Integration with Parametric Equations

Suppose we have the following parametric equations:

$$x = t^2$$
$$y = t + 1$$

$\int (t+1)\, dx$

To find the area under the curve, we want to determine to determine $\int y\, dx$.
The problem however is that y is in terms of t, not in terms of x.

Area: $$\int y\, dx = \int y \frac{dx}{dt}\, dt$$

Determine the area bound between the curve with parametric equations $x = t^2$ and $y = t + 1$, the x-axis, and the lines $x = 0$ and $x = 3$.

$3 = t^2$

$\int_0^3 y\, dx$

$x = t^2$

$\frac{dx}{dt} = 2t$

$y = t+1$

x	t
3	$\sqrt{3}$
0	0

STEP 1: Find $\frac{dx}{dt}$

STEP 2: Change limits

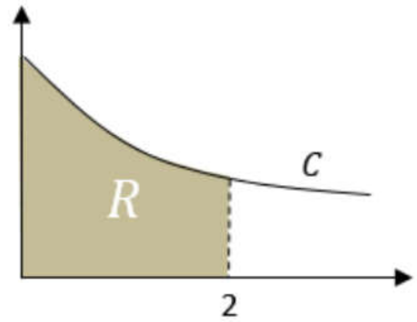
STEP 3: Integrate

$$\int_0^3 y\, dx = \int_0^{\sqrt{3}} (t+1)\, 2t\, dt$$
$$= \int_0^{\sqrt{3}} (2t^2 + 2t)\, dt = \left[\tfrac{2}{3}t^3 + t^2\right]_0^{\sqrt{3}}$$
$$= \tfrac{2}{3} \times (\sqrt{3})^3 + (\sqrt{3})^2$$
$$= 2\sqrt{3} + 3$$

The curve C has parametric equations

$$x = t(1+t), \qquad y = \frac{1}{1+t}, \qquad t \geq 0$$

Find the exact area of the region R, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.



$= t + t^2$

$\int y \frac{dx}{dt}\, dt$

$x = t(1+t)$

$\frac{dx}{dt} = 1 + 2t$

$y = \frac{1}{1+t}$

x	t
0	0
2	1

$\int_0^2 y\, dx = \int_0^1 \frac{1}{(1+t)} \times (1+2t)\, dt$

$= \int_0^1 \frac{1+2t}{1+t}\, dt$

$= \int_0^1 \left(2 - \frac{1}{t+1}\right) dt$

$= \left[2t - \ln|t+1|\right]_0^1$

$= (2 - \ln 2) - (0 - \ln 1)$

$= \underline{\underline{2 - \ln 2}}$

$$\begin{array}{r} 2 \\ t+1 \;\big)\; 2t+1 \\ 2t+2 \\ \hline -1 \end{array}$$

$x = 0 \quad 0 = t(1+t)$

$t = 0, \cancel{t = -1}$

$x = 2 \quad 2 = t(1+t)$

$2 = t + t^2$

$0 = t^2 + t - 2$

$0 = (t+2)(t-1)$

$t = 1, \cancel{t = -2}$

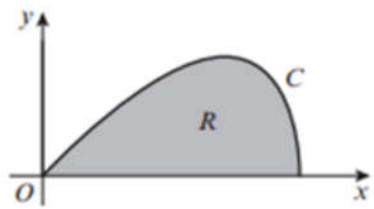
(P) **1** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

$k\sqrt[3]{2} \rightarrow k \times 2^{1/3}$

(E/P) **2** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \leqslant t \leqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis.
Find the exact area of R. **(6 marks)**

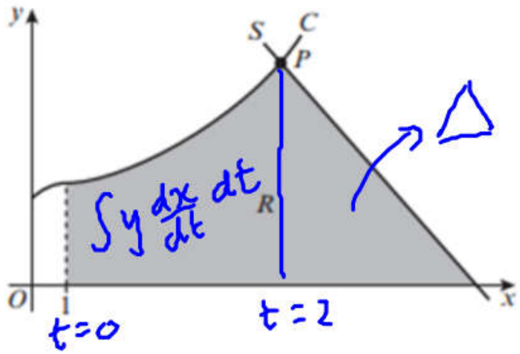


(E/P) **3** This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$
P is the point on the curve where $t = 2$.
The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**

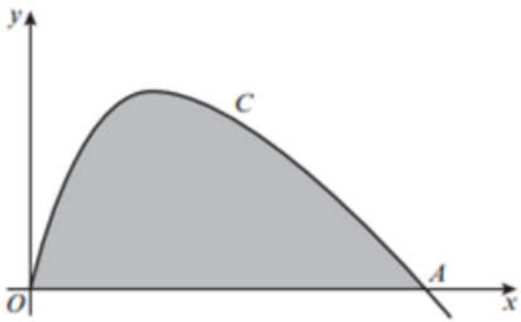


(E/P) **4** The diagram shows the curve C with parametric equations

$$x = 3t^2,\ y = \sin 2t,\ t \geqslant 0.$$

a Write down the value of t at the point A where the curve crosses the x-axis. **(1 mark)**

b Find, in terms of π, the exact area of the shaded region bounded by C and the x-axis. **(6 marks)**



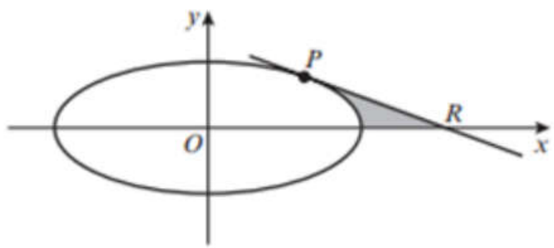
(E/P) **5** The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$

a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$ **(3 marks)**

b Find an equation of the tangent to the curve at the point P. **(3 marks)**

c Find the exact area of the shaded region bounded by the tangent PR, the curve and the x-axis. **(6 marks)**



$\frac{3}{5} \times 4^{5/3}$

$4^{5/3} = (2^2)^{5/3}$

$= 2^{10/3}$

$= 2^{9/3} \times 2^{1/3}$

$= 2^3 \times 2^{1/3}$

$= 8 \times 2^{1/3}$

$\frac{3}{5} \times 8 \times 2^{1/3}$

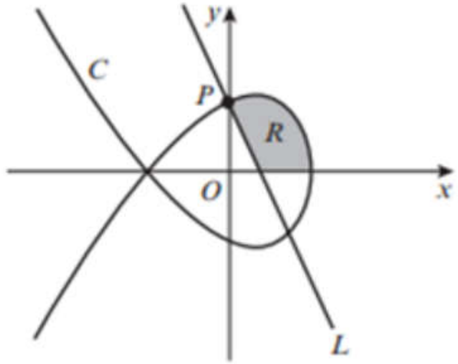
$\frac{24}{5}\sqrt[3]{2}$

$k = \frac{24}{5}$

E/P **6** The curve C has parametric equations

$$x = 1 - t^2,\ y = 2t - t^3,\ t \in \mathbb{R}$$

The line L is a normal to the curve at the point P where the curve intersects the positive y-axis. Find the exact area of the region R bounded by the curve C, the line L and the x-axis, as shown on the diagram. **(7 marks)**

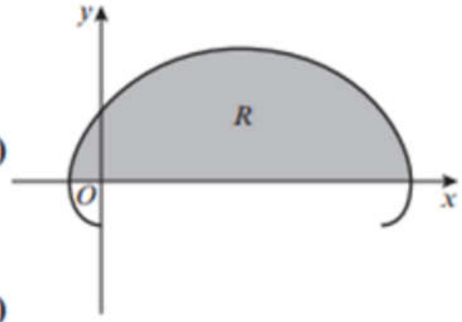


E/P **7** The curve shown in the diagram has parametric equations

$$x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \leqslant t \leqslant 2\pi$$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ **(3 marks)**

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the diagram.



b Show that the area R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, dt$ **(3 marks)**

c Use this integral to find the exact value of the shaded area. **(4 marks)**

ANSWERS

1 Area $= \int y \frac{dx}{dt} dt = \int_0^{\sqrt[3]{4}} t^2(3t^2)\ dt = \frac{3}{5}\left(\sqrt[3]{4}\right)^5 = \frac{3}{5}2^{\frac{10}{3}}$

$= \frac{3}{5}(2^3)(2^{\frac{1}{3}}) = \frac{24}{5}\sqrt[3]{2}$

2 $\frac{2}{3}$

3 **a** $x + y = 16$ **b** 61.85

4 **a** $\frac{\pi}{2}$ **b** $\frac{3\pi}{2}$

5 **a** $-\frac{4}{5}$ **b** $y - 2\sqrt{2} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$

c $10 - \frac{5\pi}{2}$

6 $\frac{41}{60}$

7 **a** $2\cos t = 1 \Rightarrow \cos t = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}$ or $t = \frac{5\pi}{3}$

b $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y \frac{dx}{dt}\, dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)(1 - 2\cos t)\, dt$

$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 dt$

c $4\pi + 3\sqrt{3}$