$$\frac{d}{dx}\left(\arccos(x^2)\right) = \frac{-2x}{\sqrt{1-x^4}} \qquad \frac{-1}{\sqrt{1-x^2}}$$

$$\boxed{\begin{array}{l} \frac{d}{dx}(\ln x) = \frac{1}{x} \\ \frac{d}{dx}\left(\ln(e^x+x^3)\right) \\ = \frac{e^x+3x^2}{e^x+x^3} \end{array}}$$

$$\int \frac{3x}{\sqrt{16-x^4}}\,dx = \int \frac{3x}{4\sqrt{1-\frac{x^4}{16}}}\,dx = \frac{3}{4}\int \frac{x}{\sqrt{1-\frac{x^4}{16}}}$$

$$= -\frac{3}{2}\arccos\left(\frac{x^2}{4}\right) + c$$

$$\frac{3}{2}\arcsin\left(\frac{x^2}{4}\right) + c$$

$u^2 = 16 - x^4$

$2u\frac{du}{dx} = -4x^3 \rightarrow dx = -u\frac{du}{2x^3}$

$$\frac{3(u^2-16)^{1/4}}{u}$$

$$\int \frac{3\cancel{x}}{\cancel{u}} \times -\cancel{u}\frac{du}{2x^{\cancel{3}2}} = \int -\frac{3}{2x^2}\,du$$

$$= \int -\frac{3}{2}(16-u^2)^{-1/2}\,du$$

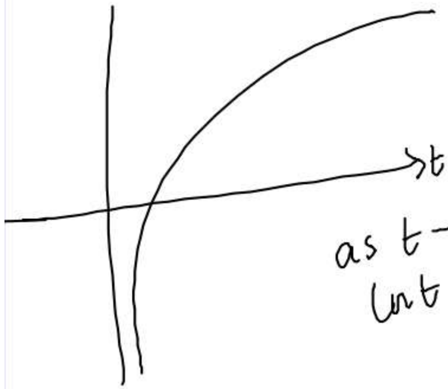
$$= \int -\frac{3}{2\sqrt{16-u^2}}\,du$$

$$= -\frac{3}{\cancel{4}8}\int \frac{1}{\sqrt{1-\frac{u^2}{16}}}\,du$$

$$= -\frac{3}{8}\arcsin\left(\frac{u}{4}\right) + c$$

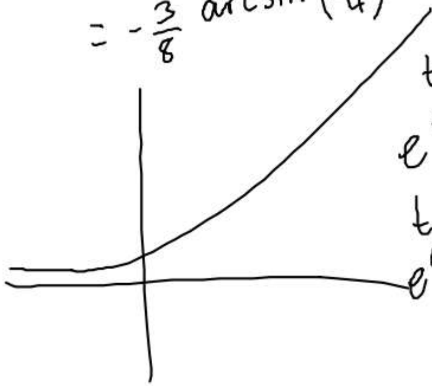
$x^2 = (16-u^2)^{1/2}$

$u = \sqrt{16-x^4}$



$\ln t$

as $t \to 0$
$\ln t \to -\infty$



$t \to \infty$
$e^t \to \infty$
$t \to -\infty$
$e^t \to 0^+$

Differentiating hyperbolic functions (Chapter 6)

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$
$$\frac{d}{dx}(\coth x) = -cosech^2 x$$

Important Memorisation Tip: They're all the same as non-hyperbolic results, other than that *cosh* is not negated and $\operatorname{sech} x$ becomes $-\operatorname{sech} x \tanh x$ (i.e. **is** negated).

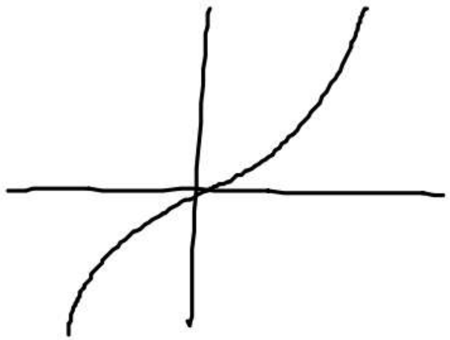
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

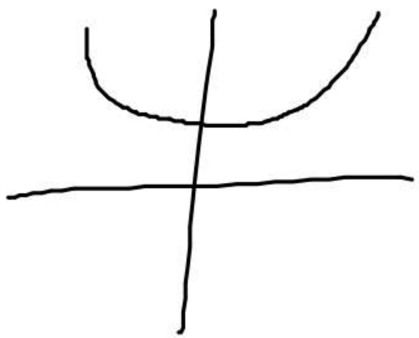
$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

Prove that $\frac{d}{dx}(\sinh x) = \cosh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right) = \frac{1}{2}(e^x + e^{-x})$$
$$= \cosh x$$





[June 2014 (R) Q3] 6. The curve C has equation

$$y = \frac{1}{2}\ln(\coth x), \qquad x > 0$$

(*a*) Show that

$$\frac{dy}{dx} = -\operatorname{cosech} 2x \quad = -\frac{1}{\sinh 2x}$$ **(3)**

Hint: chain rule?

$$y = \frac{1}{2}\ln(\coth x)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{\coth x} \times -\operatorname{cosech}^2 x$$

$$= \frac{1}{2} \times \frac{\cancel{\sinh x}}{\cosh x} \times -\frac{1}{\sinh^{\cancel{2}} x}$$

$$= -\frac{1}{2\sinh x\cosh x} = -\frac{1}{\sinh 2x} = -\operatorname{cosech} 2x$$

$$2\sinh x\cosh x = \sinh 2x$$

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

$$2\cosh^2 x - 1 = \cosh 2x$$

$$1 + 2\sinh^2 x = \cosh 2x$$

Differentiating Inverse Hyperbolic Functions

Proof?

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$$

$y = \operatorname{artanh} x$

$\tanh y = x \longrightarrow \tanh^2 y = x^2$

$1 - \operatorname{sech}^2 y = x^2$

$1 - x^2 = \operatorname{sech}^2 y$

$\operatorname{sech}^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$

$= \frac{1}{1-x^2}$

$1 + \tan^2 y = \sec^2 y$

$1 - \tanh^2 y = \operatorname{sech}^2 y$

Examples

Find $\frac{d}{dx}(artanh\ 3x)$

$$= \frac{3}{1-9x^2}$$

Given that $y = (arcosh\ x)^2$ prove that $(x^2-1)\left(\frac{dy}{dx}\right)^2 = 4y$

$y = (\operatorname{arcosh} x)^2$

$\frac{dy}{dx} = 2 \operatorname{arcosh} x \times \frac{1}{\sqrt{x^2-1}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{4(\operatorname{arcosh} x)^2}{(x^2-1)}$

$(x^2-1)\left(\frac{dy}{dx}\right)^2 = 4y$

[June 2009 Q4] Given that $y = \text{arsinh}\,(\sqrt{x})$, $x > 0$,

(*a*) find $\dfrac{dy}{dx}$, giving your answer as a simplified fraction. **(3)**

(a)	$\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \times \dfrac{1}{\sqrt{1+(\sqrt{x})^2}}$	B1, M1
	$\therefore \dfrac{dy}{dx} = \dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{1+x}} \quad \left(= \dfrac{1}{2\sqrt{x(1+x)}} \right)$	A1 (3)

[June 2010 Q5] Given that $y = (\text{arcosh } 3x)^2$, where $3x > 1$, show that

(*a*) $(9x^2 - 1)\left(\dfrac{dy}{dx}\right)^2 = 36y,$ **(5)**

(*b*) $(9x^2 - 1)\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} = 18.$ **(4)**

a) $y = (\text{arcosh } 3x)^2$

$$\frac{dy}{dx} = 2\,\text{arcosh}\,3x \times \frac{3}{\sqrt{9x^2-1}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{36(\text{arcosh}\,3x)^2}{9x^2-1}$$

$$(9x^2-1)\left(\frac{dy}{dx}\right)^2 = 36y.$$

b) $(9x^2-1)\left(\dfrac{dy}{dx}\right)^2 = 36y$

PRODUCT
IMPLICIT
CHAIN.

$$\overset{9}{\cancel{18}}x\left(\frac{dy}{dx}\right)^{\cancel{2}} + (9x^2-1)\cancel{2}\,\cancel{\frac{dy}{dx}}\,\frac{d^2y}{dx^2} = \overset{18}{\cancel{36}}\,\cancel{\frac{dy}{dx}}$$

$$9x\frac{dy}{dx} + (9x^2-1)\frac{d^2y}{dx^2} = 18$$

(a)	$\dfrac{dy}{dx} = 2\text{arcosh}(3x) \times \dfrac{3}{\sqrt{9x^2-1}}$	M1 A1 A1
	$\sqrt{9x^2-1}\,\dfrac{dy}{dx} = 6\text{arcosh}(3x)$	
	$(9x^2-1)\left(\dfrac{dy}{dx}\right)^2 = 36(\text{arcosh}(3x))^2$	dM1
	$(9x^2-1)\left(\dfrac{dy}{dx}\right)^2 = 36y$ *	A1 (5)
(b)	$\left\{18x\left(\dfrac{dy}{dx}\right)^2 + (9x^2-1)\times 2\dfrac{dy}{dx}\times\dfrac{d^2y}{dx^2}\right\} = 36\dfrac{dy}{dx}$	M1 {A1} A1
	$(9x^2-1)\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} = 18$ *	A1 (4)

Using Maclaurin expansions for approximations

(a) Show that $\frac{d}{dx}(arsinh\ x) = \frac{1}{\sqrt{1+x^2}}$ **[We did this earlier]**

(b) Find the first two non-zero terms of the series expansion of $arsinh\ x$.

The general form for the series expansion of $arsinh\ x$ is given by

$$arsinh\ x = \sum_{n=0}^{\infty}\left(\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\right)\frac{x^{2n+1}}{2n+1}$$

(c) Find, in simplest terms, the coefficient of x^5.

(d) Use your approximation up to and including the term in x^5 to find an approximate value for $arsinh\ 0.5$.

(e) Calculate the percentage error in using this approximation.

Ex 6D
odd questions

Q1, first column
Q5

b) $f(x) = arsinh x \quad f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1+x^2}} \quad f'(0) = 1$

$= (1+x^2)^{-1/2}$

$f''(x) = -\frac{1}{2}(1+x^2)^{-3/2} \times 2x \quad f''(0) = 0$

$= -x(1+x^2)^{-3/2}$

$f'''(x) = -x \times -3x(1+x^2)^{-5/2} - (1+x^2)^{-3/2} \quad f'''(0) = -1$

$arsinh x \approx f(0) + f'(0)x + f''(0)\frac{x^2}{2!}$

$\approx x - \frac{x^3}{3!} \approx x - \frac{x^3}{6}$

c) $n = 2 \quad \frac{(-1)^2(2\times 2)!}{2^4(2!)^2} \times \frac{1}{4+1} = \frac{4!}{16 \times 4 \times 5} = \frac{\cancel{4} \times 3 \times \cancel{2}}{\cancel{16}_8 \times \cancel{4} \times 5}$

$= \frac{3}{40}$

d) $arsinh x \approx x - \frac{x^3}{6} + \frac{3}{40}x^5$

$arsinh(0.5) \approx 0.5 - \frac{0.5^3}{6} + \frac{3}{40}0.5^5 = \frac{1849}{3840} \approx 0.481510\ldots$

e) $\frac{\text{change}}{\text{original}} \times 100 = \frac{\frac{1849}{3840} - arsinh 0.5}{arsinh 0.5} \times 100 = 0.062\%$ (3dp)