Edexcel A Level Further Maths: Core Pure

5.1 Volumes of Revolution

Your notes

Contents

Your notes

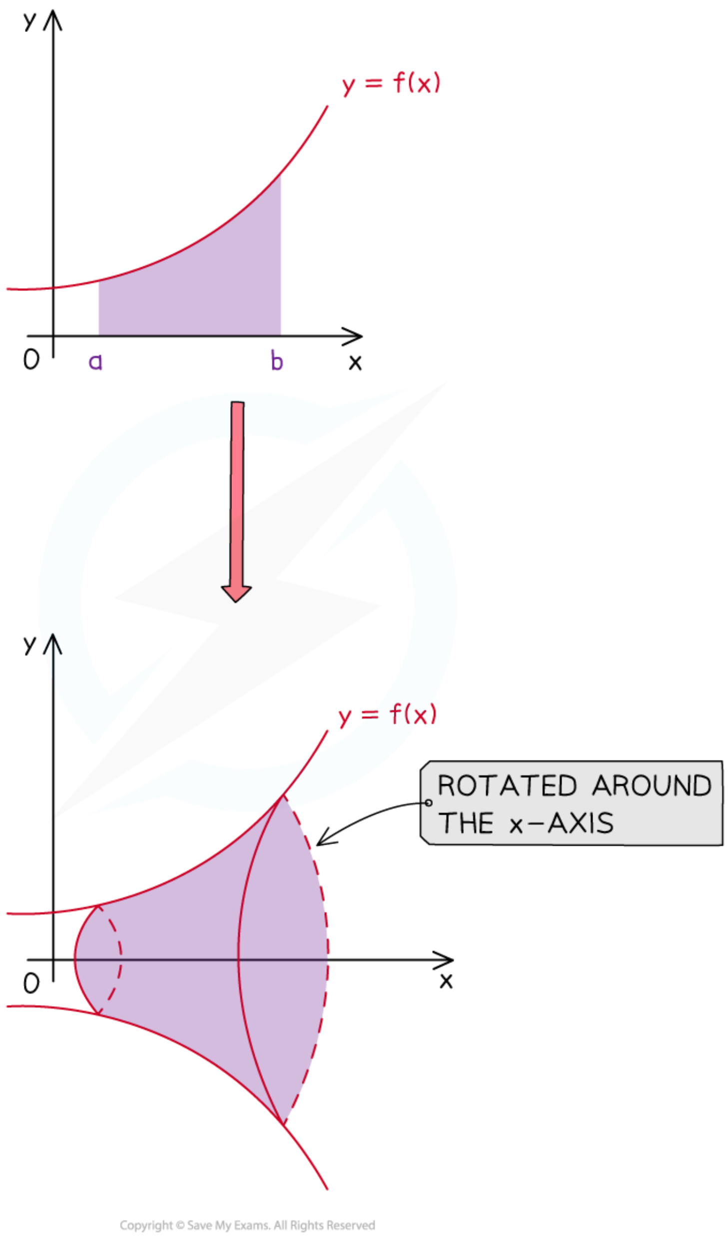
5.1.1 Volumes of Revolution

Volumes of revolution around the x-axis

What is a volume of revolution around the *x*-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = \mathrm{f}(x)$ (and other boundary equations) is rotated 360° around the *x*-axis
- A **volume** of **revolution** is the volume of this solid formed

Your notes



Example of a solid of revolution that is formed by rotating the area bounded by the function $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis $360°$ about the x-axis

How do I find the volume of revolution around the *x*-axis?

- To find the **volume** of **revolution** created when the area bounded by the function $y = \mathrm{f}(x)$, the lines $x = a$ and $x = b$, and the *x*-axis is rotated 360° about the *x*-axis use the formula

$$V = \pi \int_a^b y^2 \mathrm{d}x$$

Your notes

- The formula may look complicated or confusing at first due to the *y* and d*x*
  - remember that *y* is a function of *x*
  - once the expression for *y* is substituted in, everything will be in terms of *x*
- π is a constant so you may see this written either inside or outside the integral
- This is **not** given in the formulae booklet
  - The formulae booklet does list the volume formulae for some common 3D solids – it may be possible to use these depending on what information about the solid is available

Where does the formula for the volume of revolution come from?

- When you integrate to find the area under a curve you can see the formula by splitting the area into rectangles with small widths
  - The same method works for volumes
- Split the volume into cylinders with small widths
  - The radius will be the *y* value
  - The width will be a small interval along the *x*-axis δx
- The volume can be approximated by the sum of the volumes of these cylinders

$$V \approx \sum \pi y^2 \delta x$$

- The limit as δx goes to zero can be found by integration - just like with areas

$$\lim_{\delta x \to 0} \sum \pi y^2 \delta x = \int_a^b \pi y^2 \mathrm{d}x$$

How do I solve problems involving volumes of revolution around the *x*-axis?

- **Visualising** the solid created is helpful
  - Try sketching some functions and their solids of revolution to help
- **STEP 1** Square *y*
  - Do this first without worrying about π or the integration and limits
- **STEP 2** Identify the limits *a* and *b* (which could come from a graph)
- **STEP 3** Use the formula by evaluating the integral and multiplying by π
  - The answer may be required in **exact form** (leave in terms of π)
    - If not, round to **three significant figures** (unless told otherwise)
- Trickier questions may give you the volume and ask for the value of an unknown constant elsewhere in the problem

Your notes

Examiner Tip

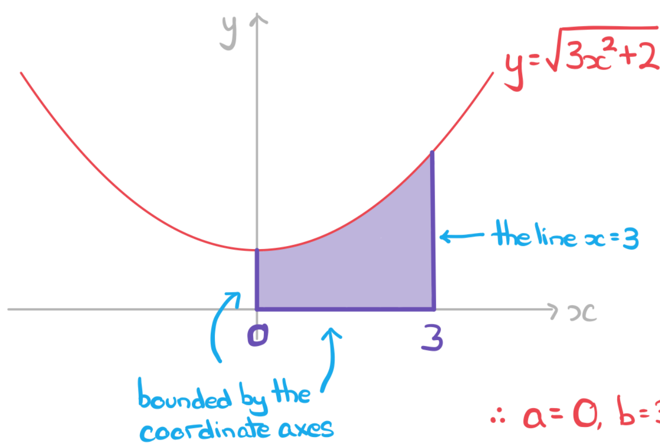
- To help remember the formula note that it is only y^2 - volume is 3D so you may have expected a cubic expression
  - If rotating a **single point** around the *x*-axis a **circle** of radius would be formed
    - The **area** of that circle would then be πy^2
    - Integration then adds up the areas of all circles between *a* and *b* creating the third dimension and **volume**
      (In 2D, integration creates area by adding up lots of 1D lines)

Your notes

Worked example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \sqrt{3x^2 + 2}$, the coordinate axes and the line $x = 3$ by 2π radians around the x-axis. Give your answer as an exact multiple of π.

STEP 1: Identify limits, sketch graph



$\therefore a = 0,\ b = 3$

STEP 2: Square y

$$y^2 = \left(\sqrt{3x^2+2}\right)^2 = 3x^2 + 2$$

STEP 3: Find the volume

$$V = \pi\int_0^3 (3x^2+2)\,dx = \pi\left[x^3 + 2x\right]_0^3$$
$$= \pi(27+6)$$

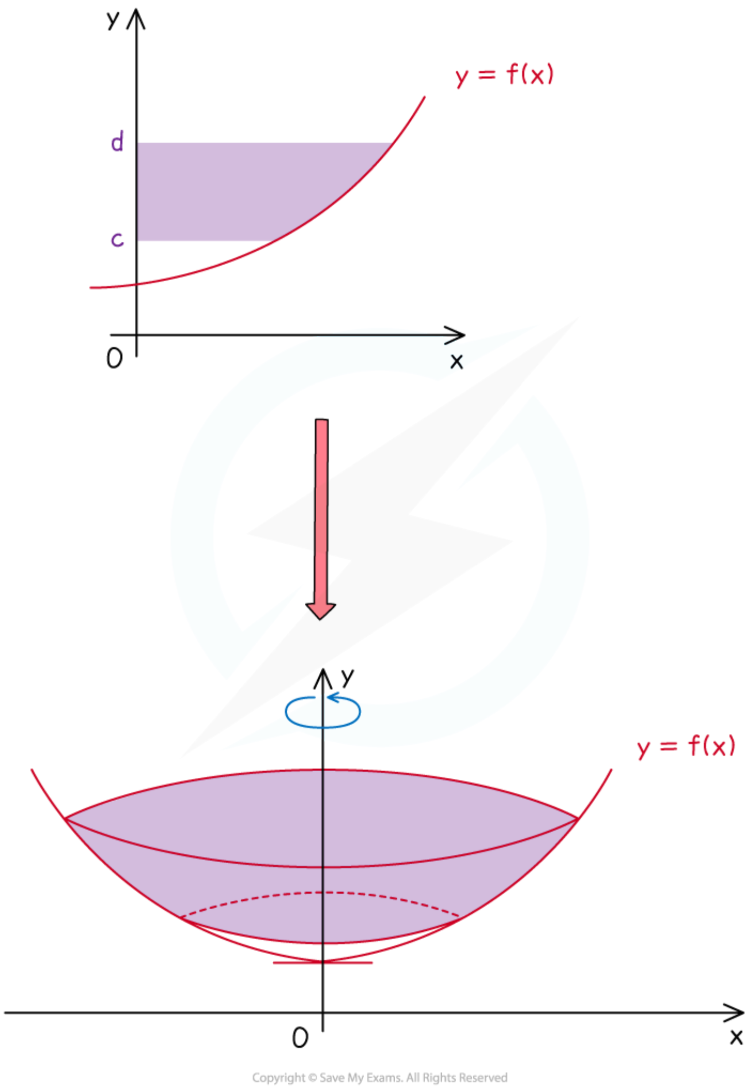
$$\boxed{\therefore V = 33\pi \text{ cubic units}}$$

Your notes

Volumes of revolution around the y-axis

What is a volume of revolution around the y-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 360° around the *y*-axis
- A **volume** of **revolution** is the volume of this solid formed



Example of a solid of revolution that is formed by rotating the area bounded by the function $y = f(x)$, the lines $y = c$ and $y = d$ and the x-axis $360°$ about the y-axis

How do I find the volume of revolution around the y-axis?

Your notes

- To find the **volume** of **revolution** created when the area bounded by the function $y = f(x)$, the lines $y = c$ and $y = d$, and the *y*-axis is rotated 360° about the *y*-axis use the formula

$$V = \pi \int_c^d x^2 \mathrm{d}y$$

- Note that although the function may be given in the form $y = f(x)$ it will first need rewriting in the form $x = g(y)$
- This is **not** given in the formulae booklet

How do I solve problems involving volumes of revolution around the *y*-axis?

- **Visualising** the solid created is helpful
  - Try sketching some functions and their solids of revolution to help
- **STEP 1** Rearrange $y = f(x)$ into the form $x = g(y)$ (if necessary)
  - This is finding the **inverse function** $f^{-1}(x)$
- **STEP 2** Square *x*
  - Do this first without worrying about π or the integration and limits
- **STEP 3** Identify the limits c and *d* (which could come from a graph)
- **STEP 4** Use the formula by evaluating the integral and multiplying by π
  - The answer may be required in **exact form** (leave in terms of π)
    - If not, round to **three significant figures** (unless told otherwise)
- Trickier questions may give you the volume and ask for the value of an unknown constant elsewhere in the problem

Examiner Tip

- Double check questions to ensure you are clear about which axis the rotation is around
- Separating the rearranging of $y = f(x)$ into $x = g(y)$ and the squaring of *x* is important for maintaining accuracy
  - In some cases it can seem as though *x* has been squared twice

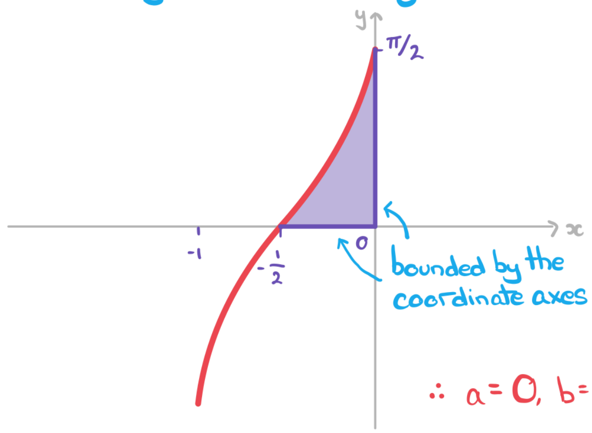
Your notes

Worked example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \arcsin(2x+1)$ and the coordinate axes by 2π radians around the y-axis. Give your answer to three significant figures.

Your notes

STEP 1: Identify limits, sketch graph



$\therefore\ a = 0,\ b = \pi/2$

STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = \arcsin(2x+1)$

$\sin y = 2x + 1$

$x = \frac{1}{2}(\sin y - 1)$

STEP 3: Square x

$x^2 = \frac{1}{4}(\sin y - 1)^2$

STEP 4: Find the volume

$$V = \pi \int_0^{\pi/2} \frac{1}{4}(\sin y - 1)^2 \, dy$$

As this is awkward, use your calculator but

- your calculator will expect the integral in terms of x
- remember π !

$V = 0.279\ 754 \ldots$

$$\boxed{\therefore V = 0.280 \text{ units}^3 \ (3 \text{ s.f.})}$$

Your notes

Volumes of Revolution using Parametric Equations

What is parametric volumes of revolution?

- **Solids** of **revolution** are formed by rotating functions about the x-axis or the y-axis
- Here though, rather than given y in terms of x, both x and y are given in terms of a parameter, t
  - $x = \mathrm{f}(t)$
  - $y = \mathrm{g}(t)$
  - Depending on the nature of the functions f and g it may not be convenient or possible to find y in terms of x

How do I find volumes of revolution when x and y are given parametrically?

- The aim is to replace everything in the 'original' integral so that it is in terms of t
- For the 'original' integral $V = \pi \int_{x_1}^{x_2} y^2 \mathrm{d}x$ or $V = \pi \int_{y_1}^{y_2} x^2 \mathrm{d}y$ and parametric equations given in the form $x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$ use the following process
- **STEP 1:** Find dx or dy in terms of t and dt
  - $\mathrm{d}x = \mathrm{f}'(t)\mathrm{d}t$ or $\mathrm{d}y = \mathrm{g}'(t)\mathrm{d}t$
- **STEP 2:** If necessary, change the limits from x values or y values to t values using
  - $x_1 = \mathrm{f}(t_1)$ or $y_1 = \mathrm{g}(t_1)$
  - $x_2 = \mathrm{f}(t_2)$ or $y_2 = \mathrm{g}(t_2)$
- **STEP 3:** Square y or x
  - $y^2 = [\mathrm{g}(t)]^2$ or $x^2 = [\mathrm{f}(t)]^2$
  - *Do this separately to avoid confusing when putting the integral together*
- **STEP 4:** Set up the integral, so everything is now in terms of t, simplify where possible and evaluate the integral to find the volume of revolution

$V = \pi \int_{t_1}^{t_2} (\mathrm{g}(t))^2 \mathrm{f}'(t)\mathrm{d}t$ (if around x-axis) or $V = \pi \int_{t_1}^{t_2} (\mathrm{f}(t))^2 \mathrm{g}'(t)\mathrm{d}t$ (if around y-axis)

Examiner Tip

- Avoid the temptation to jump straight to STEP 4
  - There could be a lot to change and simplify in exam style problems
  - Doing each step carefully helps maintain high levels of accuracy

Your notes

Worked example

The curve C is defined parametrically by $x = \sec(t)$ and $y = \sqrt{\operatorname{cosec}(t)}$. C is rotated 360° about the x-axis between the values of $x = \sqrt{2}$ and $x = 2$. Show that the volume of the solid of revolution generated by this rotation is $\pi(\sqrt{p} - q)$ cubic units where p and q are integers to be found.

STEP 1: Find "dx" in terms of t and dt

$$\frac{dx}{dt} = \sec t \tan t$$

$$dx = \sec t \tan t \; dt$$

STEP 2: If necessary, change limits

$x = \sqrt{2}$, $\sec t = \sqrt{2}$

$$t = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$ (radians)

$x = 2$, $\sec t = 2$

$$t = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

STEP 3: Square y

$$y^2 = \left(\sqrt{\operatorname{cosec} t}\right)^2 = \operatorname{cosec} t$$

STEP 4: Set up integral in terms of t, simplify and evaluate

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\operatorname{cosec} t)(\sec t \tan t) \; dt$$

$$V = \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 t \; dt = \pi \left[\tan t\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$\boxed{\therefore V = \pi\left(\sqrt{3} - 1\right) \text{ units}^3 \qquad (p = 3,\ q = 1)}$$

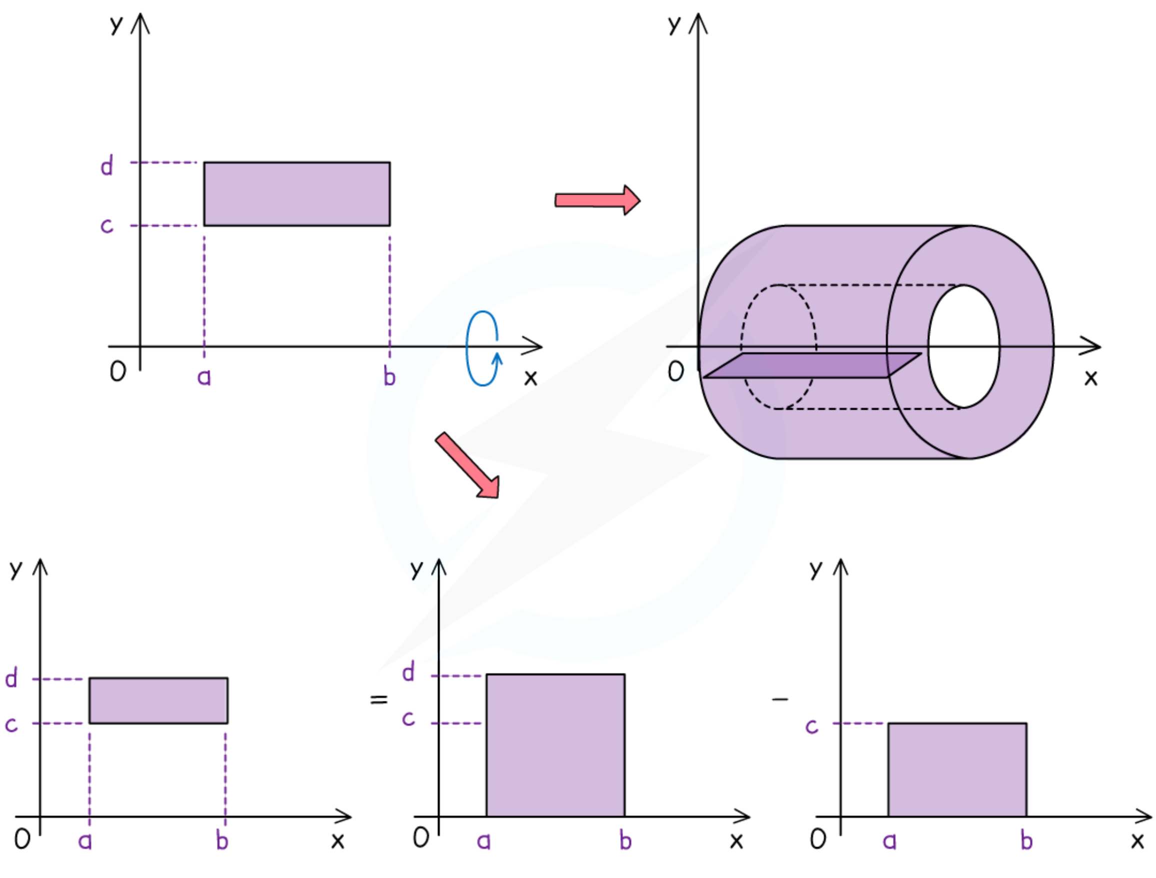
Your notes

5.1.2 Modelling with Volumes of Revolution

Adding and Subtracting Volumes

Why might I need to add or subtract volumes of revolution?

- As with the **area between a curve and a line** or the **area between 2 curves**, a required volume may be created by two functions
  - In this note we focus on volumes created by **rotation** around the ***x*-axis** but the same principles apply to rotation around the ***y*-axis**
  - Make sure you are familiar with the methods in **Volumes of Revolution**
- The volumes created here can be created from areas that do not have the *x*-axis as one its boundaries
  - A **cylinder** is created by rotating a rectangle that borders the *x*-axis around the *x*-axis by 360°
  - An **annular prism** (a cylinder with a whole through it – like a toilet roll) is created by rotating a rectangle that does **not** have a boundary with the *x*-axis around the *x*-axis by 360°
- A rectangle would be defined by two vertical and two horizontal lines

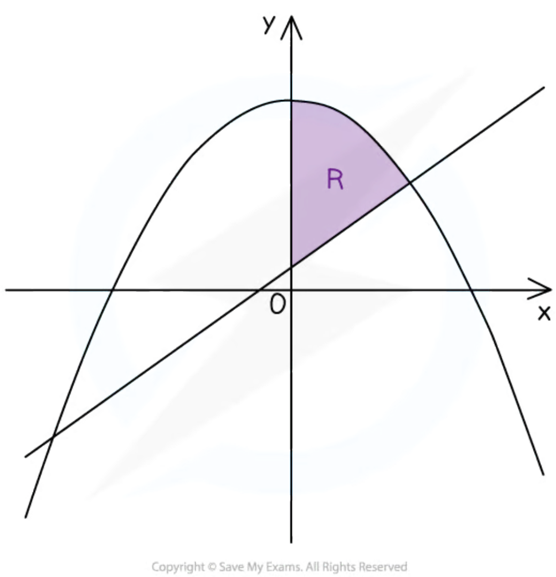


Your notes

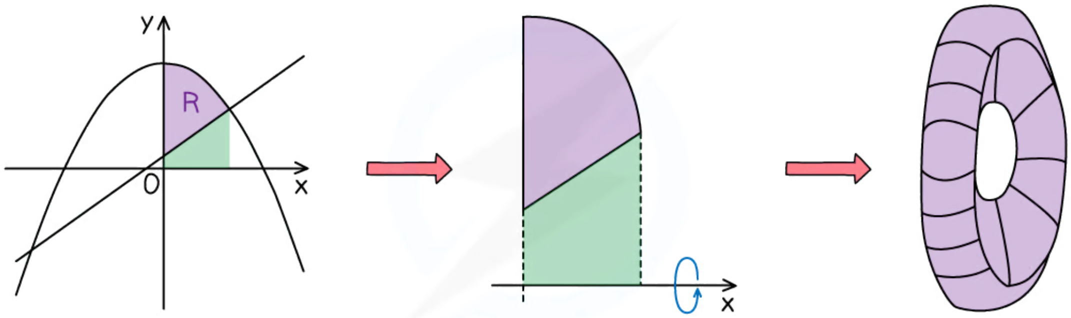
- $x = a,\ x = b,\ y = c,\ y = d$
  - Where a, b, c & d are all positive and a < b and c < d
- The volume of revolution of this rectangle would be
  - $V = \pi \int_a^b d^2 \mathrm{d}x - \pi \int_a^b c^2 \mathrm{d}x$

How do I know whether to add or subtract volumes of revolution?

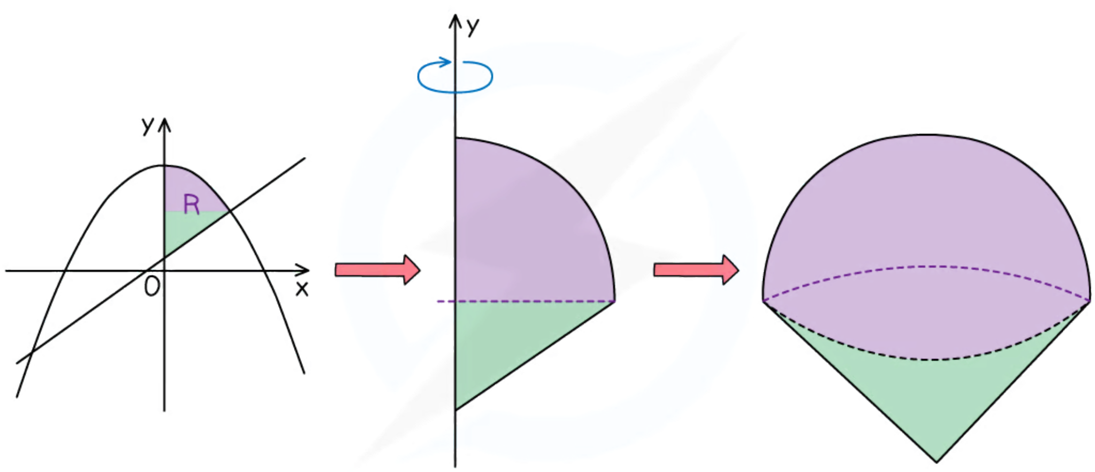
- When the **area** to be **rotated** around an axis has more than one function (and an axis) defining its boundary it can be trickier to tell whether to **add** or **subtract volumes** of **revolution**
  - It will depend on
    - The **nature** of the **functions** and their **points** of **intersection**
    - Whether **rotation** is around the ***x*-axis** or the ***y*-axis**
- Consider the region ***R***, bounded by a curve, a line and the -axis, in the diagram below



- If ***R*** is **rotated** around the X **-axis** the **solid** of **revolution** formed will have a 'hole' in its centre



- Think in 2D and area
  - "region under the curve"
    SUBTRACT
    "region under the line"
- If ***R*** is **rotated** around the y **-axis** the **solid** of **revolution** formed will look a little like a spinning top – with a 'dome top half' and a 'cone bottom half'



- Think in 2D and area
  - "top 'half' is the area 'below' the curve to the horizontal where the curve and line intersect"
    ADD
    "bottom 'half' is area 'below' the line to the horizontal where the curve and line interest"

How do I solve problems involving adding or subtracting volumes of revolution?

- Visualising the solid created becomes increasingly useful (but also trickier) for shapes generated by separate volumes of revolution

- Continue trying to sketch the functions and their solids of revolution to help

- **STEP 1:** Identify the axis that the area will be rotated around
  - Identify the functions $(y_1, y_2, \ldots)$ involved in generating the volume
  - Determine whether these will need to be added or subtracted
- **STEP 2:** If rotating around the *x*-axis, square *y* for all functions
  - If rotating around the y-axis, rearrange all the *y* functions into the form $x = \mathrm{g}(y)$ and square
  - In either case do this first without worrying about π or the integration and limits
- **STEP 3:** Identify the limits for each volume involved and form the integrals required
  - The limits could come from a graph
- **STEP 4:** Evaluate the integral for each function and add or subtract as necessary
  - The answer may be required in **exact form**
  - If not, round to **three significant figures** (unless told otherwise)

Examiner Tip

- It is possible, in subtraction questions, to combine the separate integrals into one
  - This is possible when the limits for each function are often the same in subtraction questions

$$V = \pi \int_a^b y_1^{\,2} \mathrm{d}x - \pi \int_a^b y_2^{\,2} \mathrm{d}x = \pi \int_a^b \left(y_1^{\,2} - y_2^{\,2}\right) \mathrm{d}x$$

- This doesn't really apply to addition questions as if the limits are the same, you would be adding some of the same volume twice
- If in any doubt avoid this approach as **accuracy** is far more important

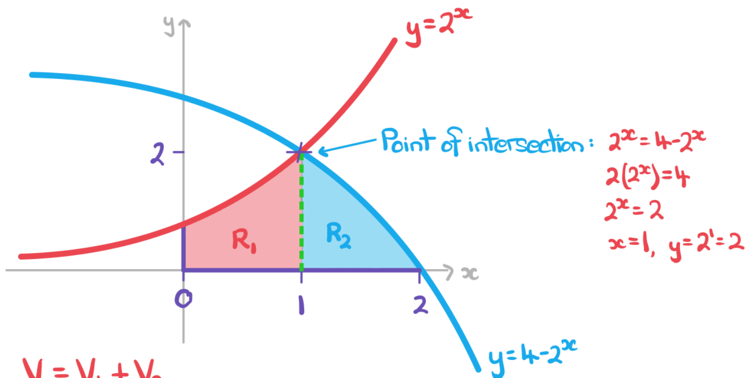
Your notes

Your notes

Worked example

Find the volume of revolution of the solid formed by rotating the region enclosed by the positive coordinate axes and the graphs of $y = 2^x$ and $y = 4 - 2^x$ by 2π radians around the x-axis. Give your answer to three significant figures.

STEP 1: Identify functions, limits and whether to add or subtract
Sketch the graphs



$V = V_1 + V_2$

For R_1, $a=0$, $b=1$

For R_2, $a=1$, $b=2$

STEP 2: Square all functions - this step is not required in this question

STEP 3: Use formula for each part, evaluate and add

$$V = \pi\int_0^1 (2^x)^2\,dx + \pi\int_1^2 (4-2^x)^2\,dx$$

Use your calculator to evaluate - to avoid typing errors evaluate each integral separately, store in memory, then add

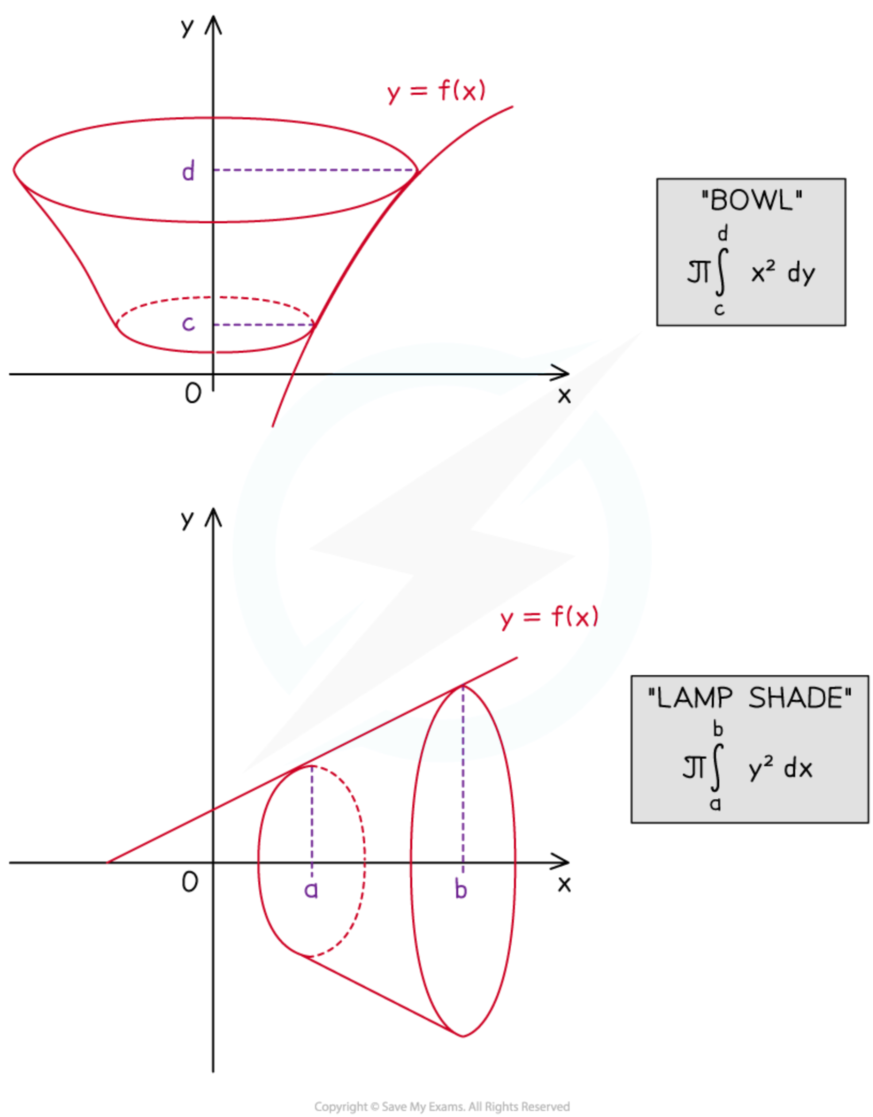
$V = 6.798\,540... + 4.941\,881... = 11.740...$

$\boxed{\therefore V = 11.7 \text{ cubic units } (3 \text{ s.f.})}$

Modelling with Volumes of Revolution

What is meant by modelling with volumes of revolution?

- Many every day objects such as buckets, beakers, vases and lamp shades can be modelled as a **solid of revolution**
- This can then be used to find the **volume** of the **solid** (**volume of revolution**) and/or other information about the solid that could be useful before an object is manufactured
- Modelling with volumes of revolution could involve rotation around the ***x*-axis** or ***y*-axis** so ensure you are familiar with both



What modelling assumptions are there with volumes of revolution?

Your notes

Your notes

- The solids formed are usually the **main shape** of the **body** of the **object**
  - For example, the handles on a vase would **not** be included
  - The lip on the top edge of a bucket would **not** be considered
- A common question or **assumption** concerns the **thickness** of a container
  - The thickness is generally ignored as it is relatively small compared to the size of the object
    - thickness will depend on the purpose of the object and the material it is made from
  - Some questions may refer to the solid formed being the 'inside' of an object or refer to the 'internal' dimensions
  - If the thickness of the material is significant it would involve two related solids of revolution (**Adding & Subtracting Volumes**)

How do I solve modelling problems with volumes of revolution?

- **Visualising** and sketching the solid formed can help with starting problems
- Familiarity with applying the volume of revolution formulae for rotations around both the *x* and *y* axes

$$x\text{-axis} \quad V = \pi \int_a^b y^2 \, \mathrm{d}x$$

$$y\text{-axis} \quad V = \pi \int_c^d x^2 \, \mathrm{d}y$$

- The volume of a solid may involve adding or subtracting different volumes of revolution
  - Subtraction would need to be used for solids formed from areas that do not have a boundary with the axis of rotation
- Questions may go on to ask related questions in context so do take notice of the context
  - A question about a **bucket** being formed may ask about its **capacity**
- This would be measured in **litres** so there may also be a mix of **units** that will need conversion (e.g. $1000\text{cm}^3 = 1$ litre)

Examiner Tip

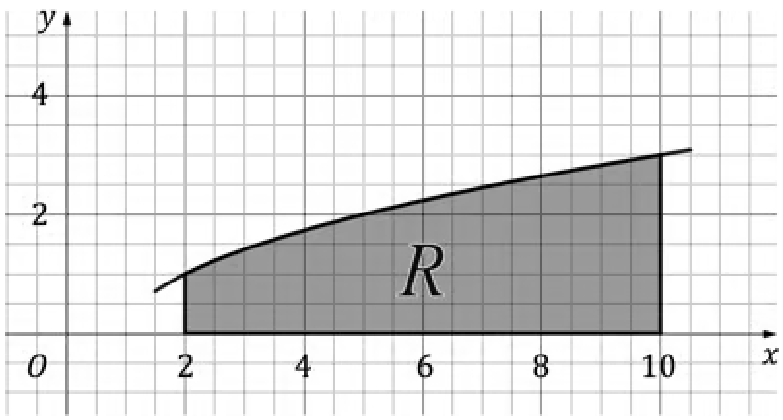
- Consider the context of the question to gauge whether your final answer is realistic
  - Look at the Worked Example below
  - a vase holding just 0.126 litres of water will not hold many flowers, but the question did state it was a *miniature* vase
- For rotation around the *y* -axis remember to rearrange $y = \mathrm{f}(x)$ into the form $x = \mathrm{g}(y)$

Your notes

Worked example

The diagram below shows the region *R*, which is bounded by the function $y = \sqrt{x-1}$, the lines $x = 2$ and $x = 10$, and the x-axis.

Dimensions are in centimetres.



A mathematical model for a miniature vase is produced by rotating the region *R* through 2π radians around the x-axis.

Find the volume of the miniature vase, giving your answer in litres to three significant figures.

Your notes

STEP 1 Identify limits

$a = 2$

$b = 10$

STEP 2 Square y

$y^2 = (\sqrt{x-1})^2 = x - 1$

STEP 3 Evaluate the integral

$$V = \pi \int_2^{10} (x-1)\, dx = \pi \left[0.5x^2 - x \right]_2^{10}$$

$$= \pi \left[(50-10) - (2-2) \right]$$

$$= 40\pi$$

Now we need to interpret this in the context of the miniature vase

$V = 40\pi \text{ cm}^3$

$V = \dfrac{40\pi}{1000}$ litres $\qquad 1000\text{cm}^3 = 1$ litre

$V = 0.125\ 663\ldots$

$\boxed{\text{Volume of the miniature vase is } 0.126 \text{ litres } (3 \text{ s.f.})}$