10. An archer shoots an arrow at 10 m s^{-1} from the origin and hits a target at $(10, -5)$ m. The initial velocity of the arrow is at an angle θ above the horizontal. The arrow is modelled as a particle moving freely under gravity.

(In this question, take $g = 10 \text{ m s}^{-2}$.)

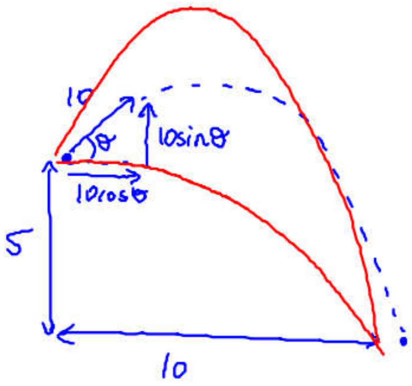
(a) Show that $(\tan \vartheta - 1)^2 = 1$.

(11)

(b) Find the possible values of θ.

(3)

(Total 14 marks)



Horiz.

$10 = 10\cos\theta \times t$

$t = \frac{10}{10\cos\theta} = \frac{1}{\cos\theta}$

Vert ↓+

$a = 10$

$u = -10\sin\theta$

$s = 5$

$t = t$

$5 = -10\sin\theta \times t + 5t^2$

$5 = -\frac{10\sin\theta}{\cos\theta} + 5 \times \frac{1}{\cos^2\theta}$

$5 = -10\tan\theta + 5\sec^2\theta$

$5 = -10\tan\theta + 5 + 5\tan^2\theta$

$0 = 5\tan^2\theta - 10\tan\theta$

$0 = \tan^2\theta - 2\tan\theta$

$0 = (\tan\theta - 1)^2 - 1$

$1 = (\tan\theta - 1)^2$

$(\tan\theta - 1)^2 = 1$

b)

$\pm 1 = \tan\theta - 1$

$\tan\theta = 1 \pm 1$

$\tan\theta = 0$

$\underline{\underline{\theta = 0^\circ}}$

$\tan\theta = 2$

$\theta = \cancel{63.4}^\circ$

$\underline{\underline{63.4^\circ}}$

Your Turn

A ladder AB, of mass m and length $4a$, has one end A resting on rough horizontal ground. The other end B rests against a smooth vertical wall. A load of mass $3m$ is fixed on the ladder at the point C, where $AC = a$. The ladder is modelled as a uniform rod in a vertical plane perpendicular to the wall and the load is modelled as a particle. The ladder rests in limiting equilibrium making an angle of 30° with the wall, as shown in Figure 2.

Find the coefficient of friction between the ladder and the ground.

(10)

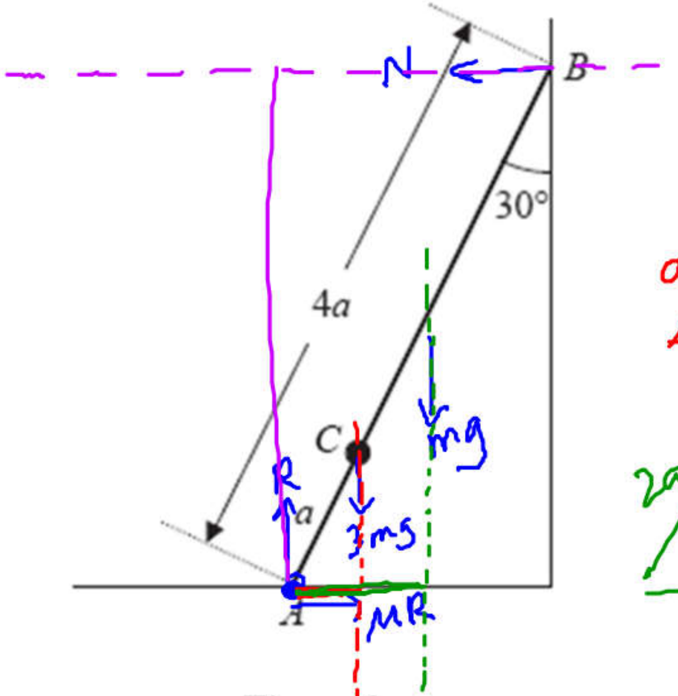


Figure 2

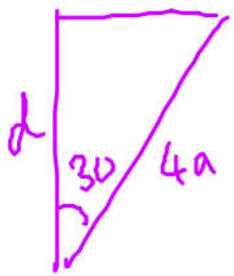
M(A) $\quad \cancel{a}\sin 30 \times 3mg + 2\cancel{a}\sin 30 \times mg = 4\cancel{a}\cos 30 \times N$

$5mg\sin 30 = 4N\cos 30$

$5mg\tan 30 = 4N$

$N = \frac{5}{4}mg\tan 30$

$N = \frac{5\sqrt{3}}{12}mg$



R↕ $\quad R = 4mg$

R↔ $\quad \mu R = N$

$\mu = \frac{N}{R}$

$\mu = \frac{\frac{5\sqrt{3}}{12}\cancel{mg}}{4\cancel{mg}} = \frac{5\sqrt{3}}{48} = 0.18$

Your Turn

A uniform rod AB of weight W has its end A freely hinged to a point on a fixed vertical wall. The rod is held in equilibrium, at angle θ to the horizontal, by a force of magnitude P. The force acts perpendicular to the rod at B and in the same vertical plane as the rod, as shown in Figure 3. The rod is in a vertical plane perpendicular to the wall. The magnitude of the vertical component of the force exerted on the rod by the wall at A is Y.

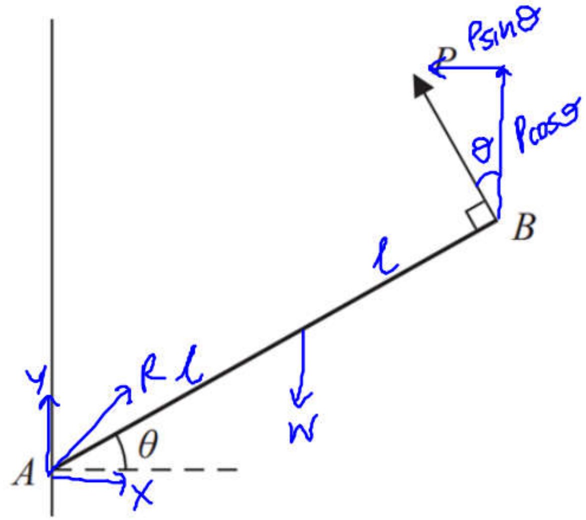
(a) Show that $Y = \dfrac{W}{2}(2 - \cos^2\theta)$.

(6)

Given that $\theta = 45°$

(b) find the magnitude of the force exerted on the rod by the wall at A, giving your answer in terms of W.

(6)



M(A) $W \times l\cos\theta = 2l\,P$

$\boxed{W\cos\theta = 2P}$ $\qquad \frac{W}{2}\cos\theta = P$

R↕ $\boxed{Y + P\cos\theta = W}$ $\qquad Y + \frac{W}{2}\cos^2\theta = W$

R↔ $X = P\sin\theta$ $\qquad Y = W - \frac{W}{2}\cos^2\theta$

$Y = \frac{W}{2}(2 - \cos^2\theta)$

$\theta = 45$

$\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

$\cos^2\theta = \frac{1}{2}$

$X = P\sin\theta$

$= \frac{W}{2}\cos\theta\sin\theta$

$X = \frac{W}{2} \times \frac{1}{2} = \frac{W}{4}$

$Y = \frac{W}{2}\left(2 - \frac{1}{2}\right)$

$Y = \frac{W}{2} \times \frac{3}{2}$

$Y = \frac{3}{4}W$.

$R = \sqrt{\left(\frac{W}{4}\right)^2 + \left(\frac{3}{4}W\right)^2}$

$= \sqrt{\frac{W^2}{16} + \frac{9W^2}{16}}$

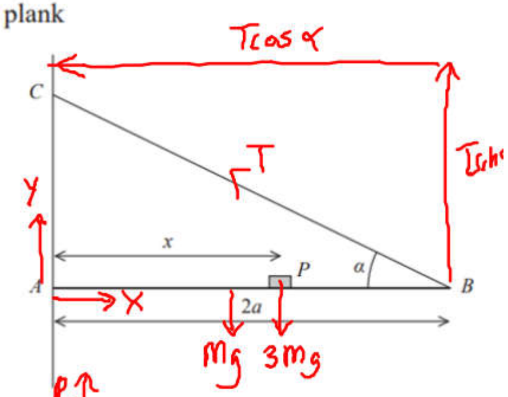
$= \sqrt{\frac{10}{16}W^2} = \frac{\sqrt{10}}{4}W$

Your Turn

A plank, AB, of mass M and length $2a$, rests with its end A against a rough vertical wall. The plank is held in a horizontal position by a rope. One end of the rope is attached to the plank at B and the other end is attached to the wall at the point C, which is vertically above A.

A small block of mass $3M$ is placed on the plank at the point P, where $AP = x$. The plank is in equilibrium in a vertical plane which is perpendicular to the wall.

The angle between the rope and the plank is α, where $\tan\alpha = \frac{3}{4}$, as shown in Figure 3.



The plank is modelled as a uniform rod, the block is modelled as a particle and the rope is modelled as a light inextensible string.

(a) Using the model, show that the tension in the rope is $\frac{5Mg(3x+a)}{6a}$ $< 5mg$ (3)

$T = 5mg\left(\frac{1}{2}\right) = \frac{5}{2}Mg$

$3x + a < 6a$

$3x < 5a$

$x < \frac{5}{3}$

The magnitude of the horizontal component of the force exerted on the plank at A by the wall is $2Mg$.

(b) Find x in terms of a. (2)

$x = \frac{2a}{3}$

$Y + T\sin\alpha = 4Mg$

$Y + \frac{5}{2}Mg \times \frac{3}{5} = 4Mg$

$Y = 4Mg - \frac{3}{2}Mg$

$Y = \frac{5}{2}Mg$

The force exerted on the plank at A by the wall acts in a direction which makes an angle β with the horizontal.

(c) Find the value of $\tan\beta$ (5)

$R \leftrightarrow \quad X = T\cos\alpha$

$= \frac{5}{2}mg \times \frac{4}{5} = 2Mg$

The rope will break if the tension in it exceeds $5Mg$.

(d) Explain how this will restrict the possible positions of P. You must justify your answer carefully. (3)

$\frac{5}{2}mg$

β

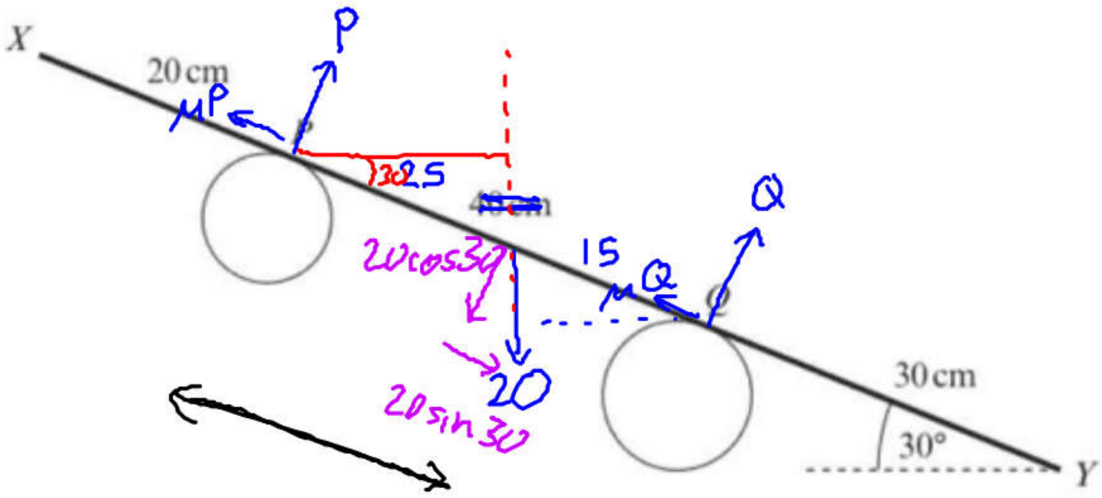
$2mg$

$\tan\beta = \frac{\frac{5}{2}\cancel{mg}}{2\cancel{mg}}$

$\tan\beta = \frac{5}{4}$

$\frac{5mg(3x+a)}{6a} \leq 5mg$

10 A uniform rod XY has weight 20 N and length 90 cm. The rod rests on two parallel pegs, with X above Y, in a vertical plane which is perpendicular to the axes of the pegs, as shown in the diagram. The rod makes an angle of 30° to the horizontal and touches the two pegs at P and Q, where $XP = 20$ cm and $XQ = 60$ cm.



a Calculate the normal components of the forces on the rod at P and at Q. ✓ **(8 marks)**

The coefficient of friction between the rod and each peg is μ.

b Given that the rod is about to slip, find μ. **(2 marks)**

m(Q) $0.4P = 0.15\cos 30 \times 20$

$P = 6.4951$

$= \underline{6.50}$ (3sf)

m(P) $0.4Q = 0.25\cos 30 \times 20$

$Q = 10.825317\ldots$

$= \underline{10.8\text{ N}}$ (3sf)

$\mu P + \mu Q = 20\sin 30$

$\mu = \dfrac{20\sin 30}{P+Q} = 0.5773$

$\approx \underline{0.58}$ (2sf)

Further Kinematics

This chapter concerns how can use **vectors to represent motion**. In the case of constant acceleration, can we still use our 'suvat' equations? And what if we have variable acceleration with expressions in terms of t?

1:: Vector equations for motion.

The velocity, $\mathbf{v}$ m s^{-1}, of a particle P at time t seconds is given by

$$\mathbf{v} = (1 - 2t)\mathbf{i} + (3t - 3)\mathbf{j}$$

(a) Find the speed of P when $t = 0$ **(3)**

(b) Find the bearing on which P is moving when $t = 2$ **(2)**

(c) Find the value of t when P is moving

(i) parallel to $\mathbf{j}$,

(ii) parallel to $(-\mathbf{i} - 3\mathbf{j})$. **(6)**

2:: Variable acceleration with vectors.

"A particle P of mass 0.8kg is acted on by a single force $\boldsymbol{F}$ N. Relative to a fixed origin O, the position vector of P at time t seconds is $\boldsymbol{r}$ metres, where

$$\boldsymbol{r} = 2t^3\boldsymbol{i} + 50t^{-\frac{1}{2}}\boldsymbol{j}, \qquad t \geq 0$$

Find (a) the speed of P when $t = 4$

(b) The acceleration of P as a vector when $t = 2$

(c) $\boldsymbol{F}$ when $t = 2$."

3:: Integration with vectors to find velocity/displacement

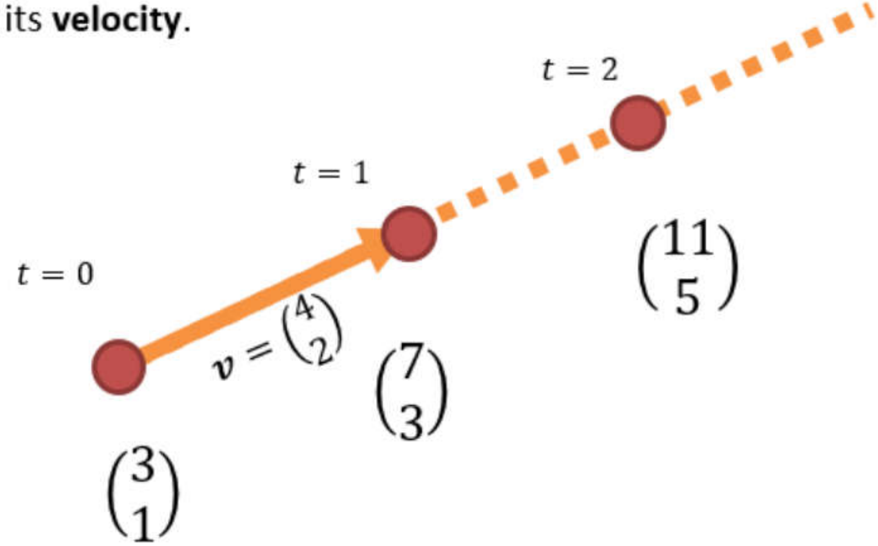
"A particle P is moving in a plane. At time t seconds, its velocity $\boldsymbol{v}$ ms^{-1} is given by $\boldsymbol{v} = 3t\boldsymbol{i} + \frac{1}{2}t^2\boldsymbol{j}$, $t \geq 0$

When $t = 0$, the position vector of P with respect to a fixed origin O is $(2\boldsymbol{i} - 3\boldsymbol{j})$ m. Find the position vector of P at time t seconds."

Vector motion

Initially, a particle is at the position vector $\begin{pmatrix}3\\1\end{pmatrix}$.

Each second, it moves $\begin{pmatrix}4\\2\end{pmatrix}$, i.e. its **velocity**.



So in general, where would the particle be after t seconds, in terms of t?

It'll be $\begin{pmatrix}3\\1\end{pmatrix}$ with t lots of $\begin{pmatrix}4\\2\end{pmatrix}$ added on, i.e.:

$$\begin{pmatrix}3\\1\end{pmatrix} + t\begin{pmatrix}4\\2\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}3+4t\\1+2t\end{pmatrix}$$

$\underline{r}_0$

✎ Position vector $\boldsymbol{r}$ of particle:

$$\boldsymbol{r} = \boldsymbol{r_0} + \boldsymbol{v}t$$

where r_0 is initial position and $\boldsymbol{v}$ is velocity.

dist = speed × time

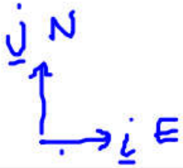
$\underline{r} = \underline{v}t$

A particle starts from the position vector $(3\boldsymbol{i} + 7\boldsymbol{j})$ m and moves with constant velocity $(2\boldsymbol{i} - \boldsymbol{j})$ ms^{-1}.

(a) Find the position vector of the particle 4 seconds later.

(b) Find the time at which the particle is due east of the origin.

NO ACCELERATION!



a) $\boxed{\underline{r} = \underline{r}_0 + \underline{v}t}$

$$\underline{r} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} 4$$

$$= \begin{pmatrix} 3 + 8 \\ 7 - 4 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 11 \\ 3 \end{pmatrix} \text{ m}$$

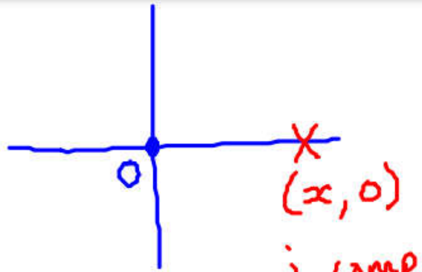
$$\underline{r} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} t$$

general position

$$= \begin{pmatrix} 3 + 2t \\ 7 - t \end{pmatrix}$$

$$7 - t = 0$$

$$\underline{\underline{t = 7}}$$



$\underline{j}$ comp $= 0$

East or West $\underline{j}$ comp $= 0$

North or South $\underline{i}$ comp $= 0$

SUVAT with but with vectors

constant acceleration

What changes?

$v = u + at$ | $\underline{v} = \underline{u} + \underline{a}t$

$s = ut + \frac{1}{2}at^2$ | $\underline{r} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$s = vt - \frac{1}{2}at^2$ | $\underline{r} = \underline{v}t - \frac{1}{2}\underline{a}t^2$

$\cancel{v^2 = u^2 + 2as}$ | NO VECTOR VERSION

$\underline{r}$ is the displacement.

$\underline{r} = \underline{r}_0 + \underline{u}t + \frac{1}{2}\underline{a}t^2$

$\underline{r}_0$ takes initial position into account.

$\underline{r} = \underline{r}_0 + \underline{v}t - \frac{1}{2}\underline{a}t^2$

A particle P has velocity $(-3\boldsymbol{i} + \boldsymbol{j})$ ms^{-1}. The particle moves with constant acceleration $\boldsymbol{a} = (2\boldsymbol{i} + 3\boldsymbol{j})$ ms^{-2}. Find (a) the speed of the particle and (b) the bearing on which it is travelling at time $t = 3$ seconds.

mag.

$\underline{u} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\underline{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$t = 3$

$\underline{v} = ?$

$\underline{v} = \underline{u} + \underline{a}t$

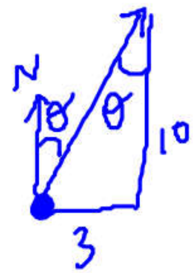
$\underline{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} 3$

$\underline{v} = \begin{pmatrix} -3+6 \\ 1+9 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$ ms^{-1}

$v = \sqrt{3^2 + 10^2} = \sqrt{109}$

$= 10.4$ (3sf) ms^{-1}

sketch



$\theta = \tan^{-1}\left(\frac{3}{10}\right)$

$= 16.69\ldots$

$= \underline{017^\circ}$

[In this question position vectors are given relative to a fixed origin O.]

Question	Scheme	Marks	AOs
1	[illegible]	B1	[illegible]
	[illegible]	M1	[illegible]
	[illegible]	A1ft	[illegible]
	[illegible]	A1	[illegible]
		(4)	
			(4 marks)

6. A particle, P, moves with constant acceleration $(\mathbf{i} - 2\mathbf{j})$ m s^{-2}.

At time t = 0 seconds, the particle is at the point A with position vector $(2\mathbf{i} + 5\mathbf{j})$ m and is moving with velocity $\mathbf{u}$ m s^{-1}.

At time t = 3 seconds, P is at the point B with position vector $(-2.5\mathbf{i} + 8\mathbf{j})$ m.

Find $\underline{\mathbf{u}}$.

(4)

$$\text{displacement} = \begin{pmatrix} -2.5 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -4.5 \\ 3 \end{pmatrix}$$

$$\underline{r} = \underline{u}t + \tfrac{1}{2}\underline{a}t^2$$

$\uparrow$ displacement

$$\underline{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \underline{r}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$t = 3$$

$$\underline{r} = \begin{pmatrix} -2.5 \\ 8 \end{pmatrix}$$

$$\underline{u} = ?$$

$$\underline{r} = \underline{r}_0 + \underline{u}t + \tfrac{1}{2}\underline{a}t^2$$

$$\begin{pmatrix} -2.5 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 3\underline{u} + \frac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix} 9$$

$$\begin{pmatrix} -4.5 \\ 3 \end{pmatrix} = 3\underline{u} + \begin{pmatrix} 4.5 \\ -9 \end{pmatrix}$$

$$\begin{pmatrix} -4.5 \\ 3 \end{pmatrix} - \begin{pmatrix} 4.5 \\ -9 \end{pmatrix} = 3\underline{u}$$

$$\begin{pmatrix} -9 \\ 12 \end{pmatrix} = 3\underline{u}$$

$$\underline{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

An ice skater is skating on a large flat ice rink. At time $t = 0$ the skater is at a fixed point O and is travelling with velocity $(2.4\boldsymbol{i} - 0.6\boldsymbol{j})$ ms^{-1}.
At time $t = 20$ s the skater is travelling with velocity $(-5.6\boldsymbol{i} + 3.4\boldsymbol{j})$ ms^{-1}.
Relative to O, the skater has position vector $\boldsymbol{s}$ at time t seconds.
Modelling the ice skater as a particle with constant acceleration, find:
(a) The acceleration of the ice skater
(b) An expression for $\boldsymbol{s}$ in terms of t
(c) The time at which the skater is directly north-east of O.
A second skater travels so that she has position vector $\boldsymbol{r} = (1.1t - 6)\boldsymbol{j}$ m relative to O at time t.
(d) Show that the two skaters will meet.

$\underline{u} = \begin{pmatrix} 2.4 \\ -0.6 \end{pmatrix}$

$t = 20$

$\underline{a} = ?$

$\underline{v} = \begin{pmatrix} -5.6 \\ 3.4 \end{pmatrix}$

$\underline{s} = ?$

a) $\underline{v} = \underline{u} + \underline{a}t$

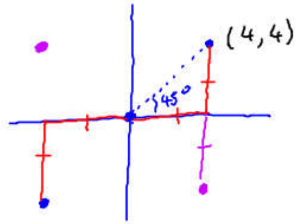
$\begin{pmatrix} -5.6 \\ 3.4 \end{pmatrix} = \begin{pmatrix} 2.4 \\ -0.6 \end{pmatrix} + 20\underline{a}$

$\begin{pmatrix} -8 \\ 4 \end{pmatrix} = 20\underline{a}$

$\underline{a} = \begin{pmatrix} -0.4 \\ 0.2 \end{pmatrix}$ ms^{-2}

SE / NW: $\underline{i}$ comp $= -\underline{j}$ comp; $-\underline{i}$ comp $= \underline{j}$ comp

NE / SW: $\underline{i}$ comp $= \underline{j}$ comp



b) $\underline{s} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$\underline{s} = \begin{pmatrix} 2.4 \\ -0.6 \end{pmatrix} t + \begin{pmatrix} -0.2 \\ 0.1 \end{pmatrix} t^2$

c) $\underline{s} = \begin{pmatrix} 2.4t - 0.2t^2 \\ -0.6t + 0.1t^2 \end{pmatrix}$

$2.4t - 0.2t^2 = -0.6t + 0.1t^2$

$0 = 0.3t^2 - 3t$

$0 = t(0.3t - 3)$

$\cancel{t=0}$ $\quad t = \frac{3}{0.3} = \underline{\underline{10}}$

d) $\underline{r} = \begin{pmatrix} 0 \\ 1.1t - 6 \end{pmatrix}$ $\quad \underline{s} = \begin{pmatrix} 2.4t - 0.2t^2 \\ -0.6t + 0.1t^2 \end{pmatrix}$

same position and same time.

$\underline{r} = \underline{s}$ $\quad \underline{i}$ comp $\quad 2.4t - 0.2t^2 = 0$

$t(2.4 - 0.2t) = 0$

$\underline{\underline{t = 0}}$ $\quad t = \frac{2.4}{0.2} = \underline{\underline{12 \text{ seconds}}}$

$t = 12$

$\underline{r} = \begin{pmatrix} 0 \\ 1.1 \times 12 - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 7.2 \end{pmatrix}$

$\underline{s} = \begin{pmatrix} 0 \\ -0.6 \times 12 + 0.1 \times 12^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7.2 \end{pmatrix}$

They have the same position at the same time, so they meet.

8. [*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively*]

A radio controlled model boat is placed on the surface of a large pond.

The boat is modelled as a particle.

At time $t = 0$, the boat is at the fixed point O and is moving due north with speed 0.6 m s^{-1}. $\underline{u} = \begin{pmatrix} 0 \\ 0.6 \end{pmatrix}$

Relative to O, the position vector of the boat at time t seconds is **r** metres.

At time $t = 15$, the velocity of the boat is $(10.5\mathbf{i} - 0.9\mathbf{j})$ m s^{-1}. $\underline{v} = \begin{pmatrix} 10.5 \\ -0.9 \end{pmatrix}$ $t = 15$.

The acceleration of the boat is constant.

(a) Show that the acceleration of the boat is $(0.7\mathbf{i} - 0.1\mathbf{j})$ m s^{-2}.

(2)

(b) Find **r** in terms of t.

(2)

(c) Find the value of t when the boat is north-east of O.
position

(3)

(d) Find the value of t when the boat is moving in a north-east direction.
velocity

(3)

a)

$\underline{u} = \begin{pmatrix} 0 \\ 0.6 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} 10.5 \\ -0.9 \end{pmatrix}$

$\underline{a} = ?$

$t = 15$

$\underline{v} = \underline{u} + \underline{a}t$

$\begin{pmatrix} 10.5 \\ -0.9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.6 \end{pmatrix} + 15\underline{a}$

$\begin{pmatrix} 10.5 \\ -1.5 \end{pmatrix} = 15\underline{a}$

$\underline{a} = \begin{pmatrix} 0.7 \\ -0.1 \end{pmatrix}$ ms^{-2}

b) $\underline{r} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$\underline{r} = \begin{pmatrix} 0 \\ 0.6 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0.7 \\ -0.1 \end{pmatrix} t^2$

$\underline{r} = \begin{pmatrix} 0.35t^2 \\ 0.6t - 0.05t^2 \end{pmatrix}$

Ex 8A Evens

c) NE when $\underline{i} = \underline{j}$ for $\underline{r}$

$0.35t^2 = 0.6t - 0.05t^2$

$0.4t^2 - 0.6t = 0$

$t(0.4t - 0.6) = 0$

$t = 1.5$.

d) $\underline{v} = \underline{u} + \underline{a}t$

$= \begin{pmatrix} 0 \\ 0.6 \end{pmatrix} + \begin{pmatrix} 0.7 \\ -0.1 \end{pmatrix} t$

$\underline{v} = \begin{pmatrix} 0.7t \\ 0.6 - 0.1t \end{pmatrix}$

So $0.7t = 0.6 - 0.1t$

$0.8t = 0.6$

$t = 0.75$.

Question	Scheme	Marks	AOs
8(a)	Use of $\mathbf{v} = \mathbf{u} + \mathbf{a}t$: $(10.5\mathbf{i} - 0.9\mathbf{j}) = 0.6\mathbf{j} + 15\mathbf{a}$	M1	3.1b
	$\mathbf{a} = (0.7\mathbf{i} - 0.1\mathbf{j})$ m s^{-2} Given answer	A1	1.1b
		(2)	
(b)	Use of $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$	M1	3.1b
	$\mathbf{r} = 0.6\mathbf{j}\,t + \frac{1}{2}(0.7\mathbf{i} - 0.1\mathbf{j})\,t^2$	A1	1.1b
		(2)	
(c)	Equating the **i** and **j** components of **r**	M1	3.1b
	$\frac{1}{2} \leftarrow 0.7\,t^2 = 0.6\,t - \frac{1}{2} \leftarrow 0.1\,t^2$	A1ft	1.1b
	$t = 1.5$	A1	1.1b
		(3)	
(d)	Use of $\mathbf{v} = \mathbf{u} + \mathbf{a}t$: $\mathbf{v} = 0.6\mathbf{j} + (0.7\mathbf{i} - 0.1\mathbf{j})\,t$	M1	3.1b
	Equating the **i** and **j** components of **v**	M1	3.1b
	$t = 0.75$	A1 **ft**	1.1b
		(3)	
			(10 marks)