Chapter 12: Differentiation

1:: Find the derivative of polynomials.

If $y = 3x^2 + \sqrt{x}$, find $\frac{dy}{dx}$

2:: Find equations of tangents and normal to curves.

The point $P(3,9)$ lies on the curve C with equation $y = x^2$. Determine the equation of the tangent to C at the point P.

3:: Identify increasing and decreasing functions.

Find the range of values for which $f(x) = x^3 - x$ is increasing.

4:: Find and understand the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$

If $y = x^4 - 3x^2$ determine $\frac{d^2y}{dx^2}$

5:: Find stationary points and determine their nature.

Find the stationary points of $y = x^3 - x$ and state whether each is a maximum or minimum point.

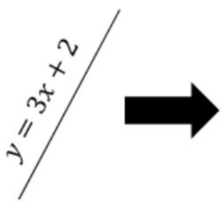
6:: Sketch a gradient function.

Draw $y = x^3$ and its gradient function on the same axes.

7:: Model real-life problems.

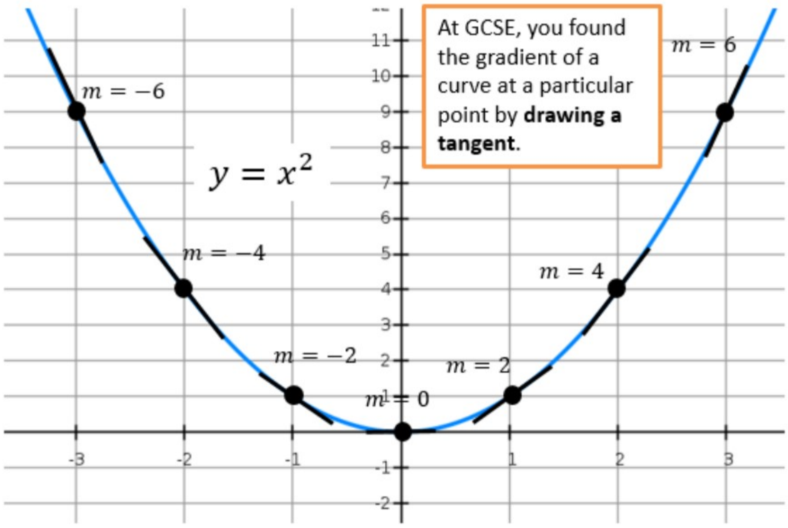
Gradient Function

For a straight line, the gradient is **constant**:



$m = 3$

However, for a curve **the gradient varies**. We can no longer have a single value for the gradient; **we ideally want an expression in terms of x** that gives us the gradient for any value of x (unsurprisingly known as the **gradient function**).



x	-3	-2	-1	0	1	2	3
Gradient	-6	-4	-2	0	2	4	6

By looking at the relationship between x and the gradient at that point, can you come up with an expression, in terms of x for the gradient?

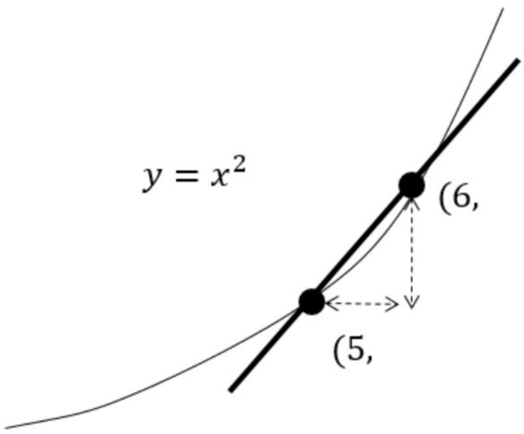
$$Gradient\ Function =$$

Finding the Gradient Function

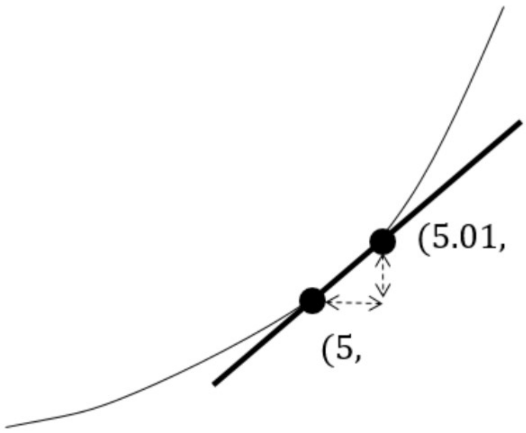
The question is then: is there a method to work out the gradient function without having to draw lots of tangents and hoping that we can spot the rule?

To approximate the gradient on the curve $y = x^2$ when $x = 5$, we could pick a point on the curve just slightly to the right, then find the gradient between the two points:

As the second point gets closer and closer, the gradient becomes a better approximation of the true gradient:

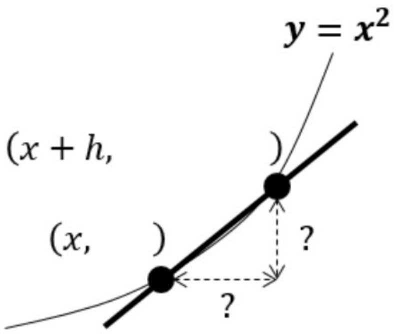


$$m = \frac{\Delta y}{\Delta x} =$$



$$m =$$

This gives us a **numerical method** to get the gradient **at a particular x**, but doesn't give us the gradient function **in general**. Let's use exactly the same method, but keep x general, and make the 'small change' (which was previously 0.01) 'h':



$$gradient =$$

The lim means "*the limit of the following expression as h tends towards 0*".
For example, $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$, because as x "tends towards" infinity, the "limiting" value of the expression is 0.

The h disappears as h tends towards 0, i.e. we can effectively treat it as 0 at this point.

Ex 12A

Differentiation by first principles

✏ The gradient function, or derivative, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

We will soon see an easier/quicker way to differentiate expressions like $y = x^3 - x$ without using 'limits'. But this method, known as **differentiating by first principles**,

Notation Note:
Whether we use $\frac{dy}{dx}$ or $f'(x)$ for the gradient function depends on whether we use $y =$ or $f(x) =$ to start with:

"Leibniz's notation"

$$y = x^2 \quad \rightarrow \quad \frac{dy}{dx} = 2x$$

$$f(x) = x^2 \quad \rightarrow \quad f'(x) = 2x$$

"Lagrange's notation"

There's in fact a third way to indicate the gradient function, notation used by Newton: (but not used at A Level)

$$y = x^2 \quad \rightarrow \quad \dot{y} = 2x$$

Advanced Notation Note:
Rather than h for the small change in x, the formal notation is δx. So actually:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

So we in fact have 3 symbols for "change in"!

- Δx: any change in x (as seen in Chp5: $m = \frac{\Delta y}{\Delta x}$)
- δx: a small change in x
- dx: an infinitesimally small change in x

So the estimated gradient using some point close by was $\frac{\delta y}{\delta x}$, but in the 'limit' as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$

The point A with coordinates $(4,16)$ lies on the curve with equation $y = x^2$. At point A the curve has gradient g.

a) Show that $g = \lim_{h \to 0}(8 + h)$

b) Deduce the value of g.

Prove **from first principles** that the derivative of x^4 is $4x^3$.

Ex 12B

Differentiating x^n

Thankfully, there's a quick way to differentiate terms of the form x^n (where n is a constant) with having to use first principles every time:

If $y = ax^n$ then $\frac{dy}{dx} = nax^{n-1}$ (where a, n are constants)
i.e. multiply by the power then reduce the power by 1

$y = x^5$

$y = x^7$

$y = 3x^{10}$

$y = 2x^6$

$f(x) = x^{\frac{1}{2}}$

$y = ax^a$

$y = \sqrt{x^6}$

$f(x) = \sqrt{49x^7}$

$f(x) = \dfrac{x}{x^4}$

$f(x) = \dfrac{x^{\frac{1}{2}}}{8x^2}$

Ex 12C

Differentiating Multiple Terms

If $y = f(x) + g(x)$ then $\frac{dy}{dx} = f'(x) + g'(x)$
i.e. **differentiate each term individually in a <u>sum/subtraction</u>**.

Differentiate $y = x^2 + 4x + 3$

Alternatively, if you compare $y = 4x$ to $y = mx + c$, it's clear that the gradient is fixed and $m = 4$.

Alternatively, if you sketch $y = 4$, the line is horizontal, so the gradient is 0.

$y = 2x^2 - 3x$

$y = 4 - 9x^3$

$y = 5x + 1$

$y = ax$
(where a is a constant)

$y = 6x - 3 + px^2$
(where p is a constant)

2. The curve C has equation

$$y = 2x^2 - 12x + 16$$

Find the gradient of the curve at the point P (5, 6).

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(4)

Let $f(x) = 4x^2 - 8x + 3$

a) Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8.

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$.

Let $f(x) = x^2 - 4x + 2$

a) Find the gradient of $y = f(x)$ at the point $(1, -1)$

b) Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 5.

c) Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 2 - x$.

Differentiating Harder Expressions

If your expression isn't a sum of x^n terms, simply manipulate it until it is!

1. Turn roots into powers:

$$y = \sqrt{x}$$

$$y = \frac{1}{\sqrt[3]{x}}$$

2. Split the numerator.

$$y = \frac{x^2 + 3}{\sqrt{x}}$$

3. Expand out brackets.

$$y = x^2(x - 3)$$

4. Beware of numbers in denominators!

$$y = \frac{1}{3x}$$

Differentiate the following.

$$y = \frac{1}{\sqrt{x}}$$

$$y = \frac{2 + x^3}{x^2}$$

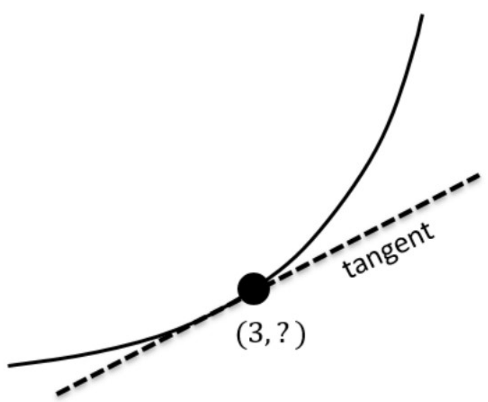
$$y = \frac{1 + 2x}{3x\sqrt{x}}$$

Ex 12E

Finding equations of tangents

Find the equation of the **tangent** to the curve $y = x^2$ when $x = 3$.

We want to use $y - y_1 = m(x - x_1)$ for the tangent (as it is a straight line!).
Therefore we need:
(a) A point (x_1, y_1)
(b) The gradient m.



Gradient function:

Gradient when $x = 3$:

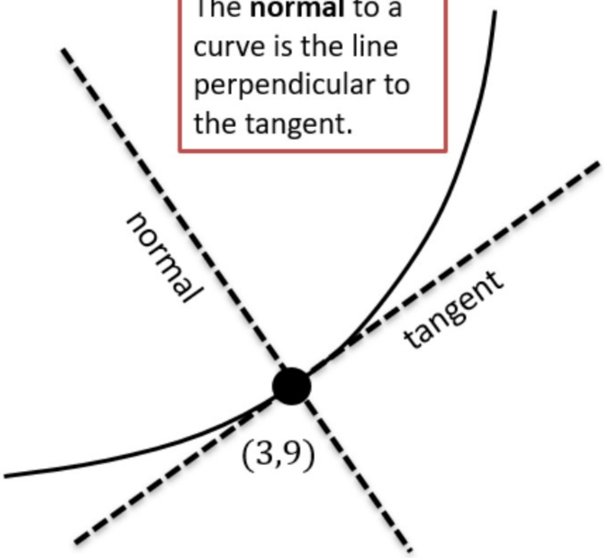
y-value when $x = 3$:

So equation of tangent:

Finding equations of normals

Find the equation of the **normal** to the curve $y = x^2$ when $x = 3$.

The **normal** to a curve is the line perpendicular to the tangent.



Equation of tangent (from earlier):

$$\boldsymbol{y - 9 = 6(x - 3)}$$

Therefore equation of normal:

Exam Tip: A **very common error** is for students to accidentally forget whether the question is asking for the tangent or for the normal.

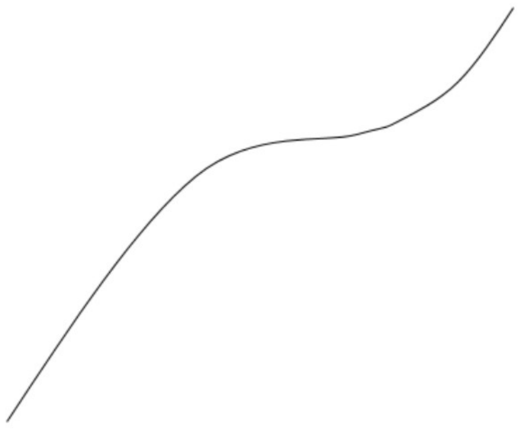
Find the equation of the **normal** to the curve $y = x + 3\sqrt{x}$ when $x = 9$.

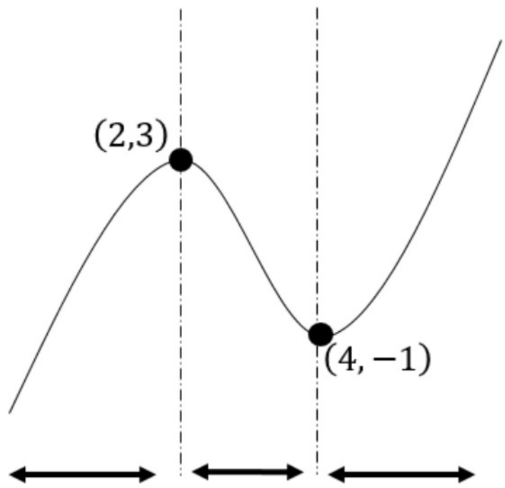
Find the equation of the **tangent** to the curve $y = x^3 - 3x^2 + 2x - 1$ when $x = 3$.

Ex 12F

Increasing and Decreasing Functions

A function can also be increasing and decreasing in certain intervals.





✐ An increasing function is one whose gradient is always at least 0.
$f'(x) \geq 0$ for all x.

It would be '**strictly increasing**' if $f(x) > 0$ for all x, i.e. is not allowed to go horizontal.

$[a, b]$ represents all the real numbers between a and b inclusive, i.e:
$[a, b] = \{x : a \leq x \leq b\}$

Show that the function
$f(x) = x^3 + 6x^2 + 21x + 2$ is increasing for all real values of x.

Find the interval on which the function $f(x) = x^3 + 3x^2 - 9x$ is decreasing.

Second Order Derivatives

When you differentiate once, the expression you get is known as the **first derivative**. Unsurprisingly, when we differentiate a second time, the resulting expression is known as the **second derivative**. And so on…

Original Function	First Derivative	Second Derivative
	Leibniz's: $\frac{dy}{dx} = 4x^3$	$\frac{d^2y}{dx^2} = 12x^2$
$y = x^4$	$y' = 4x^3$	$y'' = 12x^2$
	Newton's: $\dot{y} = 4x^3$	$\ddot{y} = 12x^2$

Lagrange's

$$f(x) = x^4 \longrightarrow f'(x) = 4x^3 \longrightarrow f''(x) = 12x^2$$

You can similarly have the third derivative ($\frac{d^3y}{dx^3}$), although this is no longer in the A Level syllabus. We'll see why might use the second derivative soon…

How does the notation $\frac{d^2y}{dx^2}$ work?

Why are the 'squareds' where they are?

If $y = 3x^5 + \frac{4}{x^2}$, find $\frac{d^2y}{dx^2}$.

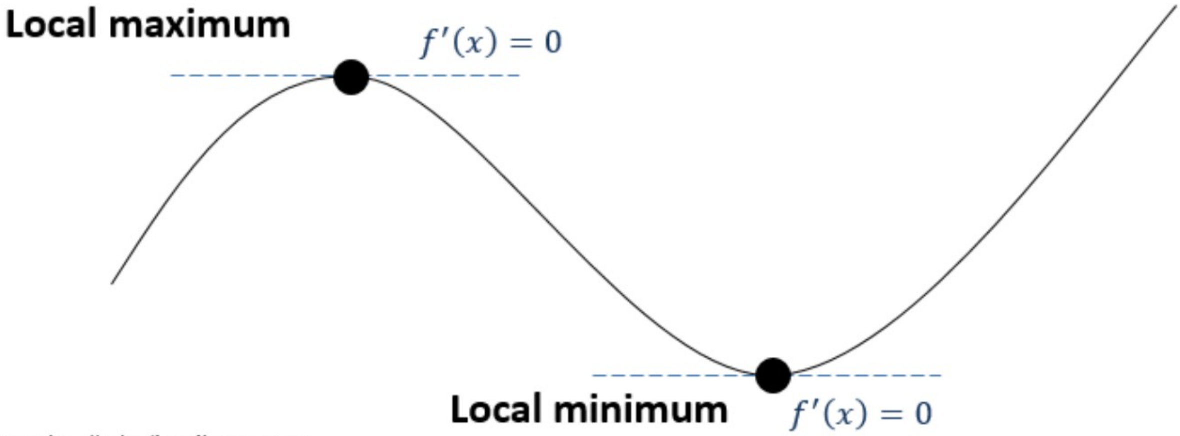
If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$.

Your Turn

If $y = 5x^3 - \frac{x}{3\sqrt{x}}$, find $\frac{d^2y}{dx^2}$.

Stationary/Turning Points

A stationary point is where the gradient is 0, i.e. $f'(x) = 0$.



Note: It's called a '**local**' maximum because it's the function's largest output within the vicinity. Functions may also have a '**global**' maximum, i.e. the maximum output across the entire function. This particular function doesn't have a global maximum because the output keeps increasing up to infinity. It similarly has no global minimum, as with all cubics.

There are three types of stationary points –

1. Maxima
2. Minima
3. Points of inflection

Find the coordinates of the turning points of $y = x^3 + 6x^2 - 135x$

Find the least value of

$$f(x) = x^2 - 4x + 9$$

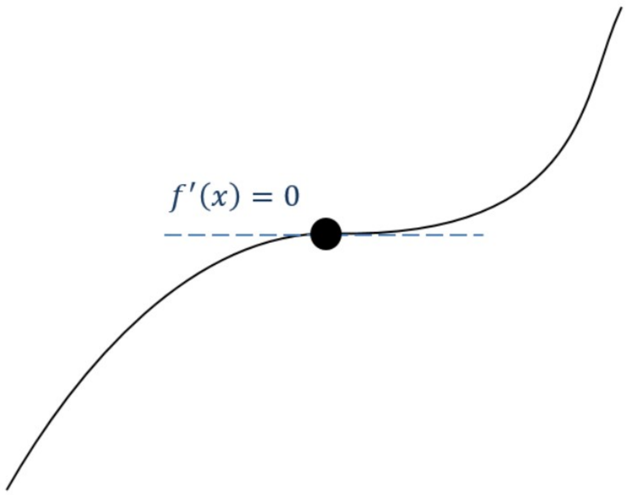
Method 1: Differentiation

Method 2:

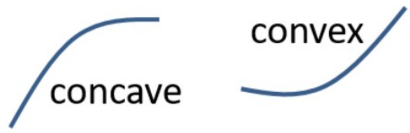
Find the turning point of

$$y = \sqrt{x} - x$$

Points of Inflection

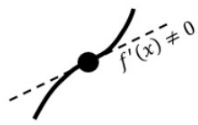


A point of inflection is where the curve changes from convex concave (or vice versa).



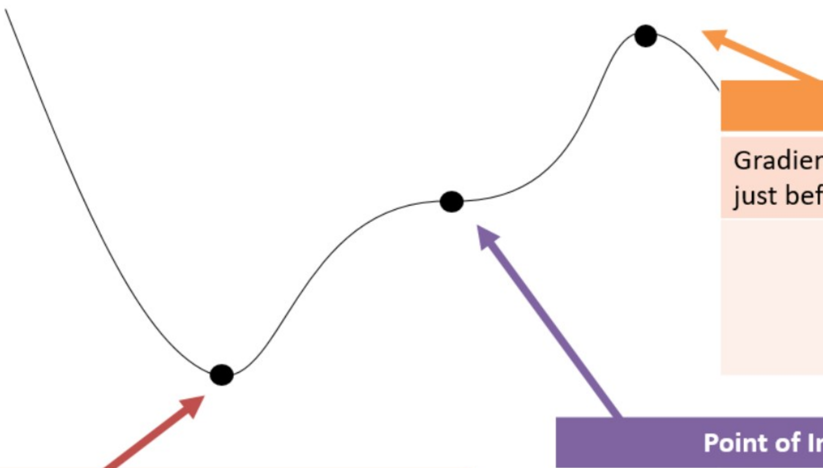
i.e. the line curves in one direction before the point of inflection, then curves in the other direction after.

Side Note: Not all points of inflection are stationary points, as can be seen in the example on the right.
A point of inflection which **<u>is</u>** a stationary point is known as a *saddle point*.



How do we tell what type of stationary point?

Method 1: Look at gradient just before and just after point.



Local Maximum		
Gradient just before	Gradient at maximum	Gradient just after

Point of Inflection		
Gradient just before	Gradient at p.o.i	Gradient just after

Local Minimum		
Gradient just before	Gradient at minimum	Gradient just after

Method 1: Look at gradient just before and just after point.

Find the stationary point on the curve with equation $y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

How do we tell what type of stationary point?

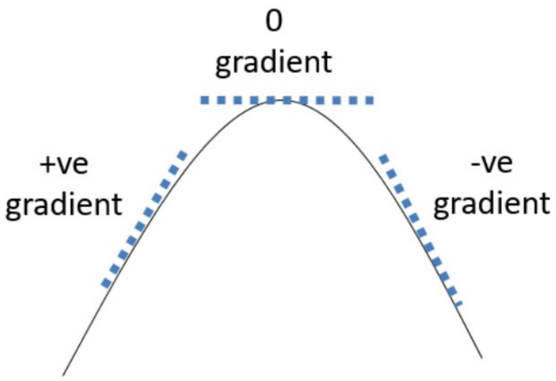
Method 2: Using the second derivative

The method of substituting values of x just before and after is a bit cumbersome.
It also has the potential for problems: what if two different types of stationary points are really close together?

Recall the gradient gives a measure of the **rate of change** of y, i.e. how much the y value changes as x changes.

Thus by differentiating the gradient function, the **<u>second derivative tells us the rate at what the gradient is changing</u>**.

Thus if the second derivative is positive, the gradient is increasing.
If the second derivative is negative, the gradient is decreasing.



At a maximum point, we can see that as x increases, the gradient is decreasing from a positive value to a negative value.

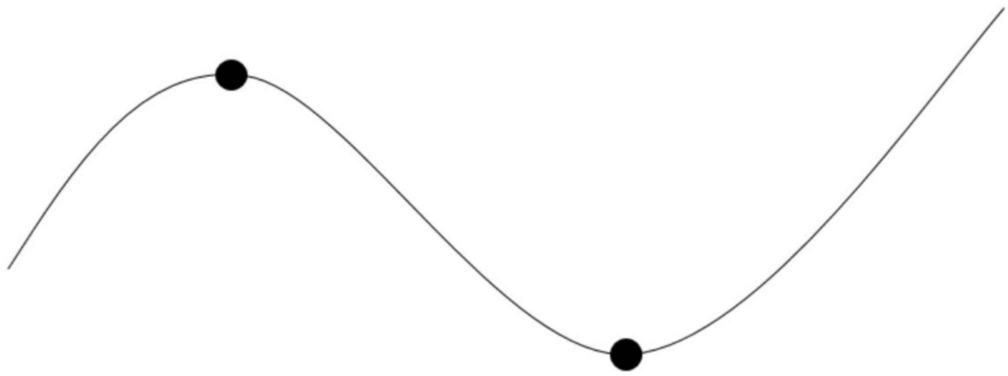
$$\therefore \frac{d^2y}{dx^2} < 0$$

✎ At a stationary point $x = a$:

- If $f''(a) > 0$ the point is a local minimum.
- If $f''(a) < 0$ the point is a local maximum.
- If $f''(a) = 0$ it could be any type of point, so resort to Method 1.

How I remember it:

Imagine being at the maximum or minimum…



Find the coordinates of the stationary points on the curve $y = 2x^3 - 15x^2 + 24x + 6$
Use the second derivative to determine the nature of these stationary points.

Sketching Graphs

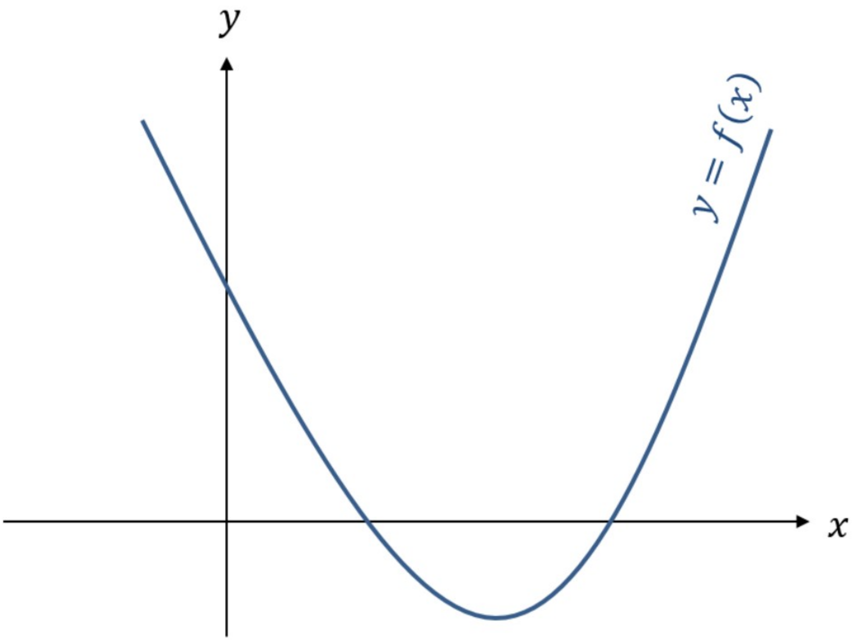
In Chapter 4 we used features such as intercepts with the axes, and behaviour when $x \to \infty$ and $x \to -\infty$ in order to sketch graphs.
Now we can also find stationary/turning points! Do you remember me mentioning this?

By first finding the stationary points, sketch the graph of $y = \frac{1}{x} + 27x^3$

Ex 12I

Sketching Gradient Functions

Draw the gradient function $f'(x)$ for the sketch of $f(x)$ provided.



1. Mark on the x-axis the points where $f(x)$ has zero gradient
2. Decide if gradient is positive or negative in between these points
3. Sketch!

Draw the gradient function $f'(x)$ for the sketch of $f(x)$ provided.

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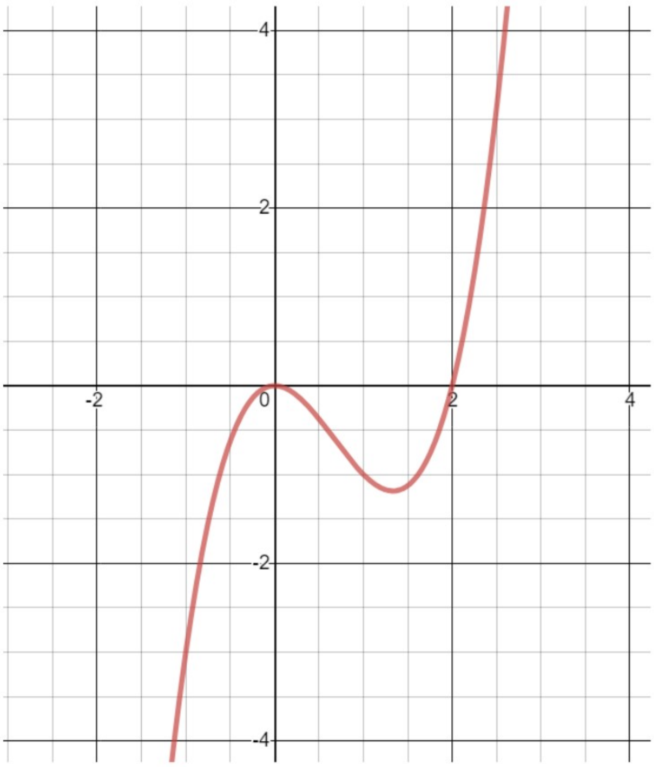

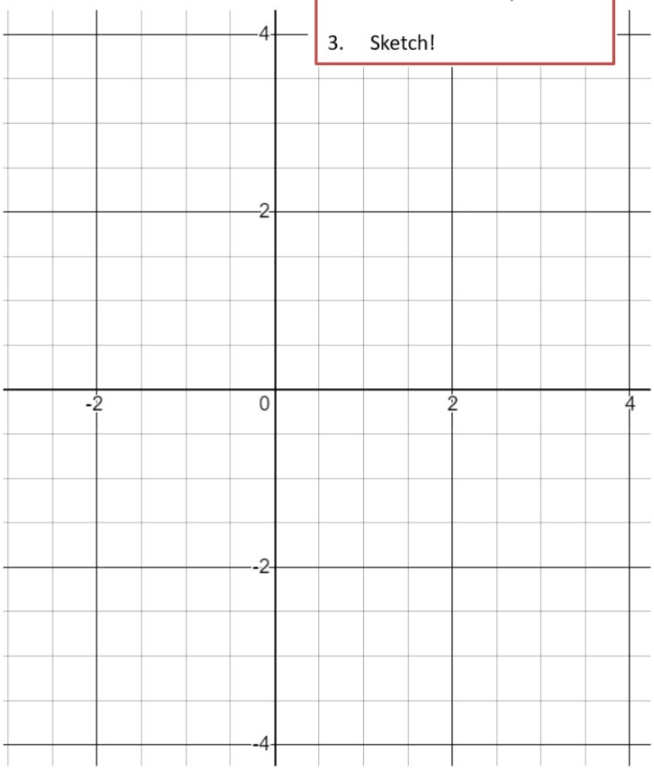

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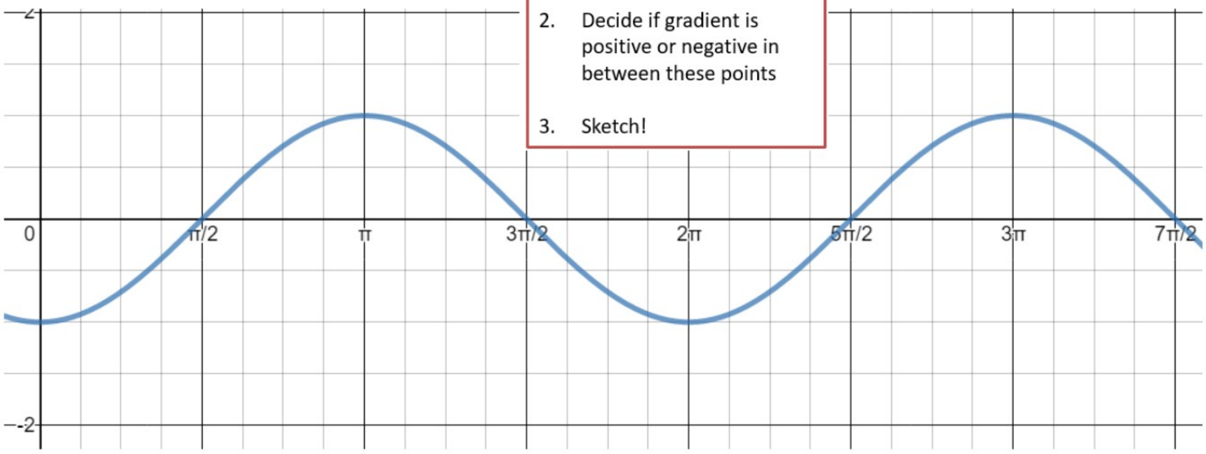

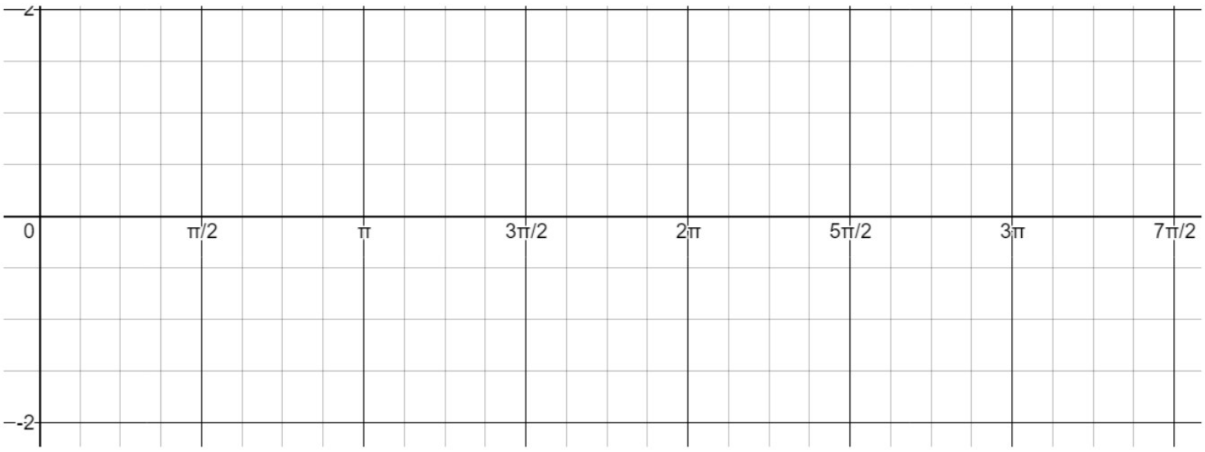

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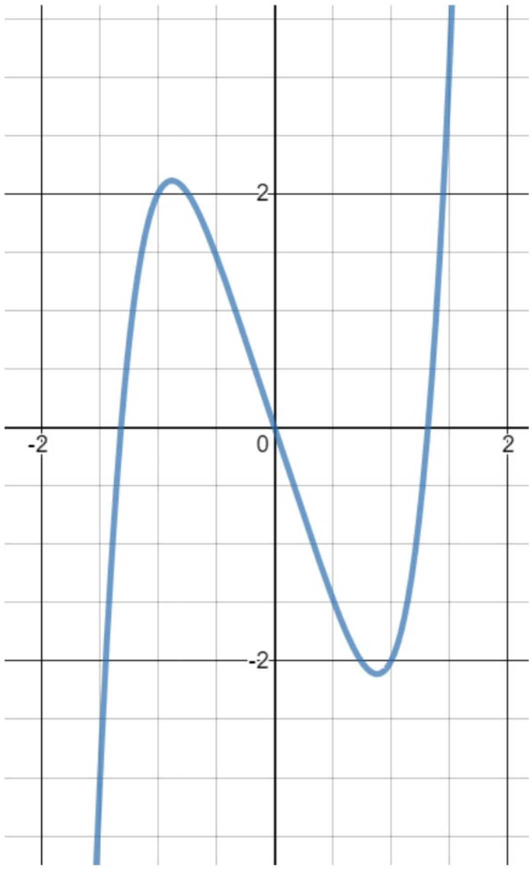

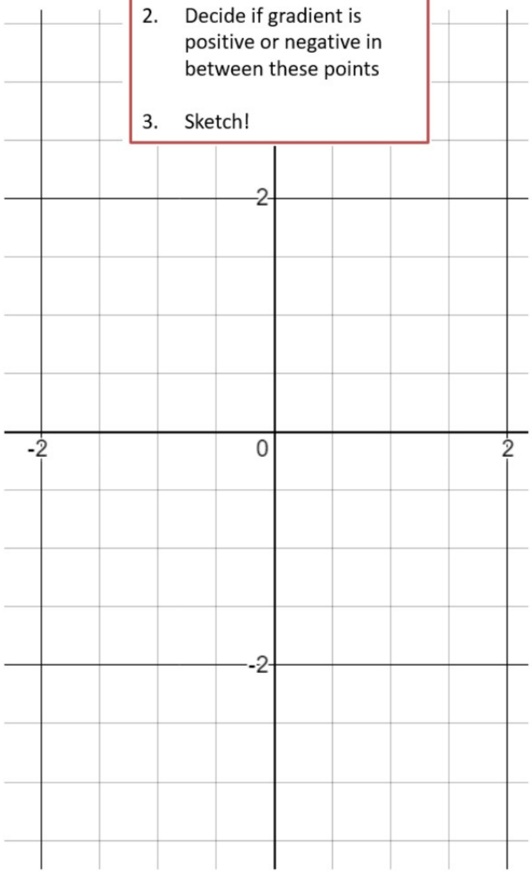

Draw the gradient function $f'(x)$ for the sketch of $f(x)$ provided.

1. Mark on the x-axis the points where $f(x)$ has zero gradient
2. Decide if gradient is positive or negative in between these points
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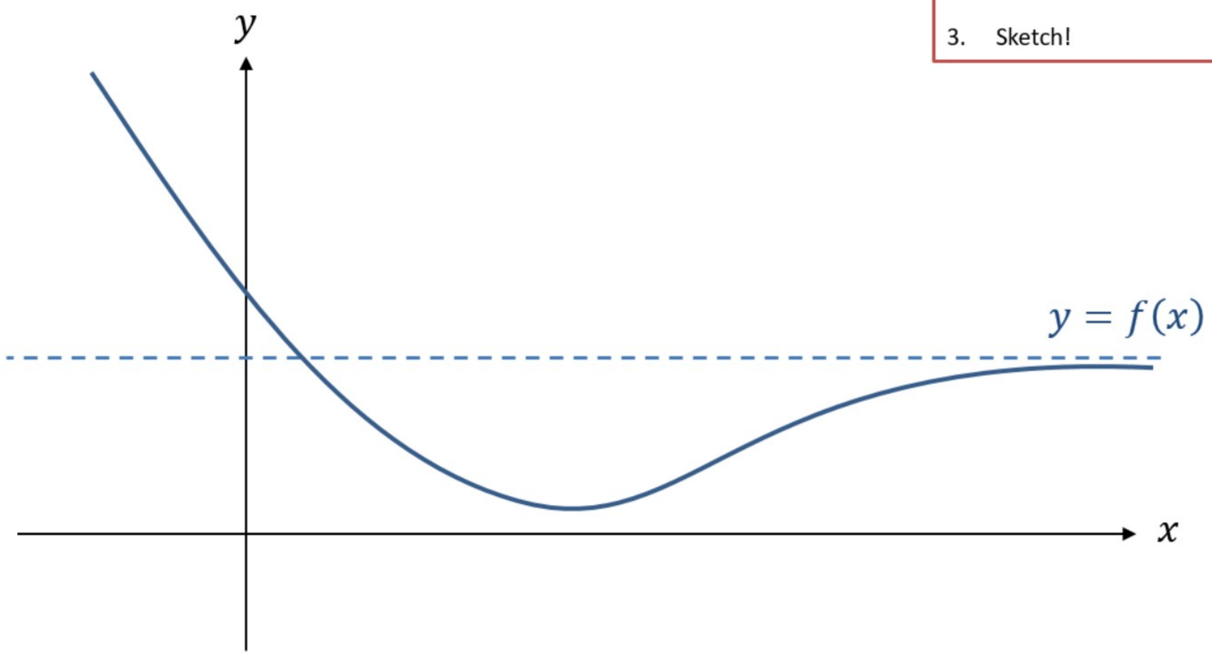
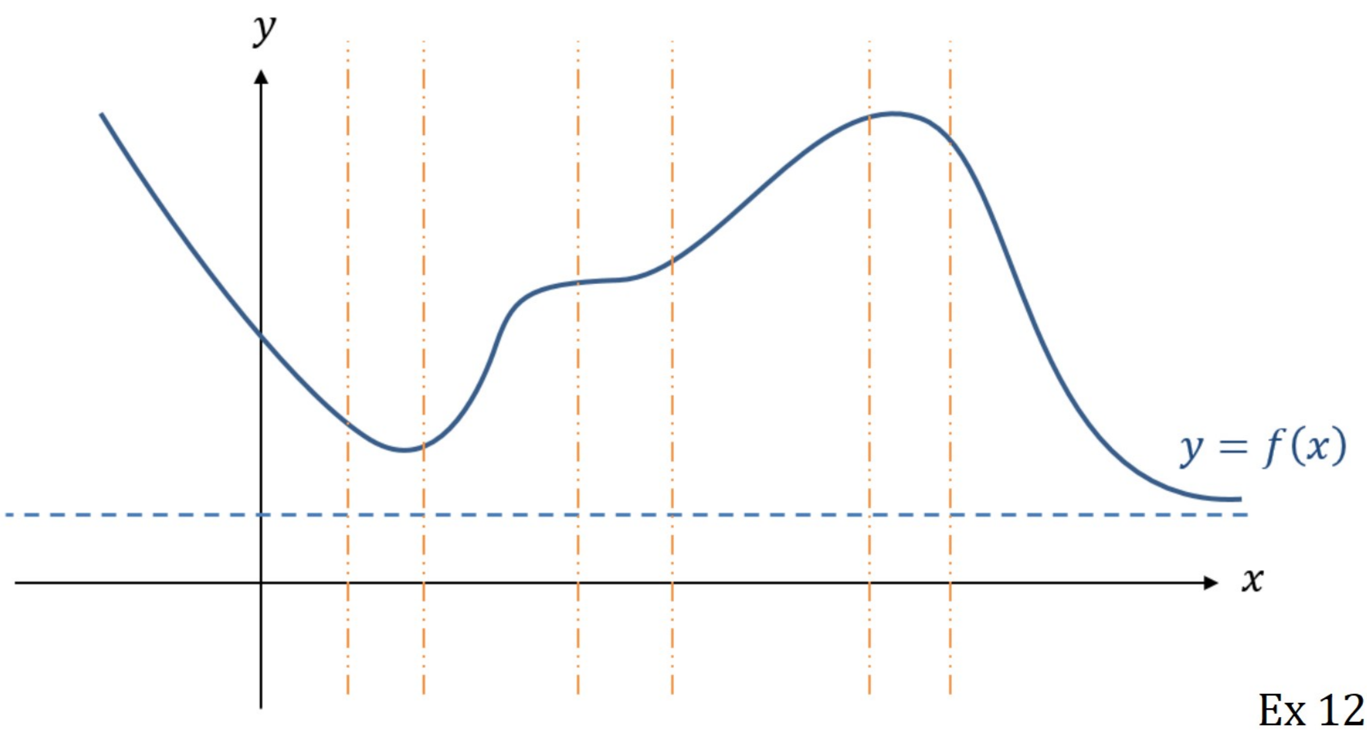


A MUCH Harder One

Draw the gradient function $f'(x)$ for the sketch of $f(x)$ provided.

1. Mark on the x-axis the points where $f(x)$ has zero gradient
2. Decide if gradient is positive or negative in between these points
3. Sketch!



Differentiating with respect to …

Up to now we've mostly had y in terms of x, and differentiated **with respect to x.**

e.g. $y = 3x^4 - 12x^2$

The $\frac{dy}{dx}$ means "the **rate** at which y changes with respect to x".

But we can also differentiate with respect to other variables if needed, not just x.

$C = 3t^4 - 12t^2$

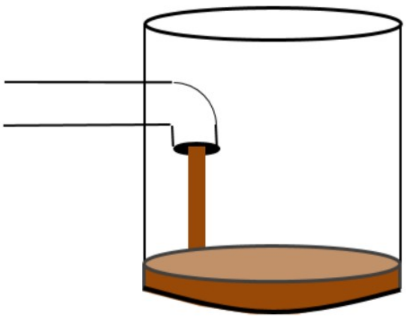
$r = 3\theta^4 - 12\theta^2$

$V = 3h^4 - 12h^2$

means "the **rate** at which changes with respect to "

means "the **rate** at which changes with respect to "

means "the **rate** at which changes with respect to "



A container fills at a rate of 20 cm^3 per second.

How could we use appropriate notation to represent this?

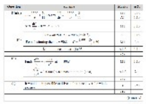
A population of rabbits increases by a rate of 300 rabbits per month.

How could we use appropriate notation to represent this?

The amount of a medicine in a patient's bloodstream decreases at a rate of 50mg per hour after injection.

How could we use appropriate notation to represent this?

Optimisation Problems/Modelling



8. A lorry is driven between London and Newcastle.

In a simple model, the cost of the journey £C when the lorry is driven at a steady speed of v kilometres per hour is

$$C = \frac{1500}{v} + \frac{2v}{11} + 60$$

(a) Find, according to this model,

(i) the value of v that minimises the cost of the journey,

(ii) the minimum cost of the journey.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

(b) Prove by using $\frac{d^2C}{dv^2}$ that the cost is minimised at the speed found in (a)(i).

(2)

(c) State one limitation of this model.

(1)

Optimisation problems in an exam usually follow the following pattern:

- There are 2 variables involved (you may have to introduce one yourself), typically lengths.
- There are expressions for **two different physical quantities**:
    - One is a **constraint**, e.g. "the surface area is 20cm²".
    - The **other we wish to maximise/minimise**, e.g. "we wish to maximise the volume".
- We use the constraint to **eliminate one of the variables** in the latter equation, so that it is then **just in terms of one variable**, and we can then use differentiation to find the turning point.

Figure 4 shows a solid brick in the shape of a cuboid measuring $2x$ cm by x cm by y cm.

The total surface area of the brick is 600 cm^2.

(a) Show that the volume, V cm^3, of the brick is given by

$$V = 200x - \frac{4x^3}{3}.$$

(4)

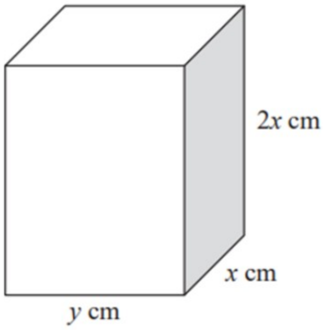


Figure 4

Given that x can vary,

(b) use calculus to find the maximum value of V, giving your answer to the nearest cm^3.

(5)

(c) Justify that the value of V you have found is a maximum.

(2)

8.

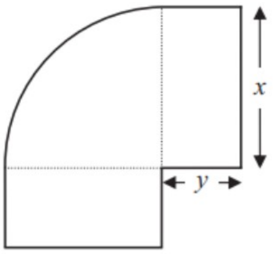


Figure 3

Figure 3 shows a flowerbed. Its shape is a quarter of a circle of radius x metres with two equal rectangles attached to it along its radii. Each rectangle has length equal to x metres and width equal to y metres.

Given that the area of the flowerbed is 4 m^2,

(a) show that

$$y = \frac{16 - \pi x^2}{8x}$$

(3)

(b) Hence show that the perimeter P metres of the flowerbed is given by the equation

$$P = \frac{8}{x} + 2x$$

(3)

(c) Use calculus to find the minimum value of P.

(5)

(d) Find the width of each rectangle when the perimeter is a minimum.
Give your answer to the nearest centimetre.

(2)

Your Turn

8.

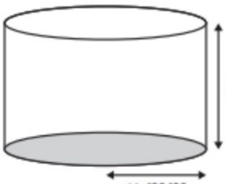


Figure 3

A manufacturer produces pain relieving tablets. Each tablet is in the shape of a solid circular cylinder with base radius x mm and height h mm, as shown in Figure 3.

Given that the volume of each tablet has to be 60 mm^3,

(a) express h in terms of x,

(1)

(b) show that the surface area, A mm^2, of a tablet is given by $A = 2\pi x^2 + \frac{120}{x}$

(3)

The manufacturer needs to minimise the surface area A mm^2, of a tablet.

(c) Use calculus to find the value of x for which A is a minimum.

(5)

(d) Calculate the minimum value of A, giving your answer to the nearest integer.

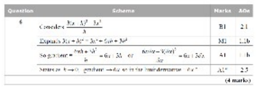
(2)

(e) Show that this value of A is a minimum.

(2)

Ex 12K

Exam Questions

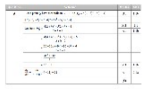


Question	Scheme	Marks	AOs
6	Considers $\frac{3(x+h)^2-3x^2}{h}$	B1	2.1
	Expands $3(x+h)^2 = 3x^2+6xh+3h^2$	M1	1.1b
	So gradient $= \frac{6xh+3h^2}{h} = 6x+3h$ or $\frac{6x\delta x+3(\delta x)^2}{\delta x} = 6x+3\delta x$	A1	1.1b
	States as $h \to 0$, gradient $\to 6x$ so in the limit derivative $= 6x$ *	A1*	2.5
			(4 marks)

6. Prove, from first principles, that the derivative of $3x^2$ is $6x$.

(4)

5. A curve has equation



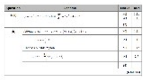
$$y = 4x^2 - 5x$$

The curve passes through the point $P(2, 6)$

Use differentiation from first principles to find the value of the gradient of the curve at P.

(5)

1. A curve has equation



$$y = 2x^3 - 2x^2 - 2x + 8.$$

(*a*) Find $\frac{dy}{dx}$.

(2)

(*b*) Hence find the range of values of x for which y is increasing.
Write your answer in set notation.

(4)

(Total for Question 1 is 6 marks)

1. The curve C has equation

$$y = 3x^4 - 8x^3 - 3$$

(a) Find (i) $\frac{dy}{dx}$

(ii) $\frac{d^2y}{dx^2}$

(3)

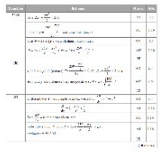
(b) Verify that C has a stationary point when $x = 2$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

16.



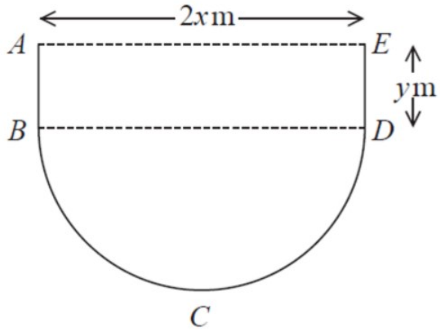


Figure 4

Figure 4 shows the plan view of the design for a swimming pool.

The shape of this pool $ABCDEA$ consists of a rectangular section $ABDE$ joined to a semicircular section BCD as shown in Figure 4.

Given that $AE = 2x$ metres, $ED = y$ metres and the area of the pool is $250\,\text{m}^2$,

(a) show that the perimeter, P metres, of the pool is given by

$$P = 2x + \frac{250}{x} + \frac{\pi x}{2}$$

(4)

(b) Explain why $0 < x < \sqrt{\frac{500}{\pi}}$

(2)

(c) Find the minimum perimeter of the pool, giving your answer to 3 significant figures.

(4)

[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

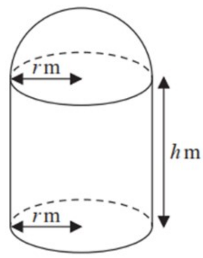


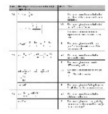
Figure 9

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.



The volume of the tank is $6\,\text{m}^3$.

(a) Show that, according to the model, the surface area of the tank, in m^2, is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2$$

(4)

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum.

(4)

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer.

(2)