Chapter 1: Complex Numbers

1:: Understand and manipulate (×,÷) complex numbers.

"Determine $\frac{4+i}{3-i}$ giving your answer in the form $a + bi$."

2:: Find complex solutions to quadratic equations.

"Solve $x^2 + 3x + 5 = 0$."

3:: Find complex solutions to cubic and quartic equations.

"Given that $-2 + i$ is one of the roots of the equation $x^3 + 3x^2 + x - 5$, determine the other two roots."

What is a number?

5

negatives

-3

What is an imaginary number?

In a way, you've been using 'imaginary' numbers for a while... just not these ones...

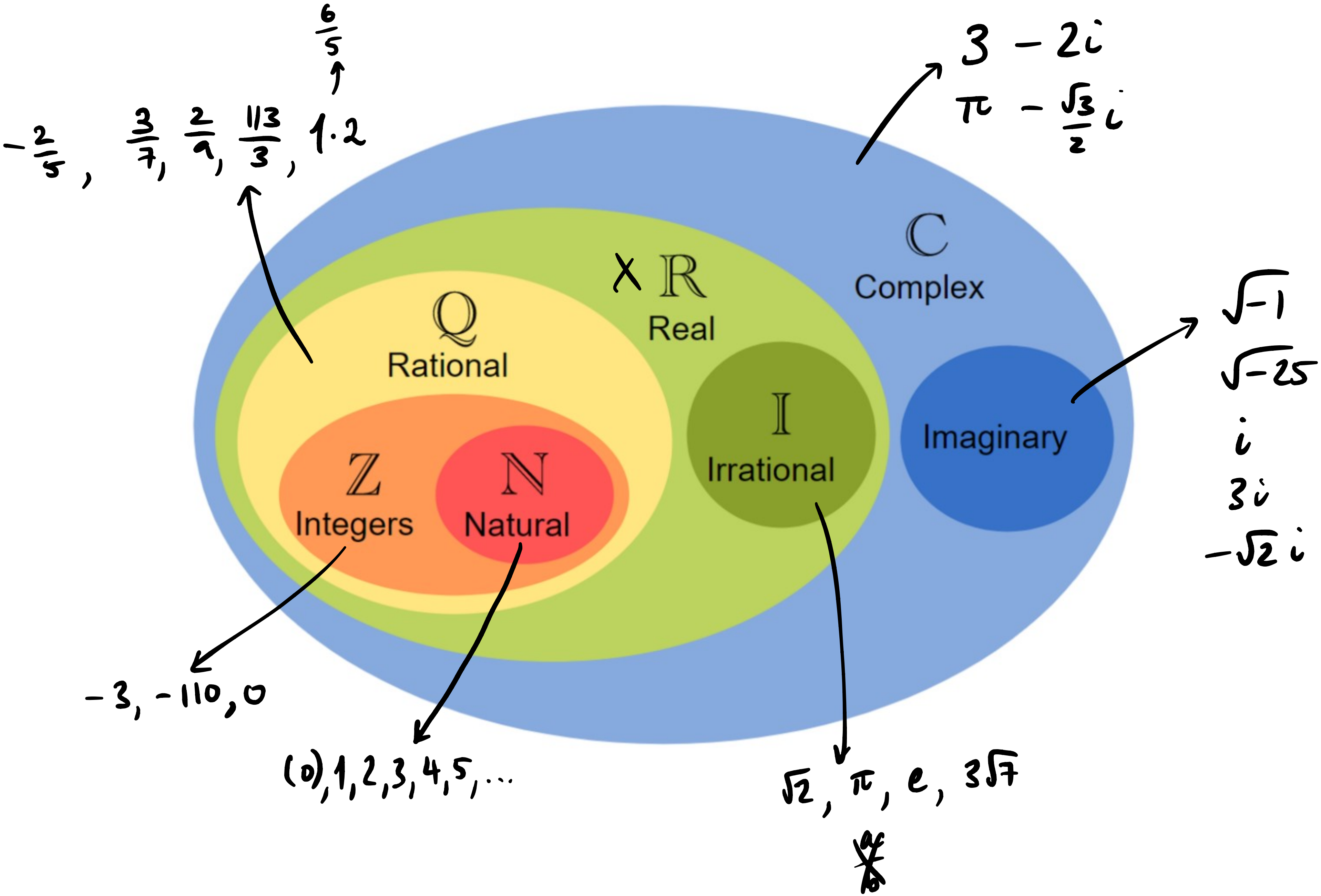
$$i = \sqrt{-1}$$

$$i^2 = -1$$

Cardano
Italian mathematician
1501-1576

- $i = \sqrt{-1}$
- An **imaginary** number is of the form bi where $b \in \mathbb{R}$, e.g. i, $3i$, $-2i$, $i\pi$
- A **complex** number is of the form $a + bi$, where $a, b \in \mathbb{R}$, e.g. $1 + i$, $3 - 2i$
- We say that a is the "real part" and b the "imaginary part" of the number.

Types of numbers



Complex Number Basics

Write the following in terms of i:

$\sqrt{-36} = \sqrt{36} \times \sqrt{-1} = 6i$

$\sqrt{-4} = \sqrt{4} \times \sqrt{-1} = 2i$

$\sqrt{-7} = \sqrt{7} \times \sqrt{-1} = \sqrt{7}i \qquad i\sqrt{7}$

$\sqrt{-45} = \sqrt{45} \times \sqrt{-1} = \sqrt{9} \times \sqrt{5}\, i = 3\sqrt{5}i$

Simplify:

$(2 + 3i) + (4 + i) = 6 + 4i$

$\dfrac{10 + 6i}{3} = \dfrac{10}{3} + \dfrac{6i}{3}$

$= \dfrac{10}{3} + 2i$

$i - 3(2 - i) = i - 6 + 3i$

$= -6 + 4i$

Ex 1A

Solving Quadratic Equations

Solve $z^2 + 25 = 0$

$$z^2 = -25$$
$$z = \pm\sqrt{-25}$$
$$z = \pm 5i$$

Notation Note: Just as we tend to use x as the default real-numbered variable and n for integers, we tend to use z (or w) as the default letter for complex numbers.

Solve $z^2 + 3z + 5 = 0$

Method 1 - complete the square

$$z^2 + 3z + 5 = 0$$
$$\left(z + \frac{3}{2}\right)^2 - \frac{9}{4} + 5 = 0$$
$$\left(z + \frac{3}{2}\right)^2 = -\frac{11}{4}$$
$$z + \frac{3}{2} = \pm\sqrt{\frac{-11}{4}}$$
$$\frac{\sqrt{-11}}{\sqrt{4}}$$
$$z + \frac{3}{2} = \pm\frac{\sqrt{11}\,i}{2} \rightarrow z = -\frac{3}{2} \pm \frac{\sqrt{11}}{2}i$$

Method 2 - the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$z$$
$$a = 1$$
$$b = 3$$
$$c = 5$$

$$z = \frac{-3 \pm \sqrt{3^2 - 4\times 1\times 5}}{2}$$
$$= \frac{-3 \pm \sqrt{9 - 20}}{2}$$
$$= \frac{-3 \pm \sqrt{-11}}{2}$$
$$= -\frac{3}{2} \pm \frac{\sqrt{11}\,i}{2}$$

Method 3 - calculator

Multiplying Complex Numbers

Given that $i = \sqrt{-1}$, it follows that $\boldsymbol{i^2} =$ -1

Express each of the following in the form $a + bi$, where a, b are integers.

1) $(2 + 3i)(3 - 2i)$

2) $(5 - 3i)^2$

1) $(2+3i)(3-2i) = 6 - 4i + 9i - 6i^2$

-6×-1

$= 6 - 4i + 9i + 6$

$= 12 + 5i$

2) $(5-3i)^2 = (5-3i)(5-3i)$

$= 25 - 15i - 15i + 9i^2$

$= 25 - 15i - 15i - 9$

$= 16 - 30i$

$$f(z) = z^2 + 6z + 13$$

Show by substitution that $z = -3 + 2i$ is a solution of $f(z) = 0$

$$(-3+2i)^2 + \underline{6(-3+2i)} + 13 = 9 - \cancel{12i} - 4 - 18 + \cancel{12i} + 13$$

$$= \underline{\underline{0}}$$

Hence $z = -3+2i$ is a solution of $f(z)=0$.

Determine the value of
i^3, i^4, i^{101} and $(3i)^5$

$i = i$

$i^2 = -1$

$i^3 = i^2 \times i = -i$

$i^4 = i^3 \times i = -i \times i$

$= -i^2$

$= -(-1)$

$= 1$

$i^5 = i^4 \times i = 1 \times i$

$= i$

$i^6 = i^5 \times i = i \times i$

$= i^2$

$= -1$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

$\vdots$

$i^{101} = i$

101 is one more than a multiple of 4.

$(3i)^5 = 3^5 \times i^5$

$= 243i$

Ex 1C

Complex Conjugation

Suppose that $x = 3 + \sqrt{2}$ and $x^* = 3 - \sqrt{2}$
Determine:

$$x + x^* = 3+\sqrt{2} + 3-\sqrt{2} = \underline{6}$$

$$xx^* = (3+\sqrt{2})(3-\sqrt{2}) = 9 - 3\sqrt{2} + 3\sqrt{2} - 2$$
$$= \underline{7}$$

What do you notice about both results?

Both our answers are rational, and no longer have a surd.

Does a similar thing happen with two complex numbers that are similarly related in this way?

$$z = 3 + 2i, \qquad z^* = 3 - 2i$$

$$z + z^* = 3 + 2i + 3 - 2i = \underline{6}$$

$$zz^* = (3+2i)(3-2i) = 9 + 6i - 6i + 4$$
$$-4i^2$$
$$= \underline{13}$$

Both our answers are now <u>real</u> numbers (there is no longer an imaginary part)

Complex Conjugation

✏ If $z = a + bi$ then $z^* = a - bi$ is the complex conjugate of z. Together, z and z^* are a **complex conjugate pair**.

Given that $z = x + iy$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$, find the value of x and the value of y such that

$$(3 - i)z^* + 2iz = 9 - i$$

where z^* is the complex conjugate of z.

(8)

collect real / im.

$z = x + iy$

$z^* = x - iy$

$$(3-i)(x-iy) + 2i(x+iy) = 9 - i$$

$$3x - 3iy - ix - y + 2ix - 2y = 9 - i$$

$$3x - 3y + i(2x - x - 3y) = 9 - i$$

$$3x - 3y + i(x - 3y) = 9 - i$$

compare the real / im. parts of both sides

$3x - 3y = 9$ $\qquad$ $x - 3y = -1$

$x = 5$

$y = 2$

'Realising' the Denominator

Write $\frac{5+4i}{2-3i}$ in the form $a+bi$.

As with rationalising denominators of surds, we multiply numerator and denominator by the conjugate of the denominator.

$$\frac{(5+4i)}{(2-3i)} \times \underbrace{\frac{(2+3i)}{(2+3i)}}_{1} = \frac{10+15i+8i-12}{4+6i-6i+9}$$

$$= \frac{-2+23i}{13}$$

$$= -\frac{2}{13} + \frac{23}{13}i$$

Speed Tip:
Difference of two squares

$$(a+b)(a-b) = a^2 - b^2$$

Difference of two squares, 'i' version

$$(a+bi)(a-bi) = a^2 + b^2$$

You can use your calculators do perform calculations with complex numbers, too!
But you must know this method in case there are algebraic terms in the expression.

Problem Solving using complex numbers

The complex number $z = \frac{3+qi}{q-5i}$, where $q \in \mathbb{R}$

demonimator needs 'realising'

Given that the real part of z is $\frac{1}{13}$,

a) Find the possible values of q

b) Write the possible values of z in the form $a + bi$ where a and b are real constants

a) $$z = \frac{(3+qi)}{(q-5i)} \times \frac{(q+5i)}{(q+5i)}$$

$$= \frac{3q + 15i + q^2 i - 5q}{q^2 + 25}$$

collect real/im.

$$z = \underbrace{\frac{-2q}{q^2+25}}_{\frac{1}{13}} + \frac{15+q^2}{q^2+25} i$$

$$\frac{-2q}{q^2+25} = \frac{1}{13}$$

$$-26q = q^2 + 25$$

$$0 = q^2 + 26q + 25$$

$$0 = (q+1)(q+25)$$

$$q = -1, \quad q = -25$$

b) If $q = -1$

$$\frac{15+(-1)^2}{(-1)^2+25} = \frac{16}{26} = \frac{8}{13}$$

$$z = \frac{1}{13} + \frac{8}{13} i$$

$q = -25$

$$\frac{15+(-25)^2}{(-25)^2+25} = \frac{64}{65}$$

$$z = \frac{1}{13} + \frac{64}{65} i$$

The square roots of complex numbers
(not covered in textbook... could be assessed?)

You might be thinking -
'if finding the square root of a negative created a whole new type of numbers, will we need *another* type of number for the square root a complex number?'

Solve $z^2 = i$

Let $z = a + bi$

$(a+bi)^2 = i$

$a^2 + 2abi - b^2 = i$

collect real/im

$a^2 - b^2 + 2abi = i$

compare real parts

$a^2 - b^2 = 0$

$\frac{1}{4b^2} - b^2 = 0$

$\frac{1}{4b^2} = b^2$

$1 = 4b^4$

compare imaginary parts

$2ab = 1$

$a = \frac{1}{2b}$

$\frac{1}{4} = b^4$

$b = \sqrt[4]{\frac{1}{4}} = \pm\frac{1}{\sqrt{2}}$

$a = \pm\frac{1}{\sqrt{2}}$

$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

$z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

Solve $z^2(1+i) = 7 - 17i$

$$z^2(1+i) = 7-17i$$

$$z^2 = \frac{7-17i}{1+i}$$

$$z^2 = -5-12i$$

$$z = \pm\sqrt{-5-12i}$$

Let $z = a+bi$

$$(a+bi)^2 = -5-12i$$

$$a^2+2abi-b^2 = -5-12i$$

Real parts $a^2-b^2 = -5$

Im. parts $2ab = -12$

$$a = \frac{-12}{2b} = -\frac{6}{b}$$

$$\left(-\frac{6}{b}\right)^2 - b^2 = -5$$

$$\frac{36}{b^2} - b^2 = -5$$

$$36 - b^4 = -5b^2$$

$$0 = b^4 - 5b^2 - 36$$

quadratic.

Let $y = b^2$

$$0 = y^2 - 5y - 36$$

$y = 9$ $\qquad$ $y = -4$

$b^2 = 9$ $\qquad$ $b^2 = -4$

$b = \pm 3$ $\qquad$ $b = \pm 2i$

$a = \mp 2$

$$z = -2+3i$$

$$z = 2-3i$$

Your Turn

Find the complex numbers w in each of these cases

a) $w^2 = 30i - 16$

b) $w^2 = -3 - 4i$

c) $w^2 - 1 = 20(1 - i)$

a) $w = 3 + 5i$ or $w = -3 - 5i$

b) $w = 1 - 2i$ or $w = -1 + 2i$

c) $w = 5 - 2i$ or $w = -5 + 2i$

Simultaneous Equations
(not covered in textbook... could be assessed?)

Solve the following simultaneous equations
$w^2 + z^2 = 0$
$z - 3w = 10$

$z = 3w + 10$

$w^2 + (3w+10)^2 = 0$

$w^2 + 9w^2 + 60w + 100 = 0$

$10w^2 + 60w + 100 = 0$

$w = -3 + i$

or

$w = -3 - i$

if $w = -3 + i$
$z = -9 + 3i + 10$
$= 1 + 3i$

$z = 1 + 3i$
$w = -3 + i$

if $w = -3 - i$
$z = -9 - 3i + 10$
$= 1 - 3i$

$z = 1 - 3i$
$w = -3 - i$

Roots of Quadratics

Lets solve $x^2 + 4x - 5 = 0$, calling its roots α and β

$-(\alpha+\beta)$ $\alpha\beta$

$x^2 + 4x - 5 = 0$

$(x+5)(x-1) = 0$

$x = -5, \quad x = 1$

$\alpha = -5,$

$\beta = 1$

How do the roots relate to the original equation?

If α and β are the roots of a quadratic $ax^2 + bx + c$ then

$$ax^2 + bx + c \equiv a(x-\alpha)(x-\beta)$$

$$\equiv a(x^2 - \alpha x - \beta x + \alpha\beta)$$

$$\equiv a(x^2 - (\alpha+\beta)x + \alpha\beta)$$

$$ax^2 + bx + c \equiv ax^2 - a(\alpha+\beta)x + a\alpha\beta$$

comparing coefficients.

$b = -a(\alpha+\beta)$ $\qquad$ $c = a\alpha\beta$

$\alpha+\beta = -\frac{b}{a}$ $\qquad$ $\alpha\beta = \frac{c}{a}$

If α and β are roots of the equation $ax^2 + bx + c = 0$, then:

- **Sum** of roots: $\alpha + \beta = -\frac{b}{a}$
- **Product** of roots: $\alpha\beta = \frac{c}{a}$

This is a preview of Chapter 4. You can use all of Chapter 4 skills to solve these types of questions, but I'll show you both methods. (I prefer Chapter 4's method!)

Roots of Quadratics - Complex Conjugate Pairs

✎ If α is the root of a quadratic equation with real coefficients and α is a complex number, then the other root must be its complex conjugate, α^*.

Why?

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<0

Given that $\alpha = 7 + 2i$ is one of the roots of a quadratic equation with real coefficients,
(a) state the value of the other root, β.
(b) find the quadratic equation.

$\alpha = 7+2i \quad \beta = 7-2i$

Chapter 4 'Roots of Polynomials' method

$\alpha+\beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$

$a=1$

$\alpha+\beta = -b$

$7+\cancel{2i}+7-\cancel{2i} = -b$

$-14 = b$

$\alpha\beta = c$

$(7+2i)(7-2i) = c$

$49+4 = c$

$53 = c$

$x^2 - 14x + 53 = 0$

$(ax^2 + bx + c = 0)$

Longer method

General idea: if α and β are roots, then $(x-\alpha)(x-\beta)=0$
(and similarly for cubics and quartics)

$(x-(7+2i))(x-(7-2i)) = 0$

$x^2 - (7-2i)x - (7+2i)x + (7+2i)(7-2i) = 0$

$x^2 - 7x + \cancel{2ix} - 7x - \cancel{2ix} + 49 + 4 = 0.$

$x^2 - 14x + 53 = 0$

Your Turn

Given that $2 - 4\mathrm{i}$ is a root of the equation

$$z^2 + pz + q = 0,$$

where p and q are real constants,

(a) write down the other root of the equation, **(1)**

(b) find the value of p and the value of q. **(3)**

An 4

$\alpha = 2 - 4i$

$\beta = 2 + 4i$

$\alpha + \beta = -\frac{b}{a}$

$\alpha + \beta = -b$ p

$2 - \cancel{4i} + 2 + \cancel{4i} = -b$

$-4 = b$

$\alpha\beta = \frac{c}{a}$

$\alpha\beta = c$ q

$(2-4i)(2+4i) = c$

$4 + 16 = c$

$c = 20$

$z^2 - 4z + 20 = 0$

$p = -4 \quad q = 20$

$(z - (2-4i))(z - (2+4i)) = 0$

$z^2 - (2-4i)z - (2+4i)z + (2-4i)(2+4i) = 0$

$z^2 - 2z + \cancel{4iz} - 2z - \cancel{4iz} + 4 + 16 = 0$

$z^2 - 4z + 20 = 0$

$p = -4$

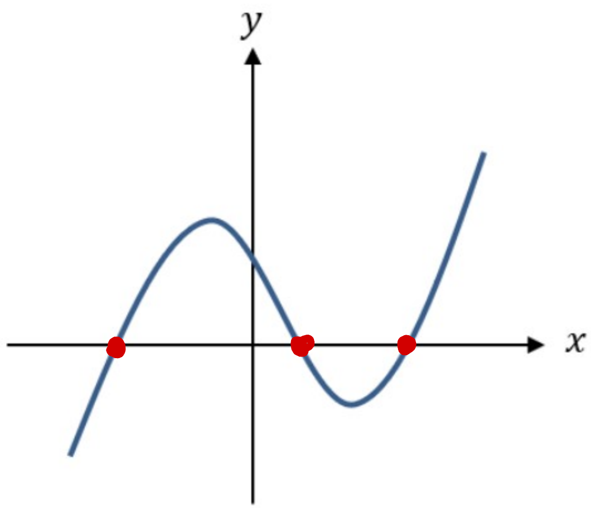
$q = 20$

(a)	$z^2 + pz + q = 0,\ z_1 = 2 - 4\mathrm{i}$ $z_2 = 2 + 4\mathrm{i}$	$2 + 4\mathrm{i}$	B1 (1)
(b)	$(z - 2 + 4\mathrm{i})(z - 2 - 4\mathrm{i}) = 0$		
	$\Rightarrow z^2 - 2z - 4\mathrm{i}z - 2z + 4 - 8\mathrm{i} + 4\mathrm{i}z - 8\mathrm{i} + 16 = 0$	An attempt to multiply out brackets of two complex factors and no i^2.	M1
	$\Rightarrow z^2 - 4z + 20 = 0$	Any one of $p = -4,\ q = 20$.	A1
		Both $p = -4,\ q = 20$.	A1
		$\Rightarrow z^2 - 4z + 20 = 0$ only 3/3	(3) [4]

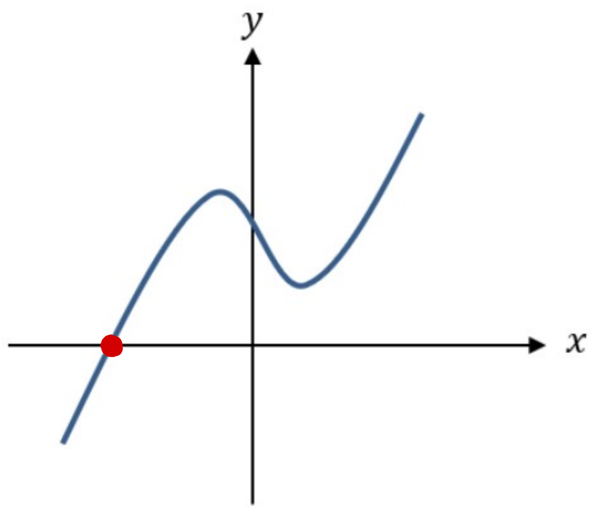
Roots of Cubic and Quartic Equations

The same principle applies to polynomials of higher degree, e.g. cubics and quartics.
All complex roots come in conjugate pairs.

A cubic equation **always has three roots** (by the Fundamental Law of Algebra). These roots may be repeated, and not all may be real roots...

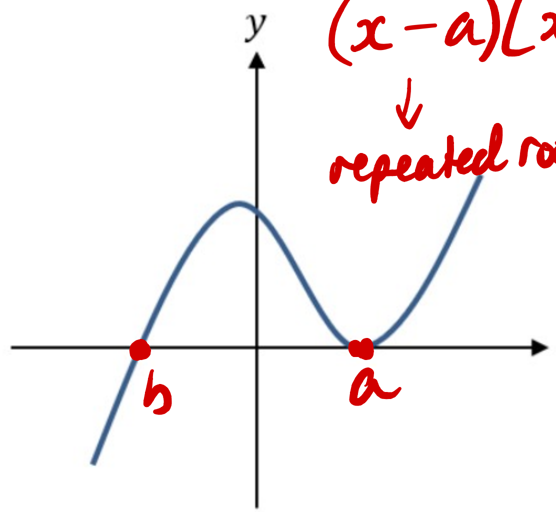


3 real roots



1 real root
2 complex roots
(conjugate pair)

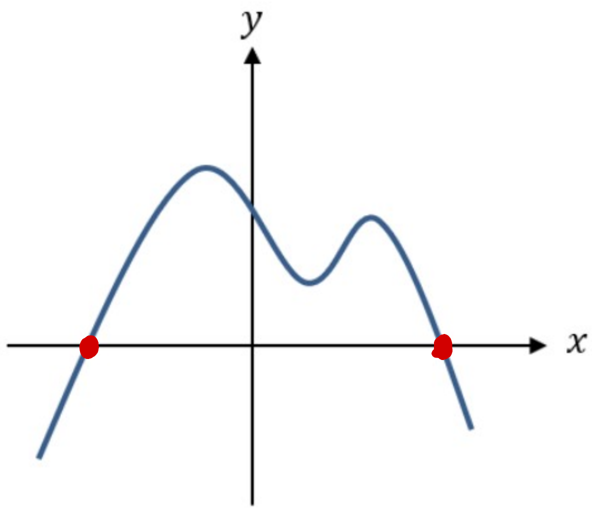
$(x-a)^2(x-b)=0$
$\downarrow$
repeated root of a

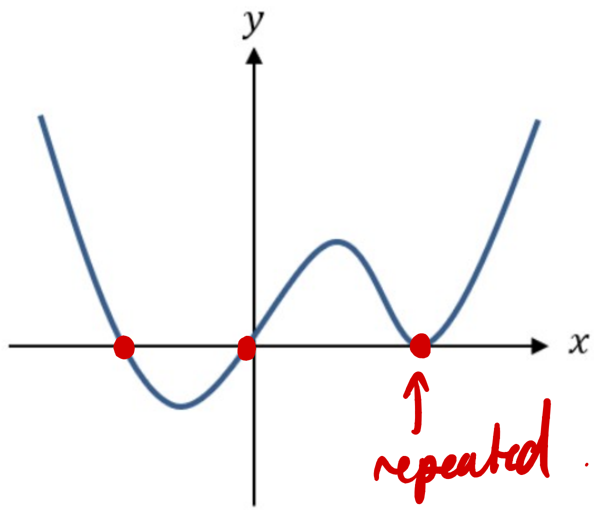


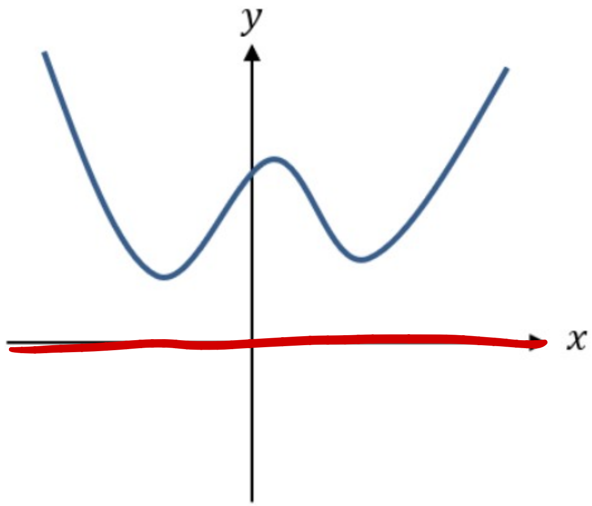
3 real roots
(2 are repeated roots).

And the same with quartics...

z^4







2 real roots
2 complex roots
(conjugate pair)

4 real roots
(2 are repeated)

4 complex roots
2 conjugate pairs.

$2 \pm 4i$

$3 \pm \frac{1}{2} i$

Given that -1 is a root of the cubic equation $z^3 - z^2 + 3z + k = 0$
Find the value of k and the other two roots of the equation.

Note that the next 3 examples can all be done using Chapter 4 techniques.
I think this method is superior, so you might like to try this after doing Chapter 4!

General idea: if α and β are roots, then $(x-\alpha)(x-\beta) = 0$
(and similarly for cubics and quartics)

$z = -1$, equation will be true/correct.

$(-1)^3 - (-1)^2 + 3(-1) + k = 0$

$-1 - 1 - 3 + k = 0$

$-5 + k = 0$

$\underline{\underline{k = 5}}$

z multiplied by a constant.

$z^3 - z^2 + 3z + 5 = 0$

$(z+1)(z^2 \underbrace{-2z}_{az} + 5) = 0$

→ algebraic division

→ expanded compared coefficients.

$z^2 - 2z + 5 = 0$

$(z-1)^2 - 1 + 5 = 0$

$(z-1)^2 = -4$

$z - 1 = \pm 2i$

$z = 1 \pm 2i$

Roots are -1, $1+2i$, $1-2i$

Given that $3+i$ is a root of the quartic equation $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$, solve the equation completely.

If $3+i$ is a root, then so is $3-i$

$$(z-(3+i))(z-(3-i))(z-\alpha)(z-\beta)=0.$$

$\alpha+\beta=-b$

$\alpha\beta=c$

"z" by a constant.

$$2z^4-3z^3-39z^2+\underline{120z}-50=0$$

$$(z^2-6z+10)(\underline{2z^2}+9z\ \underline{-5})=0$$

30z

90z

$$2z^2+9z-5=0$$

$$(2z-1)(z+5)$$

$z=\frac{1}{2}$ $\quad z=-5$

Roots:

$3+i$

$3-i$

$\frac{1}{2}$

-5

Show that z^2+4 is a factor of $z^4-2z^3+21z^2-8z+68$
Hence solve the equation $z^4-2z^3+21z^2-8z+68=0$

$z^4-2z^3+21z^2-8z+68$

$(z^2+4)(z^2-2z+17) = z^4-2z^3+17z^2+4z^2-8z+68$

$z^4-2z^3+21z^2-8z+68$ ✓

$z^2+4=0$

$z^2=-4$

$z=\pm 2i$

$z^2-2z+17=0$

$(z-1)^2-1+17=0$

$(z-1)^2=-16$

$z-1=\pm 4i$

$z=1\pm 4i$

Roots are $2i$, $-2i$, $1+4i$, $1-4i$.

Your Turn

Given that 2 and 5 + 2i are roots of the equation

$$x^3 - 12x^2 + cx + d = 0, \qquad c, d \in \mathbb{R},$$

(*a*) write down the other complex root of the equation.

(1)

(*b*) Find the value of *c* and the value of *d*.

(5)

a) $5-2i$

$(5+2i)(5-2i)$

b) $(x-2)(x-(5+2i))(x-(5-2i)) = 0$

$(x-2)(x^2-10x+29) = 0$

$x^3 - 10x^2 + 29x - 2x^2 + 20x - 58 = 0.$

$x^3 - 12x^2 + 49x - 58 = 0.$

$c = 49$

$d = -58$

Ex 1F

(a) $5-2\mathrm{i}$ is a root	B1 (1)
(b) $(x-(5+2\mathrm{i}))(x-(5-2\mathrm{i}))=x^2-10x+29$ $x^3-12x^2+cx+d=(x^2-10x+29)(x-2)$ $c=49$, $d=-58$	M1 M1 M1 A1, A1 (5)