Solving using partial fractions

Prove that $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

$$\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}$$

$$1 = A(a+x) + B(a-x)$$

$x = -a$

$1 = 2aB$

$B = \frac{1}{2a}$

$x = a$

$1 = 2aA$

$\frac{1}{2a} = A$

$\frac{1}{a^2 - x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad (\|x\| < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|$

$$\int \frac{1}{a^2-x^2}\,dx = \int\left(\frac{1/2a}{a-x} + \frac{1/2a}{a+x}\right)dx$$

$$= \frac{1}{2a}\int\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx$$

$$= \frac{1}{2a}\left(-\ln|a-x| + \ln|a+x|\right) + c$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

Partial Fractions involving Quadratic Factors

When you write as partial fractions, ensure you have the **most general possible non-top heavy fraction**, i.e. the 'order' (i.e. maximum power) of the numerator is **one less** than the denominator.

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Show that $\int \frac{1+x}{x^3+9x}\,dx = A\ln\left(\frac{x^2}{x^2+9}\right) + B\arctan\left(\frac{x}{3}\right) + c$, where A and B are constants to be found.

$$\frac{1+x}{x^3+9x} = \frac{1+x}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$1+x = A(x^2+9) + x(Bx+C)$$

comp. x^2// $\quad 0 = A + B$

x// $\quad 1 = C$

con// $\quad 1 = 9A \quad \longrightarrow A = \frac{1}{9},\ B = -\frac{1}{9},\ C = 1$

$a^2 = 9$

$a = 3$

$$\int \frac{1+x}{x^3+9x}\,dx = \int\left(\frac{1/9}{x} + \frac{-1/9\,x + 1}{x^2+9}\right)dx$$

$$= \frac{1}{9}\int \frac{1}{x}\,dx - \frac{1}{9}\int \frac{x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$$

$$= \frac{1}{9}\ln|x| - \frac{1}{9}\times\frac{1}{2}\ln|x^2+9| + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$= \frac{1}{2}\times\frac{1}{9}\times\ln|x^2| - \frac{1}{18}\ln|x^2+9| + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$= \frac{1}{18}\ln\left|\frac{x^2}{x^2+9}\right| + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C \qquad A = \frac{1}{18} \quad B = \frac{1}{3}$$

If the fraction is top-heavy, you'll have a quotient. As per Pure Year 2, if the order of numerator and denominator is the same, you'll need an extra constant term. If the power is 1 greater in the numerator, you'll need a quotient of $Ax + B$, and so on.

$$\frac{4x^2+x}{x^2+x} = \frac{4x^2+x}{x(x+1)} = A + \frac{B}{x} + \frac{C}{x+1}$$

$$\frac{4x^3+x}{x^2+x} = Ax + B + \frac{C}{x} + \frac{D}{x+1}$$

(a) Express $\frac{x^4+x}{x^4+5x^2+6}$ as partial fractions.

(b) Hence find $\int \frac{x^4+x}{x^4+5x^2+6}\, dx$.

a) $$\frac{x^4+x}{x^4+5x^2+6} = \frac{x^4+x}{(x^2+2)(x^2+3)} = A + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{x^2+3}$$

$$x^4 + x = A(x^2+2)(x^2+3) + (Bx+C)(x^2+3) + (Dx+E)(x^2+2)$$

comp co.

x^4: $1 = A$

x^3: $0 = B + D$

x^2: $0 = 5A + C + E$

x: $1 = 3B + 2D$

con: $0 = 6A + 3C + 2E$

$A = 1,\ B = 1,\ C = 4,\ D = -1,\ E = -9$

Ex 3E Even Questions

$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)$ $(\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2-a^2}\}$ $(x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right)$ $(\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

$a^2 = 3$
$a = \sqrt{3}$

$a^2 = 2$
$a = \sqrt{2}$

b) $$\int \frac{x^4+x}{x^4+5x^2+6}\, dx = \int \left(1 + \frac{x+4}{x^2+2} - \frac{x+9}{x^2+3}\right) dx$$

$$= x + \int \frac{x}{x^2+2}\, dx + \int \frac{4}{x^2+2}\, dx - \int \frac{x}{x^2+3}\, dx - \int \frac{9}{x^2+3}\, dx$$

$$= x + \frac{1}{2}\ln\lvert x^2+2\rvert + \frac{4}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2}\ln\lvert x^2+3\rvert - \frac{9}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= x + \frac{1}{2}\ln\left\lvert\frac{x^2+2}{x^2+3}\right\rvert + 2\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right) - 3\sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$