9 The straight lines l_1 and l_2 have vectors equations $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + \lambda(8\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j})$ respectively, and P is the point with coordinates $(1, 4, 2)$.

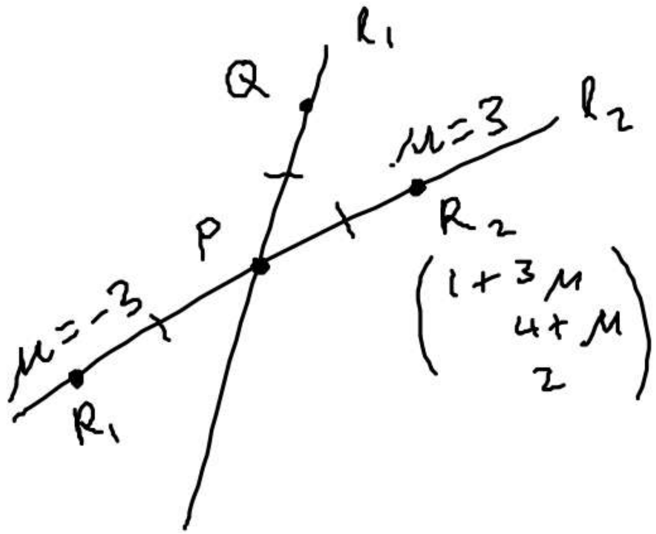
a Show that the point $Q(9, 9, 3)$ lies on l_1.

Given that l_1 and l_2 intersect, find:

b the cosine of the acute angle between l_1 and l_2

c the possible coordinates of the point R, such that R lies on l_2 and $PQ = PR$.

$$\overrightarrow{PQ} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} 1+3\mu \\ 4+\mu \\ 2 \end{pmatrix}$$

$$|\overrightarrow{PQ}| = \sqrt{64+25+1}$$

$$= \sqrt{90}$$

$$\overrightarrow{PR} = \underline{r} - \underline{p} = \begin{pmatrix} 3\mu \\ \mu \\ 0 \end{pmatrix}$$

$$|\overrightarrow{PR}| = \sqrt{90}$$

$$\sqrt{9\mu^2 + \mu^2} = \sqrt{90}$$

$$10\mu^2 = 90$$

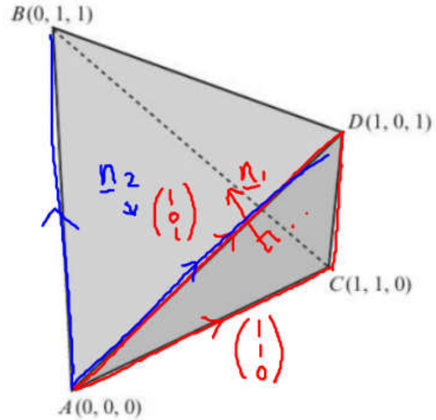
$$\mu^2 = 9$$

$$\mu = \pm 3$$

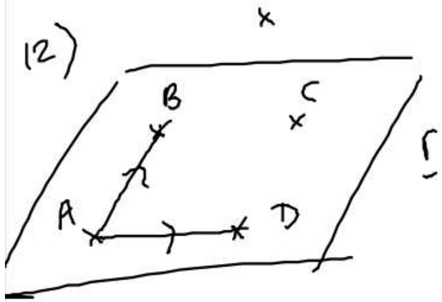
12 a Show that the points $A(3, 5, -1)$, $B(2, -2, 4)$, $C(4, 3, 0)$ and $D(1, 4, -3,)$ are not coplanar. **(6 marks)**

b Find the angle between the plane containing A, B and C and the line segment AD. **(4 marks)**

13 A regular tetrahedron has vertices A, B, C and D, with coordinates $(0, 0, 0)$, $(0, 1, 1)$, $(1, 1, 0)$ and $(1, 0, 1)$ respectively. Show that the angle between any two adjacent faces of the tetrahedron is $\arccos\left(\frac{1}{3}\right)$. **(7 marks)**



12)



$\cos^{-1}\left(\frac{1}{3}\right)$

$\cos\theta = \frac{1}{3}$

$\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$

coplanar, find an equation of a plane, containing 3 of the 4 points, investigate if 4th point is on that plane.

13) $\underline{n}_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$ $\quad \underline{n}_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$ $\quad \underline{n}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \underline{r} \cdot \underline{n}_1 = \underline{a} \cdot \underline{n}$

$x + z = 0$ $\quad x + y = 0$ $\quad -x + y + z = 0$

Let $z = 1$ $\quad x = -1$ $\quad y = 1$

$\underline{n}_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$ $\quad \underline{n}_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$

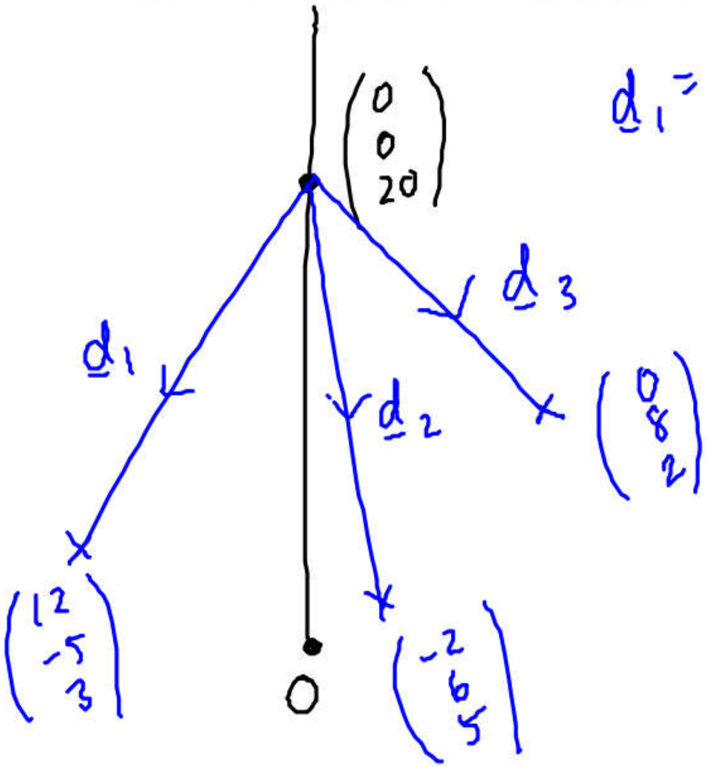
$y + z = 0$ $\quad x + z = 0$ $\quad$ Let $z = 1$, $x = -1$, $y = -1$ $\quad \underline{n}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|} = \frac{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{3} \times \sqrt{3}} = \frac{1}{3}$$

$\cos\theta = \frac{1}{3}$

$\theta = \arccos\frac{1}{3}$

14 A flagpole is supported by 3 guide ropes which are attached at a point 20 m above the base of the pole. The ends of the ropes are secured at points with position vectors (0, 8, 2), (12, −5, 3) and (−2, 6, 5) relative to the base of the pole, where the units are metres.
The flagpole will be stable if the angles between adjacent guide ropes are all greater than 15°.
Determine whether the flagpole will be stable, showing your working clearly. **(7 marks)**



$\underset{\sim}{d}_1 = \begin{pmatrix} 12 \\ -5 \\ -17 \end{pmatrix}$

$\underset{\sim}{d}_2 = \begin{pmatrix} -2 \\ 6 \\ -15 \end{pmatrix}$

$\underset{\sim}{d}_3 = \begin{pmatrix} 0 \\ 8 \\ -18 \end{pmatrix}$

$$\cos\theta = \frac{\underset{\sim}{d}_1 \cdot \underset{\sim}{d}_2}{|\underset{\sim}{d}_1||\underset{\sim}{d}_2|}$$

Intersection of two lines

The lines l_1 and l_2 have vector equations
$\boldsymbol{r} = 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + \lambda(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$ and
$\boldsymbol{r} = -2\boldsymbol{j} + 3\boldsymbol{k} + \mu(-5\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$ respectively.
Show that the two lines intersect, and find the position vector of the point of intersection.

$$\begin{pmatrix} 3 + \lambda \\ 1 - 2\lambda \\ 1 - \lambda \end{pmatrix} = \begin{pmatrix} -5\mu \\ -2 + \mu \\ 3 + 4\mu \end{pmatrix}$$

$\underline{i}$ comp.
$$3 + \lambda = -5\mu$$
$$\lambda = -5\mu - 3$$

$\underline{j}$ comp.
$$1 - 2\lambda = -2 + \mu$$
$$1 - 2(-5\mu - 3) = -2 + \mu$$
$$1 + 10\mu + 6 = -2 + \mu$$
$$9\mu = -9$$
$$\mu = -1$$
$$\lambda = -5 \times -1 - 3$$
$$\lambda = 2$$

Check $\underline{k}$ comp
$$1 - \lambda = 3 + 4\mu$$
$$1 - 2 = 3 - 4$$
$$-1 = -1$$

So true for $\underline{k}$ comp.

Hence, lines intersect when $\lambda = 2$, $\mu = -1$

intersection point $\begin{pmatrix} 3 + 2 \\ 1 - 4 \\ 1 - 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} -5 \times -1 \\ -2 - 1 \\ 3 - 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$$

The lines l_1 and l_2 have equations $\frac{x-2}{4} = \frac{y+3}{2} = \frac{z-1}{1}$ and $\frac{x+1}{5} = \frac{y}{4} = \frac{z-4}{-2}$ respectively.
Prove that l_1 and l_2 are skew.

Terminology: Two straight lines are skew lines if they do not intersect and are not parallel

3D
- intersect
- parallel
- skew
- (same line)

investigate directions

$l_1 \quad \underline{d}_1 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \qquad l_2 \quad \underline{d}_2 = \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}$

Clearly, these are not multiples, so are not parallel.

investigate intersection

$l_1 \quad \underline{r} = \begin{pmatrix} 2 + 4\lambda \\ -3 + 2\lambda \\ 1 + \lambda \end{pmatrix} \qquad l_2 \quad \underline{r} = \begin{pmatrix} -1 + 5\mu \\ 4\mu \\ 4 - 2\mu \end{pmatrix}$

$\underline{i}$ comp. $\quad 2 + 4\lambda = -1 + 5\mu$

$\underline{j}$ comp. $\quad -3 + 2\lambda = 4\mu$

$4\lambda = 8\mu + 6$

$2 + 8\mu + 6 = -1 + 5\mu$

$3\mu = -9$

$\mu = -3$

$4\lambda = -24 + 6$

$4\lambda = -18$

$\lambda = -\frac{9}{2}$

Check $\underline{k}$ comp $\quad 1 + \lambda \neq 4 - 2\mu$

$1 - \frac{9}{2} \neq 4 - 2 \times -3$

$-\frac{7}{2} \neq 10$

So they do not intersect.

Hence these lines are skew.

Intersection of line and a plane

Find the point of intersection of the line l and the plane Π where:

$$l: \quad \boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$$
$$\Pi: \quad \boldsymbol{r} \cdot (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) = 4$$

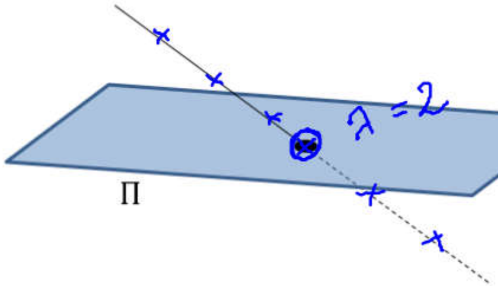
Plane MUST be in scalar dot form
Parametric is useless

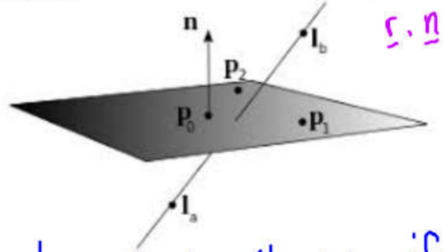
$\underline{r} = \underline{a} + \lambda\underline{b} + \mu\underline{c}$

$\underline{n} = ?$

$\underline{r} \cdot \underline{n} = p$

Ex 9E Evens





$l \quad \underline{r} = \begin{pmatrix} -1+\lambda \\ 1+\lambda \\ -5+2\lambda \end{pmatrix} \qquad \underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$

$$\begin{pmatrix} -1+\lambda \\ 1+\lambda \\ -5+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$$

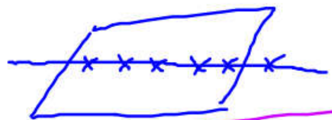
$$-1+\lambda+2+2\lambda-15+6\lambda = 4$$
$$9\lambda = 18$$
$$\lambda = 2$$

$$\underline{r} = \begin{pmatrix} -1+2 \\ 1+2 \\ -5+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \qquad (1, 3, -1)$$

What does it mean if you came up with this equation...

$$3+2\lambda+4-\lambda+3-\lambda = 10$$
$$10 = 10$$

The line is on the plane.

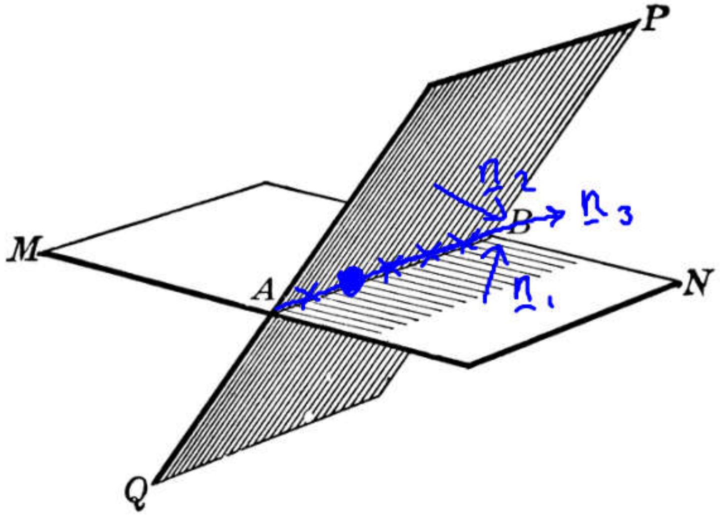


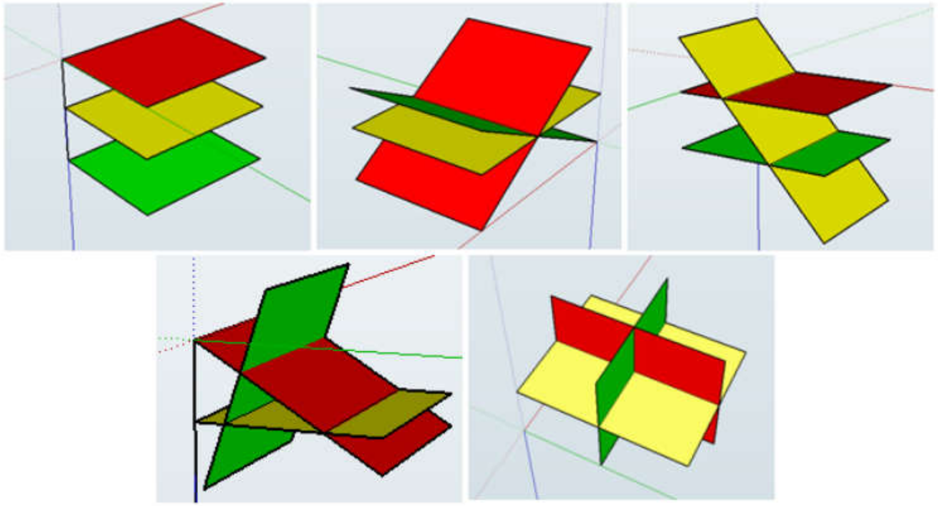
$$3+2\lambda+4-\lambda+2-\lambda = 10$$
$$9 \neq 10$$

The plane and line never intersect, so they are parallel.

$\underline{n} \cdot \underline{d} = 0$

(Intersection of two planes)





We will return to this again in FP1

2 planes intersect along a straight line.

1) The normal ($\underline{n}_3$) to the normals of both planes is the direction of the line.

Find a common point on both planes, $\underline{a}$ $\quad \underline{r} = \underline{a} + \lambda \underline{n}_3$

2) Find 2 common points on both planes, find the vector through them (the direction)

– form line equation.

7. The plane Π_1 has equation

$$\mathbf{r}.(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = -8$$

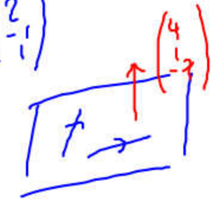
~~(a) Find the perpendicular distance from the point (8, 2, 10) to Π_1~~ **(3)**

$$\boxed{\underline{r} \cdot \begin{pmatrix}2\\-3\\4\end{pmatrix} = -8}$$

The plane Π_2 has equation

$$\mathbf{r} = \lambda(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\underline{r} = \lambda\begin{pmatrix}1\\3\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\1\end{pmatrix}$$



where λ and μ are scalar parameters.

(b) Show that the vector $4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$ is perpendicular to Π_2 **(2)**

(c) Find, to the nearest degree, the acute angle between Π_1 and Π_2 **(3)**

(d) Find a vector equation of the line of intersection of the planes Π_1 and Π_2 **(4)**

b) $\begin{pmatrix}4\\1\\-7\end{pmatrix} \cdot \begin{pmatrix}1\\3\\1\end{pmatrix} = 4 + 3 - 7 = 0$ $\quad \begin{pmatrix}4\\1\\-7\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\1\end{pmatrix} = 8 - 1 - 7 = 0$

Hence, the vector $\begin{pmatrix}4\\1\\-7\end{pmatrix}$ is perpendicular to Π_2.

c) $\cos\theta = \left|\dfrac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}\right|$

d) scalar dot form of Π_2 $\quad \boxed{\underline{r} \cdot \begin{pmatrix}4\\1\\-7\end{pmatrix} = 0}$

$2x - 3y + 4z = 8$

$4x + y - 7z = 0$

Let $z = 0$, $x = \frac{4}{7}$ $\quad y = -\frac{16}{7}$ $\quad \left(\frac{4}{7}, -\frac{16}{7}, 0\right)$.

Let $z = 1$, $x = \frac{4}{7} + \frac{17}{14} = \frac{25}{14}$ $\quad y = -\frac{16}{7} + \frac{15}{7} = -\frac{1}{7}$.

$\left(\frac{25}{14}, -\frac{1}{7}, 1\right)$

$\underline{d} = \begin{pmatrix}17/14\\15/7\\1\end{pmatrix}$ $\quad l \;\; \underline{r} = \begin{pmatrix}4/7\\-16/7\\0\end{pmatrix} + t\begin{pmatrix}17/14\\15/7\\1\end{pmatrix}$

Find the equation of the line of intersection of the planes π_1 and π_2.

Do this for homework.

π_1 has the equation $2x - 2y - z = 2$

π_2 has the equation $r \cdot (i - 3j + k) = 5$