4. A particle P moves along a straight line such that at time t seconds, $t \geq 0$, its velocity, v m s^{-1}, is given by

$$v = 16 - 3t^2$$

Find

(a) the distance travelled by P in the first second,

(3)

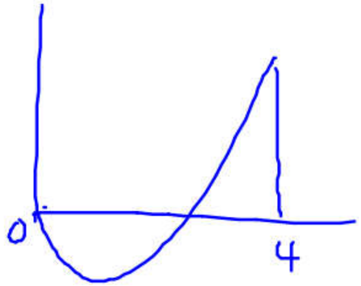
(b) the value of t at the instant when P changes its direction of motion,

(2)

(c) the value of t at the instant when P returns to its starting point.

(3)

(Total 8 marks)



$$\int_0^1 (16-3t^2)\,dt = \left[16t - t^3\right]_0^1$$

$$= 16 - 1$$

$$= 15 \text{ m}$$

b) $v = 0$ $\quad 16 - 3t^2 = 0$

$$\sqrt{\frac{16}{3}} = t$$

$v = 16 - 3t^2$

$s = 16t - t^3 + c$

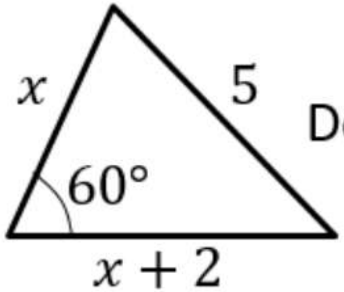
$0 = 16t - t^3$

$t = 0, 4 \qquad t = 4$

Trigonometric Ratios

There is technically no new content in this chapter since GCSE.
However, the problems might be more involved than at GCSE level.

1:: Sine/Cosine Rule



Determine x.

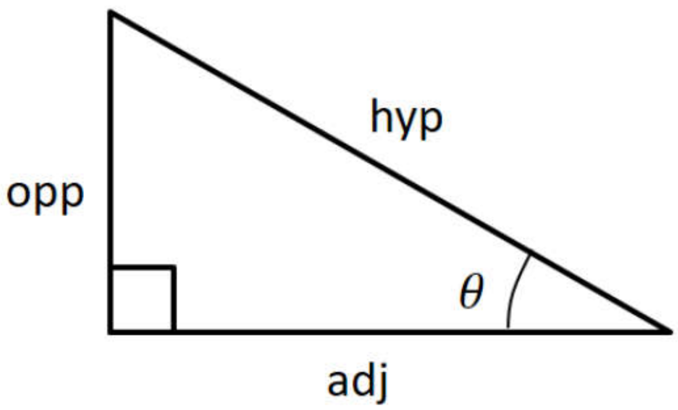
2:: Areas of Triangles

In ΔABC, $AB = 5, BC = 6$ and $\angle ABC = x$. Given that the area of $\angle ABC$ is 12cm² and that AC is the longest side, find the value of x.

3:: Graphs of Sine/Cosine/Tangent

Sketch $y = \sin(2x)$ for $0 \leq x \leq 360°$

RECAP :: Right-Angled Trigonometry

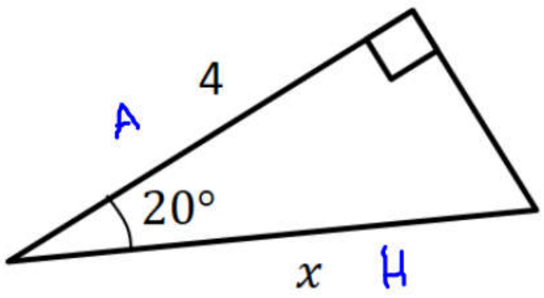


You are probably familiar with the formula: $\sin(\theta) = \frac{opp}{hyp}$

But what is the *conceptual* definition of sin ?

sin is a function which inputs an angle and gives the ratio between the opposite and hypotenuse.

Remember that a ratio just means the 'relative size' between quantities (in this case lengths). For this reason, sin/cos/tan are known as "trigonometric ratios".

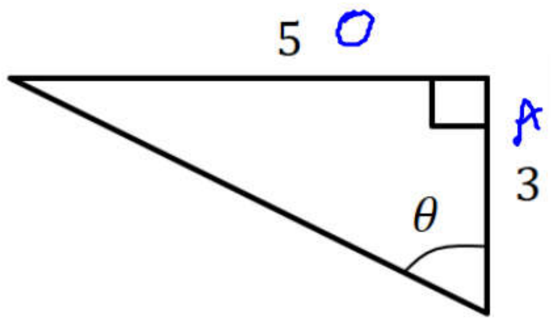


Find x.

$\cos 20 = \frac{4}{x}$

$x = \frac{4}{\cos 20} = 4.26\text{ cm}$

Tip: You can swap the thing you're dividing by and the result. e.g. $\frac{8}{2} = 4 \rightarrow \frac{8}{4} = 2$



Find θ.

$$\tan(\theta) = \frac{5}{3}$$

$$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.0°$$

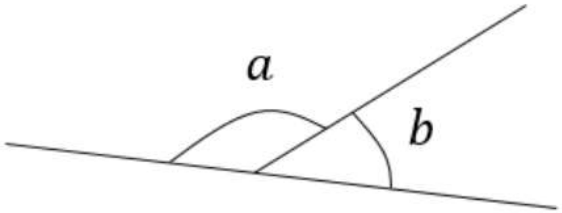
$\rightarrow \theta = \arctan\left(\frac{5}{3}\right)$

$\arcsin \rightarrow \sin^{-1}$

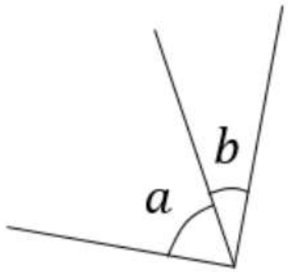
$\arccos \rightarrow \cos^{-1}$

Just for your interest…

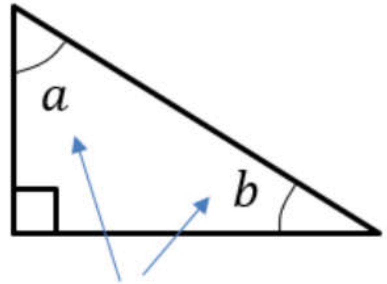
Have you ever wondered why "cosine" contains the word "sine"?



Supplementary Angles
add to 180°

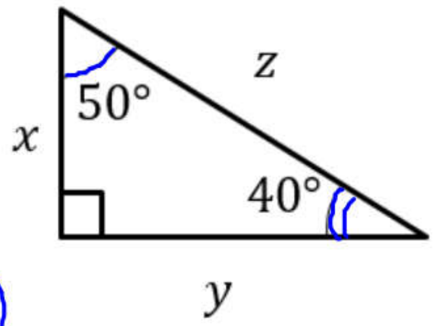


Complementary Angles
add to 90°



Therefore these angles are complementary.

i.e. The **cosine** of an angle is the **sine** of the **complementary** angle.
Hence **cosine = COMPLEMENTARY SINE**



$\cos(50) = \frac{x}{z}$

$\sin(40) = \frac{x}{z}$

$\cos(50) = \sin(40)$

$\sin 30 = \cos 60$

$\cos 10 = \sin 80$

$\sin 13.2 = \cos 76.8$

Cofunction

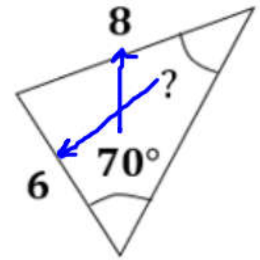
$\sin\theta = \cos(90 - \theta)$

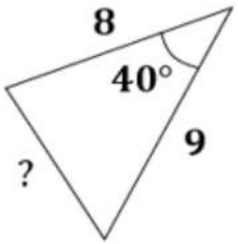
$\cos\theta = \sin(90 - \theta)$

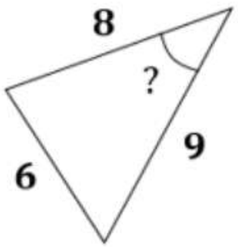
OVERVIEW: Finding missing sides and angles

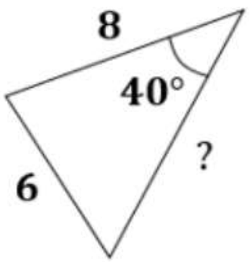
When triangles are not right-angled, we can no longer use simple trigonometric ratios, and must use the cosine and sine rules.

You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule
#2 Two sides known and a missing side opposite a known angle	Remaining side	Cosine rule
#3 All three sides	An angle	Cosine rule
#4 Two sides known and a missing side not opposite known angle	Remaining side	Cosine rule OR Sine rule twice



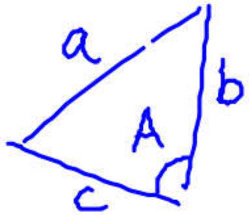


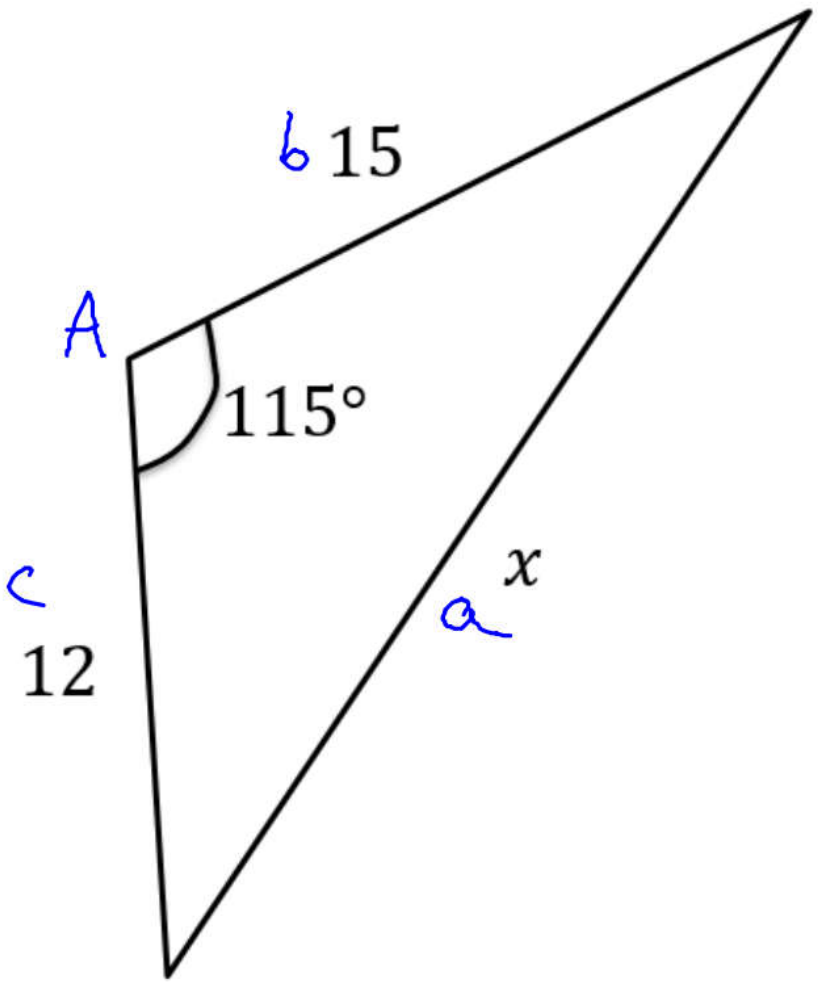




The Cosine Rule

We use the **cosine rule** whenever we have **three sides** (and an angle) involved.





Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

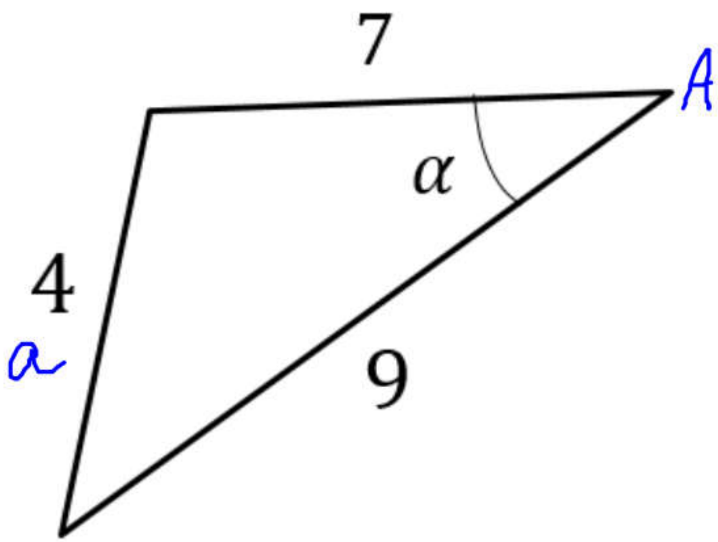
How do we label the sides?

$x^2 = 12^2 + 15^2 - 2 \times 12 \times 15 \times \cos 115°$

$x^2 = 521.14\ldots$

$x = \underline{22.8}$ (3sf)

Dealing with Missing Angles



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$4^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \cos\alpha$$

$$16 = 130 - 126\cos\alpha$$

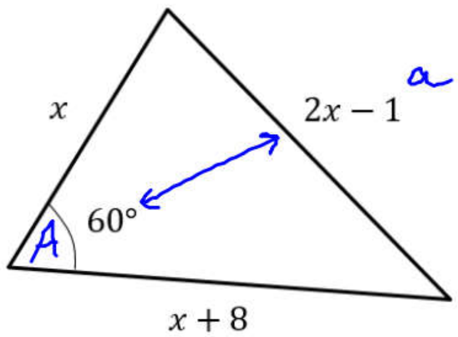
$$126\cos\alpha = 130 - 16$$

$$\cos\alpha = \frac{114}{126}$$

$$\alpha = \cos^{-1}\left(\frac{114}{126}\right) = \underline{\underline{25.2^\circ}}.$$

You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule
#2 Two sides known and a missing side opposite a known angle	Remaining side	Cosine rule
#3 All three sides	An angle	Cosine rule
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	Cosine rule OR Sine rule twice

Trickier Questions



Determine the value of x.

$$(2x-1)^2 = x^2 + (x+8)^2 - 2x(x+8)\cos 60$$

$$4x^2 - 4x + 1 = x^2 + x^2 + 16x + 64 - \cancel{2}x(x+8)\cancel{\tfrac{1}{2}}$$

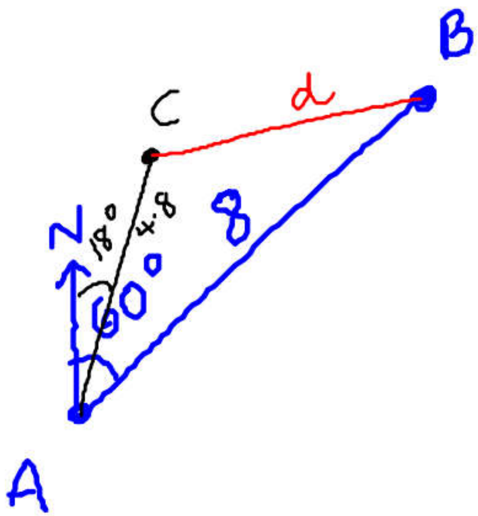
$$4x^2 - 4x + 1 = 2x^2 + 16x + 64 - x^2 - 8x$$

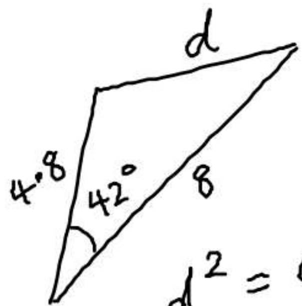
$$4x^2 - 4x + 1 = x^2 + 8x + 64$$

$$3x^2 - 12x - 63 = 0$$

$x = 7$
$x = -3$ but $x > 0$ $\underline{\underline{x = 7}}$.

Coastguard station B is 8 km, on a bearing of 060°, from coastguard station A. A ship C is 4.8 km on a bearing of 018°, away from A. Calculate how far C is from B.



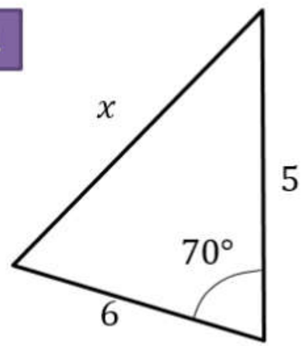


$$d^2 = 4.8^2 + 8^2 - 2 \times 4.8 \times 8 \times \cos 42$$

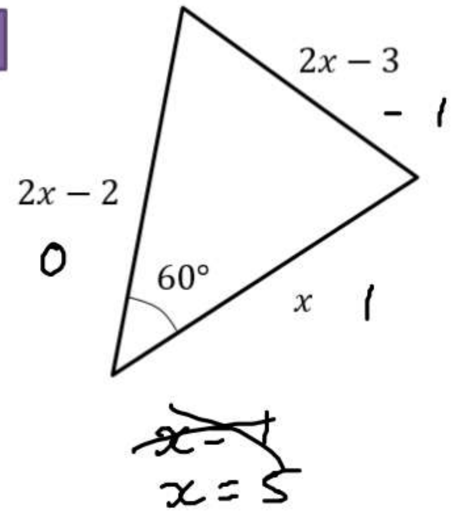
$d = \underline{\underline{5.47\text{ km}}}$ (3sf)

Your Turn

1

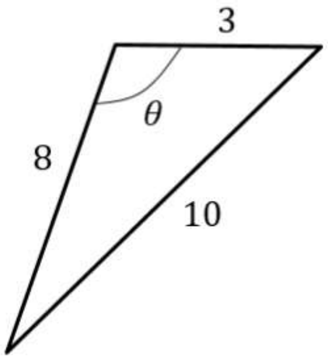


3



$x = 6.36$

2



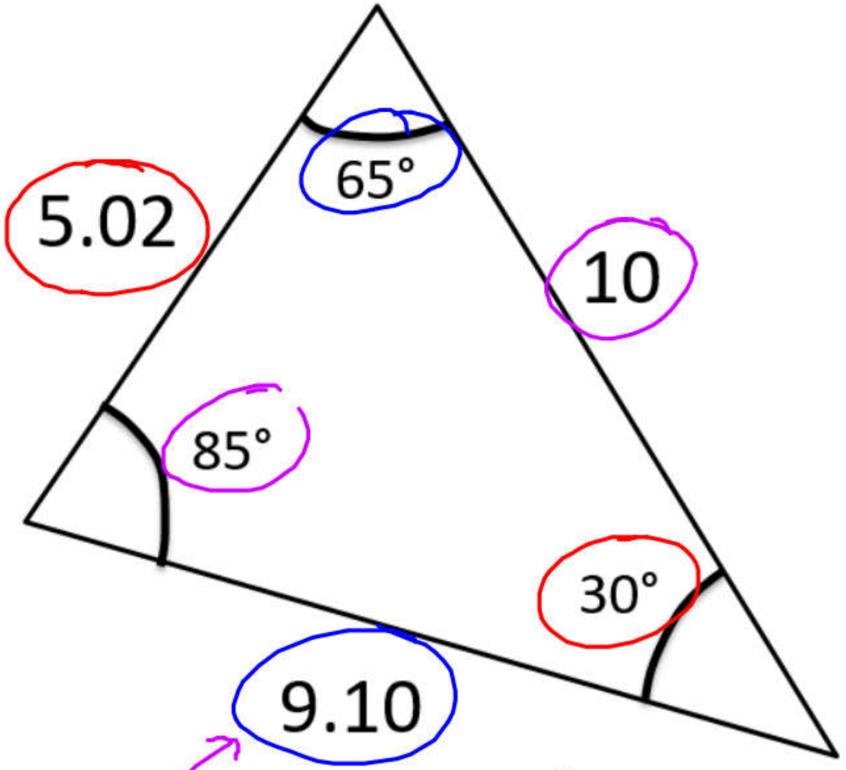
$\theta = 124.2°$

$x = 5$

Ex 9A

The Sine Rule

For this triangle, try calculating each side divided by the sin of its opposite angle. What do you notice in all three cases?



extra zero suggests it has been rounded.

$$\frac{9.10}{\sin 65} = 10.0407$$

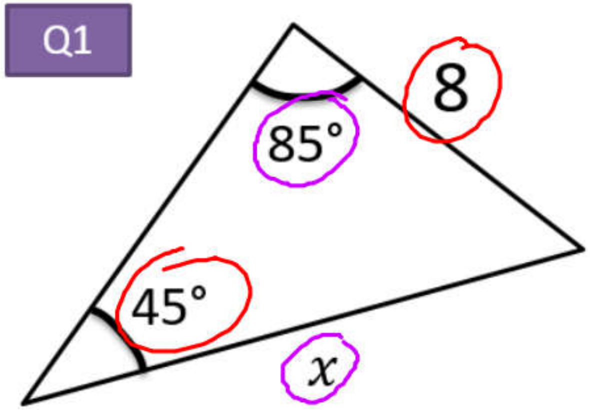
$$\frac{5.02}{\sin 30} = 10.04$$

$$\frac{10}{\sin 85} = 10.0381\ldots$$

Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q1

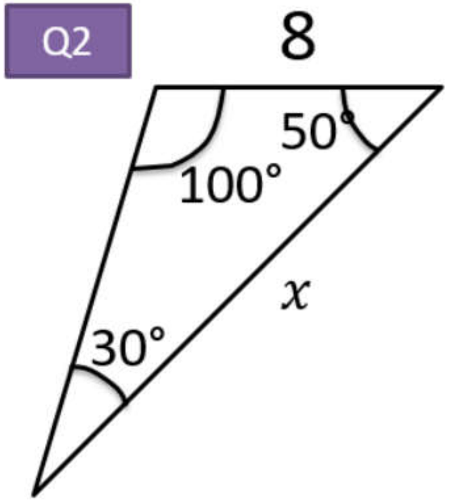


$$\frac{x}{\sin 85} = \frac{8}{\sin 45}$$

$$x = \frac{8\sin 85}{\sin 45} = \frac{8}{\sin 45} \times \sin 85$$

$$x = \underline{11.27} \text{ (2dp)}$$

Q2



$$\frac{x}{\sin 100} = \frac{8}{\sin 30}$$

$$x = \frac{8\sin 100}{\sin 30}$$

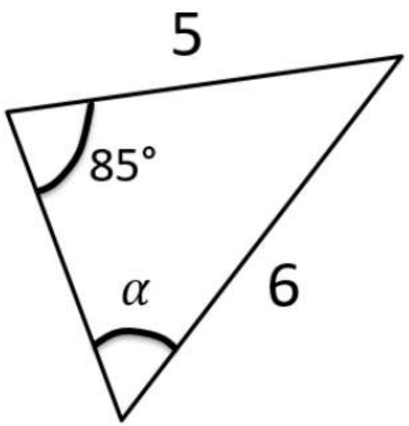
$$x = \underline{15.76} \text{ (2dp)}$$

When you have a missing angle, it's better to take reciprocals to get:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Longrightarrow \frac{\sin A}{a} = \frac{\sin B}{b}$$

i.e. in general put the missing value in the numerator.

Q3



$$\frac{\sin\alpha}{5} = \frac{\sin 85}{6}$$

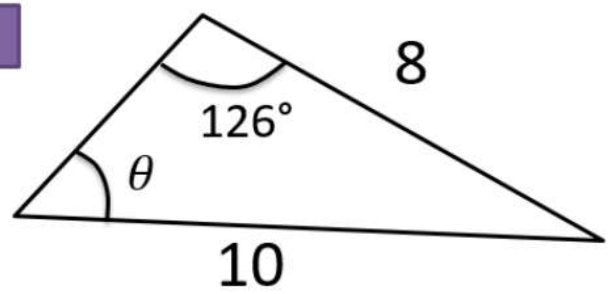
$$\sin\alpha = \frac{\sin 85}{6} \times 5 = \frac{5\sin 85}{6} = \frac{5}{6}\sin 85$$

$$\sin\alpha = 0.83\ldots$$

$$\alpha = \sin^{-1}(0.83\ldots)$$

$$\alpha = \underline{56.1^\circ}$$

Q4



$$\frac{\sin\theta}{8} = \frac{\sin 126}{10}$$

$$\sin\theta = \frac{8\sin 126}{10}$$

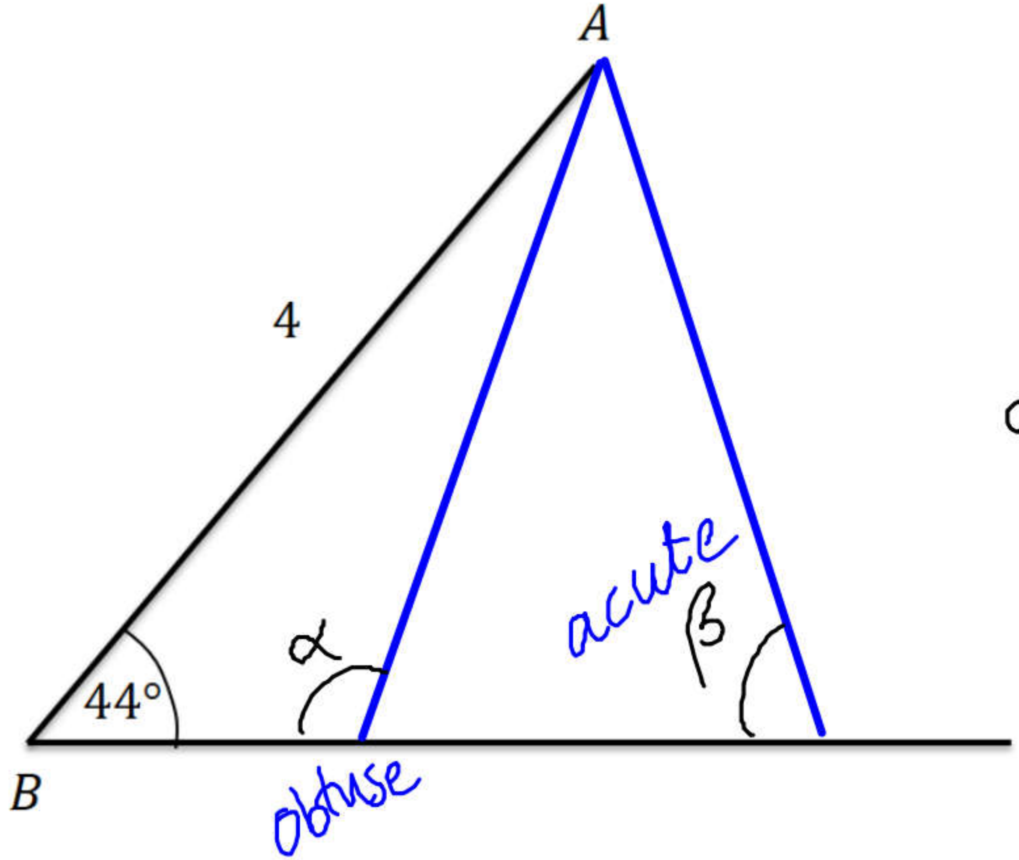
$$\sin\theta = 0.647\ldots$$

$$\theta = \sin^{-1}(0.647\ldots)$$

$$= \underline{40.3^\circ} \text{ (1dp).}$$

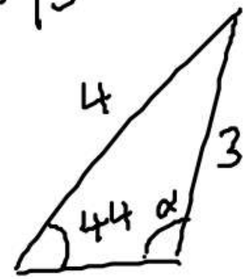
Ex 9B

The 'Ambiguous Case'



Suppose you are told that $AB = 4$, $AC = 3$ and $\angle ABC = 44°$. What are the possible values of $\angle ACB$?

$\alpha \neq \beta$



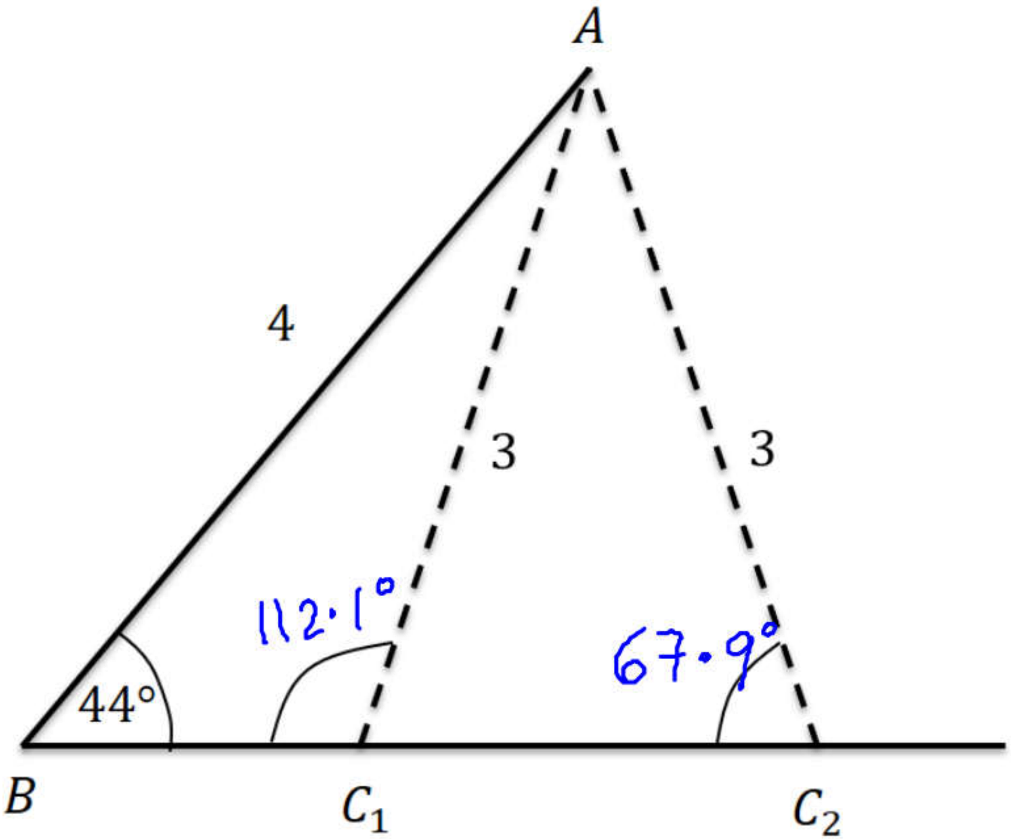
$$\frac{\sin\alpha}{4} = \frac{\sin 44}{3}$$

$$\sin\alpha = \frac{4\sin 44}{3}$$

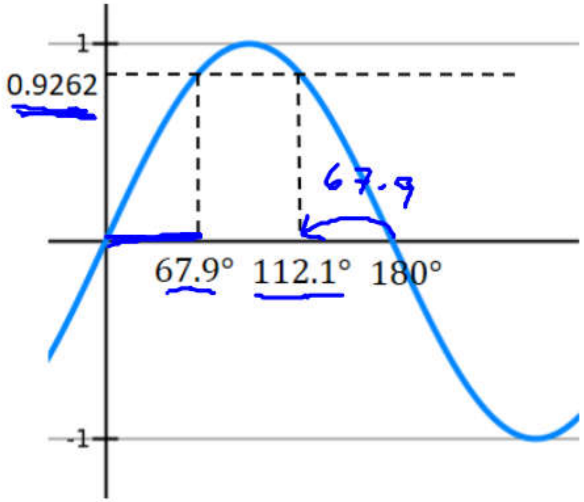
$$\alpha = \sin^{-1}(0.9262\ldots)$$

$$\frac{\sin\beta}{4} = \frac{\sin 44}{3}$$

$$\sin\beta = \frac{4\sin 44}{3}$$



180 - 67.9



Suppose you are told that $AB = 4$, $AC = 3$ and $\angle ABC = 44°$. What are the possible values of $\angle ACB$?

C is somewhere on the horizontal line. There's two ways in which the length could be 3. Using the sine rule:

$$\frac{\sin C}{4} = \frac{\sin 44}{3}$$
$$C = \sin^{-1}(0.9262)$$

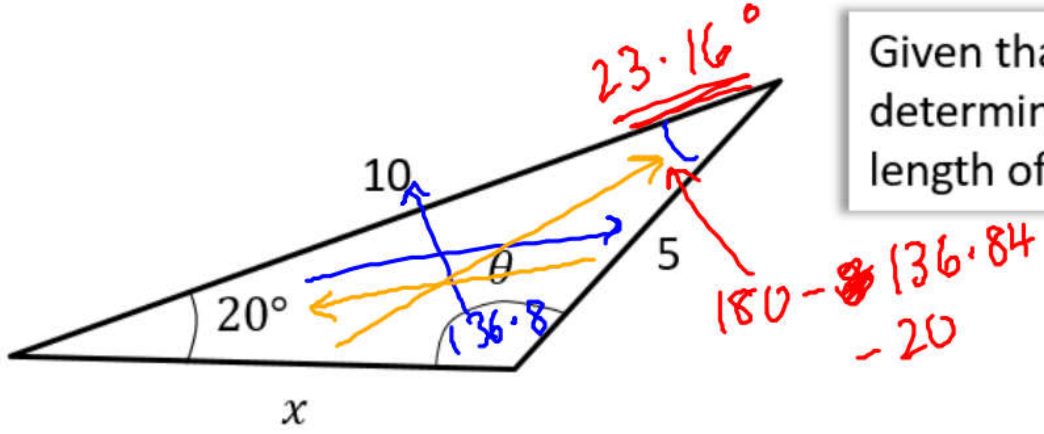
Your calculator will give the acute angle of 67.9° (i.e. C_2). But if we look at a graph of sin, we can see there's actually a second value for $\sin^{-1}(0.9262)$, corresponding to angle C_1.

✏ The sine rule produces two possible solutions for a missing angle:

$$\sin\theta = \sin(180° - \theta)$$

Whether we use the acute or obtuse angle depends on context.

$\theta \to$ acute $180-\theta \to$ obtuse



Given that the angle θ is obtuse, determine θ and hence determine the length of x.

$$\frac{\sin\theta}{10} = \frac{\sin 20}{5}$$

$$\sin\theta = \frac{10\sin 20}{5} = 2\sin 20 = 0.684\ldots$$

$$\theta = \sin^{-1}(0.684\ldots)$$

$$\theta = 43.16^\circ \ldots$$

$$180-\theta = 180-43.16$$

$$= \underline{\underline{136.84^\circ}}$$

our θ is obtuse, so $\theta = \underline{\underline{136.8^\circ}}$

$$\frac{x}{\sin 23.16} = \frac{5}{\sin 20}$$

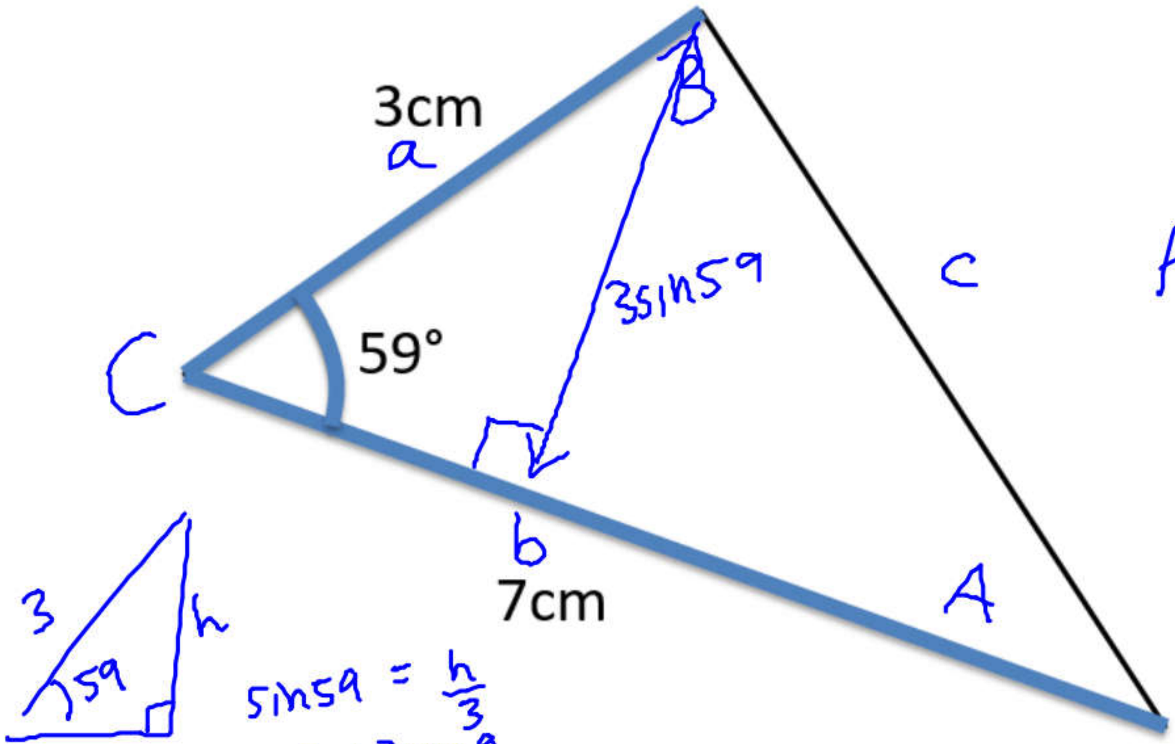
$$x = \frac{5\sin 23.16}{\sin 20}$$

$$x = \underline{\underline{5.75}}. \ (2dp)$$

Ex 9C
Q3, 4, 5.

Ex 9C

Area of Non Right-Angled Triangles



Area = $\frac{1}{2} \times 3 \times 7 \times \sin 59$

$= \underline{9.00 \text{ cm}^2}$ (2dp)

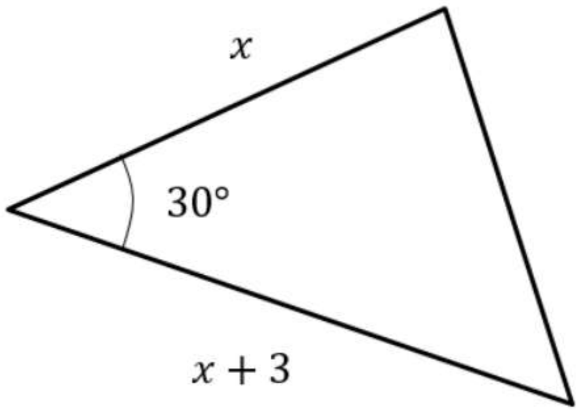
$\sin 59 = \frac{h}{3}$

$h = 3\sin 59$

$$\text{Area} = \frac{1}{2}\, a\, b \sin(C)$$

where C is the angle between two sides a and b.

Tip: You shouldn't have to label sides/angles before using the formula. Just remember that the angle is between the two sides.



The area of this triangle is 10.
Determine x.

$$10 = \frac{1}{2} \times x \times (x+3) \sin 30$$

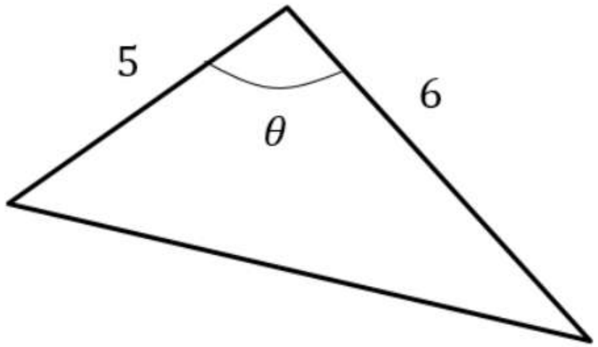
$$10 = \frac{1}{2} x(x+3) \times \frac{1}{2}$$

$$40 = x(x+3)$$

$$40 = x^2 + 3x$$

$$0 = x^2 + 3x - 40$$

$$0 = (x-5)(x+8) \quad \underline{\underline{x = 5}} \text{ or } \cancel{x = -8}$$



The area of this triangle is also 10.
If θ is obtuse, determine θ.

$$10 = \frac{1}{2} \times 5 \times 6 \times \sin\theta$$

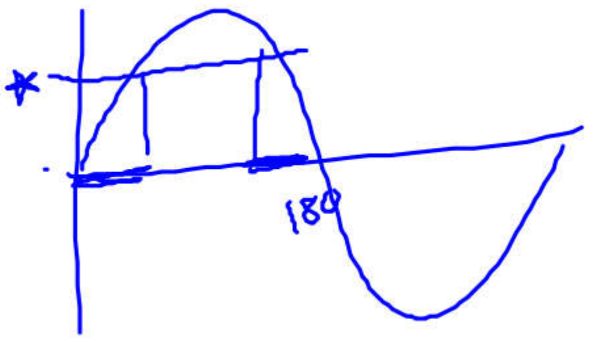
$$10 = 15 \sin\theta$$

$$\frac{2}{3} = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{2}{3}\right)$$

$$= \underline{\underline{41.8^\circ}}$$

obtuse $180 - \theta = \underline{\underline{138.2^\circ}}$



Ex 9D