Dealing with $(a + bx)^n$

Find first four terms in the binomial expansion of $\sqrt{4 + x}$
State the values of x for which the expansion is valid.

$$(4+x)^{1/2} = \left[4\left(1+\frac{x}{4}\right)\right]^{1/2}$$

$$= 4^{1/2}\left(1+\frac{x}{4}\right)^{1/2}$$

$$= 2\underline{\left(1+\frac{x}{4}\right)}^{1/2} \checkmark$$

We need it in the form $(1 + x)^n$. So factorise the 4 out.

$$2\left(1+\frac{x}{4}\right)^{1/2} = 2\left(1 + \frac{1}{2}\left(\frac{x}{4}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x}{4}\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{4}\right)^3\right)$$

$n = \frac{1}{2}$

"x" $\to \frac{x}{4}$

$$= 2\left(1 + \frac{1}{8}x - \frac{1}{128}x^2 + \frac{1}{1024}x^3\right)$$

$$\approx 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$$

$$\left|\frac{x}{4}\right| < 1$$

valid for $\underline{|x| < 4}$

Just the First Step

What would be the first step in finding the Binomial expansion of each of these?

Binomial expansion valid if:

$$\left[2\left(1+\frac{x}{2}\right)\right]^{-3}$$

$(2+x)^{-3} = 2^{-3}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$ $\quad \left|\frac{x}{2}\right| < 1 \rightarrow \underline{|x| < 2}$

$(9+2x)^{\frac{1}{2}} = 9^{1/2}\left(1+\frac{2}{9}x\right)^{1/2} = 3\left(1+\frac{2}{9}x\right)^{1/2}$ $\quad \left|\frac{2}{9}x\right| < 1 \rightarrow \underline{|x| < \frac{9}{2}}$

$(8-x)^{\frac{1}{3}} = 8^{1/3}\left(1-\frac{x}{8}\right)^{1/3} = 2\left(1-\frac{x}{8}\right)^{1/3}$ $\quad \left|-\frac{x}{8}\right| < 1 \rightarrow \underline{|x| < 8}$

$(5-2x)^{-3} = 5^{-3}\left(1-\frac{2}{5}x\right)^{-3} = \frac{1}{125}\left(1-\frac{2}{5}x\right)^{-3}$ $\quad \left|-\frac{2}{5}x\right| < 1 \rightarrow \underline{|x| < \frac{5}{2}}$

$(16+3x)^{-\frac{1}{2}} = 16^{-1/2}\left(1+\frac{3}{16}x\right)^{-1/2} = \frac{1}{4}\left(1+\frac{3}{16}x\right)^{-1/2}$ $\quad \left|\frac{3}{16}x\right| < 1 \rightarrow \underline{|x| < \frac{16}{3}}$

7. (a) Use the binomial expansion, in ascending powers of x, to show that

$$\sqrt{(4-x)} = 2 - \frac{1}{4}x + kx^2 + \ldots$$

where k is a rational constant to be found.

(4)

A student attempts to substitute $x = 1$ into both sides of this equation to find an approximate value for $\sqrt{3}$.

(b) State, giving a reason, if the expansion is valid for this value of x.

(1)

a)

$(4-x)^{1/2} = 4^{1/2}\left(1-\frac{x}{4}\right)^{1/2}$ $\quad n=\frac{1}{2} \quad x \to \left(-\frac{x}{4}\right)$

$= 2\left(1-\frac{x}{4}\right)^{1/2}$

$= 2\left(1+\frac{1}{2}\left(-\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2\right)$

$= 2\left(1 - \frac{1}{8}x - \frac{1}{128}x^2\right)$

$= 2 - \frac{1}{4}x - \frac{1}{64}x^2$ $\quad k = -\frac{1}{64}$

b) Valid for $\left|\frac{x}{4}\right| < 1$

$|x| < 4$

$1 < 4$ so

expansion will be valid.

Ex 4B

Question	Scheme	Marks	AOs
7(a)	$\sqrt{(4-x)} = 2\left(1-\frac{1}{4}x\right)^{\frac{1}{2}}$	M1	2.1
	$\left(1-\frac{1}{4}x\right)^{\frac{1}{2}} = 1+\frac{1}{2}\left(-\frac{1}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{1}{4}x\right)^2+\dots$	M1	1.1b
	$\sqrt{(4-x)} = 2\left(1-\frac{1}{8}x-\frac{1}{128}x^2+..\right)$	A1	1.1b
	$\sqrt{(4-x)} = 2-\frac{1}{4}x-\frac{1}{64}x^2+\dots$ and $k=-\frac{1}{64}$	A1	1.1b
		(4)	
(b)	The expansion is valid for $\lvert x\rvert < 4$, so $x=1$ can be used	B1	2.4
		(1)	
		(5 marks)	

2. (a) Show that the binomial expansion of

$$(4 + 5x)^{\frac{1}{2}}$$

in ascending powers of x, up to and including the term in x^2 is

$$2 + \frac{5}{4}x + kx^2$$

giving the value of the constant k as a simplified fraction.

(4)

(b) (i) Use the expansion from part (a), with $x = \frac{1}{10}$, to find an approximate value for $\sqrt{2}$

Give your answer in the form $\frac{p}{q}$ where p and q are integers.

(ii) Explain why substituting $x = \frac{1}{10}$ into this binomial expansion leads to a valid approximation.

(4)

a) $(4+5x)^{1/2} = 4^{1/2}\left(1+\frac{5}{4}x\right)^{1/2}$

$= 2\left(1+\frac{5}{4}x\right)^{1/2}$ $\quad n=\frac{1}{2}$ $\quad$ "x" $= \frac{5}{4}x$

$= 2\left(1 + \left(\frac{1}{2}\right)\left(\frac{5}{4}x\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{5}{4}x\right)^2\right)$

$$= 2\left(1 + \frac{5}{8}x - \frac{25}{128}x^2\right)$$

$$= 2 + \frac{5}{4}x - \frac{25}{64}x^2 \qquad k = -\frac{25}{64}$$

6i) $x = \frac{1}{10}$ $\qquad (4+5x)^{1/2} = 2 + \frac{5}{4}x - \frac{25}{64}x^2$

LHS $\quad x = \frac{1}{10}$

$$\left(4+\frac{1}{2}\right)^{1/2} = \left(\frac{9}{2}\right)^{1/2} = \sqrt{\frac{9}{2}} = \frac{\sqrt{9}}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

RHS $\quad x = \frac{1}{10}$ $\qquad 2 + \frac{5}{4} \times \frac{1}{10} - \frac{25}{64} \times \left(\frac{1}{10}\right)^2 = \frac{543}{256}$

$$\frac{3\sqrt{2}}{2} = \frac{543}{256}$$

$$1.414 \longleftarrow \sqrt{2} = \frac{181}{128} \longrightarrow 1.414$$

$$\Rightarrow \qquad p = 181 \quad q = 128$$

ii) Valid if $\left|\frac{5}{4}x\right| < 1$

$|x| < \frac{4}{5}$ $\qquad \frac{1}{10} < \frac{4}{5}$, so it is valid

Question	Scheme	Marks	AOs
2 (a)	$(4+5x)^{\frac{1}{2}} = (4)^{\frac{1}{2}}\left(1+\frac{5x}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{5x}{4}\right)^{\frac{1}{2}}$	B1	1.1b
	$= \{2\}\left[1+\left(\frac{1}{2}\right)\left(\frac{5x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{5x}{4}\right)^2+\ldots\right]$	M1	1.1b
		A1ft	1.1b
	$= 2+\frac{5}{4}x-\frac{25}{64}x^2+\ldots$	A1	2.1
		(4)	
(b)(i)	$\left\{x=\frac{1}{10}\Rightarrow\right\}(4+5(0.1))^{\frac{1}{2}}$	M1	1.1b
	$=\sqrt{4.5} = \frac{3}{2}\sqrt{2}$ or $\frac{3}{\sqrt{2}}$		
	$\frac{3}{2}\sqrt{2}$ or $1.5\sqrt{2}$ or $\frac{3}{\sqrt{2}} = 2+\frac{5}{4}\left(\frac{1}{10}\right)-\frac{25}{64}\left(\frac{1}{10}\right)^2+\ldots$ $\{=2.121\ldots\}$ $\Rightarrow \frac{3}{2}\sqrt{2}=\frac{543}{256}$ or $\frac{3}{\sqrt{2}}=\frac{543}{256}$ $\Rightarrow \sqrt{2}=\ldots$	M1	3.1a
	So, $\sqrt{2}=\frac{181}{128}$ or $\sqrt{2}=\frac{256}{181}$	A1	1.1b
(b)(ii)	$x=\frac{1}{10}$ satisfies $\|x\|<\frac{4}{5}$ (o.e.), so the approximation is valid.	B1	2.3
		(4)	
			(8 marks)

6. $\mathrm{f}(x) = (2 + kx)^{-4}$ where k is a positive constant

The binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^2, is

$$\frac{1}{16} + Ax + \frac{125}{32}x^2$$

where A is a constant.

(a) Find the value of A, giving your answer in simplest form.

(5)

(b) Determine, giving a reason for your answer, whether the binomial expansion for $\mathrm{f}(x)$ is valid when $x = \frac{1}{10}$

(1)

6a) $(2+kx)^{-4} = 2^{-4}\left(1 + \frac{kx}{2}\right)^{-4}$

$= \frac{1}{16}\left(1 + \frac{kx}{2}\right)^{-4}$ $n = -4$

"x" $= \frac{kx}{2}$

$= \frac{1}{16}\left(1 + (-4)\left(\frac{kx}{2}\right) + \frac{(-4)(-5)}{2}\left(\frac{kx}{2}\right)^2\right)$

$= \frac{1}{16}\left(1 - 2kx + \frac{5}{2}k^2x^2\right)$

$= \frac{1}{16} - \frac{k}{8}x + \frac{5}{32}k^2x^2$

Compare coefficients $\frac{5}{32}k^2 = \frac{125}{32}$

$k^2 = 25$

$k = \pm 5$ but $k > 0$, $k = 5$

$A = -\frac{k}{8} = -\frac{5}{8}$

b) $\left(1 + \frac{5}{2}x\right)^{-4}$ $\left|\frac{5}{2}x\right| < 1$

$|x| < \frac{2}{5}$ $\frac{1}{10} < \frac{2}{5}$

So expansion is valid for $x = \frac{1}{10}$

Question	Scheme	Marks	AOs
6	$\left\{(2+kx)^{-4} = 2^{-4}\left(1+\frac{kx}{2}\right)^{-4} = \frac{1}{16}\left(1+(-4)\left(\frac{kx}{2}\right)+\frac{(-4)(-5)}{2!}\left(\frac{kx}{2}\right)^2+\dots\right)\right\}$		
(a)	For the x^2 term: $\left(\frac{1}{16}\right)\frac{(-4)(-5)}{2!}\left(\frac{k}{2}\right)^2 \quad \left\{=\frac{5}{32}k^2\right\}$	M1	1.1b
		A1	1.1b
	$\frac{1}{16}\frac{(-4)(-5)}{2!}\left(\frac{k}{2}\right)^2 = \frac{125}{32} \Rightarrow \frac{5}{32}k^2 = \frac{125}{32} \Rightarrow k^2 = 25 \Rightarrow k = \dots \Rightarrow A = \dots$	dM1	3.1a
	$\left\{A = -\frac{4}{32}k \Rightarrow\right\} \ A = -\frac{4}{32}(5)$	M1	2.2a
	$A = -\frac{5}{8}$ or -0.625	A1	1.1b
		(5)	
(b)	$\mathrm{f}(x)$ is valid when $\left\lvert\frac{kx}{2}\right\rvert < 1 \Rightarrow \left\lvert\frac{5x}{2}\right\rvert < 1 \Rightarrow \lvert x\rvert < \frac{2}{5}$		
	E.g. • As $x = \frac{1}{10}$ lies in the interval $\lvert x\rvert < \frac{2}{5}$, the binomial expansion is valid • As $\left\lvert\left(\frac{5}{2}\right)\left(\frac{1}{10}\right)\right\rvert = \frac{1}{4} < 1$, the binomial expansion is valid	B1ft	2.3
		(1)	
			(6 marks)

4. (a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$\frac{1}{\sqrt{4-x}}$$

giving each coefficient in its simplest form.

(4)

The expansion can be used to find an approximation to $\sqrt{2}$
Possible values of x that could be substituted into this expansion are:

$\left|\frac{x}{4}\right| < 1$

$|x| < 4$

- $x = -14$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{18}} = \dfrac{\sqrt{2}}{6}$

- $x = 2$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

- $x = -\dfrac{1}{2}$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{\frac{9}{2}}} = \dfrac{\sqrt{2}}{3}$

(b) Without evaluating your expansion,

(i) state, giving a reason, which of the three values of x should not be used

$x = -14$, because expansion is only valid for $|x| < 4$ **(1)**

(ii) state, giving a reason, which of the three values of x would lead to the most accurate approximation to $\sqrt{2}$

(1)

$x = -\frac{1}{2}$ because it is the smallest value so gives most accurate.

Question 4 (Total 6 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\frac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$	M1	This mark is given for rearranging $\frac{1}{\sqrt{4-x}}$ to attempt a binomial expansion
	$\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} =$	M1	This mark is given for an attempt at a binomial expansion
	$1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)}{2}\left(-\frac{x}{4}\right)$	A1	This mark is given for a fully correct binomial expansion
	$\frac{1}{\sqrt{4-x}} = \frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2$	A1	This mark is given for a fully correct expansion with the first three terms
(b)(i)	$x = -14$, since the expansion is only valid for $\lvert x\rvert < 4$	B1	This mark is given for the correct value chosen with a correct reason
(b)(ii)	$x = -\frac{1}{2}$, since the smaller value will give the more accurate approximation	B1	This mark is given for the correct value chosen with a correct reason