Chapter 6a
Hyperbolic Functions - definitions and solving

We will see the definition and purpose of **hyperbolic functions** such as $\sinh x$, $\cosh x$, their inverses, and how we can manipulate them, such as solving equations, differentiating and integrating.

1 :: Definition of hyperbolic functions and their sketches.

"Find the exact value of: $tanh\ (ln\ 4)$"

2 :: Inverse hyperbolic functions.

Prove that

$arcosh\, x = \ln\left(x + \sqrt{x^2 - 1}\right)$

3 :: Hyperbolic Identities and Solving Equations

Solve for all real x

$2\cosh^2 x - 5 sinh\, x = 5$

Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \sinh x = \frac{1}{2}(e^x - e^{-x})$$

Using these defintions, find the definitions of:

$\operatorname{sech} x$

$\tanh x$

$\operatorname{cosech} x$

$\coth x$

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad x \in \mathbb{R}$$

Say as "shine" of x

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad x \in \mathbb{R}$$

Say as "cosh"

Hyperbolic tangent:

$$\tanh = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad x \in \mathbb{R}$$

Say as "th-an"

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad x \in \mathbb{R}$$

Say as "setch"

Hyperbolic cosecant:

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad x \in \mathbb{R}, x \neq 0$$

Say as "cosetch"

Hyperbolic cotangent:

$$\coth x = \frac{1}{\tanh x} = \frac{(e^{2x} + 1)}{e^{2x} - 1} \qquad x \in \mathbb{R}, x \neq 0$$

Say as "coth"

Equations for hyperbolic functions

Calculate (using both your $sinh$ button and using the formula)

$$\mathbf{\sinh 3}$$

Write in terms of e:

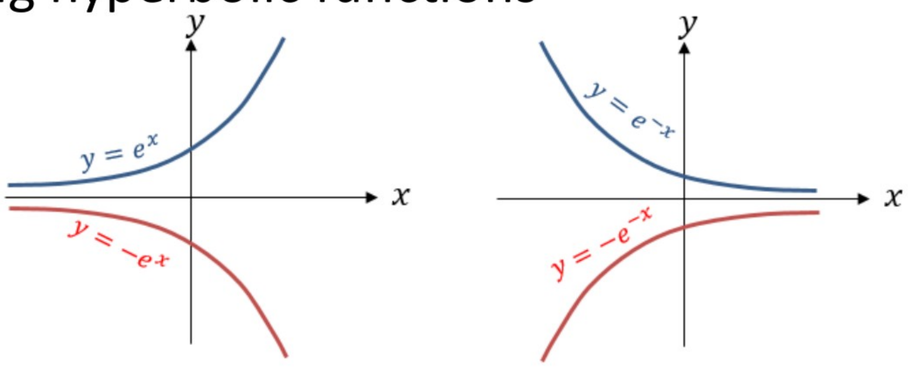
$$\mathbf{cosech\ 3}$$

Find the exact value of:

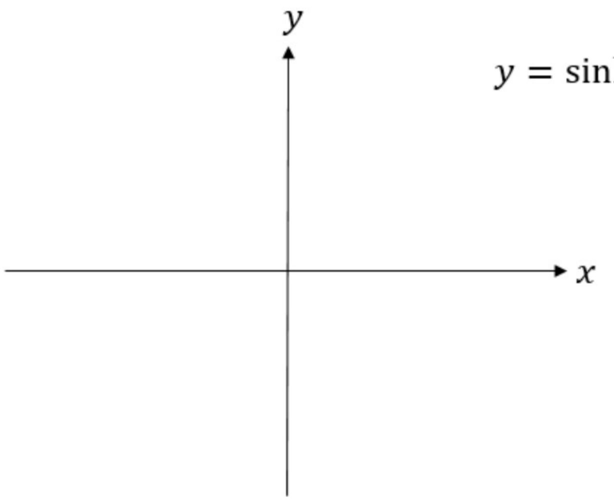
$$\boldsymbol{tanh\ (ln\ 4)}$$

Solve $sinh\ x = 5$

Sketching hyperbolic functions

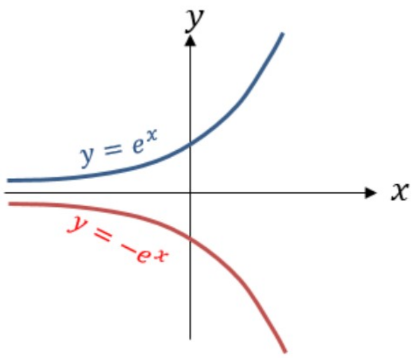


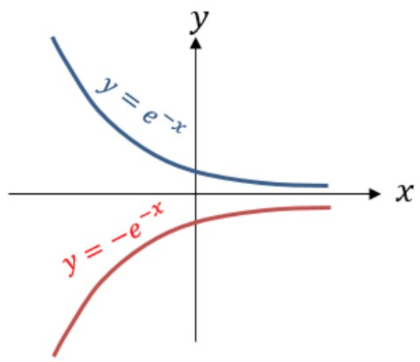
<u>Odd or Even?</u>



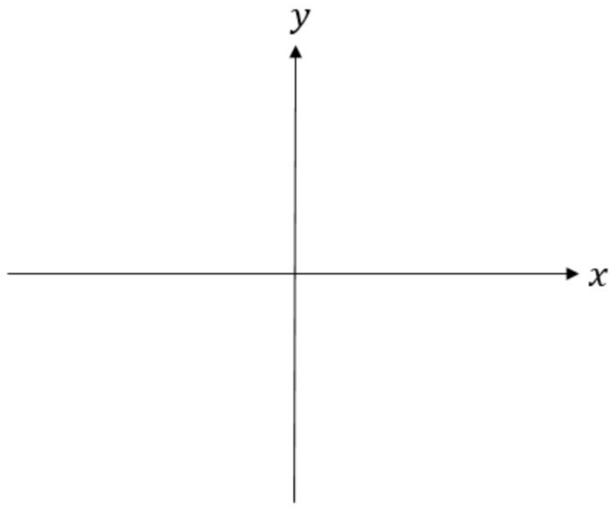
$y = \sinh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$





<u>Odd or Even?</u>



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

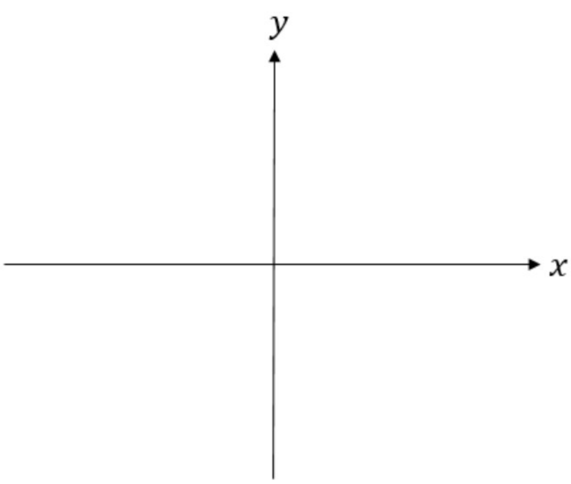
To sketch $y = \tanh x$, consider the usual features when you sketch a graph.

$$\tanh x = \frac{\sinh x}{\cosh x}$$

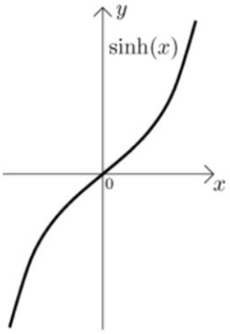
When $x = 0$,

As $x \to \infty$,

As $x \to -\infty$,

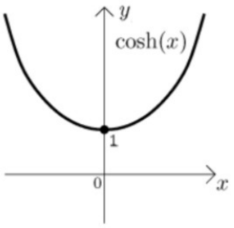


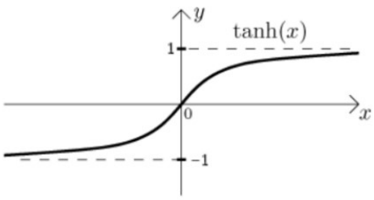
Sketching the reciprocal hyperbolic functions



cosech (x)

sech (x)





[FP3 June 2011 Q5] The curve C_1 has equation $y = 3 \sinh 2x$, and the curve C_2 has equation $y = 13 - 3e^{2x}$.

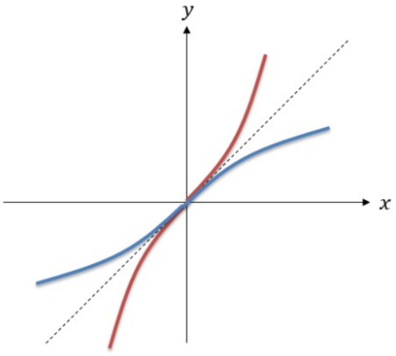
(*a*) Sketch the graph of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes.

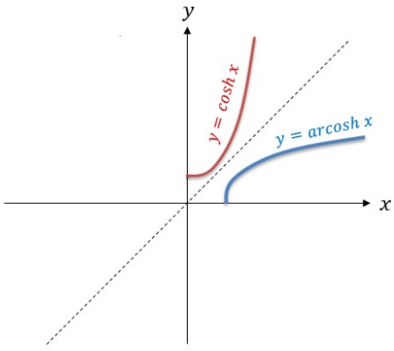
(4)

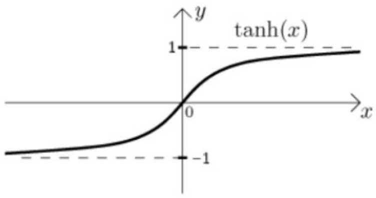
Inverse Hyperbolic Functions

$\sin^{-1} x = \arcsin x$

$\sinh^{-1} x = \text{arsinh}\, x$ etc.



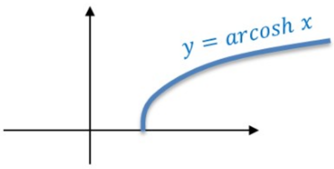




Given that hyperbolic functions can be written in terms of e, it make sense that inverse hyperbolic can be expressed in terms of ln.

Prove that $arsinh\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$

Prove that $arcosh\, x = \ln\left(x + \sqrt{x^2 - 1}\right), x \geq 1$



But recall from the graph that we only include positive values of y in the function to avoid it being one-to-many. Thus $arcosh\, x = \ln\left(x + \sqrt{x^2 - 1}\right)$ only.

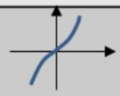
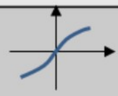
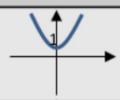
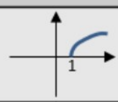
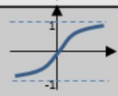
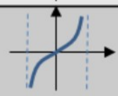
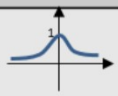
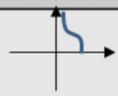
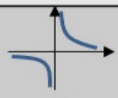
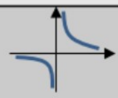
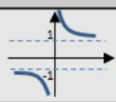
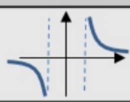
Prove that $artanh\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), |x| < 1$

Summary

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1$$

Hyperbolic	Domain	Sketch	Inverse Hyperbolic	Domain	Sketch
$y = \sinh x$	$x \in \mathbb{R}$		$y = \operatorname{arsinh} x$	$x \in \mathbb{R}$	
$y = \cosh x$	$x \geq 0$		$y = \operatorname{arcosh} x$	$x \geq 1$	
$y = \tanh x$	$x \in \mathbb{R}$		$y = \operatorname{artanh} x$	$\|x\| < 1$	
$y = \operatorname{sech} x$	$x \geq 0$		$y = \operatorname{arsech} x$	$0 < x \leq 1$	
$y = \operatorname{cosech} x$	$x \neq 0$		$y = \operatorname{arcosech} x$	$x \neq 0$	
$y = \coth x$	$x \neq 0$		$y = \operatorname{arcoth} x$	$\|x\| > 1$	

Ex 6B

Hyperbolic Identities

Use the definitions of *sinh* and *cosh* to prove that…

$$\boldsymbol{\cosh^2 x - \sinh^2 x = 1}$$

What else could we prove?

Prove the following identity using the definitions of sinh x and cosh x

$$\boldsymbol{sinh\,(A+B) = sinh\,A\,cosh\,B + cosh\,A\,sinh\,B}$$

Prove the following identity using the definitions of sinh x and cosh x

$$\boldsymbol{cosh\ 2A = 1 + 2sinh^2 A}$$

Osborn's Rule

Are these the same, or different, to the trig identities?

$$\boldsymbol{sinh\ (A \pm B) = sinh\ A\ cosh\ B \pm cosh\ A\ sinh\ B}$$

$$\boldsymbol{cosh\ (A \pm B) = cosh\ A\ cosh\ B \pm sinh\ A\ sinh\ B}$$

$$\boldsymbol{tanh\ (A \pm B) = \frac{sinh\ x}{cosh\ x} = \frac{tanh\ A + tanh\ B}{1 + tanh\ A\ tanh\ B}}$$

We can get these identities from the normal sin/cos ones by:

Osborn's Rule:
1. Replacing $sin \rightarrow sinh$ and $cos \rightarrow cosh$
2. **Negate** any explicit or implied **product of two sines**.

$$\tan(A - B) =$$

$$\sin A \sin B \rightarrow$$
$$\tan^2 A \rightarrow$$

$$cos2A =$$

Ex 6C Q1, 2

Solving Equations

You must to decide whether to use **hyperbolic identities** or **the definitions of hyperbolic functions.**

Solve for all real x
$$6 \sinh x - 2 \cosh x = 7$$

Solve for all real x

$$2\cosh^2 x - 5sinh\, x = 5$$

Solve for all real x

$$\cosh 2x \ - \ 5\cosh x \ + \ 4 \ = \ 0$$

Double Angle

$\sin 2x = 2\sin x\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 2\cos^2 x - 1$ $\cos 2x = 1 - 2\sin^2 x$
$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

If $\sinh x = \frac{3}{4}$, find the exact value of:

a) $\cosh x$
b) $\tanh x$
c) $\sinh 2x$

Ex 6C Q3-9

Exam Questions

[FP3 June 2009 Q1] Solve the equation

$$7 \operatorname{sech} x - \tanh x = 5$$

Give your answers in the form $\ln a$, where a is a rational number. **(5)**

$\frac{7}{\cosh x} - \frac{\sinh x}{\cosh x} = 5 \Rightarrow \frac{14}{e^x + e^{-x}} - \frac{(e^x - e^{-x})}{e^x + e^{-x}} = 5$	M1
$\therefore 14 - (e^x - e^{-x}) = 5(e^x + e^{-x}) \Rightarrow 6e^x - 14 + 4e^{-x} = 0$	A1
$\therefore 3e^{2x} - 7e^x + 2 = 0 \Rightarrow (3e^x - 1)(e^x - 2) = 0$	M1
$\therefore e^x = \frac{1}{3}$ or 2	A1
$x = \ln(\frac{1}{3})$ or $\ln 2$	B1ft
	(5 marks)

[FP3 June 2014 (I) Q3] Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials,

(*a*) prove that

$$\cosh^2 x - \sinh^2 x \equiv 1$$

(2)

(*b*) find algebraically the exact solutions of the equation

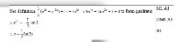
$$2 \sinh x + 7 \cosh x = 9$$

giving your answers as natural logarithms. **(5)**

[FP3 June 2011 Q5]

(*b*) Solve the equation $3 \sinh 2x = 13 - 3e^{2x}$, giving your answer in the form $\frac{1}{2} \ln k$, where k is an integer.

(5)



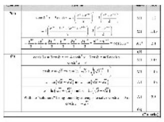
1. Solve the equation

$$6\cosh 2x + 4\sinh x = 7$$

giving your answers as exact logarithms.

(6)

7. (a) Using the definition of $\cosh x$ in terms of exponentials, prove that



$$4\cosh^3 x - 3\cosh x \equiv \cosh 3x$$

(3)

(b) Hence, or otherwise, solve the equation

$$\cosh 3x = 9\cosh x$$

(4)

1. (a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -k < x < k$$

stating the value of the constant k.

(5)

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2-3x}\right)$$

(5)