Chapter 5: Volumes of Revolution

This chapter concerns how we can find the volume of a solid when a curve is 'revolved around either the x or y axis.

1:: Find the volume when a curve is rotated around the x-axis.

"Find the volume of the solid formed by rotating the curve with equation $y = x^3$ around the x-axis, between the lines $x = 1$ and $x = 2$."

2:: Find the volume when a curve is rotated around the y-axis.

"Find the volume of the solid formed by rotating the curve with equation $y = \sqrt{x+1}$ around the x-axis, between the lines $y = 1$ and $y = 2$."

3:: Find more complex volumes by adding/subtracting.

"An area is bound between the lines with equations $y = \sqrt{x}$ and $y = x$. This area is rotated 360° around the x-axis. Find the volume of the resulting solid."

Area under a graph

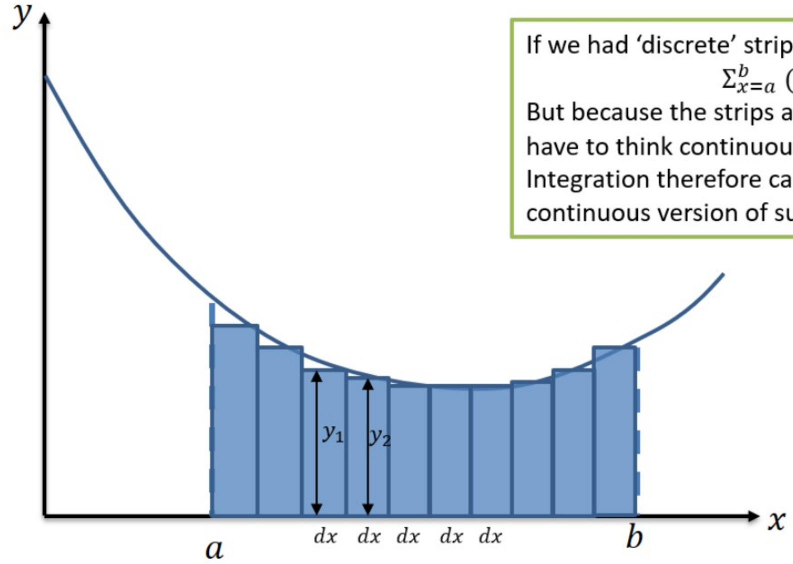
$\int_a^b y\,dx$ gives the area bounded between $y = f(x)$, $x = a$, $x = b$ and the x-axis. Why?

If we split up the area into thin rectangular strips, each with width dx and each with height the $y = f(x)$ for that particular value of x. Each has area $f(x) \times dx$.

If we had 'discrete' strips, the total area would be:

$$\Sigma_{x=a}^{b} \left(f(x)\,dx\right)$$

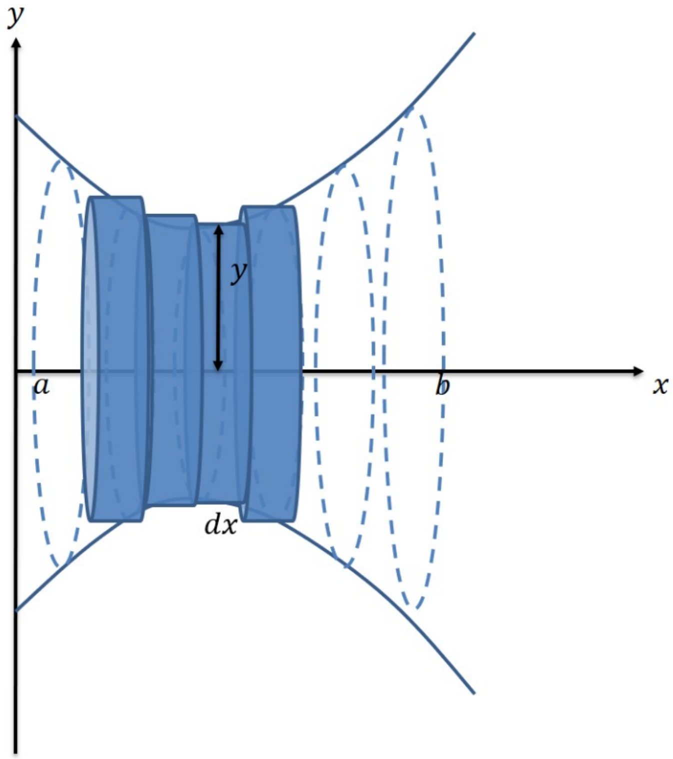
But because the strips are infinitely small and we have to think continuously, we use $\int$ instead of Σ. Integration therefore can be thought of as a continuous version of summation.

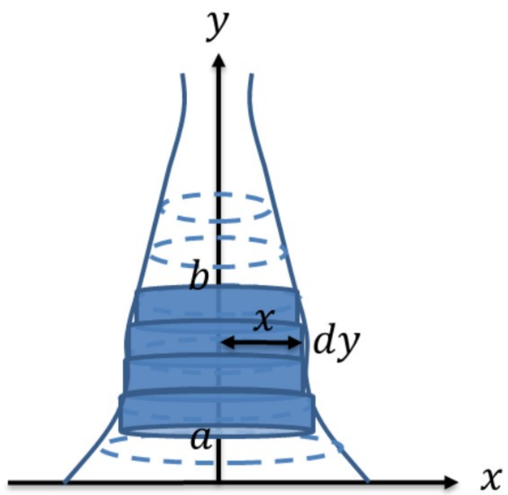


https://www.desmos.com/calculator/wiuoggdua4

Volumes of Revolution

Suppose we spun the line $y = f(x)$ about the x axis to form a solid (known as a *volume of revolution*):



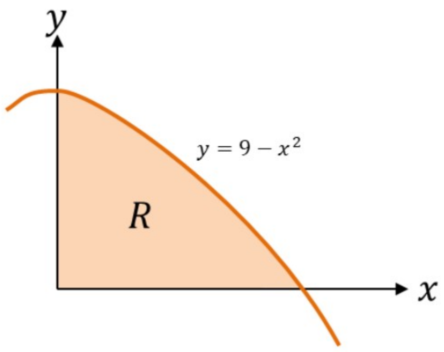


To revolve instead around the y-axis, we simply swap the roles of the x and y axes!

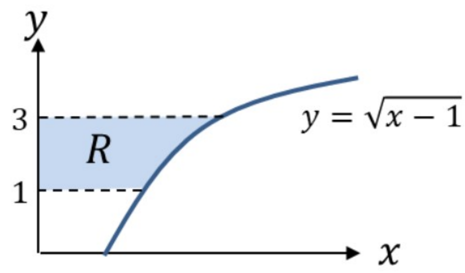
If the curve is revolved around y-axis: $V = \pi \int x^2 \, dy$

Which of these vases have NOT been created like a volume of revolution?

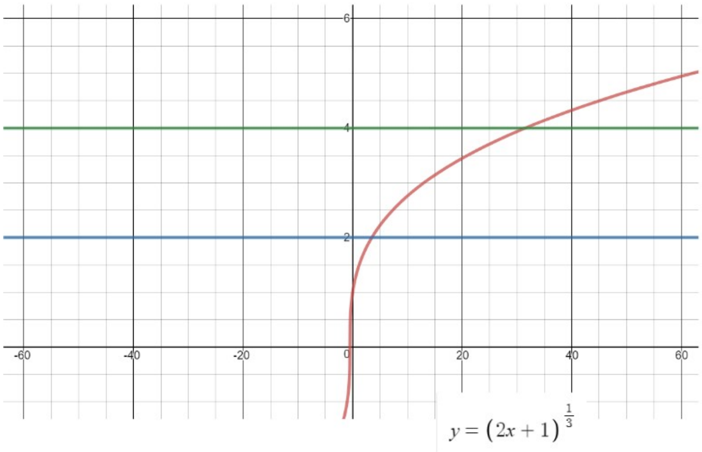
The diagram shows the region R which is bounded by the x-axis, the y-axis and the curve with equation $y = 9 - x^2$. The region is rotated through $360°$ about the x-axis. Find the exact volume of the solid generated.



The diagram shows the curve with equation $y = \sqrt{x-1}$. The region R is bounded by the curve, the y-axis and the lines $y = 1$ and $y = 3$. The region is rotated through $360°$ about the y-axis. Find the volume of the solid generated.



A curve has equation $y = \sqrt[3]{2x+1}$. The region R is bounded by the curve, the y-axis and the lines $y = 2$ and $y = 4$. The region is rotated through $360°$ about the y-axis. Find the volume of the solid generated.



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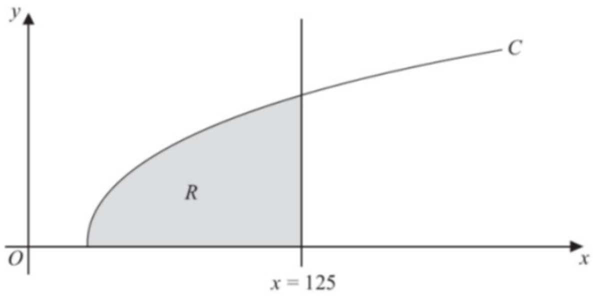
$$y = \left(x^{\frac{2}{3}} - 9\right)^{\frac{1}{2}}$$



The finite region R which is bounded by the curve C, the x-axis and the line $x = 125$ is shown shaded in Figure 3. This region is rotated through 360° about the x-axis to form a solid of revolution.

Use calculus to find the exact value of the volume of the solid of revolution. **(5)**

Ex 5A/B

When the curve isn't fully rotated...

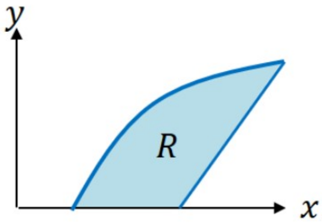
9. $$f(x) = 2x^{\frac{1}{3}} + x^{-\frac{2}{3}} \qquad x > 0$$

The finite region bounded by the curve $y = f(x)$, the line $x = \dfrac{1}{8}$, the x-axis and the line $x = 8$ is rotated through θ radians about the x-axis to form a solid of revolution.

Given that the volume of the solid formed is $\dfrac{461}{2}$ units cubed, use algebraic integration to find the angle θ through which the region is rotated.

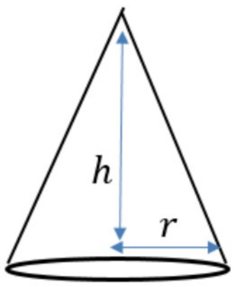
(8)

Adding and Subtracting Volumes

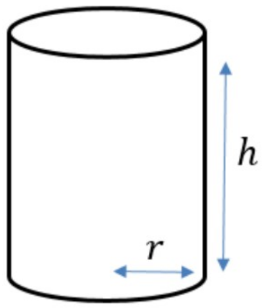


Suppose we wanted to revolve the following area around the x-axis. What strategy might we use to find the volume of this resulting solid?

GCSE Reminders:

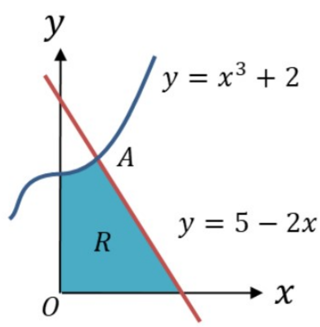


$$V = \frac{1}{3}\pi r^2 h$$



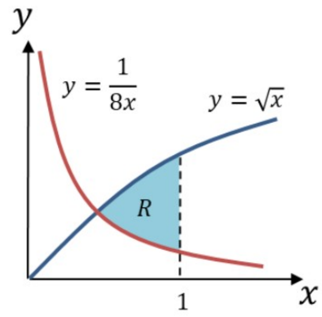
$$V = \pi r^2 h$$

The region R is bounded by the curve with equation $y = x^3 + 2$, the line $y = 5 - 2x$ and x and y-axes.
(a) Verify that the coordinates of A are $(1,3)$.
A solid is created by rotating the region $360°$ about the x-axis.
(b) Find the volume of this solid.

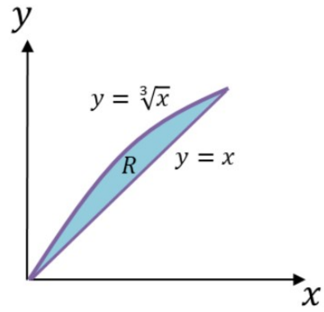


Volumes by Subtraction

The diagram shows the region R bounded by the curves with equations $y = \sqrt{x}$ and $y = \frac{1}{8x}$ and the line $x = 1$.
The region is rotated through $360°$ about the x-axis. Find the exact volume of the solid generated.



The area between the lines with equations $y = x$ and $y = \sqrt[3]{x}$, where $x \geq 0$ is rotated $360°$ about the x-axis. Determine the volume of the solid generated.



Ex 5C

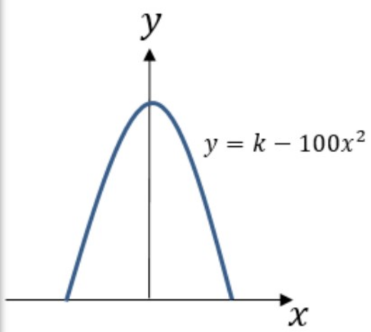
Modelling

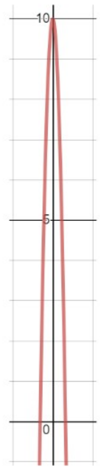
A manufacturer wants to cast a prototype for a new design for a pen barrel out of solid resin. The shaded region shown in the diagram is used as a model for the cross-section of the pen barrel. The region is bounded by the x-axis and the curve with equation $y = k - 100x^2$, and will be rotated around the y-axis. Each unit on the coordinate axes represents 1cm.

(a) Suggest a suitable value for k.

(b) Use your value of k to estimate the volume of resin needed to make the prototype.

(c) State one limitation of this model.





Ex 5D

Figure 1 shows the central cross-section *AOBCD* of a circular bird bath, which is made of concrete. Measurements of the height and diameter of the bird bath, and the depth of the bowl of the bird bath have been taken in order to estimate the amount of concrete that was required to make this bird bath.

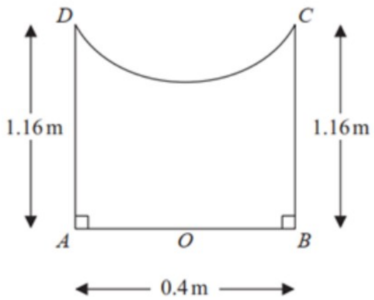


Figure 1

Using these measurements, the cross-sectional curve *CD*, shown in Figure 2, is modelled as a curve with equation

$$y = 1 + kx^2 \qquad -0.2 \leqslant x \leqslant 0.2$$

where k is a constant and where O is the fixed origin.

The height of the bird bath measured 1.16 m and the diameter, *AB*, of the base of the bird bath measured 0.40 m, as shown in Figure 1.

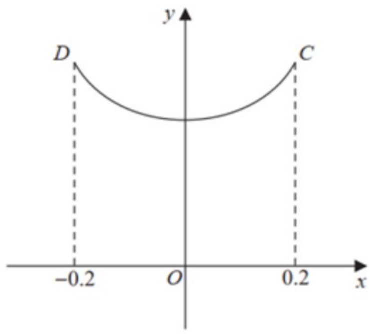


Figure 2

(a) Suggest the maximum depth of the bird bath.

(1)

(b) Find the value of k.

(2)

(c) Hence find the volume of concrete that was required to make the bird bath according to this model. Give your answer, in m^3, correct to 3 significant figures.

(7)

(d) State a limitation of the model.

(1)

It was later discovered that the volume of concrete used to make the bird bath was $0.127\,m^3$ correct to 3 significant figures.

(e) Using this information and the answer to part (c), evaluate the model, explaining your reasoning.

(1)

Limitations

a pen barrel made out of plastic resin...
'measurements may not be accurate'
'equation of curve may not fit perfectly'
'some resin may have been wasted in production'
'there may be air bubbles in the mould'

a bird bath modelled in concrete...
'measurements may not be accurate'
'inside surface may not be smooth'
'some concrete may have been wasted in production'

a glass bottle...
'measurements may not be accurate'
'equation of curve may not fit perfectly'
'the bottom of the bottle may not be flat'
'the glass may not be smooth'
'the thickness of the glass may not have been considered'

Exam Questions

- Check units - they love to switch from cm to mm part way through

- Make sure you decide whether it is about x-axis or y-axis

- Carefully consider what pieces you'll split it into

- Limitations are often asked about

Good questions to try:
Review Exercise 1, Q38, Q39

8.

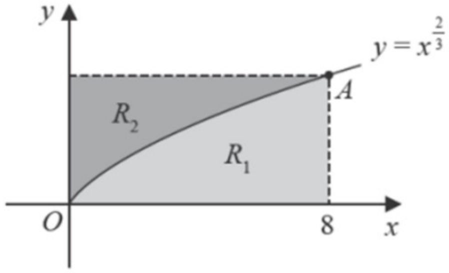


Figure 1

Figure 1 shows a sketch of the curve with equation $y = x^{\frac{2}{3}}, x \geqslant 0$

The curve passes through the point A with x coordinate 8
The region R_1 is bounded by the curve, the vertical line passing through A and the x-axis.
The region R_2 is bounded by the curve, the horizontal line passing through A and the y-axis.

The solid V_1 is formed by rotating the region R_1 through 360° about the x-axis.
The solid V_2 is formed by rotating the region R_2 through 360° about the y-axis.

(a) Show that the exact volume of the solid V_1 is $\dfrac{384\pi}{7}$

(4)

The solids V_1 and V_2 are placed in an empty container. A solid is selected at random and then replaced in the container. This is carried out 10 times.

Given that the probability of selecting each type of solid is proportional to its volume,

(b) find, to 4 decimal places, the probability that the solid V_2 is selected exactly 8 times.

(7)

A mathematics student is modelling the profile of a glass bottle of water. Figure 1 shows a sketch of a central vertical cross-section *ABCDEFGHA* of the bottle with the measurements taken by the student.

The horizontal cross-section between *CF* and *DE* is a circle of diameter 8 cm and the horizontal cross-section between *BG* and *AH* is a circle of diameter 2 cm.

The student thinks that the curve *GF* could be modelled as a curve with equation

$$y = ax^2 + b \qquad 1 \leqslant x \leqslant 4$$

where a and b are constants and O is the fixed origin, as shown in Figure 2.

(a) Find the value of a and the value of b according to the model.
(2)

(b) Use the model to find the volume of water that the bottle can contain.
(7)

(c) State a limitation of the model.
(1)

The label on the bottle states that the bottle holds approximately 750 cm^3 of water.

(d) Use this information and your answer to part (b) to evaluate the model, explaining your reasoning.
(1)

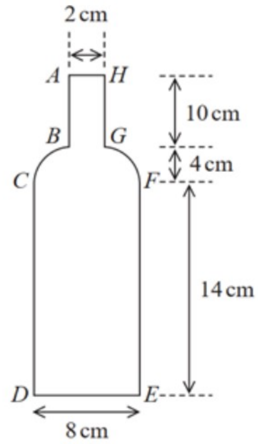


Figure 1

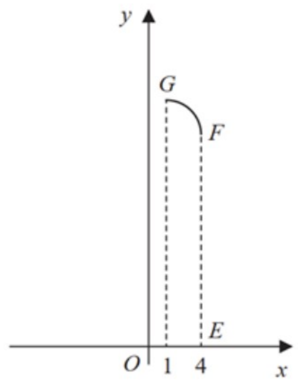


Figure 2