Differential Equations - Core Pure 2

Differential equations are equations which relate x and y with derivatives. e.g.

The rate of temperature loss is proportional to the current temperature.	$\frac{dT}{dt} = -kT$
The rate of population change is proportional to $P\left(1 - \frac{P}{M}\right)$ where P is the current population and M is the limiting size of the population (the Verhulst-Pearl Model)	$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$
Suppose x is GDP (Gross Domestic Product). Rate of change of GDP is proportional to current GDP.	$\frac{dx}{dt} = kx$

As you might imagine, they're used a lot in physics and engineering, including modelling radioactive decay, mixing fluids, cooling materials and bodies falling under gravity against resistance.

A 'first order' differential equation means the equation contains the first derivative $\left(\frac{dy}{dx}\right)$ but not the second derivative or beyond.

Separating the Variables

x and y are said to be '**separated**' because we can express the RHS as a product of two separate expressions: one in terms of just x and one in terms of just y.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Divide through by $g(y)$ and times through by dx, and slap an integral on the front!

+c

Find general solutions to $\frac{dy}{dx} = -\frac{y}{x}$

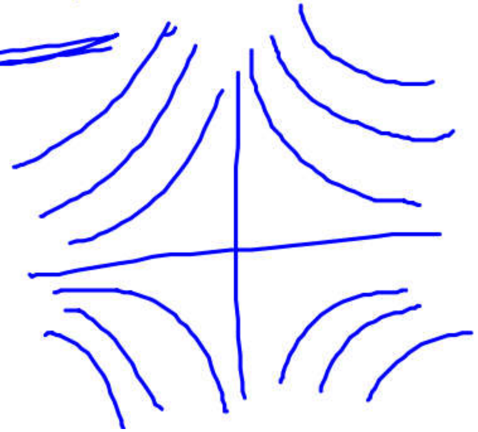
$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x}$$

$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$$

$$\ln|y| = -\ln|x| + \ln k$$

$$\ln|y| = \ln\left|\frac{k}{x}\right|$$

$$y = \frac{k}{x}$$



Using reverse product rule

We will see in a bit how to solve equations of the form $\frac{dy}{dx} + Py = Q$ (where P and Q are functions of x). We'll practice a particular part of this method before going for the full thing.

Find general solutions of the equation $x^3 \frac{dy}{dx} + 3x^2 y = \sin x$

$$\frac{d}{dx}(x^3 y) = \sin x$$

$$x^3 y = \int \sin x \, dx$$

$$x^3 y = -\cos x + c$$

$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

What is different about this equation?

$\cancel{k = \frac{c}{x^3}}$

Quickfire Questions:

$$\frac{d}{dx}(x^2 y) = x^2 \frac{dy}{dx} + 2xy$$

$$\frac{d}{dx}(y \sin(x)) = \sin x \frac{dy}{dx} + y \cos x$$

So it appears whatever term ends up on front of the $\frac{dy}{dx}$ will be on the front of the y in the integral.

$$x^4 \frac{dy}{dx} + 4x^3 y \rightarrow \frac{d}{dx}(x^4 y)$$

$$e^x \frac{dy}{dx} + e^x y \rightarrow \frac{d}{dx}(e^x y)$$

$$(\ln x) \frac{dy}{dx} + \frac{y}{x} \rightarrow \frac{d}{dx}(y \ln x)$$

Find general solutions of the equation $x^3 \frac{dy}{dx} + 3x^2 y = \sin x$

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$$y = -\frac{\cos x}{x^3} + \frac{c}{x^3}$$

Find general solutions of the equation

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = e^x$$

$$\frac{y}{x} = \int e^x \, dx$$

$$\frac{y}{x} = e^x + c$$

$$y = xe^x + cx$$

$$\frac{dy}{dx} = e^x + xe^x + c$$

$$\frac{1}{x}(e^x + xe^x + c) - \frac{1}{x^2}(xe^x + cx)$$

$$= \cancel{\frac{e^x}{x}} + e^x + \cancel{\frac{c}{x}} - \cancel{\frac{e^x}{x}} - \cancel{\frac{c}{x}}$$

$$= e^x$$

Find general solutions of the equation

$$4xy\frac{dy}{dx} + 2y^2 = x^2$$

$2y\frac{dy}{dx} \times 2x$

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

$$\frac{d}{dx}\left(2xy^2\right) = x^2$$

$$2xy^2 = \int x^2 \, dx$$

$$2xy^2 = \frac{1}{3}x^3 + c$$

$$y^2 = \frac{1}{6}x^2 + \frac{c}{2x}$$

where $k = \frac{c}{2}$

$$y^2 = \frac{1}{6}x^2 + \frac{k}{x}$$

$$y = \sqrt{\frac{1}{6}x^2 + \frac{k}{x}}$$

Ex 7A
Q3
Q5a
Q6

$dy/dx + Py = Q$ Integrating Factor . $e^{\int P\,dx}$

But what if we can't use the product rule backwards?

$\boxed{\frac{dy}{dx}} + P(x)\boxed{y} = Q(x)$

alone

Find the general solution of $\frac{dy}{dx} \underbrace{\boxed{-4}}_{}y = \boxed{e^x}$

$P(x)$ $Q(x)$

$\frac{dy}{dx} + Py = Q$ where P and Q are x functions

We can multiply through by the integrating factor $e^{\int P\,dx}$. This then produces an equation where we can use the previous reverse-product-rule trick (we'll prove this in a bit).

$P(x) = -4$

$\int P(x) = -4x$

$$I.F. = e^{\int(-4)\,dx} = e^{-4x}$$

Then multiplying through by the integrating factor:

$$e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = e^x \times e^{-4x}$$

Then we can solve in the usual way:

$$\frac{d}{dx}\left(ye^{-4x}\right) = e^{-3x}$$

$$ye^{-4x} = -\frac{1}{3}e^{-3x} + c$$

$$y = -\frac{1}{3}e^{x} + ce^{4x}$$

Proof that Integrating Factor works

Solve the general equation $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x.

Suppose $f(x)$ is the Integrating Factor. As usual we'd multiply by it:

$$f(x)\frac{dy}{dx} + \underline{f(x)Py} = f(x)Q$$

If we can use the reverse product rule trick on the LHS, then it would be of the form:

$$f(x)\frac{dy}{dx} + \underline{f'(x)y} \qquad \frac{d}{dx}\left(f(x)y\right)$$

Thus comparing the coefficients of the two LHSs:

$$f'(x) = f(x)P$$

Dividing by $f(x)$ and integrating:

$$\int \frac{f'(x)}{f(x)}\,dx = \int P\ dx$$

$$\ln|f(x)| = \int P\,dx$$

$$f(x) = e^{\int P\,dx}$$

When there's something on front of the dy/dx

$$\frac{dy}{dx} + Py = Q$$

Find the general solution of $\cos x \frac{dy}{dx} + 2y \sin x = \cos^4 x$

What shall we do first so that we have an equation like before?

$$\frac{dy}{dx} + 2y \tan x = \cos^3 x$$

I.F. $P = 2\tan x$

$$\int P\,dx = \int 2\tan x\,dx = 2\ln|\sec x|$$

$$e^{\int P\,dx} = e^{2\ln|\sec x|} = \sec^2 x$$

$$\sec^2 x \frac{dy}{dx} + 2\tan x \sec^2 x\, y = \cos x$$

$$\frac{d}{dx}\left(y\sec^2 x\right) = \cos x$$

$$y\sec^2 x = \int \cos x\,dx$$

$$y\sec^2 x = \sin x + C$$

$$\underline{\underline{y = \sin x \cos^2 x + C\cos^2 x}}$$

STEP 1: Divide by anything on front of dy/dx

STEP 2: Determine IF

STEP 3: Multiply through by IF and use product rule backwards.

STEP 4: Integrate and simplify.

Your Turn

Edexcel FP2(Old) June 2011 Q3

Find the general solution of the differential equation

$$x\frac{dy}{dx} + 5y = \frac{\ln x}{x}, \qquad x > 0$$

Remember!

- can you separate the variables? DO THIS!
- can you use reverse product rule first? THEN DO THIS!
- put it in the form dy/dx + P(x)y = Q(x), then use IF

Q13 and Q14... may not need IF...!

Ex 7A
Q7 onwards

Homework
Core Pure Year 2
Ex 7A Q7bdfhj, Q13, Q15
Mixed Exercise 7 Q1-12 (NOT 6, 7, 10)

BE WARNED! Some of them are separating the variables!