Chapter 7b: Algebraic Methods, Proof

a) Proof by deduction

Prove that the product of two odd numbers is odd.

b) Proof by exhaustion

Prove that $n^2 + n$ is even for all integers n.

c) Disproof by counter example

Disprove the statement: "$n^2 - n + 41$ is prime for all integers n."

d) Proof by contradiction (A2)

Prove that the square root of 2 is irrational.

a) Proof by Deduction

This is the simplest type, where you start from known facts and reach the desired conclusion via deductive steps.

"Prove that the product of two odd numbers is odd."

Odd number	$2n+1$	Where n is an integer
A *different* odd number	$2m+1$	Where m is an integer
Consecutive odd numbers	$2n-1, 2n+1, etc.$	Where n is an integer
Even number	$2n$	Where n is an integer
A *different* even number	$2m$	Where m is an integer
Consecutive even numbers	$2n, 2n+2, etc.$	Where n is an integer
a is a factor of b	$b = na$	Where n is an integer
A rational number	$\frac{a}{b}$	Where a and b are integers, and have a highest common factor of 1 (i.e. a fraction in its lowest terms)

Prove that the difference between the squares of any two consecutive integers is equal to the sum of these two integers.

Proof by Deduction

Exam Tip: This is quite a common last part question.

Prove that $x^2 + 4x + 5$ is positive for all values of x.

Anything squared is at least 0. This is formally known as the '*trivial inequality*'.

Test Your Understanding

Prove that the sum of the squares of two consecutive odd numbers is 2 more than a multiple of 8.

Be Warned…

Proof by Deduction requires you to **start from known facts** and end up at the conclusion. It is **not** acceptable to start with to the conclusion, and verify it works, **because you are assuming the thing you are trying to prove**.

Example: Prove that if three consecutive integers are the sides of a right-angled triangle, they must be 3, 4 and 5.

Incorrect Proof:

Correct Proof:

Ex 7D

b) Proof by Exhaustion

Proof by Exhaustion

This means breaking down the statement into **all possible smaller cases**, where we prove each individual case.
(This technique is sometimes known as 'case analysis')

Prove that $n^2 + n$ is even for all integers n.

10. (i) Prove that for all $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4

(4)

c) Disproof by Counter-Example

Disproof by Counter-Example

While to prove a statement is true, we need to prove every possible case (potentially infinitely many!), **we only need one example to disprove** a statement.
This is known as a **counterexample**.

Disprove the statement:
"$n^2 - n + 41$ is prime for all integers n."

It is suggested that for every prime number p, 2p + 1 is also prime.

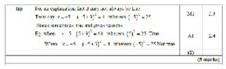
Prove that this is false.

3. (a) "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."

Disprove this statement by means of a counter example.

(2)

"Always true, sometimes true, or never true"



2. (i) Show that $x^2 - 8x + 17 > 0$ ~~for all real~~ values of x

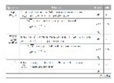
(3)

(ii) "If I add 3 to a number and square the sum, the result is greater than the square of the original number."

State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

Exam Questions



11. (a) Prove that for all positive values of x and y

$$\sqrt{xy} \leqslant \frac{x+y}{2}$$

(2)

(b) Prove by counter example that this is not true when x and y are both negative.

(1)

6. (i) Use a counterexample to show that the following statement is false.

"$n^2 - n - 1$ is a prime number, for $3 \leq n \leq 10$."

(2)

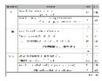
(ii) Prove that the following statement is always true.

"The difference between the cube and the square of an odd number is even."

For example, $5^3 - 5^2 = 100$ is even.

(4)

(Total for Question 6 is 6 marks)



Chapter 1a (Year 2): Algebraic Methods, Proof

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c) Disproof by counter example

Disprove the statement: "$n^2 - n + 41$ is prime for all integers n."

d) Proof by contradiction (A2)

Prove that the square root of 2 is irrational.

d) Proof by Contradiction

✎ To prove a statement is true by contradiction:

- **Assume** that the statement is in fact **false**.
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove that there is no greatest odd integer.

✎ To prove a statement is true by contradiction:

- **Assume** that the statement is in fact **false**.
- Prove that this would **lead to a contradiction**.
- Therefore we were wrong in assuming the statement was false, and therefore it must be true.

Prove by contradiction that if n^2 is even, then n must be even.

The negation of "if A then B" is "if A, then not B".

Use proof by contradiction to show that there exist no integers a and b for which $25a + 15b = 1$.

(4 marks)

Use proof by contradiction to show that, given a rational number a and an irrational number b, $a - b$ is irrational.

(4 marks)

A **rational number** is one that can be expressed in the form $\frac{a}{b}$ where a, b are integers.

An **irrational** number cannot be expressed in this form, e.g. $\pi, e, \sqrt{3}$.

The set of all rational numbers is $\mathbb{Q}$ (real numbers: $\mathbb{R}$, natural numbers: $\mathbb{N}$, integers: $\mathbb{Z}$).

Prove that the square root of 2 is irrational.

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Prove by contradiction that there are infinitely many primes.

Assume that there is a finite number of prime numbers.
Therefore we can list all the prime numbers:

$$p_1, p_2, p_3, \dots, p_n$$

Consider the number:

$$N = (p_1 \times p_2 \times \cdots \times p_n) + 1$$

When you divide N by any of $p_1, p_2, \dots, p_n$, the remainder will always be 1.
Therefore N is not divisible by any of these primes.

Therefore N must itself be prime, or its prime factorisation contains only primes not in our original list.
This contradicts the assumption that $p_1, p_2, \dots, p_n$ contained the list of all prime numbers.

Therefore, there are an infinite number of primes.

Exam Questions A2

8. (i) Show that $y^2 - 4y + 7$ is positive for all real values of y.

(2)

(ii) Bobby claims that

$$e^{3x} \geqslant e^{2x} \qquad x \in \mathbb{R}$$

Determine whether Bobby's claim is always true, sometimes true or never true, justifying your answer.

(2)

(iii) Elsa claims that

'for $n \in \mathbb{Z}^+$, if n^2 is even, then n must be even'

Use proof by contradiction to show that Elsa's claim is true.

(2)

(iv) Ying claims that

'the sum of two different irrational numbers is irrational'

Determine whether Ying's claim is always true, sometimes true or never true, justifying your answer.

(2)

14. (i) Kayden claims that

$$3^x \geqslant 2^x$$

Determine whether Kayden's claim is always true, sometimes true or never true, justifying your answer.

(2)

(ii) Prove that $\sqrt{3}$ is an irrational number.

(6)