Chapter 6: Matrices

1:: Understand matrices and perform basic operations (adding, scalar multiplication)

2:: Multiply Matrices

"Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 3 \\ 4 & 5 \end{pmatrix}$, determine the matrix $\mathbf{AB}$."

3:: Find the determinant or inverse of a matrix.

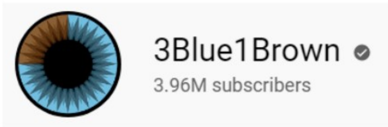
"If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$, determine $\mathbf{A}^{-1}$."

4:: Solve simultaneous equations using matrices.

"Use matrices to solve the following simultaneous equations:

$$x + 2y + z = 4$$
$$x - y + 3z = 1$$
$$2x + 5y - z = 0$$

Before studying this chapter, I **highly recommend** watching Chapters 1-8 of 3Blue1Brown's 'Essence of linear algebra' series. It will put everything we are learning about here into context, and will make Chapter 7 even easier.
No need to take notes and understand absolutely everything - but it is important to think about the ideas discussed.



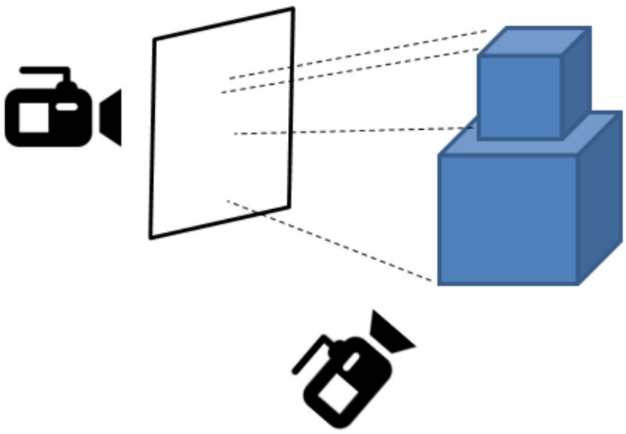
Grant Sanderson's channel is one of the best maths channels out there. Go check it out if you haven't already.

A matrix (plural: matrices) is **simply an 'array' of numbers**, e.g. $\begin{pmatrix} 1 & 0 & -2 \\ 3 & 3 & 0 \end{pmatrix}$

On a simple level, a matrix is just a way to organise values into rows and columns, and represent these multiple values as a single structure.

But the power of matrices comes from them **representing linear transformations/functions** (which we will particularly see in Chapter 7). We can

1. **Represent linear transformations** using matrices (e.g. rotations, reflections and enlargements)
2. Use them to **solve linear simultaneous equations.**



Matrices are particularly useful in 3D graphics, as matrices can be used to carry out rotations/enlargements (useful for changing the camera angle) or project into a 2D 'viewing' plane.

Matrix Fundamentals

#1 Dimensions of Matrices

The dimension of a matrix is its **size**, in terms of its number of **rows** and **columns** (in that order).

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$$

#2 Notation/Names for Matrices

A matrix can have square or curvy brackets (but the textbook only uses curvy)

$$\begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix} \qquad \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} \qquad \begin{pmatrix} 1 & 6 & 0 \end{pmatrix}$$

Matrix | Column Vector | Row Vector

A matrix with one column is **simply a vector in the usual sense**!

#3 Variables for Matrices

If the value of a variable is a matrix, we use **bold**, **capital** letters
(In contrast, vectors use bold, lowercase letters)

$$\mathbf{A} = \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

$$\mathbf{C} = \mathbf{P}^{-1}\mathbf{T}\mathbf{P}$$

#4 Adding/Subtracting Matrices

Simply add/subtract the corresponding elements of each matrix.
They must be of the same dimension.

$$\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 9 \\ 2 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} =$$

#5 Scalar Multiplication

A scalar is a number which can 'scale' the elements inside a matrix.

Side Note: You first encountered this at GCSE, in the context of vectors. $3\boldsymbol{a}$ is the vector $\boldsymbol{a}$ 'scaled' by the scalar 3.

$$3\begin{pmatrix} 1 & 3 & -7 \\ 4 & 0 & 5 \end{pmatrix} =$$

$$\boldsymbol{A} = \begin{pmatrix} q & -3 \\ 1 & 1 \\ -4 & 1 \end{pmatrix} \qquad 2\boldsymbol{A} =$$

$$\begin{pmatrix} -3 \\ k \end{pmatrix} + k\begin{pmatrix} 2k \\ 2k \end{pmatrix} = \begin{pmatrix} k \\ 6 \end{pmatrix}$$

#6 Special Matrices

A matrix is **square** if it has the same number of rows as columns.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 4 \\ 2 & 2 & 5 \\ -3 & 4 & 3 \end{pmatrix}$$

A **zero matrix** is one in which all its elements are 0. The dimensions are usually clear from the context.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A **identity matrix $\mathbf{I}$** is a square matrix which has 1's in the 'leading diagonal' (starting top-left) and 0 elsewhere. Again, the dimensions depend on the context.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We will see the significance of the identity matrix when we cover matrix multiplication imminently.

Ex 6A

Matrix Multiplication

#7 Matrix Multiplication

Matrix multiplications are not always valid: the dimensions have to agree.

Dimensions of $\mathbf{A}$	Dimension of $\mathbf{B}$	Dimensions of $\mathbf{AB}$ (if valid)
2×3	3×4	
1×3	2×3	
6×2	2×4	
1×3	3×1	
7×5	7×5	
10×10	10×9	
3×3	3×3	

Note that only **square matrices** (i.e. same width as height) can be raised to a power.

$$\begin{bmatrix} -3 & 5 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 \\ 1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5 \\ -3 & 1 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -4 & 4 \\ -2 & -4 \end{bmatrix}$$

Computing matrix multiplication on your calculator

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 8 & 4 & 3 \\ 7 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 7 \\ 0 & 3 \\ 8 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -y \\ -2x & -4 \end{bmatrix} \cdot \begin{bmatrix} -4x & 0 \\ 2y & -5 \end{bmatrix}$$

Your Turn

$\boxed{\text{a}}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} =$

$\boxed{\text{b}}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix} =$

$\boxed{\text{c}}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 =$

$\boxed{\text{d}}$ $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$

$\boxed{\text{e}}$ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} =$

$\boxed{\text{f}}$ $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^k =$

Matrix Multiplication involving $\mathbf{I}$

We earlier saw the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What do you notice about...

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

In general $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all matrices $\mathbf{A}$.

So the identity matrix is a bit like the '1' of matrix multiplication, e.g. $1 \times 3 = 3 \times 1 = 3$; multiplying by 1 has no effect, and multiplying by $\mathbf{I}$ has no effect.

For this reason, 1 is known as the 'identity' of multiplication over numbers. And 0 is known as the 'identity' of addition over numbers, given that $a + 0 = 0 + a = a$ for all a.

Matrix Multiplication:

commutative or noncommutative?

Let $\mathbf{A} = \begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix}$

Work out $\mathbf{AB}$

Work out $\mathbf{BA}$

What do you notice?
What does this tell us?

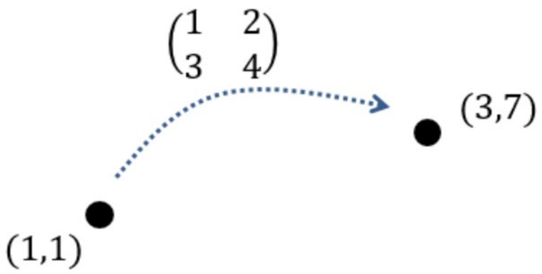
Note: Matrix multiplication is associative. This means that **A(BC) = (AB)C**

Ex 6B

Determinant of a matrix

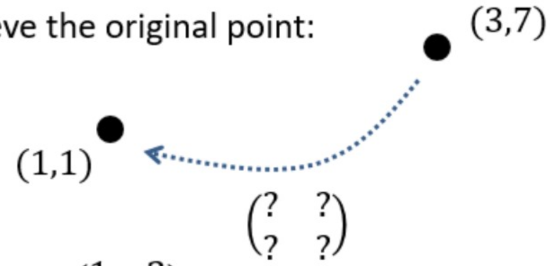
In Chapter 7, you will see that matrices can be thought of as a function that can transform a point, e.g.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$



A question might naturally be whether there is an 'inverse function/transformation' that can retrieve the original point:

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



This matrix would be known as the **inverse** of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

With quadratics, we used to the word '*discriminant*' for $b^2 - 4ac$ because it 'discriminates' between the different cases of 0, 1, 2 roots.
Analogously, the 'determinant' $|\boldsymbol{A}|$ or $\det(\boldsymbol{A})$ for a matrix $\boldsymbol{A}$ 'determines' whether it has an inverse of not.

Determinants of 2 x 2 matrices

✎ The determinant of a matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det(\mathbf{A}) = |\mathbf{A}| = ad - bc$$

✎ If $det(\mathbf{A}) = 0$, then $\mathbf{A}$ is a **singular matrix** and it does not have an inverse.
✎ If $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is a **non-singular matrix** and it has an inverse.

Quickfire Questions:

$\mathbf{A}$	$\det(\mathbf{A})$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	
$\begin{pmatrix} 0 & 3 \\ -1 & -4 \end{pmatrix}$	
$\begin{pmatrix} 10 & -2 \\ 4 & -1 \end{pmatrix}$	

$$A = \begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$$

Given that $\mathbf{A}$ is singular, find the value of p.

$$\mathbf{A} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}, \text{ where } a \text{ is real.}$$

(*a*) Find det $\mathbf{A}$ in terms of a.

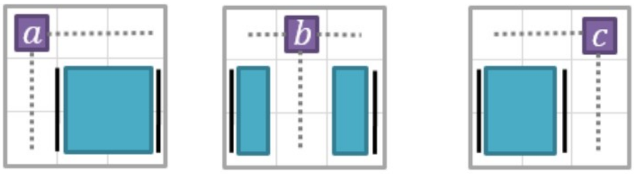
(2)

(*b*) Show that the matrix $\mathbf{A}$ is non-singular for all values of a.

(3)

Determinants of 3 x 3 matrices

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} =$$

$$\begin{vmatrix} 3 & 1 & 4 \\ 2 & 2 & 5 \\ -3 & 4 & 3 \end{vmatrix} =$$

$$\begin{vmatrix} 2 & 5 & 3 \\ 0 & -2 & -1 \\ 1 & 4 & 3 \end{vmatrix} =$$

Alternative method

$$\begin{vmatrix} 2 & 5 & 3 \\ 0 & -2 & -1 \\ 1 & 4 & 3 \end{vmatrix} =$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$$

Determine $\det(\mathbf{A})$.

Computing the determinant on your calculator

$$\boldsymbol{A} = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$$ where k is a constant.

Given that A is singular, find the possible values of k.

Minors

The **minor** of an element in a 3×3 matrix is the determinant of the 2×2 matrix that remains after the row and column containing that element have been crossed out.

Minor of 0:

$$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$$

Minor of -6:

Minor of 5:

Ex 6C

Inverting a 2 x 2 matrix

We earlier saw that the inverse of a matrix $\mathbf{M}$ (written $\mathbf{M}^{-1}$) 'undoes' the effect of the matrix. Thus:

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

as multiplying something by a matrix followed by its inverse has no overall effect (i.e. the same as the identity matrix).

✏ If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\boldsymbol{A}^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- $\mathbf{A}^{-1}$ is the 'inverse' of $\mathbf{A}$, so that if $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$, $\mathbf{A}^{-1}\boldsymbol{y} = \boldsymbol{x}$
- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Practising the Inverse

Divide by determinant.	Swap NW-SE elements.	Make SW-NE elements negative.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} =$$

Computing the inverse on your calculator

$$\begin{pmatrix} 7 & 2 \\ 1 & -3 \end{pmatrix}^{-1} =$$

For what value of p is $\begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$ singular?

Given p is not this value, find the inverse.

Matrix proofs involving inverses

If A and B are non-singular matrices such that $\mathbf{BAB} = \mathbf{I}$, prove that $\mathbf{A} = \mathbf{B}^{-1}\mathbf{B}^{-1}$

Tip: You can rid of a matrix $\mathbf{A}$ at the **front** of the expression by multiplying the **front** of each side of the equation by $\mathbf{A}^{-1}$ (to get $\mathbf{I}$). You can similarly remove an $\mathbf{A}$ at the **end** by multiplying the **end** of each side by $\mathbf{A}^{-1}$.

If $\mathbf{P}$ and $\mathbf{Q}$ are non-singular matrices, prove that $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$

Hint: Start with a simple statement of the form $\mathbf{AA}^{-1} = \mathbf{I}$

Matrix Transpose

A^T is the **transpose** of a matrix A, where the rows and columns are interchanged.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$. An $m \times n$ matrix becomes $n \times m$.

$$\begin{pmatrix} 7 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}^T$$

$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ -1 & 8 & 2 \end{pmatrix}$

Determine $\boldsymbol{A}^{\mathrm{T}}$

Finding the inverse of a 3 x 3 matrix

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find $\boldsymbol{A}^{-1}$.

The method we previously used was a specific case of **a more general method** which can be used for matrices of any size:

Step 1: Find $\det(\boldsymbol{A})$

Step 2: Form a matrix of minors, $\boldsymbol{M}$

Step 3: Form a matrix of cofactors, $\boldsymbol{C}$

Step 4: $A^{-1} = \frac{1}{\det(\boldsymbol{A})}\boldsymbol{C}^T$

If $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$, find $\boldsymbol{A}^{-1}$.

✎ Step 1: Find $\det(\boldsymbol{A})$

✎ Step 2: Form a matrix of minors, $\boldsymbol{M}$

✎ Step 3: Form a matrix of cofactors, $\boldsymbol{C}$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

✎ Step 4: $A^{-1} = \frac{1}{\det(\boldsymbol{A})}\boldsymbol{C}^T$

Computing the inverse on your calculator

$\boldsymbol{A} = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$, and the matrix $\mathbf{B}$ is such that $(\mathbf{AB})^{-1} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}$.

(a) Show that $\mathbf{A}^{-1} = \mathbf{A}$.

(b) Find $\boldsymbol{B}^{-1}$.

What can we do to avoid the long process of finding an inverse? Be smart!

... with algebra

[June 2011 Q7] The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} k & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}, \quad k \neq 1.$$

(*a*) Show that det $\mathbf{M} = 2 - 2k$. **(2)**

(*b*) Find $\mathbf{M}^{-1}$, in terms of k. **(5)**

Ex 6E

Using Matrices for Simultaneous Equations

✎ If $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{v}$ then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\boldsymbol{v}$

Use an inverse matrix to solve the simultaneous equations:

$$-x + 6y - 2z = 21$$
$$6x - 2y - z = -16$$
$$-2x + 3y + 5z = 24$$

Ex 6F
Q1

A colony of 1000 mole-rats is made up of adult males, adult females and youngsters. Originally there were 100 more adult females than adult males.

After one year:

- The number of adult males had increased by 2%
- The number of adult females had increased by 3%
- The number of youngsters had decreased by 4%
- The total number of mole-rats had decreased by 20

Form and solve a matrix equation to find out how many of each type of mole-rat were in the original colony.

Let $x =$ number of adult males
$y =$ number of adult females
$z =$ number of youngsters

Ex 6F Q3, 4
Mixed Ex Q12
Review Ex 2 Q11

Exam Questions - systems of equations

3. Tyler invested a total of £5000 across three different accounts; a savings account, a property bond account and a share dealing account.

Tyler invested £400 more in the property bond account than in the savings account.

After one year

- the savings account had increased in value by 1.5%
- the property bond account had increased in value by 3.5%
- the share dealing account had **decreased** in value by 2.5%

- the total value across Tyler's three accounts had increased by £79

Form and solve a matrix equation to find out how much money was invested by Tyler in each account.

(7)

2. A company runs three theme parks, A (Aztec Adventureland), B (Babylonian Towers) and C (Carthaginian Kingdom).

It is known that park A makes a profit of £30 per visitor, park B makes a profit of £26 per visitor and park C makes a profit of £33 per visitor.

In 2017 the Aztec Adventureland park was upgraded, which took one year to carry out. During 2017

- park A had only 50% of the number of visitors it had in 2016
- park B had 25% more than the number of visitors it had in 2016
- park C had 15% more than the number of visitors it had in 2016

In total 1 350 000 people visited the three theme parks during 2017.

The company made a total profit from the parks of £39.15 million in 2016. The profits dropped by 1% for 2017.

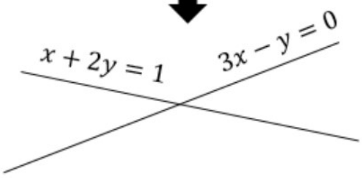
Form and solve a matrix equation to find, to 2 significant figures, the number of visitors for each of the theme parks in 2016.

(8)

Consistency of linear equations

From Pure Year 1 you are already familiar with the idea that the solution of a system of two equations (with two unknowns) can be visualised by finding the point of intersection of two lines. **A system of linear equations is known as consistent if there is at least one set of values that satisfies <u>all</u> the equations simultaneously (i.e. at least one point of intersection).**

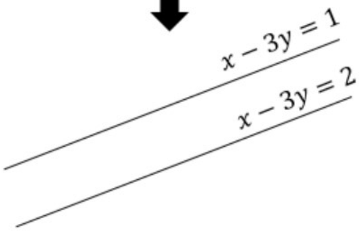
$$x + 2y = 1$$
$$3x - y = 0$$



System of equations is **consistent.** It has **one solution.**

The corresponding matrix $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ is **non-singular**.

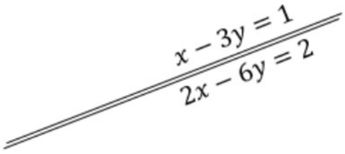
$$x - 3y = 1$$
$$x - 3y = 2$$



System of equations is **inconsistent.** It has **no solutions.**

Matrix $\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}$ is **singular.**

$$x - 3y = 1$$
$$2x - 6y = 2$$



System of equations is **consistent.** It has **infinitely many solutions.**

Matrix $\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}$ is **singular.**

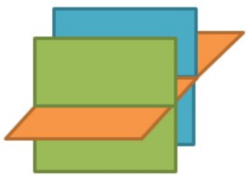
Extending consistency to 3 variables

In Chapter 9 you will learn that just as $ax + by = c$ gives the equation of a straight line, $ax + by + cz = d$ gives the equation of a plane.
Again, we get solutions to the system of linear equations **when <u>all</u> of the planes intersect:**

Scenario 1: Planes all meet at a single point. System of equations consistent, and one solution.

Scenario 2: Planes form a sheaf. They have a line of intersection consisting of infinitely many points. System of equations consistent and infinitely many solutions.

Scenario 3: Planes form a prism. While planes intersect in pairs, they don't all intersect at any point. System of equations is inconsistent.



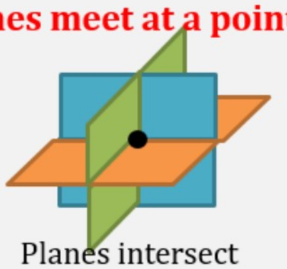
Scenario 4: Two of more planes parallel and non-identical. Again, inconsistent, as the parallel planes never intersect, and thus all equations can't be satisfied.

Any rows in the corresponding matrix which are multiples of each other will be parallel.

Scenario 5: Planes represented by equations are equivalent. System of equations consistent, and infinitely many solutions.

So, **CONSISTENT** means that there is at least one solution of all three equations – i.e. all the planes are intersecting. This could be:

Planes meet at a point

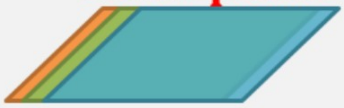


Planes intersect
Only thing that can happen if matrix is non-singular, i.e. det is not 0

A sheaf

Planes intersect along a line,
Infinitely many solutions

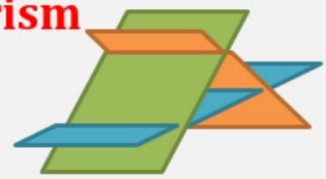
Same plane



All equations of are multiples of each other – they are the same equation.
Infinitely many solutions

So, **INCONSISTENT** means that there are NO SOLUTIONS for **<u>all three</u>** equations at the same time – i.e. they are not all intersecting together. This could be:

A prism



2 or all rows of matrix are multiples of each other

Parallel planes

Algebraic tests for consistency

Determine if the following sets of equations are consistent or inconsistent

$$x - 3y - 2z = 2$$
$$2x - 2y + 3z = 1$$
$$5x - 7y + 4z = 4$$

Create 2 new equations by eliminating x, y, or z.
Use 2 different pairs to create them.

$$\begin{aligned} x - y + 2z &= 3 \\ 4x + 2y - 2z &= 1 \\ x + 2y - 3z &= 1 \end{aligned}$$

$$\begin{aligned} 3x - y + 2z &= 1 \\ 4x + 2y + 3z &= 6 \\ 2x + y - 3z &= -2 \end{aligned}$$

Decide what geometric situation each set of equations represents.

1) Check what the determinant of the matrix is. If it is NOT zero, they meet at a point
2) Solve the equation on your calculator to decide if it has one solution (meet at a point), infinite solutions (sheaf or same plane) or no solutions (parallel planes or prism)
3) Use algebra to see if they are consistent or inconsistent

$$\begin{aligned} 3x + 4y + z &= 2 \\ 6x + 8y + 2z &= 4 \\ 9x + 12y + 3z &= 6 \end{aligned}$$

$$\begin{aligned} 4x + 3y - 2z &= 5 \\ 2x + 4y - 3z &= 8 \\ 8x + 6y - 4z &= 9 \end{aligned}$$

$$\begin{aligned} x + 2y - z &= 5 \\ 2x + 3y - 3z &= 18 \\ x + 5y + z &= 10 \end{aligned}$$

$$\begin{aligned} x - 2y + 3z &= -2 \\ 2x - 3y + 5z &= -3 \\ x + 3y - 2z &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y - 3z &= -2 \\ 2x - 3y + 5z &= -3 \\ x + 3y - 2z &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y - 11z &= -2 \\ 2x + 11y + 5z &= 11 \\ x + 3y - 2z &= -11 \end{aligned}$$

$$\begin{aligned} x - 2y + 3z &= -2 \\ 2x - 3y + 5z &= -3 \\ x + 3y - 2z &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y - 11z &= -2 \\ 2x + 11y + 5z &= 11 \\ x + 3y - 2z &= -11 \end{aligned}$$

A system of equations is shown below:

$$3x - ky - 6z = k$$
$$kx + 3y + 3z = 2$$
$$-3x - y + 3z = -2$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the plane corresponding to each value of k.

(a) $k = 0$ (b) $k = 1$ (c) $k = -6$

Ex 6F
remaining Qs

7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix $\mathbf{M}$ has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$\begin{aligned} 2x - y + z &= p \\ 3x - 6y + 4z &= 1 \\ 3x + 2y - z &= 0 \end{aligned}$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$\begin{aligned} 2x - y + z &= 1 \\ 3x - 5y + 4z &= q \\ 3x + 2y - z &= 0 \end{aligned}$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

Population Modelling with Matrices

7. The population of Zebu cattle in a particular country is modelled by two sub-populations, adults and juveniles. In this model, the only factors affecting the population of the Zebu are the birth and survival rates of the population.

Data recorded in the years preceding 2018 was used to suggest the annual birth and survival rates of the population.
The results are shown in the table below, with values to 2 significant figures. It is assumed that these rates will remain the same in future years.

	Birth rate	Survival rate
Adult population	0.23	0.97
Juvenile population	0	0.87

It is also assumed that $\frac{1}{3}$ of the surviving juvenile population become adults each year.
Let A_n and J_n be the respective sub-populations, in millions, of adults and juveniles, n years after 1st January 2018. Then the adult population in year $n + 1$ satisfies the equation

$$A_{n+1} = 0.97A_n + \frac{1}{3}(0.87)J_n = 0.97A_n + 0.29J_n$$

(a) Form the corresponding equation for the juvenile population in year $n + 1$ under this model, justifying your values.

(2)

The total population on 1st January 2018 was estimated, to 2 significant figures, as 1.5 million Zebu, with 1.2 million of these being adults.

(b) Find the value of p and the matrix $\mathbf{M}$ such that the population of Zebu can be modelled by the system

$$\begin{pmatrix} A_0 \\ J_0 \end{pmatrix} = \begin{pmatrix} 1.2 \\ p \end{pmatrix} \qquad \begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = \mathbf{M}\begin{pmatrix} A_n \\ J_n \end{pmatrix},$$

giving p to 2 significant figures and each entry of $\mathbf{M}$ to 2 decimal places.

(3)

(c) Using the model formed in part (b), find, to 3 significant figures,

(i) the **total** Zebu population that was present on 1st January 2017,

(ii) the predicted **juvenile** Zebu population on 1st January 2025.

(5)

As a result of the predictions of this model the country will export 15 000 juveniles to a neighbouring country at the end of each year.

(d) Adapt the model from 2018 onwards to include this export.

(2)

(e) State one limitation of this model.

(1)

10. The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0, 1, 2, \ldots$$

where a is a constant, and J_n and A_n are the respective numbers of juvenile and adult chimpanzees n years after the start of the study.

(a) Interpret the meaning of the constant a in the context of the model.

(1)

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

(b) (i) Find, in terms of a

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

(3)

(ii) Hence, or otherwise, find the value of a.

(3)

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

(2)

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

(1)

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear*.)

(2)