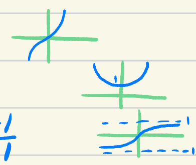
Hyperbolic Functions

Definitions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$



Plus cosech x, sech x and coth x

F $\text{arsinh}\, x = \ln(x + \sqrt{x^2+1})$

F $\text{arcosh}\, x = \ln(x \pm \sqrt{x^2-1})$ Extra – for solving equations

F $\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Identities

Same as trigonometric, Osborn's Rule

"$\sin^2$ or implied $\sin^2$ negated from trig identity"

eg $\cosh^2 x - \sinh^2 x = 1$, $1 - \tanh^2 x = \text{sech}^2 x$ F

Equations

i) Identities then inverse functions

ii) Exponential form, create a quadratic

Method of Differences

When to use...

$f(n) - f(n+1)$ or $f(n) - f(n+2)$ form

Maclaurin Expansion

Formula

F $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$

Lots of standard results in formula book

Compound Functions

Can sub into standard results eg. $\cos(2x^2)$

eg. $\ln\left(\frac{1-3x}{\sqrt{1+5x}}\right) = \ln(1-3x) - \frac{1}{2}\ln(1+5x)$

Volumes of Revolution

Same as CP1, Parametric same as P2

x-axis $\pi\int y^2\,dx$ $\quad \pi\int y^2 \frac{dx}{dt}\,dt$

y-axis $\pi\int x^2\,dy$ $\quad \pi\int x^2 \frac{dy}{dt}\,dt$

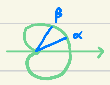
Polar Coordinates

Useful facts

$x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $r \geq 0$

Area

♥F $A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$



$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

Tangents

— $\frac{dy}{d\theta} = 0$ $\qquad$ | $\frac{dx}{d\theta} = 0$

Key:

F in the formula booklet

♥F in the formula booklet but recommend learning by heart

Complex Numbers

Check CP1!

Exponential Form

$z = re^{i\theta}$ same rules for $+, -, \times, \div$

De Moivre's Theorem

♥F $z^n = r^n e^{ni\theta} = r^n(\cos n\theta + i\sin n\theta)$

Application 1, expressing in terms of trig powers

e.g. $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

- Start with De Moivre's $(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$
- Binomial Expansion on LHS. Tip: Let $\cos\theta = c$, $\sin\theta = s$
- Compare Re for $\cos 6\theta$, Im for $\sin 6\theta$
- Use $\sin^2\theta = 1 - \cos^2\theta$ as necessary

Further Identities

$z + \frac{1}{z} = 2\cos\theta$ $\qquad z - \frac{1}{z} = 2i\sin\theta$

$z^n + \frac{1}{z^n} = 2\cos n\theta$ $\qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$

Application 2, expressing trig powers

eg. $\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

- Start with $(z - \frac{1}{z})^5 = (2i\sin\theta)^5$
- Binomial Expansion on LHS
- Reuse $z^n \pm \frac{1}{z^n}$ identities

Sums of Series Check P2 Formulae

Even further identities – "hyperbolic"

$\cos n\theta = \frac{1}{2}(e^{ni\theta} + e^{-ni\theta})$

$\sin n\theta = \frac{1}{2i}(e^{ni\theta} - e^{-ni\theta})$

$e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$

$e^{ni\theta} - e^{-ni\theta} = 2i\sin n\theta$

Simplifying complex exponential denominators 1

e.g. $\frac{}{e^{ni\theta} \pm 1}$ and $\frac{}{1 \pm e^{ni\theta}}$

Multiply by e to negated half power, $e^{-\frac{1}{2}ni\theta}$

e.g. $\frac{3}{e^{5i\theta}+1} \times \frac{e^{-\frac{5}{2}i\theta}}{e^{-\frac{5}{2}i\theta}} = \frac{3e^{-\frac{5}{2}i\theta}}{e^{\frac{5}{2}i\theta} + e^{-\frac{5}{2}i\theta}} = \frac{3e^{-\frac{5}{2}i\theta}}{2\cos\frac{5}{2}\theta}$

Simplifying complex exponential denominators 2

eg. $\frac{}{ce^{ni\theta} \pm k}$ and $\frac{}{k \pm ce^{ni\theta}}$ with either $c \neq 1$, $k \neq 1$

Multiply by same expression but with negated power

e.g. $\frac{3}{e^{4i\theta}+2} \times \frac{e^{-4i\theta}+2}{e^{-4i\theta}+2} = \frac{3e^{-4i\theta}+6}{1+4+2e^{4i\theta}+2e^{-4i\theta}} = \frac{3e^{-4i\theta}+6}{5+4\cos 4\theta}$

Infinite Series

e.g. $C + iS$ where $C = 1 + \frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta \ldots$

$S = \frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta \ldots$

- Convert to exponential form
- Use infinite series $\frac{a}{1-r}$ ♥F
- Simplify using above techniques
- Compare Re and Im parts

Roots

e.g. Solve $z^n = w$

- exp form for w
- raise to $\frac{1}{n}$ power
- $+\frac{2k\pi}{n}$ to power, ensuring $-\pi < \arg \leq \pi$

Roots of Unity: $1, \omega, \omega^2, \ldots, \omega^{n-1}$

- form vertices of regular n-polygon
- sum of roots of unity, $S_n = 0$
- multiply by ω, moves anticlockwise to next vertex
- not centred at origin? subtract centre so it is, then add it back on after finished calculations

Further Calculus

Improper Integrals

...improper if one or both limits are ∞

or... if undefined at any point of interval

Use limits eg. $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx$

Mean Value

$\bar{y} = \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Mean of $f(x) + k = \bar{f} + k$

$kf(x) = k\bar{f}$

Further Differentiation

Standard Results

$f(x)$	$f'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
* $\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$
* $\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\coth x$	$-\text{cosech}^2 x$
$\text{arsinh}\, x$	$\frac{1}{\sqrt{1+x^2}}$
$\text{arcosh}\, x$	$\frac{1}{\sqrt{x^2-1}}$
$\text{artanh}\, x$	$\frac{1}{1-x^2}$

F

- Hyperbolics: only $\cosh x$* and $\text{sech}\, x$ change from their trigonometric versions.
- These also, of course, give integration facts.
- Use the formula book!

Further Integration

Standard Results

$f(x)$	$\int f(x)\,dx$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

F

- All in the formula book, should be relied on
- All x^2 can be replaced with $(x+k)^2$, changing x to $(x+k)$ in standard result.

'Sensible' or 'Appropriate' Substitutions

Asks for substitution? Use formula book as a guide

e.g. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ Use $x = a\sin u$

e.g. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$ Use $x = a\sinh u$

Partial Fractions

i) improper?

ii) repeated factors?

iii) non-linear factors?

eg. $\frac{x^6+3}{x(x^2+2)^2} = Ax + B + \frac{C}{x} + \frac{Dx+E}{x^2+2} + \frac{Fx+G}{(x^2+2)^2}$

Completing the Square

Used for quad. denominators in ax^2+bx+c form

First Order Differential Equations

i) Separating the Variables

$\frac{dy}{dx} = f(x)g(y)$

$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

ii) Reverse Product Rule

e.g. $x^3\frac{dy}{dx} + 3x^2y = \sin x$

$x^3y = \int \sin x\,dx$

iii) Integrating Factor

Must be in $\frac{dy}{dx} + P(x)y = Q(x)$ form

Integrating Factor is $e^{\int P\,dx}$

Multiply through by I.F.

Harmonic Motion

$a = \frac{dv}{dt} = \ddot{x} = v\frac{dv}{dx}$

For $x = a\sin(\omega t + \alpha)$

amplitude is a, period is $\frac{2\pi}{\omega}$

Types

See above, top right.

Forced Harmonic Motion if $\ddot{x} + k\dot{x} + \omega^2 = f(t)$

Second Order Differential Equations

$ay'' + by' + cy = f(x)$

General Solution

$y = \text{C.F.} + \text{P.I}$

Auxiliary Equation

$am^2 + bm + c = 0$

Complementary Functions

A.E. Roots	C.F.	Harmonic?
2 real, α, β	$Ae^{\alpha x} + Be^{\beta x}$	Heavy damping
1 real repeated, α	$(A+Bx)e^{\alpha x}$	Critical damping
2 imaginary, $\pm iq$	$A\cos qx + B\sin qx$	Simple Harmonic
2 complex, $p \pm iq$	$e^{px}(A\cos qx + B\sin qx)$	Light damping

Particular Integrals

$f(x)$	P. I
a	λ
$ax + b$	$\mu x + \lambda$
$ax^2 + bx + c$	$\nu x^2 + \mu x + \lambda$
ke^{px}	λe^{px}
kxe^{px}	$\lambda x e^{px} + \mu e^{px}$
$m\cos qx + n\sin qx$	$\lambda\cos qx + \mu\sin qx$

Exceptions

If $f(x)$ appears anywhere in CF, multiply PI by x, x^2

e.g. $f(x) = e^{2x}$, C.F. $y = Ae^{2x} + Be^{3x}$, P.I. Try $y = \lambda xe^{2x}$

$f(x) = e^{3x}$, C.F. $y = (A+Bx)e^{3x}$, P.I. Try $y = \lambda x^2 e^{3x}$

$f(x) = dx + e$, C.F. $y = A + Be^{2x}$, P.I. Try $y = x(\mu x + \lambda)$

Method

Diff. P.I. twice, sub into D.E., solve to find ν, λ, μ.

Coupled First Order Differential Equations

Method $\quad$ I $\frac{dx}{dt} = ax + by + f(t)$

II $\frac{dy}{dt} = cx + dy + g(t)$

1) Make y subject in I
2) Find $\dot{y}$
3) Sub into II → creates 2nd ODE in x
4) Solve for x
5) Find $\dot{x}$
6) Sub into equation from 1, find y

Useful Notation

x	y
$\dot{x}$	$\dot{y}$
$\ddot{x}$	$\ddot{y}$

BICEN MATHS