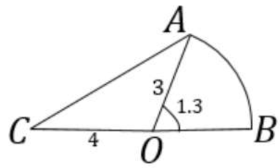
Radians

1:: Converting between degrees and radians.

"What is $45°$ in radians?"

2:: Find arc length and sector area (when using radians)

"OAB is a sector. Determine the perimeter of the shape."



3:: Solve trig equations in radians.

"Solve $\sin x = \frac{1}{2}$ for $0 \leq x < \pi$."

4:: Small angle approximations

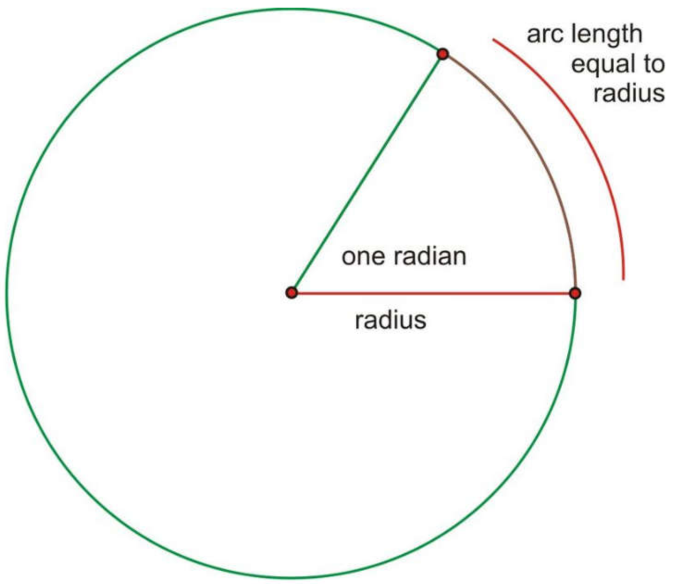
"Show that, when θ is small, $\sin 5\theta + \tan 2\theta - cos 2\theta \approx 2\theta^2 + 7\theta - 1$."

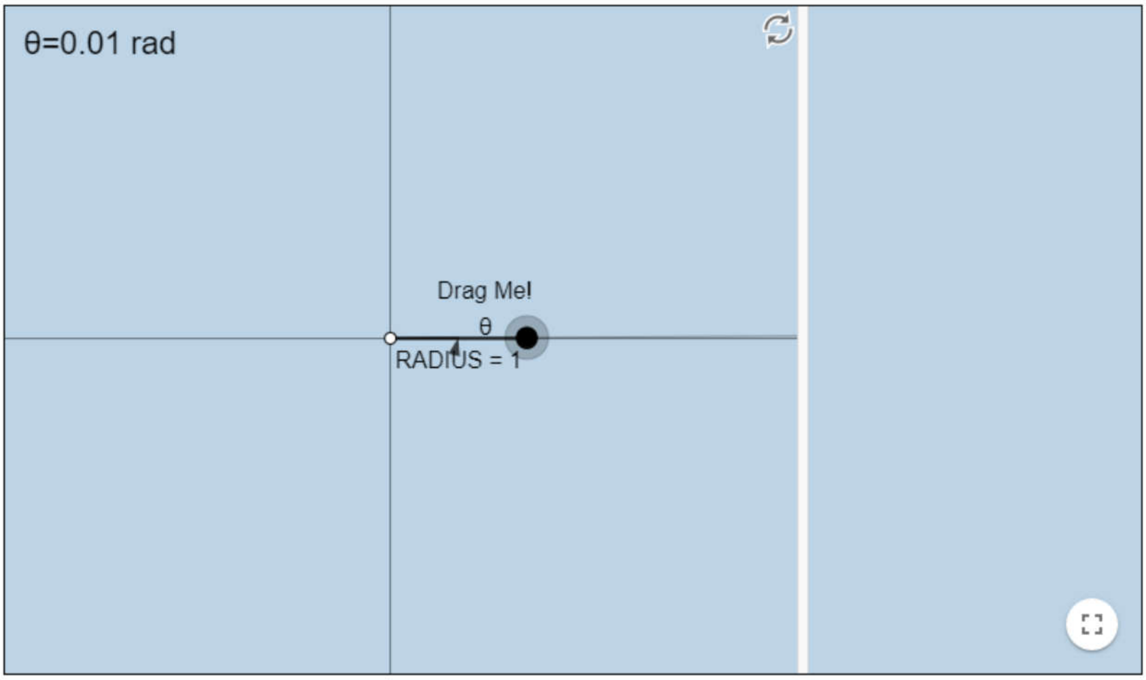
What are radians?

So far, we've used **degrees** to measure angles - with one degree as a 360th of a rotation around a full circle. Why?

One radian, however, is the movement of one radius' worth around the circumference of the circle.
In other words, if the arc of a circle is equal to its radius, then the angle subtended at the centre is one radian.

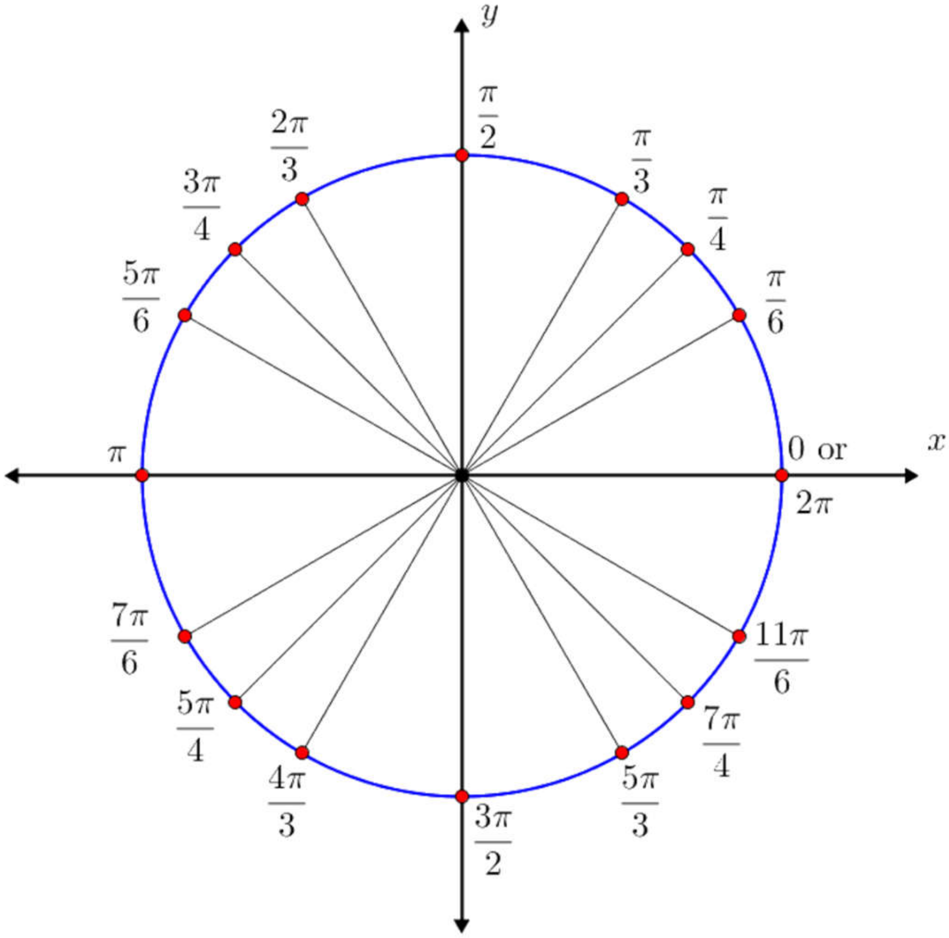
Outside geometry, mathematicians nearly always use radians - you'll have to trust me that this will make more sense why later in this chapter! It is to do with calculus.





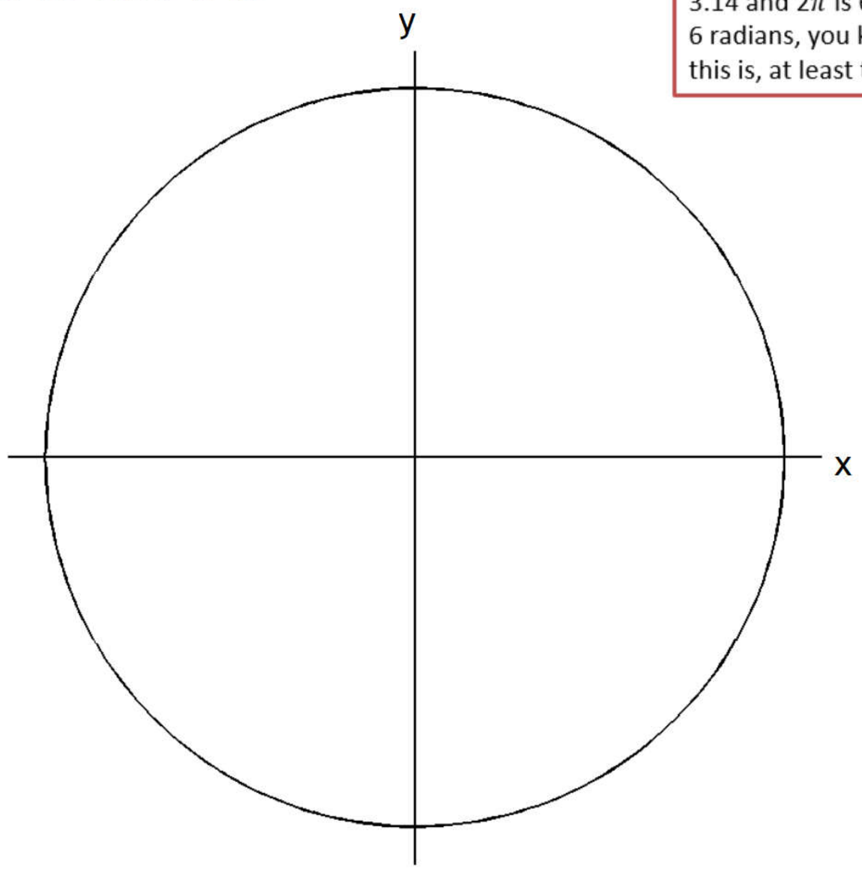
Fill in the equivalent angles in degrees around the circle

Reminder: Angles are measured anticlockwise from the positive x-axis



Roughly, where on the circle is ...

It is worth knowing that (roughly) π is 3.14 and 2π is 6.28. This means if I said 6 radians, you know roughly what angle this is, at least the quadrant it is in.



The best way to convert is to think of fractions related to 180 degrees, and to imagine the circle.

You shouldn't need to convert often to degrees or vice versa, and should NOT do this to 'make it easier' - it will slow you down and restrict you being successful.

Start thinking in radians!

We always prefer to express radians in their exact form where possible – i.e. in terms of π

$$180^\circ \underset{\div \pi \text{ and } \times 180}{\overset{\div 180 \text{ and } \times \pi}{=}} \pi$$

$90^\circ =$

$\frac{\pi}{3} =$

$45^\circ =$

$\frac{\pi}{6} =$

$135^\circ =$

$\frac{3}{2}\pi =$

$72^\circ =$

$\frac{5\pi}{6} =$

Be able to convert common angles in your head…

$45° =$ $\quad\quad$ $30° =$

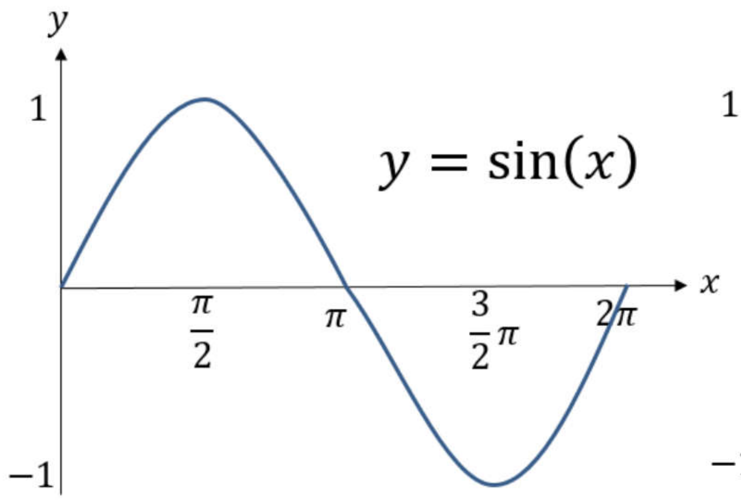
$60° =$ $\quad\quad$ $135° =$

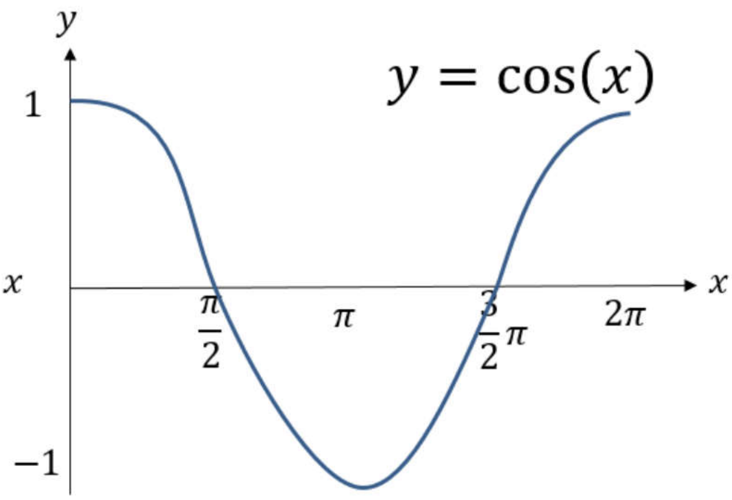
$270° =$ $\quad\quad$ $90° =$

$120° =$

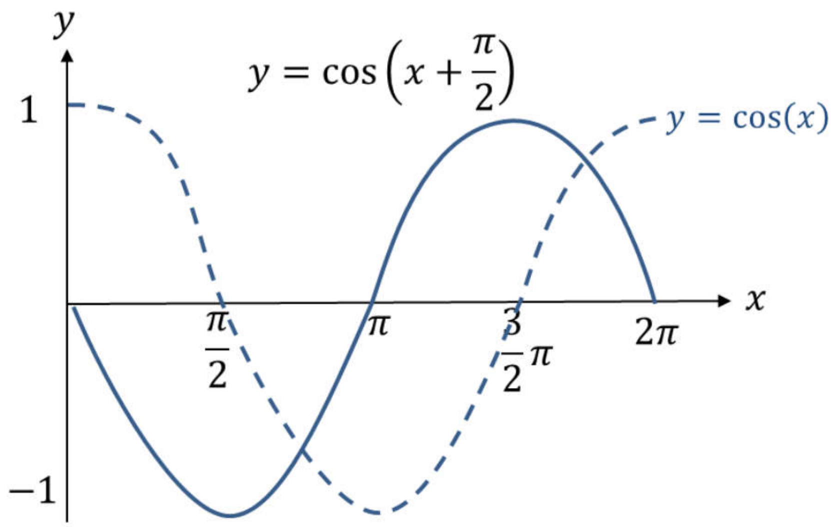
Graph Sketching with Radians

We can replace the values $90°, 180°, 270°, 360°$ on the x-axis with their equivalent value in radians.





Sketch the graph of $y = \cos\left(x + \frac{\pi}{2}\right)$ for $0 \le x < 2\pi$.



sin, cos, tan of angles in radians

Reminder of laws from Year 1:

- $\sin(x) = \sin$
- $\cos(x) = \cos$
- sin, cos repeat every ° but tan every °

IN RADIANS

To find sin/cos/tan of a '**common**' angle in radians without using a calculator, it is easiest to just **convert to degrees first**.

$\cos\left(\frac{4\pi}{3}\right) =$

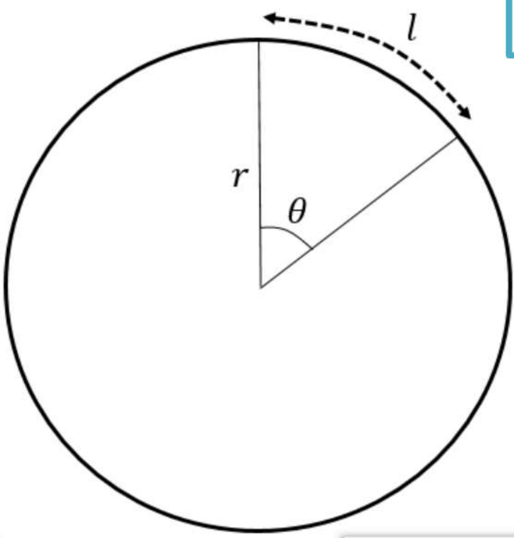
$\sin\left(-\frac{7\pi}{6}\right) =$

To find $\cos\left(\frac{4\pi}{3}\right)$ directly using your calculator, you need to switch to radians mode. Press $SHIFT \rightarrow SETUP$, then $ANGLE\ UNIT$, then $Radians$. An R will appear at the top of your screen, instead of D.

Ex 5A/B

Arc length

Reminder: to convert from radians to degrees, $\div \pi \ \times 180$



Arc length in degrees:

Arc length in radians

From before, we know that 1 radian gives an arc of 1 radius in length, so θ radians must give a length of…

Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 radians at the centre of the circle.

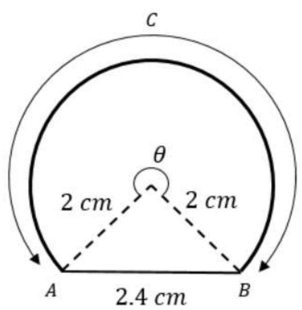
Terminology: '<u>Subtend</u>' means **opposite** or extending beneath.

An arc AB of a circle with radius 7cm and centre O has a length of 2.45 cm. Find the angle $\angle AOB$ subtended by the arc at the centre of the circle

Note: Whether your calculator is in degrees mode or radians mode is only relevant when using sin/cos/tan – it won't affect simple multiplication!

An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of P and θ.

The border of a garden pond consists of a straight edge AB of length 2.4m, and a curved part C, as shown in the diagram. The curve part is an arc of a circle, centre O and radius 2m. Find the length of C.



Tip: Trigonometry on right-angled triangles is **always** simpler than using sine/cosine rule.

Edexcel C2 Jan 2005 Q7

Figure 1

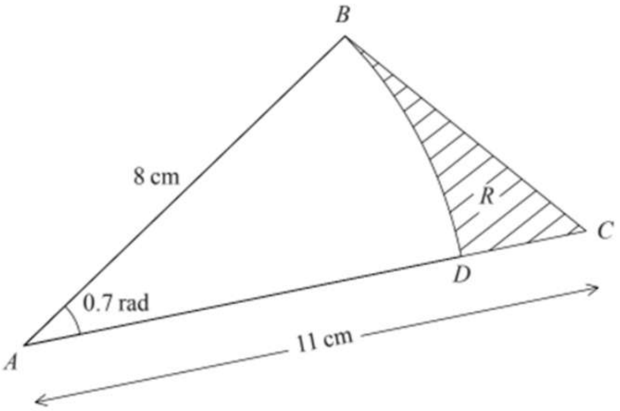
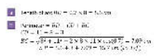


Figure 1 shows the triangle ABC, with $AB = 8\ cm$, $AC = 11\ cm$ and $\angle BAC = 0.7$ radians. The arc BD, where D lies on AC, is an arc of a circle with centre A and radius 8 cm. The region R, shown shaded in Figure 1, is bounded by the straight lines BC and CD and the arc BD.

Find

(a) The length of the arc BD.

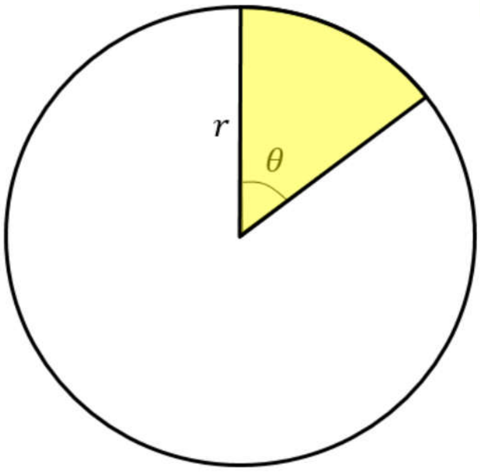
(b) The perimeter of R, giving your answer to 3 significant figures.



Ex 5C

Sector Area

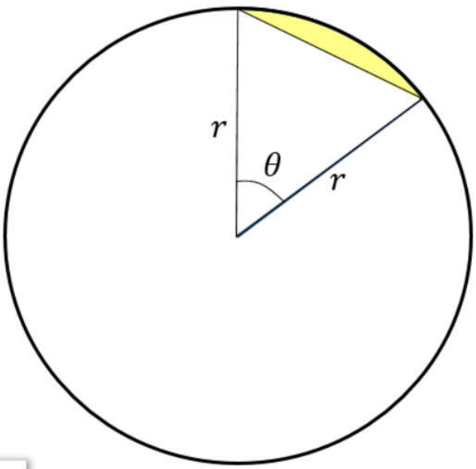
Reminder: to convert from radians to degrees, $\div \pi \times 180$



Area using Degrees

Area using Radians

Segment Area

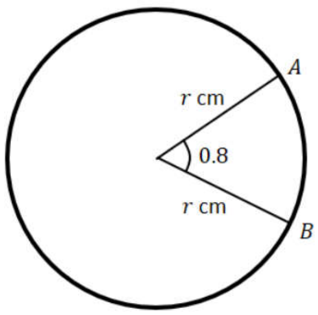


A segment is the region bound between a chord and the circumference.

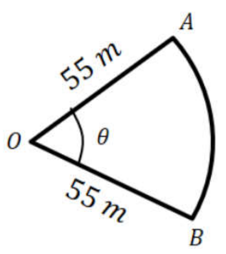
This is just a sector with a triangle cut out.

Area using radians:

In the diagram, the area of the minor sector AOB is 28.9 cm^2. Given that $\angle AOB = 0.8$ radians, calculate the value of r.

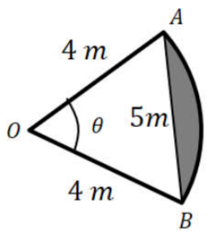


A plot of land is in the shape of a sector of a circle of radius 55 m. The length of fencing that is erected along the edge of the plot to enclose the land is 176 m. Calculate the area of the plot of land.

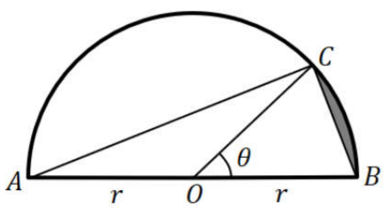


Segment Examples

In the diagram above, OAB is a sector of a circle, radius 4m. The chord AB is 5m long. Find the area of the shaded segment.



In the diagram, AB is the diameter of a circle of radius $r$cm, and $\angle BOC = \theta$ radians. Given that the area of ΔAOC is three times that of the shaded segment, show that
$3\theta - 4\sin\theta = 0$.



June 10 Q6.

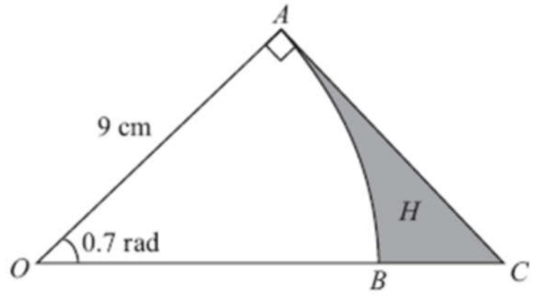


Figure 1

Figure 1 shows the sector OAB of a circle with centre O, radius 9 cm and angle 0.7 radians.

(a) Find the length of the arc AB.

(2)

(b) Find the area of the sector OAB.

(2)

The line AC shown in Figure 1 is perpendicular to OA, and OBC is a straight line.

(c) Find the length of AC, giving your answer to 2 decimal places.

(2)

The region H is bounded by the arc AB and the lines AC and CB.

(d) Find the area of H, giving your answer to 2 decimal places.

(3)

Jan 05 Q7. **Figure 1**

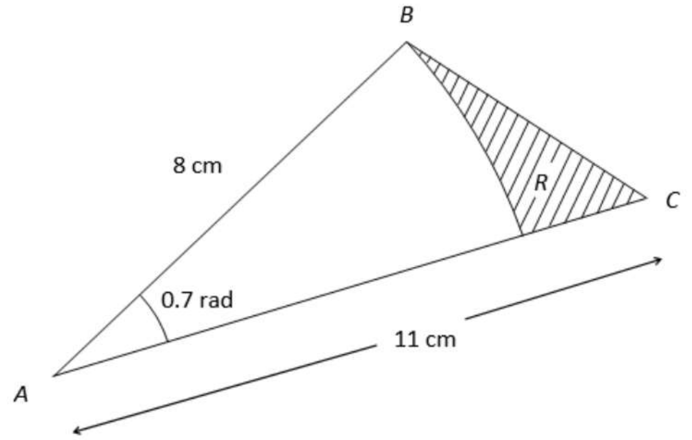


Figure 1 shows the triangle ABC, with $AB = 8$ cm, $AC = 11$ cm and $\angle BAC = 0.7$ radians. The arc BD, where D lies on AC, is an arc of a circle with centre A and radius 8 cm. The region R, shown shaded in Figure 1, is bounded by the straight lines BC and CD and the arc BD.

Find

(a) the length of the arc BD,

(2)

(b) the perimeter of R, giving your answer to 3 significant figures,

(4)

(c) the area of R, giving your answer to 3 significant figures.

(5)

Jan 09 Q7.

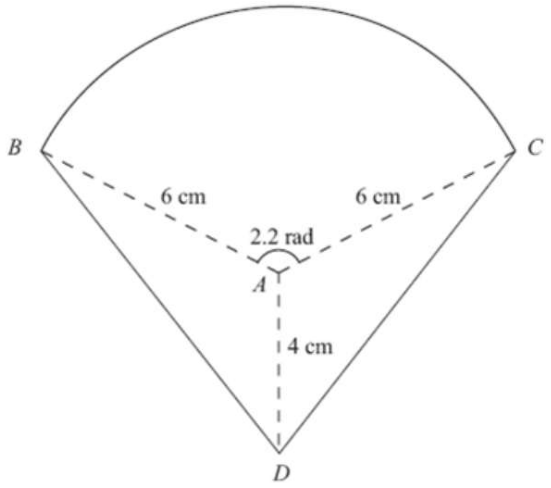


Figure 3

The shape BCD shown in Figure 3 is a design for a logo.

The straight lines DB and DC are equal in length. The curve BC is an arc of a circle with centre A and radius 6 cm. The size of $\angle BAC$ is 2.2 radians and $AD = 4$ cm.

Find

(*a*) the area of the sector BAC, in cm^2,

(2)

(*b*) the size of $\angle DAC$, in radians to 3 significant figures,

(2)

(*c*) the complete area of the logo design, to the nearest cm^2.

(4)

Jan 07 Q9.

Figure 2

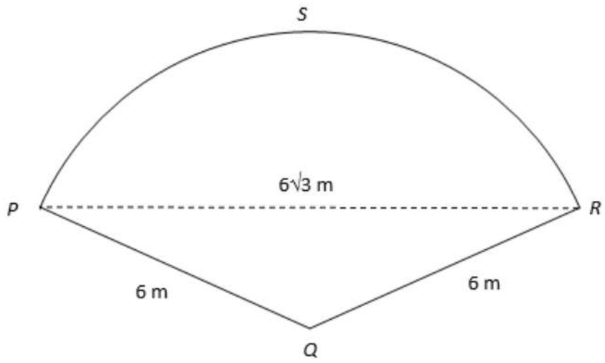


Figure 2 shows a plan of a patio. The patio *PQRS* is in the shape of a sector of a circle with centre *Q* and radius 6 m.

Given that the length of the straight line *PR* is $6\sqrt{3}$ m,

(*a*) Find the exact size of angle *PQR* in radians.

(3)

(*b*) Show that the area of the patio *PQRS* is $12\pi \text{ m}^2$.

(2)

(*c*) Find the exact area of the triangle *PQR*.

(2)

(*d*) Find, in m^2 to 1 decimal place, the area of the segment *PRS*.

(2)

(*e*) Find, in m to 1 decimal place, the perimeter of the patio *PQRS*.

(2)

Solving Trigonometric Equations

- $\sin(x) = \sin(\pi - x)$
- $\cos(x) = \cos(2\pi - x)$
- sin, cos repeat every 2π but tan every π

Solving trigonometric equations is almost the same as you did in Year 1, except:

(a) Your calculator needs to be in radians mode.

(b) We use $\pi -$ instead of $180° -$, and so on.

Solve the equation
$\sin\theta = 0.3$ in the interval $0 \leq \theta \leq 2\pi$.

Solve the equation
$4\cos\theta = 2$ in the interval $0 \leq \theta \leq 2\pi$.

Solve the equation
$5\tan\theta + 3 = 1$ in the interval $0 \leq \theta \leq 2\pi$.

Solve the equation
$\sin 3\theta = \frac{\sqrt{3}}{2}$ in the interval $0 \leq \theta \leq 2\pi$.

Solve the equation
$17\cos\theta + 2\sin^2\theta = 13$ in the interval $0 \leq \theta \leq 2\pi$.

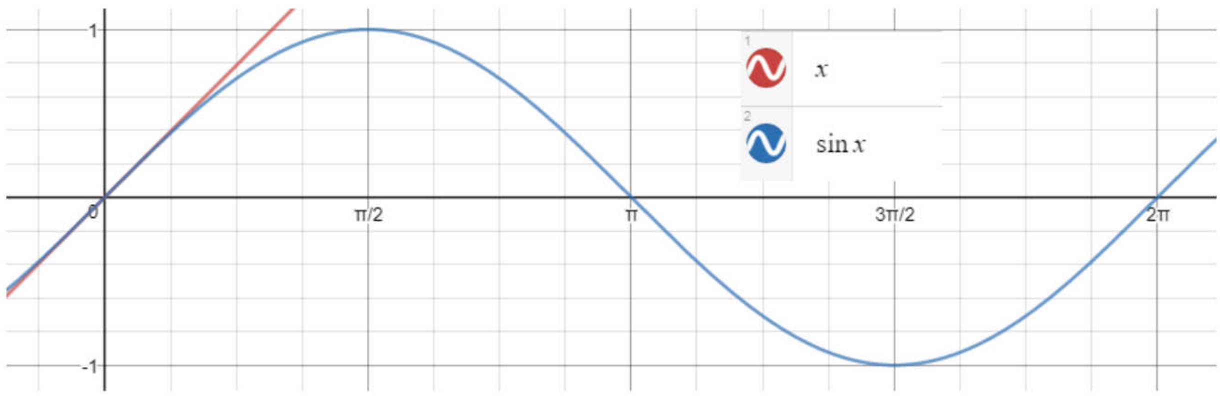
[Jan 07 Q6] Find all the solutions, in the interval $0 \leq x < 2\pi$, of the equation

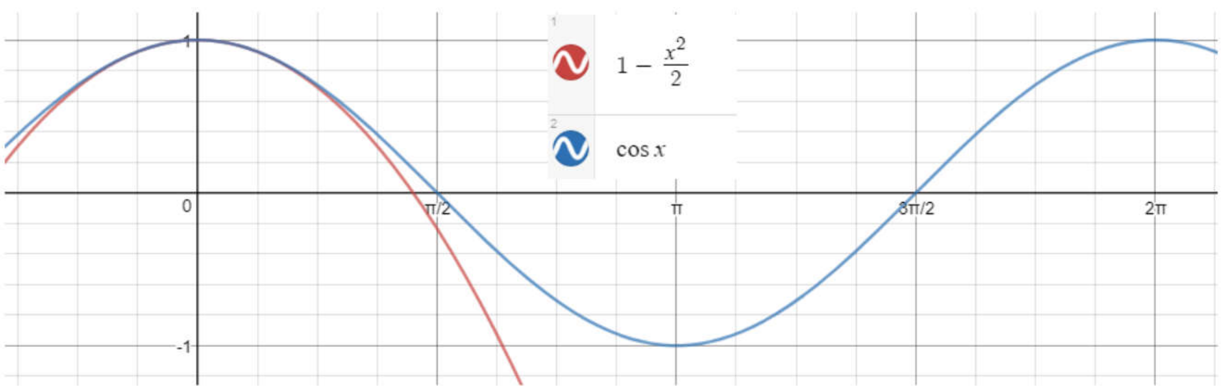
$$2\cos^2 x + 1 = 5\sin x,$$

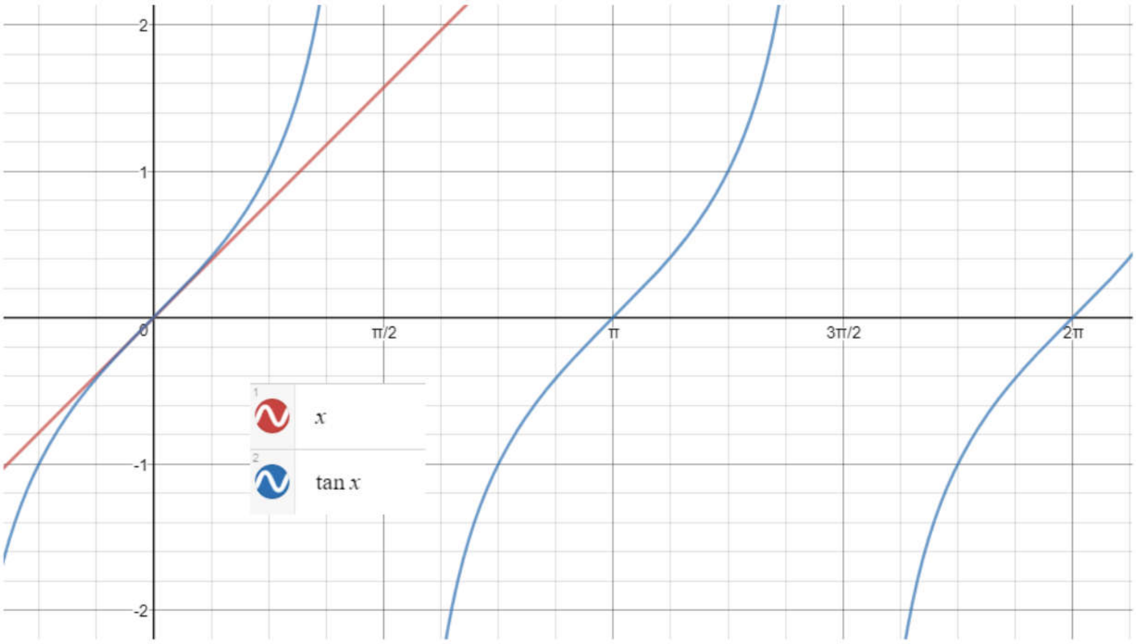
giving each solution in terms of π. **(6)**

Ex 5E

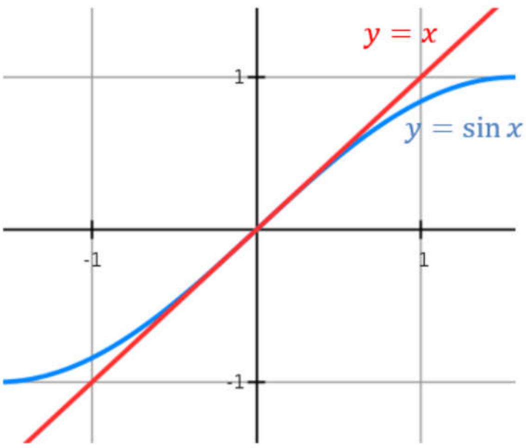
What do you notice about these graphs?

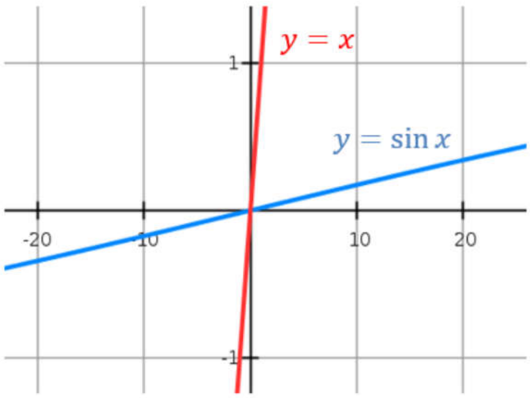






Small Angle Approximations



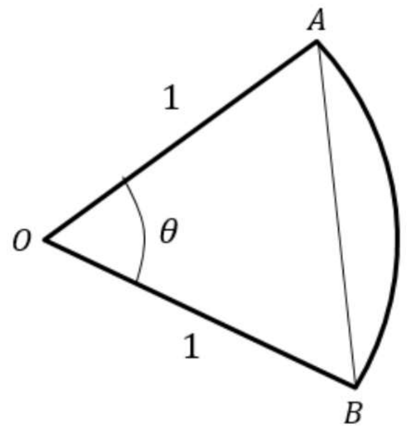


If x is in radians, we can see from the graph that as x approaches 0, the two graphs are approximately the same, i.e. $\sin x \approx x$

If x was in degrees however, then we can see this is not the case.

When θ is small and measured in radians:

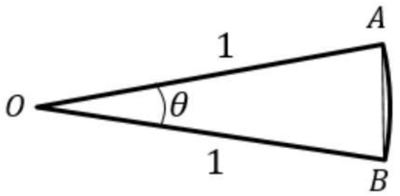
- $\sin \theta \approx \theta$
- $\tan \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{\theta^2}{2}$



Geometric Proof that $sin\ \theta \approx \theta$:

The area of sector OAB is:

The area of triangle OAB is:



As θ becomes small, the area of the triangle is approximately equal to that of the sector, so:

Note that this only works for radians, because we used the sector area formula for radians. The fact that $\sin\theta \approx \theta$ is enormously important when we come to differentiation, because we can use it to prove that $\frac{d}{dx}(\sin x) = \cos x$.

When θ is small, find the approximate value of:

a) $\frac{\sin 2\theta+\tan\theta}{2\theta}$

b) $\frac{\cos 4\theta-1}{\theta\sin 2\theta}$

a) Show that, when θ is small,

$$\sin 5\theta + \tan 2\theta - \cos 2\theta \approx 2\theta^2 + 7\theta - 1$$

b) Hence state the approximate value of $\sin 5\theta + \tan 2\theta - \cos 2\theta$ for small values of θ.

Exam Questions

1.

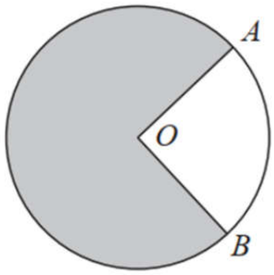


Figure 1

Figure 1 shows a circle with centre O. The points A and B lie on the circumference of the circle.

The area of the major sector, shown shaded in Figure 1, is $135\,\text{cm}^2$.

The reflex angle AOB is 4.8 radians.

Find the exact length, in cm, of the minor arc AB, giving your answer in the form $a\pi + b$, where a and b are integers to be found.

(4)

3.

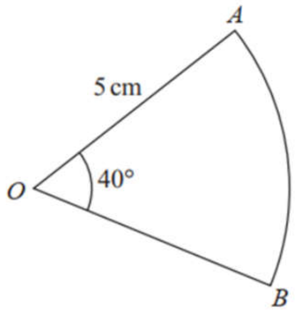


Figure 1

Figure 1 shows a sector AOB of a circle with centre O, radius 5 cm and angle $AOB = 40°$

The attempt of a student to find the area of the sector is shown below.

$$\text{Area of sector} = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 5^2 \times 40$$
$$= 500\,\text{cm}^2$$

(a) Explain the error made by this student.

(1)

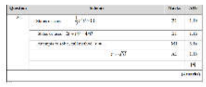
(b) Write out a correct solution.

(2)

Question 3 (Total 3 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	The formula is only valid when the angle AOB is given in radians	B1	This mark is given for a correct explanation
(b)	$\frac{40}{360} \times \pi \times 5^2$	M1	This mark is given for a correct method to find the area of the sector
	$\frac{25\pi}{9}$ cm^2	A1	This mark is given for a correct value for the area of the sector

3.



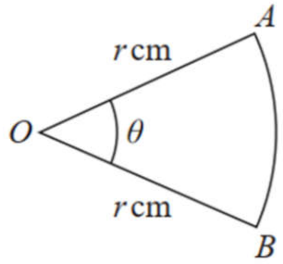


Figure 1

Figure 1 shows a sector AOB of a circle with centre O and radius r cm.

The angle AOB is θ radians.
The area of the sector AOB is 11 cm^2

Given that the perimeter of the sector is 4 times the length of the arc AB, find the exact value of r.

(4)

1. (a) Given that θ is small and in radians, show that the equation

$$\cos\theta - \sin\left(\frac{1}{2}\theta\right) + 2\tan\theta = \frac{11}{10} \qquad \text{(I)}$$

can be written as

$$5\theta^2 - 15\theta + 1 \approx 0$$

(3)

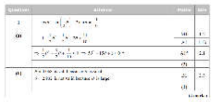
The solutions of the equation

$$5\theta^2 - 15\theta + 1 = 0$$

are 0.068 and 2.932, correct to 3 decimal places.

(b) Comment on the validity of each of these values as approximate solutions to equation (I).

(1)



2.

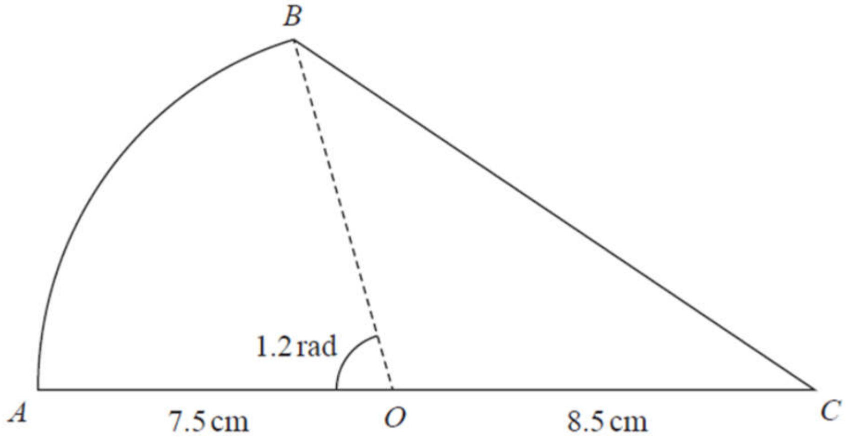


Figure 2

The shape *AOCBA*, shown in Figure 2, consists of a sector *AOB* of a circle centre *O* joined to a triangle *BOC*.

The points *A*, *O* and *C* lie on a straight line with $AO = 7.5$ cm and $OC = 8.5$ cm.

The size of angle *AOB* is 1.2 radians.

Find, in cm, the perimeter of the shape *AOCBA*, giving your answer to one decimal place.

(5)