Series - Chapter 3, Core Pure 1

Factorising Techniques

$ac + ad + bc + bd$

$x^2 + x + ax + a$

$c^3 - 4c^2 - c + 4$

$(2n + 4)(n - 1) - (n - 3)(n - 1)$

1. $xy + x + y + 1 =$

2. $2(x + 1) - x(x + 1) =$

3. $x^2 + xy + 2x + 2y =$

4. $x^2 - 2y + 2x - xy =$

5. $xy + 2x - y^2 - 2y =$

6. $(3x - 2)(x + 1) - (x + 1)(2x - 3) =$

7. $x^3 + x^2 - x - 1 =$

8. $5x^3 + x^2 - 20x - 4 =$

9. $3x^2 + 8x + 4 =$

10. $6x^2 + x - 1 =$

Factorise fully:

$(k + 1) + (k + 1)(k + 2)$

$k^2(2k - 1) + 10k - 5$

$2(k + 1)^3 + k^2(k + 1)^2 - (k + 1)^2$

Can you come up with another question that uses a similar idea to these ones?

Sigma Notation

$$\sum_{r=1}^{3}(5r + 4) =$$

$$\sum_{r=3}^{7}(r^2 - r + 1) =$$

$$\sum_{r=1}^{n} r^2 =$$

Can you express these series using sigma notation?

a) $1 + 2 + 3 + 4 + 5 + 6$

b) $7 + 8 + 9 + 10 + 11 + 12$

c) $1 + 4 + 9 + 16 + 25 + 36$

d) $5 + 8 + 11 + 14 + 17 + 20 + 23 + 26$

e) $25 + 15 + 5 - 5$

Sums of 'ones'

$$\sum_{r=1}^{n} 1$$

$$\sum_{r=1}^{n} 5$$

$$\sum_{r=1}^{n} k$$

Sums of integers

The sum of the first n natural numbers is:

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Why?

1

1 + 2

1 + 2 + 3

1 + 2 + 3 + 4

1 + 2 + 3 + 4 + 5

1 + 2 + 3 + 4 + 5 + 6

1 + 2 + 3 + 4 + 5 + 6 + 7

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Why does this formula work?

Gauss worked this out at primary school!

In your head if you can...

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

1 + 2 + 3 + ... + 10 =

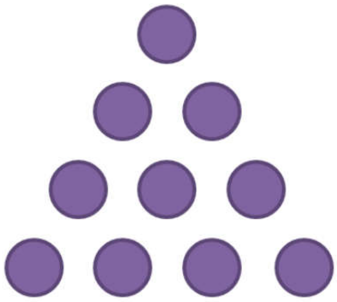
1 + 2 + 3 + ... + 99 =

11 + 12 + 13 + ... + 20 =

Sum up to 20, but get rid of everything up to 10:

$$\left(\frac{1}{2} \times 20 \times 21\right) - \left(\frac{1}{2} \times 10 \times 11\right)$$

100 + 101 + 102 + ... + 200 =



How are these two series different?

$$\sum_{r=1}^{50} r =$$

$$\sum_{r=21}^{50} r =$$

If the starting point is not 1, you will need to subtract everything BELOW the starting point

$$\sum_{r=5}^{2N-1} r = 2N^2 - N - 10, N \geq 3.$$

Show that $\displaystyle\sum_{r=n}^{3n} r = 2n(2n+1)$

Ex 3A
Q1, 2, 3, 4, 5, 6

Breaking Up Summations

How about if we wanted to find sums of more complicated series?

$$\sum_{r=1}^{13} 3r + 4 =$$

Show that:

$$\sum_{r=1}^{n+2} 4r - 6 = 2n(n+2)$$

Hence evaluate:

$$\sum_{r=10}^{102} 4r - 6 =$$

Sums of Squares and Cubes

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} k = kn$$

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Do you spot any relationship between r^2 and r^3?

Evaluate:

$$\sum_{r=13}^{24} r^2 =$$

$$\sum_{r=1}^{20} r^3 =$$

a) Show that: $$\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$$

b) Verify that the result is true for $n = 1$ and $n = 2$.

Ex 3B Q1-5

Challenge on the boards

Show that: $$\sum_{r=1}^{n}(r^2 + r - 2) = \frac{1}{3}n(n+4)(n-1)$$

Hence find the sum of the series

$$4 + 10 + 18 + 28 + 40 + \ldots + 418$$

Last example

Show that $$\sum_{r=1}^{n} r(r+3)(2r-1) = \frac{1}{6}n(n+1)(3n^2+an+b)$$

where a and b are integers to be found.

Hence calculate

$$\sum_{r=11}^{40} r(r+3)(2r-1).$$

Ex 3B Q 7 - 14

Equations with sigmas on both sides

Find the value of n that satisfies $$\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n+1} (9r+1)$$

Ex 3B Q13, 14
Mixed Ex 3 Q8-10

5. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n}(r+2)(r+3) = \frac{1}{3}n(n^2 + 9n + 26)$$

for all positive integers n.

(6)

(b) Hence show that

$$\sum_{r=n+1}^{3n}(r+2)(r+3) = \frac{2}{3}n(an^2 + bn + c)$$

where a, b and c are integers to be found.

(4)

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Harder

(*a*) Use the results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$, to prove that

$$\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n+1)(n+2)(n+7)$$

for all positive integers n. **(5)**

(*b*) Hence, or otherwise, find the value of

$$\sum_{r=20}^{50} r(r+1)(r+5).$$

(2)

6. (a) Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n,

$$\sum_{r=1}^{n} r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9)$$

(4)

(c) Hence find the value of n that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2$$

(5)

4. A company manufacturing radios agreed a 20 year contract with a retailer to supply its radios.
In the first year of the contract, 500 radios were supplied to the retailer.
In each subsequent year, the number of radios supplied to the retailer was 50 more than in the previous year.

The amount received by the company for each radio during year n of the contract was £$\left(20 + \frac{n^2}{45}\right)$

The total cost of producing the radios during year n was modelled as £$(1000 + 10n^2)$

(a) Show that, according to the model, the profit made by the company in year n, £P_n, is given by

$$P_n = \frac{10}{9}(n^3 + 900n + 7200)$$

(2)

(b) Use the standard results for summations to show that the total profit made by the company in the first N years of the contract, £T_N, is given by

$$T_N = aN(N^3 + bN^2 + cN + d)$$

where a, b, c and d are constants to be found.

(5)

At the end of the 20 years, the company found that its total profit made from this contract just exceeded £500 000.

(c) Assess the model in light of this information.

(2)