Binomial Expansion (Year 2)

In Year 1 you found the Binomial expansion of $(a+b)^n$ where n was a positive integer. This chapter allows you to extend this to when n is any rational number, i.e. could be negative or fractional.

1:: Binomial Expansion for negative/fractional powers.

"Expand $\sqrt{1+x}$ in ascending powers of x up to the x^2 term."

2:: Constant is not 1.

The same, but where the term preceding the x is not 1, e.g.

"Expand $(8+5x)^{-\frac{1}{3}}$ in ascending powers of x up to the x^3 term."

3:: Using Partial Fractions

"Show that the cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is $2-\frac{7}{2}x+\frac{11}{4}x^2-\frac{25}{8}x^3$"

Pure Year 1 Recap

Remember that for small integer n you could use a row of Pascal Triangle for the Binomial coefficients, descending powers of the first term and ascending powers of the second. If the first term is 1, we can ignore the powers of 1.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & \rightarrow 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

$$(1 + 2x)^4 = 1 + 4(2x) + 6(2x)^2 + 4(2x)^3 + (2x)^4$$

$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$$

$$(1 - 3x)^3 = 1 + 3(-3x) + 3(-3x)^2 + (-3x)^3$$

$$= 1 - 9x + 27x^2 - 27x^3$$

Binomial Coefficients - recap

Do you remember the simple way to find your Binomial coefficients?

$\binom{n}{1} = n$ $\binom{n}{2} = \frac{n(n-1)}{2!}$ $\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$ $\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4!}$

$\binom{10}{3} = \frac{10 \times 9 \times 8}{3!} = 120$ $\binom{-1}{2} = \frac{-1 \times -2}{2!} = 1$ $\binom{-2}{3} = \frac{-2 \times -3 \times -4}{3!} = -4$

$\binom{0.5}{2} = \frac{0.5 \times -0.5}{2!} = -\frac{1}{8}$

Binomial Expansion – Year 2

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Binomial expansions, when n is either negative or fractions, are infinitely long.

Use the binomial expansion to find the first four terms of $\frac{1}{1+x}$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3$$

$$n=-1 \qquad = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6$$

And the first four terms of $\sqrt{1-3x}$

$$(1\underbrace{-3x}_{x})^{1/2} = 1 + \tfrac{1}{2}(-3x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(-3x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-3x)^3$$

$$= 1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3$$

When are infinite expansions valid?

diverge

Our expansion might be an infinite number of terms. If so, the result must converge

$$\frac{1}{1+x} = (1+x)^{-1}$$
$$= 1 + (-1)x + \frac{-1 \times -2}{2!}x^2 + \frac{-1 \times -2 \times -3}{3!}x^3 + \cdots$$
$$= \mathbf{1 - x + x^2 - x^3 + \cdots}$$

What would happen in the expansion if:

$1 - 2 + 4 - 8 + 16 - 32 + 64$

$1 - 2 + 2^2 - 2^3 + 2^4$ diverges

× a) $x > 1$ $\quad x = 2 \quad \frac{1}{1+x} = \frac{1}{3}$

✓ b) $0 < x < 1$ $\quad x = 0.5 \quad \frac{1}{1+x} = \frac{2}{3} \quad 1 - 0.5 + 0.5^2 - 0.5^3 + 0.5^4 - 0.5^5 = \frac{21}{32}$

$\simeq 0.66$

✓ c) $-1 < x < 0$ $\quad x = -0.5$ The higher powers will get smaller and smaller, so it converges

× d) $x = 1$ $\quad \frac{1}{2} \quad 1 - 1 + 1^2 - 1^3 + 1^4 = 1 - 1 + 1 - 1 + 1 - 1$ No, it doesn't converge.

Therefore requirement on x: $\quad -1 < x < 1$

$|x| < 1$

Expansions are allowed to be infinite. However, the result must converge

$$\sqrt{1 \underbrace{- 3x}_{\text{"}x\text{"}}} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$= \mathbf{1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

This time, what do you think needs to be between -1 and 1 for the expansion to be valid?

$|-3x| < 1$

$|3x| < 1$

$|x| < \frac{1}{3}$

✏ An infinite expansion $(1+x)^n$ is valid if $|x| < 1$

Quickfire Examples:

Expansion of $(1+\underline{2x})^{-1}$ valid if: $|2x| < 1 \rightarrow |x| < \frac{1}{2}$

Expansion of $(1-x)^{-2}$ valid if: $|-x| < 1 \rightarrow |x| < 1$

Expansion of $\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}$ valid if: $\left|\frac{1}{4}x\right| < 1 \rightarrow |x| < 4$

Expansion of $\left(1-\frac{2}{3}x\right)^{-1}$ valid if:

$\left|-\frac{2}{3}x\right| < 1$

$\left|\frac{2}{3}x\right| < 1 \rightarrow |x| < \frac{3}{2}$

Combining Expansions

Edexcel C4 June 2013 Q2

(a) Use the binomial expansion to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1 + x + \frac{1}{2}x^2, \qquad |x| < 1$$

(6)

Firstly express as a product:

$$\left(\frac{1+x}{1-x}\right)^{1/2} = \frac{(1+x)^{1/2}}{(1-x)^{1/2}} = (1+x)^{1/2} \times (1-x)^{-1/2}$$

How many terms do we need in each expansion?

3 terms per expansion

ignore any terms higher than x^2

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$(1-x)^{-1/2} = 1 - \frac{1}{2}(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x)^2 = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

$$(1+x)^{1/2}(1-x)^{-1/2} \approx \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$\approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2$$

$$\approx 1 + x + \frac{1}{2}x^2$$

11. (a) Use binomial expansions to show that $\sqrt{\dfrac{1+4x}{1-x}} \approx 1 + \dfrac{5}{2}x - \dfrac{5}{8}x^2$ **(6)**

A student substitutes $x = \dfrac{1}{2}$ into both sides of the approximation shown in part (a) in an attempt to find an approximation to $\sqrt{6}$

(b) Give a reason why the student **should not** use $x = \dfrac{1}{2}$ **(1)**

(c) Substitute $x = \dfrac{1}{11}$ into

$$\sqrt{\frac{1+4x}{1-x}} = 1 + \frac{5}{2}x - \frac{5}{8}x^2$$

to obtain an approximation to $\sqrt{6}$. Give your answer as a fraction in its simplest form. **(3)**

a)

$$\left(\frac{1+4x}{1-x}\right)^{1/2} = (1+4x)^{1/2}(1-x)^{-1/2}$$

$|4x|<1$

$|x|<\frac{1}{4}$

$$(1+4x)^{1/2} = 1 + \frac{1}{2}(4x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(4x)^2 = 1 + 2x - 2x^2$$

$|x|<1$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

$$(1+2x-2x^2)\left(1+\frac{1}{2}x+\frac{3}{8}x^2\right) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + 2x + x^2 - 2x^2$$

$$= 1 + \frac{5}{2}x - \frac{5}{8}x^2$$

b) The student should not use $x = \frac{1}{2}$ because the expansion is only valid for $|x| < \frac{1}{4}$. $\frac{1}{2} > \frac{1}{4}$.

c) $x = \frac{1}{11}$ LHS $\sqrt{\dfrac{1+\frac{4}{11}}{1-\frac{1}{11}}} = \sqrt{\dfrac{\frac{15}{11}}{\frac{10}{11}}} = \sqrt{\dfrac{15}{10}} = \sqrt{\dfrac{3}{2}} = \dfrac{\sqrt{6}}{2}$

RHS $1 + \frac{5}{2}\left(\frac{1}{11}\right) - \frac{5}{8}\left(\frac{1}{11}\right)^2 = \dfrac{1183}{968}$ $(= 1.222)$

$$\frac{\sqrt{6}}{2} = \frac{1183}{968}$$

$$2.449 \leftarrow \sqrt{6} = \frac{1183}{484} \rightarrow 2.444$$

Question	Scheme	Marks	AOs
11 (a)	$\sqrt{\frac{1+4x}{1-x}}=(1+4x)^{0.5}\times(1-x)^{-0.5}$	B1	3.1a
	$(1+4x)^{0.5}=1+0.5\times(4x)+\frac{0.5\times-0.5}{2}\times(4x)^2$ $(1-x)^{-0.5}=1+(-0.5)(-x)+\frac{(-0.5)\times(-1.5)}{2}(-x)^2$ $(1+4x)^{0.5}=1+2x-2x^2$ **and** $(1-x)^{-0.5}=1+0.5x+0.375x^2$ oe	M1 M1 A1	1.1b 1.1b 1.1b
	$(1+4x)^{0.5}\times(1-x)^{-0.5}=\left(1+2x-2x^2......\right)\times\left(1+\frac{1}{2}x+\frac{3}{8}x^2...\right)$ $=1+\frac{1}{2}x+\frac{3}{8}x^2+2x+x^2-2x^2+...$ $=A+Bx+Cx^2$	dM1	2.1
	$=1+\frac{5}{2}x-\frac{5}{8}x^2......$ *	A1*	1.1b
		(6)	
(b)	Expression is valid $\lvert x\rvert<\frac{1}{4}$ Should not use $x=\frac{1}{2}$ as $\frac{1}{2}>\frac{1}{4}$	B1	2.3
		(1)	
(c)	Substitutes $x=\frac{1}{11}$ into $\sqrt{\frac{1+4x}{1-x}}\approx1+\frac{5}{2}x-\frac{5}{8}x^2$	M1	1.1b
	$\sqrt{\frac{3}{2}}=\frac{1183}{968}$	A1	1.1b
	(so $\sqrt{6}$ is) $\frac{1183}{484}$ or $\frac{2904}{1183}$	A1	2.1
		(3)	
			(10 marks)

Your Turn

Find the binomial expansion of $\frac{1}{(1+4x)^2}$ up to an including the term in x^3.

State the values of x for which the expansion is valid.

$$(1 + 4x)^{-2} = 1 - 2(4x) + \frac{-2 \times -3}{2!}(4x)^2 + \frac{-2 \times -3 \times -4}{3!}(4x)^3$$

$n = -2$

$x' = 4x$

$$= 1 - 8x + 48x^2 - 256x^3$$

Valid for $|4x| < 1$

$$|x| < \frac{1}{4}$$

Accuracy of an approximation

0.99009900...

0.9901

$$\frac{1}{1+x} = \underbrace{1 - x + x^2}_{} - x^3 + x^4 - \cdots$$

If $x = 0.01$, how accurate would the approximation $1 - x + x^2$ by for the value of $\frac{1}{1+x}$?

Because x is very small, the approximation will be good.

Smaller values of x provide better approximation

Common Errors

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$\boldsymbol{= 1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

What errors do you think are easy to make?

- Sign errors, e.g. $(-3x)^2 = -9x^2$
- Not putting brackets around the $-3x$, e.g. $-3x^2$ instead of $(-3x)^2$
- Dividing by say 3 instead of 3!

C4 Edexcel Jan 2010

1. (*a*) Find the binomial expansion of

$$\sqrt{(1-8x)}, \qquad |x| < \frac{1}{8},$$

in ascending powers of x up to and including the term in x^3, simplifying each term.

(6)

(*b*) Show that, when $x = \frac{1}{100}$, the exact value of $\sqrt{(1-8x)}$ is $\frac{\sqrt{23}}{5}$.

(2)

(*c*) Substitute $x = \frac{1}{100}$ into the binomial expansion in part (*a*) and hence obtain an approximation to $\sqrt{23}$. Give your answer to 5 decimal places.

(3)

(a) $(1-8x)^{\frac{1}{2}} = 1 - 4x - 8x^2; -32x^3 - \ldots$

(b) $\sqrt{(1-8x)} = \sqrt{\frac{92}{100}} = \sqrt{\frac{23}{25}} = \frac{\sqrt{23}}{5}$

(c) $= 1 - 4(0.01) - 8(0.01)^2 - 32(0.01)^3$

$= 1 - 0.04 - 0.0008 - 0.000\,032 = 0.959\,168$

$\sqrt{23} = 5 \times 0.959\,168$

$= 4.795\,84$