## General expressions

**1.** Is it a **standard result**?
*Does it need scaling?*

**2.** Can I **manipulate** it to make it a **standard result**?
*Think trig identities or expanding brackets*

**3.** Can I use the **reverse chain rule**?
*Is the numerator the derivative of the denominator?*
*Is one factor related to the derivative of the other?*

## Fractional expressions

**4a.** Can I **split the numerator**?
*Is there a single term in the denominator?*
**4b.** Can I do **partial fractions**?
*Does the denominator factorise?*
**4c.** Can I do **algebraic division**?
*Is the fraction improper?*

## Product expressions

**4.** Use **integration by parts**
*For u, choose the ln term,*
*then the polynomial*

**5.** Last resort: **substitution**

## Integrate the following expressions with respect to x

1. $\sec^2 x$ — Standard Result

2. $(3x+2)^5$ — Standard Result – scaling

3. $x\sqrt{1+x^2}$ — R.C.R.

4. $\dfrac{4\sin x\cos x}{4-8\sin^2 x}$ — Standard Result – trig identity

5. $2x\sin(3x-1)$ — by parts

6. $\dfrac{x-1}{\sqrt{x}} = \dfrac{x}{\sqrt{x}} - \dfrac{1}{\sqrt{x}}$ — split the numerator

7. $\sin x\, e^{\cos x}$ — R.C.R.

8. $\dfrac{x+1}{x-1}$ — algebraic division

9. $x\sqrt{x^2+1}$ — R.C.R.

10. $\dfrac{3x+2}{(x+1)(x-2)}$ — partial fractions

11. $\dfrac{x+1}{x}$ — split the numerator $\dfrac{x}{x} + \dfrac{1}{x}$

12. $1\times\ln x$ — SPECIAL CASE by parts $u=\ln x$, $v'=1$

13. $\frac{1}{x}\sqrt{\ln x}$ R.C.R.

$\frac{1}{x}(\ln x)^{1/2}$

14. $\frac{1}{1+e^x}$ Substitution

$u = 1+e^x$

15. $x\sec^2 x$ by parts

$u = x$

16. $\frac{2x}{2x^2-3}$ R.C.R. $\rightarrow \ln|2x^2-3|$

17. $\frac{2x}{(x^2-3)^2}$ R.C.R.

$2x(x^2-3)^{-2} \rightarrow -(x^2-3)^{-1}$

18. $x^2e^{3x}$ by parts, twice

19. $\sin x\cos^3 x$ R.C.R.

$\sin x(\cos x)^3$ $-(\cos x)^4$

20. $\tan^2 x$ Standard result

$1+\tan^2 x = \sec^2 x$ $-$ trig identity

21. $\frac{x^2}{x+1}$ Algebraic division

22. $\frac{x}{\sqrt{x+1}}$ substitution

$u = \sqrt{x+1}$

23. $\frac{2x-1}{x^2+3x+2} \rightarrow (x+2)(x+1)$ Partial fractions

24. $\sin^2 x$ Standard Result

$\frac{1}{2} - \frac{1}{2}\cos 2x$

1. $\int \sec^2 x \, dx = \tan x + C$

Standard Result

Reverse Chain Rule

Split numerator
Partial Fractions
Algebraic Division

By Parts

Substitution

2. $\int (3x+2)^5 \, dx = \frac{1}{18}(3x+2)^6 + c$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$(3x+2)^6$

$3 \times 6 \, (3x+2)^5$

3. $\int x\sqrt{1+x^2}\,dx = \int x(1+x^2)^{1/2}\,dx$

$= \frac{1}{3}(1+x^2)^{3/2} + c$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$(1+x^2)^{3/2}$

$2x \times \frac{3}{2}(1+x^2)^{1/2}$

$$4.\ \int \frac{4\sin x\cos x}{4-8\sin^2 x}\,dx = \int \frac{2\times 2\sin x\cos x}{4(1-2\sin^2 x)}\,dx$$

$$= \int \frac{2\sin 2x}{4\cos 2x}\,dx$$

$$= \int \frac{1}{2}\tan 2x\,dx.$$

$$= \frac{1}{2}\times\frac{1}{2}\ln|\sec 2x| + c$$

$$= \frac{1}{4}\ln|\sec 2x| + c.$$

Standard Result

Reverse Chain Rule

Split numerator
Partial Fractions
Algebraic Division

By Parts

Substitution

**Integration (+ constant)**

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $\sec^2 kx$ | $\frac{1}{k}\tan kx$ |
| $\tan kx$ | $\frac{1}{k}\ln\lvert\sec kx\rvert$ |

$$5.\ \int 2x\sin(3x-1)\,dx = -\frac{2}{3}x\cos(3x-1) + \int \frac{2}{3}\cos(3x-1)\,dx$$

$$= -\frac{2}{3}x\cos(3x-1) + \frac{2}{9}\sin(3x-1) + c$$

$u = 2x \quad v = -\frac{1}{3}\cos(3x-1)$

$u' = 2 \quad v' = \sin(3x-1)$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

6. $\int \frac{x-1}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) dx$

$= \int \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right) dx$

$= \int \left(x^{1/2} - x^{-1/2}\right) dx$

$= \frac{2}{3}x^{3/2} - 2x^{1/2} + c$

Standard Result
Reverse Chain Rule
Split numerator
Partial Fractions
Algebraic Division
By Parts
Substitution

7. $\int \sin x\, e^{\cos x}\, dx = -e^{\cos x} + C$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$e^{\cos x}$

$-\sin x\, e^{\cos x}$

$$8.\ \int \frac{x+1}{x-1}\,dx = \int \left(1 + \frac{2}{x-1}\right) dx$$

$$\frac{x-1}{x-1} + \frac{2}{x-1} = x + 2\ln|x-1| + c$$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$$x-1 \overline{)\,x+1\,} \quad 1$$

$$x - 1$$

$$2$$

9. $\int x\sqrt{x^2+1}\,dx = \int x(x^2+1)^{1/2}\,dx$

$= \frac{1}{3}(x^2+1)^{3/2} + c$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$(x^2+1)^{3/2}$

$2x \times \frac{3}{2}(x^2+1)^{1/2}$

10. $\int \frac{3x+2}{(x+1)(x-2)}\,dx = \int \left( \frac{1/3}{x+1} + \frac{8/3}{x-2} \right) dx$

$= \frac{1}{3}\ln|x+1| + \frac{8}{3}\ln|x-2| + C$

Standard Result
Reverse Chain Rule
Split numerator
Partial Fractions
Algebraic Division
By Parts
Substitution

$\frac{3x+2}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$

$3x+2 = A(x-2) + B(x+1)$

$x=2$ $\quad$ $x=-1$

$8 = 3B$ $\quad$ $-1 = -3A$

$B = 8/3$ $\quad$ $A = 1/3$

11. $\int \frac{x+1}{x}\,dx = \int \left(\frac{x}{x} + \frac{1}{x}\right) dx$

$= \int \left(1 + \frac{1}{x}\right) dx$

$= x + \ln|x| + C$.

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

12. $\int 1 \ln x \, dx = x \ln x - \int x \times \frac{1}{x} \, dx$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + c.$

$u = \ln x \quad v = x$

$u' = \frac{1}{x} \quad v' = 1$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

13. $\int \frac{1}{x}\sqrt{\ln x}\ dx = \int \frac{1}{x}(\ln x)^{1/2}\ dx$

$= \frac{2}{3}(\ln x)^{3/2} + c$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$(\ln x)^{3/2}$

$\frac{1}{x} \times \frac{3}{2}(\ln x)^{1/2}$

14. $\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u} \times \frac{1}{u-1}\,du$

$= \int \frac{1}{u(u-1)}\,du$

$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$1 = A(u-1) + Bu$

Let $u=1$

$1 = B$

Let $u=0$

$1 = -A$

$A = -1$

$= \int \left(\frac{1}{u-1} - \frac{1}{u}\right) du$

$= \ln|u-1| - \ln|u| + c$

$= \ln e^x - \ln|1+e^x| + c$

$= x - \ln(1+e^x) + c$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$u = 1 + e^x \rightarrow u - 1 = e^x$

$\frac{du}{dx} = e^x$

$\frac{du}{dx} = u - 1$

$\frac{1}{u-1}\,du = dx$

15. $\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx$

$= x \tan x - \ln|\sec x| + c$

$+ \ln|\cos x|$

$u = x \quad v = \tan x$

$u' = 1 \quad v' = \sec^2 x$

Standard Result
Reverse Chain Rule
Split numerator
Partial Fractions
Algebraic Division
By Parts
Substitution

16. $\int \frac{2x}{2x^2-3}\,dx = \frac{1}{2}\ln|2x^2-3| + c$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$\ln|2x^2-3|$

$\frac{4x}{2x^2-3}$

17. $\int \frac{2x}{(x^2-3)^2}\,dx = \int 2x(x^2-3)^{-2}\,dx$

$= -(x^2-3)^{-1} + C$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$(x^2-3)^{-1}$

$-(x^2-3)^{-2} \times 2x$

18. $\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \int \frac{2}{3}xe^{3x}\,dx$

$u = x^2 \quad v = \frac{1}{3}e^{3x}$

$u' = 2x \quad v' = e^{3x}$

$= \frac{1}{3}x^2e^{3x} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}\,dx\right)$

$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$

$u = \frac{2}{3}x \quad v = \frac{1}{3}e^{3x}$

$u' = \frac{2}{3} \quad v' = e^{3x}$

Standard Result
Reverse Chain Rule
Split numerator
Partial Fractions
Algebraic Division
By Parts TWICE
Substitution

19. $\int \sin x \cos^3 x \, dx = \int \sin x (\cos x)^3$

$= -\frac{1}{4}(\cos x)^4 + C$

$= -\frac{1}{4}\cos^4 x + C$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$(\cos x)^4$

$4(\cos x)^3 \times -\sin x$

20. $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$= \tan x - x + c$

Standard Result manipulation.

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$$21.\ \int \frac{x^2}{x+1}\,dx = \int \left(x - 1 + \frac{1}{x+1}\right) dx$$

$$= \frac{1}{2}x^2 - x + \ln|x+1| + c.$$

Standard Result
Reverse Chain Rule
Split numerator
Partial Fractions
Algebraic Division
By Parts
Substitution

$$\begin{array}{r} x - 1 \\ x+1 \overline{)\,x^2} \\ x^2 + x \\ \hline -x \\ -x - 1 \\ \hline 1 \end{array}$$

$$22.\ \int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u^2-1}{u} \times 2u\,du$$

$$= \int (2u^2-2)\,du$$

$$= \frac{2}{3}u^3 - 2u + c$$

$$= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + c$$

Standard Result
Reverse Chain Rule
Split numerator
Partial Fractions
Algebraic Division
By Parts
Substitution

$u = \sqrt{x+1}$

$u = (x+1)^{1/2}$

$u^2 = x+1$

$u^2 - 1 = x$

$$\frac{du}{dx} = \frac{1}{2}(x+1)^{-1/2} = \frac{1}{2\sqrt{x+1}}$$

$2\sqrt{x+1}\,du = dx.$

$$23.\int \frac{2x-1}{x^2+3x+2}\,dx = \int \left(\frac{5}{x+2} - \frac{3}{x+1}\right) dx$$

$$= 5\ln|x+2| - 3\ln|x+1| + C$$

Standard Result
Reverse Chain Rule
Split numerator
Partial Fractions
Algebraic Division
By Parts
Substitution

$$\frac{2x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$2x-1 = A(x+1) + B(x+2)$$

Let $x=-1$

$-3 = B$

Let $x=-2$

$-5 = -A$

$A = 5$

24. $\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Standard Result

Reverse Chain Rule

Split numerator

Partial Fractions

Algebraic Division

By Parts

Substitution

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\int \sin^2 x \, dx = \frac{1}{3}\sin^3 x$

↓ differentiate

$\times \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x.$