Vectors - Year 2

1:: Distance between two points.

What's the distance between $(1,0,4)$ and $(-3,5,9)$?

2:: i, j, k notation for vectors

$$\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \rightarrow \quad \boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$$

3:: Magnitude of a 3D vector and using it to find angle between vector and a coordinate axis.

"Find the angles that the vector $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ makes with each of the positive coordinate axis."

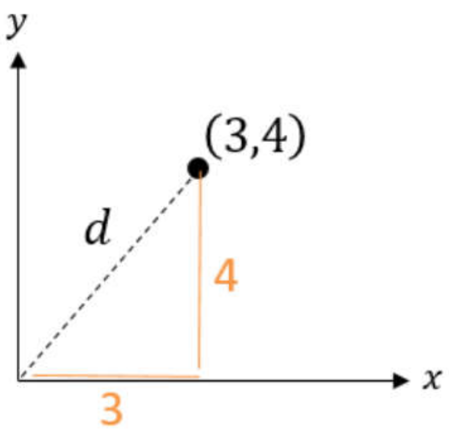
4:: Solving Geometric Problems

Same as Year 1 but with 3D vectors.

5:: Application to Mechanics

Using $F = ma$ with 3D force/acceleration vectors and understanding distance is the magnitude of the 3D displacement vector, etc.

Distance from the origin and magnitude of a vector

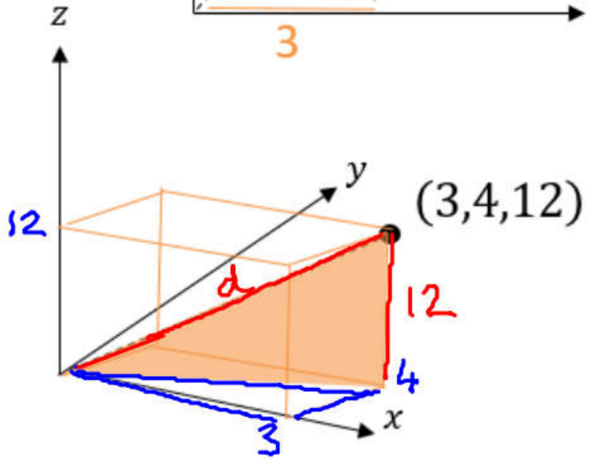


In 2D, how did we find the distance from a point to the origin?

Pythagoras

$d=\sqrt{3^2+4^2}=5$

How about in 3D then?



$\sqrt{3^2+4^2}$

$d=\sqrt{\left(\sqrt{3^2+4^2}\right)^2+12^2}$

$d=\sqrt{3^2+4^2+12^2}=\sqrt{169}=\underline{\underline{13}}.$

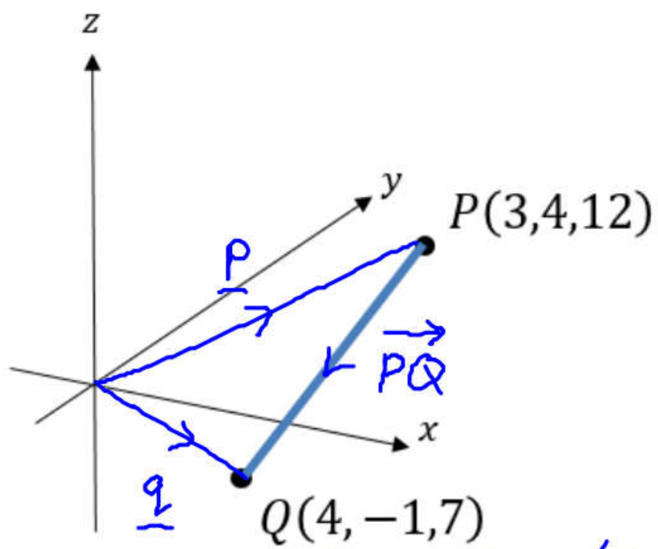
From Year 1 you will be familiar with the magnitude $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$ being its length.
We can see from above that this nicely extends to 3D:

✎ The magnitude of a vector $\boldsymbol{a}=\begin{pmatrix}x\\y\\z\end{pmatrix}$:

$$|\boldsymbol{a}|=\sqrt{x^2+y^2+z^2}$$

And the distance of (x,y,z) from the origin is $\sqrt{x^2+y^2+z^2}$

Distance between two 3D points



How do we find the distance between P and Q?

$$d = \sqrt{1^2 + 5^2 + 5^2}$$
$$= \sqrt{51}$$

$$\overrightarrow{PQ} = \underline{q} - \underline{p} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -5 \end{pmatrix}$$

✏ The distance between two points is:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

Δx means "change in x"

Quickfire Questions:

1) Distance of $(4,0,-2)$ from the origin:

$$\sqrt{4^2+2^2} = \sqrt{20} = \underline{\underline{2\sqrt{5}}}.$$

2) $\left|\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}\right| = \sqrt{42}$

3) Distance between $(0,4,3)$ and $(5,2,3)$.

$$d = \sqrt{5^2+2^2} = \sqrt{29}$$

4) Distance between $(1,1,1)$ and $(2,1,0)$.

$$d = \sqrt{1^2+1^2} = \sqrt{2}$$

5) Distance between $(-5,2,0)$ and $(-2,-3,-3)$.

$$d = \sqrt{3^2+5^2+3^2} = \underline{\underline{\sqrt{43}}}.$$

Tip: Because we're squaring, it doesn't matter whether the change is negative or positive.

Your Turn

Find the distance from the origin to the point $P(7,7,7)$.

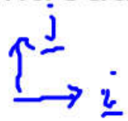
$$d = \sqrt{7^2+7^2+7^2}$$
$$= \underline{\underline{7\sqrt{3}}} \text{ units}$$

quadratic?

The coordinates of A and B are $(5,3,-8)$ and $(1,k,-3)$ respectively.
Given that the distance from A to B is $3\sqrt{10}$ units, find the possible values of k.

$$3\sqrt{10} = \sqrt{4^2+(3-k)^2+5^2}$$
$$90 = 16+9-6k+k^2+25$$
$$0 = k^2-6k-40$$
$$\underline{\underline{k=10}} \quad \underline{\underline{k=-4}}$$

In 2D you were previously introduced to $\boldsymbol{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix}0\\1\end{pmatrix}$ as unit vectors in each of the x and y directions.



It meant for example that $\begin{pmatrix}8\\-2\end{pmatrix}$ could be written as $8\boldsymbol{i} - 2\boldsymbol{j}$ since $8\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}8\\-2\end{pmatrix}$

Unsurprisingly, in 3D:

$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Quickfire Questions

1 Put in i, j, k notation:

$\begin{pmatrix}1\\2\\3\end{pmatrix} = \underline{i} + 2\underline{j} + 3\underline{k}$

$\begin{pmatrix}-7\\3\\0\end{pmatrix} = -7\underline{i} + 3\underline{j}$

$\begin{pmatrix}3\\0\\-1\end{pmatrix} = 3\underline{i} - \underline{k}$

2 Write as a column vector:

$4\boldsymbol{j} + \boldsymbol{k} = \begin{pmatrix}0\\4\\1\end{pmatrix}$ $\qquad \boldsymbol{i} - \boldsymbol{j} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

3 If $A(1,2,3), B(4,0,-1)$ then

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}4\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3\\-2\\-4\end{pmatrix}$

4 If $\boldsymbol{a} = \begin{pmatrix}2\\3\\4\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}0\\-1\\3\end{pmatrix}$ then $3\boldsymbol{a} + 2\boldsymbol{b} = 3\begin{pmatrix}2\\3\\4\end{pmatrix} + 2\begin{pmatrix}0\\-1\\3\end{pmatrix}$

$= \begin{pmatrix}6\\9\\12\end{pmatrix} + \begin{pmatrix}0\\-2\\6\end{pmatrix} = \begin{pmatrix}6\\7\\18\end{pmatrix}$

Reminder:

For position vectors a and b:

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Find the magnitude of $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

has magnitude 1.

$$|\underline{a}| = \sqrt{2^2+1^2+4^2} = \sqrt{21}$$

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

$$\hat{\underline{a}} = \frac{1}{\sqrt{21}}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{21} \\ -1/\sqrt{21} \\ 4/\sqrt{21} \end{pmatrix}$$

vectors are factors of each other.

If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$ is $2\boldsymbol{a} - 3\boldsymbol{b}$ parallel to $4\boldsymbol{i} - 5\boldsymbol{k}$?

$$2\underline{a} - 3\underline{b} = 2\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} - 3\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4-12 \\ -6+6 \\ 10+0 \end{pmatrix}$$

$$= \begin{pmatrix} -8 \\ 0 \\ 10 \end{pmatrix} \qquad \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$$

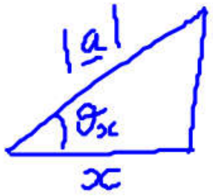
$$\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} -8 \\ 0 \\ 10 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -8 \\ 0 \\ 10 \end{pmatrix} = -2\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$$

So, they are parallel.

Ex 12B

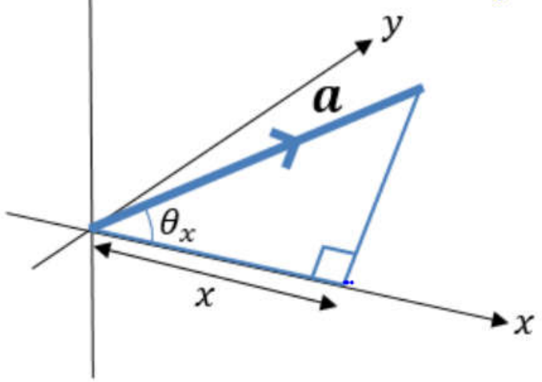
Q1, 2, 6, 8, 9, 10, 11

Angles between vectors and an axis



$\cos\theta_x = \frac{x}{|\underline{a}|}$

$\theta_x = \cos^{-1}\left(\frac{x}{|\underline{a}|}\right)$



How could you work out the angle between a vector and the x-axis?

Use SOH CAH TOA and triangles

The angle between $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the x-axis is:

$$\cos\theta_x = \frac{x}{|\boldsymbol{a}|}$$

and similarly for the y and z axes.

$\cos\theta_y = \frac{y}{|\underline{a}|}$ $\quad$ $\cos\theta_z = \frac{z}{|\underline{a}|}$

Find the angles that the vector $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ makes with each of the positive coordinate axis.

$|\underline{a}| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$

x-axis

$\cos\theta_x = \frac{2}{\sqrt{14}}$

$\theta_x = \underline{\underline{57.7^\circ}}$

y-axis

$\cos\theta_y = \frac{-3}{\sqrt{14}}$

$\theta_y = \underline{\underline{143.3^\circ}}$

z-axis

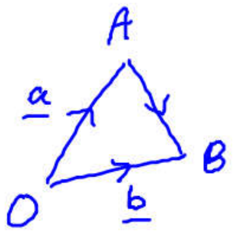
$\cos\theta_z = \frac{-1}{\sqrt{14}}$

$\theta_z = \underline{\underline{105.5^\circ}}$

(1 dp)

The points A and B have position vectors $4\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ relative to a fixed origin, O. Find $\overrightarrow{AB}$ and show that ΔOAB is isosceles.

↳ 2 sides are equal magnitude/length



$\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 3-4 \\ 4-2 \\ -1-7 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{1^2+2^2+8^2} = \sqrt{69}$

$|\underline{a}| = \sqrt{4^2+2^2+7^2} = \sqrt{69}$

$|\underline{b}| = \sqrt{3^2+4^2+1^2} = \sqrt{26}$

because $AB = OA$ the ΔOAB is isosceles.

Could include:
sine rule
cosine rule.

Find the angle that the vector $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ makes with the x-axis.
By similarly considering the angle that $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ makes with the x-axis, determine the area of OAB where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. (Hint: draw a diagram)

$\underline{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $|\underline{a}| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

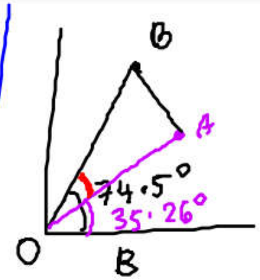
$\cos\theta_x = \frac{2}{\sqrt{6}}$

$\theta_x = 35.26°$

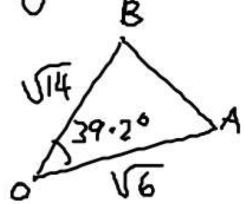
$\underline{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ $|\underline{b}| = \sqrt{1^2+3^2+2^2}$
$= \sqrt{14}$

$\cos\theta_x = \frac{1}{\sqrt{14}}$

$\theta_x = 74.50°$



red angle is $74.5 - 35.26$
$= 39.2°$



Ex 12B

Ex 12B
Q1, 2, 6, 8, 9, 10, 11
Then 13-17

Area $= \frac{1}{2} ab\sin C$

$= \frac{1}{2}\sqrt{6}\sqrt{14}\sin 39.2°$

$= 2.898\ldots = \underline{\underline{2.90}}$ units^2 (2dp)