Differentiation: Year 1

When you differentiate a function you get its

The derivative of a function tells you how it 'changes',

often called its or its

Notation

means "take the derivative with respect to x"

If you "$\frac{d}{dx}$" y, you get

Rates of Change

e.g. $\frac{dP}{dt}$ → population / → time in years

$\frac{dv}{dt}$ → velocity of a car / → time in seconds

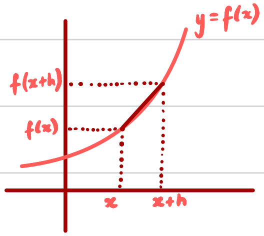
$\frac{dA}{dr}$ → area of circle / → radius of circle

$\frac{dy}{dx}$ → y-coordinate / → x-coordinate

In other words, how the top variable is changing in relation to / with respect to the bottom variable.

Gradient concept

Gradient =

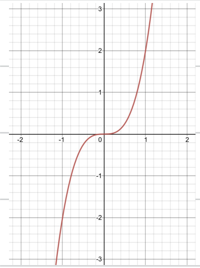


First Principles

$\frac{dy}{dx} = f'(x) =$

Example Problem

i) Find, from first principles, the derivative of $2x^3$ ii) Find the gradient of $y = 2x^3$ when $x = 1$



Differentiating ax^n → where a and n can be any real number

$\frac{d}{dx}(y) =$ $\frac{d}{dx}(ax^n) =$ . .

This only works for terms exactly in the form ax^n

Note: constants "disappear"

Consider 3

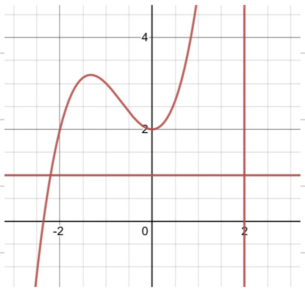
e.g. Differentiate with respect to an appropriate variable

i) $y = x^5 + 7$ ii) $y = 3x^4 - 4$ iii) $y = 2x^{-4}$

iv) $A = \pi r^2$ v) $f(x) = 4kx^2 - \frac{1}{2}x^{-3} + 2$ vi) $f(t) = \sqrt[3]{t}$

vii) $x = (y+3)(2y-1)$ viii) $x = \frac{\sqrt{t} + 4t^2}{2t^{1/3}}$ ix) $V = \frac{1 - 5x^2}{x^3} + 5$

Graphical Interpretation



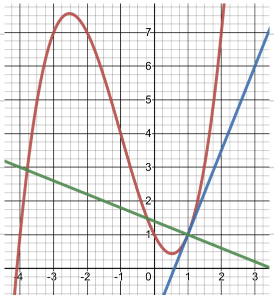
$\frac{dy}{dx} = 0$

$\frac{dy}{dx} < 0$

$\frac{dy}{dx} > 0$

$\frac{dy}{dx}$ is undefined

Tangents and Normals



- Tangent just touches curve at one point
- Normal is perpendicular to curve at a point

Need 2 things

i)

ii)

use $y - y_1 = m(x - x_1)$

Example: Find the equations of the tangent and the normal to $y = \frac{1}{2}x^3 + \frac{3}{2}x^2 - 2x + 1$ at $x = 1$

Second Derivatives (or third, etc)

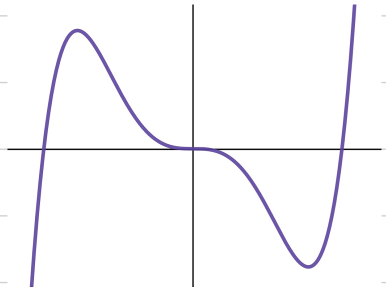
$\frac{d}{dx}\frac{d}{dx}y$

Function notation:

Using the Second Derivative

The second derivative tells you how the gradient is changing.

If $\frac{d^2y}{dx^2} < 0$ the stationary point is a



If $\frac{d^2y}{dx^2} > 0$ the stationary point is a

If $\frac{d^2y}{dx^2} = 0$,

Could be max, min, or point of inflection. Check value of $\frac{dy}{dx}$ either side of the stationary point

Optimisation Problems

Differentiation can be used to maximise or minimise any function

Exponential Functions

If $f(x) = e^x$, $f'(x) =$

$f(x) = e^{kx}$, $f'(x) =$

Example Problem

The cost, £C, of a chemical process is modelled by $C = \frac{200}{t} + \frac{1}{10}t + 12$ where t is the controlled temperature of the process, t°C.

Using differentiation, find the temperature which minimises the cost, the minimum cost, and confirm by differentiation it is a minimum.

Differentiation: Year 2 ... same ideas, but more complex functions

Standard Results

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
ax^n		$\sin x$	
a^x		$\cos x$	
e^x		$\tan x$	
$\ln x$		$\sec x$	
Product uv	u v u' v'	$\cot x$	
Quotient $\frac{u}{v}$	u v u' v'	$\operatorname{cosec} x$	

Chain Rule (Long)

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

e.g. $y = (3x^2 + x)^5$

Chain Rule (Quick)

$fg(x) \rightarrow$

$fgh(x) \rightarrow$

$f(\text{blah}) \rightarrow$

e.g. $\ln \square \rightarrow$

e.g. $\sec \square \rightarrow$

e.g. $e^{\square} \rightarrow$

e.g. $a\square^n \rightarrow$

Parametric Differentiation

If $y = f(\theta)$ $x = g(\theta)$

$\frac{dy}{dx} =$

Implicit Differentiation

Differentiate as expected, but

e.g. $\frac{d}{dx}(3y^4)$

Examples

Differentiate with respect to an appropriate variable

a) $f(x) = 2^{x^2}$

b) $y = 3\ln x$

c) $A = 2\sin 3\theta$

d) $y = \ln(4x^2 + 2x + 3)$

(Powers of trig)

e) $g(x) = 4\tan^3 x$

f) $h(t) = 2\cos^4 t$

g) $v = (12t + \cos t)^2$

h) $x = 4(5y^2 + e^y)^3$

i) $y = e^{\sin(x^2 + 4x)}$

j) $y = \sin^2(3x^{1/2})$

k) $f(x) = 3\cos 2x \sin 4x$

l) $y = e^{-x}\tan 2x$

Find dy/dx

m) $y = \frac{\sin x}{x^2}$

n) $x = 4\tan\theta$ $y = 2\sec\theta$

o) $x^2 + 3y^3 - 2xy^2 = 7$

Reciprocals

$\frac{dx}{dy} =$

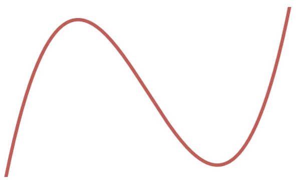
Connected Rates of Change

Rate means

Express what you are looking for in terms of other derivatives e.g. $\frac{dA}{dr} = \frac{dA}{\quad} \frac{\quad}{dr}$

eg. The volume of a cube is increasing at a constant rate of $5\text{cm}^3\text{s}^{-1}$. Assuming the solid remains a cube as it grows, find the rate of increase of the side length, x cm, when $x = 2$cm

Second Derivatives



concave if $f''(x)$

convex if $f''(x)$

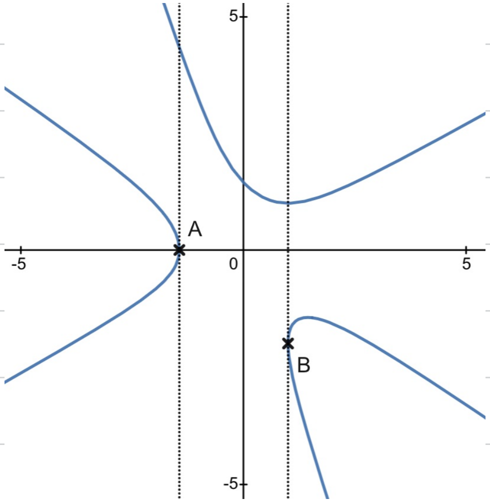
point of inflection if $f''(x)$

Note: point of inflection $\Rightarrow f''(x) =$

but $f''(x) = \quad \not\Rightarrow$ point of inflection

Exam Style Problem

i)



The figure shows the curve with equation

$$y^3 + 3xy^2 - x^3 = 3$$

a) Find an expression for $\frac{dy}{dx}$ in terms of x and y

b) Vertical tangents have been drawn at A and B, as shown on the figure. Determine the coordinates A and B.