Complex Numbers

Cartesian Form

$z = x + iy$

Modulus-Argument Form

$z = r(\cos\theta + i\sin\theta)$

Complex Conjugates

Roots come in conj. pairs, $z^* = x - iy$

Realising the Denominator

$\frac{3+2i}{7-2i} \times \frac{7+2i}{7+2i}$

Multiplying

Multiply mod, add arguments

Dividing

Divide mod, subtract arguments

Manipulating Mod-Arg Form

$\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$

Loci on Argand Diagrams

$|z - z_1| = |z - z_2|$ perp. bisector of z_1, z_2

$|z - z_1| = r$ circle radius r, centre z_1

$\arg(z - z_1) = \theta$ half line drawn from z_1, angle θ

Series

Standard Results

$\sum_{r=1}^{n} k = kn$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

F $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

F $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Starting value not r=1

$\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$

Roots of Polynomials

Sums of roots, pairs, etc.

$ax^4 + bx^3 + cx^2 + dx + e = 0$

$\sum\alpha = -b/a$

$\sum\alpha\beta = c/a$

$\sum\alpha\beta\gamma = -d/a$

$\alpha\beta\gamma\delta = e/a$

Sum of Squares

$\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

Sum of Cubes

$\sum\alpha^3 = (\sum\alpha)^3 - 3\sum\alpha\beta\sum\alpha + 3\alpha\beta\gamma$

Transformation of Roots

If roots are $a\alpha + b$, set $w = ax + b$, make x the subject, then substitute

Volumes of Revolution

About x-axis

$\pi\int_a^b y^2\,dx$

About y-axis

$\pi\int_a^b x^2\,dy$

3D Solids

Cylinder $= \pi r^2 h$

Cone $= \frac{1}{3}\pi r^2 h$

Matrices - properties

Non-commutativity

$AB \neq BA$

Associativity

$(AB)C = A(BC)$

Identity

$A = AI = IA$

Matrices - Inverses

Definition

$AA^{-1} = I = A^{-1}A$

Product of Inverses

$(AB)^{-1} = B^{-1}A^{-1}$

Determinants

2x2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3x3 $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

If $\det(M) = 0$, M is singular

Singular matrices have no inverse

Matrix of Minors

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ - draw cross through element - find determinant

Matrix of Cofactors

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ - apply pattern to matrix of minors

Transpose of a Matrix

Rows and columns swap

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

2x2 Inverse

$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

3x3 Inverse

$A^{-1} = \frac{1}{\det A}C^T$ where

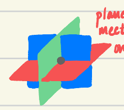
C is the matrix of cofactors

Self Inverse if $A = A^{-1}$

Matrices - Systems of Equations

Check det M

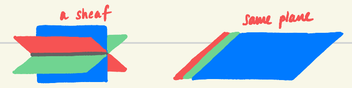
$\det M \neq 0$



$\det M = 0$

Check consistency

Consistent: solutions exist



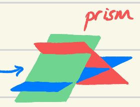
eg. $x + 2y + 3z = 1$
$2x + 4y + 6z = 2$
$-10x - 20y - 30z = -10$

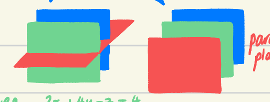
Inconsistent: no solutions exist

Check for parallel planes

2 or 3 parallel

No parallel planes





eg. $2x + 4y - z = 4$ and $2x + 4y - z = 5$ are parallel

Linear Transformations

Invariant Point, (x, y)

$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Invariant Line, $y = ax$

$M\begin{pmatrix} x \\ ax \end{pmatrix} = k\begin{pmatrix} x \\ ax \end{pmatrix}$

Invariant Line, $y = mx + c$

$M\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$

Area Scale Factor

$\det M =$ area scale factor

Successive Transformations

PQ is Q then P

Rotation θ anticlockwise

F $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Reflection in Plane

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $x=0$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $y=0$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $z=0$

Rotation θ anticlockwise...

... about x-axis $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$ ... about y-axis

... about z-axis $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Undoing Transformations

$P^{-1}(PQ)Q^{-1}$

Proof by Induction

Steps

- Base Case, $n=1$
- Assumption, $n=k$
- Inductive Case, $n=k+1$
- Conclusion: "Since true for $n=1$, and true for $n=k+1$ when assumed true for $n=k$, true for all $n \in \mathbb{N}$ or $\in \mathbb{Z}^+$.

Divisibility

For inductive, use $f(k+1) - f(k)$

(Recurrence)

If $u_{n+2} = au_{n+1} + bu_n$ must check $n=1$ and $n=2$ for base case, and assume for $n=k$ and $n=k+1$

Vectors - Definitions

Scalar Dot Product

$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$\underline{a}.\underline{b} = 0$ if $\underline{a}$ and $\underline{b}$ are perp.

Vector Line Equation

$\underline{r} = \underline{a} + \lambda\underline{b}$

Cartesian Line Equation

F $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} (= \lambda)$

Parametric Plane Equation

F $\underline{r} = \underline{a} + \lambda\underline{b} + \mu\underline{c}$

Scalar Product Plane Equation

$\underline{r}.\underline{n} = \underline{a}.\underline{n}$

Cartesian Plane Equation

F $n_1x + n_2y + n_3z = c$

Vectors - Angles

Between 2 lines

$\cos\theta = \frac{\underline{b}_1 . \underline{b}_2}{|\underline{b}_1||\underline{b}_2|}$ | | for acute

Between 2 planes

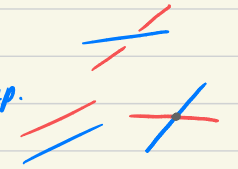
$\cos\theta = \frac{\underline{n}_1 . \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$ | | for acute.

Between a line and plane

$\sin\theta = \left|\frac{\underline{n} . \underline{b}}{|\underline{n}||\underline{b}|}\right|$

Vectors - Intersections

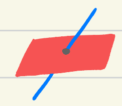
2 Lines



$\underline{r}_1 = \underline{r}_2$, find λ and μ check with $\underline{k}$ comp.

Either intersect/parallel/skew

Line and Plane



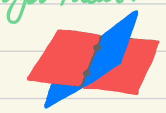
Plane must be in scalar product form

Sub line $\underline{r}$ into $\underline{r}.\underline{n} = d$

2 Planes

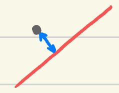
Find 2 points on both planes, find line through them.

Let $z=0$, or 1, solve x and y simultaneously



Vectors - Shortest Distances

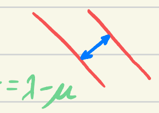
Point and Line



Find vector between point and general point on line

Make this perp to line direction

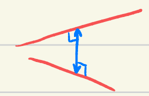
Between Parallel Lines



Find vector between 2 general points on the lines, using $t = \lambda - \mu$

Make this perp. to line direction

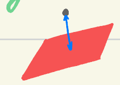
Between Skew Lines



Find vector between 2 general points on the lines.

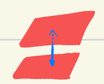
Make this perp. to both directions, solve simultaneously.

Point (α, β, γ) and Plane $n_1x + n_2y + n_3z + d = 0$



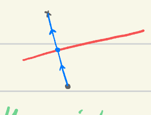
F dist $= \frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{|\underline{n}|}$

2 Parallel Planes



Find any point on one plane, then use formula.

Vectors - Reflections

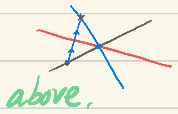


Reflect Point in Plane

Find intersection of normal through the point and the plane

Add vector onto intersection point.

Reflect Line in Plane



First reflect any point on line in the plane using above.

Find where line intersects plane. Find equation of line through intersection point and reflected point.