Integration (Year 13)

In this chapter, you'll be able to integrate a significantly greater variety of expressions, and be able to solve differential equations.

Integration by standard result
(There's certain expressions you're expected to know straight off.)

$$\int \sec^2 x\, dx = \tan x + C$$

Integration by 'reverse chain rule'
(We imagine what would have differentiated to get the expression.)

$$\int \cos 4x \;\, dx = \frac{1}{4}\sin 4x + C$$

$$\int \sin^3 x \cos x\, dx = \frac{1}{4}\sin^4 x + C$$

Integration by substitution
(We make a substitution to hopefully make the expression easier to integrate)

$$\int x\sqrt{2x+5}\, dx$$

Let $u = 2x + 5 \quad \rightarrow \quad x = \frac{u-5}{2}$

$$\frac{du}{dx} = 2 \quad \rightarrow \quad dx = \frac{1}{2}u$$

$$\int x\sqrt{2x+5}\, dx = \int \frac{u-5}{2}\frac{1}{2} du = \cdots$$

Integration by parts
(Allows us to integrate a product, just as the product rule allowed us to differentiate one)

$$\int x \cos x\, dx$$

$$u = x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 1 \qquad v = \sin x$$

$$\int x\cos x\, dx = x \sin x - \int \sin x\, dx$$

$$= x\sin x + \cos x$$

Integrating partial fractions
(We split into partial fractions first so each fraction easier to integrate)

$$\int \frac{3x+5}{(x+1)(x+2)} dx$$

Approximating areas using the trapezium rule
(Instead of integrating, we split the area under the graph into trapeziums and use these to approximate the area)

Solving Differential Equations
(Solving here means to find one variable in terms of another without derivatives present)

$$\frac{dV}{dt} = -kV$$

$$\int \frac{1}{V} dV = \int -k\; dt$$

$$\ln V = -kt + C$$

$$V = e^{-kt+C} = Ae^{-kt}$$

Integrating Parametric Equations
(Use fact that area:

$$\int y\, dx = \int y \frac{dx}{dt}\, dt)$$

SKILL #1: Integrating Standard Functions

There's certain results you should be able to integrate straight off, by just thinking about the opposite of differentiation.

y	$\int y\,dx$
x^n	
e^x	
$\frac{1}{x}$	
$\cos x$	
$\sin x$	
$\sec^2 x$	
$cosec\ x \cot x$	
$cosec^2 x$	
$\sec x \tan x$	

The $|x|$ has to do with problems when x is negative (when $\ln x$ is not defined)

$$\int \sec x \tan x \ dx =$$

$$\int cosec^2\ x\ dx =$$

$$\int \sin x \ dx =$$

$$\int -\sin x \ dx =$$

$$\int cosec^2\ x\ dx =$$

$$\int \sec x \tan x \ dx =$$

$$\int -\cos x \ dx =$$

$$\int \cos x \ dx =$$

$$\int \sec^2 x \ dx =$$

$$\int cosec\ x \cot x \ dx =$$

$$\int \frac{1}{x}\ dx =$$

$$\int -\sin x \ dx =$$

$$\int 2\cos x + \frac{3}{x} - \sqrt{x}\, dx =$$

$$\int \frac{\cos x}{\sin^2 x}\, dx =$$

Hint: What 'reciprocal' trig functions does this simplify to?

Given that $\int_a^{3a} \frac{2x+1}{x}\, dx = \ln 12$, find the exact value of a.

Important Notes:
We can simplify:

$$\frac{x+1}{x} \equiv \frac{x}{x} + \frac{1}{x} \equiv 1 + \frac{1}{x}$$

However it is **NOT** true that:

$$\frac{x}{x+1} \equiv \frac{x}{x} + \frac{x}{1}$$

In my experience students often fail to spot when they can split up a fraction to then integrate.

Ex 11A

SKILL #2: Integrating f(ax+b)

$$\frac{d}{dx}(\sin(3x+1)) =$$

Therefore:

$$\int \cos(3x+1)\ dx =$$

✏ For any expression where inner function is $ax+b$, integrate as before and $\div\, a$.

$$\int f'(ax+b)dx = \frac{1}{a} f(ax+b) + C$$

$$\int e^{3x}\, dx =$$

$$\int (3x+4)^3\, dx =$$

$$\int \frac{1}{5x+2} dx =$$

$$\int \sin(1-5x)\, dx =$$

$$\int 2\sec^2(3x-2)\, dx =$$

$$\int \frac{1}{3(4x-2)^2} dx =$$

$$\int (10x+11)^{12} =$$

Your Turn

$$\int e^{3x+1} dx =$$

$$\int \frac{1}{1-2x} dx =$$

$$\int (4-3x)^5 dx =$$

$$\int \sec(3x)\tan(3x)\, dx =$$

Ex 11B

1 **a** $\int \sin(2x+1)\, dx =$

c $\int 4e^{x+5}\, dx =$

e $\int cosec^2 3x\, dx =$

f $\int \sec 4x \tan 4x\ \ dx =$

g $\int 3\sin\left(\frac{1}{2}x+1\right) dx =$

h $\int cosec\ 2x \cot 2x\ \ dx =$

2 **a** $\int e^{2x} - \frac{1}{2}\sin(2x-1)\, dx$

$=$

b $\int (e^x+1)^2\, dx =$

c $\int \sec^2 2x\,(1+\sin 2x)\, dx =$

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)} dx =$

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\, dx$

$=$

3 **a** $\int \frac{1}{2x+1} dx =$

b $\int \frac{1}{(2x+1)^2} dx =$

c $\int (2x+1)^2\, dx =$

d $\int \frac{3}{4x-1} dx =$

f $\int \frac{3}{(1-4x)^2} dx =$

h $\int \frac{3}{(1-2x)^3} dx =$

j $\int \frac{5}{3-2x} dx =$

4 **a** $\int 3\sin(2x+1) + \frac{4}{2x+1}\ dx$

$=$

c $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2} dx$

$=$

SKILL #3: Integrating using Trig Identities

Some expressions, such as $\sin^2 x$ and $\sin x \cos x$ can't be integrated directly, but we can use one of our trig identities to replace it with an expression we can easily integrate.

Q Find $\int \sin^2 x \; dx$

Q Find $\int \sin 3x \cos 3x \; dx$

Q Find $\int \cos^2 x \; dx$

Q Find $\int \tan^2 x \; dx$

Q Find $\int (\sec x + \tan x)^2 \; dx$

Q Find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 3x \; dx$

SKILL #4: Reverse Chain Rule

There's certain more complicated expressions which look like the result of having applied the chain rule. I call this process '**consider then scale**':

1. **Consider some expression that will differentiate to something similar to it.**
2. **Differentiate, and adjust for any scale difference.**

$$\int x(x^2+5)^3\,dx$$

$$\int \cos x \sin^2 x\;dx$$

$$\int \frac{2x}{x^2+1}\,dx$$

The first x looks like it arose from differentiating the x^2 inside the brackets.

The $\cos x$ probably arose from differentiating the sin.

The $2x$ probably arose from differentiating the x^2.

✏ **Integration by Inspection/Reverse Chain Rule**: Use common sense to **consider some expression** that would differentiate to the expression given. Then **scale** appropriately.
Common patterns:

$$\int k\frac{f'(x)}{f(x)}dx \rightarrow \quad Try\ \ln|f(x)|$$

In words: "If the bottom of a fraction differentiates to give the top (forgetting scaling), try ln of the bottom".

$$\int kf'(x)[f(x)]^n \rightarrow \quad Try\ [f(x)]^{n+1}$$

$$\int \frac{x^2}{x^3+1}\,dx$$

$$\int x\,e^{x^2+1}\,dx$$

$$\int \frac{4x^3}{x^4 - 1} dx =$$

$$\int \frac{\cos x}{\sin x + 2} dx =$$

$$\int \cos x \ e^{\sin x} dx =$$

$$\int \cos x (\sin x - 5)^7 dx =$$

$$\int x^2(x^3 + 5)^7 =$$

$$\int \frac{x}{(x^2 + 5)^3} dx =$$

$\sin^n x \cos x$ vs $\sec^n x \tan x$

Notice when we differentiate $\sin^5 x$, then power decreases:

$$\frac{d}{dx}(\sin^5 x) =$$

However, when we differentiate $\sec^5 x$:

$$\frac{d}{dx}\left((\sec x)^5\right) =$$

Notice that the power of sec didn't go down. Keep this in mind when integrating.

$\int \sin^4 x \cos x \ dx$

$\int \sec^4 x \tan x \ dx$

$\int \cos x \sin^2 x \ dx$

$\int \sec^3 x \tan x \ dx$

Your Turn

$\int \sin x\,(\cos x + 1)^5\, dx$	$\int \frac{cosec^2 x}{(2+\cot x)^3} dx$

$\int \frac{\sec^2 2x}{\tan 2x + 1}\, dx$	$\int x(x^2+2)^3\, dx$	$\int 5 \tan x \sec^2 x\ dx$

Ex 11D

SKILL #5: Integration by Substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

Q Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to x, and replace it with u. We'll have to work out x and dx so that we can replace them.

STEP 1: Using substitution, work out x and dx (or variant)

STEP 2: Substitute these into expression.

STEP 3: Integrate simplified expression.

STEP 4: Write answer in terms of x.

Tip: If you have a constant factor, factor it out of the integral.

Using substitutions involving implicit differentiation

When a root is involved, it can make things tidier if we use $u^2 = \cdots$

Q Use the substitution $u^2 = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

STEP 1: Using substitution, work out x and dx (or variant)

STEP 2: Substitute these into expression.

STEP 3: Integrate simplified expression.

STEP 4: Write answer in terms of x.

This was marginally less tedious than when we used $u = 2x + 5$, as we didn't have fractional powers to deal with.

How can we tell what substitution to use?

In Edexcel you will *usually* be given the substitution!
However in some other exam boards, and in STEP, you often aren't.
There's no hard and fast rule, but it's often helpful to replace expressions inside roots, powers or the denominator of a fraction.

Sensible substitution:

$$\int \cos x \sqrt{1 + \sin x}\, dx$$

$\boldsymbol{u} =$

$$\int \frac{xe^x}{1+x} dx$$

$\boldsymbol{u} =$

$$\int e^{\frac{1-x}{1+x}}\, dx$$

$\boldsymbol{u} =$

$$\int \cos x \sin x\,(1+\sin x)^3 \;dx$$

STEP 1: Using substitution, work out x and dx **(or variant)**

STEP 2: Substitute these into expression.

STEP 3: Integrate simplified expression.

STEP 4: Write answer in terms of x.

Edexcel C4 Jan 2012 Q6c

(*c*) Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2\sin 2x}{(1+\cos x)}\,\mathrm{d}x = 4\ln(1+\cos x) - 4\cos x + k,$$

where k is a constant. **(5)**

Hint: You might want to use your double angle formula first.

Definite Integration with Substitution

Calculate $\int_0^{\frac{\pi}{2}} \cos x \sqrt{1+\sin x}\ dx$

Ex 11E

Edexcel C4 June 2011 Q4

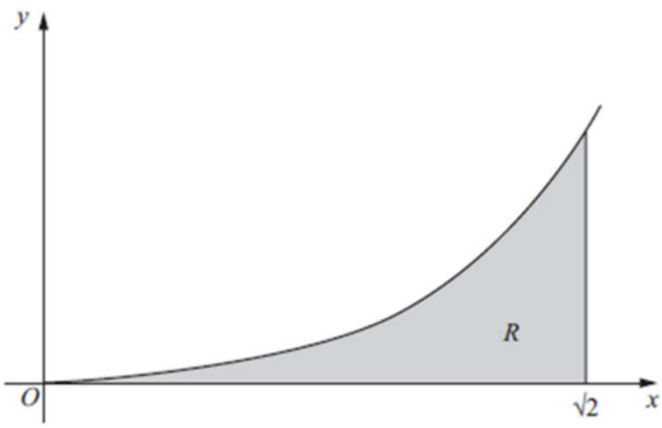

Figure 2 shows a sketch of the curve with equation $y = x^3 \ln(x^2+2)$, $x \geq 0$.

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = \sqrt{2}$.

(c) Use the substitution $u = x^2 + 2$ to show that the area of R is

$$\frac{1}{2}\int_2^4 (u-2)\ln u\ \mathrm{d}u.$$

SKILL #6: Integration by Parts

$$\int x \cos x \; dx = ???$$

Just as the Product Rule was used to **differentiate the product** of two expressions, we can often use 'Integration by Parts' to **integrate a product.**

To integrate by parts:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} \; dx$$

<u>"L-I-A-T-E" Choose 'u' to be the function that comes first in this list:</u>

L: Logrithmic Function

I: Inverse Trig Function

A: Algebraic Function

T: Trig Function

E: Exponential Function

$$\int x \cos x \; dx =$$

"L-I-A-T-E" Choose 'u' to be the function that comes first in this list:

L: Logrithmic Function
I: Inverse Trig Function
A: Algebraic Function
T: Trig Function
E: Exponential Function

Find $\int x^2 \ln x \; dx$

You will need the following standard results (given in your formula booklet) for the main exercise. We'll prove them later.

$\int \tan x \; dx = \ln|\sec x| + C \quad \int \sec x \; dx = \ln|\sec x + \tan x| + C$

$\int \cot x \; dx = \ln|sin\, x| + C \quad \int \operatorname{cosec} x \; dx = \ln|\operatorname{cosec} x + \cot x| + C$

Ex 11F
Q1
Q2 (NOT d)

Find $\int x^2 e^x \, dx$

Your Turn

Find $\int x^2 \sin x \; dx$

Ex 11F
Q3

Integrating $\ln x$ and definite integration

Find $\int \ln x \; dx$, leaving your answer in terms of natural logarithms.

Find $\int_1^2 \ln x \, dx$, leaving your answer in terms of natural logarithms.

In general:

$$\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b v \frac{du}{dx} dx$$

Your Turn

Find $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

Ex 11F Q4 onwards

One final unusual one…

$$\int e^x \sin x \, dx$$

SKILL #7: Using Partial Fractions

Find $\int \frac{2}{x^2-1}\, dx$

Find $\int \frac{x-5}{(x+1)(x-2)}\, dx$

Find $\int \frac{8x^2-19x+1}{(2x+1)(x-2)^2}\ dx$

Edexcel C4 June 2009 Q3

$$f(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}.$$

(*a*) Find the values of the constants A, B and C. **(4)**

(*b*) (i) Hence find $\int f(x)\ dx$. **(3)**

(ii) Find $\int_0^2 f(x)\ dx$ in the form $\ln k$, where k is a constant. **(3)**

SKILL #8: Integrating top-heavy algebraic fractions

$$\int \frac{x^2}{x+1}\,dx = ?$$

How would we deal with this? (the clue's in the title)

$$\int \frac{x}{x-1}\,dx$$

Contrast this with $\int \frac{x-1}{x}\,dx$ which can be integrated more simply

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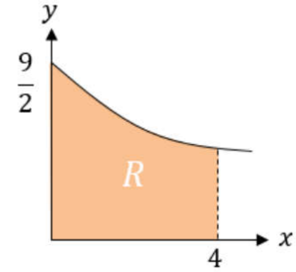
$$\int \frac{x^3+2}{x+1}\,dx$$

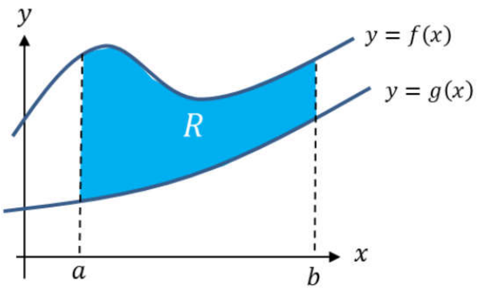
Finding Areas

You're already familiar with the idea that definite integration gives you the (signed) area bound between the curve and the x-axis.
Given your expanded integration skills, you can now find the area under a greater variety of curves.

The diagram shows part of the curve $y = \frac{9}{\sqrt{4+3x}}$
The region R is bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$, as shown in the diagram. Use integration to find the area of R.



Skill #9: Area between two curves

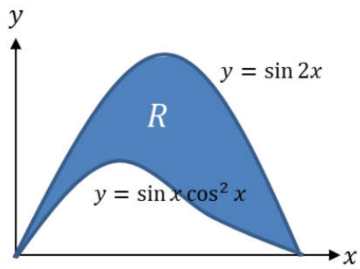


The areas under the two curves are $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$. It therefore follows the area between them (provided the curves don't overlap) is:

$$R = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \int_a^b \big(f(x) - g(x)\big)\,dx$$

Ensure you have top curve minus bottom curve.

The diagram shows part of the curves $y = \sin 2x$ and $y = \sin x \cos^2 x$ where $0 \le x \le \frac{\pi}{2}$. The region R is bounded by the two curves. Use integration to find the area of R.



Your Turn

Edexcel C4 Jan 2009 Q2

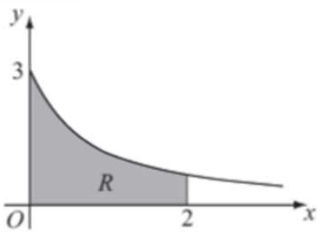


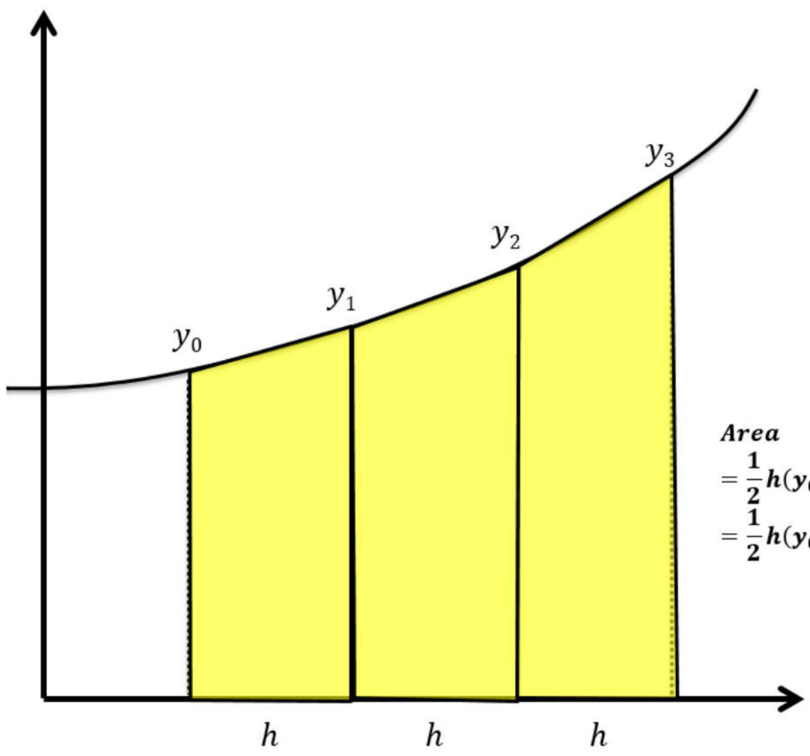
Figure 1

Figure 1 shows part of the curve $y = \frac{3}{\sqrt{(1+4x)}}$. The region R is bounded by the curve, the x-axis, and the lines $x = 0$ and $x = 2$, as shown shaded in Figure 1.

(*a*) Use integration to find the area of R. **(4)**

Ex 11H

Skill #10: Trapezium Rule



Sometimes finding the exact area under the graph via integration is difficult. You may be familiar with the idea of **approximating the area under a graph by dividing it into trapeziums of equal width.**

$$\boldsymbol{Area} = \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3)$$
$$= \frac{1}{2}h(y_0 + 2(y_1 + y_2) + y_3)$$

Overestimate?

Underestimate?

In general:

width of each trapezium

$$\int_a^b y\,dx \approx \frac{h}{2}(y_1 + 2(y_2 + \cdots + y_{n-1}) + y_n)$$

Area under curve

is approximately

Example We're approximating the region bounded between $x = 1$, $x = 3$, the x-axis the curve $y = x^2$, using 4 strips.

Dividing a gap of 2 into 4 strips means each strip will be width 0.5

x	1	1.5	2	2.5	3
y					

Edexcel C2 May 2013 (R) Q2

$$y = \frac{x}{\sqrt{(1+x)}}$$

(*a*) Complete the table below with the value of y corresponding to $x = 1.3$, giving your answer to 4 decimal places.

(1)

Tip: You can generate table with Casio calcs . $Mode \to 3$ $(Table)$. Use 'Alpha' button to key in X within the function. Press =

x	1	1.1	1.2	1.3	1.4	1.5
y	0.7071	0.7591	0.8090		0.9037	0.9487

(*b*) Use the trapezium rule, with all the values of y in the completed table, to obtain an approximate value for

$$\int_1^{1.5} \frac{x}{\sqrt{(1+x)}}\,dx$$

giving your answer to 3 decimal places.

You must show clearly each stage of your working.

(4)

Area $\approx$

Q

Given $I = \int_0^{\frac{\pi}{3}} \sec x \; dx$

a) Find the exact value of I.

b) Use the trapezium rule with two strips to estimate I.

c) Use the trapezium rule with four strips to find a second estimate of I.

d) Find the percentage error in using each estimate.

Edexcel C4 June 2014(R) Q2

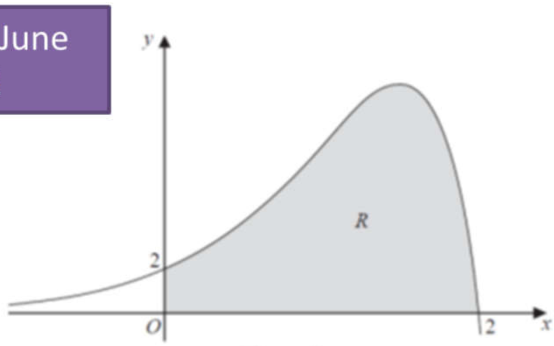


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = (2 - x)e^{2x},$$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the y-axis.

The table below shows corresponding values of x and y for $y = (2 - x)e^{2x}$.

x	0	0.5	1	1.5	2
y	2	4.077	7.389	10.043	0

(*a*) Use the trapezium rule with all the values of y in the table, to obtain an approximation for the area of R, giving your answer to 2 decimal places.

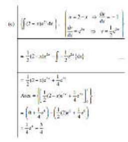
(3)

(*b*) Explain how the trapezium rule can be used to give a more accurate approximation for the area of R.

(1)

(*c*) Use calculus, showing each step in your working, to obtain an exact value for the area of R. Give your answer in its simplest form.

(5)



Ex 11I

Integration with Parametric Equations

Suppose we have the following parametric equations:

$$x = t^2$$
$$y = t + 1$$

To find the area under the curve, we want to determine to determine $\int y\, dx$.
The problem however is that y is in terms of t, not in terms of x.

Area: $$\int y\, dx = \int y \frac{dx}{dt}\, dt$$

Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t + 1$, the x-axis, and the lines $x = 0$ and $x = 3$.

STEP 1: Find $\frac{dx}{dt}$

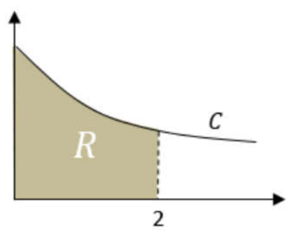
STEP 2: Change limits

STEP 3: Integrate

The curve C has parametric equations

$$x = t(1 + t), \qquad y = \frac{1}{1 + t}, \qquad t \geq 0$$

Find the exact area of the region R, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.

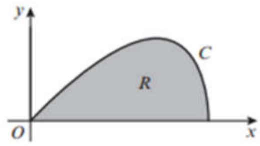


Ⓟ **1** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

(E/P) **2** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \geqslant t \geqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis. Find the exact area of R. **(6 marks)**



(E/P) **3** This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$

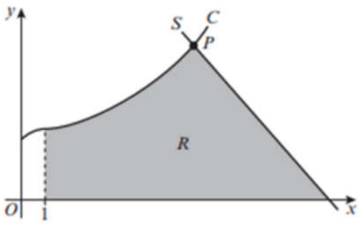
P is the point on the curve where $t = 2$.

The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**

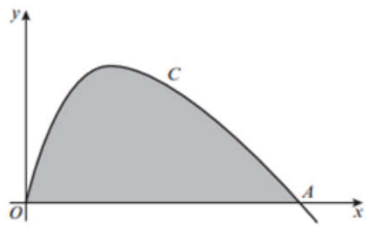


(E/P) **4** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**

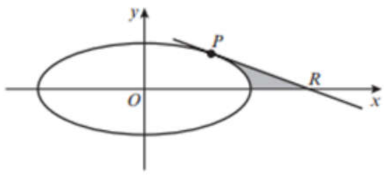


(E/P) **5** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

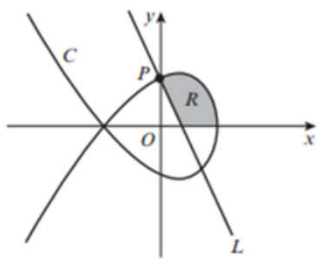


c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**

(E/P) **6** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**



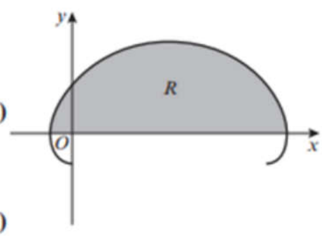
(E/P) **7** The curve shown in the diagram has parametric equations

$$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3 marks)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.

b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, dt$ **(3 marks)**



c Use this integral to find the exact value of the shaded area. **(4 marks)**

ANSWERS

1 Area $= \int y \frac{dx}{dt} dt = \int_0^{\sqrt[3]{4}} t^2(3t^2)\, dt = \frac{3}{5}\left(\sqrt[3]{4}\right)^5 = \frac{3}{5}2^{\frac{10}{3}}$

$= \frac{3}{5}(2^3)(2^{\frac{1}{3}}) = \frac{24}{5}\sqrt[3]{2}$

2 $\frac{2}{3}$

3 **a** $x + y = 16$ **b** 61.85

4 **a** $\frac{\pi}{2}$ **b** $\frac{3\pi}{2}$

5 **a** $-\frac{4}{5}$ **b** $y - 2\sqrt{2} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$

c $10 - \frac{5\pi}{2}$

6 $\frac{41}{60}$

7 **a** $2\cos t = 1 \Rightarrow \cos t = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}$ or $t = \frac{5\pi}{3}$

b $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y \frac{dx}{dt} dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)(1 - 2\cos t)\, dt$

$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 dt$

c $4\pi + 3\sqrt{3}$

Your Turn

Edexcel C4 Jan 2013 Q5

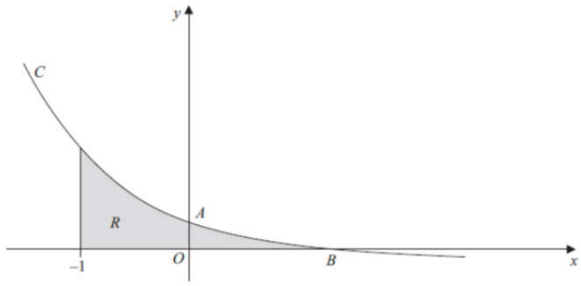


Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(*a*) Show that A has coordinates $(0, 3)$. **(2)**

(*b*) Find the x-coordinate of the point B. **(2)**

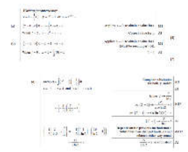
The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(*d*) Use integration to find the exact area of R. **(6)**

Helping Hand:

$$\frac{d}{dx}(a^x) = a^x(\ln a)$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + c$$



SKILL #11: Differential Equations

Differential equations are equations involving a mix of variables and derivatives, e.g. y, x and $\frac{dy}{dx}$.

'<u>Solving</u>' these equations means to get y in terms of x (with no $\frac{dy}{dx}$).

Q Find the general solution to $\frac{dy}{dx} = xy + y$

STEP 1: Get y to the side of $\frac{dy}{dx}$ by dividing and x to the other side.
(you may need to factorise to separate out y first)

STEP 2: Integrate both sides with respect to x. $\frac{dy}{dx}\ dx$ simplifies to dy (recall that (implicitly) differentiating an expression in terms of y with respect to x introduces a $\frac{dy}{dx}$, so integrating similarly would get rid of it)

STEP 3: Make y the subject, if the question asks.

Q Find the general solution to $(1+x^2)\frac{dy}{dx} = x\tan y$

$y = \arcsin\left(k\sqrt{1+x^2}\right)$

Differential Equations with Boundary Conditions

Q Find the general solution to $\frac{dy}{dx} = -\frac{3(y-2)}{(2x+1)(x+2)}$
Given that $x = 1$ when $y = 4$. Leave your answer in the form $y = f(x)$

$y - 2 = \frac{k(x+2)}{2x+1}$

Key Tips on Differential Equations

- Get y on to LHS by dividing (possibly factorising first).
- If after integrating you have ln on the RHS, make your constant of integration $\ln k$ or $\ln A$
- Be sure to combine all your ln's together just as you did in Year 12.
  e.g.:

$$2\ln|x+1| - \ln|x| \quad \rightarrow \quad \ln\left|\frac{(x+1)^2}{x}\right|$$

- Sub in boundary conditions to work out your constant – better to do sooner rather than later.
- Exam questions ♥ partial fractions combined with differential equations.

Ex 11J

Your Turn

Edexcel C4 Jan 2012 Q4

Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}.$$ **(5)**

Forming differential equations

Q The rate of increase of a rabbit population (with population P, where time is t) is **proportional to** the current population. Form a differential equation, and find its general solution.

Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

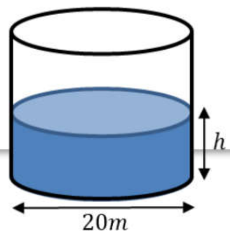
(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants.

Initially the height of the water is 27m. 10 minutes later, the height is 8m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 1m.



Ex 11K

Edexcel C4 June 2005 Q8

Liquid is pouring into a container at a constant rate of 20 cm^3 s^{-1} and is leaking out at a rate proportional to the volume of the liquid already in the container.

(*a*) Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 20 - kV,$$

where k is a positive constant. **(2)**

The container is initially empty.

(*b*) By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k. **(6)**

Given also that $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$ when $t = 5$,

(*c*) find the volume of liquid in the container at 10 s after the start. **(5)**

Jan 08 C4

Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 cm^{3}s^{-1} and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(*a*) Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.4 - k\sqrt{h},$$

where k is a positive constant.

(3)

When $h = 25$, water is leaking out of the hole at 400 cm^{3}s^{-1}.

(*b*) Show that $k = 0.02$.

(1)

(*c*) Separate the variables of the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.4 - 0.02\sqrt{h}$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\,\mathrm{d}h.$$

(2)

Using the substitution $h = (20 - x)^2$, or otherwise,

(*d*) find the exact value of $\int_0^{100} \frac{50}{20 - \sqrt{h}}\,\mathrm{d}h$.

(6)

(*e*) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$\sin x$	Standard result	$-\cos x$	No
$\cos x$	Standard result	$\sin x$	No
$\tan x$	In formula booklet, but use $\int \frac{\sin x}{\cos x} dx$ which is of the form $\int \frac{kf'(x)}{f(x)} dx$	$\ln\lvert\sec x\rvert$	Yes
$\sin^2 x$	For both $\sin^2 x$ and $\cos^2 x$ use identities for $\cos 2x$ $\cos 2x = 1 - 2\sin^2 x$ $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$	$\frac{1}{2}x - \frac{1}{4}\sin 2x$	No
$\cos^2 x$	$\cos 2x = 2\cos^2 x - 1$ $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$	$\frac{1}{2}x + \frac{1}{4}\sin 2x$	No
$\tan^2 x$	$1 + \tan^2 x \equiv \sec^2 x$ $\tan^2 x \equiv \sec^2 x - 1$	$\tan x - x$	No
$cosec\ x$	Would use substitution $u = cosec\ x + \cot x$, but too hard for exam.	$-\ln\lvert cosec\ x + \cot x\rvert$	Yes
$\sec x$	Would use substitution $u = \sec x + \tan x$, but too hard for exam.	$\ln\lvert\sec x + \tan x\rvert$	Yes
$\cot x$	$\int \frac{\cos x}{\sin x} dx$ which is of the form $\int \frac{f'(x)}{f(x)} dx$	$\ln\lvert sin\ x\rvert$	Yes

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$cosec^2 x$	By observation.	$-\cot x$	No!
$\sec^2 x$	By observation.	$\tan x$	Yes (but memorise)
$\cot^2 x$	$1 + \cot^2 x \equiv cosec^2 x$	$-\cot x - x$	No
$\sin 2x \cos 2x$	For any product of sin and cos with same coefficient of x, use double angle. $\sin 2x \cos 2x \equiv \frac{1}{2}\sin 4x$	$-\frac{1}{8}\cos 4x$	No
$\frac{1}{x}$		$\ln x$	No
$\ln x$	Use IBP, where $u = \ln x$, $\frac{dv}{dx} = \ln x$	$x\ln x - x$	No
$\frac{x}{x+1}$	Use algebraic division. $\frac{x}{x+1} \equiv 1 - \frac{1}{x+1}$	$x - \ln\lvert x+1\rvert$	
$\frac{1}{x(x+1)}$	Use partial fractions.	$\ln\lvert x\rvert - \ln\lvert x+1\rvert$	

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)
$\frac{4x}{x^2+1}$	Reverse chain rule. Of form $\int \frac{kf'(x)}{f(x)}$	$2\ln\lvert x^2+1\rvert$
$\frac{x}{(x^2+1)^2}$	Power around denominator so NOT of form $\int \frac{kf'(x)}{f(x)}$. Rewrite as product. $x(x^2+1)^{-2}$ Reverse chain rule (i.e. "Consider $y = (x^2+1)^{-1}$" and differentiate)	$-\frac{1}{2}(x^2+1)^{-1}$
e^{2x+1} $\frac{1}{1-3x}$	For any function where 'inner function' is linear expression, divide by coefficient of x	$\frac{1}{2}e^{2x+1}$ $-\frac{1}{3}\ln\lvert 1-3x\rvert$
$x\sqrt{2x+1}$	IBP or use sensible substitution. $u = 2x+1$ or even better, $u^2 = 2x+1$.	$\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$
$\sin^5 x \cos x$	Reverse chain rule.	$\frac{1}{6}\sin^6 x$

A Whole Load of Integration

This is it; where all the integration you've seen comes together. You need to find the following integrals without any clue as to how to do them! You could use 'guess and check', partial fractions, parts, substitution or more than one of the above!

1 $\int \cos(3x-1)dx$

2 $\int e^{1-x}dx$

3 $\int \frac{2x+1}{\left(x^2+x-1\right)^2}dx$

4 $\int \cos 2x dx$

5 $\int \ln 2x dx$

6 $\int \frac{x}{\left(x^2-1\right)^3}dx$

7 $\int \sqrt{2x-3}dx$

8 $\int \frac{4x-1}{(x-1)^2(x+2)}dx$

9 $\int x^3 \ln x dx$

10 $\int \frac{5}{2x^2-7x+3}dx$

11 $\int (x+1)e^{x^2+2x}dx$

12 $\int \frac{\sin x - \cos x}{\sin x + \cos x}dx$

13 $\int x^2 \sin 2x dx$

14 $\int \sin^3 2x dx$

1) $\frac{1}{3}\sin(3x-1)+c$
2) $-e^{1-x}+c$
3) $\frac{-1}{x^2+x-1}+c$
4) $\frac{1}{2}\sin(2x)+c$
5) $x\ln(2x)-x+c$
6) $\frac{-1}{4}\left(x^2-1\right)^{-2}+c$
7) $\frac{1}{3}(2x-3)^{3/2}+c$
8) $-\ln|x+2|+\ln|x-1|-(x-1)^{-1}+c$
9) $\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+c$
10) $-\ln|2x-1|+\ln|x-3|+c$
11) $\frac{1}{2}e^{x^2+2x}+c$
12) $-\ln|\sin x+\cos x|+c$
13) $\frac{1}{4}\cos 2x-\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+c$
14)