Conditional Probability

1:: Set Notation

How sets are used to describe a sample space/event and how notation like $A \cap B$ is used to combine sets.

2:: Conditional Probability in Venn Diagrams

The notation $P(A|B)$ means "the probability of A given that B happened". How we can find such probabilities using a Venn Diagram.

3:: Formula for Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

4:: Tree Diagrams

"I have 3 red and 4 green balls in a bag. I take one ball out the bag, keep it, then take another. **Given that** the second ball was green, determine the probability the first was red."

RECAP :: Using sets for sample spaces and events

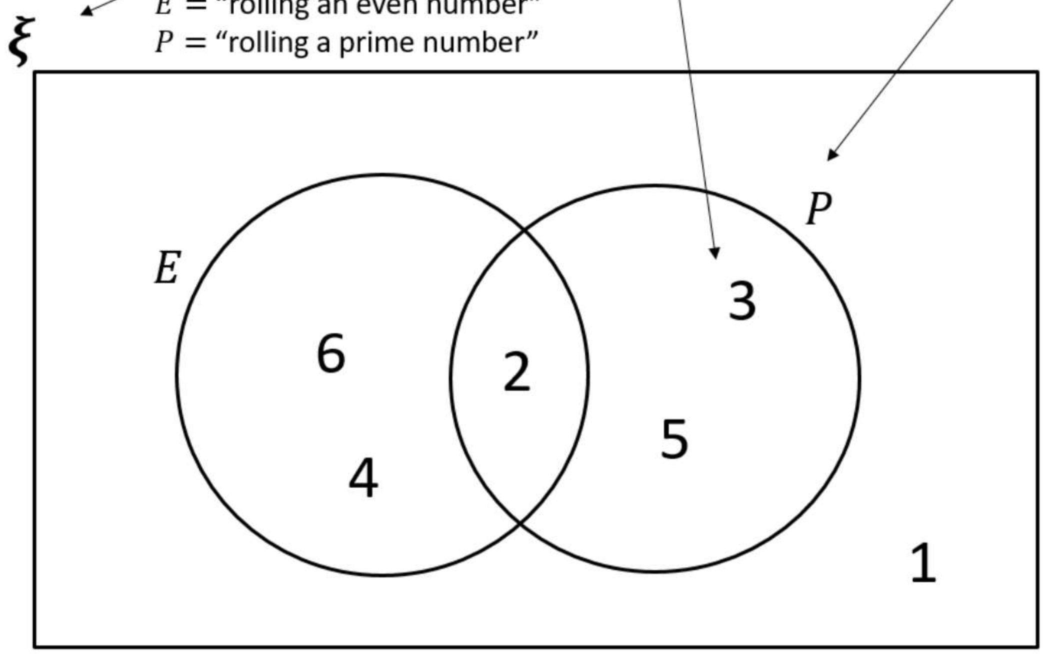
In general, sets are used to represent **collections of items**.

A **sample space** is **set of all possible outcomes**. We use ξ (Greek 'Xi'), or sometimes just S, to represent this set. We use a rectangle in a Venn Diagram.

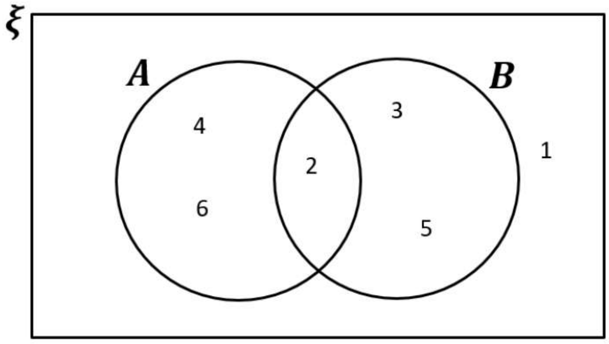
Each number represents an **outcome**.

In probability, an **event** is a **set of one or more outcomes**. These are the circles in the Venn Diagram. We use capital letters for the variables representing sets.

$E =$ "rolling an even number"
$P =$ "rolling a prime number"



Combining events/sets



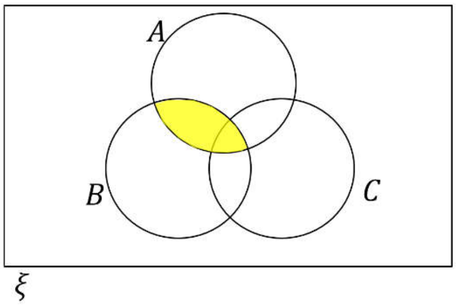
ξ = the whole sample space (1 to 6)

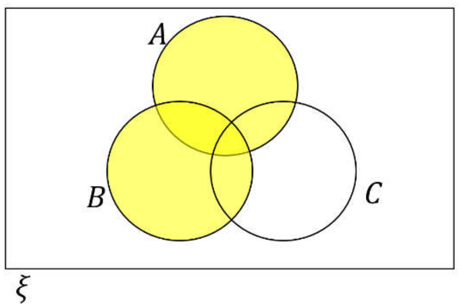
A = even number on a die thrown

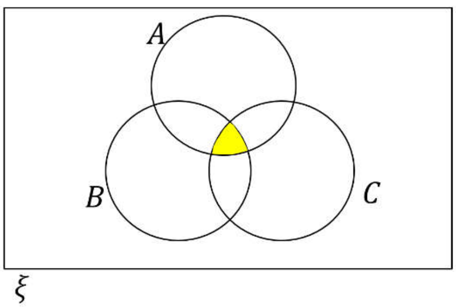
B = prime number on a die thrown

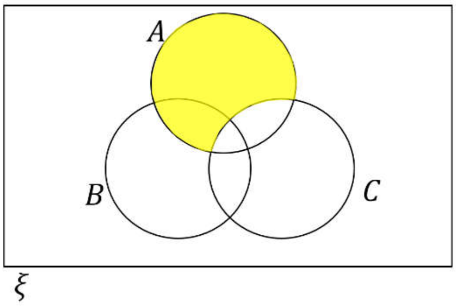
	What does it mean in this context?	What is the resulting set of outcomes?
A'	**Not A (the "<u>complement</u>" of A).** i.e. Not rolling an even number.	$\{1, 3, 5\}$
$A \cup B$	**A or B (the "<u>union</u>" of A and B).** i.e. Rolling an even or prime number.	$\{2,3,4,5,6\}$
$A \cap B$	**A and B (the "<u>intersection</u>" of A and B).** i.e. Rolling a number which is even and prime.	$\{2\}$
$A \cap B'$	**"A and not B".** Rolling a number which is even and not prime.	$\{4,6\}$
$(A \cup B)'$	Rolling a number which is not [even or prime].	$\{1\}$
$(A \cap B)'$	Rolling a number which is not [even and prime].	$\{1,3,4,5,6\}$

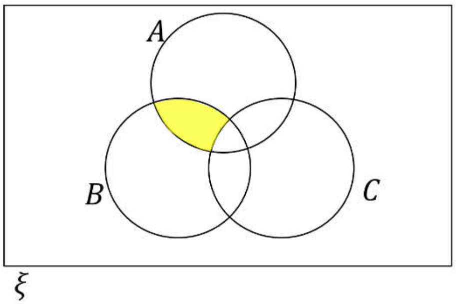
More complex Venn diagrams

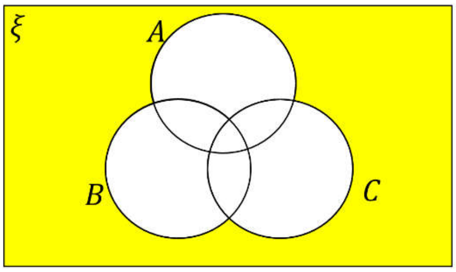


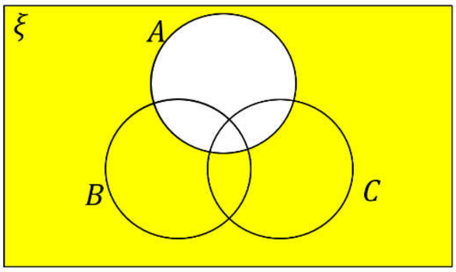


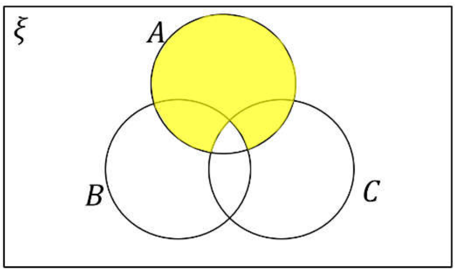


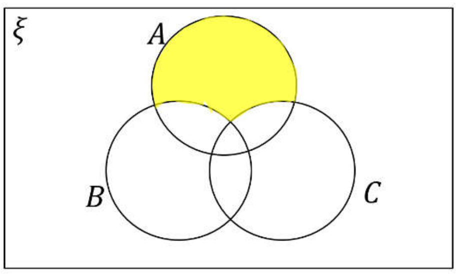

A
B
C
ξ


A
B
C
ξ


ξ
A
B
C


ξ
A
B
C


ξ
A
B
C


ξ
A
B
C

Examples

Venn Diagram can either contain:

(a) The **specific outcomes** in each set

(b) The number of items in the set (i.e. **frequencies**)

(c) The **probability** of being in that set.

This will usually be stated or made obvious from the context.

A card is selected at random from a pack of 52 playing cards. Let A be the event that the card is an ace and D the event that the card is a diamond. Find:

a) $P(A \cap D)$ b) $P(A \cup D)$ c) $P(A')$ d) $P(A' \cap D)$

Given that $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.25$,

a. Explain why events A and B are not independent.

Given also that $P(C) = 0.2$, that events A and C are mutually exclusive and that events B and C are independent,

b. Draw a Venn diagram to illustrate the events A, B and C, showing the probabilities for each region.

c. Find $P\big((A \cap B') \cup C\big)$

If events A and B are <u>independent</u>.

$$P(A \cap B) = P(A) \times P(B)$$

If events A and B are <u>mutually exclusive</u>:

$$P(A \cap B) = 0$$
$$P(A \cup B) = P(A) + P(B)$$

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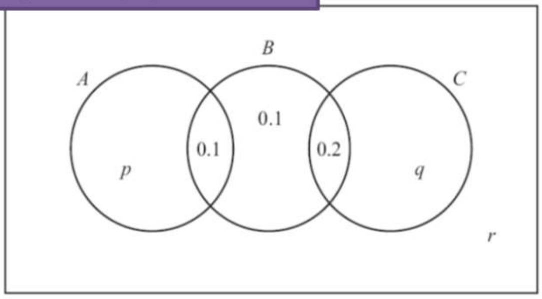


The Venn diagram in Figure 1 shows three events A, B and C and the probabilities associated with each region of B. The constants p, q and r each represent probabilities associated with the three separate regions outside B.

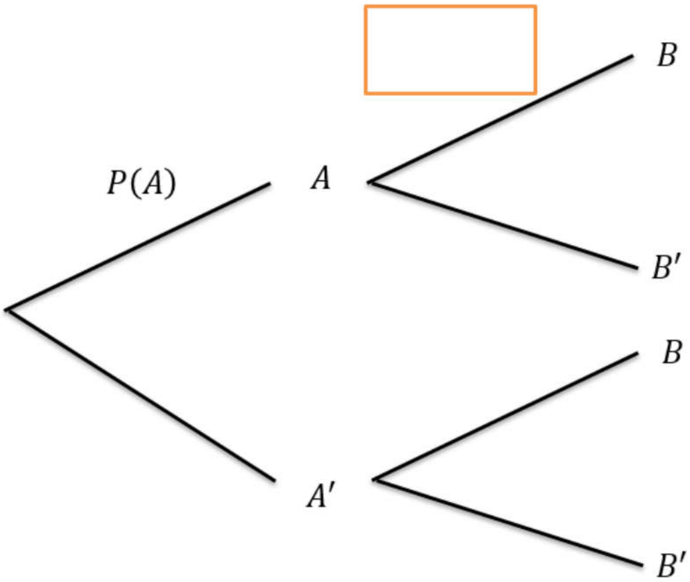
The events A and B are independent.

(a) Find the value of p. **(3)**

Ex 2A

Conditional Probability

Think about how we formed a probability tree at GCSE:



$P(A \cap B)=$

Alternatively (and more commonly):

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Memory Tip: You're dividing by the event you're conditioning on.

1 The following two-way table shows what foreign language students in Year 9 study.
B is the event that the student is a boy. F is the event they chose French as their language.

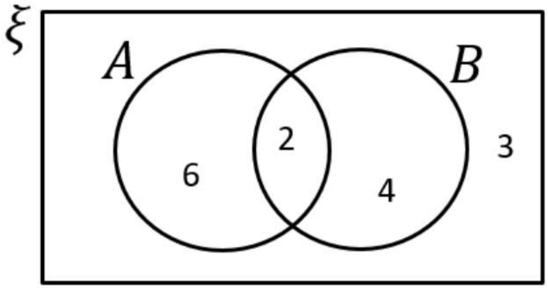
	$\boldsymbol{B}$	$\boldsymbol{B'}$	Total
$\boldsymbol{F}$	14	38	52
$\boldsymbol{F'}$	26	22	48
Total	40	60	100

a Determine the probability of:
$P(F|B')$
Method 1: Using the formula:

Method 2: Restricted sample space.

b $P(B|F') =$

2 Using the Venn Diagram, determine:



a $P(A|B)$
Method 1: Using the formula

Method 2: Restricted sample space

b $P(A'|B') =$

c $P(B|A \cup B) =$

a Given that $P(A) = 0.5$ and $P(A \cap B) = 0.3$, what is $P(B|A)$?

Tip: The 'restricted sample space' method also works for Venn Diagrams with probabilities.

b Given that $P(Y) = 0.6$ and $P(X \cap Y) = 0.4$, what is $P(X'|Y)$?
(Hint: Drawing a Venn Diagram will help!)

c Given that $P(A) = 0.5$, $P(B) = 0.5$ and $P(A \cap B) = 0.4$, what is $P(B|A')$?

Your Turn

The events E and F are such that

$$P(E) = 0.28 \quad P(E \cup F) = 0.76 \qquad P(E \cap F') = 0.11$$

Find

$a)$ $P(E \cap F) =$

$b)$ $P(F) =$

$c)$ $P(E'|F') =$

(Drawing a Venn diagram is often helpful!)

More Practice...

1 $P(A \cap B') = 0.4, P(A \cup B) = 0.75$

Then:

$$P(B) =$$
$$P(A' \cap B') =$$

2 $P(A) = 0.47$ and $P(A \cap B) = 0.12$ and $P(A' \cap B') = 0.03$

Then:

$$P(A|B') =$$

3 $P(A') = 0.7, P(B') = 0.2, P(A \cap B') = 0.1$

Then:

$$P(A \cup B') =$$

$$P(B|A') =$$

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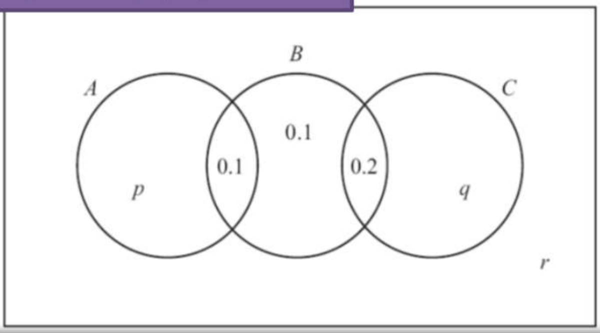


The Venn diagram in Figure 1 shows three events A, B and C and the probabilities associated with each region of B. The constants p, q and r each represent probabilities associated with the three separate regions outside B.

The events A and B are independent.

we did (a) earlier

(a) Find the value of p. **(3)**

Given that $P(B|C) = \frac{5}{11}$,

(b) find the value of q and the value of r (4)

(c) Find $P(A \cup C|B)$ (2)

Ex 2C

Full Laws of Probability

If events A and B are <u>independent</u>.

$$P(A \cap B) = P(A) \times P(B)$$
$$P(A|B) = P(A)$$

If events A and B are <u>mutually exclusive</u>:

$$P(A \cap B) = 0$$
$$P(A \cup B) = P(A) + P(B)$$

In general:

We first encountered this in the previous section.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is known as the **<u>Addition Law</u>.** **Informal Proof**: If we added the probabilities in the A and B sets in the Venn Diagram, we'd be double counting the intersection, so subtract so that it's only counted once.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

IMPORTANT TIPS

If I were to identify two tips that will possible help you the most in probability questions:

If you see the words '**given that**', Immediately write out the law for conditional probability.

Example: "Given Bob walks to school, find the probability that he's not late…"

First thing you should write:

If you see the words '**are independent**', Immediately write out the laws for independence.
(Even before you've finished reading the question!)

Example: "A is independent from B…"

First thing you should write:

If you're stuck on a question where you have to find a probability given others, it's probably because you've failed to take into account that two events are independent or mutually exclusive, or you need to use the conditional probability or additional law.

Edexcel S1

6. Explain what you understand by

(a) a sample space, **(1)**

(b) an event. **(1)**

Two events A and B are independent,

such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$

Find

(c) $P(A \cap B)$, **(2)**

(d) $P(A \mid B)$, **(2)**

(e) $P(A \cup B)$. **(2)**

a) The set of all outcomes.
b) A set of one or more outcomes (that is a subset of the sample space).

C and D are two events such that $P(C) = 0.2$, $P(D) = 0.6$ and $P(C|D) = 0.3$. Find:

a. $P(C \cap D)$ b. $P(D|C)$ c. $P(C \cup D)$

10. [Jan 2012 Q2] (a) State in words the relationship between two events R and S when $P(R \cap S) = 0$. **(1)**

The events A and B are independent with $P(A) = \frac{1}{4}$ and $P(A \cup B) = \frac{2}{3}$. Find

(*b*) $P(B)$, **(4)**

(*c*) $P(A' \cap B)$, **(2)**

(*d*) $P(B' | A)$. **(2)**

Edexcel S1

Your Turn

9. Three events A, B and C are defined in the sample space S. The events A and B are mutually exclusive and A and C are independent.

   (a) Draw a Venn diagram to illustrate the relationships between the 3 events and the sample space. **(3)**

   Given that $\mathrm{P}(A) = 0.2$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup C) = 0.7$, find

   (b) $\mathrm{P}(A \mid C)$, **(2)**

   (c) $\mathrm{P}(A \cup B)$, **(2)**

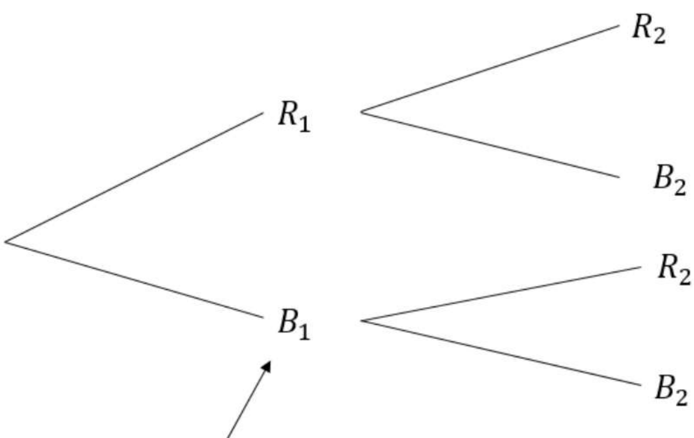
   (d) $\mathrm{P}(C)$. **(4)**

Ex 2D

Probability Trees

We saw probability trees in Year 1. The only difference here is **determining a conditional probability** using your tree.

Example: You have two bags, the first with 5 red balls and 5 blue balls, and the second with 3 red balls and 6 blue balls. You first pick a ball from the first bag, and place it in the second. You then pick a ball from the second bag. Complete the tree diagram.



Tip: Use variable subscripting to indicate what pick you're referring to.

Hence find the probability that:

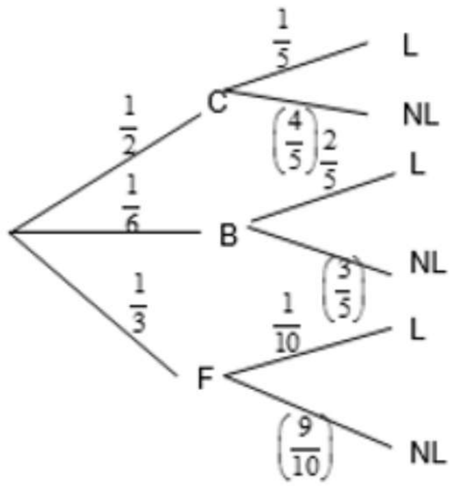
a) **You pick a red ball on your second pick.**

b) **Given that your second pick was red, the first pick was also red.**

Edexcel S1 May 2009 Q2

On a randomly chosen day the probability that Bill travels to school by car, by bicycle or on foot is $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively. The probability of being late when using these methods of travel is $\frac{1}{5}$, $\frac{2}{5}$ and $\frac{1}{10}$ respectively.

(*c*) Given that Bill is late, find the probability that he did not travel on foot. **(4)**



Edexcel S1 Your Turn

6. **[Jan 2006 Q4]** A bag contains 9 blue balls and 3 red balls. A ball is selected at random from the bag and its colour is recorded. The ball is not replaced. A second ball is selected at random and its colour is recorded.

(*a*) Draw a tree diagram to represent the information. **(3)**

Find the probability that

(*a*) the second ball selected is red, **(2)**

(*b*) both balls selected are red, given that the second ball selected is red. **(2)**

Ex 2E

EXAM PRACTICE

4. Given that

$$P(A) = 0.35 \qquad P(B) = 0.45 \qquad \text{and} \qquad P(A \cap B) = 0.13$$

find

(a) $P(A' \mid B')$

(2)

(b) Explain why the events A and B are not independent.

(1)

The event C has $P(C) = 0.20$

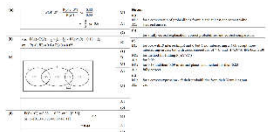
The events A and C are mutually exclusive and the events B and C are statistically independent.

(c) Draw a Venn diagram to illustrate the events A, B and C, giving the probabilities for each region.

(5)

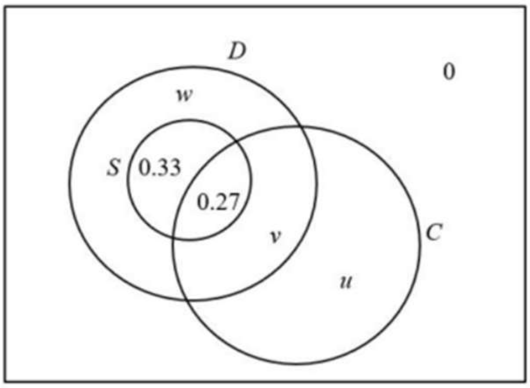
(d) Find $P([B \cup C]')$

(2)



4. The Venn diagram shows the probabilities of students' lunch boxes containing a drink, sandwiches and a chocolate bar.

D is the event that a lunch box contains a drink,
S is the event that a lunch box contains sandwiches,
C is the event that a lunch box contains a chocolate bar,
u, v and w are probabilities.



(a) Write down $P(S \cap D')$ **(1)**

One day, 80 students each bring in a lunch box.
Given that all 80 lunch boxes contain sandwiches and a drink,

(b) estimate how many of these 80 lunch boxes will contain a chocolate bar. **(3)**

Given that the events S and C are independent and that $P(D \mid C) = \frac{14}{15}$

(c) calculate the value of u, the value of v and the value of w. **(7)**

4(a)	$P(S \cap D') = 0$	B1	1.1b
		(1)	
(b)	$P(C \mid S \cap D) = \frac{0.27}{0.6} = \frac{9}{20} = 0.45$	M1	3.1b
	$\times 80 = $ "0.45"	M1	1.1b
	$= 36$	A1	1.1b
		(3)	
(c)	$[P(C) \times P(S) = P(C \cap S)]$		
	$P(S) = 0.6$, $P(C) = 0.27 + v + u$, $P(S \cap C) = 0.27$	M1	3.1a
	$0.6 \times (0.27 + u + v) = 0.27$ or $u + v = 0.18$ o.e.	A1	1.1b
	$\left[P(D \mid C) = \frac{P(D \cap C)}{P(C)}\right]$ $P(D \cap C) = 0.27 + v$	M1	3.1a
	$\frac{14}{15} = \frac{0.27 + v}{0.27 + v + u}$ or $14u - v = 0.27$ o.e.	A1	1.1b
	$15u = 0.45$	M1dd	1.1b
	$u = 0.03$ $v = 0.15$	A1	1.1b
	$w = 0.22$	A1ft	1.1b
		(7)	

(11 marks)

Notes:

(a) **B1:** correct answer only

(b) **M1:** for a correct ratio of probabilities formula with at least one correct value and multiplying by 80

A1: a correct answer

(c) **M1:** for translating the problem and realising the equation $P(C) \times P(S) = P(C \cap S)$ needs to be used with at least 2 parts correct.

A1: a correct equation

M1: for a correct probability formula with $P(D \cap C) = 0.27 + v$

A1: a second correct equation

M1dd: dependent on the previous 2 method marks being awarded. Solving the two simultaneous equations by eliminating one variable. May be implied by either u or v correct

1. Three bags, A, B and C, each contain 1 red marble and some green marbles.

Bag A contains 1 red marble and 9 green marbles only
Bag B contains 1 red marble and 4 green marbles only
Bag C contains 1 red marble and 2 green marbles only

Sasha selects at random one marble from bag A.
If he selects a red marble, he stops selecting.
If the marble is green, he continues by selecting at random one marble from bag B.
If he selects a red marble, he stops selecting.
If the marble is green, he continues by selecting at random one marble from bag C.

(a) Draw a tree diagram to represent this information.

(2)

(b) Find the probability that Sasha selects 3 green marbles.

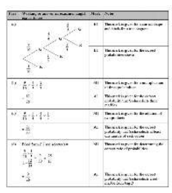
(2)

(c) Find the probability that Sasha selects at least 1 marble of each colour.

(2)

(d) Given that Sasha selects a red marble, find the probability that he selects it from bag B.

(2)



3. A company maintains machines.
It has two types of contract, a service contract and a repair contract.
The company classes its customers as new customers or existing customers.
The table gives information about the company's customers.

	Service contract	**Repair contract**
New customer	65	82
Existing customer	231	262

The company is going to survey its customers. It plans to take a sample of 100 of its customers, stratified by customer type and contract type.

(a) Work out how many new customers with repair contracts should be sampled.
(2)

The company has developed a test for a certain fault in the machines it services.
The test sometimes gives incorrect results.

The company collects information from a sample of randomly selected machines.

- 2% of the machines have the fault
- 70% of the machines with the fault test positive for the fault
- 10% of the machines without the fault test positive for the fault.

A machine is selected at random from the sample of the machines, and tests positive for the fault.

(b) (i) Calculate the probability that the machine has the fault.
(4)

(ii) Comment on the usefulness of the company's test.
Give a reason for your answer.
(1)

When the company services the machines, it checks two components, α and β, for wear and tear and replaces these if needed.

Event A is that component α needs to be replaced.

Event B is that component β needs to be replaced.

The probability that component α needs to be replaced is 0.35
The probability that component β needs to be replaced is 0.55
The probability that neither component needs to be replaced is 0.28

(c) Show that events A and B are not independent.
(2)

(d) Find the probability that component α or component β needs to be replaced, but not both.
(2)