Vectors - Core Pure

1:: Equations of straight lines in 3D

"Find an equation of the line that passes through the points $A(1,2,3)$ and $B(4,0,-2)$, giving your answer in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$"

2:: Equations of planes

"The plane Π is perpendicular to the normal vector $a = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and passes through the point P with position vector $8\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$. Find the Cartesian equation of Π."

3:: Scalar product and angles between line + line or plane + line or plane + plane.

"If the line l has equation $r = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ and point $A(3,-1,4)$ is a point on the line and B has coordinates $(5,6,6)$, find the angle between l and AB."

4:: Scalar product form of equation of plane.

$$r \cdot n = a \cdot n$$

5:: Point of intersection of two planes.

"Show that the line with equations $3i + j + k + \lambda(i - 2j - k)$ and $r = -2j + 3k + \mu(-5i + j + 4k)$ meet and find "the point of intersection."

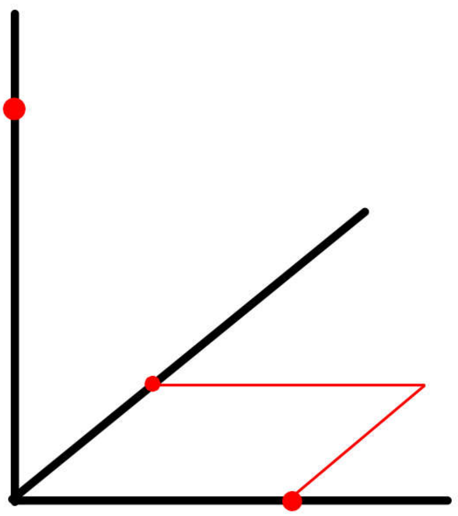
6:: Perpendicular distance between line + line or point + line or point + plane.

"Find the shortest distance between the line l with equation $\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+3}{-1}$ and the point A has coordinates $(1,2,-1)$."

Vector Basics

Distances

Find the distance from the origin to the point P(4, 2, 5)



Find the distance AB

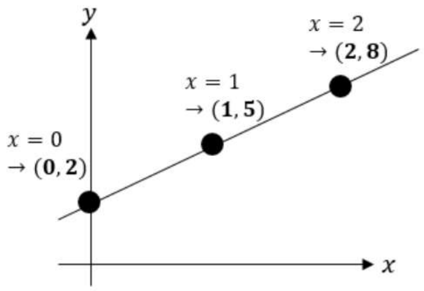
A(3, 6, -2)
B(1, 0, 5)

Find a unit vector in the direction AB.

Vector equation of a straight line in 3D

Consider the equation of a straight line in 2D:

$$y = 3x + 2$$

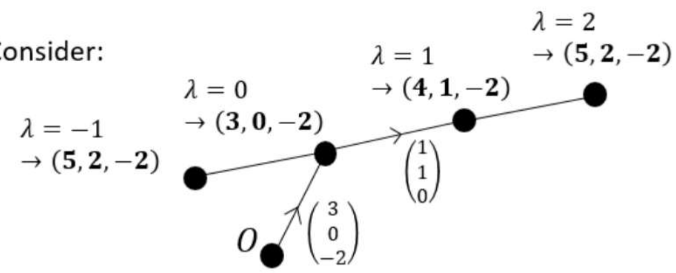


x is obviously a variable (i.e. it can vary!). As we consider different values of x, we get different points on the line.

It's worth noting that in $y = mx + c$, while x and y are variables, m and c are **constants**: after these are set for a particular line, they don't change.

Can we do something similar with vectors? Consider:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$



What happens as we vary λ?

Therefore what was the role of:

$\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$: Position vector of some arbitrary point on the line (it doesn't matter which).

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$: The **direction** of the line.

✎ Vector equation $\boldsymbol{r}$ of a straight line:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$$

where $\boldsymbol{a}$ is (the position vector of) some point on the line, $\boldsymbol{b}$ is the direction vector.

Important understanding points:

- $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants (i.e. fixed for a given line) while λ is a variable.
- It is often helpful to write as a single position vector, e.g:

$$\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3+\lambda \\ \lambda \\ -2 \end{pmatrix}$$

- It is highly important the you can distinguish between the **position vector** $\boldsymbol{r}$ of a **point on the line**, and the **direction** $\boldsymbol{b}$ of the line:

Example Problem

The equation of line l_1 is $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find the vector equation of a line parallel to l_1 which passes through the point $(2,5,1)$.

Find a vector equation of the straight line which passes through the point A, with position vector $3\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}$ and is parallel to the vector $7\boldsymbol{i} - 3\boldsymbol{k}$.

Find a vector equation of the straight line which passes through the points A and B, with coordinates $(4,5,-1)$ and $(6,3,2)$ respectively.

The straight line has vector equation
$\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + t(\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})$.
Given that the point $(a, b, 0)$ lines on l, find the value of a and the value of b.

The straight line l has vector equation
$\mathbf{r} = (2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) + \lambda(6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
Show that another vector equation of l is
$\mathbf{r} = (8\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \mu(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

Example Problem

The equation of line l_1 is $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find the coordinates of the points on l_1 which are a distance of 3 away from (3,4,4).

Ex 9A
Questions:
1ace
2ace
3
4ai, 4aiii
5ac
7
8
10
11
13

Ex 9A
Homework:
1bd
2bd
5b
9
12

If $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ is equation of straight line,

then its Cartesian form is

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Find the Cartesian equation of the line with equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$.

Find the Cartesian equation of the line with equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$.

The Cartesian equation of a line is $y = 3x + 2$. Find the vector form of the equation of the line.

The Cartesian equation of a line is $\frac{x-2}{3} = \frac{y+5}{1} = \frac{z}{4}$. Find the vector form of the equation of the line.

Ex 9A
Q4bi, biii
Q6

The line l has equation $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and the point P has position vector $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

(a) Show that P does not lie on l.

Given that a circle, centre P, intersects l at points A and B, and that A has position vector $\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$,

(b) find the position vector of B.

Q14, 15, 16, 17

16 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. A and B are the points on l_1 with $\lambda = 2$ and $\lambda = 5$ respectively.

a Find the position vectors of A and B. **(2 marks)**

The point P has position vector $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

The line l_2 passes through the point P and is parallel to the line l_1.

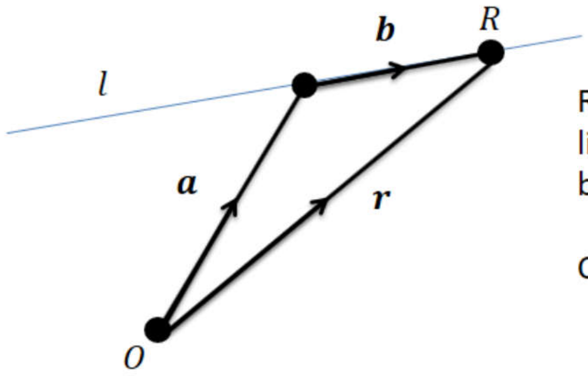
b Find a vector equation of the line l_2. **(2 marks)**

The points C and D both lie on line l_2 such that $AB = AC = AD$.

c Show that P is the midpoint of CD. **(7 marks)**

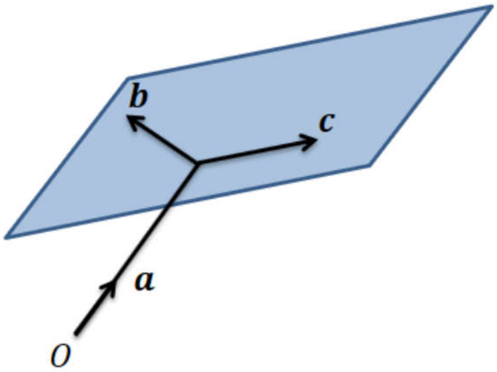
Equation of a plane - parametric vector form

$\boldsymbol{a}$ is the position vector of a point on a plane and $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-parallel vectors on the plane, how could we write the equation of the plane in vector form?



Recall the we could get to a generic point $\boldsymbol{r}$ on a line by first getting to the line using $\boldsymbol{a}$, followed by some amount of b, i.e. $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$.

Could we do a similar thing with a plane?



A plane Π passes through the points $A(2,2-1), B(3,2,-1), C(4,3,5)$
Find the equation of the plane Π in the form $\boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$

Verify that the point P with position vector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ lies in the plane with vector equation

$$r = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

[June 2015 Q5] The points A, B and C have position vectors $\begin{pmatrix}1\\3\\2\end{pmatrix}$, $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$ respectively.

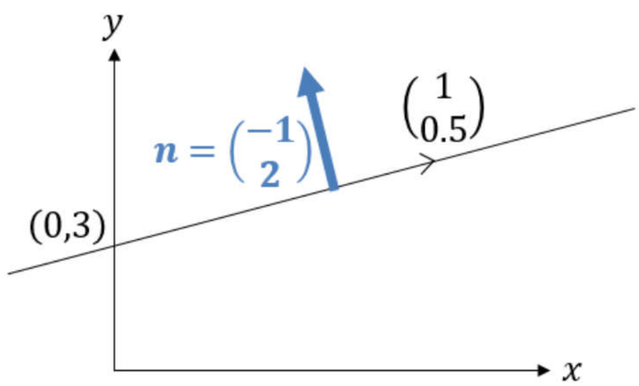
The plane Π contains the points A, B and C.

(c) Find a vector equation of Π **(4)**

Ex 9B
1ac
3ac
7

Equation of a plane - Cartesian form

An alternative approach to find the Cartesian equation of a straight line is to find a vector perpendicular to the line (known as the **normal vector $\boldsymbol{n}$**)



If $\boldsymbol{n} = \begin{pmatrix}n_1\\n_2\end{pmatrix}$, then the equation of the turns out to be $n_1x + n_2y = c$ where c is a constant to be found.

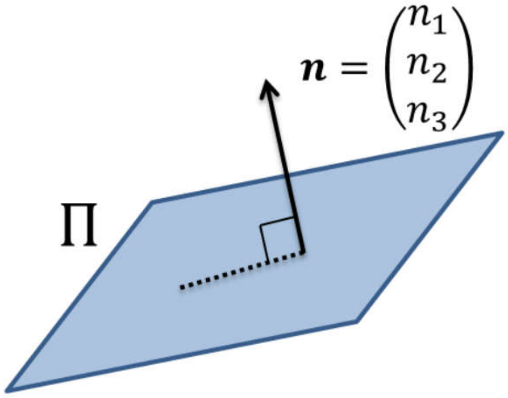
This is one reason we might want the equation of a straight line equation in the form $ax + by = c$: the **normal** to the line will be $\begin{pmatrix}a\\b\end{pmatrix}$.

For example above: $n = \begin{pmatrix}-1\\2\end{pmatrix}$ (by observation)

$$\therefore -1x + 2y = c$$

As (0,3) is on the line: $-1(0) + 2(3) = c \quad \rightarrow \quad c = 6$

$$-x + 2y = 6$$



This extends to planes:

✎ Equation of plane:

$$n_1 x + n_2 y + n_3 z = c$$

The plane Π is perpendicular to the normal $\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and passes through the point P with position vector $8\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$.
Find the Cartesian equation of Π.

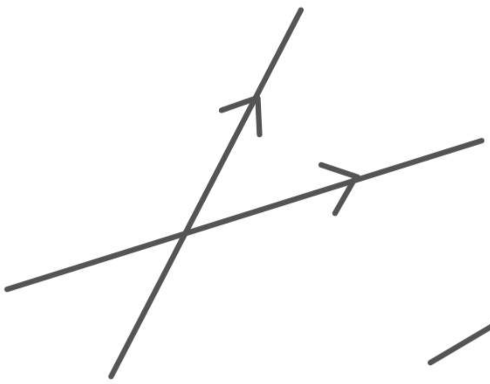
Ex 9B
2, 4, 5, 6, 9

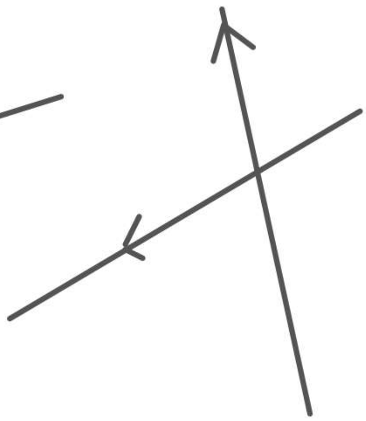
<u>Scalar Dot Product</u>

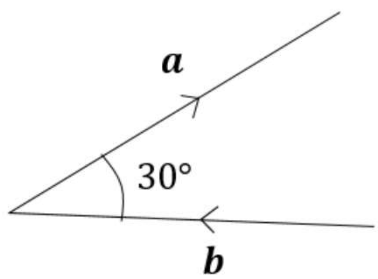
This is the scalar product of two vectors (not to be confused with the cross product, which gives a non-scalar answer)

The scalar dot product is defined as:

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$







$\cos\boxed{} = 0$ $\cos\boxed{} = 1$

Using this idea, what are the following equal to?

Hint: think about the angles between i, j, and k

$\mathbf{i}.\mathbf{i} =$	$\mathbf{j}.\mathbf{i} =$	$\mathbf{k}.\mathbf{i} =$	$2\mathbf{i}.4\mathbf{i} =$
$\mathbf{i}.\mathbf{j} =$	$\mathbf{j}.\mathbf{j} =$	$\mathbf{k}.\mathbf{j} =$	$3\mathbf{i}.7\mathbf{k} =$
$\mathbf{j}.\mathbf{k} =$	$\mathbf{j}.\mathbf{k} =$	$\mathbf{k}.\mathbf{k} =$	$2\mathbf{j}.4\mathbf{j} =$

✎ The scalar/dot product $\boldsymbol{a} \cdot \boldsymbol{b}$ of two vectors is the sum of the products of the components.

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum a_i b_i$$

$$\begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} a \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ b \\ 10 \end{pmatrix} =$$

Angle between vectors:

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Find the acute angle between the vectors $\boldsymbol{a} = \begin{pmatrix}5\\3\\1\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}1\\0\\5\end{pmatrix}$.

If $A(2,3,5)$, $B(5,0,4)$ and $C(4,-3,2)$, determine the angle ABC.

Hence find the area of triangle ABC.

If two vectors are perpendicular then:

$$\boldsymbol{a}\cdot\boldsymbol{b} = 0$$

Show that $\boldsymbol{a} = \begin{pmatrix}2\\3\\1\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$ are perpendicular.

Given that $\boldsymbol{a} = -2\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$, find a vector which is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Ex 9C
Q3, 5, 7, 9, 11, 13, 15

[June 2008 Q6] 6. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = (-9\mathbf{i} + 10\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$$
$$l_2 : \mathbf{r} = (3\mathbf{i} + \mathbf{j} + 17\mathbf{k}) + \mu(3\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$

where λ and μ are scalar parameters.

(*b*) Show that l_1 and l_2 are perpendicular to each other. **(2)**

[Jan 2012 Q7] 7. Relative to a fixed origin O, the point A has position vector $(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$,

the point B has position vector $(5\mathbf{i} + 2\mathbf{j} + 10\mathbf{k})$,

and the point D has position vector $(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$.

The line l passes through the points A and B.

(*a*) Find the vector $\overrightarrow{AB}$. **(2)**

(*b*) Find a vector equation for the line l. **(2)**

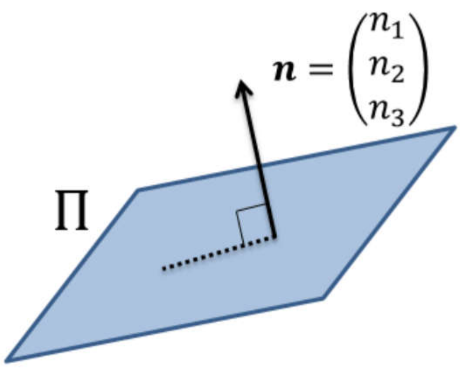
(*c*) Show that the size of the angle BAD is 109°, to the nearest degree. **(4)**

The points A, B and D, together with a point C, are the vertices of the parallelogram $ABCD$, where $\overrightarrow{AB} = \overrightarrow{DC}$.

(*d*) Find the position vector of C. **(2)**

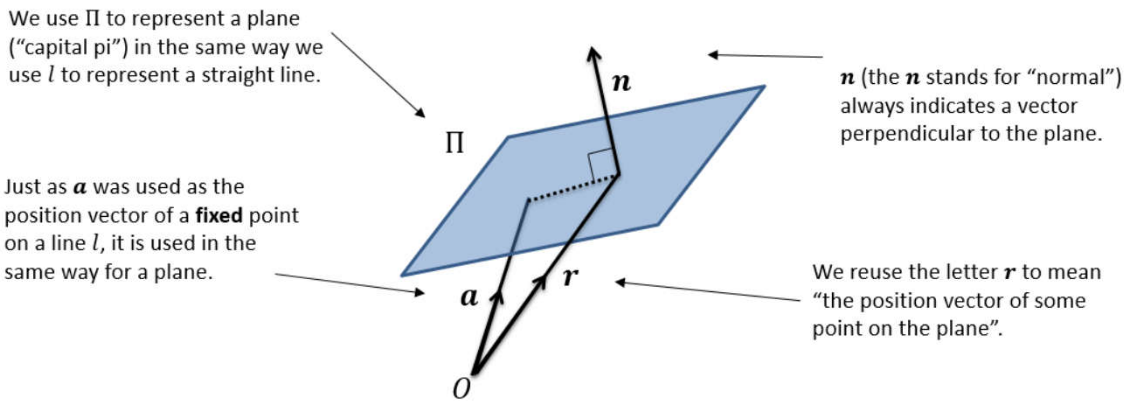
(*e*) Find the area of the parallelogram $ABCD$, giving your answer to 3 significant figures. **(3)**

Equation of Plane - scalar product form



Can we create a different equation of a plane, using our new knowledge about the scalar dot product?

Equation of Plane - scalar product form



It's important to realise here that $\boldsymbol{n}$ and $\boldsymbol{a}$ are **fixed** for a given plane (i.e. are constant vectors), whereas $\boldsymbol{r}$ can **vary** as it represents all the possible points on the plane.

How could we use the dot product to find some relationship between $\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{n}$?

A point with position vector $2\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$ lies on the plane and the vector $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ is perpendicular to the plane. Find the equation of the plane in:

a) Scalar product form.
b) Cartesian form.

✎ If $\boldsymbol{r} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = p$ is the scalar product equation of a plane, then the Cartesian form is:

$$n_1 x + n_2 y + n_3 z = p$$

Ex 9D
2ac
3
4

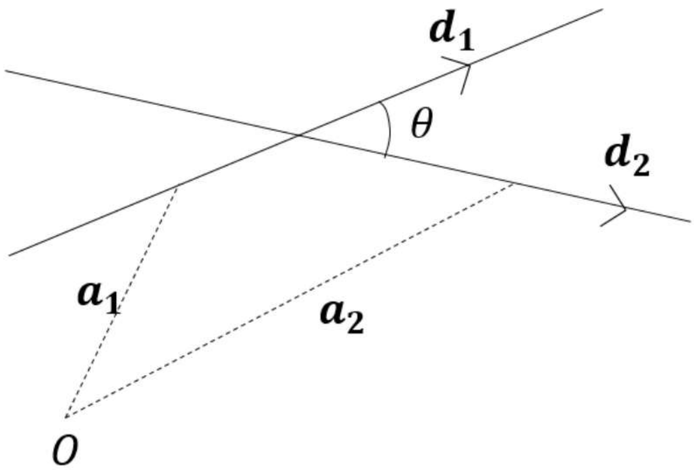
Angle between two lines
Angle between line and plane
Angle between two planes

Intersection of two lines
Intersection of line and plane
(Intersection of two planes)

Shortest distance between two parallel lines
Shortest distance between two skew lines (also in FP1)
Shortest distance between a point and a line
Shortest distance between a point and a plane
Shortest distance between two parallel planes

Reflecting a point in a plane
Reflecting a line in a plane

Angles between two lines



[Jan 2008 Q6] 6. The points A and B have position vectors $2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ respectively.

The line l_1 passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$. **(2)**

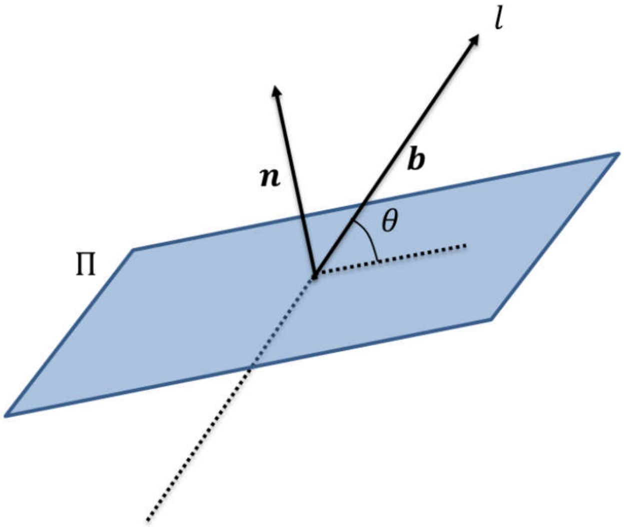
(b) Find a vector equation for the line l_1. **(2)**

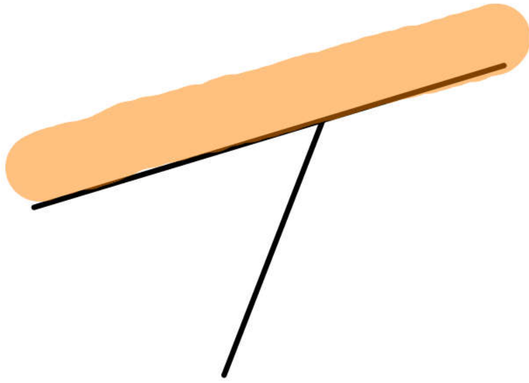
A second line l_2 passes through the origin and is parallel to the vector $\mathbf{i} + \mathbf{k}$. The line l_1 meets the line l_2 at the point C.

(c) Find the acute angle between l_1 and l_2. **(3)**

Ex 9D
1ace
9
10
11
14

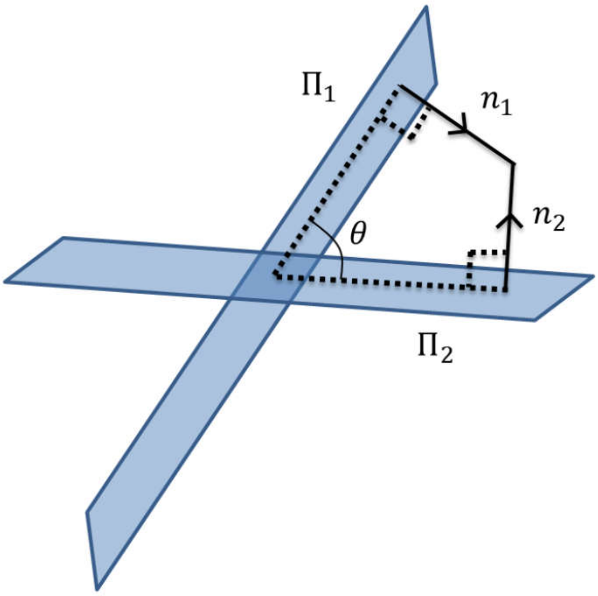
Angles between line and a plane

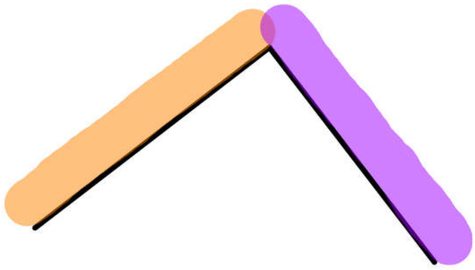




Example: Find the angle between the line $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane $2x + 3y - 7z = 5$.

Angles between two planes





Find the acute angle between the planes with Cartesian equations
$3x - 2y + 4z = 3$
and
$5x - 4y + 2z = 10$

Ex 9D
Q5, 6, 7, 8

Intersection of two lines

The lines l_1 and l_2 have vector equations
$\boldsymbol{r} = 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + \lambda(\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$ and
$\boldsymbol{r} = -2\boldsymbol{j} + 3\boldsymbol{k} + \mu(-5\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$ respectively.
Show that the two lines intersect, and find the position vector of the point of intersection.

The lines l_1 and l_2 have equations $\frac{x-2}{4} = \frac{y+3}{2} = z - 1$ and $\frac{x+1}{5} = \frac{y}{4} = \frac{z-4}{-2}$ respectively.
Prove that l_1 and l_2 are skew.

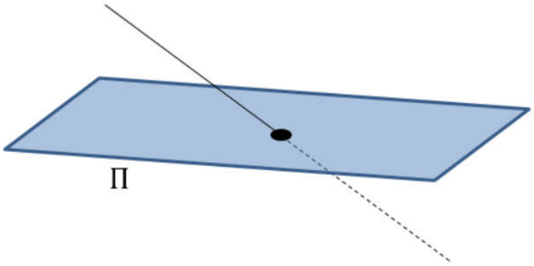
Terminology: Two straight lines are <u>skew</u> lines if they do not intersect and are not parallel

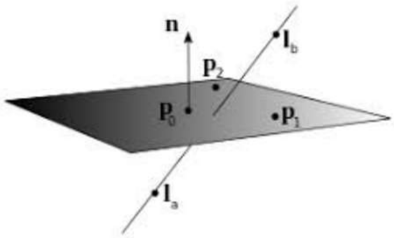
Intersection of line and a plane

Find the point of intersection of the line l and the plane Π where:

$$l: \quad \boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$$
$$\Pi: \quad \boldsymbol{r} \cdot (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) = 4$$

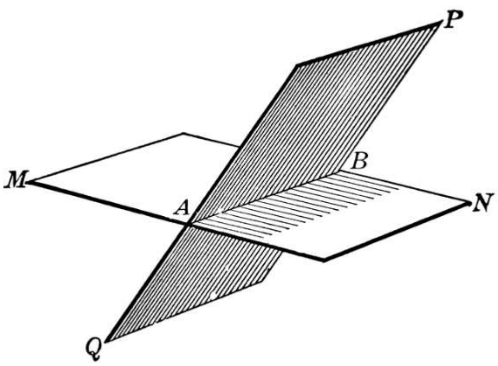
Plane MUST be in scalar dot form
Parametric is useless

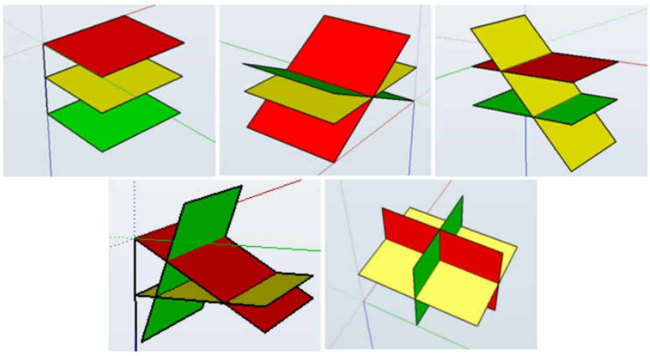




Ex 9E Evens

(Intersection of two planes)





We will return to this again in FP1

7. The plane Π_1 has equation

$$\mathbf{r}.(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = -8$$

~~(a) Find the perpendicular distance from the point (8, 2, 10) to Π_1~~ **(3)**

The plane Π_2 has equation

$$\mathbf{r} = \lambda(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$ is perpendicular to Π_2 **(2)**

(c) Find, to the nearest degree, the acute angle between Π_1 and Π_2 **(3)**

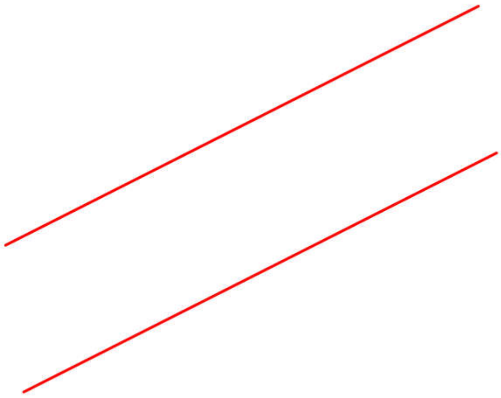
(d) Find a vector equation of the line of intersection of the planes Π_1 and Π_2 **(4)**

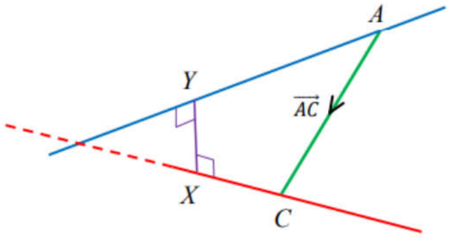
Find the equation of the line of intersection of the planes π_1 and π_2 .

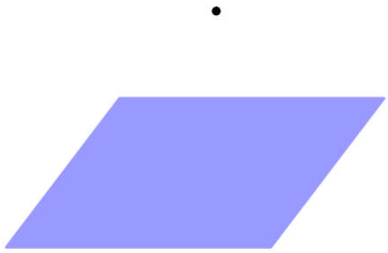
π_1 has the equation $2x - 2y - z = 2$

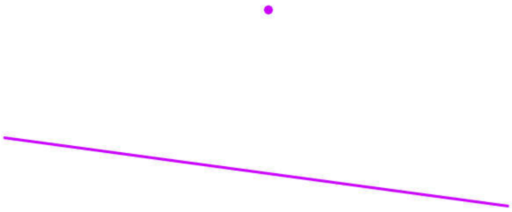
π_2 has the equation $r . (i - 3j + k) = 5$

Shortest distances

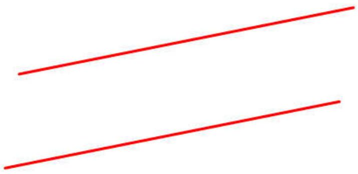








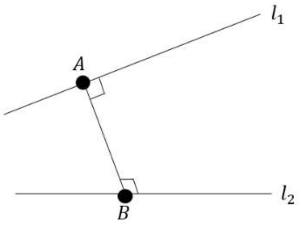
Shortest distance between two parallel lines



- find general point on l_1
- find general point on l_2
- find vector between them
- ensure that this vector is perp to l_1 (and l_2)

Show that the shortest distance between the parallel lines with equations:
$\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \lambda(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$ and $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{k} + \mu(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$,
where λ and μ are scalars, is $\frac{21\sqrt{2}}{10}$

Shortest distance between two skew lines (also in FP1)

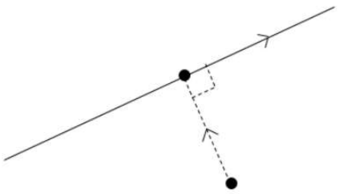


- find general point on l_1
- find general point on l_2
- find vector between them
- ensure that this vector is perp to l_1 **and l_2**

The lines l_1 and l_2 have equations $r = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $r = \begin{pmatrix}-1\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-1\end{pmatrix}$ respectively,

where λ and μ are scalars.
Find the shortest distance between these two lines.

Shortest distance between a point and a line



- find general point on l_1
- find vector between point and general point
- ensure that this vector is perp to l_1

The line l has equation $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z+3}{-1}$, and the point A has coordinates $(1,2,-1)$.
(a) Find the shortest distance between A and l.
(b) Find the Cartesian equation of the line that is perpendicular to l and passes through A.

Ex 9F Q1-4, 7, 9, 11

Shortest distance between a point and a plane

Vectors

Vector product: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{b}.(\mathbf{c} \times \mathbf{a}) = \mathbf{c}.(\mathbf{a} \times \mathbf{b})$$

If A is the point with position vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and the direction vector $\mathbf{b}$ is given by $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then the straight line through A with direction vector $\mathbf{b}$ has cartesian equation

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} (= \lambda)$$

The plane through A with normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ has cartesian equation

$n_1x + n_2y + n_3z + d = 0$ where $d = -\mathbf{a}.\mathbf{n}$

The plane through non-collinear points A, B and C has vector equation

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}) = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

The plane through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$ has equation

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

The perpendicular distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$ is $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$.

The perpendicular distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$ is $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$.

Find the perpendicular distance from the point with coordinates $(3,2,-1)$ to the plane with equation $2x - 3y + z = 5$.

[June 2013 Q8(R)] The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(*a*) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1.

(3)

Distance between (α, β, γ) and $n_1x + n_2y + n_3z + d = 0$

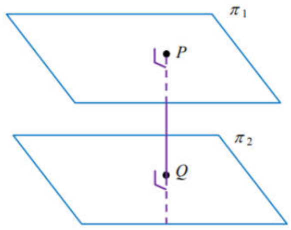
$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Shortest distance between two parallel planes

- find *any* point on the plane
- use the formula for shortest distance between point and plane

Example: Find the distance between the parallel planes

π_1: $2x - 6y + 3z = 9$ and π_2: $2x - 6y + 3z = 5$

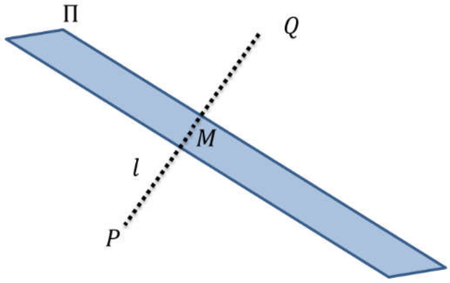


$$\left|\frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1^2 + n_2^2 + n_3^2}}\right|$$

Ex 9F Q5, 6, 10

Reflecting a point in a plane

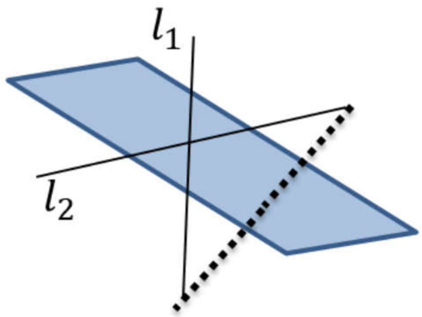
The plane Π has equation $r \cdot (i + 2j + 2k) = 5$. The point P has coordinates $(1,3,-2)$.
(a) Find the shortest distance between P and Π.
The point Q is the reflection of the point P in Π.
(b) Find the coordinates of point Q.



Reflecting a line in a plane

The line l_1 has equation $\frac{x-2}{2} = \frac{y-4}{-2} = \frac{z+6}{1}$. The plane Π has equation $2x - 3y + z = 8$. The line l_2 is the reflection of line l_1 in the plane Π. Find a vector equation of the line l_2.

The key here is that we need to reflect two points on the line through the plane, then find the equation of the line through these new points.



Ex 9F Q8, 12