Chapter 2: Argand Diagrams

1:: Represent complex numbers on an Argand Diagram.

2:: Put a complex number in modulus-argument form.

"Put $1 + i$ in modulus-argument form."

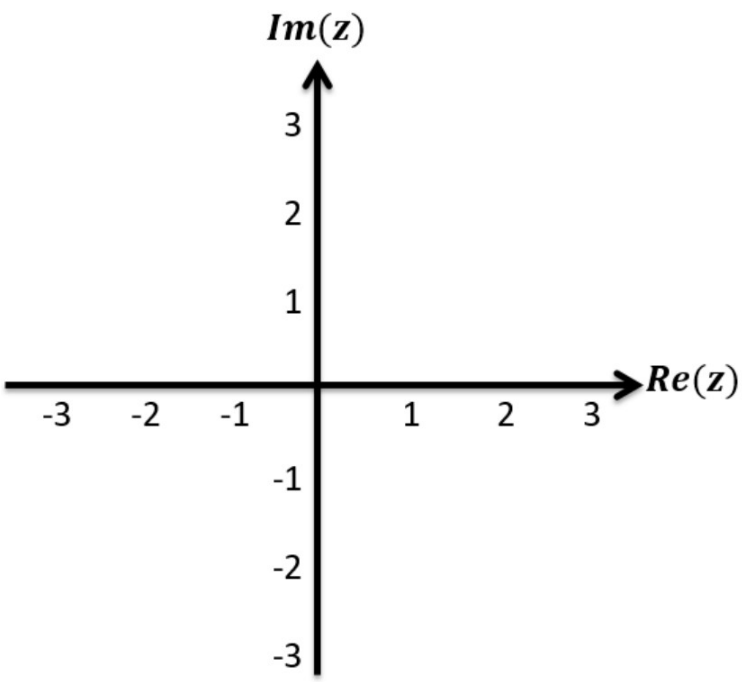
3:: Identify loci and regions.

"Give the equation of the loci of points that satisfies $|z| = |z - 1 - i|$"

Argand Diagrams

Just as x-y axes were a useful way to visualise coordinates, an Argand diagram allows us to visualise complex numbers.

Very simply, a complex number $x + iy$ can be plotted as a point (x, y).
The "x" axis is therefore the "**real axis**" and the "y" axis therefore the "**imaginary axis**".
The plane (2D space) formed by the axes is known as the "**complex plane**" or "z plane".



Plot the following complex numbers on the Argand diagram

$1 + i$

$-1 + 2i$

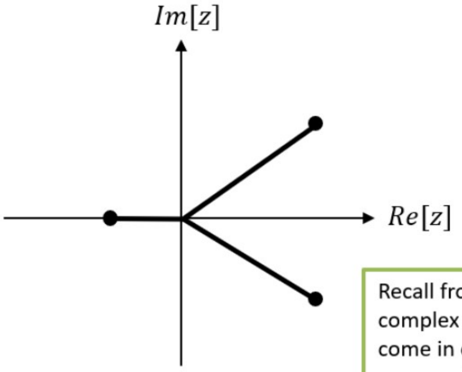
$2 - 3i$

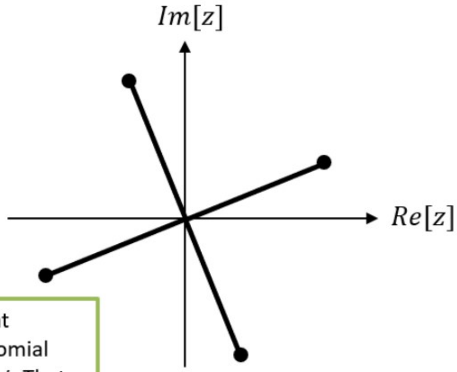
$-2i$

$-(-2i + 3)$

But *why* visualise complex numbers?

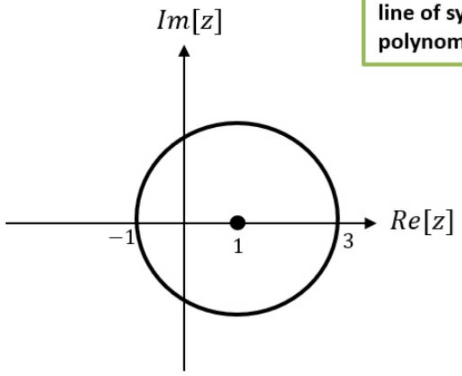
Just as with standard 2D coordinates, Argand diagrams help us interpret the relationship between complex numbers in a **geometric** way:



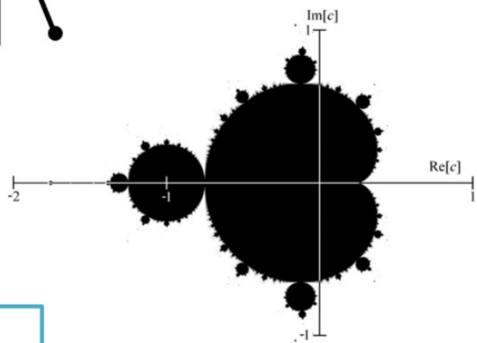


"Solve $z^4 = 1 + i$"
When you find the nth roots of a complex number, the solutions are the same distance from the origin and equally spread.

Recall from Chapter 1 that complex roots of a polynomial come in conjugates $a \pm bi$. That means when plotted on an Argand diagram, **the real axis is a line of symmetry for solutions of polynomial equations.**



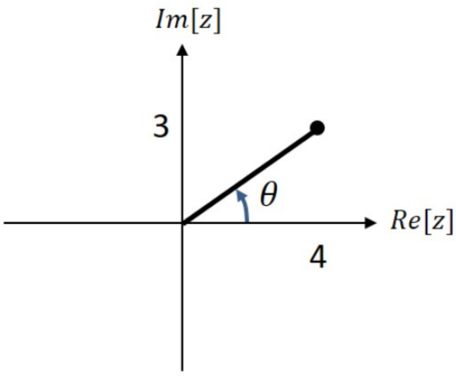
"Sketch $|z - 1| = 2$"
Later in this chapter we will see how to represent the locus of points that satisfy a given equation or inequality.



You may recognise images like the ones above. They are Mandelbrot sets, and are plotted on an Argand diagram.

Ex 2A

Modulus and Argument

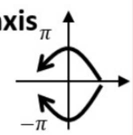


$4 + 3i$ is plotted on an Argand diagram.
(a) What is its distance from the origin?
(b) What is its anti-clockwise angle from the positive real axis? (in radians)

These are respectively known as the modulus $|z|$ and argument $\arg(z)$ of a complex number.

(The former | … | you would have seen used in the same way for the magnitude of a vector)

If $z = x + iy$ then
- $|z| = \sqrt{x^2 + y^2}$ is the modulus of z.
- $\arg(z)$ is the argument of z: the **anti-clockwise** rotation, in **radians**, from the **positive real axis**
  $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ in the 1st and 4th quadrants – *but drawing a diagram is easiest*!
  $\arg(z)$ is usually given in range $-\pi < \theta \leq \pi$ and is known as the **principal argument**.



Determine the modulus and argument of:

(a) $5 + 12i$
(b) $-1 + i$
(c) $-2i$
(d) $-1 - 3i$

$$z = 2 - 3i$$

(*a*) Show that $z^2 = -5 - 12i$. **(2)**

Find, showing your working,

(*b*) the value of $|z^2|$, **(2)**

(*c*) the value of arg (z^2), giving your answer in radians to 2 decimal places. **(2)**

(*d*) Show z and z^2 on a single Argand diagram. **(1)**

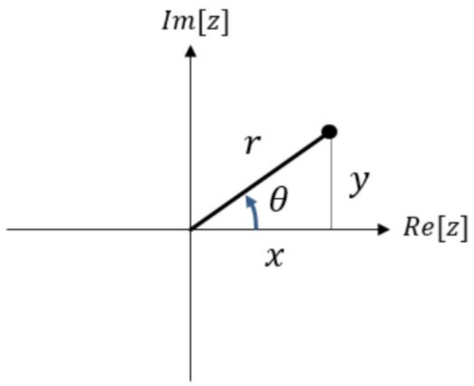
$$z = \frac{a + 3i}{2 + ai} \qquad a \in \mathbb{R}$$

(a) Given that $a = 4$, find $|z|$

(b) Show that there is only one value of a for which $\arg z = \frac{\pi}{4}$, and find this value.

Ex 2B

Modulus-Argument Form



If we let $r = |z|$ and $\theta = \arg(z)$, can you think of a way of expressing z in terms of just r and θ?

✏ The modulus-argument form of z is $r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg z$

Context: (r, θ) is known as a polar coordinate and you will learn about these in Core Pure Year 2. Instead of coordinates being specified by their x and y position (known as a Cartesian coordinate), they are specified by their distance from the origin (the 'pole') and their rotation.

Express $z = -\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$

Express $z = -1 - \sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$ where $-\pi < \theta \leq \pi$

Express $3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ in the form $x + iy$, where $x, y \in \mathbb{R}$

The modulus of a complex number is 4, and its argument is -0.3 radians.
Express the complex number in the form $x + iy$, where $x, y \in \mathbb{R}$

Ex 2C

Multiplying and Dividing Complex Numbers

Find the modulus and argument of $1 + i$ and $1 + \sqrt{3}$.
When you multiply these complex numbers together, do you notice anything about the modulus and argument of the result?

	$\mathbf{1+i}$	$\mathbf{1+\sqrt{3}i}$	$(\mathbf{1+i})(\mathbf{1+\sqrt{3}i})$ $=$
r			
θ			

Observation:

If $\quad z_1 = 3(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}), \quad z_2 = 4(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$

Express $z_1 z_2$ in the form $r\,(\cos\theta + i\sin\theta)$ and $x + iy$

✏ Multiplying complex numbers:

$$|z_1 z_2| = |z_1||z_2|$$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

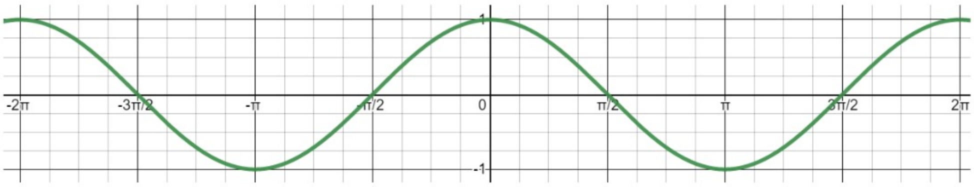
Dividing complex numbers:

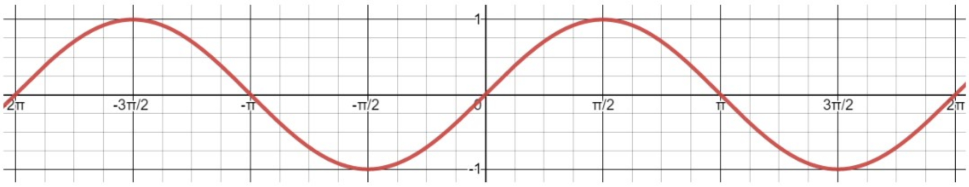
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Express $\frac{\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in the form $x + iy$

Manipulating $r(\cos\theta - i\sin\theta)$





$$\cos\theta = \cos(-\theta)$$
$$\sin\theta = -\sin(-\theta)$$

More generally, if $f(x) = f(-x)$, then the function is ***EVEN***
If $f(x) = -f(-x)$, then the function is ***ODD***

Even functions are symmetrical along the y-axis
Odd functions have rotational symmetry order 2

Express $r(\cos\theta - i\sin\theta)$ in correct modulus-argument form

Express $2(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$ in correct modulus-argument form

In other words, negate the argument, and negate *only* the sine term

Express $2(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}) \times 3(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5})$ in the form $x + iy$

Ex 2D

Challenges

Simplify $$\frac{\left(\cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}\right)^5}{\left(\cos\frac{2\pi}{17}-i\sin\frac{2\pi}{17}\right)^3}$$

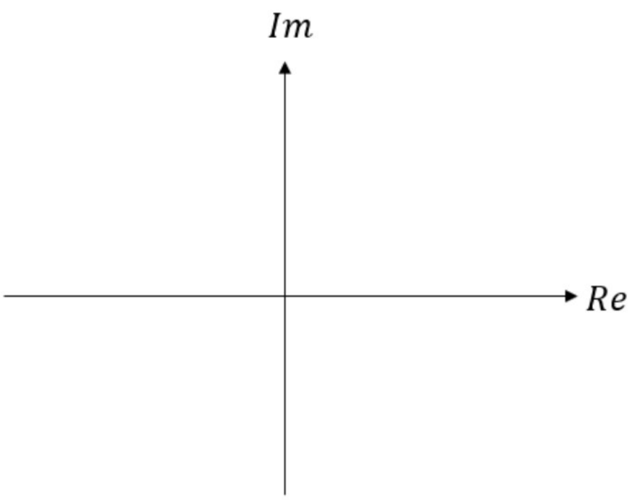
Complete the following table

	Modulus	Argument
z	r	θ
z^*		
z^2		
zz^*		
$\frac{z}{z^*}$		

Loci

You have already encountered loci at GCSE as **a set of points (possibly forming a line or region) which satisfy some restriction**.

The definition of a circle for example is “a set of points equidistant from a fixed centre”.



$|z| = 3$ means that the modulus of the complex number has to be 3. What points does this give us on the Argand diagram?

A quick reminder…

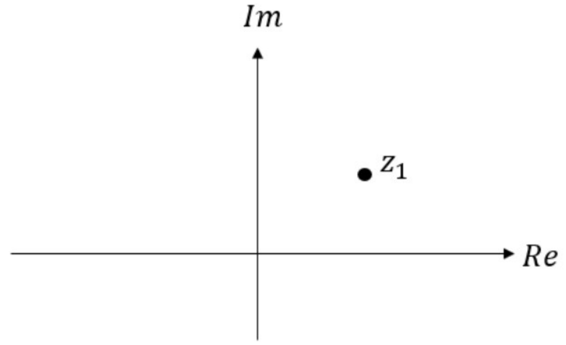
$|z| =$

$|z - 3| =$

$|z + 2 - 4i| =$

Loci of form $|z - z_1| = r$

What does $|z - z_1| = r$ mean?

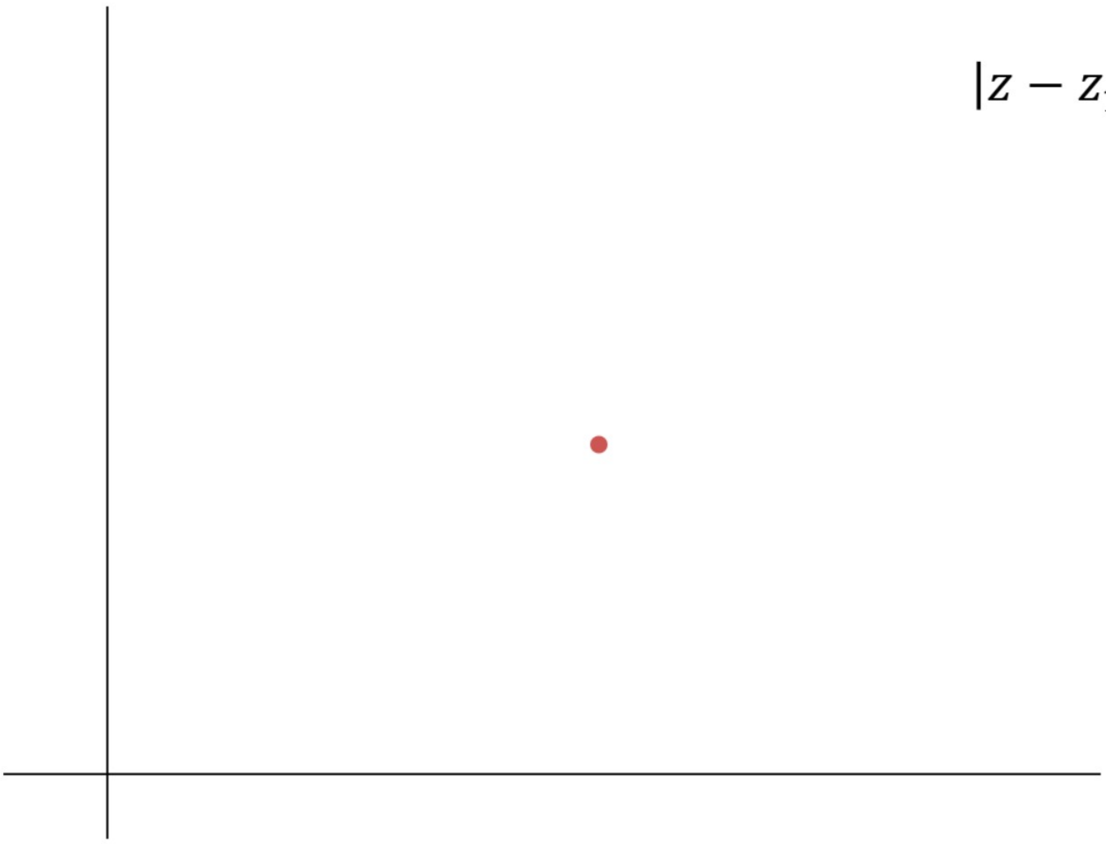


✏ $|z - z_1| = r$ is represented by a circle centre (x_1, y_1) with radius r, where $z_1 = x_1 + iy_1$

https://www.geogebra.org/m/sTbQFb59

Exploring vector connections...

$$|z - z_1| = 2$$



Sketch the locus of points represented by $|z - 5 - 3i| = 3$

Find the Cartesian equation of this locus.

Method 1 – from definition of modulus

Method 2 – from circle sketch

Sketch the locus of points represented by $|z - 3 + 4i| = 5$

Sketch the locus of points represented by $|4i + 2 - z| = 4$

Minimising/Maximising $\arg(z)$ and $|z|$

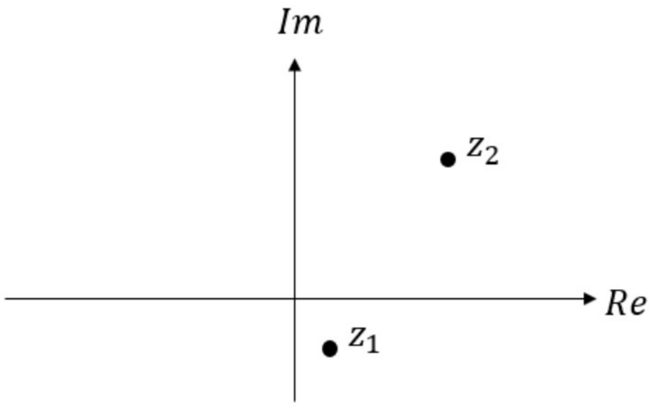
A complex number z is represented by the point P. Given that
$|z - 5 - 3i| = 3$

(a) Sketch the locus of P
(b) Find the Cartesian equation of the locus.
(c) Find the maximum value of $\arg z$ in the interval $(-\pi, \pi)$
(d) Find the minimum and maximum values of $|z|$

Given that the complex number z satisfies the equation $|z - 12 - 5i| = 3$, find the minimum value of $|z|$ and the maximum.

Loci of form $|z - z_1| = |z - z_2|$

What does $|z - z_1| = |z - z_2|$ mean?



✏ $|z - z_1| = |z - z_2|$ is represented is represented by a perpendicular bisector of the line segment joining the points z_1 to z_2.

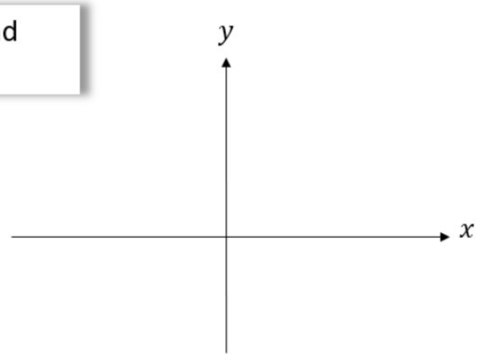
https://www.geogebra.org/m/sTbQFb59

Sketch the locus of points represented by $|z| = |z - 6i|$. Write its equation.

Method 1 – from definition of modulus

Method 2 – from graph properties of sketch

Find the Cartesian equation of the locus of z if $|z - 3| = |z + i|$, and sketch the locus of z on an Argand diagram.



Method 1 – from definition of modulus

Method 2 – from graph properties of sketch

What if we also required that $Re(z) = 0$?

Minimising $|z|$ with perpendicular bisectors

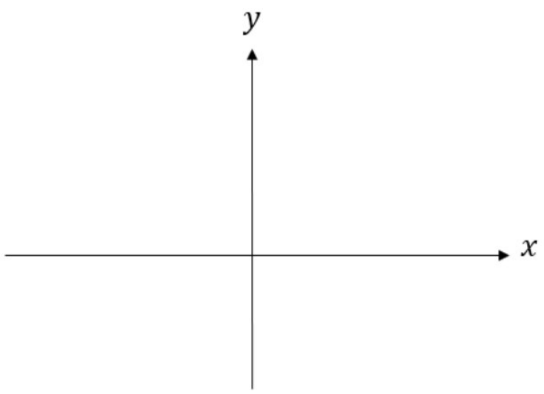
Find the Cartesian equation of the locus of z if $|z-3| = |z+i|$, and sketch the locus of z on an Argand diagram.
Hence, find the least possible value of $|z|$.

Same question as before!

Ex 2E
Q1-9

$$\arg(z - z_1) = \theta$$

Sketch $\arg(z) = \frac{\pi}{6}$
Find its Cartesian equation

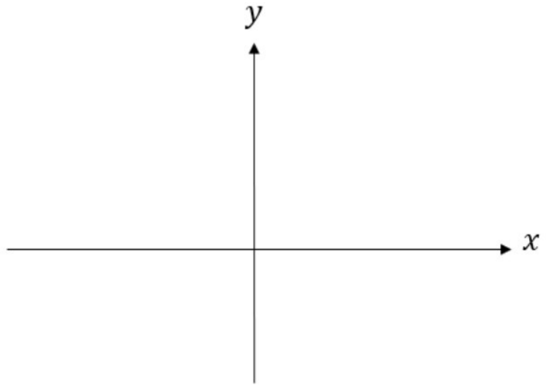


Method 1 – from definition of argument

Method 2 – from graph properties of sketch

https://www.geogebra.org/m/sTbQFb59

Sketch $\arg(z + 3 + 2i) = \frac{3\pi}{4}$
Find its Cartesian equation



Method 1 – from definition of argument

Method 2 – from graph properties of sketch

Ex 2E
Q10

Intersections of complex loci

Find the complex number z that satisfies both
$|z + 3 + 2i| = 10$ and $\arg(z + 3 + 2i) = \frac{3\pi}{4}$

Same question as before!

Method 1 – coordinate geometry

There's nearly always 2 methods - if you're good with geometry, this method is superior!

Method 2 – Cartesian equations

Ex 2E Q11, Q13-19

Range of values with complex loci

A complex number z is represented by the point P. Given that $|z-5|=4$

(a) Sketch the locus of P

(b) Given that $\arg(z+5)=\theta$ and $|z-5|=4$ have no common solutions, find the range of possible values of θ in the interval $(-\pi,\pi)$

Ex 2E Q12

Regions

Shade each of the regions on an Argand diagram.

$|z - 4 - 2i| \leq 2$

$|z - 4| < |z - 6|$

$0 < \arg(z - 2 - 2i) \leq \frac{\pi}{4}$

Shade the region for which
$|z - 1| \leq 1$ and $\frac{\pi}{12} \leq \arg(z + 1) < \frac{\pi}{2}$

8. (a) Shade on an Argand diagram the set of points

$$\left\{z \in \mathbb{C} : |z - 4\mathrm{i}| \leqslant 3\right\} \cap \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z + 3 - 4\mathrm{i}) \leqslant \frac{\pi}{4}\right\}$$

(6)

The complex number w satisfies

$$|w - 4\mathrm{i}| = 3$$

(b) Find the maximum value of $\arg w$ in the interval $(-\pi, \pi]$.
Give your answer in radians correct to 2 decimal places.

(2)

2. (a) Sketch, on an Argand diagram, the set of points

$$X = \{z \in \mathbb{C} : |z - 4 - 2\mathrm{i}| < 3\} \cap \left\{z \in \mathbb{C} : 0 \leqslant \arg(z) \leqslant \frac{\pi}{4}\right\}$$

On your diagram

- shade the part of the diagram that is included in the set
- use solid lines to show the parts of the boundary that are included in the set, and use dashed lines to show the parts of the boundary that are not included in the set

(3)

(b) Show that the complex number $z = 5 + 4\mathrm{i}$ is in the set X.

(3)

3. (a) Shade on an Argand diagram the set of points

$$\{z \in \mathbb{C} : |z - 1 - \mathrm{i}| \leqslant 3\} \cap \left\{z \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(z - 2) \leqslant \frac{3\pi}{4}\right\}$$

(5)

The complex number w satisfies

$$|w - 1 - \mathrm{i}| = 3 \text{ and } \arg(w - 2) = \frac{\pi}{4}$$

(b) Find, in simplest form, the exact value of $|w|^2$

(4)

3. (a) Show on an Argand diagram the locus of points given by

$$|z - 10 - 12\mathrm{i}| = 8$$

(2)

Set A is defined by

$$A = \left\{z : 0 \leqslant \arg(z - 10 - 10\mathrm{i}) \leqslant \frac{\pi}{2}\right\} \cap \{z : |z - 10 - 12\mathrm{i}| \leqslant 8\}$$

(b) Shade the region defined by A on your Argand diagram.

(2)

(c) Determine the area of the region defined by A.

(5)