Chapter 1: Complex Numbers (Year 2)

1 :: Exponential form of a complex number

$cos\ 2\theta + i\ sin\ 2\theta$
$\rightarrow \quad e^{2i\theta}$

2 :: Multiplying and dividing complex numbers

If z_1 and z_2 are two complex numbers, what happens to their moduli when we find $z_1 z_2$. What happens to their arguments when we find $\frac{z_1}{z_2}$?

3 :: De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$,
$z^n = r^n(\cos n\theta + i\sin n\theta)$

4 :: De Moivre's for Trigonometric Identities

"Express $\cos 3\theta$ in terms of powers of $\cos\theta$"

5 :: Roots

"Solve $z^4 = 3 + 2i$"

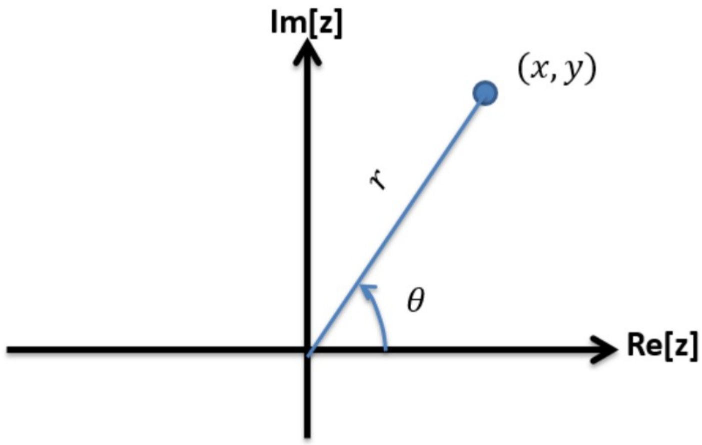
6 :: Sums of series

"Given that $z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, where n is a positive integer, show that

$$1 + z + z^2 + \cdots + z^{n-1}$$
$$= 1 + i\cot\left(\frac{\pi}{2n}\right)$$

"

RECAP :: Modulus-Argument Form



If $\boldsymbol{z = x + iy}$ (and suppose in this case z is in the first quadrant), what was:

$$r = |z| = \sqrt{\boldsymbol{x^2 + y^2}}$$
$$\theta = \arg(z) = \boldsymbol{\tan^{-1}\left(\frac{y}{x}\right)}$$

When $-\pi < \theta < \pi$ it is known as the **principal argument**.

Then in terms of r and θ:

$$\boldsymbol{x = r\cos\theta}$$
$$\boldsymbol{y = r\sin\theta}$$
$$\boldsymbol{z = x + iy}$$
$$\boldsymbol{= r\ cos\ \theta + i\ r sin\ \theta}$$
$$\boldsymbol{= r(\cos\theta + i\sin\theta)}$$

This is known as the **modulus-argument form** of z.

$x+iy$	r	θ	Modulus-argument form
-1			
i			
$1+i$			
$-\sqrt{3}+i$			

Exponential Form

We've seen the Cartesian form a complex number $z = x + yi$ and the modulus-argument form $z = r(\cos\theta + i\sin\theta)$. But, there's a third form!

Later in Chapter 3 on Taylor expansions, you'll see that you that you can write functions as an infinitely long polynomial:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} +$$

It looks like the $\cos x$ and $\sin x$ somehow add to give e^x. The one problem is that the signs don't quite match up. But i changes sign as we raise it to higher powers.

$e^{i\theta} =$

Exponential form

$$z = re^{i\theta}$$

You need to be able to convert to and from exponential form.

$x + iy$	Mod-arg form	Exp Form
-1		
$2-3i$		
	$\sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$	
		$z = \sqrt{2}e^{\frac{3\pi i}{4}}$
		$z = 2e^{\frac{23\pi i}{5}}$

$\boldsymbol{e^{i\pi} + 1 = 0}$
This is Euler's identity. It relates the five most fundamental constants in maths!

To get Cartesian form, put in modulus-argument form first.

Notice this is not a principal argument.

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

Multiplying and Dividing Complex Numbers

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
Then:

$$z_1 z_2 = \boldsymbol{r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))}$$

$$\frac{z_1}{z_2} = \boldsymbol{\frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))}$$

i.e. If you multiply two complex numbers, you **multiply the moduli** and **add the arguments**, and if you divide them, you divide the moduli and subtract the arguments.

Similarly if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$
Then:

$$z_1 z_2 = \boldsymbol{r_1 r_2 e^{i(\theta_1 + \theta_2)}}$$

$$\frac{z_1}{z_2} = \boldsymbol{\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}}$$

$$3\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) \times 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Write in the form $re^{i\theta}$:

$$\frac{2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)} =$$

$$2\left(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}\right) \times 3\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$$

$\cos\theta - i\sin\theta$ can be written as $\cos(-\theta) + i\sin(-\theta)$

Edexcel FP2(Old) June 2013 Q2

$$z = 5\sqrt{3} - 5i$$

Find

(a) $|z|$ (1)

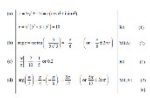
(b) $\arg(z)$ in terms of π (2)

$$w = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Find

(c) $\left|\frac{w}{z}\right|$ (1)

(d) $\arg\left|\frac{w}{z}\right|$ (2)



De Moivre's Theorem

We saw that: $z_1 z_2 = \boldsymbol{r_1 r_2(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))}$

Can you think therefore what z^n is going to be?

If $z = r(cos\ \theta + i\ sin\ \theta)$
$\boldsymbol{z^n = r^n(\cos n\theta + i \sin n\theta)}$
This is known as **De Moivre's Theorem**.

Edexcel FP2(Old) June 2013 Q4

Prove by induction that $z^n = r^n(\cos n\theta + i \sin n\theta)$

De Moivre's Theorem for Exponential Form

If $z = re^{i\theta}$ then $\boldsymbol{z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}}$

Alternative: Using Euler's form

$z = r(\cos\theta + i\sin\theta) = r\mathrm{e}^{i\theta}$
$z^{k+1} = z^k \times z = \left(r\mathrm{e}^{i\theta}\right)^k \times r\mathrm{e}^{i\theta} = r^k \mathrm{e}^{ik\theta} \times r\mathrm{e}^{i\theta}$
$= r^{k+1}\mathrm{e}^{i(k+1)\theta}$
$= r^{k+1}(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta)$
$k = 1 \quad z^1 = r^1(\cos\theta + \mathrm{i}\sin\theta)$
True for $n = 1$ $\therefore$ true for all n etc

Simplify $\dfrac{\left(\cos\frac{9\pi}{17}+i\,\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17}-i\,\sin\frac{2\pi}{17}\right)^3}$

Express $\left(1+\sqrt{3}\,i\right)^7$ in the form $x+iy$ where $x, y \in \mathbb{R}$.

Edexcel FP2(Old) June 2010 Q4

$$z = -8 + (8\sqrt{3})\text{i}$$

(*a*) Find the modulus of z and the argument of z. **(3)**

Using de Moivre's theorem,

(*b*) find z^3, **(2)**

(a)	Modulus = 16	B1
	Argument = $\arctan(-\sqrt{3}) = \frac{2\pi}{3}$	M1A1 (3)
(b)	$z^3 = 16^3(\cos(\frac{2\pi}{3}) + \text{i}\sin(\frac{2\pi}{3}))^3 = 16^3(\cos 2\pi + \text{i}\sin 2\pi) = 4096$ or 16^3	M1 A1 (2)

Ex 1C

Applications of de Moivre #1: Trig identities

Express $\cos 3\theta$ in terms of powers of $\cos\theta$

1) Create a 'de Moivre' statement that includes a $\cos 3\theta$ on RHS
2) Binomial expansion
3) Compare real/imaginary parts

Express

(a) $\cos 6\theta$ in terms of $\cos\theta$.

(b) $\frac{\sin 6\theta}{\sin\theta}, \theta \neq n\pi$, in terms of $\cos\theta$.

Edexcel FP2(Old) June 2011 Q7

(a) Use de Moivre's theorem to show that **(5)**

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

Hence, given also that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(b) Find all the solutions of

$$\sin 5\theta = 5\sin 3\theta$$

in the interval $0 \le \theta < 2\pi$. Give your answers to 3 decimal places. **(6)**

(a)	$\sin 5\theta = \text{Im}(\cos\theta + i\sin\theta)^5$	B1
	$5\cos^4\theta(i\sin\theta) - 10\cos^2\theta(i^3\sin^3\theta) + i^5\sin^5\theta$	M1
	$= i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$	A1
	$[\text{Im}(\cos\theta + i\sin\theta)^5] = 5\sin\theta(1-\sin^2\theta)^2 - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta$	M1
	$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (*)	A1cso
(b)	$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 5(3\sin\theta - 4\sin^3\theta)$	M1
	$16\sin^5\theta - 10\sin\theta = 0$	M1
	$\sin^4\theta = \frac{5}{8}$ $\theta = 1.095$	A1
	Inclusion of solutions from $\sin\theta = -\sqrt[4]{\frac{5}{8}}$	M1
	Other solutions: $\theta = 2.046, 4.237, 5.188$	A1
	$\sin\theta = 0 \Rightarrow \theta = 0, \theta = \pi\ (3.142)$	B1

Finding identities for $\sin^n \theta$ and $\cos^n \theta$

The technique we've seen allows us to write say $\cos 3\theta$ in terms of powers of $\cos \theta$ (e.g. $\cos^3 \theta$).
Is it possible to do the opposite, to say express $\cos^3 \theta$ in terms of a linear combination of $\cos 3\theta$ and $\cos \theta$ (with no powers)?

If $z = \cos \theta + i \sin \theta$, what is $z + \frac{1}{z}$ and $z - \frac{1}{z}$?

And what is $z^n + \frac{1}{z^n}$? $z^n - \frac{1}{z^n}$?

Express $\cos^5 \theta$ in the form $a \cos 5\theta + b \cos 3\theta + c \cos \theta$

1) Raise RHS to the required power – careful of the '2' or '2i'
2) Raise LHS to same power
3) Binomial expansion
4) Use the identities once again
5) Remember to isolate by dividing by any coefficients on LHS

Results you need to use

$$z + \frac{1}{z} = 2\cos\theta \qquad z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z - \frac{1}{z} = 2i\sin\theta \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Prove that $\sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$

1) Raise RHS to the required power – careful of the ‘2’ or ‘2i’
2) Raise LHS to same power
3) Binomial expansion
4) Use the identities once again
5) Remember to isolate by dividing by any coefficients on LHS

✎ Results you need to use

$$z + \frac{1}{z} = 2\cos\theta \qquad z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z - \frac{1}{z} = 2i\sin\theta \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Your Turn

(a) Express $\sin^4 \theta$ in the form $a \cos 4\theta + b \cos 2\theta + c$

(b) Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4 \theta \ d\theta$

Ex 1D

Sums of Series

The formula for the sum of a geometric series also applies to complex numbers:

For $w, z \in \mathbb{C}$,

$$\sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + \cdots + wz^{n-1} = \frac{w(z^n - 1)}{z - 1}$$

$$\sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + \cdots + wz^{n-1} = \frac{w}{1 - z}$$

provided $|z| < 1$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_\infty = \frac{a}{1 - r}$$

Geometric series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_\infty = \frac{a}{1 - r} \text{ for } |r| < 1$$

Show that if $z = e^{\frac{\pi i}{4}}$, then $\sum_{r=0}^{8} z^r = 1$

Show that $\sum_{r=0}^{5}\left(1+i\sqrt{3}\right)^r = -21\sqrt{3}\,i$

Ex1E Q2, 3

Some very useful further identities

✎ Results you need to use

$$z+\frac{1}{z}=2\cos\theta \qquad z^n+\frac{1}{z^n}=2\cos n\theta$$

$$z-\frac{1}{z}=2i\sin\theta \qquad z^n-\frac{1}{z^n}=2i\sin n\theta$$

1) Rewrite these in exponential form
2) Make $\sin n\theta$ and $\cos n\theta$ the subject

Things to note:
- Indices are same but negated
- $\cos n\theta$ goes with +
- $\sin n\theta$ goes with –
    - Hyperbolic connection...

Creating expressions in the hyperbolic form **PARTS 1**

$$\cos n\theta = \frac{1}{2}(e^{ni\theta} + e^{-ni\theta})$$

$$\sin n\theta = \frac{1}{2i}(e^{ni\theta} - e^{-ni\theta})$$

$$2\cos n\theta = e^{ni\theta} + e^{-ni\theta}$$

$$2i\sin n\theta = e^{ni\theta} - e^{-ni\theta}$$

when there is a 1+, 1-, or -1, with coefficient of $e^{in\theta}$ as 1

$$\frac{3}{e^{2i\theta} - 1}$$

$$\frac{e^{i\theta}}{e^{\frac{i\theta}{3}} - 1}$$

$$\frac{e^{5i\theta} + 1}{e^{4i\theta} - 1}$$

Tricky example using several skills we have learned

Given that $z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, where n is a positive integer, show that

$$1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\left(\frac{\pi}{2n}\right)$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Ex1E Q1

Using mod-arg form to split summation

$e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{ni\theta}$ is a geometric series,

$$\therefore e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{ni\theta} = \frac{e^{i\theta}(e^{ni\theta} - 1)}{e^{i\theta} - 1}$$

Converting each exponential term to modulus-argument form would allow us to consider the real and imaginary parts of the series separately:

$$\begin{aligned} &e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{ni\theta} \\ &= (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \cdots \\ &= (\cos\theta + \cos 2\theta + \cdots) + i(\sin\theta + \sin 2\theta + \cdots) \end{aligned}$$

Thus $\cos\theta + \cos 2\theta + \cdots$ is the real part of $\frac{e^{i\theta}(e^{ni\theta}-1)}{e^{i\theta}-1}$ and $\sin\theta + \sin 2\theta + \cdots$ the imaginary part.

$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{8i\theta}$, for $\theta \neq 2n\pi$, where n is an integer.

(a) Show that $S = \dfrac{e^{\frac{9i\theta}{2}} \sin 4\theta}{\sin\frac{\theta}{2}}$

Let $P = \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos 8\theta$ and $Q = \sin\theta + \sin 2\theta + \cdots + \sin 8\theta$

(b) Use your answer to part **a** to show that $P = \cos\frac{9\theta}{2} \sin 4\theta \ \ cosec\frac{\theta}{2}$ and find similar expressions for Q and $\frac{Q}{P}$

Ex1E Q5
Rev 1 Q6
Ex1E Q6 *hard

Creating expressions in the hyperbolic form **PART 2**

$$\cos n\theta = \frac{1}{2}(e^{ni\theta} + e^{-ni\theta})$$

$$\sin n\theta = \frac{1}{2i}(e^{ni\theta} - e^{-ni\theta})$$

$$2\cos n\theta = e^{ni\theta} + e^{-ni\theta}$$

$$2i\sin n\theta = e^{ni\theta} - e^{-ni\theta}$$

when there is a k+, k-, or -k where k is a constant, or if there is a $ke^{in\theta}$ instead of $e^{in\theta}$

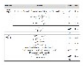
multiply by same expression by with power negated

$$\frac{3}{e^{4i\theta} + 2}$$

$$\frac{1}{3e^{i\theta} - 1}$$

$$\frac{e^{i\theta}}{3 - e^{2i\theta}}$$

4. The infinite series C and S are defined by



$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

Ex1E Q4, 7
Mix 1 Q13

Applications of de Moivre #2: Roots

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

We have so far used de Moivre's theorem when n was an integer.
It also works however when n is a rational number! (proof not required)

Solve $z^3 = 1$
Plot these roots on an Argand diagram

Solve $z^4 = 2 + 2\sqrt{3}\, i$

Edexcel FP2(Old) June 2012 Q3

a) Express the complex number $-2 + (2\sqrt{3})i$ in the form $r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$. **(3)**

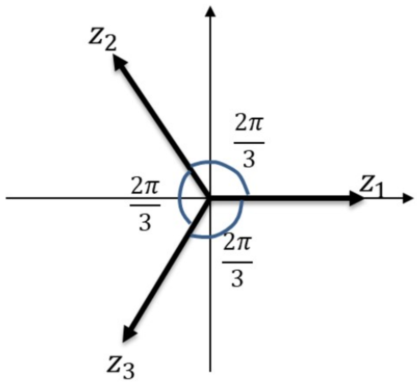
b) Solve the equation

$$z^4 = -2 + (2\sqrt{3})\mathrm{i}$$

giving the roots in the form $r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$. **(5)**

Roots of Unity

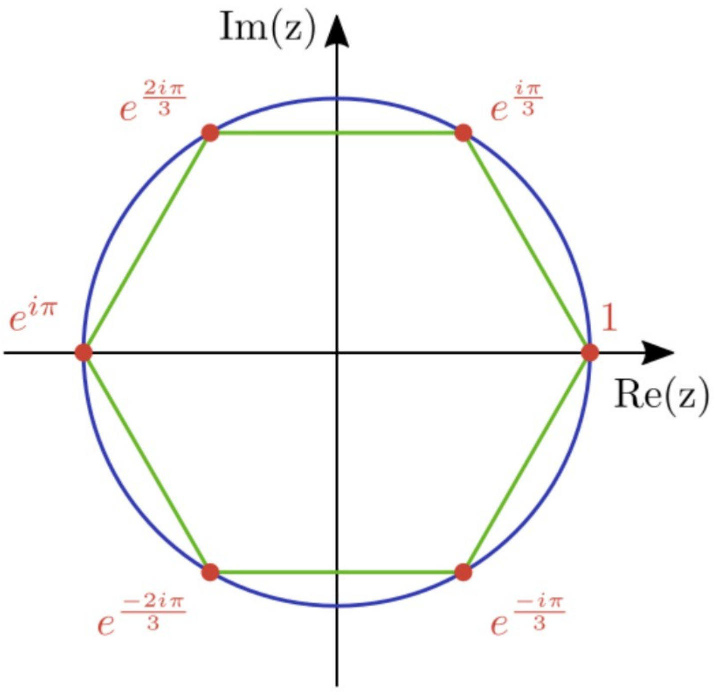
Solve $z^3 = 1$
Plot these roots on an Argand diagram



Solve $z^4 = 1$
Plot these roots on an Argand diagram

Solve $z^6 = 1$
Plot these roots on an Argand diagram

How would the solutions to $z^n = 1$ look on an Argand diagram?



- 1 will always be a root
- Suppose the next root around the polygon is ω… then…

The roots of unity sum to zero

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

Solve $z^5 = 1$

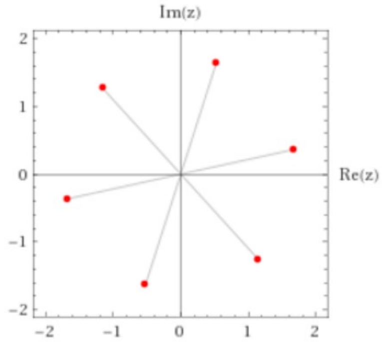
Hence show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

Ex1F

Solving Geometric Problems

$$z^6 = 7 + 24i$$



Recall that ω is the first root of unity: $\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$.
This has modulus 1 and argument $\frac{2\pi}{n}$.

✎ If z_1 is one root of the equation $z^n = s$, and $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1\omega, z_1\omega^2, \dots, z_1\omega^{n-1}$.

The point $P(\sqrt{3}, 1)$ lies at one vertex of an equilateral triangle. The centre of the triangle is at the origin.

(a) Find the coordinates of the other vertices of the triangle.

(b) Find the area of the triangle.

Find the coordinates of the vertices of an equilateral triangle with centre (5, 5) and one vertex at (3, 4)

6. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\text{i}$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \text{i}y$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

Ex1G