Trigonometric Identities and Equations

1:: Know exact trig values for $30°, 45°, 60°$ and understand unit circle.

2:: Use identities $\frac{\sin x}{\cos x} \equiv \tan x$ and $\sin^2 x + \cos^2 x \equiv 1$

Show that $3\sin^2 x + 7\sin x = \cos^2 x - 4$ can be written in the form $4\sin^2 x + 7\sin x + 3 = 0$

3:: Solve equations of the form $\sin(n\theta) = k$ and $\sin(\theta \pm \alpha) = k$

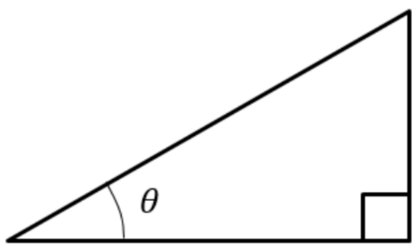
Solve $\sin(2x)(\cos 2x + 1) = 0$, for $0 \leq x < 360°$.

4:: Solve equations which are quadratic in sin/cos/tan.

Solve, for $0 \leq x < 360°$, the equation

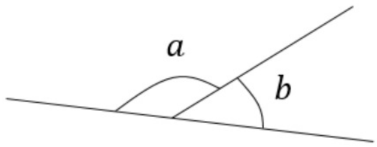
$$4\sin^2 x + 9\cos x - 6 = 0$$

RECAP :: What actually ***are*** the trigonometric ratios?

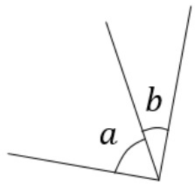


RECAP :: Co-function identity

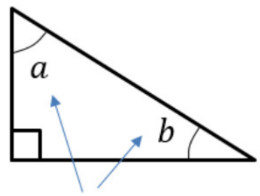
Have you ever wondered why "cosine" contains the word "sine"?



Supplementary Angles add to 180°

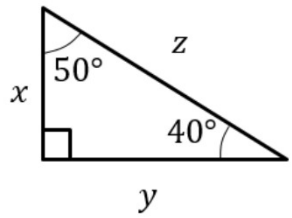


Complementary Angles add to 90°



Therefore these angles are complementary.

i.e. The **cosine** of an angle is the **sine** of the **complementary** angle.
Hence **cosine = COMPLEMENTARY SINE**



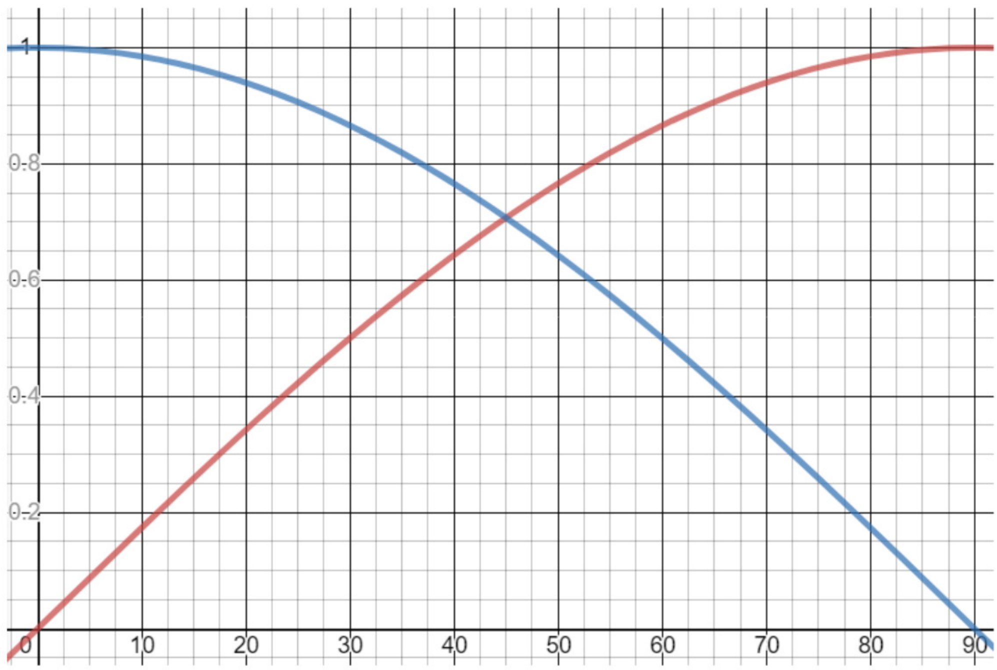
$\cos(50) =$

$\sin(40) =$

Cofunction identity

$$sin\theta = \cos(90 - \theta)$$

$$cos\theta = \sin(90 - \theta)$$

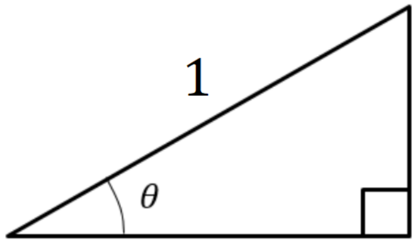


Cofunction identity

$$sin\theta = \cos(90 - \theta)$$

$$cos\theta = \sin(90 - \theta)$$

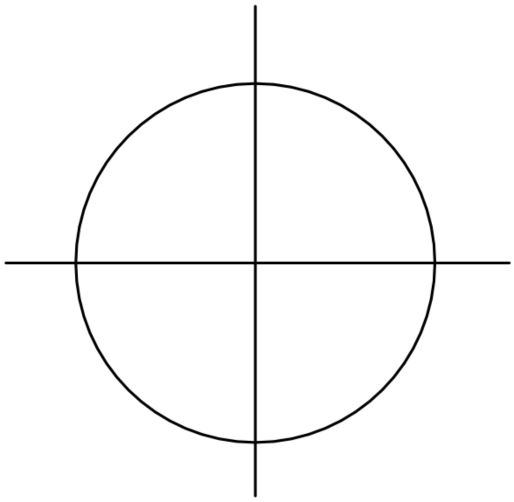
Considering a triangle with hypotenuse length 1 unit



How are are the sine, cosine and tangent functions related to this triangle?

Different triangles - compare sine, cosine and tangent of the angle

The Unit Circle - going beyond 90°

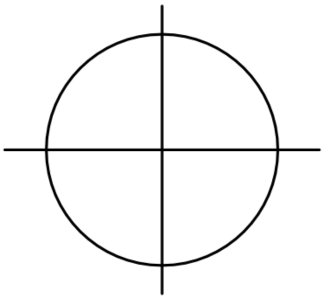
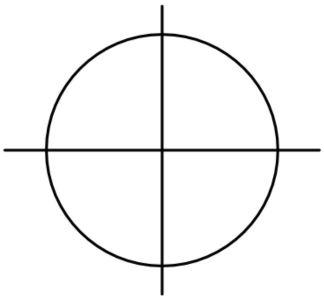


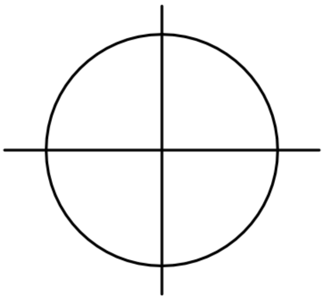
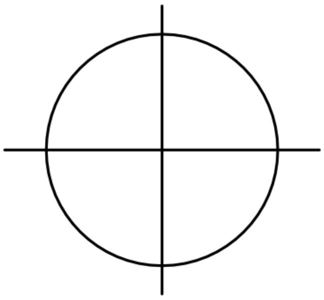
Angles are measured anticlockwise from the positive x-axis.

Quadrants are the 4 'sections' of the graph. They are similarly labelled anticlockwise

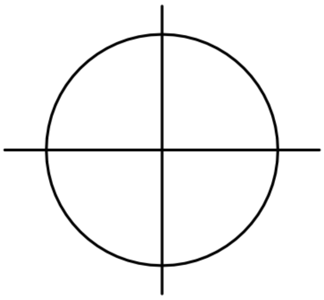
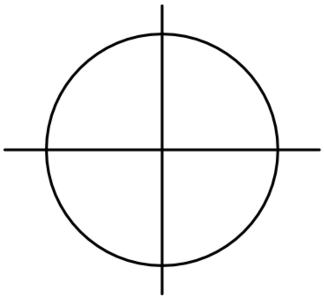
What can we say about the sine, cosine, and tangents of obtuse angles?

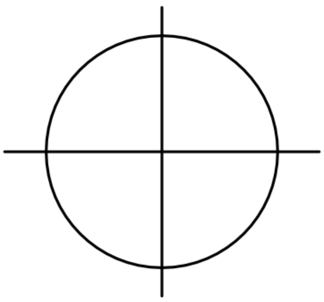
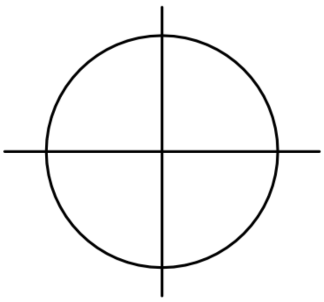
Exploring the Unit Circle - analysing multiples of 90°

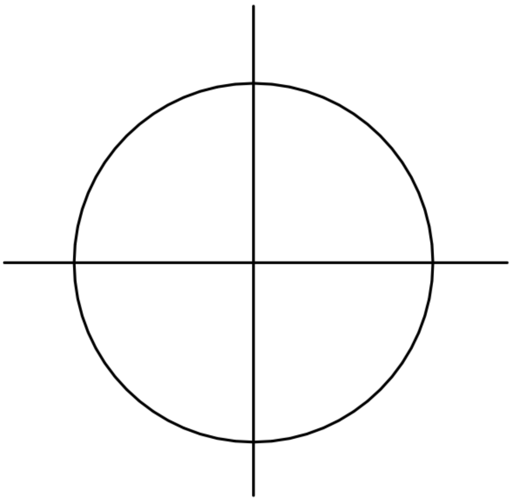



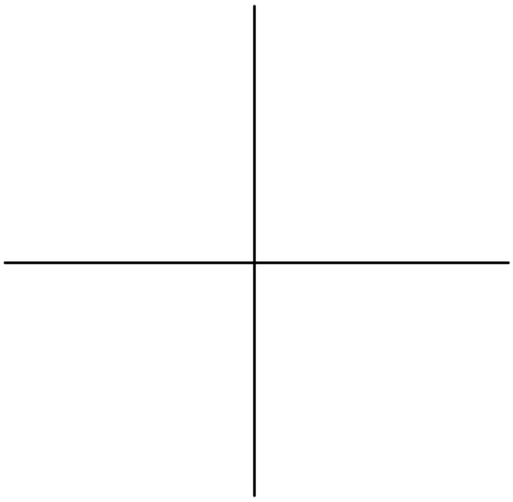
Exploring the Unit Circle - analysing the quadrants

Exploring the Unit Circle - negative angles





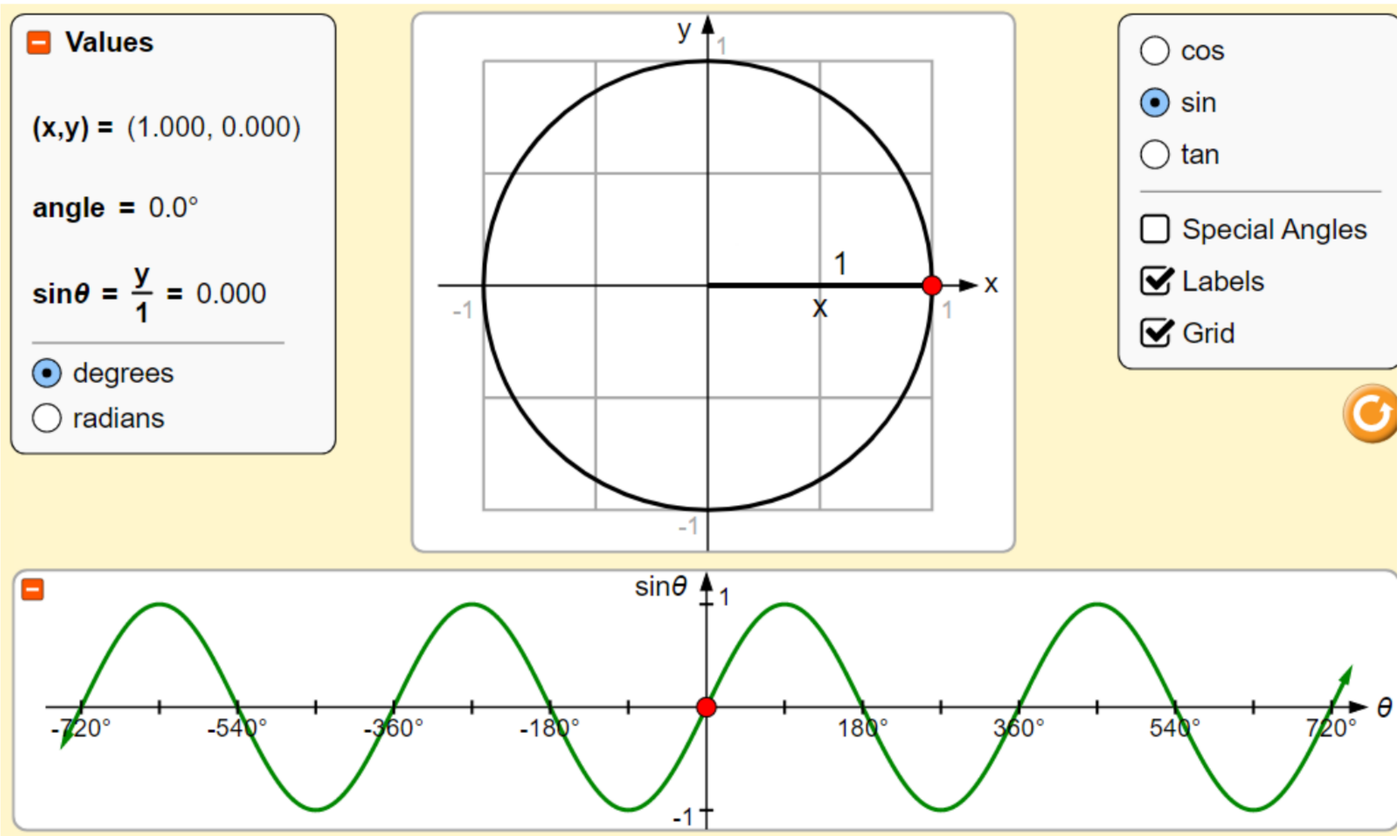
What can we say about

a) -50°

b) -120°

c) -270°

Exploring the Unit Circle - links with the graphs



https://phet.colorado.edu/sims/html/trig-tour/latest/trig-tour_en.html

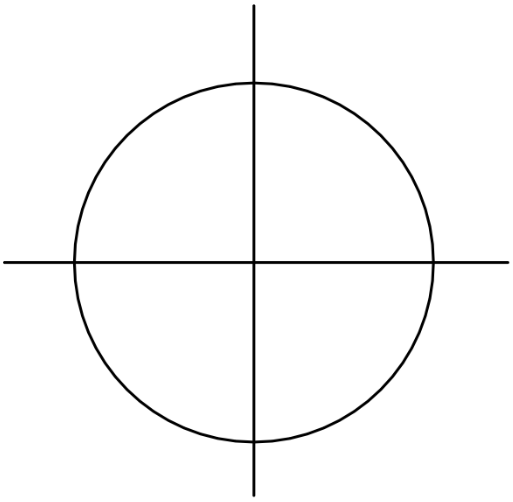
Search: 'phet interactive unit circle'

Writing trig functions as acute angle made with x-axis

The sine/cosine/tangent of **any positive or negative angle** can be expressed as the sine/cosine/tangent of a **positive acute angle**

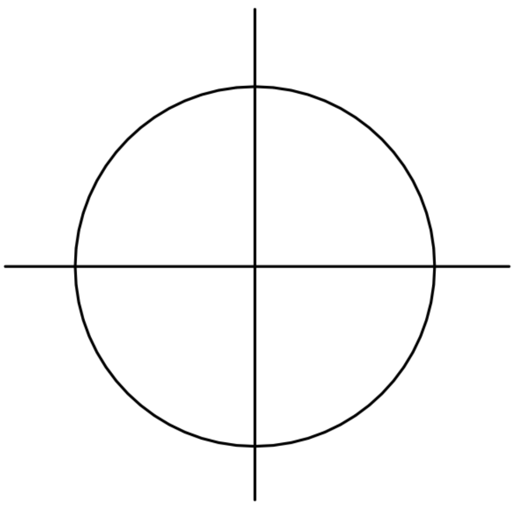
Let's look at, for example, $\sin 50^{\circ}$

Where else on the unit circle appears to relate to $\sin 50^{\circ}$?

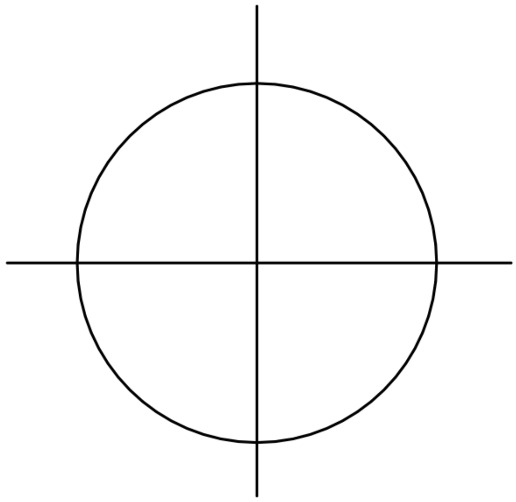


Let's look at, for example, $\cos 50^{\circ}$

Where else on the unit circle appears to relate to $\cos 50^{\circ}$?



Let's look at, for example, $\tan 50^{\circ}$
Where else on the unit circle appears to relate to $\tan 50^{\circ}$?



Examples

These will help with Ex 10A Q3

Without a calculator, write down the values of:

i. $\sin 180°$

ii. $\tan 270°$

iii. $\cos 540°$

Examples

These will help with Ex 10A Q4

Express the following in terms of trigonometric ratios of positive acute angles:

i. $sin\,150°$

ii. $sin(-25°)$

Express the following in terms of trigonometric ratios of positive acute angles:

iii. $cos\,480°$

iv. $cos\,130°$

Express the following in terms of trigonometric ratios of positive acute angles:

v. $tan\,110°$

vi. $tan(-150)°$

Examples

These will help with Ex 10A Q5 and Q6

Given that θ is an acute angle, express in terms of sin θ

i. $sin(180+\theta)°$

ii. $sin(-180-\theta)°$

Given that θ is an acute angle, express in terms of cos θ

iii. $cos(-\theta)°$

iv. $cos(\theta - 540)°$

Given that θ is an acute angle, express in terms of tan θ

v. $tan(-\theta)°$

vi. $tan(-180 + \theta)°$

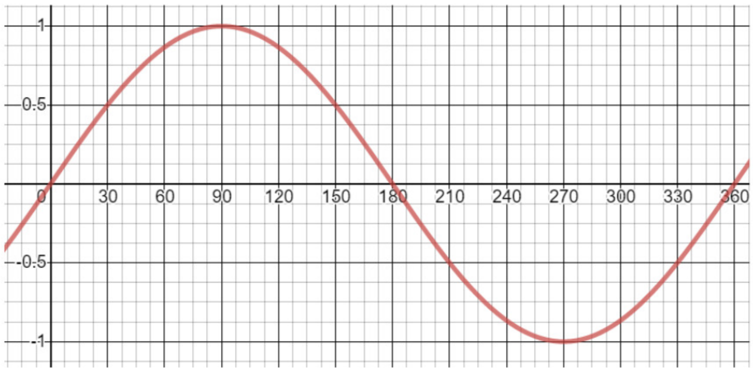
sin/cos/tan of $30°, 45°, 60°$

You will frequently encounter angles of $30°, 60°, 45°$ in geometric problems.
This is because we see these angles in equilateral triangles and half squares.

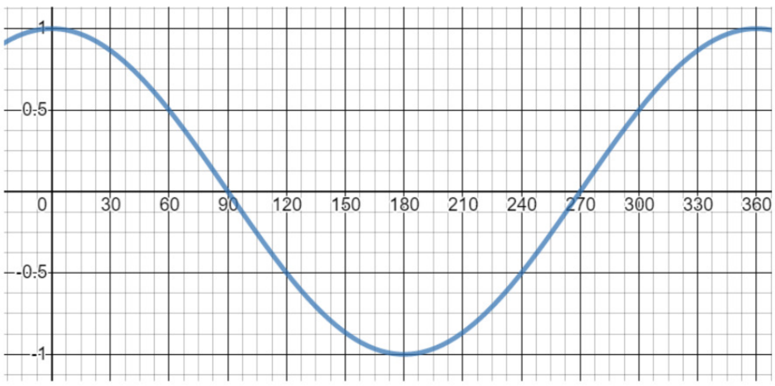
Although you will always have a calculator, you need to know how to derive these.
All you need to remember:
✏ **Draw half a unit square and half an equilateral triangle of side 2.**

A Few Trigonometric Angle Laws

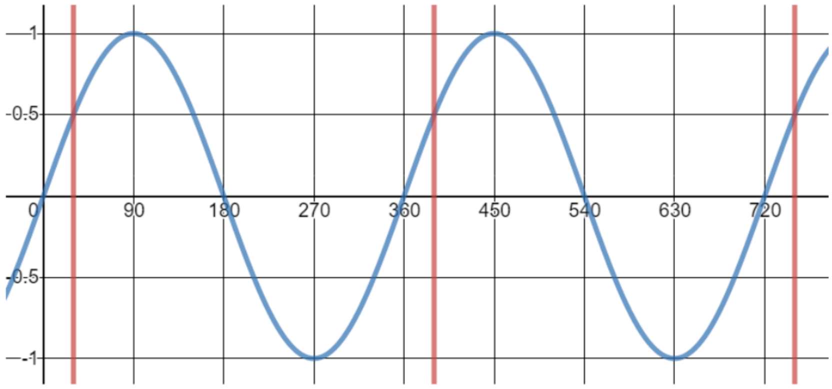
- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every 360° but tan every 180°



- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every 360° but tan every 180°



- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every $360°$ but tan every $180°$



Without a calculator, work out the value of each below.

$\tan(225°) =$

$\tan(210°) =$

$\sin(150°) =$

$\cos(300°) =$

$\sin(-45°) =$

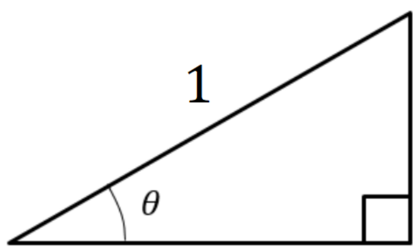
$\cos(750°) =$

$\cos(120°) =$

Ex 10B

Trigonometric Identities

What else can we learn from this triangle?



$\sin^2\theta$ is a shorthand for $(\sin\theta)^2$. It does NOT mean the sin is being squared – this does not make sense as sin is a function, and not a quantity that we can square!

Using the Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

Given that $\sin\alpha = \frac{2}{5}$ and that α is obtuse, find the exact value of $\cos\alpha$ and $\tan\alpha$

Given that $\cos\theta = -\frac{3}{5}$ and that θ is reflex, find the exact value of $\sin\theta$ and $\tan\theta$

Proof

$$\tan x = \frac{\sin(x)}{\cos(x)}$$
$$\sin^2 x + \cos^2 x = 1$$

Prove that $1 - \tan\theta \sin\theta \cos\theta \equiv \cos^2\theta$

Tip #1: Turn any tan's into sin's and cos's.

Recall that $\equiv$ means 'equivalent to', and just means the LHS is **always** equal to the RHS for all values of θ.
Usually the best method is to manipulate one side (e.g. LHS) until we get to the other (RHS).

Prove that $\tan\theta + \frac{1}{\tan\theta} \equiv \frac{1}{\sin\theta\cos\theta}$

Tip #2: In any addition/subtraction involving at least one fraction (with trig functions), always combine algebraically into one.

Simplify $5 - 5\sin^2(3\theta)$

Tip #3: Look out for $1 - \sin^2\theta$ and $1 - \cos^2\theta$. Students often don't spot that these can be simplified.

Prove that $\frac{\tan x \cos x}{\sqrt{1-\cos^2 x}} \equiv 1$

Prove that $\frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$

Prove that $\tan^2\theta \equiv \frac{1}{\cos^2\theta} - 1$

Solving Trigonometric Equations

Reminder of 'trig laws':
- $\sin(x) = \sin(180 - x)$
- $\cos(x) = \cos(360 - x)$
- sin, cos repeat every $360°$ but tan every $180°$

Solve $\cos\theta = \cos 20°$ in the interval $0 \leq \theta \leq 360°$.

Solve $\sin\theta = \frac{1}{2}$ in the interval $0 \leq \theta \leq 360°$.

Calculator Note:
When you do sin^{-1}, cos^{-1} and tan^{-1} on a calculator, it gives you only one value, known as the **principal value**.

Solve $5\tan\theta = 10$ in the interval $-180° \leq \theta < 180°$

Tip: Look out for the solution range required. $-180 \leq \theta < 180°$ is a particularly common one.

Solve $\sin\theta = -\frac{1}{2}$ in the interval $0 \leq \theta \leq 360°$.

Solve $\sin\theta = \sqrt{3}\cos\theta$ in the interval $0 \leq \theta \leq 360°$.

Your Turn

Solve $2\cos\theta = \sqrt{3}$ in the interval $0 \leq \theta \leq 360°$.

Solve $\sqrt{3}\sin\theta = \cos\theta$ in the interval $-180° \leq \theta \leq 180°$.

Ex 10D

Harder Equations

Harder questions replace the angle θ with a linear expression.

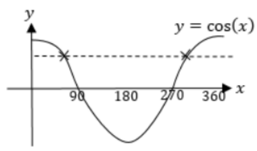
Solve $\cos 3x = -\frac{1}{2}$ in the interval $0 \leq x \leq 360°$.

STEP 1: Adjust the range of values for θ to match the expression inside the cos.

STEP 2: Immediately after applying an inverse trig function (and BEFORE dividing by 3!), find all solutions up to the end of the interval.

STEP 3: Then do final manipulation to each value.

Reflections: As mentioned before, in general you tend to get a pair of values per $360°$ (for any of sin/cos/tan), except for $\cos\theta = \pm 1$ or $\sin\theta = \pm 1$:

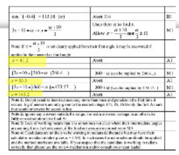


Thus once getting your first pair of values (e.g. using $\sin(180-\theta)$ or $\cos(360-\theta)$ to get the second value), keep adding $360°$ to generate new pairs.

Solve $\sin(2x + 30°) = \frac{1}{\sqrt{2}}$ in the interval $0 \leq x \leq 360°$.

Solve $\sin\frac{1}{2}x = 2\cos\frac{1}{2}x$ in the interval $0 \leq x < 1080°$

Your Turn



Solve, for $0 \leq x < 180°$,

$$\cos(3x - 10°) = -0.4,$$

giving your answers to 1 decimal place. You should show each step in your working.

(7)

Quadratics in sin/cos/tan

We saw that an equation can be 'quadratic in' something, e.g. $x - 2\sqrt{x} + 1 = 0$ is 'quadratic in $\sqrt{x}$', meaning that $\sqrt{x}$ could be replaced with another variable, say y, to produce a quadratic equation $y^2 - 2y + 1 = 0$.

Solve $5\sin^2 x + 3\sin x - 2 = 0$ in the interval $0 \leq x \leq 360°$.

Solve $\tan^2 \theta = 4$ in the interval $0 \leq x \leq 360°$.

Solve $2\cos^2 x + 9\sin x = 3\sin^2 x$ in the interval $-180° \leq x \leq 180°$.

Solve $2\cos^2 2x - \cos 2x - 1 = \sin^2 2x$ in the interval $0° \leq x \leq 180°$.

Your Turn

(*a*) Show that the equation

$$5 \sin x = 1 + 2 \cos^2 x$$

can be written in the form

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(2)

(*b*) Solve, for $0 \leq x < 360°$,

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(4)

(a)	$5\sin x = 1 + 2(1 - \sin^2 x)$	M1
	$2\sin^2 x + 5\sin x - 3 = 0$ (*)	A1cso (2)
(b)	$(2s-1)(s+3) = 0$ **giving** $x =$	M1
	$[\sin x = -3$ has no solution$]$ **so** $\sin x = \frac{1}{2}$	A1
	$\therefore\ x = 30,\ 150$	B1, B1ft (4) [6]

Your Turn - Exam Questions

Question	Scheme	Marks	AOs
9	Uses $\sin^2 x = 1 - \cos^2 x \Rightarrow 12(1-\cos^2 x) + 7\cos x - 13 = 0$	M1	3.1a
	$\Rightarrow 12\cos^2 x - 7\cos x + 1 = 0$	A1	1.1b
	Uses solution of quadratic to give $\cos x =$	M1	1.1b
	Uses inverse cosine on their values, giving two correct follow through values (see note)	M1	1.1b
	$\Rightarrow x = 430.5°, 435.5°$	A1	1.1b
			(5 marks)

9. Solve, for $360° \leqslant x < 540°$,

$$12\sin^2 x + 7\cos x - 13 = 0$$

Give your answers to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

11. (i) Solve, for $-90° \leq \theta < 270°$, the equation,

$$\sin(2\theta + 10°) = -0.6,$$

giving your answers to one decimal place.

(5)

Question	Scheme	Marks	AOs
11(i)	$(2\theta + 10°) = \arcsin(-0.6)$	M1	1.1b
	$(2\theta + 10°) = -143.13°, -36.87°, 216.87°, 323.13°$ (Any two)	A1	1.1b
	Correct order to find $\theta = \ldots$	dM1	1.1b
	Two of $\theta = -76.6°, -23.4°, 103.4°, 156.6°$.	A1	1.1b
	$\theta = -76.6°, -23.4°, 103.4°, 156.6°$, only	A1	2.1
		(5)	

12. (a) Show that the equation

$$4\cos\theta - 1 = 2\sin\theta\tan\theta$$

can be written in the form

$$6\cos^2\theta - \cos\theta - 2 = 0$$

(4)

(b) Hence solve, for $0 \leqslant x < 90°$

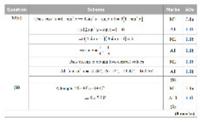
$$4\cos 3x - 1 = 2\sin 3x\tan 3x$$

giving your answers, where appropriate, to one decimal place.
(*Solutions based entirely on graphical or numerical methods are not acceptable*.)

(4)

Question	Scheme	Marks	AOs
12 (a)	$4\cos\theta - 1 = 2\sin\theta\tan\theta \rightarrow 4\cos\theta - 1 = 2\sin\theta \times \frac{\sin\theta}{\cos\theta}$	M1	1.2
	$\Rightarrow 4\cos^2\theta - \cos\theta = 2\sin^2\theta$ oe	A1	1.1b
	$\rightarrow 4\cos^2\theta - \cos\theta = 2(1 - \cos^2\theta)$	M1	1.1b
	$6\cos^2\theta - \cos\theta - 2 = 0$ *	A1*	2.1
		(4)	
(b)	For attempting to solve given quadratic	M1	1.1b
	$(\cos 3x) = \frac{2}{3}, -\frac{1}{2}$	B1	1.1b
	$x = \frac{1}{3}\arccos\left(\frac{2}{3}\right)$ or $\frac{1}{3}\arccos\left(-\frac{1}{2}\right)$	M1	1.1b
	$x = 40°, 80°$, awrt $16.1°$	A1	2.2a
		(4)	
			(8 marks)

12. (a) Solve, for $-180° \leqslant x < 180°$, the equation



$$3 \sin^2 x + \sin x + 8 = 9 \cos^2 x$$

giving your answers to 2 decimal places.

(6)

(b) Hence find the smallest positive solution of the equation

$$3\sin^2(2\theta - 30°) + \sin(2\theta - 30°) + 8 = 9 \cos^2(2\theta - 30°)$$

giving your answer to 2 decimal places.

(2)