20) $u = \frac{2}{3}\sinh x$

$$\frac{du}{dx} = \frac{2}{3}\cosh x$$

$$\frac{1}{\frac{2}{3}\cosh x}\,du = dx$$

$$\frac{3}{2}u = \sinh x$$

$$\frac{9}{4}u^2 = \sinh^2 x$$

$$\frac{9}{4}u^2 + \frac{9}{4} = \sinh^2 x + \frac{9}{4}$$

$$\frac{9}{4}(u^2+1) = \sinh^2 x + \frac{9}{4}.$$

$x=0 \quad u = \frac{2}{3}\sinh 0 = 0$

$x=1 \quad u = \frac{2}{3}\sinh 1$

$$\int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x+9}}\,dx = \frac{1}{2}\int \frac{\cosh x}{\sqrt{\sinh^2 x+\frac{9}{4}}}$$

$$= \frac{1}{2}\int_0^{\frac{2}{3}\sinh 1} \frac{\cancel{\cosh x}}{\sqrt{\frac{9}{4}(u^2+1)}} \times \frac{1}{\frac{2}{3}\cancel{\cosh x}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\frac{3}{2}\sqrt{u^2+1}} \times \frac{1}{\frac{2}{3}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u^2+1}}\,du$$

$$= \frac{1}{2}\Big[\operatorname{arsinh} u\Big]_0^{\frac{2}{3}\sinh 1}$$

$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2}{3}\sinh 1\right)$$

$$x = 2\cosh u$$

$$2\sinh u\cosh u - 2u + c$$

$$\left(\frac{x}{2}\right)^2 = \cosh^2 u$$

$$\left(\frac{x}{2}\right)^2 = 1 + \sinh^2 u$$

$$\sqrt{\left(\frac{x}{2}\right)^2 - 1} = \sinh u$$

$$\cancel{2}\sqrt{\frac{x^2}{4} - 1} \times \frac{x}{\cancel{2}} \qquad \sqrt{x^2 - 4}$$

$$x\sqrt{\frac{x^2}{4} - 1}$$

$$x \times \sqrt{\frac{1}{4}}\sqrt{x^2 - 4}$$

$$\frac{x}{2}\sqrt{x^2 - 4}$$

Integrating by Completing the Square

Determine $\int \frac{1}{x^2-8x+8}\,dx$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right), \quad \ln\{x+\sqrt{x^2-a^2}\} \quad (x>a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right), \quad \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\rightarrow \underline{\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert}$

$$x^2-8x+8 = (x-4)^2-16+8$$
$$=(x-4)^2-8$$

$$\int \frac{1}{x^2-8x+8}\,dx = \int \frac{1}{(x-4)^2-8}\,dx$$

Let $u = x-4$ $\quad = \int \frac{1}{u^2-8}\,du$

$\frac{du}{dx}=1$

$du = dx$ $\quad = \frac{1}{4\sqrt{2}}\ln\left|\frac{u-2\sqrt{2}}{u+2\sqrt{2}}\right| + c$

$a^2=8$

$a=2\sqrt{2}$ $\quad = \frac{1}{4\sqrt{2}}\ln\left|\frac{(x-4)-2\sqrt{2}}{(x-4)+2\sqrt{2}}\right| + c$

$$\frac{1}{4\sqrt{2}}\ln\left|\frac{(x-4)-2\sqrt{2}}{(x-4)+2\sqrt{2}}\right| + c$$

why it works.

eg: $\int \cos(x-2)\,dx = \sin(x-2) + c$

Determine $\int \frac{1}{\sqrt{12x+2x^2}}dx$

$$12x + 2x^2 = 2(x^2 + 6x)$$
$$= 2[(x+3)^2 - 9]$$

$$\int \frac{1}{\sqrt{12x+2x^2}}dx = \int \frac{1}{\sqrt{2}\sqrt{(x+3)^2-9}}dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x+3)^2-9}}dx \qquad a^2 = 9, \quad a = 3$$
$$= \frac{1}{\sqrt{2}}\operatorname{arcosh}\left(\frac{x+3}{3}\right) + c$$

$$\int \frac{1}{(x+2)(x+4)}dx = \int \frac{1}{x^2+6x+8}dx = \int \frac{1}{(x+3)^2-1}dx = \frac{1}{2}\ln\left|\frac{x+3-1}{x+3+1}\right| + c$$
$$= \frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + c$$

$$\frac{1}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4}$$
$$1 = A(x+4) + B(x+2)$$
$$0 = A + B \qquad A = \tfrac{1}{2}$$
$$1 = 4A + 2B \qquad B = -\tfrac{1}{2}$$

$$= \int \left(\frac{1/2}{x+2} - \frac{1/2}{x+4}\right)dx$$
$$= \frac{1}{2}\ln|x+2| - \frac{1}{2}\ln|x+4| + c$$
$$= \frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + c$$

Ex 6E Q21-23

[June 2014(R) Q2]

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c$$

(*a*) Find the values of the constants *a*, *b* and *c*. **(3)**

Hence, or otherwise, find

(*b*) $\displaystyle\int \frac{1}{9x^2+6x+5}\,dx$ **(2)**

(*c*) $\displaystyle\int \frac{1}{\sqrt{9x^2+6x+5}}\,dx$ **(2)**

(a)	$9x^2+6x+5 \equiv a(x+b)^2+c$		
	$a=9,\ b=\frac{1}{3},\ c=4$		B1, B1, B1
			(3)
(b)	$\displaystyle\int \frac{1}{9(x+\frac{1}{3})^2+4}\,dx = \frac{1}{6}\arctan\left(\frac{3x+1}{2}\right)(+c)$	M1: $k\arctan\left(\dfrac{x+"\frac{1}{3}"}{\sqrt{\frac{"4"}{"9"}}}\right)$ A1: $\dfrac{1}{6}\arctan\left(\dfrac{3x+1}{2}\right)$ oe	M1A1
			(2)
(c)	$\displaystyle\int \frac{1}{\sqrt{9(x+\frac{1}{3})^2+4}}\,dx = \frac{1}{3}\operatorname{arsinh}\left(\frac{3x+1}{2}\right)(+c)$	M1: $k\operatorname{arsinh}\left(\dfrac{x+"\frac{1}{3}"}{\sqrt{\frac{"4"}{"9"}}}\right)$ A1: $\dfrac{1}{3}\operatorname{arsinh}\left(\dfrac{3x+1}{2}\right)$ oe Allow $\dfrac{1}{\sqrt{9}}$	M1A1
			(2)
			Total 7