping** (no oscillations)	**Critical damping** (the limit for which there are no oscillations)	**Light damping** (oscillates)
Sketch of x against t:			

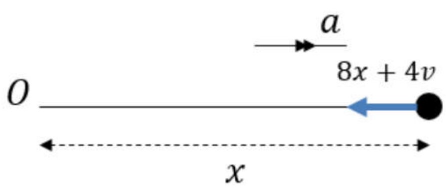


https://www.desmos.com/calculator/ksdarm3ftq

A particle P of mass 0.5 kg moves in a horizontal straight line. At time t seconds, the displacement of P from a fixed point, O, on the line is x m and the velocity of P is v ms^{-1}. A force of magnitude $8x$ N acts on P in the direction PO. The particle is also subject to a resistance of magnitude $4v$ N. When $t = 0, x = 1.5$ and P is moving in the direction of increasing x with speed 4 ms^{-1},

(a) Show that $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$

(b) Find the value of x when $t = 1$.



A particle P hangs freely in equilibrium attached to one end of a light elastic string. The other end of the string is attached to a fixed point A. The particle is now pulled down and held at rest in a container of liquid which exerts a resistance to motion on P. P is then released from rest. While the string remains taut and the particle in the liquid, the motion can be modelled using the equation

$\frac{d^2x}{dt^2} + 6k\frac{dx}{dt} + 5k^2x = 0$, where k is a positive real constant

Find the general solution to the differential equation and state the type of damping that the particle is subject to.

One end of a light elastic spring is attached to a fixed point A. A particle P is attached to the other end and hangs in equilibrium vertically below A. The particle is pulled vertically down from its equilibrium position and released from rest. A resistance proportional to the speed of P acts on P. The equation of motion of P is given as

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0$$

where k is a positive real constant and x is the displacement of P from its equilibrium position.

(a) Find the general solution to the differential equation.

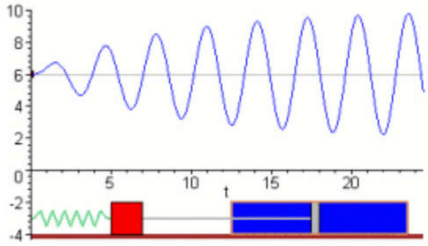
(b) Write down the period of oscillation in terms of k.

Ex 8C
Q1-5

Forced Harmonic Motion

In addition to the 'natural' forces acting on the particle, i.e. damping force and restoring force, there may be a further a further force acting on the particle.

This is known as **forced harmonic motion**.



Forced harmonic motion

$$\frac{dx^2}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$$

A particle P of mass 1.5 kg is moving on the x-axis. At time t the displacement of P from the origin O is x metres and the speed of P is v ms^{-1}. Three forces act on P, namely a restoring force of magnitude $7.5x$ N, a resistance to the motion of P of magnitude $6v$ N and a force of magnitude $12 \sin t$ N acting in the direction OP.

When $t = 0, x = 5$ and $\frac{dx}{dt} = 2$.

(a) Show that $\frac{dx^2}{dt^2} + 4\frac{dx}{dt} + 5x = 8 \sin t$

(b) Find x as a function of t.

(c) Describe the motion when t is large.

A particle P is attached to end A of a light elastic spring AB. Initially the particle and the string lie at rest on a smooth horizontal plane. At time $t = 0$, the end B of the string is set in motion and moves with constant speed U in the direction AB, and the displacement of P from A is x. Air resistance acting on P is proportional to its speed. The subsequent motion can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + k^2x = 2kU$$

Find an expression for x in terms of U, k and t

6. A damped spring is part of a car suspension system. In tests for the system, a mass is attached to the damped spring and is made to move upwards in a vertical line.

The motion of the system is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 2e^{-3t}$$

where x cm is the vertical displacement of the mass above its equilibrium position and t is the time, in seconds, after motion begins.

In one particular test, the mass is moved to a position 20 cm above its equilibrium position and given an initial velocity of 1 ms^{-1} upwards. For this test, use the model to

(a) find an equation for x in terms of t,

(9)

(b) find, to the nearest mm, the maximum displacement of the mass from its equilibrium position.

(3)

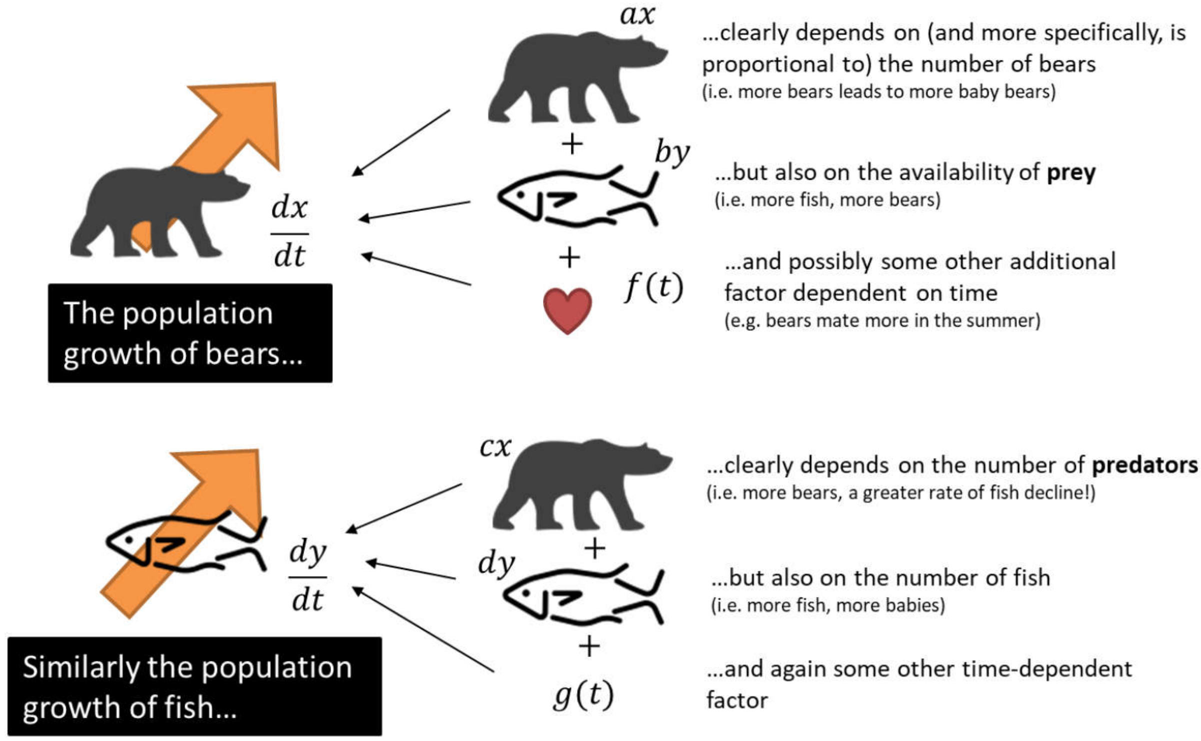
In this test, the time taken for the mass to return to its equilibrium position was measured as 2.86 seconds.

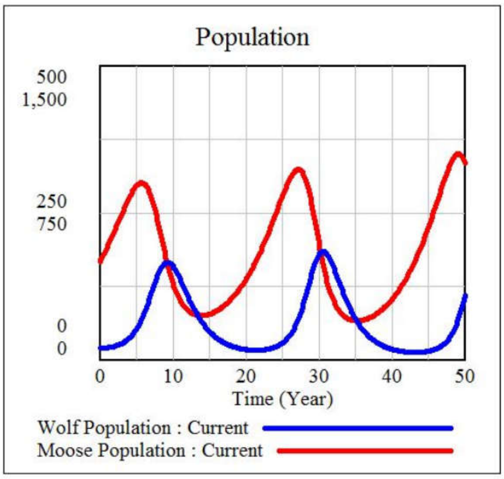
(c) State, with justification, whether or not this supports the model.

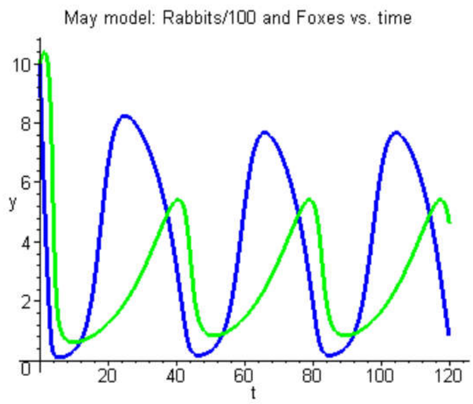
(1)

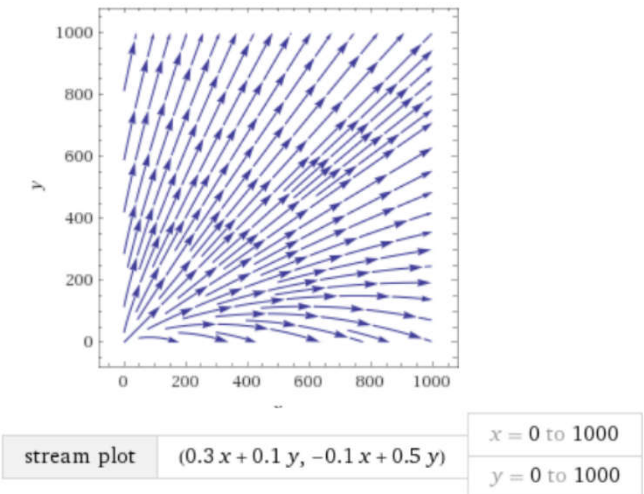
Coupled First-Order Linear Differential Equations

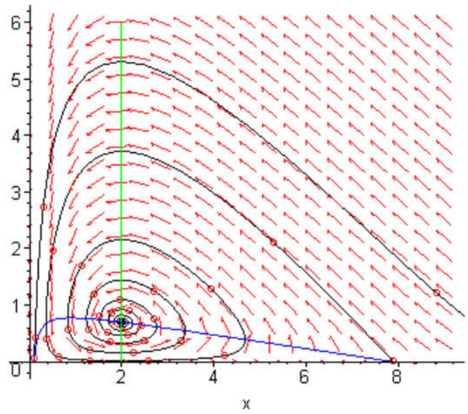
In Biology, **Lotka-Volterra equations**, also known as **predator-prey equations**, describe how two species interact, in terms of their populations.
Suppose there are x bears and y fish:











Coupled first-order linear differential equations:

$$\frac{dx}{dt} = ax + by + f(t)$$
$$\frac{dy}{dt} = cx + dy + g(t)$$

Homogeneous if $f(t) = g(t) = 0$ for all t.

Possible strategy to solve for x:

1. Make y the subject of first equation then differentiate to find $\frac{dy}{dt}$.
2. Substitute into second equation to get single **second-order** differential equation just in terms of x, and solve.
3. To solve for y, no need to repeat whole process. Differentiate x from Step 2 and sub x and $\frac{dx}{dt}$ into y from Step 1.

At the start of the year 2010, a survey began on the numbers of bears and fish on a remote island in Northern Canada. After t years the number of bears, x, and the number of fish, y, on the island are modelled by the differential equations

$$\frac{dx}{dt} = 0.3x + 0.1y \qquad (1)$$
$$\frac{dy}{dt} = -0.1x + 0.5y \qquad (2)$$

(a) Show that $\frac{d^2x}{dt^2} - 0.8\frac{dx}{dt} + 0.16x = 0$

(b) Find the general solution for the number of bears on the island at time t.

(c) Find the general solution for the number of fish on the island at time t.

(d) At the start of 2010 there were 5 bears and 20 fish on the island.
Use this information to find the number of bears predicted to be on the island in 2020.

(e) Comment on the suitability of the model.

Two barrels contain contaminated water. At time t seconds, the amount of contaminant in barrel A is x ml and the amount of contaminant in barrel B is y ml. Additional contaminated water flows into barrel A at a rate of 5ml per second. Contaminated water flows from barrel A to barrel B and from barrel B to barrel A through two connecting hoses, and drains out of barrel A to leave the system completely.
The system is modelled using the differential equations

$$\frac{dx}{dt} = 5 + \frac{4}{9}y - \frac{1}{7}x \quad (1)$$

$$\frac{dy}{dt} = \frac{3}{70}x - \frac{4}{9}y \quad (2)$$

Show that $630\frac{d^2y}{dt^2} + 370\frac{dy}{dt} + 28y = 135$

Use strategy as per previous slide, but now need to make x subject in (2) and sub into (1).

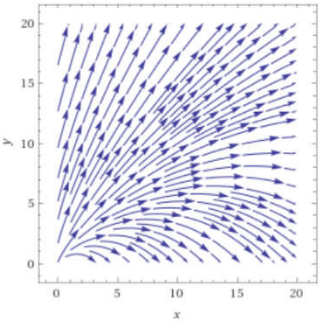
Your Turn - exam question

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r$$



stream plot	$(0.2x + 0.1y, -0.2x + 0.4y)$	$x = 0$ to 20
		$y = 0$ to 20

(a) Show that $\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\dfrac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

Your Turn - exam question

8. A doctor is studying the concentration of an antibiotic in the blood and the body tissue of a patient.
Let x be the number of micrograms of the antibiotic in the blood.
Let y be the number of micrograms of the antibiotic in the body tissue.
The doctor models her results by the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5x + y + 51$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 12x - 6y$$

where t is the time in hours after a dose of the antibiotic has been administered to the patient.

(a) Show that

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 11\frac{\mathrm{d}x}{\mathrm{d}t} + 18x = 306$$

(3)

(b) Find a general solution for the number of micrograms of the antibiotic in the blood at time t hours.

(6)

(c) Hence find a general solution for the number of micrograms of the antibiotic in the body tissue at time t hours.

(2)

Initially there is none of this antibiotic in the blood and none of this antibiotic in the body tissu

(d) Find, in minutes, to 2 decimal places, the time when the rate of increase of the antibiotic in the blood is equal to the rate of increase of the antibiotic in the body tissue.

(5)

(e) Evaluate the model.

(1)

Your Turn - exam question

9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 200\cos t, \qquad t \geqslant 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be $10\,\mathrm{ms^{-2}}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)