Edexcel A Level Further Maths: Core Pure

Your notes

5.2 Methods in Calculus

Contents

5.2.1 Improper Integrals

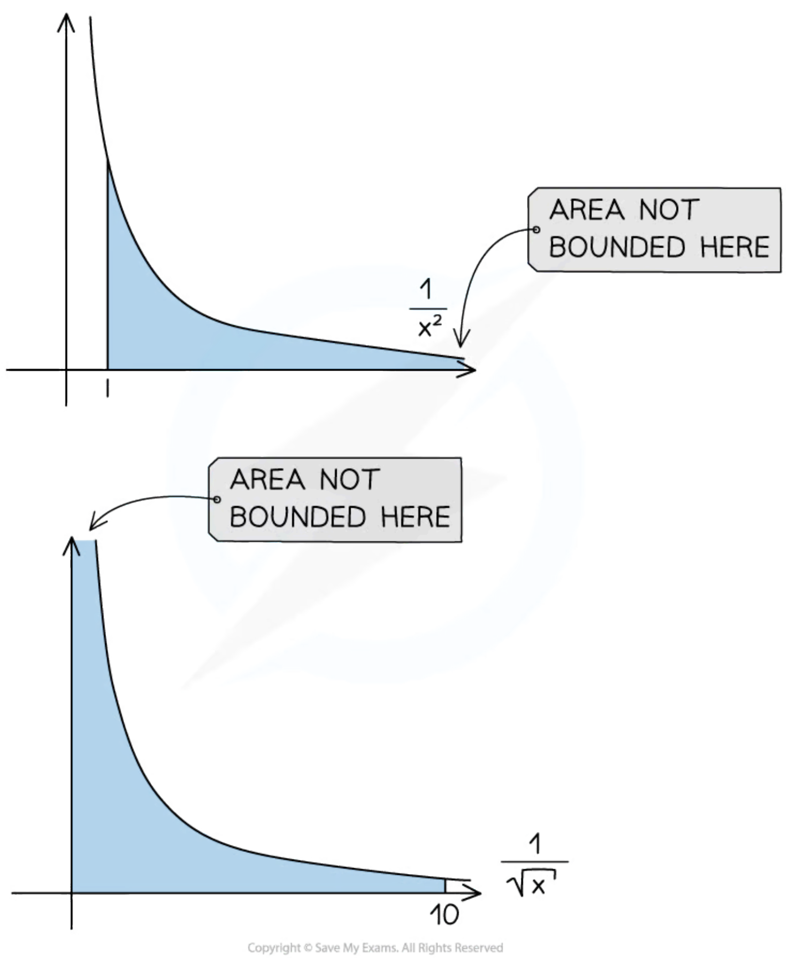
Your notes

Improper Integrals

What are improper integrals?

- An **improper integral** is a definite integral where one or both of the limits is either:
  - Positive or minus infinity
  - A point where the function is undefined
    - Consider the graph of $y = \frac{1}{\sqrt{x}}$
    - It is undefined at the point $x = 0$
    - The integral of $y = \frac{1}{\sqrt{x}}$ with a **limit of zero** would be an improper integral
- Examples include:
  - $\int_0^5 \frac{1}{\sqrt[3]{x}} \mathrm{d}x = \lim_{a \to 0} \int_a^5 \frac{1}{\sqrt[3]{x}} \mathrm{d}x$
  - $\int_1^\infty \frac{1}{x^3} \mathrm{d}x = \lim_{b \to \infty} \int_1^b \frac{1}{x^3} \mathrm{d}x$

Your notes



How do we find the value of an improper integral?

- Use algebra to **replace the limit** which cannot be found with a variable
  - E.g. let the undefined limit of zero be a or the infinite limit be b
- **Evaluate the integral** and substitute your chosen variable into the expression
- Consider what will happen to your answer as the value of your chosen variable **tends towards the limit**
  - E.g. what happens as a gets closer to zero or as b gets closer to infinity?
- Your final answer will be the value you get if you substitute this into your answer
  - E.g. as a tends to zero a^2 tends to zero and so this part of your solution will be zero
- It is useful to remember as a tends to infinity then $\frac{1}{a}$ tends to 0

Examiner Tip

- Be careful if a limit of your integral is zero, always check to see if the function is defined at zero and if not treat it as an improper integral.
- Infinite limits will always be treated as improper integrals.

Your notes

Your notes

Worked example

Find the following improper integrals, give your answers as exact values,

a) $$\int_0^3 \frac{1}{2\sqrt{x}}\,dx$$

$$\lim_{a\to 0}\int_a^3 \frac{1}{2\sqrt{x}}\,dx$$ undefined at zero, so use a as limit rather than zero

$$\lim_{a\to 0}\int_a^3 \frac{1}{2}x^{-\frac{1}{2}}\,dx = \lim_{a\to 0}\left[x^{\frac{1}{2}}\right]_a^3 = \lim_{a\to 0}\left(3^{\frac{1}{2}} - a^{\frac{1}{2}}\right)$$

Consider limit of expression as a tends to zero

As $a \to 0$, $a^{\frac{1}{2}} \to 0$ $\therefore$ $\lim_{a\to 0}\left(3^{\frac{1}{2}} - a^{\frac{1}{2}}\right) = 3^{\frac{1}{2}}$

$$\boxed{\int_0^3 \frac{1}{2\sqrt{x}}\,dx = \sqrt{3}}$$

b) $$\int_3^\infty \frac{1}{2x^2}\,dx$$

Your notes

$$\lim_{b\to\infty}\int_3^b \frac{1}{2x^2}\,dx$$ Limit at infinity, so use b as limit rather than infinity

$$\lim_{b\to\infty}\int_3^b \frac{1}{2}x^{-2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}x^{-1}\right]_3^b$$

$$= \lim_{b\to\infty}\left(\left(-\frac{1}{2b}\right) - \left(-\frac{1}{6}\right)\right)$$

$$= \lim_{b\to\infty}\left(\frac{1}{6} - \frac{1}{2b}\right)$$

Consider limit of expression as b tends to infinity

As $b\to\infty$, $\frac{1}{2b}\to 0$ $\therefore \lim_{b\to\infty}\left(\frac{1}{6}-\frac{1}{2b}\right) = \frac{1}{6}$

$$\boxed{\int_3^\infty \frac{1}{2x^2}\,dx = \frac{1}{6}}$$

5.2.2 Mean Value of a Function

Your notes

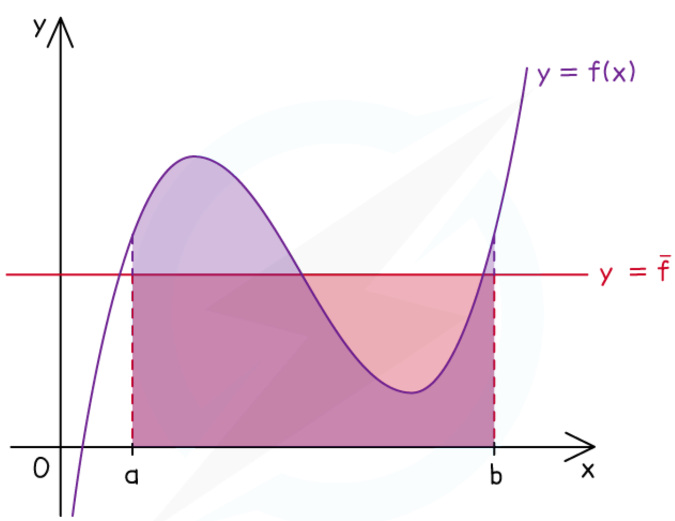
Mean Value of a Function

What is the mean value of a function?

- The **mean value of a function** may be thought of as the 'average' value of a function over a given interval
- For a function f(x), **the mean value of the function over the interval [a, b]** is given by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

  - Note that the mean value $\bar{f}$ is simply a real number – it is not a function
  - The mean value depends on the interval chosen – if the interval [a, b] changes, then the mean value may change as well
- Because $\bar{f}$ is a real number, the graph of $y = \bar{f}$ is a horizontal line
  - This gives a **geometrical interpretation** of the mean value of a function over a given interval
  - If A is the area bounded by the curve y = f(x), the x-axis and the lines $x = a$ and $x = b$, then the rectangle with its base on the interval [a, b] and with height also has area A
    - i.e. $(b-a)\bar{f} = \int_a^b f(x)\,dx$



AREA OF RED RECTANGLE EQUALS PURPLE AREA UNDER THE CURVE

What are the properties of the mean value of a function?

Your notes

- If $\overline{\mathrm{f}}$ is the mean value of a function f(*x*) over the interval [*a*, *b*], and *k* is a real constant, then:
  - f(*x*) + *k* has mean value $\overline{\mathrm{f}} + k$ over the interval [*a*, *b*]
  - *k*f(*x*) has mean value $k\overline{\mathrm{f}}$ over the interval [*a*, *b*]
  - -f(*x*) has mean value $-\overline{\mathrm{f}}$ over the interval [*a*, *b*]
- If $\overline{\mathrm{f}} = 0$ then the area that is above the *x*-axis and under the curve is equal to the area that is below the *x*-axis and above the curve

Your notes

Worked example

Let f be the function defined by $f(x) = \frac{1}{x+1}$.

a) Find the exact mean value of f over the interval $[0, 1]$.

Using $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\bar{f} = \frac{1}{1-0}\int_0^1 \frac{1}{x+1}\,dx = \int_0^1 \frac{1}{x+1}\,dx$$

Using the result $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$$= \left[\ln|x+1|\right]_0^1 = (\ln 2) - (\ln 1) = \ln 2$$

$$\boxed{\ln 2}$$

b) Write down the exact mean value of each of the following functions over the interval $[0, 1]$:

(i) $f(x) + 3$

(ii) $-f(x)$

(iii) $6f(x)$

Your notes

(i) $f(x)+3$ translates the function upwards by 3 units, so the mean value of $f(x)$ will also increase by 3

$\boxed{3+\ln 2}$

(ii) $-f(x)$ is a reflection in the x-axis of $f(x)$, so the mean value of $f(x)$ will be multiplied by -1

$\boxed{-\ln 2}$

(iii) $6f(x)$ is a vertical stretch of scale factor 6 so the mean value of $f(x)$ will be multiplied by 6

$\boxed{6\ln 2}$

5.2.3 Integrating with Partial Fractions

Your notes

Integrating with Partial Fractions

What is meant by partial fractions with quadratic denominators?

- For **linear denominators** the denominator of the original fraction can be factorised such that the denominator becomes a product of linear terms of the form $(ax + b)$
- With **squared linear denominators**, the same applies, except that some (usually just one) of the factors on the denominator may be squared, i.e. $(ax + b)^2$
- In both the above cases it can be shown that the **numerators** of each of the partial fractions will be a **constant** (A, B, C, etc)
- For this course, **quadratic denominators** refer to fractions that contain a **quadratic factor** (that **cannot** be **factorised**) on the denominator
  - the denominator of the quadratic partial fraction will be of the form $(ax^2 + bx + c)$; very often $b = 0$ leaving it as $(ax^2 + c)$
  - the numerator of the quadratic partial fraction could be of linear form, $(Ax + B)$

How do I find partial fractions involving quadratic denominators?

- STEP 1 **Factorise** the denominator as far as possible (if not already done so)
  - Sometimes the numerator can be factorised too
- STEP 2 **Split** the fraction into a **sum** with
  - the **linear denominator** having an (unknown) **constant numerator**
  - the **quadratic denominator** having an (unknown) **linear numerator**
- STEP 3 Multiply through by the denominator to eliminate fractions
- STEP 4 Substitute values into the identity and solve for the unknown constants
  - Use the root of the **linear factor** as a value of x to find one of the unknowns
  - Use any two values for x to form two equations to solve simultaneously
    - $x = 0$ is a good choice if this has not already been used with the linear factor
- STEP 5 Write the **original** as partial fraction

How do I integrate the fraction with the quadratic denominator?

- The quadratic denominator will be of the form $ax^2 + c$
  - If it is not then you can get it to look like this by **completing the square**
- Split into to fraction $\dfrac{Ax + B}{ax^2 + c} = \dfrac{Ax}{ax^2 + c} + \dfrac{B}{ax^2 + c}$
- Integrate $\dfrac{Ax}{ax^2 + c}$ using logarithms to get $\dfrac{A}{2a}\ln|ax^2 + c|$

Your notes

- Integrate $\frac{B}{ax^2 + c}$ using the **formula booklet** or using a **trigonometric or hyperbolic** substitution
  - If *a* and c have the same sign then use $x = \sqrt{\frac{c}{a}}\tan(u)$
  - If *a* and c have different signs then use $x = \sqrt{-\frac{c}{a}}\tanh(u)$
    - Or in this case you can factorise using surds and then use partial fractions

Your notes

Worked example

Find $\int \frac{8x^2 - 9x}{(x-3)(4x^2+9)}\,dx$

factorise numerator and split into partial fractions

$$\frac{x(8x-9)}{(x-3)(4x^2+9)} = \frac{A}{x-3} + \frac{Bx+C}{4x^2+9}$$

Unfactorisable quadratic as denominator ∴ linear numerator

$$x(8x-9) = A(4x^2+9) + (Bx+C)(x-3)$$

let $x=3$: $45 = 45A + 0$ ∴ $A = 1$

let $x=0$: $0 = 9A - 3C$
$0 = 9(1) - 3C$ ∴ $C = 3$

let $x=1$: $-1 = 13A + (B+C)(-2)$
$-1 = 13(1) + (B+3)(-2)$ ∴ $B = 4$

Rewriting question $\int \frac{1}{x-3} + \frac{4x+3}{4x^2+9}\,dx$

Split up the term with the linear numerator $\int \frac{1}{x-3} + \frac{4x}{4x^2+9} + \frac{3}{4x^2+9}\,dx$

As $\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$ we can find each part of this integral separately

Using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$ for first two integrals

$$\int \frac{1}{x-3}\,dx = \ln|x-3| + c_1$$

$$\int \frac{4x}{4x^2+9}\,dx = \frac{1}{2}\int \frac{8x}{4x^2+9}\,dx = \frac{1}{2}\ln|4x^2+9| + c_2$$

Your notes

For the final integral we will use the following result from the formula book $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

$$\int \frac{3}{4x^2+9}\,dx = \frac{3}{4}\int \frac{1}{x^2+\frac{9}{4}}\,dx$$

so $a^2 = \frac{9}{4}$

and $a = \frac{3}{2}$

$$= \frac{3}{4}\left(\frac{1}{\frac{3}{2}}\arctan\left(\frac{x}{\frac{3}{2}}\right)\right) + c_3 = \frac{1}{2}\arctan\left(\frac{2x}{3}\right) + c_3$$

Combine all 3 results back together. The 3 constants of integration can be merged into one

$$\boxed{\ln|x-3| + \frac{1}{2}\ln|4x^2+9| + \frac{1}{2}\arctan\left(\frac{2x}{3}\right) + c}$$

This answer could be further simplified by factoring out the $\frac{1}{2}$ and/or using laws of logarithms to combine terms if necessary

Your notes

5.2.4 Calculus involving Inverse Trig

Differentiating Inverse Trig Functions

What are the inverse trigonometric functions?

- **arcsin**, **arccos** and **arctan** are functions defined as the inverse functions of sine, cosine and tangent respectively
  - $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ which is equivalent to $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
  - $\arctan(-1) = \frac{3\pi}{4}$ which is equivalent to $\tan\left(\frac{3\pi}{4}\right) = -1$

What are the derivatives of the inverse trigonometric functions?

- $f(x) = \arcsin x$
  - $f'(x) = \frac{1}{\sqrt{1-x^2}}$
- $f(x) = \arccos x$
  - $f'(x) = -\frac{1}{\sqrt{1-x^2}}$
- $f(x) = \arctan x$
  - $f'(x) = \frac{1}{1+x^2}$
- Unlike other derivatives these look completely unrelated at first
  - their derivation involves use of the identity $\cos^2 x + \sin^2 x \equiv 1$
  - hence the squares and square roots!
- All three are given in the **formula booklet**
- Note with the derivative of $\arctan x$ that $(1+x^2)$ is the same as (x^2+1)

How do I show or prove the derivatives of the inverse trigonometric functions?

- For $y = \arcsin x$
  - Rewrite, $\sin y = x$
  - Differentiate implicitly, $\cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$
  - Rearrange, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y}$

Your notes

- Using the identity $\cos^2 y \equiv 1 - \sin^2 y$ rewrite, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-\sin^2 y}}$
- Since, $\sin y = x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$
- Similarly, for $y = \arccos x$
  - $\cos y = x$
  - $-\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y}$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{1-\cos^2 y}}$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}}$
- Notice how the derivative of $y = \arcsin x$ is positive but is negative for $y = \arccos x$
  - This subtle but crucial difference can be seen in their graphs
    - $y = \arcsin x$ has a positive gradient for all values of x in its domain
    - $y = \arccos x$ has a negative gradient for all values of x in its domain

Examiner Tip

- For $f(x) = \arctan x$ the terms on the denominator can be reversed (as they are being added rather than subtracted)
  - $f'(x) = \frac{1}{1+x^2} = \frac{1}{x^2+1}$
  - Don't be fooled by this, it sounds obvious but on awkward "show that" questions it can be off-putting!

Your notes

Worked example

a) Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$

$y = \arctan x$

$\tan y = x$

Differentiate implicitly

$\sec^2 y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

Using the identity $\tan^2 y + 1 \equiv \sec^2 y$

$\dfrac{dy}{dx} = \dfrac{1}{\tan^2 y + 1}$

Since $\tan y = x$

$\dfrac{dy}{dx} = \dfrac{1}{x^2+1}$

$$\boxed{\therefore \dfrac{dy}{dx} = \dfrac{1}{1+x^2}}$$

b) Find the derivative of $\arctan(5x^3 - 2x)$.

Your notes

$5x^3 - 2x$ is not a linear function of x, use chain rule

$y = \arctan(5x^3 - 2x)$

$$\frac{dy}{dx} = \frac{1}{1+(5x^3-2x)^2} \times (15x^2 - 2)$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{15x^2-2}{1+(5x^3-2x)^2}}$$

Integrating Inverse Trig Functions

How do I integrate inverse trig functions?

- Use **integration by parts** in the same way you would integrate $\ln x$
- These can be integrated using parts however
  - rewrite as the product '$1 \times f(x)$' and choose $u = f(x)$ and $\frac{dv}{dx} = 1$
  - 1 is easy to integrate and the inverse trig functions have standard derivatives listed in the formula booklet
- The expression $\frac{x}{1+x^2}$ integrates to $\frac{1}{2}\ln(1+x^2)$
- The expression $\pm\frac{x}{\sqrt{1-x^2}}$ integrates to $\mp\sqrt{1-x^2}$

$$\int \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2} + c$$

$$\int \arccos(x)\,dx = x\arccos(x) - \sqrt{1-x^2} + c$$

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c$$

Your notes

5.2.5 Integration by Substitution

Integrating using Trigonometric Substitutions

The integrals covered in this revision note are based on the standard results

$$\int \frac{1}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + \mathrm{c}$$

and

$$\int \frac{1}{a^2 + x^2}\ \mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + \mathrm{c},\ \ \mathrm{a} > 0,\ \ |x| > \mathrm{a}$$

These **are** given in the **formulae booklet**

How do I know when to use a trigonometric substitution in integration?

There are three main types of problem

- Type 1
  Showing the standard results using a substitution (a may have a value)
  The substitution will **not** be given in such cases

  e.g. Use a suitable substitution to show that $\int \frac{1}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \arcsin(\frac{x}{a}) + c$

  Let $x = a\sin u$, so $\frac{\mathrm{d}x}{\mathrm{d}u} = a\cos u$ and $u = \arcsin\left(\frac{x}{a}\right)$

  $$\int \frac{1}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \int \frac{1}{\sqrt{a^2 - (a\sin u)^2}}\ (a\cos u)\ \mathrm{d}u = \int \frac{a\cos u}{\sqrt{a^2(1 - \sin^2 u)}}\ \mathrm{d}u = \int 1\ \ \mathrm{d}u = u + c$$

  $$\therefore \int \frac{1}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + c$$

  The general idea in these types of problems is to reduce the denominator to a single term, often involving the identity , so it can be integrated using standard results or techniques

- Type 2
  Reverse chain rule, possibly involving some factorising in the denominator and using 'adjust' and 'compensate' if necessary

  e.g. Find $\int \frac{3}{9^2 + (2x)^2}\ \mathrm{d}x$

Your notes

$$\int \frac{3}{9^2+(2x)^2}\ \mathrm{d}x = 3 \times \frac{1}{2}\int \frac{2}{9^2+(2x)^2}\ \mathrm{d}x = \frac{3}{2}\left(\frac{1}{9}\arctan\left(\frac{2x}{9}\right)\right) + c = \frac{1}{6}\arctan\left(\frac{2x}{9}\right) + c$$

- Type 3
  The denominator contains a three-term quadratic expression – i.e. there is an *x* term
  In such cases complete the square and use reverse chain rule
  e.g. Find $\int \frac{1}{25+x^2-6x}\ \mathrm{d}x$
  $$\int \frac{1}{25+x^2-6x}\ \mathrm{d}x = \int \frac{1}{25+(x-3)^2-9}\ \mathrm{d}x = \int \frac{1}{4^2+(x-3)^2}\ \mathrm{d}x = \frac{1}{4}\arctan\left(\frac{x-3}{4}\right) + c$$
  (This works since $\frac{\mathrm{d}}{\mathrm{d}x}[x-3] = 1$, so effectively there is no reverse chain rule involved)
- A fourth type of problem may involve a given substitution but the skills to solve these are covered in the A Level Mathematics course

How do I use a trigonometric substitution to find integrals?

- **STEP 1**
  Identify the type of problem and if a substitution is required
  Determine the substitution if needed
- **STEP 2**
  For Type 1 problems, differentiate and rearrange the substitution; change everything in the integral
  For Type 2 problems, 'adjust' and 'compensate' as necessary
  For Type 3 problems complete the square
- **STEP 3**
  Integrate using standard techniques and results, possibly from the formulae booklet
  For definite integration, a calculator may be used but look out for ***exact*** values being required, a calculator may give an approximation
- **STEP 4**
  Substitute the original variable back in if necessary – this shouldn't be necessary for definite integration
  For indefinite integration, simplify where obvious and/or rearrange into a required format

Why is arccos *x* not involved in any of the integration results?

- $\frac{\mathrm{d}}{\mathrm{d}x}\left[\arccos\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2-x^2}}$
  For integration the "-" at the start can be treated as the constant "-1" and so integrating would lead to "-arcsin ..."

- i.e. $\int -\frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = -\arcsin\left(\frac{x}{a}\right) + c$

Your notes

Examiner Tip

- The general form of the functions involving trigonometric and hyperbolic functions are very similar
- Be clear about which form needs a trigonometric substitution and which form need a hyperbolic substitution
- Always have a copy of the formula booklet to hand when practising these problems

Your notes

Worked example

(a) Use an appropriate substitution to show that

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$$

Let $x = a\tan u$

so $\frac{dx}{du} = a\sec^2 u$ $\qquad$ ("$dx = a\sec^2 u\,du$")

and $u = \arctan\left(\frac{x}{a}\right)$

$$\therefore \int \frac{1}{a^2+x^2}\,dx = \int \frac{a\sec^2 u}{a^2+a^2\tan^2 u}\,du$$

$$= \int \frac{a\sec^2 u}{a^2(\underbrace{\cancel{1+\tan^2 u}}_{\sec^2 u})}\,du \qquad \text{"}1+\tan^2 A \equiv \sec^2 A\text{"}$$

$$= \frac{1}{a}\int du$$

$$= \frac{1}{a}u + c$$

$$\boxed{\therefore \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c}$$

(b) Find

Your notes

$$\int \frac{3}{16+9x^2}\,dx$$

STEP 1 This is a 'type 2' problem

STEP 2: 'Adjust' and 'compensate' as necessary

$$I = \int \frac{3}{16+9x^2}\,dx = \int \frac{3}{4^2+(3x)^2}\,dx$$

no 'adjust' and 'compensate' needed

STEP 3: Integrate using result from formulae booklet

$$\boxed{I = \frac{1}{4}\arctan\left(\frac{3x}{4}\right) + c}$$

Your notes

Integrating using Hyperbolic Substitutions

The integrals covered in this revision note are based on the standard results

$$\int \frac{1}{\sqrt{a^2 + x^2}}\ \mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right) + c$$

and

$$\int \frac{1}{\sqrt{x^2 - a^2}}\ \mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right) + c,\ x > a$$

These **are** given in the **formulae booklet**

How do I know when to use a hyperbolic substitution in integration?

There are three main types of problem

- Type 1
  Showing the standard results using a substitution (a may have a value)
  The substitution will **not** be given in such cases
  e.g. Use a suitable substitution to show that $\int \frac{1}{\sqrt{x^2 - a^2}}\ \mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right) + c$
  Let $x = a\cosh u$, so $\frac{\mathrm{d}x}{\mathrm{d}u} = a\sinh u$ and $u = \operatorname{arcosh}\left(\frac{x}{a}\right)$
  $$\int \frac{1}{\sqrt{x^2 - a^2}}\ \mathrm{d}x = \int \frac{1}{\sqrt{(a\cosh u)^2 - a^2}}\ (a\sinh u)\ \mathrm{d}u = \int \frac{a\sinh u}{\sqrt{a^2(\cosh^2 u - 1)}}\ \mathrm{d}u = \int 1\ \mathrm{d}u = u + c$$
  $$\therefore \int \frac{1}{\sqrt{x^2 - a^2}}\ \mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right) + c$$
  The general idea in these types of problems is to reduce the denominator to a single term, often involving the identity $\cosh^2 x - \sinh^2 x \equiv 1$, so it can be integrated using standard results or techniques

- Type 2
  Reverse chain rule, possibly involving some factorising in the denominator and using 'adjust' and 'compensate' if necessary
  e.g. Find $\int \frac{3}{\sqrt{4x^2 + 9}}\ \mathrm{d}x$

Your notes

$$\int \frac{3}{\sqrt{(2x)^2+3^2}}\,\mathrm{d}x = 3 \times \frac{1}{2}\int \frac{2}{\sqrt{(2x)^2+3^2}}\,\mathrm{d}x = \frac{3}{2}\left(\operatorname{arsinh}\left(\frac{2x}{3}\right)\right) + c = \frac{3}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right) + c$$

- Type 3

  The denominator contains a three-term quadratic expression – i.e. there is an x term

  In such cases complete the square and use reverse chain rule

  e.g. Find $\int \frac{1}{\sqrt{x^2-6x+25}}\,\mathrm{d}x$

  $$\int \frac{1}{\sqrt{x^2-6x+25}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x-3)^2-9+25}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x-3)^2+4^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x-3}{4}\right) + c$$

  (This works since $\frac{\mathrm{d}}{\mathrm{d}x}[x-3]=1$,so effectively there is no reverse chain rule involved

- A fourth type of problem may involve a given substitution but the skills to solve these are covered in the A Level Mathematics course, although hyperbolic functions are not

How do I use a hyperbolic substitution to find integrals?

- **STEP 1**

  Identify the type of problem and if a substitution is required

  Determine the substitution if needed

- **STEP 2**

  For Type 1 problems, differentiate and rearrange the substitution; change everything in the integral

  For Type 2 problems, 'adjust' and 'compensate' as necessary

  For Type 3 problems complete the square

- **STEP 3**

  Integrate using standard techniques and results, possibly from the formulae booklet

  For definite integration, a calculator may be used but look out for ***exact*** values being required, a calculator may give an approximation

- **STEP 4**

  Substitute the original variable back in if necessary – this shouldn't be necessary for definite integration

  For indefinite integration, simplify where obvious and/or rearrange into a required format

Is artanh x involved in integration?

- The standard result, given in the formulae booklet is

  $$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) + c,\ |x| < a$$

Your notes

with the alternative result $\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+c$ also given

- Problems involving these often involve partial fractions (since $a^2 - x^2$ is the difference of two squares) leading to the 'ln' result
- If you happen to recognise the integral and can use the formulae booklet result involving "artanh" to solve a problem, then do so!

Examiner Tip

- The general form of the functions involving trigonometric and hyperbolic functions are very similar
- Be clear about which form needs a trigonometric substitution and which form need a hyperbolic substitution
- Always have a copy of the formula booklet to hand when practising these problems

Your notes

Worked example

(a) Use an appropriate substitution to show that

$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \operatorname{arsinh}\left(\frac{x}{a}\right) + c$$

Let $x = a\sinh u$

so $\frac{dx}{du} = a\cosh u$ ("$dx = a\cosh u\, du$")

and $u = \operatorname{arsinh}\left(\frac{x}{a}\right)$

$$\therefore \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \int \frac{a\cosh u}{\sqrt{(a\sinh u)^2 + a^2}}\,du$$

$$= \int \frac{a\cosh u}{\sqrt{a^2(\underbrace{\cancel{\sinh^2 u + 1}}_{\cosh^2 u})}}\,du$$

"$\cosh^2 A - \sinh^2 A \equiv 1$"

$$= \int du$$

$$= u + c$$

$$\boxed{\therefore \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{a}\right) + c}$$

(b) Find

$$\int \frac{5}{\sqrt{9x^2 - 25}}\ \mathrm{d}x$$

Your notes

STEP 1 This is a 'type 2' problem

STEP 2: 'Adjust' and 'compensate' as necessary

$$I = \int \frac{5}{\sqrt{9x^2-25}}\,dx = 5 \times \underset{\text{'compensate'}}{\frac{1}{3}} \int \frac{\overset{\text{'adjust'}}{3}}{\sqrt{(3x)^2-5^2}}\,dx$$

STEP 3: Integrate using result from formulae booklet

$$\boxed{I = \frac{5}{3} \operatorname{arcosh}\left(\frac{3x}{5}\right) + c}$$