Particular Integrals

So far we've always had 0 in the RHS of the differential equation.
What if we have some function in terms of x?

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \boldsymbol{f(x)}$$

When the RHS is not 0, we have a **non-homogeneous** second order differential equation.

Solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \mathbf{0}$ first to obtain what is known as the **complementary function**. (C.F.)

Then solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \boldsymbol{f(x)}$ which can be found using appropriate substitution and comparing coefficients. Solution known as particular integral. (P.I.)

$$\boldsymbol{y = C.F. + P.I.}$$

This is because $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy$ for the C.F. is 0 and $f(x)$ for the P.I., which sum to $f(x)$

Forms of PI's to use

Form of f(x)	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m \cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

$= 5e^{3x}$

WARNING!
The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an x^2 in front of your usual PI form

$y = Ae^{2x} + Be^{3x}$

Find the **particular integral** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3}$

P.I. Try

$$y = \lambda$$

$$\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = 0$$

$$0 - 5\times 0 + 6\lambda = 3$$
$$6\lambda = 3$$
$$\lambda = \frac{1}{2}$$

P.I. is $y = \frac{1}{2}$

Hence find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{3}$

Find the C.F.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

A.E. $m^2 - 5m + 6 = 0$

$$(m-3)(m-2) = 0$$

$$m = 3 \quad m = 2$$

C.F. $y = Ae^{3x} + Be^{2x}$

G.S. $y = CF + PI$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}$$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{2x}$

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

$m = 2,\ m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

P.I. Try $y = \lambda x + \mu$

$\frac{dy}{dx} = \lambda$

$\frac{d^2y}{dx^2} = 0$

$0 - 5\lambda + 6\lambda x + 6\mu = 2x$

compare coefficients

x: $6\lambda = 2$

$\lambda = \frac{1}{3}$

const: $-5\lambda + 6\mu = 0$

$-\frac{5}{3} + 6\mu = 0$

$6\mu = \frac{5}{3}$

$\mu = \frac{5}{18}$

G.S

$y = Ae^{2x} + Be^{2x} + \frac{1}{3}x + \frac{5}{18}$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \boldsymbol{3x^2}$

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

P.I. Try $y = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$

$\frac{d^2y}{dx^2} = 2a$

Sub in $2a - 10ax - 5b + 6ax^2 + 6bx + 6c = 3x^2$

compare x^2: $6a = 3$

$a = \frac{1}{2}$

x: $-10a + 6b = 0$

$-5 + 6b = 0$

$b = \frac{5}{6}$

con: $2a - 5b + 6c = 0$

$1 - \frac{25}{6} + 6c = 0$

$6c = \frac{19}{6}$

$c = \frac{19}{36}$

Form of $f(x)$
k
$ax +$
$ax^2 + bx$
ke^{px}
$m\cos$
$m\sin$
$m\cos\omega x +$

G.S. $y = Ae^{2x} + Be^{3x} + \frac{1}{2}x^2 + \frac{5}{6}x + \frac{19}{36}$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x$

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

$m = 2,\ m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

P.I.

Try $y = \lambda e^x$

$\frac{dy}{dx} = \lambda e^x$

$\frac{d^2y}{dx^2} = \lambda e^x$

$\lambda e^x - 5\lambda e^x + 6\lambda e^x = e^x$

$2\lambda = 1$

$\lambda = \frac{1}{2}$

G.S. $y = Ae^{2x} + Be^{3x} + \frac{1}{2}e^x$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \mathbf{13\ sin\ 3x}$

$m^2 - 5m + 6 = 0$
$(m - 2)(m - 3) = 0$
$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

P.I. Try $y = a\sin 3x + b\cos 3x$

$\frac{dy}{dx} = 3a\cos 3x - 3b\sin 3x$

$\frac{d^2y}{dx^2} = -9a\sin 3x - 9b\cos 3x$

$-9a\sin 3x - 9b\cos 3x - 15a\cos 3x + 15b\sin 3x + 6a\sin 3x + 6b\cos 3x = 13\sin 3x$

comp. sin $\quad -9a + 15b + 6a = 13$

$\boxed{-3a + 15b = 13}$

comp. cos $\quad -9b - 15a + 6b = 0$

$\boxed{-15a - 3b = 0}$

$a = -\frac{1}{6} \quad b = \frac{5}{6}.$

G.S. $y = Ae^{2x} + Be^{3x} - \frac{1}{6}\sin 3x + \frac{5}{6}\cos 3x$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$

$f(x)$

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2, m = 3$

CF is $y = Ae^{2x} + Be^{3x}$

NOT GONNA WORK

Try P.I. $y = \lambda e^{2x}$

$\frac{dy}{dx} = 2\lambda e^{2x}$

$\frac{d^2y}{dx^2} = 4\lambda e^{2x}$

$4\cancel{\lambda e^{2x}} - 10\cancel{\lambda e^{2x}} + 6\cancel{\lambda e^{2x}} = e^{2x}$

$0 = e^{2x}$

Suppose we did use $y = \lambda e^{2x}$ for the particular integral. What goes wrong?

Then the general solution might appear to be $y = Ae^{2x} + Be^{3x} + \lambda e^{2x}$

$$= (A+\lambda)e^{2x} + Be^{2x}$$

But $A + \lambda$ is still just an arbitrary constant, so we have exactly the same as the complementary function, which we know gives 0 when subbed into $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y$.

Thus we end up with $0 = e^{2x}$. Oh dear!

CORRECT WAY

Try $y = \lambda x e^{2x}$

$\frac{dy}{dx} = \underline{2\lambda x e^{2x}} + \lambda e^{2x}$

$\frac{d^2y}{dx^2} = 4\lambda x e^{2x} + 2\lambda e^{2x} + 2\lambda e^{2x}$

$= 4\lambda x e^{2x} + 4\lambda e^{2x}$

$4\cancel{\lambda x e^{2x}} + 4\lambda e^{2x} - 10\cancel{\lambda x e^{2x}} - 5\lambda e^{2x} + 6\cancel{\lambda x e^{2x}} = e^{2x}$

$-\lambda e^{2x} = e^{2x}$

$\underline{\lambda = -1}$

G.S. $y = Ae^{2x} + Be^{3x} - xe^{2x}$

Find the **general solution** of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$

A.E.

$m^2 - 2m = 0$

$m = 0$

$m = 2$

C.F. $y = Ae^{0x} + Be^{2x}$

$y = A + Be^{2x}$

P.I. Try $y = ax$

$\frac{dy}{dx} = a$

$\frac{d^2y}{dx^2} = 0$

$0 - 2a = 3$

$a = -\frac{3}{2}$.

G.S.

$y = A + Be^{2x} - \frac{3}{2}x$

x x^2?

$y = (A + Bx)e^{3x}$

$\frac{d^2y}{dx^2} \quad \frac{dy}{dx} + y = e^{3x}$

Try P.I. $y = \lambda x^2 e^{3x}$

June 2010 Q8

It's that cheeky little x.

(a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation **(4 marks)**

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(b) Using your answer to part (a), find the general solution of the differential equation **(3 marks)**

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

8(a)	Differentiate twice and obtaining $\frac{dy}{dx} = \lambda \sin 5x + 5\lambda x \cos 5x$ and $\frac{d^2y}{dx^2} = 10\lambda \cos 5x - 25\lambda x \sin 5x$	M1 A1
	Substitute to give $\lambda = \frac{3}{10}$	M1 A1
(b)	Complementary function is $y = A\cos 5x + B\sin 5x$ or $Pe^{5ix} + Qe^{-5ix}$	M1 A1
	So general solution is $y = A\cos 5x + B\sin 5x + \frac{3}{10}x\sin 5x$ or in exponential form	A1ft

June 2012 Q4

Be warned: x is being used here as y was previous used.

Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t$$

AE

$m^2+5m+6=0$

$m=-2,\ m=-3$

CF.

$x = Ae^{-2t} + Be^{-3t}$

P.I Try $x = a\cos t + b\sin t$

$\frac{dx}{dt} = -a\sin t + b\cos t$

$\frac{d^2x}{dt^2} = -a\cos t - b\sin t$.

$-a\cos t - b\sin t - 5a\sin t + 5b\cos t + 6a\cos t + 6b\sin t = 2\cos t - \sin t$.

$-a+5b+6a=2$

$5a+5b=2$

$-b-5a+6b=-1$

$-5a+5b=-1$

$10b=1$

$b=\frac{1}{10}$

$a=\frac{3}{10}$

Hence

$x = Ae^{-2t} + Be^{-3t} + \frac{3}{10}\cos t + \frac{1}{10}\sin t$

Exercise 7C First column of 1, then rest of the questions