Worked Solutions to Lots of Lovely Integrals

1. By Parts: $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

$u = x \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x \quad v = \sin x$

2. By Parts: $\int_0^2 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^2 + \int_0^2 e^{-x}\,dx$

$u = x \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \quad v = -e^{-x}$

$= \left[-xe^{-x} - e^{-x}\right]_0^2$

$= (-2e^{-2} - e^{-2}) - (0 - 1)$

$= 1 - 3e^{-2}$

3. By Substitution: $\int_3^6 \frac{x}{\sqrt{x-2}}\,dx = \int_1^4 \frac{u+2}{\sqrt{u}}\,du = \int_1^4 (u^{1/2} + 2u^{-1/2})\,du$

(NB. By Parts, with $u = x$ and $\frac{dv}{dx} = \frac{1}{\sqrt{x-2}}$ would also work!)

$u = x - 2$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u + 2$

if $x = 6$, $u = 4$; if $x = 3$, $u = 1$

$= \left[\frac{2}{3}u^{3/2} + 4u^{1/2}\right]_1^4$

$= \left(\frac{2}{3}\times 8 + 4\times 2\right) - \left(\frac{2}{3} + 4\right)$

$= \frac{14}{3} + 4 = \frac{26}{3}$

4. By Substitution: $\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{x^2}{\sqrt{1-x^2}}\,x\,dx = \int \frac{1-u}{\sqrt{u}}\times\left(-\frac{1}{2}du\right) = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du$

$u = 1 - x^2$

$\frac{du}{dx} = -2x$ so $x\,dx = -\frac{1}{2}du$

$x^2 = 1 - u$

$= \frac{1}{2}\int (u^{1/2} - u^{-1/2})\,du$

$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right] + C$

$= \frac{1}{3}(1-x^2)^{3/2} - (1-x^2)^{1/2} + C$

(A more bizarre method that works [can you see why?!] would be to do integrate by Parts, using $u = x^2$ and $\frac{dv}{dx} = \frac{x}{\sqrt{1-x^2}}$)

5. By Inspection: $\int \frac{1}{\cos^2 2\theta}\,d\theta = \frac{1}{2}\tan 2\theta + C$

By Substitution: $\int \frac{1}{\cos^2 2\theta}\,d\theta = \int \frac{1}{\cos^2 u}\times\frac{1}{2}du = \frac{1}{2}\int \frac{1}{\cos^2 u}\,du = \frac{1}{2}\tan u + C$

$= \frac{1}{2}\tan 2\theta + C$

$u = 2\theta$

$\frac{du}{d\theta} = 2$ so $d\theta = \frac{1}{2}du$

6. By Inspection: $\int \frac{1}{(2x-3)^3}\,dx = -\frac{1}{4}(2x-3)^{-2} + C$

By Substitution: $\int \frac{1}{(2x-3)^3}\,dx = \int \frac{1}{u^3}\times\frac{1}{2}du = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\times\frac{u^{-2}}{-2} + C$

$u = 2x - 3$. $\frac{du}{dx} = 2$ so $dx = \frac{1}{2}du$

$= -\frac{1}{4}u^{-2} + C = -\frac{1}{4}(2x-3)^{-2} + C$

7. By Rewriting: $\int \frac{1-x}{\sqrt{x}}\,dx = \int\left(\frac{1}{\sqrt{x}} - \frac{x}{\sqrt{x}}\right)dx = \int\left(x^{-1/2} - x^{1/2}\right)dx$

$$= 2x^{1/2} - \tfrac{2}{3}x^{3/2} + C$$

8. By Inspection: $\int \frac{x^2-1}{x^3-3x+1}\,dx = \frac{1}{3}\ln\left|x^3-3x+1\right| + C$

By Substitution: $\int \frac{x^2-1}{x^3-3x+1}\,dx = \int \frac{1}{u}\times\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C$

$$= \tfrac{1}{3}\ln\left|x^3-3x+1\right| + C$$

$u = x^3-3x+1$

$\frac{du}{dx} = 3x^2-3 = 3(x^2-1)$

so $(x^2-1)dx = \frac{1}{3}du$

9. By Inspection: $\int \frac{1}{x}\sqrt{\ln x}\,dx = \frac{2}{3}(\ln x)^{3/2} + C$

By Substitution: $\int \frac{1}{x}\sqrt{\ln x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\ln x)^{3/2} + C$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ so $\frac{1}{x}dx = du$

10. By Substitution: $\int_1^2 \frac{2x-1}{(x+1)^2}\,dx = \int_2^3 \frac{2u-3}{u^2}\,du = \int_2^3\left(\frac{2u}{u^2} - \frac{3}{u^2}\right)du$

$u = x+1$

$\frac{du}{dx} = 1$ so $dx = du$

$x = u-1$

so $2x-1 = 2(u-1)-1 = 2u-3$

if $x=2$, $u=3$; if $x=1$, $u=2$.

$$= \int_2^3\left(\frac{2}{u} - 3u^{-2}\right)du$$

$$= \left[2\ln|u| + 3u^{-1}\right]_2^3$$

$$= \left(2\ln 3 + 3\times\tfrac{1}{3}\right) - \left(2\ln 2 + 3\times\tfrac{1}{2}\right)$$

$$= 2(\ln 3 - \ln 2) - \tfrac{1}{2}$$

$$= 2\ln\left(\tfrac{3}{2}\right) - \tfrac{1}{2}$$

(Another possibility would be to rewrite the integral:
$\int_1^2 \frac{2x-1}{(x+1)^2}\,dx = \int_1^2\left(\frac{2x+2}{(x+1)^2} - \frac{3}{(x+1)^2}\right)dx = \left[\ln\left((x+1)^2\right) + 3(x+1)^{-1}\right]_1^2$ etc.)

11. By Substitution: $\int_{\frac{1}{2}}^{2\frac{1}{2}} x\sqrt{2x-1}\,dx = \int_0^4 \frac{u+1}{2}\sqrt{u}\times\frac{1}{2}du = \frac{1}{4}\int_0^4 (u+1)\sqrt{u}\,du$

$u = 2x-1$

$\frac{du}{dx} = 2$ so $dx = \frac{1}{2}du$

$2x = u+1$ so $x = \frac{u+1}{2}$

if $x = 2\frac{1}{2}$, $u=4$; if $x=\frac{1}{2}$, $u=0$

$$= \tfrac{1}{4}\int_0^4\left(u^{3/2} + u^{1/2}\right)du$$

$$= \tfrac{1}{4}\left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^4$$

$$= \tfrac{1}{4}\left[\left(\tfrac{2}{5}\times 32 + \tfrac{2}{3}\times 8\right) - 0\right]$$

$$= \tfrac{16}{5} + \tfrac{4}{3} = \tfrac{48}{15} + \tfrac{20}{15} = \tfrac{68}{15}$$

12. By Substitution: $\int \frac{x}{(2x-3)^2}\,dx = \int \frac{\left(\frac{u+3}{2}\right)}{u^2}\times\frac{1}{2}du = \frac{1}{4}\int \frac{u+3}{u^2}du = \frac{1}{4}\int\left(\frac{u}{u^2} + \frac{3}{u^2}\right)du$

$u = 2x-3$ so $x = \frac{u+3}{2}$

$\frac{du}{dx} = 2$ so $dx = \frac{1}{2}du$

$$= \tfrac{1}{4}\int\left(\tfrac{1}{u} + 3u^{-2}\right)du$$

$$= \tfrac{1}{4}\left(\ln|u| - 3u^{-1}\right) + C = \tfrac{1}{4}\ln|2x-3| - \tfrac{3}{4(2x-3)} + C$$

(You could also do this by Parts, with $u=x$ and $\frac{dv}{dx}=\frac{1}{(2x-3)^2}$)

13. By Inspection: $\int \frac{1}{\sqrt{x}} \sin\sqrt{x}\, dx = -2\cos\sqrt{x} + C$

By Substitution: $\int \frac{1}{\sqrt{x}} \sin\sqrt{x}\, dx = \int \sin u \times 2du = 2\int \sin u\, du = -2\cos u + C$
$= -2\cos\sqrt{x} + C$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

so $2du = \frac{1}{\sqrt{x}}dx$

14. By Inspection: $\int \frac{1}{x}\ln x\, dx = \frac{(\ln x)^2}{2} + C$

By Substitution: $\int \frac{1}{x}\ln x\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ so $du = \frac{1}{x}dx$

(By Parts, bizarrely, also works!:

$\int \frac{1}{x}\ln x\, dx = (\ln x)^2 - \int \frac{1}{x}\ln x\, dx$ so $2\int \frac{1}{x}\ln x\, dx = (\ln x)^2 + C$

$u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$ so $\int \frac{1}{x}\ln x\, dx = \frac{(\ln x)^2}{2} + D$

$\frac{dv}{dx} = \frac{1}{x}$ $\quad v = \ln x$ ← don't worry about $\ln|x|$ here... the integral is only defined on $x>0$ $\quad D = \frac{C}{2}$)

15. By Substitution: $\int x\sqrt{x+1}\, dx = \int (u-1)\sqrt{u}\, du = \int (u^{3/2} - u^{1/2})du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

$u = x+1$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u-1$

(A brilliant spot by a student is that you can also get this by rewriting the integral: $\int x\sqrt{x+1}\, dx = \int ((x+1)\sqrt{x+1} - \sqrt{x+1})dx$

$= \int ((x+1)^{3/2} - (x+1)^{1/2})\, dx$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$ by Inspection or Substitution)

16. By Parts: $\int_0^{\pi} x\sin x\, dx = \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} \cos x\, dx = \left[-x\cos x + \sin x\right]_0^{\pi}$

$u = x$ $\quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x$ $\quad v = -\cos x$

$= (-\pi \times -1 + 0) - 0 = \pi$

17. By Parts, twice!: $\int_0^{1/2} x^2 e^{2x} dx = \left[\frac{1}{2}x^2 e^{2x}\right]_0^{1/2} - \int_0^{1/2} x e^{2x} dx$

$u = x^2$ $\quad \frac{du}{dx} = 2x$ $\qquad u = x$ $\quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{2x}$ $\quad v = \frac{1}{2}e^{2x}$ $\qquad \frac{dv}{dx} = e^{2x}$ $\quad v = \frac{1}{2}e^{2x}$

$= \left[\frac{1}{2}x^2 e^{2x}\right]_0^{1/2} - \left(\left[\frac{1}{2}x e^{2x}\right]_0^{1/2} - \int_0^{1/2} \frac{1}{2}e^{2x} dx\right)$

$$= \left[\tfrac{1}{2}x^2e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x}\right]_0^{1/2}$$
$$= \left(\tfrac{1}{2}\times\tfrac{1}{4}\times e - \tfrac{1}{2}\times\tfrac{1}{2}\times e + \tfrac{1}{4}e\right) - \left(0 - 0 + \tfrac{1}{4}\right)$$
$$= \tfrac{1}{8}e - \tfrac{1}{4} = \frac{e-2}{8}$$

18. By Rewriting: $\int_0^1 (x+e^x)^2\,dx = \int_0^1 (x^2 + 2xe^x + e^{2x})\,dx$

$$= \left[\frac{x^3}{3} + \tfrac{1}{2}e^{2x}\right]_0^1 + 2\int_0^1 xe^x\,dx$$

By Parts: $u = x \quad \frac{du}{dx} = 1$
$\frac{dv}{dx} = e^x \quad v = e^x$

$$= \left[\frac{x^3}{3} + \tfrac{1}{2}e^{2x}\right]_0^1 + 2\left(\left[xe^x\right]_0^1 - \int_0^1 e^x\,dx\right)$$
$$= \left[\frac{x^3}{3} + \tfrac{1}{2}e^{2x} + 2xe^x - 2e^x\right]_0^1$$
$$= \left(\tfrac{1}{3} + \tfrac{1}{2}e^2 + 2e - 2e\right) - \left(0 + \tfrac{1}{2} + 0 - 2\right)$$
$$= \tfrac{1}{3} + \tfrac{1}{2}e^2 - \tfrac{1}{2} + 2 = \tfrac{11}{6} + \tfrac{1}{2}e^2 = \frac{11+3e^2}{6}$$

19. By Parts twice!: $\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$

$u = x^2 \quad \frac{du}{dx} = 2x$ $\qquad$ $u = x \quad \frac{du}{dx} = 1$
$\frac{dv}{dx} = \cos x \quad v = \sin x$ $\qquad$ $\frac{dv}{dx} = \sin x \quad v = -\cos x$

$$= x^2\sin x - 2\left(-x\cos x + \int \cos x\,dx\right)$$
$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

20. By Inspection: $\int e^{7x}\,dx = \frac{1}{7}e^{7x} + C$

By Substitution: $\int e^{7x}\,dx = \int e^u \times \frac{1}{7}du = \frac{1}{7}\int e^u\,du = \frac{1}{7}e^u + c = \frac{1}{7}e^{7x} + C$

$u = 7x \quad \frac{du}{dx} = 7$ so $dx = \frac{1}{7}du$

21. By Inspection: $\int \cos 11x\,dx = \frac{1}{11}\sin 11x + C$

By Substitution: $\int \cos 11x\,dx = \int \cos u \times \frac{1}{11}du = \frac{1}{11}\int \cos u\,du = \frac{1}{11}\sin u + C$

$u = 11x \quad \frac{du}{dx} = 11$ so $dx = \frac{1}{11}du$ $\qquad = \frac{1}{11}\sin 11x + C$

22. By Inspection: $\int xe^{5x^2}\,dx = \frac{1}{10}e^{5x^2} + C$

By Substitution: $\int xe^{5x^2}\,dx = \int e^u \times \frac{1}{10}du = \frac{1}{10}\int e^u\,du = \frac{1}{10}e^u + C = \frac{1}{10}e^{5x^2} + C$

$u = 5x^2$
$\frac{du}{dx} = 10x$ so $x\,dx = \frac{1}{10}du$

23. By Inspection: $\int \sin x\sin(\cos x)\,dx = \cos(\cos x) + C$

By Substitution: $\int \sin x\sin(\cos x)\,dx = \int \sin u \times -du = -\int \sin u\,du$

$u = \cos x$ $\qquad = \cos u + C$
$\frac{du}{dx} = -\sin x$ so $\sin x\,dx = -du$ $\qquad = \cos(\cos x) + C$

24. By Substitution Version #1: $\int_1^4 \frac{1}{1+\sqrt{x}}\,dx$

$u = 1+\sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $dx = 2\sqrt{x}\,du$

$= 2(u-1)du$

if $= 4$, $u = 1+\sqrt{4} = 3$

if $x = 1$, $u = 1+\sqrt{1} = 2$

so $\int_1^4 \frac{1}{1+\sqrt{x}}dx = \int_2^3 \frac{1}{u} \times 2(u-1)\,du = 2\int_2^3 \frac{u-1}{u}\,du = 2\int_2^3 \left(1-\frac{1}{u}\right)du$

$= 2\left[u - \ln|u|\right]_2^3$

$= 2\left[(3-\ln 3)-(2-\ln 2)\right]$

$= 2\left(1+\ln\left(\frac{2}{3}\right)\right)$

By Substitution: Version #2 (Version #1 is better!) $\int_1^4 \frac{1}{1+\sqrt{x}}dx = \int_1^2 \frac{1}{1+u} \times 2u\,du = 2\int_1^2 \frac{u}{1+u}\,du$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

so $dx = 2\sqrt{x}\,du = 2u\,du$

if $x = 4$, $u = 2$; if $x = 1$, $u = 1$

$= 2\int_1^2 \left(\frac{1+u}{1+u} - \frac{1}{1+u}\right)du$

$= 2\int_1^2 \left(1 - \frac{1}{1+u}\right)du$

$= 2\left[u - \ln|1+u|\right]_1^2$

$= 2\left[(2-\ln 3)-(1-\ln 2)\right]$

$= 2\left(1+\ln\left(\frac{2}{3}\right)\right)$

25. By Substitution: $\int_1^2 x^2\sqrt{x-1}\,dx = \int_0^1 (u+1)^2\sqrt{u}\,du = \int_0^1 (u^2+2u+1)\sqrt{u}\,du$

$u = x-1$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u+1$ so $x^2 = (u+1)^2$

if $x = 2$, $u = 1$; if $x = 1$, $u = 0$

$= \int_0^1 \left(u^{5/2} + 2u^{3/2} + u^{1/2}\right)du$

$= \left[\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1$

$= \left(\frac{2}{7}+\frac{4}{5}+\frac{2}{3}\right) - 0$

$= \frac{30}{105} + \frac{84}{105} + \frac{70}{105} = \frac{184}{105}$

26. $\int \frac{1}{x+\sqrt{x}}dx = \int \frac{1}{\sqrt{x}+1} \times \frac{1}{\sqrt{x}}dx = \int \frac{1}{u+1} \times 2du = 2\int \frac{1}{u+1}du$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $\frac{1}{\sqrt{x}}dx = 2du$

$= 2\ln|u+1| + C$

$= 2\ln(\sqrt{x}+1) + C$

27. By Substitution: $\int \frac{x^2}{\sqrt{1+x}}dx = \int \frac{(u-1)^2}{\sqrt{u}}du = \int \frac{u^2-2u+1}{\sqrt{u}}du$

$u = 1+x$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u-1$ so $x^2 = (u-1)^2$

$= \int \left(u^{3/2} - 2u^{1/2} + u^{-1/2}\right)du$

$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2} + C$

$= \frac{2}{5}(1+x)^{5/2} - \frac{4}{3}(1+x)^{3/2} + 2(1+x)^{1/2} + C$

(You could also do Integration by Parts twice, starting with $u = x^2$ and $\frac{dv}{dx} = \frac{1}{\sqrt{1+x}}$, but above is much better!)

28. By Substitution: $\int \frac{x}{1+2x}\,dx = \int \frac{\frac{u-1}{2}}{u}\,\frac{1}{2}du = \frac{1}{4}\int \frac{u-1}{u}\,du$

$u = 1+2x$ $\qquad = \frac{1}{4}\int\left(1-\frac{1}{u}\right)du$

$\frac{du}{dx} = 2$ so $dx = \frac{1}{2}du$

$u-1 = 2x$ so $x = \frac{u-1}{2}$ $\qquad = \frac{1}{4}\left(u - \ln|u|\right) + C$

$$= \frac{1}{4}(1+2x) - \frac{1}{4}\ln|1+2x| + C$$

By Rewriting and then Inspection/Substitution:

$$\int \frac{x}{1+2x}\,dx = \frac{1}{2}\int \frac{2x}{1+2x}\,dx = \frac{1}{2}\int\left(\frac{1+2x}{1+2x} - \frac{1}{1+2x}\right)dx = \frac{1}{2}\int\left(1 - \frac{1}{1+2x}\right)dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\ln|1+2x|\right) + C = \frac{1}{2}x - \frac{1}{4}\ln|1+2x| + C$$

These two answers 'seem' to be out by $\frac{1}{4}$ ... no matter! Any constants can be 'swallowed up' in the '+C'!!

29. By Substitution: $\int \frac{x^2}{1+x}\,dx = \int \frac{(u-1)^2}{u}\,du = \int \frac{u^2-2u+1}{u}\,du$

$u = 1+x$

$\frac{du}{dx} = 1$ so $du = dx$ $\qquad = \int\left(u - 2 + \frac{1}{u}\right)du$

$x = u-1$ so $x^2 = (u-1)^2$ $\qquad = \frac{u^2}{2} - 2u + \ln|u| + C$

$$= \frac{1}{2}(1+x)^2 - 2(1+x) + \ln|1+x| + C$$

By Rewriting: $\int \frac{x^2}{1+x}\,dx = \int \frac{x(1+x)-x}{1+x}\,dx = \int\left(x - \frac{x}{1+x}\right)dx$

$$= \int\left(x - \frac{1+x-1}{1+x}\right)dx = \int\left(x - \frac{1+x}{1+x} + \frac{1}{1+x}\right)dx$$

$$= \int\left(x - 1 + \frac{1}{1+x}\right)dx = \frac{x^2}{2} - x + \ln|1+x| + C$$

Comparing with the first answer,

$\frac{1}{2}(1+x)^2 - 2(1+x) = \frac{1}{2}(1+2x+x^2) - 2 - 2x = \frac{1}{2}x^2 - x - \frac{3}{2}$ This just alters the '+C'.

30. By Inspection: $\int \frac{1}{4x+7}\,dx = \frac{1}{4}\ln|4x+7| + C$

By Substitution: $\int \frac{1}{4x+7}\,dx = \int \frac{1}{u} \times \frac{1}{4}du = \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| + C$

$u = 4x+7$ $\qquad = \frac{1}{4}\ln|4x+7| + C$

$\frac{du}{dx} = 4$ so $dx = \frac{1}{4}du$

31. By Inspection: $\int \frac{x-2}{x^2-4x+11}\,dx = \frac{1}{2}\ln(x^2-4x+11) + C$

By Substitution: $\int \frac{x-2}{x^2-4x+11}\,dx = \int \frac{1}{u} \times \frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$u = x^2 - 4x + 11$ $\qquad = \frac{1}{2}\ln(x^2-4x+11) + C$

$\frac{du}{dx} = 2x - 4 = 2(x-2)$ so $(x-2)dx = \frac{1}{2}du$

32. By Rewriting and Inspection/Substitution:

$$\int \frac{x^2-x}{x^2-3x+3}\,dx = \int \frac{x^2-3x+3+2x-3}{x^2-3x+3}\,dx = \int\left(1+\frac{2x-3}{x^2-3x+3}\right)dx$$

$$= x+\ln(x^2-3x+3)+C$$

by Inspection.

by Substitution.

$$\int \frac{2x-3}{x^2-3x+3}\,dx = \int \frac{1}{u}\,du = \ln|u|+C$$
$$= \ln(x^2-3x+3)+C$$

$u = x^2-3x+3$

$\frac{du}{dx} = 2x-3$ so $(2x-3)dx = du$

so $\int\left(1+\frac{2x-3}{x^2-3x+3}\right)dx = x+\ln(x^2-3x+3)+C$

33. By Substitution: $\int_0^5 \frac{x}{\sqrt{x+4}}\,dx = \int_4^9 \frac{u-4}{\sqrt{u}}\,du = \int_4^9 (u^{1/2}-4u^{-1/2})\,du$

$u = x+4$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u-4$

if $x=5$, $u=9$; if $x=0$, $u=4$

$$= \left[\tfrac{2}{3}u^{3/2}-8u^{1/2}\right]_4^9$$
$$= \left(\tfrac{2}{3}\times27-8\times3\right)-\left(\tfrac{2}{3}\times8-8\times2\right)$$
$$= \tfrac{38}{3}-8 = \tfrac{14}{3}$$

By Parts: $\int_0^5 \frac{x}{\sqrt{x+4}}\,dx = \left[2x(x+4)^{1/2}\right]_0^5 - 2\int_0^5 (x+4)^{1/2}\,dx$

$u = x$ $\quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \frac{1}{\sqrt{x+4}}$ $\quad v = 2(x+4)^{1/2}$

$$= \left[2x(x+4)^{1/2}-\tfrac{4}{3}(x+4)^{3/2}\right]_0^5$$
$$= \left(10\times3-\tfrac{4}{3}\times27\right)-\left(0-\tfrac{4}{3}\times8\right)$$
$$= 30-\tfrac{76}{3} = \tfrac{14}{3}$$

34. By Substitution: $\int 2x(x+2)^5\,dx = \int 2(u-2)u^5\,du = 2\int(u^6-2u^5)\,du$

$u = x+2$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u-2$

$$= 2\left(\tfrac{1}{7}u^7-\tfrac{1}{3}u^6\right)+C$$
$$= \tfrac{2}{7}(x+2)^7-\tfrac{2}{3}(x+2)^6+C$$

By Parts: $\int 2x(x+2)^5\,dx = \frac{\cancel{2}x(x+2)^6}{\cancel{6}\,3} - \int \frac{\cancel{2}(x+2)^6}{\cancel{6}\,3}\,dx$

$u = 2x$ $\quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = (x+2)^5$ $\quad v = \frac{(x+2)^6}{6}$

$$= \tfrac{x}{3}(x+2)^6-\tfrac{1}{3}\int(x+2)^6\,dx$$
$$= \tfrac{x}{3}(x+2)^6-\tfrac{1}{21}(x+2)^7+C$$

How are these answers the same?!

$$\tfrac{x}{3}(x+2)^6-\tfrac{1}{21}(x+2)^7 = \tfrac{2}{7}(x+2)^7-\tfrac{1}{3}(x+2)^7+\tfrac{x}{3}(x+2)^6$$
$$= \tfrac{2}{7}(x+2)^7-\tfrac{1}{3}(x+2)^6(x+2-x)$$
$$= \tfrac{2}{7}(x+2)^7-\tfrac{2}{3}(x+2)^6$$

35. $\int\sqrt{3x-2}\,dx = \frac{2}{9}(3x-2)^{3/2}+C$ by Inspection

By Substitution: $\int\sqrt{3x-2}\,dx = \int\sqrt{u}\times\frac{1}{3}du = \frac{1}{3}\int\sqrt{u}\,du = \frac{1}{3}\times\frac{2}{3}u^{3/2}+C$

$$= \tfrac{2}{9}(3x-2)^{3/2}+C$$

$u = 3x-2$

$\frac{du}{dx} = 3$ so $dx = \frac{1}{3}du$

36. By Inspection: $\int 6x\sin(x^2-4)dx = -3\cos(x^2-4)+C$

By Substitution: $\int 6x\sin(x^2-4)dx = \int \sin u \times 3du = 3\int \sin u\, du$

$u = x^2-4$ $\quad = -3\cos u + C$

$\frac{du}{dx} = 2x$ $\quad = -3\cos(x^2-4)+C$

so $2x\,dx = du$ and $6x\,dx = 3du$

37. By Inspection: $\int 5x\cos(5-x^2)dx = -\frac{5}{2}\sin(5-x^2)+C$

By Substitution: $\int 5x\cos(5-x^2)dx = \int \cos u \times -\frac{5}{2}du = -\frac{5}{2}\int \cos u\, du$

$u = 5-x^2$

$\frac{du}{dx} = -2x$ $\quad = -\frac{5}{2}\sin u + C$

so $-2x\,dx = du$ and $5x\,dx = -\frac{5}{2}du$ $\quad = -\frac{5}{2}\sin(5-x^2)+C$

38. By Substitution: $\int x(x+2)^9dx = \int (u-2)u^9du = \int (u^{10}-2u^9)du$

$u = x+2$ $\quad = \frac{1}{11}u^{11} - \frac{1}{5}u^{10} + C$

$\frac{du}{dx} = 1$ so $du = dx$ $\quad = \frac{1}{11}(x+2)^{11} - \frac{1}{5}(x+2)^{10} + C$

$x = u-2$

By Parts: $\int x(x+2)^9dx = \frac{1}{10}x(x+2)^{10} - \int \frac{1}{10}(x+2)^{10}dx$

$u = x$ $\quad \frac{du}{dx} = 1$ $\quad = \frac{1}{10}x(x+2)^{10} - \frac{1}{110}(x+2)^{11} + C$

$\frac{dv}{dx} = (x+2)^9$ $\quad v = \frac{1}{10}(x+2)^{10}$

How on <u>earth</u> are these two answers the same?!

$\frac{1}{11}(x+2)^{11} - \frac{1}{5}(x+2)^{10} = -\frac{1}{110}(x+2)^{11} + \frac{1}{10}(x+2)^{11} - \frac{1}{5}(x+2)^{10}$

$= -\frac{1}{110}(x+2)^{11} + \frac{1}{10}(x+2)^{10}((x+2)-2)$

$= -\frac{1}{110}(x+2)^{11} + \frac{1}{10}x(x+2)^{10}$

39. By Inspection: $\int (x+2)^9dx = \frac{1}{10}(x+2)^{10}+C$

By Substitution: $\int (x+2)^9dx = \int u^9du = \frac{u^{10}}{10}+C = \frac{(x+2)^{10}}{10}+C$

$u = x+2$

$\frac{du}{dx} = 1$ so $du = dx$

40. By Inspection: $\int (3x+2)^9dx = \frac{1}{30}(3x+2)^{10}+C$

By Substitution: $\int (3x+2)^9dx = \int u^9 \times \frac{1}{3}du = \frac{1}{3}\int u^9du = \frac{1}{3}\times\frac{u^{10}}{10}+C$

$u = 3x+2$

$\frac{du}{dx} = 3$ so $dx = \frac{1}{3}du$ $\quad = \frac{u^{10}}{30}+C = \frac{(3x+2)^{10}}{30}+C$

41. By Substitution: $\int \frac{3x}{\sqrt{2x+3}}dx = \int \frac{\frac{3}{2}(u-3)}{\sqrt{u}}\times\frac{1}{2}du = \frac{3}{4}\int \frac{u-3}{\sqrt{u}}du$

$u = 2x+3$

$\frac{du}{dx} = 2$ so $dx = \frac{1}{2}du$ $\quad = \frac{3}{4}\int (u^{1/2} - 3u^{-1/2})du$

$2x = u-3$ so $3x = \frac{3}{2}(u-3)$ $\quad = \frac{3}{4}\left(\frac{2}{3}u^{3/2} - 6u^{1/2}\right)+C$

$= \frac{1}{2}u^{3/2} - \frac{9}{2}u^{1/2} + C$

(You could also do parts with $u = 3x$ and $\frac{dv}{dx} = \frac{1}{\sqrt{2x+3}}$ ... much harder!)

$$= \frac{1}{2}(2x+3)^{3/2} - \frac{9}{2}(2x+3)^{1/2} + C$$

42. By Substitution: $\int_1^2 (x+2)(x-1)^5 dx = \int_0^1 (u+3)u^5 du = \int_0^1 (u^6 + 3u^5) du$

$u = x-1$

$\frac{du}{dx} = 1$ so $du = dx$ $\qquad = \left[\frac{u^7}{7} + \frac{u^6}{2}\right]_0^1$

$x = u+1$ so $x+2 = u+3$

if $x=2$, $u=1$; if $x=1$, $u=0$ $\qquad = \left(\frac{1}{7} + \frac{1}{2}\right) - 0 = \frac{9}{14}$

By Parts: $\int_1^2 (x+2)(x-1)^5 dx = \left[\frac{1}{6}(x+2)(x-1)^6\right]_1^2 - \frac{1}{6}\int_1^2 (x-1)^6 dx$

$u = x+2$ $\quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = (x-1)^5$ $\quad v = \frac{(x-1)^6}{6}$ $\qquad = \left[\frac{1}{6}(x+2)(x-1)^6 - \frac{1}{42}(x-1)^7\right]_1^2$

$= \left(\frac{1}{6} \times 4 \times 1 - \frac{1}{42} \times 1\right) - 0$

$= \frac{27}{42} = \frac{9}{14}$

43. By Substitution: $\int_0^1 4x(2x-1)^4 dx = \int_{-1}^1 2(u+1)u^4 \frac{1}{2} du$

$u = 2x-1$ $\qquad = \int_{-1}^1 (u^5 + u^4) du$

$\frac{du}{dx} = 2$ so $dx = \frac{1}{2}du$ $\qquad = \left[\frac{u^6}{6} + \frac{u^5}{5}\right]_{-1}^1$

$2x = u+1$ so $4x = 2(u+1)$ $\qquad = \left(\frac{1}{6} + \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{5}\right)$

if $x=1$, $u=1$; if $x=0$, $u=-1$ $\qquad = \frac{2}{5}$

By Parts: $\int_0^1 4x(2x-1)^4 dx = \left[\frac{2}{5}x(2x-1)^5\right]_0^1 - \int_0^1 \frac{2}{5}(2x-1)^5 dx$

$u = 4x$ $\quad \frac{du}{dx} = 4$ $\qquad = \left[\frac{2}{5}x(2x-1)^5 - \frac{1}{30}(2x-1)^6\right]_0^1$

$\frac{dv}{dx} = (2x-1)^4$ $\quad v = \frac{1}{10}(2x-1)^5$ $\qquad = \left(\frac{2}{5} \times 1 \times 1 - \frac{1}{30} \times 1\right) - \left(0 - \frac{1}{30} \times 1\right)$

$= \frac{2}{5} - \frac{1}{30} + \frac{1}{30} = \frac{2}{5}$

44. By Parts: $\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

$u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^3$ $\quad v = \frac{x^4}{4}$

45. By Parts: $\int \ln x \, dx = \int 1 \times \ln x \, dx = x\ln x - \int 1 \, dx = x \ln x - x + C$

$u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1$ $\quad v = x$

46. By Parts: $\int 2x \sin(3x-1) dx = -\frac{2}{3}x\cos(3x-1) + \frac{2}{3}\int \cos(3x-1) dx$

$u = 2x$ $\quad \frac{du}{dx} = 2$ $\qquad = -\frac{2}{3}x\cos(3x-1) + \frac{2}{9}\sin(3x-1) + C$

$\frac{dv}{dx} = \sin(3x-1)$ $\quad v = -\frac{1}{3}\cos(3x-1)$

47. By Parts: $\int 2x \ln(x+1) dx = x^2 \ln(x+1) - \int \frac{x^2}{x+1} dx$

$u = \ln(x+1)$ $\quad \frac{du}{dx} = \frac{1}{x+1}$

$\frac{dv}{dx} = 2x$ $\quad v = x^2$

By Rewriting: $\int \frac{x^2}{x+1} dx = \int \frac{x(x+1) - x}{x+1} dx = \int \frac{x(x+1) - (x+1) + 1}{x+1} dx$

$= \int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{x^2}{2} - x + \ln|x+1| + C$

so $\int 2x\ln(x+1)\,dx = x^2\ln(x+1) - \int \frac{x^2}{x+1}\,dx$

$$= x^2\ln(x+1) - \left(\frac{x^2}{2} - x + \ln|x+1|\right) + C$$

$$= x^2\ln(x+1) - \frac{x^2}{2} + x - \ln|x+1| + C$$

48. By Parts: $\int x\ln(2x+1)\,dx \quad = \frac{1}{2}x^2\ln(2x+1) - \int \frac{x^2}{2x+1}\,dx$

$u = \ln(2x+1) \quad \frac{du}{dx} = \frac{2}{2x+1}$

$\frac{dv}{dx} = x \quad v = \frac{x^2}{2}$

By Rewriting: $\int \frac{x^2}{2x+1}\,dx = \int \frac{\frac{1}{2}x(2x+1) - \frac{1}{2}x}{2x+1}\,dx = \int \frac{\frac{1}{2}x(2x+1) - \frac{1}{4}(2x+1) + \frac{1}{4}}{2x+1}\,dx$

$$= \int\left(\frac{1}{2}x - \frac{1}{4} + \frac{1}{4} \times \frac{1}{2x+1}\right)dx = \frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{8}\ln|2x+1| + C$$

so $\int x\ln(2x+1)\,dx = \frac{1}{2}x^2\ln(2x+1) - \int \frac{x^2}{2x+1}\,dx$

$$= \frac{1}{2}x^2\ln(2x+1) - \left(\frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{8}\ln|2x+1|\right) + C$$

$$= \frac{1}{2}x^2\ln(2x+1) - \frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8}\ln|2x+1| + C$$

49. By Parts: $\int \frac{1}{x^3}\ln x\,dx \quad = -\frac{1}{2} \times \frac{\ln x}{x^2} + \frac{1}{2}\int \frac{1}{x^3}\,dx = \frac{-\ln x}{2x^2} + \frac{1}{2} \times \frac{x^{-2}}{-2} + C$

$u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{x^3} \quad v = -\frac{x^{-2}}{2}$

$$= \frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C$$

50. By Parts, twice!: $\int x^2 e^{3x}\,dx \quad = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx$

$u = x^2 \quad \frac{du}{dx} = 2x \qquad\qquad u = x \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{3x} \quad v = \frac{1}{3}e^{3x} \qquad\qquad \frac{dv}{dx} = e^{3x} \quad v = \frac{1}{3}e^{3x}$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx\right)$$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

51. By Parts, twice and rearrange!:

$\int e^x\sin x\,dx \quad = e^x\sin x - \int e^x\cos x\,dx$

$u = \sin x \quad \frac{du}{dx} = \cos x \qquad\qquad u = \cos x \quad \frac{du}{dx} = -\sin x$

$\frac{dv}{dx} = e^x \quad v = e^x \qquad\qquad \frac{dv}{dx} = e^x \quad v = e^x$

$$= e^x\sin x - \left(e^x\cos x + \int e^x\sin x\,dx\right)$$

so $\int e^x\sin x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$

so $2\int e^x\sin x\,dx = e^x(\sin x - \cos x) + C$

so $\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + D$ $\leftarrow D = \frac{1}{2}C$!!

52. By Parts, twice and rearrange!:

$\int e^x\cos 2x\,dx \quad = e^x\cos 2x + 2\int e^x\sin 2x\,dx$

$u = \cos 2x \quad \frac{du}{dx} = -2\sin 2x \qquad\qquad u = \sin 2x \quad \frac{du}{dx} = 2\cos 2x$

$\frac{dv}{dx} = e^x \quad v = e^x \qquad\qquad \frac{dv}{dx} = e^x \quad v = e^x$

$$= e^x\cos 2x + 2\left(e^x\sin 2x - 2\int e^x\cos 2x\,dx\right)$$

So $\int e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x - 4\int e^x\cos 2x\,dx$

so $5\int e^x\cos 2x\,dx = e^x(\cos 2x + 2\sin 2x) + C$

so $\int e^x\cos 2x\,dx = \frac{1}{5}e^x(\cos 2x + 2\sin 2x) + D$ ← $D = \frac{1}{5}C$!!

53. By Parts: $\int \ln(2x+1)\,dx = \int 1\times\ln(2x+1)\,dx = x\ln(2x+1) - \int\frac{2x}{2x+1}\,dx$

$u = \ln(2x+1) \quad \frac{du}{dx} = \frac{2}{2x+1}$

$\frac{dv}{dx} = 1 \quad v = x$

By Rewriting: $\int\frac{2x}{2x+1}\,dx = \int\frac{2x+1-1}{2x+1}\,dx = \int\left(1 - \frac{1}{2x+1}\right)dx = x - \frac{1}{2}\ln|2x+1| + C$

so $\int\ln(2x+1)\,dx = x\ln(2x+1) - \left(x - \frac{1}{2}\ln|2x+1|\right) + C$

$= x\ln(2x+1) - x + \frac{1}{2}\ln|2x+1| + C$

54. By Parts: $\int_1^4 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^4 + \int_1^4\frac{1}{x^2}\,dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^4$

$u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{x^2} \quad v = -\frac{1}{x}$

$= \left(-\frac{\ln 4}{4} - \frac{1}{4}\right) - (0 - 1)$

$= -\frac{2\ln 2}{4} - \frac{1}{4} + 1 = \frac{3}{4} - \frac{1}{2}\ln 2$

55. $\int\tan x\,dx = \int\frac{\sin x}{\cos x}\,dx$

By Substitution: $\int\frac{\sin x}{\cos x}\,dx = \int\frac{1}{u}\times -du = -\int\frac{1}{u}\,du = -\ln|u| + C$

$= -\ln|\cos x| + C$

You might also get this just by Inspection!

$u = \cos x$

$\frac{du}{dx} = -\sin x$ so $\sin x\,dx = -du$

56. Hopefully straightforward: $\int\frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} + C = -\frac{1}{x} + C$

57. Ditto!: $\int\frac{2}{x}\,dx = 2\int\frac{1}{x}\,dx = 2\ln|x| + C$

58. By Inspection: $\int\frac{8x}{(x^2-4)^5}\,dx = -(x^2-4)^{-4} + C$

By Substitution: $\int\frac{8x}{(x^2-4)^5}\,dx = \int\frac{1}{u^5}\times 4\,du = 4\int u^{-5}\,du = 4\times\frac{u^{-4}}{-4} + C$

$u = x^2 - 4$

$= -u^{-4} + C$

$\frac{du}{dx} = 2x$ so $2x\,dx = du$

$= -(x^2-4)^{-4} + C$

so $8x\,dx = 4\,du$

59. By Rewriting: $\int 4x(3x-5)\,dx = \int(12x^2 - 20x)\,dx = 4x^3 - 10x^2 + C$

(you could do this by Parts... but is there really any need!)

60. By Inspection: $\int 9\cos 3x\,dx = 3\sin 3x + C$

By Substitution: $\int 9\cos 3x\,dx = \int 9\cos u\times\frac{1}{3}\,du = \int 3\cos u\,du$

$u = 3x$

$= 3\sin u + C$

$\frac{du}{dx} = 3$ so $dx = \frac{1}{3}\,du$

$= 3\sin 3x + C$

61. By Inspection: $\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\ln|x^3+1| + C$

By Substitution: $\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+1| + C$

$u = x^3+1$

$\frac{du}{dx} = 3x^2$ so $3x^2dx = du$ and $x^2dx = \frac{1}{3}du$

62. By Inspection: $\int \frac{\cos(2x+3)}{\sin^4(2x+3)}\,dx = -\frac{1}{6}(\sin(2x+3))^{-3} + C$

By Substitution: $\int \frac{\cos(2x+3)}{\sin^4(2x+3)}\,dx = \frac{1}{2}\int u^{-4}\,du = \frac{1}{2}\times\frac{u^{-3}}{-3} + C = -\frac{1}{6}u^{-3} + C$

$= -\frac{1}{6}(\sin(2x+3))^{-3} + C$

or

$-\frac{1}{6\sin^3(2x+3)} + C$

$u = \sin(2x+3)$

$\frac{du}{dx} = 2\cos(2x+3)$ so $\cos(2x+3)\,dx = \frac{1}{2}du$

63. By Inspection: $\int x\cos(x^2+3)\,dx = \frac{1}{2}\sin(x^2+3) + C$

By Substitution: $\int x\cos(x^2+3)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C$

$= \frac{1}{2}\sin(x^2+3) + C$

$u = x^2+3$

$\frac{du}{dx} = 2x$ so $x\,dx = \frac{1}{2}du$

64. By Inspection: $\int \frac{2x}{\cos^2(x^2+3)}\,dx = \tan(x^2+3) + C$

By Substitution: $\int \frac{2x}{\cos^2(x^2+3)}\,dx = \int \frac{1}{\cos^2 u}\,du = \tan u + C$

$= \tan(x^2+3) + C$

$u = x^2+3$

$\frac{du}{dx} = 2x$ so $2x\,dx = du$

65. By Inspection: $\int_0^{\pi} \sin\left(\frac{3x}{2}\right)dx = \left[-\frac{2}{3}\cos\left(\frac{3x}{2}\right)\right]_0^{\pi} = -\frac{2}{3}\times 0 + \frac{2}{3}\times 1 = \frac{2}{3}$

By Substitution: $\int_0^{\pi} \sin\left(\frac{3x}{2}\right)dx = \frac{2}{3}\int_0^{\frac{3\pi}{2}} \sin u\,du = \frac{2}{3}\left[-\cos u\right]_0^{3\pi/2}$

$= \frac{2}{3}(0+1) = \frac{2}{3}$

$u = \frac{3x}{2}$

$\frac{du}{dx} = \frac{3}{2}$ so $\frac{3}{2}dx = du$

so $dx = \frac{2}{3}du$

if $x = \pi$, $u = \frac{3\pi}{2}$

if $x = 0$, $u = 0$

66. $\int_0^{\pi/4} \frac{1}{\cos^2 x}\,dx = \left[\tan x\right]_0^{\pi/4} = 1 - 0 = 1$

67. By Substitution: $\int_4^5 \frac{2x-1}{(x-3)^{3/2}}\,dx = \int_1^2 \frac{2u+5}{u^{3/2}}\,du = \int_1^2 \left(2u^{-1/2} + 5u^{-3/2}\right)du$

$= \left[4u^{1/2} - 10u^{-1/2}\right]_1^2$

$= \left(4\sqrt{2} - \frac{10}{\sqrt{2}}\right) - (4-10) = 6 - \sqrt{2}$

$u = x-3$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u+3$ so $2x-1 = 2u+5$

if $x=4$, $u=1$; if $x=5$, $u=2$

By Parts: $\int_4^5 \frac{2x-1}{(x-3)^{3/2}}\,dx = \left[-2(2x-1)(x-3)^{-1/2}\right]_4^5 + 4\int_4^5 (x-3)^{-1/2}\,dx$

$u = 2x-1 \quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = \frac{1}{(x-3)^{3/2}} \quad v = -2(x-3)^{-1/2}$

$= \left[-2(2x-1)(x-3)^{-1/2} + 8(x-3)^{1/2}\right]_4^5$

$= \left(-2\times 9\times\frac{1}{\sqrt{2}} + 8\sqrt{2}\right) - (-2\times 7\times 1 + 8\times 1)$

$= -18\times\frac{\sqrt{2}}{2} + 8\sqrt{2} + 14 - 8$

$= 6 - \sqrt{2}$

68. By Rewriting: $\int \left(x - \frac{2}{x}\right)^2 dx = \int (x^2 - 4 + 4x^{-2})\,dx$

$= \frac{x^3}{3} - 4x - 4x^{-1} + C$

69. By Parts: $\int_0^\pi x\sin\left(\frac{1}{3}x\right)dx = \left[-3x\cos\left(\frac{1}{3}x\right)\right]_0^\pi + 3\int_0^\pi \cos\left(\frac{1}{3}x\right)dx$

$u = x \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin\left(\frac{1}{3}x\right) \quad v = -3\cos\left(\frac{1}{3}x\right)$

$= \left[-3x\cos\left(\frac{1}{3}x\right) + 9\sin\left(\frac{1}{3}x\right)\right]_0^\pi$

$= \left(-3\pi\times\frac{1}{2} + 9\times\frac{\sqrt{3}}{2}\right) - (0+0)$

$= \frac{1}{2}(9\sqrt{3} - 3\pi) = \frac{3}{2}(3\sqrt{3} - \pi)$

70. By Inspection: $\int_e^{e^3} \frac{1}{x(\ln x)^2}\,dx = \left[-(\ln x)^{-1}\right]_e^{e^3} = \frac{-1}{\ln(e^3)} + \frac{1}{\ln e} = -\frac{1}{3} + 1 = \frac{2}{3}$

By Substitution: $\int_e^{e^3} \frac{1}{x(\ln x)^2}\,dx = \int_1^3 u^{-2}\,du = \left[-u^{-1}\right]_1^3$

$= -\frac{1}{3} + 1 = \frac{2}{3}$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ so $\frac{1}{x}dx = du$

if $x = e^3$, $u = 3$; if $x = e$, $u = 1$

71. By Inspection: $\int \frac{\sqrt{\ln x}}{x}\,dx = \frac{2}{3}(\ln x)^{3/2} + C$

By Substitution: $\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\ln x)^{3/2} + C$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ so $\frac{1}{x}dx = du$

72. By Parts: $\int_0^1 2xe^{5x}\,dx = \left[\frac{2}{5}xe^{5x}\right]_0^1 - \frac{2}{5}\int_0^1 e^{5x}\,dx$

$u = 2x \quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = e^{5x} \quad v = \frac{1}{5}e^{5x}$

$= \left[\frac{2}{5}xe^{5x} - \frac{2}{25}e^{5x}\right]_0^1$

$= \left(\frac{2}{5}e^5 - \frac{2}{25}e^5\right) - \left(0 - \frac{2}{25}\right)$

$= \frac{8}{25}e^5 + \frac{2}{25} = \frac{2}{25}(4e^5 + 1)$

73 $\int \tan 2x\,dx = \int \frac{\sin 2x}{\cos 2x}\,dx$

By Inspection: $\int \frac{\sin 2x}{\cos 2x}\,dx = -\frac{1}{2}\ln|\cos 2x| + C$

By Substitution: $\int \frac{\sin 2x}{\cos 2x} dx = -\frac{1}{2}\int \frac{1}{u} du = -\frac{1}{2}\ln|u| + C$
$= -\frac{1}{2}\ln|\cos 2x| + C$

$u = \cos 2x$

$\frac{du}{dx} = -2\sin 2x$ so $\sin 2x\, dx = -\frac{1}{2} du$

74. By Inspection: $\int e^{2x}(e^{2x}+1)^6 dx = \frac{1}{14}(e^{2x}+1)^7 + C$

By Substitution: $\int e^{2x}(e^{2x}+1)^6 dx = \frac{1}{2}\int u^6 du = \frac{1}{2}\times\frac{u^7}{7} + C$
$= \frac{1}{14}(e^{2x}+1)^7 + C$

$u = e^{2x}+1$

$\frac{du}{dx} = 2e^{2x}$ so $e^{2x} dx = \frac{1}{2} du$

75. By Substitution: $\int_0^1 \frac{2x-1}{(x-3)^2} dx = \int_{-3}^{-2} \frac{2u+5}{u^2} du = \int_{-3}^{-2}\left(\frac{2}{u} + \frac{5}{u^2}\right) du$
$= \left[2\ln|u| - 5u^{-1}\right]_{-3}^{-2}$
$= \left(2\ln 2 + \frac{5}{2}\right) - \left(2\ln 3 + \frac{5}{3}\right)$
$= 2\ln\left(\frac{2}{3}\right) + \frac{5}{6}$

$u = x-3$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u+3$ so $2x-1 = 2u+5$

if $x=1$, $u=-2$; if $x=0$, $u=-3$.

By Parts: $\int_0^1 \frac{2x-1}{(x-3)^2} dx = \left[-(2x-1)(x-3)^{-1}\right]_0^1 + 2\int_0^1 \frac{1}{x-3} dx$
$= \left[-(2x-1)(x-3)^{-1} + 2\ln|x-3|\right]_0^1$
$= \left(-1\times -\frac{1}{2} + 2\ln|-2|\right) - \left(-(-1)\times -\frac{1}{3} + 2\ln|-3|\right)$
$= \frac{1}{2} + 2\ln 2 + \frac{1}{3} - 2\ln 3 = 2\ln\left(\frac{2}{3}\right) + \frac{5}{6}$

$u = 2x-1$ $\quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = (x-3)^{-2}$ $\quad v = -(x-3)^{-1}$

76. By Inspection: $\int \sin x\, e^{\cos x} dx = -e^{\cos x} + C$

By Substitution: $\int \sin x\, e^{\cos x} dx = -\int e^u du = -e^u + C$
$= -e^{\cos x} + C$

$u = \cos x$

$\frac{du}{dx} = -\sin x$ so $\sin x\, dx = -du$

77. By Rewriting: $\int \frac{x^3}{1+x} dx = \int \frac{x^2(1+x) - x^2}{1+x} dx = \int \frac{x^2(1+x) - x(1+x) + x}{1+x} dx$

$= \int \frac{x^2(1+x) - x(1+x) + (1+x) - 1}{1+x} dx$

$= \int \left(x^2 - x + 1 - \frac{1}{1+x}\right) dx$

$= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|1+x| + C$

By Substitution: $\int \frac{x^3}{1+x} dx = \int \frac{(u-1)^3}{u} du = \int \frac{u^3 - 3u^2 + 3u - 1}{u} du$
$= \int \left(u^2 - 3u + 3 - \frac{1}{u}\right) du$
$= \frac{u^3}{3} - 3\frac{u^2}{2} + 3u - \ln|u| + C$
$= \frac{(1+x)^3}{3} - \frac{3(1+x)^2}{2} + 3(1+x)$
$- \ln|1+x| + C$

$u = 1+x$

$\frac{du}{dx} = 1$ so $du = dx$

$x = u-1$ so $x^3 = (u-1)^3$

bad layout from the teacher, sorry!

N.B. that $\frac{(1+x)^3}{3} - \frac{3(1+x)^2}{2} + 3(1+x) = \frac{1+3x+3x^2+x^3}{3} - \frac{3(1+2x+x^2)}{2} + 3 + 3x$

$= x^3\left(\frac{1}{3}\right) + x^2\left(1-\frac{3}{2}\right) + x(1-3+3) + \frac{1}{3} - \frac{3}{2} + 3$

$= \frac{x^3}{3} - \frac{x^2}{2} + x + \frac{11}{6}$

so the two answers do agree with each other, just with an adjustment of the integration constant!

78. By Inspection: $\int_0^{3/4} x\sqrt{1+x^2}\,dx = \left[\frac{1}{3}(1+x^2)^{3/2}\right]_0^{3/4} = \frac{1}{3}\left(\frac{25}{16}\right)^{3/2} - \frac{1}{3}\times 1$

$= \frac{1}{3}\times\frac{125}{64} - \frac{1}{3} = \frac{1}{3}\left(\frac{125}{64}-1\right)$

$= \frac{1}{3}\times\frac{61}{64} = \frac{61}{192}$

By Substitution: $\int_0^{3/4} x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_1^{\frac{25}{16}} \sqrt{u}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^{\frac{25}{16}}$

$u = 1+x^2$

$\frac{du}{dx} = 2x$ so $x\,dx = \frac{1}{2}du$

if $x = \frac{3}{4}$, $u = 1+\frac{9}{16} = \frac{25}{16}$; if $x = 0$, $u = 1$

$= \frac{1}{2}\left(\frac{2}{3}\times\frac{125}{64} - \frac{2}{3}\times 1\right)$

$= \frac{1}{3}\times\frac{125}{64} - \frac{1}{3} = \frac{1}{3}\left(\frac{125}{64}-1\right)$

$= \frac{1}{3}\times\frac{61}{64} = \frac{61}{192}$

79. By Inspection: $\int_1^2 (1-2x)^{-2}\,dx = \left[\frac{1}{2}(1-2x)^{-1}\right]_1^2 = \frac{1}{2}\times -\frac{1}{3} - \frac{1}{2}\times -1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

By Substitution: $\int_1^2 (1-2x)^{-2}\,dx = -\frac{1}{2}\int_{-1}^{-3} u^{-2}\,du = -\frac{1}{2}\left[-u^{-1}\right]_{-1}^{-3}$

$u = 1-2x$

$\frac{du}{dx} = -2$ so $dx = -\frac{1}{2}du$

if $x = 2$, $u = -3$; if $x = 1$, $u = -1$

$= -\frac{1}{2}\left(\frac{1}{3} - 1\right) = -\frac{1}{2}\times -\frac{2}{3} = \frac{1}{3}$

80. By Inspection: $\int_1^2 (1-2x)^{-3}\,dx = \left[\frac{1}{4}(1-2x)^{-2}\right]_1^2 = \frac{1}{4}\times\frac{1}{9} - \frac{1}{4}\times 1 = \frac{1}{4}\left(\frac{1}{9}-1\right) = -\frac{8}{9}\times\frac{1}{4}$

$= -\frac{2}{9}$

By Substitution: $\int_1^2 (1-2x)^{-3}\,dx = -\frac{1}{2}\int_{-1}^{-3} u^{-3}\,du = -\frac{1}{2}\left[-\frac{1}{2}u^{-2}\right]_{-1}^{-3}$

$u = 1-2x$

$\frac{du}{dx} = -2$ so $dx = -\frac{1}{2}du$

if $x = 2$, $u = -3$; if $x = 1$, $u = -1$

$= -\frac{1}{2}\left(-\frac{1}{2}\times\frac{1}{9} + \frac{1}{2}\times 1\right)$

$= \frac{1}{4}\times\frac{1}{9} - \frac{1}{4} = \frac{1}{4}\left(\frac{1}{9}-1\right)$

$= -\frac{8}{9}\times\frac{1}{4} = -\frac{2}{9}$

81. By Substitution: $\int_1^2 x(1-2x)^{-3}\,dx = -\frac{1}{2}\int_{-1}^{-3} \frac{1-u}{2}u^{-3}\,du$

$u = 1-2x$

$\frac{du}{dx} = -2$ so $dx = -\frac{1}{2}du$

$2x = 1-u$ so $x = \frac{1-u}{2}$

if $x = 2$, $u = -3$; if $x = 1$, $u = -1$

$= -\frac{1}{4}\int_{-1}^{-3} \frac{1-u}{u^3}\,du$

$= -\frac{1}{4}\int_{-1}^{-3} (u^{-3} - u^{-2})\,du$

$= -\frac{1}{4}\left[-\frac{1}{2}u^{-2} + u^{-1}\right]_{-1}^{-3}$

$= -\frac{1}{4}\left(\left(-\frac{1}{2}\times\frac{1}{9} - \frac{1}{3}\right) - \left(-\frac{1}{2}\times 1 - 1\right)\right)$

$= -\frac{1}{4}\left(-\frac{1}{18} - \frac{1}{3} + \frac{1}{2} + 1\right) = -\frac{1}{4}\times\frac{20}{18} = \frac{-5}{18}$

By Parts: $\int_1^2 x(1-2x)^{-3}\,dx = \left[\frac{1}{4}x(1-2x)^{-2}\right]_1^2 - \frac{1}{4}\int_1^2 (1-2x)^{-2}\,dx$

$u = x$ $\quad \frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = (1-2x)^{-3}$ $\quad v = \frac{1}{4}(1-2x)^{-2}$

$$= \left[\tfrac{1}{4}x(1-2x)^{-2} - \tfrac{1}{4}\times\tfrac{1}{2}(1-2x)^{-1}\right]_1^2$$

$$=\left(\tfrac{1}{4}\times 2\times\tfrac{1}{9} - \tfrac{1}{8}\times -\tfrac{1}{3}\right) - \left(\tfrac{1}{4}\times 1\times 1 - \tfrac{1}{8}\times -1\right)$$

$$= \frac{1}{18} + \frac{1}{24} - \frac{1}{4} - \frac{1}{8} = \frac{4+3-18-9}{72} = \frac{-20}{72} = \frac{-5}{18}$$

82. By Inspection: $\int \frac{\ln(x+1)}{x+1}\,dx = \frac{1}{2}(\ln(x+1))^2 + C$

By Substitution: $\int \frac{\ln(x+1)}{x+1}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln(x+1))^2}{2} + C$

$u = \ln(x+1)$

$\frac{du}{dx} = \frac{1}{x+1}$ so $\frac{1}{x+1}dx = du$

By Parts: $\int \frac{\ln(x+1)}{x+1}\,dx = (\ln(x+1))^2 - \int \frac{\ln(x+1)}{x+1}\,dx$

$u = \ln(x+1)$ $\frac{du}{dx} = \frac{1}{x+1}$

$\frac{dv}{dx} = \frac{1}{x+1}$ $v = \ln(x+1)$

So $2\int \frac{\ln(x+1)}{x+1}\,dx = (\ln(x+1))^2 + C$

so $\int \frac{\ln(x+1)}{x+1}\,dx = \frac{1}{2}(\ln(x+1))^2 + D$

$= \frac{1}{2}C$!!

83. By Inspection: $\int x^6 e^{x^7-1}\,dx = \frac{1}{7}e^{x^7-1} + C$

By Substitution: $\int x^6 e^{x^7-1}\,dx = \frac{1}{7}\int e^u\,du = \frac{1}{7}e^u + c = \frac{1}{7}e^{x^7-1} + C$

$u = x^7 - 1$

$\frac{du}{dx} = 7x^6$ so $x^6\,dx = \frac{1}{7}du$

84. By Substitution: $\int x(3-x)^{10}\,dx = -\int(3-u)u^{10}\,du = -\int(3u^{10} - u^{11})\,du$

$u = 3-x$ $\quad = -\left(\frac{3}{11}u^{11} - \frac{1}{12}u^{12}\right) + C$

$\frac{du}{dx} = -1$ so $dx = -du$ $\quad = -\frac{3}{11}(3-x)^{11} + \frac{1}{12}(3-x)^{12} + C$

$x = 3-u$

By Parts: $\int x(3-x)^{10}\,dx = -\frac{1}{11}x(3-x)^{11} + \frac{1}{11}\int(3-x)^{11}\,dx$

$u = x$ $\frac{du}{dx} = 1$ $\quad = -\frac{1}{11}x(3-x)^{11} - \frac{1}{132}(3-x)^{12} + C$

$\frac{dv}{dx} = (3-x)^{10}$ $v = -\frac{1}{11}(3-x)^{11}$

Both answers are correct since

$$-\tfrac{1}{11}x(3-x)^{11} - \tfrac{1}{132}(3-x)^{12} = \tfrac{1}{12}(3-x)^{12} - \tfrac{1}{11}(3-x)^{12} - \tfrac{1}{11}x(3-x)^{11}$$

$$= -\tfrac{1}{12}(3-x)^{12} - \tfrac{1}{11}(3-x)^{11}(3-x+x)$$

$$= -\tfrac{1}{12}(3-x)^{12} - \tfrac{3}{11}(3-x)^{11}$$

85. $\int_{-\pi}^{0} x\sin x\,dx = \left[-x\cos x\right]_{-\pi}^{0} + \int_{-\pi}^{0}\cos x\,dx$

By Parts:

$u = x$ $\frac{du}{dx} = 1$ $\quad = \left[-x\cos x + \sin x\right]_{-\pi}^{0}$

$\frac{dv}{dx} = \sin x$ $v = -\cos x$ $\quad = (0+0) - (-(-\pi)\times(-1) + 0) = \pi$

This is the same answer as the answer to Q16. The reason for this is that

$f(x) = x\sin x$ is an even function:

$$f(-x) = -x\sin(-x) = -x \times -\sin x = x\sin x = f(x)$$

So the graph of $f(x) = x\sin x$ is symmetrical in the y-axis

Therefore the area from $x=-\pi$ to $x=0$ will be the same as the area from $x=0$ to $x=\pi$.

86. By Inspection: $\int \frac{2(\ln x)^2 + 3\ln x - 1}{x}\,dx = \frac{2}{3}(\ln x)^3 + \frac{3}{2}(\ln x)^2 - \ln x + C$

By Substitution: $\int \frac{2(\ln x)^2 + 3\ln x - 1}{x}\,dx = \int (2u^2 + 3u - 1)\,du$

$= \frac{2}{3}u^3 + \frac{3}{2}u^2 - u + C$

$u = \ln x$

$= \frac{2}{3}(\ln x)^3 + \frac{3}{2}(\ln x)^2 - \ln x + C$

$\frac{du}{dx} = \frac{1}{x}$ so $\frac{1}{x}dx = du$

87. By Inspection: $\int \frac{1}{x}\cos(\ln x)\,dx = \sin(\ln x) + C$

By Substitution: $\int \frac{1}{x}\cos(\ln x)\,dx = \int \cos u\,du = \sin u + C = \sin(\ln x) + C$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$ so $\frac{1}{x}dx = du$

88. By Inspection: $\int e^{5-2x}\,dx = -\frac{1}{2}e^{5-2x} + C$

By Substitution: $\int e^{5-2x}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{5-2x} + C$

$u = 5-2x$

$\frac{du}{dx} = -2$ so $dx = -\frac{1}{2}du$

89. By Rewriting: $\int (x+\frac{1}{x})^2\,dx = \int (x^2 + 2 + x^{-2})\,dx = \frac{x^3}{3} + 2x - x^{-1} + C$

90. By Parts: $\int x^3\cos(x^2)\,dx = \int x^2 \times x\cos(x^2)\,dx = \frac{1}{2}x^2\sin(x^2) - \int x\sin(x^2)\,dx$

$u = x^2$ $\quad \frac{du}{dx} = 2x$

$= \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C$

$\frac{dv}{dx} = x\cos(x^2)$ $\quad v = \frac{1}{2}\sin(x^2)$

By Substitution then Parts: $\int x^3\cos(x^2)\,dx = \frac{1}{2}\int u\cos u\,du$

$u = x^2$

$\frac{du}{dx} = 2x$ so $x\,dx = \frac{1}{2}du$

so $x^3dx = \frac{1}{2}x^2du = \frac{1}{2}u\,du$

Then $\frac{1}{2}\int u\cos u\,du = \frac{1}{2}\left(u\sin u - \int \sin u\,du\right) = \frac{1}{2}(u\sin u + \cos u) + C$

$a = u$ $\quad \frac{da}{du} = 1$

$= \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C$

$\frac{db}{du} = \cos u$ $\quad b = \sin u$

91. By Parts: $\int_1^2 x^5\ln x\,dx = \left[\frac{1}{6}x^6\ln x\right]_1^2 - \frac{1}{6}\int_1^2 x^5\,dx$

$u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$

$= \left[\frac{1}{6}x^6\ln x - \frac{1}{36}x^6\right]_1^2$

$\frac{dv}{dx} = x^5$ $\quad v = \frac{1}{6}x^6$

$= \left(\frac{1}{6}\times 64 \times \ln 2 - \frac{1}{36}\times 64\right) - \left(0 - \frac{1}{36}\right)$

$$= \frac{32}{3}\ln 2 - \frac{64}{36} + \frac{1}{36}$$
$$= \frac{32}{3}\ln 2 - \frac{63}{36} = \frac{32}{3}\ln 2 - \frac{7}{4}$$

92. By Inspection: $\int (2x+1)(3x^2+3x-1)^{5/2}\,dx = \frac{2}{21}(3x^2+3x-1)^{7/2} + C$

By Substitution: $\int (2x+1)(3x^2+3x-1)^{5/2}\,dx = \frac{1}{3}\int u^{5/2}\,du = \frac{1}{3}\times\frac{2}{7}u^{7/2} + C$

$u = 3x^2+3x-1$ $\qquad = \frac{2}{21}(3x^2+3x-1)^{7/2} + C$

$\frac{du}{dx} = 6x+3 = 3(2x+1)$

so $(2x+1)dx = \frac{1}{3}du$

93. By Rewriting: $\int \frac{x^3}{x-2}\,dx = \int \frac{x^2(x-2)+2x^2}{x-2}\,dx = \int \frac{x^2(x-2)+2x(x-2)+4x}{x-2}\,dx$

$$= \int \frac{x^2(x-2)+2x(x-2)+4(x-2)+8}{x-2}\,dx$$
$$= \int \left(x^2+2x+4+\frac{8}{x-2}\right)dx$$
$$= \frac{x^3}{3} + x^2 + 4x + 8\ln|x-2| + C$$

By Substitution: $\int \frac{x^3}{x-2}\,dx = \int \frac{(u+2)^3}{u}\,du = \int \frac{u^3+6u^2+12u+8}{u}\,du$

$u = x-2$ $\qquad = \int \left(u^2+6u+12+\frac{8}{u}\right)du$

$\frac{du}{dx} = 1$ so $du = dx$ $\qquad = \frac{u^3}{3} + 3u^2 + 12u + 8\ln|u| + C$

$x = u+2$ $\qquad = \frac{(x-2)^3}{3} + 3(x-2)^2 + 12(x-2) + 8\ln|x-2| + C$

These answers are both correct since

$$\frac{(x-2)^3}{3} + 3(x-2)^2 + 12(x-2) = \frac{x^3-6x^2+12x-8}{3} + 3(x^2-4x+4) + 12x - 24$$
$$= \frac{x^3}{3} + x^2(-2+3) + x(4-12+12) - \frac{8}{3} + 12 - 24$$
$$= \frac{x^3}{3} + x^2 + 4x \left(-\frac{44}{3}\right)$$

This is just soaked up in the integration constant!

94. By Inspection: $\int x e^{4x^2-1}\,dx = \frac{1}{8}e^{4x^2-1} + C$

By Substitution: $\int x e^{4x^2-1}\,dx = \frac{1}{8}\int e^u\,du = \frac{1}{8}e^u + C = \frac{1}{8}e^{4x^2-1} + C$

$u = 4x^2-1$

$\frac{du}{dx} = 8x$ so $x\,dx = \frac{1}{8}du$

95. By Inspection: $\int_1^3 \frac{1}{3x-1}\,dx = \left[\frac{1}{3}\ln|3x-1|\right]_1^3 = \frac{1}{3}\ln 8 - \frac{1}{3}\ln 2 = \frac{1}{3}\times 2\ln 2 - \frac{1}{3}\ln 2$

$= \frac{1}{3}\ln 2$

By Substitution: $\int_1^3 \frac{1}{3x-1}\,dx = \frac{1}{3}\int_2^8 \frac{1}{u}\,du = \frac{1}{3}\left[\ln|u|\right]_2^8$

$u = 3x-1$ $\qquad = \frac{1}{3}(\ln 8 - \ln 2) = \frac{1}{3}(2\ln 2 - \ln 2)$

$\frac{du}{dx} = 3$ so $dx = \frac{1}{3}du$ $\qquad = \frac{1}{3}\ln 2$

if $x=3$, $u=8$; if $x=1$, $u=2$

96. By Inspection: $\int \frac{3+x}{x^2+6x-5}\,dx = \frac{1}{2}\ln(x^2+6x-5) + C$

By Substitution: $\int \frac{3+x}{x^2+6x-5}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$

$u = x^2+6x-5$ $\qquad = \frac{1}{2}\ln(x^2+6x-5) + C$

$\frac{du}{dx} = 2x+6 = 2(3+x)$

so $(3+x)dx = \frac{1}{2}du$

97. By Substitution: $\int (2x-1)(2x+1)^7\,dx = \frac{1}{2}\int (u-2)u^7\,du = \frac{1}{2}\int (u^8-2u^7)\,du$

$u = 2x+1$ $\qquad = \frac{1}{2}\left(\frac{u^9}{9} - \frac{u^8}{4}\right) + C$

$\frac{du}{dx} = 2$ so $dx = \frac{1}{2}du$ $\qquad = \frac{1}{18}(2x+1)^9 - \frac{1}{8}(2x+1)^8 + C$

$u-2 = 2x-1$

By Parts: $\int (2x-1)(2x+1)^7\,dx = \frac{1}{16}(2x-1)(2x+1)^8 - \frac{1}{8}\int (2x+1)^8\,dx$

$u = 2x-1$ $\quad \frac{du}{dx} = 2$ $\qquad = \frac{1}{16}(2x-1)(2x+1)^8 - \frac{1}{8}\times\frac{1}{18}(2x+1)^9 + C$

$\frac{dv}{dx} = (2x+1)^7$ $\quad v = \frac{1}{16}(2x+1)^8$ $\qquad = \frac{1}{16}(2x-1)(2x+1)^8 - \frac{1}{144}(2x+1)^9 + C$

These answers are both correct since

$\frac{1}{16}(2x-1)(2x+1)^8 - \frac{1}{144}(2x+1)^9 = \frac{1}{18}(2x+1)^9 - \frac{\cancel{9}^{\,1}}{\cancel{144}_{\,16}}(2x+1)^9 + \frac{1}{16}(2x-1)(2x+1)^8$

$= \frac{1}{18}(2x+1)^9 + \frac{1}{16}(2x+1)^8\left(-(2x+1) + 2x-1\right)$

$= \frac{1}{18}(2x+1)^9 - \frac{\cancel{2}^{\,1}}{\cancel{16}_{\,8}}(2x+1)^8$

98. By Inspection: $\int \sin 2x \cos^2 2x\,dx = -\frac{1}{6}\cos^3 2x + C$

By Substitution: $\int \sin 2x \cos^2 2x\,dx = -\frac{1}{2}\int u^2\,du = -\frac{1}{2}\times\frac{u^3}{3} + C = -\frac{1}{6}u^3 + C$

$u = \cos 2x$ $\qquad = -\frac{1}{6}\cos^3 2x + C$

$\frac{du}{dx} = -2\sin 2x$ so $\sin 2x\,dx = -\frac{1}{2}du$

99. By Inspection: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C$

By Substitution: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $\frac{1}{\sqrt{x}}dx = 2du$

100. By Rewriting: $\int \ln(2e^x)\,dx = \int (\ln 2 + \ln e^x)\,dx = \int (\ln 2 + x)\,dx$

$= x\ln 2 + \frac{x^2}{2} + C$

By Parts: $\int \ln(2e^x)\,dx = \int 1\times\ln(2e^x)\,dx = x\ln(2e^x) - \int x\,dx = x\ln(2e^x) - \frac{x^2}{2} + C$

$u = \ln(2e^x)$ $\quad \frac{du}{dx} = \frac{1}{2e^x}\times 2e^x = 1$ $\quad$ of course, $u = \ln(2e^x) = \ln 2 + x$ also gives $\frac{du}{dx} = 1$.

$\frac{dv}{dx} = 1$ $\quad v = x$

Both answers are correct since

$x\ln(2e^x) - \frac{x^2}{2} = x(\ln 2 + \ln e^x) - \frac{x^2}{2} = x(\ln 2 + x) - \frac{x^2}{2}$

$= x\ln 2 + x^2 - \frac{x^2}{2}$

$= x\ln 2 + \frac{x^2}{2}$