9 With respect to a fixed origin O the lines l_1 and l_2 are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix}8\\2\\-12\end{pmatrix} + \lambda\begin{pmatrix}-1\\3\\2\end{pmatrix} \qquad l_2: \mathbf{r} = \begin{pmatrix}-4\\10\\p\end{pmatrix} + \mu\begin{pmatrix}q\\2\\-1\end{pmatrix}$$

where λ and μ are parameters and p and q are constants. Given that l_1 and l_2 are perpendicular,

a show that $q = 4$. **(2 marks)**

Given further that l_1 and l_2 intersect, find:

b the value of p **(6 marks)**

c the coordinates of the point of intersection. **(2 marks)**

The point A lies on l_1 and has position vector $\begin{pmatrix}9\\-1\\-14\end{pmatrix}$. The point C lies on l_2.

Given that a circle, with centre C, cuts the line l_1 at the points A and B,

d find the position vector of B. **(3 marks)**

Problem-solving

Draw a diagram showing the lines l_1 and l_2 and the circle, and use circle properties.

a) $\begin{pmatrix}-1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}q\\2\\-1\end{pmatrix} = 0$

$-q + 6 - 2 = 0$

$q = 4$

b) $\begin{pmatrix}8-\lambda\\2+3\lambda\\-12+2\lambda\end{pmatrix} = \begin{pmatrix}-4+4\mu\\10+2\mu\\p-\mu\end{pmatrix}$

$8-\lambda = -4+4\mu$ $\quad \mu = 2, \lambda = 4$

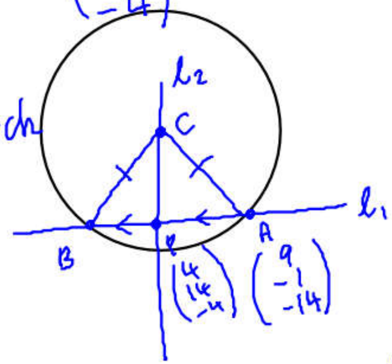
$2+3\lambda = 10+2\mu$

$-12+8 = p-2$

$-4+2 = p, \; p = -2$

c) $\lambda = 4$ $\quad \begin{pmatrix}4\\14\\-4\end{pmatrix}$

d) Sketch



$\vec{AP} = \vec{PB}$

$\underline{p} - \underline{a} = \underline{b} - \underline{p}$

$2\underline{p} - \underline{a} = \underline{b}$

$2\begin{pmatrix}4\\14\\-4\end{pmatrix} - \begin{pmatrix}9\\-1\\-14\end{pmatrix} = \underline{b}$

$\begin{pmatrix}-1\\29\\6\end{pmatrix} = \underline{b}$

11 The line l has a Cartesian equation $\frac{x-3}{5}=\frac{y+2}{3}=\frac{4-z}{1}$

The plane Π has Cartesian equation $4x+3y-2z=-10$.

The line intersects the plane at the point P.

a Find the position vector of P. **(5 marks)**

b Find the acute angle between the line and the plane at the point of intersection. **(5 marks)**

$\frac{4-z}{1}\quad \frac{x-1}{x-1} = \frac{z-4}{-1}$

$l\quad \underline{r} = \begin{pmatrix}3\\-2\\4\end{pmatrix} + \lambda\begin{pmatrix}5\\3\\-1\end{pmatrix}$

$= \begin{pmatrix}3+5\lambda\\-2+3\lambda\\4-\lambda\end{pmatrix}$

$\Pi\quad \underline{r}\cdot\begin{pmatrix}4\\3\\-2\end{pmatrix} = -10$

$\begin{pmatrix}3+5\lambda\\-2+3\lambda\\4-\lambda\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-2\end{pmatrix} = 10$

$12+20\lambda-6+9\lambda-8+2\lambda=-10$

$31\lambda-2=-10$

$31\lambda=-8$

$\lambda=-\frac{8}{31}$

$\underline{p} = \begin{pmatrix}3+5\times-\frac{8}{31}\\-2+3\times-\frac{8}{31}\\4--\frac{8}{31}\end{pmatrix} = \begin{pmatrix}53/31\\-86/31\\132/31\end{pmatrix}$

b) $\sin\theta = \left|\frac{\underline{d}\cdot\underline{n}}{|\underline{d}||\underline{n}|}\right|$

$= \frac{\begin{pmatrix}5\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-2\end{pmatrix}}{\sqrt{35}\sqrt{29}} = \frac{31}{\sqrt{35}\sqrt{29}}$

$\theta = \underline{76.7^\circ}$.

Find the equation of the line of intersection of the planes π_1 and π_2 .

π_1 has the equation $2x - 2y - z = 2$
π_2 has the equation $r \cdot (i - 3j + k) = 5$

$$2x - 2y - z = 2$$
$$x - 3y + z = 5$$

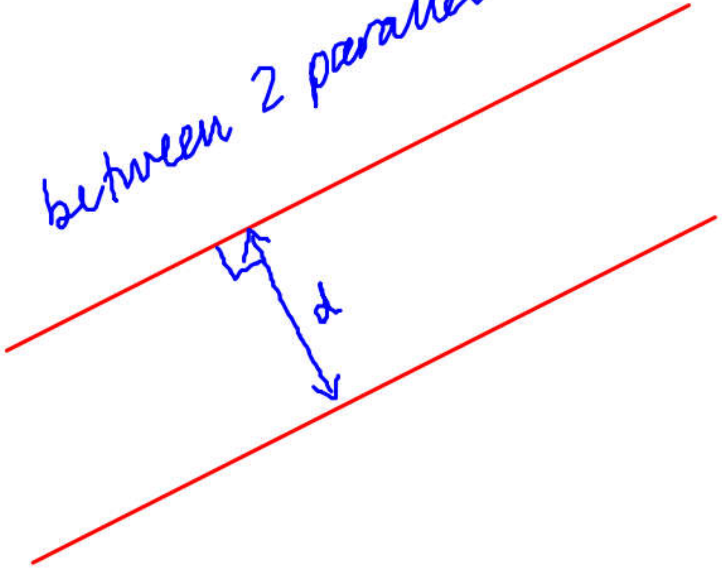
2 common points: $(4, 1, 4)$ and $(-1, -2, 0)$

$$\underline{d} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

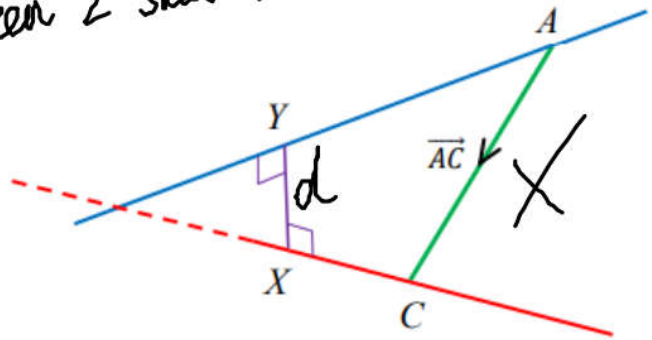
$$\underline{r} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

Shortest distances

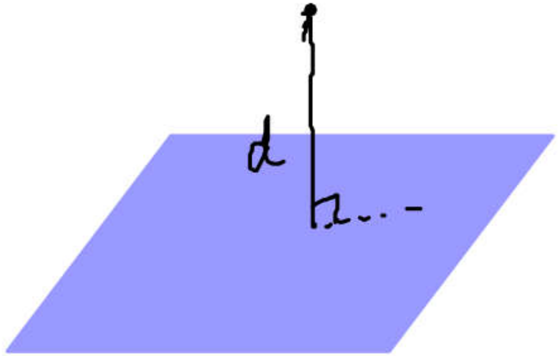
between 2 parallel lines



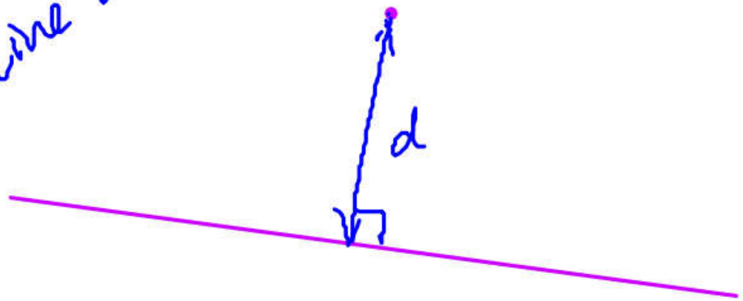
between 2 skew lines



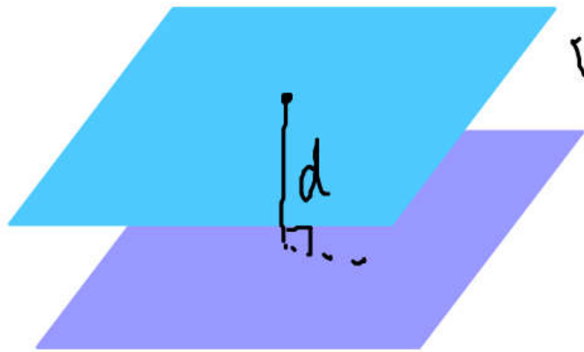
plane and a point.



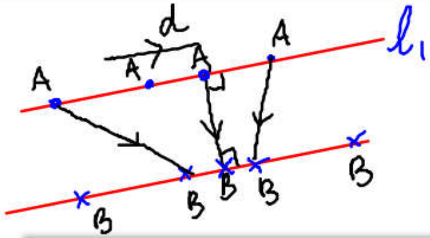
line and a point



plane and plane which are parallel



Shortest distance between two parallel lines



- find general point on l_1 $\underset{\sim}{a}$
- find general point on l_2 $\underset{\sim}{b}$
- find vector between them $\overrightarrow{AB}$
- ensure that this vector is perp to l_1 (and l_2)

Show that the shortest distance between the parallel lines with equations:
$\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \lambda(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$ and $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{k} + \mu(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$,
where λ and μ are scalars, is $\frac{21\sqrt{2}}{10}$

$$\underset{\sim}{r} = \begin{pmatrix} 1 + 5\lambda \\ 2 + 4\lambda \\ -1 + 3\lambda \end{pmatrix} = \underset{\sim}{a} \qquad \underset{\sim}{r} = \begin{pmatrix} 2 + 5\mu \\ 4\mu \\ 1 + 3\mu \end{pmatrix} = \underset{\sim}{b}$$

general point on l_1 general point on l_2

"Trick"

Let $t = \mu - \lambda$

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} 2 + 5\mu & -1 - 5\lambda \\ 4\mu & -2 - 4\lambda \\ 1 + 3\mu & +1 - 3\lambda \end{pmatrix} = \begin{pmatrix} 1 + 5\mu - 5\lambda \\ -2 + 4\mu - 4\lambda \\ 2 + 3\mu - 3\lambda \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1 + 5t \\ -2 + 4t \\ 2 + 3t \end{pmatrix}$$

$\overrightarrow{AB}$ and $\underset{\sim}{d}$ are perp.

$$\overrightarrow{AB} \cdot \underset{\sim}{d} = 0 \qquad \begin{pmatrix} 1 + 5t \\ -2 + 4t \\ 2 + 3t \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = 0$$

$$5 + 25t - 8 + 16t + 6 + 9t = 0$$

$$50t = -3$$

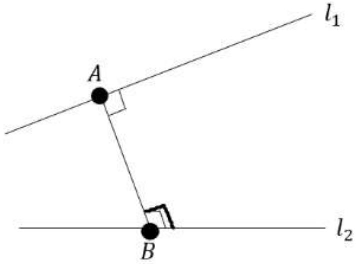
$$t = -\frac{3}{50}$$

$$\overrightarrow{AB} = \begin{pmatrix} 7/10 \\ -56/25 \\ 91/50 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 2.24 \\ 1.82 \end{pmatrix}$$

$$AB = \sqrt{0.7^2 + 2.24^2 + 1.82^2}$$

$$= \frac{21\sqrt{2}}{10}$$

Shortest distance between two skew lines (also in FP1)



- find general point on l_1 $\underline{a}$
- find general point on l_2 $\underline{b}$
- find vector between them $\overrightarrow{AB}$
- ensure that this vector is perp to l_1 **and l_2**

The lines l_1 and l_2 have equations $r = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $r = \begin{pmatrix}-1\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-1\end{pmatrix}$ respectively,

where λ and μ are scalars.
Find the shortest distance between these two lines.

general point on l_1

$$\underline{a} = \begin{pmatrix}1\\ \lambda\\ \lambda\end{pmatrix}$$

general point on l_2

$$\underline{b} = \begin{pmatrix}-1+2\mu\\ 3-\mu\\ -1-\mu\end{pmatrix}$$

vector between the lines

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}-1+2\mu-1\\ 3-\mu-\lambda\\ -1-\mu-\lambda\end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix}-2+2\mu\\ 3-\mu-\lambda\\ -1-\mu-\lambda\end{pmatrix}$$

$\overrightarrow{AB}$ is perp to l_1

$$\overrightarrow{AB}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} = 0$$

$$\begin{pmatrix}-2+2\mu\\ 3-\mu-\lambda\\ -1-\mu-\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} = 0$$

$$3-\mu-\lambda-1-\mu-\lambda = 0$$

$$-2\mu-2\lambda = -2$$

$\overrightarrow{AB}$ is perp to l_2

$$\overrightarrow{AB}\cdot\begin{pmatrix}2\\-1\\-1\end{pmatrix} = 0$$

$$-4+4\mu-3+\mu+\lambda+1+\mu+\lambda = 0$$

$$6\mu+2\lambda = 6$$

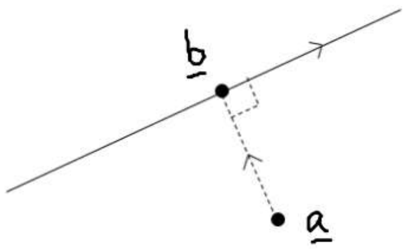
sim. eq. $\mu = 1,\ \lambda = 0$

$$\overrightarrow{AB} = \begin{pmatrix}0\\2\\-2\end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{2^2+2^2}$$

$$= \underline{\underline{2\sqrt{2}}}$$

Shortest distance between a point and a line



- find general point on l_1
- find vector between point and general point $\vec{AB} = \underline{b} - \underline{a}$
- ensure that this vector is perp to l_1

The line l has equation $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z+3}{-1}$, and the point A has coordinates $(1,2,-1)$.

(a) Find the shortest distance between A and l.

(b) Find the Cartesian equation of the line that is perpendicular to l and passes through A.

a)

$l \quad \underline{r} = \begin{pmatrix} 1+2\lambda \\ 1-2\lambda \\ -3-\lambda \end{pmatrix} = \underline{b}$

general point on the line

$\underline{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

fixed point.

$\vec{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 1+2\lambda-1 \\ 1-2\lambda-2 \\ -3-\lambda+1 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -1-2\lambda \\ -2-\lambda \end{pmatrix}$

$\vec{AB}$ and line are perpendicular

$\vec{AB} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 0$

$\begin{pmatrix} 2\lambda \\ -1-2\lambda \\ -2-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 0$

$4\lambda + 2 + 4\lambda + 2 + \lambda = 0$

$9\lambda = -4$

$\lambda = -\frac{4}{9}$

$\vec{AB} = \begin{pmatrix} -8/9 \\ -1/9 \\ -14/9 \end{pmatrix}$

$|\vec{AB}| = \sqrt{\left(\frac{8}{9}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(\frac{14}{9}\right)^2}$

$= \frac{\sqrt{29}}{3}$ units

b) direction is $\begin{pmatrix} 8 \\ 1 \\ 14 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 1 \\ 14 \end{pmatrix}$

$\frac{x-1}{8} = \frac{y-2}{1} = \frac{z+1}{14}$

Ex 9F Q1-4, 7, 9, 11