Edexcel C2 Jan 2013 Q9c

Ex 13E

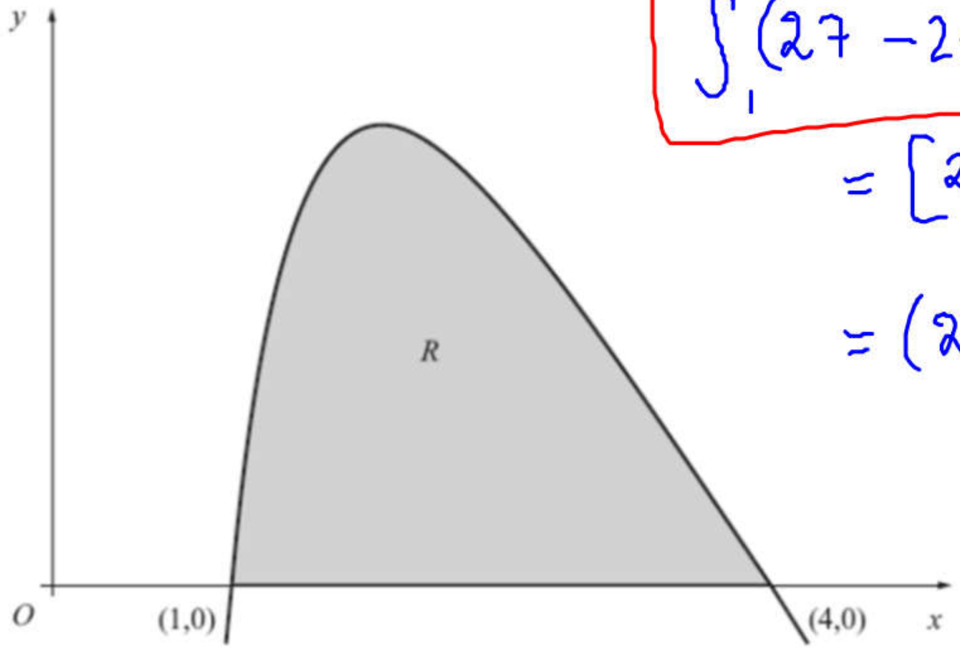


Figure 2

The finite region R, as shown in Figure 2, is bounded by the x-axis and the curve with equation

$$y = 27 - 2x - 9\sqrt{x} - \frac{16}{x^2}, \qquad x > 0.$$

The curve crosses the x-axis at the points (1, 0) and (4, 0).

(*c*) Use integration to find the exact value for the area of R.

~~(6)~~

$$\boxed{\int_1^4 (27 - 2x - 9x^{1/2} - 16x^{-2})\,dx}$$

$$= \left[27x - x^2 - 6x^{3/2} + 16x^{-1}\right]_1^4$$

$$= \left(27(4) - (4)^2 - 6(4)^{3/2} + 16(4)^{-1}\right) - (27 - 1 - 6 + 16)$$

$$\boxed{= 12}$$

$$\int_1^k x^2\,dx = \frac{1}{3}$$

$$= \left[\frac{1}{3}x^3\right]_1^k$$

$$= \left(\frac{1}{3}k^3\right) - \left(\frac{1}{3}\right)$$

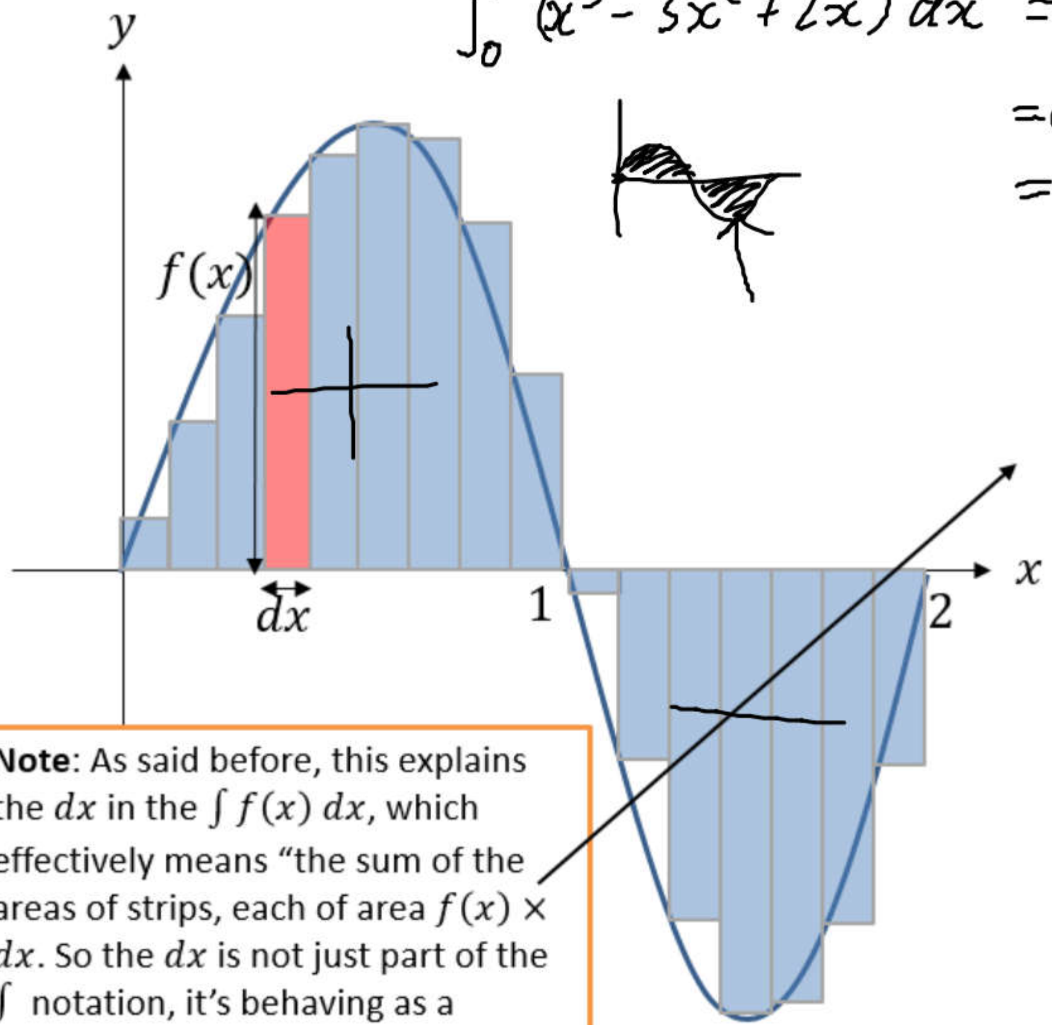
$$= \frac{1}{3}k^3 - \frac{1}{3}$$

$$\frac{1}{3}k^3 - \frac{1}{3} = \frac{1}{3}$$

‘Negative Areas’

Sketch the curve $y = x(x-1)(x-2)$ (which expands to give $y = x^3 - 3x^2 + 2x$).
Now calculate $\int_0^2 x(x-1)(x-2)\ dx$. Why is this result surprising?

$$\int_0^2 (x^3 - 3x^2 + 2x)\,dx = \left[\tfrac{1}{4}x^4 - x^3 + x^2\right]_0^2$$
$$= (4 - 8 + 4) - (0)$$
$$= 0$$



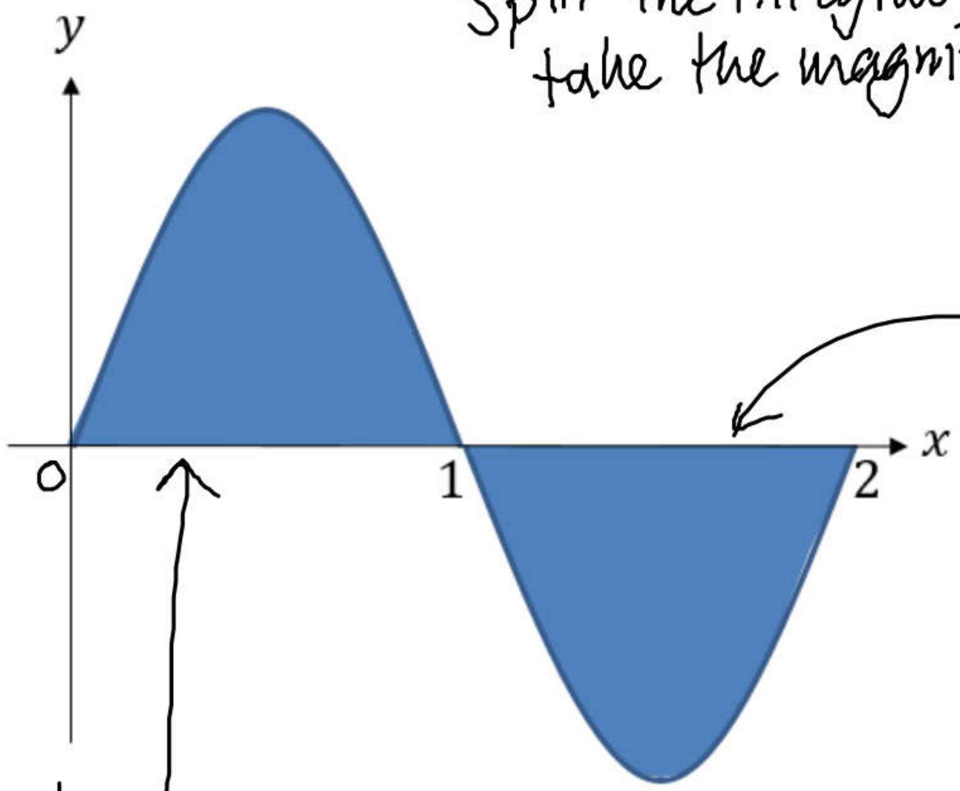
Integration $\int f(x)\ dx$ is just the sum of areas of infinitely thin rectangles, where the current y value (i.e. $f(x)$) is each height, and the widths are dx.
i.e. The area of each is $f(x) \times dx$

The problem is, when $f(x)$ is negative, then $f(x) \times dx$ is negative, i.e. a negative area!
The result is that the ‘positive area’ from 0 to 1 is cancelled out by the ‘negative area’ from 1 to 2, giving an overall ‘area’ of 0.
So how do we resolve this?

Note: As said before, this explains the dx in the $\int f(x)\ dx$, which effectively means “the sum of the areas of strips, each of area $f(x) \times dx$. So the dx is not just part of the $\int$ notation, it’s behaving as a physical quantity! (i.e. length)

Find the total area bound between the curve $y = x(x-1)(x-2)$ and the x-axis.

Split the integral into 2 separate pieces, and take the magnitude of any negative areas.



$$\int_1^2 (x^3-3x^2+2x)\,dx$$
$$= \left[\frac{1}{4}x^4 - x^3 + x^2\right]_1^2$$
$$= \left(\frac{1}{4}(2)^4 - 2^3 + 2^2\right) - \frac{1}{4}$$
$$= (4 - 8 + 4) - \frac{1}{4}$$
$$= -\frac{1}{4}$$

So area is $\frac{1}{4}$

$$\int_0^1 (x^3-3x^2+2x)\,dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^1$$
$$= \frac{1}{4} - 1 + 1$$
$$= \frac{1}{4}$$

Total area bound between curve and x-axis $= \frac{1}{2}$

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Ex 13F

1e
2
3 ~~x~~

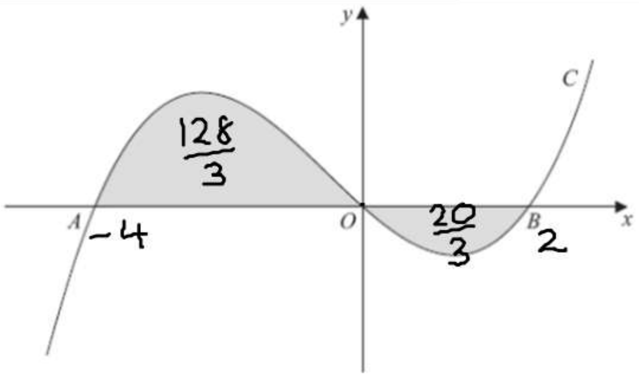


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = x(x+4)(x-2).$$

The curve C crosses the x-axis at the origin O and at the points A and B.

(*a*) Write down the x-coordinates of the points A and B. **(1)**

The finite region, shown shaded in Figure 3, is bounded by the curve C and the x-axis.

(*b*) Use integration to find the total area of the finite region shown shaded in Figure 3. **(7)**

a)

$y = x(x+4)(x-2)$

$x=0,\ x=2,\ x=-4$

A has x-coord -4

B has x-coord 2

b)

$$x(x+4)(x-2) = x(x^2+2x-8)$$
$$= x^3+2x^2-8x$$

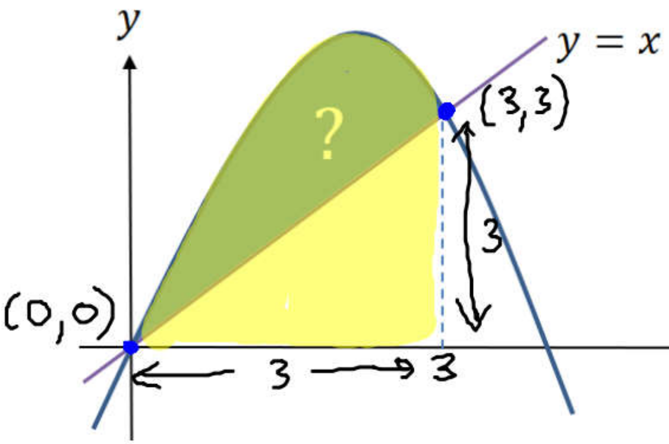
$$\int_{-4}^{0}(x^3+2x^2-8x)\,dx = \left[\tfrac{1}{4}x^4+\tfrac{2}{3}x^3-4x^2\right]_{-4}^{0}$$
$$= 0-\left(\tfrac{1}{4}(-4)^4+\tfrac{2}{3}(-4)^3-4(-4)^2\right)$$
$$= -\left(64-\tfrac{128}{3}-64\right)$$
$$= \tfrac{128}{3}$$

$$\int_{0}^{2}(x^3+2x^2-8x)\,dx = \left[\tfrac{1}{4}x^4+\tfrac{2}{3}x^3-4x^2\right]_{0}^{2}$$
$$= \left(\tfrac{1}{4}\times 2^4+\tfrac{2}{3}(2)^3-4(2^2)\right)-0$$
$$= 4+\tfrac{16}{3}-16$$
$$= -\tfrac{20}{3}$$

So total area $= \frac{128}{3}+\frac{20}{3}$

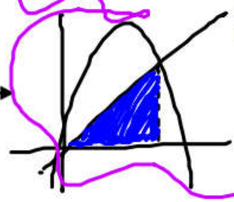
$= \frac{148}{3}$.

Areas between curves and lines



How could we find the area between the line and the curve?

$\int_0^3 x(4-x)\,dx$ − area of triangle = blue area



$\int_0^3 x \, dx = \left[\frac{1}{2}x^2\right]_0^3$

Determine the area between the lines with equations $y = x(4-x)$ and $y = x$

$y = x(4-x)$

$y = x$

Solve simultaneously

$x = x(4-x)$

$x = 4x - x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x=0 \quad x=3$

$y=3$

Area of triangle $= 3 \times 3 \times \frac{1}{2} = \frac{9}{2}$

$\int_0^3 x(4-x)\,dx = \int_0^3 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^3$

$= (2 \times 3^2 - \frac{1}{3} \times 3^3) - 0$

$= 9$

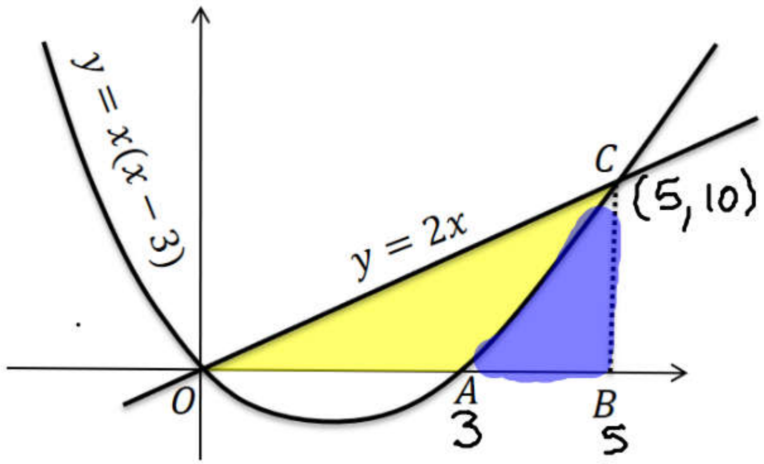
$\int_0^3 (x(4-x) - x)\,dx$

$\int$ (top curve − bottom curve) dx

Area $= 9 - \frac{9}{2}$

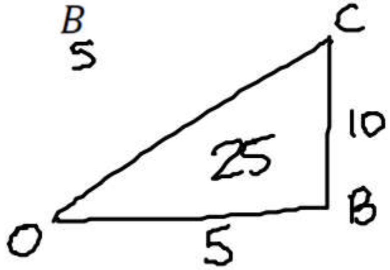
$= \frac{9}{2}$ units^2

A Harder One



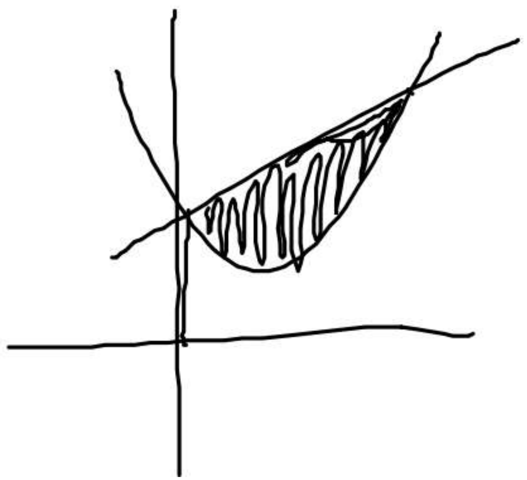
The diagram shows a sketch of the curve with equation $y = x(x-3)$ and the line with equation $y = 2x$.
Find the area of the shaded region OAC.

$x(x-3) = 2x$
$x^2 - 3x = 2x$
$x^2 - 5x = 0$
$x(x-5) = 0$
$x = 0, x = 5$
$y = 10$



Area $= 5 \times 10 \times \frac{1}{2} = 25$

$$\int_3^5 x(x-3)\,dx = \int_3^5 (x^2 - 3x)\,dx$$
$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_3^5$$
$$= \left(\frac{1}{3}(5)^3 - \frac{3}{2}(5)^2\right) - \left(\frac{1}{3}(3)^3 - \frac{3}{2}(3)^2\right)$$
$$= \frac{25}{6} - \left(-\frac{9}{2}\right) = \frac{26}{3}$$



Area = area of $\Delta - \frac{26}{3} = 25 - \frac{26}{3} = \frac{49}{3}$ units^2

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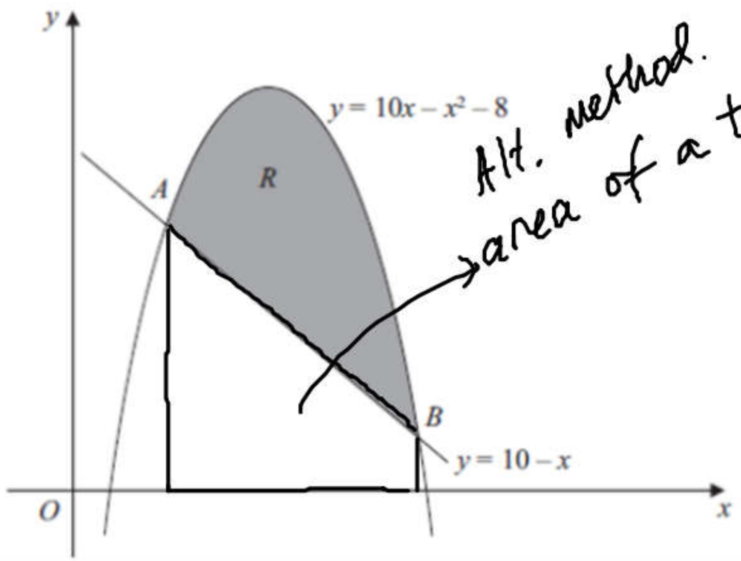

Alt. method
area of a trapezium

Figure 2 shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$.

The line and the curve intersect at the points A and B, and O is the origin.

(*a*) Calculate the coordinates of A and the coordinates of B. **(5)**

The shaded area R is bounded by the line and the curve, as shown in Figure 2.

(*b*) Calculate the exact area of R. **(7)**

$10 - x = 10x - x^2 - 8$

$0 = -x^2 + 11x - 18$

$x = 2, \ x = 9$

$\int_2^9 (10x - x^2 - 8 - (10 - x))\, dx$

$= \int_2^9 (10x - x^2 - 8 - 10 + x)\, dx$

$= \int_2^9 (-x^2 + 11x - 18)\, dx$

$= \left[-\frac{1}{3}x^3 + \frac{11}{2}x^2 - 18x\right]_2^9 = \left(-\frac{1}{3}\times 9^3 + \frac{11}{2}\times 9^2 - 18\times 9\right) - \left(-\frac{1}{3}\times 2^3 + \frac{11}{2}\times 2^2 - 18\times 2\right) = \frac{81}{2} - \left(-\frac{50}{3}\right) = \frac{343}{6}$

Preferred Method?

If the top curve has equation $y = f(x)$ and the bottom curve $y = g(x)$, the area between them is:

a and b are intersections

$$\int_b^a \left(f(x) - g(x)\right) dx$$

This means you can integrate a single expression to get the final area, without any adjustment required after.

Ex 13G