AS Core Pure

Complex Numbers	Argand Diagrams	Series	Roots of Polynomials	Volumes of Revolution
Matrices	Linear Transformations	Proof by Induction	Vectors	

Contains:
SAMs
2018
2019
2020
2021
2022

A2 Topics

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A2 Core Pure

Complex Numbers pt. 2	Series – Method of Differences	Maclaurin Series	Methods in Calculus (includes hyperbolics)	Volumes of Revolution pt. 2
Polar Coordinates	Hyperbolic Functions	Differential Equations		

Contains:
SAMs
2019
2020
2021
2022

Complex Numbers

2. Given that

$$z_1 = 2 + 3\mathrm{i}$$

$$|z_1 z_2| = 39\sqrt{2}$$

$$\arg(z_1 z_2) = \frac{\pi}{4}$$

where z_1 and z_2 are complex numbers,

(a) write z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$

Give the exact value of r and give the value of θ in radians to 4 significant figures.

(2)

(b) Find z_2 giving your answer in the form $a + \mathrm{i}b$ where a and b are integers.

(6)

2(a)	$\lvert z_1\rvert = \sqrt{13}$ and $\arg z_1 = \tan^{-1}\left(\frac{3}{2}\right)$	B1	1.1b
	$z_1 = \sqrt{13}\left(\cos 0.9828 + \mathrm{i}\sin 0.9828\right)$	B1ft	1.1b
		(2)	
(b)	A complete method to find the modulus of z_2 e.g. $\lvert z_1\rvert = \sqrt{13}$ and uses $\lvert z_1 z_2\rvert = \lvert z_1\rvert \times \lvert z_2\rvert = 39\sqrt{2} \Rightarrow \lvert z_2\rvert = 3\sqrt{26}$ or $\sqrt{234}$	M1 A1	3.1a 1.1b
	A complete method to find the argument of z_2 e.g. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{4} \Rightarrow \arg(z_2) = \ldots$ $\arg(z_2) = \frac{\pi}{4} - \tan^{-1}\left(\frac{3}{2}\right)$ or $\frac{\pi}{4} - 0.9828$ or $-0.1974\ldots$	M1 A1	3.1a 1.1b
	$z_2 = 3\sqrt{26}\left(\cos('-0.1974\ldots') + \mathrm{i}\sin('-0.1974\ldots')\right)$ or $z_2 = a + b\mathrm{i} \Rightarrow a^2 + b^2 = 234$ and $\tan^{-1}(-0.1974) = \frac{b}{a} \Rightarrow \frac{b}{a} = -0.2$ $\Rightarrow a = \ldots$ and $b = \ldots$	ddM1	1.1b
	Deduces that $z_2 = 15 - 3\mathrm{i}$ only	A1	2.2a

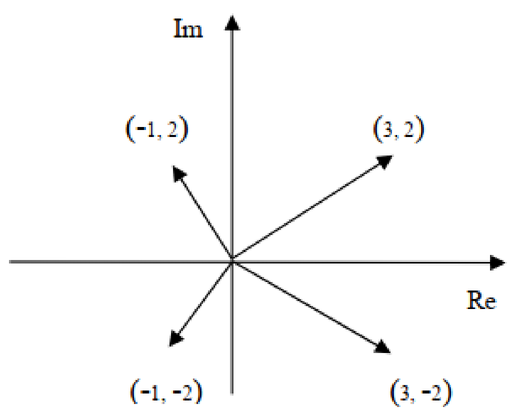
3. $$f(z) = z^4 + az^3 + 6z^2 + bz + 65$$

where a and b are real constants.

Given that $z = 3 + 2\mathbf{i}$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

(9)

A2 SAMs Paper 1

Question	Scheme	Marks	AOs
3	$z = 3 - 2\mathrm{i}$ is also a root	B1	1.2
	$(z-(3+2\mathrm{i}))(z-(3-2\mathrm{i})) = \ldots$ **or** Sum of roots = 6, Product of roots = 13 $\Rightarrow \ldots$	M1	3.1a
	$= z^2 - 6z + 13$	A1	1.1b
	$(z^4 + az^3 + 6z^2 + bz + 65) = (z^2 - 6z + 13)(z^2 + cz + 5) \Rightarrow c = \ldots$	M1	3.1a
	$z^2 + 2z + 5 = 0$	A1	1.1b
	$z^2 + 2z + 5 = 0 \Rightarrow z = \ldots$	M1	1.1a
	$z = -1 \pm 2\mathrm{i}$	A1	1.1b
		B1 $3 \pm 2\mathrm{i}$ Plotted correctly	1.1b
		B1ft $-1 \pm 2\mathrm{i}$ Plotted correctly	1.1b
		(9 marks)	

1.

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

Given that $-1 + 2i$ and $3 - i$ are two roots of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(4)

(b) find the values of a, b, c and d.

(5)

BICEN MATHS

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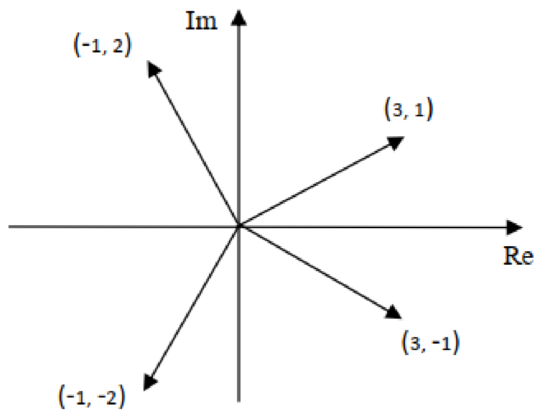
A2 2019 Paper 1

Complex Numbers

1(a)	$z=-1-2\text{i}$ or $z=3+\text{i}$		M1	1.2
	$z=-1-2\text{i}$ and $z=3+\text{i}$		A1	1.1b
			B1	1.1b
			B1	1.1b
			(4)	
(b) Way 1	$\left(z-(-1+2\text{i})\right)\left(z-(-1-2\text{i})\right)=...$ or $\left(z-(3+\text{i})\right)\left(z-(3-\text{i})\right)=...$	$\text{f}(z)=\left(z-(-1+2\text{i})\right)\left(z-(-1-2\text{i})\right)\left(z-(3+\text{i})\right)\left(z-(3-\text{i})\right)=...$	M1	3.1a
	z^2+2z+5 or $z^2-6z+10$	e.g. $\text{f}(z)=\left(z^2+2z+5\right)(...)$	A1	1.1b
	z^2+2z+5 and $z^2-6z+10$	$\text{f}(z)=\left(z^3+z^2(-1-\text{i})+z(-1+2\text{i})-15-5\text{i}\right)(...)$	A1	1.1b
	$\text{f}(z)=\left(z^2+2z+5\right)\left(z^2-6z+10\right)$	Expands the brackets to forms a quartic	M1	3.1a
	$\text{f}(z)=z^4-4z^3+3z^2-10z+50$ or States $a=-4, b=3, c=-10, d=50$		A1	1.1b
			(5)	

1. Given that

$$z_1 = 3\left(\cos\left(\frac{\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{3}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) - \text{i}\sin\left(\frac{\pi}{12}\right)\right)$$

(a) write down the exact value of

(i) $|z_1 z_2|$

(ii) $\arg(z_1 z_2)$

(2)

Given that $w = z_1 z_2$ and that $\arg(w^n) = 0$, where $n \in \mathbb{Z}^+$

(b) determine

(i) the smallest positive value of n

(ii) the corresponding value of $|w^n|$

(3)

A2 2021 Paper 2

Complex Numbers

Question	Scheme	Marks	AOs
1(a) (i)	$\lvert z_1z_2\rvert = 3\sqrt{2}$	B1	1.1b
(ii)	$\arg(z_1z_2) = \frac{\pi}{3} + \left(-\frac{\pi}{12}\right) = \frac{\pi}{4}$ o.e.	B1	1.1b
		(2)	
(b) (i)	$n = 8$	B1ft	2.2a
(ii)	$\lvert w^n\rvert = (\text{'their}\lvert z_1z_2\rvert\text{'})^{\text{their } n}$	M1	1.1b
	$\lvert w^n\rvert = 104\,976$	A1	1.1b
		(3)	
			(5 marks)

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A2 2022 Paper 1

Complex Numbers

1. $f(z) = z^3 + az + 52$ where a is a real constant

Given that $2 - 3i$ is a root of the equation $f(z) = 0$

(a) write down the other complex root.

(1)

(b) Hence

(i) solve completely $f(z) = 0$

(ii) determine the value of a

(4)

(c) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram.

(1)

BICEN MATHS

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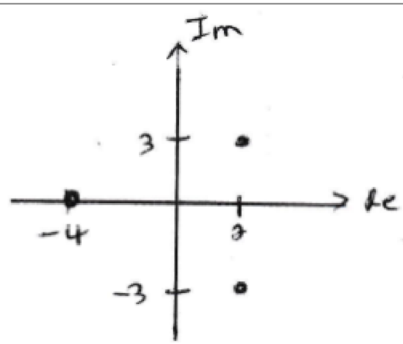
A2 2022 Paper 1

Complex Numbers

1(a)	$2+3i$	B1	1.1b
		(1)	
(b) (i)	$z*=2+3i$ so $z+z*=4$, $zz*=13$ $z+z*+\alpha=0 \Rightarrow \alpha=\ldots$ or $\alpha zz*=-52 \Rightarrow \alpha=-\frac{52}{\text{"13"}}=\ldots$ or $z^2-(\text{sum roots})z+(\text{product roots})=0$ or $\left(z-(2+3i)\right)\left(z-(2-3i)\right)=\ldots$ $\Rightarrow (z^2-4z+13)\underline{(z+4)} \Rightarrow z=\ldots$	M1	3.1a
	$z=2\pm 3\mathrm{i},\ -4$	A1	1.1b
(ii)	$(z^2-4z+13)(z+4)$ expands the brackets to find value for a Or a = pair sum $=-4(2+3i+2-3i)+13=\ldots$ Or $f(-4)/f(2\pm 3i)=0 \Rightarrow \ldots \Rightarrow a=\ldots$	M1	1.1b
	$a=-3$	A1	2.2a
		(4)	
(c)		B1ft	1.1b
		(1)	
			(6 marks)

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7. Given that $z = a + b\text{i}$ is a complex number where a and b are real constants,

(a) show that zz^* is a real number.

(2)

Given that

- $zz^* = 18$
- $\dfrac{z}{z^*} = \dfrac{7}{9} + \dfrac{4\sqrt{2}}{9}\text{i}$

(b) determine the possible complex numbers z

(5)

A2 2022 Paper 1

Complex Numbers

7(a)	$z* = a - bi$ then $zz* = (a+bi)(a-bi) = \ldots$	M1	1.1b
	$zz* = a^2 + b^2$ therefore, a real number	A1	2.4
		(2)	
(b)	$\frac{z}{z*} = \frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{(a^2-b^2)+2abi}{a^2+b^2} = \frac{7}{9} + \frac{4\sqrt{2}i}{9}$ or $\frac{z}{z^*} = \frac{z^2}{zz^*} = \frac{z^2}{18} \Rightarrow$ $z^2 = 14 + 8\sqrt{2}i$ or $a + bi = \left(\frac{7}{9} + \frac{4\sqrt{2}i}{9}\right)(a - bi) = .. + .. i$	M1	1.1b
	Forms two equations from $a^2 + b^2 = 18$ or $\frac{a^2-b^2}{18} = \frac{7}{9}$ or $\frac{a^2-b^2}{a^2+b^2} = \frac{7}{9}$ or $\frac{2ab}{18} = \frac{4\sqrt{2}}{9}$ or $\frac{2ab}{a^2+b^2} = \frac{4\sqrt{2}}{9}$ or $a = \frac{7}{9}a + \frac{4\sqrt{2}}{9}b$ oe	M1 A1	3.1a 1.1b
	Solves the equations simultaneously e.g. $a^2 + b^2 = 18$ and $a^2 - b^2 = 14$ leading to a value for a or b	dM1	1.1b
	$z = \pm(4 + \sqrt{2}i)$	A1	2.2a
		(5)	
			(7 marks)

Argand Diagrams

8. (a) Shade on an Argand diagram the set of points

$$\left\{z \in \mathbb{C} : |z - 4\mathrm{i}| \leqslant 3\right\} \cap \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z + 3 - 4\mathrm{i}) \leqslant \frac{\pi}{4}\right\}$$

(6)

The complex number w satisfies

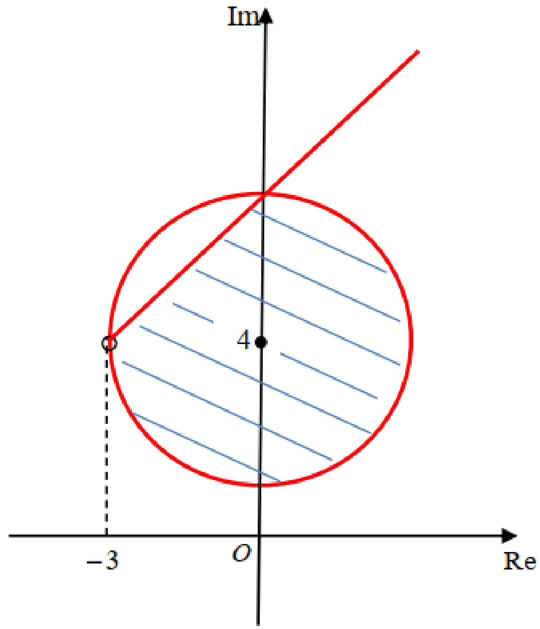
$$|w - 4\mathrm{i}| = 3$$

(b) Find the maximum value of $\arg w$ in the interval $(-\pi, \pi]$.
Give your answer in radians correct to 2 decimal places.

(2)

AS SAMs

Argand Diagrams

Question	Scheme	Marks	AOs
8(a)		M1	1.1b
		A1	1.1b
		M1	1.1b
		A1	2.2a
		M1	3.1a
		A1	1.1b
		(6)	
(b)	$(\arg w)_{\max} = \frac{\pi}{2} + \arcsin\left(\frac{3}{4}\right)$	M1	3.1a
	$= 2.42$ (2dp) **cao**	A1	1.1b
		(2)	
		(8 marks)	

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3. (a) Shade on an Argand diagram the set of points

$$\{z \in \mathbb{C} : |z-1-\mathrm{i}| \leqslant 3\} \cap \left\{z \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(z-2) \leqslant \frac{3\pi}{4}\right\}$$

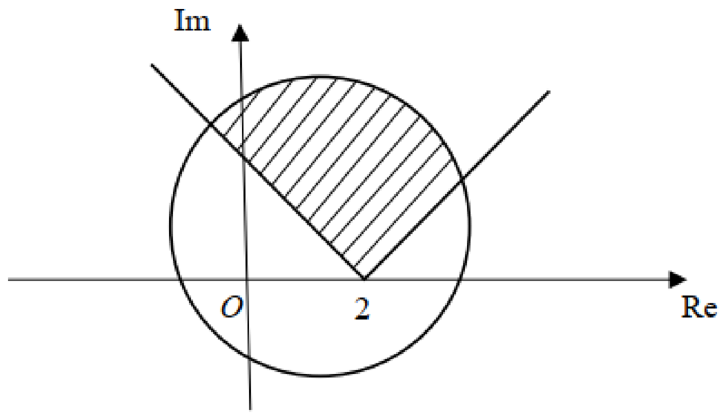
(5)

The complex number w satisfies

$$|w-1-\mathrm{i}| = 3 \text{ and } \arg(w-2) = \frac{\pi}{4}$$

(b) Find, in simplest form, the exact value of $|w|^2$

(4)

Question	Scheme	Marks	AOs
3(a)		M1	1.1b
		M1	1.1b
		A1	2.2a
		M1	3.1a
		A1	1.1b
		(5)	
(b)	$(x-1)^2+(y-1)^2=9,\ \ y=x-2 \Rightarrow x=\ldots,$ or $y=\ldots$	M1	3.1a
	$x=2+\dfrac{\sqrt{14}}{2}, y=\dfrac{\sqrt{14}}{2}$	A1	1.1b
	$\lvert w\rvert^2=\left(2+\dfrac{\sqrt{14}}{2}\right)^2+\left(\dfrac{\sqrt{14}}{2}\right)^2$	M1	1.1b
	$=11+2\sqrt{14}$	A1	1.1b
		(4)	
		(9 marks)	

5.

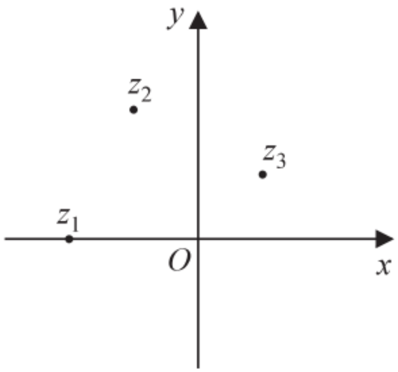


Figure 1

The complex numbers $z_1 = -2$, $z_2 = -1 + 2\mathrm{i}$ and $z_3 = 1 + \mathrm{i}$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + \mathrm{i}y$

(a) Explain why z_1, z_2 and z_3 cannot all be roots of a quartic polynomial equation with real coefficients.

(2)

(b) Show that $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \dfrac{\pi}{4}$

(3)

(c) Hence show that $\arctan(2) - \arctan\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$

(2)

A copy of Figure 1, labelled Diagram 1, is given on page 12.

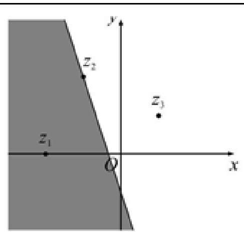
(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leqslant |z - 1 - \mathrm{i}|$$

(2)

AS 2019

Argand Diagrams

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	The complex roots of a real polynomial occur in conjugate pairs	M1	This mark is given for a correct statement that complex roots of a real polynomial occur in conjugate pairs
	A polynomial with roots z_1, z_2 and z_3 also needs roots z_2* and z_3* (five in total) However, a quartic has at most four roots, so no quartic can have z_1, z_2 and z_3 roots.	A1	This mark is given for s correct statement referencing that a quartic has four roots but would need five if z_1, z_2 and z_3 were to all be roots.
(b)	$\frac{z_2 - z_1}{z_3 - z_1} = \frac{-1+2\mathrm{i}-(-2)}{1+\mathrm{i}-(-2)} = \frac{1+2\mathrm{i}}{3+\mathrm{i}} \times \frac{3-\mathrm{i}}{3-\mathrm{i}}$	M1	This mark is given for substituting into the expression and multiplying the numerator and denominator by the conjugate of the denominator to find the quotient
	$= \frac{3-\mathrm{i}+6\mathrm{i}+2}{9+1} = \frac{5\mathrm{i}}{10} = \frac{1}{2} + \frac{1}{2}\mathrm{i}$	A1	This mark is given for simplifying to find $\frac{1}{2} + \frac{1}{2}\mathrm{i}$
	$\arctan\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = \arctan(1)$ and $\frac{1}{2} + \frac{1}{2}\mathrm{i}$ is in the first quadrant; hence $\arg \frac{z_2 - z_1}{z_3 - z_1} = \frac{\pi}{4}$	A1	This mark is given for using arctan (1) and making reference to the first quadrant to justify the argument
(c)	$\arg \frac{z_2 - z_1}{z_3 - z_1}$ $= \arg(z_2 - z_1) - \arg(z_3 - z_1)$ $= \arg(-1 + 2\mathrm{i} - (-2)) - \arg(1 + \mathrm{i} - (-2))$ $= \arg(1 + 2\mathrm{i}) - \arg(3 + \mathrm{i})$	M1	This mark is given for applying the formula for the argument of a difference of complex numbers
	Hence $\arctan(2) - \arctan\left(\frac{1}{3}\right)$ $= \arg\left(\frac{1}{2} + \frac{1}{2}\mathrm{i}\right) = \frac{\pi}{4}$	A1	This mark is given for a complete proof identifying the required arguments
(d)		B1	This mark is given for a line passing through z_2 with one side shaded
		B1	This mark is given for the area below and left of the line shaded

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7.

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

The equation $f(z) = 0$ has complex roots z_1, z_2, z_3 and z_4
When plotted on an Argand diagram, the points representing z_1, z_2, z_3 and z_4 form the vertices of a square, with one vertex in each quadrant.
Given that $z_1 = 2 + 3\text{i}$, determine the values of a, b, c and d.

(6)

7	$z_2 = 2-3\text{i}$	B1
	$(z_3 =)\, p-3\text{i}$ and $(z_4 =)\, p+3\text{i}$ May be seen in an Argand diagram	M1
	$(z_3 =) -4-3\text{i}$ and $(z_4 =) -4+3\text{i}$ May be seen in an Argand diagram, but the complex numbers used in their method takes precedence	A1
	$(z^2-4z+13)(z^2+8z+25)$ or $(z-(2-3\text{i}))(z-(2+3\text{i}))(z-(-4-3\text{i}))(z-(-4+3\text{i}))$ or $a=-[(2-3\text{i})+(2+3\text{i})+(-4-3\text{i})+(-4+3\text{i})]$ and $b=(2-3\text{i})(2+3\text{i})+(2-3\text{i})(-4-3\text{i})+(2-3\text{i})(-4+3\text{i})$ $+(2+3\text{i})(-4-3\text{i})+(2+3\text{i})(-4+3\text{i})+(-4-3\text{i})(-4+3\text{i})$ and $c=-\begin{bmatrix}(2-3\text{i})(2+3\text{i})(-4-3\text{i})+(2-3\text{i})(2+3\text{i})(-4+3\text{i}) \\ +(2-3\text{i})(-4-3\text{i})(-4+3\text{i})+(2+3\text{i})(-4-3\text{i})(-4+3\text{i})\end{bmatrix}$ and $d=(2-3\text{i})(2+3\text{i})(-4-3\text{i})(-4+3\text{i})$ or Substitutes in one root from each conjugate pair and equates real and imaginary parts and solves simultaneously $(2\pm3\text{i})^4+a(2\pm3\text{i})^3+b(2\pm3\text{i})^2+c(2\pm3\text{i})+d=0$ $(-4\pm3\text{i})^4+a(-4\pm3\text{i})^3+b(-4\pm3\text{i})^2+c(-4\pm3\text{i})+d=0$	dM1
	$a=4,\ b=6,\ c=4,\ d=325$ $\text{f}(z)=z^4+4z^3+6z^2+4z+325$	A1 A1
		(6)

10. Given that there are two distinct complex numbers z that satisfy

$$\{z : |z - 3 - 5\mathrm{i}| = 2r\} \cap \left\{z : \arg(z - 2) = \frac{3\pi}{4}\right\}$$

determine the exact range of values for the real constant r.

(7)

10	$(x-3)^2+(y-5)^2=(2r)^2$ and $y=-x+2$	B1
	$(x-3)^2+(-x+2-5)^2=(2r)^2$ or $(-y+2-3)^2+(y-5)^2=(2r)^2$	M1
	$2x^2+18-4r^2=0$ or $2y^2-8y+26-4r^2=0$	A1
	$b^2-4ac>0 \Rightarrow 0^2-4(2)(18-4r^2)>0 \Rightarrow r>\ldots$ or $x^2=9-2r^2 \Rightarrow 9-2r^2>0 \Rightarrow r>\ldots$ or $b^2-4ac>0 \Rightarrow (-8)^2-4(2)(26-4r^2)>0 \Rightarrow r>\ldots$	dM1
	Finds a maximum value for r $(2r)^2=5^2+(3-2)^2 \Rightarrow r=\ldots$	M1
	$\dfrac{3\sqrt{2}}{2}<r<\dfrac{\sqrt{26}}{2}$ o.e.	A1 A1

5.

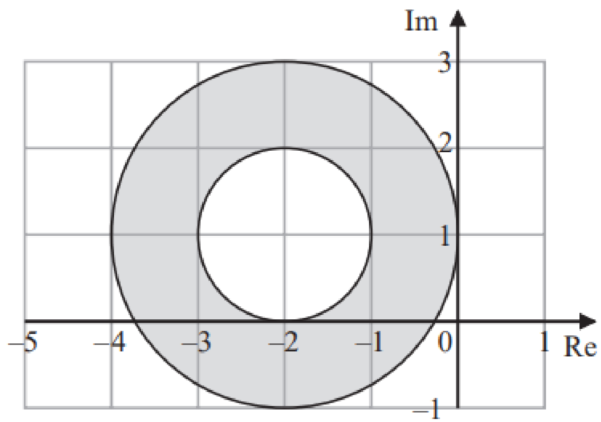


Figure 1

Figure 1 shows an Argand diagram.

The set P, of points that lie within the shaded region including its boundaries, is defined by

$$P = \{z \in \mathbb{C} : a \leqslant |z + b + c\mathrm{i}| \leqslant d\}$$

where a, b, c and d are integers.

(a) Write down the values of a, b, c and d.

(3)

The set Q is defined by

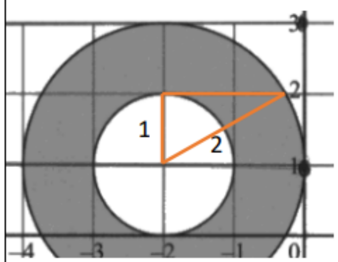
$$Q = \{z \in \mathbb{C} : a \leqslant |z + b + c\mathrm{i}| \leqslant d\} \cap \{z \in \mathbb{C} : |z - \mathrm{i}| \leqslant |z - 3\mathrm{i}|\}$$

(b) Determine the exact area of the region defined by Q, giving your answer in simplest form.

(7)

AS 2021

Argand Diagrams

Question	Scheme	Marks	AOs
5(a)	$a = 1, d = 2$	B1	1.1b
	$b = 2$	B1	1.1b
	$c = -1$	B1	1.1b
		(3)	
(b)	$\lvert z - \text{i}\rvert = \lvert z - 3\text{i}\rvert \Rightarrow y = 2$	B1	2.2a
	Area between the circles $= \pi \times 2^2 - \pi \times 1^2$	M1	1.1a
	 Angle subtended at centre = $2 \times \cos^{-1}\left(\frac{1}{2}\right)$ Alternatively $(x+2)^2 + (y-1)^2 = 4,\ \ y = 2 \Rightarrow x = \ldots$ Or $x = \sqrt{2^2 - 1^2}$ Leading to Angle subtended at centre = $2 \times \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$	M1	3.1a
	Segment area = $\frac{1}{2} \times \frac{2\pi}{3} \times 2^2 - \frac{1}{2} \times 2^2 \times \sin\left(\text{"}\frac{2\pi}{3}\text{"}\right)\left\{= \frac{4}{3}\pi - \sqrt{3}\right\}$	M1 A1	2.1 1.1b
	Area of Q: $\pi \times 2^2 - \pi \times 1^2 - \left(\frac{1}{2} \times \frac{2\pi}{3} \times 2^2 - \frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right)\right)$	M1	3.1a
	$= \frac{5\pi}{3} + \sqrt{3}$	A1	1.1b
		(7)	
			(10 marks)

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2. (a) Express the complex number $w = 4\sqrt{3} - 4\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$ where $r > 0$ and $-\pi < \theta \leqslant \pi$

(4)

(b) Show, on a single Argand diagram,

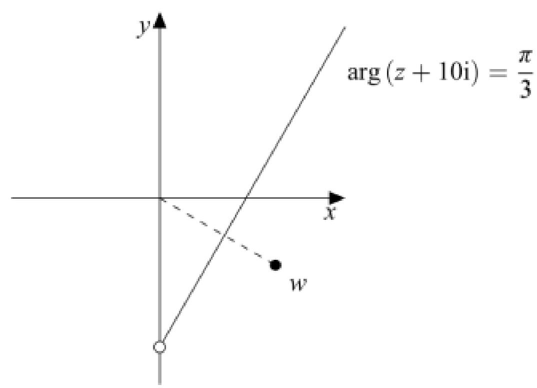
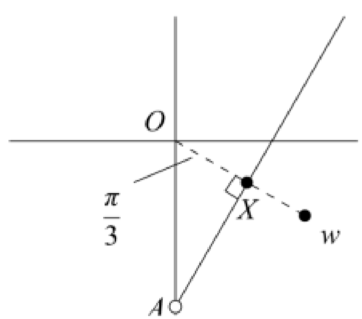
(i) the point representing w

(ii) the locus of points defined by $\arg(z + 10\text{i}) = \frac{\pi}{3}$

(3)

(c) Hence determine the minimum distance of w from the locus $\arg(z + 10\text{i}) = \frac{\pi}{3}$

(3)

Question	Scheme	Notes	Marks
2(a)	$\lvert w\rvert=\sqrt{\left(4\sqrt{3}\right)^2+(-4)^2}=8$		B1
	$\arg w=\arctan\left(\frac{\pm 4}{4\sqrt{3}}\right)=\arctan\left(\pm\frac{1}{\sqrt{3}}\right)$		M1
	$=-\frac{\pi}{6}$		A1
	So $(w=)8\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$		A1
			(4)
(b)		(i) w in 4th quadrant with either $\left(4\sqrt{3},-4\right)$ seen or $-\frac{\pi}{4}<\arg w<0$	B1
		(ii) half line with positive gradient emanating from imaginary axis.	M1
		The half line should pass between O and w starting from a point on the imaginary axis below w	A1
			(3)
(c)		ΔOAX is right angled at X so $OX=10\sin\frac{\pi}{6}=5$ (oe)	M1
		So shortest distance is $WX=OW-OX=$ '8' $-5=\ldots$	M1
		So min distance is 3	A1

HOME

2. In an Argand diagram, the points A and B are represented by the complex numbers $-3 + 2\text{i}$ and $5 - 4\text{i}$ respectively. The points A and B are the end points of a diameter of a circle C.

(a) Find the equation of C, giving your answer in the form

$$|z - a| = b \qquad a \in \mathbb{C},\ b \in \mathbb{R}$$

(3)

The circle D, with equation $|z - 2 - 3\text{i}| = 2$, intersects C at the points representing the complex numbers z_1 and z_2

(b) Find the complex numbers z_1 and z_2

(6)

A2 2020 Paper 2

Argand Diagrams

2(a)	Centre of circle C is $(1, -1)$	B1	1.1b
	$r=\sqrt{(5-1)^2+(-4+1)^2}=5$ or $r=\sqrt{(-3-1)^2+(2+1)^2}=5$ or $r=\frac{1}{2}\sqrt{(-3-5)^2+(2+4)^2}=5$	M1	3.1a
	$\lvert z-1+\mathrm{i}\rvert=5$ or $\lvert z-(1-\mathrm{i})\rvert=5$	A1	2.5
		(3)	
(b)	$(x-1)^2+(y+1)^2=25,\quad (x-2)^2+(y-3)^2=4$ $x^2-2x+1+y^2+2y+1=25$ $x^2-4x+4+y^2-6y+9=4$ $\Rightarrow 2x+8y=32$	M1	3.1a
	$(16-4y)^2-4(16-4y)+4+y^2-6y+9=4$ or $x^2-4x+4+\left(\frac{16-x}{4}\right)^2-6\left(\frac{16-x}{4}\right)+9=4$	M1	1.1b
	$17y^2-118y+201=0$ or $17x^2-72x+16=0$	A1	1.1b
	$17y^2-118y+201=0\Rightarrow(17y-67)(y-3)=0\Rightarrow y=\frac{67}{17},3$ or $17x^2-72x+16=0\Rightarrow(17x-4)(x-4)=0\Rightarrow x=\frac{4}{17},4$	M1	1.1b
	$y=\frac{67}{17},3\Rightarrow x=\frac{4}{17},4$ or $x=\frac{4}{17},4\Rightarrow y=\frac{67}{17},3$	M1	2.1
	$4+3\mathrm{i},\ \frac{4}{17}+\frac{67}{17}\mathrm{i}$	A1	2.2a
		(6)	
			(9 marks)

BICEN MATHS

HOME

1. A student was asked to answer the following:

For the complex numbers $z_1 = 3 - 3\mathrm{i}$ and $z_2 = \sqrt{3} + \mathrm{i}$, find the value of $\arg\left(\frac{z_1}{z_2}\right)$

The student's attempt is shown below.

Line 1 → $\arg(z_1) = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$

Line 2 → $\arg(z_2) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Line 3 → $\arg\left(\frac{z_1}{z_2}\right) = \frac{\arg(z_1)}{\arg(z_2)}$

Line 4 → $= \left(\frac{\pi}{4}\right) \Big/ \left(\frac{\pi}{6}\right) = \frac{3}{2}$

The student made errors in line 1 and line 3

Correct the error that the student made in

(a) (i) line 1

(ii) line 3

(2)

(b) Write down the correct value of $\arg\left(\frac{z_1}{z_2}\right)$

(1)

A2 2022 Paper 2

Argand Diagrams

Question	Scheme	Marks
1(a) (i) (a) (ii)	$\{arg(z_1) =\} tan^{-1}\left(\frac{-3}{3}\right)$ or $\{arg(z_1) =\} tan^{-1}(-1)$ or $\{arg(z_1) =\} - tan^{-1}\left(\frac{3}{3}\right)$ or $\{arg(z_1) =\} -\frac{\pi}{4}$ or $\{arg(z_1) =\} 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ or states should be -3 not 3 on top	B1
	States that $\left\{arg\left(\frac{z_1}{z_2}\right) =\right\} arg(z_1) - arg(z_2)$ Or states that the arguments should be subtracted	B1
		(2)
(b)	$\left\{arg\left(\frac{z_1}{z_2}\right) = \left(\text{their } -\frac{\pi}{4}\right) - \frac{\pi}{6} =\right\} - \frac{5\pi}{12}$ Or $\left\{arg\left(\frac{z_1}{z_2}\right) = \left(\text{their} \frac{7\pi}{4}\right) - \frac{\pi}{6}\right\} = \frac{19\pi}{12}$	B1ft
		(1)

HOME

Series

6. (a) Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n,

$$\sum_{r=1}^{n} r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9)$$

(4)

(c) Hence find the value of n that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2$$

(5)

(b)	$\sum_{r=1}^{n} r(r+6)(r-6) = \sum_{r=1}^{n}(r^3 - 36r)$		
	$= \frac{1}{4}n^2(n+1)^2 - \frac{36}{2}n(n+1)$	M1 A1	2.1 1.1b
	$= \frac{1}{4}n(n+1)\left[n(n+1) - 72\right]$	M1	1.1b
	$= \frac{1}{4}n(n+1)(n-8)(n+9)$ * cso	A1*	1.1b
		(4)	
(c)	$\frac{1}{4}n(n+1)(n-8)(n+9) = \frac{17}{6}n(n+1)(2n+1)$	M1	1.1b
	$\frac{1}{4}(n-8)(n+9) = \frac{17}{6}(2n+1)$	M1	1.1b
	$3n^2 - 65n - 250 = 0$	A1	1.1b
	$(3n+10)(n-25) = 0$	M1	1.1b
	(As n must be a positive integer,) $n = 25$	A1	2.3
		(5)	
			(15 marks)

6. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n}(3r-2)^2 = \frac{1}{2}n\left[6n^2 - 3n - 1\right]$$

for all positive integers n.

(5)

(b) Hence find any values of n for which

$$\sum_{r=5}^{n}(3r-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$$

(5)

Question	Scheme	Marks	AOs
6(a)	$(3r-2)^2 = 9r^2 - 12r + 4$	B1	1.1b
	$\sum_{r=1}^{n}(9r^2 - 12r + 4) = 9 \times \frac{1}{6}n(n+1)(2n+1) - 12 \times \frac{1}{2}n(n+1) + \ldots$	M1	2.1
	$= 9 \times \frac{1}{6}n(n+1)(2n+1) - 12 \times \frac{1}{2}n(n+1) + 4n$	A1	1.1b
	$= \frac{1}{2}n\left[3(n+1)(2n+1) - 12(n+1) + 8\right]$	dM1	1.1b
	$= \frac{1}{2}n\left[6n^2 - 3n - 1\right]$*	A1*	1.1b
		(5)	
(b)	$\sum_{r=5}^{n}(3r-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1) - \frac{1}{2}(4)\left(6(4)^2 - 3 \times 4 - 1\right)$	M1	3.1a
	$\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 0 - 2 + 0 + 4 + 0 - 6 + 0 + 8 + 0 - 10 + 0 + 12 + \ldots$	M1	3.1a
	$3n^3 - \frac{3}{2}n^2 - \frac{1}{2}n - 166 + 103 \times 14 = 3n^3$ $\Rightarrow 3n^2 + n - 2552 = 0$	A1	1.1b
	$\Rightarrow 3n^2 + n - 2552 = 0 \Rightarrow n = \ldots$	M1	1.1b
	$n = 29$	A1	2.3
		(5)	
		(10 marks)	

6. An art display consists of an arrangement of n marbles.

When arranged in ascending order of mass, the mass of the first marble is 10 grams.
The mass of each subsequent marble is 3 grams more than the mass of the previous one, so that the rth marble has mass $(7 + 3r)$ grams.

(a) Show that the mean mass, in grams, of the marbles in the display is given by

$$\frac{1}{2}(3n+17)$$

(3)

Given that there are 85 marbles in the display,

(b) use the standard summation formulae to find the standard deviation of the mass of the marbles in the display, giving your answer, in grams, to one decimal place.

(6)

AS 2019

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Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\bar{x} = \frac{1}{n}\sum_{r=1}^{n}(7+3r)$	M1	This mark is given for finding a correct expression for the mean mass of the marbles, $\bar{x}$
	$\sum_{r=1}^{n}(7+3r)$ $= 7\sum_{r=1}^{n}1 + 3\sum_{r=1}^{n}r$ $= 7n + 3\frac{n}{2}(n+1)$	M1	This mark is given for correctly splitting the sum and using the arithmetic series formula
	$\bar{x} = 7 + 3\frac{3}{2}(n+1)$ $= \frac{14+3n+3}{2}$ $= \frac{1}{2}(3n+17)$	A1	This mark is given for correct working to arrive at the answer shown
(b)	Standard deviation = $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$	B1	This mark is given for using the formula for standard deviation (given in the formulae booklet)
	When $n = 85$, $\bar{x} = (3 \times 85 + 17) = 136$	B1	This mark is given for finding the mean mass of the marbles when $n = 85$
	$\sum x^2 = \sum_{r=1}^{n}(7+3r)^2 = \sum_{r=1}^{n}49 + 42r + 9r^2$	M1	This mark is given for a method to find an expression for $\sum x^2$
	$= 49n + 42 \times \frac{1}{2}n(n+1) + 9 \times \frac{1}{6}n(n+1)(2n+1)$	B1	This mark is given for an expression for $\sum x^2$ in terms of n
	When $n = 85$, $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$ $= \sqrt{\frac{4165 + 153510 + 1875015}{85} - 136^2}$ $= \sqrt{5418}$	M1	This mark is given for substituting to find a value for the standard deviation for the mass of the marbles when $n = 85$
	$= 73.6$ g	A1	This mark is given for a value of the standard deviation of the mass of the marbles (accept any answer which rounds to 74, units not needed)

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5.

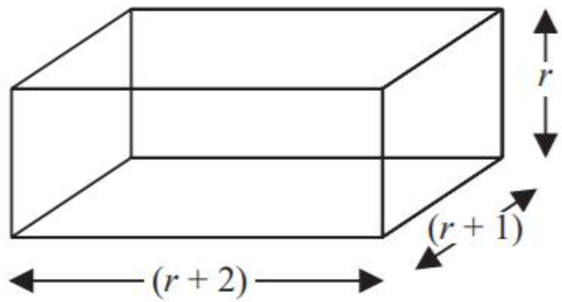

Figure 2

A block has length $(r + 2)$ cm, width $(r + 1)$ cm and height r cm, as shown in Figure 2.

In a set of n such blocks, the first block has a height of 1 cm, the second block has a height of 2 cm, the third block has a height of 3 cm and so on.

(a) Use the standard results for $\sum_{r=1}^{n} r^3, \sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that the **total** volume, V, of all n blocks in the set is given by

$$V = \frac{n}{4}(n+1)(n+2)(n+3) \qquad n \geqslant 1$$

(5)

Given that the total volume of all n blocks is

$$(n^4 + 6n^3 - 11\,710)\ \text{cm}^3$$

(b) determine how many blocks make up the set.

(2)

HOME

5(a)	Volume = $r\times(r+1)\times(r+2)$	B1	1.1b
	A complete method for finding the total volume of n blocks and expressing it in sigma notation. This can be implied by later work. $\sum_{r=1}^{n}\left(r^3+3r^2+2r\right)$	M1	3.1b
	$V=\frac{1}{4}n^2(n+1)^2+3\times\frac{1}{6}n(n+1)(2n+1)+2\times\frac{n}{2}(n+1)$	M1	2.1
	$V=\frac{1}{4}n(n+1)\left[n(n+1)+2(2n+1)+4\right]$	dM1	1.1b
	$V=\frac{1}{4}n(n+1)\left[n^2+5n+6\right]$ $\Rightarrow V=\frac{1}{4}n(n+1)(n+2)(n+3)$ *	A1*	1.1b
		(5)	
(b)	Sets $\frac{1}{4}n(n+1)(n+2)(n+3)=n^4+6n^3-11710$ $\frac{1}{4}n^4+\frac{3}{2}n^3+\frac{11}{4}n^2+\frac{3}{2}n=n^4+6n^3-11710$ simplifies $\left(3n^4+18n^3-11n^2-6n-46840=0\right)$ and solves $n=$ …	M1	1.1b
	There are 10 blocks or n = 10	A1	3.2a
		(2)	

(7 marks)

HOME

3. (a) Use the standard results for summations to show that for all positive integers n

$$\sum_{r=1}^{n}(5r-2)^2 = \frac{1}{6}n(an^2+bn+c)$$

where a, b and c are integers to be determined.

(5)

(b) Hence determine the value of k for which

$$\sum_{r=1}^{k}(5r-2)^2 = 94k^2$$

(4)

Question	Scheme	Marks	AOs
3(a)	$(5r-2)^2 = 25r^2 - 20r + 4$	B1	1.1b
	$\sum_{r=1}^{n} 25r^2 - 20r + 4 = \frac{25}{6}n(n+1)(2n+1) - \frac{20}{2}n(n+1) + \ldots$	M1	2.1
	$= \frac{25}{6}n(n+1)(2n+1) - \frac{20}{2}n(n+1) + 4n$	A1	1.1b
	$= \frac{1}{6}n\left[25\left(2n^2+3n+1\right) - 60(n+1) + 24\right]$	dM1	1.1b
	$= \frac{1}{6}n\left[50n^2 + 15n - 11\right]$	A1	1.1b
		(5)	
(b)	$\frac{1}{6}k\left[50k^2 + 15k - 11\right] = 94k^2$	M1	1.1b
	$50k^3 - 549k^2 - 11k = 0$ or $50k^2 - 549k - 11 = 0$	A1	1.1b
	$(k-11)(50k+1) = 0 \Rightarrow k = \ldots$	M1	1.1b
	$k = 11$(only)	A1	2.3
		(4)	
			(9 marks)

5. (a) Use the standard summation formulae to show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{n} \left(3r^2 - 17r - 25\right) = n\left(n^2 - An - B\right)$$

where A and B are integers to be determined.

(4)

(b) Explain why, for $k \in \mathbb{N}$,

$$\sum_{r=1}^{3k} r\tan\left(60r\right)^\circ = -k\sqrt{3}$$

(2)

Using the results from part (a) and part (b) and showing all your working,

(c) determine any value of n that satisfies

$$\sum_{r=5}^{n} \left(3r^2 - 17r - 25\right) = 15\left[\sum_{r=6}^{3n} r\tan\left(60r\right)^\circ\right]^2$$

(6)

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5(a)	$\sum_{r=1}^{n}\left(3r^2-17r-25\right)=3\times\frac{n}{6}(n+1)(2n+1)-17\times\frac{1}{2}n(n+1)-\ldots$	M1
	$=3\times\frac{n}{6}(n+1)(2n+1)-17\times\frac{1}{2}n(n+1)-25n$	A1
	$=n\left(\frac{1}{2}\left(2n^2+3n+1\right)-\frac{17}{2}(n+1)-25\right)$ or $=\frac{n}{2}\left(\left(2n^2+3n+1\right)-17(n+1)-50\right)$	M1
	$=n\left(n^2-7n-33\right)$ cso (so $A=7$ and $B=33$)	A1 cso
		(4)
(b)	$\sum_{r=1}^{3k} r\tan\left(60r\right)^\circ$ $=\tan\left(60\right)^\circ+2\tan\left(120\right)^\circ+3\tan\left(180\right)^\circ+4\tan\left(240\right)^\circ+5\tan\left(300\right)^\circ+6\tan\left(360\right)^\circ+$ $=\left(\sqrt{3}-2\sqrt{3}+0\right)+\left(4\sqrt{3}-5\sqrt{3}+0\right)+\ldots$	M1
	Since tan has period 180° we see $\tan\left(60r\right)^\circ$ repeats every three terms and **each group of three terms results in** $-\sqrt{3}$ **as a sum**, so with ***k*** **groups** of terms the sum is $-k\sqrt{3}$	A1
		(2)

(c)	$\sum_{r=5}^{n}\left(3r^2-17r-25\right)=\sum_{r=1}^{n}\left(3r^2-17r-25\right)-\sum_{r=1}^{4}\left(3r^2-17r-25\right)$	M1
	$=n\left(n^2-7n-33\right)-4\left(4^2-7\times4-33\right)$ $\left(=n\left(n^2-7n-33\right)+180\right)$	A1
	$\sum_{r=6}^{3n} r\tan\left(60r\right)^\circ=-n\sqrt{3}+2\sqrt{3}$ (allow for $-n\sqrt{3}--2\sqrt{3}$)	B1
	$\Rightarrow n\left(n^2-7n-33\right)+180=15\left[-n\sqrt{3}+2\sqrt{3}\right]^2$ $\Rightarrow n^3-7n^2-33n+180=15\left(3n^2-12n+12\right)$ $\Rightarrow n^3-52n^2+147n=0$	M1
	$\Rightarrow n^3-52n^2+147n=0\Rightarrow n=\ldots$	M1
	But need $n>5$ for sums to be valid, so $n=49$ (allow if $n=0$ also given but $n=3$ must be rejected).	A1
		(6)

HOME

4. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3, \; \sum_{r=1}^{n} r^2 \text{ and } \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2 - 1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

A2 2021 Paper 2

Series

Question	Scheme	Marks	
4(a)	A complete attempt to find the sum of the cubes of the first n odd numbers using three of the standard summation formulae. Attempts to find $\sum(2r+1)^3$ or $\sum(2r-1)^3$ by expanding and using summation formulae	M1	
	$\sum_{r=1}^{n}(2r-1)^3=\sum_{r=1}^{n}(8r^3-12r^2+6r-1)=8\sum_{r=1}^{n}r^3-12\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$ or $\sum_{r=0}^{n-1}(2r+1)^3=\sum_{r=0}^{n-1}(8r^3+12r^2+6r+1)=8\sum_{r=0}^{n-1}r^3+12\sum_{r=0}^{n-1}r^2+6\sum_{r=0}^{n-1}r+\sum_{r=0}^{n-1}1$	M1	
	$=8\frac{n^2}{4}(n+1)^2-12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)-n$ or $=8\frac{(n-1)^2}{4}(n)^2+12\frac{(n-1)}{6}(n)(2n-1)+6\frac{(n-1)}{2}(n)+n$	M1 A1	
	Multiplies out to achieve a correct intermediate line for example $n\ \ n+1\ \ 2n^2-2n+1\ \ -n=2n^4-2n^3+n^2+2n^3-2n^2+n-n$ $2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$ leading to $=n^2(2n^2-1)$ cso *	A1 *	
		(5)	
(b)	$\sum_{r=n}^{n+9}(2r-1)^3=\sum_{r=1}^{n+9}(2r-1)^3-\sum_{r=1}^{n-1}(2r-1)^3$ $=(n+9)^2(2(n+9)^2-1)-(n-1)^2(2(n-1)^2-1)=99800$ or $\sum_{r=n+1}^{n+10}(2r-1)^3=\sum_{r=1}^{n+10}(2r-1)^3-\sum_{r=1}^{n}(2r-1)^3$ $=(n+10)^2(2(n+10)^2-1)-(n)^2(2n^2-1)=99800$ or $\sum_{r=n-9}^{n}(2r-1)^3=\sum_{r=1}^{n}(2r-1)^3-\sum_{r=1}^{n-10}(2r-1)^3$ $=(n)^2(2(n)^2-1)-(n-10)^2(2(n-10)^2-1)=99800$	M1	3.1a
	$80n^3+960n^2+5820n-86760=0$ or $80n^3+1200n^2+7980n-79900=0$ or $80n^3-1200n^2+7980n-119700=0$	A1	1.1b
	Solves cubic equation	dM1	1.1b
	Achieves $n=6$ and the smallest number as 11 or Achieves $n=5$ and the smallest number as 11 or Achieves $n=15$ and the smallest number as 11	A1	2.3
		(4)	
			(9 marks)

HOME

Roots of Polynomials

1. $$f(z) = z^3 + pz^2 + qz - 15$$

where p and q are real constants.

Given that the equation $f(z) = 0$ has roots

$$\alpha, \ \frac{5}{\alpha} \text{ and } \left(\alpha + \frac{5}{\alpha} - 1\right)$$

(a) solve completely the equation $f(z) = 0$

(5)

(b) Hence find the value of p.

(2)

Question	Scheme	Marks	AOs
1(a)	$\alpha\left(\frac{5}{\alpha}\right)\left(\alpha+\frac{5}{\alpha}-1\right)=15$	M1	1.1b
		A1	1.1b
	$\Rightarrow 5\alpha+\frac{25}{\alpha}-5=15 \Rightarrow \alpha^2-4\alpha+5=0$ $\Rightarrow \alpha=\frac{--4\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$ or $(\alpha-2)^2-4+5=0 \Rightarrow \alpha=\ldots$	M1	3.1a
	$\Rightarrow \alpha=2\pm\mathrm{i}$	A1	1.1b
	Hence the roots of $\mathrm{f}(z)=0$ are $2+\mathrm{i}, 2-\mathrm{i}$ and 3	A1	2.2a
		(5)	
(b)	$p=-\left(\text{"}(2+\mathrm{i})\text{"}+\text{"}(2-\mathrm{i})\text{"}+\text{"}3\text{"}\right) \Rightarrow p=\ldots$	M1	3.1a
	$\Rightarrow p=-7$ cso	A1	1.1b
		(2)	
	1(b) alternative		
	$\mathrm{f}(z)=(z-3)(z^2-4z+5) \Rightarrow p=\ldots$	M1	3.1a
	$\Rightarrow p=-7$ cso	A1	1.1b
		(2)	
			(7 marks)

4. The cubic equation

$$x^3 + 3x^2 - 8x + 6 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

AS SAMs

Roots of Polynomials

Question	Scheme	Marks	AOs
4	$\{w = x - 1 \Rightarrow\}\ x = w + 1$	B1	3.1a
	$(w+1)^3 + 3(w+1)^2 - 8(w+1) + 6 = 0$	M1	3.1a
	$w^3 + 3w^2 + 3w + 1 + 3(w^2 + 2w + 1) - 8w - 8 + 6 = 0$		
	$w^3 + 6w^2 + w + 2 = 0$	M1	1.1b
		A1	1.1b
		A1	1.1b
		(5)	
	Alternative		
	$\alpha + \beta + \gamma = -3,\ \alpha\beta + \beta\gamma + \alpha\gamma = -8,\ \alpha\beta\gamma = -6$	B1	3.1a
	sumroots $= \alpha - 1 + \beta - 1 + \gamma - 1$	M1	3.1a
	$= \alpha + \beta + \gamma - 3 = -3 - 3 = -6$		
	pairsum $= (\alpha-1)(\beta-1) + (\alpha-1)(\gamma-1) + (\beta-1)(\gamma-1)$		
	$= \alpha\beta + \alpha\gamma + \beta\gamma - 2(\alpha+\beta+\gamma) + 3$		
	$= -8 - 2(-3) + 3 = 1$		
	product $= (\alpha - 1)(\beta - 1)(\gamma - 1)$		
	$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha+\beta+\gamma) - 1$		
	$= -6 - (-8) - 3 - 1 = -2$		
	$w^3 + 6w^2 + w + 2 = 0$	M1	1.1b
		A1	1.1b
		A1	1.1b
		(5)	
	(5 marks)		

HOME

2. The cubic equation

$$z^3 - 3z^2 + z + 5 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

AS 2018

Roots of Polynomials

Question	Scheme	Marks	AOs
2	$w = 2z + 1 \Rightarrow z = \dfrac{w-1}{2}$	B1	3.1a
	$\left(\dfrac{w-1}{2}\right)^3 - 3\left(\dfrac{w-1}{2}\right)^2 + \left(\dfrac{w-1}{2}\right) + 5 = 0$	M1	3.1a
	$\dfrac{1}{8}\left(w^3 - 3w^2 + 3w - 1\right) - \dfrac{3}{4}\left(w^2 - 2w + 1\right) + \dfrac{w-1}{2} + 5 = 0$		
	$w^3 - 9w^2 + 19w + 29 = 0$	M1 A1 A1	1.1b 1.1b 1.1b
		(5)	

7.

$$f(z) = z^3 + z^2 + pz + q$$

where p and q are real constants.

The equation $f(z) = 0$ has roots z_1, z_2 and z_3
When plotted on an Argand diagram, the points representing z_1, z_2 and z_3 form the vertices of a triangle of area 35

Given that $z_1 = 3$, find the values of p and q.

(7)

Question	Scheme	Marks	AOs
7	Complex roots are e.g. $\alpha \pm \beta i$ or $(z^3+z^2+pz+q)\div(z-3)=z^2+4z+p+12$ or $f(3)=0 \Rightarrow 3^3+3^2+3p+q=0$ or One of: $3+z_2+z_3=-1$, $3z_2z_3=-q$, $3z_2+3z_3+z_2z_3=p$	B1	3.1a
	Sum of roots $\alpha+\beta i+\alpha-\beta i+3=-1 \Rightarrow \alpha=\ldots$ or $\alpha+\beta i+\alpha-\beta i=-4 \Rightarrow \alpha=\ldots$	M1	1.1b
	$\alpha=-2$	A1	1.1b
	So $\frac{1}{2}\times 2\beta\times 5=35 \Rightarrow \beta=7$	M1	1.1b
	$q=-3(\text{"}-2+7i\text{"})(\text{"}-2-7i\text{"})=\ldots$ or $p=3(\text{"}-2+7i\text{"})+3(\text{"}-2-7i\text{"})+(\text{"}-2+7i\text{"})(\text{"}-2-7i\text{"})$ or $(z-3)(z-(\text{"}-2+7i\text{"}))(z-(\text{"}-2-7i\text{"}))=\ldots$	M1	3.1a
	$q=-159$ or $p=41$	A1	1.1b
	$3p+q=-36 \Rightarrow p=\frac{-36-q}{3}=41$ and $q=-159$	A1	1.1b
		(7)	

2. The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where p, q, r and s are integers to be found.

(5)

Question 2 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	$w = x + 3$, so $x = w - 3$	B1	This mark is given for finding the relationship between x and w
	$2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12 = 0$	M1	This mark is given for substituting $w - 3$ into the cubic equation
	$2w^3 - 18w^2 + 54w + 6w^2 - 36w + 54 - 3w + 9 + 12 = 0$	M1	This mark is given for multiplying out all the terms correctly
	$2w^3 - 12w^2 + 15w + 21 = 0$	A1	This mark is given for at least two of p, q, r and s correct
	(so $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	A1	This mark is given for all four terms p, q, r and s correct

7.

$$f(z) = z^3 - 8z^2 + pz - 24$$

where p is a real constant.

Given that the equation $f(z) = 0$ has distinct roots

$$\alpha, \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $f(z) = 0$

(6)

(b) Hence find the value of p.

(2)

Question 7 (Total 8 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\alpha + \beta + \left(\alpha + \frac{12}{\alpha} - \beta\right) = 8$	M1	This mark is given for finding the sum of roots and equating them to 8
	$2\alpha + \frac{12}{\alpha} = 8$	A1	This mark is given for finding an equation in terms of α only.
	$2\alpha^2 + 12 = 8\alpha$ so $2\alpha^2 - 8\alpha + 12 = 0$ $\alpha = \frac{4 \pm \sqrt{(-4)^2 - 4\times1\times6}}{2\times1}$	M1	This mark is given for rearranging to find a quadratic equation to solve
	$\alpha = 2 \pm i\sqrt{2}$	A1	This mark is given for finding the two complex roots
	Product of roots = 24 Third root = $\frac{24}{(2+i\sqrt{2})(2-i\sqrt{2})} = \frac{24}{4+2}$	M1	This mark is given for a method to find the third root of f(z)
	Hence the roots of f(z) are $2 + i\sqrt{2}$, $2 - i\sqrt{2}$ and 4	A1	This mark is given for finding the three roots of f(z)
(b)	$f(4) = 0 \Rightarrow 4^3 - (8 \times 4^2) + 4p - 24 = 0$ $4p = 88$	M1	This mark is given for a method to find a value of p
	$p = 22$	A1	This mark is given for a correct value for p

9. The cubic equation

$$3x^3 + x^2 - 4x + 1 = 0$$

has roots α, β, and γ.

Without solving the cubic equation,

(a) determine the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(3)

(b) find a cubic equation that has roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$, giving your answer in the form

$x^3 + ax^2 + bx + c = 0$, where a, b and c are integers to be determined.

(3)

9(a)	$\alpha\beta\gamma=-\frac{1}{3}$ and $\alpha\beta+\alpha\gamma+\beta\gamma=-\frac{4}{3}$	B1
	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=\frac{-4/3}{-1/3}$	M1
	$=4$	A1
		(3)
(b)	$\left\{\alpha+\beta+\gamma=-\frac{1}{3}\right\}$ New product $=\frac{1}{\alpha}\times\frac{1}{\beta}\times\frac{1}{\gamma}=\frac{1}{\alpha\beta\gamma}=\frac{1}{-1/3}=...(-3)$ New pair sum $\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma}=\frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}=\frac{-1/3}{-1/3}=...(1)$	M1
	$x^3-(\text{part (a)})x^2+(\text{new pair sum})x-(\text{new product})(=0)$	M1
	$x^3-4x^2+x+3=0$	A1
		(3)

2. The cubic equation

$$9x^3 - 5x^2 + 4x + 7 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(3\alpha - 2)$, $(3\beta - 2)$ and $(3\gamma - 2)$, giving your answer in the form $aw^3 + bw^2 + cw + d = 0$, where a, b, c and d are integers to be determined.

(5)

Question	Scheme	Marks	AOs
2	$w = 3x - 2 \Rightarrow x = \dfrac{w+2}{3}$	B1	3.1a
	$9\left(\dfrac{w+2}{3}\right)^3 - 5\left(\dfrac{w+2}{3}\right)^2 + 4\left(\dfrac{w+2}{3}\right) + 7 = 0$	M1	3.1a
	$\dfrac{1}{3}\left(w^3 + 6w^2 + 12w + 8\right) - \dfrac{5}{9}\left(w^2 + 4w + 4\right) + \dfrac{4}{3}(w+2) + 7 = 0$		
	$3w^3 + 13w^2 + 28w + 91 = 0$	dM1 A1 A1	1.1b 1.1b 1.1b
		(5)	
	Alternative:		
	$\alpha + \beta + \gamma = \dfrac{5}{9}, \alpha\beta + \beta\gamma + \alpha\gamma = \dfrac{4}{9}, \alpha\beta\gamma = -\dfrac{7}{9}$	B1	3.1a
	New sum $= 3(\alpha + \beta + \gamma) - 6 = -\dfrac{13}{3}$	M1	3.1a
	New pair sum $= 9(\alpha\beta + \beta\gamma + \gamma\alpha) - 12(\alpha + \beta + \gamma) + 12 = \dfrac{28}{3}$		
	New product $= 27\alpha\beta\gamma - 18(\alpha\beta + \beta\gamma + \gamma\alpha) + 12(\alpha + \beta + \gamma) - 8 = -\dfrac{91}{3}$		
	$w^3 - \left(-\dfrac{13}{3}\right)w^2 + \dfrac{28}{3}w - \left(-\dfrac{91}{3}\right) = 0$	dM1	1.1b
	$3w^3 + 13w^2 + 28w + 91 = 0$	A1 A1	1.1b 1.1b
		(5)	
			(5 marks)

7. $$f(z) = z^4 - 6z^3 + pz^2 + qz + r$$

where p, q and r are real constants.

The roots of the equation $f(z) = 0$ are α, β, γ and δ where $\alpha = 3$ and $\beta = 2 + \mathrm{i}$

Given that γ is a complex root of $f(z) = 0$

(a) (i) write down the root γ,

(ii) explain why δ must be real.

(2)

(b) Determine the value of δ.

(2)

(c) Hence determine the values of p, q and r.

(3)

(d) Write down the roots of the equation $f(-2z) = 0$

(2)

HOME

Question	Scheme	Marks	AOs
7(a)(i)	$2-\mathrm{i}$	B1	1.2
(ii)	Roots of polynomials with real coefficients occur in conjugate pairs, β and γ form a conjugate pair, α is real so δ must also be real. or Quartics have either 4 real roots, 2 real roots and 2 complex roots or 4 complex roots. As 2 complex roots and 1 real root therefore so δ must also be real. or As α real and only one root δ remaining, if complex it would need to have a complex conjugate, which it can't have so must be real	B1	2.4
		(2)	
(b)	$\alpha+\beta+\gamma+\delta=6$ $\Rightarrow 3+2+\mathrm{i}+2-\mathrm{i}+\delta=6 \Rightarrow \delta=\ldots$	M1	3.1a
	$\delta=-1$	A1	1.1b
		(2)	
(c)	$\mathrm{f}(z)=(z-3)(z+1)(z-(2+\mathrm{i}))(z-(2-\mathrm{i}))=\ldots$ Alternative pair sum $=(3)(2+\mathrm{i})+(3)(2-\mathrm{i})+(3)(-1)+(-1)(2+\mathrm{i})$ $+(-1)(2-\mathrm{i})+(2+\mathrm{i})(2-\mathrm{i})=\ldots\{10\}$ triple sum $=(3)(2+\mathrm{i})(2-\mathrm{i})+(3)(-1)(2+\mathrm{i})$ $+(3)(-1)(2-\mathrm{i})+(-1)(2+\mathrm{i})(2-\mathrm{i})=\ldots\{-2\}$ product $=(3)(2+\mathrm{i})(2-\mathrm{i})(-1)=\ldots\{-15\}$	M1	3.1a
	$=(z^2-2z-3)(z^2-4z+5)$ $=z^4-6z^3+10z^2+2z-15$ $p=10, q=2, r=-15$	A1 A1	1.1b 1.1b
		(3)	
(d)	$z=\frac{1}{2}, -\frac{3}{2}$	B1ft	1.1b
	$z=-1\pm\frac{\mathrm{i}}{2}$	B1ft	1.1b
		(2)	
			(9 marks)

HOME

4. The roots of the quartic equation

$$3x^4 + 5x^3 - 7x + 6 = 0$$

are α, β, γ and δ

Making your method clear and without solving the equation, determine the exact value of

(i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(3)

(ii) $\dfrac{2}{\alpha} + \dfrac{2}{\beta} + \dfrac{2}{\gamma} + \dfrac{2}{\delta}$

(3)

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)(3 - \delta)$

(3)

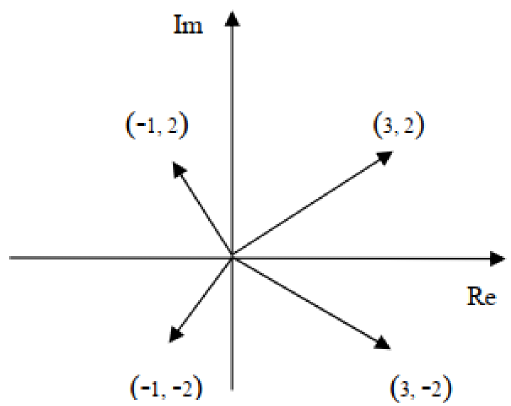
4(i)	$\sum \alpha_i = -\frac{5}{3}$ and $\sum \alpha_i \alpha_j = 0$ This mark can be awarded if seen in part (ii) or part (iii)	B1
	So $\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2\left(\sum \alpha_i\alpha_j\right) = \ldots$	M1
	$= \frac{25}{9} - 2\times 0 = \frac{25}{9}$	A1
		(3)
(ii)	$\sum \alpha_i\alpha_j\alpha_k = \frac{7}{3}$ and $\prod \alpha_i = 2$ or for $x = \frac{2}{w}$ used in equation This mark can be awarded if seen in part (i) or part (iii)	B1
	So $2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right) = 2\times\frac{\sum \alpha_i\alpha_j\alpha_k}{\alpha\beta\gamma\delta} = 2\times\frac{'\frac{7}{3}'}{'\frac{6}{3}'}$ or for $3\left(\frac{16}{w^4}\right)+5\left(\frac{8}{w^3}\right)-7\left(\frac{2}{w}\right)+6=0 \Rightarrow 6w^4-14w^3+\ldots=0$ leading to $\frac{14}{6}$	M1
	$\left(=2\times\frac{7/3}{2}\right)\left(=\frac{14}{6}\right)=\frac{7}{3}$	A1
		(3)
(iii)	$(3-\alpha)(3-\beta)(3-\gamma)(3-\delta) = \ldots$ expands all four brackets Or equation with these roots is $3(3-x)^4+5(3-x)^3-7(3-x)+6=0$	M1
	$= 81-27\left(\sum \alpha_i\right)+9\left(\sum \alpha_i\alpha_j\right)-3\left(\sum \alpha_i\alpha_j\alpha_k\right)+\prod \alpha_i$ $= 81-27\left(-\frac{5}{3}\right)+9(0)-3\left(\frac{7}{3}\right)+2$ Or expands to fourth power and constant terms and attempts product of roots $3x^4+\ldots+3\times3^4+5\times3^3-7\times3+6 \to \prod \alpha_i = \frac{"363"}{3}$	dM1
	$=121$	A1
		(3)

3.
$$f(z) = z^4 + az^3 + 6z^2 + bz + 65$$

where a and b are real constants.

Given that $z = 3 + 2\mathbf{i}$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

(9)

Question	Scheme	Marks	AOs
3	$z = 3 - 2\text{i}$ is also a root	B1	1.2
	$\left(z-(3+2\text{i})\right)\left(z-(3-2\text{i})\right) = \dots$ **or** Sum of roots = 6, Product of roots = 13 $\Rightarrow \dots$	M1	3.1a
	$= z^2 - 6z + 13$	A1	1.1b
	$\left(z^4 + az^3 + 6z^2 + bz + 65\right) = \left(z^2 - 6z + 13\right)\left(z^2 + cz + 5\right) \Rightarrow c = \dots$	M1	3.1a
	$z^2 + 2z + 5 = 0$	A1	1.1b
	$z^2 + 2z + 5 = 0 \Rightarrow z = \dots$	M1	1.1a
	$z = -1 \pm 2\text{i}$	A1	1.1b
		B1 $3 \pm 2\text{i}$ Plotted correctly	1.1b
		B1ft $-1 \pm 2\text{i}$ Plotted correctly	1.1b
		(9 marks)	

HOME

1. The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

A2 SAMs Paper 2

Roots of Polynomials

Question	Scheme	Marks	AOs
1(i)	$\alpha+\beta+\gamma=8, \quad \alpha\beta+\beta\gamma+\gamma\alpha=28, \quad \alpha\beta\gamma=32$	B1	3.1a
	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$	M1	1.1b
	$=\frac{7}{8}$	A1ft	1.1b
		(3)	
(ii)	$(\alpha+2)(\beta+2)(\gamma+2)=(\alpha\beta+2\alpha+2\beta+4)(\gamma+2)$	M1	1.1b
	$=\alpha\beta\gamma+2(\alpha\beta+\alpha\gamma+\beta\gamma)+4(\alpha+\beta+\gamma)+8$	A1	1.1b
	$=32+2(28)+4(8)+8=128$	A1	1.1b
		(3)	
	Alternative:		
	$(x-2)^3-8(x-2)^2+28(x-2)-32=0$	M1	1.1b
	$=\ldots-8+\ldots-32+\ldots-56-32=-128$	A1	1.1b
	$\therefore(\alpha+2)(\beta+2)(\gamma+2)=128$	A1	1.1b
		(3)	
(iii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$	M1	3.1a
	$=8^2-2(28)=8$	A1ft	1.1b
		(2)	
		(8 marks)	

HOME

2. The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are p, q and r.

Without solving the equation, find the value of

(i) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

(8)

A2 2019 Paper 2

Roots of Polynomials

2(i)	$p+q+r=2, \quad pq+pr+qr=4, \quad pqr=5$	B1	3.1a
	$\frac{2}{p}+\frac{2}{q}+\frac{2}{r}=\frac{2(pq+pr+qr)}{pqr}$	M1	1.1b
	$=\frac{8}{5}$	A1ft	1.1b
		(3)	
(ii)	$(p-4)(q-4)(r-4)=(pq-4p-4q+16)(r-4)$ $=pqr-4pq-4pr-4qr+16p+16q+16r-64$	M1 A1	1.1b 1.1b
	$(=pqr-4(pq+pr+qr)+16(p+q+r)-64)$		
	$=5-4(4)+16(2)-64=-43$	A1	1.1b
		(3)	
(iii)	E.g. $p^3+q^3+r^3=$ $=(p+q+r)^3-3(p+q+r)(pq+pr+qr)+3pqr$ or $=(p+q+r)((p+q+r)^2-2(pq+pr+qr)-pq-pr-qr)+3pqr$ or $=2((p+q+r)^2-2(pq+pr+qr))-4(p+q+r)+3pqr$ $\Rightarrow p^3+q^3+r^3=\ldots$	M1	3.1a
	$=2^3-3(2)(4)+3(5)=-1$ $=2(2^2-3(4))+3(5)=-1$ $=2(2^2-2(4))-4(2)+3(5)=-1$	A1	1.1b
		(2)	

HOME

1. $$f(z) = 3z^3 + pz^2 + 57z + q$$

where p and q are real constants.

Given that $3 - 2\sqrt{2}\text{i}$ is a root of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

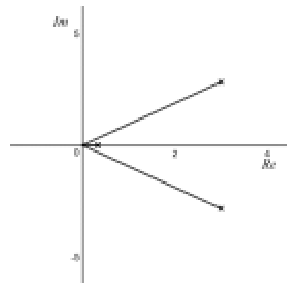
(7)

(b) find the value of p and the value of q.

(3)

A2 2020 Paper 1

Roots of Polynomials

Question	Scheme	Marks	AOs
1(a)	$\beta = 3+2\sqrt{2}\mathrm{i}$ is also a root	B1	1.2
	$\alpha\beta = 17, \alpha+\beta = 6$	B1	1.1b
	$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{57}{3}$	M1	1.1b
	$\alpha\gamma+\beta\gamma = \frac{57}{3}-17 = \gamma(\alpha+\beta) = 6\gamma \Rightarrow \gamma = \ldots$	M1	3.1a
	$\gamma = \frac{1}{3}$	A1	2.2a
		B1	1.1b
		B1ft	1.1b
		(7)	
	(a) Alternative:		
	$\beta = 3+2\sqrt{2}\mathrm{i}$ is also a root	B1	1.2
	$\left(z-\left(3+2\sqrt{2}\mathrm{i}\right)\right)\left(z-\left(3-2\sqrt{2}\mathrm{i}\right)\right) = z^2-6z+17$	B1	1.1b
	$\mathrm{f}(z) = \left(z^2-6z+17\right)(3z+a) = 3z^3+az^2-18z^2-6az+51z+17a$	M1	1.1b
	$\Rightarrow 51-6a = 57 \Rightarrow a = -1 \Rightarrow \gamma = \ldots$	M1	3.1a
	$\gamma = \frac{1}{3}$	A1	2.2a
	Then B1 B1ft as above		
		(7)	
(b)	$3-2\sqrt{2}\mathrm{i}+3+2\sqrt{2}\mathrm{i}+\frac{1}{3} = -\frac{p}{3} \Rightarrow p = \ldots$ or $\left(3-2\sqrt{2}\mathrm{i}\right)\left(3+2\sqrt{2}\mathrm{i}\right)\times\frac{1}{3} = -\frac{q}{3} \Rightarrow q = \ldots$	M1	3.1a
	$p = -19$ **or** $q = -17$	A1	1.1b
	$p = -19$ **and** $q = -17$	A1	1.1b

HOME

3. The cubic equation

$$ax^3 + bx^2 - 19x - b = 0$$

where a and b are constants, has roots α, β and γ

The cubic equation

$$w^3 - 9w^2 - 97w + c = 0$$

where c is a constant, has roots $(4\alpha - 1)$, $(4\beta - 1)$ and $(4\gamma - 1)$

Without solving either cubic equation, determine the value of a, the value of b and the value of c.

(6)

Question	Scheme	Marks	AOs
3	$w = 4x - 1 \Rightarrow x = \frac{w+1}{4}$	B1	3.1a
	$a\left(\frac{w+1}{4}\right)^3 + b\left(\frac{w+1}{4}\right)^2 - 19\left(\frac{w+1}{4}\right) - b\ (=0)$ or $(4x-1)^3 - 9(4x-1)^2 - 97(4x-1) + c\ (=0)$	M1	3.1a
	$aw^3 + (3a+4b)w^2 + (3a+8b-304)w + (a-60b-304) = 0$ or $64x^3 - 192x^2 - 304x + 87 + c = 0$	M1	1.1b
	Divides by a and equates the coefficients of w^2 and w $\frac{3a+4b}{a} = -9 \quad \frac{3a+8b-304}{a} = -97$ and solves simultaneously to find a value for a or a value for b <u>**Note:**</u> $12a + 4b = 0$ and $100a + 8b = 304$ or Divides through by '16' leading to values of a and b $4x^3 - 12x^2 - 19x + \frac{87+c}{19} = 0$	M1	3.1a
	$c = \frac{a-60b-304}{a} = \ldots$ or $\frac{87+c}{19} = 12 \Rightarrow c = \ldots$	M1	1.1b
	$a = 4 \quad b = -12 \quad c = 105$	A1	1.1b
		(6)	
		(6 marks)	

6. The cubic equation

$$4x^3 + px^2 - 14x + q = 0$$

where p and q are real positive constants, has roots α, β and γ

Given that $\alpha^2 + \beta^2 + \gamma^2 = 16$

(a) show that $p = 12$

(3)

Given that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{14}{3}$

(b) determine the value of q

(3)

Without solving the cubic equation,

(c) determine the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(4)

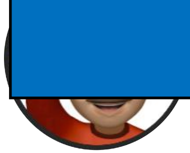
A2 2022 Paper 2

Roots of Polynomials

6(a)	$4x^3+px^2-14x+q=0 \Rightarrow x^3+\frac{p}{4}x^2-\frac{14}{4}x+\frac{q}{4}=0$ $\alpha+\beta+\gamma=-\frac{p}{4}$ $\alpha\beta+\alpha\gamma+\beta\gamma=-\frac{14}{4}$ or $-\frac{7}{2}$	B1
	$(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $\left(-\frac{p}{4}\right)^2=16+2\left(-\frac{7}{2}\right)\Rightarrow p=\ldots$ or $(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=\alpha^2+\beta^2+\gamma^2$ $\left(-\frac{p}{4}\right)^2-2\left(-\frac{7}{2}\right)=16\Rightarrow p=\ldots$	M1
	$p=12$ * cso	A1*
		(3)
(b)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$	M1
	$\dfrac{\left(-\frac{7}{2}\right)}{\left(\frac{-q}{4}\right)}=\frac{14}{3}\Rightarrow q=\ldots$	M1
	$q=3$	A1
		(3)
(c)	$(\alpha-1)(\beta-1)(\gamma-1)=\ldots$ $=\alpha\beta\gamma-(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)-1$	M1 A1
	$=\left(-\frac{\text{their } 3}{4}\right)-\left(-\frac{7}{2}\right)+\left(-\frac{12}{4}\right)-1=\ldots$	dM1
	$=-\frac{5}{4}$	A1
		(4)

HOME

Volumes of Revolution



AS SAMs

7.

Diagrams not drawn to scale

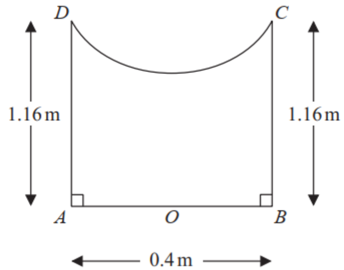


Figure 1

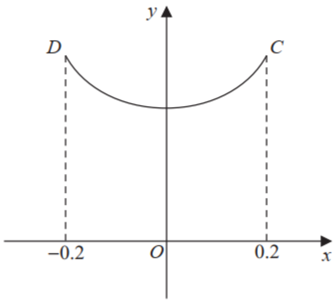


Figure 2

Figure 1 shows the central cross-section *AOBCD* of a circular bird bath, which is made of concrete. Measurements of the height and diameter of the bird bath, and the depth of the bowl of the bird bath have been taken in order to estimate the amount of concrete that was required to make this bird bath.

Using these measurements, the cross-sectional curve *CD*, shown in Figure 2, is modelled as a curve with equation

$$y = 1 + kx^2 \qquad -0.2 \leqslant x \leqslant 0.2$$

where k is a constant and where O is the fixed origin.

The height of the bird bath measured 1.16 m and the diameter, *AB*, of the base of the bird bath measured 0.40 m, as shown in Figure 1.

(a) Suggest the maximum depth of the bird bath.

(1)

(b) Find the value of k.

(2)

(c) Hence find the volume of concrete that was required to make the bird bath according to this model. Give your answer, in m^3, correct to 3 significant figures.

(7)

(d) State a limitation of the model.

(1)

It was later discovered that the volume of concrete used to make the bird bath was $0.127\,\text{m}^3$ correct to 3 significant figures.

(e) Using this information and the answer to part (c), evaluate the model, explaining your reasoning.

(1)

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AS SAMs

Volumes of Revolution

Question	Scheme		Marks	AOs
7(a)	Depth = 0.16 (m)		B1	2.2b
			(1)	
(b)	$y=1+kx^2 \Rightarrow 1.16=1+k(0.2)^2 \Rightarrow k=...$		M1	3.3
	$\Rightarrow k=4$ cao $\{\text{So } y=1+4x^2\}$		A1	1.1b
			(2)	
(c)	$\frac{\pi}{4}\int(y-1)\mathrm{d}y$	$\frac{\pi}{4}\int y\,\mathrm{d}y$	B1ft	1.1a
	$=\left\{\frac{\pi}{4}\right\}\int_1^{1.16}(y-1)\mathrm{d}y$	$=\left\{\frac{\pi}{4}\right\}\int_0^{0.16}y\,\mathrm{d}y$	M1	3.3
	$=\left\{\frac{\pi}{4}\right\}\left[\frac{y^2}{2}-y\right]_1^{1.16}$	$=\left\{\frac{\pi}{4}\right\}\left[\frac{y^2}{2}\right]_0^{0.16}$	M1 A1	1.1b 1.1b
	$=\frac{\pi}{4}\left(\left(\frac{1.16^2}{2}-1.16\right)-\left(\frac{1}{2}-1\right)\right)\ \{=0.0032\pi\}$	$=\frac{\pi}{4}\left(\left(\frac{0.16^2}{2}\right)-(0)\right)\ \{=0.0032\pi\}$		
	$V_{\text{cylinder}}=\pi(0.2)^2(1.16)\ \{=0.0464\pi\}$			
	Volume $=0.0464\pi-0.0032\pi\ \{=0.0432\pi\}$			
	$=0.1357168026...=0.136(\text{m}^3)$ (3sf)			
(d)	Any one of e.g. the measurements may not be accurate the inside surface of the bowl may not be smooth there may be wastage of concrete when making the bird bath		B1	3.5b
			(1)	
(e)	Some comment consistent with their values. We do need a reason e.g. $\left[\left(\frac{0.136-0.127}{0.127}\right)\times 100=7.0866...\right]$ so not a good estimate because the volume of concrete needed to make the bird bath is approximately 7% lower than that predicted by the model **or** We might expect the actual amount of concrete to exceed that which the model predicts due to wastage, so the model does not look suitable since it predicts more concrete than was used		B1ft	3.5a
			(1)	
			(12 marks)	

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AS 2018

9.

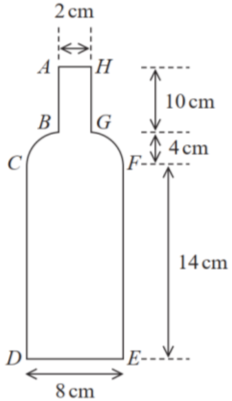

Figure 1

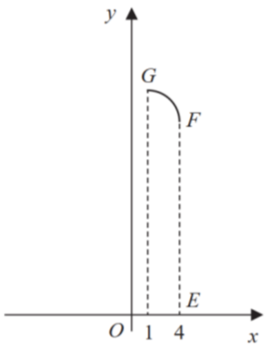

Figure 2

A mathematics student is modelling the profile of a glass bottle of water. Figure 1 shows a sketch of a central vertical cross-section $ABCDEFGHA$ of the bottle with the measurements taken by the student.

The horizontal cross-section between CF and DE is a circle of diameter 8 cm and the horizontal cross-section between BG and AH is a circle of diameter 2 cm.

The student thinks that the curve GF could be modelled as a curve with equation

$$y = ax^2 + b \qquad 1 \leqslant x \leqslant 4$$

where a and b are constants and O is the fixed origin, as shown in Figure 2.

(a) Find the value of a and the value of b according to the model.

(2)

(b) Use the model to find the volume of water that the bottle can contain.

(7)

(c) State a limitation of the model.

(1)

The label on the bottle states that the bottle holds approximately 750 cm^3 of water.

(d) Use this information and your answer to part (b) to evaluate the model, explaining your reasoning.

(1)

AS 2018

Volumes of Revolution

Question	Scheme	Marks	AOs
9(a)	$(4, 14),\ (1, 18) \Rightarrow 14 = a(4)^2 + b,\ 18 = a(1)^2 + b \Rightarrow a = \ldots, b = \ldots$	M1	3.3
	$a = -\frac{4}{15}, b = \frac{274}{15}$	A1	1.1b
		(2)	
(b)	$\pi \times 4^2 \times 14$ and $\pi \times 1^2 \times 10$	B1	1.1b
	$\pi \int x^2\,dy = \frac{\pi}{4}\int(274 - 15y)\,dy$	B1ft	1.1a
	$= \frac{\pi}{4}\int_{14}^{18}(274 - 15y)\,dy$	M1	3.3
	$= \frac{\pi}{4}\left[274y - \frac{15y^2}{2}\right]_{14}^{18}$	M1 A1	1.1b 1.1b
	$V = 234\pi + \frac{\pi}{4}\left[274(18) - \frac{15(18)^2}{2} - \left(274(14) - \frac{15(14)^2}{2}\right)\right]$	ddM1	3.4
	$V = 268\pi \approx 842\,\text{cm}^3$	A1	2.2b
		(7)	
(c)	Any one of e.g. The measurements may not be accurate The equation of the curve may not be a suitable model The bottom of the bottle may not be flat The thickness of the glass may not have been considered The glass may not be smooth This part asks for a limitation of the model so their answer must refer to e.g. : • The measuring of the dimensions • The model used for the curve • The simplified model (the thickness of glass, the simplified shape, smoothness of the glass etc.)	B1	3.5b
		(1)	
(d)	There are 2 criteria for this mark: • A comparison of their value to 750 e.g. larger, smaller, about the same or a difference **demonstrated** e.g. 810 – 750 = … but not **just** a percentage error or **just** a difference with no calculation • A conclusion that is consistent with their values e.g. this is not a good model, this is a good model etc. If they reach an answer that is less than 750, they need to conclude that it is not a good model If they reach an answer that is greater than 750 then look for a sensible comment that is consistent with their value	B1ft	3.5a
		(1)	
		(11 marks)	

9. $$f(x) = 2x^{\frac{1}{3}} + x^{-\frac{2}{3}} \qquad x > 0$$

The finite region bounded by the curve $y = f(x)$, the line $x = \frac{1}{8}$, the x-axis and the line $x = 8$ is rotated through θ radians about the x-axis to form a solid of revolution.

Given that the volume of the solid formed is $\frac{461}{2}$ units cubed, use algebraic integration to find the angle θ through which the region is rotated.

(8)

Question 9 (Total 8 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	$\frac{\theta}{2}\int y^2 \ \mathrm{d}x$	M1	This mark is given for an attempt at integrating y^2 with respect to x in volume of revolution formulae.
	$y^2 = (2x^{\frac{1}{3}} + x^{-\frac{2}{3}})(2x^{\frac{1}{3}} + x^{-\frac{2}{3}})$ $= 4x^{\frac{2}{3}} + 4x^{-\frac{1}{3}} + x^{-\frac{4}{3}}$	M1	This mark is given for $x^{\frac{2}{3}}$, $x^{-\frac{1}{3}}$ and $x^{-\frac{4}{3}}$ seen, regardless of the coefficients
		A1	This mark is given for a fully correct expression for y^2
	$\int y^2 \ \mathrm{d}x = \int 4x^{\frac{2}{3}} + 4x^{-\frac{1}{3}} + x^{-\frac{4}{3}} \ \mathrm{d}x$ $= \frac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - 3x^{-\frac{1}{3}}$	M1	This mark is given for two of $x^{\frac{5}{3}}$, $x^{\frac{2}{3}}$ and $x^{-\frac{1}{3}}$ seen, regardless of coefficients
		A1	This mark is given for two terms of the integral given correctly
		A1	This mark is given for a fully correct integral
	$\frac{\theta}{2}\left[\frac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - 3x^{-\frac{1}{3}}\right]_{\frac{1}{8}}^{8} = \frac{461}{2}$ $\Rightarrow \frac{\theta}{2}(99.3 + 4.425) = \frac{461}{2}$	M1	This mark is given for enumerating the integral to solve for θ
	$\Rightarrow \theta = 2 \times \frac{230.5}{103.725} = 4.444\ldots$ $\theta = \frac{40}{9}$ (radians)	A1	This mark is given for the correct angle found

3.

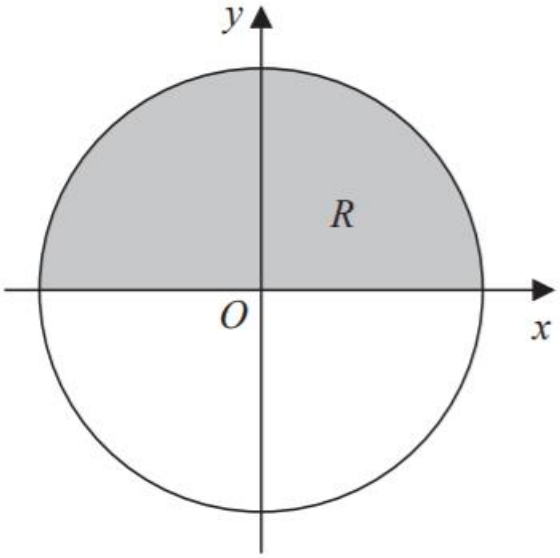

Figure 1

Figure 1 shows a circle with radius r and centre at the origin.
The region R, shown shaded in Figure 1, is bounded by the x-axis and the part of the circle for which $y > 0$
The region R is rotated through 360° about the x-axis to create a sphere with volume V

Use integration to show that $V = \dfrac{4}{3}\pi r^3$

(5)

3	$x^2+y^2=r^2$	B1	1.2
	$\{V\}=\pi\int_{-r}^{r} r^2-x^2\ \mathrm{d}x$ or $\{V\}=2\pi\int_{0}^{r} r^2-x^2\ \mathrm{d}x$	B1	2.1
	Integrates to the form $\alpha x\pm\beta x^3$ $\left[\text{note: the correct integration gives } r^2x-\frac{1}{3}x^3\right]$	M1	1.1b
	Substitutes limits of $-r$ and r and subtracts the correct way round $\left(r^2(r)-\frac{1}{3}(r)^3\right)-\left(r^2(-r)-\frac{1}{3}(-r)^3\right)$ or Substitutes limits of 0 and r and subtracts the correct way round with twice the volume. Note the limit of 0 can be implied if gives and answer of 0 $\left(r^2(r)-\frac{1}{3}(r)^3\right)-(0)$	dM1	1.1b
	$V=\frac{4}{3}\pi r^3$ * cso	A1*	1.1b
		(5)	

(5 marks)

Figure 2 shows the vertical cross-section, *AOBCDE*, through the centre of a wax candle.

In a model, the candle is formed by rotating the region bounded by the y-axis, the line OB, the curve BC, and the curve CD through 360° about the y-axis.

The point B has coordinates (3, 0) and the point C has coordinates (5, 15).

The units are in centimetres.

The curve BC is represented by the equation

$$y = \frac{\sqrt{225x^2 - 2025}}{a} \qquad 3 \leqslant x < 5$$

where a is a constant.

(a) Determine the value of a according to this model.

(2)

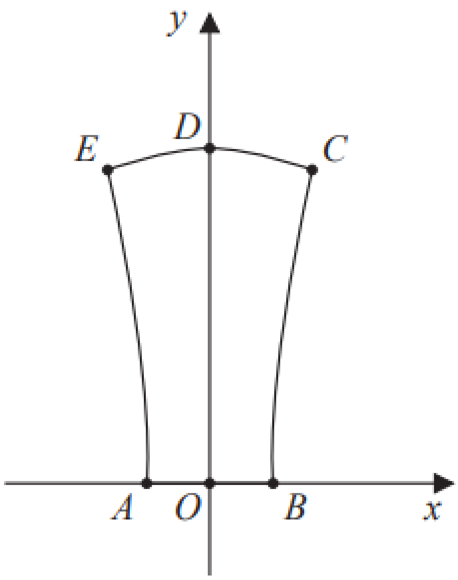

Figure 2

The curve CD is represented by the equation

$$y = 16 - 0.04x^2 \qquad 0 \leqslant x < 5$$

(b) Using algebraic integration, determine, according to the model, the exact volume of wax that would be required to make the candle.

(9)

(c) State a limitation of the model.

(1)

When the candle was manufactured, $700\,\text{cm}^3$ of wax were required.

(d) Use this information and your answer to part (b) to evaluate the model, explaining your reasoning.

(1)

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Volumes of Revolution

Question	Scheme	Marks	AOs
9(a)	$(5, 15) \Rightarrow 15 = \dfrac{\sqrt{225\times 5^2 - 2025}}{a} \Rightarrow a = \ldots$	M1	3.3
	$a = 4$	A1	1.1b
		(2)	
(b)	Evidence of the use of $\pi\int x^2\,dy$ for the curve BC or the curve CD	M1	3.1b
	For BC $V_1 = \dfrac{\pi}{225}\int\left(16y^2+2025\right)dy$ or $\pi\int\left(\dfrac{16}{225}y^2+9\right)dy$	A1ft	1.1b
	For CD $V_2 = 25\pi\int(16-y)\,dy$ or $\pi\int(400-25y)\,dy$	A1	1.1b
	$V_1 = \dfrac{\pi}{225}\int_0^{15}\left(16y^2+2025\right)dy$ or $\pi\int_0^{15}\left(\dfrac{16}{225}y^2+9\right)dy$	M1	3.3
	$V_2 = 25\pi\int_{15}^{16}(16-y)\,dy$ or $\pi\int_{15}^{16}(400-25y)\,dy$	M1	3.3
	$V_1 = \dfrac{\{\pi\}}{225}\left[\dfrac{16y^3}{3}+2025y\right]_0^{15}$ or $\{\pi\}\left[\dfrac{16y^3}{675}+9y\right]_0^{15}$	A1ft	1.1b
	$V_2 = 25\{\pi\}\left[16y-\dfrac{y^2}{2}\right]_{15}^{16}$ or $\{\pi\}\left[400y-\dfrac{25y^2}{2}\right]_{15}^{16}$	A1ft	1.1b
	$V = V_1+V_2 = \dfrac{\pi}{225}(18000+30375)+25\pi\left(128-\dfrac{255}{2}\right)$ $V = V_1+V_2 = 215\pi+12.5\pi$	M1	3.4
	$V = \dfrac{455\pi}{2}\,\text{cm}^3$ or $227.5\pi\,\text{cm}^3$	A1	2.2b
		(9)	
(c)	E.g. • The equation of the curve may not be a suitable model • The sides of the candle will not be perfectly curved/smooth • There will be a whole in the middle for the wick	B1	3.5b
		(1)	
(d)	Makes an appropriate comment that is consistent with their value for the volume and 700 cm^3. E.g. a good estimate as 700 cm^3 is only 15 cm^3 less than 715 cm^3	B1ft	3.5a
		(1)	
		(13 marks)	

AS 2022

Volumes of Revolution

Figure 1 shows a sketch of a 16 cm tall vase which has a flat circular base with diameter 8 cm and a circular opening of diameter 8 cm at the top.

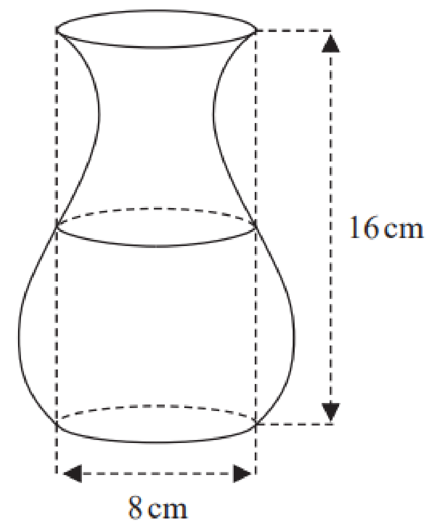

Figure 1

A student measures the circular cross-section halfway up the vase to be 8 cm in diameter.

The student models the shape of the vase by rotating a curve, shown in Figure 2, through 360° about the x-axis.

(a) State the value of a that should be used when setting up the model. **(1)**

Two possible equations are suggested for the curve in the model.

$$\text{Model A} \qquad y = a - 2\sin\left(\frac{45}{2}x\right)^{\circ}$$

$$\text{Model B} \qquad y = a + \frac{x(x-8)(x+8)}{100}$$

For each model,

(b) (i) find the distance from the base at which the widest part of the vase occurs,

(ii) find the diameter of the vase at this widest point. **(7)**

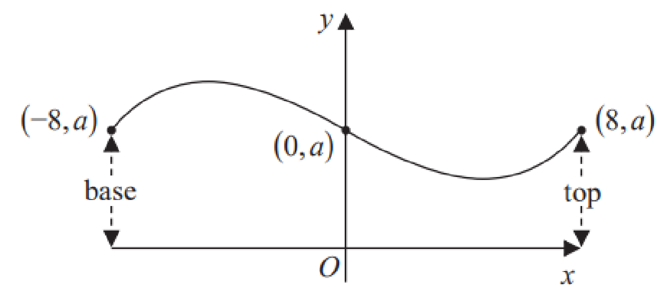

Figure 2

The widest part of the vase has diameter 12 cm and is just over 3 cm from the base.

(c) Using this information and making your reasoning clear, suggest which model is more appropriate. **(1)**

(d) Using algebraic integration, find the volume for the vase predicted by Model B. You must make your method clear. **(5)**

The student pours water from a full one litre jug into the vase and finds that there is 100 ml left in the jug when the vase is full.

(e) Comment on the suitability of Model B in light of this information. **(1)**

HOME

8(a)	$a = 4$	B1
		(1)
(b)	Model A: (i) Widest point will be 4 (cm) from the base	B1
	(ii) Width at widest point is 12 (cm) $\left(2\times('a'+2)\text{ ft}\right)$	B1ft
	Model B: (i) $y = 4 + \frac{x^3 - 64x}{100} \Rightarrow \frac{dy}{dx} = \frac{3x^2 - 64}{100}$	M1
	$\frac{dy}{dx} = 0 \Rightarrow x = \pm\sqrt{\frac{64}{3}} = \pm\frac{8\sqrt{3}}{3} = \pm\text{awrt}4.62$	A1
	So max width is a distance $8 - \frac{8}{\sqrt{3}} = 8 - \frac{8\sqrt{3}}{3} \approx 3.38$ (cm) from base.	A1
	(ii) $y\big\rvert_{-4.61..} = 4 + \frac{(-4.62...)^3 - 64(-4.62...)}{100} = ...$	dM1
	$= 5.97...$ so diameter is approximately 11.9 (cm) $\left[2a + 3.94...\text{ ft}\right]$	A1ft
		(7)
(c)	Model A and model B both have diameters closed to 12 Model B distance from base is closer to 3 than Model A so is more appropriate.	B1ft
		(1)

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(d)	$V_B = \pi \int_{-8}^{8} y^2 \, dx = \pi \int_{-8}^{8} \left(4 + \frac{x^3 - 64x}{100}\right)^2 dx = \ldots$	B1
	$= \frac{\{\pi\}}{10000} \int_{(-8)}^{(8)} 400^2 + x^6 + 64^2 x^2 + 2\left(400x^3 - 400 \times 64x - 64x^4\right) dx$ $= \frac{\{\pi\}}{10000} \int_{(-8)}^{(8)} 160000 + x^6 + 4096x^2 + 800x^3 - 51200x - 128x^4 \, dx$ $= \{\pi\} \int_{(-8)}^{(8)} 16 + \frac{x^6}{10000} + \frac{4096}{10000}x^2 + \frac{8}{100}x^3 - \frac{512}{100}x - \frac{128}{10000}x^4 \, dx$ $= \{\pi\} \int_{(-8)}^{(8)} 16 + \frac{x^6}{1000} + \frac{256}{625}x^2 + \frac{2}{25}x^3 - \frac{128}{25}x - \frac{8}{625}x^4 \, dx$ $= \{\pi\} \int_{(-8)}^{(8)} 16 + \frac{8x(x-8)(x+8)}{100} + \left(\frac{x(x-8)(x+8)}{100}\right)^2 dx$	M1
	$= \frac{\{\pi\}}{10000} \left[160000x + \frac{x^7}{7} + 4096\frac{x^3}{3} + 800\frac{x^4}{4} - 51200\frac{x^2}{2} - 128\frac{x^5}{5}\right]_{(-8)}^{(8)}$	dM1
	$= \{\pi\} \left[16x + \frac{x^7}{70000} + \frac{256}{1875}x^3 + \frac{1}{50}x^4 - \frac{64}{25}x^2 - \frac{8}{3125}x^5\right]_{(-8)}^{(8)}$	
	$= \frac{\{\pi\}}{10000}(620583.00\ldots. - -2258983.01\ldots) \approx \frac{2879566\pi}{10000}$	M1
	$= \text{awrt } 905\left(\text{cm}^3\right)$ cso	A1
		(5)
(e)	Compares their volume to 900 or compares their volume + 100 to 1 litre or 1000 and comments appropriately.	B1ft
		(1)

Matrices

3. Tyler invested a total of £5000 across three different accounts; a savings account, a property bond account and a share dealing account.

Tyler invested £400 more in the property bond account than in the savings account.

After one year

- the savings account had increased in value by 1.5%
- the property bond account had increased in value by 3.5%
- the share dealing account had **decreased** in value by 2.5%

- the total value across Tyler's three accounts had increased by £79

Form and solve a matrix equation to find out how much money was invested by Tyler in each account.

(7)

Question	Scheme	Marks	AOs
3	x = value of savings account, y = value of property bond account, z = value of share dealing account	M1	3.1b
	$x+y+z=5000$ $x+400=y$ $0.015x+0.035y-0.025z=79$ **or** $1.015x+1.035y+0.975z=5079$	A1	1.1b
	Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0.015 & 0.035 & -0.025 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.015 & 1.035 & 0.975 \end{pmatrix}$		
	e.g. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0.015 & 0.035 & -0.025 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5000 \\ -400 \\ 79 \end{pmatrix}$	M1 A1	3.1a 1.1b
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0.015 & 0.035 & -0.025 \end{pmatrix}^{-1}\begin{pmatrix} 5000 \\ -400 \\ 79 \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1800 \\ 2200 \\ 1000 \end{pmatrix}$	A1	1.1b
	Tyler invested £1800 in the savings account, £2200 in the property bond account and £1000 in the share dealing account	A1ft	3.2a
		(7 marks)	

1.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}$$

(a) Find $\mathbf{M}^{-1}$ giving each element in exact form.

(2)

(b) Solve the simultaneous equations

$$\begin{aligned} 2x + y - 3z &= -4 \\ 4x - 2y + z &= 9 \\ 3x + 5y - 2z &= 5 \end{aligned}$$

(2)

(c) Interpret the answer to part (b) geometrically.

(1)

AS 2018

Matrices

Question	Scheme	Marks	AOs
1(a)	$$\mathbf{M}^{-1} = \frac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}$$	B1 B1	1.1b 1.1b
		(2)	
(b)	$$\frac{1}{69}\begin{pmatrix} 1 & 13 & 5 \\ -11 & -5 & 14 \\ -26 & 7 & 8 \end{pmatrix}\begin{pmatrix} -4 \\ 9 \\ 5 \end{pmatrix} = \ldots$$	M1	1.1b
	$x = 2, y = 1, z = 3$ or $(2, 1, 3)$ or $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ or $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$	A1	1.1b
		(2)	
(c)	The **point** where three **planes** meet	B1ft	2.2a
		(1)	
			(5 marks)

10. The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0, 1, 2, \ldots$$

where a is a constant, and J_n and A_n are the respective numbers of juvenile and adult chimpanzees n years after the start of the study.

(a) Interpret the meaning of the constant a in the context of the model.

(1)

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

(b) (i) Find, in terms of a

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

(3)

(ii) Hence, or otherwise, find the value of a.

(3)

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

(2)

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

(1)

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear.*)

(2)

AS 2019

Matrices

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	a represents the proportion of juvenile chimpanzees that survive and remain juvenile chimpanzees next year	B1	This mark is given for a correct interpretation of a
(b)(i)	Determinant $= 0.82a - (0.08 \times 0.15)$	M1	This mark is given for finding the determinant of the matrix
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \ldots \begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	M1	This mark is given for a method to find the inverse, with swapped leading diagonals and signs changed on the off diagonals
	$= \dfrac{1}{0.82a - 0.012} \begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	A1	This mark is given for a fully correct inverse matrix
(b)(ii)	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} \begin{pmatrix} 15360 \\ 43008 \end{pmatrix} =$ $\dfrac{1}{0.82a - 0.012} \begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix} \begin{pmatrix} 15360 \\ 43008 \end{pmatrix} =$ $\dfrac{1}{0.82a - 0.012} \begin{pmatrix} 0.82 \times 15360 - 0.15 \times 43008 \\ -0.08 \times 15360 + 43008a \end{pmatrix}$	M1	This mark is given for using the inverse matrix to find the initial juvenile and adult populations of chimpanzees
	$\dfrac{1}{0.82a - 0.012}[6144 + (43008a - 1228.8) = 64\,000$ $4915.2 + 43008a = 64\,000(0.82a - 0.012)$	M1	This mark is given for using the sum of initial populations adding up to 64000 to find the value of a
	$a = \dfrac{5683.2}{9472} = 0.60$	A1	This mark is given for finding the correct value of a

(b)(iii)	Initial juvenile population $= \dfrac{(0.82\times 15360)-(0.15\times 43008)}{(0.82\times 0.60)-0.012}$ $= \dfrac{6144}{0.48} = 12800$	M1	This mark is given for using the value of a to find J_0
	$15\,360 - 12\,800 = 2560$	A1	This mark is given for finding the change (increase) in the number of juvenile chimpanzees in the first year of the study
(c)	Since the number of juvenile chimpanzees has increased, the model is not initially predicting a decline so is not suitable in the short term	B1	This mark is given for a valid comment
(d)	A third category needs to be introduced for chimpanzees aged 40 and above and for which there is no birth rate (no new juveniles produced)	M1	This mark is given for identifying a third category to be added to the model
	Matrix model with form $\begin{pmatrix} J_{n+1} \\ A_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ 0.08 & c & 0 \\ 0 & d & e \end{pmatrix}\begin{pmatrix} J_n \\ A_n \\ M_n \end{pmatrix}$ since mature chimpanzees (M_n) cannot become juvenile or adult, and juveniles cannot become mature in one year	A1	This mark is given for a matrix model in the correct form with a valid explanation

HOME

1. A system of three equations is defined by

$$\begin{aligned} kx + 3y - \ \ z &= 3 \\ 3x - \ \ y + \ \ z &= -k \\ -16x - ky - kz &= k \end{aligned}$$

where k is a positive constant.

Given that there is no unique solution to all three equations,

(a) show that $k = 2$

(2)

Using $k = 2$

(b) determine whether the three equations are consistent, justifying your answer.

(3)

(c) Interpret the answer to part (b) geometrically.

(1)

Question	Scheme	Marks
1(a)	$\begin{vmatrix} k & 3 & -1 \\ 3 & -1 & 1 \\ -16 & -k & -k \end{vmatrix} = k(k+k) - 3(-3k+16) - 1(-3k-16)$	M1
	Solves det = 0 $\Rightarrow 2k^2 + 12k - 32 = 0$ or $k^2 + 6k - 16 = 0$ To achieve $k = 2$ ($k = -8$ must be rejected)	A1
		(2)
	Special case $\begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & 1 \\ -16 & -2 & -2 \end{vmatrix} = 2(2+2) - 3(-3\times 2+16) - 1(-3\times 2-16)$ Shows det = 0, therefore when $k = 2$ there is no unique solution	M1 A0
(b)	Eliminates z to achieve **two** equations in x and y e.g. $5x + 2y = 1$ $-10x - 4y = -2$ $20x + 8y = 4$ Eliminates x to achieve **two** equations in y and z e.g. $11y - 5z = 13$ $22y - 10z = 26$ $-22y - 10z = -26$ Eliminates y to achieve **two** equations in x and z e.g. $11x + 2z = -3$ $22x + 4z = -6$ $-44x - 8z = 12$	M1 A1
	Must give a **reason:** e.g. Two equations are a linear multiple of each other e.g. shows they are the same equation therefore the equations are **consistent**.	A1
(c)	The three **planes** form a **sheaf.**	B1
		(1)

HOME

4. $$\mathbf{M} = \begin{pmatrix} 2 & 1 & 4 \\ k & 2 & -2 \\ 4 & 1 & -2 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} k-7 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}$$

where k is a constant.

(a) Determine, in simplest form in terms of k, the matrix $\mathbf{MN}$.

(2)

(b) Given that $k = 5$

(i) write down $\mathbf{MN}$

(ii) hence write down $\mathbf{M}^{-1}$

(2)

(c) Solve the simultaneous equations

$$\begin{aligned} 2x + y + 4z &= 2 \\ 5x + 2y - 2z &= 3 \\ 4x + y - 2z &= -1 \end{aligned}$$

(2)

(d) Interpret the answer to part (c) geometrically.

(1)

AS 2021

Matrices

Question	Scheme	Marks	AOs
4(a)	$\mathbf{MN} = \begin{pmatrix} 2k-24 & 0 & 0 \\ k^2-7k+10 & 6k-44 & -10k+50 \\ 4k-20 & 0 & -14 \end{pmatrix}$	B1 B1	1.1b 1.1b
		(2)	
(b)(i)	$\mathbf{MN} = \begin{pmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{pmatrix}$	B1ft	1.1b
(ii)	$\mathbf{M}^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}$	B1	1.1b
		(2)	
(c)	$\mathbf{M}^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \ldots$	M1	1.1b
	$\left(-\frac{12}{7}, \frac{40}{7}, -\frac{1}{14}\right)$	A1	1.1b
		(2)	
(d)	The coordinates of the only point at which the **planes** represented by the equations in (c) meet.	B1	2.2a
		(1)	
			(7 marks)

1.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 7 & 2 \\ -5 & 8 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 6 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -5 & 2 & 1 \\ 4 & 3 & 8 \\ -6 & 11 & 2 \end{pmatrix}$$

Given that $\mathbf{I}$ is the 3×3 identity matrix,

(a) (i) show that there is an integer k for which

$$\mathbf{AB} - 3\mathbf{C} + k\mathbf{I} = \mathbf{0}$$

stating the value of k

(ii) explain why there can be no constant m such that

$$\mathbf{BA} - 3\mathbf{C} + m\mathbf{I} = \mathbf{0}$$

(4)

(b) (i) Show how the matrix $\mathbf{C}$ can be used to solve the simultaneous equations

$$\begin{aligned} -5x + 2y + z &= -14 \\ 4x + 3y + 8z &= 3 \\ -6x + 11y + 2z &= 7 \end{aligned}$$

(ii) Hence use your calculator to solve these equations.

(3)

1(a)

(i)

$$\mathbf{AB}=\begin{pmatrix}4&-1\\7&2\\-5&8\end{pmatrix}\begin{pmatrix}2&3&2\\-1&6&5\end{pmatrix}=\begin{pmatrix}8+1&12-6&8-5\\14-2&21+12&14+10\\-10-8&-15+48&-10+40\end{pmatrix}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}$$

M1

So $$\mathbf{AB}-3\mathbf{C}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}-\begin{pmatrix}-15&6&3\\12&9&24\\-18&33&6\end{pmatrix}=\begin{pmatrix}24&0&0\\0&24&0\\0&0&24\end{pmatrix}$$

or

$$\mathbf{AB}-3\mathbf{C}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}+\begin{pmatrix}15&-6&-3\\-12&-9&-24\\18&-33&-6\end{pmatrix}=\begin{pmatrix}24&0&0\\0&24&0\\0&0&24\end{pmatrix}$$

and states a value for k

M1

Hence $\mathbf{AB}-3\mathbf{C}-24\mathbf{I}=\mathbf{0}$ so $k=-24$

A1

(ii) Need two things

One of:

- **BA** is a 2×2 matrix
- Finds the matrix **BA** (must be a 2×2 matrix)

AND

One of:

- cannot subtract a 3×3 matrix
- finds matrix **3C** and comments that they have different dimensions / can't be done
- can't subtract matrices of different sizes
- **3C** or **C** is a 3×3 matrix
- **BA** needs to be a 3×3 matrix

B1

(4)

(b)(i)

$$\begin{pmatrix}-5&2&1\\4&3&8\\-6&11&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-14\\3\\7\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-5&2&1\\4&3&8\\-6&11&2\end{pmatrix}^{-1}\begin{pmatrix}-14\\3\\7\end{pmatrix}$$

Or states $\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{C}^{-1}\begin{pmatrix}-14\\3\\7\end{pmatrix}$

Or states $\begin{pmatrix}x\\y\\z\end{pmatrix}=\dfrac{1}{360}\begin{pmatrix}-82&7&13\\-56&-4&44\\62&43&-23\end{pmatrix}\begin{pmatrix}-14\\3\\7\end{pmatrix}$

M1

(ii)

$$=\frac{1}{360}\begin{pmatrix}-82&7&13\\-56&-4&44\\62&43&-23\end{pmatrix}\begin{pmatrix}-14\\3\\7\end{pmatrix}=\ldots$$

M1

$$=\begin{pmatrix}-\frac{41}{180}&\frac{7}{360}&\frac{13}{360}\\-\frac{7}{45}&-\frac{1}{90}&\frac{11}{90}\\\frac{31}{180}&\frac{43}{360}&-\frac{23}{360}\end{pmatrix}\begin{pmatrix}-14\\3\\7\end{pmatrix}=\ldots$$

$$\mathbf{C}^{-1}\begin{pmatrix}-14\\3\\7\end{pmatrix}=\ldots$$

So solution is $x=\frac{7}{2}, y=3, z=-\frac{5}{2}$ or $(3.5,\ 3,\ -2.5)$

A1

(3)

3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix $\mathbf{M}$ have an inverse?

(2)

Given that $\mathbf{M}$ is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

HOME

3(i)(a)	$\lvert\mathbf{M}\rvert = 2(1+2) - a(-1-1) + 4(2-1) = 0 \Rightarrow a = \ldots$		M1	2.3
	The matrix $\mathbf{M}$ has an inverse when $a \neq -5$		A1	1.1b
			(2)	
(b)	$\text{Minors}: \begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & a+4 \\ 4-a & -6 & -2-a \end{pmatrix}$ or $\text{Cofactors}: \begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -a-4 \\ 4-a & 6 & -2-a \end{pmatrix}$		B1	1.1b
	$\mathbf{M}^{-1} = \dfrac{1}{\lvert\mathbf{M}\rvert}\operatorname{adj}(\mathbf{M})$		M1	1.1b
	$\mathbf{M}^{-1} = \dfrac{1}{2a+10}\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$	2 correct rows or columns. Follow through their det**M**	A1ft	1.1b
		All correct. Follow through their det**M**	A1ft	1.1b
			(4)	

A2 2019 Paper 2

Matrices

7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix $\mathbf{M}$ has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$\begin{aligned} 2x - y + z &= p \\ 3x - 6y + 4z &= 1 \\ 3x + 2y - z &= 0 \end{aligned}$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$\begin{aligned} 2x - y + z &= 1 \\ 3x - 5y + 4z &= q \\ 3x + 2y - z &= 0 \end{aligned}$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

HOME

A2 2019 Paper 2

Matrices

7(a)	$\lvert\mathbf{M}\rvert = 2(-k-8)+1(-3-12)+1(6-3k)=0 \Rightarrow k=\ldots$	M1	1.1b
	$k \neq -5$	A1	2.4
		(2)	
(b) Way 1	$\mathbf{M}=\begin{pmatrix}2 & -1 & 1\\ 3 & -6 & 4\\ 3 & 2 & -1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}p\\1\\0\end{pmatrix}$	M1	3.1a
	$\mathbf{M}^{-1}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}$	B1	1.1b
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}\begin{pmatrix}p\\1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$	M1	2.1
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2p+1\\ 15p-5\\ 24p-7\end{pmatrix}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1,\ \frac{24p-7}{5}\right)$	A1ft	2.5
		(5)	
(c)(i)	For consistency: E.g. $5x+y=4-q$ and $15x+3y=q$	M1	3.1a
	$4-q=\frac{q}{3} \Rightarrow q=\ldots$	M1	2.1
	$q=3$	A1	1.1b
	Alternative for (c)(i): $x=1 \Rightarrow 2-y+z=1,\ 3+2y-z=0 \Rightarrow y=\ldots, z=\ldots$ M1 for allocating a number to one variable and solves for the other 2 $x=1, y=-4, z=-5 \Rightarrow 3+20-20=q$ M1 substitutes into the second equation and solves for q A1: $q=3$		
(ii)	Three **planes** that intersect in a **line** Or Three **planes** that form a **sheaf** allow **sheath**!	B1	2.4
		(4)	
			(11 marks)

HOME

6.

$$\mathbf{M} = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Given that $k \neq 4$, find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(4)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect.

$$\begin{aligned} 2x + 5y + 7z &= 1 \\ x + y + z &= p \\ 2x + y - z &= 2 \end{aligned}$$

(3)

(c) (i) Find the value of q for which the following planes intersect in a straight line.

$$\begin{aligned} 4x + 5y + 7z &= 1 \\ x + y + z &= q \\ 2x + y - z &= 2 \end{aligned}$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)

A2 2020 Paper 2

Matrices

6(a)	$\lvert\mathbf{M}\rvert = k(-1-1)-5(-1-2)+7(1-2)$ $\{=8-2k\}$	M1	1.1b
	Minors: $\begin{pmatrix} -2 & -3 & -1 \\ -12 & -k-14 & k-10 \\ -2 & k-7 & k-5 \end{pmatrix}$ Cofactors: $\begin{pmatrix} -2 & 3 & -1 \\ 12 & -k-14 & 10-k \\ -2 & 7-k & k-5 \end{pmatrix}$	M1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{8-2k}\begin{pmatrix} -2 & 12 & -2 \\ 3 & -k-14 & 7-k \\ -1 & 10-k & k-5 \end{pmatrix}$	M1 A1	2.1 1.1b
		(4)	
(b)	$\mathbf{M}^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 12 & -2 \\ 3 & -16 & 5 \\ -1 & 8 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$ Solve the equations simultaneously to achieve values for x, y and z $y+3z=2p-2$ and $4y+8z=-1 \Rightarrow x=\ldots, y=\ldots, z=\ldots$	M1	3.1a
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}+3p-1 \\ \frac{3}{4}-4p+\frac{5}{2} \\ -\frac{1}{4}+2p-\frac{3}{2} \end{pmatrix}$	A1ft	1.1b
	$\left(\frac{12p-6}{4}, \frac{13-16p}{4}, \frac{8p-7}{4}\right)$ $\left(3p-\frac{3}{2}, \frac{13}{4}-4p, 2p-\frac{7}{4}\right)$	A1	2.2a
		(3)	

A2 2020 Paper 2

Matrices

(c)(i)				
	For consistency: E.g. eliminates z to find two equations from $3x+2y=q+2$ $3x+2y=7q-1$ $18x+12y=15$	For consistency: E.g. eliminates x to find two equations from $y+3z=1-4q$ $3y+9z=-3$ $y+3z=2q-2$	For consistency: E.g. eliminates y to find two equations from $x-2z=2-q$ $-x+2z=1-5q$ $-6x+12z=-9$	M1
	e.g. $q+2=7q-1$ $\Rightarrow q=\ldots$	e.g. $-3=3(1-4q)$ $\Rightarrow q=\ldots$	e.g. $-9=6(1-5q)$ $\Rightarrow q=\ldots$	M1
	$q=\frac{1}{2}$			A1
(ii)	For example: $x=\lambda \Rightarrow 3\lambda+2y=\frac{5}{2},\ \lambda-2z=\frac{3}{2} \Rightarrow y=\mathrm{f}(\lambda), z=\mathrm{f}(\lambda)$ $y=\lambda \Rightarrow 3x+2\lambda=\frac{5}{2},\ \lambda+3z=1 \Rightarrow x=\mathrm{f}(\lambda), z=\mathrm{f}(\lambda)$ $z=\lambda \Rightarrow 3y+9\lambda=-3,\ -6x+12\lambda=-9 \Rightarrow x=\mathrm{f}(\lambda), y=\mathrm{f}(\lambda)$			M1
	Let $x=\lambda$, $\lambda=\frac{y-\frac{5}{4}}{-\frac{3}{2}}=\frac{z+\frac{3}{4}}{\frac{1}{2}}$ or $y=\frac{5}{4}-\frac{3}{2}\lambda, z=-\frac{3}{4}+\frac{1}{2}\lambda$ Let $y=\lambda$, $\lambda=\frac{x-\frac{5}{6}}{-\frac{2}{3}}=\frac{z+\frac{1}{3}}{-\frac{1}{3}}$ or $x=\frac{5}{5}-\frac{2}{3}\lambda, z=-\frac{1}{3}-\frac{1}{3}\lambda$ Let $z=\lambda$, $\lambda=\frac{x-\frac{3}{2}}{2}=\frac{y+1}{-3}$ or $x=\frac{3}{2}+2\lambda, y=-1-3\lambda$			A1
	$\mathbf{r}=\frac{5}{4}\mathbf{j}-\frac{3}{4}\mathbf{k}+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ o.e. $\mathbf{r}=\frac{5}{6}\mathbf{i}-\frac{1}{3}\mathbf{k}+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ o.e. $\mathbf{r}=\frac{3}{2}\mathbf{i}-\mathbf{j}+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ o.e.			M1 A1

HOME

4. (i) **A** is a 2 by 2 matrix and **B** is a 2 by 3 matrix.

Giving a reason for your answer, explain whether it is possible to evaluate

(a) **AB**

(b) **A** + **B**

(2)

(ii) Given that

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \lambda \mathbf{I}$$

where a, b and λ are constants,

(a) determine

- the value of λ
- the value of a
- the value of b

(b) Hence deduce the inverse of the matrix $\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}$

(3)

(iii) Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \qquad \text{where } 0 \leqslant \theta < \pi$$

determine the values of θ for which the matrix **M** is singular.

(4)

A2 2021 Paper 1

Matrices

Question	Scheme	Marks	AOs
4(i) (a)	It is possible as the number of columns of matrix **A** matches the number of rows of matrix **B**.	B1	2.4
(b)	It is not possible as matrix **A** and matrix **B** have different dimensions o.e. different number of columns	B1	2.4
		(2)	
(ii) (a)	$\lambda = 5$	B1	2.2a
	$a = 1,\ b = 2$	B1	2.2a
(b)	Inverse matrix $= \frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix}$	B1 ft	3.1a
		(3)	
(iii)	A complete method to find the determinant of the matrix and set equal to zero.	M1	1.1b
	Determinant = $1(\sin\theta\sin 2\theta - \cos\theta\cos 2\theta) - 1(0) + 1(0) = 0$	A1	1.1b
	Uses compound angle formula to achieve $\cos 3\theta = 0$ leading to $\theta = \ldots$ or use of $\sin 2q = 2\sin q\cos q$ and $\cos 2q = 1 - 2\sin^2 q$ (e.g. to achieve $\cos q(4\sin^2 q - 1) = 0$) leading to $\theta = \ldots$ or use of $\sin 2q = 2\sin q\cos q$ and $\cos 2q = 2\cos^2 q - 1$ (e.g. to achieve $4\cos^3 q - 3\cos q = 0$) leading to $\theta = \ldots$	M1	3.1a
	$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$	A1	1.1b
		(4)	
			(9 marks)

HOME

5.

$$\mathbf{M} = \begin{pmatrix} a & 2 & -3 \\ 2 & 3 & 0 \\ 4 & a & 2 \end{pmatrix} \qquad \text{where } a \text{ is a constant}$$

(a) Show that $\mathbf{M}$ is non-singular for all values of a.

(2)

(b) Determine, in terms of a, $\mathbf{M}^{-1}$

(4)

A2 2022 Paper 1

Matrices

5(a)	$det(\boldsymbol{M}) = a(6) - 2(4) - 3(2a - 12)$	M1
	$det(\boldsymbol{M}) = 28 \neq 0$ therefore, non-singular for all values of a	A1
		(2)
(b)	Finds the matrix of minors $\begin{pmatrix} 6 & 4 & 2a-12 \\ 4+3a & 2a+12 & a^2-8 \\ 9 & 6 & 3a-4 \end{pmatrix}$	M1
	Finds the matrix of cofactors and transposes. $\begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}$	M1
	$\frac{1}{28}\begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}$	M1 A1
		(4)

BICEN MATHS

HOME

2. **In this question you must show all stages of your working.**

A college offers only three courses: Construction, Design and Hospitality.

Each student enrols on just one of these courses.

In 2019, there was a total of 1110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

In 2020 the number of students enrolled on

- Construction **increased** by 1.25%
- Design **increased** by 2.5%
- Hospitality **decreased** by 2%

In 2020, the total number of students at the college increased by 0.27% to 2 significant figures.

(a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019.

(ii) Using your variables from part (a)(i), write down **three** equations that model this situation.

(4)

(b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019.

(4)

2(a)(i)	x / C = number of **Construction** students y / D = number of **Design** students z / H = number of **Hospitality** students	B1
(ii)	The increase in number of students in 2020 $1110 \times 0.0027\{= 2.997 \approx 3\}$ Or The number of students in 2020 $1110 \times 1.0027 = \{1112.997 \approx 1113\}$	M1
	$x + y + z = 1110$ $C + D + H = 1110$ $x - z = 370$ o.e. $C - H = 370$ o.e. $0.0125C + 0.025D - 0.02H = 3$or 2.997 o.e $1.0125C + 1.025D + 0.98H = 1113$ or 1112.997o.e. $0.0125x + 0.025y - 0.02z = 3$ or 2.997o.e$1.0125x + 1.025y + 0.98z = 1113$ or 1112.997 o.e.	M1 A1
		(4)
(b)	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 3 \end{pmatrix}$	M1 A1ft
	$\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}^{-1}\begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$ or $\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}^{-1}\begin{pmatrix} 1110 \\ 370 \\ 3 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$	dM1
	So in 2019, **720** students studied **C**onstruction, **40** students studied **D**esign and **350** students studied **H**ospitality	A1
		(4)

HOME

Linear Transformations

5.
$$\mathbf{M} = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

(a) Show that $\mathbf{M}$ is non-singular.

(2)

The hexagon R is transformed to the hexagon S by the transformation represented by the matrix $\mathbf{M}$.

Given that the area of hexagon R is 5 square units,

(b) find the area of hexagon S.

(1)

The matrix $\mathbf{M}$ represents an enlargement, with centre (0, 0) and scale factor k, where $k > 0$, followed by a rotation anti-clockwise through an angle θ about (0, 0).

(c) Find the value of k.

(2)

(d) Find the value of θ.

(2)

HOME

Question	Scheme	Marks	AOs
5(a)	$\det(\mathbf{M}) = (1)(1) - \left(\sqrt{3}\right)\left(-\sqrt{3}\right)$	M1	1.1a
	$\mathbf{M}$ is non-singular because $\det(\mathbf{M}) = 4$ and so $\det(\mathbf{M}) \neq 0$	A1	2.4
		(2)	
(b)	$\text{Area}(S) = 4(5) = 20$	B1ft	1.2
		(1)	
(c)	$k = \sqrt{(1)(1) - \left(\sqrt{3}\right)\left(-\sqrt{3}\right)}$	M1	1.1b
	$= 2$	A1ft	1.1b
		(2)	
(d)	$\cos\theta = \frac{1}{2}$ or $\sin\theta = \frac{\sqrt{3}}{2}$ or $\tan\theta = \sqrt{3}$	M1	1.1b
	$\theta = 60°$ or $\frac{\pi}{3}$	A1	1.1b
		(2)	
			(7 marks)

5.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **A**.

(3)

The transformation V, represented by the 2×2 matrix **B**, is a reflection in the line $y = -x$

(b) Write down the matrix **B**.

(1)

Given that U followed by V is the transformation T, which is represented by the matrix **C**,

(c) find the matrix **C**.

(2)

(d) Show that there is a real number k for which the point $(1, k)$ is invariant under T.

(4)

AS 2018

Linear Transformations

Question	Scheme	Marks	AOs
5(a)	Rotation	B1	1.1b
	120 degrees (anticlockwise) or $\frac{2\pi}{3}$ radians (anticlockwise) Or 240 degrees clockwise or $\frac{4\pi}{3}$ radians clockwise	B1	2.5
	About (from) the origin. Allow (0, 0) or O for origin.	B1	1.2
		(3)	
(b)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	B1	1.1b
		(1)	
(c)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}=\begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$		
	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}=\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$		

(d)	$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix}=\begin{pmatrix} 1 \\ k \end{pmatrix}=\dots$ or $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\dots$ Note: $\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix}=\begin{pmatrix} -\frac{\sqrt{3}}{2}+\frac{1}{2}k \\ \frac{1}{2}+\frac{\sqrt{3}}{2}k \end{pmatrix}=\begin{pmatrix} 1 \\ k \end{pmatrix}$ can score M1 (for the matrix equation) but needs an equation to be "extracted" to score the next A1	M1	3.1a
	$-\frac{\sqrt{3}}{2}+\frac{1}{2}k=1$ or $\frac{1}{2}+\frac{\sqrt{3}}{2}k=k$ or $x=-\frac{\sqrt{3}}{2}x+\frac{1}{2}y$ or $y=\frac{1}{2}x+\frac{\sqrt{3}}{2}y$ (Note that candidates may then substitute x = 1 which is acceptable)	A1ft	1.1b
	$-\frac{\sqrt{3}}{2}+\frac{1}{2}k=1$ or $x=-\frac{\sqrt{3}}{2}x+\frac{1}{2}y \Rightarrow k=2+\sqrt{3}\left(\text{or } \frac{1}{2-\sqrt{3}}\right)$	A1	1.1b
	$\frac{1}{2}+\frac{\sqrt{3}}{2}k=k$ or $y=\frac{1}{2}x+\frac{\sqrt{3}}{2}y \Rightarrow k=2+\sqrt{3}\left(\text{or } \frac{1}{2-\sqrt{3}}\right)$	B1	1.1b
		(4)	
			(10 marks)

HOME

1.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(a) Show that the matrix $\mathbf{M}$ is non-singular.

(2)

The transformation T of the plane is represented by the matrix $\mathbf{M}$.

The triangle R is transformed to the triangle S by the transformation T.

Given that the area of S is 63 square units,

(b) find the area of R.

(2)

(c) Show that the line $y = 2x$ is invariant under the transformation T.

(2)

AS 2019

Question 1 (Total 6 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	det $\mathbf{M} = (4 \times -7) - (2 \times -5) = -18$	M1	This mark is given for a method to find the determinant of **M**
	$\mathbf{M}$ is non-singular since det $\mathbf{M} \neq 0$	A1	This mark is given for a correct reason and an understanding that a non-singular matrix has a non-zero determinant.
(b)	$\lvert \det \mathbf{M} \rvert \times \text{Area } R = \text{Area } S$	M1	This mark is given for recognising the relationship between the size of det **M** and the areas of triangles R and S
	Area $R = \dfrac{63}{\lvert -18 \rvert} = \dfrac{7}{2}$	A1	This mark is given for finding the area of triangle R
(c)	$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 4x - 10x \\ 2x - 14x \end{pmatrix}$	M1	This mark is given for mapping the line $y = 2x$ under the transformation T
	$= \begin{pmatrix} -6x \\ -12x \end{pmatrix} = -6\begin{pmatrix} x \\ 2x \end{pmatrix}$ All points of $y = 2x$ map to points on $y = 2x$, hence the line $y = 2x$ is invariant	A1	This mark is given correct multiplication and working leading to the conclusion that the line is invariant under the transformation T

6. (i)

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}$$

where a and b are non-zero constants.

Given that the matrix $\mathbf{A}$ is self-inverse,

(a) determine the value of b and the possible values for a.

(5)

The matrix $\mathbf{A}$ represents a linear transformation M.

Using the smaller value of a from part (a),

(b) show that the invariant points of the linear transformation M form a line, stating the equation of this line.

(3)

(ii)

$$\mathbf{P} = \begin{pmatrix} p & 2p \\ -1 & 3p \end{pmatrix}$$

where p is a positive constant.

The matrix $\mathbf{P}$ represents a linear transformation U.
The triangle T has vertices at the points with coordinates (1, 2), (3, 2) and (2, 5).
The area of the image of T under the linear transformation U is 15

(a) Determine the value of p.

(4)

The transformation V consists of a stretch scale factor 3 parallel to the x-axis with the y-axis invariant followed by a stretch scale factor –2 parallel to the y-axis with the x-axis invariant. The transformation V is represented by the matrix $\mathbf{Q}$.

(b) Write down the matrix $\mathbf{Q}$.

(2)

Given that U followed by V is the transformation W, which is represented by the matrix $\mathbf{R}$,

(c) find the matrix $\mathbf{R}$.

(2)

6(i) (a)

Multiplies the matrix **A** by itself and sets equal to **I** to form one equation in a only and another equation involving both a and b. $\begin{pmatrix}2 & a\\ a-4 & b\end{pmatrix}\begin{pmatrix}2 & a\\ a-4 & b\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Rightarrow 4+a(a-4)=1$ and either $2a+ab=0$ or $2(a-4)+b(a-4)=0$ or $a(a-4)+b^2=1$		M1
Solves a 3TQ involving only the constant a. This could come after a value of b is found and this value substituted into an equation involving both a and b $a^2-4a+3=0\Rightarrow(a-3)(a-1)=0\Rightarrow a=...$		dM1
$a=1, a=3$		A1
Substitutes a value for a into an equation involving both a and b and solves for b. e.g. $2(1)+(1)b\Rightarrow b=...$ $2(1-4)b+(1-4)=0\Rightarrow b=...$ $(1)(1-4)+b^2=1\Rightarrow b=...$	Alternatively uses $2a+ab=0$ $a(2+b)=0$ As $a\neq 0$ $2+b=0\Rightarrow b=...$	dM1
$b=-2$		A1
		(5)

(b)

Uses their smallest value of **a** and their value for **b** to form two equations $\begin{pmatrix}2 & 'a'\\ 'a-4' & 'b'\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}\Rightarrow 2x+ay=x$ and $(a-4)x+by=y$ $\begin{pmatrix}2 & 1\\ -3 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}\Rightarrow 2x+y=x$ and $-3x-2y=y$	M1
$2x+y=x\Rightarrow x+y=0$ o.e. and $-3x-2y=y\Rightarrow x+y=0$ o.e.	M1
$x+y=0$ o.e.	A1
	(3)

HOME

(ii)(a)	Area of the triangle $T = 3$	B1
	Complete method to find a value for p. Need to see an attempt at the determinant and setting equal to 15 divided by their area of T. The resulting 3TQ needs to be solved to find a value of p. Determinant $3p \times p - (-1) \times 2p = \dfrac{15}{\text{'their area'}} \Rightarrow p = \ldots$	M1
	$3p^2 + 2p - 5 (= 0)$	A1
	$p = 1$ must reject $p = -\dfrac{5}{3}$	A1
		(4)
(b)	$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$	B1 B1
		(2)
(c)	$(\text{their matrix found in part (b)})\begin{pmatrix} 'p' & 2'p' \\ -1 & 3'p' \end{pmatrix} = \begin{pmatrix} \ldots & \ldots \\ \ldots & \ldots \end{pmatrix}$ $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \ldots & \ldots \\ \ldots & \ldots \end{pmatrix}$	M1
	$\begin{pmatrix} 3 & 6 \\ 2 & -6 \end{pmatrix}$	A1ft
		(2)

1. $$\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(a) (i) Describe fully the single geometrical transformation P represented by the matrix $\mathbf{P}$.

(ii) Describe fully the single geometrical transformation Q represented by the matrix $\mathbf{Q}$.

(4)

The transformation P followed by the transformation Q is the transformation R, which is represented by the matrix $\mathbf{R}$.

(b) Determine $\mathbf{R}$.

(1)

(c) (i) Evaluate the determinant of $\mathbf{R}$.

(ii) Explain how the value obtained in (c)(i) relates to the transformation R.

(2)

Question	Scheme	Marks	AOs
1(a)(i)	Rotation	B1	1.1b
	90 degrees anticlockwise about the origin	B1	1.1b
(ii)	Stretch	B1	1.1b
	Scale factor 3 parallel to the y-axis	B1	1.1b
		(4)	
(b)	$\mathbf{QP} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$	B1	1.1b
		(1)	
(c)(i)	$\lvert\mathbf{R}\rvert = 3$	B1ft	1.1b
(ii)	The **area** scale factor of the transformation	B1	2.4
		(2)	
		(7 marks)	

Notes

(a)(i)
B1: Identifies the transformation as a rotation
B1: Correct angle (allow equivalents in degrees or radians), direction and centre the origin
(ii)
B1: Identifies the transformation as a stretch
B1: Correct scale factor and parallel to/in/along the y-axis/y direction
(b)
B1: Correct matrix
(c)(i)
B1ft: Correct value for the determinant (follow through their **R**)
(ii)
B1: Correct explanation, must include **area**
Note: scale factor of the transformation is B0

6. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\mathrm{i}$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

A2 2019 Paper 2

Linear Transformations

6(a)	Examples: $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix} = \ldots$ **or** $(6+2\mathrm{i})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\mathrm{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\mathrm{e}^{\mathrm{i}\arctan\left(\frac{2}{6}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$	M1	3.1a
	$\left(-3-\sqrt{3}\right)$ **or** $\left(3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
	$\left(-3-\sqrt{3}\right)+\left(3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
	Examples: $\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix} = \ldots$ **or** $(6+2\mathrm{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\mathrm{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\mathrm{e}^{\mathrm{i}\arctan\left(\frac{2}{6}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$	M1	3.1a
	$\left(-3+\sqrt{3}\right)$ **or** $\left(-3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
	$\left(-3+\sqrt{3}\right)+\left(-3\sqrt{3}-1\right)\mathrm{i}$	A1	1.1b
		(6)	
(b) **Way 1**	Area $ABC = 3\times\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$ or Area $AOB = \frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$	M1	2.1
	Area $DEF = \frac{1}{4}ABC$ or $\frac{3}{4}AOB$	dM1	3.1a
	$=\frac{3}{8}\times 40\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
		(3)	

1. The transformation P is an enlargement, centre the origin, with scale factor k, where $k > 0$

The transformation Q is a rotation through angle θ degrees anticlockwise about the origin.

The transformation P followed by the transformation Q is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

(a) Determine

(i) the value of k,

(ii) the smallest value of θ

(4)

A square S has vertices at the points with coordinates (0, 0), (a, –a), (2a, 0) and (a, a) where a is a constant.

The square S is transformed to the square S' by the transformation represented by **M**.

(b) Determine, in terms of a, the area of S'

(2)

A2 2021 Paper 1

Linear Transformations

Question	Scheme	Marks	AOs
1(a) **Way 1**	$\det \mathbf{M} = -4\times -4 - 4\sqrt{3}\times -4\sqrt{3} = \ldots \Rightarrow k = \sqrt{\det \mathbf{M}} = \ldots$	M1	3.1a
	$k = 8$	A1	1.1b
	$\Rightarrow \mathbf{Q} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \ldots$	M1	1.1b
	$(\cos\theta < 0, \sin\theta > 0 \Rightarrow$ Quadrant 2 so$)$ $\theta = 120°$	A1	1.1b
		(4)	
Way 2	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = k\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$	M1	3.1a
	Achieves both the equations $k\cos\theta = -4$ and $k\sin\theta = 4\sqrt{3}$	A1	1.1b
	$\frac{k\sin\theta}{k\cos\theta} = \frac{4\sqrt{3}}{-4} \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = \ldots$	M1	1.1b
	$\theta = 120°$ and $k = 8$	A1	1.1b
		(4)	
(b)	Area of S' = area of $S \times k^2$ (The area of the square $S = 2a^2$)	M1	1.1b
	Area of S' = $128a^2$	A1ft	2.2a
		(2)	
		(6 marks)	

HOME

2.

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix $\mathbf{A}$ represents the linear transformation M.

Prove that, for the linear transformation M, there are no invariant lines.

(5)

A2 2021 Paper 2

Linear Transformations

Question	Scheme		Marks	AOs
2	$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix}=\begin{pmatrix} X \\ mX+c \end{pmatrix}$ leading to an equation in x, m, c and X		M1	3.1a
	$4x-2(mx+c)=X$ and $5x+3(mx+c)=mX+c$		A1	1.1b
	$5x+3(mx+c)=m\big(4x-2(mx+c)\big)+c$ leading to $5+3m=4m-2m^2$ $\quad(3c=-2mc+c)$		M1	2.1
	$2m^2-m+5=0\Rightarrow b^2-4ac=(-1)^2-4(2)(5)=\ldots$	Solves $3c=-2mc+c\Rightarrow m=\ldots$	dM1	1.1b
	Correct expression for the discriminant $=\{-39\}<0$ therefore there are no invariant lines.	$m=-1$ and shows a contradiction in $5+3m=4m-2m^2$ therefore there are no invariant lines.	A1	2.4
	Alternative $\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix}=\begin{pmatrix} X \\ mX \end{pmatrix}$ leading to an equation in x, m and X		M1	3.1a
	$4x-2(mx)=X$ and $5x+3(mx)=mX$		A1	1.1b
	$5x+3(mx)=m\big(4x-2(mx)\big)$ leading to $5+3m=4m-2m^2$		M1	2.1
	$2m^2-m+5=0\Rightarrow b^2-4ac=(-1)^2-4(2)(5)=\ldots$		dM1	1.1b
	Correct expression for the discriminant $=\{-39\}<0$ therefore there are no invariant lines that pass through the origin no invariant lines.		A1	2.4
			(5)	
				(5 marks)

HOME

3. $$\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \qquad \text{where } a \text{ is a constant}$$

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$$

(6)

Triangle T has vertices A, B and C.

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$ where $n \in \mathbb{N}$

Given that

- triangle T has an area of $5\,\text{cm}^2$
- triangle T' has an area of $1215\,\text{cm}^2$
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$

(b) determine

(i) the value of n

(ii) the value of a

(5)

A2 2022 Paper 2

Linear Transformations

3(a)	$n=1 \Rightarrow \mathbf{M}^1 = \begin{pmatrix}3^1 & \frac{a}{2}(3^1-1)\\ 0 & 1\end{pmatrix} = \begin{pmatrix}3 & a\\ 0 & 1\end{pmatrix}$ {So the result is true for $n=1$}	B1
	Assume true for $n=k$ Or assume $\mathbf{M}^n$ or $\begin{pmatrix}3 & a\\ 0 & 1\end{pmatrix}^k = \begin{pmatrix}3^k & \frac{a}{2}(3^k-1)\\ 0 & 1\end{pmatrix}$	M1
	A correct method to find an expression for $n=k+1$ $\begin{pmatrix}3 & a\\ 0 & 1\end{pmatrix}^{k+1} = \begin{pmatrix}3^k & \frac{a}{2}(3^k-1)\\ 0 & 1\end{pmatrix}\begin{pmatrix}3 & a\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}3 & a\\ 0 & 1\end{pmatrix}^{k+1} = \begin{pmatrix}3 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}3^k & \frac{a}{2}(3^k-1)\\ 0 & 1\end{pmatrix}$	M1
	$\begin{pmatrix}3(3^k) & a(3^k)+\frac{a}{2}(3^k-1)\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}3(3^k) & 3\times\frac{a}{2}(3^k-1)+a\\ 0 & 1\end{pmatrix}$	A1
	$\begin{pmatrix}3^{k+1} & \frac{a}{2}[2(3^k)+(3^k-1)]\\ 0 & 1\end{pmatrix} =$ $\begin{pmatrix}3^{k+1} & \frac{a}{2}[3(3^k)-1]\\ 0 & 1\end{pmatrix} = \begin{pmatrix}3^{k+1} & \frac{a}{2}[3^{k+1}-1]\\ 0 & 1\end{pmatrix}$ $\begin{pmatrix}3(3^k) & 3\times\frac{a}{2}(3^k-1)+a\\ 0 & 1\end{pmatrix} =$ $\begin{pmatrix}3^{k+1} & \frac{a}{2}(3(3^k-1)+2)\\ 0 & 1\end{pmatrix}$ $= \begin{pmatrix}3^{k+1} & \frac{a}{2}(3^{k+1}-1)\\ 0 & 1\end{pmatrix}$	A1
	If true for $n=k$ **then** true for $n=k+1$ and as it is **true** for $n=1$ the statement is **true for** all (positive integers) $\boldsymbol{n}$	A1
		(6)

(b)(i)	$det(\mathbf{M}^n) = 3^n$ or $det(\mathbf{M}) = 3$	B1
	Uses $5\times det(\mathbf{M}^n) = 1215 \Rightarrow p^n = q \Rightarrow n = \ldots$ $5\times 3^n = 1215 \Rightarrow 3^n = 243 \Rightarrow n = \ldots$	M1
	$n=5$	A1
(ii)	$\begin{pmatrix}3^n & \frac{a}{2}(3^n-1)\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ -2\end{pmatrix} = \begin{pmatrix}123\\ -2\end{pmatrix} \Rightarrow 2(3^n) - 2\frac{a}{2}(3^n-1) = 123$ $\Rightarrow a = \ldots$	M1
	$\begin{pmatrix}243 & \frac{a}{2}(243-1)\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ -2\end{pmatrix} = \begin{pmatrix}123\\ -2\end{pmatrix} \Rightarrow 2(243) - 2\frac{a}{2}(243-1)$ $= 123 \Rightarrow a = \ldots$ $\frac{1}{243}\begin{pmatrix}1 & -\frac{a}{2}(243-1)\\ 0 & a\end{pmatrix}\begin{pmatrix}123\\ -2\end{pmatrix} = \begin{pmatrix}2\\ -2\end{pmatrix} \Rightarrow \frac{123 - 2\frac{a}{2}(243-1)}{243}$ $= -2 \Rightarrow a = \ldots$	
	$a = 1.5$	A1
		(5)

Proof by Induction

6. (a) Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n,

$$\sum_{r=1}^{n} r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9)$$

(4)

(c) Hence find the value of n that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2$$

(5)

Question	Scheme	Marks	AOs
6(a)	$n=1$, $\sum_{r=1}^{1} r^2 = 1$ and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(1)(2)(3) = 1$	B1	2.2a
	Assume general statement is true for $n=k$ So assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true	M1	2.4
	$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$	M1	2.1
	$= \frac{1}{6}(k+1)(2k^2+7k+6)$	A1	1.1b
	$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$	A1	1.1b
	Then the general result is <u>true for $n=k+1$</u> As the general result has been shown to be <u>true for $n=1$,</u> then the general result <u>is true for all n</u> $\in \mathbb{Z}^+$	A1	2.4
		(6)	

8. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

(6)

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

Question	Scheme	Marks	AOs
8(i)	$n=1, \begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^1=\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}, \quad \begin{pmatrix}4\times1+1 & -8(1)\\ 2\times1 & 1-4(1)\end{pmatrix}=\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$ So the result is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^k=\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}$	M1	2.4
	$\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^{k+1}=\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$ or $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^{k+1}=\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}$	M1	1.1b
	$\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}=\begin{pmatrix}5(4k+1)-16k & -8(4k+1)+24k\\ 10k+2(1-4k) & -16k-3(1-4k)\end{pmatrix}$ or $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}=\begin{pmatrix}5(4k+1)-16k & -40k-8(1-4k)\\ 2(1+4k)-6k & -16k-3(1-4k)\end{pmatrix}$	A1	1.1b
	$=\begin{pmatrix}4(k+1)+1 & -8(k+1)\\ 2(k+1) & 1-4(k+1)\end{pmatrix}$	A1	2.1
	<u>If true for $n=k$ then true for $n=k+1$</u>, <u>true for $n=1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	
(ii) **Way 1**	$f(k+1)-f(k)$		
	When $n=1,\ 4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $4^{k+1}+5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1)-f(k)=4^{k+2}+5^{2k+1}-4^{k+1}-5^{2k-1}$	M1	2.1
	$=4\times4^{k+1}+25\times5^{2k-1}-4^{k+1}-5^{2k-1}$		
	$=3f(k)+21\times5^{2k-1}$ or e.g. $=24f(k)-21\times4^{k+1}$	A1	1.1b
	$f(k+1)=4f(k)+21\times5^{2k-1}$ or e.g. $f(k+1)=25f(k)-21\times4^{k+1}$	A1	1.1b
	<u>If true for $n=k$ then true for $n=k+1$</u>, <u>true for $n=1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	

3. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

(6)

AS 2019

Proof by Induction

Question 3 (Total 6 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	When $n = 1$, $\sum_{r=1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{1\times 3} = \frac{1}{3}$ and $\frac{n}{2n+1} = \frac{1}{(2\times 1)+1} = \frac{1}{3}$	B1	This mark is given for checking both sides of the statement for $n = 1$
	Assume the statement is true for $n = k$, then $\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$ is true	M1	This mark is given for assuming the general statement $n = k$ is true
	$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)}$ $= \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$ $= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$	M1	This mark is given for adding the $(k+1)$th term to the sum of k terms
	$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$	M1	This mark is given for combining the two fractions over a common denominator
	$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$ $= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$ $= \frac{(k+1)}{(2k+3)}$ $= \frac{(k+1)}{2(k+1)+1}$	A1	This mark is given for correct algebra leading to the term $\frac{(k+1)}{2(k+1)+1}$
	Thus the general result is true for $n = k + 1$ Since the general result is true for $n = 1$ and true for $n = k$ implies true for $n = k + 1$, the result is true for all $n \in \mathbb{N}$	A1	This mark is given for a fully correct induction statement

HOME

8. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

(6)

8	**Way 1:** $\mathbf{f}(\boldsymbol{k}+1)-\mathbf{f}(\boldsymbol{k})$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ Shows the statement is true for $n=1$, allow 5(7)	B1
	Assume true for $\boldsymbol{n}=\boldsymbol{k}$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1
	$\mathrm{f}(k+1)-\mathrm{f}(k)=2^{k+3}+3^{2k+3}-\left(2^{k+2}+3^{2k+1}\right)$	M1
	$=2\times2^{k+2}+9\times3^{2k+1}-2^{k+2}-3^{2k+1}$ $=2^{k+2}+8\times3^{2k+1}$ $=\mathrm{f}(k)+7\times3^{2k+1}$ or $8\mathrm{f}(k)-7\times2^{k+2}$	A1
	$\mathrm{f}(k+1)=2\mathrm{f}(k)+7\times3^{2k+1}$ or $9\mathrm{f}(k)-7\times2^{k+2}$	A1
	If **true for** $\boldsymbol{n}=\boldsymbol{k}$ then **true for** $\boldsymbol{n}=\boldsymbol{k}+1$ and as it is **true for** $\boldsymbol{n}=1$ the statement is **true for all** (positive integers) $\boldsymbol{n}$	A1
		(6)

8. (a) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

(6)

(b) Hence, show that, for all positive integers n,

$$\sum_{r=n}^{2n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$$

where a, b, c and d are integers to be determined.

(3)

AS 2021

Proof by Induction

Question	Scheme	Marks	AOs
8(a)	$n=1,\quad \text{lhs}=1(2)(3)=6,\quad \text{rhs}=\frac{1}{2}(1)(2)^2(3)=6$ (true for $n=1$)	B1	2.2a
	Assume true for $n=k$ so $\sum_{r=1}^{k} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)$	M1	2.4
	$\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)+(k+1)(k+2)(2k+3)$	M1	2.1
	$=\frac{1}{2}(k+1)(k+2)\left[k(k+1)+2(2k+3)\right]$	dM1	1.1b
	$=\frac{1}{2}(k+1)(k+2)\left[k^2+5k+6\right]=\frac{1}{2}(k+1)(k+2)(k+2)(k+3)$ Shows that $=\frac{1}{2}(\underline{k+1})(\underline{k+1}+1)^2(\underline{k+1}+2)$ Alternatively shows that $\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}(k+1)(k+1+1)^2(k+1+2)$ $=\frac{1}{2}(k+1)(k+2)^2(k+3)$ Compares with their summation and concludes true for $n=k+1$, may be seen in the conclusion.	A1	1.1b
	If the statement is **true for $n=k$ then** it has been shown **true for $n=k+1$** and as it is **true for $n=1$**, the statement is **true** for all **positive integers n.**	A1	2.4
		(6)	
(b)	$\sum_{r=n}^{2n} r(r+1)(2r+1)=\frac{1}{2}(2n)(2n+1)^2(2n+2)-\frac{1}{2}(n-1)n^2(n+1)$	M1	3.1a
	$=\frac{1}{2}n(n+1)\left[4(2n+1)^2-n(n-1)\right]$	M1	1.1b
	$=\frac{1}{2}n(n+1)(15n^2+17n+4)$ $=\frac{1}{2}n(n+1)(3n+1)(5n+4)$	A1	1.1b
		(3)	
			(9 marks)

7. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^n = \begin{pmatrix} 1-6n & 9n \\ -4n & 1+6n \end{pmatrix}$$

(6)

7	For $n=1$: $\begin{pmatrix}1-6\times1 & 9\times1\\ -4\times1 & 1+6\times1\end{pmatrix}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^{1}$ So the statement is true for $n=1$	B1
	Assume true for $\boldsymbol{n=k}$, or **Assume** $\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^{k}=\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}$	M1
	$\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^{k+1}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^{k}\times\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}$ OR $\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}\times\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^{k}$	M1
	$=\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}\times\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}=\begin{pmatrix}-5+30k-36k & 9-54k+63k\\ 20k-4-24k & -36k+7+42k\end{pmatrix}$ OR $=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}\times\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}=\begin{pmatrix}-5+30k-36k & -45k+9+54k\\ -4+24k-28k & -36k+7+42k\end{pmatrix}$	M1
	Achieves from fully correct working $=\begin{pmatrix}-5-6k & 9+9k\\ -4-4k & 7+6k\end{pmatrix}$	A1
	$=\begin{pmatrix}1-6(k+1) & 9(k+1)\\ -4(k+1) & 1+6(k+1)\end{pmatrix}$ Hence the result is true for $n=k+1$. Since it is true for $n=1$, and if true for $n=k$ then true for $n=k+1$, thus by mathematical induction the result holds for all $n \in \mathbb{N}$	A1cso
		(6)

HOME

2. Prove by induction that for all positive integers n,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

(6)

A2 SAMs Paper 1

Proof by Induction

Question	Scheme	Marks	AOs
2	When $n=1$, $2^{3n+1}+3\left(5^{2n+1}\right)=16+375=391$ $391 = 17 \times 23$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $2^{3k+1}+3\left(5^{2k+1}\right)$ is divisible by 17	M1	2.4
	$\mathrm{f}(k+1)-\mathrm{f}(k)=2^{3k+4}+3\left(5^{2k+3}\right)-2^{3k+1}-3\left(5^{2k+1}\right)$	M1	2.1
	$=7\times 2^{3k+1}+7\times 3\left(5^{2k+1}\right)+17\times 3\left(5^{2k+1}\right)$		
	$=7\mathrm{f}(k)+17\times 3\left(5^{2k+1}\right)$	A1	1.1b
	$\mathrm{f}(k+1)=8\mathrm{f}(k)+17\times 3\left(5^{2k+1}\right)$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n	A1	2.4
		(6)	
			(6 marks)

HOME

3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix $\mathbf{M}$ have an inverse?

(2)

Given that $\mathbf{M}$ is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

HOME

A2 SAMs Paper 2

Proof by Induction

(ii)	When $n = 1$, lhs $= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$, rhs $= \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ So the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$	M1	2.4
	$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$	M1	2.1
	$= \begin{pmatrix} 3 \times 3^k & 0 \\ 3 \times 3(3^k - 1) + 6 & 1 \end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n	A1	2.4
		(6)	

HOME

6. Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

A2 2019 Paper 1

Proof by Induction

6	**Way 1** $f(k+1)-f(k)$		
	When $n=1$, $3^{2n+4}-2^{2n}=729-4=725$ $(725=145\times 5)$ so the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$f(k+1)-f(k)=3^{2k+6}-2^{2k+2}-3^{2k+4}+2^{2k}$	M1	2.1
	either $8f(k)+5\times 2^{2k}$ or $3f(k)+5\times 3^{2k+4}$	A1	1.1b
	$f(k+1)=9f(k)+5\times 2^{2k}$ or $f(k+1)=4f(k)+5\times 3^{2k+4}$ o.e.	A1	1.1b
	If true for $n=k$ then it is true for	A1	2.4
	$n=k+1$ and as it is true for $n=1$, the statement is true for all (positive integers) n. (Allow 'for all values')		
		(6)	

HOME

6. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} (3r+1)(r+2) = n(n+2)(n+3)$$

(6)

(ii) Prove by induction that for all positive **odd** integers n

$$f(n) = 4^n + 5^n + 6^n$$

is divisible by 15

(6)

A2 2020 Paper 1

Proof by Induction

6. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(3r+1)(r+2) = n(n+2)(n+3)$$

(6)

(ii) Prove by induction that for all positive **odd** integers n

$$f(n) = 4^n + 5^n + 6^n$$

is divisible by 15

(6)

HOME

A2 2020 Paper 1

Proof by Induction

6(i)	When $n = 1$, $\sum_{r=1}^{1}(3r+1)(r+2) = 4\times 3 = 12$ $1(1+2)(1+3) = 12$ (so the statement is true for $n = 1$)	B1	2.2a
	Assume true for $n = k$ so $\sum_{r=1}^{k}(3r+1)(r+2) = k(k+2)(k+3)$	M1	2.4
	$\sum_{r=1}^{k+1}(3r+1)(r+2) = k(k+2)(k+3)+(3k+4)(k+3)$	M1	2.1
	$=(k+3)(k^2+5k+4)$	A1	1.1b
	$\sum_{r=1}^{k+1}(3r+1)(r+2) = (k+1)(k+3)(k+4)$	A1	1.1b
	$\sum_{r=1}^{k+1}(3r+1)(r+2) = (\underline{k+1})(\underline{k+1}+2)(\underline{k+1}+3)$ If the statement is <u>true for $n = k$ then</u> it has been shown <u>true for $n = k + 1$</u> and as it is <u>true for $n = 1$</u>, the statement <u>is true for all (positive integers) n.</u>	A1	2.4
		(6)	

HOME

A2 2020 Paper 1

Proof by Induction

(ii) Way 1	When $n = 1$, $4^1 + 5^1 + 6^1 = 15$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $4^k + 5^k + 6^k$ is divisible by 15	M1	2.4
	$\mathrm{f}(k+2) = 4^{k+2} + 5^{k+2} + 6^{k+2}$	M1	2.1
	$= 16\times4^k + 16\times5^k + 16\times6^k + 9\times5^k + 20\times6^k$ $= 16\mathrm{f}(k) + 45\times5^{k-1} + 120\times6^{k-1}$	A1 A1	1.1b 1.1b
	E.g As 15 divides f(k), 45 and 120, so 15 divides f(k+1). <u>If true for $n = k$</u> then <u>true for $n = k + 2$</u>, <u>true for $n = 1$</u> so <u>true for all positive odd integers n</u>	A1	2.4
		(6)	
(ii) Way 2	When $n = 1$, $4^1 + 5^1 + 6^1 = 15$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $4^k + 5^k + 6^k$ is divisible by 15	M1	2.4
	$\mathrm{f}(k+2) - \mathrm{f}(k) = 4^{k+2} + 5^{k+2} + 6^{k+2} - 4^k - 5^k - 6^k$	M1	2.1
	$= 15\times4^k + 24\times5^k + 35\times6^k$ $= 15\mathrm{f}(k) + 45\times5^{k-1} + 120\times6^{k-1}$	A1	1.1b
	$\mathrm{f}(k+2) = 16\mathrm{f}(k) + 45\times5^{k-1} + 120\times6^{k-1}$	A1	1.1b
	E.g $\mathrm{f}(k+2) = 16\mathrm{f}(k) + 15\left(3\times5^{k-1} + 8\times6^{k-1}\right)$ so <u>if true for $n = k$</u> then <u>true for $n = k + 2$</u>, <u>true for $n = 1$</u> so <u>true for all positive odd integers n</u>	A1	2.4
		(6)	

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3. $$\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \qquad \text{where } a \text{ is a constant}$$

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$$

(6)

Triangle T has vertices A, B and C.

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$ where $n \in \mathbb{N}$

Given that

- triangle T has an area of 5 cm^2
- triangle T' has an area of 1215 cm^2
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$

(b) determine

(i) the value of n

(ii) the value of a

(5)

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Proof by Induction

3(a)	$n=1 \Rightarrow \ \mathbf{M}^1 = \begin{pmatrix} 3^1 & \frac{a}{2}(3^1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ {So the result is true for $n=1$}	B1
	Assume true for $n=k$ Or assume $\mathbf{M}^n$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$	M1
	A correct method to find an expression for $n=k+1$ $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$	M1
	$\begin{pmatrix} 3(3^k) & a(3^k)+\frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3(3^k) & 3\times\frac{a}{2}(3^k-1)+a \\ 0 & 1 \end{pmatrix}$	A1
	$\begin{pmatrix} 3^{k+1} & \frac{a}{2}[2(3^k)+(3^k-1)] \\ 0 & 1 \end{pmatrix} =$ $\begin{pmatrix} 3^{k+1} & \frac{a}{2}[3(3^k)-1] \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & \frac{a}{2}[3^{k+1}-1] \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 3(3^k) & 3\times\frac{a}{2}(3^k-1)+a \\ 0 & 1 \end{pmatrix} =$ $\begin{pmatrix} 3^{k+1} & \frac{a}{2}(3(3^k-1)+2) \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} & \frac{a}{2}(3^{k+1}-1) \\ 0 & 1 \end{pmatrix}$	A1
	If true for $n=k$ **then** true for $n=k+1$ and as it is **true** for $n=1$ the statement is **true for** all (positive integers) $\boldsymbol{n}$	A1
		(6)

(b)(i)	$det(\mathbf{M}^n) = 3^n$ or $det(\mathbf{M}) = 3$	B1
	Uses $5\times det(\mathbf{M}^n) = 1215 \Rightarrow p^n = q \Rightarrow n = \ldots$ $5\times 3^n = 1215 \Rightarrow 3^n = 243 \Rightarrow n = \ldots$	M1
	$n=5$	A1
(ii)	$\begin{pmatrix} 3^n & \frac{a}{2}(3^n-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix} \Rightarrow 2(3^n) - 2\frac{a}{2}(3^n-1) = 123$ $\Rightarrow a = \ldots$	M1
	$\begin{pmatrix} 243 & \frac{a}{2}(243-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix} \Rightarrow 2(243) - 2\frac{a}{2}(243-1)$ $= 123 \Rightarrow a = \ldots$ $\frac{1}{243}\begin{pmatrix} 1 & -\frac{a}{2}(243-1) \\ 0 & a \end{pmatrix}\begin{pmatrix} 123 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \Rightarrow \frac{123 - 2\frac{a}{2}(243-1)}{243}$ $= -2 \Rightarrow a = \ldots$	
	$a = 1.5$	A1
		(5)

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Vectors

2. The plane Π passes through the point A and is perpendicular to the vector $\mathbf{n}$

Given that

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

where O is the origin,

(a) find a Cartesian equation of Π.

(2)

With respect to the fixed origin O, the line l is given by the equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}$$

The line l intersects the plane Π at the point X.

(b) Show that the acute angle between the plane Π and the line l is 21.2° correct to one decimal place.

(4)

(c) Find the coordinates of the point X.

(4)

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Question	Scheme	Marks	AOs
2(a)	$\mathbf{r}\langle\begin{pmatrix}3\\-1\\2\end{pmatrix}=\begin{pmatrix}5\\-3\\-4\end{pmatrix}\langle\begin{pmatrix}3\\-1\\2\end{pmatrix}$	M1	1.1b
	$3x-y+2z=10$	A1	2.5
		(2)	
(b)	$\begin{pmatrix}3\\-1\\2\end{pmatrix}\langle\begin{pmatrix}-1\\-5\\3\end{pmatrix}=8$	B1	1.1b
	$\sqrt{(3)^2+(-1)^2+(2)^2}\,.\sqrt{(-1)^2+(-5)^2+(3)^2}\ \cos\alpha = \text{"}-3+5+6\text{"}$	M1	1.1b
	$\theta=90^\circ-\arccos\left(\dfrac{8}{\sqrt{14}.\sqrt{35}}\right)$ or $\sin\theta=\dfrac{8}{\sqrt{14}.\sqrt{35}}$	M1	2.1
	$\theta=21.2^\circ$ (1 dp) * cso	A1*	1.1b
		(4)	
(c)	$3(7-\lambda)-(3-5\lambda)+2(-2+3\lambda)=10\Rightarrow\lambda=\ldots$	M1	3.1a
	$\lambda=-\dfrac{1}{2}$	A1	1.1b
	$\overrightarrow{OX}=\begin{pmatrix}7\\3\\-2\end{pmatrix}-\dfrac{1}{2}\begin{pmatrix}-1\\-5\\3\end{pmatrix}=\begin{pmatrix}\ldots\\\ldots\\\ldots\end{pmatrix}$	M1	1.1b
	$X(7.5, 5.5, -3.5)$	A1ft	1.1b
		(4)	
		(10 marks)	

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9. An octopus is able to catch any fish that swim within a distance of 2 m from the octopus's position.

A fish F swims from a point A to a point B.

The octopus is modelled as a fixed particle at the origin O.

Fish F is modelled as a particle moving in a straight line from A to B.

Relative to O, the coordinates of A are $(-3, 1, -7)$ and the coordinates of B are $(9, 4, 11)$, where the unit of distance is metres.

(a) Use the model to determine whether or not the octopus is able to catch fish F.

(7)

(b) Criticise the model in relation to fish F.

(1)

(c) Criticise the model in relation to the octopus.

(1)

Question	Scheme	Marks	AOs
9(a)	$\overrightarrow{AB} = \begin{pmatrix} 9 \\ 4 \\ 11 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \left\{ = \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} \right\}$ or $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix}$	M1	3.1a
	$\left\{ \overrightarrow{OF} = \mathbf{r} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$	M1	1.1b
	$\left\{ \overrightarrow{OF} \bullet \overrightarrow{AB} = 0 \Rightarrow \right\} \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix} \bullet \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = 0$ $\Rightarrow -36+144\lambda+3+9\lambda-126+324\lambda = 0 \Rightarrow 477\lambda - 159 = 0$	dM1	1.1b
	$\Rightarrow \lambda = \frac{1}{3}$	A1	1.1b
	$\left\{ \overrightarrow{OF} = \right\} \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and minimum distance $= \sqrt{(1)^2 + (2)^2 + (-1)^2}$	dM1	3.1a
	$= \sqrt{6}$ **or** $2.449\ldots$	A1	1.1b
	> 2, so the octopus is not able to catch the fish F	A1ft	3.2a
		(7)	

4. Part of the mains water system for a housing estate consists of water pipes buried beneath the ground surface. The water pipes are modelled as straight line segments. One water pipe, W, is buried beneath a particular road. With respect to a fixed origin O, the road surface is modelled as a plane with equation $3x - 5y - 18z = 7$, and W passes through the points $A(-1, -1, -3)$ and $B(1, 2, -3)$. The units are in metres.

(a) Use the model to calculate the acute angle between W and the road surface.

(5)

A point $C(-1, -2, 0)$ lies on the road. A section of water pipe needs to be connected to W from C.

(b) Using the model, find, to the nearest cm, the shortest length of pipe needed to connect C to W.

(6)

Question	Scheme	Marks	AOs
4(a)	Attempts the scalar product between the direction of W and the normal to the road and uses trigonometry to find an angle.	M1	3.1a
	$\left(\begin{pmatrix}1\\2\\-3\end{pmatrix}-\begin{pmatrix}-1\\-1\\-3\end{pmatrix}\right)\bullet\begin{pmatrix}3\\-5\\-18\end{pmatrix}=-9$ or $\left(\begin{pmatrix}-1\\-1\\-3\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}\right)\bullet\begin{pmatrix}3\\-5\\-18\end{pmatrix}=9$	M1 A1	1.1b 1.1b
	$\sqrt{(2)^2+(3)^3+(0)^2}\sqrt{(3)^2+(-5)^3+(-18)^2}\cos\alpha=\text{"}-9\text{"}$ $\theta=90-\arccos\left(\frac{9}{\sqrt{13}\sqrt{358}}\right)$ or $\theta=\arcsin\left(\frac{9}{\sqrt{13}\sqrt{358}}\right)$ Angle between pipe and road $=7.58°$ (3sf) or 0.132 radians (3sf) (Allow $-7.58°$ or -0.132 radians)	M1 A1	1.1b 3.2a
		(5)	
(b)	$W:\begin{pmatrix}-1\\-1\\-3\end{pmatrix}+t\begin{pmatrix}2\\3\\0\end{pmatrix}$ or $\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\0\end{pmatrix}$	B1ft	1.1b
	$C \text{ to } W:\left\{\begin{pmatrix}-1\\-1\\-3\end{pmatrix}+t\begin{pmatrix}2\\3\\0\end{pmatrix}-\begin{pmatrix}-1\\-2\\0\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\0\end{pmatrix}-\begin{pmatrix}-1\\-2\\0\end{pmatrix}\right\}$	M1	3.4
	$\begin{pmatrix}2t\\3t+1\\-3\end{pmatrix}\bullet\begin{pmatrix}2\\3\\0\end{pmatrix}=0\Rightarrow t=\ldots$ or $\begin{pmatrix}2+2\lambda\\4+3\lambda\\-3\end{pmatrix}\bullet\begin{pmatrix}2\\3\\0\end{pmatrix}=0\Rightarrow\lambda=\ldots$ or $(2t)^2+(3t+1)^2+(-3)^2=\ldots$ or $(2+2t)^2+(4+3t)^2+(-3)^2=\ldots$	M1	3.1b
	$t=-\frac{3}{13}$ or $\lambda=-\frac{16}{13}\Rightarrow(C\text{ to }W)_{\min}$ is $-\frac{6}{13}\mathbf{i}+\frac{4}{13}\mathbf{j}-3\mathbf{k}$ or $(2t)^2+(3t+1)^2+(-3)^2=13\left(t+\frac{3}{13}\right)^2+\frac{121}{13}$ or $(2+2t)^2+(4+3t)^2+(-3)^2=13\left(\lambda+\frac{16}{13}\right)^2+\frac{121}{13}$ or $\frac{d\left((2t)^2+(3t+1)^2+(-3)^2\right)}{dt}=0\Rightarrow t=-\frac{3}{13}\Rightarrow C\text{ to }W\text{ is }-\frac{6}{13}\mathbf{i}+\frac{4}{13}\mathbf{j}-3\mathbf{k}$ Or $\frac{d\left((2+2t)^2+(4+3t)^2+(-3)^2\right)}{dt}=0\Rightarrow t=-\frac{16}{13}\Rightarrow(C\text{ to }W)_{\min}\text{ is }-\frac{6}{13}\mathbf{i}+\frac{4}{13}\mathbf{j}-3\mathbf{k}$	A1	1.1b
	$d=\sqrt{\left(-\frac{6}{13}\right)^2+\left(\frac{4}{13}\right)^2+(-3)^2}$ or $d=\sqrt{\frac{121}{13}}$	ddM1	1.1b
	Shortest length of pipe needed is 305 or 305 cm or 3.05 m	A1	3.2a
		(6)	
			(11 marks)

4. The line l has equation

$$\frac{x+2}{1} = \frac{y-5}{-1} = \frac{z-4}{-3}$$

The plane Π has equation

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -7$$

Determine whether the line l intersects Π at a single point, or lies in Π, or is parallel to Π without intersecting it.

(5)

Question 4 (Total 5 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
	$\mathbf{r} = \begin{pmatrix} -2+\lambda \\ 5-\lambda \\ 4-3\lambda \end{pmatrix}$	M1	This mark is given for finding a parametric form for the line l
	The line l and the plane Π meet if $\begin{pmatrix} -2+\lambda \\ 5-\lambda \\ 4-3\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -7$	M1	This mark is given for substituting into the equation of the plane to find any points of intersection
	$\Rightarrow (-2+\lambda)\times 1 + (5-\lambda)\times -2 + (4-3\lambda)\times 1 = -7$	A1	This mark is given for finding a correct equation in λ
	$\Rightarrow 0\lambda - 8 = -7$ $\Rightarrow -8 = -7$, a contradiction	A1	This mark is given for simplifying the equation and deriving a contradiction
	Hence the line l is parallel to the plane Π but not in it, so there is no intersection	A1	This mark is given for a correct deduction following correct working

8. A gas company maintains a straight pipeline that passes under a mountain.

The pipeline is modelled as a straight line and one side of the mountain is modelled as a plane.

There are accessways from a control centre to two access points on the pipeline.

Modelling the control centre as the origin O, the two access points on the pipeline have coordinates $P(-300, 400, -150)$ and $Q(300, 300, -50)$, where the units are metres.

(a) Find a vector equation for the line PQ, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a scalar parameter.

(2)

The equation of the plane modelling the side of the mountain is $2x + 3y - 5z = 300$

The company wants to create a new accessway from this side of the mountain to the pipeline.

The accessway will consist of a tunnel of shortest possible length between the pipeline and the point $M(100, k, 100)$ on this side of the mountain, where k is a constant.

(b) Using the model, find

(i) the coordinates of the point at which this tunnel will meet the pipeline,

(ii) the length of this tunnel.

(7)

It is only practical to construct the new accessway if it will be significantly shorter than both of the existing accessways, OP and OQ.

(c) Determine whether the company should build the new accessway.

(2)

(d) Suggest one limitation of the model.

(1)

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Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\mathbf{b} = \pm(OQ - OP)$ $= \pm\left[\begin{pmatrix}300\\300\\-50\end{pmatrix} - \begin{pmatrix}-300\\400\\-150\end{pmatrix}\right] = \pm\begin{pmatrix}600\\-100\\100\end{pmatrix}$	M1	This mark is given for finding the positions between P and Q
	$\mathbf{r} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \lambda\begin{pmatrix}600\\-100\\100\end{pmatrix}$	A1	This mark is given for a correct vector equation in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$
(b)(i)	$200 + 3k - 500 = 300$ $k = 200$	B1	This mark is given for substituting the coordinates of M into the equation of the plane modelling the side of the mountain and deducing a correct value of k
	If M is the point on the mountain and X a general point on the line, then $\overline{MX} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \lambda\begin{pmatrix}600\\-100\\100\end{pmatrix} - \begin{pmatrix}100\\200\\100\end{pmatrix}$ $= \begin{pmatrix}-400+600\lambda\\200-100\lambda\\-250+100\lambda\end{pmatrix}$	M1	This mark is given for finding the distance from the point M on the mountain to a general point on the line
	$\begin{pmatrix}-400+600\lambda\\200-100\lambda\\-250+100\lambda\end{pmatrix} . \begin{pmatrix}600\\-100\\100\end{pmatrix} = 0 \Rightarrow \lambda = \frac{3}{4}$	M1	This mark is given for taking the dot product with the direction vector of the line and equating to zero to find a value of λ
	$\overline{OX} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \frac{3}{4}\begin{pmatrix}600\\-100\\100\end{pmatrix}$	M1	This mark is given for substituting $\lambda = \frac{3}{4}$
	$\begin{pmatrix}150\\325\\-75\end{pmatrix}$	A1	This mark is given for finding the correct coordinates of the point at which the tunnel meets the pipeline
(b)(ii)	$\sqrt{(150-100)^2 + (325-200)^2 + (-75-100)^2}$	M1	This mark is given for a method to find the length of the tunnel
	$= \sqrt{48750} \approx 221$ m	A1	This mark is given for correct length of the tunnel (including units)

Part	Working or answer an examiner might expect to see	Mark	Notes
(c)	$\left\lvert\overline{OP}\right\rvert = \sqrt{(-300)^2 + 400^2 + (-150)^2} \approx 522$ $\left\lvert\overline{OQ}\right\rvert = \sqrt{300^2 + 300^2 + 50^2} \approx 427$	M1	This mark is given for calculating the two lengths of the existing accessways OP and OQ
	The new accessway length is considerably shorter than 522 m and 427 m, so the company should build the new accessway	A1	This mark is given for an appropriate conclusion
(d)	For example: The mountainside in not likely to be flat, so a plane might not be a good model The tunnel and pipelines will not have negligible thickness so modelling as mines might not be appropriate The shortest length for a tunnel might not be practicable depending on the type of rock in the mountain	B1	This mark is given for a valid limitation of the model stated

4. **All units in this question are in metres.**

A lawn is modelled as a plane that contains the points $L(-2, -3, -1)$, $M(6, -2, 0)$ and $N(2, 0, 0)$, relative to a fixed origin O.

(a) Determine a vector equation of the plane that models the lawn, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

(3)

(b) (i) Show that, according to the model, the lawn is perpendicular to the vector $\begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix}$

(ii) Hence determine a Cartesian equation of the plane that models the lawn.

(4)

There are two posts set in the lawn.
There is a washing line between the two posts.
The washing line is modelled as a straight line through points at the top of each post with coordinates $P(-10, 8, 2)$ and $Q(6, 4, 3)$.

(c) Determine a vector equation of the line that models the washing line.

(2)

(d) State a limitation of one of the models.

(1)

The point $R(2, 5, 2.75)$ lies on the washing line.

(e) Determine, according to the model, the shortest distance from the point R to the lawn, giving your answer to the nearest cm.

(2)

Given that the shortest distance from the point R to the lawn is actually 1.5 m,

(f) use your answer to part (e) to evaluate the model, explaining your reasoning.

(1)

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Question	Scheme	Marks	AOs
4(a)	Finds any two vectors $\pm\overrightarrow{LM}, \pm\overrightarrow{LN}$ or $\pm\overrightarrow{MN}$ $\pm\begin{pmatrix}8\\1\\1\end{pmatrix}$ or $\pm\begin{pmatrix}4\\3\\1\end{pmatrix}$ or $\pm\begin{pmatrix}-4\\2\\0\end{pmatrix}$ two out of three values correct is sufficient to imply the correct method	M1	3.3
	Applies the vector equation of the plane formula $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$ Where **a** is any coordinate from L, M & N and vectors **b** and **c** come from an attempt at finding any two vectors that lie on the plane.	M1	1.1b
	A correct equation for the plane $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$ $\mathbf{a}=\begin{pmatrix}-2\\-3\\-1\end{pmatrix}$ or $\begin{pmatrix}6\\-2\\0\end{pmatrix}$ or $\begin{pmatrix}2\\0\\0\end{pmatrix}$ **b** and **c** are any two vectors from $\pm\begin{pmatrix}8\\1\\1\end{pmatrix}$ or $\pm\begin{pmatrix}4\\3\\1\end{pmatrix}$ or $\pm\begin{pmatrix}-4\\2\\0\end{pmatrix}$	A1	1.1b
		(3)	
(b)(i) (ii)	Applies 'their' $\mathbf{b}.\begin{pmatrix}1\\2\\-10\end{pmatrix}$ **AND** 'their' $\mathbf{c}.\begin{pmatrix}1\\2\\-10\end{pmatrix}$ **Alternative 1** Finds 'their **b**' – 'their **c**' or vice versa and applies the dot product with $\begin{pmatrix}1\\2\\-10\end{pmatrix}$ **AND** one of their **b** or **c** **Alternative 2** Applies 'their' $\mathbf{b}.\begin{pmatrix}x\\y\\z\end{pmatrix}$ **AND** 'their' $\mathbf{c}.\begin{pmatrix}x\\y\\z\end{pmatrix}$ and solves to find values of x, y and z **Alternative 3** Applies the dot product between their answer to part (a) and the vector $\begin{pmatrix}1\\2\\-10\end{pmatrix}$	M1	1.1b
	Show that both dot product(s) = **0** therefore the lawn is **perpendicular** **Alternative 1** Shows results is **parallel** to $\begin{pmatrix}1\\2\\-10\end{pmatrix}$ therefore the lawn is **perpendicular** **Alternative 2** Achieves the value 2 and concludes as a **constant** therefore the lawn is **perpendicular**	A1	2.4

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(c)	Finds the vector $\overrightarrow{PQ}$ or $\overrightarrow{QP}$ and uses it as the direction vector in the formula $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ Two out three values correct is sufficient to imply the correct method	M1
	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where $\mathbf{a} = \begin{pmatrix} -10 \\ 8 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \pm\begin{pmatrix} 16 \\ -4 \\ 1 \end{pmatrix}$	A1
		(2)
(d)	For example: The lawn will not be flat The washing line will not be straight	B1
		(1)
(e)	Applies the distance formula $\dfrac{\lvert(2\times1)+5\times2+(2.75\times-10)-2\rvert}{\sqrt{1^2+2^2+(-10)^2}}$	M1
	= 1.71 m or 171 cm	A1
		(2)
(f)	Must have an answer to part (e). Compares their answer to part (e) with 1.5 m and makes an appropriate comment about the model that is consistent with their answer to part (e). If their answer to part (e) is close to 1.5 (e.g. 1.4 to 1.6) they must compare and conclude that the model therefore is good If their answer to part (e) is significantly different to 1.5 they must compare and conclude that the model therefore it is not a good model.	B1ft

6. A mining company has identified a mineral layer below ground.

The mining company wishes to drill down to reach the mineral layer and models the situation as follows.

With respect to a fixed origin O,

- the ground is modelled as a horizontal plane with equation $z = 0$
- the mineral layer is modelled as part of the plane containing the points $A(10, 5, -50)$, $B(15, 30, -45)$ and $C(-5, 20, -60)$, where the units are in metres

(a) Determine an equation for the plane containing A, B and C, giving your answer in the form $\mathbf{r}.\mathbf{n} = d$

(5)

(b) Determine, according to the model, the acute angle between the ground and the plane containing the mineral layer. Give your answer to the nearest degree.

(3)

The mining company plans to drill vertically downwards from the point $(5, 12, 0)$ on the ground to reach the mineral layer.

(c) Using the model, determine, in metres to 1 decimal place, the distance the mining company will need to drill in order to reach the mineral layer.

(2)

(d) State a limitation of the assumption that the mineral layer can be modelled as a plane.

(1)

Question	Scheme	Marks	AOs
6(a)	Any two of: $\begin{cases} \pm k\overrightarrow{AB} = \pm k(5\mathbf{i}+25\mathbf{j}+5\mathbf{k}), \\ \pm k\overrightarrow{AC} = \pm k(-15\mathbf{i}+15\mathbf{j}-10\mathbf{k}), \\ \pm k\overrightarrow{BC} = \pm k(-20\mathbf{i}-10\mathbf{j}-15\mathbf{k}) \end{cases}$	M1	3.3
	Let normal vector be $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ $(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\bullet(\mathbf{i}+5\mathbf{j}+\mathbf{k})=0,\ (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\bullet(-3\mathbf{i}+3\mathbf{j}-2\mathbf{k})=0$ $\Rightarrow a+5b+c=0,\ -3a+3b-2c=0 \Rightarrow a=\ldots,\ b=\ldots,\ c=\ldots$ Alternative: cross product $\begin{vmatrix} 1 & 5 & 1 \\ -3 & 3 & -2 \end{vmatrix} = (-10-3)\mathbf{i}-(-2+3)\mathbf{j}+(3+15)\mathbf{k}$	M1	1.1b
	$\mathbf{n}=k(-13\mathbf{i}-\mathbf{j}+18\mathbf{k})$	A1	1.1b
	$(-13\mathbf{i}-\mathbf{j}+18\mathbf{k})\bullet(10\mathbf{i}+5\mathbf{j}-50\mathbf{k})=\ldots$	M1	1.1b
	$\mathbf{r}\bullet(13\mathbf{i}+\mathbf{j}-18\mathbf{k})=1035$ o.e. $\mathbf{r}\bullet(-13\mathbf{i}-\mathbf{j}+18\mathbf{k})=-1035$ $\mathbf{r}\bullet(325\mathbf{i}+25\mathbf{j}-450\mathbf{k})=25875$	A1	2.5
		(5)	
(b)	Attempts the scalar product between their normal vector and the vector **k** and uses trigonometry to find an angle	M1	3.1b
	$(-13\mathbf{i}-\mathbf{j}+18\mathbf{k})\bullet\mathbf{k}=-18=\sqrt{13^2+1^2+18^2}\cos\alpha$	M1	1.1b
	$\cos\alpha=\dfrac{-18}{\sqrt{494}} \Rightarrow \alpha=144.08\ldots \Rightarrow \theta=36°$	A1	3.2a
		(3)	
(c)	Distance required is $\lvert\lambda\rvert$ where $\begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix}\bullet\begin{pmatrix} 5 \\ 12 \\ \lambda \end{pmatrix}=1035$	M1	3.4
	$\lvert\lambda\rvert = 53.2\text{m}$	A1	1.1b
		(2)	
(d)	E.g. • The mineral layer will **not** be perfectly **flat/smooth** and will not form a plane • The mineral layer will have a **depth** and this should be taken into account	B1	3.5b

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3. *With respect to the **right-hand rule**, a rotation through $\theta°$ anticlockwise about the y-axis is represented by the matrix*

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

The point P has coordinates $(8, 3, 2)$

The point Q is the image of P under the transformation reflection in the plane $y = 0$

(a) Write down the coordinates of Q

(1)

The point R is the image of P under the transformation rotation through 120° anticlockwise about the y-axis, with respect to the **right-hand rule**.

(b) Determine the exact coordinates of R

(2)

(c) Hence find $|\overrightarrow{PR}|$ giving your answer as a simplified surd.

(2)

(d) Show that $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ are perpendicular.

(1)

(e) Hence determine the exact area of triangle PQR, giving your answer as a surd in simplest form.

(2)

3(a)	Coordinates of Q are $(8,-3,2)$	B1
		(1)
(b)	Coordinates of R are $\begin{pmatrix} \cos 120° & 0 & \sin 120° \\ 0 & 1 & 0 \\ -\sin 120° & 0 & \cos 120° \end{pmatrix}\begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \ldots$ or $\begin{pmatrix} -0.5 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}\begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \ldots$	M1
	So R is $\left(-4+\sqrt{3},\ 3, -4\sqrt{3}-1\right)$	A1
		(2)
(c)	Finds the distance $PR = \sqrt{\left(8-'\left(-4+\sqrt{3}\right)'\right)^2 + (3-'3')^2 + \left(2-'\left(-4\sqrt{3}-1\right)'\right)^2}$ Alternatively finds their $\overrightarrow{PR}$ or their $\overrightarrow{RP}$ then applies length of a vector formula. $\sqrt{\left(12-\sqrt{3}\right)^2 + \left(3+4\sqrt{3}\right)^2}$ or $\sqrt{\left(-12+\sqrt{3}\right)^2 + \left(-3-4\sqrt{3}\right)^2}$	M1
	$= \sqrt{204} \quad \left(= 2\sqrt{51}\right)$ cso	A1
		(2)
(d)	$\overrightarrow{PR}.\overrightarrow{PQ} = \left(-12+\sqrt{3}, 0, -3-4\sqrt{3}\right).(0,-6,0) = 0$ hence perpendicular	B1ft
		(1)
(e)	PQ is perpendicular to PR so Area $= \frac{1}{2} \times PQ \times PR$	M1
	$= \frac{1}{2} \times 6 \times \sqrt{204} = 6\sqrt{51}$ cso	A1
		(2)

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6. The surface of a horizontal tennis court is modelled as part of a horizontal plane, with the origin on the ground at the centre of the court, and

- **i** and **j** are unit vectors directed across the width and length of the court respectively
- **k** is a unit vector directed vertically upwards
- units are metres

After being hit, a tennis ball, modelled as a particle, moves along the path with equation

$$\mathbf{r} = (-4.1 + 9\lambda - 2.3\lambda^2)\mathbf{i} + (-10.25 + 15\lambda)\mathbf{j} + (0.84 + 0.8\lambda - \lambda^2)\mathbf{k}$$

where λ is a scalar parameter with $\lambda \geqslant 0$

Assuming that the tennis ball continues on this path until it hits the ground,

(a) find the value of λ at the point where the ball hits the ground.

(2)

The direction in which the tennis ball is moving at a general point on its path is given by

$$(9 - 4.6\lambda)\mathbf{i} + 15\mathbf{j} + (0.8 - 2\lambda)\mathbf{k}$$

(b) Write down the direction in which the tennis ball is moving as it hits the ground.

(1)

(c) Hence find the acute angle at which the tennis ball hits the ground, giving your answer in degrees to one decimal place.

(4)

The net of the tennis court lies in the plane $\mathbf{r}.\mathbf{j} = 0$

(d) Find the position of the tennis ball at the point where it is in the same plane as the net.

(3)

The maximum height above the court of the top of the net is 0.9 m.

Modelling the top of the net as a horizontal straight line,

(e) state whether the tennis ball will pass over the net according to the model, giving a reason for your answer.

(1)

With reference to the model,

(f) decide whether the tennis ball will actually pass over the net, giving a reason for your answer.

(2)

6(a)	Need **k** component to be zero at ground, so $0.84+0.8\lambda-\lambda^2=0 \Rightarrow \lambda=...$	M1
	$\lambda=-\frac{3}{5},\frac{7}{5}$, but $\lambda\geqslant 0$ so $\lambda=\frac{7}{5}$	A1
		(2)
(b)	Direction is $(9-4.6\times1.4)\mathbf{i}+15\mathbf{j}+(0.8-2\times1.4)$ $=2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k}$ or $\frac{64}{25}\mathbf{i}+15\mathbf{j}-2\mathbf{k}$	B1ft
		(1)
(c)	Direction perpendicular to ground is $a\mathbf{k}$, so angle to perpendicular is given by $(\cos\theta)=\frac{a\mathbf{k}.(2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k})}{a\times\|2.56\mathbf{i}+15\mathbf{j}-2\mathbf{k}\|}$ or $\frac{\begin{pmatrix}2.56\\15\\-2\end{pmatrix}\bullet\begin{pmatrix}0\\0\\a\end{pmatrix}}{\left\|\begin{matrix}2.56\\15\\-2\end{matrix}\right\|\left\|\begin{matrix}0\\0\\a\end{matrix}\right\|}$ or angle between $\begin{pmatrix}2.56\\15\\-2\end{pmatrix}$ and $\begin{pmatrix}2.56\\15\\0\end{pmatrix}$ is given by $(\cos\theta)=\frac{\begin{pmatrix}2.56\\15\\-2\end{pmatrix}\bullet\begin{pmatrix}2.56\\15\\0\end{pmatrix}}{\left\|\begin{matrix}2.56\\15\\-2\end{matrix}\right\|\left\|\begin{matrix}2.56\\15\\0\end{matrix}\right\|}$	M1
	$=\frac{-2}{\sqrt{2.56^2+15^2+(-2)^2}}(=-0.130\ldots)$ Or $=\frac{231.5536}{\sqrt{2.56^2+15^2+(-2)^2}\sqrt{2.56^2+15^2+(0)^2}}=0.991\ldots$	M1
	$90°-\arccos('-0.130...')=-7.48...$ or $\arccos(0.991...)$	ddM1
	So the tennis ball hits ground at angle of $7.5°$ (1d.p.) cao	A1
	Alternative Finds the length of the vector in the **ij** plane $=\sqrt{2.56^2+15^2}$	M1
	$\tan\theta=\frac{2}{\sqrt{2.56^2+15^2}}$	M1
	$\theta=\arctan\left(\frac{2}{\sqrt{2.56^2+15^2}}\right)$ or $\theta=90-\arctan\left(\frac{\sqrt{2.56^2+15^2}}{2}\right)$	ddM1
	So the tennis ball hits ground at angle of $7.5°$ (1d.p.)	A1
		(4)
(d)	In same plane as net when $\mathbf{r}.\mathbf{j}=0$, $\begin{pmatrix}-4.1+9\lambda-2.3\lambda^2\\-10.25+15\lambda\\0.84+0.8\lambda-\lambda^2\end{pmatrix}\bullet\begin{pmatrix}0\\1\\0\end{pmatrix}$ leading to $-10.25+15\lambda=0\Rightarrow\lambda=...$ $\left(=\frac{41}{60}=0.683333...\right)$	M1
	So is at position $\left(-4.1+9\times\frac{41}{60}-2.3\left(\frac{41}{60}\right)^2\right)\mathbf{i}+0\mathbf{j}+\left(0.84+0.8\times\frac{41}{60}-\left(\frac{41}{60}\right)^2\right)\mathbf{k}$	M1
	$=$ awrt $0.976\mathbf{i}+$ awrt $0.920\mathbf{k}$ or $=$ awrt $0.976\mathbf{i}+0.92\mathbf{k}$ (to 3 s.f.) or $=$ awrt $0.976\mathbf{i}+\frac{3311}{3600}\mathbf{k}$	A1
		(3)
(e)	Modelling as a line, height of net is 0.9m along its length so as **0.92 > 0.9** the ball **will** pass over the net according to the model.	B1ft
		(1)
(f)	Identifies a suitable feature of the model that affects the outcome And uses it to draw a compatible conclusion. For example • The ball is **not a particle** and will have **diameter/radius**, therefore it will hit the net and not pass over. • As above, but so the **ball will clip the net** but it's momentum will take it over as it is mostly above the net. • The **model** says that the ball will clear the net by 2cm which may be smaller than the balls **diameter** • The **net will not be a straight line/taut** so will not be 0.9m high, so the ball **will** have enough clearance to **pass** over the net.	M1 A1
		(2)

8. The line l_1 has equation $\dfrac{x-2}{4} = \dfrac{y-4}{-2} = \dfrac{z+6}{1}$

The plane Π has equation $x - 2y + z = 6$

The line l_2 is the reflection of the line l_1 in the plane Π.

Find a vector equation of the line l_2

(7)

Question	Scheme	Marks	AOs
8	$2+4\lambda-2(4-2\lambda)-6+\lambda=6\Rightarrow\lambda=\ldots$	M1	1.1b
	$\lambda=2\Rightarrow$ Required point is $(2+2(4),\ 4+2(-2),\ -6+2(1))$ $(10,\ 0,\ -4)$	A1	1.1b
	$2+t-2(4-2t)-6+t=6\Rightarrow t=\ldots$	M1	3.1a
	$t=3$ so reflection of $(2,4,-6)$ is $(2+6(1),4+6(-2),-6+6(1))$	M1	3.1a
	$(8,\ -8,\ 0)$	A1	1.1b
	$\begin{pmatrix}10\\0\\-4\end{pmatrix}-\begin{pmatrix}8\\-8\\0\end{pmatrix}=\begin{pmatrix}2\\8\\-4\end{pmatrix}$	M1	3.1a
	$\mathbf{r}=\begin{pmatrix}10\\0\\-4\end{pmatrix}+k\begin{pmatrix}1\\4\\-2\end{pmatrix}$ or equivalent e.g. $\left(\mathbf{r}-\begin{pmatrix}10\\0\\-4\end{pmatrix}\right)\times\begin{pmatrix}1\\4\\-2\end{pmatrix}=\mathbf{0}$	A1	2.5
		(7)	
			(7 marks)

2. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1

(3)

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to Π_2

(2)

(c) Show that the acute angle between Π_1 and Π_2 is 52° to the nearest degree.

(3)

2(a)	$\begin{pmatrix}3\\-4\\2\end{pmatrix}\cdot\begin{pmatrix}6\\2\\12\end{pmatrix}=18-8+24$	M1	3.1a
	$d=\dfrac{18-8+24-5}{\sqrt{3^2+4^2+2^2}}$	M1	1.1b
	$=\sqrt{29}$	A1	1.1b
		(3)	
(b)	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\5\end{pmatrix}=\ldots$ and $\begin{pmatrix}-1\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\ldots$	M1	2.1
	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\5\end{pmatrix}=0$ and $\begin{pmatrix}-1\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=0$ $\therefore\ -\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is perpendicular to Π_2	A1	2.2a
		(2)	
(c)	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\2\end{pmatrix}=-3+12+2$	M1	1.1b
	$\sqrt{(-1)^2+(-3)^2+1^2}\sqrt{(3)^2+(-4)^2+2^2}\cos\theta=11$ $\Rightarrow\cos\theta=\dfrac{11}{\sqrt{(-1)^2+(-3)^2+1^2}\sqrt{(3)^2+(-4)^2+2^2}}$	M1	2.1
	So angle between planes $\theta=52°$ *	A1*	2.4
		(3)	
			(8 marks)

7. The line l_1 has equation

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$$

The line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

where t is a scalar parameter.

(a) Show that l_1 and l_2 lie in the same plane.

(3)

(b) Write down a vector equation for the plane containing l_1 and l_2

(1)

(c) Find, to the nearest degree, the acute angle between l_1 and l_2

(3)

A2 2019 Paper 1

Vectors

7(a) **Way 1**	$1+2\lambda=1+t$ $-1-\lambda=-t$ $4+3\lambda=3+2t$ $\Rightarrow t=\ldots \text{or } \lambda=\ldots$	M1	3.1a
	Checks the third equation with $t=2$ and $\lambda=1$ Or shows that the coordinate (3, -2, 7) lies on both lines	A1	1.1b
	As the lines intersect at a point the lines lie in the same plane.	A1	2.4
		(3)	
(b)	e.g. $\mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+p\begin{pmatrix}2\\-1\\3\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $\mathbf{r}=\begin{pmatrix}1\\-1\\4\end{pmatrix}+p\begin{pmatrix}2\\-1\\3\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $\mathbf{r}=\begin{pmatrix}3\\-2\\7\end{pmatrix}+p\begin{pmatrix}2\\-1\\3\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $\mathbf{r}=\begin{pmatrix}3\\-2\\7\end{pmatrix}+p\begin{pmatrix}0\\-1\\1\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $\mathbf{r}.k\begin{pmatrix}1\\-1\\-1\end{pmatrix}=-2k$	B1	2.5
		(1)	
(c) **Way 1**	$\begin{pmatrix}2\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\2\end{pmatrix}=2+1+6$	M1	1.1b
	$\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+(-1)^2+2^2}\cos\theta=9$ $\Rightarrow \cos\theta=\dfrac{9}{\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+(-1)^2+2^2}}$	dM1	2.1
	$\theta=11$ cao	A1	1.1b
		(3)	

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4. The plane Π_1 has equation

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(a) Find a Cartesian equation for Π_1

(4)

The line l has equation

$$\frac{x-1}{5} = \frac{y-3}{-3} = \frac{z+2}{4}$$

(b) Find the coordinates of the point of intersection of l with Π_1

(3)

The plane Π_2 has equation

$$\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 5$$

(c) Find, to the nearest degree, the acute angle between Π_1 and Π_2

(2)

A2 2020 Paper 1

4(a)	Attempts normal vector: E.g. let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + \mathbf{k}$ then $a + 2b - 3 = 0,\ -a + 2b + 1 = 0$ $\Rightarrow a = \ldots, b = \ldots$ or $\mathbf{n} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \times (-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$	M1
	$\mathbf{n} = k(4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$	A1
	$(4\mathbf{i} + \mathbf{j} + 2\mathbf{k}).(2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = \ldots$	M1
	$4x + y + 2z = 10$	A1
		(4)
(b)	$\dfrac{x-1}{5} = \dfrac{y-3}{-3} = \dfrac{z+2}{4} \Rightarrow \mathbf{r} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ $4(1+5\lambda) + 3 - 3\lambda + 2(4\lambda - 2) = 10 \Rightarrow \lambda = \ldots$	M1
	$\lambda = \dfrac{7}{25} \Rightarrow \mathbf{r} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \dfrac{7}{25}(5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$	dM1
	$\left(\dfrac{12}{5}, \dfrac{54}{25}, -\dfrac{22}{25}\right)$	A1
		(3)
	Alternative: $4x + \left(-\dfrac{3}{5}(x-1) + 3\right) + 2\left(\dfrac{4}{5}(x-1) - 2\right) = 10 \Rightarrow x = \ldots$	M1
	$\Rightarrow y = \ldots,\ z = \ldots$	M1
	$\left(\dfrac{12}{5}, \dfrac{54}{25}, -\dfrac{22}{25}\right)$	A1
		(3)
(c)	$(4\mathbf{i} + \mathbf{j} + 2\mathbf{k}).(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 8 - 1 + 6 = 13$ $13 = \sqrt{14}\sqrt{21}\cos\theta \Rightarrow \theta = \ldots$	M1
	$\theta = 41°$	A1
		(2)

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7. The plane Π has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Show that vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ is perpendicular to Π.

(2)

(b) Hence find a Cartesian equation of Π.

(2)

The line l has equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

where t is a scalar parameter.

The point A lies on l.

Given that the shortest distance between A and Π is $2\sqrt{29}$

(c) determine the possible coordinates of A.

(4)

A2 2021 Paper 1

Vectors

Question	Scheme	Marks	AOs
7(a)	$\begin{pmatrix}-1\\2\\1\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=-2+6-4=0$ and $\begin{pmatrix}2\\0\\1\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=4+0-4=0$ Alt: $\begin{pmatrix}-1\\2\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}2\times1-1\times0\\-(-1\times1-1\times2)\\-1\times0-2\times2\end{pmatrix}=\ldots$	M1	1.1b
	As $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ is perpendicular to both direction vectors (two non-parallel vectors) of Π then it must be perpendicular to Π	A1	2.2a
		(2)	
(b)	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}3\\3\\2\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}\Rightarrow\ldots$	M1	1.1a
	$2x+3y-4z=7$	A1	2.2a
		(2)	
(c)	$\dfrac{\lvert 2(4+t)+3(-5+6t)-4(2-3t)-7\rvert}{\sqrt{2^2+3^2+(-4)^2}}=2\sqrt{29}\Rightarrow t=\ldots$	M1	3.1a
	$t=-\dfrac{9}{8}$ and $t=\dfrac{5}{2}$	A1	1.1b
	$\boldsymbol{r}=\begin{pmatrix}4\\-5\\2\end{pmatrix}-\dfrac{9}{8}\begin{pmatrix}1\\6\\-3\end{pmatrix}=\ldots$ or $\boldsymbol{r}=\begin{pmatrix}4\\-5\\2\end{pmatrix}+\dfrac{5}{2}\begin{pmatrix}1\\6\\-3\end{pmatrix}=\ldots$	M1	1.1b
	$\left(\dfrac{23}{8},-\dfrac{47}{4},\dfrac{43}{8}\right)$ and $\left(\dfrac{13}{2},10,-\dfrac{11}{2}\right)$	A1	2.2a
		(4)	
		(8 marks)	

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8. Two birds are flying towards their nest, which is in a tree.

Relative to a fixed origin, the flight path of each bird is modelled by a straight line.

In the model, the equation for the flight path of the first bird is

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$$

and the equation for the flight path of the second bird is

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

where λ and μ are scalar parameters and a is a constant.

In the model, the angle between the birds' flight paths is 120°

(a) Determine the value of a.

(4)

(b) Verify that, according to the model, there is a common point on the flight paths of the two birds and find the coordinates of this common point.

(5)

The position of the nest is modelled as being at this common point.

The tree containing the nest is in a park.

The ground level of the park is modelled by the plane with equation

$$2x - 3y + z = 2$$

(c) Hence determine the shortest distance from the nest to the ground level of the park.

(3)

(d) By considering the model, comment on whether your answer to part (c) is reliable, giving a reason for your answer.

(1)

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A2 2022 Paper 2

Vectors

8(a)	A complete method to use the scalar product of the direction vectors and the angle 120° to form an equation in a $$\frac{\begin{pmatrix}2\\a\\0\end{pmatrix}\bullet\begin{pmatrix}0\\1\\-1\end{pmatrix}}{\sqrt{2^2+a^2}\sqrt{1^2+(-1)^2}}=cos\,1\,20$$	M1
	$$\frac{a}{\sqrt{4+a^2}\sqrt{2}}=-\frac{1}{2}$$	A1
	$2a=-\sqrt{4+a^2}\sqrt{2}\Rightarrow 4a^2=8+2a^2\Rightarrow a^2=4\Rightarrow a=\ldots$	M1
	$a=-2$	A1
		(4)
(b)	Any two of **i**: $-1+2\lambda=4$ (1) **j**: $5+$'their -2'$\lambda=-1+\mu$ (2) **k**: $2=3-\mu$ (3)	M1
	Solves the equations to find a value of $\lambda\left\{=\frac{5}{2}\right\}$ **and** $\mu\{=1\}$	M1
	$r_1=\begin{pmatrix}-1\\5\\2\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\\text{'their}-2\text{'}\\0\end{pmatrix}$ or $r_2=\begin{pmatrix}4\\-1\\3\end{pmatrix}+1\begin{pmatrix}0\\1\\-1\end{pmatrix}$	dM1
	$(4,0,2)$ or $\begin{pmatrix}4\\0\\2\end{pmatrix}$	A1
	Checks the third equation e.g. $\lambda=\frac{5}{2}$: **LHS** $=5-2\lambda=5-5=0$ $\mu=1$: **RHS** $=-1+\mu=-1+1=0$ therefore **common point/intersect/consistent/tick** or substitutes the values of λ and μ into the relevant lines and achieves the same coordinate	B1
		(5)
(c)	Full attempt to find the minimum distance from the point of intersection (nest) to the plane (ground) E.g. Minimum distance $=\frac{\lvert 2\times\text{'4'}+(-3)\times\text{'0'}+1\times\text{'2'}-2\rvert}{\sqrt{2^2+(-3)^2+1)^2}}=\ldots$ **Alternatively** $\mathbf{r}=\begin{pmatrix}\text{'4'}\\\text{'0'}\\\text{'2'}\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}$ $2(\text{'4'}+2\lambda)-3(\text{'0'}-3\lambda)+(\text{'2'}+\lambda)=2\Rightarrow$ $\lambda=\ldots\left\{-\frac{4}{7}\right\}$	M1 A1ft
	$\mathbf{r}=\begin{pmatrix}\text{'4'}\\\text{'0'}\\\text{'2'}\end{pmatrix}+\text{'}-\frac{4}{7}\text{'}\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}\frac{20}{7}\\\frac{12}{7}\\\frac{10}{7}\end{pmatrix}$ Minimum distance $=\sqrt{\left(2\times-\frac{4}{7}\text{'}\right)^2+\left(-3\times-\frac{4}{7}\text{'}\right)^2+\left(1\times-\frac{4}{7}\right)^2}=\ldots$ $=\sqrt{\left(\text{'4'}-\text{'}\frac{20}{7}\text{'}\right)^2+\left(\text{'0'}-\text{'}\frac{12}{7}\text{'}\right)^2+\left(\text{'2'}-\text{'}\frac{10}{7}\text{'}\right)^2}=\ldots$	
	$\frac{8}{\sqrt{14}}$ or $\frac{4\sqrt{14}}{7}$ or awrt 2.1	A1
		(3)
	Alternative Find perpendicular distance from plane to the origin $2x-3y+z=2$ $\lvert n\rvert=\sqrt{2^2+(-3)^2+1^2}=\sqrt{14}$ shortest distance $=\frac{2}{\sqrt{14}}$ Find perpendicular distance from the plane containing the point of intersection to the origin $2x-3y+z=\begin{pmatrix}4\\0\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix}=10$ shortest distance $=\frac{10}{\sqrt{14}}$ Minimum distance $=\frac{10}{\sqrt{14}}-\frac{2}{\sqrt{14}}$	M1 A1ft
	$\frac{8}{\sqrt{14}}$ or $\frac{4\sqrt{14}}{7}$ or awrt 2.1	A1
		(3)
(d)	For example Not reliable as the birds will not fly in a straight line Not reliable as angle between flights paths will not always be 120° Not reliable/reliable as the ground will not be flat/smooth Not reliable as bird's nest is not a point	B1
		(1)

Complex Numbers pt. 2

4. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

A2 SAMs Paper 2

Complex Numbers pt. 2

Question	Scheme	Marks	AOs
4(a)	$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$	M1	2.1
	$= 2\cos n\theta$ *	A1*	1.1b
		(2)	
(b)	$\left(z + z^{-1}\right)^4 = 16\cos^4\theta$	B1	2.1
	$\left(z + z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$	M1	2.1
	$= z^4 + z^{-4} + 4\left(z^2 + z^{-2}\right) + 6$	A1	1.1b
	$= 2\cos 4\theta + 4\left(2\cos 2\theta\right) + 6$	M1	2.1
	$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$*	A1*	1.1b
		(5)	
			(7 marks)

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A2 Mock Paper (Set 1) Paper 1

Complex Numbers pt. 2

2. (a) Use de Moivre's theorem to show that

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(5)

(b) Hence solve, for $\pi < \theta < 2\pi$, the equation

$$\cos\theta - \cos 5\theta = 5\cos 3\theta$$

(5)

HOME

A2 Mock Paper (Set 1) Paper 1

Complex Numbers pt. 2

Question	Scheme	Marks	AOs
2(a)	$z=\cos\theta+\mathrm{i}\sin\theta \Rightarrow \frac{1}{z}=\cos\theta-\mathrm{i}\sin\theta$ $\Rightarrow \left(z+\frac{1}{z}\right)^5=(2\cos\theta)^5=32\cos^5\theta$	M1	2.1
	$\left(z+\frac{1}{z}\right)^5=z^5+\frac{1}{z^5}+5\left(z^3+\frac{1}{z^3}\right)+10\left(z+\frac{1}{z}\right)$	M1 A1	2.1 1.1b
	$=2\cos5\theta+10\cos3\theta+20\cos\theta$	M1	2.1
	$\cos^5\theta=\frac{1}{16}(\cos5\theta+5\cos3\theta+10\cos\theta)$*	A1*	1.1b
		(5)	
(b)	$\cos\theta-\cos5\theta=5\cos3\theta\Rightarrow\cos\theta=5\cos3\theta+\cos5\theta=16\cos^5\theta-10\cos\theta$	M1	3.1a
	$16\cos^5\theta-11\cos\theta=0$	A1	1.1b
	$\cos\theta(16\cos^4\theta-11)=0\Rightarrow\cos\theta=0,\ \pm\sqrt[4]{\frac{11}{16}}$	M1	1.1b
	$\theta=3.57,\frac{3\pi}{2}(\text{or } 4.71), 5.86$	A1 A1	1.1b 1.1b
		(5)	
		(10 marks)	

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4. The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \dots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \dots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

A2 2019 Paper 2

Complex Numbers pt. 2

Question	Scheme	Marks	AOs
4(a) **Way 1**	$C+iS=\cos\theta+i\sin\theta+\frac{1}{2}(\cos 5\theta+i\sin 5\theta)\left(+\frac{1}{4}(\cos 9\theta+i\sin 9\theta)+\ldots\right)$	M1	1.1b
	$=e^{i\theta}+\frac{1}{2}e^{5i\theta}\left(+\frac{1}{4}e^{9i\theta}+\ldots\right)$	A1	2.1
	$C+iS=\dfrac{e^{i\theta}}{1-\frac{1}{2}e^{4i\theta}}$	M1	3.1a
	$=\dfrac{2e^{i\theta}}{2-e^{4i\theta}}$ *	A1*	1.1b
		(4)	
(b) **Way 1**	$\dfrac{2e^{i\theta}}{2-e^{4i\theta}}\times\dfrac{2-e^{-4i\theta}}{2-e^{-4i\theta}}$	M1	3.1a
	$\dfrac{4e^{i\theta}-2e^{-3i\theta}}{4-2e^{-4i\theta}-2e^{4i\theta}+1}$	A1	1.1b
	$\dfrac{4\cos\theta+4i\sin\theta-2\cos 3\theta+2i\sin 3\theta}{5-2\cos 4\theta+2i\sin 4\theta-2\cos 4\theta-2i\sin 4\theta}$ **Dependent on the first M**	dM1	2.1
	$S=\dfrac{4\sin\theta+2\sin 3\theta}{5-4\cos 4\theta}$ *	A1*	1.1b
		(4)	

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6. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\text{i}$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \text{i}y$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)

A2 2019 Paper 2

Complex Numbers pt. 2

6(a)	Examples: $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix} = \ldots$ **or** $(6+2\text{i})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\text{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{2\pi}{3}\right)+\text{i}\sin\left(\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)+\text{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\text{e}^{\text{i}\arctan\left(\frac{2}{6}\right)}\text{e}^{\text{i}\left(\frac{2\pi}{3}\right)}$	M1	3.1a
	$\left(-3-\sqrt{3}\right)$ **or** $\left(3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	$\left(-3-\sqrt{3}\right)+\left(3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	Examples: $\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix} = \ldots$ **or** $(6+2\text{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\text{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{4\pi}{3}\right)+\text{i}\sin\left(\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)+\text{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\text{e}^{\text{i}\arctan\left(\frac{2}{6}\right)}\text{e}^{\text{i}\left(\frac{4\pi}{3}\right)}$	M1	3.1a
	$\left(-3+\sqrt{3}\right)$ **or** $\left(-3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	$\left(-3+\sqrt{3}\right)+\left(-3\sqrt{3}-1\right)\text{i}$	A1	1.1b
		(6)	
(b) Way 1	Area $ABC = 3\times\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$ or Area $AOB = \frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$	M1	2.1
	Area $DEF = \frac{1}{4}ABC$ or $\frac{3}{4}AOB$	dM1	3.1a
	$= \frac{3}{8}\times 40\times\frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$	A1	1.1b
		(3)	

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4. (a) Use de Moivre's theorem to prove that

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

(5)

(b) Hence find the distinct roots of the equation

$$1 + 7x - 56x^3 + 112x^5 - 64x^7 = 0$$

(5)

giving your answer to 3 decimal places where appropriate.

A2 2020 Paper 2

Complex Numbers pt. 2

4(a)	$(\cos\theta + i\sin\theta)^7 = \cos^7\theta + \binom{7}{1}\cos^6\theta(i\sin\theta) + \binom{7}{2}\cos^5\theta(i\sin\theta)^2 + \dots$ Some simplification may be done at this stage e.g. $c^7 + 7c^6 is - 21c^5 s^2 - 35c^4 is^3 + 35c^3 s^4 + 21c^2 is^5 - 7cs^6 - is^7$	M1	1.1b
	$i\sin 7\theta = {}^7C_1 c^6 is + {}^7C_3 c^4 i^3 s^3 + {}^7C_5 c^2 i^5 s^5 + i^7 s^7$ or $= 7c^6 is + 35c^4 i^3 s^3 + 21c^2 i^5 s^5 + i^7 s^7$	M1	2.1
	$\sin 7\theta = 7c^6 s - 35c^4 s^3 + 21c^2 s^5 - s^7$	A1	1.1b
	$= 7(1-s^2)^3 s - 35(1-s^2)^2 s^3 + 21(1-s^2)s^5 - s^7$ $= 7(1-3s^2+3s^4-s^6)s - 35(1-2s^2+s^4)s^3 + 21(1-s^2)s^5 - s^7$	M1	2.1
	$\{7s - 21s^3 + 21s^5 - 7s^7 - 35s^3 + 70s^5 - 35s^7 + 21s^5 - 21s^7 - s^7\}$ leading to $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$ *	A1*	1.1b
		(5)	
(b)	$1 + \sin 7\theta = 0 \Rightarrow \sin 7\theta = -1$	M1	3.1a
	$7\theta = -450, -90, 270, 630, \dots$ or $7\theta = -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \dots$	A1	1.1b
	$\theta = -\frac{450}{7}, -\frac{90}{7}, \frac{270}{7}, \frac{630}{7}, \dots \Rightarrow \sin\theta = \dots$ or $\theta = -\frac{5\pi}{14}, -\frac{\pi}{14}, \frac{3\pi}{14}, \frac{7\pi}{14}, \dots \Rightarrow \sin\theta = \dots$	M1	2.2a
	$x = \sin\theta = -0.901, -0.223, 0.623, 1$	A1 A1	1.1b 2.3
		(5)	
			(10 marks)

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8. (i) The point P is one vertex of a regular pentagon in an Argand diagram. The centre of the pentagon is at the origin.

Given that P represents the complex number $6 + 6\text{i}$, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $r\text{e}^{\text{i}\theta}$

(5)

(ii) (a) On a single Argand diagram, shade the region, R, that satisfies both

$$|z - 2\text{i}| \leqslant 2 \qquad \text{and} \qquad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$

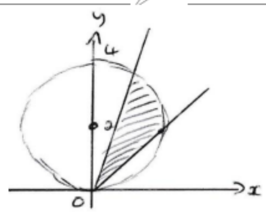
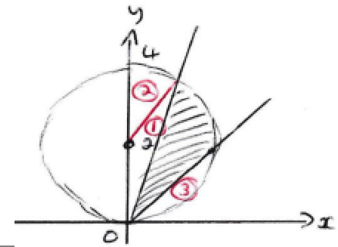
(2)

(b) Determine the exact area of R, giving your answer in simplest form.

(4)

A2 2021 Paper 2

Complex Numbers pt. 2

8(i)	$\lvert z\rvert=\sqrt{6^2+6^2}=\ldots\ 6\sqrt{2}$ or $\sqrt{72}$ and arg $z=\tan^{-1}\left(\frac{6}{6}\right)=\ldots\left\{\frac{\pi}{4}\right\}$ Can be implied by $r=6\sqrt{2}e^{\frac{\pi}{4}i}$	M1 A1	3.1a 1.1b
	Adding multiples of $\frac{2\pi}{5}$ to their argument $z=6\sqrt{2}e^{\frac{\pi}{4}i}\times e^{\frac{2\pi k}{5}i}$ or $z=6\sqrt{2}\left[\cos\left(\frac{\pi}{4}+\frac{2\pi k}{5}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\pi k}{5}\right)\right]$	M1	1.1b
	$z=re^{\left(\theta+\frac{2\pi}{5}\right)i},\ re^{\left(\theta+\frac{4\pi}{5}\right)i},\ re^{\left(\theta+\frac{6\pi}{5}\right)i},\ re^{\left(\theta+\frac{8\pi}{5}\right)i}$ o.e. or $z=re^{\left(\theta+\frac{2\pi}{5}\right)i},\ re^{\left(\theta-\frac{2\pi}{5}\right)i},\ re^{\left(\theta-\frac{6\pi}{5}\right)i},\ re^{\left(\theta-\frac{8\pi}{5}\right)i}$ o.e.	A1ft	1.1b
	$z=6\sqrt{2}e^{\frac{13\pi}{20}i},\ 6\sqrt{2}e^{\frac{21\pi}{20}i},\ 6\sqrt{2}e^{\frac{29\pi}{20}i},\ 6\sqrt{2}e^{\frac{37\pi}{20}i}$ o.e. or $z=6\sqrt{2}e^{\frac{13\pi}{20}i},\ 6\sqrt{2}e^{-\frac{19\pi}{20}i},\ 6\sqrt{2}e^{-\frac{11\pi}{20}i},\ 6\sqrt{2}e^{-\frac{3\pi}{20}i}$ o.e.	A1	1.1b
		(5)	
(ii)(a)	Circle centre (0, 2) and radius 2 or  with the point on the origin	B1	1.1b
	Fully correct  	B1	1.1b
		(2)	
(ii)(b)	area $=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(4\sin\theta\right)^2\,d\theta$ or area $=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\alpha\sin\theta\right)^2\,d\theta$	M1	3.1a
	Uses $\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$ and integrates to the form $A\theta+B\sin 2\theta$ area $=8\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin^2\theta\,d\theta=4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}1-\cos 2\theta\,d\theta=4\theta-2\sin 2\theta$	M1	3.1a
	Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around $\left[4\left(\frac{\pi}{3}\right)-2\sin\left(\frac{2\pi}{3}\right)\right]-\left[4\left(\frac{\pi}{4}\right)-2\sin\left(\frac{2\pi}{4}\right)\right]$	M1	1.1b
	Area $=\frac{\pi}{3}-\sqrt{3}+2$	A1	1.1b
		(4)	
	Alternative  		
	Finds either the areas 1 or 2 Area 1 $=\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\left\{=\sqrt{3}\right\}$ Area 2 $=\frac{1}{2}\times 2^2\times\frac{\pi}{3}\left\{=\frac{2\pi}{3}\right\}$	M1	1.1b
	A complete method to find area 3 Area 3 $=\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\left\{=\pi-2\right\}$	M1	3.1a
	A complete method to find the required area Shaded area $=$ Area of semi circle $-$ area 1 $-$ area 2 $-$ area 3 $=\left[\frac{1}{2}\pi\times 2^2\right]-\left[\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{2}\times 2^2\times\frac{\pi}{3}\right]-\left[\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\right]$ $=2\pi-\sqrt{3}-\frac{2\pi}{3}-(\pi-2)$ Or Shaded area $=$ Area of sector $-$ area 1 $-$ area 3 $=\left[\frac{1}{2}\times 4\times\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\right]$ $=\frac{4\pi}{3}-\sqrt{3}-(\pi-2)$	M1	3.1a
	Area $=\frac{\pi}{3}-\sqrt{3}+2$	A1	1.1b

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9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots \quad = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

(2)

A2 2021 Paper 2

Complex Numbers pt. 2

Question	Scheme	Marks	AOs
9(a)	$\frac{1}{1-z}$	B1	2.2a
		(1)	
(b)(i)	$1+z+z^2+z^3+\dots$ $=1+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^3+\dots$ $=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos2\theta+i\sin2\theta)+\frac{1}{8}(\cos3\theta+i\sin3\theta)+\dots$	M1	3.1a
	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}$ or $\frac{1}{1-z}=\frac{2}{2-(\cos\theta+i\sin\theta)}\times\frac{2-(\cos\theta-i\sin\theta)}{2-(\cos\theta-i\sin\theta)}$	M1	3.1a
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin2\theta)+\frac{1}{8}(\sin3\theta)+\dots\right\}=\frac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin2\theta)+\frac{1}{8}(\sin3\theta)+\dots\right\}=\frac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2=1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2=4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin2\theta+\frac{1}{8}\sin3\theta+\dots=\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
	Alternative $1+z+z^2+z^3+\dots$ $=1+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^3+\dots$ $=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos2\theta+i\sin2\theta)+\frac{1}{8}(\cos3\theta+i\sin3\theta)+\dots$	M1	3.1a
	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}e^{i\theta}}\times\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}}=\frac{4-2e^{-i\theta}}{5-2\left(e^{i\theta}+e^{-i\theta}\right)}=\frac{4-2(\cos\theta-i\sin\theta)}{5-2(2\cos\theta)}$	M1	2.1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin2\theta+\frac{1}{8}\sin3\theta+\dots=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
		(5)	
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}=0\Rightarrow\cos\theta=2$	M1	3.1a
	As $(-1\le)\cos\theta\le1$ therefore there is no solution to $\cos\theta=2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1	2.4
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta}=\frac{1}{2}+\frac{3}{2(5-4\cos\theta)}$	M1	3.1a
	$-1\le\cos\theta\le1$ therefore $\frac{1}{6}<\frac{3}{2(5-4\cos\theta)}<\frac{3}{2}$ so sum must contain real part	A1	2.4
	Alternative 2 $\frac{1}{1-z}=ki\Rightarrow z=1+\frac{i}{k}$	M1	3.1a
	mod z >1 contradiction hence cannot be purely imaginary	A1	2.4
		(2)	
		(8 marks)	

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4. (i) Given that

$$z_1 = 6e^{\frac{\pi}{3}i} \quad \text{and} \quad z_2 = 6\sqrt{3}e^{\frac{5\pi}{6}i}$$

show that

$$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$$

(3)

(ii) Given that

$$\arg(z - 5) = \frac{2\pi}{3}$$

determine the least value of $|z|$ as z varies.

(3)

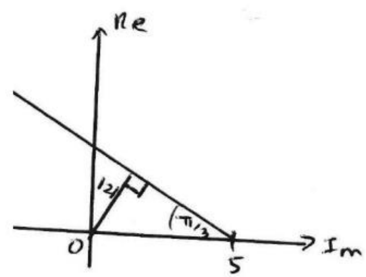
BICEN MATHS

HOME

A2 2022 Paper 2

Complex Numbers pt. 2

Question	Scheme		Marks
4(i)	$z_1 = 6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = \ldots\{3 + 3\sqrt{3}i\}$ $z_2 = 6\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = \ldots\{-9 + 3\sqrt{3}i\}$ $\{z_1 + z_2 =\}(3 + 3\sqrt{3}i) + (-9 + 3\sqrt{3}i) = \ldots\{-6 + 6\sqrt{3}i\}$ Or $\{z_1 + z_2 =\}6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] + 6\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = a + bi$ where a and b are constants, the trig function must be evaluated		M1
	Clearly show the method to find modulus **and** argument for $z_1 + z_2$ $arg(z_1 + z_2) = \pi - \tan^{-1}\left(\frac{6\sqrt{3}}{6}\right)$ or $\tan^{-1}\left(\frac{6\sqrt{3}}{-6}\right) = \ldots\left\{\frac{2\pi}{3}\right\}$ **and** $\lvert z_1 + z_2\rvert = \sqrt{6^2 + (6\sqrt{3})^2} = \ldots\{12\}$	**Alternative 1** $-6 + 6\sqrt{3}i = 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $= 12\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ **Alternative 2** $12e^{\frac{2\pi}{3}i} = 12\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $= \ldots\{-6 + 6\sqrt{3}i\}$	dM1
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$ *	$12e^{\frac{2\pi}{3}i} = -6 + 6\sqrt{3}i$ Therefore $z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$*	A1*
			(3)

Question	Scheme	Marks
(ii)		M1
	$\sin\left(\frac{\pi}{3}\right) = \frac{\lvert z\rvert}{5} \Rightarrow \lvert z\rvert = \ldots$	M1
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1
		(3)

Series – Method of Differences

1. Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)}$$

where a and b are constants to be found.

(5)

Series – method of diff

Question	Scheme	Marks	AOs
1	$\frac{1}{(r+1)(r+3)} \equiv \frac{A}{(r+1)} + \frac{B}{(r+3)} \Rightarrow A = \ldots,\ B = \ldots$	M1	3.1a
	$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} =$ $\frac{1}{2\times2} - \frac{1}{2\times4} + \frac{1}{2\times3} - \frac{1}{2\times5} + \ldots + \frac{1}{2n} - \frac{1}{2(n+2)} + \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$	M1	2.1
	$= \frac{1}{4} + \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$	A1	2.2a
	$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{12(n+2)(n+3)}$	M1	1.1b
	$= \frac{n(5n+13)}{12(n+2)(n+3)}$	A1	1.1b
		(5)	

4. Prove that, for $n \in \mathbb{Z}$, $n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(5)

A2 2019 Paper 1

Series – method of diff

Question	Scheme	Marks	AOs
4	$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{A}{r+1}+\frac{B}{r+2}+\frac{C}{r+3} \Rightarrow A=\ldots,\ B=\ldots,\ C=\ldots$ $\left(\text{NB } A=\frac{1}{2} \quad B=-1 \quad C=\frac{1}{2}\right)$	M1	3.1a
	$r=0$ $\quad \frac{1}{2}\left[\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right]$ or $\frac{1}{2\ 1}-\frac{1}{2}+\frac{1}{2\ 3}$ or $\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$	M1	2.1
	$r=1$ $\quad \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$ or $\frac{1}{2\ 2}-\frac{1}{3}+\frac{1}{2\ 4}$ or $\frac{1}{4}-\frac{1}{3}+\frac{1}{8}$		
	$r=n-1$ $\quad \frac{1}{2}\left[\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right]$ or $\frac{1}{2\ n}-\frac{1}{n+1}+\frac{1}{2\ n+2}$ or $\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2n+4}$		
	$r=n$ $\quad \frac{1}{2}\left[\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}\right]$ or $\frac{1}{2\ n+1}-\frac{1}{n+2}+\frac{1}{2\ n+3}$ or $\frac{1}{2n+2}-\frac{1}{n+2}+\frac{1}{2n+6}$		
	$\frac{1}{2}-\frac{1}{2}+\frac{1}{4}+\frac{1}{2(n+2)}-\frac{1}{n+2}+\frac{1}{2(n+3)}$ or $\frac{1}{4}-\frac{1}{2(n+2)}+\frac{1}{2(n+3)}$	A1	1.1b
	$=\frac{n^2+5n+6+2n+6-4n-12+2n+4}{4(n+2)(n+3)}$	M1	1.1b
	$=\frac{(n+1)(n+4)}{4(n+2)(n+3)}$	A1	2.2a
		(5)	
			(5 marks)

HOME

4. (a) Use the method of differences to prove that for $n > 2$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$$

(4)

(b) Hence find the exact value of

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

Give your answer in the form $a\ln\left(\frac{b}{c}\right)$ where a, b and c are integers to be determined.

(3)

A2 2022 Paper 1

Series – method of diff

4(a)	Applies $ln\left(\frac{r+1}{r-1}\right) = ln(r+1) - ln(r-1)$ to the problem in order to apply differences.	M1	3.1a
	$\sum_{r=2}^{n}(ln(r+1) - ln(r-1))$ $= (ln(3) - ln(1)) + (ln(4) - ln(2)) + (ln(5) - ln(3)) + \ldots$ $+ (ln(n) - ln(n-2)) + (ln(n+1) - ln(n-1))$	dM1	1.1b
	$ln(n) + ln(n+1) - ln\,2$	A1	1.1b
	$ln\left(\frac{n(n+1)}{2}\right)$ * cso	A1 *	2.1
		(4)	
(b)	$\sum_{r=51}^{100} ln\left(\frac{r+1}{r-1}\right) = \sum_{r=2}^{100} ln\left(\frac{r+1}{r-1}\right) - \sum_{r=2}^{50} ln\left(\frac{r+1}{r-1}\right)$ $= ln\left(\frac{100 \times 101}{2}\right) - ln\left(\frac{50 \times 51}{2}\right)$	M1	1.1b
	$\sum_{r=51}^{100} ln\left(\frac{r+1}{r-1}\right)^{35} = 35\,ln\left(\frac{100\times101}{2} \div \frac{50\times51}{2}\right)$	M1	3.1a
	$= 35\,ln\left(\frac{202}{51}\right)$	A1	1.1b
		(3)	
			(7 marks)

BICEN MATHS

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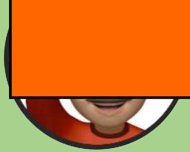
Maclaurin Series



5.

$$y = \sin x \sinh x$$

(a) Show that $\dfrac{d^4y}{dx^4} = -4y$

(4)

(b) Hence find the first three non-zero terms of the Maclaurin series for y, giving each coefficient in its simplest form.

(4)

(c) Find an expression for the nth non-zero term of the Maclaurin series for y.

(2)

AS 2019 Paper 2

Maclaurin Series

5(a)	$\dfrac{dy}{dx} = \sin x \cosh x + \cos x \sinh x$	M1	1.1a
	$\dfrac{d^2y}{dx^2} = \cos x \cosh x + \sin x \sinh x + \cos x \cosh x - \sin x \sinh x$ $(= 2\cos x \cosh x)$	M1	1.1b
	$\dfrac{d^3y}{dx^3} = 2\cos x \sinh x - 2\sin x \cosh x$	M1	1.1b
	$\dfrac{d^4y}{dx^4} = -4\sinh x \sin x = -4y$ *	A1*	2.1
		(4)	
(b)	$\left(\dfrac{d^2y}{dx^2}\right)_0 = 2,\ \left(\dfrac{d^6y}{dx^6}\right)_0 = -8,\ \left(\dfrac{d^{10}y}{dx^{10}}\right)_0 = 32$	B1	3.1a
	Uses $y = y_0 + xy_0' + \dfrac{x^2}{2!}y_0'' + \dfrac{x^3}{3!}y_0''' + \ldots$ with their values	M1	1.1b
	$= \dfrac{x^2}{2!}(2) + \dfrac{x^6}{6!}(-8) + \dfrac{x^{10}}{10!}(32)$	A1	1.1b
	$= x^2 - \dfrac{x^6}{90} + \dfrac{x^{10}}{113400}$	A1	1.1b
		(4)	
(c)	$2(-4)^{n-1}\dfrac{x^{4n-2}}{(4n-2)!}$	M1 A1	3.1a 2.2a
		(2)	
			(10 marks)

HOME

2. (a) Use the Maclaurin series expansion for $\cos x$ to determine the series expansion of $\cos^2\left(\frac{x}{3}\right)$ in ascending powers of x, up to and including the term in x^4

Give each term in simplest form.

(2)

(b) Use the answer to part (a) and calculus to find an approximation, to 5 decimal places, for

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) dx$$

(3)

(c) Use the integration function on your calculator to evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) dx$$

Give your answer to 5 decimal places.

(1)

(d) Assuming that the calculator answer in part (c) is accurate to 5 decimal places, comment on the accuracy of the approximation found in part (b).

(1)

A2 2021 Paper 1

Maclaurin Series

Question	Scheme	Marks	AOs
2 (a)	$\cos^2\frac{x}{3} = \left(1 - \frac{\left(\frac{x}{3}\right)^2}{2} + \frac{\left(\frac{x}{3}\right)^4}{24} - \ldots\right)^2 = \ldots$ or $\left(1 - \frac{x^2}{18} + \frac{x^4}{1944} - \ldots\right)^2 = \ldots$ or $\frac{1}{2}\left(1 \pm \cos\frac{2x}{3}\right) = \frac{1}{2}\left(1 \pm \left(1 - \frac{1}{2}\left(\frac{2x}{3}\right)^2 + \frac{1}{4!}\left(\frac{2x}{3}\right)^4 - \right)\ldots\right)$	M1	2.2a
	$= 1 - \frac{x^2}{9} + \frac{1}{243}x^4$	A1	1.1b
		(2)	
(b)	$\int \frac{1 - \frac{x^2}{9} + \frac{1}{243}x^4}{x} = \int \frac{1}{x} - \frac{x}{9} + \frac{1}{243}x^3 = A\ln x + Bx^2 + Cx^4$ where A, B and $C \neq 0$	M1	3.1a
	$\ln x - \frac{x^2}{18} + \frac{1}{972}x^4$	A1ft	1.1b
	= awrt 0.98295	A1	2.2a
		(3)	
(c)	Calculator = awrt 0.98280	B1	1.1b
		(1)	
(d)	E.g. the approximation is correct to 3 d.p.	B1	3.2b
		(1)	
			(7 marks)

HOME

A2 2021 Paper 2

Maclaurin Series

3. $$f(x) = \arcsin x \qquad -1 \leqslant x \leqslant 1$$

(a) Determine the first two non-zero terms, in ascending powers of x, of the Maclaurin series for $f(x)$, giving each coefficient in its simplest form.

(4)

(b) Substitute $x = \dfrac{1}{2}$ into the answer to part (a) and hence find an approximate value for π

Give your answer in the form $\dfrac{p}{q}$ where p and q are integers to be determined.

(2)

A2 2021 Paper 2

Maclaurin Series

Question	Scheme	Marks	AOs
3(a)	$\mathrm{f}'(x) = A\left(1-x^2\right)^{-\frac{1}{2}}$ $\quad \mathrm{f}''(x) = Bx\left(1-x^2\right)^{-\frac{3}{2}}$ and $\mathrm{f}'''(x) = C\left(1-x^2\right)^{-\frac{3}{2}} + Dx^2\left(1-x^2\right)^{-\frac{5}{2}}$ or $\dfrac{C\left(1-x^2\right)^{\frac{3}{2}} + Dx^2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^3}$	M1	2.1
	$\mathrm{f}'(x) = \left(1-x^2\right)^{-\frac{1}{2}}$ or $\dfrac{1}{\sqrt{1-x^2}}$ $\quad \mathrm{f}''(x) = x\left(1-x^2\right)^{-\frac{3}{2}}$ or $\dfrac{x}{\left(1-x^2\right)^{\frac{3}{2}}}$ and $\mathrm{f}'''(x) = \left(1-x^2\right)^{-\frac{3}{2}} + 3x^2\left(1-x^2\right)^{-\frac{5}{2}}$ or $\dfrac{1}{\left(1-x^2\right)^{\frac{3}{2}}} + \dfrac{3x^2}{\left(1-x^2\right)^{\frac{5}{2}}}$ from quotient rule $\dfrac{\left(1-x^2\right)^{\frac{3}{2}} + 3x^2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^3}$	A1	1.1b
	Finds $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and $\mathrm{f}'''(0)$ and applies the formula $\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \mathrm{f}''(0)\dfrac{x^2}{2} + \mathrm{f}'''(0)\dfrac{x^3}{6}$ $\{\mathrm{f}(0) = 0,\ \mathrm{f}'(0) = 1,\ \mathrm{f}''(0) = 0,\ \mathrm{f}'''(0) = 1\}$	M1	1.1b
	$\mathrm{f}(x) = x + \dfrac{x^3}{6}$ cso	A1	1.1b
		(4)	
(b)	$\arcsin\left(\dfrac{1}{2}\right) = \dfrac{1}{2} + \dfrac{\left(\frac{1}{2}\right)^3}{6} = \dfrac{\pi}{6} \Rightarrow \pi = \ldots$	M1	1.1b
	$\pi = \dfrac{25}{8}$ o.e.	A1ft	2.2b
		(2)	
	(6 marks)		

HOME

A2 2022 Paper 2

Maclaurin Series

9. $$y = \cosh^n x \qquad n \geqslant 5$$

(a) (i) Show that

$$\frac{d^2 y}{dx^2} = n^2 \cosh^n x - n(n-1)\cosh^{n-2} x$$

(4)

(ii) Determine an expression for $\frac{d^4 y}{dx^4}$

(2)

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form.

(2)

A2 2022 Paper 2

Maclaurin Series

Question	Scheme	Marks	AOs
9(a)(i)	$\frac{dy}{dx} = \ldots \cosh^{n-1} x \sinh x$ $\frac{d^2y}{dx^2} = \ldots \cosh^{n-2} x \sinh^2 x + \ldots \cosh^{n-1} x \cosh x$ **Alternatively** $y = \left(\frac{e^x+e^{-x}}{2}\right)^n$ leading to $\frac{dy}{dx} = \ldots \left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right)$ $\frac{d^2y}{dx^2} = \ldots \left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 + \ldots \left(\frac{e^x+e^{-x}}{2}\right)^n$	M1	1.1b
	$\frac{dy}{dx} = n \cosh^{n-1} x \sinh x$ $\frac{d^2y}{dx^2} = n(n-1)\cosh^{n-2} x \sinh^2 x + n\cosh^n x$ **Alternatively** $\frac{dy}{dx} = n\left(\frac{e^x+e^{-x}}{2}\right)^{n-1}\left(\frac{e^x-e^{-x}}{2}\right)$ $\frac{d^2y}{dx^2} = n(n-1)\left(\frac{e^x+e^{-x}}{2}\right)^{n-2}\left(\frac{e^x-e^{-x}}{2}\right)^2 + n\left(\frac{e^x+e^{-x}}{2}\right)^n$	A1	2.1
	$\frac{d^2y}{dx^2} = n(n-1)\cosh^{n-2} x\,(\cosh^2 x - 1) + n\cosh^n x$	M1	2.1
	$\frac{d^2y}{dx^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2} x$* cso	A1*	1.1b
		(4)	
(a)(ii)	$\frac{d^3y}{dx^3} = \ldots \cosh^{n-1} x \sinh x - \ldots \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4}$ $= \ldots \cosh^{n-2} x \sinh^2 x + \ldots \cosh^n x - \ldots \cosh^{n-4} x \sinh^2 x - \ldots \cos$	M1	1.1b
	$\frac{d^3y}{dx^3} = n^3\cosh^{n-1} x \sinh x - n(n-1)(n-2)\cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4} = n^3(n-1)\cosh^{n-2} x \sinh^2 x + n^3\cosh^n x$ $-n(n-1)(n-2)(n-3)\cosh^{n-4} x \sinh^2 x - n(n-1)(n$ $-2)\cosh^{n-2} x$	A1	1.1b
		(2)	
(b)	When $x = 0$ $y = 1,\ y' = 0,\ y'' = n^2 - n(n-1),\ y^{(3)} = 0,$ $y^{(4)} = n^3 - n(n-1)(n-2)$ Uses their values in the expansion$y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \frac{x^4}{4!}y^{(4)}(0) + \ldots$	M1	
	$y = 1 + \frac{nx^2}{2} + \frac{(3n^2-2n)x^4}{24} + \ldots$ cso	A1	
		(2)	

HOME

Methods in Calculus

6.

$$f(x) = \frac{x+2}{x^2+9}$$

(a) Show that

$$\int f(x)\,dx = A\ln(x^2+9) + B\arctan\left(\frac{x}{3}\right) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence show that the mean value of $f(x)$ over the interval $[0, 3]$ is

$$\frac{1}{6}\ln 2 + \frac{1}{18}\pi$$

(3)

(c) Use the answer to part (b) to find the mean value, over the interval $[0, 3]$, of

$$f(x) + \ln k$$

where k is a positive constant, giving your answer in the form $p + \frac{1}{6}\ln q$, where p and q are constants and q is in terms of k.

(2)

A2 SAMs Paper 1

Question	Scheme	Marks	AOs
6(a)	$\mathrm{f}(x)=\dfrac{x+2}{x^2+9}=\dfrac{x}{x^2+9}+\dfrac{2}{x^2+9}$	B1	3.1a
	$\displaystyle\int\frac{x}{x^2+9}\,\mathrm{d}x=k\ln\left(x^2+9\right)(+c)$	M1	1.1b
	$\displaystyle\int\frac{2}{x^2+9}\,\mathrm{d}x=k\arctan\left(\frac{x}{3}\right)(+c)$	M1	1.1b
	$\displaystyle\int\frac{x+2}{x^2+9}\,\mathrm{d}x=\frac{1}{2}\ln\left(x^2+9\right)+\frac{2}{3}\arctan\left(\frac{x}{3}\right)+c$	A1	1.1b
		(4)	
(b)	$\displaystyle\int_0^3\mathrm{f}(x)\,\mathrm{d}x=\left[\frac{1}{2}\ln\left(x^2+9\right)+\frac{2}{3}\arctan\left(\frac{x}{3}\right)\right]_0^3$ $=\dfrac{1}{2}\ln 18+\dfrac{2}{3}\arctan\left(\dfrac{3}{3}\right)-\left(\dfrac{1}{2}\ln 9+\dfrac{2}{3}\arctan(0)\right)$ $=\dfrac{1}{2}\ln\dfrac{18}{9}+\dfrac{2}{3}\arctan\left(\dfrac{3}{3}\right)$	M1	1.1b
	Mean value $=\dfrac{1}{3-0}\left(\dfrac{1}{2}\ln 2+\dfrac{\pi}{6}\right)$	M1	2.1
	$\dfrac{1}{6}\ln 2+\dfrac{1}{18}\pi$ *	A1*	2.2a
		(3)	
(c)	$\dfrac{1}{6}\ln 2+\dfrac{1}{18}\pi+\ln k$	M1	2.2a
	$\dfrac{1}{6}\ln 2k^6+\dfrac{1}{18}\pi$	A1	1.1b
		(2)	
		(9 marks)	

HOME

2. Show that

$$\int_0^\infty \frac{8x - 12}{(2x^2 + 3)(x + 1)}\,\mathrm{d}x = \ln k$$

where k is a rational number to be found.

(7)

A2 2019 Paper 1

Methods in Calculus

2	$\dfrac{8x-12}{(2x^2+3)(x+1)} = \dfrac{Ax+B}{2x^2+3} + \dfrac{C}{x+1}$	M1	3.1a
	$8x-12 = (Ax+B)(x+1) + C(2x^2+3)$ E.g. $x=-1 \Rightarrow C=-4, x=0 \Rightarrow B=0, x=1 \Rightarrow A=8$ Or Compares coefficients and solves $(A+2C=0 \quad A+B=8 \quad B+3C=-12)$ $\Rightarrow A=..., B=..., C=...$	dM1	1.1b
	$A=8 \quad B=0 \quad C=-4$	A1	1.1b
	$\int\left(\dfrac{8x}{2x^2+3} - \dfrac{4}{x+1}\right)\mathrm{d}x = 2\ln(2x^2+3) - 4\ln(x+1)$	A1ft	1.1b
	$2\ln(2x^2+3) - 4\ln(x+1) = \ln\left(\dfrac{(2x^2+3)^2}{(x+1)^4}\right)$ or $2\ln(2x^2+3) - 4\ln(x+1) = 2\ln\left(\dfrac{(2x^2+3)}{(x+1)^2}\right)$	M1	2.1
	$\lim\limits_{x\to\infty}\left\{\ln\dfrac{(2x^2+3)^2}{(x+1)^4}\right\} = \ln 4$ or $\lim\limits_{x\to\infty}\left\{2\ln\dfrac{(2x^2+3)}{(x+1)^2}\right\} = 2\ln 2$	B1	2.2a
	$\Rightarrow \int_0^\infty \dfrac{8x-12}{(2x^2+3)(x+1)}\mathrm{d}x = \ln\dfrac{4}{9}$ cao	A1	1.1b
		(7)	

3.

$$f(x) = \frac{1}{\sqrt{4x^2 + 9}}$$

(a) Using a substitution, that should be stated clearly, show that

$$\int f(x)\,dx = A\sinh^{-1}(Bx) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence find, in exact form in terms of natural logarithms, the mean value of $f(x)$ over the interval [0, 3].

(2)

A2 2019 Paper 2

Methods in Calculus

Question	Scheme	Marks	AOs
3(a) Way 1	$x = \frac{3}{2}\sinh u$	B1	2.1
	$\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}} = \int \frac{1}{\sqrt{4\left(\frac{9}{4}\right)\sinh^2 u + 9}} \times \frac{3}{2}\cosh u \; \mathrm{d}u$	M1	3.1a
	$= \int \frac{1}{2}\, \mathrm{d}u$	A1	1.1b
	$= \int \frac{1}{2}\, \mathrm{d}u = \frac{1}{2}u = \frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right) + c$	A1	1.1b
		(4)	
(b)	Mean value = $\frac{1}{3(-0)}\left[\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)\right]_0^3 = \frac{1}{3} \times \frac{1}{2}\sinh^{-1}\left(\frac{2\times 3}{3}\right)(-0)$	M1	2.1
	$= \frac{1}{6}\ln\left(2+\sqrt{5}\right)$ **(Brackets are required)**	A1ft	1.1b
		(2)	
			(6 marks)

HOME

2. (a) Explain why $\int_1^{\infty} \frac{1}{x(2x+5)}\,\mathrm{d}x$ is an improper integral.

(1)

(b) Prove that

$$\int_1^{\infty} \frac{1}{x(2x+5)}\,\mathrm{d}x = a\ln b$$

where a and b are rational numbers to be determined.

(6)

A2 2020 Paper 1

Methods in Calculus

Question	Scheme	Marks	AOs
2(a)	E.g. • Because the interval being integrated over is unbounded • Accept because the upper limit is infinity • Accept because a limit is required to evaluate it	B1	2.4
		(1)	
(b)	$\dfrac{1}{x(2x+5)}=\dfrac{A}{x}+\dfrac{B}{2x+5}\Rightarrow A=\ldots, B=\ldots$	M1	3.1a
	$\dfrac{1}{x(2x+5)}=\dfrac{1}{5x}-\dfrac{2}{5(2x+5)}$	A1	1.1b
	$\displaystyle\int \frac{1}{5x}-\frac{2}{5(2x+5)}\,\mathrm{d}x=\frac{1}{5}\ln x-\frac{1}{5}\ln(2x+5)$	A1ft	1.1b
	$\dfrac{1}{5}\ln x-\dfrac{1}{5}\ln(2x+5)=\dfrac{1}{5}\ln\dfrac{x}{(2x+5)}$	M1	2.1
	$\lim\limits_{x\to\infty}\left\{\dfrac{1}{5}\ln\dfrac{x}{2x+5}\right\}=\dfrac{1}{5}\ln\dfrac{1}{2}$	B1	2.2a
	$\displaystyle\Rightarrow\int_1^{\infty}\frac{1}{x(2x+5)}\,\mathrm{d}x=\frac{1}{5}\ln\frac{1}{2}-\frac{1}{5}\ln\frac{1}{7}=\frac{1}{5}\ln\frac{7}{2}$	A1	1.1b
		(6)	
		(7 marks)	

5. (a)

$$y = \tan^{-1} x$$

Assuming the derivative of $\tan x$, prove that

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

(3)

$$f(x) = x\tan^{-1} 4x$$

(b) Show that

$$\int f(x)\,dx = Ax^2\tan^{-1} 4x + Bx + C\tan^{-1} 4x + k$$

where k is an arbitrary constant and A, B and C are constants to be determined.

(5)

(c) Hence find, in exact form, the mean value of $f(x)$ over the interval $\left[0, \frac{\sqrt{3}}{4}\right]$

(2)

A2 2020 Paper 2

5(a)	$y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \dfrac{dx}{dy} = \sec^2 y$ $y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \dfrac{dy}{dx}\sec^2 y = 1$	M1	3.1a
	$\dfrac{dx}{dy} = 1 + \tan^2 y$ or $\dfrac{dy}{dx}\left(1 + \tan^2 y\right) = 1$	M1	1.1b
	$\dfrac{dy}{dx} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$ *	A1*	2.1
		(3)	
(b)	$\dfrac{d\left(\tan^{-1} 4x\right)}{dx} = \dfrac{4}{1+16x^2}$	B1	1.1b
	$\displaystyle\int x\tan^{-1} 4x\,dx = \alpha x^2 \tan^{-1} 4x - \int \alpha x^2 \times \text{'}\frac{4}{1+16x^2}\text{'}\,dx$	M1	2.1
	$\displaystyle\int x\tan^{-1} 4x\,dx = \frac{x^2}{2}\tan^{-1} 4x - \int \frac{x^2}{2} \times \frac{4}{1+16x^2}\,dx$	A1	1.1b
	$\displaystyle = \ldots - \frac{1}{8}\int \frac{16x^2+1-1}{1+16x^2}\,dx = \ldots - \frac{1}{8}\int \left(1 - \frac{1}{1+16x^2}\right)dx$ or let $4x = \tan u \Rightarrow \dfrac{1}{8}\displaystyle\int \frac{\tan^2 u}{1+\tan^2 u} \times \frac{1}{4}\sec^2 u\,du$ $\Rightarrow \dfrac{1}{32}\displaystyle\int \tan^2 u\,du = \frac{1}{32}\int \sec^2 u - u\,du$	M1	3.1a
	$= \dfrac{x^2}{2}\tan^{-1} 4x - \dfrac{1}{8}x + \dfrac{1}{32}\tan^{-1} 4x + k$	A1	2.1
		(5)	
(c)	Mean value $= \left(\dfrac{1}{\frac{\sqrt{3}}{4} - 0}\right)\left[\dfrac{x^2}{2}\tan^{-1} 4x - \dfrac{1}{8}x + \dfrac{1}{32}\tan^{-1} 4x\right]_0^{\frac{\sqrt{3}}{4}}$ $= \dfrac{4}{\sqrt{3}}\left(\left(\dfrac{3}{32}\times\dfrac{\pi}{3} - \dfrac{1}{8}\times\dfrac{\sqrt{3}}{4} + \dfrac{1}{32}\times\dfrac{\pi}{3}\right) - 0\right)$	M1	2.1
	$= \dfrac{\sqrt{3}}{72}\left(4\pi - 3\sqrt{3}\right)$ or $\dfrac{\sqrt{3}}{18}\pi - \dfrac{1}{8}$ oe	A1	1.1b
		(2)	
		(10 marks)	

HOME

5. (i) Evaluate the improper integral

$$\int_1^{\infty} 2e^{-\frac{1}{2}x}\,dx$$

(3)

(ii) The air temperature, θ °C, on a particular day in London is modelled by the equation

$$\theta = 8 - 5\sin\left(\frac{\pi}{12}t\right) - \cos\left(\frac{\pi}{6}t\right) \qquad 0 \leqslant t \leqslant 24$$

where t is the number of hours after midnight.

(a) Use calculus to show that the mean air temperature on this day is 8 °C, according to the model.

(3)

Given that the actual mean air temperature recorded on this day was higher than 8 °C,

(b) explain how the model could be refined.

(1)

Question	Scheme	Marks	AOs
5(i)	$\int 2e^{-\frac{1}{2}x}\,dx = -4e^{-\frac{1}{2}x}$	B1	1.1b
	$\int_1^{\infty} 2e^{-\frac{1}{2}x}\,dx = \lim_{t \to \infty}\left[\left(-4e^{-\frac{1}{2}t}\right)-\left(-4e^{-\frac{1}{2}}\right)\right]$	M1	2.1
	$= 4e^{-\frac{1}{2}}$	A1	1.1b
		(3)	
(ii)(a)	Mean temperature $= \frac{1}{24}\int_0^{24}\left(8-5\sin\left(\frac{\pi}{12}t\right)-\cos\left(\frac{\pi}{6}t\right)\right)dt$	B1	1.2
	$= \frac{1}{24}\left[\left(8t+\frac{60}{\pi}\cos\left(\frac{\pi}{12}t\right)-\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\right)\right]_0^{24} = \frac{1}{24}[...]$	M1	1.1b
	$= \frac{1}{24}\left[\left(8(24)+\frac{60}{\pi}-\frac{6}{\pi}\times 0\right)-\left(\frac{60}{\pi}\right)\right] = 8$ * cso	A1*	2.1
		(3)	
(ii)(b)	E.g. increase the value of the constant 8 / adapt the constant 8 to a function which takes values greater than 8.	B1	3.5c
		(1)	
			(7 marks)

HOME

9. (a) Use a hyperbolic substitution and calculus to show that

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh} x\right] + k$$

where k is an arbitrary constant.

(6)

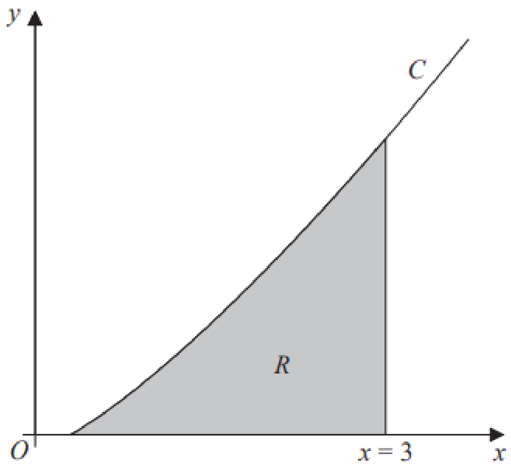


Figure 1

Figure 1 shows a sketch of part of the curve C with equation

$$y = \frac{4}{15}x \operatorname{arcosh} x \qquad x \geqslant 1$$

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Using algebraic integration and the result from part (a), show that the area of R is given by

$$\frac{1}{15}\left[17\ln\left(3 + 2\sqrt{2}\right) - 6\sqrt{2}\right]$$

(5)

9(a)	$\int \frac{x^2}{\sqrt{x^2-1}}\,dx \to \int f(u)\,du$ Uses the substitution $x=\cosh u$ fully to achieve an integral in terms of u only, including replacing the dx	M1	3.1a
	$\int \frac{\cosh^2 u}{\sqrt{\cosh^2 u - 1}} \sinh u\,(du)$	A1	1.1b
	Uses correct identities $\cosh^2 u - 1 = \sinh^2 u$ and $\cosh 2u = 2\cosh^2 u - 1$ to achieve an integral of the form $A\int(\cosh 2u \pm 1)\,du \qquad A>0$	M1	3.1a
	Integrates to achieve $A\left(\pm\frac{1}{2}\sinh 2u \pm u\right)(+c) \qquad A>0$	M1	1.1b
	Uses the identity $\sinh 2u = 2\sinh u \cosh u$ and $\cosh^2 u - 1 = \sinh^2 u$ $\to \sinh 2u = 2x\sqrt{x^2-1}$	M1	2.1
	$\frac{1}{2}\left[x\sqrt{x^2-1} + \text{ar}\cosh x\right] + k$ * cso	A1*	1.1b
		(6)	
(b)	Uses integration by parts the correct way around to achieve $\int \frac{4}{15} x \text{ arcosh}x dx = Px^2\text{arcosh}x - Q\int \frac{x^2}{\sqrt{x^2-1}}\,dx$	M1	2.1
	$= \frac{4}{15}\left(\frac{1}{2}x^2\text{arcosh}x - \frac{1}{2}\int \frac{x^2}{\sqrt{x^2-1}}\,dx\right)$	A1	1.1b
	$= \frac{4}{15}\left(\frac{1}{2}x^2\text{arcosh}x - \frac{1}{2}\left(\frac{1}{2}\left[x\sqrt{x^2-1} + \text{arcosh}x\right]\right)\right)$	B1ft	2.2a
	Uses the limits $x=1$ and $x=3$ the correct way around and subtracts $= \frac{4}{15}\left(\frac{1}{2}(3)^2\text{ arcosh}3 - \frac{1}{2}\left(\frac{1}{2}\left[3\sqrt{(3)^2-1} + \text{arcosh}3\right]\right)\right) - \frac{4}{15}(0)$	dM1	1.1b
	$= \frac{4}{15}\left(\frac{9}{2}\ln\left(3+\sqrt{8}\right) - \frac{3\sqrt{8}}{4} - \frac{1}{4}\ln\left(3+\sqrt{8}\right)\right)$ $= \frac{1}{15}\left[17\ln\left(3+2\sqrt{2}\right) - 6\sqrt{2}\right]$*	A1*	1.1b
		(5)	

BICEN MATHS

HOME

5. The curve C has equation

$$y = \arccos\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2$$

(a) Show that C has no stationary points.

(3)

The normal to C, at the point where $x = 1$, crosses the x-axis at the point A and crosses the y-axis at the point B.

Given that O is the origin,

(b) show that the area of the triangle OAB is $\frac{1}{54}\left(p\sqrt{3} + q\pi + r\sqrt{3}\,\pi^2\right)$ where p, q and r are integers to be determined.

(5)

A2 2021 Paper 2

Methods in Calculus

Question	Scheme	Marks	AOs
5(a)	$\dfrac{dy}{dx}=\dfrac{-\lambda}{\sqrt{1-\beta x^2}}$ where $\lambda>0$ and $\beta>0$ and $\beta\neq 1$ Alternatively $2\cos y = x \Rightarrow \dfrac{dx}{dy}=\alpha\sin y \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\alpha\sin y}$	M1	1.1b
	$\dfrac{dy}{dx}=\dfrac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}x^2}}$ or $\dfrac{dy}{dx}=\dfrac{-1}{2\sqrt{1-\frac{1}{4}x^2}}$ o.e. or $\dfrac{dy}{dx}=-\dfrac{1}{2\sin y}$ or	A1	1.1b
	States that $\dfrac{dy}{dx}\neq 0$ therefore C has no stationary points. Tries to solve $\dfrac{dy}{dx}=0$ and ends up with a contradiction e.g. -1 = 0 therefore C has no stationary points. As cosec $y>1$ therefore C has no stationary points.	A1	2.4
		(3)	
(b)	$\dfrac{dy}{dx}=\dfrac{-1}{2\sqrt{1-\frac{1}{4}\times 1^2}}=\left\{-\dfrac{1}{\sqrt{3}}\right\}$	M1	1.1b
	Normal gradient $=-\dfrac{1}{m}$ and $y-\dfrac{\pi}{3}=m_n(x-1)$ Alternatively $\dfrac{\pi}{3}=m_n(1)+c \Rightarrow c=\ldots\left\{\dfrac{\pi}{3}-\sqrt{3}\right\}$ and then $y=m_n x+c$	M1	1.1b
	$y=0 \Rightarrow 0-\dfrac{\pi}{3}=\sqrt{3}(x_A-1)\Rightarrow x_A=\ldots\left\{1-\dfrac{\pi}{3\sqrt{3}} \text{ or } 1-\dfrac{\pi\sqrt{3}}{9}\right\}$ and $x=0 \Rightarrow y_B-\dfrac{\pi}{3}=\sqrt{3}(0-1)\Rightarrow y_B=\ldots\left\{\dfrac{\pi}{3}-\sqrt{3}\right\}$	M1	3.1a
	Area $=\dfrac{1}{2}\times x_A\times -y_B=\dfrac{1}{2}\left(1-\dfrac{\pi}{3\sqrt{3}}\right)\left(\sqrt{3}-\dfrac{\pi}{3}\right)$	M1	1.1b
	Area $\dfrac{1}{54}\left(27\sqrt{3}-18\pi+\sqrt{3}\pi^2\right)$ $(p=27, q=-18, r=1)$	A1	2.1
		(5)	

HOME

6. (a) Express as partial fractions

$$\frac{2x^2 + 3x + 6}{(x+1)(x^2+4)}$$

(3)

(b) Hence, show that

$$\int_0^2 \frac{2x^2 + 3x + 6}{(x+1)(x^2+4)}\,\mathrm{d}x = \ln\left(a\sqrt{2}\right) + b\pi$$

where a and b are constants to be determined.

(4)

6(a)	$\frac{2x^2+3x+6}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} \Rightarrow 2x^2+3x+6 = A(x^2+4)+(Bx+C)(x+1)$	M1	1.1b
	e.g. $x=-1 \Rightarrow A = \ldots,\ x=0 \Rightarrow C = \ldots,$ coeff $x^2 \Rightarrow B = \ldots$ or Compares coefficients and solves to find values for A, B and C $2 = A+B,\ 3 = B+C,\ 6 = 4A+C$	dM1	1.1b
	$A=1,\quad B=1,\quad C=2$	A1	1.1b
		(3)	
(b)	$\int_0^2 \frac{1}{x+1} + \frac{x+2}{x^2+4}\,\mathrm{d}x = \int_0^2 \frac{1}{x+1} + \frac{x}{x^2+4} + \frac{2}{x^2+4}\,\mathrm{d}x$ $= \left[\alpha\ln(x+1) + \beta\ln(x^2+4) + \lambda\arctan\left(\frac{x}{2}\right)\right]_0^2$	M1	3.1a
	$= \left[ln(x+1) + \frac{1}{2}ln(x^2+4) + arctan\left(\frac{x}{2}\right)\right]_0^2$	A1	2.1
	$= \left[ln(3) + \frac{1}{2}ln(8) + arctan\,1\right] - \left[ln(1) + \frac{1}{2}ln(4) + arctan(0)\right]$ $=$ $= \left[ln(3) + \frac{1}{2}ln(8) + arctan(1)\right] - \left[\frac{1}{2}ln\,4\right] = \underline{ln\left(\frac{3\sqrt{8}}{2}\right)} + \frac{\pi}{4}$	dM1	2.1
	$ln(3\sqrt{2}) + \frac{\pi}{4}$	A1	2.2a
		(4)	
			(7 marks)

HOME

9. (i) (a) Explain why $\int_0^{\infty} \cosh x \, dx$ is an improper integral.

(1)

(b) Show that $\int_0^{\infty} \cosh x \, dx$ is divergent.

(3)

(ii) $$4 \sinh x = p \cosh x \qquad \text{where } p \text{ is a real constant}$$

Given that this equation has real solutions, determine the range of possible values for p

(2)

A2 2022 Paper 1

Methods in Calculus

9(i) (a)	E. g. • Because the interval being integrated over is unbounded. • $\cosh x$ is undefined at the limit of ∞ • the upper limit is infinite	B1
		(1)
(i) (b)	$\int_0^\infty \cosh x \, dx = \lim_{t\to\infty} \int_0^t \cosh x \, dx$ or $\lim_{t\to\infty} \int_0^t \frac{1}{2}(e^x + e^{-x})dx$	B1
	$\int_0^t \cosh x \, dx = [\sinh x]_0^t = \sinh t \; (-0)$ or $\frac{1}{2}\int_0^t e^x + e^{-x} \, dx = \frac{1}{2}[e^x - e^{-x}]_0^t = \frac{1}{2}[e^t - e^{-t}] \left(-\frac{1}{2}[e^0 - e^0]\right)$	M1
	When $t \to \infty$ $e^t \to \infty$ and $e^{-t} \to 0$ therefore the integral is divergent	A1
		(3)
(ii)	$4\,sinh\,x = p\,cosh\,x \Rightarrow tanh\,x = \frac{p}{4}$ or $4\,tanh\,x = p$ **Alternative** $\frac{4}{2}(e^x - e^{-x}) = \frac{p}{2}(e^x + e^{-x}) \Rightarrow 4e^x - 4e^{-x} = pe^x + pe^{-x}$ $e^{2x}(4-p) = p + 4 \Rightarrow e^{2x} = \frac{p+4}{4-p}$	M1
	$\left\{-1 < \frac{p}{4} < 1 \Rightarrow\right\} - 4 < p < 4$	A1
		(2)

5. (a) Given that

$$y = \arcsin x \qquad -1 \leqslant x \leqslant 1$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 - x^2}}$$

(3)

(b)

$$\mathrm{f}(x) = \arcsin\left(\mathrm{e}^x\right) \qquad x \leqslant 0$$

Prove that $\mathrm{f}(x)$ has no stationary points.

(3)

A2 2022 Paper 2

Methods in Calculus

5(a)	$\sin y = x \Rightarrow \cos y \dfrac{dy}{dx} = 1$	$\sin y = x \Rightarrow \dfrac{dx}{dy} = \cos y$	M1
	Uses $\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \sqrt{1 - \sin^2 y} \Rightarrow \sqrt{1 - x^2}$		M1
	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ * cso		A1*
			(3)
(b)	Using the answer to (a) $f'(x) = \dfrac{1}{\sqrt{1-e^{2x}}} \times \ldots$	Restart $\sin y = e^x \Rightarrow \cos y \dfrac{dy}{dx} = e^x$	M1
	$f'(x) = \dfrac{1}{\sqrt{1-e^{2x}}} \times e^x$	$f'(x) = \dfrac{e^x}{\cos y}$	A1
	$e^x \neq 0$ (or $e^x > 0$) therefore, there are no stationary points Alternatively, $e^x = 0$ leading to $x = \ln 0$ which is impossible/undefined therefore there are no stationary points.		A1
			(3)

HOME

Volumes of Revolution pt. 2

7.

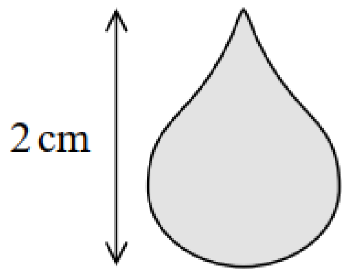


Figure 2

Figure 2 shows the image of a gold pendant which has height 2 cm. The pendant is modelled by a solid of revolution of a curve C about the y-axis. The curve C has parametric equations

$$x = \cos\theta + \frac{1}{2}\sin 2\theta, \qquad y = -(1 + \sin\theta) \qquad 0 \leqslant \theta \leqslant 2\pi$$

(a) Show that a Cartesian equation of the curve C is

$$x^2 = -(y^4 + 2y^3)$$

(4)

(b) Hence, using the model, find, in cm^3, the volume of the pendant.

(4)

A2 SAMs Paper 1

Volumes of Revolution pt. 2

Question	Scheme	Marks	AOs
7(a)	$x = \cos\theta + \sin\theta\cos\theta = -y\cos\theta$	M1	2.1
	$\sin\theta = -y - 1$	M1	2.1
	$\left(\dfrac{x}{-y}\right)^2 = 1 - (-y-1)^2$	M1	2.1
	$x^2 = -\left(y^4 + 2y^3\right)$*	A1*	1.1b
		(4)	
(b)	$V = \pi\int x^2\,\mathrm{d}y = \pi\int -\left(y^4 + 2y^3\right)\,\mathrm{d}y$	M1	3.4
	$= \pi\left[-\left(\dfrac{y^5}{5} + \dfrac{y^4}{2}\right)\right]$	A1	1.1b
	$= -\pi\left[\left(\dfrac{(0)^5}{5} + \dfrac{(0)^4}{2}\right) - \left(\dfrac{(-2)^5}{5} + \dfrac{(-2)^4}{2}\right)\right]$	M1	3.4
	$= 1.6\pi\ \text{cm}^3$ **or** awrt $5.03\ \text{cm}^3$	A1	1.1b
		(4)	
	(8 marks)		

HOME

A2 2019 Paper 2

Volumes of Revolution pt. 2

Figure 1 shows the central vertical cross section $ABCD$ of a paddling pool that has a circular horizontal cross section. Measurements of the diameters of the top and bottom of the paddling pool have been taken in order to estimate the volume of water that the paddling pool can contain.

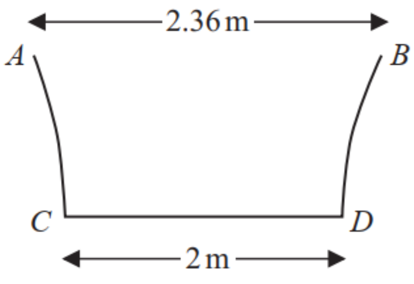


Figure 1

Using these measurements, the curve BD is modelled by the equation

$$y = \ln(3.6x - k) \qquad 1 \leqslant x \leqslant 1.18$$

as shown in Figure 2.

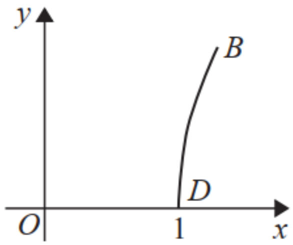


Figure 2

(a) Find the value of k. **(1)**

(b) Find the depth of the paddling pool according to this model. **(2)**

The pool is being filled with water from a tap.

(c) Find, in terms of h, the volume of water in the pool when the pool is filled to a depth of h m. **(5)**

Given that the pool is being filled at a constant rate of 15 litres every minute,

(d) find, in $\text{cm}\,\text{h}^{-1}$, the rate at which the water level is rising in the pool when the depth of the water is 0.2 m. **(3)**

HOME

A2 2019 Paper 2

Volumes of Revolution pt. 2

8(a)	$k = 2.6$	B1	3.4
		(1)	
(b)	$x = 1.18 \Rightarrow \ln\left(3.6\times1.18 - \text{"}2.6\text{"}\right) = \ldots$	M1	1.1b
	$h = 0.4995\ldots$ m	A1	2.2b
		(2)	
(c)	$y = \ln(3.6x - 2.6) \Rightarrow x = \dfrac{e^y + 2.6}{3.6}$ or $\dfrac{5e^y + 13}{18}$	B1ft	1.1a
	$V = \pi\displaystyle\int \left(\frac{e^y + 2.6}{3.6}\right)^2 dy = \frac{\pi}{3.6^2}\int \left(e^{2y} + 5.2e^y + 6.76\right)dy$ or $\dfrac{\pi}{324}\displaystyle\int\left(25e^{2y} + 130e^y + 169\right)dy$	M1	3.3
	$= \dfrac{\pi}{3.6^2}\left[\dfrac{1}{2}e^{2y} + 5.2e^y + 6.76y\right]$ $\left(\text{or } \dfrac{\pi}{324}\left[\dfrac{25}{2}e^{2y} + 130e^y + 169y\right]\right)$	A1	1.1b
	$= \dfrac{\pi}{3.6^2}\left\{\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h\right) - \left(\dfrac{1}{2}e^0 + 5.2e^0 + 6.76(0)\right)\right\}$ or e.g. $= \dfrac{\pi}{324}\left\{\left(\dfrac{25}{2}e^{2h} + 130e^h + 169h\right) - \left(\dfrac{25}{2}e^0 + 130e^0 + 6.76(0)\right)\right\}$	M1	2.1
	$= \dfrac{\pi}{3.6^2}\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h - 5.7\right)$	A1	1.1b
		(5)	
(d)	$\dfrac{dV}{dh} = \dfrac{\pi}{3.6^2}\left(e^{2h} + 5.2e^h + 6.76\right) = \dfrac{\pi}{3.6^2}\left(e^{0.4} + 5.2e^{0.2} + 6.76\right)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{1}{3.539\ldots}\times 0.015\times 60$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm}\,\text{h}^{-1}$	A1	3.2a
		(3)	
(d) Way 2	$y = 0.2 \Rightarrow x = \dfrac{2.6 + e^{0.2}}{3.6} \Rightarrow A = \pi\left(\dfrac{2.6 + e^{0.2}}{3.6}\right)^2 (= 3.54)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{0.015\times 60}{3.54}$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm}\,\text{h}^{-1}$	A1	3.2a
			(11 marks)

HOME

A2 2020 Paper 2

Volumes of Revolution pt. 2

7.

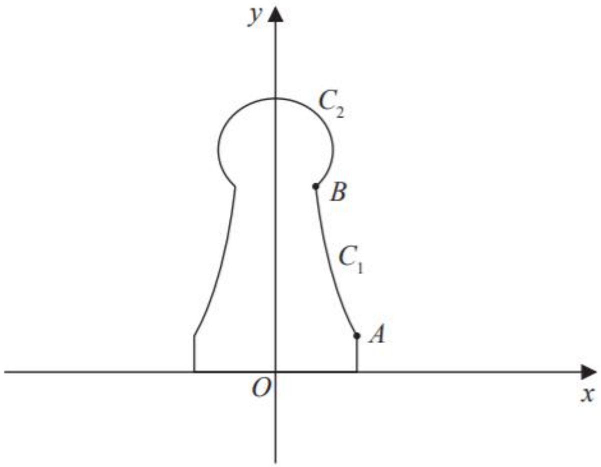


Figure 1

A student wants to make plastic chess pieces using a 3D printer. Figure 1 shows the central vertical cross-section of the student's design for one chess piece. The plastic chess piece is formed by rotating the region bounded by the y-axis, the x-axis, the line with equation $x = 1$, the curve C_1 and the curve C_2 through 360° about the y-axis.

The point A has coordinates (1, 0.5) and the point B has coordinates (0.5, 2.5) where the units are centimetres.

The curve C_1 is modelled by the equation

$$x = \frac{a}{y + b} \qquad 0.5 \leqslant y \leqslant 2.5$$

(a) Determine the value of a and the value of b according to the model.

(2)

The curve C_2 is modelled to be an arc of the circle with centre (0, 3).

(b) Use calculus to determine the volume of plastic required to make the chess piece according to the model.

(9)

HOME

Volumes of Revolution pt. 2

7(a)	$1=\dfrac{a}{0.5+b},\ 0.5=\dfrac{a}{2.5+b}\Rightarrow a=\ldots,\ b=\ldots$	M1	3.3
	$a=2,\ b=1.5$	A1	1.1b
		(2)	
(b)	$V_1=\pi\int x^2\mathrm{d}y=\pi\int\left(\dfrac{\text{"2"}}{y+\text{"1.5"}}\right)^2\mathrm{d}y$	B1ft	3.4
	$\pi\int_{0.5}^{2.5}\left(\dfrac{\text{"2"}}{y+\text{"1.5"}}\right)^2\mathrm{d}y$	M1	1.1a
	$=\{4\pi\}\left[-(y+1.5)^{-1}\right]_{0.5}^{2.5}(=\pi)$	M1	1.1b
	$x^2+(y-3)^2=0.5$	B1	2.2a
	$V_2=\pi\int x^2\mathrm{d}y=\pi\int\left(0.5-(y-3)^2\right)\mathrm{d}y$ or $\pi\int\left(-y^2+6y-8.5\right)\mathrm{d}y$	M1	1.1b
	$=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}\left(0.5-(y-3)^2\right)\mathrm{d}y$ or $=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}\left(-y^2+6y-8.5\right)\mathrm{d}y$	M1	3.3
	$=\{\pi\}\left[0.5y-\dfrac{1}{3}(y-3)^3\right]_{2.5}^{3+\frac{1}{\sqrt{2}}}$ or $=\{\pi\}\left[-\dfrac{1}{3}y^3+3y^2-8.5y\right]_{2.5}^{3+\frac{1}{\sqrt{2}}}$	A1	1.1b
	$V_1+V_2+\text{cylinder}=\pi+\pi\left(\dfrac{5}{24}+\dfrac{\sqrt{2}}{6}\right)+\dfrac{1}{2}\pi$	dM1	3.4
	$=\pi\left(\dfrac{41}{24}+\dfrac{\sqrt{2}}{6}\right)\approx 6.11\text{cm}^3$	A1	2.2b
		(9)	
			(11 marks)

BICEN MATHS

HOME

7. **Solutions based entirely on graphical or numerical methods are not acceptable.**

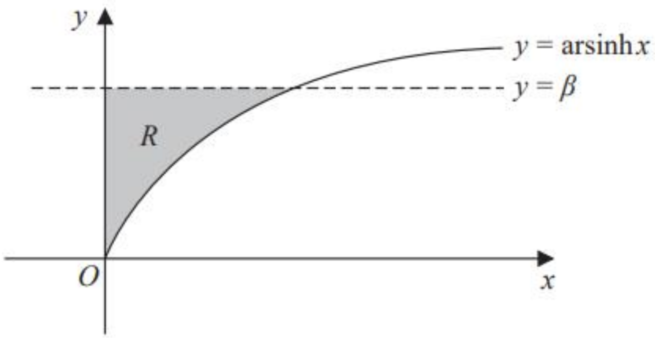


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = \text{arsinh}\,x \qquad x \geqslant 0$$

and the straight line with equation $y = \beta$

The line and the curve intersect at the point with coordinates (α, β)

Given that $\beta = \frac{1}{2}\ln 3$

(a) show that $\alpha = \frac{1}{\sqrt{3}}$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the curve with equation $y = \text{arsinh}\,x$, the y-axis and the line with equation $y = \beta$

The region R is rotated through 2π radians about the y-axis.

(b) Use calculus to find the exact value of the volume of the solid generated.

(6)

A2 2021 Paper 2

Volumes of Revolution pt. 2

Question	Scheme		Marks	AOs
7(a)	Using $\text{arsinh}\,\alpha = \frac{1}{2}\ln 3$ $\alpha = \dfrac{e^{\frac{1}{2}\ln 3} - e^{-\frac{1}{2}\ln 3}}{2}$	$\ln\left(\alpha + \sqrt{\alpha^2+1}\right) = \frac{1}{2}\ln 3$	B1	1.2
	$\alpha = \dfrac{\sqrt{3} - \frac{1}{\sqrt{3}}}{2} \Rightarrow \alpha = \ldots$	$\alpha + \sqrt{\alpha^2+1} = \sqrt{3}$ $\sqrt{\alpha^2+1} = \sqrt{3} - \alpha$ $\alpha^2 + 1 = 3 - 2\sqrt{3}\alpha + \alpha^2 \Rightarrow \alpha = \ldots$	M1	1.1b
	$\alpha = \dfrac{\sqrt{3}}{3}$ or $\dfrac{1}{\sqrt{3}}$		A1	2.2a
			(3)	
(b)	Volume = $\pi\int_0^{\frac{1}{2}\ln 3} \sinh^2 y \, dy$		B1	2.5
	$\{\pi\}\int\left(\dfrac{e^y - e^{-y}}{2}\right)^2 dy = \{\pi\}\int\left(\dfrac{e^{2y} - 2 + e^{-2y}}{4}\right) dy$ or $\{\pi\}\int \frac{1}{2}\cosh 2y - \frac{1}{2}\, dy$		M1	3.1a
	$\frac{1}{4}\left(\frac{1}{2}e^{2y} - 2y - \frac{1}{2}e^{-2y}\right)$ or $\frac{1}{4}\sinh 2y - \frac{1}{2}y$		dM1 A1	1.1b 1.1b
	Use limits $y = 0$ and $y = \frac{1}{2}\ln 3$ and subtracts the correct way round		M1	1.1b
	$\dfrac{\pi}{4}\left(\dfrac{4}{3} - \ln 3\right)$ or exact equivalent		A1	1.1b
			(6)	
			(9 marks)	

HOME

8. (a) Given

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad n \in \mathbb{N}$$

show that

$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

(5)

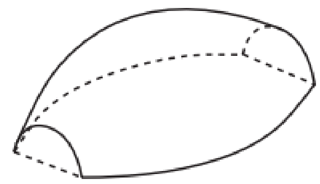


Figure 1

Figure 1 shows a solid paperweight with a flat base.

Figure 2 shows the curve with equation

$$y = H\cos^3\left(\frac{x}{4}\right) \qquad -4 \leqslant x \leqslant 4$$

where H is a positive constant and x is in radians.

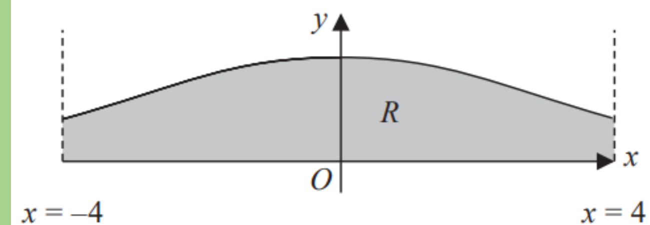


Figure 2

The region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = -4$, the line with equation $x = 4$ and the x-axis.

The paperweight is modelled by the solid of revolution formed when R is rotated **180°** about the x-axis.

Given that the maximum height of the paperweight is 2 cm,

(b) write down the value of H.

(1)

(c) Using algebraic integration and the result in part (a), determine, in cm^3, the volume of the paperweight, according to the model. Give your answer to 2 decimal places.

[*Solutions based entirely on calculator technology are not acceptable.*]

(5)

(d) State a limitation of the model.

(1)

A2 2022 Paper 1

Volumes of Revolution pt. 2

8(a)	$\left(z+\frac{1}{z}\right)^6 = 64\cos^6\theta$	B1
	$\left(z+\frac{1}{z}\right)^6 = z^6 + 6(z^5)\left(\frac{1}{z}\right) + 15(z^4)\left(\frac{1}{z^2}\right) + 20(z^3)\left(\frac{1}{z^3}\right) + 15(z^2)\left(\frac{1}{z^4}\right) + 6(z)\left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$	M1
	$= \left[z^6+\frac{1}{z^6}\right] + 6\left[z^4+\frac{1}{z^4}\right] + 15\left[z^2+\frac{1}{z^2}\right] + 20$	A1
	Uses $z^n + \frac{1}{z^n} = 2\cos n\theta$ $\{64\cos^6\theta\} = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	M1
	$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ * cso	A1 *
		(5)
(b)	$\boldsymbol{H = 2}$	B1
		(1)
(c)	$\text{vol} = \left\{\frac{1}{2}\right\}\pi\int\left(2\cos^3\left(\frac{x}{4}\right)\right)^2 \mathrm{d}x$	B1ft
	$\text{vol} = \{2\pi\}\int\cos^6\left(\frac{x}{4}\right)\mathrm{d}x$ $= \{2\pi\}\int\frac{1}{32}\left(\cos\left(\frac{6x}{4}\right)+6\cos\left(\frac{4x}{4}\right)+15\cos\left(\frac{2x}{4}\right)+10\right)\mathrm{d}x = \ldots$	M1
	$= \{2\pi\}\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3x}{2}\right) + 6\sin(x) + 30\sin\left(\frac{x}{2}\right) + 10x\right)\right]$	A1
	$= 2\times 2\pi\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times 4\right) + 6\sin(4) + 30\sin\left(\frac{4}{2}\right) + (10\times 4)\right) - 0\right] = \ldots$ or $=$ $2\pi\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times 4\right) + 6\sin(4) + 30\sin\left(\frac{4}{2}\right) + (10\times 4)\right) - \frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times -4\right) + 6\sin(-4) + 30\sin\left(-\frac{4}{2}\right) + (10\times -4)\right)\right]$ …	dM1
	$= 24.56$	A1

(d)	The equation of the curve may not be suitable The measurements may not be accurate The paperweight may not be smooth	B1

HOME

Polar Coordinates

A2 SAMs Paper 1

Polar Coordinates

4.

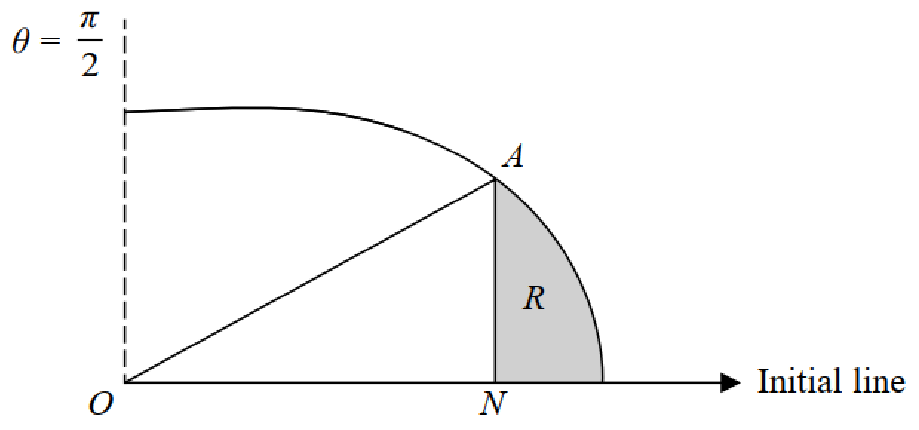


Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 4 + \cos 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{9}{2}$

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

Find the exact area of the shaded region R, giving your answer in the form $p\pi + q\sqrt{3}$ where p and q are rational numbers to be found.

(9)

HOME

Question	Scheme	Marks	AOs
4	$4+\cos 2\theta=\frac{9}{2}\Rightarrow\theta=\ldots$	M1	3.1a
	$\theta=\frac{\pi}{6}$	A1	1.1b
	$\frac{1}{2}\int(4+\cos 2\theta)^2\mathrm{d}\theta=\frac{1}{2}\int\left(16+8\cos 2\theta+\cos^2 2\theta\right)\mathrm{d}\theta$	M1	3.1a
	$\cos^2 2\theta=\frac{1}{2}+\frac{1}{2}\cos 4\theta\Rightarrow A=\frac{1}{2}\int\left(16+8\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$	M1	3.1a
	$=\frac{1}{2}\left[16\theta+4\sin 2\theta+\frac{\sin 4\theta}{8}+\frac{\theta}{2}\right]$	A1	1.1b
	Using limits 0 and their $\frac{\pi}{6}$: $\frac{1}{2}\left[\frac{33\pi}{12}+2\sqrt{3}+\frac{\sqrt{3}}{16}-(0)\right]$	M1	1.1b
	Area of triangle $=\frac{1}{2}(r\cos\theta)(r\sin\theta)=\frac{1}{2}\times\frac{81}{4}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}$	M1	3.1a
	Area of $R=\frac{33\pi}{24}+\frac{33\sqrt{3}}{32}-\frac{81\sqrt{3}}{32}$	M1	1.1b
	$=\frac{11}{8}\pi-\frac{3\sqrt{3}}{2}\left(p=\frac{11}{8},\ q=-\frac{3}{2}\right)$	A1	1.1b
		(9 marks)	

HOME

6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathbf{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

The set of points A is defined by

$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \{z : |z - 4 - 3\mathbf{i}| \leqslant 5\}$$

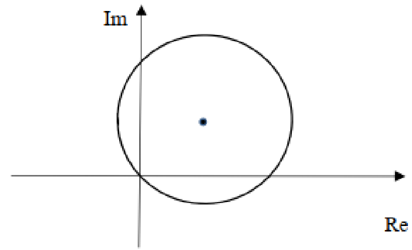
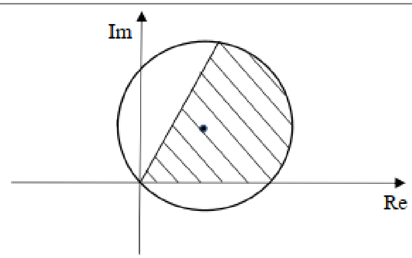
(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

A2 SAMs Paper 2

Polar Coordinates

6(a)(i)		M1	1.1b
		A1	1.1b
(a)(ii)	$\lvert z-4-3\mathrm{i}\rvert=5 \Rightarrow \lvert x+\mathrm{i}y-4-3\mathrm{i}\rvert=5 \Rightarrow (x-4)^2+(y-3)^2=\ldots$	M1	2.1
	$(x-4)^2+(y-3)^2=25$ or any correct form	A1	1.1b
	$(r\cos\theta-4)^2+(r\sin\theta-3)^2=25$ $\Rightarrow r^2\cos^2\theta-8r\cos\theta+16+r^2\sin^2\theta-6r\sin\theta+9=25$ $\Rightarrow r^2-8r\cos\theta-6r\sin\theta=0$	M1	2.1
	$\therefore r=8\cos\theta+6\sin\theta$ *	A1*	2.2a
		(6)	
(b)(i)		B1	1.1b
		B1ft	1.1b
(b)(ii)	$A=\frac{1}{2}\int r^2\,\mathrm{d}\theta=\frac{1}{2}\int (8\cos\theta+6\sin\theta)^2\,\mathrm{d}\theta$ $=\frac{1}{2}\int (64\cos^2\theta+96\sin\theta\cos\theta+36\sin^2\theta)\,\mathrm{d}\theta$	M1	3.1a
	$=\frac{1}{2}\int \big(32(\cos 2\theta+1)+96\sin\theta\cos\theta+18(1-\cos 2\theta)\big)\,\mathrm{d}\theta$	M1	1.1b
	$=\frac{1}{2}\int (14\cos 2\theta+50+48\sin 2\theta)\,\mathrm{d}\theta$	A1	1.1b
	$=\frac{1}{2}\left[7\sin 2\theta+50\theta-24\cos 2\theta\right]_0^{\frac{\pi}{3}}=\frac{1}{2}\left\{\left(\frac{7\sqrt{3}}{2}+\frac{50\pi}{3}+12\right)-(-24)\right\}$	M1	2.1
	$=\frac{7\sqrt{3}}{4}+\frac{25\pi}{3}+18$	A1	1.1b
		(7)	

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A2 2019 Paper 1

Polar Coordinates

3.

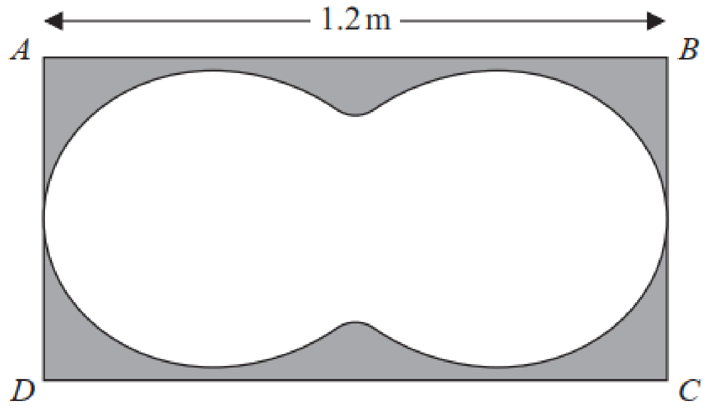


Diagram not to scale

Figure 1

Figure 1 shows the design for a table top in the shape of a rectangle $ABCD$. The length of the table, AB, is 1.2 m. The area inside the closed curve is made of glass and the surrounding area, shown shaded in Figure 1, is made of wood.

The perimeter of the glass is modelled by the curve with polar equation

$$r = 0.4 + a\cos 2\theta \qquad 0 \leqslant \theta < 2\pi$$

where a is a constant.

(a) Show that $a = 0.2$

(2)

Hence, given that $AD = 60\,\text{cm}$,

(b) find the area of the wooden part of the table top, giving your answer in m^2 to 3 significant figures.

(8)

HOME

A2 2019 Paper 1

Polar Coordinates

Question	Scheme	Marks	AOs
3(a)(i)	$2(0.4+a)=1.2$ or $0.4+a=0.6$ or $0.4+a\cos 0=0.6$ $\Rightarrow a=\ldots$	M1	3.4
	$a=0.2$ * cso	A1*	1.1b
		(2)	
(b)	Area of rectangle is $1.2\times0.6(=0.72)$	B1	1.1b
	Area enclosed by curve = $\frac{1}{2}\int (0.4+0.2\cos 2\theta)^2 (\mathrm{d}\theta)$	M1	3.1a
	$(0.4+0.2\cos 2\theta)^2=0.16+0.16\cos 2\theta+0.04\cos^2 2\theta$ $=0.16+0.16\cos 2\theta+0.04\left(\frac{\cos 4\theta+1}{2}\right)$	M1	2.1
	$\frac{1}{2}\int (0.4+0.2\cos 2\theta)^2 \mathrm{d}\theta=\frac{1}{2}\left[0.18\theta+0.08\sin 2\theta+0.005\sin 4\theta(+c)\right]$ $=0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta(+c)$ o.e.	A1ft	1.1b
	Area enclosed by curve = $\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{2\pi}$ or Area enclosed by curve = $2\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{\pi}$ or Area enclosed by curve = $4\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{\pi/2}$	dM1	3.1a
	$=\frac{9}{50}\pi$ or $0.18\pi(=0.5654\ldots)$	A1	1.1b
	Area of wood $=1.2\times0.6-0.18\pi$	M1	1.1b
	$=$ awrt 0.155 (m^2)	A1	1.1b
		(8)	
		(10 marks)	

HOME

3.

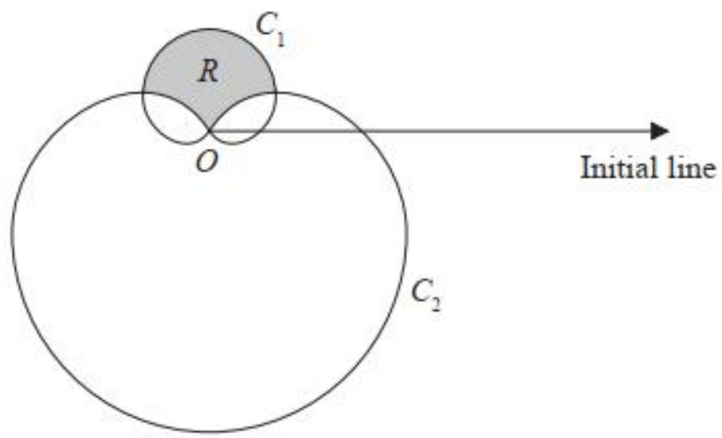


Figure 1

Figure 1 shows a sketch of two curves C_1 and C_2 with polar equations

$$C_1 : r = (1 + \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

$$C_2 : r = 3(1 - \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

The region R lies inside C_1 and outside C_2 and is shown shaded in Figure 1.

Show that the area of R is

$$p\sqrt{3} - q\pi$$

where p and q are integers to be determined.

(9)

A2 2020 Paper 1

Polar Coordinates

Question	Scheme	Marks	AOs
3	$3(1-\sin\theta)=1+\sin\theta \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\ldots$	M1	3.1a
	$\theta=\frac{\pi}{6}\left(\text{or } \frac{5\pi}{6}\right)$	A1	1.1b
	Use of $\frac{1}{2}\int(1+\sin\theta)^2\,\mathrm{d}\theta$ or $\frac{1}{2}\int\{3(1-\sin\theta)\}^2\,\mathrm{d}\theta$	M1	1.1a
	$\left(\frac{1}{2}\right)\int\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta$ $=\left(\frac{1}{2}\right)\int\left[1+2\sin\theta+\sin^2\theta-9+18\sin\theta-9\sin^2\theta\right]\mathrm{d}\theta$ or $\int(1+\sin\theta)^2\,\mathrm{d}\theta=\int(1+2\sin\theta+\sin^2\theta)\,\mathrm{d}\theta$ **and** $\int 9(1-\sin\theta)^2\,\mathrm{d}\theta=9\int(1-2\sin\theta+\sin^2\theta)\,\mathrm{d}\theta$	M1 A1	2.1 1.1b
	$\int\sin^2\theta\,\mathrm{d}\theta=\frac{1}{2}\int(1-\cos2\theta)\,\mathrm{d}\theta\Rightarrow$ $\int\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta=2\sin2\theta-12\theta-20\cos\theta$	M1 A1	3.1a 1.1b
	$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta$ or $A=2\times\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta$ $=\frac{1}{2}\left\{\left(-\sqrt{3}-10\pi+10\sqrt{3}\right)-\left(\sqrt{3}-2\pi-10\sqrt{3}\right)\right\}=\ldots$	DM1	3.1a
	$=9\sqrt{3}-4\pi$	A1	1.1b
		(9)	

HOME

6. The curve C has equation

$$r = a(p + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where a and p are positive constants and $p > 2$

There are exactly four points on C where the tangent is perpendicular to the initial line.

(a) Show that the range of possible values for p is

$$2 < p < 4$$

(5)

(b) Sketch the curve with equation

$$r = a(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi \qquad \text{where } a > 0$$

(1)

John digs a hole in his garden in order to make a pond.

The pond has a uniform horizontal cross section that is modelled by the curve with equation

$$r = 20(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where r is measured in centimetres.

The depth of the pond is 90 centimetres.

Water flows through a hosepipe into the pond at a rate of 50 litres per minute.

Given that the pond is initially empty,

(c) determine how long it will take to completely fill the pond with water using the hosepipe, according to the model. Give your answer to the nearest minute.

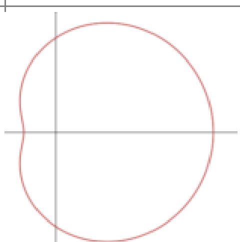
(7)

(d) State a limitation of the model.

(1)

A2 2021 Paper 1

Polar Coordinates

Question	Scheme	Marks	AOs
6(a)	$x = r\cos\theta = a(p+2\cos\theta)\cos\theta$ Leading to $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta(p+2\cos\theta)$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta$ or $x = a(p\cos\theta + 2\cos^2\theta) = a(\cos 2\theta + p\cos\theta + 1)$ leading to $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin 2\theta + \beta\sin\theta$	M1	3.1a
	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a[-2\sin\theta\cos\theta - \sin\theta(p+2\cos\theta)]$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -4a\sin\theta\cos\theta - ap\sin\theta$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2a\sin 2\theta - ap\sin\theta$	A1	1.1b
	$a[-2\sin\theta\cos\theta - \sin\theta(p+2\cos\theta)] = 0$ $\pm a(4\sin\theta\cos\theta + p\sin\theta) = 0$ $a\sin\theta(4\cos\theta + p) = 0$ Either $\sin\theta = 0$ or $\cos\theta = -\frac{p}{4}$	M1	3.1a
	$\sin\theta = 0$ implies 2 solutions (tangents which are perpendicular to the initial line) e.g. $\theta = 0, \pi$	B1	2.2a
	Therefore two solutions to $\cos\theta = -\frac{p}{4}$ are required $-\frac{p}{4} > -1 \Rightarrow p < 4$ as p is a positive constant $2 < p < 4$ *	A1*	2.4
		(5)	
(b)	 Correct shape and position. Condone cusp	B1	2.2a

Question	Scheme	Marks	AOs
(c)	Area = $2\times\frac{1}{2}\int_0^{\pi}[20(3+2\cos\theta)]^2\,\mathrm{d}\theta = 400\int_0^{\pi}(9+12\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta$ or $=\int_0^{\pi}(3600+4800\cos\theta+1600\cos^2\theta)\,\mathrm{d}\theta$ or $\frac{1}{2}\int_0^{2\pi}[20(3+2\cos\theta)]^2\,\mathrm{d}\theta = 200\int_0^{2\pi}(9+12\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta$ or $=\int_0^{2\pi}(1800+2400\cos\theta+800\cos^2\theta)\,\mathrm{d}\theta$	M1	3.4
	$\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta \Rightarrow$ $A = ...\int(9+12\cos\theta+2+2\cos 2\theta)\,\mathrm{d}\theta = \alpha\theta \pm \beta\sin\theta \pm \lambda\sin 2\theta$	M1	3.1a
	$=400[11\theta+12\sin\theta+\sin 2\theta]$ or $=200[11\theta+12\sin\theta+\sin 2\theta]$	A1	1.1b
	Using limits $\theta=0$ and $\theta=\pi$ or $\theta=0$ and $\theta=2\pi$ as appropriate and subtracts the correct way round provided there is an attempt at integration $=400[11\pi-0]=4400\pi=13823.0\,(\text{cm}^2)$ or $=200[11(2\pi)-0]=4400\pi=13823.0\,(\text{cm}^2)$	M1	1.1b
	Volume $=$ area $\times 90 = 396\,000\pi = 1\,244\,070.691\,(\text{cm}^3)$	M1	3.4
	time $=\frac{1\,244\,070.691}{50\,000}=...$ or volume $=1244$ litres therefore time $=\frac{1244}{50}=...$	M1	2.2b
	25 (minutes)	A1	3.2a
		(7)	
(d)	For example Polar equation is not likely to be accurate. Some comment that the sides will not be smooth and draws an appropriate conclusion. The hole may not be uniform depth The pond may leak/ ground may absorb some water	B1	3.5b
		(1)	

Figure 1 shows a sketch of the curve C with equation

$$r = 1 + \tan\theta \qquad\qquad 0 \leqslant \theta < \frac{\pi}{3}$$

Figure 1 also shows the tangent to C at the point A.
This tangent is perpendicular to the initial line.

(a) Use differentiation to prove that the polar coordinates of A are $\left(2, \frac{\pi}{4}\right)$

(4)

The finite region R, shown shaded in Figure 1, is bounded by C, the tangent at A and the initial line.

(b) Use calculus to show that the exact area of R is $\frac{1}{2}(1 - \ln 2)$

(6)

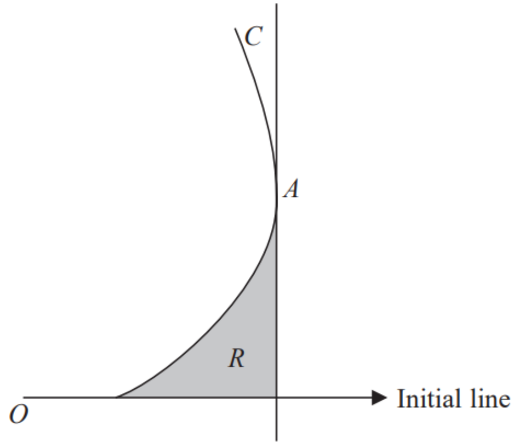


Figure 1

A2 2022 Paper 2

Polar Coordinates

7(a)	$x = r\cos\theta = (1+\tan\theta)\cos\theta = \cos\theta + \sin\theta$ $= \cos\theta + \tan\theta\cos\theta$ $\frac{dx}{d\theta} = \alpha(1+\tan\theta)\sin\theta + \beta\sec^2\theta\cos\theta$ or $\frac{dx}{d\theta} = \alpha\sin\theta + \beta\cos\theta$ $\frac{dx}{d\theta} = \alpha\sin\theta + \beta\sec^2\theta\cos\theta + \delta\tan\theta\sin\theta$	M1
	$\frac{dx}{d\theta} = -(1+\tan\theta)\sin\theta + \sec^2\theta\cos\theta$ or $\frac{dx}{d\theta} = -\sin\theta + \cos\theta$ $\frac{dx}{d\theta} = -\sin\theta + \sec^2\theta\cos\theta - \tan\theta\sin\theta$ or $\frac{dx}{d\theta} = -\sin\theta + \sec\theta - \tan\theta\sin\theta$	A1
	For example $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta + \cos\theta = 0 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \ldots$ $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta + \cos\theta = 0 \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \ldots$ $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta + \cos\theta = \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) = \theta = \ldots$ or $\left\{\frac{dx}{d\theta} =\right\} -(1+\tan\theta)\sin\theta + \sec^2\theta\cos\theta = 0$ $\Rightarrow -\sin\theta - \frac{\sin^2\theta}{\cos\theta} + \frac{1}{\cos\theta} = 0 \Rightarrow -\sin\theta + \frac{1-\sin^2\theta}{\cos\theta} = 0$ $\Rightarrow -\sin\theta + \cos\theta = 0 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \ldots$ or $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta - \tan\theta\sin\theta + \sec\theta = 0$ $\Rightarrow -\frac{1}{2}\sin 2\theta - \sin^2\theta + 1 = 0 \Rightarrow \sin 2\theta + 2\sin^2\theta - 1 = 1$ $\Rightarrow \sin 2\theta - \cos 2\theta = 1 \Rightarrow \sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right) = 1 \Rightarrow \theta = \ldots$ or $\left\{\frac{dx}{d\theta} =\right\} -\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = 0$ $\left\{\frac{dx}{d\theta} =\right\} -\left(1+\tan\left(\frac{\pi}{4}\right)\right)\sin\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 0$ $\left\{\frac{dx}{d\theta} =\right\} -\sin\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = 0$	dM1
	$r = 1 + \tan\left(\frac{\pi}{4}\right) = 2$ therefore $A\left(2, \frac{\pi}{4}\right)$*	A1*
		(4)
	Area bounded by the curve $= \frac{1}{2}\int(1+\tan\theta)^2 \ \{d\theta\}$	M1

HOME

A2 2022 Paper 2

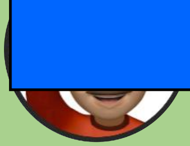
Polar Coordinates

(b)	$=\frac{1}{2}\int(1+2\tan\theta+\tan^2\theta)\ \{d\theta\}$ $=\frac{1}{2}\int(1+2\tan\theta+[\sec^2\theta-1])\ \{d\theta\}=\ldots$	
	$=\frac{1}{2}[2\ln\lvert\sec\theta\rvert+\tan\theta]$ or $\ln\lvert\sec\theta\rvert+\frac{1}{2}\tan\theta$ or $-\ln\cos\theta+\frac{1}{2}\tan\theta$ or $=\frac{1}{2}[-2\ln\lvert\cos\theta\rvert+\tan\theta]$	A1
	$=\frac{1}{2}\left[2\ln\left\lvert\sec\left(\frac{\pi}{4}\right)\right\rvert+\tan\left(\frac{\pi}{4}\right)\right]-\frac{1}{2}[2\ln\lvert\sec(0)\rvert+\tan(0)]$ $=\left(\ln\left\lvert\sec\left(\frac{\pi}{4}\right)\right\rvert+\frac{1}{2}\tan\left(\frac{\pi}{4}\right)\right)-\left(\ln\lvert\sec 0\rvert+\frac{1}{2}\tan 0\right)$ $\left\{=\ln\sqrt{2}+\frac{1}{2}\right\}$	dM1
	Area of triangle $=\frac{1}{2}xy=\frac{1}{2}\left(2\cos\frac{\pi}{4}\right)\left(2\sin\frac{\pi}{4}\right)=\ldots\left\{\frac{1}{2}\times\sqrt{2}\times\sqrt{2}=1\right\}$ The equation of the tangent is $r=\sqrt{2}\sec\theta$ then applies Area bounded of triangle $=\frac{1}{2}\int_0^{\frac{\pi}{4}}(\sqrt{2}\sec\theta)^2\ \{d\theta\}$	M1
	Finds the required area = area of triangle – area bounded by the curve $=1-\left[\ln\sqrt{2}+\frac{1}{2}\right]$ May be seen within an integral $=\frac{1}{2}\int(\sqrt{2}\sec\theta)^2\ \{d\theta\}-\frac{1}{2}\int(1+\tan\theta)^2\ \{d\theta\}$	M1
	$=\frac{1}{2}(1-\ln 2)$ * cso	A1*
		(6)
	Alternative Area bounded by the curve $=\frac{1}{2}\int(1+\tan\theta)^2\ \{d\theta\}$ $=\frac{1}{2}\int(1+2\tan\theta+\tan^2\theta)\ \{d\theta\}$ let $u=\tan\theta\Rightarrow\frac{du}{d\theta}=\sec^2\theta$ Leading to $=\frac{1}{2}\int\frac{(1+2u+u^2)}{1+u^2}\{du\}=\frac{1}{2}\int 1+\frac{2u}{1+u^2}\{du\}=\ldots$	M1
	$\frac{1}{2}[u+\ln(1+u^2)]$	A1
	$\frac{1}{2}[(1+\ln(1+(1)^2))-(0+\ln 1)]$ or $\frac{1}{2}\left[\left(\tan\left(\frac{\pi}{4}\right)+\ln\left(1+\tan^2\left(\frac{\pi}{4}\right)\right)\right)-(\tan(0)+\ln(1+\tan^2(0)))\right]$ $\left\{=\frac{1}{2}\ln 2+\frac{1}{2}\right\}$	dM1
	Area of triangle $=\frac{1}{2}xy=\frac{1}{2}\left(2\cos\frac{\pi}{4}\right)\left(2\sin\frac{\pi}{4}\right)=\ldots\left\{\frac{1}{2}\times\sqrt{2}\times\sqrt{2}=1\right\}$	M1

Finds the required area = area of triangle – area bounded by the curve $=1-\left[\ln\sqrt{2}+\frac{1}{2}\right]$	M1
$=\frac{1}{2}(1-\ln 2)$ *	A1*
	(6)

HOME

Hyperbolic Functions



1. (a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -k < x < k$$

stating the value of the constant k.

(5)

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2-3x}\right)$$

(5)

A2 2019 Paper 2

Hyperbolic Functions

1(a)	$y=\tanh^{-1}(x)\Rightarrow \tanh y=x\Rightarrow x=\dfrac{\sinh y}{\cosh y}=\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ or $\tanh x=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ Allow this to score M1A1		
	$x\left(e^{2y}+1\right)=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x\Rightarrow e^{2y}=\dfrac{1+x}{1-x}$	M1	1.1b
	$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)*$	A1*	2.1
	Note that $e^{2y}(x-1)+x+1=0$ can be solved as a quadratic in e^{y}: $e^{y}=\dfrac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)}=\dfrac{-\sqrt{4(1-x)(x+1)}}{2(x-1)}=\dfrac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $=\dfrac{\sqrt{(x+1)}}{\sqrt{(1-x)}}\Rightarrow y=\dfrac{1}{2}\ln\dfrac{(x+1)}{(1-x)}*$ Score **M1** for an attempt at the quadratic formula to make e^{y} the subject (condone $\pm\sqrt{\ldots}$) and **A1*** for a correct solution that rejects the positive root at some point and deals with the $(x-1)$ bracket correctly		
	$k=1$ or $-1<x<1$	B1	1.1b
		(5)	
(b)	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow \tanh^{-1}(2x)=\ln\sqrt{2-3x}$	M1	3.1a
	$\dfrac{1}{2}\ln\left(\dfrac{1+2x}{1-2x}\right)=\dfrac{1}{2}\ln(2-3x)\Rightarrow\dfrac{1+2x}{1-2x}=2-3x$	M1	2.1
	$6x^{2}-9x+1=0$	A1	1.1b
	$6x^{2}-9x+1=0\Rightarrow x=\ldots$	M1	1.1b
	$x=\dfrac{9-\sqrt{57}}{12}$	A1	3.2a
		(5)	

HOME

1. The curve C has equation

$$y = 31\sinh x - 2\sinh 2x \qquad x \in \mathbb{R}$$

Determine, in terms of natural logarithms, the exact x coordinates of the stationary points of C.

(7)

A2 2020 Paper 2

Hyperbolic Functions

1	$\frac{\mathrm{d}y}{\mathrm{d}x} = 31\cosh x - 4\cosh 2x$	B1	1.1b
	$\frac{\mathrm{d}y}{\mathrm{d}x} = 31\cosh x - 4\left(2\cosh^2 x - 1\right)$	M1	3.1a
	$8\cosh^2 x - 31\cosh x - 4 = 0$	A1	1.1b
	$(8\cosh x + 1)(\cosh x - 4) = 0 \Rightarrow \cosh = \ldots$	M1	1.1b
	$\cosh x = 4, \left(-\frac{1}{8}\right)$	A1	1.1b
	$\cosh x = \alpha \Rightarrow x = \ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $\ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $-\ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ or $\ln\left(\alpha - \sqrt{\alpha^2 - 1}\right)$ or $\frac{e^x + e^{-x}}{2} = 4 \text{Þ } e^{2x} - 8e^x + 7 = 0 \text{Þ } e^x = \ldots \text{Þ } x = \ln(\ldots)$	M1	1.2
	$\pm\ln\left(4 + \sqrt{15}\right)$ or $\ln\left(4 \pm \sqrt{15}\right)$	A1	2.2a
		(7)	

HOME

A2 2022 Paper 1

Hyperbolic Functions

2. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Determine the values of x for which

$$64\cosh^4 x - 64\cosh^2 x - 9 = 0$$

Give your answers in the form $q\ln 2$ where q is rational and in simplest form.

(4)

BICEN MATHS

HOME

A2 2022 Paper 1

Hyperbolic Functions

2	Solves the quadratic equation for $cosh^2 x$ e.g. $(8\,cosh^2 x - 9)(8\,cosh^2 + 1) = 0 \Rightarrow cosh^2 x = \ldots$	M1	3.1a
	$cosh^2 x = \frac{9}{8}\ \left\{-\frac{1}{8}\right\}$	A1	1.1b
	$cosh\,x = \frac{3}{4}\sqrt{2} \Rightarrow x = ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\left(\frac{3}{4}\sqrt{2}\right)^2 - 1}\right]$ **Alternatively** $cosh\,x = \frac{3}{4}\sqrt{2} \Rightarrow \frac{1}{2}(e^x + e^{-x}) \Rightarrow e^{2x} - \frac{3}{2}\sqrt{2}e^x + 1 = 0$ $\Rightarrow e^x = \sqrt{2}$ or $\frac{\sqrt{2}}{2} \Rightarrow x = \ldots$	M1	1.1b
	$x = \pm\frac{1}{2}ln\,2$	A1	2.2a
		(4)	
			(4 marks)

HOME

Differential Equations

5. A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are x grams of the pollutant in the pond after t days,

(a) show that the situation can be modelled by the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 50 - \frac{4x}{200 + t}$$

(4)

(b) Hence find the number of grams of pollutant in the pond after 8 days.

(5)

(c) Explain how the model could be refined.

(1)

A2 SAMs Paper 1

Differential Equations

Question	Scheme	Marks	AOs
5(a)	Pond contains 1000 + 5t litres after t days	M1	3.3
	If x is the amount of pollutant in the pond after t days Rate of pollutant out = $20\times\dfrac{x}{1000+5t}$ g per day	M1	3.3
	Rate of pollutant in = 25 × 2 g = 50g per day	B1	2.2a
	$\dfrac{dx}{dt}=50-\dfrac{4x}{200+t}$ *	A1*	1.1b
		(4)	
(b)	$I=e^{\int\frac{4}{200+t}dt}=(200+t)^4 \Rightarrow x(200+t)^4=\int 50(200+t)^4\,dt$	M1	3.1b
	$x(200+t)^4=10(200+t)^5+c$	A1	1.1b
	$x=0,\ t=0 \Rightarrow\ c=-3.2\times10^{12}$	M1	3.4
	$t=8 \Rightarrow\ x=\ 10(200+8)-\dfrac{3.2\times10^{12}}{(200+8)^4}$	M1	1.1b
	$=370$g	A1	2.2b
		(5)	
(c)	e.g. • The model should take into account the fact that the pollutant does not dissolve throughout the pond upon entry • The rate of leaking could be made to vary with the volume of water in the pond	B1	3.5c
		(1)	
		(10 marks)	

HOME

9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 200\cos t, \qquad t \geqslant 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be $10\,\mathrm{ms}^{-2}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)

9(a)(i)	Weight = mass × g $\Rightarrow m = \dfrac{30000}{g} = 3000$ But mass is in thousands of kg, so $m = 3$	M1	3.3
(ii)	$\dfrac{dx}{dt} = 40\cos t + 20\sin t,\ \dfrac{d^2x}{dt^2} = -40\sin t + 20\cos t$	M1	1.1b
	$3(-40\sin t + 20\cos t) + 4(40\cos t + 20\sin t)$ $+40\sin t - 20\cos t = \ldots$	M1	1.1b
	$= 200\cos t$ so PI is $x = 40\sin t - 20\cos$	A1*	2.1
	or		
	Let $x = a\cos t + b\sin t$ $\dfrac{dx}{dt} = -a\sin t + b\cos t,\ \dfrac{d^2x}{dt^2} = -a\cos t - b\sin t$	M1	1.1b
	$4b - 2a = 200,\ -2b - 4a = 0 \Rightarrow a = \ldots, b = \ldots$	M1	2.1
	$x = 40\sin t - 20\cos t$	A1*	1.1b
(iii)	$3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = -1, -\frac{1}{3}$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t}$	A1	1.1b
	$x = PI + CF$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
		(8)	
(b)	$t = 0, x = 0 \Rightarrow A + B = 20$	M1	3.4
	$x = 0, \dfrac{dx}{dt} = -Ae^{-t} - \frac{1}{3}Be^{-\frac{1}{3}t} + 40\cos t + 20\sin t = 0$ $\Rightarrow A + \frac{1}{3}B = 40$	M1	3.4
	$x = 50e^{-t} - 30e^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
	$t = 9 \Rightarrow x = 33\text{m}$	A1	3.4
		(4)	

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r$$

(a) Show that $\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\dfrac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

HOME

A2 SAMs Paper 2

Differential Equations

7(a)	$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = 10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t}$	M1	2.1
	$10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t} = -0.2f + 0.4\left(10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f\right)$	M1	2.1
	$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$ *	A1*	1.1b
		(3)	
(b)	$m^2 - 0.6m + 0.1 = 0 \Rightarrow m = \frac{0.6 \pm \sqrt{0.6^2 - 4\times 0.1}}{2}$	M1	3.4
	$m = 0.3 \pm 0.1\mathrm{i}$	A1	1.1b
	$f = \mathrm{e}^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$	M1	3.4
	$f = \mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	A1	1.1b
		(4)	
(c)	$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.3\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right) + 0.1\mathrm{e}^{0.3t}\left(B\cos 0.1t - A\sin 0.1t\right)$	M1	3.4
	$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f$ $= \mathrm{e}^{0.3t}\left(\left(3A+B\right)\cos 0.1t + \left(3B - A\right)\sin 0.1t\right) - 2\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	M1	3.4
	$r = \mathrm{e}^{0.3t}\left(\left(A+B\right)\cos 0.1t + \left(B - A\right)\sin 0.1t\right)$	A1	1.1b
		(3)	
(d)(i)	$t = 0, f = 6 \Rightarrow A = 6$	M1	3.1b
	$t = 0, r = 20 \Rightarrow B = 14$	M1	3.3
	$r = \mathrm{e}^{0.3t}\left(20\cos 0.1t + 8\sin 0.1t\right) = 0$	M1	3.1b
	$\tan 0.1t = -2.5$	A1	1.1b
	2019	A1	3.2a
(d)(ii)	3750 foxes	B1	3.4
(d)(iii)	e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible	B1	3.5a
		(7)	
			(17 marks)

HOME

5. A tank at a chemical plant has a capacity of 250 litres. The tank initially contains 100 litres of pure water.

Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt.

It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a valve is opened at the bottom of the tank and the solution in the tank flows out at a rate of 2 litres per minute.

Given that there are S grams of salt in the tank after t minutes,

(a) show that the situation can be modelled by the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - \frac{2S}{100 + t}$$

(4)

(b) Hence find the number of grams of salt in the tank after 10 minutes.

(5)

When the concentration of salt in the tank reaches 0.9 grams per litre, the valve at the bottom of the tank must be closed.

(c) Find, to the nearest minute, when the valve would need to be closed.

(3)

(d) Evaluate the model.

(1)

A2 2019 Paper 1

Differential Equations

5(a)	The tank initially contains 100L. 3 L are entering every minute and 2 L are leaving every minute so overall 1 L increase in volume each minute so the tank contains 100 + t litres after t minutes	M1	
	2 L leave the tank each minute and if there are Sg of salt in the tank, the concentration will be $\frac{S}{100+t}$ g/L so salt leaves the tank at a rate of $2\times\frac{S}{100+t}$ g per minute	M1	
	Salt enters the tank at a rate of $3\times 1g$ per minute	B1	
	$\therefore \frac{dS}{dt}=3-\frac{2S}{100+t}$ * cso	A1*	
		(4)	
(b)	$\frac{dS}{dt}+\frac{2S}{100+t}=3$		
	$I=e^{\int\frac{2}{100+t}dt}=(100+t)^2 \Rightarrow S(100+t)^2=\int 3(100+t)^2 dt$	M1	
	$S(100+t)^2=(100+t)^3\ (+c)$ OR $S(100+t)^2=30\,000t+300t^2+t^3\ (+c)$	A1	
	$t=0,\ S=0 \Rightarrow c=-10^6$	M1	
	$t=10 \Rightarrow S=100+10-\frac{10^6}{(100+10)^2}$	dM1	
	OR $S(100+10)^2=(100+10)^3(+c)\Rightarrow S=...$		
	$=$ awrt 27 (g) or $\frac{3310}{121}$ (g)	A1	2.2b
		(5)	
(c)	Concentration is $\left(100+t-\frac{10^6}{(100+t)^2}\right)\div(100+t)=0.9$ OR $S=0.9\ 100+t \Rightarrow 0.9\ 100+t\ =\ 100+t\ -\frac{10^6}{100+10\ ^2}$ OR $S=0.9\ 100+t \Rightarrow 0.9\ 100+t\ ^3=\ 100+t\ ^3-10^6$	M1	3.4
	$(100+t)^3=10^7 \Rightarrow t=...$ OR $t^3+300t^2+30\,000t-9\,000\,000=0 \Rightarrow t=...$	dM1	1.1b
	$t=$ awrt 115 (minutes)	A1	2.2b
		(3)	
(d)	E.g. • It is unlikely that mixing is instantaneous • The model will only be valid when the tank is not full • When the valve is closed, the model is not valid • It is unlikely that the concentration of salt water entering the tank remains exactly the same	B1	3.5a
		(1)	
			(13 marks)

HOME

8. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.
At time t years, the number of white-clawed crayfish, w, and the number of signal crayfish, s, are modelled by the differential equations

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{5}{2}(w - s)$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2}{5}w - 90\mathrm{e}^{-t}$$

(a) Show that

$$2\frac{\mathrm{d}^2w}{\mathrm{d}t^2} - 5\frac{\mathrm{d}w}{\mathrm{d}t} + 2w = 450\mathrm{e}^{-t}$$

(3)

(b) Find a general solution for the number of white-clawed crayfish at time t years. **(6)**

(c) Find a general solution for the number of signal crayfish at time t years. **(2)**

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of T, giving your answer to 3 decimal places. **(6)**

(e) Suggest a limitation of the model. **(1)**

A2 2019 Paper 1

Differential Equations

8(a)	$\frac{d^2w}{dt^2}=\frac{5}{2}\left(\frac{dw}{dt}-\frac{ds}{dt}\right)$ or $\frac{ds}{dt}=\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}$ o.e.	B1
	$\frac{ds}{dt}=\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}\Rightarrow\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}=\frac{2}{5}w-90e^{-t}$	M1
	$2\frac{d^2w}{dt^2}-5\frac{dw}{dt}+2w=450e^{-t}$ *	A1*
		(3)
(b)	$2m^2-5m+2=0\Rightarrow m=\ldots$	M1
	$m=2,\frac{1}{2}$	A1
	$(w)=Ae^{\alpha t}+Be^{\beta t}$	M1
	$(w)=Ae^{0.5t}+Be^{2t}$	A1
	PI: Try $w=ke^{-t}\Rightarrow\frac{dw}{dt}=-ke^{-t}\Rightarrow\frac{d^2w}{dt^2}=ke^{-t}$ $2ke^{-t}+5ke^{-t}+2ke^{-t}=450e^{-t}\Rightarrow k=\ldots$	M1
	$w=$ 'their C.F.' $+\ 50e^{-t}$ $\left(w=Ae^{0.5t}+Be^{2t}+50e^{-t}\right)$	A1ft
		(6)
(c)	$s=w-\frac{2}{5}\frac{dw}{dt}=Ae^{0.5t}+Be^{2t}+50e^{-t}-\frac{2}{5}\left(\frac{A}{2}e^{0.5t}+2Be^{2t}-50e^{-t}\right)$	M1
	$s=\frac{4A}{5}e^{0.5t}+\frac{B}{5}e^{2t}+70e^{-t}$	A1

(d)	$65=A+B+50,\ 85=\frac{4A}{5}+\frac{B}{5}+70\Rightarrow A=\ldots,B=\ldots$ $(\text{NB } A=20\ \ B=-5)$	M1	
	$w=0\Rightarrow 20e^{0.5t}-5e^{2t}+50e^{-t}=0$	dM1	
	$e^{3t}-4e^{1.5t}-10(=0)$ or a multiple	A1	
	$e^{1.5t}=\frac{4\pm\sqrt{4^2-4\times(1)(-10)}}{2}$	M1	
	$1.5t=\ln\left(\frac{4+\sqrt{56}}{2}\right)$	M1	
	$T=\frac{2}{3}\ln\left(\frac{4+\sqrt{56}}{2}\right)=$ awrt 1.165	A1	
		(6)	
(e)	E.g. • Either population becomes negative which is not possible • When the white-clawed crayfish have died out, the model will not be valid	B1	3.5b
		(1)	

HOME

A2 2019 Paper 2

Differential Equations

5. An engineer is investigating the motion of a sprung diving board at a swimming pool.
Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation

$$4\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 4\frac{\mathrm{d}h}{\mathrm{d}t} + 37h = 0$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.

(2)

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of $55\,\mathrm{cm\,s^{-1}}$.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E.

(8)

(c) Comment on the suitability of the model for large values of t.

(2)

HOME

A2 2019 Paper 2

5(a)	$4m^2+4m+37=0 \Rightarrow m=-\dfrac{1}{2}\pm 3\text{i}$	M1	1.1b
	$h=\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)$	A1	1.1b
		(2)	
(b)	$t=0,\ h=-20 \Rightarrow A=-20$	M1	3.4
	$\dfrac{\text{d}h}{\text{d}t}=-0.5\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)+\text{e}^{-0.5t}\left(-3A\sin 3t+3B\cos 3t\right)$ $t=0,\ \dfrac{\text{d}h}{\text{d}t}=55 \Rightarrow B=\ldots$ (NB $B=15$)	M1	3.4
	$(h=)\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)$	A1	1.1b
	$-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\text{e}^{-0.5t}\left(60\sin 3t+45\cos 3t\right)=0$ or e.g. $-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\dfrac{25\sqrt{37}}{2}\text{e}^{-0.5t}\sin\left(3t+\arctan\dfrac{22}{21}\right)=0$ $\Rightarrow t=\ldots$	M1	3.1b
	$\tan 3t=-\dfrac{22}{21}$ or e.g. $3t+\tan^{-1}\dfrac{22}{21}=0$	A1 **M1 on ePEN**	2.1
	$t=0.778$ s	A1	1.1b
	$h=\text{e}^{-0.5\times\text{"}0.778\text{"}}\left(15\sin\left(3\times\text{"}0.778\text{"}\right)-20\cos\left(3\times\text{"}0.778\text{"}\right)\right)$	dM1	1.1b
	$=16.7$ cm	A1	3.2a
		(8)	
(c)	E.g. considers large values of t in the model for h **or** states that for large values of t, h becomes smaller or becomes zero	M1	3.4
	E.g. • The value of h is very small when t is large and this is likely to be correct (as the displacement of end of the board should get smaller and smaller) • This suggests the model is suitable • This is realistic • This is suitable as the board will tend towards its equilibrium position • When t is large the value of h is never zero so the model is not really appropriate for large values of t	A1 **B1 on ePEN**	3.2b
		(2)	

HOME

5. Two compounds, X and Y, are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations

$$\frac{dx}{dt} = -5x + 10y - 30$$

$$\frac{dy}{dt} = -2x + 3y - 4$$

(a) Show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$

(3)

(b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(6)

(c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

(3)

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for x,

(ii) the particular solution for y.

(4)

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

(e) Explain whether this is supported by the model.

(1)

A2 2020 Paper 1

Differential Equations

5(a)	$\frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10\frac{dy}{dt}$ oe e.g. $\frac{dy}{dt} = \frac{1}{10}\left(\frac{d^2x}{dt^2} + 5\frac{dx}{dt}\right)$	B1	1.1b
	$\frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10(-2x + 3y - 4)$ $= -5\frac{dx}{dt} - 20x + \frac{30}{10}\left(\frac{dx}{dt} + 5x + 30\right) - 40$ Or $\frac{1}{10}\left(\frac{d^2x}{dt^2} + 5\frac{dx}{dt}\right) = -2x + \frac{3}{10}\left(30 + 5x + \frac{dx}{dt}\right) - 4$	M1	2.1
	$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$*	A1*	1.1b
		(3)	
(b)	$m^2 + 2m + 5 = 0 \Rightarrow m = \ldots$	M1	3.4
	$m = -1 \pm 2\mathrm{i}$	A1	1.1b
	$m = \alpha \pm \beta\mathrm{i} \Rightarrow x = e^{\alpha t}(A\cos\beta t + B\sin\beta t) = \ldots$	M1	3.4
	$x = e^{-t}(A\cos 2t + B\sin 2t)$	A1	1.1b
	PI: Try $x = k \Rightarrow 5k = 50 \Rightarrow k = 10$	M1	3.4
	$GS: x = e^{-t}(A\cos 2t + B\sin 2t) + 10$	A1ft	1.1b
		(6)	

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(c)	$\frac{dx}{dt} = e^{-t}\left(2B\cos 2t - 2A\sin 2t\right) - e^{-t}\left(A\cos 2t + B\sin 2t\right)$	B1ft	1.1b
	$(y =)\frac{1}{10}\left(\frac{dx}{dt} + 5x + 30\right) = \ldots$	M1	3.4
	$y = \frac{1}{10}e^{-t}\left(\left(4A + 2B\right)\cos 2t + \left(4B - 2A\right)\sin 2t\right)\right) + 8$	A1	1.1b
		(3)	
(d)	$t = 0, x = 2 \Rightarrow 2 = A + 10 \Rightarrow A = -8$	M1	3.1b
	$t = 0, y = 5 \Rightarrow 5 = \frac{1}{10}\left(2B - 32\right) + 8 \Rightarrow B = 1$	M1	3.3
	$x = e^{-t}\left(\sin 2t - 8\cos 2t\right) + 10$	A1	2.2a
	$y = e^{-t}\left(2\sin 2t - 3\cos 2t\right) + 8$	A1	2.2a
		(4)	
(e)	E.g When $t > 8$, the amount of compound X and the amount of compound Y remain (approximately) constant at 10 and 8 respectively, which suggests that the chemical reaction has stopped. This supports the scientist's claim.	B1	3.5a
		(1)	
			(17 marks)

HOME

7. A sample of bacteria in a sealed container is being studied.

The number of bacteria, P, in thousands, is modelled by the differential equation

$$(1+t)\frac{\mathrm{d}P}{\mathrm{d}t} + P = t^{\frac{1}{2}}(1+t)$$

where t is the time in hours after the start of the study.

Initially, there are exactly 5000 bacteria in the container.

(a) Determine, according to the model, the number of bacteria in the container 8 hours after the start of the study.

(6)

(b) Find, according to the model, the rate of change of the number of bacteria in the container 4 hours after the start of the study.

(4)

(c) State a limitation of the model.

(1)

A2 2020 Paper 1

Differential Equations

7(a)	$(1+t)\dfrac{\mathrm{d}P}{\mathrm{d}t}+P=t^{\frac{1}{2}}(1+t)\Rightarrow\dfrac{\mathrm{d}P}{\mathrm{d}t}+\dfrac{P}{1+t}=t^{\frac{1}{2}}$	B1	1.1b
	$I=\mathrm{e}^{\int\frac{1}{1+t}\mathrm{d}t}=1+t\Rightarrow P(1+t)=\int t^{\frac{1}{2}}(1+t)\mathrm{d}t=\ldots$	M1	3.1b
	$P(1+t)=\dfrac{2}{3}t^{\frac{3}{2}}+\dfrac{2}{5}t^{\frac{5}{2}}+c$	A1	1.1b
	$t=0,\ P=5\Rightarrow c=5$	M1	3.4
	$P=\dfrac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}=\dfrac{\frac{2}{3}8^{\frac{3}{2}}+\frac{2}{5}8^{\frac{5}{2}}+5}{9}=\ldots$	M1	1.1b
	= 10 277 bacteria (allow awrt 10 300)	A1	2.2b
		(6)	
(b)	$P=\dfrac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}\Rightarrow\dfrac{\mathrm{d}P}{\mathrm{d}t}=\dfrac{(1+t)\left(t^{\frac{1}{2}}+t^{\frac{3}{2}}\right)-\left(\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5\right)}{(1+t)^2}$ Alt: $P+(1+t)\dfrac{\mathrm{d}P}{\mathrm{d}t}=t^{\frac{1}{2}}+t^{\frac{3}{2}}\Rightarrow\dfrac{\mathrm{d}P}{\mathrm{d}t}=\dfrac{t^{\frac{1}{2}}+t^{\frac{3}{2}}-\dfrac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}}{(1+t)}$	M1 A1ft	3.4 1.1b
	$\left(\dfrac{\mathrm{d}P}{\mathrm{d}t}\right)_{t=1}=\dfrac{\mathrm{d}P}{\mathrm{d}t}=\dfrac{5\times10-\left(\frac{16}{3}+\frac{64}{5}+5\right)}{(5)^2}=\dfrac{403}{375}$	dM1	3.1a
	$\dfrac{403}{375}\times1000=\dfrac{3224}{3}(=awrt\,1070)$ bacteria per hour	A1	3.2a
		(4)	

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	(b) Alternative:	
	$P=\dfrac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}=\dfrac{\frac{16}{3}+\frac{64}{5}+5}{(1+4)}$	M1
	$=\dfrac{347}{75}$	A1ft
	$(1+t)\dfrac{\mathrm{d}P}{\mathrm{d}t}+P=t^{\frac{1}{2}}(1+t)\Rightarrow 5\dfrac{\mathrm{d}P}{\mathrm{d}t}+\dfrac{347}{75}=2\times 5\Rightarrow\dfrac{\mathrm{d}P}{\mathrm{d}t}=\dfrac{403}{375}$	dM1
	$\dfrac{403}{375}\times 1000=\dfrac{3224}{3}(=1075)$ bacteria per hour	A1
		(4)
(c)	E.g. • The number of bacteria increases indefinitely which is not realistic	B1
		(1)

3. A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination.
The concentration of antibodies, x, measured in micrograms (μg) per millilitre (ml) of blood, is modelled by the differential equation

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$$

where t is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

(4)

Initially,

- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 μg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

(8)

A second dose of the vaccine has to be given to try to ensure that it is fully effective.
It is only safe to give the second dose if the concentration of antibodies in the bloodstream of the patient is less than 5 μg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

(2)

A2 2020 Paper 2

Differential Equations

3(a)	$100m^2+60m+13=0 \Rightarrow m=-0.3\pm 0.2\mathrm{i}$	M1	1.1b
	$x=\mathrm{e}^{-0.3t}\left(A\cos 0.2t+B\sin 0.2t\right)$	A1	1.1b
	PI: $x=2$	B1	1.1b
	$x=\mathrm{e}^{-0.3t}\left(A\cos 0.2t+B\sin 0.2t\right)+2$	A1ft	2.2a
		(4)	
(b)	$t=0,\ x=0 \Rightarrow A=-2$	M1	3.4
	$\frac{\mathrm{d}x}{\mathrm{d}t}=-0.3\mathrm{e}^{-0.3t}\left(-2\cos 0.2t+B\sin 0.2t\right)+\mathrm{e}^{-0.3t}\left(0.4\sin 0.2t+0.2B\cos 0.2t\right)$ $t=0,\ \frac{\mathrm{d}x}{\mathrm{d}t}=10 \Rightarrow B=\ldots$ (NB $B=47$)	M1	3.4
	$x=\mathrm{e}^{-0.3t}\left(47\sin 0.2t-2\cos 0.2t\right)+2$	A1	1.1b
	$-0.3\mathrm{e}^{-0.3t}\left(47\sin 0.2t-2\cos 0.2t\right)+\mathrm{e}^{-0.3t}\left(9.4\cos 0.2t+0.4\sin 0.2t\right)=0$ $\Rightarrow t=\ldots$ or $x=\sqrt{2213}\mathrm{e}^{-0.3t}\sin(0.2t-0.0425)+2$ Þ $\frac{\mathrm{d}x}{\mathrm{d}t}=-0.3\sqrt{2213}\mathrm{e}^{-0.3t}\sin(0.2t-0.0425)$ $+0.2\sqrt{2213}\mathrm{e}^{-0.3t}\cos(0.2t-0.0425)$ Þ $t=\ldots$	M1	3.1b
	$\tan 0.2t=\frac{100}{137} \Rightarrow 0.2t=0.630\ldots$ or $\tan(0.2t-0.0425)=\frac{2}{3}$ Þ $0.2t=0.630$	M1	2.1
	$t=3.15\ldots$ weeks	A1	1.1b
	$x=\mathrm{e}^{-0.3\times\text{"}3.15\ldots\text{"}}\left(47\sin(0.2\times\text{"}3.15\ldots\text{"})-2\cos(0.2\times\text{"}3.15\ldots\text{"})\right)+2$	M1	3.4
	= awrt 12.1 {µg/ml}	A1	3.2a
		(8)	
(c)	$t=10 \Rightarrow x=\mathrm{e}^{-3}\left(47\sin(2)-2\cos(2)\right)+2=4.16\ldots$	M1	3.4
	The model suggests that it would be safe to give the second dose	A1ft	2.2a
		(2)	
			(14 marks)

6. A tourist decides to do a bungee jump from a bridge over a river.
One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.
The tourist jumps off the bridge.

At time t seconds after the tourist reaches their lowest point, their vertical displacement is x metres above a fixed point 30 metres vertically above the river.

When $t = 0$

- $x = -20$
- the velocity of the tourist is $0\,\mathrm{m\,s^{-1}}$
- the acceleration of the tourist is $13.6\,\mathrm{m\,s^{-2}}$

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the differential equation

$$5k\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + 17x = 0 \qquad t \geqslant 0$$

where k is a positive constant.

(a) Determine the value of k

(2)

(b) Determine the particular solution to the differential equation.

(7)

(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point.

(2)

(d) Give a limitation of the model.

(1)

Question	Scheme	Marks	AOs
6(a)	$5k(13.6)+2k(0)+17(-20)=0 \Rightarrow k=\ldots$	M1	3.3
	$k=5$	A1	1.1b
		(2)	
(b)	Solves their $25m^2+10m+17=0 \Rightarrow m=\ldots$	M1	3.1b
	$m=-0.2\pm0.8\mathrm{i}$	A1	1.1b
	$x=\mathrm{e}^{-0.2t}\left(A\cos0.8t+B\sin0.8t\right)$	A1ft	1.1b
	$t=0,\ x=-20 \Rightarrow A=\ldots\ (=-20)$	M1	3.4
	$\frac{\mathrm{d}x}{\mathrm{d}t}=-0.2\mathrm{e}^{-0.2t}\left(A\cos0.8t+B\sin0.8t\right)$ $+\mathrm{e}^{-0.2t}\left(-0.8A\sin0.8t+0.8B\cos0.8t\right)$	M1	1.1b
	$t=0\ \frac{\mathrm{d}x}{\mathrm{d}t}=0 \Rightarrow -0.2A+0.8B=0 \Rightarrow B=\ldots\ (=-5)$	dM1	3.4
	$x=\mathrm{e}^{-0.2t}\left(-20\cos0.8t-5\sin0.8t\right)$ o.e.	A1	1.1b
		(7)	
(c)	Vertical height $=30+\left[\mathrm{e}^{-0.2\times15}\left(-20\cos(0.8\times15)-5\sin(0.8\times15)\right)\right]$	M1	3.4
	Vertical height = awrt 29.3 m	A1	2.2b
		(2)	
(d)	For example It is unlikely that the rope will remain taut The model predicts the tourist will continue to move up and down, (but in fact they will lose momentum) The tourist is modelled as a particle	B1	3.5b
		(1)	
			(12 marks)

HOME

8. Two different colours of paint are being mixed together in a container.

The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.

Initially the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.

The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second
- adding blue paint to the container at a rate of 1 litre per second
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let r litres be the amount of red paint in the container at time t seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{dr}{dt} = 2 - \frac{r}{\alpha}$$

where α is a positive constant to be determined.

(2)

(b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model. Give your answer to the nearest second.

(6)

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red paint and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer.

(1)

Question	Scheme		Marks	AOs
8(a)	Volume of paint = 30 litres therefore Rate of paint out $= 3 \times \frac{r}{30}$ litres per second		M1	3.3
	$\frac{\mathrm{d}r}{\mathrm{d}t} = 2 - \frac{r}{10}$		A1	1.1b
			(2)	
(b)	Rearranges $\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{r}{10} = 2$ and attempts integrating factor IF $= \mathrm{e}^{\int \frac{1}{10}\mathrm{d}t} = \ldots$	Separates the variables $\int \frac{1}{20-r}\,\mathrm{d}r = \frac{1}{10}\mathrm{d}t$ $\Rightarrow \ldots$	M1	3.1a
	$r\mathrm{e}^{\frac{t}{10}} = \int 2\mathrm{e}^{\frac{t}{10}}\,\mathrm{d}t \Rightarrow r\mathrm{e}^{\frac{t}{10}} = \lambda \mathrm{e}^{\frac{t}{10}}(+c)$	Integrates to the form $\lambda \ln(20-r) = \frac{1}{10}t(+c)$	M1	1.1b
	$r\mathrm{e}^{\frac{t}{10}} = 20\mathrm{e}^{\frac{t}{10}} + c$	$-\ln(20-r) = \frac{1}{10}t + c$	A1ft	1.1b
	$t = 0,\ r = 10 \Rightarrow c = \ldots$		M1	3.4
	$r = \frac{20\mathrm{e}^{\frac{t}{10}} - 10}{\mathrm{e}^{\frac{t}{10}}} = 15$ rearranges to achieve $\mathrm{e}^{\frac{t}{10}} = \alpha$ and solves to find a value for t or $r = 20 - 10\mathrm{e}^{-\frac{t}{10}} = 15$ rearranges to achieve $\mathrm{e}^{-\frac{t}{10}} = \beta$ and solves to find a value for t	$-\ln(20-15) = \frac{1}{10}t - \ln 10$ Leading to a value for t	M1	3.4
	$t =$ awrt 7 seconds		A1	2.2b
			(6)	
(c)	The model predicts 7 seconds but it actually takes 9 seconds so (over) 2 seconds out (over 20%), therefore it is not a good model		B1ft	3.5a
			(1)	
				(9 marks)

BICEN MATHS

3. (a) Determine the general solution of the differential equation

$$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x = \mathrm{e}^{2x} \cos^2 x$$

giving your answer in the form $y = \mathrm{f}(x)$

(3)

Given that $y = 3$ when $x = 0$

(b) determine the smallest positive value of x for which $y = 0$

(3)

3(a)	$\frac{dy}{dx} + y\tan x = e^{2x}\cos x$ $\text{IF} = e^{\int \tan x\, dx} = e^{\ln \sec x} = \sec x \Rightarrow \sec x\frac{dy}{dx} + y\sec x\tan x = e^{2x}$ $\Rightarrow y\sec x = \int e^{2x}\ dx$	M1	3.1a
	$y\sec x = \frac{1}{2}e^{2x}(+c)$	A1	1.1b
	$y = \left(\frac{1}{2}e^{2x} + c\right)\cos x$	A1	1.1b
		(3)	
(b)	$x = 0,\ y = 3 \Rightarrow c = \ldots\{2.5\}$	M1	3.1a
	$y = \left(\frac{1}{2}e^{2x} + \frac{5}{2}\right)\cos x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \ldots$	M1	1.1b
	$x = \frac{\pi}{2}$	A1	1.1b
		(3)	
			(6 marks)

A2 2022 Paper 1

Differential Equations

The motion of a pendulum, shown in Figure 3, is modelled by the differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

where θ is the angle, in radians, that the pendulum makes with the downward vertical, t seconds after it begins to move.

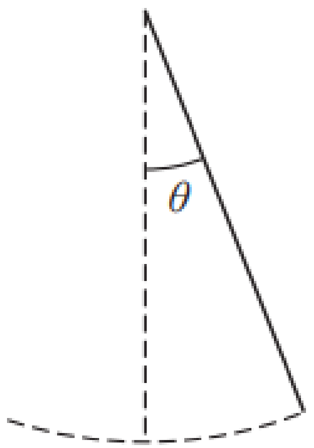

Figure 3

(a) (i) Show that a particular solution of the differential equation is

$$\theta = \frac{1}{12}t\sin 3t$$

(4)

(ii) Hence, find the general solution of the differential equation.

(4)

Initially, the pendulum

- makes an angle of $\frac{\pi}{3}$ radians with the downward vertical
- is at rest

Given that, 10 seconds after it begins to move, the pendulum makes an angle of α radians with the downward vertical,

(b) determine, according to the model, the value of α to 3 significant figures.

(4)

Given that the true value of α is 0.62

(c) evaluate the model.

(1)

The differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion.

(1)

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A2 2022 Paper 1

Differential Equations

10(a)(i)	$\frac{d\theta}{dt} = \alpha \sin 3t + \beta t \cos 3t$ and $\frac{d^2\theta}{dt^2} = \delta \cos 3t + \gamma t \sin 3t$	Let $\theta = \lambda t \sin 3t$ $\frac{d\theta}{dt} = \alpha \sin 3t + \beta t \cos 3t$ and $\frac{d^2\theta}{dt^2} = \delta \cos 3t + \gamma t \sin 3t$	M1
	$\frac{d\theta}{dt} = \frac{1}{12}\sin 3t + \frac{1}{4}t\cos 3t$ and $\frac{d^2\theta}{dt^2} = \frac{1}{4}\cos 3t + \frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t$ $= \frac{1}{2}\cos 3t - \frac{3}{4}t\sin 3t$	$\frac{d\theta}{dt} = \lambda \sin 3t + 3\lambda t\cos 3t$ and $\frac{d^2\theta}{dt^2} = 3\lambda\cos 3t + 3\lambda\cos 3t - 9\lambda t\sin 3t$ $= 6\lambda\cos 3t - 9\lambda t\sin 3t$	A1
	$\frac{1}{2}\cos 3t - \frac{3}{4}t\sin 3t + 9\left(\frac{1}{12}t\sin 3t\right) = \ldots$	$6\lambda\cos 3t - 9\lambda t\sin 3t + 9(\lambda t\sin 3t) = \frac{1}{2}\cos 3t \Rightarrow \lambda = \ldots$	dM1
	$= \frac{1}{2}\cos 3t$ so PI is $\theta = \frac{1}{12}t\sin 3t$ *	$\theta = \frac{1}{12}t\sin 3t$ *	A1*
			(4)
(a)(ii)	$m^2 + 9 = 0 \Rightarrow m = \pm 3i$		M1
	$\theta = A\cos 3t + B\sin 3t$		A1
	$(\theta =) CF + PI$		dM1
	$\theta = A\cos 3t + B\sin 3t + \frac{1}{12}t\sin 3t$		A1
			(4)
(b)	$t = 0,\ \theta = \frac{\pi}{3} \Rightarrow A = \ldots\left\{\frac{\pi}{3}\right\}$		M1
	$t = 0,\ \frac{d\theta}{dt} = -3A\sin 3t + 3B\cos 3t + \frac{1}{12}\sin 3t + \frac{1}{4}t\cos 3t = 0 \Rightarrow B = \ldots\{0\}$		M1
	$\alpha = \frac{\pi}{3}\cos(3\times 10) + \frac{1}{12}(10)\sin(3\times 10) = \ldots$		ddM1
	$\alpha = \pm$awrt 0.662		A1
			(4)
(c)	0.662 is close to 0.62 so a good model (at $t = 10$)		B1ft
			(1)
(d)	$\frac{d^2\theta}{dt^2} + 9\theta = 0$ oe		B1
			(1)

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