# Functions and Graphs - Chapter 2, Pure Year 2

Describe the following transformations to graphs:

$f(x) + 2$

$f(x + 2)$

$f(x) - 2$

$f(x - 2)$

$2f(x)$

$f(2x)$

$\frac{1}{2} f(x)$

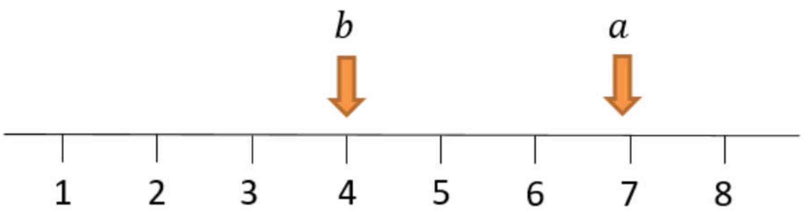
$f(\frac{1}{2}x)$

$-f(x)$

$f(-x)$

## The Modulus Function

The modulus of a number $a$, written $|a|$, is its **non-negative** numerical value.
e.g. $|6| = 6$ and $|-7.1| = 7.1$




The modulus function is particularly useful in expressing a **difference**. We generally like to quote differences as positive values, but $b - a$ may be negative if $b$ is smaller than $a$. By using $|b - a|$, we get round this problem!

More fundamentally, the modulus of a value gives us its '**magnitude**', i.e. size; from Mechanics, you should also be used to the notion the distances and speeds are quoted as positive values.
And in Pure Year 1 we saw the same notation used for vectors: $|\boldsymbol{a}|$ gives us the magnitude/length of the vector $\boldsymbol{a}$. It's the same function!

If $f(x) = |2x - 3| + 1$, find
a) $f(5)$
b) $f(-2)$
c) $f(1)$

Sketch
$y = |x|$

| $x$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

To sketch $y = |ax + b|$, sketch $y = ax + b$ then reflect up any section below the $x$-axis.

Sketch $y = |2x - 3|$

Solve $|2x - 3| = 5$

Solve $|3x - 5| = 2 - \frac{1}{2}x$

Solve $|3x - 5| > 2 - \frac{1}{2}x$

Your Turn

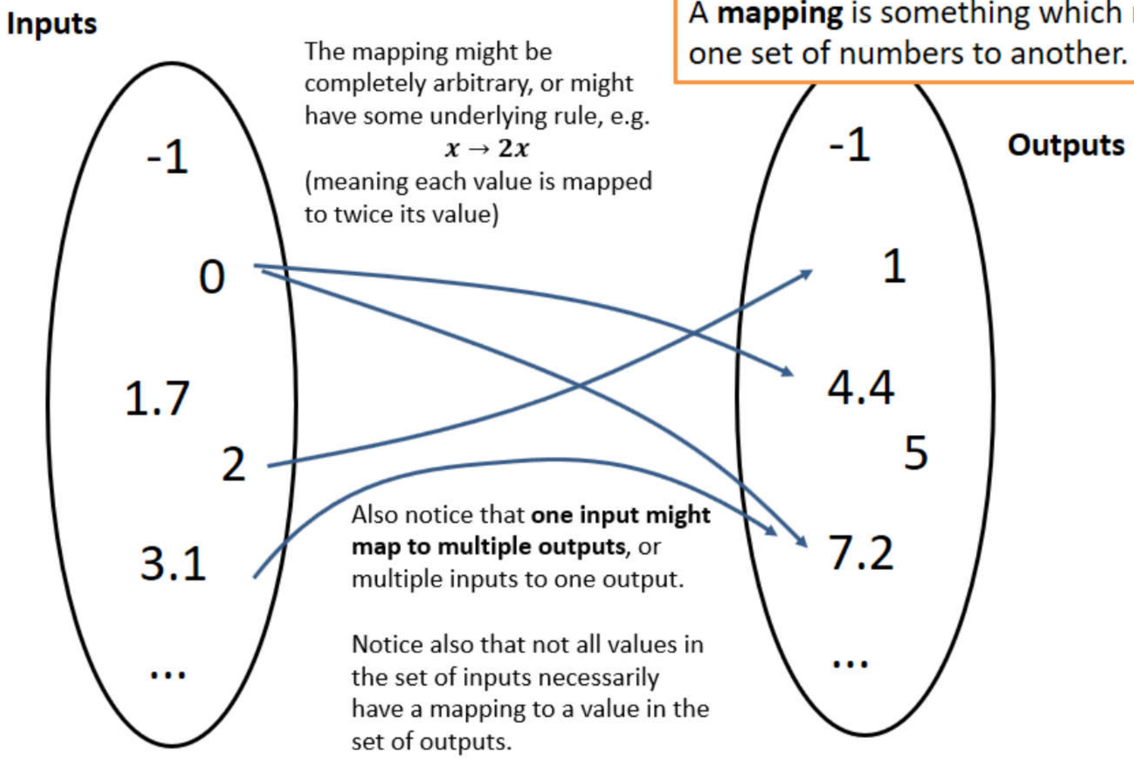
Solve $|x+1| = 2x+5$
(be careful – there's only one solution!)

Solve $|4x-1| < 2x$

Ex 2A

# What is a mapping?

A **mapping** is something which maps one set of numbers to another.




The mapping might be completely arbitrary, or might have some underlying rule, e.g.
$$x \to 2x$$
(meaning each value is mapped to twice its value)

Also notice that **one input might map to multiple outputs**, or multiple inputs to one output.

Notice also that not all values in the set of inputs necessarily have a mapping to a value in the set of outputs.

✎ The **domain** is the set of possible inputs.

✎ The **range** is the set of possible outputs.

# What is a function?

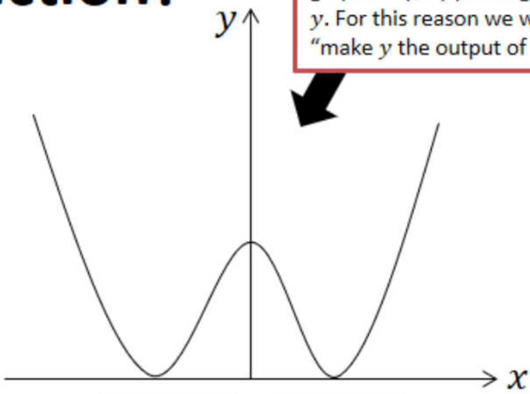
A function is: a mapping such that every element of the domain is mapped to exactly one element of the range.

Notation: $f(x) = 2x + 1$ $\quad$ $f: x \to 2x + 1$

$f(x)$ refers to the **output** of the function.

# Function?

**Note:** We can illustrate a mapping/function graphically, by plotting a point $(x, y)$ if $x$ maps to $y$. For this reason we write $y = f(x)$ to mean "make $y$ the output of the function".



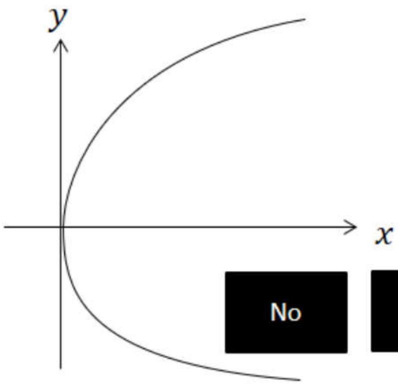

No Yes

**Tip**: Use the 'vertical ray test'. If a vertically fired ray can hit the curve multiple times, it is NOT a function.

## Functions?

$f(x) = 2^x$ Domain: $x \in \mathbb{R}$ (i.e. all real values)

No Yes




No Yes

$f(x) = \sqrt{x}$ Domain: $x \in \mathbb{R}$

No Yes

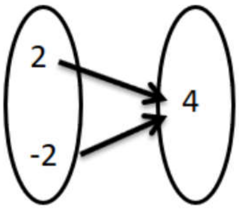
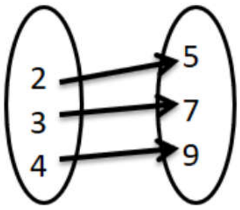
*A function maps every member of the domain to exactly one element of the range*

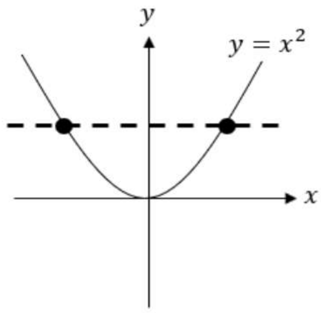
$f(x) = \pm\sqrt{x}$ Domain: $x \geq 0$

No Yes

# One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

| Type | Description | Example |
|---|---|---|
| Many-to-one function | Multiple inputs can map to the same output.   | $f(x) = x^2$<br>e.g. $f(2) = 4$<br>$f(-2) = 4$ |
| One-to-one function | Each output has one input and vice versa.   | $f(x) = 2x + 1$ |

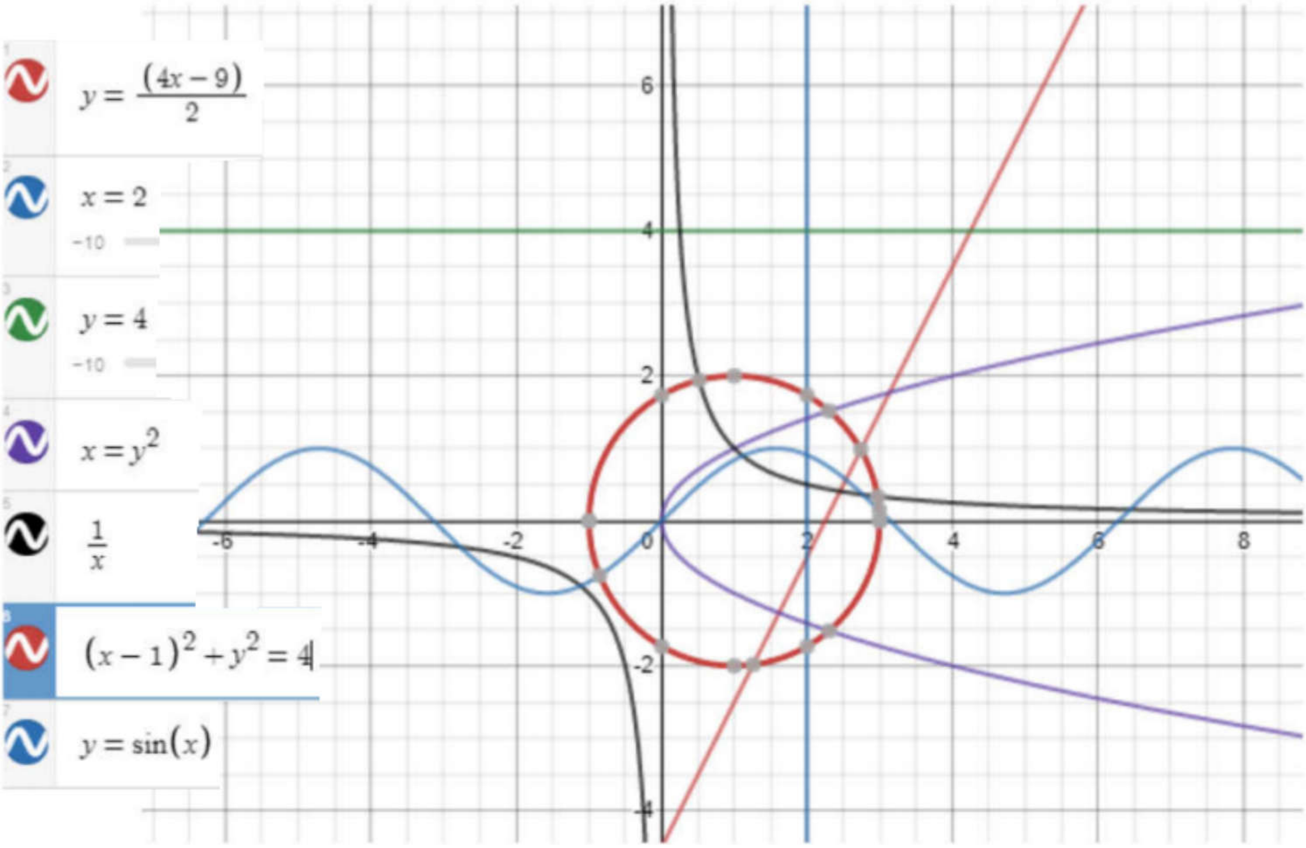



You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

It is often helpful to sketch the function to reason about the range.

Find the range of each of the following functions.
a) $f(x) = 3x - 2$, domain $\{1,2,3,4\}$
b) $g(x) = x^2$, domain $\{x \in \mathbb{R}, -5 \leq x \leq 5\}$
c) $h(x) = \frac{1}{x}$, domain $\{x \in \mathbb{R}, 0 < x \leq 3\}$
State if the functions are one-to-one or many-to-one.

We use $x$ to refer to the input, and $f(x)$ to refer to the output.
**Thus your ranges should be in terms of $f(x)$.**




Decide if the mapping is:
one-to-one, one-to-many, many-to-one, many-to-many

## Piecewise Functions

A 'piecewise function' is one which is defined in parts: we can use different rules for different intervals within the domain.

The function $f(x)$ is defined by

$$f: x \rightarrow \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geq 1 \end{cases}$$

a) Sketch $y = f(x)$, and state the range of $f(x)$.
b) Solve $f(x) = 19$

**Edexcel C4 June 2012 Q6a**

The function $f$ is defined by

$$f: x \to e^x + 2, \qquad x \in \mathbb{R}$$

State the range of $f$.

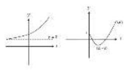
**Edexcel C4 June 2010 Q4d**

The function $g$ is defined by

$$g: x \to x^2 - 4x + 1, \qquad x \in \mathbb{R}, 0 \leq x \leq 5$$

Find the range of $g$.

**Hint:** Identify the minimum point first, as this may or may not affect the range.
**Extra Hint**: Carefully consider the stated domain.



Ex 2B

# Summary of Domain and Range

It is important that you can identify the range for **<u>common graphs</u>, using a suitable sketch**:

$f(x) = x^2, \qquad x \in \mathbb{R}$
**Range:**

$f(x) = \frac{1}{x}, \qquad x \in \mathbb{R}, x \neq 0$
**Range:**

$f(x) = \ln x, \qquad x \in \mathbb{R}, x > 0$
**Range:**

$f(x) = e^x, \qquad x \in \mathbb{R},$
**Range:**

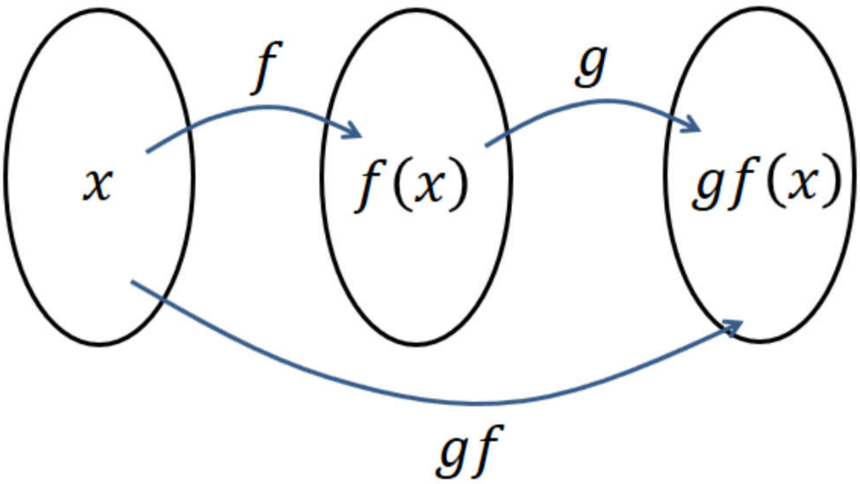
$f(x) = x^2 + 2x + 9, \qquad x \in \mathbb{R}$
**Range:**

Be careful in noting the domain – it may be 'restricted', which similarly restricts the range. Again, use a sketch!

$f(x) = x^2, \qquad x \in \mathbb{R}, -1 \leq x \leq 4$
**Range:**

# Composite Functions

Sometimes we may apply multiple functions in succession to an input.
These combined functions are known as a **composite function**.




✎ $gf(x)$ means $g(f(x))$, i.e. $f$ is applied first, then $g$.

Let $f(x) = x^2 + 1$, and $g(x) = 4x - 2$.
What is...

$fg(2)$?

$fg(x)$?

$gf(x)$?

$f^2(x)$?

$f^2(x)$ means $ff(x)$

Solve $gf(x) = 38$

The functions $f$ and $g$ are defined by

$$f: x \to |2x - 8|$$

$$g: x \to \frac{x+1}{2}$$

a) Find $fg(3)$

b) Solve $fg(x) = x$

*Your Turn*

**Edexcel C4 June 2013(R) Q4**

The functions and $f$ and $g$ are defined by

$$f: x \to 2|x| + 3, \qquad x \in \mathbb{R}$$

$$g: x \to 3 - 4x, \qquad x \in \mathbb{R}$$

b) Find $fg(1)$

d) Solve the equation

$$gg(x) + [g(x)]^2 = 0$$

**Edexcel C4 June 2012 Q6**

The functions $f$ and $g$ are defined by

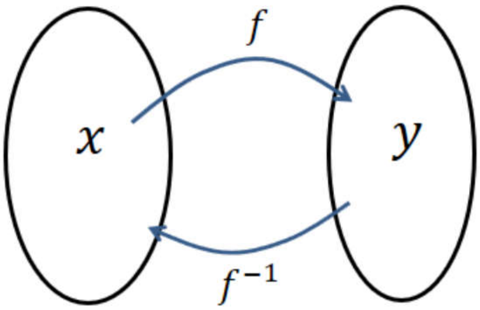
$$f: x \to e^x + 2, \qquad x \in \mathbb{R}$$

$$g: x \to \ln x, \qquad x > 0$$

b) Find $fg(x)$, giving your answer in its simplest form.

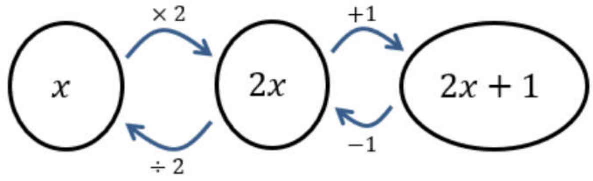
*Ex 2C*

# Inverse Functions




An inverse function $f^{-1}$ **does the opposite of the original function**. For example, if $f(4) = 2$, then $f^{-1}(2) = 4$.

If $f(x) = 2x + 1$, we could do the opposite operations within the function in reverse order to get back to the original input:

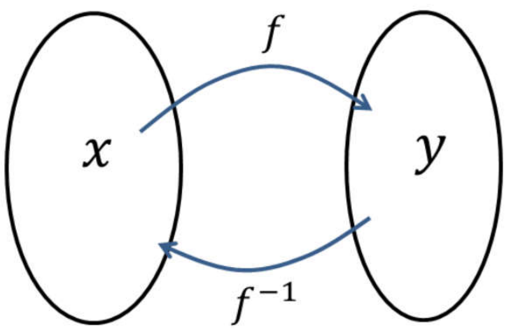



Thus $f^{-1}(x) = \frac{x-1}{2}$

**Notation:** Just like $f^2$ means "apply $f$ twice", $f^{-1}$ means "apply $f$ -1 times", i.e. once backwards!
This is why we write $\sin^{-1}(x)$ to mean "inverse sin".

This has appeared in exams before.

**Explain why the function must be one-to-one for an inverse function to exist:**




In the original function, we have the **output $y$ in terms of the input $x$**, e.g. $y = 2x + 1$

Therefore if we **change the subject to get $x$ in terms of $y$**, then we have the input in terms of the output, i.e. the inverse function!

$$x = \frac{y-1}{2}$$

However, we tend to write a function in terms of $x$, so would write;

$$f^{-1}(x) = \frac{x-1}{2}$$

If $f(x) = 3 - 4x$, find $f^{-1}(x)$

If $f(x) = \frac{x+2}{2x-1}$, $x \neq \frac{1}{2}$, determine $f^{-1}(x)$

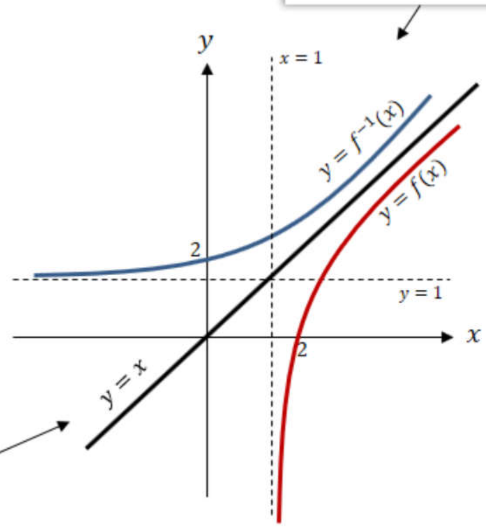
# Graphing Inverse Functions

We saw that the inverse function effectively swaps the input $x$ and output $y$. Thus the **$x$ and $y$ axis are swapped** when sketching the original function and its inverse.

And since the set of inputs and set of outputs is swapped…

✏ The domain of $f(x)$ is the range of $f^{-1}(x)$ and vice versa.

Notice that $x$-intercepts become $y$-intercepts, and vertical asymptotes become horizontal ones.




$y = f(x)$ and $y = f^{-1}(x)$ have the line $y = x$ as a line of symmetry.

Domain of $f$: $\boldsymbol{x > 1}$

Range of $f^{-1}$: $\boldsymbol{f^{-1}(x) > 1}$

The domain of the function is the same as the range of the inverse, but remember that we write a domain in terms of $x$, but a range in terms of $f(x)$ or $f^{-1}(x)$.

If $g(x)$ is defined as $g(x) = \sqrt{x-2}\ \{x \in \mathbb{R}, x \geq 2\}$

a) Find the range of $g(x)$.
b) Calculate $g^{-1}(x)$
c) Sketch the graphs of both functions.
d) State the domain and range of $g^{-1}(x)$.

The function is defined by $f(x) = x^2 - 3, x \in \mathbb{R}, x \geq 0$.

a) Find $f^{-1}(x)$

b) Sketch $y = f(x)$ and $y = f^{-1}(x)$ and state the domain of $f^{-1}$.

c) Solve the equation $f(x) = f^{-1}(x)$.

Edexcel C4 June 2012 Q6

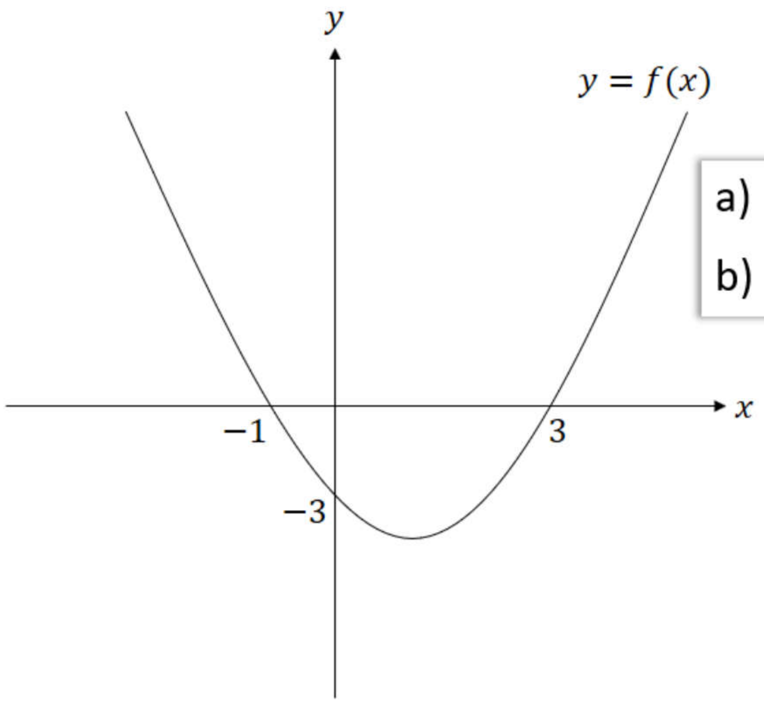
The function $f$ is defined by

$$f: x \to e^x + 2, \quad x \in \mathbb{R}$$

(d) Find $f^{-1}$, the inverse function of $f$, stating its domain.

(e) On the same axe sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes.

# Sketching y=|f(x)| and y=f(|x|)




This is a sketch of $y = f(x)$
where $f(x) = (x - 3)(x + 1)$

| | |
|---|---|
| a) Sketch $y = \|f(x)\|$ | Sketch |
| b) Sketch $y = f(\|x\|)$ | Sketch |

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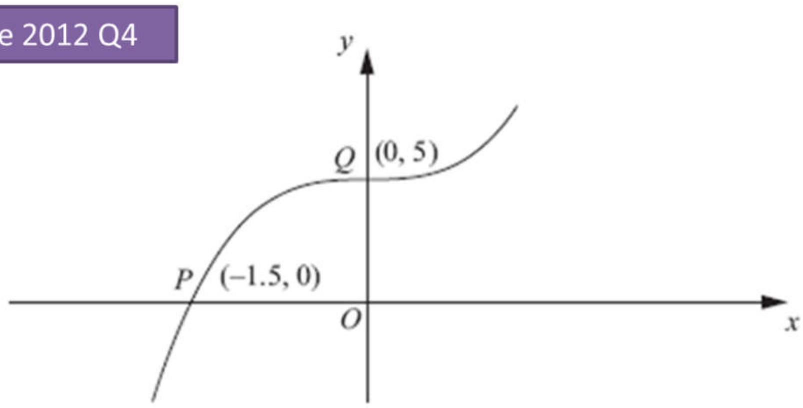



Figure 2 shows part of the curve with equation $y = \mathrm{f}(x)$.
The curve passes through the points $P(-1.5, 0)$ and $Q(0, 5)$ as shown.

On separate diagrams, sketch the curve with equation

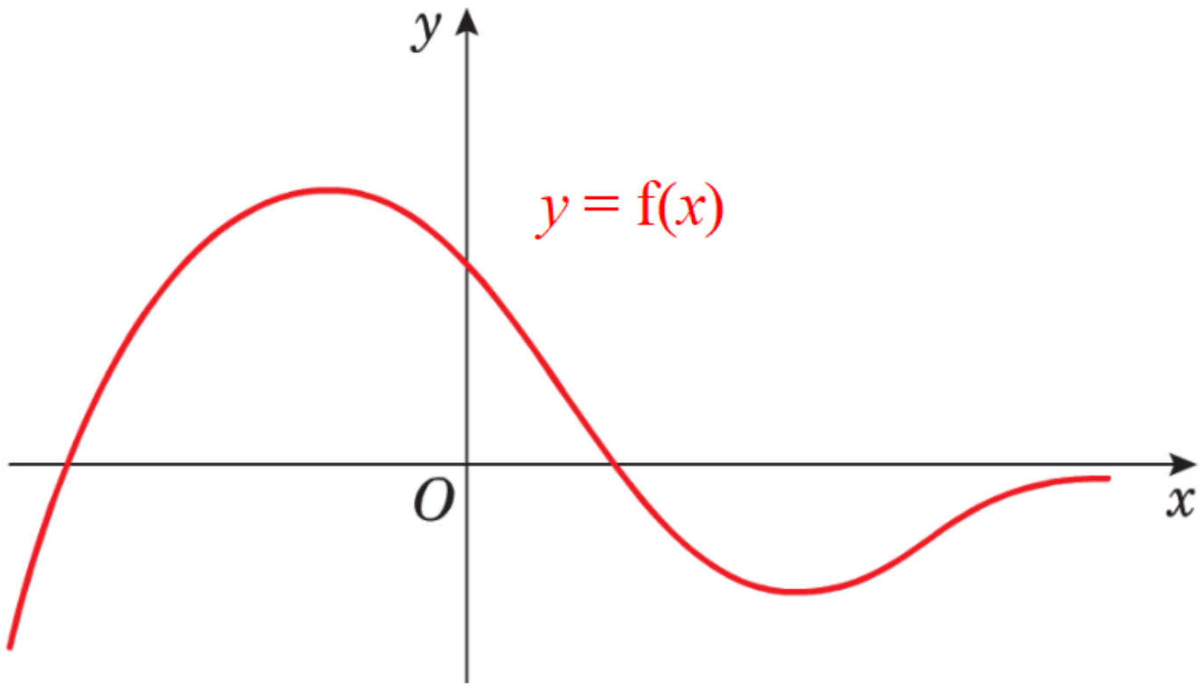
(*a*) $y = |\mathrm{f}(x)|$

**(2)**

(*b*) $y = \mathrm{f}(|x|)$

**(2)**

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

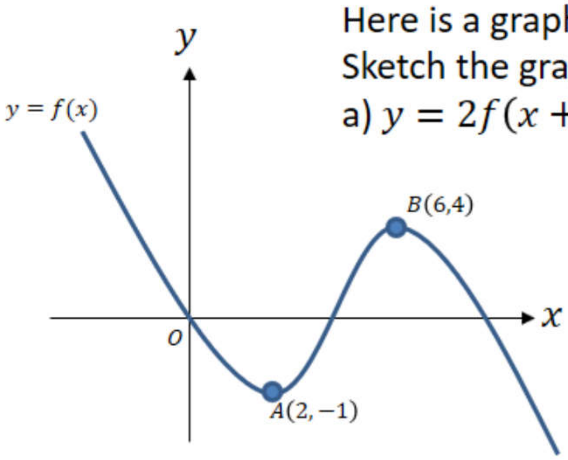



Sketch for $-2\pi \leq x \leq 2\pi$:

a) $y = |\sin(x)|$

b) $y = \sin(|x|)$

*Ex 2E*




Here is a graph of $y = f(x)$.
Sketch the graph of:

a) $y = 2f(x + 2)$

b) $y = -f(2x)$

c) $y = |f(-x)|$

*Ex 2F*

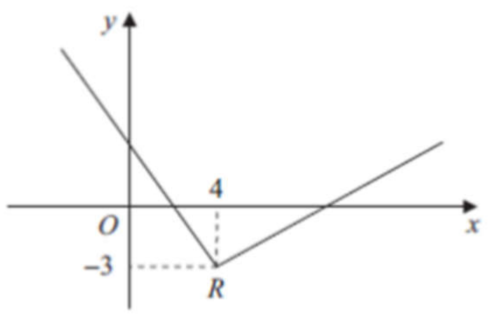



**Figure 1**

Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point $R\ (4, -3)$, as shown in Figure 1.

Sketch, on separate diagrams, the graphs of

(*a*) $y = 2\mathrm{f}(x + 4)$, **(3)**

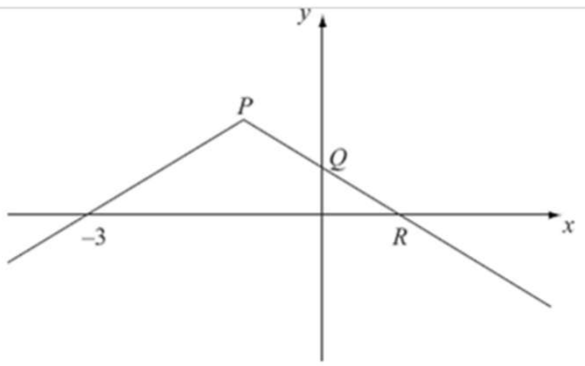
(*b*) $y = |\mathrm{f}(-x)|$ . **(3)**

On each diagram, show the coordinates of the point corresponding to $R$.

# Solving Modulus Problems

Given the function $f(x) = 3|x-1| - 2, x \in \mathbb{R}$,
(a) Sketch the graph of $y = f(x)$
(b) State the range of $f$.
(c) Solve the equation $f(x) = \frac{1}{2}x + 3$

(d) Find the range of values of $k$ for which $f(x) = \frac{1}{2}x + k$ has no solutions




Given that $f(x) = 2 - |x+1|$,

(*c*) find the coordinates of the points $P$, $Q$ and $R$, **(3)**

(*d*) solve $f(x) = \frac{1}{2}x$. **(5)**