Differentiating inverse trigonometric functions

Show that if $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\sin y = x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$x = \sin y$$

$$x^2 = \sin^2 y$$

$$x^2 = 1 - \cos^2 y$$

$$\cos^2 y = 1 - x^2$$

$$\cos y = \sqrt{1-x^2}$$

blah

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

use these!

Given that $y = \arcsin x^2$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^4}} \times 2x = \frac{2x}{\sqrt{1-x^4}}$$

$$y = \arccos(e^x + x^3)$$

$$\frac{dy}{dx} = \frac{-(e^x + 3x^2)}{\sqrt{1-(e^x+x^3)^2}}$$

Your Turn

Given that $y = \text{arcsec}\, 2x$,
show that $y = \frac{1}{x\sqrt{4x^2-1}}$

$y = \text{arcsec}\, 2x$

$\sec y = 2x$

$\sec y \tan y \frac{dy}{dx} = 2$

$\frac{dy}{dx} = \frac{2}{\sec y \tan y}$

$\sec y = 2x$

$\sec^2 y = 4x^2$

$1 + \tan^2 y = 4x^2$

$\tan y = \sqrt{4x^2 - 1}$

Hence $\frac{dy}{dx} = \frac{2}{2x\sqrt{4x^2-1}}$

$= \frac{1}{x\sqrt{4x^2-1}}$

Ex 3C

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$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

So because of these differentiation facts, what else do we know?

Differentiation

f(x)	**f′(x)**
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Integration (+ constant; $a > 0$ where relevant)

f(x)	$\int$ **f(x) dx**
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right), \quad \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right), \quad \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

Integrating with inverse trigonometric functions

Use an appropriate substitution to show that

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

$x = a\tan u$

$\frac{dx}{du} = a\sec^2 u$

$dx = a\sec^2 u \, du$

$\frac{x}{a} = \tan u$

$\arctan\frac{x}{a} = u$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2+a^2\tan^2 u} \times a\sec^2 u \, du$$

$$= \int \frac{1}{a^2(1+\tan^2 u)} \times a\sec^2 u \, du$$

$$= \int \frac{a\sec^2 u}{a^2\sec^2 u}\, du$$

$$= \int \frac{1}{a}\, du$$

$$= \frac{1}{a}u + C$$

$$= \underline{\frac{1}{a}\arctan\frac{x}{a} + C}.$$

$x = a\sin u$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$ $(\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2-a^2}\}$ $(x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2+a^2}\}$

$x = a\tan u$

$x = a\cosh u$

$x = a\sinh u$

Find $\int \frac{4}{5+x^2}\,dx$

$$\int \frac{4}{5+x^2}\,dx = 4\int \frac{1}{5+x^2}\,dx$$

$a^2 = 5$

$a = \sqrt{5}$

$$= \frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + c$$

Find $\int \frac{1}{25+9x^2}\,dx$

$$\int \frac{1}{25+9x^2}\,dx = \int \frac{1}{9\left(\frac{25}{9}+x^2\right)}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\frac{25}{9}+x^2}\,dx$$

$a^2 = \frac{25}{9}$

$a = \frac{5}{3}$

$$= \frac{1}{9} \times \frac{3}{5}\arctan\left(\frac{3x}{5}\right) + c$$

$$= \frac{1}{15}\arctan\left(\frac{3x}{5}\right) + c$$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\}$	$(x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$	

Find $\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx$

$$\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx = \int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{4}\sqrt{\frac{3}{4}-x^2}}$$

$a^2 = \frac{3}{4}$

$a = \frac{\sqrt{3}}{2}$

$$= \frac{1}{2}\int_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{\frac{3}{4}-x^2}}$$

$$= \frac{1}{2}\left[\arcsin\left(\frac{2x}{\sqrt{3}}\right)\right]_{-\frac{\sqrt{3}}{4}}^{\frac{\sqrt{3}}{4}}$$

$$= \frac{1}{2}\left(\arcsin\frac{1}{2} - \text{arc}\left(-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - -\frac{\pi}{6}\right)$$

$$= \frac{\pi}{6}$$

Find $\int \frac{x+4}{\sqrt{1-4x^2}}\,dx$

$$\int \frac{x+4}{\sqrt{1-4x^2}}\,dx = \int \frac{x}{\sqrt{1-4x^2}}\,dx + \int \frac{4}{\sqrt{1-4x^2}}\,dx$$

$x(1-4x^2)^{-1/2}$

$$= -\frac{1}{4}(1-4x^2)^{1/2} + \frac{4}{\sqrt{4}}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx$$

$a^2 = \frac{1}{4}$

$a = \frac{1}{2}$

$$= -\frac{1}{4}(1-4x^2)^{1/2} + 2\arcsin 2x + c$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + 2\arcsin 2x + c.$$

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$ $(\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2-a^2}\}$ $(x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2+a^2}\}$

Ex 3D Even