Edexcel A Level Further Maths: Decision Maths 1

Your notes

Graphical Solution of LP problems

Contents

Solving a Linear Programming Problem Graphically

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Introduction to Solving an LP Problem Graphically

How do I solve a linear programming problem graphically?

- For problems with two decision variables
  - the **constraints** (inequalities) are plotted accurately on a graph
  - this leads to the **feasible region**
  - the **optimal solution** will be one of the vertices of the feasible region
- In harder problems
  - there may be a **third** decision variable
    - but there will be a connection to one (or both) of the other decision variables such that **all** constraints can be rewritten in terms of just two of them

      e.g. x, y, z are the numbers of chairs, tables and desks made by a furniture manufacturer but the number of desks produced is twice the number of tables (i.e. $z=2y$)
  - the optimal solution may not give **integer** values for the decision variables but the context demands they are
    - e.g. is it possible to make 3.65 chairs and 4.2 tables per day?

Feasible Region

What is the feasible region?

- Technically, the **feasible region** is the set of all values that satisfy **all** the **constraints** in a linear programming problem
- In practice, this is the **area** on a **graph** that **satisfies** all of the **inequalities**
  - including the **non-negativity** constraint/inequality

How do I find the feasible region?

- To find the **feasible region**
  - **accurately** plot each **inequality** (constraint) on a **graph**
    - plot each inequality as a **straight line**
      - rearranging to the form $y \leq mx + c$ or $y \geq mx + c$ may help
      - but it can be easier to determine two points that lie on each line, plot and join them up
    - draw the line solid for inequalities involving ≤ or ≥, or dotted for inequalities involving > or <
      - (< and > are rare in linear programming problems)
  - **shade** the part of the graph **not satisfied** by each inequality
    - be careful with graphing software - these will often shade the part of the graph that **does satisfiy** an inequality
    - but it is easier to see a 'blank' area rather than an area shaded several times
    - a workaround to this is to reverse the inequality sign when typing the inequality into the software
  - the **feasible region** is the **area** on the **graph** left **unshaded**
    - it is the area that **satisfies all** of the **inequalities**
    - it is usually labelled with the capital letter R
- Label each inequality/line around the edge of the graph

Examiner Tip

- Exam questions will provide a graph for you to accurately plot the inequalities on or provide an accurate graph with some or all of the inequalities already plotted

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Worked example

A linear programming problem is formulated as

Maximise

$$P = 30x + 40y$$

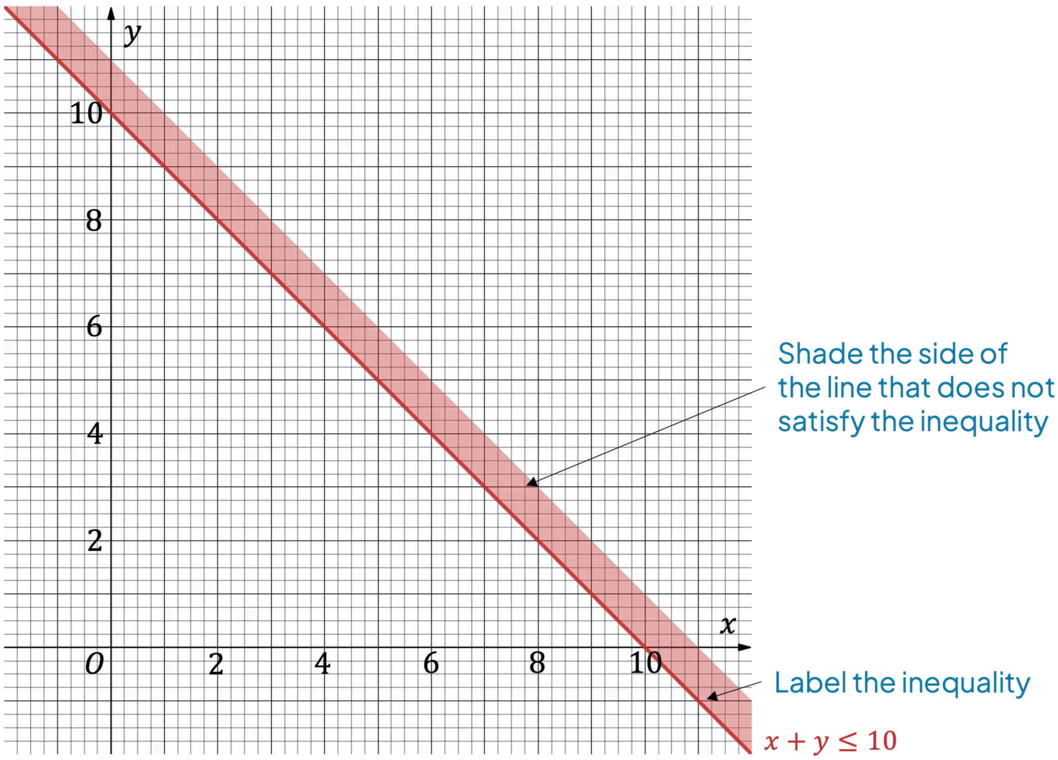
subject to

$$\begin{aligned} x + y &\le 10 \\ 3x + 2y &\le 24 \\ x + 2y &\le 18 \\ x,\ y &\ge 0 \end{aligned}$$

Show graphically the feasible region, R, of the linear programming problem.

(The objective function is not needed to plot the feasible region)

Plot each inequality as a straight line graph with the 'unwanted' side shaded - all will be solid lines
The first constraint will be the line $y = -x + 10$ (gradient -1 and y-axis intercept 10)
(You may find it easier to 'see' that points like (0, 10) and (10, 0) lie on the line, which you can plot and join up)



Plot and label the rest of the inequalities in the same way

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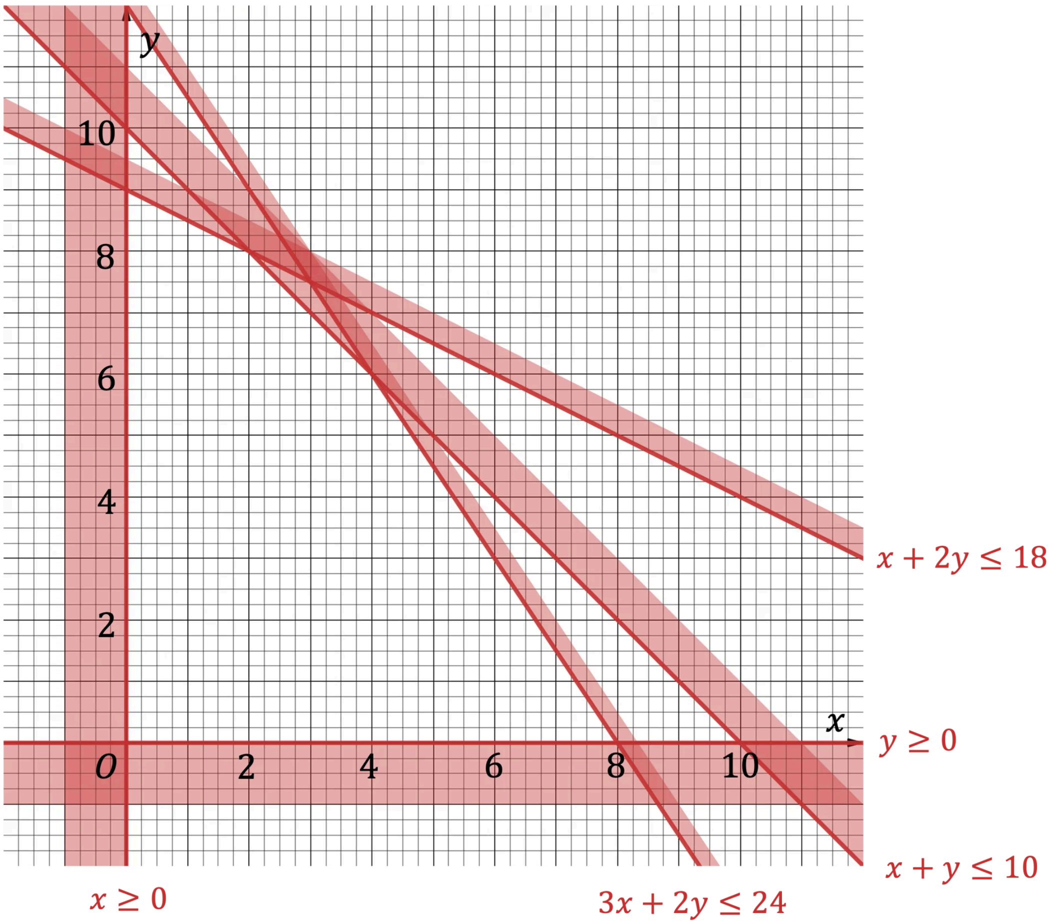
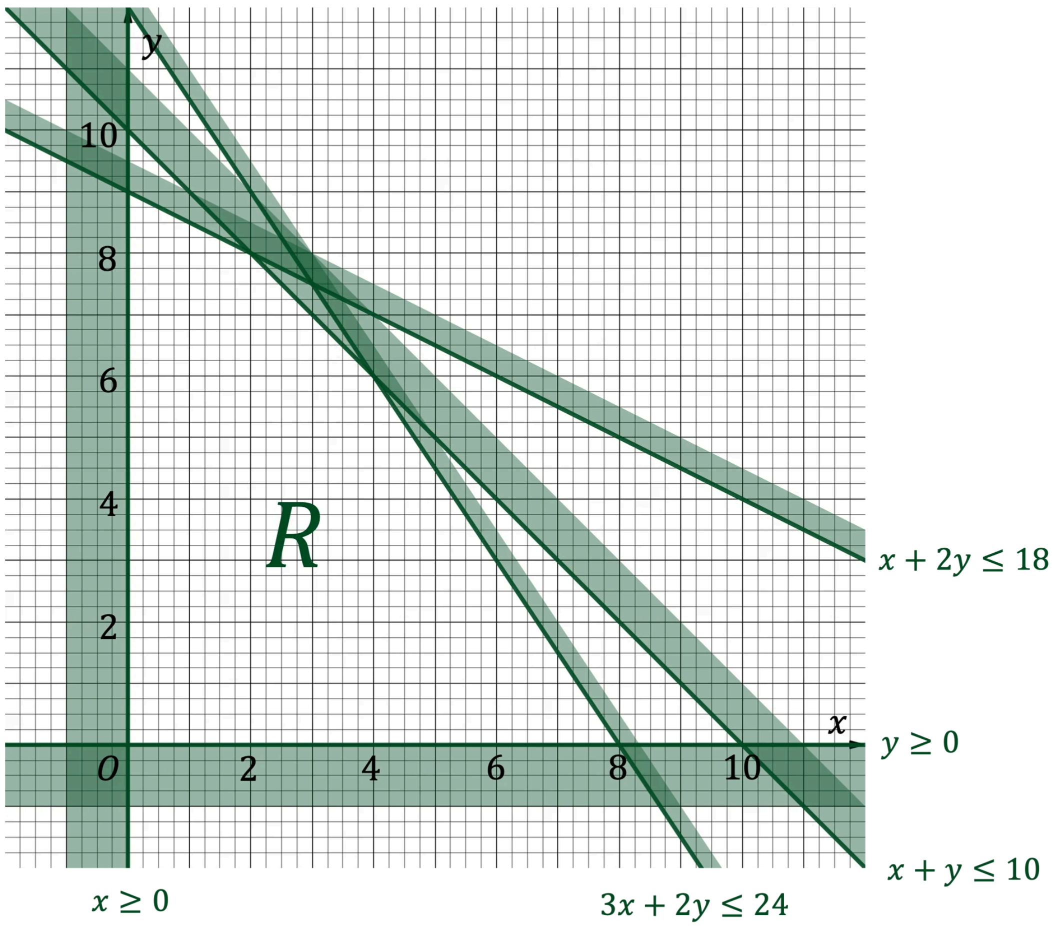


Label the feasible region with R


y
10
8
6
4
2
O
2
4
6
8
10
x
R
$x + 2y \leq 18$
$y \geq 0$
$x + y \leq 10$
$x \geq 0$
$3x + 2y \leq 24$

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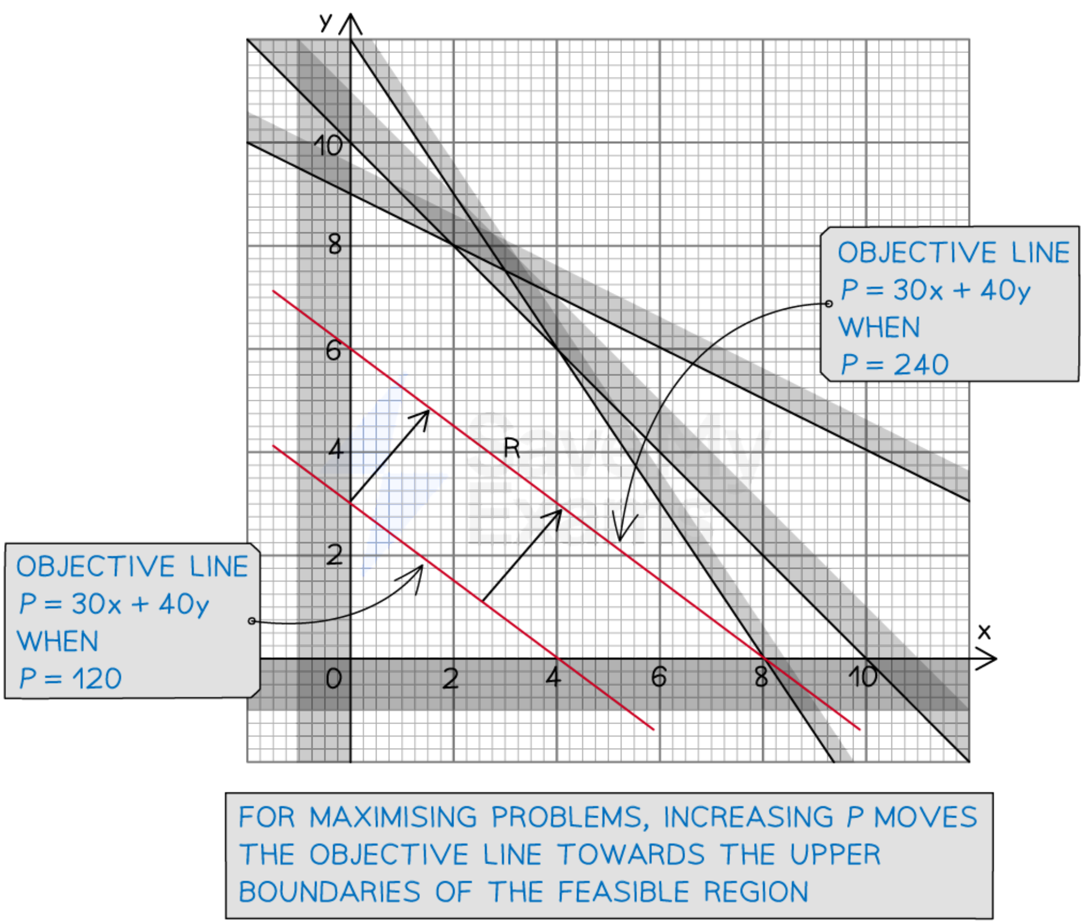
Objective Line

What is the objective line?

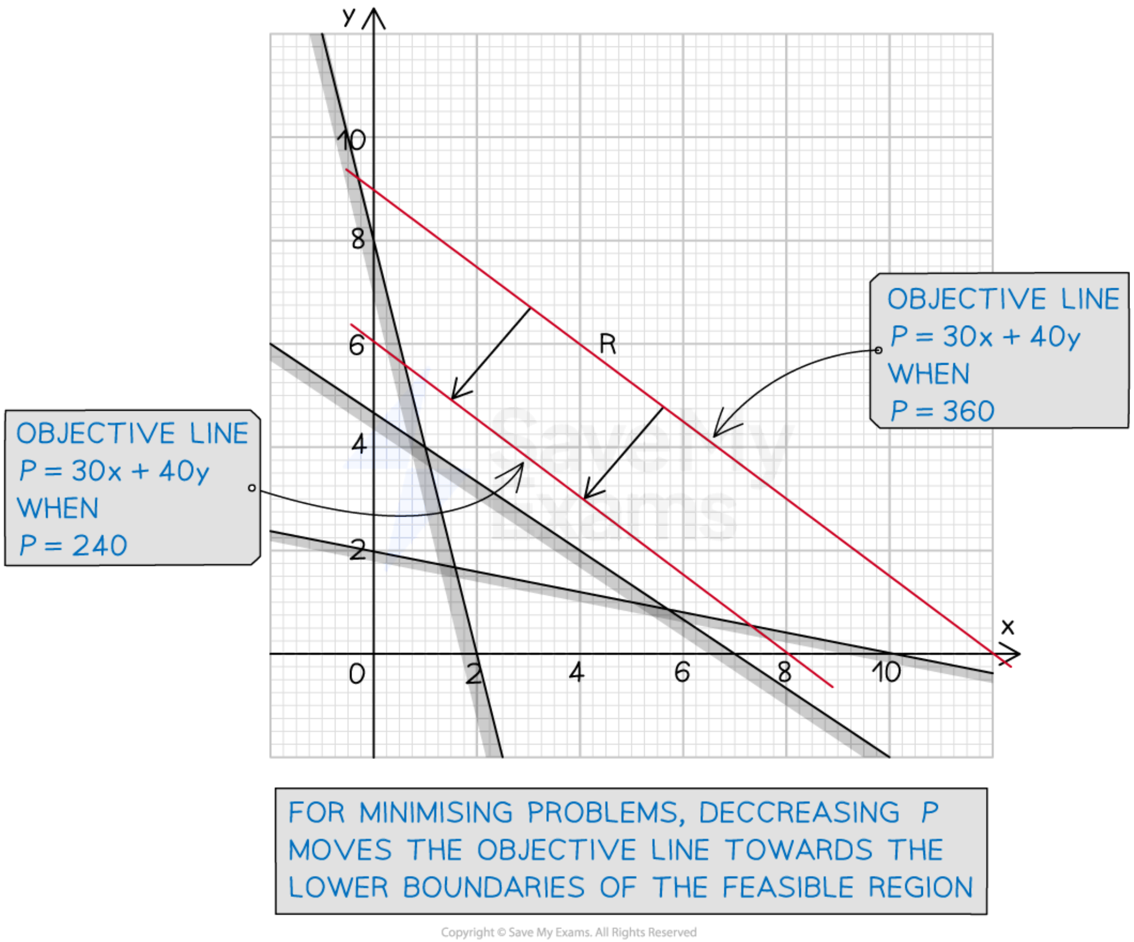
- The **objective function** (for an LP problem with two decision variables) is of the form $P = ax + by$
  - Rearranged, this is of the form '$y = mx + c$'
  - So for a particular value of P, there is a straight line graph
    - this is the **objective line**

How does an objective line indicate where the optimal solution is?

- In a linear programming problem, P is usually **unknown** - it is the quantity that is to be **maximised** or **minimised**
  - In **maximisation** problems **increasing** the value of P 'moves' the **objective line away** from the **origin** and towards the **upper boundaries** of the **feasible region**



- Similarly, in **minimisation** problems, **decreasing** the value of P 'moves' the **objective line closer** to the **origin** and towards the **lower boundaries** of the **feasible region**



- It therefore follows that the **optimal solution** to a linear programming problem occurs when an objective line passes through a **vertex** of the **feasible region**
  - which vertex this is will depend on the gradient of the objective line

How do I find the optimal solution to a linear programming problem using an objective line?

- Whether maximising or minimising, for the objective function $P = ax + by$
  - choose a value of P that is a multiple of a and b
  - plot the objective line $P = ax + by$
    - this is usually easiest by considering the two points where $x = 0$ and where $y = 0$
  - using your ruler, and keeping it **parallel** to the **objective line** just drawn move it away (for maximisation) or towards (for minimisation) the origin

- the **last** vertex of the feasible region that your ruler passes through will be the **optimal solution** to the problem

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Examiner Tip

- To show your working (and understanding) draw an objective line each time your ruler passes through a vertex of the feasible region

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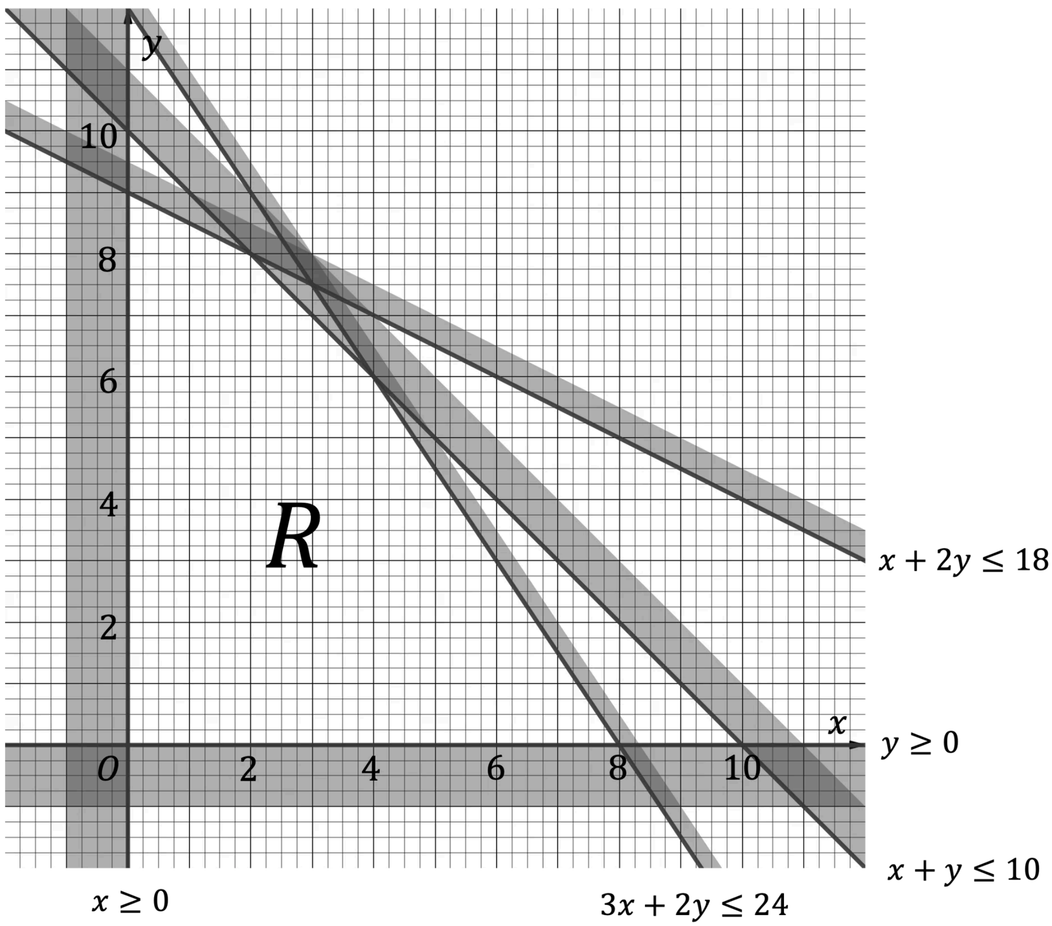
Worked example

The constraints of a linear programming problem and the feasible region (labelled R) are shown in the graph below.



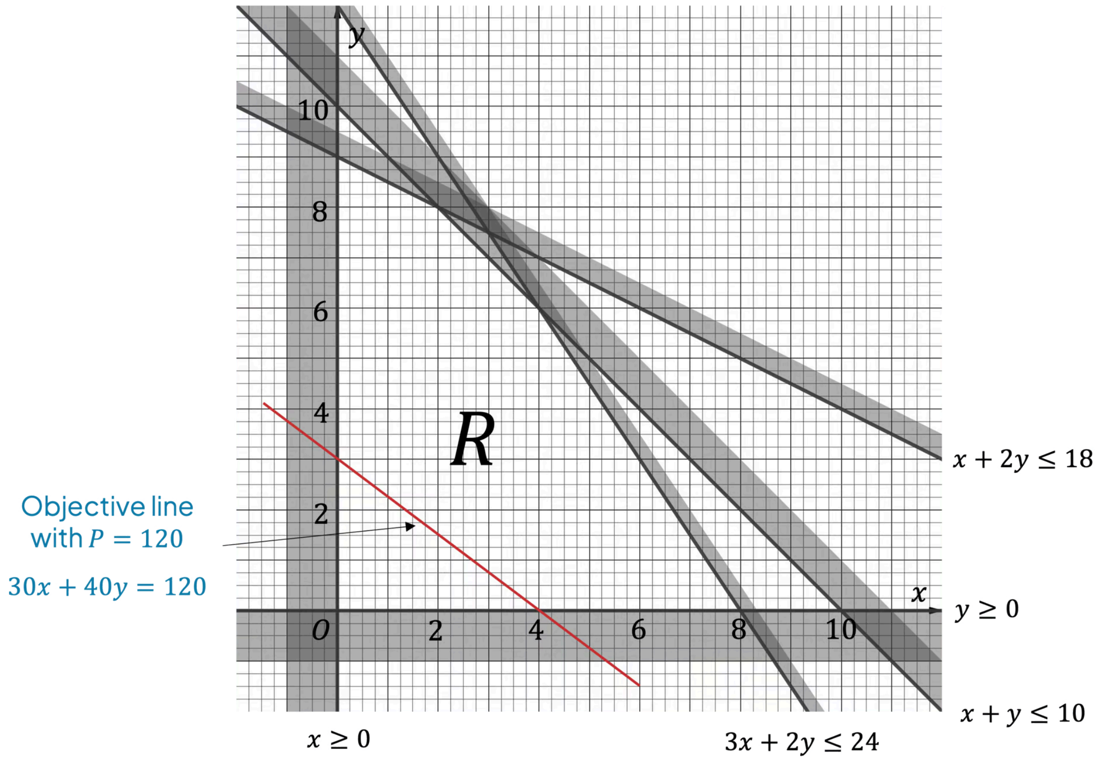
The objective function, $P = 30x + 40y$ is to be maximised.

Showing your method clearly, use the objective line method to determine the optimal solution to the problem.

To get started, choose a value of P that is both a multiple of 30 and 40. We've started with 120.

Now plot the objective line with equation $120 = 30x + 40y$

You can rearrange if you prefer, but by choosing a multiple of 30 and 40, it is easy to see this line will pass through the points (0, 3) and (4, 0)



After plotting an initial line, slide your ruler parallel and 'up' the graph (away from the origin, as it is a maximising problem)

Draw an objective line when your ruler passes through a vertex of the feasible region - (8, 0), (4, 6) and (2, 8)

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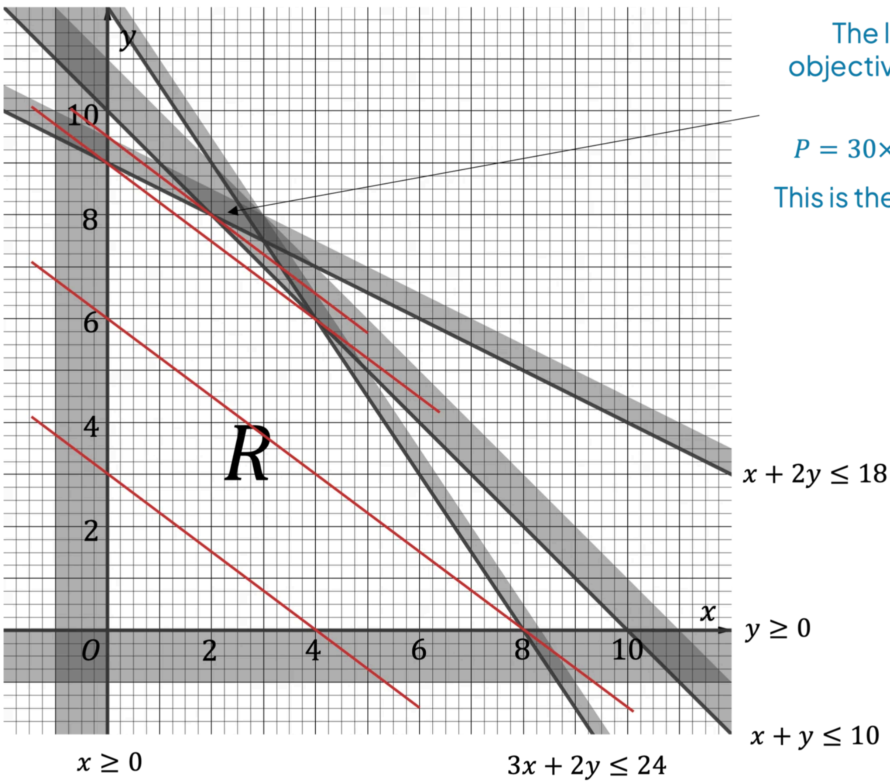

The last vertex an objective line intersects $(2, 8)$

$P = 30\times2 + 40\times8 = 380$

This is the optimal solution

The optimal solution is the last vertex the objective line passes through - which in this case is (2, 8)

The optimal solution is $x = 2,\ y = 8$ and P is maximised at $P = 30 \times 2 + 40 \times 8 = 380$

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Vertex Method

What is the vertex method?

- The vertex method is a way to find the **optimal solution** to a linear programming problem
- The optimal solution to a linear programming problem lies on a vertex of the **feasible region**
- By finding the coordinates of these vertices the values for the **decision variables** can be deduced
- Substituting each of these sets of decision variables into the **objective function** allows the **maximum** or **minimum** objective function to be determined

How do I find the optimal solution from the vertex method?

- **STEP 1**
  Find the coordinates of each **vertex** of the **feasible region**
  - If an accurate plot of the region is provided or drawn, these may be able to be read directly from the graph
  - On less accurate diagrams, some vertices may be obvious - such as the **origin** or any vertices along an axis
  - Otherwise find the vertices by solving each **pair** of the **inequalities** as **simultaneous equations**
    e.g. For the **inequalities** $x+y\le 8$ and $x+4y\le 17$ solve the **simultaneous equations** $x+y=8$ and $x+4y=17$
- **STEP 2**
  Substitute the coordinates of each vertex (i.e. the decision variables) into the **objective function** and **evaluate** it
- **STEP 3**
  Determine which set of decision variables lead to the **maximum** or **minimum** objective function as required by the problem
  - This will be the **optimal solution**

Examiner Tip

- In maximising problems where the origin is a vertex of the feasible region
  - it is usually obvious that the origin will not be the optimal solution
    - it is still a vertex of the feasible region however, so should be included in your list of vertices

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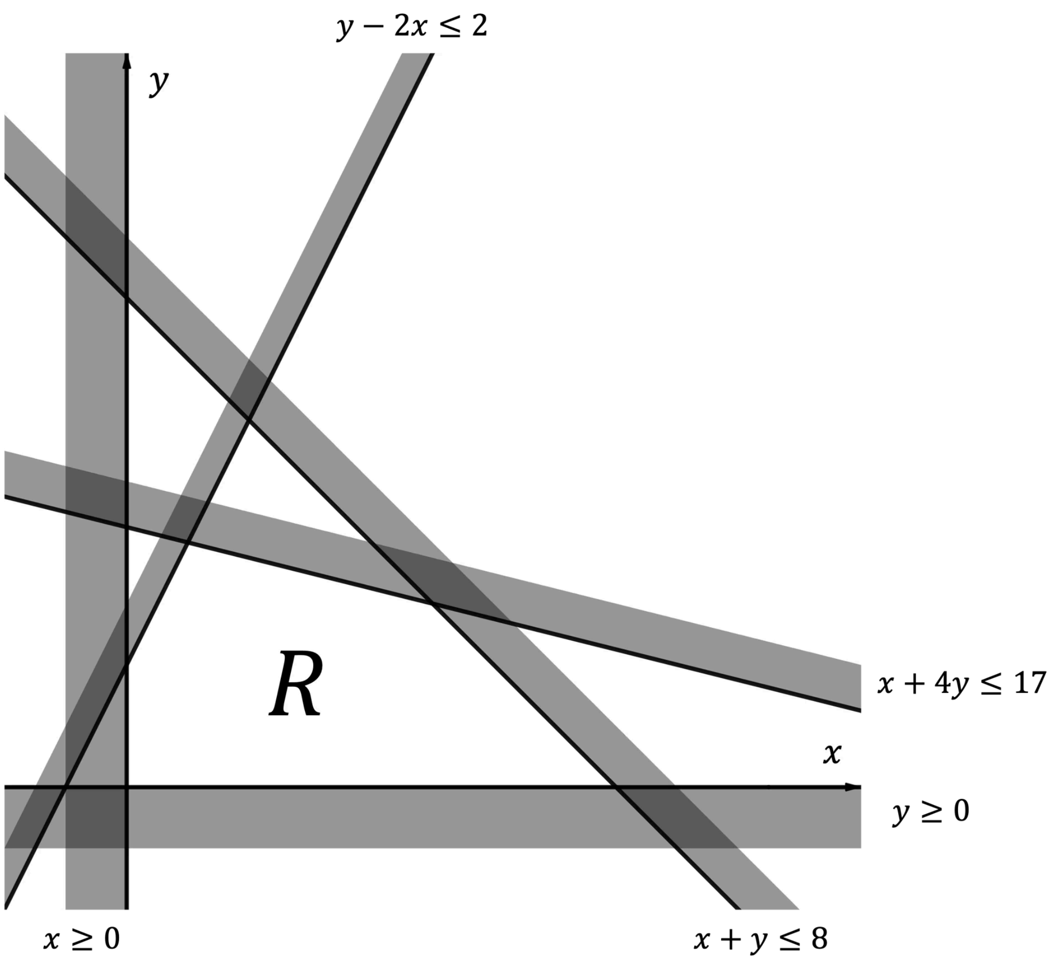
Worked example

The graph below shows the feasible region, labelled R, of a linear programming problem where the objective function is to maximise $P = 2x + 5y$ subject to the constraints shown on the graph.



Use the vertex method to solve the linear programming problem.

- **STEP 1**
  Find the coordinates of each vertex of the feasible region
  Three of them should be obvious to spot!

$$x = 0,\ y = 0$$
$$(0,\ 0)$$

$$x = 0,\ y - 2x = 2$$
$$(0,\ 2)$$

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$$\mathbf{x+y=8,\ y=0}$$
$$\mathbf{(8,\ 0)}$$

For the two vertices that are not obvious, solve the appropriate simultaneous equations
(Your calculator may have a simultaneous equation solver that you can use)

$$\mathbf{y-2x=2,\ x+4y=17}$$
$$y=2+2x$$
$$x+4(2+2x)=17$$
$$9x=9$$
$$x=1,\ y=2+2(1)=4$$
$$\mathbf{(1,\ 4)}$$

$$\mathbf{x+4y=17,\ x+y=8}$$
$$x=8-y$$
$$8-y+4y=17$$
$$3y=9$$
$$y=3,\ x=8-3=5$$
$$\mathbf{(5,\ 3)}$$

- **STEP 2**
  Find $P=2x+5y$ for each pair of x and y values
  Writing them out in a table can help keep track and make the optimal solution stand out

x	y	$P=2x+5y$
0	0	0
0	2	5 × 2 = 10
8	0	2 × 8 = 16
1	4	2 × 1 + 5 × 4 = 22
5	3	2 × 5 + 5 × 3 = 25

- **STEP 3**
  The maximum value is 25, which occurs when $x=5$ and $y=3$

The optimal solution is $x=5,\ y=3$ giving a maximum value of $P=25$

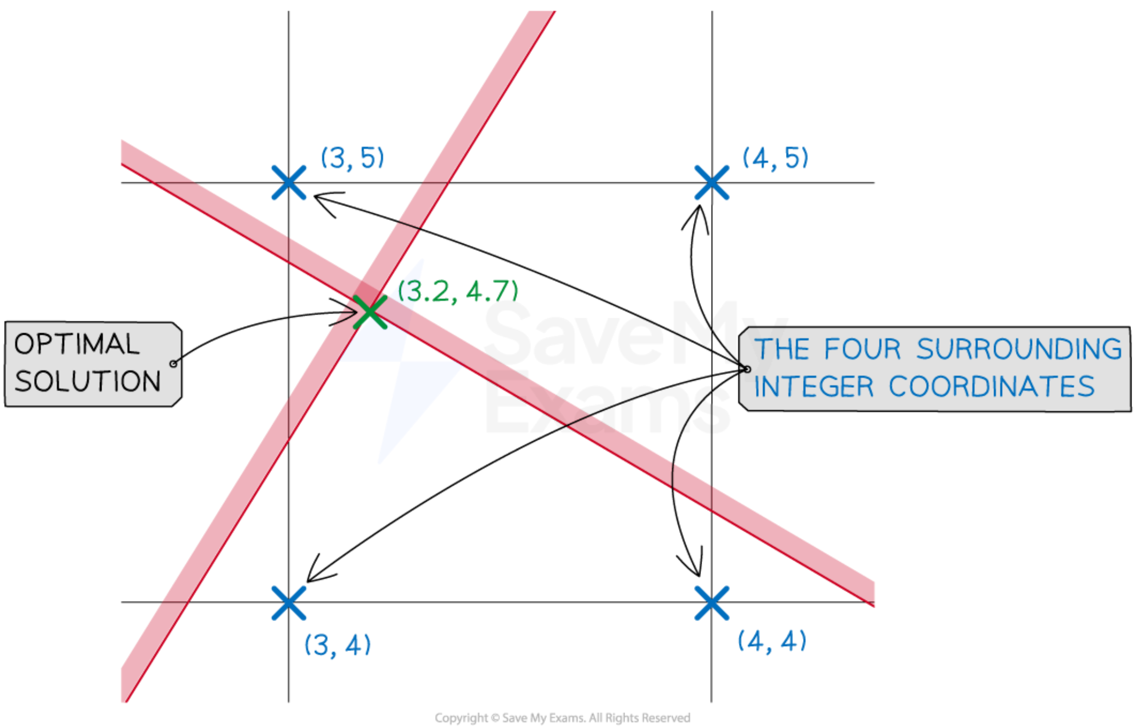
Integer Solutions

What do we mean by integer solutions?

- The **optimal solution** to a linear programming problem lies on a vertex of the feasible region
- The values of the **decision variables** at this vertex may **not** take **integer** values
- However the context of the problem may demand that the decision variables take integer values
  - Decision variables are often a 'number of things'
    - is it possible for the furniture manufacturer to make 3.65 chairs per day?
    - for public health reasons, it would not be appropriate for a food factory to leave a tin of beans partially produced overnight !
- This is what is meant by the phrase **integer solutions**

How do I find the integer solutions to a linear programming problem?

- Find the optimal solution of the linear programming problem as usual
  - using the objective line or vertex method
- Consider the four points with integer coordinates that surround the optimal solution
  - e.g. for an optimal solution of $x = 3.2,\ y = 4.7$, the four surrounding points would be (3, 4), (3, 5), (4, 5) and (4, 4)



- Check whether each of these four points satisfies all of the **constraints**
  - it may be obvious that one (or more) do not but they should still be mentioned
- For those coordinates that do satisfy all the constraints

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- evaluate the objective function (P) at each of the coordinates
- the integer solution will be the point that maximises or minimises the objective function as required
- The **integer solution** may not be the **optimal solution**
  - Depending on the exact nature (gradient) of the objective line
    - The objective line 'moves away' from the boundary of the feasible region when an integer solution is found
    - So there could be another integer solution inside (or on the boundary of) the feasible region some way from the optimal solution
    - This other integer solution may be closer to the boundary of the feasible region than the one just found
  - You will not be expected to find this other integer solution, just to recognise that the integer solution found using the above process is not necessarily optimal

Examiner Tip

- Questions won't necessarily indicate if integer solutions are required
  - Use common sense and think carefully about the context of the problem

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Worked example

The linear programming problem formulated as

Maximise

$$P = 5x + 10y$$

subject to

$$\begin{aligned} 13x + 22y &\le 145 \\ 10x - 20y &\le 3 \\ 13x - 8y &\ge 4 \\ 6x + 5y &\le 50 \\ x,\ y &\ge 0 \end{aligned}$$

has optimal solution $x = 3.2,\ y = 4.7\ (P = 63)$.

However, the decision variables may only take integer values.
Find the solution closest to the optimal solution, stating the values of the decision variables and the resulting value of P.

The four surrounding integer coordinates to (3.2, 4.7) are

(3, 4), (3, 5), (4, 5), (4, 4)

Check that these satisfy all the constraints and if so, evaluate P
Once a point fails to satisfy an inequality we do not need to make any further checks

x	y	$13x + 22y \le 145$	$10x - 20y \le 3$	$13x - 8y \ge 4$	$6x + 5y \le 50$	$x, y \ge 0$	$P = 5x + 10y$
3	4	✓	✓	✓	✓	✓	$P = 55$
3	5	✗					
4	5	✗					
4	4	✓	✓	✓	✓	✓	$P = 60$

The integer solution closest to the optimal solution is $x = 4,\ y = 4$ and $P = 60$