Your notes

Edexcel A Level Further Maths: Core Pure

8.1 First Order Differential Equations

Contents

8.1.1 Intro to Differential Equations

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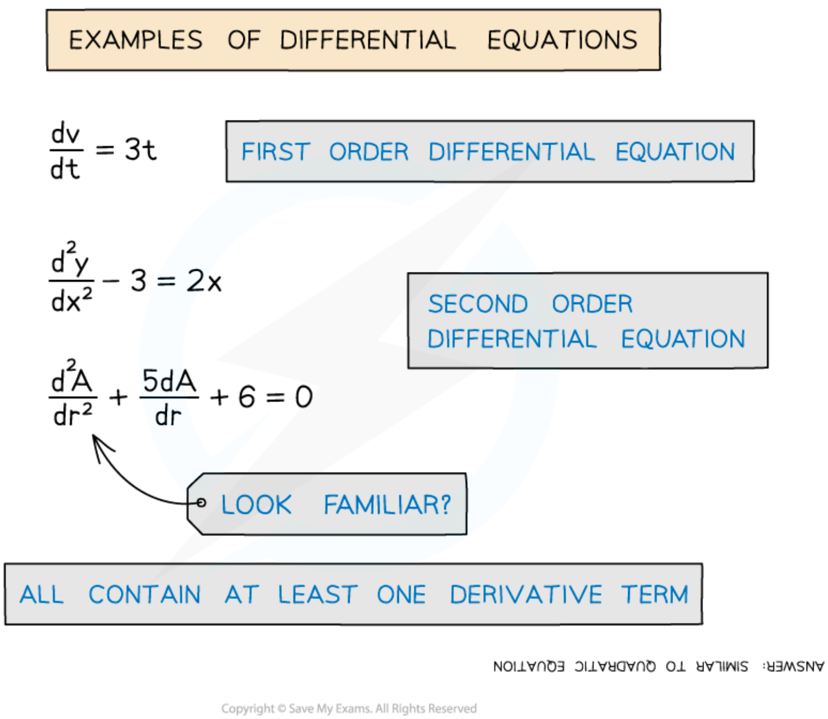
General Solutions

What is a differential equation?



- Any equation, involving a **derivative term**, is a **differential equation**
- Equations involving only first derivative terms are called **first order differential equations**
- Equations involving second derivative terms are called **second order differential equations**

What is a general solution?

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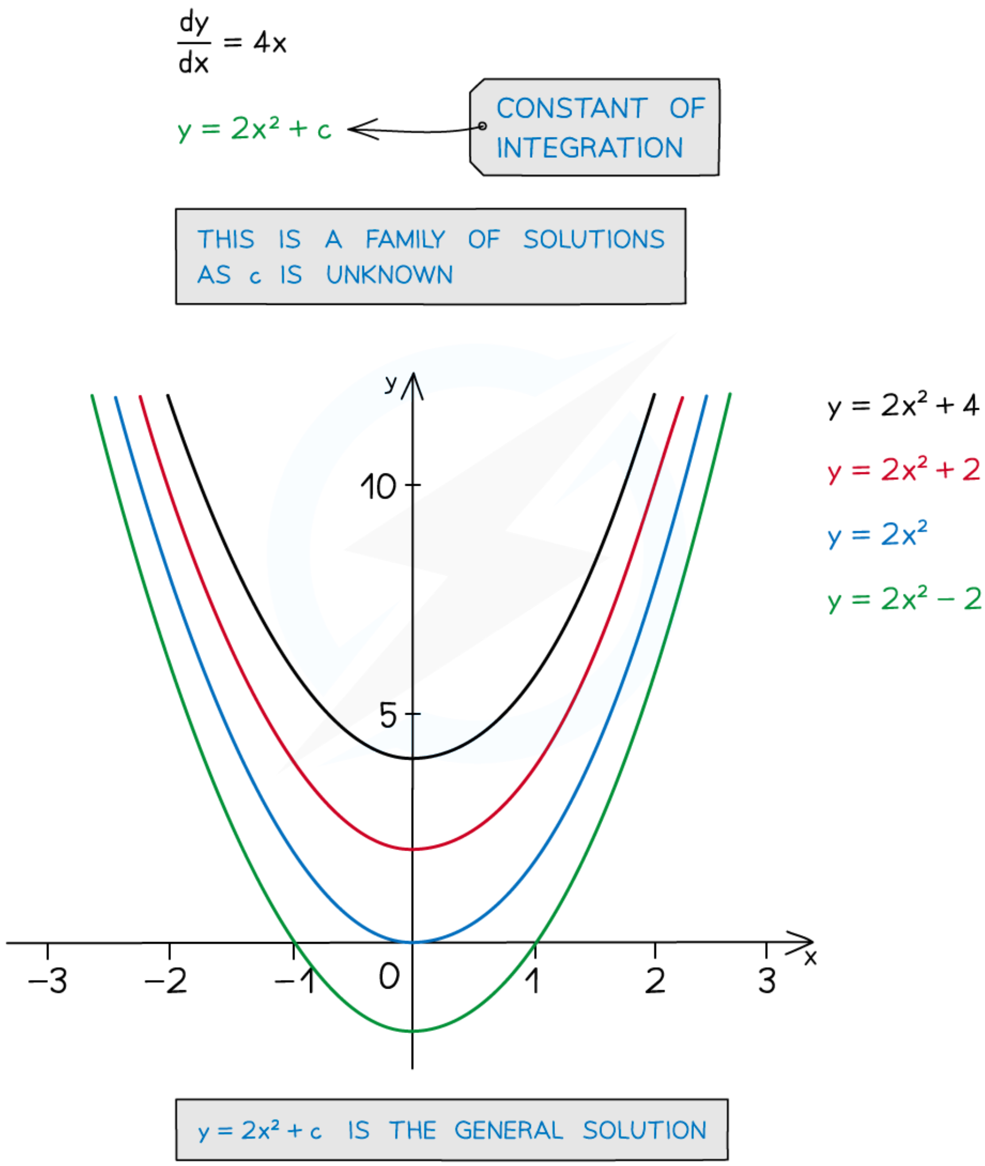

- Integration will be involved in **solving** the differential equation
  - ie working back to "**y = f(x)**"
- A constant of integration, **c** is produced
- This gives an infinite number of solutions to the differential equation, each of the form **y = g(x) + c** (ie **y = f(x)** where **f(x) = g(x) + c**)
-
  - ... and the solution **y= g(x) + c** is called the **general solution**These are often called a **family of solutions** ...

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Worked example

Find the general solution to the differential equation $\frac{d^2y}{dx^2} = 4$.

$\frac{d^2y}{dx^2} = 4$ integrate to find $\frac{dy}{dx}$

$\frac{dy}{dx} = \int 4 \, dx = 4x + c$ integrate again to find y

$y = \int 4x + c \, dx = \frac{4x^2}{2} + cx + d$

integrating twice will lead to two unknown constants of integration

$$\boxed{y = 2x^2 + cx + d}$$

Particular Solutions

What is a particular solution?

- Ensure you are familiar with **General Solutions** first
- With extra information, the constant of integration, **c**, can be found
- This means the **particular solution** (from the family of solutions) can be found

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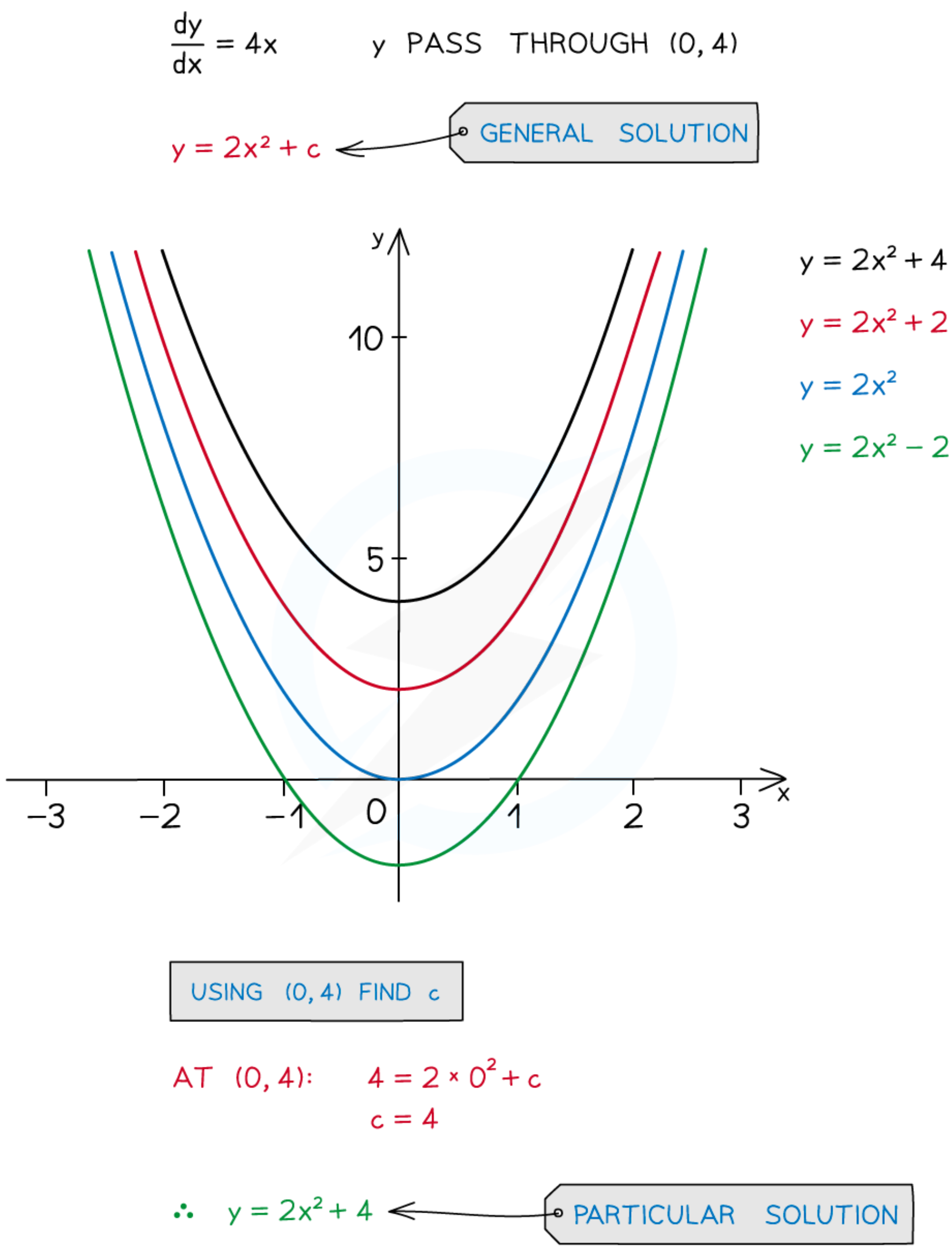

What is a boundary condition/initial condition?

A **boundary condition** is a piece of extra information that lets you find the particular solution

- For example knowing **y = 4** when **x = 0** in the preceding example
- In a model this could be a particle coming to rest after a certain time, ie **v = 0** at time **t**

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- Differential equations are used in modelling, experiments and real-life situations
- A boundary condition is often called an **initial condition** when it gives the situation at the start of the model or experiment
- This is often linked to time, so **t = 0**
- It is possible to have two boundary conditions
  - eg a particle initially at rest has velocity, **v = 0** and acceleration, **a = 0** at time, **t = 0**
  - for a **second order** differential equation you need **two** boundary conditions to find the particular solution

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Worked example

The velocity of a particle, initially at rest, is modelled by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} = 2t - 2$,

where v is the velocity of the particle and t is the time since the particle began moving.

a) Find the velocity of the particle after 3 seconds.

$\frac{dv}{dt} = 2t - 2$ Integrate with respect to t to find v

$v = \int 2t - 2\,dt = \frac{2t^2}{2} - 2t + c = t^2 - 2t + c$

general solution

'Initially at rest' means v = 0 when t = 0 so substitute t = 0 and v = 0 to find c:

$v = t^2 - 2t + c$

when $t = 0$, $v = 0$ $\Rightarrow$ $0 = 0^2 - 2(0) + c \Rightarrow c = 0$

$\therefore v = t^2 - 2t$

particular solution

Substitute $t = 3$, $v = 3^2 - 2(3) = 3$

$\boxed{v = 3\,\text{ms}^{-1}}$

b) Find the time at which the particle comes to rest for the second time.

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At rest means $v = 0$, so find the values of t for which $v = 0$.

$$t^2 - 2t = 0$$

$$t(t-2) = 0$$

$$t_1 = 0, \quad t_2 = 2$$

initially

The particle is at rest for the second time after 2 seconds.

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8.1.2 Solving First Order Differential Equations

First Order Differential Equations

What is a differential equation?

- A **differential equation** is simply an equation that contains derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a differential equation
  - And so is $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$

What is a first order differential equation?

- A **first order differential equation** is a differential equation that contains first derivatives but no second (or higher) derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a first order differential equation
  - But $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$ is **not** a first order differential equation, because it contains the second derivative $\frac{d^2x}{dt^2}$
- The general solution to a first order differential equation will have one unknown constant
- To find the particular solution you will need to know an initial condition or a boundary condition

Wait – haven't I seen first order differential equations before?

- Yes you have!
  - For example $\frac{dy}{dx} = 3x^2$ is also a first order differential equation, because it contains a first derivative and no second (or higher) derivatives
  - But for that equation you can just integrate to find the solution $y = x^3 + c$ (where c is a constant of integration)
- In A Level Maths you will have solved some first order differential equations using the method of separation of variables

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Integrating Factors

What is an integrating factor?

- An **integrating factor** can be used to solve a differential equation that can be written in the **standard form** $\frac{\mathrm{d}y}{\mathrm{d}x}+p(x)y=q(x)$
  - Be careful – the 'functions of x' $p(x)$ and $q(x)$ may just be constants!
    - For example in $\frac{\mathrm{d}y}{\mathrm{d}x}+6y=\mathrm{e}^{-2x}$, $p(x) = 6$ and $q(x) = \mathrm{e}^{-2x}$
    - While in $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{2x}=12$, $p(x)=\frac{1}{2x}$ and $q(x) = 12$
- For an equation in standard form, the integrating factor is $\mathrm{e}^{\int p(x)\ \mathrm{d}x}$

How do I use an integrating factor to solve a differential equation?

- STEP 1: If necessary, rearrange the differential equation into standard form
- STEP 2: Find the integrating factor
  - Note that you don't need to include a constant of integration here when you integrate $\int p(x)\ \mathrm{d}x$
- STEP 3: Multiply both sides of the differential equation by the integrating factor
- This will turn the equation into an **exact differential equation** of the form $\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int p(x)\ \mathrm{d}x}\right)=q(x)\mathrm{e}^{\int p(x)\ \mathrm{d}x}$
- STEP 4: Integrate both sides of the equation with respect to x
  - The left side will automatically integrate to $y\mathrm{e}^{\int p(x)\ dx}$
  - For the right side, integrate $\int q(x)\mathrm{e}^{\int p(x)\mathrm{d}x}\mathrm{d}x$ using your usual techniques for integration
  - Don't forget to include a constant of integration
    - Although there are two integrals, you only need to include one constant of integration
- STEP 5: Rearrange your solution to get it in the form $y = f(x)$

What else should I know about using an integrating factor to solve differential equations?

- After finding the general solution using the steps above you may be asked to do other things with the solution
  - For example you may be asked to find the solution corresponding to certain initial or boundary conditions

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Worked example

Consider the differential equation $\frac{dy}{dx} = 2xy + 5e^{x^2}$ where $y = 7$ when $x = 0$.

Use an integrating factor to find the solution to the differential equation with the given boundary condition.

STEP 1: $\frac{dy}{dx} - 2xy = 5e^{x^2}$ $\quad p(x) = -2x \quad q(x) = 5e^{x^2}$

STEP 2: $e^{\int -2x\, dx} = e^{-x^2}$

STEP 3: $\left(\frac{dy}{dx} - 2xy\right)e^{-x^2} = \left(5e^{x^2}\right)e^{-x^2}$

$$e^{-x^2}\frac{dy}{dx} - 2xye^{-x^2} = 5$$

$$\frac{d}{dx}\left(ye^{-x^2}\right) = 5$$

Exact differential equation

STEP 4: $ye^{-x^2} = \int 5\, dx = 5x + c$

Don't forget constant of integration

STEP 5: $y = e^{x^2}(5x + c)$

Now use the boundary condition

$$7 = e^0(5(0) + c) \implies c = 7$$

$$\boxed{y = e^{x^2}(5x + 7)}$$

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8.1.3 Modelling using First Order Differential Equations

Modelling using First Order Differential Equations

Why are differential equations used to model real-world situations?

- A **differential equation** is an equation that contains one or more derivatives
- Derivatives deal with rates of change, and with the way that variables change with respect to one another
- Therefore differential equations are a natural way to model real-world situations involving change
  - Most frequently in real-world situations we are interested in how things change over time, so the derivatives used will usually be with respect to time t

How do I set up a differential equation to model a situation?

- An exam question may require you to create a differential equation from information provided
- The question will provide a context from which the differential equation is to be created
- Most often this will involve the rate of change of a variable being proportional to some function of the variable
    - For example, the rate of change of a population of bacteria, P, at a particular time may be proportional to the size of the population at that time
- The expression 'rate of' ('rate of change of...', 'rate of growth of...', etc.) in a modelling question is a strong hint that a differential equation is needed, involving derivatives with respect to time t
    - So with the bacteria example above, the equation will involve the derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$
- Recall the basic equation of proportionality
  - If y is proportional to x, then $y = kx$ for some **constant of proportionality** k
    - So for the bacteria example above the differential equation needed would be $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$
  - The precise value of k will generally not be known at the start, but will need to be found as part of the process of solving the differential equation
  - It can often be useful to assume that $k > 0$ when setting up your equation
    - In this case, $-k$ will be used in the differential equation in situations where the rate of change is expected to be negative
    - So in the bacteria example, if it were known that the population of bacteria was decreasing, then the equation could instead be written $\frac{\mathrm{d}P}{\mathrm{d}t} = -kP$
- Often scenarios will involves multiple things than affect the rate
  - If something causes the variable to increase then that term will be added to the rate of change
    - Such as water flowing into a space
  - If something causes the variable to decrease then that term will be subtracted from the rate of change
    - Such as water flowing out of space

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Worked example

a) In a particular pond, the rate of change of the area covered by algae, A, at any time t is directly proportional to the square root of the area covered by algae at that time. Write down a differential equation to model this situation.

$$\frac{dA}{dt} = k\sqrt{A}$$ (where k is a constant of proportionality)

b) Newton's Law of Cooling states that the rate of change of the temperature of an object, T, at any time t is proportional to the difference between the temperature of the object and the ambient temperature of its surroundings, T_a, at that time. Assuming that the object starts off warmer than its surroundings, write down the differential equation implied by Newton's Law of Cooling.

The object is assumed to be warmer than its surroundings, so $T - T_a > 0$

$$\frac{dT}{dt} = -k(T - T_a)$$

(where $k > 0$ is a constant of proportionality)

We expect the temperature to be decreasing, so $-k$ in the equation combined with $k > 0$ assures that $\frac{dT}{dt}$ is negative.