Edexcel A Level Further Maths: Decision Maths 1

The Route Inspection Algorithm

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Route Inspection

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Chinese Postman Problem

What is the Chinese postman problem?

- The **route inspection problem** is also known as the **Chinese postman problem**
- You are required to find the **route of least weight** that traverses **every edge** in the graph such that the **route starts** and **finishes** at the **same** vertex
- Some **edges** may need to be traversed twice and the challenge is to **minimise** the total weight of these **repeated edges**
- Variations to the route inspection problem could involve
  - the start and finish vertices being different
  - certain edges being disregarded
    - e.g. a road closure
  - the route requiring repetition
    - e.g. a road sweeper covering both sides of the road

Your notes

Networks with 0 or 2 Odd Nodes

How do I solve the Chinese postman problem?

- If the graph is **Eulerian**
  - **all** of the vertices in the graph are **even**
    - i.e. 0 odd nodes
  - it will be possible to find an **Eulerian circuit**
    - i.e. a route that traverses each edge once only, starting and finishing at the same vertex
  - no route/edges will need repetition
  - the shortest route will be the **sum of the weights** of the **edges** in the network
- If the graph is **semi-Eulerian**
  - there will be **one pair** of **odd** vertices
    - i.e. 2 odd nodes
  - if the route has to start and end at the **same vertex**
    - the **shortest path** between the two odd nodes will need to be found
      - the **edges** making that route will need to be repeated
    - the **shortest route** will be the **sum of the weights** of the **edges** in the network **plus the weight of the repeated edges**
  - if the route starts at one of the odd vertices and ends at the other
    - no edges will need repetition
    - the shortest route will be the sum of the weights of the edges in the network

What are the steps of the Chinese postman algorithm?

- **STEP 1**
  Inspect the degree of all of the vertices and identify any odd nodes

- **STEP 2**
  If all of the vertices are even go to STEP 3
  If there is one pair of odd nodes, find the shortest path between them - the edges making this path will need repeating so add them to the graph

- **STEP 3**
  Write down an Eulerian circuit of the adjusted graph to find a possible route
  - Find the sum of the edges traversed to find the total weight

Examiner Tip

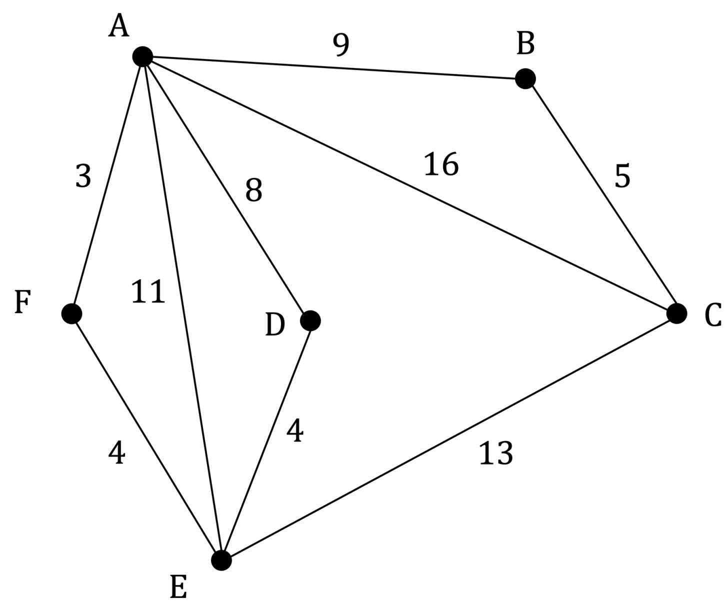
- Look carefully for the shortest path between two vertices
  - exam questions often have graphs where a path made up of several edges will create a shorter distance than a direct connection

Your notes

Worked example

The network shown below displays the distance, in metres, of the cables in a network between different connection points A, B, C, D, E and F.

Each length of cable must be inspected.

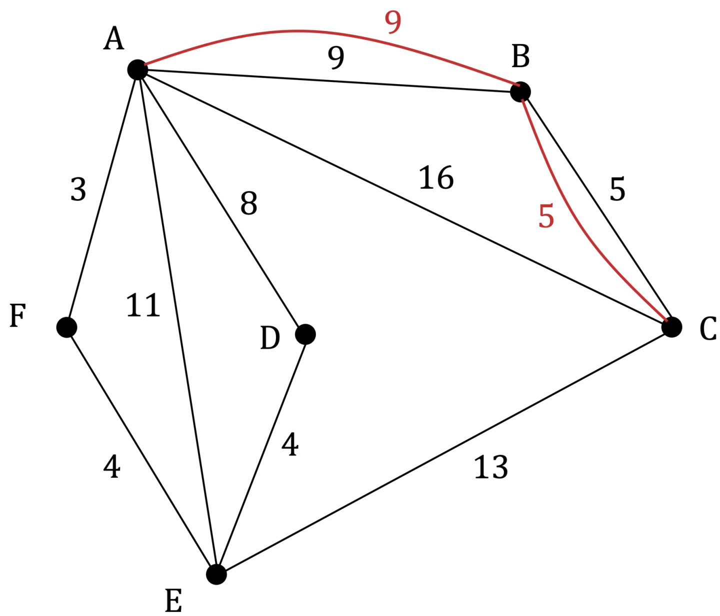


a) Find the shortest route that starts and finishes at connection point E.

- **STEP 1**
  Inspect the order of the nodes

A: 5 (odd)
B: 2 (even)
C: 3 (odd)
D: 2 (even)
E: 4 (even)
F: 2 (even)

- **STEP 2**
  There is exactly one pair of odd vertices, A and C
  The shortest *route* between A and C is ABC so the *edges* AB and BC need repeating
  An Eulerian circuit, starting and ending at connection point E, is now possible



- **STEP 3**
  Find an Eulerian circuit, starting and ending at connection point E

Shortest route: EFAEDABACBCE

b) State the total length of the shortest route.

Add together the lengths of the edges in the original graph

$$3 + 4 + 4 + 5 + 8 + 9 + 11 + 13 + 16 = 73$$

Add the result to the lengths of the repeated edges

$$73 + 5 + 9 = 87$$

Shortest route = 87 km

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Route Inspection & Repeated Routes

Your notes

Networks with more than 2 Odd Nodes

How do I solve the Chinese postman problem when the network has more than 2 odd nodes?

- In any graph there will always be an **even number of odd nodes**
  - the **total sum of the degrees** of the nodes is **double** the **number of edges**
- If there are **more than two odd nodes**, the **shortest route** between **each possible pairing** of the odd nodes must be considered in order to find the **minimum weight** of the **routes/edges** that need to be repeated
  - In cases of **four odd nodes,** there will be **three** such pairings
  - For example in a graph where the four odd nodes are P, Q, R and S the pairings would be
    - PQ and RS
    - PR and QS
    - PS and QR
  - In questions where there are **more than four** odd vertices, additional information will be given (such as a different start and end vertex) that essentially reduces the problem to four odd nodes
    - (There are 15 pairings to consider for 6 odd vertices!)

What are the steps of the Chinese postman algorithm for networks with more than 2 odd nodes?

- **STEP 1**
  Inspect the degree of all nodes and identify any odd nodes

- **STEP 2**
  Find all the possible pairings between the odd nodes

- **STEP 3**
  For each possible pairing of odd nodes, find (by inspection) the shortest *route* between them
  The shortest of these routes will be repeated so add any repeated edges required to the network

- **STEP 4**
  Write down an Eulerian circuit of the adjusted network to find a possible route
  - Find the sum of the edges traversed to find the total weight

What variations may there be on the Chinese postman algorithm?

- A variation on the **4 odd nodes** problem is that the **start** and **end** nodes can be any two of the odd nodes
  - The problem is to find the **shortest route and** the corresponding **start** and **end** nodes
  - To solve this problem
    - find the length of the routes for **all** possible pairings of the odd nodes

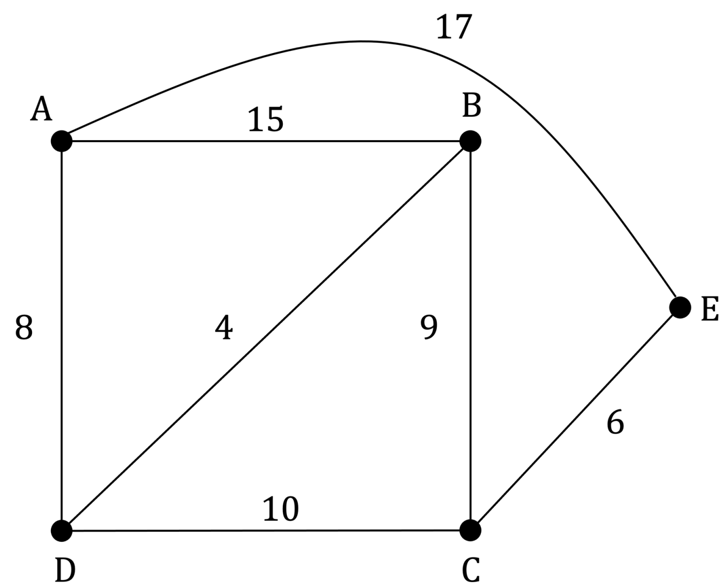
    - choose the **shortest route** between any 2 of them to be **repeated**
    - the other two odd vertices will be your start and finish points
- Another variation may be that the **weighting** of an edge between a pair of nodes may be different depending on if it is the **first time** it is being traversed or a **repeat**
  - For example, if an inspector was checking a pipeline for defects then the first time going along a section of pipeline could take longer during inspection than if it is being repeated simply to get from one node to another

Your notes

Your notes

Worked example

The network shown below displays the distances, in kilometres, of the main roads between towns A, B, C, D and E.
Each road is to be inspected for potholes.



a) Explain why the network does not contain an Eulerian circuit.

Inspect the degree of each node

A: 3 (odd)
B: 3 (odd)
C: 3 (odd)
D: 3 (odd)
E: 2 (even)

The graph does not contain an Eulerian circuit as some of the vertices are odd

b) Find the shortest route that starts and finishes at town A and allows for each road to be inspected.

- **STEP 1**
  There are 4 odd nodes, A, B, C and D
- **STEP 2**
  The possible pairings between the odd nodes are AB and CD, AC and BD and AD and BC

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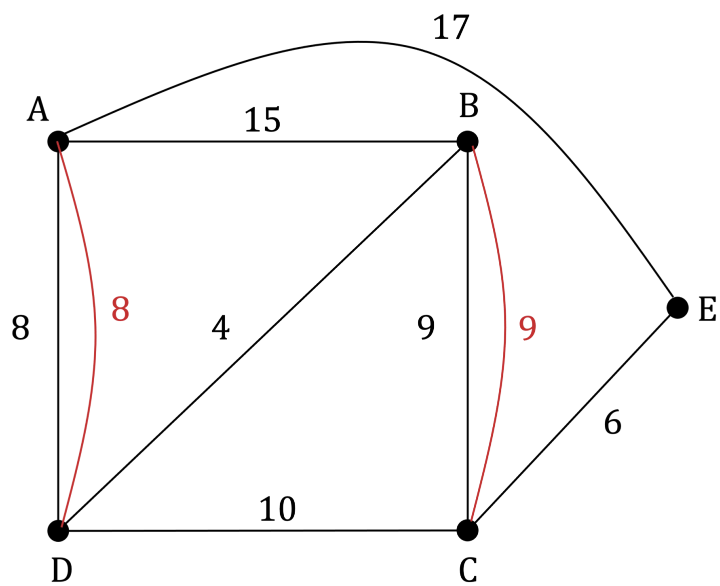
- **STEP 3**
  By inspection, find the shortest route (and edges involved) between each pairing

AB + CD = (ABD) + (CD) = 12 + 10 = 22

AC + BD = (ADC) + (BD) = 18 + 4 = 22

AD + BC = (AD) + (BC) = 8 + 9 = 17 ⟵ shortest

The shortest of these routes is AD and BC, so add the edges AD and BC to the graph



- **STEP 4**
  Write down an Eulerian circuit (from the adjusted graph) starting at vertex A

Eulerian circuit: ADABDCBCEA

There are other possible Eulerian circuits that you could find

c) State the total length of the shortest route.

Add the length of each edge in the graph, then add the weight of the repeated edges.

15 + 17 + 8 + 4 + 9 + 10 + 6 = 69 (edges in the original graph)

17 (repeated edges)

Total length of the shortest route: 86 km

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