Integration (Year 1)

This chapter is roughly divided into two parts: the first, **indefinite integration**, is the **opposite of differentiation**. The second, **definite integration**, allows us to **find areas under graphs** (as well as surface areas and volumes) or areas between two graphs.

1:: Find y given $\frac{dy}{dx}$

A curve has the gradient function

$$\frac{dy}{dx} = 3x + 1$$

If the curve goes through the point $(2,3)$, determine y.

2:: Evaluate definite integrals, and hence the area under a curve.

Find the area bounded between the curve with equation $y = x^2 - 2x$ and the x-axis.

3:: Find areas bound between two different lines.

Find the points of intersection of $y = x^2 - 4x + 3$ and $y = x + 9$, and hence find the area bound between the two lines.

Integrating x^n terms

Integration is the **opposite of differentiation.**
(For this reason it is also called '*antidifferentiation*')

Function

$5x^3$

multiply by power → reduce power by 1

Gradient Function

$15x^2$

increase power by 1 → divide by <u>new</u> power

However, there's one added complication...

Find y when $\frac{dy}{dx} = 3x^2$

Adding 1 to the power and dividing by this power give us:

$$y = x^3$$

However, other functions would also have differentiated to $3x^2$:

$$y = x^3 + 1, \qquad y = x^3 - 4, \qquad \ldots$$

Clearly we could have had any constant, as it disappears upon differentiation.

$$\therefore \boldsymbol{y = x^3 + c}$$

c is known as a **constant of integration**

Find y when:

$$\frac{dy}{dx} = 4x^3$$

$y = \frac{4}{4}x^4 = x^4 + c$ $+c$

Exam Note: You should **always** include it for indefinite integration

$$\frac{dy}{dx} = x^5$$

$y = \frac{1}{6}x^6 + c$

You could also write as $\frac{x^6}{6}$. It's a matter of personal preference.

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$y = 3 \times \frac{2}{3}x^{3/2} + c = 2x^{3/2} + c$

$\cancel{\dfrac{3x^{3/2}}{3/2}}$

$\div \frac{3}{2}$

$\times \frac{2}{3}$ (multiply by the reciprocal)

Tip: Many students are taught to write $\frac{3x^{\frac{3}{2}}}{\frac{3}{2}}$ (as does textbook!). This is ugly and students then often struggle to simplify it. Instead remember back to GCSE: **When you divide by a fraction, you multiply by the reciprocal.**

Find y when:

When we divide by $\frac{1}{2}$ we multiply by the reciprocal, i.e. 2.
Tip: I recommend eventually doing this in your head when the simplification would be simple.

$$\frac{dy}{dx} = \frac{4}{\sqrt{x}} = 4x^{-1/2}$$

$$y = 4 \times 2x^{1/2} + c$$
$$y = 8x^{1/2} + c$$

$$\frac{dy}{dx} = 5x^{-2}$$

$$y = -5x^{-1} + c$$

$$\frac{dy}{dx} = 4x^{\frac{2}{3}}$$

$$y = 4 \times \frac{3}{5}x^{5/3} + c = \frac{12}{5}x^{5/3} + c$$

$$\frac{dy}{dx} = \frac{12}{\cancel{8}} \times \frac{\cancel{8}}{3} x^{2/3}$$
$$= 4x^{2/3}$$

$$\frac{dy}{dx} = 10x^{-\frac{2}{7}}$$

$$y = \overset{2}{\cancel{10}} \times \frac{7}{\cancel{5}} x^{5/7} + c$$
$$= 14x^{5/7} + c$$

Your Turn

$f'(x)$ is $f(x)$ differentiated
$\frac{dy}{dx}$ y

Find $f(x)$ when:

$f'(x) = 2x + 7$ $f(x) = x^2 + 7x + c$

$f'(x) = x^2 - 1$ $f(x) = \frac{1}{3}x^3 - x + c$

$f'(x) = \frac{2}{x^7} = 2x^{-7}$ $f(x) = -\frac{1}{3}x^{-6} + c$

$f'(x) = \sqrt[3]{x} = x^{1/3}$ $f(x) = \frac{3}{4}x^{4/3} + c$

$f'(x) = 33x^{\frac{5}{6}}$ $f(x) = \frac{33 \times 6}{11}x^{11/6} + c = 18x^{11/6} + c$

Note: In case you're wondering what happens if $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$, the problem is that after adding 1 to the power, we'd be dividing by 0. You will learn how to integrate $\frac{1}{x}$ in Year 2.

Exercise 13A

1 Find an expression for y when $\frac{dy}{dx}$ is the following:

a x^5 **b** $10x^4$ **c** $-x^{-2}$ **d** $-4x^{-3}$ **e** $x^{\frac{2}{3}}$ **f** $4x^{\frac{1}{2}}$

g $-2x^6$ **h** $x^{-\frac{1}{2}}$ **i** $5x^{-\frac{3}{2}}$ **j** $6x^{\frac{1}{3}}$ **k** $36x^{11}$ **l** $-14x^{-8}$

m $-3x^{-\frac{2}{3}}$ **n** -5 **o** $6x$ **p** $2x^{-0.4}$

2 Find y when $\frac{dy}{dx}$ is given by the following expressions. In each case simplify your answer.

a $x^3 - \frac{3}{2}x^{-\frac{1}{2}} - 6x^{-2}$ **b** $4x^3 + x^{-\frac{2}{3}} - x^{-2}$ **c** $4 - 12x^{-4} + 2x^{-\frac{1}{2}}$

d $5x^{\frac{2}{3}} - 10x^4 + x^{-3}$ **e** $-\frac{4}{3}x^{-\frac{4}{3}} - 3 + 8x$ **f** $5x^4 - x^{-\frac{3}{2}} - 12x^{-5}$

3 Find $f(x)$ when $f'(x)$ is given by the following expressions. In each case simplify your answer.

a $12x + \frac{3}{2}x^{-\frac{3}{2}} + 5$ **b** $6x^5 + 6x^{-7} - \frac{1}{6}x^{-\frac{7}{6}}$ **c** $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

d $10x^4 + 8x^{-3}$ **e** $2x^{-\frac{1}{3}} + 4x^{-\frac{5}{3}}$ **f** $9x^2 + 4x^{-3} + \frac{1}{4}x^{-\frac{1}{2}}$

E/P **4** Find y given that $\frac{dy}{dx} = (2x + 3)^2$. **(4 marks)**

Problem-solving

Start by expanding the brackets.

Exercise 13A

1 **a** $y = \frac{1}{6}x^6 + c$ **b** $y = 2x^5 + c$

c $y = x^{-1} + c$ **d** $y = 2x^{-2} + c$

e $y = \frac{3}{5}x^{\frac{5}{3}} + c$ **f** $y = \frac{8}{3}x^{\frac{3}{2}} + c$

g $y = -\frac{2}{7}x^7 + c$ **h** $y = 2x^{\frac{1}{2}} + c$

i $y = -10x^{-\frac{1}{2}} + c$ **j** $y = \frac{9}{2}x^{\frac{4}{3}} + c$

k $y = 3x^{12} + c$ **l** $y = 2x^{-7} + c$

m $y = -9x^{\frac{1}{3}} + c$ **n** $y = -5x + c$

o $y = 3x^2 + c$ **p** $y = \frac{10}{3}x^{0.6} + c$

$x^2 = \frac{1}{3}x^3 + c$

$\frac{dy}{dx} = x^2$

$y = \frac{1}{3}x^3 + c$

2 **a** $y = \frac{1}{4}x^4 - 3x^{\frac{1}{2}} + 6x^{-1} + c$ **b** $y = x^4 + 3x^{\frac{1}{3}} + x^{-1} + c$

c $y = 4x + 4x^{-3} + 4x^{\frac{1}{2}} + c$ **d** $y = 3x^{\frac{4}{3}} - 2x^5 - \frac{1}{2}x^{-2} + c$

e $y = 4x^{-\frac{1}{3}} - 3x + 4x^2 + c$ **f** $y = x^5 + 2x^{-\frac{1}{2}} + 3x^{-4} + c$

3 **a** $f(x) = 6x^2 - 3x^{-\frac{1}{2}} + 5x + c$ **b** $f(x) = x^6 - x^{-6} + x^{-\frac{1}{6}} + c$

c $f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + c$ **d** $f(x) = 2x^5 - 4x^{-2} + c$

e $f(x) = 3x^{\frac{2}{3}} - 6x^{-\frac{2}{3}} + c$

f $f(x) = 3x^3 - 2x^{-2} + \frac{1}{2}x^{\frac{1}{2}} + c$

4 $y = \dfrac{4x^3}{3} + 6x^2 + 9x + c$

Integration notation

The following notation could be used to differentiate an expression:

The dx here means differentiating "with respect to x".

$$\frac{d}{dx}(5x^2) = 10x$$

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

$y = x^2$

$\frac{dy}{dx} = 2x$

There is similarly notation for integrating an expression:

$$\int 10x \ dx = 5x^2 + c$$

"Integrate..."

"...this expression"

"...with respect to x"

(the dx is needed just as it was needed in the differentiation notation at the top of this slide)

This is known as **indefinite integration**, in contrast to definite integration, which we'll see later in the chapter.

It is called 'indefinite' because the exact expression is unknown (due to the $+c$).

Note: The brackets are required if there's multiple terms.

Find $\int (x^{-\frac{3}{2}} + 2)\, dx$

$\int (x^{-3/2} + 2)\, dx = -2x^{-1/2} + 2x + c$

Find $\int (6t^2 - 1)\, dt$

Note the dt instead of dx.

$\int (6t^2 - 1)\, dt = 2t^3 - t + c$

Find $\int (px^3 + q)\, dx$ where p and q are constants.

behaves like a number, not a variable

$\int (px^3 + q)\, dx = \frac{p}{4}x^4 + qx + c$

or $= \frac{1}{4}px^4 + qx + c$

Edexcel C1 May 2014(R) Q4b

Given that $y = 2x^5 + \frac{6}{\sqrt{x}}$, $x > 0$, find in their simplest form

(b) $\int y \, dx$ **(3)**

Ex 13B

Q4, 5, 6, 7

$$\int y \, dx = \int (2x^5 + 6x^{-1/2}) \, dx$$

$$= \frac{1}{3}x^6 + 12x^{1/2} + c$$

4 Find the following integrals:

a $\int(4x^3 - 3x^{-4} + r)\mathrm{d}x$

b $\int(x + x^{-\frac{1}{2}} + x^{-\frac{3}{2}})\mathrm{d}x$

c $\int(px^4 + 2t + 3x^{-2})\mathrm{d}x$

Hint In **Q4** part **c** you are integrating with respect to x, sc treat p and t as constants.

5 Find the following integrals:

a $\int(3t^2 - t^{-2})\mathrm{d}t$

b $\int(2t^2 - 3t^{-\frac{3}{2}} + 1)\mathrm{d}t$

c $\int(pt^3 + q^2 + px^3)\mathrm{d}t$

6 Find the following integrals:

a $\int\frac{(2x^3 + 3)}{x^2}\mathrm{d}x$

b $\int(2x + 3)^2\,\mathrm{d}x$

c $\int(2x + 3)\sqrt{x}\,\mathrm{d}x$

$x^{1/2}(2x+3)$

$2x^{3/2} + 3x^{1/2}$

7 Find $\int\mathrm{f}(x)\mathrm{d}x$ when $\mathrm{f}(x)$ is given by the following:

a $\left(x + \frac{1}{x}\right)^2$

b $(\sqrt{x} + 2)^2$

c $\left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right)$

$\frac{2x^3}{x^2} + \frac{3}{x^2}$

$2x + 3x^{-2}$

Each term must be in the form kx^n before integrating!

Homework Questions

6 Find the following integrals:

a $\int \frac{(2x^3 + 3)}{x^2} \, dx$ **b** $\int (2x + 3)^2 \, dx$ **c** $\int (2x + 3)\sqrt{x} \, dx$

7 Find $\int f(x)dx$ when $f(x)$ is given by the following:

a $\left(x + \frac{1}{x}\right)^2$ **b** $(\sqrt{x} + 2)^2$ **c** $\left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right)$

8 Find the following integrals:

a $\int \left(x^{\frac{2}{3}} + \frac{4}{x^3}\right) dx$ **b** $\int \left(\frac{2 + x}{x^3} + 3\right) dx$ **c** $\int (x^2 + 3)(x - 1)dx$

d $\int \frac{(2x + 1)^2}{\sqrt{x}} \, dx$ **e** $\int \left(3 + \frac{\sqrt{x} + 6x^3}{x}\right) dx$ **f** $\int \sqrt{x}(\sqrt{x} + 3)^2 \, dx$

9 Find the following integrals:

a $\int \left(\frac{A}{x^2} - 3\right) dx$ **b** $\int \left(\sqrt{Px} + \frac{2}{x^3}\right) dx$ **c** $\int \left(\frac{p}{x^2} + q\sqrt{x} + r\right) dx$

10 Given that $f(x) = \frac{6}{x^2} + 4\sqrt{x} - 3x + 2$, $x > 0$, find $\int f(x)dx$.