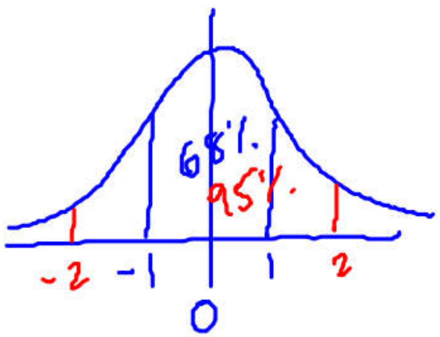
# Standard Normal Distribution

$Z$ is the number of standard deviations above the mean.

z-value

If again we use IQ distributed as $X \sim N(100, 15^2)$ then: (in your head!)

| IQ | Z |
|---|---|
| 100 | 0 |
| 130 | 2 |
| 85 | -1 |
| 165 | 4.333 |
| 62.5 | -2.5 |




This formula makes sense if you think about the definition above. For an IQ of 130: $Z = \frac{130-100}{15} = 2$ as expected.

$Z$ represents the coding:

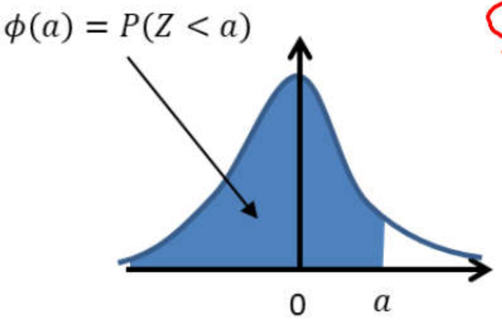
$$Z = \frac{X - \mu}{\sigma}$$

and $Z \sim N(0, 1^2)$. $Z$ is known as a **standard** normal distribution.

The 0 and 1 of $Z \sim N(0, 1^2)$ are the result of the coding. If we've subtracted $\mu$ from each value the mean of the normal distribution is now 0. If we've divided all the values by $\sigma$ the standard deviation is now $\frac{\sigma}{\sigma} = 1$

The point of coding in this context is that all different possible normal distributions become a single unified distribution where we no longer have to worry about the mean and standard deviation.
It means for example when we calculate $P(Z < 3)$, this will always give the same probability regardless of the original distribution.

It also means we can look up probabilities in a ***z*-table**:




$\Phi(a)$

✏ $\Phi(a) = P(Z < a)$ is the cumulative distribution for the standard normal distribution. The values of $\Phi(a)$ can be found in a $z$-table.

This is a traditional z-table in the old A Level syllabus (but also found elsewhere). You no longer get given this and are expected to use your calculator.

This is from the new formula booklet. This is sometimes known as a 'reverse z-table', because you're looking up the z-value for a probability. Beware: $p$ here it the probability of **<u>exceeding</u>** $z$ rather than being up to $z$. Let's use it...

THE NORMAL DISTRIBUTION FUNCTION

The function tabulated below is $\Phi(z)$, defined as $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} \, dt$.

| $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.5000 | 0.50 | 0.6915 | 1.00 | 0.8413 | 1.50 | 0.9332 | 2.00 | 0.9772 |
| 0.01 | 0.5040 | 0.51 | 0.6950 | 1.01 | 0.8438 | 1.51 | 0.9345 | 2.02 | 0.9783 |
| 0.02 | 0.5080 | 0.52 | 0.6985 | 1.02 | 0.8461 | 1.52 | 0.9357 | 2.04 | 0.9793 |
| 0.03 | 0.5120 | 0.53 | 0.7019 | 1.03 | 0.8485 | 1.53 | 0.9370 | 2.06 | 0.9803 |
| 0.04 | 0.5160 | 0.54 | 0.7054 | 1.04 | 0.8508 | 1.54 | 0.9382 | 2.08 | 0.9812 |
| 0.05 | 0.5199 | 0.55 | 0.7088 | 1.05 | 0.8531 | 1.55 | 0.9394 | 2.10 | 0.9821 |
| 0.06 | 0.5239 | 0.56 | 0.7123 | 1.06 | 0.8554 | 1.56 | 0.9406 | 2.12 | 0.9830 |
| 0.07 | 0.5279 | 0.57 | 0.7157 | 1.07 | 0.8577 | 1.57 | 0.9418 | 2.14 | 0.9838 |
| 0.08 | 0.5319 | 0.58 | 0.7190 | 1.08 | 0.8599 | 1.58 | 0.9429 | 2.16 | 0.9846 |
| 0.09 | 0.5359 | 0.59 | 0.7224 | 1.09 | 0.8621 | 1.59 | 0.9441 | 2.18 | 0.9854 |
| 0.10 | 0.5398 | 0.60 | 0.7257 | 1.10 | 0.8643 | 1.60 | 0.9452 | 2.20 | 0.9861 |

## Percentage Points of The Normal Distribution

The values $z$ in the table are those which a random variable $Z \sim N(0, 1)$ exceeds with probability $p$; that is, $P(Z > z) = 1 - \Phi(z) = p$.

| $p$ | $z$ | $p$ | $z$ |
|---|---|---|---|
| 0.5000 | 0.0000 | 0.0500 | 1.6449 |
| 0.4000 | 0.2533 | 0.0250 | 1.9600 |
| 0.3000 | 0.5244 | 0.0100 | 2.3263 |
| 0.2000 | 0.8416 | 0.0050 | 2.5758 |
| 0.1500 | 1.0364 | 0.0010 | 3.0902 |
| 0.1000 | 1.2816 | 0.0005 | 3.2905 |

The random variable $X \sim N(50, 4^2)$. Write in terms of $\Phi(z)$ for some value of $z$.

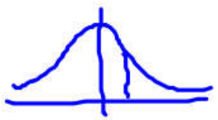
(a) $P(X < 53)$ (b) $P(X \geq 55)$

$$\frac{X-\mu}{\sigma} = Z$$

a) $P(X<53) = P\left(Z < \frac{53-50}{4}\right)$

$= P(Z < 0.75)$

$= \Phi(0.75)$



b) $P(X \geq 55) = P\left(Z > \frac{55-50}{4}\right)$

$= P(Z > 1.25)$

$= 1 - \Phi(1.25)$


The systolic blood pressure of an adult population, $S$ mmHg, is modelled as a normal distribution with mean 127 and standard deviation 16. A medical research wants to study adults with blood pressures higher than the 95th percentile. Find the minimum blood pressure for an adult included in her study.

$X \sim N(127, 16^2)$

$P(X < a) = 0.95$

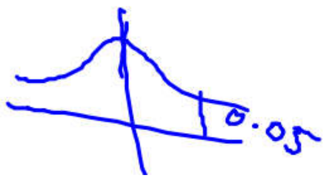
$a = \underline{153.3 \text{ mmHg}}$

$P(Z > 1.6449) = 0.05$

$$\frac{a-127}{16} = 1.6449$$

$a = \underline{153.3 \text{ mmHg}}$

| $p$ | $z$ | $p$ | $z$ |
|---|---|---|---|
| 0.5000 | 0.0000 | 0.0500 | 1.6449 |
| 0.4000 | 0.2533 | 0.0250 | 1.9600 |
| 0.3000 | 0.5244 | 0.0100 | 2.3263 |
| 0.2000 | 0.8416 | 0.0050 | 2.5758 |
| 0.1500 | 1.0364 | 0.0010 | 3.0902 |
| 0.1000 | 1.2816 | 0.0005 | 3.2905 |



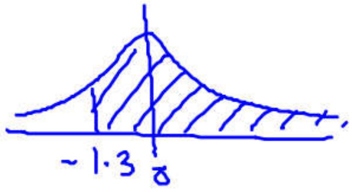
(a) Determine $P(Z > -1.3)$
(b) Determine $P(-2 < Z < 1)$
(c) Determine the $a$ such that $P(Z > a) = 0.7$
(d) Determine the $a$ such that $P(-a < Z < a) = 0.6$

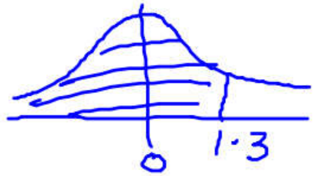
**The aim here is to rewrite using standardised form first – this will be helpful in the next few exercises, trust me!**

| $p$ | $z$ | $p$ | $z$ |
|---|---|---|---|
| 0.5000 | 0.0000 | 0.0500 | 1.6449 |
| 0.4000 | 0.2533 | 0.0250 | 1.9600 |
| 0.3000 | 0.5244 | 0.0100 | 2.3263 |
| 0.2000 | 0.8416 | 0.0050 | 2.5758 |
| 0.1500 | 1.0364 | 0.0010 | 3.0902 |
| 0.1000 | 1.2816 | 0.0005 | 3.2905 |

Sketches.

a) $P(Z > -1.3) = P(Z < 1.3)$

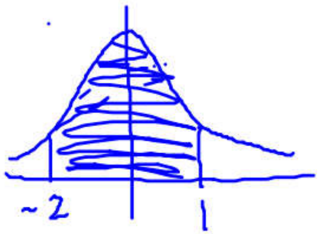



**Tip**: Either changing < to/from > or changing the sign (+ to/from -) has the effect of "1 –". However, if you change both, the "1 –"s cancel out!

b) $P(-2 < Z < 1) = P(Z < 1) - P(Z < -2)$

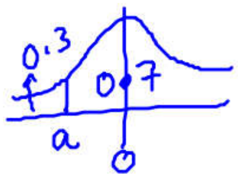
$P(a < Z < b)$
$= P(Z < b) - P(Z < a)$



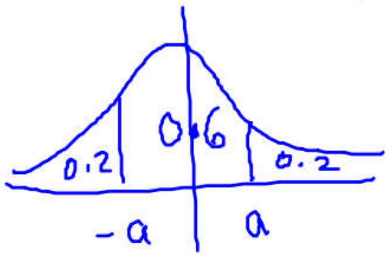

c) $P(Z > a) = 0.7$

$P(Z < a) = 0.3$

$a = \underline{-0.5224}$




d) $P(-a < Z < a) = 0.6$




$P(Z < a) = 0.8$

$a = \underline{0.8416}$

IQ is distributed with mean 100 and standard deviation 15. Using an appropriate table, determine the IQ corresponding to the
(a) top 10% of people.
(b) bottom 20% of people.

| $p$ | $z$ | $p$ | $z$ |
|---|---|---|---|
| 0.5000 | 0.0000 | 0.0500 | 1.6449 |
| 0.4000 | 0.2533 | 0.0250 | 1.9600 |
| 0.3000 | 0.5244 | 0.0100 | 2.3263 |
| 0.2000 | 0.8416 | 0.0050 | 2.5758 |
| 0.1500 | 1.0364 | 0.0010 | 3.0902 |
| 0.1000 | 1.2816 | 0.0005 | 3.2905 |

$P(Z>a)=0.1$

$P(Z>1.2816)=0.1$

$\frac{a-100}{15}=1.2816$

$=\underline{\underline{119.2}}$

$X\sim N(100,15^2).$

$P(X>a)=0.1$

$P(X<a)=0.9$

$a=\underline{\underline{119.2}}.$

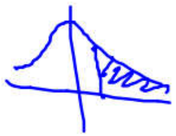
$P(X<a)=0.2$

$a=\underline{\underline{87.4}}.$

$\Phi(a)=P(Z<a)$

If $X\sim N(100,15^2)$, determine, in terms of $\Phi$:
(a) $P(X>115)$
(b) $P(77.5<X<112)$

a) $P(X>115)=P\left(Z>\frac{115-100}{15}\right)$

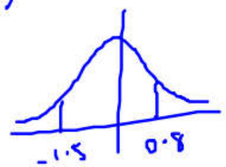


$=P(Z>1)$

$=1-P(Z<1)$

$=1-\Phi(1)$

b) $P(77.5<X<112)=P\left(\frac{77.5-100}{15}<Z<\frac{112-100}{15}\right)$



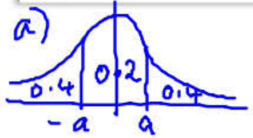

$=P(-1.5<Z<0.8)$

$=P(Z<0.8)-P(Z<-1.5)$

$=\Phi(0.8)-\Phi(-1.5)$

Find the $a$ such that:
(a) $P(-a<Z<a)=0.2$
(b) $P(0<Z<a)=0.35$

a)

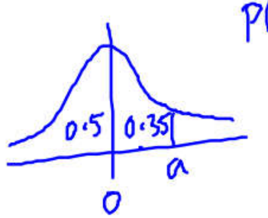



$P(Z<a)=0.6$

$a=\underline{\underline{0.2533}}.$

b) $P(0<Z<a)=0.35$

$P(Z<a)=0.85$




$a=\underline{\underline{1.0364}}$ ~~Ex 3D~~

# Missing $\mu$ and $\sigma$

In the last section, you may have thought, “what’s the point of standardising to $Z$ when I can just use the DISTRIBUTION mode on my calculator?”
Fair point, but both forward and reverse normal lookups on the calculator **required you to specify $\mu$ and $\sigma$.**

???

We use "Z" to standardise.

$X \sim N(\mu, 3^2)$. Given that $P(X > 20) = 0.2$, find the value of $\mu$.

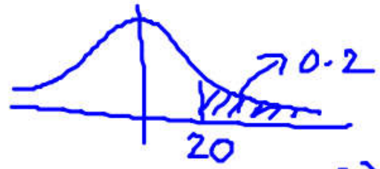
| $p$ | $z$ | $p$ | $z$ |
|---|---|---|---|
| 0.5000 | 0.0000 | 0.0500 | 1.6449 |
| 0.4000 | 0.2533 | 0.0250 | 1.9600 |
| 0.3000 | 0.5244 | 0.0100 | 2.3263 |
| 0.2000 | 0.8416 | 0.0050 | 2.5758 |
| 0.1500 | 1.0364 | 0.0010 | 3.0902 |
| 0.1000 | 1.2816 | 0.0005 | 3.2905 |

$P(X > 20) = 0.2$

$P(X < 20) = 0.8$

$P(Z < a) = 0.8$

$a = 0.8416$




$X \to Z \sim N(0, 1^2)$

$\frac{20 - \mu}{3} = 0.8416$

$20 - \mu = 3 \times 0.8416$

$20 - 3 \times 0.8416 = \mu$

$\mu = 17.4752$

$= 17.5$ (1dp)

A machine makes metal sheets with width, $X$ cm, modelled as a normal distribution such that $X \sim N(50, \sigma^2)$.

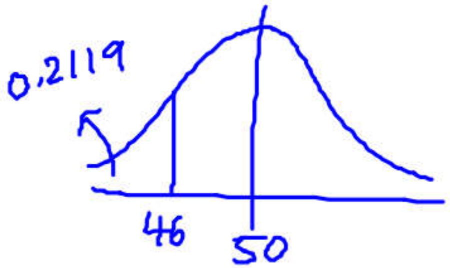
(a) Given that $P(X < 46) = 0.2119$, find the value of $\sigma$.

(b) Find the 90th percentile of the widths.

The method here is exactly the same as before:

1. Using a sketch, determine whether you're expecting a positive or negative $z$ value.
2. Look up $z$ value, using tables if you can (otherwise your calculator). Make negative if in bottom half.
3. Use $Z = \frac{X-\mu}{\sigma}$

$X \sim N(50, \sigma^2)$




$P(X < 46) = 0.2119$

$\mu = 0, \sigma = 1$

$P(Z < a) = 0.2119$

$a = -0.7998$

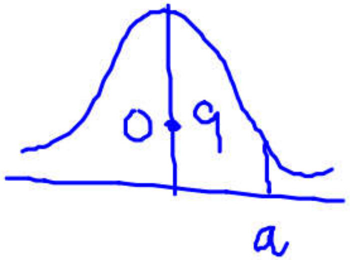
When I standardise 46 I get $-0.7998$

$$\frac{46-50}{\sigma} = -0.7998$$

$$\frac{-4}{-0.7998} = \sigma$$

$$\sigma = 5.0096$$
$$= 5.0\text{cm (1dp)}$$

b)




$P(X < a) = 0.9$

$a = \underline{\underline{56.4\text{cm}}}$ (1dp)

# When both are missing

If both $\mu$ and $\sigma$ are missing, we end up with simultaneous equations which we must solve.

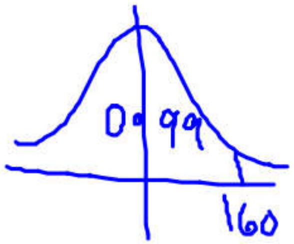
**Edexcel S1 Jan 2011**

The weight, $Y$ grams, of soup put into a carton by machine $B$ is normally distributed with mean $\mu$ grams and standard deviation $\sigma$ grams.

(*c*) Given that $\mathrm{P}(Y < 160) = 0.99$ and $\mathrm{P}(Y > 152) = 0.90$, find the value of $\mu$ and the value of $\sigma$. **(6)**

$Y \sim N(\mu, \sigma^2)$

$P(Y < 160) = 0.99$

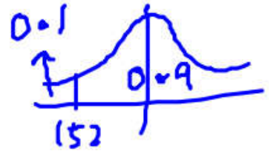



$P(Z < a) = 0.99$

$a = 2.3263$

$\frac{160 - \mu}{\sigma} = 2.3263$

$160 = \mu + 2.3263\sigma$




$P(Y > 152) = 0.9$

$P(Y < 152) = 0.1$

$\rightarrow P(Z < a) = 0.1$

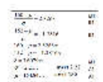
$a = -1.2816$

$\frac{152 - \mu}{\sigma} = -1.2816$

$152 = \mu - 1.2816\sigma$

$\mu = 154.89$

$\sigma = 2.229$



**4.** A company has a customer services call centre. The company believes that the time taken to complete a call to the call centre may be modelled by a normal distribution with mean 16 minutes and standard deviation $\sigma$ minutes. $X \sim N(16, \sigma^2)$

Given that 10% of the calls take longer than 22 minutes, $P(X > 22) = 0.1$

(a) show that, to 3 significant figures, the value of $\sigma$ is 4.68

$P(X < 22) = 0.9$

$\frac{22-16}{\sigma} = 1.2816$ **(3)**

$\sigma = 4.68$ (2dp)

(b) Calculate the percentage of calls that take less than 13 minutes.

$P(X < 13) = 0.2607 = 26.07\%$ **(1)**

A supervisor in the call centre claims that the mean call time is less than 16 minutes. He collects data on his own call times.

- 20% of the supervisor's calls take more than 17 minutes to complete. $P(X > 17) = 0.2$ or $P(X < 17) = 0.8$
- 10% of the supervisor's calls take less than 8 minutes to complete. $P(X < 8) = 0.1$

Assuming that the time the supervisor takes to complete a call may be modelled by a normal distribution,

(c) estimate the mean and the standard deviation of the time taken by the supervisor to complete a call.


**(6)**

(d) State, giving a reason, whether or not the calculations in part (c) support the supervisor's claim.

**(1)**

$P(X < 17) = 0.8$

$P(Z < a) = 0.8$

$a = 0.8416$

$\frac{17-\mu}{\sigma} = 0.8416$

$17 = \mu + 0.8416\sigma$

$P(X < 8) = 0.1$

$P(Z < a) = 0.1$

$a = -1.2816$

$\frac{8-\mu}{\sigma} = -1.2816$

$8 = \mu - 1.2816\sigma$

Sim. eqn.

$\mu = 13.4$ mins $\sigma = 4.24$ mins

d) The mean has decreased to 13.4, so manager's claim is supported.

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 4(a) | $[\mathrm{P}(T > 22) > 0.1]$<br>$\frac{22-16}{\sigma} = their\ z\ value$ | M1 | 3.4 |
| | 1.28155.... | B1 | 1.1b |
| | $\frac{22-16}{1.28155...} = 4.6818\ ...$<br>$\cong 4.68$ | A1 | 1.1b |
| | | (3) | |
| (b) | $\mathrm{P}(L < 13) = \mathrm{P}\left(Z < \frac{13-16}{4.68}\right)$<br>= 0.2607... 26.1% | B1 | 1.1b |
| | | (1) | |
| (c) | $\mathrm{P}(S > 17) = 0.2$ or $\mathrm{P}(S < 8) = 0.1$ | | |
| | $\therefore \frac{17-\mu}{\sigma} = 0.8416$ or $\therefore \frac{8-\mu}{\sigma} = -1.2816$ | M1 | 3.4 |
| | $0.8416$ and $-1.2816$ | B1 | 1.1b |
| | $\therefore \frac{17-\mu}{\sigma} = 0.8416$ and $\therefore \frac{8-\mu}{\sigma} = -1.2816$ | A1 | 1.1b |
| | $17 - \mu = 0.8416\sigma$<br>$-(8 - \mu = -1.2816\sigma)$ | M1 | 1.1b |
| | $\sigma = 4.238\ ...$ | A1 | 1.1b |
| | $\mu = 13.432\ ...$ | A1 | 1.1b |
| | | (6) | |
| (d) | $\mu = 13.4 < 16$ | B1 | 2.4 |
| | Yes, supports supervisor's belief | | |
| | | (1) | |
| | | | **(11 marks)** |

**Notes:**

(a)

M1: for a suitable equation to find σ with attempt at a z value

B1: for awrt 1.28

A1: for a complete solution showing that σ is 4.68 to 3 significant figures cso

(c)

B1: for 0.842 and $-1.28$ or better

2nd M1: for a method to solve simultaneous equations

A1: for awrt $\sigma = 4.24$

A1: for awrt $\mu = 13.4$

Ignore units

(d)

B1: for a suitable comparison of mean and conclusion

**5.** The duration of the pregnancy of a certain breed of cow is normally distributed with mean $\mu$ days and standard deviation $\sigma$ days. Only 2.5% of all pregnancies are shorter than 235 days and 15% are longer than 286 days.

(*a*) Show that $\mu - 235 = 1.96\sigma$.

**(2)**

(*b*) Obtain a second equation in $\mu$ and $\sigma$.

**(3)**

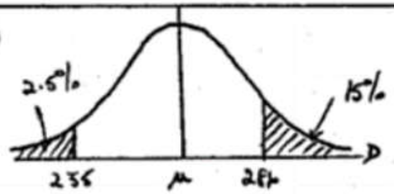
(*c*) Find the value of $\mu$ and the value of $\sigma$.

**(4)**

(*d*) Find the values between which the middle 68.3% of pregnancies lie.

**(2)**

5. (a)




$P(D < 235) = 0.025$

$\therefore \frac{235-\mu}{\sigma} = -1.96$ — $\frac{235-\mu}{\sigma}$ = z — M1

$* \therefore \mu - 235 = 1.96\sigma *$ — A1 (2)

(b) $P(D > 286) = 0.15$ — $\frac{286-\mu}{\sigma}$ = z — M1

$\therefore \frac{286-\mu}{\sigma} = 1.0364$ $\therefore 286-\mu = 1.0364\sigma$ — 1.0364 — B1

A1√ (3)

(c) Solving for $\mu$ or $\sigma$ — M1

Substituting for other unknown — M1

$\mu = 268.360\ldots$ $\sigma = 17.0204\ldots$ — AWRT 268 — A1

AWRT 17 — A1 (4)

(d) $\mu \pm \sigma = 268.36 \pm 17.02$ — $\mu$ + their $\sigma$ — M1

$= (251, 285)$ — 3sf — A1√ (2)