Differentiation: Year 1

When you differentiate a function you get its derivative

The derivative of a function tells you how it 'changes', often called its rate of change or its gradient

Notation

$\frac{d}{dx}$ means "take the derivative with respect to x"

If you "$\frac{d}{dx}$" y, you get $\frac{dy}{dx}$

Rates of Change

e.g. $\frac{dP}{dt}$ → population / → time in years

$\frac{dv}{dt}$ → velocity of a car / → time in seconds

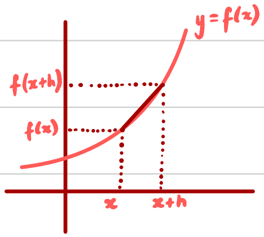
$\frac{dA}{dr}$ → area of circle / → radius of circle

$\frac{dy}{dx}$ → y-coordinate / → x-coordinate

In other words, how the top variable is changing in relation to / with respect to the bottom variable.

Gradient concept

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$



First Principles

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Example Problem

i) Find, from first principles, the derivative of $2x^3$ ii) Find the gradient of $y = 2x^3$ when $x = 1$

Let $f(x) = 2x^3$

$f(x+h) = 2(x+h)^3$

$$f'(x) = \lim_{h\to 0} \frac{2(x+h)^3 - 2x^3}{h}$$

$$= \lim_{h\to 0} \frac{2(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{2x^3}}{h}$$

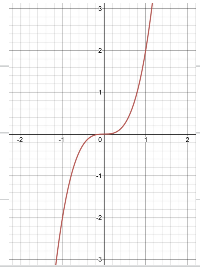
$$= \lim_{h\to 0} \frac{6x^2h + 6xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} (6x^2 + 6xh + h^2)$$

As $h \to 0$

$6xh, h^2 \to 0$

$$= \underline{\underline{6x^2}}$$



$$\frac{dy}{dx} = 6x^2 = \underline{\underline{6}}$$

Differentiating ax^n → where a and n can be any real number

$$\frac{d}{dx}(y) = \frac{dy}{dx} \qquad \frac{d}{dx}(ax^n) = anx^{n-1}$$

- "Pull" the power down
- Reduce the power by 1

This only works for terms exactly in the form ax^n

Note: constants "disappear"

Consider $3 = 3x^0 \rightarrow 3\times 0x^{-1} = 0$

e.g. Differentiate with respect to an appropriate variable

i) $y = x^5 + 7$

$\frac{dy}{dx} = 5x^4$

ii) $y = 3x^4 - 4$

$\frac{dy}{dx} = 12x^3$

iii) $y = 2x^{-4}$

$\frac{dy}{dx} = -8x^{-5}$

iv) $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

v) $f(x) = 4kx^2 - \frac{1}{2}x^{-3} + 2$

$f'(x) = 8kx + \frac{3}{2}x^{-4}$

vi) $f(t) = \sqrt[3]{t} = t^{1/3}$

$f'(t) = \frac{1}{3}t^{-2/3}$

vii) $x = (y+3)(2y-1)$

$x = 2y^2 + 5y - 3$

$\frac{dx}{dy} = 4y + 5$

viii) $x = \frac{\sqrt{t} + 4t^2}{2t^{1/3}}$

$\frac{t^{1/2}}{t^{1/3}}$

$x = \frac{1}{2}t^{1/6} + 2t^{5/3}$

$\frac{dx}{dt} = \frac{1}{12}t^{-5/6} + \frac{10}{3}t^{2/3}$

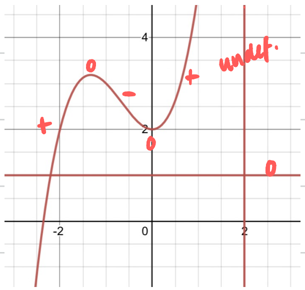
ix) $V = \frac{1 - 5x^2}{x^3} + 5$

$V = \frac{1}{x^3} - \frac{5}{x} + 5$

$V = x^{-3} - 5x^{-1} + 5$

$\frac{dV}{dx} = -3x^{-4} + 5x^{-2}$

Graphical Interpretation



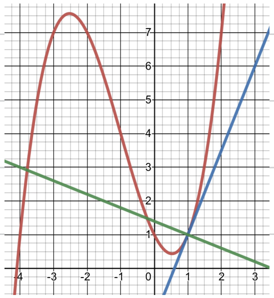
$\frac{dy}{dx} = 0$ — stationary

$\frac{dy}{dx} < 0$ \ decreasing function

$\frac{dy}{dx} > 0$ / increasing function

$\frac{dy}{dx}$ is undefined | vertical

Tangents and Normals



- Tangent just touches curve at one point
- Normal is perpendicular to curve at a point

Need 2 things

i) (x,y) coordinate

ii) $\frac{dy}{dx}$ at that point

$\Big\}$ use $y - y_1 = m(x - x_1)$

Example: Find the equations of the tangent and the normal to $y = \frac{1}{2}x^3 + \frac{3}{2}x^2 - 2x + 1$ at $x = 1$

$x = 1, \quad y = \frac{1}{2} + \frac{3}{2} - 2 + 1 = 1 \qquad (1, 1)$

$\frac{dy}{dx} = \frac{3}{2}x^2 + 3x - 2$, when $x = 1$, $\frac{dy}{dx} = \frac{3}{2} + 3 - 2 = \frac{5}{2}$

Tangent: $y - 1 = \frac{5}{2}(x - 1)$

Normal: $y - 1 = -\frac{2}{5}(x - 1)$

Second Derivatives (or third, etc)

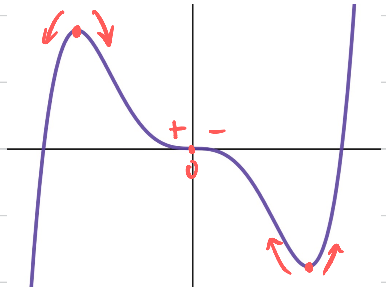
$\frac{d}{dx}\frac{d}{dx}y = \frac{d^2y}{dx^2}$

Function notation: $f''(x)$

Using the Second Derivative

The second derivative tells you how the gradient is changing.

If $\frac{d^2y}{dx^2} < 0$ the stationary point is a local maximum



If $\frac{d^2y}{dx^2} > 0$ the stationary point is a local minimum

If $\frac{d^2y}{dx^2} = 0$, investigate further! Could be max, min, or point of inflection. Check value of $\frac{dy}{dx}$ either side of the stationary point

Optimisation Problems

Differentiation can be used to maximise or minimise any function

Exponential Functions

If $f(x) = e^x$, $f'(x) = e^x$

$f(x) = e^{kx}$, $f'(x) = ke^{kx}$

Example Problem

The cost, £C, of a chemical process is modelled by $C = \frac{200}{t} + \frac{1}{10}t + 12$ where t is the controlled temperature of the process, t°C.

Using differentiation, find the temperature which minimises the cost, the minimum cost, and confirm by differentiation it is a minimum.

$C = 200t^{-1} + \frac{1}{10}t + 12$

$\frac{dC}{dt} = -200t^{-2} + \frac{1}{10}$

$\frac{dC}{dt} = 0, \quad -\frac{200}{t^2} + \frac{1}{10} = 0$

$-\frac{200}{t^2} =$

$\frac{200}{t^2} = \frac{1}{10}$

$2000 = t^2$

$\underline{\underline{t = 44.7\text{°C}}}$

$C = \frac{200}{44.7\ldots} + \frac{1}{10} \times 44.7\ldots + 12$

$= \underline{\underline{£20.94}}$

$\frac{d^2C}{dt^2} = 400t^{-3}$ when $t = 44.7$

$\frac{d^2C}{dt^2} = 0.0044\ldots > 0$

Hence it is a $\underline{\underline{\text{minimum}}}$.

Differentiation: Year 2

... same ideas, but more complex functions

Standard Results

$f(x)$	$f'(x)$		$f(x)$	$f'(x)$
ax^n	anx^{n-1}		$\sin x$	$\cos x$
a^x	$\ln a \times a^x$		$\cos x$	$-\sin x$
e^x	e^x		$\tan x$	$\sec^2 x$
$\ln x$	$\frac{1}{x}$		$\sec x$	$\sec x \tan x$
Product uv	$uv' + vu'$	$\begin{matrix} u & & v \\ & \times & \\ u' & & v' \end{matrix}$	$\cot x$	$-\operatorname{cosec}^2 x$
Quotient $\frac{u}{v}$	$\frac{vu' - uv'}{v^2}$	$\begin{matrix} u & & v \\ & \propto & \\ u' & & v' \end{matrix}$	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Chain Rule (Long)

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

e.g. $y = (3x^2 + x)^5$

$u = 3x^2 + x$, $y = u^5$

$\frac{du}{dx} = 6x + 1$, $\frac{dy}{du} = 5u^4$

$$\frac{dy}{dx} = 5u^4(6x+1)$$

$$= 5(3x^2+x)^4(6x+1)$$

Chain Rule (Quick)

$fg(x) \rightarrow f'g(x) \times g'(x)$

$fgh(x) \rightarrow f'gh(x) \times g'h(x) \times h'(x)$

$f(\text{blah}) \rightarrow f'(\text{blah}) \times \text{blah}'$

e.g. $\ln \square \rightarrow \frac{1}{\square} \times \square' = \frac{\square'}{\square}$

e.g. $\sec \square \rightarrow \square' \sec \square \tan \square$

e.g. $e^{\square} \rightarrow \square' e^{\square}$

e.g. $a\square^n \rightarrow an\square^{n-1} \times \square'$

Parametric Differentiation

If $y = f(\theta)$ $x = g(\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

Implicit Differentiation

Differentiate as expected, but $\times \frac{dy}{dx}$

e.g. $\frac{d}{dx}(3y^4) = \frac{d}{dy}(3y^4)\frac{dy}{dx}$

$= 12y^3 \frac{dy}{dx}$

Examples

Differentiate with respect to an appropriate variable

a) $f(x) = 2^{x^2}$

$f'(x) = 2x \ln 2 \times 2^{x^2}$

b) $y = 3\ln x$

$\frac{dy}{dx} = 3 \times \frac{1}{x} = \frac{3}{x}$

c) $A = 2\sin 3\theta$

$\frac{dA}{d\theta} = 6\cos 3\theta$

d) $y = \ln(4x^2 + 2x + 3)$

$\frac{dy}{dx} = \frac{8x+2}{4x^2+2x+3}$

(Powers of trig)

e) $g(x) = 4\tan^3 x$

$g(x) = 4(\tan x)^3$

$g'(x) = 12\tan^2 x \sec^2 x$

f) $h(t) = 2\cos^4 t$

$h(t) = 2(\cos t)^4$

$h'(t) = -8\cos^3 t \sin t$

g) $v = (12t + \cos t)^2$

$\frac{dv}{dt} = 2(12t + \cos t)(12 - \sin t)$

h) $x = 4(5y^2 + e^y)^3$

$\frac{dx}{dy} = 12(5y^2 + e^y)^2(10y + e^y)$

i) $y = e^{\sin(x^2+4x)}$

$\frac{dy}{dx} = (2x+4)e^{\sin(x^2+4x)}\cos(x^2+4x)$

j) $y = \sin^2(3x^{\frac{1}{2}}) = (\sin(3x^{\frac{1}{2}}))^2$

$\frac{dy}{dx} = 2\sin(3x^{\frac{1}{2}})\cos(3x^{\frac{1}{2}}) \times \frac{3}{2}x^{-\frac{1}{2}}$

$= 3x^{-\frac{1}{2}}\sin(3x^{\frac{1}{2}})\cos(3x^{\frac{1}{2}})$

k) $f(x) = 3\cos 2x \sin 4x$

$\begin{matrix} u = 3\cos 2x & & v = \sin 4x \\ & \times & \\ u' = -6\sin 2x & & v' = 4\cos 4x \end{matrix}$

$f'(x) = 12\cos 2x \cos 4x - 6\sin 2x \sin 4x$

l) $y = e^{-x}\tan 2x$

$\begin{matrix} u = e^{-x} & & v = \tan 2x \\ & \times & \\ u' = -e^{-x} & & v' = 2\sec^2 2x \end{matrix}$

$\frac{dy}{dx} = 2e^{-x}\sec^2 2x - e^{-x}\tan 2x$

Find dy/dx

m) $y = \frac{\sin x}{x^2}$

$\begin{matrix} u = \sin x & & v = x^2 \\ & \propto & \\ u' = \cos x & & v' = 2x \end{matrix}$

$\frac{dy}{dx} = \frac{x^2\cos x - 2x\sin x}{x^4} = \frac{x\cos x - 2\sin x}{x^3}$

n) $x = 4\tan\theta$ $y = 2\sec\theta$

$\frac{dx}{d\theta} = 4\sec^2\theta$ $\frac{dy}{d\theta} = 2\sec\theta\tan\theta$

$\frac{dy}{dx} = \frac{2\cancel{\sec\theta}\tan\theta}{4\sec^{\cancel{2}}\theta} = \frac{1}{2}\tan\theta\cos\theta = \frac{1}{2}\sin\theta$

o) $x^2 + 3y^3 - 2xy^2 = 7$

$\begin{matrix} u = -2x & & v = y^2 \\ & \times & \\ u' = -2 & & v' = 2y\frac{dy}{dx} \end{matrix}$

$2x + 9y^2\frac{dy}{dx} - 2y^2 - 4xy\frac{dy}{dx} = 0$

$\frac{dy}{dx}(9y^2 - 4xy) = 2y^2 - 2x \longrightarrow \frac{dy}{dx} = \frac{2y^2 - 2x}{9y^2 - 4xy}$

Reciprocals

$\frac{dx}{dy} = 1 / \frac{dy}{dx}$

Connected Rates of Change

Rate means $\frac{d}{dt}$

Express what you are looking for in terms of other derivatives e.g. $\frac{dA}{dr} = \frac{dA}{dx}\frac{dx}{dr}$

eg. The volume of a cube is increasing at a constant rate of $5cm^3s^{-1}$. Assuming the solid remains a cube as it grows, find the rate of increase of the side length, x cm, when $x = 2cm$

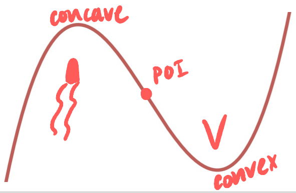
$V = x^3 \qquad \frac{dV}{dt} = 5$

$\frac{dV}{dx} = 3x^2 \qquad \frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt}$

$\frac{dx}{dV} = \frac{1}{3x^2} \qquad = \frac{1}{3x^2} \times 5$

When $x=2$, $\frac{dx}{dt} = \frac{5}{3 \times 2^2} = \frac{5}{12}\ cms^{-1}$

Second Derivatives



concave if $f''(x) \leq 0$

convex if $f''(x) = 0$

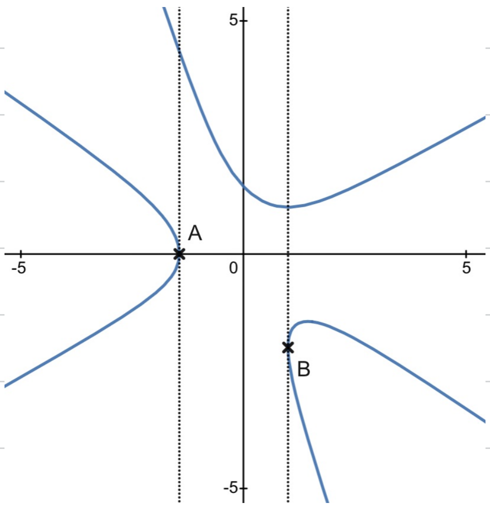
point of inflection if $f''(x) \geq 0$

Note: point of inflection $\Rightarrow f''(x) = 0$

but $f''(x) = 0 \not\Rightarrow$ point of inflection

Exam Style Problem

i)



The figure shows the curve with equation

$$y^3 + 3xy^2 - x^3 = 3$$

a) Find an expression for $\frac{dy}{dx}$ in terms of x and y

b) Vertical tangents have been drawn at A and B, as shown on the figure. Determine the coordinates A and B.

a) $y^3 + 3xy^2 - x^3 = 3$

$3y^2\frac{dy}{dx} + 3y^2 + 6xy\frac{dy}{dx} - 3x^2 = 0$

$\frac{dy}{dx}(3y^2 + 6xy) = 3x^2 - 3y^2$

$\frac{dy}{dx} = \frac{3x^2 - 3y^2}{3y^2 + 6xy} = \frac{x^2 - y^2}{y^2 + 2xy}$

b) A $\quad y = 0$

$0 + 0 - x^3 = 3$

$x^3 = -3$

$x = -\sqrt[3]{3}$

$A(-\sqrt[3]{3}, 0)$

b) cont. gradient at B is undefined

$\frac{dy}{dx}$ is undefined when denominator $= 0$

$y^2 + 2xy = 0$

$y(y + 2x) = 0$

$y = 0 \qquad y = -2x$

A

When $y = -2x$

$(-2x)^3 + 3x(-2x)^2 - x^3 = 3$

$-8x^3 + 12x^3 - x^3 = 3$

$3x^3 = 3$

$x^3 = 1$

$x = 1,\ y = -2 \quad B(1, -2)$