Sequences and Series

1:: Arithmetic Series

Determine the value of
$2 + 4 + 6 + \cdots + 100$

2:: Geometric Series

The first term of a geometric sequence is 3 and the second term 1. Find the sum to infinity.

3:: Sigma Notation

Determine the value of

$$\sum_{r=1}^{100} (3r + 1)$$

4:: Recurrence Relations

If $a_1 = k$ and $a_{n+1} = 2a_n - 1$, determine a_3 in terms of k.

Types of sequences

common difference d

$+3 \quad +3 \quad +3$

$$2, 5, 8, 11, 14, \ldots$$

Arithmetic Sequence

✏ An arithmetic sequence is one which has a common difference between terms.

common ratio r

$\times 2 \quad \times 2 \quad \times 2$

$$3, 6, 12, 24, 48, \ldots$$

Geometric Sequence

(We will explore these later in the chapter)

$$1, 1, 2, 3, 5, 8, \ldots$$

This is the **Fibonacci Sequence**. The terms follow a **recurrence relation** because each term can be generated using the previous ones.
We will encounter recurrence relations later in the chapter.

The fundamentals of sequences

u_n The n^{th} **term**. So u_3 would refer to the 3rd term.

n The **position** of the term in the sequence.

$n = 3$ $u_3 = 8$

$n=2$

$$2, 5, 8, 11, 14, \ldots$$

$u_5 = 14$

$u_6 = 17$

$u_n = 3n - 1$

n^{th} term of an arithmetic sequence

We use a to denote the **first term**. d is the **difference** between terms, and n is the **position** of the term we're interested in. Therefore:

1st Term	2nd Term	3rd Term	...	n^{th} term
a	$a+d$	$a+2d$	...	$a+(n-1)d$
2	$2+3=5$	$2+2\times3=8$		

n^{th} term of arithmetic sequence:

$$u_n = a + (n-1)d$$

memory

Example 1

The nth term of an arithmetic sequence is $u_n = 55 - 2n$.

a) Write down the first 3 terms of the sequence.

b) Find the first term in the sequence that is negative.

a) $u_1 = 55 - 2\times1$
$= 53$

$u_2 = 51$, $u_3 = 49$

b) $55 - 2n < 0$
$55 < 2n$
$27.5 < n$
$n = 28$

$u_{28} = 55 - 2\times28$
$= -1$

Example 2

Find the nth term of each arithmetic sequence.

a) 6, 20, 34, 48, 62

b) 101, 94, 87, 80, 73

Tip: Always write out $a =, d =, n =$ first.

a) $a = 6$

$d = 14$

$u_n = a + (n-1)d$

$= 6 + (n-1)14$

$= 14n - 8$

b) $a = 101$

$d = -7$

$u_n = 101 + (n-1)(-7)$

$= 101 - 7n + 7$

$= 108 - 7n$

~~-4~~, -1, 2, 5, 8, 11, 14, 17, 20

A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.

Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b.

$u_3 = 5$

$n = 3$

$5 = a + (3-1)d$

$\boxed{5 = a + 2d}$ (2)

$u_8 = 20$

$n = 8$

$20 = a + (8-1)d$

$\boxed{20 = a + 7d}$ (1)

(1) − (2)

$15 = 5d$

$d = 3$

$5 = a + 6$

$a = -1$

$u_n = -1 + (n-1)3$

$= -1 + 3n - 3 = 3n - 4$

$a = 3$, $b = -4$

For which values of x would the expression $-8, x^2$ and $17x$ form the first three terms of an arithmetic sequence.

$$x^2 - -8 = 17x - x^2$$

$$x^2 + 8 = 17x - x^2$$

$$2x^2 - 17x + 8 = 0$$

$$x = \underline{8} \quad \text{or} \quad \underline{0.5}$$

11. The second, third and fourth terms of an arithmetic sequence are $2k$, $5k - 10$ and $7k - 14$ respectively, where k is a constant.

12, 20, 28

~~Show that the sum of the first n terms of the sequence is a square number.~~

(5)

$$5k - 10 - 2k = 7k - 14 - (5k - 10)$$

$$3k - 10 = 7k - 14 - 5k + 10$$

$$3k - 10 = 2k - 4$$

$$k - 10 = -4$$

$$\underline{k = 6}$$

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Xin has been given a 14 day training schedule by her coach.

Xin will run for A minutes on day 1, where A is a constant.

She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.

(a) Show that on day 14, Xin will run for

$n = 14$

$$(A + 13d + 13) \text{ minutes.}$$

(2)

Yi has also been given a 14 day training schedule by her coach.

Yi will run for $(A - 13)$ minutes on day 1.

She will then increase her running time by $(2d - 1)$ minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

(b) find the value of d.

(3)

$a = A$

"d" $= d + 1$

$u_n = a + (n-1)d$

$u_{14} = A + 13(d+1)$

$= A + 13d + 13$

$a = A - 13$

$d = 2d - 1$

$u_{14} = A - 13 + 13(2d - 1)$

$= A - 13 + 26d - 13$

$= A + 26d - 26$

$\cancel{A} + 13d + 13 = \cancel{A} + 26d - 26$

$39 = 13d$

$d = 3$

Ex 3A

Arithmetic Series

A **series** is a **sum** of terms in a sequence.
You will encounter 'series' in many places in A Level Maths and Further Maths:
Arithmetic Series, Geometric Series, Binomial Series, Taylor Series…

n^{th} term	✎ Sum of first n terms
$u_n = a + (n-1)d$	$S_n = \frac{n}{2}(2a + (n-1)d)$

Example:

Take an arithmetic sequence $2, 5, 8, 11, 14, 17, \ldots$

$$S_5 = 2 + 5 + 8 + 11 + 14$$

$$+ \quad S_5 = 14 + 11 + 8 + 5 + 2$$

$$2S_5 = 5 \times 16$$

$$S_5 = \frac{5}{2} \times 16 = 40$$

Let's prove it!

$$a + d + a + (n-2)d$$

$$2a + d + dn - 2d$$

$$2a + dn - d$$

$$2a + d(n-1)$$

Proving more generally:

$$S_n = a + a + d + \ldots + a + (n-2)d + a + (n-1)d$$

$$S_n = a + (n-1)d + a + (n-2)d + \ldots + a + d + a$$

$$2S_n = n(2a + (n-1)d)$$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Exam Note: The proof has been an exam question before. It's also a university interview favourite!

Alternative Formula

$$a + (a + d) + \cdots + L$$

Suppose last term was L.
We saw earlier that each opposite pair of terms (first and last, second and second last, etc.) added to the same total, in this case $a + L$.

There are $\frac{n}{2}$ pairs, therefore:

first

last

$$\boxed{S_n = \frac{n}{2}(a + L)}$$

$S_n = \frac{n}{2}(2a + (n-1)d)$

Find the sum of the first 30 terms of the following arithmetic sequences...

$2 + 5 + 8 + 11 + 14 \ldots$

$n = 30$
$a = 2$
$d = 3$

$S_{30} = \frac{30}{2}(2 \times 2 + (30-1)3)$
$= 15(4 + 29 \times 3)$
$= \underline{\underline{1365}}$

$100 + 98 + 96 + \cdots$

$n = 30$
$a = 100$
$d = -2$

$S_{30} = \frac{30}{2}(2 \times 100 + (30-1)(-2))$
$= 15(200 - 2 \times 29) = \underline{\underline{2130}}$

$p + 2p + 3p + \cdots$

$n = 30$
$a = p$
$d = p$

$S_{30} = \frac{30}{2}(2p + 29p)$
$= 15(31p) = \underline{\underline{465p}}$

Find the minimum number of terms for the sum of $4 + 9 + 14 + \cdots$ to exceed 2000.

n

$S_n > 2000$
$a = 4$
$d = 5$

$\frac{n}{2}(2 \times 4 + (n-1)5) > 2000$
$\frac{n}{2}(8 + 5n - 5) > 2000$
$n(3 + 5n) > 4000$
$5n^2 + 3n - 4000 > 0$

$n = 27.98$
$= -\cancel{28.58}$

$n > 27.98$
$\underline{\underline{n = 28}}$

Worded Arithmetic Series

Edexcel C1 Jan 2012 Q9

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P.
Salary increases by £($2T$) each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £(P + 1800).
Salary increases by £T each year, forming an arithmetic sequence.

(*a*) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$£(10P + 90T).$$

(2)

For the 10-year period, the total earned is the same for both salary schemes.

(*b*) Find the value of T.

(4)

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850.

(*c*) Find the value of P.

(3)

a) $n=10$

$a=P$

$d=2T$

$S_{10}=\frac{10}{2}(2P+9\times 2T)$

$=5(2P+18T)$

$=10P+90T$

b) Scheme 2

$n=10 \quad a=P+1800$

$d=T$

$S_{10}=5(2P+3600+9T)$

$=10P+18000+45T$

$\cancel{10P}+90T=\cancel{10P}+18000+45T$

$45T=18000$

$T=400$

$u_n=a+(n-1)d$

c) $u_{10}=P+1800+9\times 400$

$29850=P+1800+3600$

$P=£24450$

Ex 3B