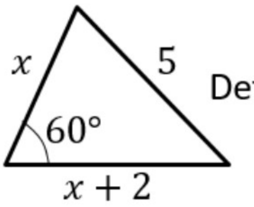
Chapter 9: Trigonometric Ratios

There is technically no new content in this chapter since GCSE.
However, the problems might be more involved than at GCSE level.

1:: Sine/Cosine Rule



Determine x.

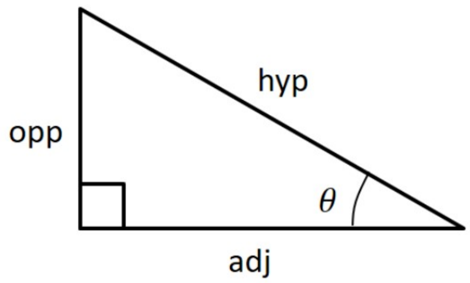
2:: Areas of Triangles

In ΔABC, $AB = 5, BC = 6$ and $\angle ABC = x$. Given that the area of $\angle ABC$ is 12cm² and that AC is the longest side, find the value of x.

3:: Graphs of Sine/Cosine/Tangent

Sketch $y = \sin(2x)$ for $0 \leq x \leq 360°$

RECAP :: Right-Angled Trigonometry

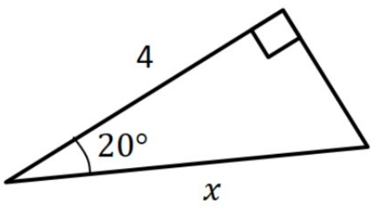


You are probably familiar with the formula: $\sin(\theta) = \frac{opp}{hyp}$

But what is the *conceptual* definition of sin ?

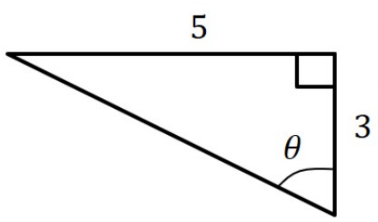
sin is a <u>function</u> which <u>inputs an angle</u> and gives the <u>ratio</u> between the opposite and hypotenuse.

Remember that a ratio just means the 'relative size' between quantities (in this case lengths). For this reason, sin/cos/tan are known as "trigonometric ratios".



Find x.

Tip: You can swap the thing you're dividing by and the result. e.g. $\frac{8}{2} = 4 \rightarrow \frac{8}{4} = 2$



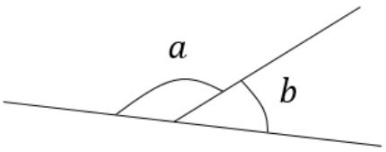
Find θ.

$$\tan(\theta) = \frac{5}{3}$$

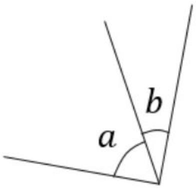
$$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.0°$$

Just for your interest…

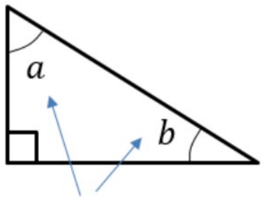
Have you ever wondered why “cosine” contains the word “sine”?



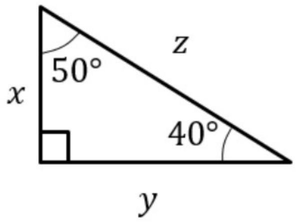
Supplementary Angles
add to 180°



Complementary Angles
add to 90°



Therefore these angles are complementary.



$\cos(50) =$

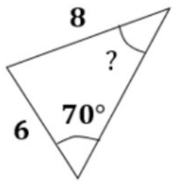
$\sin(40) =$

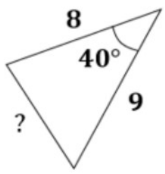
i.e. The **cosine** of an angle is the **sine** of the **complementary** angle.
Hence **cosine = COMPLEMENTARY SINE**

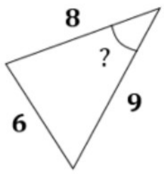
OVERVIEW: Finding missing sides and angles

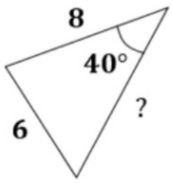
When triangles are not right-angled, we can no longer use simple trigonometric ratios, and must use the cosine and sine rules.

You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule
#2 Two sides known and a missing side opposite a known angle	Remaining side	Cosine rule
#3 All three sides	An angle	Cosine rule
#4 Two sides known and a missing side not opposite known angle	Remaining side	Cosine rule OR Sine rule twice



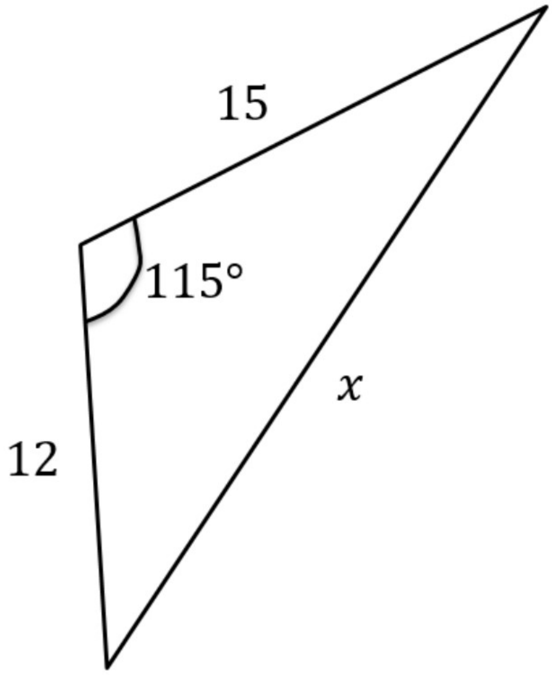






The Cosine Rule

We use the **cosine rule** whenever we have **three sides** (and an angle) involved.

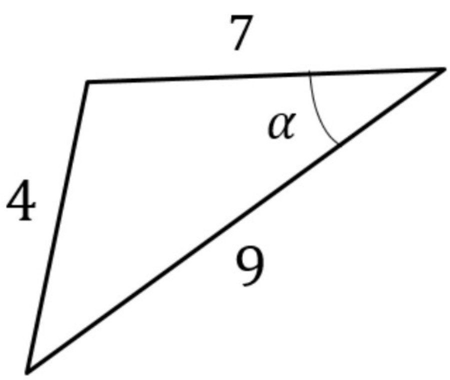


Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

How do we label the sides?

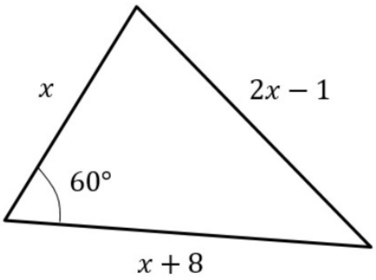
Dealing with Missing Angles



You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule
#2 Two sides known and a missing side opposite a known angle	Remaining side	Cosine rule
#3 All three sides	An angle	Cosine rule
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	Cosine rule OR Sine rule twice

Trickier Questions

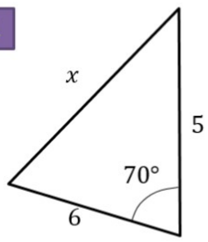
Determine the value of x.



Coastguard station B is 8 km, on a bearing of $060°$, from coastguard station A. A ship C is 4.8 km on a bearing of $018°$, away from A. Calculate how far C is from B.

Your Turn

1



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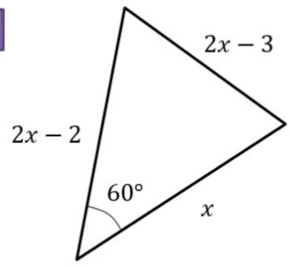

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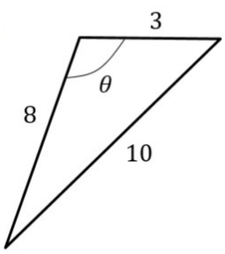


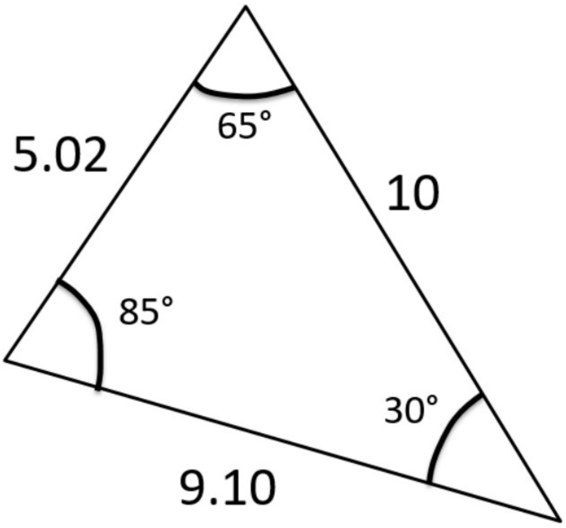
$x = 6.36$

$\theta = 124.2°$

$x = 5$

Ex 9A

The Sine Rule

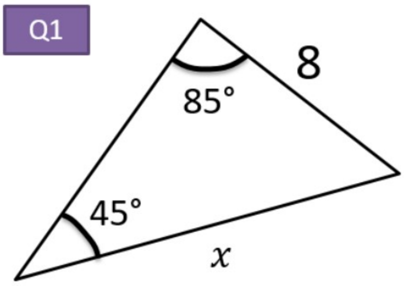


For this triangle, try calculating each side divided by the sin of its opposite angle. What do you notice in all three cases?

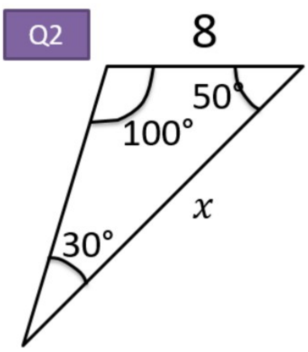
Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q1



Q2

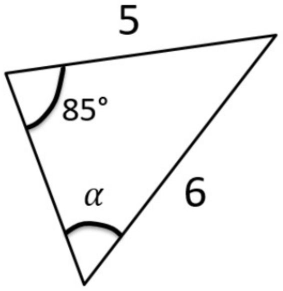


When you have a missing angle, it's better to take reciprocals to get:

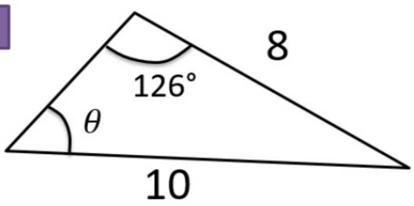
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

i.e. in general put the missing value in the numerator.

Q3

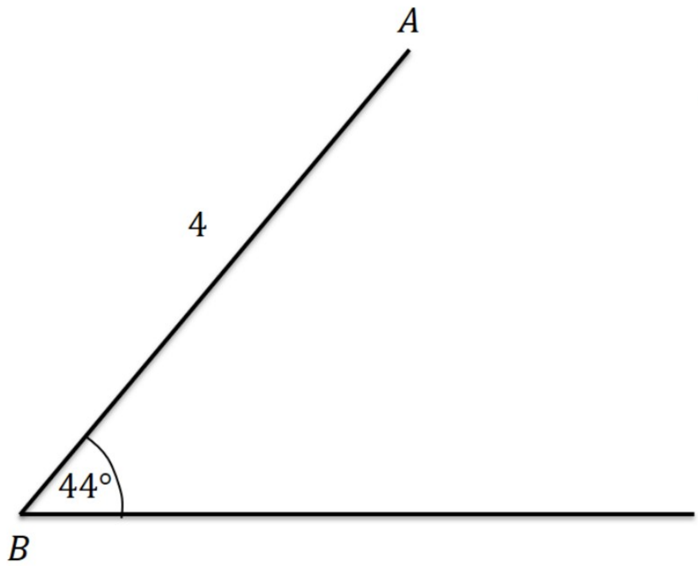


Q4

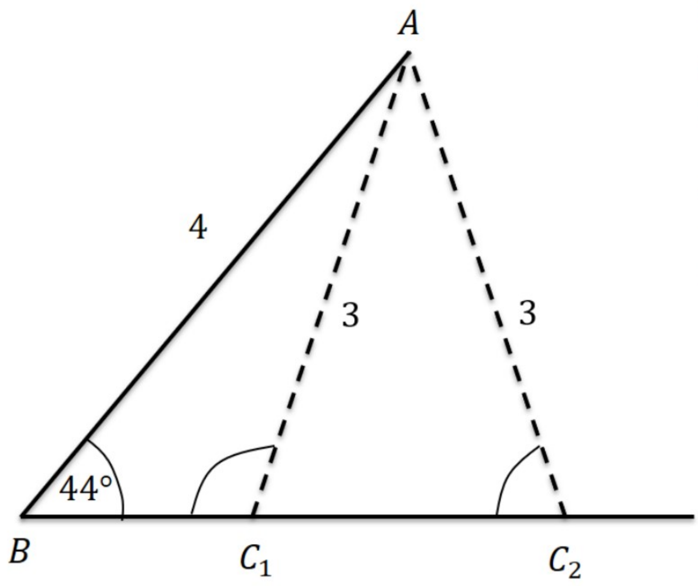


Ex 9B

The 'Ambiguous Case'



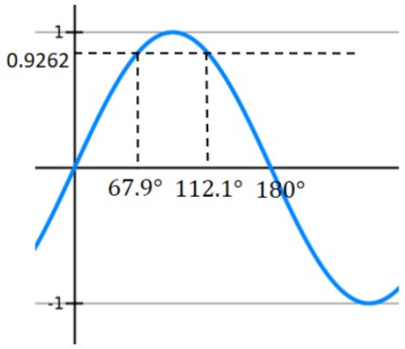
Suppose you are told that $AB = 4$, $AC = 3$ and $\angle ABC = 44°$. What are the possible values of $\angle ACB$?



Suppose you are told that $AB = 4$, $AC = 3$ and $\angle ABC = 44°$. What are the possible values of $\angle ACB$?

C is somewhere on the horizontal line. There's two ways in which the length could be 3. Using the sine rule:

$$\frac{\sin C}{4} = \frac{\sin 44}{3}$$
$$C = \sin^{-1}(0.9262)$$

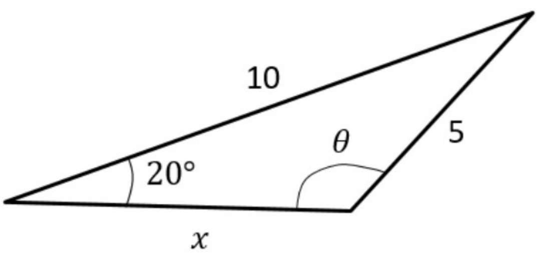


Your calculator will give the acute angle of 67.9° (i.e. C_2). But if we look at a graph of sin, we can see there's actually a second value for $\sin^{-1}(0.9262)$, corresponding to angle C_1.

The sine rule produces two possible solutions for a missing angle:

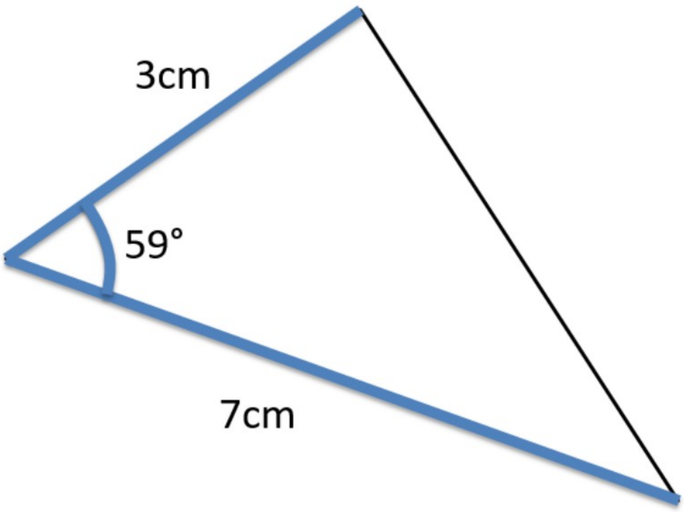
$$\sin\theta = \sin(180° - \theta)$$

Whether we use the acute or obtuse angle depends on context.



Given that the angle θ is obtuse, determine θ and hence determine the length of x.

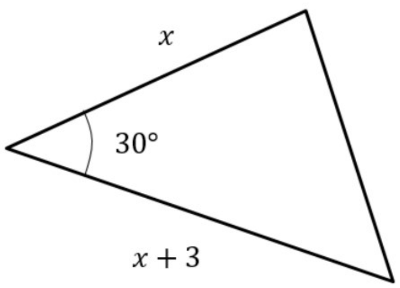
Area of Non Right-Angled Triangles



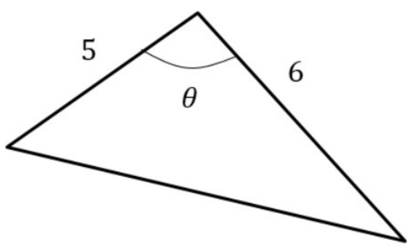
$$\text{Area} = \frac{1}{2} a\ b \sin(C)$$

where C is the angle between two sides a and b.

Tip: You shouldn't have to label sides/angles before using the formula. Just remember that the angle is <u>between the two sides</u>.



The area of this triangle is 10.
Determine x.

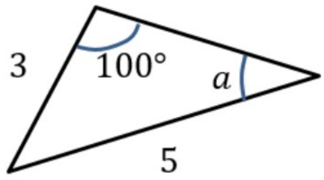


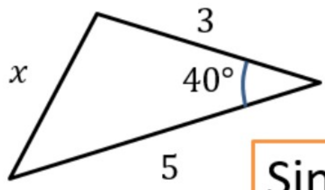
The area of this triangle is also 10.
If θ is obtuse, determine θ.

Sine or cosine rule?

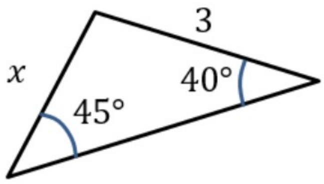
Recall that whenever we have **two "side-angle pairs"** involved, use sine rule. If there's **3 sides** involved, we can use cosine rule.

Sine | Cosine

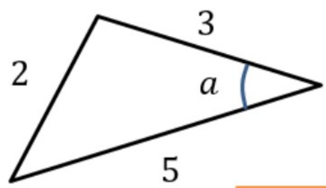




Sine | Cosine

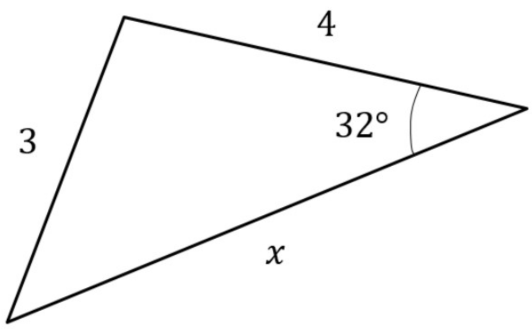


Sine | Cosine



Sine | Cosine

Sine or cosine rule?



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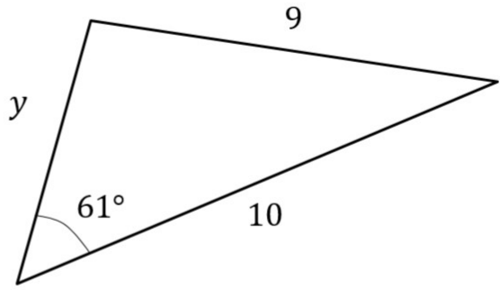
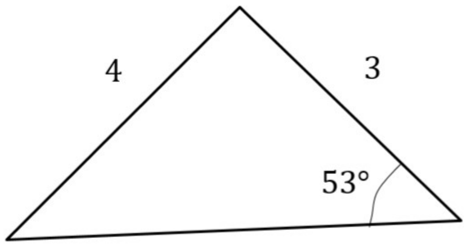


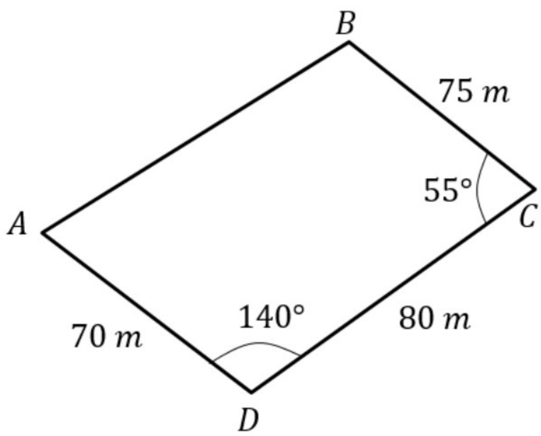


Find the area

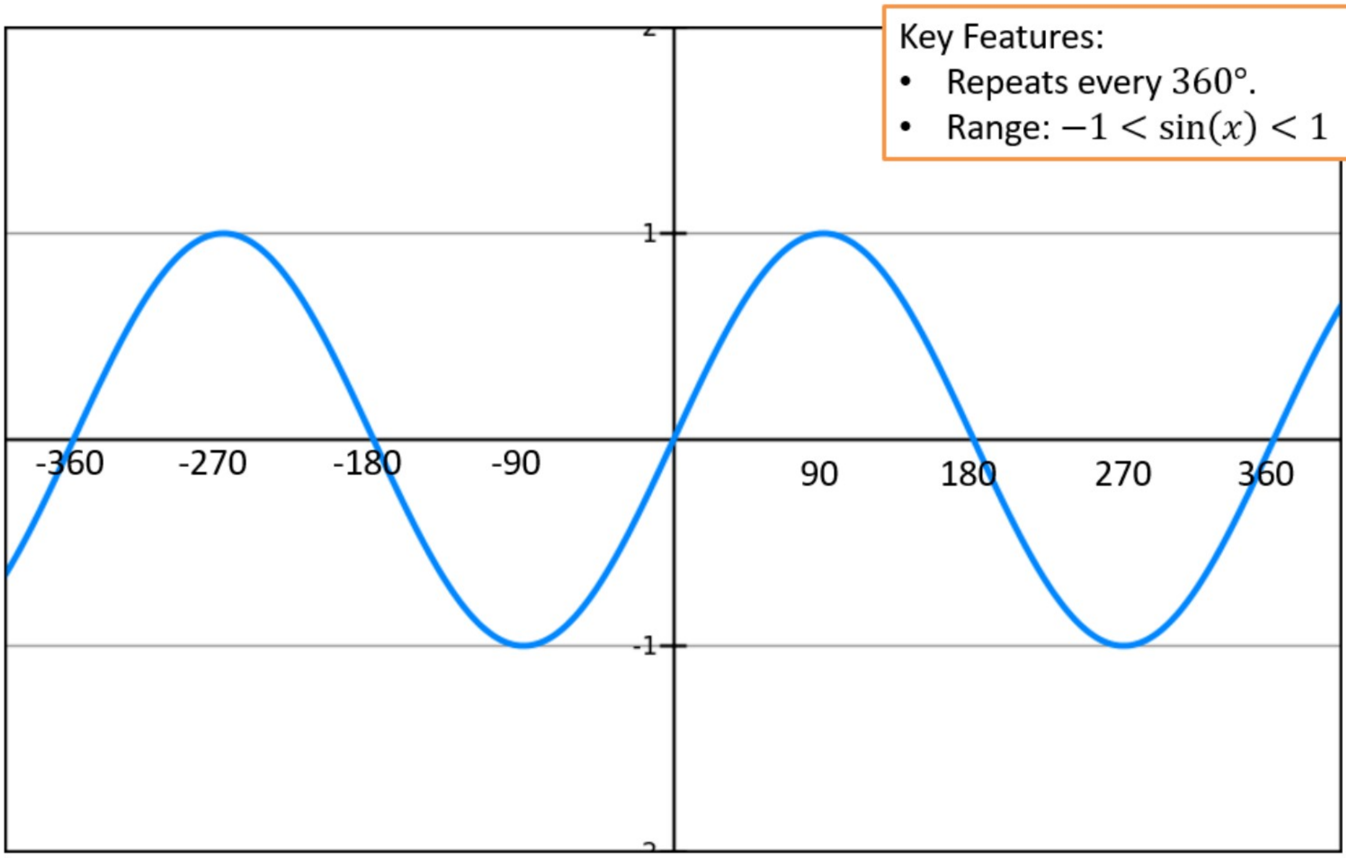
Problem Solving With Sine/Cosine Rule

The diagram shows the locations of four mobile phone masts in a field, $BC = 75\ m$.
$CD = 80m$, angle $BCD = 55°$ and angle $ADC = 140°$.
In order that the masts do not interfere with each other, they must be at least 70m apart.
Given that A is the minimum distance from D, find:

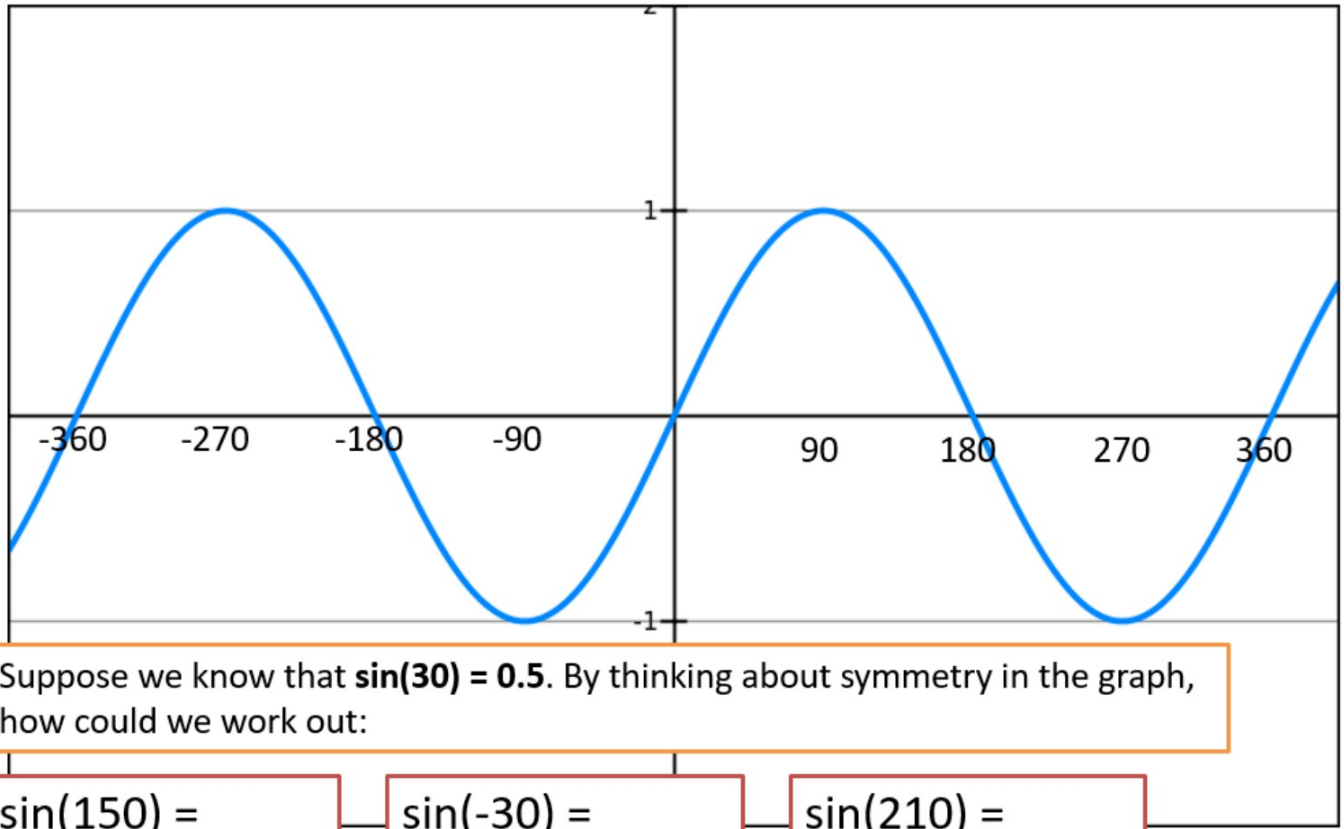
a) The distance A is from B
b) The angle BAD
c) The area enclosed by the four masts.



Sine Graph



Key Features:
- Repeats every $360°$.
- Range: $-1 < \sin(x) < 1$



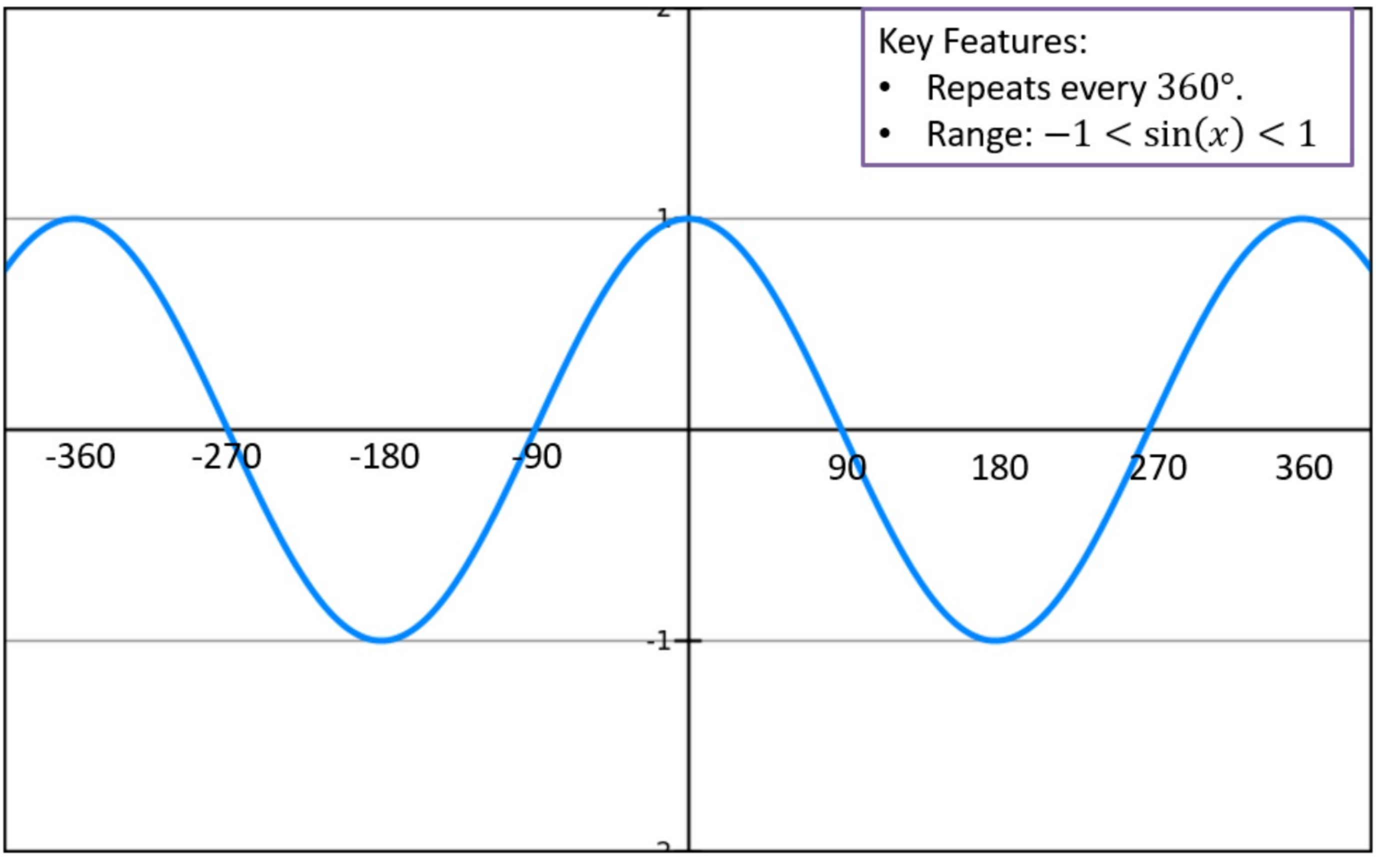
Suppose we know that **sin(30) = 0.5**. By thinking about symmetry in the graph, how could we work out:

sin(150) =

sin(-30) =

sin(210) =

Cosine Graph



Key Features:

- Repeats every $360°$.
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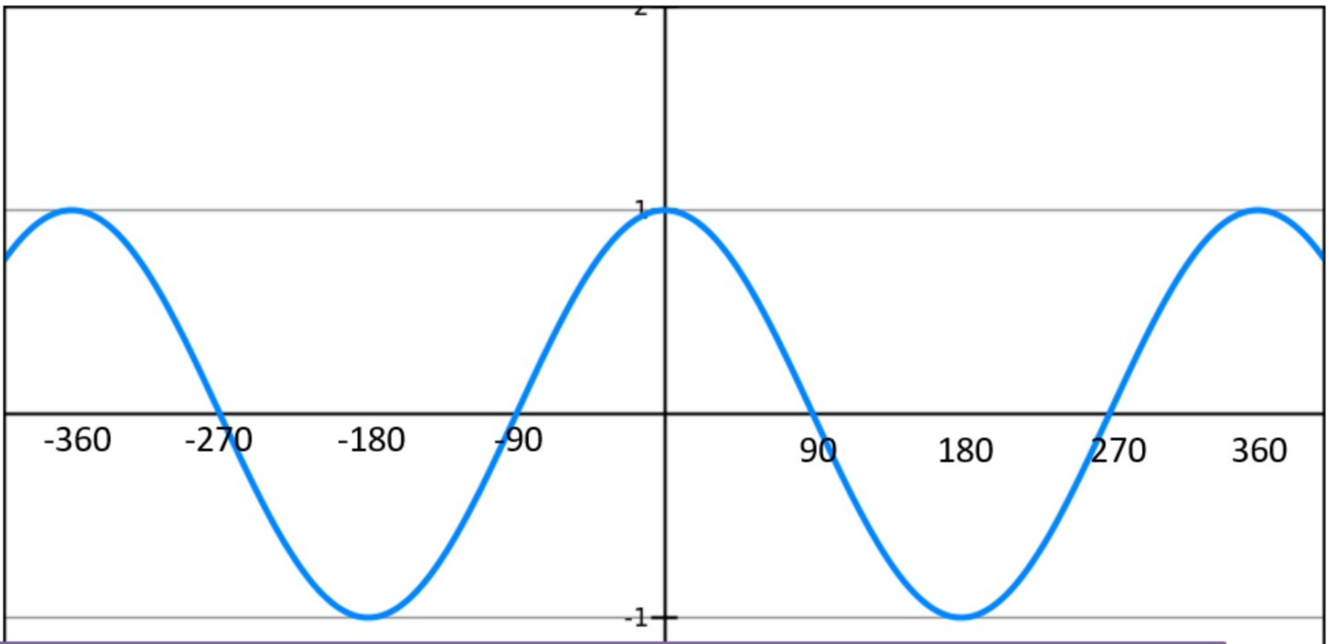
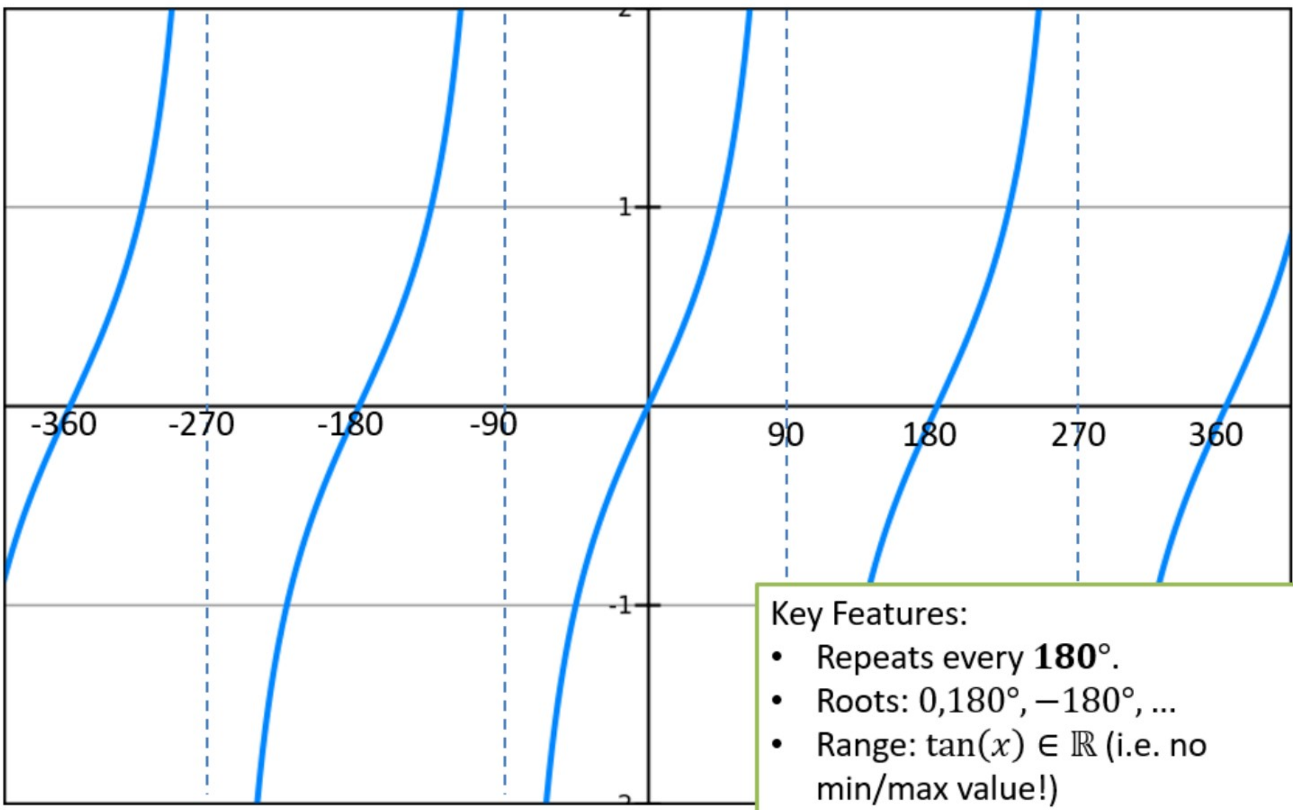


Suppose we know that **cos(60) = 0.5**. By thinking about symmetry in the graph, how could we work out:

cos(120) =

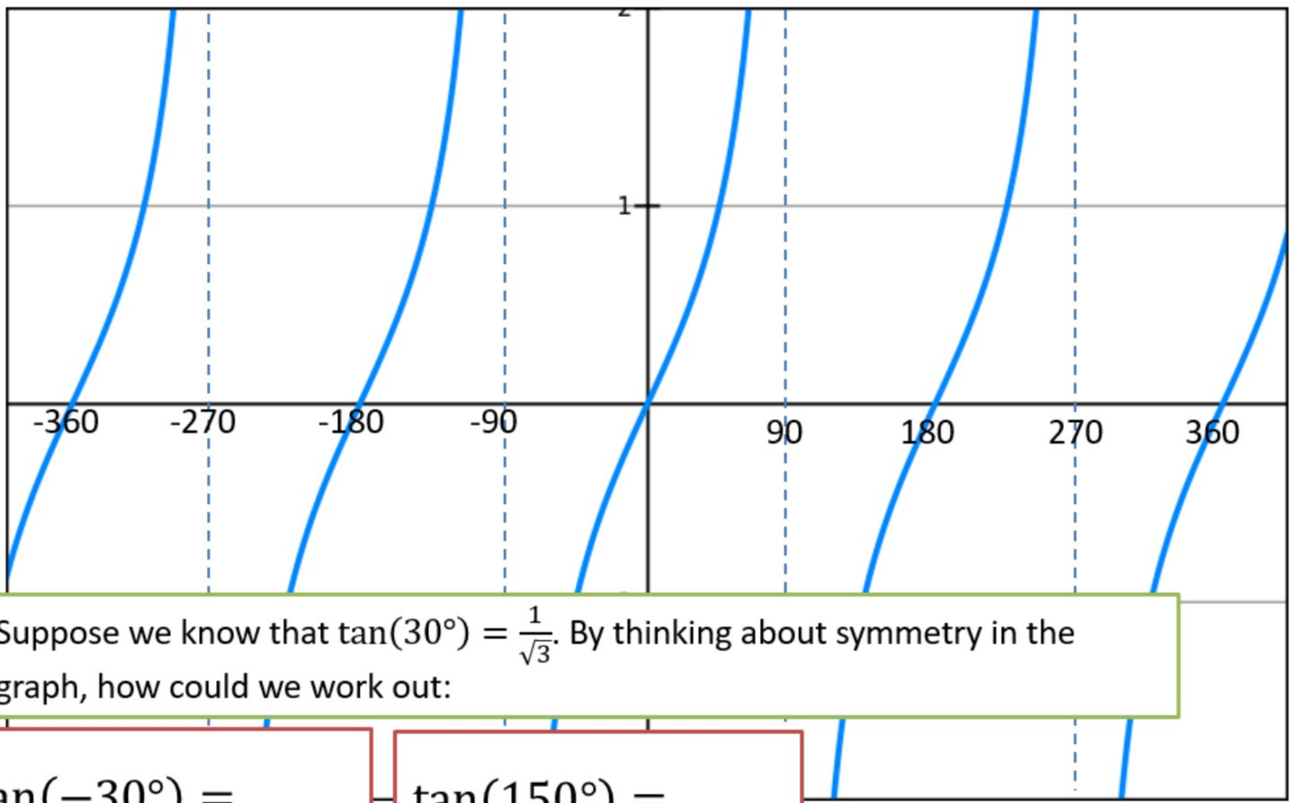
cos(-60) =

cos(240) =



Key Features:

- Repeats every $\mathbf{180}°$.
- Roots: $0, 180°, -180°, \ldots$
- Range: $\tan(x) \in \mathbb{R}$ (i.e. no min/max value!)
- Asymptotes: $x = \pm 90°, \pm 270°, \ldots$



Suppose we know that $\tan(30°) = \frac{1}{\sqrt{3}}$. By thinking about symmetry in the graph, how could we work out:

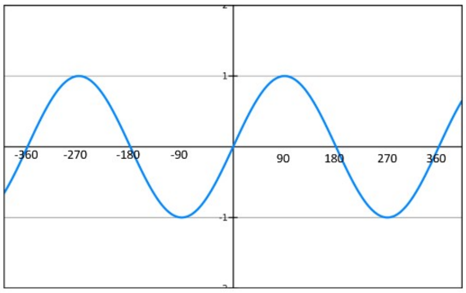
$\tan(-30°) =$

$\tan(150°) =$

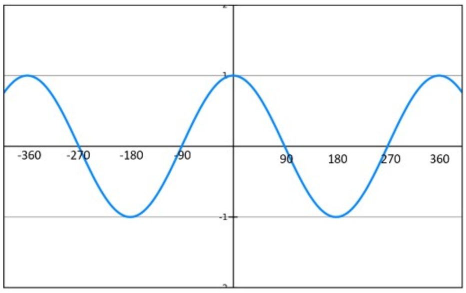
Transforming Trigonometric Graphs

There is no new theory here: just use your knowledge of transforming graphs, i.e. whether the transformation occurs 'inside' the function (i.e. input modified) or 'outside' the function (i.e. output modified).

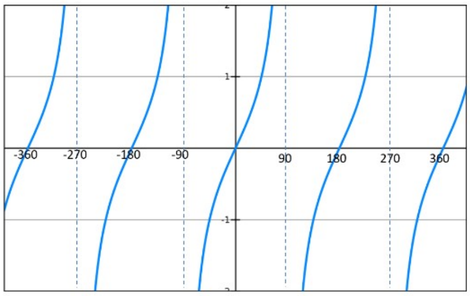
Sketch $y = 4\sin x, 0 \leq x \leq 360°$



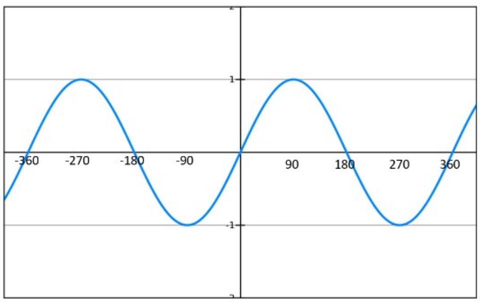
Sketch $y = \cos(x + 45°), 0 \leq x \leq 360°$



Sketch $y = -\tan x, 0 \leq x \leq 360°$

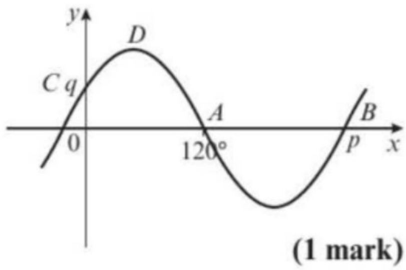


Sketch $y = \sin\left(\frac{x}{2}\right), 0 \leq x \leq 360°$



Ex 9F/G

14 The diagram shows part of the graph of $y = \mathrm{f}(x)$.
It crosses the x-axis at $A(120°, 0)$ and $B(p, 0)$.
It crosses the y-axis at $C(0, q)$ and has a maximum value at D, as shown.



Given that $\mathrm{f}(x) = \sin(x + k)$, where $k > 0$, write down

a the value of p **(1 mark)**

b the coordinates of D **(1 mark)**

c the smallest value of k **(1 mark)**

d the value of q. **(1 mark)**

When triangles are not right-angled, we can no longer use simple trigonometric ratios, and must use the cosine and sine rules.

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#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule
#2 Two sides known and a missing side opposite a known angle	Remaining side	Cosine rule
#3 All three sides	An angle	Cosine rule
#4 Two sides known and a missing side not opposite known angle	Remaining side	Cosine rule OR Sine rule twice