Decide what geometric situation each set of equations represents.

1) Check what the determinant of the matrix is. If it is NOT zero, they meet at a point
2) Solve the equation on your calculator to decide if it has one solution (meet at a point), infinite solutions (sheaf or same plane) or no solutions (parallel planes or prism)
3) Use algebra to see if they are consistent or inconsistent

$$\begin{aligned} 3x + 4y + z &= 2 \\ 6x + 8y + 2z &= 4 \\ 9x + 12y + 3z &= 6 \end{aligned}$$

$$\begin{aligned} 4x + 3y - 2z &= 5 \\ 2x + 4y - 3z &= 8 \\ 8x + 6y - 4z &= 9 \end{aligned}$$

$$\begin{aligned} x + 2y - z &= 5 \\ 2x + 3y - 3z &= 18 \\ x + 5y + z &= 10 \end{aligned}$$

$$\begin{aligned} x - 2y + 3z &= -2 \\ 2x - 3y + 5z &= -3 \\ x + 3y - 2z &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y - 3z &= -2 \\ 2x - 3y + 5z &= -3 \\ x + 3y - 2z &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y - 11z &= -2 \\ 2x + 11y + 5z &= 11 \\ x + 3y - 2z &= -11 \end{aligned}$$

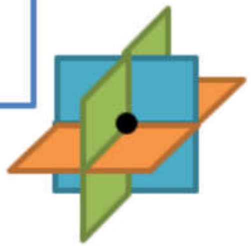
$$
\begin{aligned}
3x + 4y + z &= 2 \\
6x + 8y + 2z &= 4 \\
9x + 12y + 3z &= 6
\end{aligned}
$$

Same plane!

$$
\begin{aligned}
4x + 3y - 2z &= 5 \\
2x + 4y - 3z &= 8 \\
8x + 6y - 4z &= 9
\end{aligned}
$$

(1) and (3) are parallel

$$
\begin{aligned}
x + 2y - z &= 5 \\
2x + 3y - 3z &= 18 \\
x + 5y + z &= 10
\end{aligned}
$$

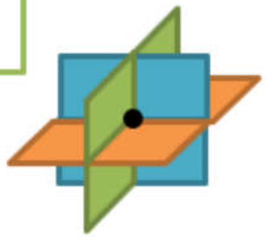


Planes meet at a point, det is not 0

$$
\begin{aligned}
x - 2y + 3z &= -2 \\
2x - 3y + 5z &= -3 \\
x + 3y - 2z &= 3
\end{aligned}
$$

Sheaf, none are parallel or multiples, infinite solutions

$$
\begin{aligned}
x - 2y - 3z &= -2 \\
2x - 3y + 5z &= -3 \\
x + 3y - 2z &= 3
\end{aligned}
$$



Planes meet at a point, det is not 0

$$
\begin{aligned}
x - 2y - 11z &= -2 \\
2x + 11y + 5z &= 11 \\
x + 3y - 2z &= -11
\end{aligned}
$$

Prism, no solutions, none are parallel or multiples

$$\boxed{\begin{aligned} x - 2y + 3z &= -2 \quad ①\\ 2x - 3y + 5z &= -3 \quad ②\\ x + 3y - 2z &= 3 \quad ③ \end{aligned}}$$

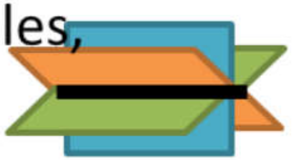
Infinitely many

$x = -z$

$y = -1 + z$

$z = z$

Sheaf, none are parallel or multiples, infinite solutions



$(-10, 11, 10)$

① − ③

$-5y + 5z = -5$

$\boxed{-y + z = -1}$ → consistent.

① × 2 − ②

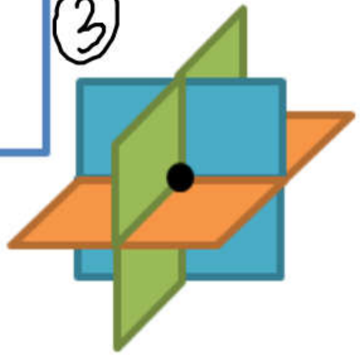
$2x - 4y + 6z = -4$

$2x - 3y + 5z = -3$

$\boxed{-y + z = -1}$

$$\left| \begin{pmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 1 & 3 & -2 \end{pmatrix} \right| = 0$$

$$x + 2y - z = 5 \quad (1)$$
$$2x + 3y - 3z = 18 \quad (2)$$
$$x + 5y + z = 10 \quad (3)$$



Planes meet at a point, det is not 0

$(3) - (1)$

$3y + 2z = 5$

$(1) \times 2 - (2)$

$2x + 4y - 2z = 10$

$2x + 3y - 3z = 18$

$y + z = -8$

$3y + 2z = 5$

$y + z = -8$

$y = -8 - z$

$3(-8 - z) + 2z = 5$

$-24 - 3z + 2z = 5$

$-29 = z$

$y = -8 - -29$

$= 21$

$x = -66$

Example

[Textbook] A system of equations is shown below:

$$3x - ky - 6z = k$$
$$kx + 3y + 3z = 2$$
$$-3x - y + 3z = -2$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the plane corresponding to each value of k.

(a) $k = 0$ (b) $k = 1$ (c) $k = -6$

Remember that the system of equations is **consistent** if the corresponding matrix is non-singular, i.e. its determinant is non-0.

a

$$k = 0: \begin{vmatrix} 3 & 0 & -6 \\ 0 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = -18$$

Matrix non-singular so a unique solution, i.e. planes meet at single point.

Example

[Textbook] A system of equations is shown below:

$$3x - ky - 6z = k$$
$$kx + 3y + 3z = 2$$
$$-3x - y + 3z = -2$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the plane corresponding to each value of k.

(a) $k = 0$ (b) $k = 1$ (c) $k = -6$

b

$$k = 1: \begin{vmatrix} 3 & -1 & -6 \\ 1 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = 0$$

$3x - y - 6z = 1$ (1)
$x + 3y + 3z = 2$ (2)
$-3x - y + 3z = -2$ (3)
$(1) + 2 \times (2)$: $5x + 5y = 5$ (4)
$(2) - (3)$: $4x + 4y = 4$ (5)

Equations (4) and (5) are consistent so system is consistent and has an infinity of solutions. Planes meet at a sheaf.

If the matrix is singular, the system of equations could still be consistent: recall that we might have a sheaf (i.e. planes intersect at a line) or equations represent same plane.
Eliminate one of the variables. If resulting two equations are consistent, then system will be consistent.

Example

[Textbook] A system of equations is shown below:

$$3x - ky - 6z = k$$
$$kx + 3y + 3z = 2$$
$$-3x - y + 3z = -2$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the plane corresponding to each value of k.

(a) $k = 0$ (b) $k = 1$ (c) $k = -6$

$$k = -6: \begin{vmatrix} 3 & 6 & -6 \\ -6 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = 0$$

$$3x + 6y - 6z = -6$$
$$-6x + 3y + 3z = 2$$
$$-3x - y + 3z = -2$$

$① + ③ \quad 5y - 3z = -8 \rightarrow$ inconsistent, so form a prism.

$2 \times ① + ② \quad 15y - 9z = -10$

$5y - 3z = -\frac{10}{3}$

If the matrix is singular, the system of equations could still be consistent: recall that we might have a sheaf (i.e. planes intersect at a line) or equations represent same plane.

Eliminate one of the variables. If resulting two equations are consistent, then system will be consistent.

Test Your Understanding

The system of equations is consistent and has a single solution. Determine the possible values of k.

$$2x + 3y - z = 13$$
$$3x - y + kz = 11$$
$$x + y + z = 7$$

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & k \\ 1 & 1 & 1 \end{vmatrix} \neq 0$$

$$2(-1-k) - 3(3-k) - 1(3+1) \neq 0$$

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & k \\ 1 & 1 & 1 \end{vmatrix} = k - 15$$

To have a solution, we require that $k - 15 \neq 0$, thus $k \neq 15$.

The population of the Zebu cattle in a particular country is modelled by two sub-populations, adults and juveniles. In this model, the only factors affecting the population of the Zebu are the birth and survival rates of the population.

Data recorded in the years preceding 2018 was used to suggest the annual birth and survival rates. The results are shown in the table below, with values to 2 significant figures. It is assumed that these rates will remain the same in future years.

	Birth rate	Survival Rate
Adult population	0.23	0.97
Juvenile population	0	0.87

It is also assumed that $\frac{1}{3}$ of the surviving juvenile population become adults each year.

Let A_n and J_n be these respective sub-populations, in millions of adults and juveniles, n years after 1st January 2018. Then the adult population in year $n + 1$ satisfies the equation

$$A_{n+1} = 0.97A_n + \frac{1}{3}(0.87)J_n = 0.97A_n + 0.29J_n\,.$$

(a) Form the corresponding equation for the juvenile population in year $n + 1$ under this model, justifying your values.

(2)

$$J_{n+1} = 0.23A_n + \frac{2}{3}(0.87)J_n$$

$$= 0.23A_n + 0.58J_n$$

The total population on 1st January 2018 was estimated, to 2 significant figures, as 1.5 million Zebu, with 1.2 million of these being adults. Total = 1.5 $A_0 = 1.2$ $J_0 = 0.3$

(b) Find the value of p and the matrix $\mathbf{M}$ such that the population of Zebu can be modelled by the system

$$\begin{pmatrix} A_0 \\ J_0 \end{pmatrix} = \begin{pmatrix} 1.2 \\ p \end{pmatrix} \qquad \begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = \mathbf{M}\begin{pmatrix} A_n \\ J_n \end{pmatrix},$$

giving p to 2 significant figures and each entry of $\mathbf{M}$ to 2 decimal places.

(3)

$A_{n+1} = 0.97A_n + 0.29J_n$ $\qquad$ $\underline{\underline{p = 0.3}}$ ✓

$J_{n+1} = 0.23A_n + 0.58J_n$

$$M = \begin{pmatrix} 0.97 & 0.29 \\ 0.23 & 0.58 \end{pmatrix}$$ ✓✓

$$\begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = \begin{pmatrix} 0.97 & 0.29 \\ 0.23 & 0.58 \end{pmatrix}\begin{pmatrix} A_n \\ J_n \end{pmatrix}$$

(c) Using the model formed in part (b), find, to 3 significant figures,

(i) the **total** Zebu population that was present on 1st January 2017, A_{-1}, J_{-1}

(ii) the predicted **juvenile** Zebu population on 1st January 2025.

(5)

i) $$\begin{pmatrix} A_0 \\ J_0 \end{pmatrix} = \begin{pmatrix} 0.97 & 0.29 \\ 0.23 & 0.58 \end{pmatrix} \begin{pmatrix} A_{-1} \\ J_{-1} \end{pmatrix}$$

$$\begin{pmatrix} 0.97 & 0.29 \\ 0.23 & 0.58 \end{pmatrix}^{-1} \begin{pmatrix} 1.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} A_{-1} \\ J_{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1.22807\ldots \\ 0.030\ldots \end{pmatrix} = \begin{pmatrix} A_{-1} \\ J_{-1} \end{pmatrix}$$

TOTAL in 2017 $= 1.228\ldots + 0.030$
$= 1.23$ million (3sf)

ii) $$\begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = \begin{pmatrix} 0.97 & 0.29 \\ 0.23 & 0.58 \end{pmatrix} \begin{pmatrix} A_n \\ J_n \end{pmatrix}$$

$n = 0$, 2018

$n = 7$, 2025

$$\begin{pmatrix} A_7 \\ J_7 \end{pmatrix} = \begin{pmatrix} 0.97 & 0.29 \\ 0.23 & 0.58 \end{pmatrix}^7 \begin{pmatrix} 1.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2.117 \\ 0.938 \end{pmatrix}$$

Juveniles $= 0.938$ million (3sf).

As a result of the predictions of this model the country will export 15 000 juveniles to a neighbouring country at the end of each year.

(d) Adapt the model from 2018 onwards to include this export.

(2)

(e) State one limitation of this model.

(1)

d)
$$\begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = M \begin{pmatrix} A_n \\ J_n \end{pmatrix} - \begin{pmatrix} 0 \\ 0.015 \end{pmatrix}$$

e) We would expect survival rates and birth rates to vary year on year.

It doesn't take into account how populations of predators or rivals may affect their population.