Chapter 9b: Differentiation - Applications (Year 2)

3:: Differentiate parametric equations.

If $x = \sin t$ and $y = e^t$, determine the equation of the tangent at the point $(0,1)$.

4:: Implicit Differentiation

An implicit relationship is where y is not in terms of x.

"If $xy + y^2 = 3$, determine $\frac{dy}{dx}$."

5:: Rates of change

"A circle's radius increases at a rate of $5\ cm/s$. Determine the rate at which area changes when $r = 3$."

Parametric Differentiation

✏ If x and y are given as functions of a parameter t, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Proof?

Find the gradient at the point P where $t = 2$, on the curve given parametrically by

$x = t^3 + t, \qquad y = t^2 + 1, \qquad t \in \mathbb{R}$

Find the equation of the normal at the point P where $\theta = \frac{\pi}{6}$, to the curve with parametric equations $x = 3\sin\theta$, $y = 5\cos\theta$

June 05 Q6. A curve has parametric equations

$$x = 2\cot t, \quad y = 2\sin^2 t, \quad 0 < t \leq \frac{\pi}{2}.$$

(*a*) Find an expression for $\frac{dy}{dx}$ in terms of the parameter t. **(4)**

(*b*) Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$. **(4)**

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

C4 June 2012 Q6

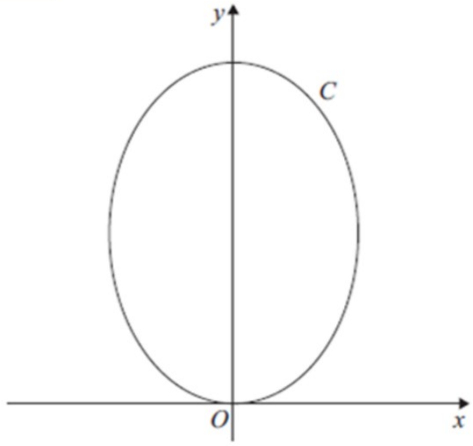


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = \sqrt{3}\sin 2t, \qquad y = 4\cos^2 t, \qquad 0 \le t \le \pi.$$

(*a*) Show that $\dfrac{dy}{dx} = k\sqrt{3}\tan 2t$, where k is a constant to be determined.

(5)

(*b*) Find an equation of the tangent to C at the point where $t = \dfrac{\pi}{3}$.

Give your answer in the form $y = ax + b$, where a and b are constants.

(4)

(a)	$\dfrac{dx}{dt} = 2\sqrt{3}\cos 2t$	B1
	$\dfrac{dy}{dt} = -8\cos t\sin t$	M1 A1
	$\dfrac{dy}{dx} = \dfrac{-8\cos t\sin t}{2\sqrt{3}\cos 2t}$	M1
	$= -\dfrac{4\sin 2t}{2\sqrt{3}\cos 2t}$	
	$\dfrac{dy}{dx} = -\dfrac{2}{3}\sqrt{3}\tan 2t \qquad \left(k = -\dfrac{2}{3}\right)$	A1
(b)	When $t = \dfrac{\pi}{3}$ $\quad x = \dfrac{3}{2},\ y = 1$ $\quad$ can be implied	B1
	$m = -\dfrac{2}{3}\sqrt{3}\tan\left(\dfrac{2\pi}{3}\right) \quad (= 2)$	M1
	$y - 1 = 2\left(x - \dfrac{3}{2}\right)$	M1
	$y = 2x - 2$	A1

Ex 9F Q12

Implicit Differentiation

Explicit Functions

Implicit Functions

$$y = x^2$$

$$\frac{d}{dx}\downarrow \qquad \downarrow\frac{d}{dx}$$

$$\frac{dy}{dx} = 2x$$

When seeing $y = x^2$ and differentiating, you probably think you're just differentiating the x^2. But in fact, you're differentiating **both** sides of the equation! (with respect to x)
y (by definition) differentiates to $\frac{dy}{dx}$

Remember that y differentiated with respect to x is, by definition, $\frac{dy}{dx}$

Differentiate y^2 with respect to x

In general, when differentiating a function of y, but with respect to x, multiply by $\frac{dy}{dx}$

$$\frac{d}{dx}\big(f(y)\big) = f'(y)\frac{dy}{dx}$$

Differentiate the following with respect to x

$\sin y$

e^y

Implicit Differentiation - trickier examples

Differentiate the following with respect to x

$x^2 + \cos y$

$\tan(x + y)$

Implicit Differentiation - using the product rule

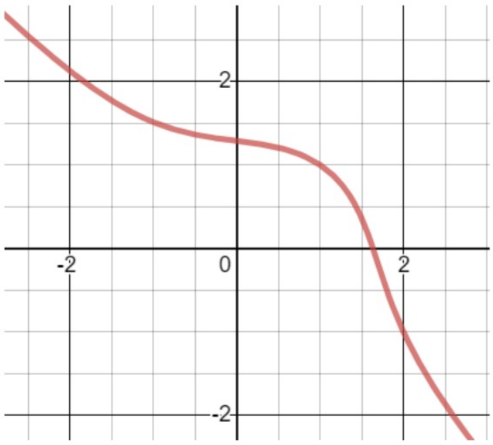
Differentiate the following with respect to x

xy

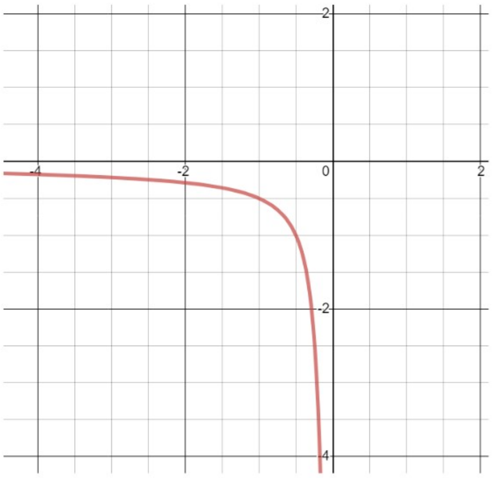
$e^{x^2 y}$

Implicit Differentiation - problems

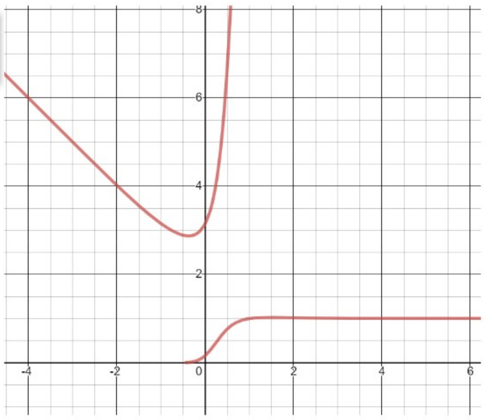
Find $\frac{dy}{dx}$ in terms of x and y where $x^3 + x + y^3 + 3y = 6$



Find $\frac{dy}{dx}$ in terms of x and y where $e^{2x} + e^{2y} = xy$



Find the value of $\frac{dy}{dx}$ at the point $(1,1)$, where $e^{2x} \ln y = x + y - 2$



Tip: Substitute in sooner rather than later. (i.e. No need to make $\frac{dy}{dx}$ the subject first)

Note: In Year 1 differentiation, you only ever needed the x value to calculate the gradient a particular point. In Year 2 the gradient can depend on x and y.

Your Turn

A curve is described by the equation

$$x^3 - 4y^2 = 12xy.$$

(*a*) Find the coordinates of the two points on the curve where $x = -8$. **(3)**

(*b*) Find the gradient of the curve at each of these points. **(6)**

a) $-512 - 4y^2 = -96y$

$y^2 - 24y + 128 = 0$

$(y - 16)(y - 8) = 0$

Gives point $(-8, 16)$, $(-8, 8)$

b) Implicitly differentiating:

$3x^2 - 8y\frac{dy}{dx} = 12x\frac{dy}{dx} + 12y$

$\frac{dy}{dx}$ not needed, but it's $\frac{3x^2 - 12y}{12x + 8y}$

If $x = -8, y = 16, \frac{dy}{dx} = -3$

If $x = -8, y = 8, \frac{dy}{dx} = 0$

Ex 9H

Implicit Differentiation - with simultaneous equations

$$x^2 + y^2 + 10x + 2y - 4xy = 10$$

(*a*) Find $\frac{dy}{dx}$ in terms of x and y, fully simplifying your answer. **(5)**

(*b*) Find the values of y for which $\frac{dy}{dx} = 0$. **(5)**

Implicit Differentiation - parallel to axes

12. A curve C is given by the equation

$$\sin x + \cos y = 0.5 \qquad -\frac{\pi}{2} \leqslant x < \frac{3\pi}{2}, -\pi < y < \pi$$

A point P lies on C.

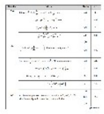
What if it were parallel to the y-axis instead?

The tangent to C at the point P is parallel to the x-axis.

Find the exact coordinates of all possible points P, justifying your answer.
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(7)

Your Turn



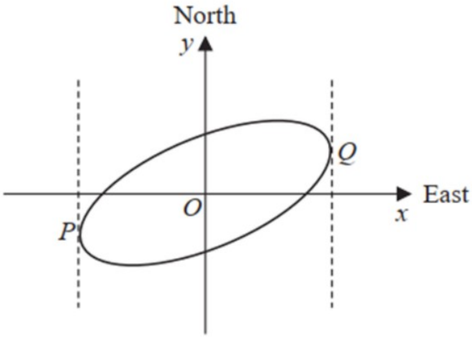


Figure 4

Figure 4 shows a sketch of the curve with equation $x^2 - 2xy + 3y^2 = 50$

(a) Show that $\dfrac{dy}{dx} = \dfrac{y - x}{3y - x}$

(4)

The curve is used to model the shape of a cycle track with both x and y measured in km.

The points P and Q represent points that are furthest west and furthest east of the origin O, as shown in Figure 4.

Using part (a),

(b) find the exact coordinates of the point P.

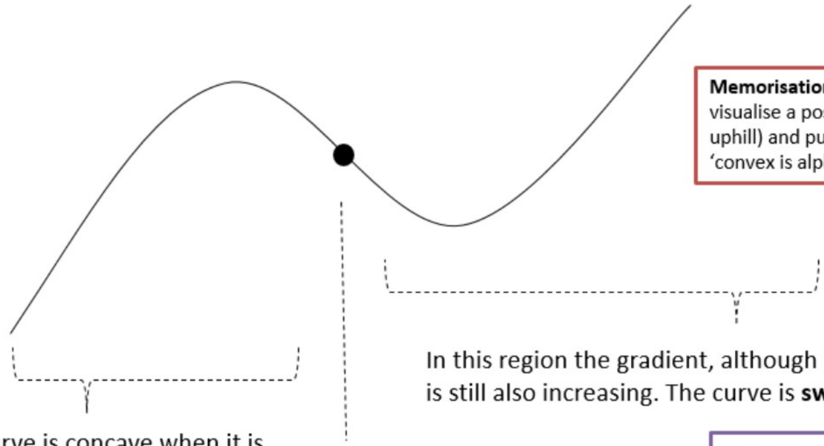
(5)

(c) Explain briefly how to find the coordinates of the point that is furthest north of the origin O. (You **do not** need to carry out this calculation).

(1)

Try Mixed Ex 9 Q32

Using the second derivative



Memorisation tip: Mentally visualise a positive cubic (i.e. uphill) and put 'concave' and 'convex is alphabetical order.

In this region the gradient, although initially negative, is still also increasing. The curve is **swerving left**.

✎ $f(x)$ convex when $f''(x) \leq 0$

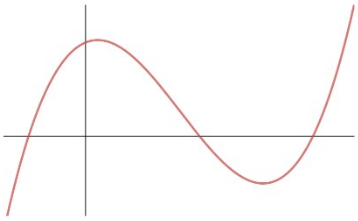
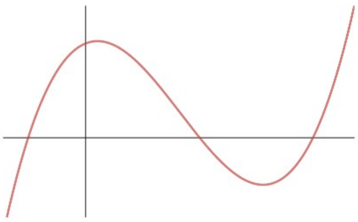
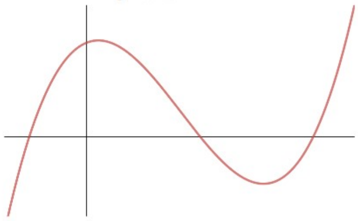
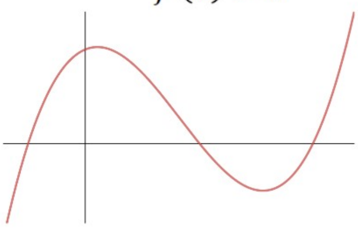
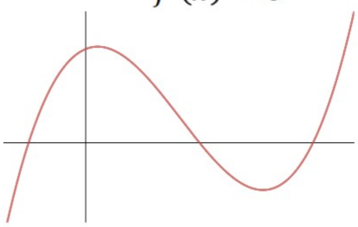
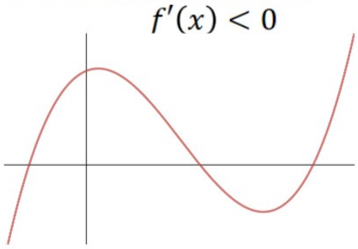
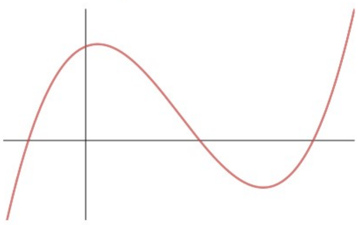
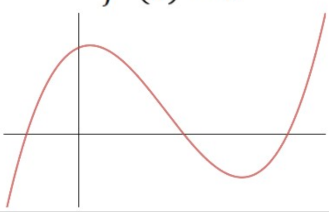
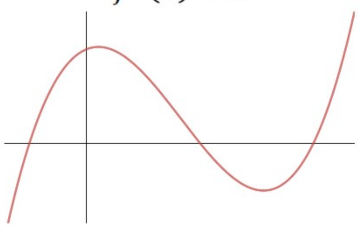
We say the curve is concave when it is **swerving right**: we can see the gradient is always decreasing, i.e. the change in the gradient is negative.

✎ $f(x)$ concave when $f''(x) \leq 0$

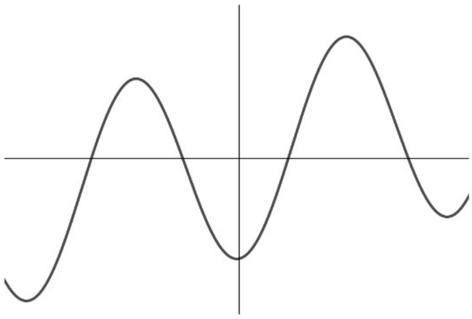
At point of inflection, the gradient is not changing, i.e. the gradient of the gradient is 0. Note that points of inflection (as with this example) are not necessarily stationary points.

✎ Point of inflection when $f''(x) = 0$

Recapping the features of a graph

The function is <u>**above the x axis**</u> when $f(x) > 0$ 	The function is <u>**on the x axis**</u> when $f(x) = 0$ 	The function is <u>**below the x axis**</u> when $f(x) < 0$ 
The function is <u>**increasing**</u> when $f'(x) > 0$ 	The function is <u>**stationary**</u> when $f'(x) = 0$ 	The function is <u>**decreasing**</u> when $f'(x) < 0$ 
The function is <u>**convex**</u> when $f''(x) > 0$ 	The function has a <u>**point of inflection**</u> (changes from convex to concave) when $f''(x) = 0$ 	The function is <u>**concave**</u> when $f''(x) < 0$ 

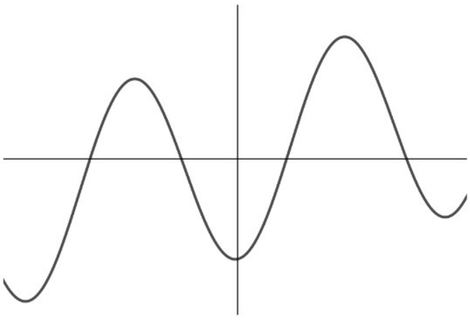
Indicate on the diagrams where…



$f(x) > 0$

$f(x) = 0$

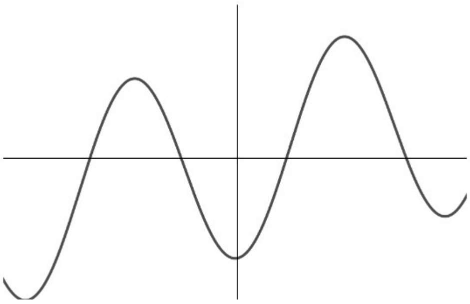
$f(x) < 0$



$f'(x) > 0$

$f'(x) = 0$

$f'(x) < 0$

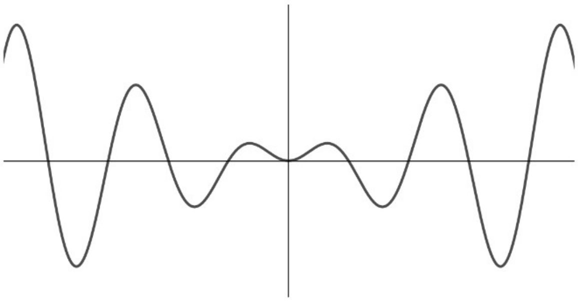


$f''(x) > 0$

$f''(x) = 0$

$f''(x) < 0$

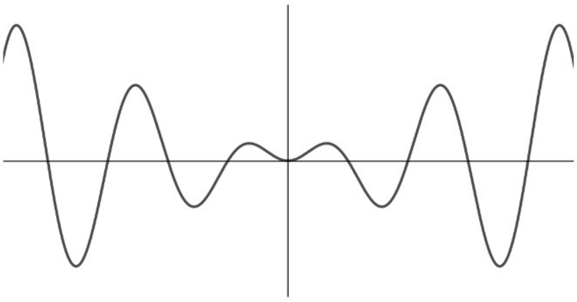
Indicate on the diagrams where…



$f(x) > 0$

$f(x) = 0$

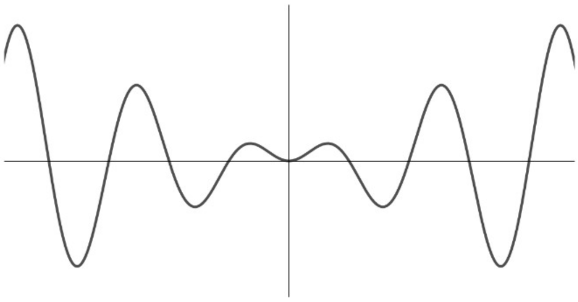
$f(x) < 0$



$f'(x) > 0$

$f'(x) = 0$

$f'(x) < 0$



$f''(x) > 0$

$f''(x) = 0$

$f''(x) < 0$

Find the range of values of x on which the function $f(x) = x^3 + 4x + 3$ is concave.

Show that $f(x) = e^{2x} + x^2$ is convex for all real values of x.

The curve C has equation $y = x^3 - 2x^2 - 4x + 5$
Find the range of values of x where the curve is convex,
and find the coordinates of the point of inflection.

Connected Rates of Change

Determine the rate of change of the area A of a circle when the radius $r = 3$cm, given that the radius is changing at a rate of $5\ cm\ s^{-1}$.

Firstly, how would we represent…

"the rate of change of the area A"

"the rate of change of the radius r is 5"

"the area A of a circle"

Tip: Whenever you see the word 'rate', think $/dt$

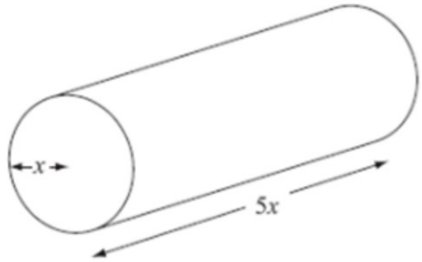


Figure 2 shows a right circular cylindrical metal rod which is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm.
The cross-sectional area of the rod is increasing at the constant rate of 0.032 $\text{cm}^2\,\text{s}^{-1}$.

(*a*) Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when the radius of the rod is 2 cm, giving your answer to 3 significant figures.

(4)

(*b*) Find the rate of increase of the volume of the rod when $x = 2$. **(4)**

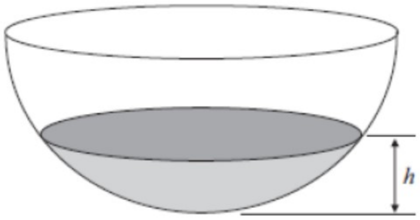


Figure 1

A hollow hemispherical bowl is shown in Figure 1. Water is flowing into the bowl.

When the depth of the water is h m, the volume V m^3 is given by

$$V = \frac{1}{12}\pi h^2(3 - 4h), \qquad 0 \leq h \leq 0.25.$$

(*a*) Find, in terms of π, $\frac{\mathrm{d}V}{\mathrm{d}h}$ when $h = 0.1$.

(4)

Water flows into the bowl at a rate of $\frac{\pi}{800}$ m^3 s^{-1}.

(*b*) Find the rate of change of h, in m s^{-1}, when $h = 0.1$.

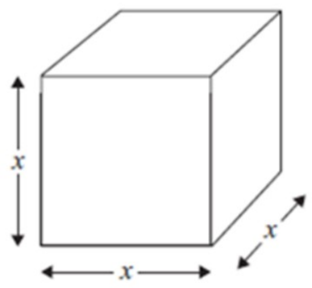


Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated.

At time t seconds, the length of each edge of the cube is x cm, and the volume of the cube is V cm^3.

(*a*) Show that $\dfrac{dV}{dx} = 3x^2$.

(1)

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^3 s^{-1},

(*b*) find $\dfrac{dx}{dt}$ when $x = 8$,

(2)

(*c*) find the rate of increase of the total surface area of the cube, in cm^2 s^{-1}, when $x = 8$.

(3)

Triple Connected Rates of Change

The volume of a hemisphere V cm^3 is related to its radius r cm by the formula $V = \frac{2}{3}\pi r^3$ and the total surface area $S = 3\pi r^2$

Given that the rate of **increase** of volume is $6\ \text{cm}^2\text{s}^{-1}$, find the rate of increase of surface area, when $r = 9$ cm

Note: I have skipped out the content on 'setting up differential equations' as I think it's better to do this in the Integration chapter. You therefore won't be able to yet do Q5, Q8, Q9, Q12, Q14.

Ex 9J

Exam Questions

15.

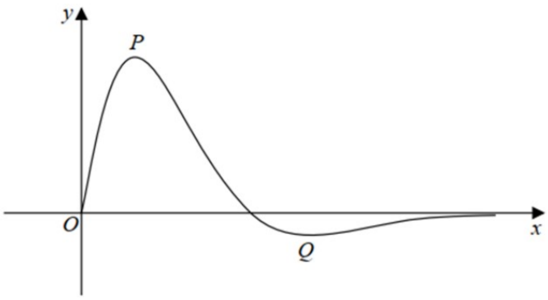


Figure 5

Figure 5 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{4\sin 2x}{e^{\sqrt{2}x-1}}, \quad 0 \leqslant x \leqslant \pi$$

The curve has a maximum turning point at P and a minimum turning point at Q as shown in Figure 5.

(a) Show that the x coordinates of point P and point Q are solutions of the equation

$$\tan 2x = \sqrt{2}$$

(4)

(b) Using your answer to part (a), find the x-coordinate of the minimum turning point on the curve with equation

(i) $y = \mathrm{f}(2x)$.

(ii) $y = 3 - 2\mathrm{f}(x)$.

(4)

Figure 5 shows a sketch of the curve with equation $y = f(x)$, where

$$f(x) = \frac{\sin 2x}{-3 + \cos 2x} \qquad 0 \leqslant x \leqslant \pi$$

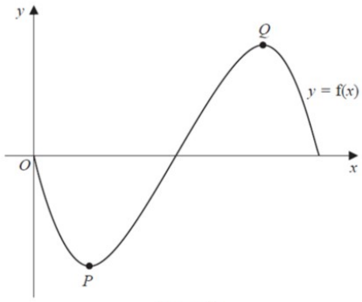


Figure 5

The curve has a minimum turning point at P and a maximum turning point at Q, as shown in Figure 5.

(a) Show that the x coordinate of P and the x coordinate of Q are solutions of the equation

$$\cos 2x = \frac{1}{3}$$

(4)

(b) Hence find, to 2 decimal places, the x coordinate of the maximum turning point on the curve with equation

(i) $y = f(3x) + 5 \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$

(ii) $y = -f\left(\frac{1}{4}x\right) \qquad 0 \leqslant x \leqslant 4\pi$

(4)

Figure 4 shows a bowl with a circular cross-section.

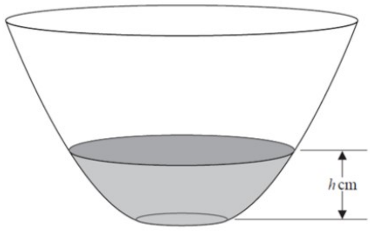


Figure 4

Initially the bowl is empty. Water begins to flow into the bowl.

At time t seconds after the water begins to flow into the bowl, the height of the water in the bowl is h cm.

The volume of water, V cm^3, in the bowl is modelled as

$$V = 4\pi h(h + 6) \qquad 0 \leqslant h \leqslant 25$$

The water flows into the bowl at a constant rate of 80π cm^3 s^{-1}

(a) Show that, according to the model, it takes 36 seconds to fill the bowl with water from empty to a height of 24 cm.

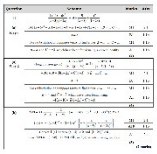
(1)

(b) Find, according to the model, the rate of change of the height of the water, in cm s^{-1}, when $t = 8$

(8)

11.

$$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}$$



(a) Find the values of the constants A, B and C.

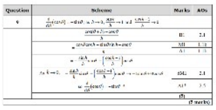
(4)

$$\mathrm{f}(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)} \qquad x > 3$$

(b) Prove that $\mathrm{f}(x)$ is a decreasing function.

(3)

9. Given that θ is measured in radians, prove, from first principles, that



Question	Scheme	Marks	AOs
9	[illegible]		
	[illegible]	B1	2.1
	[illegible]	M1	1.1b
		A1	1.1b
	[illegible]		
	[illegible]	dM1	2.1
	[illegible]	A1*	2.5
		(5)	
			(5 marks)

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$$

You may assume the formula for $\cos(A \pm B)$ and that as $h \to 0$, $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to 0$

(5)