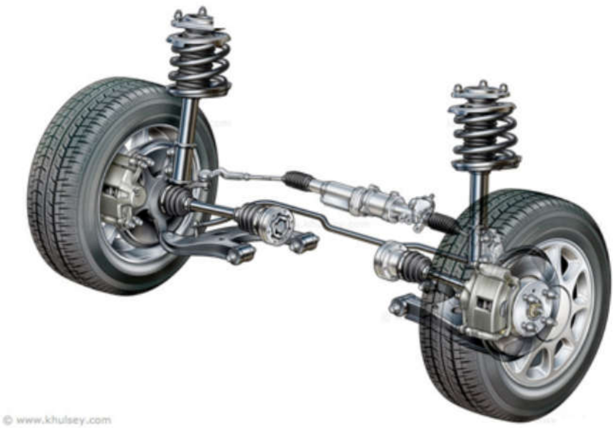
Damped Harmonic Motion



We have seen so far that the extension/compression of the spring leads to a force, and hence an **acceleration, which is proportional to the displacement from some central point**, i.e. the more you stretch the spring, the greater the force, and hence the greater the acceleration. With this force alone, we saw this resulted in the displacement that follows a sine curve as time increases.

But we also know from practice that the amplitude gradually decreases over time (until perhaps the spring no longer oscillates), so there must be some other force at work.

This is known as the **damping force**, and is **proportional to velocity** (k is positive).
In contrast the force we previously saw, caused by the elasticity of the spring, is known as the **restoring force**.

Such motion is known as **damped harmonic motion**.

Note that k is positive

✎ For particle moving with damped harmonic motion:

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - \omega^2 x$$

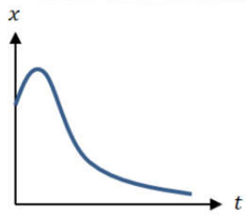
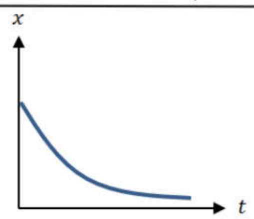
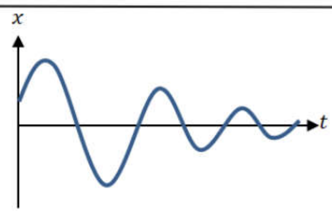
$$\Rightarrow \quad \frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

The type of motion seen will depend on the roots to the auxiliary equation...

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

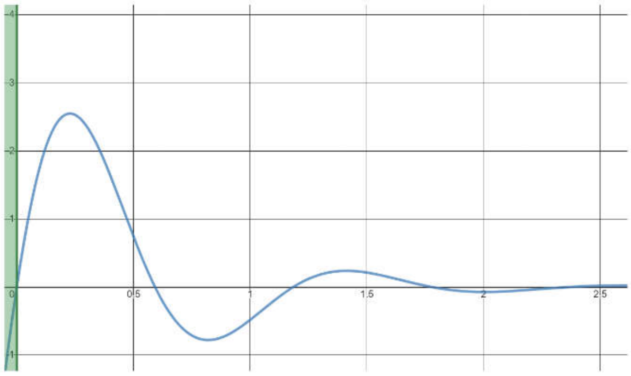
Auxiliary equation: $\boldsymbol{m^2 + km + \omega^2 = 0}$

The solution to the second-order differential equation depends on the number of roots (and hence the discriminant) of this auxiliary equation:

Roots of auxiliary:	**Distinct roots**: $k^2 - 4\omega^2 > 0$	**Equal roots**: $k^2 - 4\omega^2 = 0$	**No roots**: $k^2 - 4\omega^2 < 0$
Form of resulting solution to differential equation:	$x = Ae^{-\alpha t} + Be^{-\beta t}$	$x = (A + Bt)e^{-\alpha t}$	$x = Ae^{-at}\sin bt$
Type of damping:	**Heavy damping** (no oscillations)	**Critical damping** (the limit for which there are no oscillations)	**Light damping** (oscillates)
Sketch of x against t:			

real roots

repeated roots



complex roots.

https://www.desmos.com/calculator/ksdarm3ftq

A particle P of mass 0.5 kg moves in a horizontal straight line. At time t seconds, the displacement of P from a fixed point, O, on the line is x m and the velocity of P is v ms^{-1}. A force of magnitude $8x$ N acts on P in the direction PO. The particle is also subject to a resistance of magnitude $4v$ N. When $t = 0$, $x = 1.5$ and P is moving in the direction of increasing x with speed 4 ms^{-1},

(a) Show that $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$

(b) Find the value of x when $t = 1$.

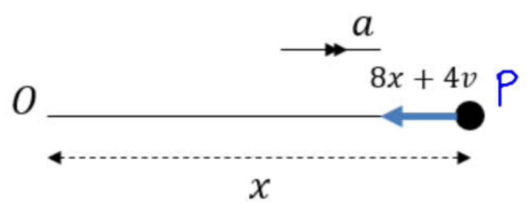
a)

$R \rightarrow,\ F = ma$

$-8x - 4v = 0.5a$

$-8x - 4\frac{dx}{dt} = 0.5\frac{d^2x}{dt^2}$

$0 = \frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x$



b) A.E. $m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$m = -4$

G.S. $x = (A + Bt)e^{-4t}$

$1.5 = A$

$\frac{dx}{dt} = -4(A+Bt)e^{-4t} + Be^{-4t}$

$4 = -4A + B$

$4 + 4 \times 1.5 = B$

$B = 10$

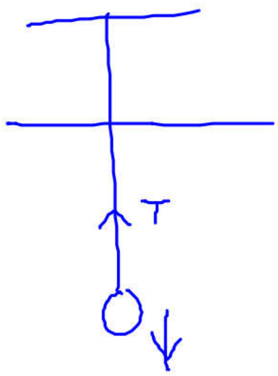
P.S. $x = (1.5 + 10t)e^{-4t}$

$x = (1.5 + 10)e^{-4} = \underline{\underline{0.211\text{ m}}}$ (3sf)

A particle P hangs freely in equilibrium attached to one end of a light elastic string. The other end of the string is attached to a fixed point A. The particle is now pulled down and held at rest in a container of liquid which exerts a resistance to motion on P. P is then released from rest. While the string remains taut and the particle in the liquid, the motion can be modelled using the equation

$\frac{d^2x}{dt^2} + 6k\frac{dx}{dt} + 5k^2x = 0$, where k is a positive real constant

Find the general solution to the differential equation and state the type of damping that the particle is subject to.



$\ddot{x} + 6k\dot{x} + 5k^2x = 0$

A.E. $m^2 + 6km + 5k^2 = 0$

$(m + 5k)(m + k) = 0$

$m = -k \quad m = -5k.$

$x = Ae^{-kt} + Be^{-5kt}$

Heavy damping.

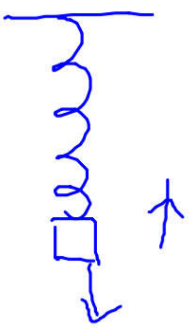
One end of a light elastic spring is attached to a fixed point A. A particle P is attached to the other end and hangs in equilibrium vertically below A. The particle is pulled vertically down from its equilibrium position and released from rest. A resistance proportional to the speed of P acts on P. The equation of motion of P is given as

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0$$

where k is a positive real constant and x is the displacement of P from its equilibrium position.

(a) Find the general solution to the differential equation.

(b) Write down the period of oscillation in terms of k.



a) $\ddot{x} + 2k\dot{x} + 2k^2x = 0$

A.E. $m^2 + 2km + 2k^2 = 0$

$(m+k)^2 - k^2 + 2k^2 = 0$

$(m+k)^2 = -k^2$

$m+k = \pm ki$

$m = -k \pm ki$

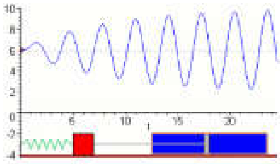
G.S. $x = e^{-kt}(P\cos kt + Q\sin kt)$

b) period $= \frac{2\pi}{\omega} = \frac{2\pi}{k}$ seconds.

Ex 8C
Q1-5

Forced Harmonic Motion

In addition to the 'natural' forces acting on the particle, i.e. damping force and restoring force, there may be ~~a further~~ a further force acting on the particle. This is known as **forced harmonic motion**.



Forced harmonic motion

$$\frac{dx^2}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$$

A particle P of mass 1.5 kg is moving on the x-axis. At time t the displacement of P from the origin O is x metres and the speed of P is v ms⁻¹. Three forces act on P, namely a restoring force of magnitude $7.5x$ N, a resistance to the motion of P of magnitude $6v$ N and a force of magnitude $12 \sin t$ N acting in the direction OP.

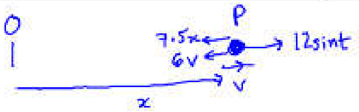
When $t = 0, x = 5$ and $\frac{dx}{dt} = 2$.

(a) Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 8\sin t$

(b) Find x as a function of t.

(c) Describe the motion when t is large.

$m = 1.5$



a) $R \rightarrow, F = ma$

$12\sin t - 7.5x - 6v = 1.5a$

$12\sin t - 7.5x - 6\frac{dx}{dt} = 1.5\frac{d^2x}{dt}$

$8\sin t = \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x$

b) A.E. $m^2 + 4m + 5 = 0$

$m = -2 \pm i$

C.F. $x = e^{-2t}(P\cos t + Q\sin t)$.

P.I. Try $x = \lambda \sin t + \mu \cos t$

$\frac{dx}{dt} = \lambda\cos t - \mu \sin t$

$\frac{d^2x}{dt^2} = -\lambda \sin t - \mu\cos t$.

$-\lambda\sin t - \mu\cos t + 4\lambda\cos t - 4\mu\sin t + 5\lambda\sin t + 5\mu\cos t = 8\sin t$.

compare $\sin t$ $\quad -\lambda - 4\mu + 5\lambda = 8$

$4\lambda - 4\mu = 8$

$\boxed{\lambda - \mu = 2}$

compare $\cos t$ $\quad -\mu + 4\lambda + 5\mu = 0$

$4\lambda + 4\mu = 0$

$\boxed{\lambda + \mu = 0}$

$2\lambda = 2, \ \lambda = 1$

$\mu = -1$

P.I $\quad x = \sin t - \cos t$

G.S. $x = CF + PI$ $\qquad t=0, \ x=5, \ \frac{dx}{dt} = 2$

$x = e^{-2t}(P\cos t + Q\sin t) + \sin t - \cos t$.

$5 = P - 1$

$P = 6$

$\frac{dx}{dt} = -2e^{-2t}(P\cos t + Q\sin t) + e^{-2t}(-P\sin t + Q\cos t) + \cos t + \sin t$

$2 = -2(P) + Q + 1$

$2 = -12 + Q + 1$

$Q = 13$

$x = e^{-2t}(6\cos t + 13\sin t) + \sin t - \cos t$.

c) When $t \to \infty$ $\quad e^{-2t} \to 0$ so $x \to (\sin t - \cos t)$

i.e. as t becomes large, motion is SHM.

A particle P is attached to end A of a light elastic spring AB. Initially the particle and the string lie at rest on a smooth horizontal plane. At time $t = 0$, the end B of the string is set in motion and moves with constant speed U in the direction AB, and the displacement of P from A is x. Air resistance acting on P is proportional to its speed. The subsequent motion can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + k^2x = 2kU$$

Find an expression for x in terms of U, k and t

A.E. $m^2 + 2km + k^2 = 0$

$(m+k)^2 = 0$

$m = -k$

C.F. $x = (A+Bt)e^{-kt}$

P.I. Try $x = \lambda$

$\dot{x} = 0$

$\ddot{x} = 0$

$0 + 0 + k^2\lambda = 2kU$

$\lambda = \frac{2U}{k}$

G.S. $x = (A+Bt)e^{-kt} + \frac{2U}{k}$

$t=0 \quad v = U$

$x = 0 \quad \frac{dx}{dt} = U$

$0 = A + \frac{2U}{k}$

$A = -\frac{2U}{k}$

$\frac{dx}{dt} = -k(A+Bt)e^{-kt} + Be^{-kt}$

$U = -kA + B$

$U = -k\left(-\frac{2U}{k}\right) + B$

$U = 2U + B$

$B = -U$

$x = \left(-\frac{2U}{k} - Ut\right)e^{-kt} + \frac{2U}{k}.$

6. A damped spring is part of a car suspension system. In tests for the system, a mass is attached to the damped spring and is made to move upwards in a vertical line.

The motion of the system is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 2e^{-3t}$$

where x cm is the vertical displacement of the mass above its equilibrium position and t is the time, in seconds, after motion begins.

In one particular test, the mass is moved to a position 20 cm above its equilibrium position and given an initial velocity of 1 ms^{-1} upwards. For this test, use the model to

$x = 20$
$\dot{x} = 100$
$t = 0$

(a) find an equation for x in terms of t,

(9)

(b) find, to the nearest mm, the maximum displacement of the mass from its equilibrium position.

(3)

In this test, the time taken for the mass to return to its equilibrium position was measured as 2.86 seconds.

(c) State, with justification, whether or not this supports the model.

(1)

a) A.E. $m^2 + 6m + 9 = 0$
$(m+3)^2 = 0$
$m = -3$

C.F. $x = (A + Bt)e^{-3t}$

P.I. Try $x = \lambda t^2 e^{-3t}$
$\dot{x} = 2\lambda t e^{-3t} - 3\lambda t^2 e^{-3t}$
$\ddot{x} = 2\lambda e^{-3t} - 6\lambda t e^{-3t} - 6\lambda t e^{-3t} + 9\lambda t^2 e^{-3t}$
$\ddot{x} = 2\lambda e^{-3t} - 12\lambda t e^{-3t} + 9\lambda t^2 e^{-3t}$

$\ddot{x} + 6\dot{x} + 9x = 2e^{-3t}$
$2\lambda - \cancel{12\lambda t} + \cancel{9\lambda t^2} + \cancel{12\lambda t} - \cancel{18\lambda t^2} + \cancel{9\lambda t^2} = 2$
$2\lambda = 2$
$\lambda = 1$ Hence P.I is $x = t^2 e^{-3t}$

G.S. $x = (A + Bt)e^{-3t} + t^2 e^{-3t}$

When $t = 0$, $x = 20$, $\dot{x} = 100$
$20 = A$
$\dot{x} = -3(A + Bt)e^{-3t} + Be^{-3t} + 2te^{-3t} - 3t^2 e^{-3t}$
$100 = -3A + B$
$100 + 60 = B$
$B = 160$ Hence $x = (20 + 160t)e^{-3t} + t^2 e^{-3t}$

b) $\dot{x} = 0$ $\quad 0 = -3(20 + 160t)e^{-3t} + 160e^{-3t} + 2te^{-3t} - 3t^2 e^{-3t}$
$0 = -60 - 480t + 160 + 2t - 3t^2$
$0 = -3t^2 - 478t + 100$
$t = 0.2089...$ or -159.5 but $t > 0$
$x = (20 + 160 \times 0.2089)e^{-3 \times 0.2089} + 0.2089^2 e^{-3 \times 0.2089}$
$x = 28.57$ cm $= \underline{286 \text{ mm}}$

c) $t = 2.86$ $\quad x = (20 + 160 \times 2.86)e^{-3 \times 2.86} + 2.86^2 e^{-3 \times 2.86} = 0.0912$ cm

which is very close to 0, so a good model.

Question	Scheme	Marks	AOs
6(a)	$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=2e^{-3t}$		
	AE: $m^2+6m+9=0 \Rightarrow (m+3)^2=0 \Rightarrow m=\ldots\ (=-3)$	M1	1.1b
	So C.F. is $x_{CF}=(A+Bt)e^{-3t}$	A1	2.2a
	For P.I. try $x_{PI}=kt^2e^{-3t}$	B1	2.2a
	$\dot{x}_{PI}=2kte^{-3t}-3kt^2e^{-3t}\quad (=k(2t-3t^2)e^{-3t})$ $\ddot{x}_{PI}=2ke^{-3t}-6kte^{-3t}-6kte^{-3t}+9kt^2e^{-3t}\quad (=k(2-12t+9t^2)e^{-3t})$ $\Rightarrow k(2-12t+9t^2)e^{-3t}+6k(2t-3t^2)e^{-3t}+9kt^2e^{-3t}=2e^{-3t}\Rightarrow k=\ldots$	M1	1.1b
	So $k=1$ ie $x_{PI}=t^2e^{-3t}$	A1	1.1b
	General solution is $x=(A+Bt)e^{-3t}+t^2e^{-3t}$ (their C.F. + their P.I.)	M1	1.1a
	$x(0)=20\Rightarrow A=20$	M1	3.4
	$\dot{x}=Be^{-3t}-3(A+Bt)e^{-3t}+2te^{-3t}-3t^2e^{-3t}=(B-3A+(2-3B)t-3t^2)e^{-3t}$ $\dot{x}(0)=100\Rightarrow B=100+3A=\ldots(=160)$	M1	3.4
	So $x=(20+160t+t^2)e^{-3t}$	A1	1.1b
		(9)	
(b)	From above $\dot{x}=(B-3A+(2-3B)t-3t^2)e^{-3t}=(100-478t-3t^2)e^{-3t}$		
	$\dot{x}=0\Rightarrow 100-478t-3t^2=0\Rightarrow t=\ldots\ (=-159.5\ldots$ or $0.2089\ldots)$	M1	3.1a
	$t>0$, so $t_{max}=0.2089\ldots\Rightarrow$ $x_{max}=\left(20+160\times0.2089\ldots+(0.2089\ldots)^2\right)e^{-3\times0.2089\ldots}=\ldots$	M1	3.4
	$x_{max}=$ awrt 28.6 cm (3 s.f.) (28.57055381741878)	A1	1.1b
		(3)	
(c)	$x(2.86)=0.0912\ldots$ which is close to zero (less than 1mm), which can be accounted for by inaccuracies in measurements. So the model is supported by this measurement.	B1ft	2.2b
		(1)	
			(13 marks)