Chapter 6: Matrices

1:: Understand matrices and perform basic operations (adding, scalar multiplication)

2:: Multiply Matrices

"Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix}$, determine the matrix $\mathbf{AB}$."

3:: Find the determinant or inverse of a matrix.

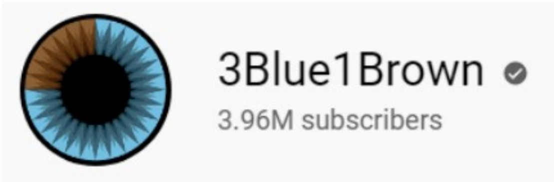
"If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$, determine $\mathbf{A}^{-1}$."

4:: Solve simultaneous equations using matrices.

"Use matrices to solve the following simultaneous equations:

$$x + 2y + z = 4$$
$$x - y + 3z = 1$$
$$2x + 5y - z = 0$$

Before studying this chapter, I **highly recommend** watching Chapters 1-8 of 3Blue1Brown's 'Essence of linear algebra' series. It will put everything we are learning about here into context, and will make Chapter 7 even easier.
No need to take notes and understand absolutely everything - but it is important to think about the ideas discussed.



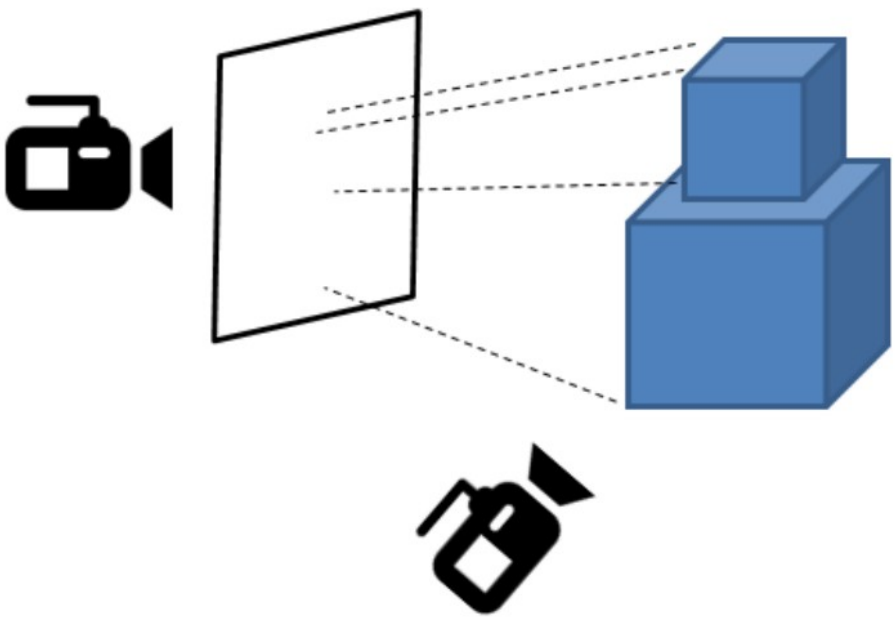
Grant Sanderson's channel is one of the best maths channels out there. Go check it out if you haven't already.

A matrix (plural: matrices) is **simply an 'array' of numbers**, e.g. $\begin{pmatrix} 1 & 0 & -2 \\ 3 & 3 & 0 \end{pmatrix}$

On a simple level, a matrix is just a way to organise values into rows and columns, and represent these multiple values as a single structure.

But the power of matrices comes from them **representing linear transformations/functions** (which we will particularly see in Chapter 7). We can

1. **Represent linear transformations** using matrices (e.g. rotations, reflections and enlargements)
2. Use them to **solve linear simultaneous equations**.



Matrices are particularly useful in 3D graphics, as matrices can be used to carry out rotations/enlargements (useful for changing the camera angle) or project into a 2D 'viewing' plane.

Matrix Fundamentals

#1 Dimensions of Matrices

x y The dimension of a matrix is its **size**, in terms of its number of **rows** and **columns** (in that order).

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} \qquad 2 \times 3$$

$$\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix} \qquad 3 \times 1$$

$$\begin{pmatrix} 1 & 6 & 0 \end{pmatrix} \qquad 1 \times 3$$

#2 Notation/Names for Matrices

A matrix can have square or curvy brackets (but the textbook only uses curvy)

$\begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}$	$\begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix}$	$\begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$
Matrix	**Column Vector**	**Row Vector**

A matrix with one column is **simply a vector in the usual sense**!

#3 Variables for Matrices

If the value of a variable is a matrix, we use **bold**, **capital** letters
(In contrast, vectors use bold, lowercase letters)

$$\mathbf{A} = \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

$$\mathbf{C} = \mathbf{P}^{-1}\mathbf{T}\mathbf{P}$$

#4 Adding/Subtracting Matrices

Simply add/subtract the corresponding elements of each matrix.
They must be of the same dimension.

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 9 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 3-q & 3 \\ -2 & 1 \\ 4 & 2 \end{pmatrix}$$

#5 Scalar Multiplication

A scalar is a number which can 'scale' the elements inside a matrix.

$$3\begin{pmatrix}1 & 3 & -7\\ 4 & 0 & 5\end{pmatrix} = \begin{pmatrix}3 & 9 & -21\\ 12 & 0 & 15\end{pmatrix}$$

Side Note: You first encountered this at GCSE, in the context of vectors. 3$\boldsymbol{a}$ is the vector $\boldsymbol{a}$ 'scaled' by the scalar 3.

$$\boldsymbol{A} = \begin{pmatrix}q & -3\\ 1 & 1\\ -4 & 1\end{pmatrix} \qquad 2\boldsymbol{A} = \begin{pmatrix}2q & -6\\ 2 & 2\\ -8 & 2\end{pmatrix}$$

Find k

$$\begin{pmatrix}-3\\ k\end{pmatrix} + \underline{k\begin{pmatrix}2k\\ 2k\end{pmatrix}} = \begin{pmatrix}k\\ 6\end{pmatrix}$$

$$\begin{pmatrix}-3\\ k\end{pmatrix} + \begin{pmatrix}2k^2\\ 2k^2\end{pmatrix} = \begin{pmatrix}k\\ 6\end{pmatrix}$$

$k + 2k^2 = 6$

$2k^2 + k - 6 = 0$

$(2k-3)(k+2) = 0$

$k = \frac{3}{2}$ or $\cancel{k = -2}$

$-3 + 2k^2 = k$

$2k^2 - k - 3 = 0$

$(2k-3)(k+1) = 0$

$k = \frac{3}{2}$ or $\cancel{k = -1}$

$k = \frac{3}{2}$

#6 Special Matrices

A matrix is **square** if it has the same number of rows as columns.

$$\overset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \qquad \begin{pmatrix} 3 & 1 & 4 \\ 2 & 2 & 5 \\ -3 & 4 & 3 \end{pmatrix} \; 3\times 3$$

A **zero matrix** is one in which all its elements are 0. The dimensions are usually clear from the context.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A **identity matrix $\mathbf{I}$** is a square matrix which has 1's in the 'leading diagonal' (starting top-left) and 0 elsewhere. Again, the dimensions depend on the context.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We will see the significance of the identity matrix when we cover matrix multiplication imminently.

Ex 6A

Matrix Multiplication

#7 Matrix Multiplication

Matrix multiplications are not always valid: the dimensions have to agree.

Dimensions of $\mathbf{A}$	Dimension of $\mathbf{B}$	Dimensions of $\mathbf{AB}$ (if valid)
2×3	3×4	2×4
1×3	2×3	Not valid.
6×2	2×4	6×4
1×3	3×1	1×1 ( )
7×5	7×5	Not valid
10×10	10×9	10×9
3×3	3×3	3×3

Note that only **square matrices** (i.e. same width as height) can be raised to a power.

$$\begin{bmatrix} -3 & 5 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} -13 & -19 \\ -11 & -1 \end{bmatrix}$$

$-18 + 5 = -13$

$6 - 25 = -19$

$-12 + 1 = -11$

$4 - 5 = -1$

$$\begin{bmatrix} 0 & 5 \\ -3 & 1 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -4 & 4 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} -10 & -20 \\ 10 & -16 \\ 18 & -24 \end{bmatrix}$$

$3 \times \underline{2}$ $\qquad$ $\underline{2} \times 2$

Computing matrix multiplication on your calculator

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 8 & 4 & 3 \\ 7 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 7 \\ 0 & 3 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} -11 & 16 \\ 42 & 61 \\ 50 & -6 \end{bmatrix}$$

3×4 $\qquad$ 4×2

$$\begin{bmatrix} -4 & -y \\ -2x & -4 \end{bmatrix} \cdot \begin{bmatrix} -4x & 0 \\ 2y & -5 \end{bmatrix} = \begin{bmatrix} 16x - 2y^2 & 5y \\ 8x^2 - 8y & 20 \end{bmatrix}$$

2×2 $\qquad$ 2×2

Your Turn

$\boxed{\text{a}}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} =$

$\boxed{\text{b}}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix} =$

$\boxed{\text{c}}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 =$

$\boxed{\text{d}}$ $\underset{}{\overset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}}\overset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = (14)$

$\boxed{\text{e}}$ $\overset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}\overset{1\times 3}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}} = \begin{pmatrix} \quad \end{pmatrix}$

$\boxed{\text{f}}$ $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

$\boxed{\text{a}}$ $$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}$$

$\boxed{\text{b}}$ $$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}2 & 0 & -1\\3 & 2 & 1\end{pmatrix}=\begin{pmatrix}8 & 4 & 1\\18 & 8 & 1\end{pmatrix}$$

$\boxed{\text{c}}$ $$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}^2=\begin{pmatrix}7 & 10\\15 & 22\end{pmatrix}$$

$\boxed{\text{d}}$ $$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}14\end{pmatrix}$$

$\boxed{\text{e}}$ $$\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{pmatrix}$$

$\boxed{\text{f}}$ $$\begin{pmatrix}1 & a\\0 & 1\end{pmatrix}^k=\begin{pmatrix}1 & ak\\0 & 1\end{pmatrix}$$

Matrix Multiplication involving $\mathbf{I}$

We earlier saw the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What do you notice about…

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}\underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}}_{= A}$$

$$\underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} = \underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}}_{= A}$$

In general $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all matrices $\mathbf{A}$.

So the identity matrix is a bit like the '1' of matrix multiplication, e.g. $1 \times 3 = 3 \times 1 = 3$; multiplying by 1 has no effect, and multiplying by $\mathbf{I}$ has no effect.

For this reason, 1 is known as the 'identity' of multiplication over numbers. And 0 is known as the 'identity' of addition over numbers, given that $a + 0 = 0 + a = a$ for all a.

Matrix Multiplication:

commutative or noncommutative?

$3 \times 6 = 18$
$6 \times 3 = 18$

order doesn't matter

Let $\mathbf{A} = \begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix}$

$3 \times (2 \times 4) = 24$
$(3 \times 2) \times 4 = 24$

Work out **AB** $\begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 35 & 5 \\ 5 & -4 \end{pmatrix}$

Work out **BA** $\begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 37 & 19 \\ -3 & -6 \end{pmatrix}$

What do you notice? Answers are different

What does this tell us? $AB \neq BA$ Matrix multiplication is non commutative

ORDER MATTERS!

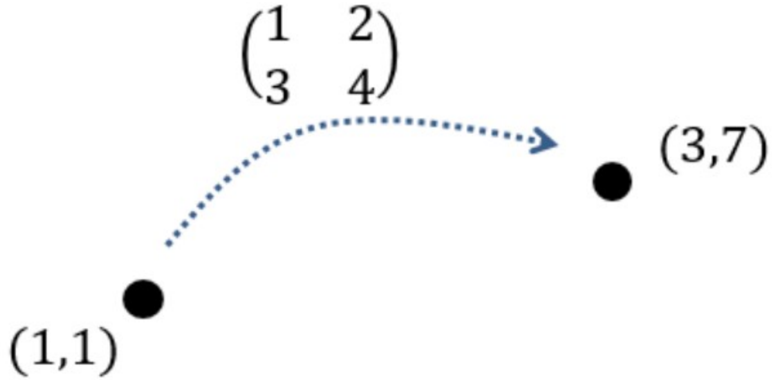
Note: Matrix multiplication is associative. This means that **A(BC) = (AB)C**

Ex 6B

Determinant of a matrix

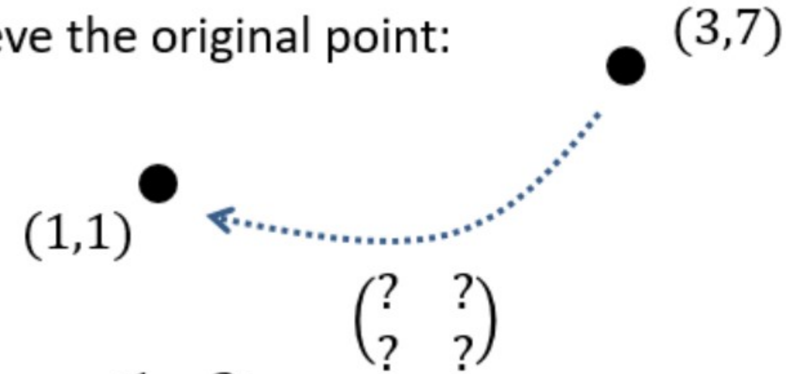
In Chapter 7, you will see that matrices can be thought of as a function that can transform a point, e.g.

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\7\end{pmatrix}$$



A question might naturally be whether there is an 'inverse function/transformation' that can retrieve the original point:

$$\begin{pmatrix}? & ?\\? & ?\end{pmatrix}\begin{pmatrix}3\\7\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$



This matrix would be known as the **inverse** of $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$.

With quadratics, we used to the word '*discriminant*' for b^2-4ac because it 'discriminates' between the different cases of 0, 1, 2 roots.
Analogously, the 'determinant' $|\boldsymbol{A}|$ or $\det(\boldsymbol{A})$ for a matrix $\boldsymbol{A}$ 'determines' whether it has an inverse of not.

Determinants of 2 x 2 matrices

The determinant of a matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det(\mathbf{A}) = |\mathbf{A}| = ad - bc$$

If $det(\mathbf{A}) = 0$, then $\mathbf{A}$ is a **singular matrix** and it does not have an inverse.

If $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is a **non-singular matrix** and it has an inverse.

Quickfire Questions:

$\mathbf{A}$	$\det(\mathbf{A})$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	$4 - 6 = -2$
$\begin{pmatrix} 0 & 3 \\ -1 & -4 \end{pmatrix}$	$0 - -3 = 3$
$\begin{pmatrix} 10 & -2 \\ 4 & -1 \end{pmatrix}$	$-10 - -8 = -2$

$$A = \begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$$

Given that **A** is singular, find the value of p.

$\det(A) = 0$

$$4(3-p) - -(p+2) = 0$$
$$12 - 4p + p + 2 = 0$$
$$14 = 3p$$
$$\frac{14}{3} = p$$

$$\mathbf{A} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}, \text{ where } a \text{ is real.}$$

(*a*) Find det **A** in terms of a. **(2)**

$\det \neq 0$

(*b*) Show that the matrix **A** is non-singular for all values of a. **(3)**

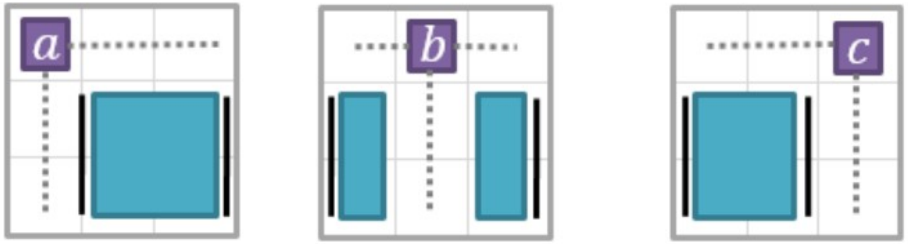
a) $\det A = a(a+4) + 10$
$= a^2 + 4a + 10$

b) $a^2 + 4a + 10 = (a+2)^2 - 4 + 10$
$= (a+2)^2 + 6 \geq 6$

Because $\det A \geq 6$, $\det A \neq 0$ for any value of a, hence, it is non singular for all values of a.

Determinants of 3 x 3 matrices

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

$$\begin{vmatrix} 3 & 1 & 4 \\ 2 & 2 & 5 \\ -3 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ -3 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 2 \\ -3 & 4 \end{vmatrix}$$

$$= 3(6-20) - 1(6+15) + 4(8+6)$$

$$= \underline{\underline{-7}}$$

$$\begin{vmatrix} 2 & 5 & 3 \\ 0 & -2 & -1 \\ 1 & 4 & 3 \end{vmatrix} = 2(-6+4) - 5(0+1) + 3(0+2)$$

$$= -4 - 5 + 6$$

$$= \underline{\underline{-3}}$$

Alternative method

$$\begin{vmatrix} 2 & 5 & 3 \\ 0 & -2 & -1 \\ 1 & 4 & 3 \end{vmatrix} = 2(-6+4) + 0 + 1(-5+6)$$

$$= -4 + 1$$

$$= \underline{\underline{-3}}$$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$$

Determine det(**A**).

Computing the determinant on your calculator

$\det A = 54$

$$A = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$$ where k is a constant.

Given that A is singular, find the possible values of k.

$\det A = 0$

$3(k+3) - k(-2(k+3) - 10) + 0 = 0$

$3k + 9 - k(-2k - 6 - 10) = 0$

$3k + 9 + 2k^2 + 16k = 0$

$2k^2 + 19k + 9 = 0$

$k = -\frac{1}{2}$ or $k = -9$

Minors

The **minor** of an element in a 3×3 matrix is the determinant of the 2×2 matrix that remains after the row and column containing that element have been crossed out.

Minor of 0: $4 \times 8 - -1 \times 5 = 32 + 5 = \underline{\underline{37}}$

$$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$$

Minor of -6: $8 + 2 = \underline{\underline{10}}$

Minor of 5: $\underline{\underline{2}}$

Ex 6C

Inverting a 2 x 2 matrix

We earlier saw that the inverse of a matrix $\mathbf{M}$ (written $\mathbf{M}^{-1}$) 'undoes' the effect of the matrix. Thus:

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

$AMM^{-1} = AI$

as multiplying something by a matrix followed by its inverse has no overall effect (i.e. the same as the identity matrix).

If a matrix is self inverse, then

$A = A^{-1} \Rightarrow AA^{-1} = I$

$AA = I$

$A^2 = I$

✎ If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\boldsymbol{A}^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- $\mathbf{A}^{-1}$ is the 'inverse' of $\mathbf{A}$, so that if $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$, $\mathbf{A}^{-1}\boldsymbol{y} = \boldsymbol{x}$
- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Practising the Inverse

Divide by determinant.

Swap NW-SE elements.

Make SW-NE elements negative.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Computing the inverse on your calculator

$$\begin{pmatrix} 7 & 2 \\ 1 & -3 \end{pmatrix}^{-1} = \frac{1}{23}\begin{pmatrix} 3 & 2 \\ 1 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 3/23 & 2/23 \\ 1/23 & -7/23 \end{pmatrix}$$

det Matrix $= 0$

For what value of p is $\begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$ singular?

Given p is not this value, find the inverse.

$$\begin{vmatrix} 4 & p+2 \\ -1 & 3-p \end{vmatrix} = 0$$

$$4(3-p) + (p+2) = 0$$

$$12 - 4p + p + 2 = 0$$

$$14 = 3p$$

$$p = \frac{14}{3}$$

$$\begin{vmatrix} 4 & p+2 \\ -1 & 3-p \end{vmatrix} = 14 - 3p$$

$$\begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}^{-1} = \frac{1}{14-3p}\begin{pmatrix} 3-p & -p-2 \\ 1 & 4 \end{pmatrix}$$

Matrix proofs involving inverses

If A and B are non-singular matrices such that $\mathbf{BAB} = \mathbf{I}$, prove that $\mathbf{A} = \mathbf{B}^{-1}\mathbf{B}^{-1}$

$$\cancel{B^{-1}}\cancel{B}AB = B^{-1}I$$

$$IAB = B^{-1}$$

$$A\cancel{B}\cancel{B^{-1}} = B^{-1}B^{-1}$$

$$A = B^{-1}B^{-1}$$

$$BAB = I$$

$$B^{-1}BAB = B^{-1}I$$

$$AB = B^{-1}$$

$$ABB^{-1} = B^{-1}B^{-1}$$

$$A = B^{-1}B^{-1}$$

Tip: You can rid of a matrix $\mathbf{A}$ at the **front** of the expression by multiplying the **front** of each side of the equation by $\mathbf{A}^{-1}$ (to get $\mathbf{I}$). You can similarly remove an $\mathbf{A}$ at the **end** by multiplying the **end** of each side by $\mathbf{A}^{-1}$.

If $\mathbf{P}$ and $\mathbf{Q}$ are non-singular matrices, prove that $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$

Hint: Start with a simple statement of the form $\mathbf{AA}^{-1} = \mathbf{I}$

$$(PQ)(PQ)^{-1} = I$$

$$\left(P \times Q \times (PQ)^{-1} = I \right)$$

$$P^{-1}PQ(PQ)^{-1} = P^{-1}I$$

$$Q(PQ)^{-1} = P^{-1}$$

$$Q^{-1}Q(PQ)^{-1} = Q^{-1}P^{-1}$$

$$(PQ)^{-1} = Q^{-1}P^{-1}$$

Matrix Transpose

A^T is the **transpose** of a matrix A, where the rows and columns are interchanged.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$. An $m \times n$ matrix becomes $n \times m$.

2×3 $\quad$ 3×2

$$\begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}^T = \begin{pmatrix} 7 & 6 \\ 1 & 1 \\ 2 & 5 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$$

Determine $\boldsymbol{A}^{\mathrm{T}}$

$$A^T = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 5 & 8 \\ 0 & -6 & 2 \end{pmatrix}$$

Finding the inverse of a 3 x 3 matrix

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find A^{-1}.

$$= \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

The method we previously used was a specific case of **a more general method** which can be used for matrices of any size:

Step 1: Find $\det(A)$

$$\det A = 4 - 6 = -2$$

Step 2: Form a matrix of minors, M

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad M = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

Step 3: Form a matrix of cofactors, C

$$C = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Step 4: $A^{-1} = \frac{1}{\det(A)} C^T$

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

If $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$, find $\boldsymbol{A}^{-1}$.

Step 1: Find $\det(\boldsymbol{A})$

$$\det A = 1(0+1) + 0 + 2(3-4)$$
$$= 1 - 2 = \underline{\underline{-1}}$$

Step 2: Form a matrix of minors, $\boldsymbol{M}$

$$M = \begin{pmatrix} 1 & -2 & -8 \\ 1 & -2 & -7 \\ -1 & 1 & 4 \end{pmatrix}$$

Step 3: Form a matrix of cofactors, $\boldsymbol{C}$

$$C = \begin{pmatrix} 1 & 2 & -8 \\ -1 & -2 & 7 \\ -1 & -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Step 4: $A^{-1} = \frac{1}{\det(\boldsymbol{A})} \boldsymbol{C}^T$

$$A^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -1 \\ -8 & 7 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 8 & -7 & -4 \end{pmatrix}$$

Computing the inverse on your calculator

$\mathbf{A} = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$, and the matrix **B** is such that $(\mathbf{AB})^{-1} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}$.

(a) Show that $\mathbf{A}^{-1} = \mathbf{A}$.

(b) Find $\boldsymbol{B}^{-1}$.

What can we do to avoid the long process of finding an inverse? Be smart!

a) If $A^{-1} = A$ then A is self inverse.

$A^{-1}A = I$

If $AA = I$, $A^2 = I$, then $A^{-1} = A$

Calculate $A^2 = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

As $AA^{-1} = I$, $A = A^{-1}$.

b) $(PQ)^{-1} = Q^{-1}P^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

$B^{-1}\cancel{A^{-1}}\cancel{A} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix} A$

$B^{-1} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}\begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -7 & -2 & -6 \\ 4 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

... with algebra

[June 2011 Q7] The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} k & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}, \quad k \neq 1.$$

(*a*) Show that det **M** = 2 − 2*k*. **(2)**

(*b*) Find $\mathbf{M}^{-1}$, in terms of *k*. **(5)**

a) $\det M = k(\underline{-2}) + 1(\underline{1+3}) + 1(\underline{-2})$

$= -2k + 4 - 2$

$= \underline{\underline{2 - 2k}}$

$$\begin{pmatrix} & - & \\ - & & - \\ & - & \end{pmatrix}$$

b) the matrix of minors, $A = \begin{pmatrix} -2 & 4 & -2 \\ 1 & k-3 & -2k+3 \\ 1 & -k-1 & 1 \end{pmatrix}$

$C = \begin{pmatrix} -2 & -4 & -2 \\ -1 & k-3 & 2k-3 \\ 1 & k+1 & 1 \end{pmatrix}$

$\frac{1}{\det} C^T$

$$M^{-1} = \frac{1}{2-2k} \begin{pmatrix} -2 & -1 & 1 \\ -4 & k-3 & k+1 \\ -2 & 2k-3 & 1 \end{pmatrix}$$

Ex 6E