More invariant points

1.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix **M** is non-singular.

(2)

The transformation T of the plane is represented by the matrix **M**.

The triangle R is transformed to the triangle S by the transformation T.

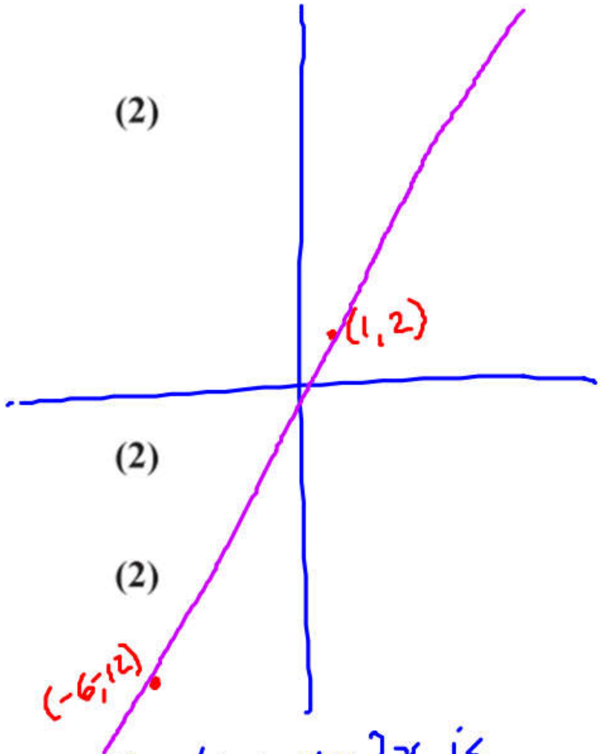
Given that the area of S is 63 square units,

(b) find the area of R.

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation T.

(2)



$$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 4x - 10x \\ 2x - 14x \end{pmatrix} = \begin{pmatrix} -6x \\ -12x \end{pmatrix}$$

$$= -6\begin{pmatrix} x \\ 2x \end{pmatrix}$$

So the line $y = 2x$ is invariant.

Combined Transformations

$\begin{pmatrix} x \\ y \end{pmatrix}$ $\begin{pmatrix} a & b \\ c & c \end{pmatrix}$

We know that for a position vector $\boldsymbol{x}$ and a matrix $\mathbf{A}$ representing some transformation, then $\mathbf{A}\boldsymbol{x}$ is the transformed point.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

If we wanted to apply a transformation represented by a matrix $\mathbf{A}$ followed by another represented by $\mathbf{B}$, what transformation matrix do we use to represent the combined transformation?

$\underline{BA}x$

$(2 \times 3) \times 5$

$2 \times (3 \times 5)$

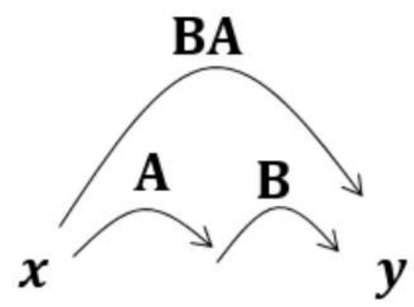
This is because to apply the effect of A followed by B, we have:

$$B(A\boldsymbol{x}) = (BA)\boldsymbol{x}$$

(because matrix multiplication is 'associative'*)

$\otimes$

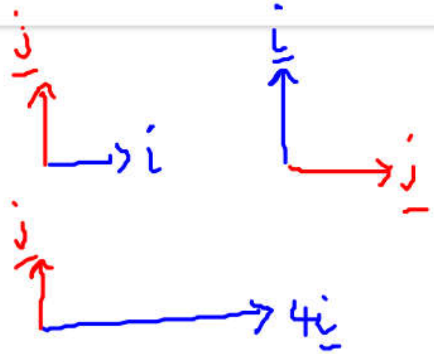
Tip: Ensure that you put these matrices in the right order – the first that gets applied is on the right!



* A binary operator $\otimes$ is **associative** if $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, i.e. when we multiply matrices, the order in which we multiply them doesn't matter.
Similarly addition on real numbers is associative, e.g. $1 + (2 + 3) = (1 + 2) + 3$.
However subtraction and division are not, e.g. $(16 \div 2) \div 8 \neq 16 \div (2 \div 8)$.

Represent as a single matrix the transformation representing a reflection in the line $y = x$ followed by a stretch on the x axis by a factor of 4.

A   B



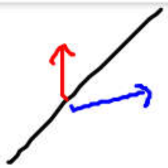
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

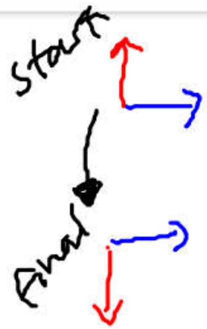
$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \qquad BA = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

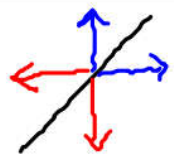
Represent as a single matrix the transformation representing a rotation $90°$ anticlockwise about the point $(0,0)$ followed by a reflection in the line $y = x$.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$y = x$







$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

reflection in x-axis

What single transformation is this?

Edexcel Jan 2013 Q4

The transformation U, represented by the 2×2 matrix $\mathbf{P}$, is a rotation through 90° anticlockwise about the origin.

(*a*) Write down the matrix $\mathbf{P}$. **(1)**

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line $y = -x$.

(*b*) Write down the matrix $\mathbf{Q}$. **(1)**

Given that U followed by V is transformation T, which is represented by the matrix $\mathbf{R}$,

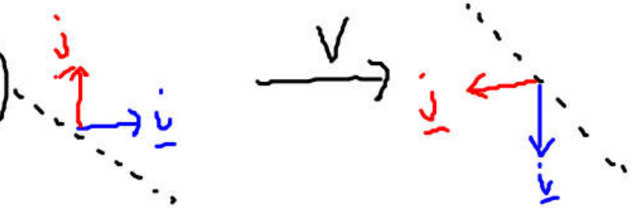
(*c*) express $\mathbf{R}$ in terms of $\mathbf{P}$ and $\mathbf{Q}$, **(1)**

(*d*) find the matrix $\mathbf{R}$, **(2)**

(*e*) give a full geometrical description of T as a single transformation. **(2)**

a) 

$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) 

$Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

c) $R = QP$

d) $= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

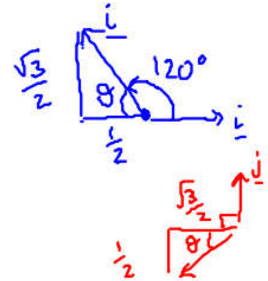
d) reflection in the y-axis.

More invariant points!

5.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$



$\tan^{-1}(\sqrt{3}) = 60°$

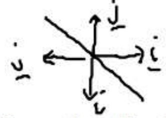
$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$

(a) Describe fully the single geometrical transformation U represented by the matrix **A**. (3)

rotation 120° about (0,0)

The transformation V, represented by the 2×2 matrix **B**, is a reflection in the line $y = -x$

(b) Write down the matrix **B**. (1)



$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Given that U followed by V is the transformation T, which is represented by the matrix **C**,

(c) find the matrix **C**. (2)

$$C = BA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T. (4)

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

$-\frac{\sqrt{3}}{2} + \frac{1}{2}k = 1$

$\frac{1}{2} + \frac{\sqrt{3}}{2}k = k$

$\frac{1}{2}k = 1 + \frac{\sqrt{3}}{2}$

$k = 2 + \sqrt{3}$

$\frac{1}{2} + \frac{\sqrt{3}}{2}k = k$

$\frac{1}{2} = k - \frac{\sqrt{3}}{2}k$

$1 = 2k - \sqrt{3}k$

$1 = k(2 - \sqrt{3})$

$\frac{1}{2 - \sqrt{3}} = k$

$\frac{2 + \sqrt{3}}{1} = k$

$k = 2 + \sqrt{3}$

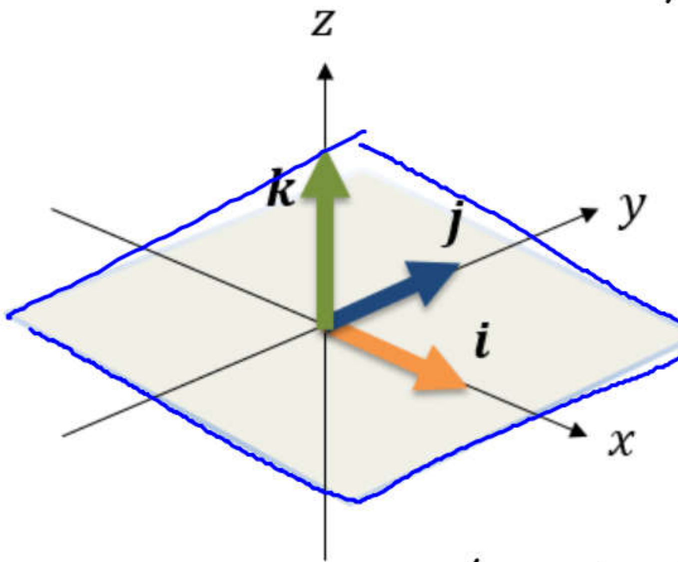
Check

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix}$$

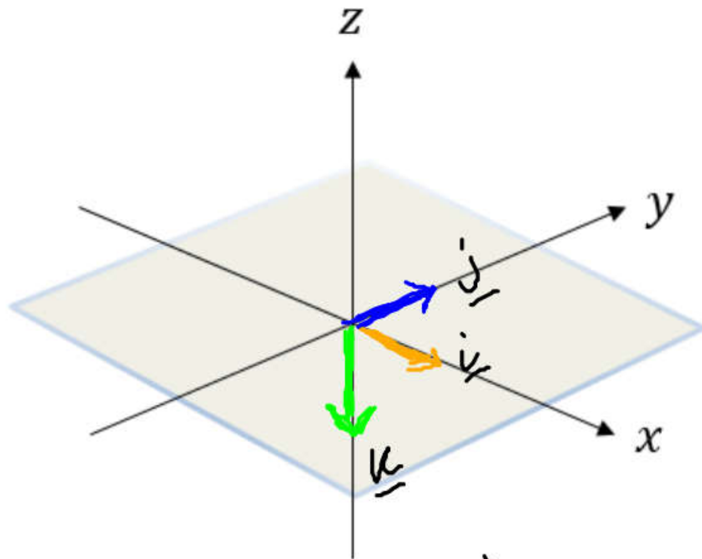
Linear transformations in 3D

We saw earlier that we could determine the matrix corresponding to a transformation by transforming each of the unit vectors (i.e. the axes) and using these as the columns of the matrix. This works in 3D too!

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Reflect in the plane $z = 0$.

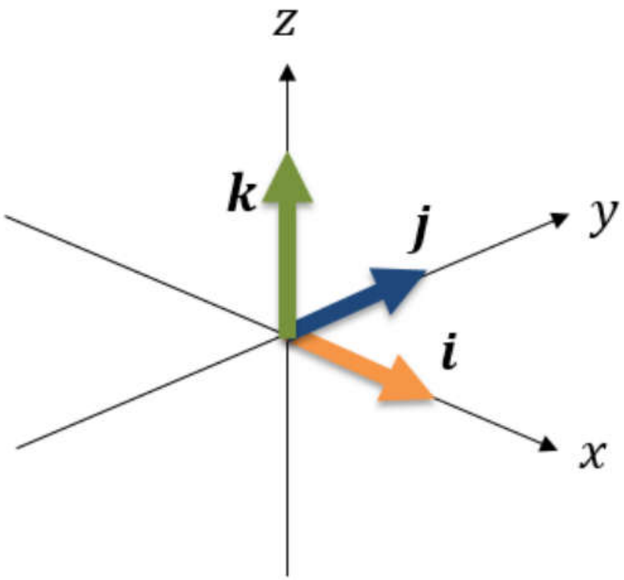


$(2, 3, \underline{0})$

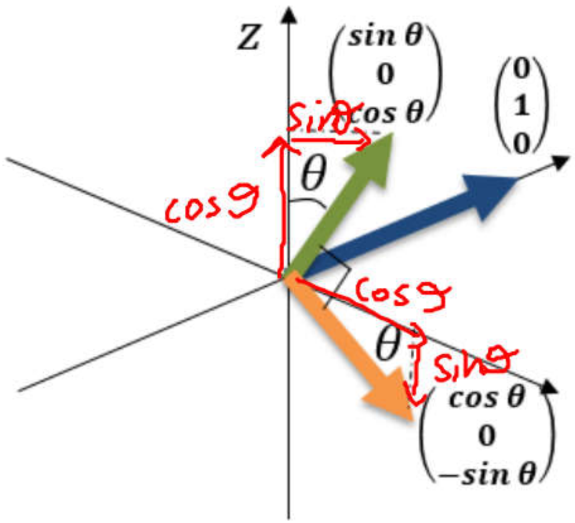
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Reminder: The rotation is anticlockwise relative to the positive y axis.



Rotate by angle θ about the y-axis.



$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

✎ Rotation θ about x-axis: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

Rotation θ about y-axis: $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Rotation θ about z-axis: $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Tip: You can tell whether it's a rotation in the x, y or z axes by looking whether the 1 is in the 1st, 2nd of 3rd row/column.

Ex 7E

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

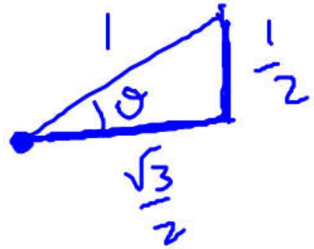
(a) Describe the transformation represented by $\mathbf{M}$.

(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by $\mathbf{M}$.

a) rotation about y-axis 30°.

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$

$\sin\theta = \frac{1}{2}$ $\quad \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$



b) $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{3}}{2} \\ -2 \\ \frac{1+\sqrt{3}}{2} \end{pmatrix}$$

$\left(\frac{1-\sqrt{3}}{2}, -2, \frac{1+\sqrt{3}}{2}\right)$.