

Chapter 6a

Hyperbolic Functions - definitions and solving

We will see the definition and purpose of **hyperbolic functions** such as $\sinh x$, $\cosh x$, their inverses, and how we can manipulate them, such as solving equations, differentiating and integrating.

1 :: Definition of hyperbolic functions and their sketches.

"Find the exact value of: $\tanh (\ln 4)$ "

2 :: Inverse hyperbolic functions.

Prove that

$$\operatorname{arcosh} x = \ln \left(x + \sqrt{x^2 - 1} \right)$$

3 :: Hyperbolic Identities and Solving Equations

Solve for all real x

$$2 \cosh^2 x - 5 \sinh x = 5$$

Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh x = \frac{1}{2} (e^x + e^{-x})$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh x = \frac{1}{2} (e^x - e^{-x})$$

Using these definitions, find the definitions of:

$\operatorname{sech} x$

$\tanh x$

$\operatorname{cosech} x$

$\coth x$

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad x \in \mathbb{R}$$

Say as “shine” of x

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad x \in \mathbb{R}$$

Say as “cosh”

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1} \quad x \in \mathbb{R}$$

Say as “th-an”

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad x \in \mathbb{R}$$

Say as “setch”

Hyperbolic cosecant:

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad x \in \mathbb{R}, x \neq 0$$

Say as “cosetch”

Hyperbolic cotangent:

$$\operatorname{coth} x = \frac{1}{\tanh x} = \frac{(e^{2x} + 1)}{e^{2x} - 1} \quad x \in \mathbb{R}, x \neq 0$$

Say as “coth”

Equations for hyperbolic functions

Calculate (using both your *sinh* button and using the formula)

$\sinh 3$

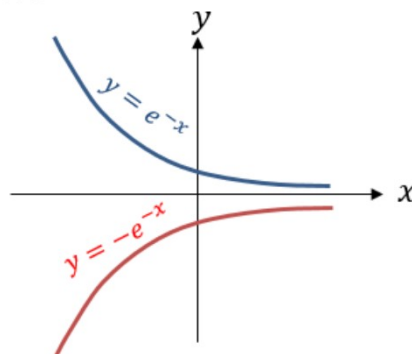
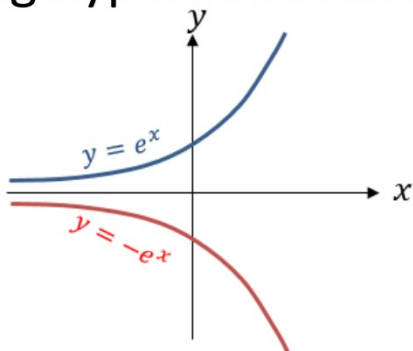
Write in terms of e :

$\operatorname{cosech} 3$

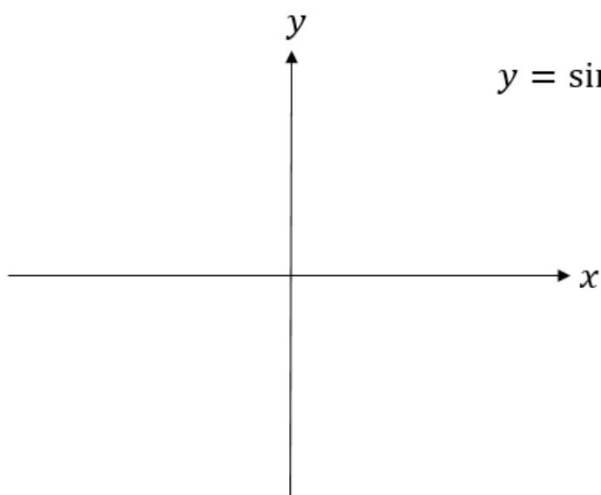
Find the exact value of:
 $\tanh(\ln 4)$

Solve $\sinh x = 5$

Sketching hyperbolic functions

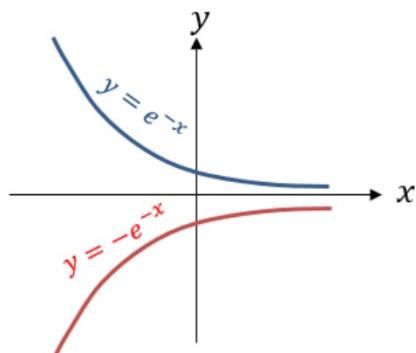
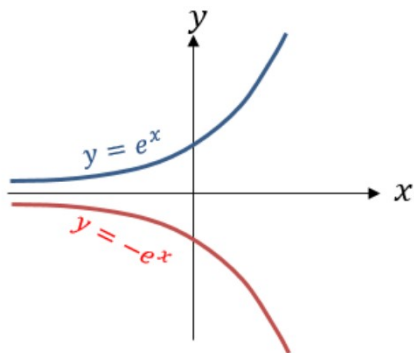


Odd or Even?

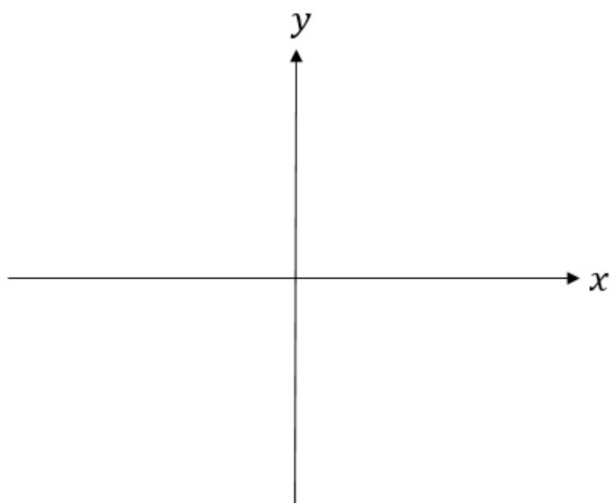


$y = \sinh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Odd or Even?



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

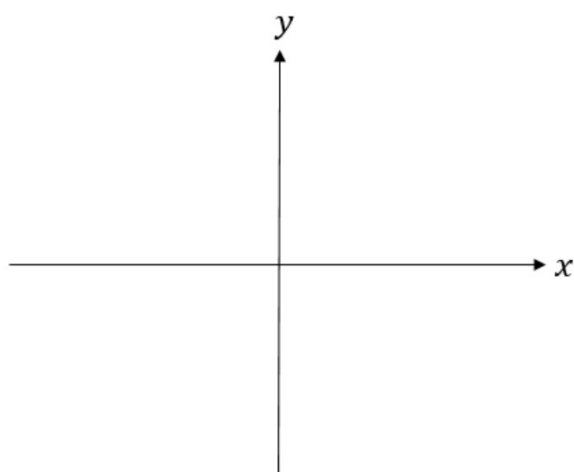
To sketch $y = \tanh x$, consider the usual features when you sketch a graph.

$$\tanh x = \frac{\sinh x}{\cosh x}$$

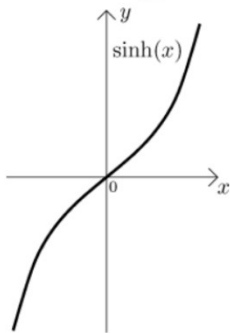
When $x = 0$,

As $x \rightarrow \infty$,

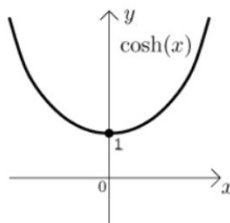
As $x \rightarrow -\infty$,



Sketching the reciprocal hyperbolic functions

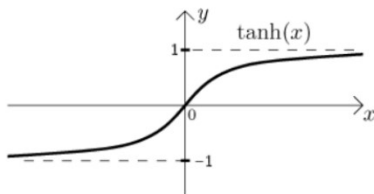


$\operatorname{cosech}(x) \uparrow$



$\operatorname{sech}(x) \uparrow$

$y \uparrow 1$



[FP3 June 2011 Q5] The curve C_1 has equation $y = 3 \sinh 2x$, and the curve C_2 has equation $y = 13 - 3e^{2x}$.

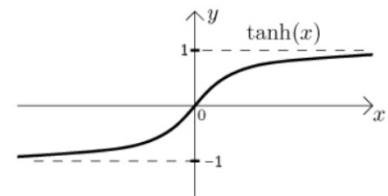
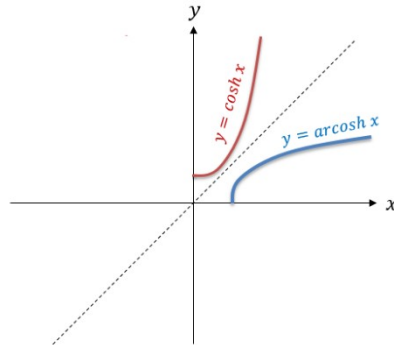
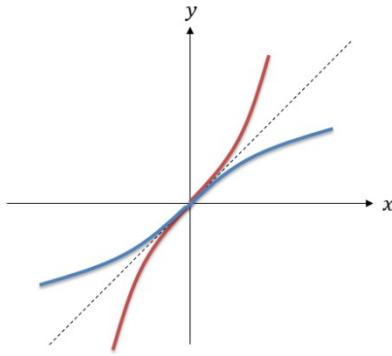
(a) Sketch the graph of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes.

(4)

Inverse Hyperbolic Functions

$$\sin^{-1} x = \arcsin x$$

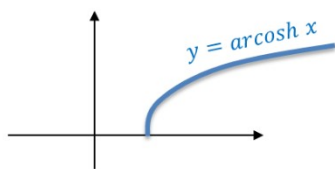
$$\sinh^{-1} x = \operatorname{arsinh} x \quad \text{etc.}$$



Given that hyperbolic functions can be written in terms of e , it makes sense that inverse hyperbolic can be expressed in terms of \ln .

Prove that $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$

Prove that $\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$

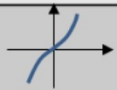
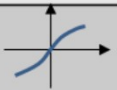


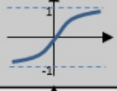
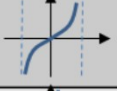
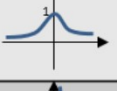




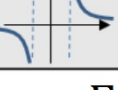


But recall from the graph that we only include positive values of y in the function to avoid it being one-to-many. Thus $\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$ only.

Prove that $\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), |x| < 1$

Summary

$$\begin{aligned} \operatorname{arsinh} x &= \ln \left(x + \sqrt{x^2 + 1} \right) \\ \operatorname{arcosh} x &= \ln \left(x + \sqrt{x^2 - 1} \right), \quad x \geq 1 \\ \operatorname{artanh} x &= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \quad |x| < 1 \end{aligned}$$

Hyperbolic	Domain	Sketch	Inverse Hyperbolic	Domain	Sketch
$y = \sinh x$	$x \in \mathbb{R}$		$y = \operatorname{arsinh} x$	$x \in \mathbb{R}$	
$y = \cosh x$	$x \geq 0$		$y = \operatorname{arcosh} x$	$x \geq 1$	
$y = \tanh x$	$x \in \mathbb{R}$		$y = \operatorname{artanh} x$	$ x < 1$	
$y = \operatorname{sech} x$	$x \geq 0$		$y = \operatorname{arsech} x$	$0 < x \leq 1$	
$y = \operatorname{cosech} x$	$x \neq 0$		$y = \operatorname{arcosech} x$	$x \neq 0$	
$y = \coth x$	$x \neq 0$		$y = \operatorname{arcoth} x$	$ x > 1$	

Ex 6B

Hyperbolic Identities

Use the definitions of \sinh and \cosh to prove that...

$$\cosh^2 x - \sinh^2 x = 1$$

What else could we prove?

Prove the following identity using the definitions of $\sinh x$ and $\cosh x$

$$\sinh (A + B) = \sinh A \cosh B + \cosh A \sinh B$$

Prove the following identity using the definitions of $\sinh x$ and $\cosh x$

$$\cosh 2A = 1 + 2\sinh^2 A$$

Osborn's Rule

Are these the same, or different, to the trig identities?

$$\sinh (A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$$

$$\cosh (A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$

$$\tanh (A \pm B) = \frac{\sinh x}{\cosh x} = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

We can get these identities from the normal sin/cos ones by:

Osborn's Rule:

1. Replacing $\sin \rightarrow \sinh$ and $\cos \rightarrow \cosh$
2. **Negate** any explicit or implied **product of two sines**.

$$\tan(A - B) =$$

$$\begin{array}{l} \sin A \sin B \rightarrow \\ \tan^2 A \quad \rightarrow \end{array}$$

$$\cos 2A =$$

Ex 6C Q1, 2

Solving Equations

You must to decide whether to use **hyperbolic identities** or **the definitions of hyperbolic functions**.

Solve for all real x

$$6 \sinh x - 2 \cosh x = 7$$

Solve for all real x

$$2 \cosh^2 x - 5 \sinh x = 5$$

Solve for all real x

$$\cosh 2x - 5 \cosh x + 4 = 0$$

Double Angle

$\sin 2x = 2 \sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 2 \cos^2 x - 1$ $\cos 2x = 1 - 2 \sin^2 x$
$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

If $\sinh x = \frac{3}{4}$, find the exact value of:

- $\cosh x$
- $\tanh x$
- $\sinh 2x$

Ex 6C Q3-9

Exam Questions

[FP3 June 2009 Q1] Solve the equation

$$7 \operatorname{sech} x - \tanh x = 5$$

Give your answers in the form $\ln a$, where a is a rational number.

(5)

$\frac{7}{\sin 2x} \cdot \frac{\sin 2x}{\cos 2x} = 5 \Rightarrow \frac{14}{e^2 + e^{-2}} = \frac{(e^2 - e^{-2})}{e^2 + e^{-2}} = 5$	3d
$\Rightarrow 14 = (e^2 - e^{-2}) = 5(e^2 + e^{-2}) \Rightarrow 9e^2 - 14 = 5e^{-2} = 0$	3e
$\Rightarrow 9e^2 - 5e^{-2} = 2 = 0 \Rightarrow 9e^4 - 5(e^2 - 2) = 0$	3f
$\Rightarrow e^2 = \frac{1}{3} \approx 2$	3g
$x = \ln(2)$ or $\ln 2$	3h
	(5 marks)

10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
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[FP3 June 2014 (I) Q3] Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials,

(a) prove that

$$\cosh^2 x - \sinh^2 x \equiv 1$$

(2)

(b) find algebraically the exact solutions of the equation

$$2 \sinh x + 7 \cosh x = 9$$

giving your answers as natural logarithms.

(5)

[FP3 June 2011 Q5]

(b) Solve the equation $3 \sinh 2x = 13 - 3e^{2x}$, giving your answer in the form $\frac{1}{2} \ln k$, where k is an integer.

(5)

10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
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1. Solve the equation

$$6 \cosh 2x + 4 \sinh x = 7$$

giving your answers as exact logarithms.

(6)

7. (a) Using the definition of $\cosh x$ in terms of exponentials, prove that

$$4 \cosh^3 x - 3 \cosh x \equiv \cosh 3x$$

(3)

(b) Hence, or otherwise, solve the equation

$$\cosh 3x = 9 \cosh x$$

(4)

1. (a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad -k < x < k$$

stating the value of the constant k .

(5)

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2-3x}\right)$$

(5)

$\frac{1}{x^2} = x^{-2}$	$\frac{d}{dx} x^{-2} = -2x^{-3} = -\frac{2}{x^3}$	$-\frac{2}{x^3}$
$\frac{1}{x^3} = x^{-3}$	$\frac{d}{dx} x^{-3} = -3x^{-4} = -\frac{3}{x^4}$	$-\frac{3}{x^4}$
$\frac{1}{x^4} = x^{-4}$	$\frac{d}{dx} x^{-4} = -4x^{-5} = -\frac{4}{x^5}$	$-\frac{4}{x^5}$
$\frac{1}{x^5} = x^{-5}$	$\frac{d}{dx} x^{-5} = -5x^{-6} = -\frac{5}{x^6}$	$-\frac{5}{x^6}$
$\frac{1}{x^6} = x^{-6}$	$\frac{d}{dx} x^{-6} = -6x^{-7} = -\frac{6}{x^7}$	$-\frac{6}{x^7}$
$\frac{1}{x^7} = x^{-7}$	$\frac{d}{dx} x^{-7} = -7x^{-8} = -\frac{7}{x^8}$	$-\frac{7}{x^8}$
$\frac{1}{x^8} = x^{-8}$	$\frac{d}{dx} x^{-8} = -8x^{-9} = -\frac{8}{x^9}$	$-\frac{8}{x^9}$
$\frac{1}{x^9} = x^{-9}$	$\frac{d}{dx} x^{-9} = -9x^{-10} = -\frac{9}{x^{10}}$	$-\frac{9}{x^{10}}$
$\frac{1}{x^{10}} = x^{-10}$	$\frac{d}{dx} x^{-10} = -10x^{-11} = -\frac{10}{x^{11}}$	$-\frac{10}{x^{11}}$
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$\frac{1}{x^{50}} = x^{-50}$	$\frac{d}{dx} x^{-50} = -50x^{-51} = -\frac{50}{x^{51}}$	$-\frac{50}{x^{51}}$
$\frac{1}{x^{51}} = x^{-51}$	$\frac{d}{dx} x^{-51} = -51x^{-52} = -\frac{51}{x^{52}}$	$-\frac{51}{x^{52}}$
$\frac{1}{x^{52}} = x^{-52}$	$\frac{d}{dx} x^{-52} = -52x^{-53} = -\frac{52}{x^{53}}$	$-\frac{52}{x^{53}}$
$\frac{1}{x^{53}} = x^{-53}$	$\frac{d}{dx} x^{-53} = -53x^{-54} = -\frac{53}{x^{54}}$	$-\frac{53}{x^{54}}$
$\frac{1}{x^{54}} = x^{-54}$	$\frac{d}{dx} x^{-54} = -54x^{-55} = -\frac{54}{x^{55}}$	$-\frac{54}{x^{55}}$
$\frac{1}{x^{55}} = x^{-55}$	$\frac{d}{dx} x^{-55} = -55x^{-56} = -\frac{55}{x^{56}}$	$-\frac{55}{x^{56}}$
$\frac{1}{x^{56}} = x^{-56}$	$\frac{d}{dx} x^{-56} = -56x^{-57} = -\frac{56}{x^{57}}$	$-\frac{56}{x^{57}}$
$\frac{1}{x^{57}} = x^{-57}$	$\frac{d}{dx} x^{-57} = -57x^{-58} = -\frac{57}{x^{58}}$	$-\frac{57}{x^{58}}$
$\frac{1}{x^{58}} = x^{-58}$	$\frac{d}{dx} x^{-58} = -58x^{-59} = -\frac{58}{x^{59}}$	$-\frac{58}{x^{59}}$
$\frac{1}{x^{59}} = x^{-59}$	$\frac{d}{dx} x^{-59} = -59x^{-60} = -\frac{59}{x^{60}}$	$-\frac{59}{x^{60}}$
$\frac{1}{x^{60}} = x^{-60}$	$\frac{d}{dx} x^{-60} = -60x^{-61} = -\frac{60}{x^{61}}$	$-\frac{60}{x^{61}}$
$\frac{1}{x^{61}} = x^{-61}$	$\frac{d}{dx} x^{-61} = -61x^{-62} = -\frac{61}{x^{62}}$	$-\frac{61}{x^{62}}$
$\frac{1}{x^{62}} = x^{-62}$	$\frac{d}{dx} x^{-62} = -62x^{-63} = -\frac{62}{x^{63}}$	$-\frac{62}{x^{63}}$
$\frac{1}{x^{63}} = x^{-63}$	$\frac{d}{dx} x^{-63} = -63x^{-64} = -\frac{63}{x^{64}}$	$-\frac{63}{x^{64}}$
$\frac{1}{x^{64}} = x^{-64}$	$\frac{d}{dx} x^{-64} = -64x^{-65} = -\frac{64}{x^{65}}$	$-\frac{64}{x^{65}}$
$\frac{1}{x^{65}} = x^{-65}$	$\frac{d}{dx} x^{-65} = -65x^{-66} = -\frac{65}{x^{66}}$	$-\frac{65}{x^{66}}$
$\frac{1}{x^{66}} = x^{-66}$	$\frac{d}{dx} x^{-66} = -66x^{-67} = -\frac{66}{x^{67}}$	$-\frac{66}{x^{67}}$
$\frac{1}{x^{67}} = x^{-67}$	$\frac{d}{dx} x^{-67} = -67x^{-68} = -\frac{67}{x^{68}}$	$-\frac{67}{x^{68}}$
$\frac{1}{x^{68}} = x^{-68}$	$\frac{d}{dx} x^{-68} = -68x^{-69} = -\frac{68}{x^{69}}$	$-\frac{68}{x^{69}}$
$\frac{1}{x^{69}} = x^{-69}$	$\frac{d}{dx} x^{-69} = -69x^{-70} = -\frac{69}{x^{70}}$	$-\frac{69}{x^{70}}$
$\frac{1}{x^{70}} = x^{-70}$	$\frac{d}{dx} x^{-70} = -70x^{-71} = -\frac{70}{x^{71}}$	$-\frac{70}{x^{71}}$
$\frac{1}{x^{71}} = x^{-71}$	$\frac{d}{dx} x^{-71} = -71x^{-72} = -\frac{71}{x^{72}}$	$-\frac{71}{x^{72}}$
$\frac{1}{x^{72}} = x^{-72}$	$\frac{d}{dx} x^{-72} = -72x^{-73} = -\frac{72}{x^{73}}$	$-\frac{72}{x^{73}}$
$\frac{1}{x^{73}} = x^{-73}$	$\frac{d}{dx} x^{-73} = -73x^{-74} = -\frac{73}{x^{74}}$	$-\frac{73}{x^{74}}$
$\frac{1}{x^{74}} = x^{-74}$	$\frac{d}{dx} x^{-74} = -74x^{-75} = -\frac{74}{x^{75}}$	$-\frac{74}{x^{75}}$
$\frac{1}{x^{75}} = x^{-75}$	$\frac{d}{dx} x^{-75} = -75x^{-76} = -\frac{75}{x^{76}}$	$-\frac{75}{x^{76}}$
$\frac{1}{x^{76}} = x^{-76}$	$\frac{d}{dx} x^{-76} = -76x^{-77} = -\frac{76}{x^{77}}$	$-\frac{76}{x^{77}}$
$\frac{1}{x^{77}} = x^{-77}$	$\frac{d}{dx} x^{-77} = -77x^{-78} = -\frac{77}{x^{78}}$	$-\frac{77}{x^{78}}$
$\frac{1}{x^{78}} = x^{-78}$	$\frac{d}{dx} x^{-78} = -78x^{-79} = -\frac{78}{x^{79}}$	$-\frac{78}{x^{79}}$
$\frac{1}{x^{79}} = x^{-79}$	$\frac{d}{dx} x^{-79} = -79x^{-80} = -\frac{79}{x^{80}}$	$-\frac{79}{x^{80}}$
$\frac{1}{x^{80}} = x^{-80}$	$\frac{d}{dx} x^{-80} = -80x^{-81} = -\frac{80}{x^{81}}$	$-\frac{80}{x^{81}}$
$\frac{1}{x^{81}} = x^{-81}$	$\frac{d}{dx} x^{-81} = -81x^{-82} = -\frac{81}{x^{82}}$	$-\frac{81}{x^{82}}$
$\frac{1}{x^{82}} = x^{-82}$	$\frac{d}{dx} x^{-82} = -82x^{-83} = -\frac{82}{x^{83}}$	$-\frac{82}{x^{83}}$
$\frac{1}{x^{83}} = x^{-83}$	$\frac{d}{dx} x^{-83} = -83x^{-84} = -\frac{83}{x^{84}}$	$-\frac{83}{x^{84}}$
$\frac{1}{x^{84}} = x^{-84}$	$\frac{d}{dx} x^{-84} = -84x^{-85} = -\frac{84}{x^{85}}$	$-\frac{84}{x^{85}}$
$\frac{1}{x^{85}} = x^{-85}$	$\frac{d}{dx} x^{-85} = -85x^{-86} = -\frac{85}{x^{86}}$	$-\frac{85}{x^{86}}$
$\frac{1}{x^{86}} = x^{-86}$	$\frac{d}{dx} x^{-86} = -86x^{-87} = -\frac{86}{x^{87}}$	$-\frac{86}{x^{87}}$
$\frac{1}{x^{87}} = x^{-87}$	$\frac{d}{dx} x^{-87} = -87x^{-88} = -\frac{87}{x^{88}}$	$-\frac{87}{x^{88}}$
$\frac{1}{x^{88}} = x^{-88}$	$\frac{d}{dx} x^{-88} = -88x^{-89} = -\frac{88}{x^{89}}$	$-\frac{88}{x^{89}}$
$\frac{1}{x^{89}} = x^{-89}$	$\frac{d}{dx} x^{-89} = -89x^{-90} = -\frac{89}{x^{90}}$	$-\frac{89}{x^{90}}$
$\frac{1}{x^{90}} = x^{-90}$	$\frac{d}{dx} x^{-90} = -90x^{-91} = -\frac{90}{x^{91}}$	$-\frac{90}{x^{91}}$
$\frac{1}{x^{91}} = x^{-91}$	$\frac{d}{dx} x^{-91} = -91x^{-92} = -\frac{91}{x^{92}}$	$-\frac{91}{x^{92}}$
$\frac{1}{x^{92}} = x^{-92}$	$\frac{d}{dx} x^{-92} = -92x^{-93} = -\frac{92}{x^{93}}$	$-\frac{92}{x^{93}}$
$\frac{1}{x^{93}} = x^{-93}$	$\frac{d}{dx} x^{-93} = -93x^{-94} = -\frac{93}{x^{94}}$	$-\frac{93}{x^{94}}$
$\frac{1}{x^{94}} = x^{-94}$	$\frac{d}{dx} x^{-94} = -94x^{-95} = -\frac{94}{x^{95}}$	$-\frac{94}{x^{95}}$
$\frac{1}{x^{95}} = x^{-95}$	$\frac{d}{dx} x^{-95} = -95x^{-96} = -\frac{95}{x^{96}}$	$-\frac{95}{x^{96}}$
$\frac{1}{x^{96}} = x^{-96}$	$\frac{d}{dx} x^{-96} = -96x^{-97} = -\frac{96}{x^{97}}$	$-\frac{96}{x^{97}}$
$\frac{1}{x^{97}} = x^{-97}$	$\frac{d}{dx} x^{-97} = -97x^{-98} = -\frac{97}{x^{98}}$	$-\frac{97}{x^{98}}$
$\frac{1}{x^{98}} = x^{-98}$	$\frac{d}{dx} x^{-98} = -98x^{-99} = -\frac{98}{x^{99}}$	$-\frac{98}{x^{99}}$
$\frac{1}{x^{99}} = x^{-99}$	$\frac{d}{dx} x^{-99} = -99x^{-100} = -\frac{99}{x^{100}}$	$-\frac{99}{x^{100}}$
$\frac{1}{x^{100}} = x^{-100}$	$\frac{d}{dx} x^{-100} = -100x^{-101} = -\frac{100}{x^{101}}$	$-\frac{100}{x^{101}}$