Partial Fractions - dealing with improper fractions

In Pure Year 1, we saw that the '**degree**' of a polynomial is the highest power, e.g. a quadratic has degree 2.

An algebraic fraction is **improper** if the degree of the numerator is **at least** the degree of the denominator.

$$\frac{x^2-3}{x+2} \qquad \frac{x+1}{x-1} \qquad \frac{x^3-x^2+3}{x^2-x}$$

A partial fraction is still improper if the degree is the same top and bottom.

Questions might take one of two forms:

- Do the division to express as a quotient and a remainder, e.g. $\frac{x+1}{x-1} \rightarrow 1+\frac{2}{x-1}$
- Express as partial fractions, e.g. $\frac{x^2+x}{(x+1)(x-2)} = A+\frac{B}{x+1}+\frac{C}{x-2}$

You know for example that as $7 \div 3 = 2\ rem\ 1$, we could write:

$$\frac{7}{3} = 2 + \frac{1}{3}$$

Similarly in general:

$$\frac{F(x)}{divisor} = Q(x) + \frac{remainder}{divisor}$$

Quotient

If $\frac{x^2+5x-9}{x+2} = Ax + B + \frac{C}{x+2}$, determine the values of A, B and C.

Polynomial Division

$$\begin{array}{r} x + 3 \\ x+2 \overline{)\ x^2+5x-9} \\ -(x^2+2x) \\ \hline 3x-9 \\ -(3x+6) \\ \hline -15 \end{array}$$

$$\frac{x^2+5x-9}{x+2} = x + 3 - \frac{15}{x+2}$$

Your Turn

Edexcel C4 June 2013 Q1

Given that

$$\frac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 - 4}, \quad x \neq \pm 2$$

find the values of the constants a, b, c, d and e.

(4)

Tip: There's a missing x term in the numerator and missing x term in the denominator. Use $+0x$ to avoid gaps.

$$\begin{array}{r} 3x^2 - 2x + 7 \\ x^2 + 0x - 4 \overline{)\,3x^4 - 2x^3 - 5x^2 + 0x - 4} \\ -\,(3x^4 + 0x^3 - 12x^2) \\ \hline -2x^3 + 7x^2 + 0x - 4 \\ -\,(-2x^3 + 0x^2 + 8x + 0) \\ \hline 7x^2 - 8x - 4 \\ -\,(7x^2 + 0x - 28) \\ \hline -8x + 24 \end{array}$$

$$= 3x^2 - 2x + 7 + \frac{-8x + 24}{x^2 - 4}$$

$a = 3$

$b = -2$

$c = 7$

$d = -8$

$e = 24$

Q1–4

Ex 1F

Dealing with Improper Fractions

Q Split $\frac{3x^2-3x-2}{(x-1)(x-2)}$ into partial fractions.

Method 1: Algebraic Division

Check if the fraction is improper.

$$\frac{3x^2-3x-2}{x^2-3x+2} = 3 + \frac{6x-8}{x^2-3x+2}$$

$$\begin{array}{r} 3 \\ x^2-3x+2 \overline{)\,3x^2-3x-2} \\ \underline{3x^2-9x+6} \\ 6x-8 \end{array}$$

$$\frac{6x-8}{(x-1)(x-2)} = \frac{B}{x-1} + \frac{C}{x-2}$$

$$6x-8 = B(x-2) + C(x-1)$$

$x=2$: $4 = C$

$x=1$: $-2 = -B$, $B = 2$

$$\frac{3x^2-3x-2}{(x-1)(x-2)} = 3 + \frac{2}{x-1} + \frac{4}{x-2}$$

Q Split $\frac{3x^2-3x-2}{(x-1)(x-2)}$ into partial fractions.

5

3

$$\frac{3x^5+2x^2}{(x+1)(x+2)(x-3)} = Ax^2+Bx+C + \frac{D}{x+1} + \frac{E}{x+2} + \frac{F}{x-3}$$

Method 2: Using one identity

(2)

$$\frac{3x^2-3x-2}{(x-1)(x-2)} = A + \frac{B}{x-1} + \frac{C}{x-2}$$

(2)

$$3x^2-3x-2 = A(x-1)(x-2) + B(x-2) + C(x-1)$$

Compare x^2 coefficient $3 = A$.

$x = 1$ $\quad -2 = -B$

$B = 2$

$x = 2$

$4 = C$

$$\frac{3x^2-3x-2}{(x-1)(x-2)} = 3 + \frac{2}{x-1} + \frac{4}{x-2}$$

Your Turn

C4 Jan 2013 Q3

Express $\dfrac{9x^2+20x-10}{(x+2)(3x-1)}$ in partial fractions.

(4)

$$\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv A + \frac{B}{x+2} + \frac{C}{3x-1}$$

$$9x^2+20x-10 \equiv A(x+2)(3x-1) + B(3x-1) + C(x+2)$$

compare x^2 $\quad 9 = 3A$

$$A = 3$$

Sub $x = -2$

$$36 - 40 - 10 = -7B$$

$$-14 = -7B$$

$$B = 2$$

Sub $x = \frac{1}{3}$

$$1 + \frac{20}{3} - 10 = \frac{7}{3}C$$

$$-\frac{7}{3} = \frac{7}{3}C$$

$$C = -1$$

$$\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv 3 + \frac{2}{x+2} - \frac{1}{3x-1}$$

Ex 1G