

Sequences and Series

1:: Arithmetic Series

Determine the value of
 $2 + 4 + 6 + \dots + 100$

2:: Geometric Series

The first term of a geometric sequence is 3 and the second term 1. Find the sum to infinity.

3:: Sigma Notation

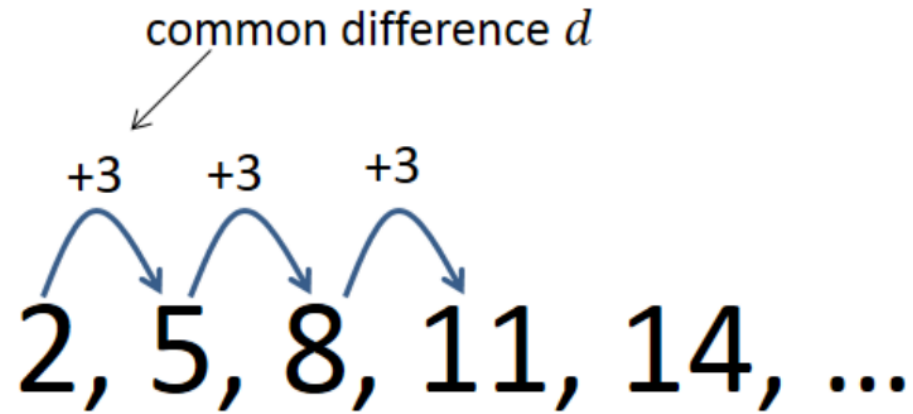
Determine the value of

$$\sum_{r=1}^{100} (3r + 1)$$


4:: Recurrence Relations

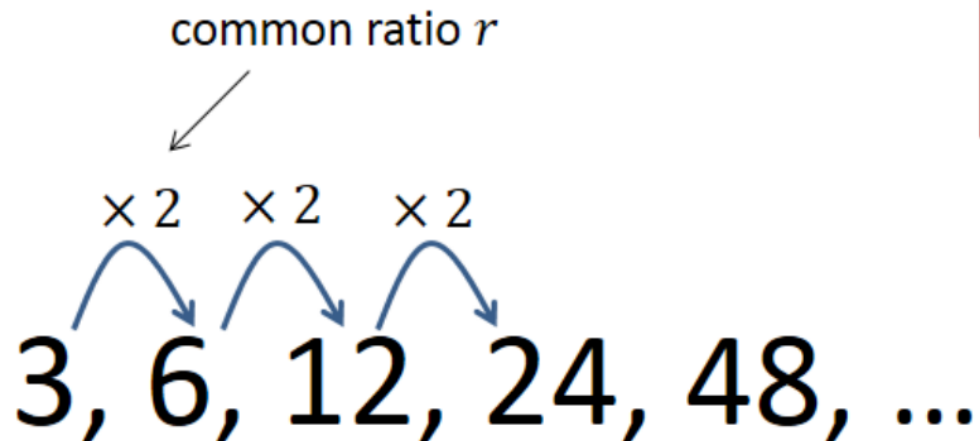
If $a_1 = k$ and $a_{n+1} = 2a_n - 1$,
determine a_3 in terms of k .

Types of sequences



Arithmetic Sequence

 An arithmetic sequence is one which has a common difference between terms.



Geometric Sequence

(We will explore these later in the chapter)

1, 1, 2, 3, 5, 8, ...

This is the **Fibonacci Sequence**. The terms follow a **recurrence relation** because each term can be generated using the previous ones. We will encounter recurrence relations later in the chapter.

The fundamentals of sequences

u_n The n^{th} **term**. So u_3 would refer to the 3rd term.

n The **position** of the term in the sequence.

$n = 3$ $u_3 = 8$

2, 5, 8, 11, 14, ...

$$u_n = \underline{\underline{3n - 1}}$$

$$n = 2$$

$$u_5 = 14$$

$$u_6 = 17$$

n^{th} term of an arithmetic sequence

We use a to denote the **first term**. d is the **difference** between terms, and n is the **position** of the term we're interested in. Therefore:

1 st Term	2 nd Term	3 rd Term	...	n^{th} term
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$$\begin{array}{ccccccc} a & & a + d & & a + 2d & & \dots & & a + (n - 1)d \\ 2 & & 2 + 3 = 5 & & 2 + 2 \times 3 = 8 & & & & \end{array}$$

 n^{th} term of arithmetic sequence:

$$u_n = a + (n - 1)d$$

memory

Example 1

The n th term of an arithmetic sequence is
 $u_n = 55 - 2n$.

- Write down the first 3 terms of the sequence.
- Find the first term in the sequence that is negative.

$$\begin{aligned} \text{a) } u_1 &= 55 - 2 \times 1 \\ &= 53 \end{aligned}$$

$$u_2 = 51, \quad u_3 = 49$$

$$\begin{aligned} \text{b) } 55 - 2n &< 0 \\ 55 &< 2n \\ 27.5 &< n \end{aligned}$$

$$n = 28$$

$$\begin{aligned} u_{28} &= 55 - 2 \times 28 \\ &= -1 \end{aligned}$$

Example 2

Find the n th term of each arithmetic sequence.

a) 6, 20, 34, 48, 62

b) 101, 94, 87, 80, 73

Tip: Always write out $a =$, $d =$, $n =$ first.

a) $a = 6$

$d = 14$

$$\begin{aligned} u_n &= a + (n-1)d \\ &= 6 + (n-1)14 \\ &= 14n - 8 \end{aligned}$$

b) $a = 101$
 $d = -7$

$$\begin{aligned} u_n &= 101 + (n-1)(-7) \\ &= 101 - 7n + 7 \\ &= 108 - 7n \end{aligned}$$

~~1~~ -1, 2, ⑤, 8, 11, 14,
17, ②0

A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.

Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b .

$u_3 = 5$
 $n = 3$

$5 = a + (3-1)d$
 $5 = a + 2d$ ②

① - ②

$u_n = -1 + (n-1)3$

$-1 + 3n - 3 = 3n - 4$

$u_8 = 20$
 $n = 8$

$15 = 5d$
 $d = 3$

$20 = a + (8-1)d$

$20 = a + 7d$ ①

$5 = a + 6$
 $a = -1$

$a = 3$, $b = -4$

For which values of x would the expression $-8, x^2$ and $17x$ form the first three terms of an arithmetic sequence.

$$x^2 - -8 = 17x - x^2$$

$$x^2 + 8 = 17x - x^2$$

$$2x^2 - 17x + 8 = 0$$

$$x = \underline{8} \text{ or } \underline{0.5}$$

11. The second, third and fourth terms of an arithmetic sequence are $2k$, $5k - 10$ and $7k - 14$ respectively, where k is a constant.

$$12, 20, 28$$

~~Show that the sum of the first n terms of the sequence is a square number.~~

(5)

$$5k - 10 - 2k = 7k - 14 - (5k - 10)$$

$$3k - 10 = 7k - 14 - 5k + 10$$

$$3k - 10 = 2k - 4$$

$$k - 10 = -4$$

$$\underline{k = 6}$$

Edexcel C1 May 2014(R) Q10

Xin has been given a 14 day training schedule by her coach.

Xin will run for A minutes on day 1, where A is a constant.

She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.

(a) Show that on day 14, Xin will run for

$$n=14 \quad \underline{\underline{(A + 13d + 13) \text{ minutes.}}}$$

(2)

Yi has also been given a 14 day training schedule by her coach.

Yi will run for $(A - 13)$ minutes on day 1.

She will then increase her running time by $(2d - 1)$ minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

(b) find the value of d .

(3)

$$\cancel{A} + 13d + 13 = \cancel{A} + 26d - 26$$

$$39 = 13d$$

$$\underline{\underline{d = 3}}$$

$$a = A$$

$$d = d + 1$$

$$u_n = a + (n-1)d$$

$$u_{14} = A + 13(d+1)$$

$$= A + 13d + 13$$

$$a = A - 13$$

$$d = 2d - 1$$

$$u_{14} = A - 13 + 13(2d - 1)$$

$$= A - 13 + 26d - 13$$

$$= A + 26d - 26$$

Arithmetic Series

A **series** is a sum of terms in a sequence.

You will encounter 'series' in many places in A Level Maths and Further Maths:

Arithmetic Series, Geometric Series, Binomial Series, Taylor Series...

n^{th} term

$$u_n = a + (n - 1)d$$

 **Sum of first n terms**

$$S_n = \frac{n}{2} (2a + (n - 1)d)$$

Example:

Let's prove it!

Take an arithmetic sequence 2, 5, 8, 11, 14, 17, ...

$$\begin{array}{r} S_5 = 2 + 5 + 8 + 11 + 14 \\ + S_5 = 14 + 11 + 8 + 5 + 2 \end{array}$$

$$2S_5 = 5 \times 16$$

$$S_5 = \frac{5}{2} \times 16 = 40$$

Proving more generally:

$$\begin{array}{r} S_n = a + a + d + \dots + a + (n-2)d + a + (n-1)d \\ S_n = a + (n-1)d + a + (n-2)d + \dots + a + d + a \end{array}$$

$$2S_n = n(2a + (n-1)d)$$

$$S_n = \frac{n}{2} (2a + (n-1)d)$$

$$a + d + a + (n-2)d$$

$$2a + d + dn - 2d$$

$$2a + dn - d$$

$$2a + d(n-1)$$

Exam Note: The proof has been an exam question before. It's also a university interview favourite!

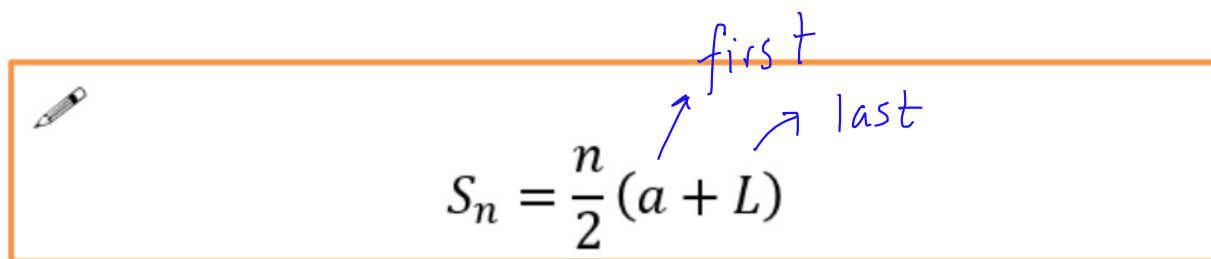
Alternative Formula

$$a + (a + d) + \cdots + L$$

Suppose last term was L .

We saw earlier that each opposite pair of terms (first and last, second and second last, etc.) added to the same total, in this case $a + L$.

There are $\frac{n}{2}$ pairs, therefore:



$S_n = \frac{n}{2}(a + L)$

$$S_n = \frac{n}{2} (2a + (n-1)d)$$

Find the sum of the first 30 terms of the following arithmetic sequences...

$$2 + 5 + 8 + 11 + 14 \dots$$

$$n = 30$$

$$a = 2$$

$$d = 3$$

$$S_{30} = \frac{30}{2} (2 \times 2 + (30-1)3)$$

$$= 15(4 + 29 \times 3)$$

$$= \underline{\underline{1365}}$$

$$100 + 98 + 96 + \dots$$

$$n = 30$$

$$a = 100$$

$$d = -2$$

$$S_{30} = \frac{30}{2} (2 \times 100 + (30-1)(-2))$$

$$= 15(200 - 2 \times 29) = \underline{\underline{2130}}$$

$$p + 2p + 3p + \dots$$

$$n = 30$$

$$a = p$$

$$d = p$$

$$S_{30} = \frac{30}{2} (2p + 29p)$$

$$= 15(31p) = \underline{\underline{465p}}$$

Find the minimum number of terms for the sum of $4 + 9 + 14 + \dots$ to exceed 2000.

$$S_n > 2000$$

$$a = 4$$

$$d = 5$$

$$\frac{n}{2} (2 \times 4 + (n-1)5) > 2000$$

$$\frac{n}{2} (8 + 5n - 5) > 2000$$

$$n(3 + 5n) > 4000$$

$$5n^2 + 3n - 4000 > 0$$

$$n = 27.98$$

$$= -\cancel{28.58}$$

$$n > 27.98$$

$$\underline{\underline{n = 28}}$$

Worted Arithmetic Series

Edexcel C1 Jan 2012 Q9

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is $\pounds P$.

Salary increases by $\pounds(2T)$ each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is $\pounds(P + 1800)$.

Salary increases by $\pounds T$ each year, forming an arithmetic sequence.

- (a) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$\pounds(10P + 90T).$$

For the 10-year period, the total earned is the same for both salary schemes.

- (b) Find the value of T .

For this value of T , the salary in Year 10 under Salary Scheme 2 is $\pounds 29\,850$.

- (c) Find the value of P .

$$\begin{aligned} a) \quad n &= 10 \\ a &= P \\ d &= 2T \end{aligned}$$

$$\begin{aligned} S_{10} &= \frac{10}{2} (2P + 9 \times 2T) \\ &= 5(2P + 18T) \\ &= \underline{10P + 90T} \end{aligned}$$

$$(4) \quad b) \text{ Scheme 2}$$

$$\begin{aligned} n &= 10 \quad a = P + 1800 \\ d &= T \end{aligned}$$

$$\begin{aligned} S_{10} &= 5(2P + 3600 + 9T) \\ &= \underline{10P + 18000 + 45T} \end{aligned}$$

$$\begin{aligned} 10P + 90T &= 10P + 18000 + 45T \\ 45T &= 18000 \\ T &= \underline{400} \end{aligned}$$

$$\begin{aligned} u_n &= a + (n-1)d \\ c) \quad u_{10} &= P + 1800 + 9 \times 400 \\ 29850 &= P + 1800 + 3600 \\ P &= \underline{\pounds 24450} \end{aligned}$$

Ex 3B