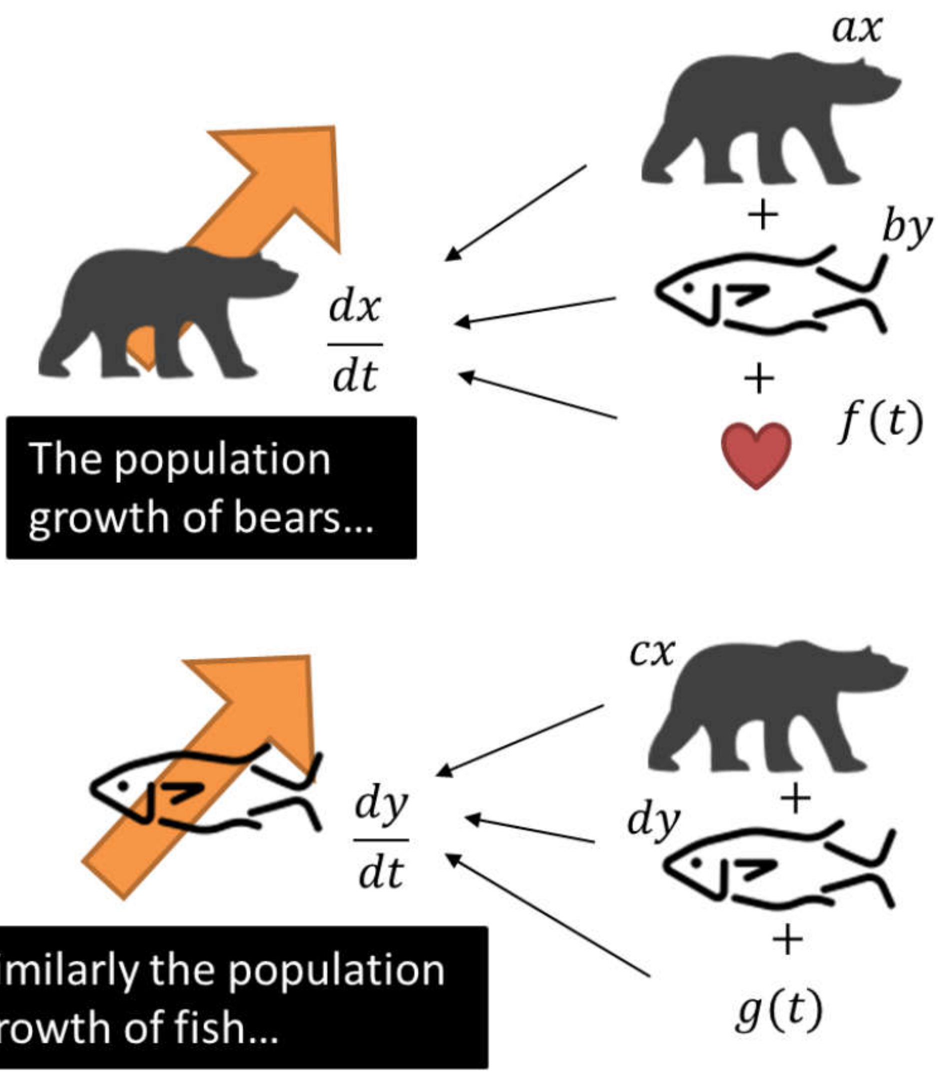
Coupled First-Order Linear Differential Equations

In Biology, **Lotka-Volterra equations**, also known as **predator-prey equations**, describe how two species interact, in terms of their populations.
Suppose there are x bears and y fish:



…clearly depends on (and more specifically, is proportional to) the number of bears
(i.e. more bears leads to more baby bears)

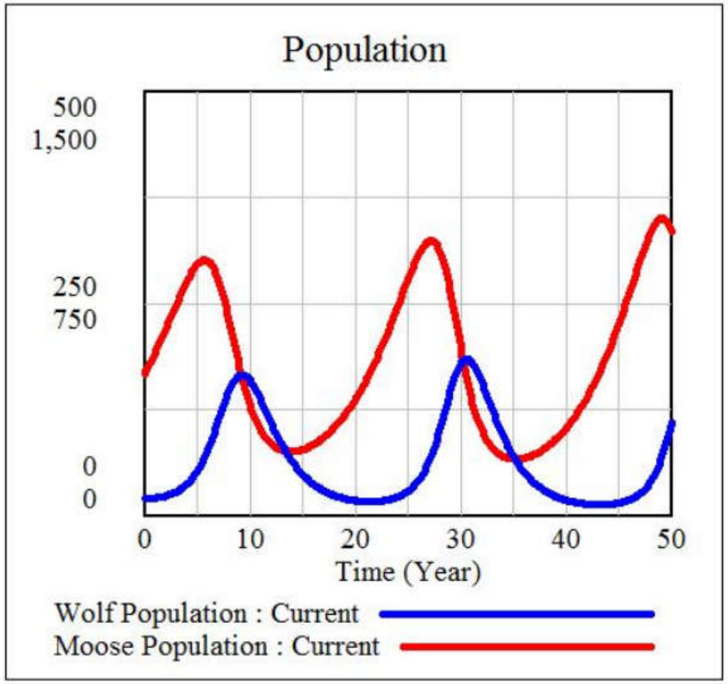
…but also on the availability of **prey**
(i.e. more fish, more bears)

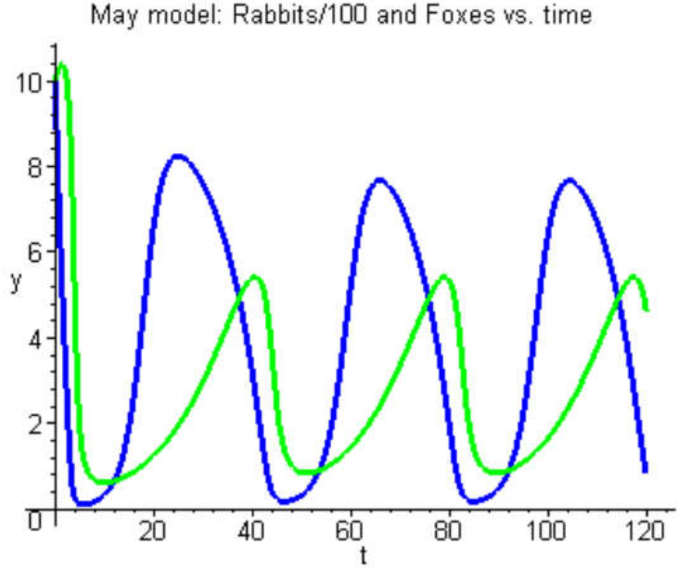
…and possibly some other additional factor dependent on time
(e.g. bears mate more in the summer)

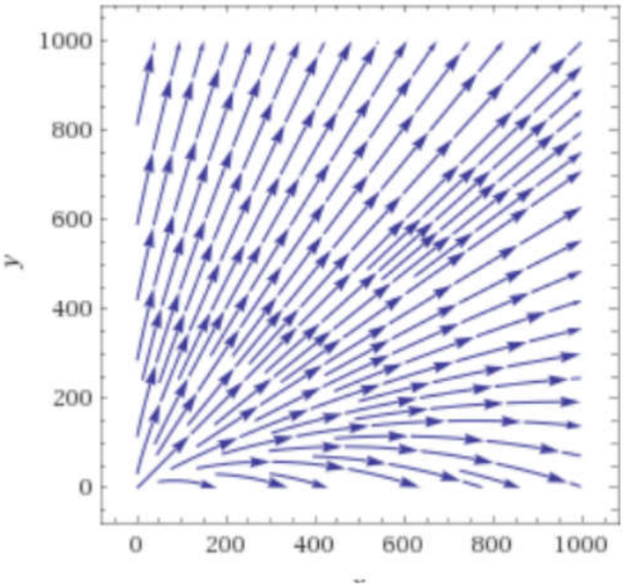
…clearly depends on the number of **predators**
(i.e. more bears, a greater rate of fish decline!)

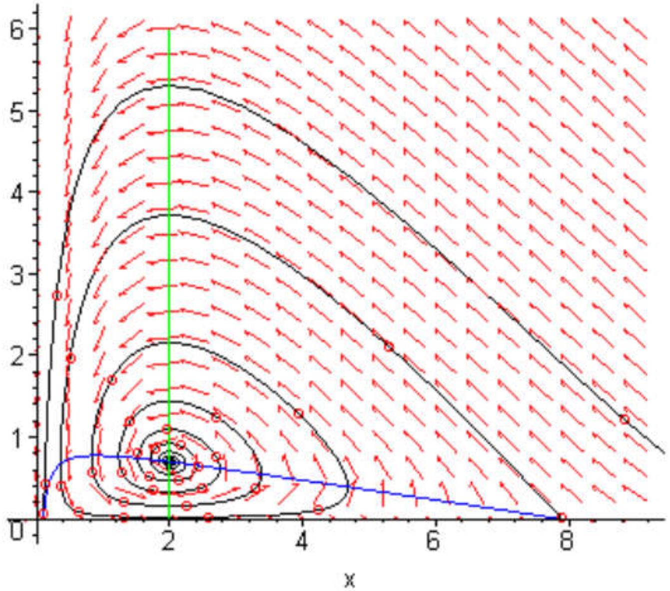
…but also on the number of fish
(i.e. more fish, more babies)

…and again some other time-dependent factor









stream plot	$(0.3\,x + 0.1\,y,\ -0.1\,x + 0.5\,y)$	$x = 0$ to 1000 $y = 0$ to 1000

Coupled first-order linear differential equations:

$$\frac{dx}{dt} = ax + by + f(t)$$
$$\frac{dy}{dt} = cx + dy + g(t)$$

Homogeneous if $f(t) = g(t) = 0$ for all t.

Possible strategy to solve for x:

1. Make y the subject of first equation then differentiate to find $\frac{dy}{dt}$.
2. Substitute into second equation to get single **second-order** differential equation just in terms of x, and solve.
3. To solve for y, no need to repeat whole process. Differentiate x from Step 2 and sub x and $\frac{dx}{dt}$ into y from Step 1.

At the start of the year 2010, a survey began on the numbers of bears and fish on a remote island in Northern Canada. After t years the number of bears, x, and the number of fish, y, on the island are modelled by the differential equations

$$\frac{dx}{dt} = 0.3x + 0.1y \quad (1)$$
$$\frac{dy}{dt} = -0.1x + 0.5y \quad (2)$$

(a) Show that $\frac{d^2x}{dt^2} - 0.8\frac{dx}{dt} + 0.16x = 0$ ✓

(b) Find the general solution for the number of bears on the island at time t.

(c) Find the general solution for the number of fish on the island at time t.

(d) At the start of 2010 there were 5 bears and 20 fish on the island.
Use this information to find the number of bears predicted to be on the island in 2020.

(e) Comment on the suitability of the model.

a)
$\dot{x} = 0.3x + 0.1y$

$\dot{x} - 0.3x = 0.1y$

$\boxed{10\dot{x} - 3x = y}$ $\frac{d}{dt}$

$\boxed{10\ddot{x} - 3\dot{x} = \dot{y}}$

$\frac{dx}{dt} \rightarrow \frac{d^2x}{dt^2}$

$\dot{y} = -0.1x + 0.5y$

$10\ddot{x} - 3\dot{x} = -0.1x + 0.5(10\dot{x} - 3x)$

$10\ddot{x} - 8\dot{x} + 1.6x = 0$

$\ddot{x} - 0.8\dot{x} + 0.16x = 0.$

$\frac{d^2x}{dt^2} - 0.8\frac{dx}{dt} + 0.16x = 0$

b) A.E. $m^2 - 0.8m + 0.16 = 0$

$m = 0.4$

Bears.
G.S. $x = (A+Bt)e^{0.4t}$

c) $\dot{x} = 0.4(A+Bt)e^{0.4t} + Be^{0.4t}$

Use (1) $\dot{x} = 0.3x + 0.1y$,

$y = 10\dot{x} - 3x$

$y = 4(A+Bt)e^{0.4t} + 10Be^{0.4t} - 3(A+Bt)e^{0.4t}$

Fish $y = (A+Bt)e^{0.4t} + 10Be^{0.4t}$

d) $x = 5,\ y = 20,\ t = 0$

$5 = A$

$20 = A + 10B$

$15 = 10B$

$B = 1.5$

Bears in 2020, $t = 10$, $x = ?$

$x = (5 + 1.5t)e^{0.4t}$

$x = (5 + 1.5\times10)e^{4}$

$= \underline{1092}$ (nearest bear)

e) As $t \to \infty$, population of bears and fish also tend to ∞. i.e. the populations keep growing. This is unrealistic – the model should take into account other factors like other species; illnesses/disease; resources available (like fish food?); space available

Two barrels contain contaminated water. At time t seconds, the amount of contaminant in barrel A is x ml and the amount of contaminant in barrel B is y ml. Additional contaminated water flows into barrel A at a rate of 5ml per second. Contaminated water flows from barrel A to barrel B and from barrel B to barrel A through two connecting hoses, and drains out of barrel A to leave the system completely.
The system is modelled using the differential equations

$$\frac{dx}{dt} = 5 + \frac{4}{9}y - \frac{1}{7}x \quad (1)$$
$$\frac{dy}{dt} = \frac{3}{70}x - \frac{4}{9}y \quad (2)$$

Show that $630\frac{d^2y}{dt^2} + 370\frac{dy}{dt} + 28y = 135$

Use strategy as per previous slide, but now need to make x subject in (2) and sub into (1).

Use (2) to find x and $\dot{x}$

$$\dot{y} = \frac{3}{70}x - \frac{4}{9}y$$
$$\dot{y} + \frac{4}{9}y = \frac{3}{70}x$$
$$\frac{70}{3}\dot{y} + \frac{280}{27}y = x$$
$$\frac{70}{3}\ddot{y} + \frac{280}{27}\dot{y} = \dot{x}$$

$\frac{d}{dt}$

Sub into (1).

$$\frac{70}{3}\ddot{y} + \frac{280}{27}\dot{y} = 5 + \frac{4}{9}y - \frac{1}{7}\left(\frac{70}{3}\dot{y} + \frac{280}{27}y\right)$$
$$\frac{70}{3}\ddot{y} + \frac{280}{27}\dot{y} = 5 + \frac{4}{9}y - \frac{10}{3}\dot{y} - \frac{40}{27}y$$
$$\frac{70}{3}\ddot{y} + \frac{370}{27}\dot{y} + \frac{28}{27}y = 5$$
$$630\ddot{y} + 370\dot{y} + 28y = 135$$
$$630\frac{d^2y}{dt^2} + 370\frac{dy}{dt} + 28y = 135$$

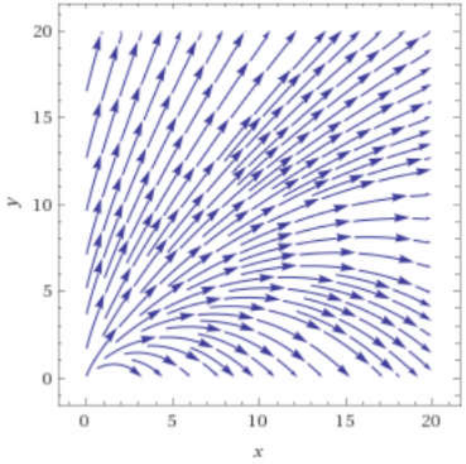
A.E.
C.F.
P.I.
G.S.
P.S.

$y = f(t)$
$\dot{y} = f'(t)$
Sub into (2)

To find $x(t)$

Ex 8D

Your Turn - exam question



stream plot	$(0.2\,x + 0.1\,y,\ -0.2\,x + 0.4\,y)$	$x = 0$ to 20
		$y = 0$ to 20

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r$$

(a) Show that $\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\dfrac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

7(a)	$r = 10\frac{df}{dt} - 2f \Rightarrow \frac{dr}{dt} = 10\frac{d^2f}{dt^2} - 2\frac{df}{dt}$	M1	2.1
	$10\frac{d^2f}{dt^2} - 2\frac{df}{dt} = -0.2f + 0.4\left(10\frac{df}{dt} - 2f\right)$	M1	2.1
	$\frac{d^2f}{dt^2} - 0.6\frac{df}{dt} + 0.1f = 0$ *	A1*	1.1b
		(3)	
(b)	$m^2 - 0.6m + 0.1 = 0 \Rightarrow m = \frac{0.6 \pm \sqrt{0.6^2 - 4\times 0.1}}{2}$	M1	3.4
	$m = 0.3 \pm 0.1\mathrm{i}$	A1	1.1b
	$f = \mathrm{e}^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$	M1	3.4
	$f = \mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	A1	1.1b
		(4)	
(c)	$\frac{df}{dt} = 0.3\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right) + 0.1\mathrm{e}^{0.3t}\left(B\cos 0.1t - A\sin 0.1t\right)$	M1	3.4
	$r = 10\frac{df}{dt} - 2f$ $= \mathrm{e}^{0.3t}\left(\left(3A+B\right)\cos 0.1t + \left(3B - A\right)\sin 0.1t\right) - 2\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	M1	3.4
	$r = \mathrm{e}^{0.3t}\left(\left(A+B\right)\cos 0.1t + \left(B - A\right)\sin 0.1t\right)$	A1	1.1b
		(3)	
(d)(i)	$t = 0, f = 6 \Rightarrow A = 6$	M1	3.1b
	$t = 0, r = 20 \Rightarrow B = 14$	M1	3.3
	$r = \mathrm{e}^{0.3t}\left(20\cos 0.1t + 8\sin 0.1t\right) = 0$	M1	3.1b
	$\tan 0.1t = -2.5$	A1	1.1b
	2019	A1	3.2a
(d)(ii)	3750 foxes	B1	3.4
(d)(iii)	e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible	B1	3.5a
		(7)	
			(17 marks)

Your Turn - exam question

8. A doctor is studying the concentration of an antibiotic in the blood and the body tissue of a patient.
Let x be the number of micrograms of the antibiotic in the blood.
Let y be the number of micrograms of the antibiotic in the body tissue.
The doctor models her results by the differential equations

$$\frac{dx}{dt} = -5x + y + 51$$

$$\frac{dy}{dt} = 12x - 6y$$

where t is the time in hours after a dose of the antibiotic has been administered to the patient.

(a) Show that

$$\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + 18x = 306$$

(3)

(b) Find a general solution for the number of micrograms of the antibiotic in the blood at time t hours.

(6)

(c) Hence find a general solution for the number of micrograms of the antibiotic in the body tissue at time t hours.

(2)

Initially there is none of this antibiotic in the blood and none of this antibiotic in the body tissu

(d) Find, in minutes, to 2 decimal places, the time when the rate of increase of the antibiotic in the blood is equal to the rate of increase of the antibiotic in the body tissue.

(5)

(e) Evaluate the model.

(1)

8(a)	$y=\frac{\mathrm{d}x}{\mathrm{d}t}+5x-51 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+5\frac{\mathrm{d}x}{\mathrm{d}t}$	B1	2.1
	$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2}+5\frac{\mathrm{d}x}{\mathrm{d}t}=12x-6\left(\frac{\mathrm{d}x}{\mathrm{d}t}+5x-51\right)$	M1	2.1
	$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2}+11\frac{\mathrm{d}x}{\mathrm{d}t}+18x=306$*	A1*	1.1b
		(3)	
(b)	$m^2+11m+18=0 \Rightarrow m=\ldots$	M1	3.4
	$m=-2,-9$	A1	1.1b
	$x=A\mathrm{e}^{\alpha t}+B\mathrm{e}^{\beta t}$	M1	3.4
	$x=A\mathrm{e}^{-9t}+B\mathrm{e}^{-2t}$	A1	1.1b
	PI: Try $x=k \Rightarrow 18k=306$ $\Rightarrow k=17$	M1	3.4
	$GS: x=A\mathrm{e}^{-9t}+B\mathrm{e}^{-2t}+17$	A1ft	1.1b
		(6)	
(c)	$y=\frac{\mathrm{d}x}{\mathrm{d}t}+5x-51 \Rightarrow y=-9A\mathrm{e}^{-9t}-2B\mathrm{e}^{-2t}+5A\mathrm{e}^{-9t}+5B\mathrm{e}^{-2t}+85-51$	M1	3.4
	$y=3B\mathrm{e}^{-2t}-4A\mathrm{e}^{-9t}+34$	A1	1.1b
		(2)	
(d)	$0=A+B+17,\ 0=3B-4A+34 \Rightarrow A=\ldots, B=\ldots$ (NB $A=-\frac{17}{7}, B=-\frac{102}{7}$)	M1	3.3
	$x=17-\frac{17}{7}\mathrm{e}^{-9t}-\frac{102}{7}\mathrm{e}^{-2t}, y=34+\frac{68}{7}\mathrm{e}^{-9t}-\frac{306}{7}\mathrm{e}^{-2t}$	A1	1.1b
	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}t} \Rightarrow \frac{153}{7}\mathrm{e}^{-9t}+\frac{204}{7}\mathrm{e}^{-2t}=-\frac{612}{7}\mathrm{e}^{-9t}+\frac{612}{7}\mathrm{e}^{-2t} \Rightarrow \mathrm{e}^{k}=\alpha$	M1	3.1b
	$\mathrm{e}^{7t}=\frac{15}{8} \Rightarrow 7t=\ln\left(\frac{15}{8}\right) \Rightarrow t=\frac{1}{7}\ln\left(\frac{15}{8}\right)$	M1	1.1b
	= 5.39 minutes	A1	3.2a
		(5)	
(e)	E.g. • The model suggests that, in the long term, the amount of antibiotic in the blood (and/or the body tissue) will remain constant and this is unlikely	B1	3.5a
		(1)	

Your Turn - exam question

9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 200\cos t, \qquad t \geqslant 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be $10\,\mathrm{ms}^{-2}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)

9(a)(i)	Weight = mass × g $\Rightarrow m = \dfrac{30000}{g} = 3000$ But mass is in thousands of kg, so $m = 3$	M1	3.3
(ii)	$\dfrac{dx}{dt} = 40\cos t + 20\sin t,\ \dfrac{d^2x}{dt^2} = -40\sin t + 20\cos t$	M1	1.1b
	$3(-40\sin t + 20\cos t) + 4(40\cos t + 20\sin t)$ $+ 40\sin t - 20\cos t = \ldots$	M1	1.1b
	$= 200\cos t$ so PI is $x = 40\sin t - 20\cos$	A1*	2.1
	or		
	Let $x = a\cos t + b\sin t$ $\dfrac{dx}{dt} = -a\sin t + b\cos t,\ \dfrac{d^2x}{dt^2} = -a\cos t - b\sin t$	M1	1.1b
	$4b - 2a = 200,\ -2b - 4a = 0 \Rightarrow a = \ldots, b = \ldots$	M1	2.1
	$x = 40\sin t - 20\cos t$	A1*	1.1b
(iii)	$3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = -1, -\dfrac{1}{3}$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t}$	A1	1.1b
	$x = PI + CF$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
		(8)	
(b)	$t = 0, x = 0 \Rightarrow A + B = 20$	M1	3.4
	$x = 0, \dfrac{dx}{dt} = -Ae^{-t} - \dfrac{1}{3}Be^{-\frac{1}{3}t} + 40\cos t + 20\sin t = 0$ $\Rightarrow A + \dfrac{1}{3}B = 40$	M1	3.4
	$x = 50e^{-t} - 30e^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
	$t = 9 \Rightarrow x = 33\text{m}$	A1	3.4
		(4)	