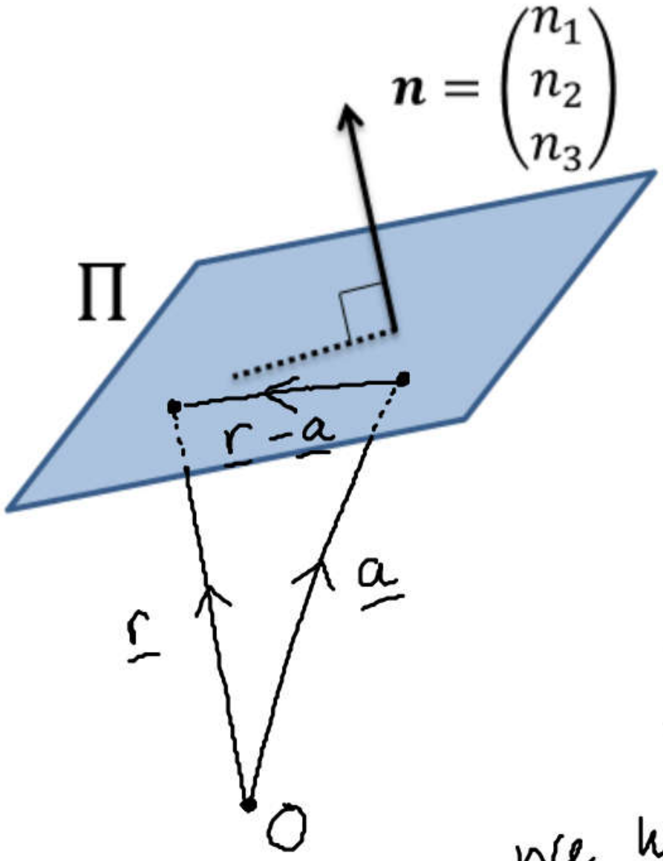
Equation of Plane - scalar product form



Can we create a different equation of a plane, using our new knowledge about the scalar dot product?

If $\underline{a}$ is a position vector of a point <u>on</u> the plane, and so is $\underline{r}$, $(\underline{r} - \underline{a})$ is a vector parallel to the plane.

we know $(\underline{r} - \underline{a})$ is $\perp$ar to $\underline{n}$

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$

$$\underline{r} \cdot \underline{n} - \underline{a} \cdot \underline{n} = 0$$

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

$$\underline{r} \cdot \underline{n} = p$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = p$$

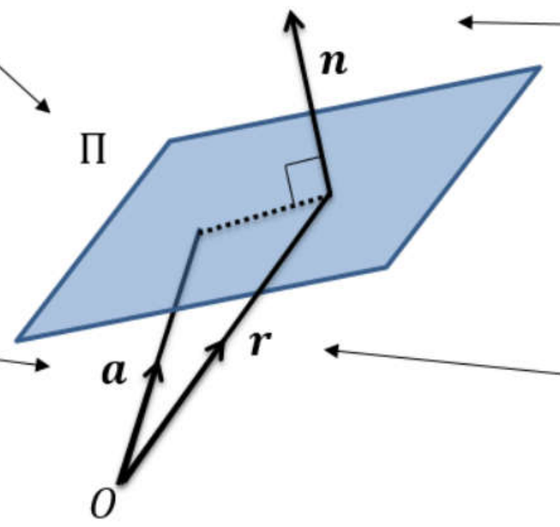
$$n_1 x + n_2 y + n_3 z = p$$

Cartesian form.

Equation of Plane - scalar product form

We use Π to represent a plane ("capital pi") in the same way we use l to represent a straight line.

$\boldsymbol{n}$ (the $\boldsymbol{n}$ stands for "normal") always indicates a vector perpendicular to the plane.



Just as $\boldsymbol{a}$ was used as the position vector of a **fixed** point on a line l, it is used in the same way for a plane.

We reuse the letter $\boldsymbol{r}$ to mean "the position vector of some point on the plane".

It's important to realise here that $\boldsymbol{n}$ and $\boldsymbol{a}$ are **fixed** for a given plane (i.e. are constant vectors), whereas $\boldsymbol{r}$ can **vary** as it represents all the possible points on the plane.

How could we use the dot product to find some relationship between $\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{n}$?

$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$ $\quad$ $\underline{r} \cdot \underline{n} = p$ $\quad$ $n_1 x + n_2 y + n_3 z = p$

A point with position vector $2\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$ lies on the plane and the vector $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ is perpendicular to the plane. Find the equation of the plane in:

a) Scalar product form.

b) Cartesian form.

$(2, 3, -5)$

If $\boldsymbol{r} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = p$ is the scalar product equation of a plane, then the Cartesian form is:

$$n_1 x + n_2 y + n_3 z = p$$

a) $\underline{a} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ $\quad \underline{n} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

$\underline{r} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

$\underline{r} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 14$

b) $3x + y - z = 14$

Cart.

$3x + y - z = c$

$3 \times 2 + 1 \times 3 - 1 \times (-5) = c$

$c = 14$

$30x + 10y - 10z = 140$

Ex 9D

2ac

3

4

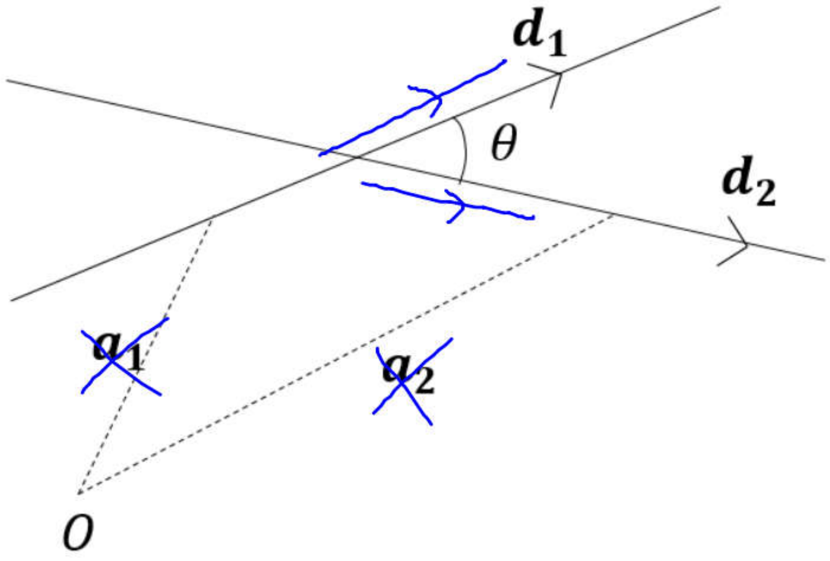
Angle between two lines
Angle between line and plane
Angle between two planes

Intersection of two lines
Intersection of line and plane
(Intersection of two planes)

Shortest distance between two parallel lines
Shortest distance between two skew lines (also in FP1)
Shortest distance between a point and a line
Shortest distance between a point and a plane
Shortest distance between two parallel planes

Reflecting a point in a plane
Reflecting a line in a plane

Angles between two lines



✏ To find angle between two lines:
Find the angle between their <u>direction</u> vectors.

i.e. we only care about the directional part of the line, not how we got to the line.

[Jan 2008 Q6] 6. The points A and B have position vectors $2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ respectively.

The line l_1 passes through the points A and B.

(*a*) Find the vector $\overrightarrow{AB}$. **(2)**

$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}3\\4\\1\end{pmatrix} - \begin{pmatrix}2\\6\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$

(*b*) Find a vector equation for the line l_1. **(2)**

$\underline{r} = \begin{pmatrix}2\\6\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}$

A second line l_2 passes through the origin and is parallel to the vector $\mathbf{i} + \mathbf{k}$. The line l_1 meets the line l_2 at the point C.

$\begin{pmatrix}1\\0\\1\end{pmatrix}$

(*c*) Find the acute angle between l_1 and l_2. **(3)**

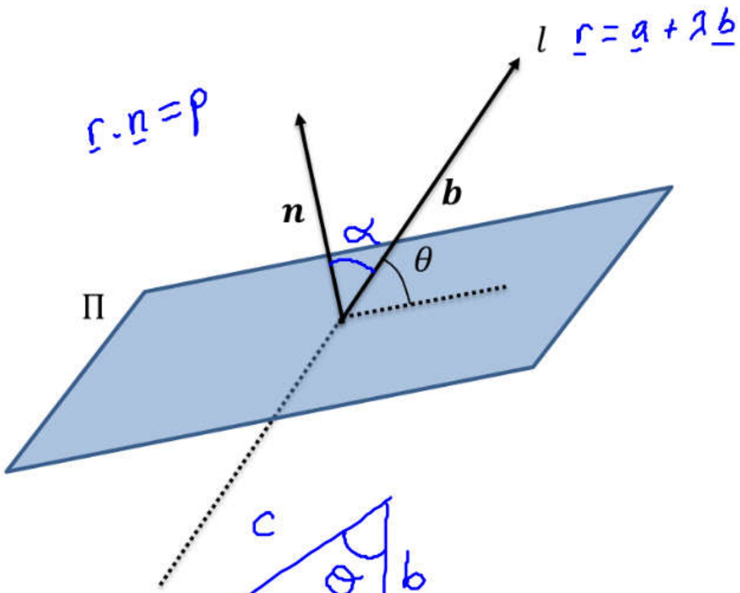
$\cos\theta = \dfrac{\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}}{\sqrt{2}\times 3}$

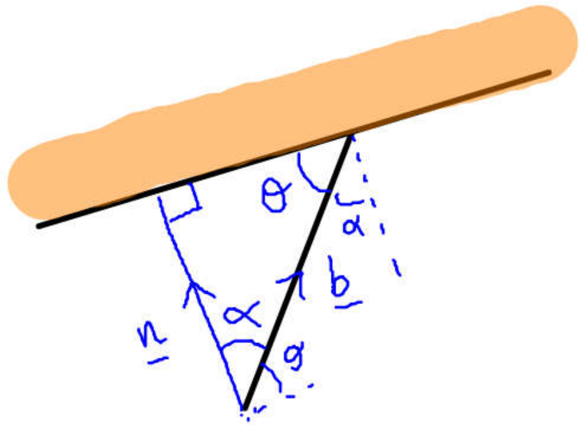
" $\cos\theta = \left|\dfrac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}\right|$ "

$\cos\theta = \dfrac{3}{\sqrt{2}\times 3} = \dfrac{1}{\sqrt{2}}$ $\theta = 45^\circ$

Ex 9D
1ace
9
10
11
14

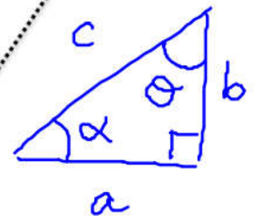
Angles between line and a plane





$$\cos\alpha = \left|\frac{\underline{n}\cdot\underline{b}}{|\underline{n}||\underline{b}|}\right|$$

$$\sin\theta = \left|\frac{\underline{n}\cdot\underline{b}}{|\underline{n}||\underline{b}|}\right|$$



$$\cos\alpha = \frac{a}{c} = \sin\theta$$

If $\alpha + \theta = 90$

Example: Find the angle between the line $\dfrac{x+1}{2} = \dfrac{y-2}{1} = \dfrac{z-3}{-2}$ and the plane $2x + 3y - 7z = 5$.

$$\underline{r} = \begin{pmatrix}-1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

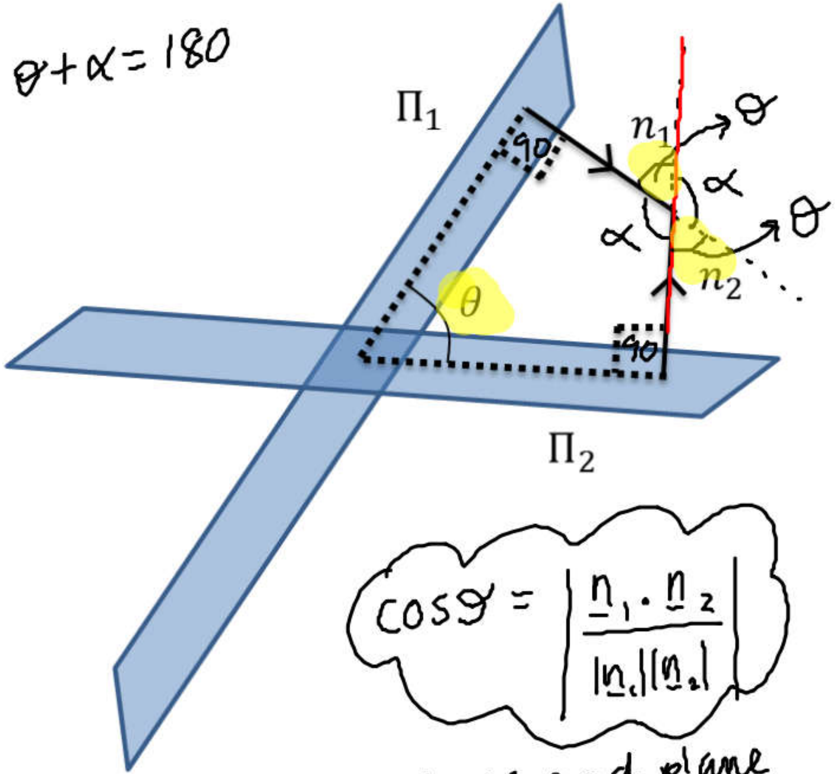
$$\underline{n} = \begin{pmatrix}2\\3\\-7\end{pmatrix}$$

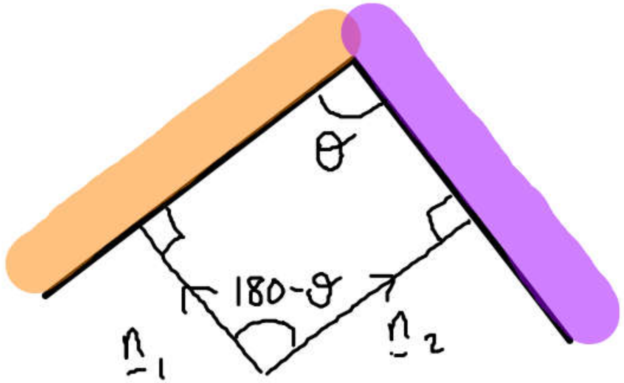
$$\sin\theta = \frac{\begin{pmatrix}2\\3\\-7\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}}{\sqrt{62}\times\sqrt{9}}$$

$$\sin\theta = \frac{21}{3\sqrt{62}} \qquad \theta = \underline{\underline{62.7^\circ}}$$

Angles between two planes

$\theta + \alpha = 180$





$$\cos\theta = \left|\frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}\right|$$

plane and plane

$$\cos\theta = \frac{\underline{d}_1 \cdot \underline{d}_2}{|\underline{d}_1||\underline{d}_2|}$$

(add in | | for acute
line and line.

$$\sin\theta = \left|\frac{\underline{n} \cdot \underline{b}}{|\underline{n}||\underline{b}|}\right|$$

line and plane

Find the acute angle between the planes with Cartesian equations

3x - 2y + 4z = 3

and

5x - 4y + 2z = 10

$\underline{n}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ $\underline{n}_2 = \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}}{\sqrt{29}\sqrt{45}} = \frac{31}{\sqrt{29}\sqrt{45}}$$

$\theta = \underline{\underline{30.9^\circ}}$

Ex 9D
Q5, 6, 7, 8