Ex 11C

Q2h)

$$\int (\cos x - \sec x)^2 \, dx = \int (\cos^2 x - 2\cos x \sec x + \sec^2 x) \, dx$$

$$= \int (\cos^2 x - 2 + \sec^2 x) \, dx$$

$$= \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x - 2 + \sec^2 x\right) dx$$

$$= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - 2x + \tan x + c$$

$$= \tfrac{1}{4}\sin 2x + \tan x - \tfrac{3}{2}x + c$$

11D)

Q4)

$$\int_0^k kx^2 e^{x^3} \, dx = \tfrac{2}{3}(e^8 - 1)$$

$$\int_0^k kx^2 e^{x^3} = \left[\tfrac{k}{3} e^{x^3}\right]_0^k = \tfrac{k}{3}e^{k^3} - \tfrac{k}{3}$$

$$= \tfrac{k}{3}(e^{k^3} - 1)$$

$$8 = k^3$$

$$2 = k$$

$\sin^n x \cos x$ vs $\sec^n x \tan x$

Notice when we differentiate $\sin^5 x$, then power decreases:

$$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cos x$$

$(\sin x)^5$

However, when we differentiate $\sec^5 x$:

$$\frac{d}{dx}\left((\sec x)^5\right) = 5\sec^4 x \sec x \tan x$$

$$= 5\sec^5 x \tan x$$

Notice that the power of sec didn't go down. Keep this in mind when integrating.

$$\int \sin^4 x \cos x \, dx = \frac{1}{5}\sin^5 x + c$$

$$\int \sec^4 x \tan x \, dx = \frac{1}{4}\sec^4 x + C$$

$$\frac{1}{4} \times 4\sec^3 x \times \sec x \tan x$$

$$\int \cos x \sin^2 x \, dx = \frac{1}{3}\sin^3 x + C$$

$$\int \sec^3 x \tan x \, dx = \frac{1}{3}\sec^3 x + C$$

Your Turn

$$\int \sin x\,(\cos x + 1)^5\,dx$$

consider $(\cos x+1)^6$

differ. $6(\cos x+1)^5(-\sin x)$

scale : $-\frac{1}{6}(\cos x+1)^6$

$= -\frac{1}{6}(\cos x+1)^6$

$$\int \frac{cosec^2 x}{(2+\cot x)^3}\,dx$$

$cosec^2 x(2+\cot x)^{-3}$

Consider $(2+\cot x)^{-2}$

diff $-2(2+\cot x)^{-3}(-cosec^2 x)$

scale : $\frac{1}{2}(2+\cot x)^{-2}$

$= \frac{1}{2(2+\cot x)^2} + c$

$$\int \frac{\sec^2 2x}{\tan 2x + 1}\,dx$$

consider: $\ln|\tan 2x+1|$

diff : $\frac{2\sec^2 2x}{\tan 2x+1}$

scale : $\frac{1}{2}$

$= \frac{1}{2}\ln|\tan 2x+1| + c$

$$\int x(x^2+2)^3\,dx$$

consider : $(x^2+2)^4$

diff : $4(x^2+2)^3(2x)$

scale : $\frac{1}{8}$

$= \frac{1}{8}(x^2+2)^4 + c$

$$\int 5\tan x \sec^2 x\,dx$$

consider : $\sec^2 x$

diff : $2\sec x \sec x \tan x$

scale : $\frac{5}{2}$

$= \frac{5}{2}\sec^2 x + c$ $= \frac{5}{2}(1+\tan^2 x) + c$

$= \frac{5}{2}\tan^2 x + \frac{5}{2} + c$

consider : $\tan^2 x$

diff : $2\tan x \sec^2 x$

scale : $\frac{5}{2}$

$= \frac{5}{2}\tan^2 x + c$

Ex 11D

SKILL #5: Integration by Substitution

For some integrations involving a complicated expression, we can make a substitution to turn it into an equivalent integration that is simpler. We wouldn't be able to use 'reverse chain rule' on the following:

Q Use the substitution $u = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

The aim is to completely remove any reference to x, and replace it with u. We'll have to work out x and dx so that we can replace them.

STEP 1: Using substitution, work out x and dx (or variant)

$$\int x\underbrace{\sqrt{2x+5}}_{\sqrt{u}}\ dx$$

$$u = 2x+5$$

$$\frac{u-5}{2} = x$$

$$u = 2x+5$$

$$\frac{du}{dx} = 2$$

$$\frac{du}{2} = dx$$

$$\frac{1}{2}du = dx$$

STEP 2: Substitute these into expression.

$$\int x\sqrt{2x+5}\ dx = \int \frac{u-5}{2} \times u^{1/2} \times \frac{1}{2}\ du$$

$$= \int \frac{(u-5)u^{1/2}}{4}\ du$$

Tip: If you have a constant factor, factor it out of the integral.

$$= \frac{1}{4}\int (u-5)u^{1/2}\ du$$

STEP 3: Integrate simplified expression.

$$= \frac{1}{4}\int \left(u^{3/2} - 5u^{1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{10}{3}u^{3/2}\right) + C$$

STEP 4: Write answer in terms of x.

$$= \frac{1}{10}u^{5/2} - \frac{5}{6}u^{3/2} + C$$

$$= \frac{1}{10}(2x+5)^{5/2} - \frac{5}{6}(2x+5)^{3/2} + C$$

Using substitutions involving implicit differentiation

When a root is involved, it can make things tidier if we use $u^2 = \cdots$

Q Use the substitution $u^2 = 2x + 5$ to find $\int x\sqrt{2x+5}\ dx$

STEP 1: Using substitution, work out x and dx (or variant)

$u^2 = 2x + 5$

$\frac{1}{2}(u^2 - 5) = x$

$u^2 = 2x+5$

$u = \sqrt{2x+5}$

$u = (2x+5)^{1/2}$

$u^2 = 2x + 5$

$2u \frac{du}{dx} = 2$

$u\ du = dx$

STEP 2: Substitute these into expression.

STEP 3: Integrate simplified expression.

STEP 4: Write answer in terms of x.

$$\int x\sqrt{2x+5}\ dx = \int \tfrac{1}{2}(u^2-5)u \times u\ du$$

$$= \frac{1}{2}\int (u^2-5)u^2\ du$$

$$= \frac{1}{2}\int (u^4 - 5u^2)\ du$$

$$= \frac{1}{2}\left(\frac{1}{5}u^5 - \frac{5}{3}u^3\right) + c$$

$$= \frac{1}{10}u^5 - \frac{5}{6}u^3 + c$$

$$= \frac{1}{10}(2x+5)^{5/2} - \frac{5}{6}(2x+5)^{3/2} + c$$

This was marginally less tedious than when we used $u = 2x + 5$, as we didn't have fractional powers to deal with.

How can we tell what substitution to use?

In Edexcel you will *usually* be given the substitution!
However in some other exam boards, and in STEP, you often aren't.
There's no hard and fast rule, but it's often helpful to replace ~~to replace~~ expressions inside roots, powers or the denominator of a fraction.

Sensible substitution:

$$\int \cos x \sqrt{1 + \sin x} \, dx \qquad \boldsymbol{u} = 1 + \sin x$$

Ex 11E
Q 1a
Q 3ab

$$\int \frac{xe^x}{1 + x} \, dx \qquad \boldsymbol{u} = 1 + x$$

$$\int e^{\frac{1-x}{1+x}} \, dx \qquad \boldsymbol{u} = \frac{1-x}{1+x}$$

(very messy)