Chapter 4 - Roots of Polynomials

This chapter is about the **underlying relationship** between the **coefficients** of a polynomial (e.g. the a, b, c in $ax^2 + bx + c$) and the **roots** of a polynomial (i.e. the values of x which make the polynomial 0).

1:: Use relationships between coefficients and roots of a quadratic, cubic or quartic equation.

"Given that $kx^2 + (k-3)x - 2 = 0$, find the value of k if the sum of the roots is 4."

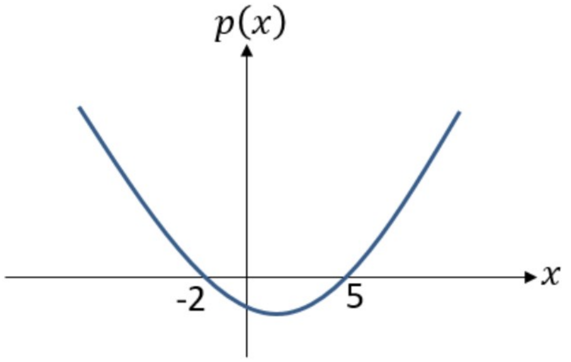
2:: Find the value of expressions based on the roots of a polynomial.

"Without explicitly finding the roots of $x^3 - 3x^2 - 3x + 1$, determine the sum of the squares of its roots."

3:: Find the new polynomial when the roots undergo a linear transformation.

"The quartic equation $x^4 - 3x^3 + 15x + 1 = 0$ has roots α, β, γ and δ. Find the equation with roots $(2\alpha + 1), (2\beta + 1), (2\gamma + 1)$ and $(2\delta + 1)$."

Introduction



$\Longleftrightarrow$ $p(x) = x^2 - 3x - 10$

The purpose of this chapter is to understand the underlying relationship between the **roots** of a polynomial, and the **coefficients** of each term.

What do you notice about:

The **sum** of the roots, i.e. $-2 + 5 = 3$?

It's the negation of the coefficient of the x term

The **product** of the roots, i.e. $-2 \times 5 = -10$?

It's the constant term in the polynomial

This is not yet exactly surprising: At GCSE you found the factorisation of a quadratic by finding two numbers that added to give the middle coefficient and multiplied to give the last number; this in turn allowed you to find the roots...

Let's more formally determine this relationship:

If α and β are the roots of a quadratic $ax^2 + bx + c$ then

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

✏ If α and β are roots of the equation $ax^2 + bx + c = 0$, then:

- **Sum** of roots: $\alpha + \beta = -\frac{b}{a}$
- **Product** of roots: $\alpha\beta = \frac{c}{a}$

Does this generalise to higher-order polynomials?

We will see there are very similar relationship between roots of higher order polynomials, and their coefficients:

Polynomial	Sum of roots	Sum of possible products of pairs of roots	Sum of products of triples
Quadratic $ax^2 + bx + c$ (Roots: α, β)	$\alpha + \beta = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$	$\alpha\beta = \frac{\boldsymbol{c}}{\boldsymbol{a}}$	N/A
Cubic $ax^3 + bx^2 + cx + d$ (Roots: α, β, γ)	$\alpha + \beta + \gamma = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$	$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{\boldsymbol{c}}{\boldsymbol{a}}$	$\alpha\beta\gamma = -\frac{\boldsymbol{d}}{\boldsymbol{a}}$
Quartic $ax^4 + bx^3 + cx^2 + dx + e$ (Roots: $\alpha, \beta, \gamma, \delta$)	$\alpha + \cdots + \delta = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$	$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{\boldsymbol{c}}{\boldsymbol{a}}$	$\alpha\beta\gamma + \alpha\beta\delta + \cdots = -\frac{\boldsymbol{d}}{\boldsymbol{a}}$

We can see the sum of the roots for example always has the same relationship with the coefficients.
There's a pattern! What would be the product of the roots of a quartic?
$\frac{e}{a}$. Then $-\frac{f}{a}$ for the product of roots of a quintic, $\frac{g}{a}$ for a sextic, oscillating between positive and negative. These are known as Vieta's formulas.

Back to roots of quadratics…

For the quadratic $x^2 + 2x + 3$, find:
(a) The sum of the roots.
(b) The product of the roots.

The roots of the quadratic equation $2x^2 - 5x - 4 = 0$ are α and β.
Without solving the equation, find the values of:
(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\frac{1}{\alpha} + \frac{1}{\beta}$ (d) $\alpha^2 + \beta^2$

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{3}{2}$ and $\beta = \frac{5}{4}$.
Find integer values for a, b and c.

If the roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{5}$, determine integer values for a, b, c.

The roots of the equation $6x^2 + 36x + k = 0$ are reciprocals of each other. Find the value of k.

Ex 4A

Roots of Cubics

A cubic equation $ax^3 + bx^2 + cx + d = 0$ always has 3 (potentially repeated) roots, α, β, γ. We saw in the previous chapters that these could all be real, or one real and two complex roots (which are complex conjugates).

Let's again try to determine the relationship between roots and coefficients of the polynomial:

$$\boldsymbol{ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)}$$

If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:

- **Sum** of roots: $\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a}$
- Sum of **product pairs**: $\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
- **Product** of all roots: $\alpha\beta\gamma = -\frac{d}{a}$

α, β and γ are the roots of the cubic equation $2x^3 + 3x^2 - 4x + 2 = 0$. Without solving the equation, find the values of:

(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \beta\gamma + \gamma\alpha$ (c) $\alpha\beta\gamma$ (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 2i, \beta = 1 + 2i$ and $\gamma = 2$. Find integers values for a, b, c and d.

1.
$$f(z) = z^3 + pz^2 + qz - 15$$

where p and q are real constants.

Given that the equation $f(z) = 0$ has roots

$$\alpha, \ \frac{5}{\alpha} \text{ and } \left(\alpha + \frac{5}{\alpha} - 1\right)$$

(a) solve completely the equation $f(z) = 0$

(5)

(b) Hence find the value of p.

(2)

7.

$$f(z) = z^3 + z^2 + pz + q$$

where p and q are real constants.

The equation $f(z) = 0$ has roots z_1, z_2 and z_3
When plotted on an Argand diagram, the points representing z_1, z_2 and z_3 form the vertices of a triangle of area 35

Given that $z_1 = 3$, find the values of p and q.

(7)

Roots of Quartics

The pattern continues!

Polynomial	Sum of roots	Sum of possible <u>products of pairs</u> of roots	Sum of <u>products of triples</u>	Sum of products of <u>quadruples</u>
Quadratic ax^2+bx+c (Roots: α, β)	$\alpha+\beta=-\frac{\boldsymbol{b}}{\boldsymbol{a}}$	$\alpha\beta=\frac{\boldsymbol{c}}{\boldsymbol{a}}$	N/A	N/A
Cubic ax^3+bx^2+cx+d (Roots: α, β, γ)	$\alpha+\beta+\gamma=-\frac{\boldsymbol{b}}{\boldsymbol{a}}$	$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{\boldsymbol{c}}{\boldsymbol{a}}$	$\alpha\beta\gamma=-\frac{\boldsymbol{d}}{\boldsymbol{a}}$	N/A
Quartic $ax^4+bx^3+cx^2+dx+e$ (Roots: $\alpha, \beta, \gamma, \delta$)	$\alpha+\cdots+\delta=-\frac{\boldsymbol{b}}{\boldsymbol{a}}$	$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{\boldsymbol{c}}{\boldsymbol{a}}$	$\alpha\beta\gamma+\alpha\beta\delta+\cdots=-\frac{\boldsymbol{d}}{\boldsymbol{a}}$	$\alpha\beta\gamma\delta=\frac{\boldsymbol{e}}{\boldsymbol{a}}$

In general:

$$\sum\alpha=-\frac{b}{a} \qquad \sum\alpha\beta=\frac{c}{a} \qquad \sum\alpha\beta\gamma=-\frac{d}{a} \qquad \sum\alpha\beta\gamma\delta=\frac{e}{a}$$

$$3x^5+2x^4-7x^3+x^2-2x+4=0$$

The equation $x^4 + 2x^3 + px^2 + qx - 60 = 0, x \in \mathbb{C}, p, q \in \mathbb{R}$, has roots $\alpha, \beta, \gamma, \delta$.
Given that $\gamma = -2 + 4i$ and $\delta = \gamma^*$.

$\mathbb{C}$ is the set of all complex numbers.

(a) Show that $\alpha + \beta - 2 = 0$ and that $\alpha\beta + 3 = 0$

(b) Hence find all the roots of the quartic equation and find the values of p and q.

Ex 4C

Expressions related to the roots of a polynomial

We have seen that we can calculate the **sum of the roots** $\alpha + \beta + \cdots$ and the **product of the roots** $\alpha\beta$ ... of a polynomial without needing to find the roots themselves.

We also saw earlier that for quadratic equations, we could find expressions for the sum of the squares of the roots, or the sum of the reciprocals of the roots, both in terms of $\alpha + \beta$ and $\alpha\beta$ (whose values could both be easily evaluated):

Sum of squares of roots: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Sum of reciprocals: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

Such identities can be extended to cubics and quartics: (You can use these results without proof)

Sums of squares:

- Quadratic: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- Cubic: $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
- Quartic: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\beta + \alpha\gamma + \beta\gamma + \cdots)$

REWRITING WITH SIGMA NOTATION

Sums of squares:

- Quadratic: $\alpha^2 + \beta^2 =$
- Cubic: $\alpha^2 + \beta^2 + \gamma^2 =$
- Quartic: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 =$

Sums of cubes:

- Quadratic: $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- Cubic: $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$

REWRITING WITH SIGMA NOTATION

Sums of cubes:

- Quadratic: $\alpha^3 + \beta^3 =$
- Cubic: $\alpha^3 + \beta^3 + \gamma^3 =$

(We can see these cubes formulae **<u>don't</u>** generalise nicely as we increase the order of the polynomial. For this reason you are not required to know the sum of cubes for quartics)

Sums of reciprocals:

- Quadratic: $\frac{1}{\alpha} + \frac{1}{\beta} =$
- Cubic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} =$
- Quartic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} =$

Products of powers:

- Quadratic: $\alpha^n \times \beta^n = (\alpha\beta)^n$
- Cubic: $\alpha^n \times \beta^n \times \gamma^n = (\alpha\beta\gamma)^n$
- Quartic: $\alpha^n \times \beta^n \times \gamma^n \times \delta^n = (\alpha\beta\gamma\delta)^n$

The three roots of a cubic equation are α, β and γ.

Given that $\alpha + \beta + \gamma = \frac{3}{2}$, $\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{4}{3}$ and $\alpha\beta\gamma = \frac{1}{2}$, find the value of:

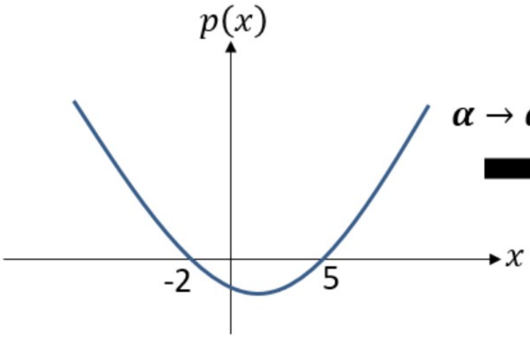
a) The sum of the reciprocals of the roots
b) The sum of the squares of the roots
c) The sum of the cubes of the roots
d) $\alpha^3\beta^3\gamma^3$

The three roots of a cubic equation are α, β and γ.
Given that $\alpha\beta + \beta\gamma + \gamma\alpha = 7$ and $\alpha + \beta + \gamma = -3$, find the value of $(\alpha + \beta + \gamma)^2$

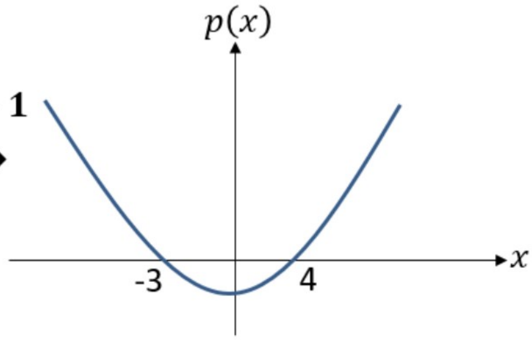
The three roots of a cubic equation are α, β and γ. Given that $\alpha\beta\gamma = 4$, $\alpha\beta + \beta\gamma + \gamma\alpha = -5$ and $\alpha + \beta + \gamma = 3$, find the value of $(\alpha + 3)(\beta + 3)(\gamma + 3)$

Linear Transformations of Roots

Suppose we transform the roots of a polynomial.



$\boldsymbol{\alpha \to \alpha - 1}$



$p(x) = (x+2)(x-5)$
$= x^2 - 3x - 10$

$p(x) = (x+3)(x-4)$
$= x^2 - x - 12$

If the polynomial was in factorised form, then the transformation is obvious: we can just replace each root with 1 less in the equation.

However, it's not so obvious how this affects the polynomial in $ax^2 + bx + c$ form. How do the coefficients change?

On the previous slide, we had the polynomial $x^2 - 3x - 10 = 0$ which has the roots α and β. Without finding the roots, determine the equation with roots $\alpha - 1$ and $\beta - 1$.

Use a substitution
(I recommended this method, there are others)

Let $w = x - 1$

Use a new variable to represent the transformed value.

Note: The above is effectively a "Pure Year 1"/GCSE-style method involving function transformations.
We know the roots have been translated 1 left.
Therefore if $f(x) = x^2 - 3x - 10$, we know we can find $f(x+1)$ to have this effect.

The quartic equation $x^4 - 3x^3 + 15x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$, $(2\gamma + 1)$ and $(2\delta + 1)$.

The cubic equation $x^3 - 2x^2 + 4 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equation with roots $(3\alpha - 1)$, $(3\beta - 1)$, $(3\gamma - 1)$ and $(3\delta - 1)$.

Exam Questions

4. The cubic equation

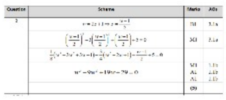
$$x^3 + 3x^2 - 8x + 6 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

2. The cubic equation

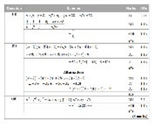


$$z^3 - 3z^2 + z + 5 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

1. The roots of the equation 

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

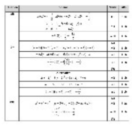
Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

1. The roots of the equation 

$$2x^3 - 3x^2 + 4x + 7 = 0$$

are α, β and γ

Without solving the equation, determine the value of

(i) $\dfrac{3}{\alpha} + \dfrac{3}{\beta} + \dfrac{3}{\gamma}$

(ii) $(\alpha - 2)(\beta - 2)(\gamma - 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

6.

$$f(x) = kx^2 + 3x - 11 \qquad\qquad g(x) = mx^3 - 2x^2 + 3x - 9$$

where k and m are real constants.

Given that

- the sum of the roots of f is equal to the product of the roots of g
- g has at least one root on the imaginary axis

(a) solve completely

(i) $f(x) = 0$

(ii) $g(x) = 0$

(7)

(b) Plot the roots of f and the roots of g on a single Argand diagram.

(2)

2. The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are p, q and r.

Without solving the equation, find the value of

(i) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

(8)