Edexcel A Level Further Maths: Core Pure

8.3 Simple Harmonic Motion

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8.3.1 Simple Harmonic Motion

Simple Harmonic Motion (SHM) Equations

What is simple harmonic motion?

- A particle undergoing **simple harmonic motion** moves along a straight line subject to the following constraints:
  - The **acceleration** of the particle is always **directed towards a fixed point** on the line of motion
  - The **acceleration** is **proportional** to the **displacement** of the particle **from the fixed point**
- As a result the particle **oscillates back and forth** along the line around the fixed point
- Many physical systems can be modelled using simple harmonic motion
  - One example is an object attached to a spring oscillating in one dimension, when friction, air resistance and other such resistive forces are disregarded

What is the equation that describes simple harmonic motion?

- The standard form of the **simple harmonic motion** equation is $\ddot{x} = -\omega^2 x$
  - x is the displacement of the particle from the fixed point
    - The fixed point is normally indicated by O and is the origin (i.e. zero point) of the coordinate system
    - The fixed point O is known as the **centre of oscillation**
  - ω^2 is the **constant of proportionality**, and represents the strength of the force accelerating the particle back towards point O
    - The negative sign means that the acceleration is always directed back towards O
    - We use ω^2 to assure that the constant is positive, and also to simplify the notation for the solution to the equation
  - $\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$ is the **acceleration** of the particle
    - With simple harmonic motion, Newton's 'dot notation' is often used for the derivatives
    - In this notation, $\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$ and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$

Solutions to SHM Equations

Your notes

What is the solution to the standard simple harmonic motion equation?

- The SHM equation may be solved using the standard techniques for **second order differential equations**
  - $\ddot{x} = -\omega^2 x$ may be rearranged to give the **homogeneous** second order equation $\ddot{x} + \omega^2 x = 0$
  - That has **auxiliary equation** $m^2 + \omega^2 = 0$ with roots $m = \pm \omega \mathrm{i}$
  - Therefore the **general solution** is $x = A\cos\omega t + B\sin\omega t$
    - Initial or boundary conditions given with a question may allow you to find the precise values of the arbitrary constants *A* and *B*
- The **linear combination of sine and cosine terms** in the general solution may be **rewritten** in the form $x = R\sin(\omega t + \alpha)$
  - This can sometimes be a more useful form for the general solution
  - This form of the equation shows that the particle oscillates around the fixed point O (i.e., around the centre of oscillation)
  - The **amplitude** of the motion is *R*
    - $R = \sqrt{A^2 + B^2}$
    - This is the **maximum distance** that the particle moves away from the fixed point *O*
  - The **period** of the motion is $\dfrac{2\pi}{\omega}$
    - This is the **length of time** it takes the particle to complete one oscillation
  - The constant α is a '**phase constant**'
    - $\alpha = \tan^{-1}\left(\dfrac{A}{B}\right)$
    - If $\alpha = 0$, then the particle is at point O (i.e., $x = 0$) when $t = 0$
    - If $\alpha = \dfrac{\pi}{2}$ then the particle is at its **maximum displacement** (i.e., $x = R$) when $t = 0$

How can I solve the simple harmonic motion equation to link displacement *x* and velocity *v*?

- Because **acceleration is the derivative of velocity**, and **velocity is the derivative of displacement**, we may use the **chain rule** to write

$$\ddot{x} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

- Substituting that into the standard SHM equation gives

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\omega^2 x$$

- That form of the SHM equation may be solved using **separation of variables**:

Your notes

$$\int v\,\mathrm{d}v = \int -\omega^2 x\,\mathrm{d}x \Rightarrow \frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$$

- But the **velocity is momentarily zero** when the particle reaches its **maximum displacement**
  - After this the particle's velocity changes direction and the particle heads back towards the centre of oscillation
- This gives us a boundary condition (because we know that $v = 0$ when $x = R$) and allows us to find the value of the constant of integration:

$$\frac{1}{2}(0)^2 = -\frac{1}{2}\omega^2(R)^2 + c \Rightarrow c = \frac{1}{2}\omega^2 R^2$$

- Substituting that value of c into the solution and rearranging gives

$$v^2 = \omega^2\left(R^2 - x^2\right)$$

  - where R again is the **amplitude** of the simple harmonic motion
- Note that this version of the solution connects the velocity v to the displacement x, and is independent of the time t
  - Taking square roots gives $v = \pm\,\omega\sqrt{R^2 - x^2}$
  - Be careful when using this to answer questions
    - The **direction of the velocity** (plus or minus) will depend on the displacement (plus or minus) and whether the particle is **moving towards or away from the centre** of oscillation

Examiner Tip

- Even though you may have memorised the forms of the solutions for the SHM equation, it is important on an exam question to derive the solution 'from scratch', showing your method and working

Your notes

Worked example

A particle is moving along a straight line. At time t seconds its displacement x metres from a fixed point O is such that $\frac{d^2x}{dt^2} = -9x$. At time $t = 0$, $x = 2$ and the velocity of the particle is 9 ms^{-1}.

a) Find an expression for the displacement of the particle at time t seconds.

$$\frac{d^2x}{dt^2} = -9x \implies \frac{d^2x}{dt^2} + 9x = 0$$

$$m^2 + 9 = 0 \implies m = \pm 3i$$ auxiliary equation

$$x = A\cos 3t + B\sin 3t$$ general solution

$$\implies \frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t$$ velocity

Now use the boundary conditions:

$$A\cos 0 + B\sin 0 = A = 2$$ $x = 2$ when $t = 0$

$$-3A\sin 0 + 3B\cos 0 = 3B = 9$$ $\frac{dx}{dt} = 9$ when $t = 0$

$$\implies B = 3$$

$$\boxed{x = 2\cos 3t + 3\sin 3t}$$

b) Hence determine the maximum displacement of the particle from O.

Your notes

Rewrite $x = 2\cos 3t + 3\sin 3t$:

$$R = \sqrt{2^2 + 3^2} = \sqrt{13} \qquad \alpha = \tan^{-1}\left(\frac{2}{3}\right) = 0.588002\ldots$$

$$x = \sqrt{13}\sin\left(3t + \tan^{-1}\left(\frac{2}{3}\right)\right)$$

The maximum displacement is $\sqrt{13}$ metres.

c) Show that the relationship between the velocity *v* and displacement *x* of the particle may be described by the equation $v^2 = 117 - 9x^2$.

$$a = \frac{d^2x}{dt^2} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}$$

$$\Rightarrow v\frac{dv}{dx} = \frac{d^2x}{dt^2} \quad \Rightarrow v\frac{dv}{dx} = -9x$$

Now solve by separation of variables :

constant of integration

$$\int v \, dv = \int -9x \, dx \quad \Rightarrow \frac{1}{2}v^2 = -\frac{9}{2}x^2 + c$$

And we know $v = 9$ when $x = 2$, so :

$$\frac{1}{2}(9)^2 = -\frac{9}{2}(2)^2 + c \quad \Rightarrow c = \frac{117}{2}$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{117}{2} - \frac{9}{2}x^2$$

$$\Rightarrow v^2 = 117 - 9x^2$$

Your notes

8.3.2 Damped or Forced Harmonic Motion

Damped or Forced Harmonic Motion

What is damped harmonic motion?

- If we add a term representing a resistive force to the simple harmonic motion equation, the new equation describes a particle undergoing **damped harmonic motion**
  - Depending on the situation being modelled, this resistive force may represent such phenomena as friction or air resistance that resist the motion of the particle
- The standard damped harmonic motion equation is of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = 0$$

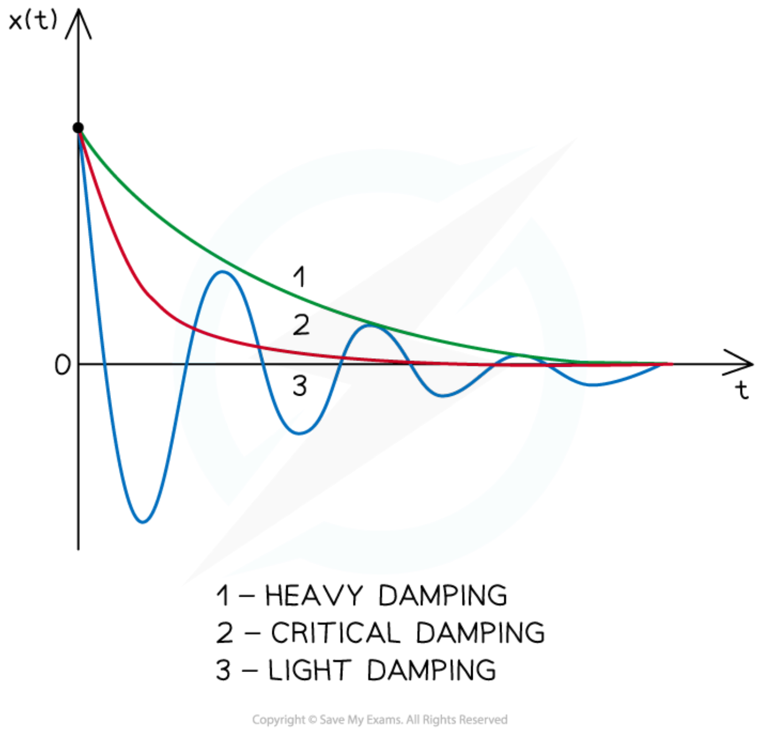
  - Note that that is the same as the simple harmonic motion equation, except for the addition of the damping term $k\frac{\mathrm{d}x}{\mathrm{d}t}$
  - x is the displacement of the particle from a fixed point O at time t
  - k is a positive constant representing the strength of the damping force
  - ω^2 is a positive constant representing the strength of the restoring force that accelerates the particle back towards point O
- The damped harmonic motion equation is a second order homogeneous differential equation, and may be solved using the standard methods for such equations
  - This will involve using the **auxiliary equation** to find the **complementary function** for the equation
- You should, however, be familiar with the three main cases:
  - CASE 1: $k^2 > 4\omega^2$
    - The auxiliary equation has **two distinct real roots**, both of which are negative
    - This is known as **heavy damping** (sometimes also referred to as **overdamping**)
    - The general solution will be of the form $x = Ae^{\alpha t} + Be^{\beta t}$ where α and β are the roots $\frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$ of the auxiliary equation
    - Because α and β are both negative, the two exponentials will decay to zero as t increases
    - Therefore the particle's displacement will also decay to zero, without any oscillations occurring
    - However the decay to zero will not happen as quickly as in Case 2 (critical damping) below
  - CASE 2: $k^2 = 4\omega^2$
    - The auxiliary equation has a **single repeated root**, which is negative
    - This is known as **critical damping**

Your notes

- The general solution will be of the form $x = (A + Bt)e^{\alpha t}$ where $\alpha = -\frac{k}{2}$ is the repeated root of the auxiliary equation
- Because α is negative, the exponential will decay to zero as *t* increases
- Therefore the particle's displacement will also decay to zero, without any oscillations occurring (although depending on the values of *A* and *B* it is possible that the particle will change direction once as the decay to zero occurs
- For a given value of ω^2, the displacement of the critical damping case will decay to zero faster than any instance of the heavy damping case

- CASE 3: $k < 4\omega^2$
  - The auxiliary equation has **complex roots** which form a complex conjugate pair
  - This known as **light damping** (sometimes also referred to as **underdamping**)
  - The general solution will be of the form $x = e^{pt}(A\cos qt + B\sin qt)$, where $p = -\frac{k}{2}$ and $q = \frac{\sqrt{4\omega^2 - k^2}}{2}$
  - Because *p* is negative, the exponential will decay to zero as *t* increases
  - Therefore the particle's displacement will also decay to zero
  - However the cosine and sine terms mean that the particle will continue to oscillate with decreasing amplitude as the decay to zero occurs
- In all three cases, initial or boundary conditions given in a question may allow you to work out the precise values of the arbitrary constants *A* and *B*

- The following displacement-time graph illustrates the behaviour displayed by a particle for each of the three cases



What is forced harmonic motion?

- If we add a term representing an external 'driving' force to the damped harmonic motion equation, the new equation describes a particle undergoing **forced harmonic motion**
- The standard forced harmonic motion equation is of the form

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$$

  - Note that it is the same as the damped harmonic motion equation, except for the addition of the driving term f(t) on the right-hand side of the equation
  - x is the displacement of the particle from a fixed point O at time t
  - k is a non-negative constant representing the strength of the damping force
    - If $k = 0$ then there is no damping force
  - ω^2 is a positive constant representing the strength of the restoring force that accelerates the particle back towards point O
- The forced harmonic motion equation is a second order non-homogeneous differential equation, and may be solved using the standard methods for such equations
  - This will involve using the **auxiliary equation** to find the **complementary function** for the equation
  - It will also involve finding the **particular integral** for the equation, based on the form of f(t)
  - Initial or boundary conditions given in a question may allow you to work out the precise values of any arbitrary constants in your general solution

- If $k \neq 0$ then the long-term behaviour of the system will be predominantly determined by the driving force $f(t)$
  - If $k = 0$, then the long-term behaviour will be a combination of the effects of the driving force and of the system's natural oscillation

Examiner Tip

- Even though you may have memorised the forms of the solutions for the damped harmonic motion equation, it is important on an exam question to derive the solution 'from scratch', showing your method and working

Your notes

Your notes

Worked example

A particle is moving along a straight line. At time t seconds its displacement x metres from a fixed point O is such that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\frac{\mathrm{d}x}{\mathrm{d}t}+9x=0$. At time $t=0$, $x=1$ and the velocity of the particle is 3 ms^{-1}.

(a) Given that $k=10$, find an expression for the displacement of the particle at time t seconds and describe the type of damping that is present.

Your notes

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 0 \qquad k = 10 \quad \omega^2 = 9$$

$$10^2 = 100 > 36 = 4 \times 9 \Rightarrow \boxed{\text{heavy damping}}$$

$$m^2 + 10m + 9 = (m+1)(m+9) = 0 \Rightarrow m = -1, -9$$ auxiliary equation

$$x = Ae^{-t} + Be^{-9t}$$ general solution

$$\Rightarrow \frac{dx}{dt} = -Ae^{-t} - 9Be^{-9t}$$ velocity

Now use the boundary conditions:

$$Ae^0 + Be^0 = A + B = 1$$ $x = 1$ when $t = 0$

$$-Ae^0 - 9Be^0 = -A - 9B = 3$$ $\frac{dx}{dt} = 3$ when $t = 0$

$$\Rightarrow A = \frac{3}{2}, \quad B = -\frac{1}{2}$$ solve simultaneous equations

$$\boxed{x = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-9t}}$$

(b) Given that $k = 6$, find an expression for the displacement of the particle at time t seconds and describe the type of damping that is present.

Your notes

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0 \qquad k = 6 \quad \omega^2 = 9$$

$$6^2 = 36 = 4 \times 9 \implies \boxed{\text{critical damping}}$$

$$m^2 + 6m + 9 = (m+3)^2 = 0 \implies m = -3$$ auxiliary equation

repeated root

$$x = (A + Bt)e^{-3t}$$ general solution

$$\implies \frac{dx}{dt} = (-3A + B - 3Bt)e^{-3t}$$ velocity

Now use the boundary conditions:

$$(A + B(0))e^0 = A = 1 \qquad x = 1 \text{ when } t = 0$$

$$(-3A + B - 3B(0))e^0 = -3A + B = 3 \qquad \frac{dx}{dt} = 3 \text{ when } t = 0$$

$$\implies A = 1, \quad B = 6$$ solve simultaneous equations

$$\boxed{x = (6t + 1)e^{-3t}}$$

(c) Given that $k = 4$, find an expression for the displacement of the particle at time t seconds and describe the type of damping that is present.

Your notes

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 0 \qquad k = 4 \qquad \omega^2 = 9$$

$$4^2 = 16 < 36 = 4 \times 9 \implies \boxed{\text{light damping}}$$

$$m^2 + 4m + 9 = (m+2)^2 + 5 = 0 \implies m = -2 \pm \sqrt{5}\,i$$ auxiliary equation

$$x = e^{-2t}\left(A\cos\sqrt{5}t + B\sin\sqrt{5}t\right)$$ general solution

$$\implies \frac{dx}{dt} = e^{-2t}\left(\left(-2A + \sqrt{5}B\right)\cos\sqrt{5}t + \left(-\sqrt{5}A - 2B\right)\sin\sqrt{5}t\right)$$ velocity

Now use the boundary conditions:

$$e^0\left(A\cos 0 + B\sin 0\right) = A = 1$$ $x = 1$ when $t = 0$

$$e^0\left(\left(-2A + \sqrt{5}B\right)\cos 0 + \left(-\sqrt{5}A - 2B\right)\sin 0\right) = -2A + \sqrt{5}B = 3$$ $\frac{dx}{dt} = 3$ when $t = 0$

$$\implies A = 1, \quad B = \sqrt{5}$$ solve simultaneous equations

$$\boxed{x = e^{-2t}\left(\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t\right)}$$

The system is now modified so that at time *t* seconds the particle's displacement *x* metres from the fixed point O is such that . At time $t = 0$, $x = 1$ and the velocity of the particle is 3 ms^{-1}.

(d) Find an expression for the displacement of the particle at time *t* seconds and describe the long-term behaviour of the system.

Your notes

From part (c), we already know the complementary function is $x = e^{-2t}(A\cos\sqrt{5}t + B\sin\sqrt{5}t)$.

$x = \lambda\cos t + \mu\sin t$ test form of particular integral

$\Rightarrow \frac{dx}{dt} = \mu\cos t - \lambda\sin t \Rightarrow \frac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$

Substitute those into the differential equation:

$(-\lambda\cos t - \mu\sin t) + 4(\mu\cos t - \lambda\sin t) + 9(\lambda\cos t + \mu\sin t)$

$= (8\lambda + 4\mu)\cos t + (8\mu - 4\lambda)\sin t = \sin t$

$\Rightarrow 8\lambda + 4\mu = 0, \quad -4\lambda + 8\mu = 1$ compare coefficients

$\Rightarrow \lambda = -\frac{1}{20}, \quad \mu = \frac{1}{10}$ solve simultaneous equations

$\Rightarrow x = -\frac{1}{20}\cos t + \frac{1}{10}\sin t$ particular integral

$x = e^{-2t}(A\cos\sqrt{5}t + B\sin\sqrt{5}t) - \frac{1}{20}\cos t + \frac{1}{10}\sin t$ general solution

$\frac{dx}{dt} = e^{-2t}\left((-2A + \sqrt{5}B)\cos\sqrt{5}t + (-\sqrt{5}A - 2B)\sin\sqrt{5}t\right) + \frac{1}{10}\cos t + \frac{1}{20}\sin t$

velocity

Now use the boundary conditions:

$e^0(A\cos 0 + B\sin 0) - \frac{1}{20}\cos 0 + \frac{1}{10}\sin 0 = A - \frac{1}{20} = 1$ $x = 1$ when $t = 0$

$e^0\left((-2A + \sqrt{5}B)\cos 0 + (-\sqrt{5}A - 2B)\sin 0\right) + \frac{1}{10}\cos 0 + \frac{1}{20}\sin 0$

$= -2A + \sqrt{5}B + \frac{1}{10} = 3$ $\frac{dx}{dt} = 3$ when $t = 0$

$\Rightarrow A = \frac{21}{20}, \quad B = \sqrt{5}$ solve simultaneous equations

$x = e^{-2t}\left(\frac{21}{\ }\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t\right) - \frac{1}{\ }\cos t + \frac{1}{\ }\sin t$

$x = e^{-2t}\left(\frac{21}{20}\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t\right) - \frac{1}{20}\cos t + \frac{1}{10}\sin t$

Your notes

As $t \to \infty$, $e^{-2t} \to 0$.

And $-\frac{1}{20}\cos t + \frac{1}{10}\sin t = \frac{\sqrt{5}}{20}\sin\left(t - \tan^{-1}\left(\frac{1}{2}\right)\right)$.

So as $t \to \infty$, $x \to \frac{\sqrt{5}}{20}\sin\left(t - \tan^{-1}\left(\frac{1}{2}\right)\right)$.

In the long term the $e^{-2t}\left(\frac{21}{20}\cos\sqrt{5}t + \sqrt{5}\sin\sqrt{5}t\right)$ part of the solution will decay to zero, and the particle will oscillate with amplitude $\frac{\sqrt{5}}{20}$ metres and period 2π seconds.