# Edexcel A Level Further Maths: Core Pure

## 3.1 Roots of Polynomials

Your notes

### Contents

Your notes

# 3.1.1 Roots of Polynomials

## Roots of Quadratics

### How are the roots of a quadratic linked to its coefficients?

- Because a quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) has roots $\alpha$ and $\beta$, you can write this equation instead in the form $a(x-\alpha)(x-\beta) = 0$
    - Note that $(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$
    - It is possible that the roots are repeated, i.e. that $\alpha = \beta$
  - You can then equate the two forms:
    - $ax^2 + bx + c = a(x-\alpha)(x-\beta)$
  - Then (because $a \neq 0$) you can divide both sides of that by *a* and expand the brackets:
    - $x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (\alpha+\beta)x + \alpha\beta$
  - Finally, compare the coefficients
    - Coefficients of *x*: $-\frac{b}{a} = (\alpha+\beta)$
    - Constant terms: $\frac{c}{a} = \alpha\beta$
- Therefore for a quadratic equation $ax^2 + bx + c = 0$ :
  - The sum of the roots $\alpha + \beta$ is equal to $-\frac{b}{a}$
  - The product of the roots $\alpha\beta$ is equal to $\frac{c}{a}$
  - Unless an exam question specifically asks you to prove these results, you can always use them without proof to answer questions about quadratics

**Related Roots**

- You may be asked to consider two quadratic equations, with the roots of the second quadratic linked to the roots of the first quadratic in some way
  - You are usually required to find the sum or product of the roots of the second equation
- The strategy is to use identities which contain $\alpha\beta$ and $\alpha + \beta$ (where $\alpha$ and $\beta$ are the roots of the first quadratic)
  - If you know the values of $\alpha$ and $\beta$ from the first quadratic, you can use them to help find the sum or product of the new roots
  - If the second quadratic has roots $\alpha^2$ and $\beta^2$, then use the identities:
    - $(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$ or $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

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- $(\alpha\beta)^2 = \alpha^2\beta^2$
- If the second quadratic has roots $\alpha^3$ and $\beta^3$, then use the identities:
  - $(\alpha+\beta)^3 = \alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)$
  - $(\alpha\beta)^3 = \alpha^3\beta^3$
- If the second quadratic has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, then use the identities:
  - $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$
  - $\frac{1}{\alpha}\times\frac{1}{\beta} = \frac{1}{\alpha\beta}$
- You can then form a new equation for a quadratic with the new roots
  - This is done by recalling that a quadratic with a given pair of roots can be written in the form $x^2$ – (sum of the roots)$x$ + (product of the roots) = 0
  - Be aware that this will not give a unique answer
    - This is because multiplying an entire quadratic by a constant does not change its roots
    - You can use this fact, for example, to find a quadratic that has a particular pair of roots AND has all integer coefficients
- See the worked example below for an example of how to do some of this!

Your notes

## Worked example

The roots of an equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{1}{2}$ and $\beta = 3$.

a) Find integer values of *a*, *b*, and *c*.

For a quadratic: $ax^2 + bx + c = 0$

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\alpha + \beta = -\frac{b}{a}$ $\quad -\frac{b}{a} = -\frac{1}{2} + 3 = \frac{5}{2}$

$\alpha\beta = \frac{c}{a}$ $\quad \frac{c}{a} = \left(-\frac{1}{2}\right)(3) = \frac{-3}{2}$

$\Rightarrow x^2 - \frac{5}{2}x - \frac{3}{2} = 0$

× by 2 to get integers $\quad 2x^2 - 5x - 3 = 0$

$2x^2 - 5x - 3 = 0$
or any multiple of this
with integer a, b, c

b) Hence find a quadratic equation whose roots are $\alpha^2$ and $\beta^2$.

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The 'new' quadratic will have $-\frac{b}{a} = \alpha^2 + \beta^2$

$\frac{c}{a} = \alpha^2 \beta^2$

$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$\alpha^2 + \beta^2 = \left(-\frac{1}{2} + 3\right)^2 - 2\left(-\frac{1}{2}\right)(3)$$

$$\alpha^2 + \beta^2 = \frac{37}{4}$$

$(\alpha\beta)^2 = \alpha^2\beta^2$

$$\alpha^2\beta^2 = \left(-\frac{1}{2} \times 3\right)^2 = \frac{9}{4}$$

$$\Rightarrow x^2 - \frac{37}{4}x + \frac{9}{4} = 0$$

$\times$ by 4 to get integers $\quad 4x^2 - 37x + 9 = 0$

$\boxed{4x^2 - 37x + 9 = 0 \text{ or any multiple of this}}$

Your notes

## Roots of Cubics

### How are the roots of a cubic linked to its coefficients?

- Because a cubic equation $ax^3 + bx^2 + cx + d = 0$ (where $a \neq 0$) has roots $\alpha$, $\beta$ and $\gamma$, you can write this equation instead in the form $a(x-\alpha)(x-\beta)(x-\gamma) = 0$
  - Note that $(x-\alpha)(x-\beta)(x-\gamma) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\alpha\gamma)x - \alpha\beta\gamma$
  - It is possible that some of the roots are repeated, i.e. that some or all of them are equal to each other
 - You can then equate the two forms:
   - $ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$
 - Then (because $a \neq 0$) you can divide both sides of that by *a* and expand the brackets:
   - $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\alpha\gamma)x - \alpha\beta\gamma$
 - Finally, compare the coefficients
   - Coefficients of $x^2$: $-\frac{b}{a} = \alpha+\beta+\gamma$
   - Coefficients of $x$: $\frac{c}{a} = \alpha\beta+\beta\gamma+\alpha\gamma$
   - Constant terms: $-\frac{d}{a} = \alpha\beta\gamma$
- Therefore for a cubic equation $ax^3 + bx^2 + cx + d = 0$:
  - The sum of the roots $\alpha+\beta+\gamma$ is equal to $-\frac{b}{a}$
    - The sum of roots $\alpha+\beta+\gamma$ can also be denoted by $\sum\alpha$
  - The sum of the product pairs of roots $\alpha\beta+\beta\gamma+\alpha\gamma$ is equal to $\frac{c}{a}$
    - This 'sum of pairs' $\alpha\beta+\beta\gamma+\alpha\gamma$ can also be denoted by $\sum\alpha\beta$
  - The product of the roots $\alpha\beta\gamma$ is equal to $-\frac{d}{a}$
    - The product of roots $\alpha\beta\gamma$ can also be denoted by $\sum\alpha\beta\gamma$
    - See quartic equations where using this 'sum of triples' notation makes more sense!

- Unless an exam question specifically asks you to prove these results, you can always use them without proof to answer questions about cubics

Your notes

**Related Roots**

- You may be asked to consider two cubic equations, with the roots of the second cubic linked to the roots of the first cubic in some way
  - You are usually required to find the sum or product of the roots of the second equation
- The strategy is to use identities which contain $\sum \alpha$, $\sum \alpha\beta$, and $\sum \alpha\beta\gamma$ (where $\alpha$, $\beta$ and $\gamma$ are the roots of the first cubic)
  - If you know the values of $\alpha$, $\beta$, and $\gamma$ from the first cubic, you can use them to help find the sum or product of the new roots
  - If the second cubic has roots $\alpha^2$, $\beta^2$, and $\gamma^2$, then use the identities:
    - $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\alpha\gamma)$
    - i.e., $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2\sum\alpha\beta$
    - $(\alpha\beta\gamma)^2 = \alpha^2\beta^2\gamma^2$
  - If the second cubic has roots $\alpha^3, \beta^3, \gamma^3$, then use the identities:
    - $(\alpha+\beta+\gamma)^3 = \alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\alpha\gamma)-3\alpha\beta\gamma$
    - i.e., $(\alpha+\beta+\gamma)^3 = \alpha^3+\beta^3+\gamma^3+3\sum\alpha\sum\alpha\beta-3\sum\alpha\beta\gamma$
    - $(\alpha\beta\gamma)^3 = \alpha^3\beta^3\gamma^3$
  - If the second cubic has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$, then use the identities:
    - $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
    - $\frac{1}{\alpha}\times\frac{1}{\beta}\times\frac{1}{\gamma} = \frac{1}{\alpha\beta\gamma}$
- You can then form a new equation for a cubic with the new roots
  - This is done by recalling that a cubic with three given roots can be written in the form $x^3-(\text{sum of roots})x^2+(\text{sum of product pairs})x-(\text{product of roots})=0$
  - Be aware that this will not give a unique answer
    - This is because multiplying an entire cubic by a constant does not change its roots
    - You can use this fact, for example, to find a cubic that has a particular pair of roots AND has all integer coefficients

Your notes

## Worked example

a) Given the cubic equation $x^3 + 3x^2 - 10x - 24 = 0$, find $\sum\alpha$, $\sum\alpha\beta$, and $\sum\alpha\beta\gamma$

For a cubic: $ax^3 + bx^2 + cx + d = 0$

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$

For this cubic:

$a = 1 \quad \frac{b}{a} = 3 \quad \frac{c}{a} = -10 \quad \frac{d}{a} = -24$

$\sum\alpha = -\frac{b}{a}$

$\sum\alpha\beta = \frac{c}{a}$

$\sum\alpha\beta\gamma = \frac{-d}{a}$

$$\boxed{\begin{aligned}\sum\alpha &= -3\\ \sum\alpha\beta &= -10\\ \sum\alpha\beta\gamma &= 24\end{aligned}}$$

b) Another cubic has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$. Find $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

Can either remember this result or work it out as algebraic fractions by using common denominator $\alpha\beta\gamma$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

$$= \frac{\sum \alpha\beta}{\sum \alpha\beta\gamma}$$

Using previous results $= \dfrac{-10}{24} = \dfrac{-5}{12}$

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{-5}{12}}$$

Your notes

Your notes

## Roots of Quartics

### How are the roots of a quartic linked to its coefficients?

- Because a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ (where $a \neq 0$) has roots $\alpha$, $\beta$, $\gamma$ and $\delta$, you can write this equation instead in the form $a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0$
    - Note that $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ expands to $x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\beta\gamma+\alpha\gamma+\gamma\delta+\alpha\delta+\beta\delta)x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta$
    - It is possible that some of the roots are repeated, i.e. that some or all of them are equal to each other
  - You can then equate the two forms:
    - $ax^4 + bx^3 + cx^2 + dx + e = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
  - Then (because $a \neq 0$) you can divide both sides of that by *a* and expand the brackets:
    - $x^3 + \dfrac{b}{a}x^2 + \dfrac{c}{a}x + \dfrac{d}{a} =$ $x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\beta\gamma+\alpha\gamma+\gamma\delta+\alpha\delta+\beta\delta)x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta$
  - Finally, compare the coefficients
    - Coefficients of $x^3$: $-\dfrac{b}{a} = \alpha+\beta+\gamma+\delta$
    - Coefficients of $x^2$: $\dfrac{c}{a} = \alpha\beta+\beta\gamma+\alpha\gamma+\gamma\delta+\alpha\delta+\beta\delta$
    - Coefficients of x: $-\dfrac{d}{a} = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$
    - Constant terms: $\dfrac{e}{a} = \alpha\beta\gamma\delta$
- Therefore for a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$:
  - The sum of the roots $\alpha+\beta+\gamma+\delta$ is equal to $-\dfrac{b}{a}$
    - The sum of roots $\alpha+\beta+\gamma+\delta$ can also be denoted by $\sum\alpha$
  - The sum of the product pairs of roots $\alpha\beta+\beta\gamma+\alpha\gamma+\gamma\delta+\alpha\delta+\beta\delta$ is equal to $\dfrac{c}{a}$
    - This 'sum of pairs' $\alpha\beta+\beta\gamma+\alpha\gamma+\gamma\delta+\alpha\delta+\beta\delta$ can also be denoted by $\sum\alpha\beta$
  - The sum of the product triples of roots $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$ is equal to $-\dfrac{d}{a}$
    - This 'sum of triples' can also be denoted by $\sum\alpha\beta\gamma$

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- The product of the roots $\alpha\beta\gamma\delta$ is equal to $\frac{e}{a}$
  - The product of roots (or 'product of fours') $\alpha\beta\gamma\delta$ can also be denoted by $\sum\alpha\beta\gamma\delta$
- Unless an exam question specifically asks you to prove these results, you can always use them without proof to answer questions about quartics

**Related Roots**

- You may be asked to consider two quartic equations, with the roots of the second quartic linked to the roots of the first quartic in some way
  - You are usually required to find the sum or product of the roots of the second equation
- The strategy is to use identities which contain $\sum\alpha$, $\sum\alpha\beta$, $\sum\alpha\beta\gamma$, and $\sum\alpha\beta\gamma\delta$ (where $\alpha, \beta, \gamma$ and $\delta$ are the roots of the first quartic)
  - If you know the values of $\alpha$, $\beta$, $\gamma$, and $\delta$ from the first quartic, you can use them to help find the sum or product of the new roots
  - If the second quartic has roots $\alpha^2$, $\beta^2$, $\gamma^2$ and $\delta^2$, then use the identities:
    - $(\alpha+\beta+\gamma+\delta)^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\beta\gamma+\alpha\gamma+\gamma\delta+\alpha\delta+\beta\delta)$
    - i.e., $(\alpha+\beta+\gamma+\delta)^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+2\sum\alpha\beta$
    - $(\alpha\beta\gamma\delta)^2 = \alpha^2\beta^2\gamma^2\delta^2$
  - (Note that you will not be asked about a quartic with roots $\alpha^3, \beta^3, \gamma^3$ and $\delta^3$)
  - If the second quartic has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$, then use the identities:
    - $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} = \frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$
    - $\frac{1}{\alpha}\times\frac{1}{\beta}\times\frac{1}{\gamma}\times\frac{1}{\delta} = \frac{1}{\alpha\beta\gamma\delta}$
- You can then form a new equation for a quartic with the new roots
  - This is done by recalling that a quartic with four given roots can be written in the form $x^4 - (\text{sum of roots})x^3 + (\text{sum of product pairs})x^2 - (\text{sum of product triples})x + (\text{product of roots}) = 0$
  - Be aware that this will not give a unique answer
    - This is because multiplying an entire quartic by a constant does not change its roots
    - You can use this fact, for example, to find a quartic that has a particular pair of roots AND has all integer coefficients

Your notes

### Worked example

a) The roots of $x^4 - 12x^3 + 33x^2 + 38x - 168 = 0$ are $\alpha,\ \beta,\ \gamma$ and $\delta$. Find

$\sum\alpha,\ \sum\alpha\beta,\ \sum\alpha\beta\gamma$, and $\sum\alpha\beta\gamma\delta$.

For a quartic: $ax^4 + bx^3 + cx^2 + dx + e = 0$

$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$

For this quartic:

$a = 1 \quad \frac{b}{a} = -12 \quad \frac{c}{a} = 33 \quad \frac{d}{a} = 38 \quad \frac{e}{a} = -168$

$\sum\alpha = -\frac{b}{a}$ $\quad \sum\alpha = 12$

$\sum\alpha\beta = \frac{c}{a}$ $\quad \sum\alpha\beta = 33$

$\sum\alpha\beta\gamma = \frac{-d}{a}$ $\quad \sum\alpha\beta\gamma = -38$

$\sum\alpha\beta\gamma\delta = \frac{e}{a}$ $\quad \sum\alpha\beta\gamma\delta = -168$

b) Another quartic has roots $\alpha^2,\ \beta^2,\ \gamma^2$ and $\delta^2$. Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

Your notes

Need to find a link between $\alpha^2+\beta^2+\gamma^2+\delta^2$ and the information we already know

Expanding $(\alpha+\beta+\gamma+\delta)^2$

$$=\alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\beta\gamma+\alpha\gamma+\gamma\delta+\alpha\delta+\beta\delta)$$

$$=\alpha^2+\beta^2+\gamma^2+\delta^2+2\sum\alpha\beta$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2\sum\alpha\beta$$

$$=\left(\sum\alpha\right)^2-2\sum\alpha\beta$$

Fill in previously found information

$$=12^2-2(33)$$

$$=78$$

$$\boxed{\alpha^2+\beta^2+\gamma^2+\delta^2=78}$$

## Roots of Polynomials

### What is the general pattern linking the roots to the coefficients of a polynomial?

- By looking at the links between the coefficients and the roots of quadratics, cubics, and quartics, you can see that a pattern emerges, which also holds true for higher order polynomials
- It is useful to use sigma notation to keep expressions for sums of roots concise
  - For a quartic with roots $\alpha, \beta, \gamma, \delta$, for example:
    - The sum of the roots $\alpha + \beta + \gamma + \delta$ is denoted by $\sum \alpha$
    - The sum of the pairs of roots $\alpha\beta + \beta\gamma + \alpha\gamma + \gamma\delta + \alpha\delta + \beta\delta$ is denoted by $\sum \alpha\beta$
    - The sum of the triples of roots $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ is denoted by $\sum \alpha\beta\gamma$
    - The sum of the sets of fours (in this case just one term) $\alpha\beta\gamma\delta$ is denoted by $\sum \alpha\beta\gamma\delta$
- The table below summarises the relationships between the coefficients and roots of quadratics, cubics, and quartics:

| | $ax^2 + bx + c = 0$ Roots $\alpha, \beta$ | $ax^3 + bx^2 + cx + d = 0$ Roots $\alpha, \beta, \gamma$ | $ax^4 + bx^3 + cx^2 + dx + e = 0$ Roots $\alpha, \beta, \gamma, \delta$ |
|---|---|---|---|
| $\sum \alpha$ | $\alpha + \beta$ $= -\frac{b}{a}$ | $\alpha + \beta + \gamma$ $= -\frac{b}{a}$ | $\alpha + \beta + \gamma + \delta$ $= -\frac{b}{a}$ |
| $\sum \alpha\beta$ | $\alpha\beta$ $= \frac{c}{a}$ | $\alpha\beta + \beta\gamma + \alpha\gamma$ $= \frac{c}{a}$ | $\alpha\beta + \beta\gamma + \alpha\gamma + \gamma\delta + \alpha\delta + \beta\delta$ $= \frac{c}{a}$ |
| $\sum \alpha\beta\gamma$ | | $\alpha\beta\gamma$ $= -\frac{d}{a}$ | $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ $= -\frac{d}{a}$ |
| $\sum \alpha\beta\gamma\delta$ | | | $\alpha\beta\gamma\delta$ $= \frac{e}{a}$ |

### How can I find sums and products of related roots?

Your notes

Your notes

- You may be asked to consider a second equation, that has roots linked to the roots of the first equation in some way
  - You are usually required to find the sum or product of the roots of the second equation
- The strategy is to use identities containing $\sum \alpha$, $\sum \alpha\beta$, $\sum \alpha\beta\gamma$, and/or $\sum \alpha\beta\gamma\delta$ (depending on the question and the degree of the polynomial)
  - If you know the value of the roots from the first equation, these identities can help you find the sum or product of the roots of the second equation
- The table below shows useful identities for finding a new quadratic equation whose roots are related to the roots $\alpha$ and $\beta$ of the original quadratic equation
  - In each case the sum or the product of the 'new roots' can be linked back to $\alpha\beta$ or $\alpha + \beta$ for the original equation

| Roots of new Equation | Useful Identities to find sums and products of new roots |
|---|---|
| $\frac{1}{\alpha}, \frac{1}{\beta}$ | $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$<br>$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta}$ |
| $\alpha^2, \beta^2$ | $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$<br>$\alpha^2\beta^2 = (\alpha\beta)^2$ |
| $\alpha^3, \beta^3$ | $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$<br>$\alpha^3\beta^3 = (\alpha\beta)^3$ |

- Similar identities that could be useful for cubics and quartics are listed earlier in this revision note in the cubics and quartics sections
- A good place to start if the new roots are squared, is by considering $(\sum \alpha)^2$
  - or if the new roots are cubed, then start by considering $(\sum \alpha)^3$
  - or if the new roots are reciprocals (i.e., $\frac{1}{\alpha}, \frac{1}{\beta}$, etc.), then start by adding the new roots together to form a single algebraic fraction

### Examiner Tip

- Although you may be asked to tackle questions on this topic showing full working and without relying on a calculator, you can still use your calculator to *check* your work by finding the roots of a polynomial in the polynomial solver
- You can then use these to check your answers for sums of roots, products of roots, etc.

Your notes

Your notes

## Worked example

a) Given a polynomial equation of order 5 (a quintic); $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, make 5 conjectures linking the coefficients *a, b, c, d, e, f* to its roots $\alpha, \beta, \gamma, \delta, \varepsilon$.

Considering the patterns for the sums of roots, sums of pairs of roots, etc:

For order 5 (quintic): $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

$$\boxed{\begin{aligned}
&\sum \alpha = \alpha + \beta + \gamma + \delta + \varepsilon = \frac{-b}{a} \\
&\sum \alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \alpha\varepsilon + \beta\gamma + \beta\delta + \beta\varepsilon + \gamma\delta + \gamma\varepsilon + \delta\varepsilon = \frac{c}{a} \\
&\sum \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\beta\varepsilon + \alpha\gamma\delta + \alpha\gamma\varepsilon + \alpha\delta\varepsilon + \beta\gamma\delta + \beta\gamma\varepsilon + \gamma\delta\varepsilon + \beta\delta\varepsilon = \frac{-d}{a} \\
&\sum \alpha\beta\gamma\delta = \alpha\beta\gamma\delta + \alpha\beta\gamma\varepsilon + \alpha\beta\delta\varepsilon + \beta\gamma\delta\varepsilon + \gamma\delta\varepsilon\alpha = \frac{e}{a} \\
&\sum \alpha\beta\gamma\delta\varepsilon = \alpha\beta\gamma\delta\varepsilon = \frac{-f}{a}
\end{aligned}}$$

b) Test your conjectures on the example: $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = 0$ which has roots $x = 1, 2, 3, 4, 5$.

Your notes

Testing each conjecture from (a) on given quintic

$\sum \alpha = -\frac{b}{a} \quad 1 + 2 + 3 + 4 + 5 = \frac{15}{1}$

$\boxed{15 = 15 \checkmark}$

$\sum \alpha\beta = \frac{c}{a}$

$(1\times 2) + (1\times 3) + (1\times 4) + (1\times 5) + (2\times 3) + (2\times 4) + (2\times 5)$
$+ (3\times 4) + (3\times 5) + (4\times 5) = \frac{85}{1}$

$\boxed{85 = 85 \checkmark}$

$\sum \alpha\beta\gamma = \frac{-d}{a}$

$(1\times 2\times 3) + (1\times 2\times 4) + (1\times 2\times 5) + (1\times 3\times 4) + (1\times 3\times 5) + (1\times 4\times 5)$
$+ (2\times 3\times 4) + (2\times 3\times 5) + (3\times 4\times 5) + (2\times 4\times 5) = \frac{225}{1}$

$\boxed{225 = 225 \checkmark}$

$\sum \alpha\beta\gamma\delta\varepsilon = \frac{-f}{a} \quad 1\times 2\times 3\times 4\times 5 = \frac{120}{1}$

$\boxed{120 = 120 \checkmark}$

Note that this does not prove the conjectures; it just shows they are true for this particular equation

## 3.1.2 Linear Transformations of Roots

Your notes

# Linear Transformations of Roots

### What is a Linear Transformation?

- A linear transformation can be expressed as $w = px + q$ where:
  - *w* is the new root
  - *x* is the original root
  - *p* and *q* are constants
- If you perform a linear transformation on a polynomial equation, the transformed equation's roots will be linked directly to the original roots
- You can think of this as a translation and/or a stretch of the polynomial and its roots

### How do I find the new transformed polynomial?

- STEP 1

  Rewrite the transformation $w = px + q$ as $x = \dfrac{w - q}{p}$
- STEP 2

  Make the substitution $x = \dfrac{w - q}{p}$ into the original polynomial
- STEP 3
  Expand and multiply by a constant to make all the coefficients integers (if necessary or desirable) and swap the *w* back for an *x* in your final answer
- Remember that your solution is not unique. Multiplying the entire polynomial by a constant will produce a different polynomial, but will not affect the roots

### Examiner Tip

- Check the question to see if you are required to expand and simplify your final answer or not, as it can be time consuming!
- If you are required to expand and simplify, make use of the binomial expansion to make the process much quicker
- Use your calculator's polynomial solver to check the solutions of the original equation and your new equation, to make sure they are related to each other as described

Your notes

## Worked example

The cubic equation $x^3 - 7x^2 + 2x + 40 = 0$ has roots $\alpha, \beta, \gamma$. Find a polynomial equation with roots:

a) $3\alpha,\ 3\beta,\ 3\gamma$

Transformation is $w = 3x$ (w is new root, x is old root)

rearrange for x $\quad x = \frac{w}{3}$

Substitute into original $\quad \left(\frac{w}{3}\right)^3 - 7\left(\frac{w}{3}\right)^2 + 2\left(\frac{w}{3}\right) + 40 = 0$

expand $\quad \frac{1}{27}w^3 - \frac{7}{9}w^2 + \frac{2}{3}w + 40 = 0$

Multiply by 27 to turn into integers (optional) and swap variable back to an x

$$\boxed{x^3 - 21x^2 + 18x + 1080 = 0}$$
(or any multiple)

b) $(2\alpha - 1),\ (2\beta - 1),\ (2\gamma - 1)$

Your notes

Transformation is $w = 2x - 1$ ($w$ is new root, $x$ is old root)

rearrange for $x$ $\quad x = \dfrac{w+1}{2}$

Substitute into original $\quad \left(\dfrac{w+1}{2}\right)^3 - 7\left(\dfrac{w+1}{2}\right)^2 + 2\left(\dfrac{w+1}{2}\right) + 40 = 0$

expand $\quad \dfrac{1}{8}(w+1)^3 - \dfrac{7}{4}(w+1)^2 + (w+1) + 40 = 0$

and again! $\quad \dfrac{1}{8}(w^3 + 3w^2 + 3w + 1) - \dfrac{7}{4}(w^2 + 2w + 1) + w + 1 + 40 = 0$

Simplify $\quad \dfrac{1}{8}w^3 - \dfrac{11}{8}w^2 - \dfrac{17}{8}w + \dfrac{315}{8} = 0$

Multiply by 8 to turn into integers (optional) and swap variable back to an $x$

$$\boxed{x^3 - 11x^2 - 17x + 315 = 0}$$

(or any multiple)