Integration (Year 13)

In this chapter, you'll be able to integrate a significantly greater variety of expressions, and be able to solve differential equations.

Integration by standard result
(There's certain expressions you're expected to know straight off.)

$$\int \sec^2 x\, dx = \tan x + C$$

Integration by 'reverse chain rule'
(We imagine what would have differentiated to get the expression.)

$$\int \cos 4x \;\, dx = \frac{1}{4}\sin 4x + C$$

$$\int \sin^3 x \cos x\, dx = \frac{1}{4}\sin^4 x + C$$

Integration by substitution
(We make a substitution to hopefully make the expression easier to integrate)

$$\int x\sqrt{2x+5}\, dx$$

Let $u = 2x + 5 \quad \rightarrow \quad x = \frac{u-5}{2}$

$$\frac{du}{dx} = 2 \quad \rightarrow \quad dx = \frac{1}{2}u$$

$$\int x\sqrt{2x+5}\, dx = \int \frac{u-5}{2}\frac{1}{2}\, du = \cdots$$

Integration by parts
(Allows us to integrate a product, just as the product rule allowed us to differentiate one)

$$\int x \cos x\, dx$$

$$u = x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 1 \qquad v = \sin x$$

$$\int x\cos x\, dx = x \sin x - \int \sin x\, dx$$

$$= x \sin x + \cos x$$

Integrating partial fractions

(We split into partial fractions first so each fraction easier to integrate)

$$\int \frac{3x+5}{(x+1)(x+2)}\,dx$$

Approximating areas using the trapezium rule

(Instead of integrating, we split the area under the graph into trapeziums and use these to approximate the area)

Solving Differential Equations

(Solving here means to find one variable in terms of another without derivatives present)

$$\frac{dV}{dt} = -kV$$

$$\int \frac{1}{V}\,dV = \int -k\,dt$$

$$\ln V = -kt + C$$

$$V = e^{-kt+C} = Ae^{-kt}$$

Integrating Parametric Equations

(Use fact that area:

$$\int y\,dx = \int y\frac{dx}{dt}\,dt)$$

SKILL #1: Integrating Standard Functions

There's certain results you should be able to integrate straight off, by just thinking about the opposite of differentiation.

y	$\int y\,dx$
x^n	$\frac{1}{n+1}x^{n+1} + C$
e^x	$e^x + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$cosec\ x \cot x$	$-cosec\ x + C$
$cosec^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$

The $\lvert x\rvert$ has to do with problems when x is negative (when $\ln x$ is not defined)

$$\int \sec x \tan x \; dx = \sec x + c$$

$$\int cosec^2 x \, dx = -\cot x + c$$

$$\int \sin x \; dx = -\cos x + c$$

$$\int -\sin x \; dx = \cos x + c$$

$$\int cosec^2 x \, dx = -\cot x + c$$

$$\int \sec x \tan x \; dx = \sec x + c$$

$$\int -\cos x \; dx = -\sin x + c$$

$$\int \cos x \; dx = \sin x + c$$

$$\int \sec^2 x \; dx = \tan x + c$$

$$\int cosec\, x \cot x \; dx = -cosec\, x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

$$\int -\sin x \; dx = \cos x + c$$

$$\int 2\cos x + \frac{3}{x} - \sqrt{x}\, dx = 2\sin x + 3\ln|x| - \frac{2}{3}x^{3/2} + c$$

$x^{1/2}$

$$\int \frac{\cos x}{\sin^2 x}\, dx = \int \frac{\cos x}{\sin x} \times \frac{1}{\sin x}\, dx = \int \cot x \operatorname{cosec} x\, dx = -\operatorname{cosec} x + c$$

Hint: What 'reciprocal' trig functions does this simplify to?

Given that $\int_a^{3a} \frac{2x+1}{x}\, dx = \ln 12$, find the exact value of a.

$$\int_a^{3a} \frac{2x+1}{x}\, dx = \int_a^{3a} \frac{2x}{x} + \frac{1}{x}\, dx$$

$$= \int_a^{3a} \left(2 + \frac{1}{x}\right) dx$$

$$= \left[2x + \ln x\right]_a^{3a}$$

$$= 6a + \ln 3a - 2a - \ln a$$

$$= 4a + \ln\frac{3a}{a}$$

$$= 4a + \ln 3$$

$$4a + \ln 3 = \ln 12$$

$$4a = \ln 12 - \ln 3$$

$$4a = \ln 4$$

$$a = \frac{1}{4}\ln 4$$

Important Notes:

We can simplify:

$$\frac{x+1}{x} \equiv \frac{x}{x} + \frac{1}{x} \equiv 1 + \frac{1}{x}$$

However it is **NOT** true that:

$$\frac{x}{x+1} \equiv \frac{x}{x} + \frac{x}{1}$$

In my experience students often fail to spot when they can split up a fraction to then integrate.

Ex 11A

Q 1acg

Q 2 first column

Q4 Q5

SKILL #2: Integrating f(ax+b)

$$\frac{d}{dx}(\sin(3x+1)) = 3\cos(3x+1)$$

Therefore:

$$\int \cos(3x+1)\ dx = \frac{1}{3}\sin(3x+1) + c$$

$$\hookrightarrow \frac{1}{3} \times 3\cos(3x+1)$$

For any expression where inner function is $ax+b$, integrate as before and $\div\ a$.

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$$

$$\int e^{3x}\ dx = \frac{1}{3}e^{3x} + c$$

$$\int \frac{1}{5x+2} dx = \frac{1}{5}\ln|5x+2| + c$$

$$\int 2\sec^2(3x-2)\ dx = \frac{2}{3}\tan(3x-2) + c$$

$$\int (3x+4)^3\ dx = \frac{1}{3} \times \frac{1}{4}(3x+4)^4$$

$$= \frac{1}{12}(3x+4)^4 + c$$

$$\int \sin(1-5x)\ dx = -\frac{1}{-5}\cos(1-5x)$$

$$= \frac{1}{5}\cos(1-5x) + c$$

$$\int \frac{1}{3(4x-2)^2} dx = -\frac{1}{3}(4x-2)^{-1} \times \frac{1}{4} + c$$

$$\frac{1}{3}(4x-2)^{-2}$$

$$= -\frac{1}{12}(4x-2)^{-1} + c$$

$$\int (10x+11)^{12} = \frac{1}{13}(10x+11)^{13} \times \frac{1}{10} + c$$

$$= \frac{1}{130}(10x+11)^{13} + c$$

Your Turn

Only works for "ax+b"

$$\int e^{3x+1}dx = \frac{1}{3}e^{3x+1} + c$$

$$\int \frac{1}{1-2x}dx = -\frac{1}{2}\ln|1-2x| + c$$

$$\int (4-3x)^5 dx = \frac{1}{6}(4-3x)^6 \times -\frac{1}{3} + c = -\frac{1}{18}(4-3x)^6 + c$$

$$\int \sec(3x)\tan(3x)\,dx = \frac{1}{3}\sec 3x + c$$

Incorrect things people do

"$ax+b$"

$\frac{1}{a}$

$$\int (e^x+1)^2\,dx = \cancel{\frac{1}{3e^x}(e^x+1)^3}$$

$$(e^x+1)(e^x+1) = e^{2x}+2e^x+1$$

$$\int (e^{2x}+2e^x+1)\,dx = \frac{1}{2}e^{2x}+2e^x+x+c.$$

$$\int \sin^2 x\,dx = \int (\sin x)^2\,dx = \cancel{\frac{1}{3}\sin^3 x \times \frac{1}{\cos x}}$$

$$\cos 2x = 1-2\sin^2 x$$

$$2\sin^2 x = 1-\cos 2x$$

$$\sin^2 x = \frac{1}{2}-\frac{1}{2}\cos 2x$$

1 a $\int \sin(2x+1)\,dx =$

c $\int 4e^{x+5}\,dx =$

e $\int cosec^2 3x\,dx =$

f $\int \sec 4x \tan 4x\;dx =$

g $\int 3\sin\left(\frac{1}{2}x+1\right)dx =$

h $\int cosec\,2x \cot 2x\;dx =$

2 a $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

$=$

b $\int (e^x+1)^2\,dx =$

c $\int \sec^2 2x\,(1+\sin 2x)\,dx =$

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}dx =$

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

$=$

3 a $\int \frac{1}{2x+1}dx =$

b $\int \frac{1}{(2x+1)^2}dx =$

c $\int (2x+1)^2\,dx =$

d $\int \frac{3}{4x-1}dx =$

f $\int \frac{3}{(1-4x)^2}dx =$

h $\int \frac{3}{(1-2x)^3}dx =$

j $\int \frac{5}{3-2x}dx =$

4 a $\int 3\sin(2x+1) + \frac{4}{2x+1}\,dx$

$=$

c $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}dx$

$=$

1 a $\int \sin(2x+1)\,dx = -\frac{1}{2}\cos(2x+1) + C$

c $\int 4e^{x+5}\,dx = 4e^{x+5} + C$

e $\int cosec^2 3x\,dx = -\frac{1}{3}\cot 3x + C$

f $\int \sec 4x \tan 4x\,dx = \frac{1}{4}\sec 4x + C$

g $\int 3\sin\left(\frac{1}{2}x+1\right)dx = -6\cos\left(\frac{1}{2}x+1\right)$

h $\int cosec\,2x \cot 2x\,dx = -\frac{1}{2}cosec\,2x + C$

2 a $\int e^{2x} - \frac{1}{2}\sin(2x-1)\,dx$

$$= \frac{1}{2}e^{2x} + \frac{1}{4}\cos(2x-1) + C$$

b $\int (e^x+1)^2\,dx = \frac{1}{2}e^{2x} + 2e^x + x + C$

c $\int \sec^2 2x\,(1+\sin 2x)\,dx = \frac{1}{2}\tan 2x + \frac{1}{2}\sec 2x$

d $\int \frac{3-2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)}dx = -6\cot\left(\frac{1}{2}x\right) + 4cosec\left(\frac{1}{2}x\right)$

e $\int e^{3-x} + \sin(3-x) + \cos(3-x)\,dx$

$$= -e^{3-x} + \cos(3-x) - \sin(3-x) + C$$

3 a $\int \frac{1}{2x+1}dx = \frac{1}{2}\ln|2x+1| + C$

b $\int \frac{1}{(2x+1)^2}dx = -\frac{1}{2(2x+1)} + C$

c $\int (2x+1)^2\,dx = \frac{1}{6}(2x+1)^3 + C$

d $\int \frac{3}{4x-1}dx = \frac{3}{4}\ln|4x-1| + C$

f $\int \frac{3}{(1-4x)^2}dx = \frac{3}{4(1-4x)} + C$

h $\int \frac{3}{(1-2x)^3}dx = \frac{3}{4(1-2x)^2} + C$

j $\int \frac{5}{3-2x}dx = -\frac{5}{2}\ln|3-2x| + C$

4 a $\int 3\sin(2x+1) + \frac{4}{2x+1}\,dx$

$$= -\frac{3}{2}\cos(2x+1) + 2\ln|2x+1| + C$$

c $\int \frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2}dx$

$$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| - \frac{1}{2(1+2x)}$$