Boundary Conditions

Sometimes you're given certain conditions, which allows us to find constants (just as we could with first order differential equations).

Find y in terms of x, given that $\frac{d^2y}{dx^2} - y = 2e^x$, and that $\frac{dy}{dx} = 0$ and $y = 0$ at $x = 0$.

we also need to diff. the G.S.

General solution: $\boldsymbol{y = Ae^x + Be^{-x} + xe^x}$

$$\boxed{0 = A + B}$$

$$\frac{dy}{dx} = Ae^x - Be^{-x} + e^x + xe^x$$

$$0 = A - B + 1$$

$$\boxed{B = A + 1}$$

$$0 = A + A + 1$$

$$A = -\frac{1}{2} \qquad B = \frac{1}{2}$$

$$y = -\frac{1}{2}e^x + \frac{1}{2}e^{-x} + xe^x$$

June 2010 Q8 (revisited!)

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(c) Given that at $x = 0$, $y = 0$ and $\frac{dy}{dx} = 5$, find the particular solution to this differential equation, giving your solution in the form $y = f(x)$ **(5)**

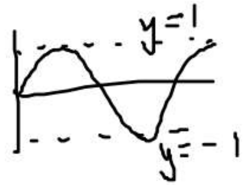
(d) Sketch the curve with equation $y = f(x)$ for $0 \le x \le \pi$ **(2)**

You previously found the general solution in (b) as $y = A\cos 5x + B\sin 5x + \frac{3}{10}x\sin 5x$

$y = A\cos 5x + B\sin 5x + \frac{3}{10}x\sin 5x$

$0 = A$

$\frac{dy}{dx} = -5A\sin 5x + 5B\cos 5x + \frac{3}{10}\sin 5x + \frac{3}{2}x\cos 5x$



$5 = 5B$

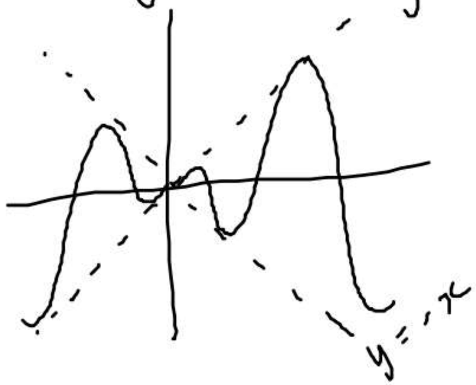
$B = 1$

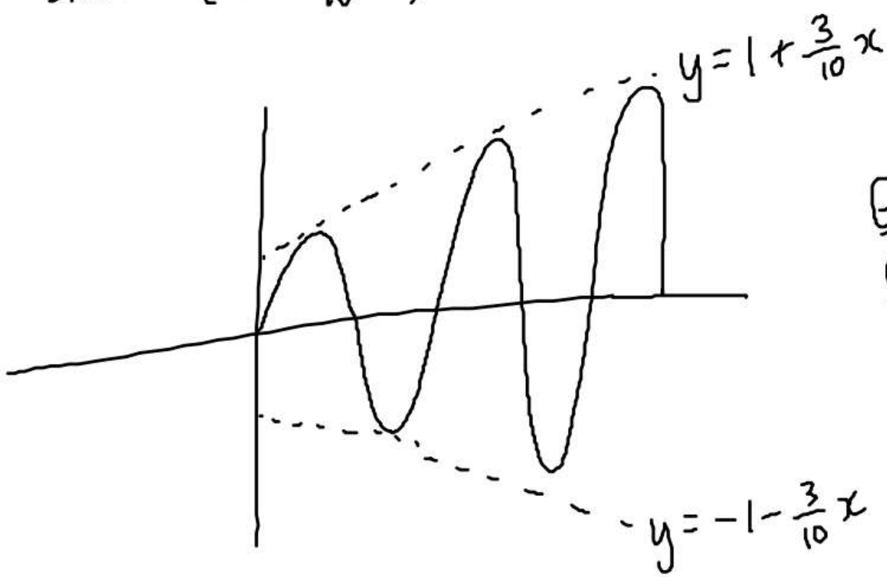
$y = \sin 5x + \frac{3}{10}x\sin 5x$

$y = \sin 5x\left(1 + \frac{3}{10}x\right)$

$f(5x)$

d) $y = x\sin x$





Ex 7D

Odd Q

Modelling with Differential Equations

This chapter looks at applied settings in which 1st and 2nd order differential equations might emerge.

1:: Modelling with 1st order differential equations.

"A particle P is moving along a straight line. At time t seconds, the acceleration of the particle is given by $a = t + \frac{3}{t}v,\ t \geq 0$
Given that $v = 0$ when $t = 2$, show that the velocity of the particle at time t is given by the equation $v = ct^3 - t^2$ where c is a constant to be found."

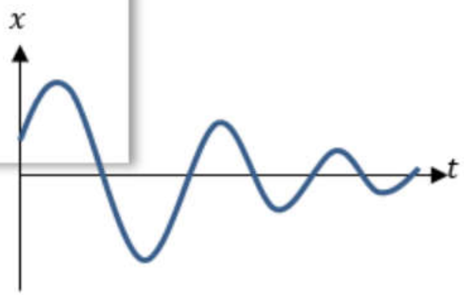
2:: Simple Harmonic Motion

Modelling the motion of a particle which has a acceleration towards a central point proportional to its displacement from this centre.

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

3:: Damped and Force Harmonic Motion

How to model the damping force on a spring (damped harmonic motion) and how to model additional forces (forced harmonic motion).



4:: Coupled First-Order Differential Equations

The differential-equation version of "simultaneous equations". e.g. Prey-predator models found in Biology.

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

At the end of Pure Year 2 (Integration), we saw how we could form and solve 1st order differential equations from context. In Core Pure Year 2 this differential equation may be of the form $\frac{dy}{dx} + Py = Q$. One common example of first order differential equations is with displacement, velocity and acceleration, e.g. $a = \frac{dv}{dt}$

$v = \frac{dx}{dt}$

$t = 0,\ x = 0,\ v = 0$

A particle P starts from rest at a point O and moves along a straight line. At time t seconds the acceleration, a ms^{-2}, of P is given by

$$a = \frac{6}{(t+2)^2}, \qquad t \geq 0$$

(a) Find the velocity of P at time t seconds.

(b) Show that the displacement of P from O when $t = 6$ is $(18 - 12\ln 2)$m

a) $\frac{dv}{dt} = \frac{6}{(t+2)^2}$

$v = \int \frac{6}{(t+2)^2}\,dt$ $\quad 6(t+2)^{-2}$

$v = -6(t+2)^{-1} + c$

$0 = -\frac{6}{2} + c$

$c = 3$

$v = -\frac{6}{t+2} + 3$

b) $\frac{dx}{dt} = -\frac{6}{t+2} + 3$

$x = \int \left(-\frac{6}{t+2} + 3\right) dt$

$x = -6\ln|t+2| + 3t + c$

$0 = -6\ln 2 + c$

$x = -6\ln|t+2| + 3t + 6\ln 2$

$x = -6\ln 8 + 18 + 6\ln 2$

$x = -6 \times 3\ln 2 + 18 + 6\ln 2$

$x = \underline{18 - 12\ln 2}$

A particle P is moving along a straight line. At time t seconds, the acceleration of the particle is given by $\boxed{a = t + \frac{3}{t}v,}$ $t \geq 0$

Given that $v = 0$ when $t = 2$, show that the velocity of the particle at time t is given by the equation $v = ct^3 - t^2$ where c is a constant to be found.

$$\frac{dv}{dt} = t + \frac{3}{t}v$$

"$\frac{dy}{dx} + Py = Q$"

Use integrating factor

$$\boxed{\frac{dv}{dt} - \frac{3}{t}v = t}$$

×

I.F.

$$P(t) = -\frac{3}{t}$$

$$\int P(t)\,dt = -3\ln t$$

$$\text{I.F} = e^{-3\ln t} = t^{-3}$$

$$t^{-3}\frac{dv}{dt} - 3t^{-4}v = t^{-2}$$

$$\frac{d}{dt}\left(t^{-3}v\right) = t^{-2}$$

$$t^{-3}v = \int t^{-2}\,dt$$

$$t^{-3}v = -t^{-1} + c$$

$$v = -t^2 + ct^3$$

$v = 0, t = 2$

$$0 = -4 + 8c$$

$$8c = 4$$

$$c = \frac{1}{2}$$

$$\underline{v = \frac{1}{2}t^3 - t^2}.$$

Ex 8A Q1-6

Classic 'Filling a Container' Example

A storage tank initially ($t=0$) containers 1000 litres of pure water ($x=0$). Liquid is removed from the tank at a constant rate of 30 litres per hour and a chemical solution is added to the tank at a constant rate of 40 litres per hour. The chemical solution contains 4 grams of copper sulphate per litre of water. Given that there are x grams of copper sulphate in the tank after t hours and that the copper sulphate immediately disperses throughout the tank on entry,

(a) Show that the situation can be modelled by the differential equation

$$\underbrace{\frac{dx}{dt}}_{\text{rate of change of cop. sul.}} = \underbrace{160}_{\text{in}} - \underbrace{\frac{3x}{100+t}}_{\text{out}}$$

(b) Hence find the number of grams of copper sulphate in the tank after 6 hours. $t=6$

(c) Explain how the model could be refined.

40 litres/hour

30 litres/hour

Litres of liquid in tank after t hours: $1000 + 40t - 30t = 1000 + 10t$

Concentration (fraction) of copper sulphate after t hours (grams/volume): $\frac{x}{1000+10t}$ (grams per litre)

Rate copper sulphate **in**: per hour $4 \times 40 = \underline{160}$

Rate copper sulphate **out**:

30 litres and concentration of cop. sul.

Rate out $30 \times \frac{x}{1000+10t} = \frac{3x}{100+t}$

So overall rate $\boxed{\frac{dx}{dt} = 160 - \frac{3x}{100+t}}$

b) "$\frac{dy}{dx} + Py = Q$" $\quad \frac{dx}{dt} + \frac{3x}{100+t} = 160$

$P(t) = \frac{3}{100+t}$

$\int P(t)\,dt = 3\ln|100+t|$

I.F. is $e^{3\ln|100+t|} = (100+t)^3$

$(100+t)^3\frac{dx}{dt} + 3(100+t)^2x = 160(100+t)^3$

$\frac{d}{dt}\left((100+t)^3x\right) = 160(100+t)^3$

$(100+t)^3x = \int 160(100+t)^3\,dt$

$(100+t)^3x = 40(100+t)^4 + C$ $\qquad t=0, x=0$ pure water

$0 = 40 \times 100^4 + C$

$C = -40 \times 100^4$

$x = 40(100+t) - \frac{40 \times 100^4}{(100+t)^3}$

$t=6$ $\qquad x = 40(106) - \frac{40 \times 100^4}{106^3}$

$x = 881.5$ grams (1dp)

$= 882$ grams

c) Add in the fact that the cop. sul. would not immediately disperse.

8. A large container initially contains 3 litres of pure water.
Contaminated water starts pouring into the container at a constant rate of 250 ml per minute and you may assume the contaminant dissolves completely.

At the same time, the container is drained at a constant rate of 125 ml per minute.
The water in the container is continually mixed.

The amount of contaminant in the water pouring into the container, at time t minutes after pouring began, is modelled to be $(5 - e^{-0.1t})$ mg per litre.

Let m be the amount of contaminant, in milligrams, in the container at time t minutes after the contaminated water begins pouring into the container.

(a) (i) Write down an expression for the total volume of water in litres in the container at time t.

(ii) Hence show that the amount of contaminant in the container can be modelled by the differential equation

$$\frac{dm}{dt} = \frac{5 - e^{-0.1t}}{4} - \frac{m}{24 + t}$$

(4)

(b) By solving the differential equation, find an expression for the amount of contaminant, in milligrams, in the container t minutes after the contaminated water begins to be poured into the container.

(8)

After 30 minutes, the concentration of contaminant in the water was measured as 3.79 mg per litre.

(c) Assess the model in light of this information, giving a reason for your answer.

(2)

8(a)(i)	Container contains $3+0.25t-0.125t = 3+0.125t$ litres after t minutes	B1	3.3
(ii)	Rate of contaminant out $=0.125\times\dfrac{m}{3+0.125t}$ mg per minute	M1	3.3
	Rate of contaminant in $= 0.25\times(5-e^{-0.1t})$ mg per minute	B1	2.2a
	$\dfrac{dm}{dt}=\dfrac{5-e^{-0.1t}}{4}-\dfrac{m}{24+t}$ *	A1*	1.1b
		(4)	
(b)	Rearranges to form $\dfrac{dm}{dt}+\dfrac{m}{24+t}=\dfrac{5-e^{-0.1t}}{4}$ and attempts integrating factor (may be by recognition).	M1	3.1a
	I.F. $=\left(e^{\int\frac{1}{24+t}\,dt}=e^{\ln(24+t)}\right)=24+t$	A1	1.1b
	$(24+t)m=\dfrac{1}{4}\int(24+t)(5-e^{-0.1t})\,dt=\dfrac{1}{4}\int 120+5t-24e^{-0.1t}-te^{-0.1t}\,dt=..$	M1	3.1a
	$=\dfrac{1}{4}\left(120t+\dfrac{5t^2}{2}-\dfrac{24e^{-0.1t}}{-0.1}+\ldots\right)$	A1	1.1b
	$\int te^{-0.1t}\,dt=t\dfrac{e^{-0.1t}}{-0.1}-\int 1\times\dfrac{e^{-0.1t}}{-0.1}\,dt=t\dfrac{e^{-0.1t}}{-0.1}-\dfrac{e^{-0.1t}}{(-0.1)^2}$	M1 A1	1.1b 1.1b
	So $(24+t)m=\dfrac{5}{8}t^2+30t+85e^{-0.1t}+\dfrac{5}{2}te^{-0.1t}+c$		
	When $t=0$, $m=0$ as initially no contaminant in the container, so $0=0+0+85+0+c\Rightarrow c=-85$	M1	3.4
	$m=\dfrac{1}{24+t}\left(\dfrac{5}{8}t^2+30t+85e^{-0.1t}+\dfrac{5}{2}te^{-0.1t}-85\right)$	A1	2.2b
		(8)	
(c)	When $t=30$ $m=25.65677...$ and $V=6.75$, hence the concentration is 3.80 mg per litre.	M1	3.4
	This resembles the measured value very closely and could easily be explained by minor inaccuracies in measurements, so the model seems to be suitable over this timeframe.	A1	3.5a
		(2)	