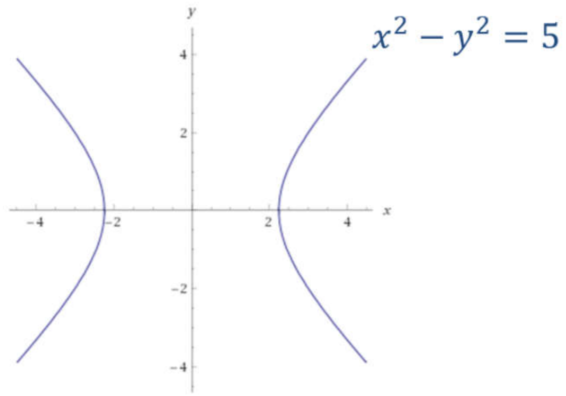
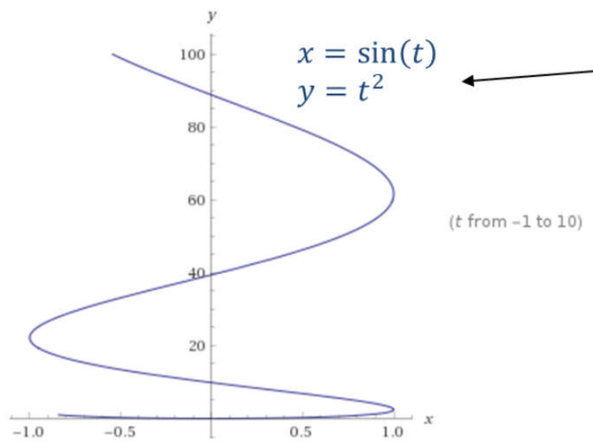
Parametric Equations



Typically, with two variables x and y, we can relate the two by **a single equation involving just x and y**.
This is known as a **<u>Cartesian equation</u>**.

The line shows all points (x, y) which satisfy the Cartesian equation.



However, in Mechanics for example, we might want each of the x and y values to be some function of time t, as per this example.

This would allow us to express the position of a particle at time t as the vector:

$$\begin{pmatrix} \sin t \\ t^2 \end{pmatrix}$$

These are known as **parametric equations**, because **each of x and y are defined in terms of some other variable, known as the parameter** (in this case t).

Finding Cartesian form

How could we convert these parametric equations into a single Cartesian one?

$$x = 2t, \qquad y = t^2, \qquad -3 < t < 3$$

What is the domain and range of the function?

If $x = p(t)$ and $y = q(t)$ can be written as $y = f(x)$, then the domain of f is the range of p...

and the range of f is the range of q.

A curve has the parameter equations

$$x = \ln(t+3), \qquad y = \frac{1}{t+5}, \qquad t > -2$$

a) Find a Cartesian equation of the curve of the form $y = f(x)$, $x > k$, where k is a constant to be found.

b) Write down the range of $f(x)$.

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The curve C has parametric equations

$$x = \ln(t+2), \quad y = \frac{1}{(t+1)}, \quad t > -1.$$

(c) Find a cartesian equation of the curve C, in the form $y = \mathrm{f}(x)$. **(4)**

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6. The curve C has parametric equations

$$x = \ln t, \qquad y = t^2 - 2, \qquad t > 0.$$

(*b*) a cartesian equation of C.

(3)

Finding Cartesian form - trig functions

It's often helpful to use $\sin^2 t + \cos^2 t \equiv 1$ or $1 + \tan^2 t \equiv \sec^2 t$ to turn parametric equations into a single Cartesian one.

A curve has the parametric sequences $x = \sin t + 2$, $y = \cos t - 3$, $t \in \mathbb{R}$.

a) Find a Cartesian equation for the curve.

b) Hence sketch the curve.

A curve is defined by the parametric equations

$$x = \sin t, \qquad y = \sin 2t, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

a) Find a Cartesian equation of the curve in the form $y = f(x)$, $-k \leq x \leq k$, stating the value of the constant k.

b) Write down the range of $f(x)$.

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which double angle formula would be best here?

4. A curve C has parametric equations

$$x = 2\sin t, \qquad y = 1 - \cos 2t, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

(*b*) Find a cartesian equation for C in the form

$$y = f(x), \qquad -k \leq x \leq k,$$

stating the value of the constant k.

A curve C has parametric equations

$$x = \cot t + 2, \qquad y = cosec^2\, t - 2, \qquad 0 < t < \pi$$

a) Find the equation of the curve in the form $y = f(x)$ and state the domain of x for which the curve is defined.

b) Hence, sketch the curve.

Ex 8B

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Figure 2 shows a sketch of the curve C with parametric equations

$$x = \sqrt{3}\sin 2t, \qquad y = 4\cos^2 t, \qquad 0 \le t \le \pi.$$

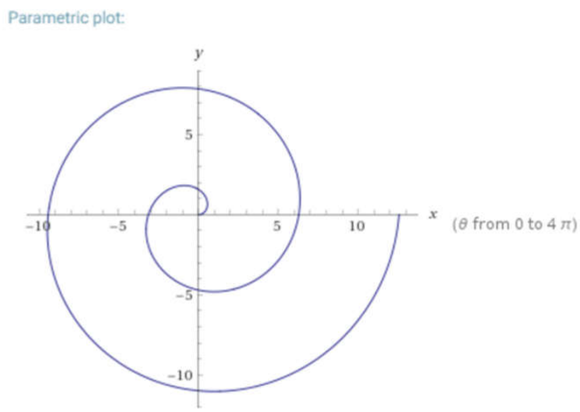
(c) Find a cartesian equation of C. **(3)**

We saw that one strategy for sketching parametric curves is to convert into a Cartesian equation, and hope this is a form we recognise (e.g. quadratic or equation of circle) to appropriately sketch.
However, some parametric equations can't easily be turned into Cartesian form:

Input interpretation:

plot	$x = \theta\cos(\theta)$ $y = \theta\sin(\theta)$	$\theta = 0$ to 4π

These parametric equations in Cartesian form would be $\sqrt{x^2+y^2} = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$; this would obviously be incredibly hard to sketch!

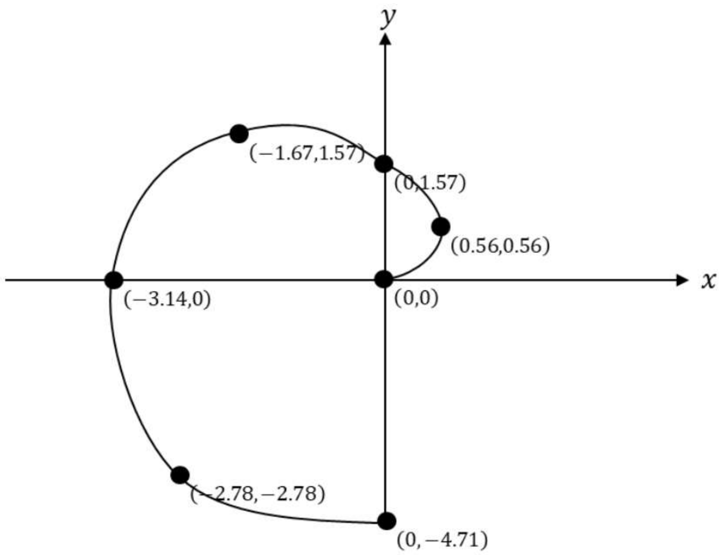


Instead we can try different values of t and determine the point (x, y) for each value to get a sequence of points...

Input interpretation:

plot	$x = \theta\cos(\theta)$ $y = \theta\sin(\theta)$	$\theta = 0$ to 4π

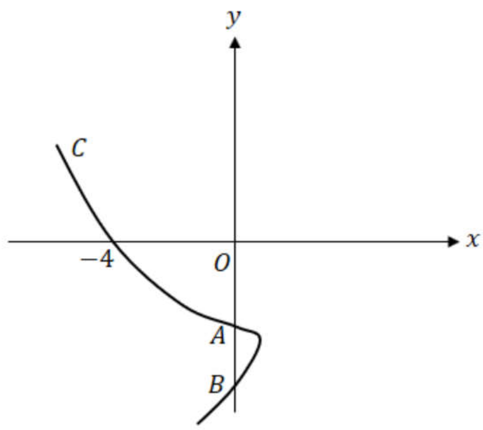
$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$\boldsymbol{x}$	0	0.56	0	-1.67	-3.14	-2.78	0	3.89	6.28
$\boldsymbol{y}$	0	0.56	1.57	1.67	0	-2.78	-4.71	-3.89	0



Points of Intersection

We can find where a parametric curve crosses a particular axis or where curves cross each other.
The key is to first find the value of the parameter t.

The diagram shows a curve C with parametric equations $x = at^2 + t$, $\quad y = a(t^3 + 8)$, $\ t \in \mathbb{R}$, where a is a non-zero constant. Given that C passes through the point $(-4,0)$,
a) find the value of a.
b) find the coordinates of the points A and B where the curve crosses the y-axis.



A curve is given parametrically by the equations $x = t^2,\ y = 4t$.
The line $x + y + 4 = 0$ meets the curve at A. Find the coordinates of A.

Whenever you want to solve a Cartesian equation and pair of parametric equations simultaneously, substitute the parametric equations into the Cartesian one.

The diagram shows a curve C with parametric equations

$$x = \cos t + \sin t\,, \qquad y = \left(t - \frac{\pi}{6}\right)^2, \qquad -\frac{\pi}{2} < t < \frac{4\pi}{3}$$

a) Find the point where the curve intersects the line $y = \pi^2$.
b) Find the coordinates of the points A and B where the curve cuts the y-axis.

Ex 8D

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5.

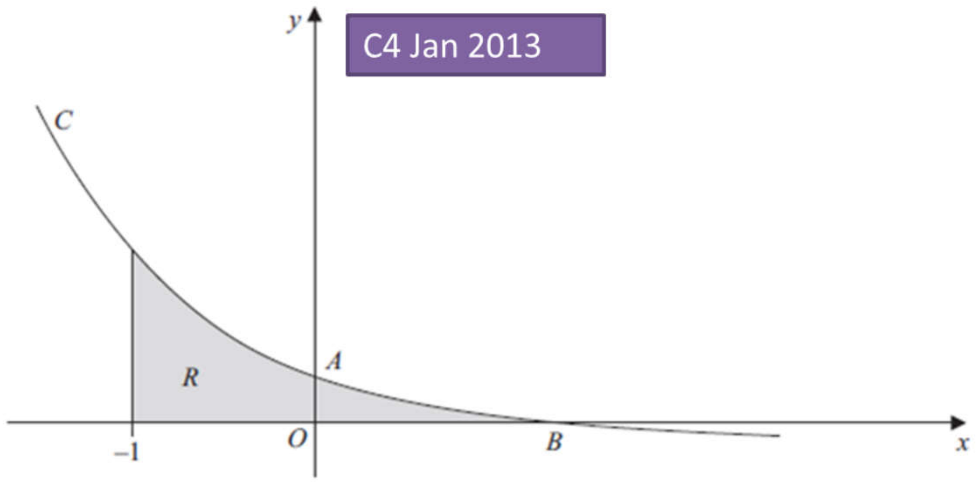


Figure 2

Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \qquad y = 2^t - 1.$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(*a*) Show that A has coordinates (0, 3).
(2)

(*b*) Find the x-coordinate of the point B.
(2)

Modelling with Parametric Equations

As we saw at the start of this chapter, parametric equations are frequently used in mechanics, particularly where the (x, y) position (the Cartesian variables) depends on time t (the parameter).

A plane's position at time t seconds after take-off can be modelled with the following parametric equations:

$$x = (v\cos\theta)t \text{ m}, \qquad y = (v\sin\theta)t \text{ m}, \qquad t > 0$$

where v is the speed of the plane, θ is the angle of elevation of its path, x is the horizontal distance travelled and y is the vertical distance travelled, relative to a fixed origin.

When the plane has travelled 600m horizontally, it has climbed 120m.

a. find the angle of elevation, θ.

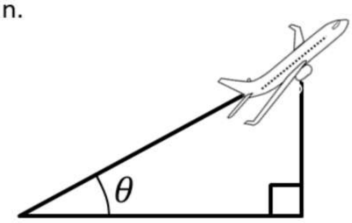
Given that the plane's speed is 50 m s^{-1},

b. find the parametric equations for the plane's motion.

c. find the vertical height of the plane after 10 seconds.

d. show that the plane's motion is a straight line.

e. explain why the domain of t, $t > 0$, is not realistic.



The motion of a figure skater relative to a fixed origin, O, at time t minutes is modelled using the parametric equations

$$x = 8\cos 20t, \qquad y = 12\sin\left(10t - \frac{\pi}{3}\right), \qquad t \geq 0$$

where x and y are measured in metres.

a) Find the coordinates of the figure skater at the beginning of his motion.

b) Find the coordinates of the point where the figure skater intersects his own path.

c) Find the coordinates of the points where the path of the figure skater crosses the y-axis.

d) Determine how long it takes the figure skater to complete one complete figure-of-eight motion.