Chapter 9a: Differentiation - The Rules (Year 2)

In Year 1 the expressions you could differentiation were somewhat limited: only terms of the form ax^n.

1:: Differentiate trigonometric, exponential and log functions.

Find $\frac{d}{dx}(\sin x)$.

$\frac{d}{dx}(\ln x)$

$\frac{d}{dx}(2^x)$

2:: Use chain, product and quotient rules.

These allow you to different composite functions, a product of two functions or division of two functions respectively.

$$\frac{d}{dx}(x \sin x) \qquad \frac{d}{dx}\left(e^{2x^2}\right)$$

Differentiating trigonometric functions

$$\frac{d}{dx}(\sin x) = \boldsymbol{\cos x}$$

Helpful facts:

- As $x \to 0$, $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$
- $\sin(a+b) = \sin a \cos b + \cos a \sin b$

Proof

If $y = f(x)$ then$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Differentiation

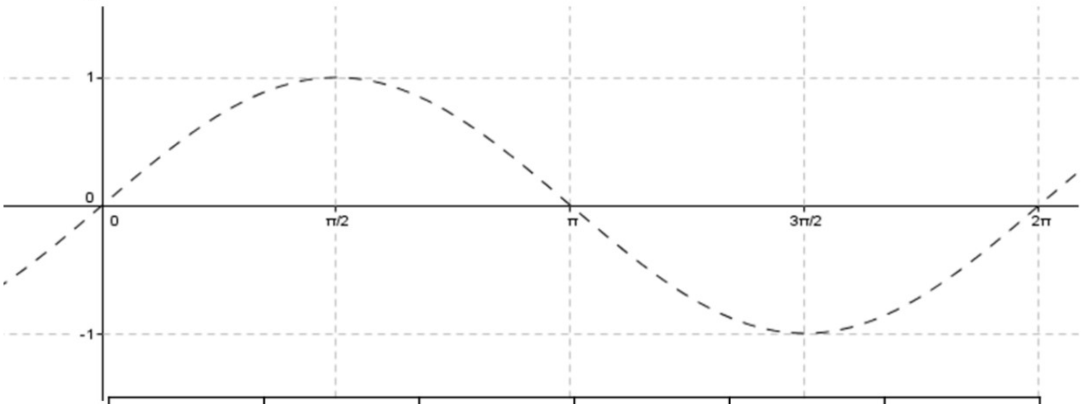
First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

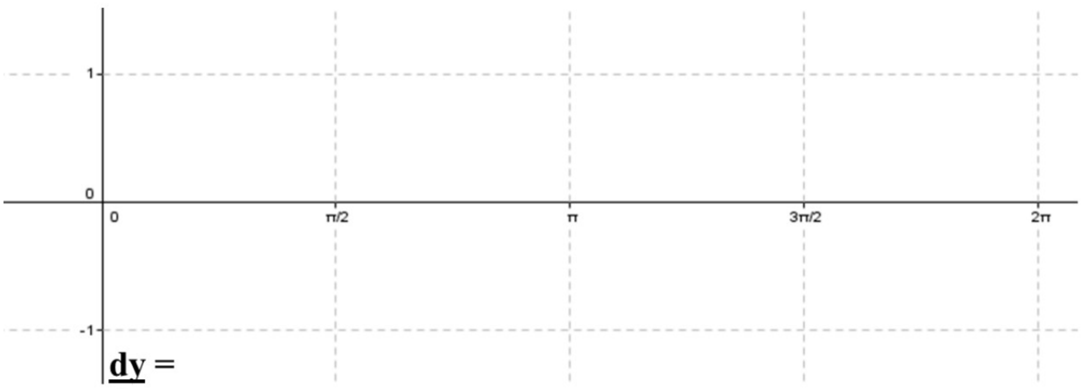
Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

y =



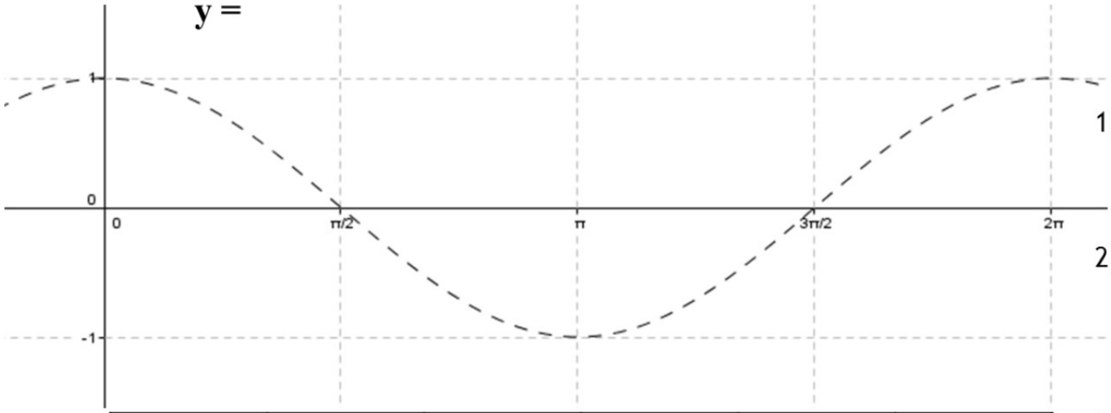
x	0	π/2	π	3π/2	2π
Gradient					



$\frac{dy}{dx}$ =

1) Name the function in the top graph
2) Draw tangents to the curve at 0, π/2, π, 3π/2, and 2π (see example)
3) Estimate the gradients of these tangents
4) Plot these gradients on the second set of axes (this is the derivative of the function)
5) What function does the derivative resemble?

y =



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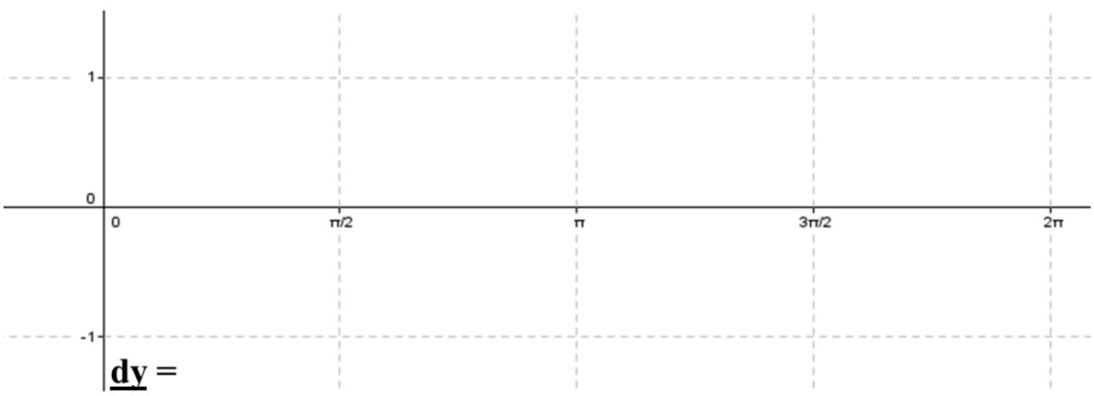

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Differentiating $\sin kx$ and $\cos kx$

$$\frac{d}{dx}(\sin x) = \boldsymbol{\cos x}$$

$$\frac{d}{dx}(\cos x) = -\boldsymbol{\sin x}$$

$$\frac{d}{dx}(\sin kx) = \boldsymbol{k\cos kx}$$

$$\frac{d}{dx}(\cos kx) = \boldsymbol{-k\sin kx}$$

Note: This is not a rule in itself but a specific case of the '*chain rule*' which we'll see later this chapter.

$$\frac{d}{dx}(\sin 3x) =$$

$$\frac{d}{dx}(4\cos 3x) =$$

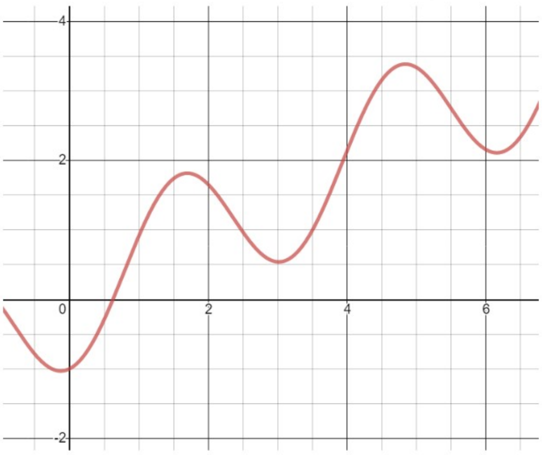
$$\frac{d}{dx}(\cos 5x) =$$

$$\frac{d}{dx}\left(-\frac{1}{2}\sin x\right) =$$

$$\frac{d}{dx}(3\sin 5x) =$$

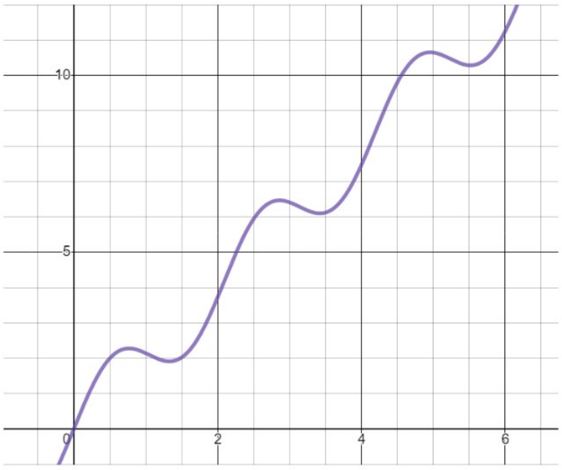
$$\frac{d}{dx}\left(-\frac{2}{3}\cos\frac{1}{2}x\right) =$$

A curve has equation $y = \frac{1}{2}x - \cos 2x$. Find the stationary points on the curve in the interval $0 \leq x \leq \pi$.



Your Turn

A curve has equation $y = \sin 3x + 2x$. Find the stationary points on the curve in the interval $0 \le x \le \frac{2}{3}\pi$.



Ex 9A

Differentiation exponential and log functions

$$\frac{d}{dx}(e^x) = \boldsymbol{e^x}$$

$$\frac{d}{dx}(a^x) = \boldsymbol{\ln a \, . \, a^x}$$

i.e. When we differentiate an exponential function, we multiply by ln of the base. We will prove this after we cover implicit differentiation.

$$\frac{d}{dx}\left(e^{kx}\right) =$$

$$\frac{d}{dx}\left(a^{kx}\right) = \boldsymbol{k \ln a \, . \, a^{kx}}$$

$$\frac{d}{dx}\left(e^{kx}\right) = \boldsymbol{k e^{kx}}$$

$$\frac{d}{dx}(\ln x) = \frac{\mathbf{1}}{\boldsymbol{x}}$$

$$\frac{d}{dx}(\ln kx) =$$

$\frac{d}{dx}(e^x) = \boldsymbol{e^x}$ $\frac{d}{dx}(e^{kx}) = \boldsymbol{ke^{kx}}$	$\frac{d}{dx}(a^x) = \boldsymbol{lna\,.\,a^x}$ $\frac{d}{dx}(a^{kx}) = \boldsymbol{klna\,.\,a^{kx}}$	$\frac{d}{dx}(\ln x) = \frac{\mathbf{1}}{\boldsymbol{x}}$ $\frac{d}{dx}(\ln kx) = \frac{\mathbf{1}}{\boldsymbol{x}}$

$$\frac{d}{dx}(3^x) =$$

$$\frac{d}{dx}(x^3) =$$

$$\frac{d}{dx}(\ln(3x)) =$$

$$\frac{d}{dx}(3^{2x}) =$$

$$\frac{d}{dx}(2^{3x}) =$$

$$\frac{d}{dx}(5\ln x) =$$

$$\frac{d}{dx}\left(e^{\frac{1}{2}x}\right) =$$

$$\frac{d}{dx}(5\ln(2x)) =$$

$$\frac{d}{dt}(9^t) =$$

$$\frac{d}{dx}\left(5(4^x)\right) =$$

$$\frac{d}{dx}(x^4) =$$

$$\frac{d}{dx}(\ln 6x) =$$

$$\frac{d}{dx}(6\ln x) =$$

$$\frac{d}{dx}(3e^{2x}) =$$

$$\frac{d}{dx}(e^{-x}) =$$

Differentiate $y = (e^x + 2)^2$ *(Hint: Expand first)*

A child has headlice and his parents treat it using a special shampoo. The population P of headlice after t days can be modelled using $P = 460(3^{-2t})$

a) Determine how many days have elapsed before the child has 20 headlice left.

b) Determine the rate of change of headlice after 3 days.

A rabbit population P after t years can be modelled using $P = 1000(2^t)$. Determine after how many years the rate of population increase will reach 20,000 rabbits per year.

Ex 9B

Differentiating combinations of functions

Functions can interact in different ways...

How to differentiate

1 **Composite Function**
i.e. of form $y = f(g(x))$

$$y = \sqrt{1+3x}$$

The 'outer' function here is the $\sqrt{}$ and the inner function the $1+3x$.
i.e. $f(x) = \sqrt{x}$ and $g(x) = 1 + 3x$

$\Rightarrow$ The Chain Rule (Ex9C)

2 **Product of Two Functions**
i.e. of form $y = f(x)g(x)$

$$y = x \sin 2x$$

$\Rightarrow$ The Product Rule (Ex9D)

3 **Division (i.e. "Quotient") of Two Functions**
i.e. of form $y = \frac{f(x)}{g(x)}$

$$y = \frac{\ln x}{x}$$

$\Rightarrow$ The Quotient Rule (Ex9E)

$$f(x) = (4x+3)^4$$

$$g(x) = (2x+1)sinx$$

$$h(x) = \sqrt{x^2 + cosx}$$

$$m(x) = e^x\sqrt{x^2 + x}$$

$$y = 2x\,sin(x)$$

$$y = 3sinxcosx$$

$$y = (4x+2)^2(1-x)^3$$

$$y = xe^x$$

$$f(y) = e^{y^2}$$

$$g(y) = \sin(3y^6)$$

$$h(y) = \sqrt[3]{\sin(y)}$$

Which of these are **composite** functions?
Which of these are **products** of functions?
Which of these are a combination - i.e. they are a **product of composite** functions?

The Chain Rule

The chain rule allows us to differentiate a composite function, i.e. a function within a function.

$$y = (3x^4 + x)^5$$

The Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Note: Notice how the du's sort of 'cancel' top and bottom on the RHS. This is not a valid proof of the chain rule, but the d's sort of behave as quantities which can often be manipulated in this way.

Full Method:

$$y = (3x^4 + x)^5$$

Doing it mentally in one go:
(aka the 'blah method')

$$y = (3x^4 + x)^5$$

f(x)	f'(x)	f(x)	f'(x)
a $(1 + 2x)^4$		**b** $(3 - 2x^2)^{-5}$	
c $(3 + 4x)^{\frac{1}{2}}$		**d** $(6x + x^2)^7$	
e $\dfrac{1}{3 + 2x}$		**f** $\sqrt{7 - x}$	
g $4(2 + 8x)^4$		**h** $3(8 - x)^{-6}$	

Practice using the Chain Rule

$y = (x^2 + 1)^3$

$y = \sqrt{x + 1}$

$y = \sin 5x$

What do we expect the answer to be from earlier on?

$y = (\ln x)^3$

$y = e^{x^2+x}$

$y = \ln(\sin x)$

$y = (2^x + 1)^2$

$y = e^{e^x}$

$$\frac{d}{dx}\left(-\frac{1}{2}\sin(x^2)\right)$$

$$\frac{d}{dx}\left(-\frac{2}{3}\cos\frac{1}{2}x^4\right)$$

Differentiate with respect to x

(*a*) $\ln(x^2 + 3x + 5)$, **(2)**

In general:
$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

The Chain Rule - powers of trig

Tip: When differentiating a trig function to a power, always rewrite using a bracket first.

$$y = \sin^2 x$$

$$y = \cos^3 x$$

$$y = -5\cos^6 x$$

$$y = -\sin^{-2} x$$

Using the Chain Rule Twice... or more?

$y = \cos^3 2x$

$y = -5\sin^3(x^2)$

Ex 9C Q1-5, 11-13

Challenge #1

$\cos 2x = 1 - 2\sin^2 x$

Differentiate $\sin^2 x$ using the Chain Rule.

Use the double angle formula to find $\sin^2 x$ in terms of $\cos 2x$. Differentiate. Do you get the same thing?

$y = \sin^2 x$

Challenge #2

Differentiate $ln(x^3)$

i) Using the Chain Rule

ii) Using log laws

Ex 9C Q6-10

dx/dy

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

Sometimes we might have x in terms of y, but we want to find $\frac{dy}{dx}$.

Find $\frac{dy}{dx}$ when $x = 2y^2 + y$

Find the gradient of $x = (1 + 2y)^3$ when $y = 1$

The Product Rule

✏ The product rule:

If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Alternatively $y' = uv' + vu'$

This is quite easy to remember. Differentiate one of the things but leave the other. Then do the other way round. Then add!
Since addition is commutative, it doesn't matter which way round we do it.

If $y = x^2 \sin x$, determine $\frac{dy}{dx}$

Tip: With the product rule I want you to write out u= and v=. This is because each of u and v can be more complicated to differentiate. If you're doing Further Maths, you'll need to work towards doing it all mentally

If $y = xe^{2x}$, determine the coordinates of the turning point.

The Product Rule combined with Chain Rule

Given that $f(x) = x^2\sqrt{3x-1}$, find $f'(x)$

If $y = e^{4x}\sin^2 3x$, show that $\frac{dy}{dx} = e^{4x}\sin 3x\,(A\cos 3x + B\sin 3x)$, where A and B are constants to be determined.

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Differentiate with respect to x, giving your answer in its simplest form,

(*a*) $x^2 \ln(3x)$, **(4)**

3.

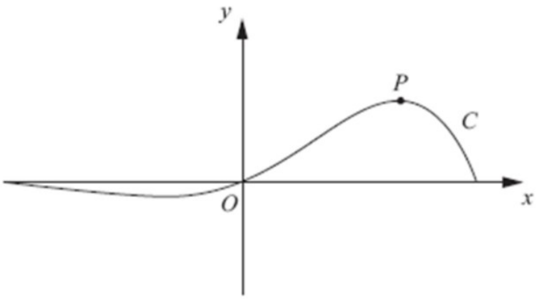

Figure 1

Figure 1 shows a sketch of the curve C which has equation

$$y = e^{x\sqrt{3}} \sin 3x, \quad -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}.$$

(*a*) Find the x-coordinate of the turning point P on C, for which $x > 0$.
Give your answer as a multiple of π.

(6)

(*b*) Find an equation of the normal to C at the point where $x = 0$.

(3)

The Quotient Rule

$$\text{If } y = \frac{u}{v}, \text{then} \frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

Want to prove it?
Let u = yv, then differentiate with respect to x using the product rule and rearrange to make y' the subject

If $y = \frac{x}{2x+5}$, find $\frac{dy}{dx}$

Find the stationary point of $y = \frac{\sin x}{e^{2x}}$, $0 < x < \pi$

Your Turn

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Differentiate with respect to x, giving your answer in its simplest form,

(b) $\dfrac{\sin 4x}{x^3}$. **(5)**

Ex 9E
1ace
2ace
3, 5, 7, 9, 11

Differentiating other trigonometric functions

Differentiate $y = \tan x$

More generally: $\frac{d}{dx}(\tan kx) = k \sec^2 kx$

Differentiate $y = \sec x$

✎ More generally: $\frac{d}{dx}(\sec kx) = k \sec kx \tan kx$

Standard Results - differentiating trigonometric functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$$
$$\frac{d}{dx}(\mathrm{cosec}\, x) = -\,cosec\, x \cot x$$

Memory Tip: To memorise the two on the right, notice that when we ‘co’ the two on the left, it ‘co’s each term in the derivative, but also negates it.

Differentiate

(a) $y = \frac{cosec\,2x}{x^2}$

(b) $y = \sec^3 x$

Observation: Notice that when differentiating a power of *sec*, the power stays the same. This will be relevant when we come to integration.

Ex 9F Q1-7

Note Q8-11 has moved to Further Maths

Edexcel C3 June 2013(R) Q5b

(*b*) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 3x)$ can be written in the form

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant. **(3)**

Recap #1: differentiate

$f(x) = \ln(\sec x)$

$g(x) = \sin^4 x$

$h(x) = \tan x \sin x$

$m(x) = 3x^2 \cot x$

$n(x) = e^{\operatorname{cosec} 2x}$

Often in exam questions, you will be given x in terms of y, but want to find $\frac{dy}{dx}$ in terms of x.

The key is to make use of an appropriate trig identity, e.g:

$$\sin^2 x + \cos^2 x \equiv 1 \qquad 1 + \tan^2 x \equiv \sec^2 x$$

Given that $x = \tan y$, express $\frac{dy}{dx}$ in terms of x.

Given that $x = 2\sin y$, express $\frac{dy}{dx}$ in terms of x.

Ex 9F Q12

Recap #2: differentiate

$f(x) = \ln(\sin x)$

$g(x) = \sec(x^3)$

$h(x) = 2xe^x$

$m(x) = (x^2 + 1)\cot x$

$n(x) = e^{(2x)}\ln x$