Edexcel A Level Further Maths: Further Mechanics 1

Your notes

Elastic Collisions in 2D

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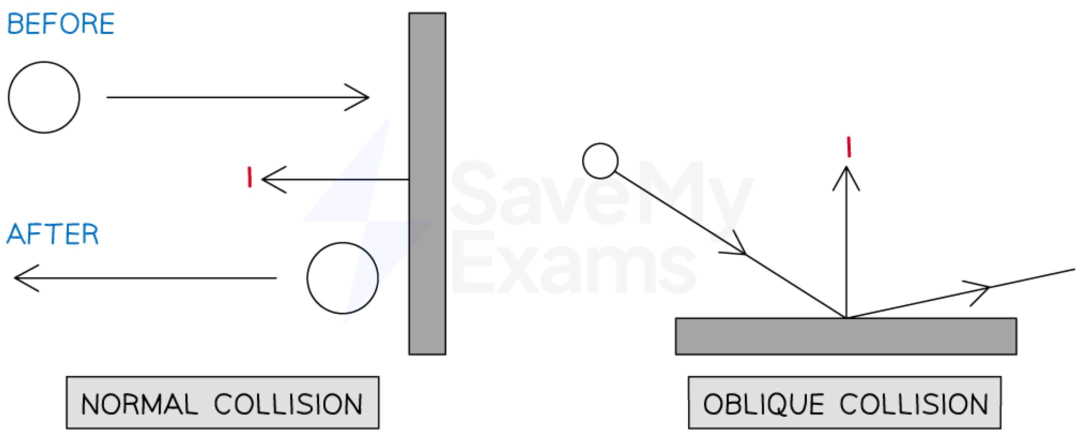
Oblique Collisions with a Surface

Your notes

Oblique Collisions with a Surface

What are oblique collisions (with a surface)?

- In a **normal** collision a particle collides with a surface at right angles
- In an **oblique collision** the angle at which the particle collides with the surface is not 90°
- In oblique collisions
  - there are **two** dimensions of motion of the particle to consider
  - the **velocity** of the particle will change and so its **momentum** will change
  - this is caused by an **impulse** from the surface to the particle
    - the impulse acts **perpendicular** to the surface



What modelling assumptions are used for oblique collisions?

- Problems are usually presented with a diagram in plan view (i.e. from above)
- Modelling assumptions
  - the surface the particle is moving across ('the floor') is horizontal
  - the surface the particle will collide with ('the wall') is flat and fixed
  - the 'floor' and 'wall' are smooth (no friction)
  - particles are usually smooth spheres
    - this is so that the impact of the collision can be considered as occurring at a single point in space

What equations are needed to solve oblique collision problems?

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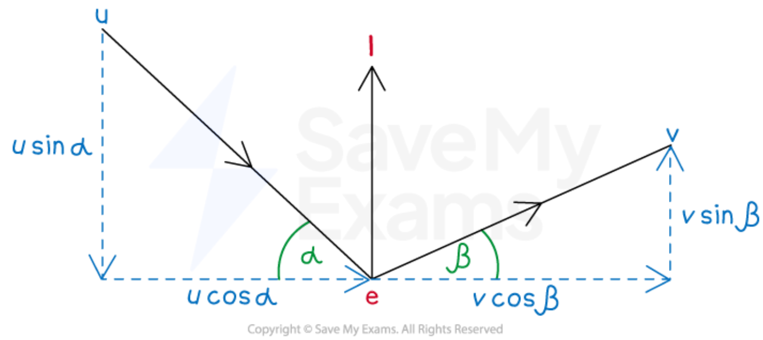


- In the diagram
  - $\mathbf{u}$ m s^{-1} is the velocity **before** impact, $\mathbf{v}$ m s^{-1} is the velocity **after** impact
  - α° is the angle of **approach,** β° is the angle of **rebound**
  - $\mathbf{I}$ N is the **impulse** (which always acts perpendicular to the surface)
  - e is the coefficient of restitution (between the particle and the surface)
- The component of velocity **parallel** to the surface remains unchanged
    - $v \cos \beta = u \cos \alpha$
- The component of velocity **perpendicular** to the surface can be found by applying Newton's Law of Restitution
  - $e = \frac{v \sin \beta}{u \sin \alpha}$
  - Rearranging
    - $v \sin \beta = eu \sin \alpha$
- Dividing the above two equations eliminates u and v
  - $\tan \beta = e \tan \alpha$
  - Since $0 \le e \le 1$ it follows that $\tan \beta \le \tan \alpha$ and so $\beta \le \alpha$
    (i.e. angle of rebound is less than or equal to the angle of approach)

How do I solve oblique collision problems?

- STEP 1
  Draw a diagram (or add to a given one) showing important information in the question such as velocity/speed of approach/rebound, angle of approach/rebound, impulse
- STEP 2
  Write an equation for the motion parallel to the surface using $v \cos \beta = u \cos \alpha$
  Write an equation for the motion perpendicular to the surface using $v \sin \beta = eu \sin \alpha$
- STEP 3
  Using "square and add" and/or "division" to eliminate unwanted quantities
  ($v \tan \beta = eu \tan \alpha$ can be used directly)
- STEP 4
  Answer the question by solving the relevant equation(s) for the required quantity

Examiner Tip

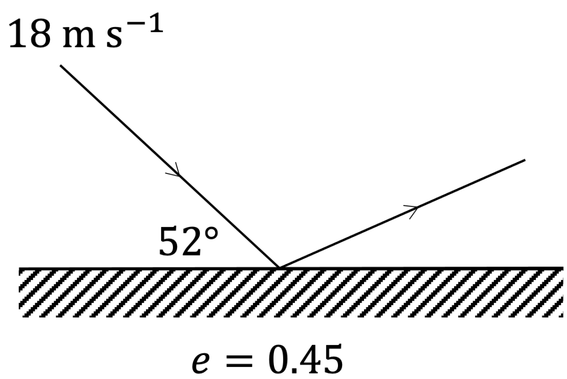
- Problems will often refer to the **speed** (rather than velocity) of the particle before and/or after the collision
    - This can be confusing but speed is the magnitude of velocity and so has components parallel and perpendicular to the surface
- Do not assume all surfaces ('walls') are orientated in a 'nice' direction
    - e.g. parallel to the *x*- or *y*-axes in an *xy* plane, or parallel to the **i** or **j** vectors in a vector problem
    - In questions given in vector form the direction of the impulse is often needed before the orientation of the surface can be deduced
    - impulse is perpendicular to the surface
    - draw, and if necessary, redraw, a diagram to help visualise the problem

Your notes

Your notes

Worked example

A smooth sphere is rolling across a smooth horizontal floor with speed 18 m s^{-1} when it collides with a smooth, fixed vertical wall. The angle of the collision with the wall is 52° and the coefficient of restitution between the floor and the wall is 0.45.



Find, the speed of the sphere immediately after the collision.

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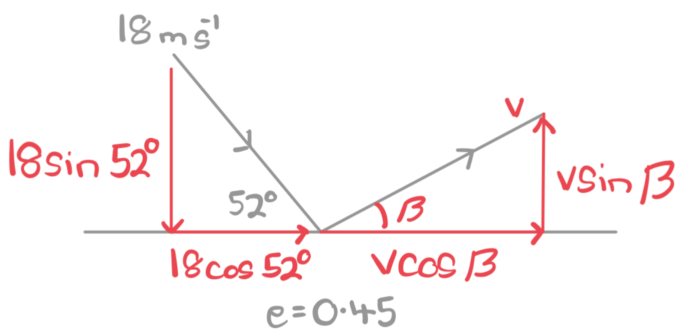
STEP 1: Add components of speed to the diagram



STEP 2: Write equations parallel and perpendicular to the wall

$$v\cos\beta = 18\cos 52^\circ$$
$$v\sin\beta = 0.45 \times 18 \times \sin 52^\circ$$

STEP 3: The speed after impact, v, is required, so eliminate β by squaring and adding

$$v^2\left(\underbrace{\cancel{\cos^2\beta + \sin^2\beta}}_{1}\right) = (18\cos 52^\circ)^2 + (8.1\sin 52^\circ)^2$$

STEP 4: Solve the equation

$$v^2 = 163.549\,900\ldots$$

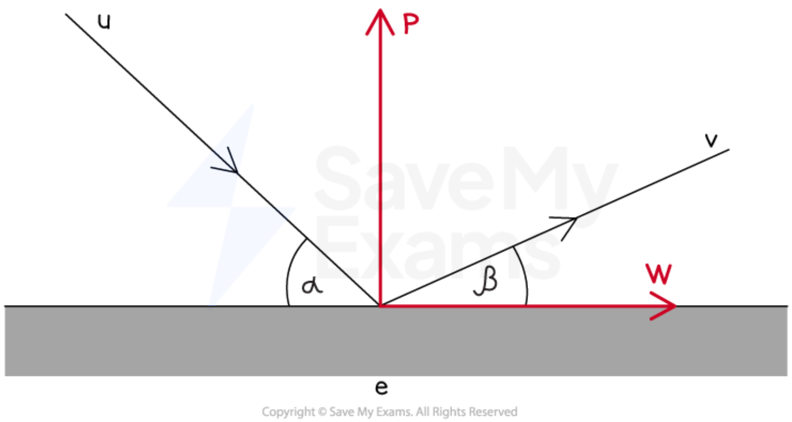
$$\boxed{\therefore v = 12.8 \text{ m s}^{-1} \text{ (3 s.f.)}}$$

Your notes

Using Scalar Product with Collisions

The scalar product

The scalar product is defined as $\mathbf{a} \cdot \mathbf{b} = a\,b\cos\theta$ where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, and a and b are the magnitudes of those vectors respectively.



In the diagram above

- $\mathbf{P}$ is a vector that is **perpendicular** to the surface and so is in the **direction** of the impulse
  $\mathbf{P}$ could, but isn't necessarily, equal to the impulse $\mathbf{I}$
- $\mathbf{W}$ is a vector that is parallel to (in the direction of) the surface

Applying the scalar product perpendicular to the surface

- Consider $\mathbf{u} \cdot \mathbf{P}$
  - $\mathbf{u} \cdot \mathbf{P} = uP\cos(90 + \alpha)$
  - Since $\cos(90 + \alpha) = -\sin\alpha$ this leads to the result
    $\mathbf{u} \cdot \mathbf{P} = -uP\sin\alpha$
- Similarly, for $\mathbf{v}.\mathbf{P}$
  - $\mathbf{v} \cdot \mathbf{P} = vP\cos(90 - \beta)$
  - Since $\cos(90 - \beta) = \sin\beta$ this leads to the result
    $\mathbf{v} \cdot \mathbf{P} = vP\sin\beta$
- Combining these two results with $v\sin\beta = eu\sin\alpha$ gives
  - $\dfrac{\mathbf{v} \cdot \mathbf{P}}{P} = -e\dfrac{\mathbf{u} \cdot \mathbf{P}}{P}$
  - So, $\mathbf{v} \cdot \mathbf{P} = -e\,\mathbf{u} \cdot \mathbf{P}$

Your notes

- As the impulse ($\mathbf{I}$) is always **perpendicular** to the surface, if the **direction** of the surface ($\mathbf{W}$) is known then the **direction** of the **impulse** ($\mathbf{P}$) can be written down, and vice versa

Applying the scalar product parallel to the surface

- Now consider $\mathbf{u} \cdot \mathbf{W}$
  - $\mathbf{u} \cdot \mathbf{W} = uW\cos\alpha$
- Similarly, for $\mathbf{v} \cdot \mathbf{W}$
  - $\mathbf{v} \cdot \mathbf{W} = vW\cos\beta$
- Combining these two results with $v\cos\beta = u\cos\alpha$ gives
  - $\dfrac{\mathbf{v} \cdot \mathbf{W}}{W} = \dfrac{\mathbf{u} \cdot \mathbf{W}}{W}$
  - So, $\mathbf{v} \cdot \mathbf{W} = \mathbf{u} \cdot \mathbf{W}$

How do I use the scalar product to solve oblique collision problems?

- The scalar product approach should be used when questions give velocities/impulse/etc in vector form
  - If magnitude (speed) and angles are given, using the techniques in the revision notes above are usually easier to apply
- Using the scalar product is particularly suited to problems where the surface is **not** simply parallel to $\mathbf{i}$ or $\mathbf{j}$
- Ensure you are familiar with the two formulae
  $\mathbf{v} \cdot \mathbf{P} = -e\,\mathbf{u} \cdot \mathbf{P}$ and
  $\mathbf{v} \cdot \mathbf{W} = \mathbf{u} \cdot \mathbf{W}$
- For problems where the direction of the surface is **not** parallel to $\mathbf{i}$ or $\mathbf{j}$ use the fact that the **direction of the impulse** ($\mathbf{P}$) and the **direction of the fixed surface** ($\mathbf{W}$) are **perpendicular**
  - So if $\mathbf{P} = x\mathbf{i} + y\mathbf{j}$ then $\mathbf{W} = y\mathbf{i} - x\mathbf{j}$ (and vice versa)
  - To find an unknown $\mathbf{P}$ or $\mathbf{W}$, use both dot product equations to set up simultaneous equations in x and y
- In some problems it may be necessary to use the impulse-momentum principle to find the impulse, $\mathbf{I}$ (and so its direction $\mathbf{P}$)
  $\mathbf{I} = m(\mathbf{v} - \mathbf{u})$

- STEP 1
  Draw a diagram, or add to a given one
  Consider whether the direction of the fixed surface ($\mathbf{W}$) and/or the direction of the impulse ($\mathbf{P}$) are known
  One can be determined from the other

Depending on the information given, $\mathbf{I} = m(\mathbf{v} - \mathbf{u})$ may be needed to find the direction of the impulse ($\mathbf{P}$)

- STEP 2
  Determine which information is given and required, and use this to select one or both of the scalar product equations
- STEP 3
  Use the scalar product to set up equation(s) in the unknown(s) required
- STEP 4
  Solve the equations and hence solve the problem

Your notes

Examiner Tip

- The use of scalar product may seem very complicated at first, especially with the notation used
  - It is worth spending some time learning and becoming familiar with these formulae though as they can greatly reduce the amount of work required to gain lots of marks!

Your notes

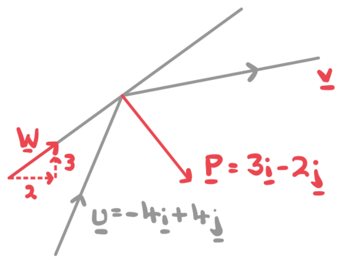
Worked example

A small smooth sphere is moving with velocity $(-4\mathbf{i}+4\mathbf{j})\ \mathrm{m\ s^{-1}}$ across a smooth horizontal plane. It collides with a smooth fixed, vertical plane that lies in the direction $(2\mathbf{i}+3\mathbf{j})$. The coefficient of restitution between the sphere and the vertical plane is 0.1.

Find the velocity of the sphere immediately after it collides with the vertical plane.

Your notes

STEP 1: Draw a diagram - the direction of the wall ($\underline{W}$) is known so the direction of the impulse ($\underline{I}$) can be written down.



$$\underline{W} = 2\underline{i} + 3\underline{j}$$

$$\underline{P} = 3\underline{i} - 2\underline{j}$$

STEP 2: $\underline{v}$ is required and $\underline{W}$ is not parallel to $\underline{i}$ or $\underline{j}$ so both dot product equations are needed

$$\underline{v} \cdot (3\underline{i} - 2\underline{j}) = -0.1(-4\underline{i} + 4\underline{j}) \cdot (3\underline{i} - 2\underline{j}) \qquad \underline{v} \cdot \underline{P} = -e\underline{u} \cdot \underline{P}$$

$$\underline{v} \cdot (2\underline{i} + 3\underline{j}) = (-4\underline{i} + 4\underline{j}) \cdot (2\underline{i} + 3\underline{j}) \qquad \underline{v} \cdot \underline{W} = \underline{u} \cdot \underline{W}$$

STEP 3: Set up simultaneous equations

Let $\underline{v} = x\underline{i} + y\underline{j}$

$$3x - 2y = 2$$
$$2x + 3y = 4$$

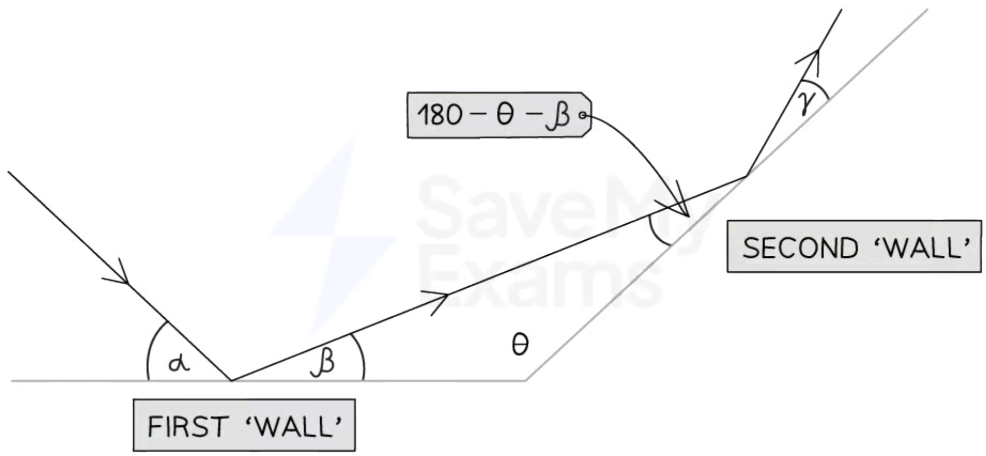
STEP 4: Solve and answer the question

$$x = \frac{14}{13} \qquad y = \frac{8}{13}$$

$$\boxed{\therefore \ \underline{v} = \left(\frac{14}{13}\underline{i} + \frac{8}{13}\underline{j}\right) \text{ m s}^{-1}}$$

Successive Collisions in 2D

How do I solve problems involving successive collisions in 2D?



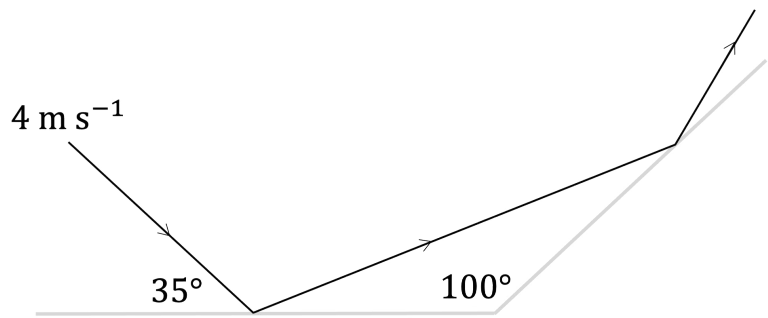
- **Successive collisions** are where a particle collides with one fixed surface, then another
- The modelling assumption that the particle, the plane it is travelling in, and the fixed surface(s) it collides with are all smooth mean that the velocity/speed of **rebound** from the **first** collision will be the velocity/speed of the **approach** in the **second** collision
- The two **fixed surfaces** the particle collides with may, or may not, be perpendicular to each other
  - the cushions on a snooker table are perpendicular
- The **coefficients** of **restitution** between the particle and each of the fixed surfaces may or may not be equal
  - read the information given in the question carefully
- Separate the collisions into two single collision problems
  - Draw and label a diagram for each collision
- Use the scalar product approach where possible
  - This is often easiest when velocities, etc have been given as vectors

Your notes

Your notes

Worked example

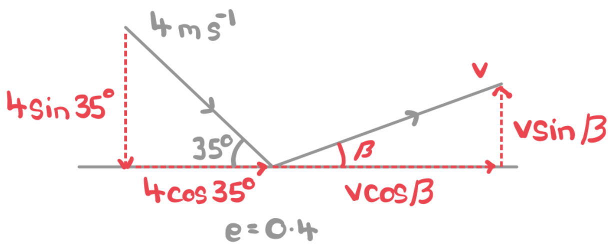
Two fixed, smooth vertical walls meet at an angle of 100° on a smooth horizontal surface. A smooth sphere is moving across the surface with speed 4 m s^{-1} at an angle of 35° to the first wall and towards the intersection of the two walls. The coefficient of restitution between the sphere and the first wall is 0.4; for the second wall it is 0.8.



Find the speed and direction (relative to the second wall) of the sphere after its collision with the second wall.

Your notes

Draw a diagram for the first collision only



Form and solve equations parallel and perpendicular to the (first) wall

$v\cos\beta = 4\cos 35°$
$v\sin\beta = 0.4 \times 4 \times \sin 35°$

"$v\cos\beta = u\cos\alpha$"
"$v\sin\beta = eu\sin\alpha$"

Square and add to solve for v

$v^2 = 16\cos^2 35° + 2.56\sin^2 35°$
$v = 3.402\ 701\ ...$

Store v and β in your calculator's memory to keep accuracy later

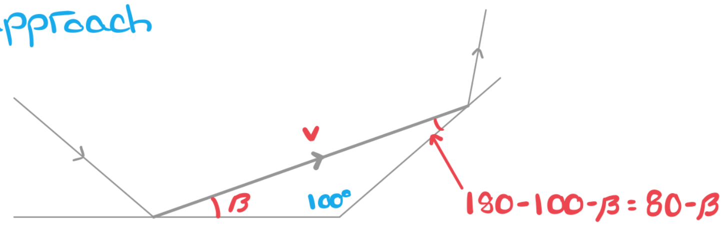
Substitute v to find β

$\cos\beta = \dfrac{4\cos 35°}{3.402...}$

$\beta = 15.646\ 656\ ...$

β can also be found using "$\tan\beta = e\tan\alpha$"

For the second collision first find the angle of approach



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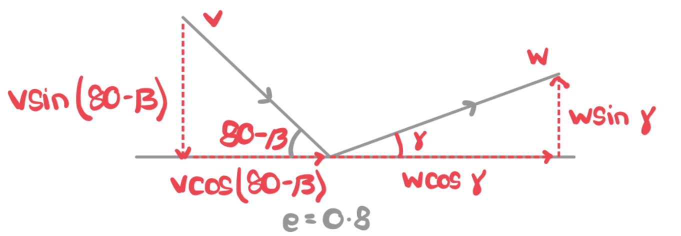
Draw a diagram for the second collision



Form and solve the equations for the second collision

$$w\cos\gamma = v\cos(80-\beta)$$
$$w\sin\gamma = 0.8v\sin(80-\beta)$$

Using stored values for v and β, square and add to find w

$$w^2 = [(3.4...)\cos(64.3...)]^2 + [0.64(3.4...)\sin(64.3...)]^2$$

80 − 15.6...

$$w = 2.861\,993...$$

Substite w to find γ (or use $\tan\gamma = e\tan(80-\gamma)$)

$$\gamma = 59.029\,852...$$

∴ After the second collision the sphere is moving with speed 2.86 m s⁻¹ (3 s.f.) at an angle of 59.0° (1 d.p.) to the second wall.

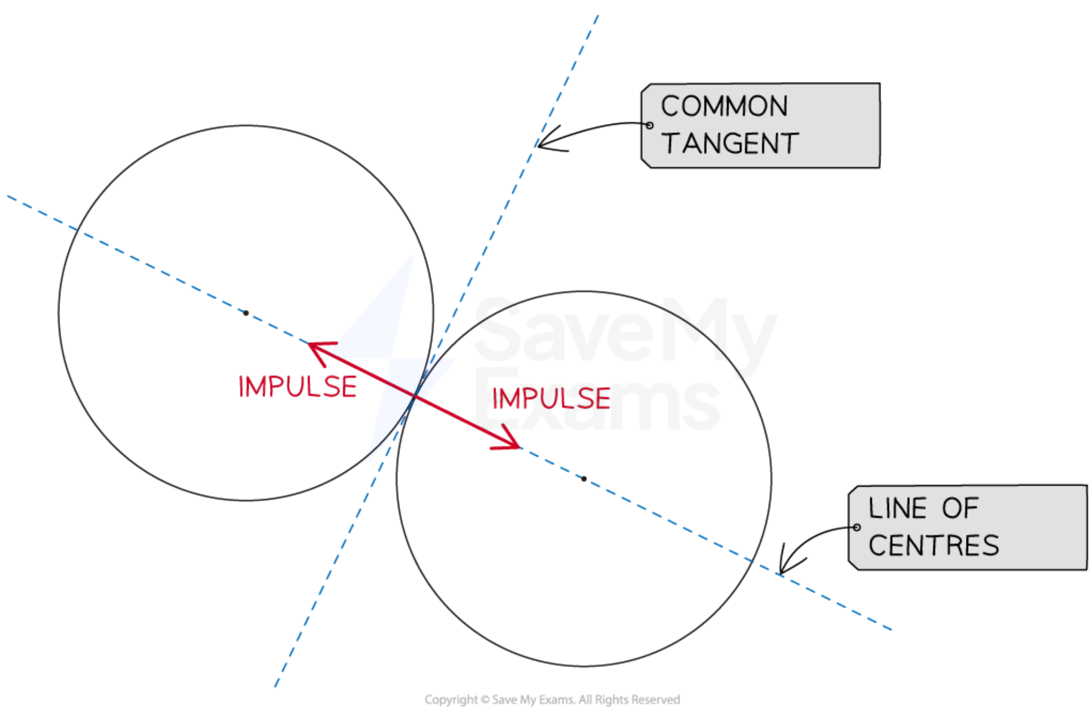
Oblique Collisions of Two Spheres

Your notes

Oblique Collisions of Two Spheres

What is an oblique collision between two spheres?

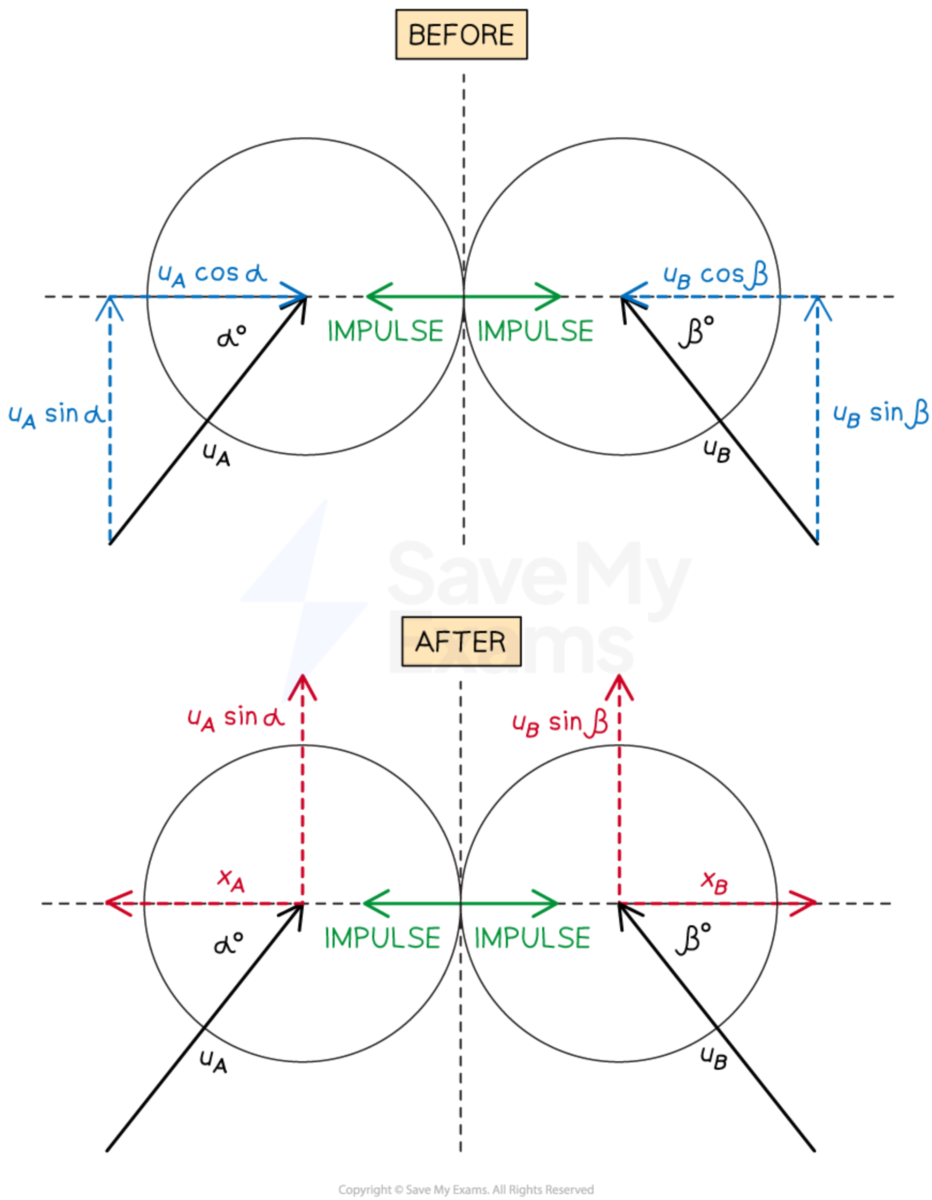
- An oblique collision between two spheres is when two spheres which are not travelling along the same straight line collide
  - For the purposes of this course, the spheres are modelled as smooth, and as having equal radii
- When they collide, they touch at a single point, and so share a **common tangent**
- Perpendicular to the common tangent, is the **line of centres**, which passes through the centre of both spheres
- Similar to when colliding with a surface, the objects only meet at a single point,
  - so the impulse only acts perpendicular to the 'surface',
    - in this case the 'surface' is another sphere
- Therefore **the impulse acts along the line of centres**



How do I solve problems involving oblique collisions between two spheres?

The diagram below shows how to model the collision of spheres *A* and *B*, moving with speeds u_A and u_B at angles $\alpha°$ and $\beta°$ respectively to the line of centres. (Note that the line of centres has been drawn horizontally in this diagram, but all the expressions shown will be exactly the same whatever the direction of the line of centres is.)

Your notes



- As the impulse only acts along the line of centres, it is only this component of the velocity which will be affected
  - In the diagram it can be seen that for the components parallel to the line of centres:
    - $u_A \cos\alpha$ becomes x_A; the 'line of centres' component of *A* after the collision
    - $u_B \cos\beta$ becomes x_B; the 'line of centres' component of *B* after the collision
    - x_A and x_B can be calculated later
- The component of the velocity perpendicular to this, which is in the direction of the common tangent, is unaffected by the collision

Your notes

- In the diagram it can be seen that for the components parallel to the common tangent:
  - $u_A \sin\alpha$ is still the 'common tangent' component of *A* after the collision
  - $u_B \sin\beta$ is still the 'common tangent' component of *B* after the collision
- To find the components which have been affected by the impulse, x_A and x_B, we can:
  - Apply conservation of linear momentum in the direction of the impulse
    - Conservation of linear momentum:
    - $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$
    - Applied to the diagram above in the direction of the impulse, assuming *A* and *B* have masses m_A and m_B respectively:
    - $(\rightarrow) \quad m_A u_A \cos\alpha + m_B(-u_B \cos\beta) = m_A(-x_A) + m_B x_B$
  - Apply Newton's law of restitution
    - Newton's law of restitution:
    - $e = \dfrac{\text{Speed of separation of the objects}}{\text{Speed of approach of the objects}}$
    - Applied to the diagram above, in the direction of the impulse:
    - $e = \dfrac{x_A + x_B}{u_A \cos\alpha + u_B \cos\beta}$
  - This will lead to a pair or simultaneous equations which can be solved to find x_A and x_B
  - Be careful with positive and negative signs when forming these equations, the signs of the velocities will depend on how they are modelled in the diagram
    - e.g. If the arrows are drawn pointing away from each other in the "after" diagram, the speed of separation will be $x_A + x_B$
    - If the arrows were drawn both pointing to the right, the speed of separation would be $x_B - x_A$
- Once all the components of the velocities of the spheres after the collision are known,
  - the speeds can be calculated using Pythagoras
  - the angle each sphere travels at can be calculated using right-angled trigonometry

Examiner Tip

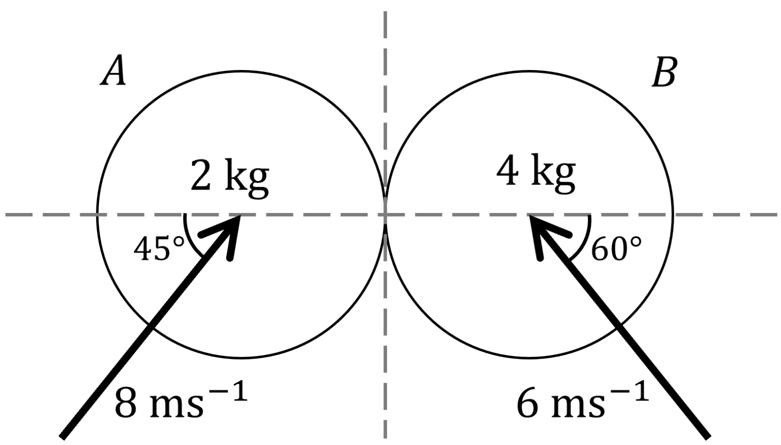
- Drawing a bigger diagram will enable you to have more room for all the information and workings
  - This can help reduce errors caused by squashed writing!
- You can choose whether to draw separate "before" and "after" diagrams, or combine them into one
  - Try both to see which you prefer

Your notes

Worked example

A small smooth sphere *A* of mass 2 kg collides with a small smooth sphere *B* of mass 4 kg. Immediately before the impact *A* is moving with a speed of 8 ms^{-1} in a direction 45° to the line of centres, and *B* is moving with speed 6 ms^{-1} in a direction 60° to the line of centres, as shown in the diagram. The coefficient of restitution between the two spheres is $\frac{1}{2}$.

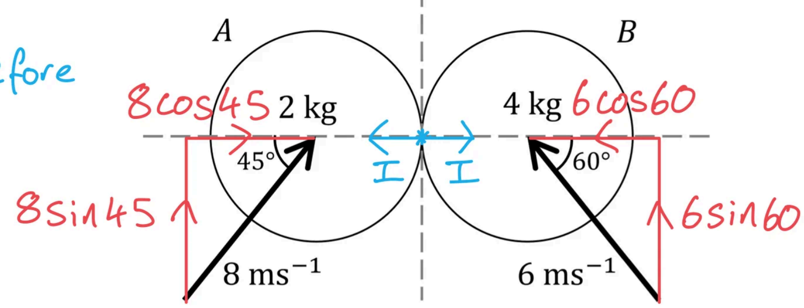
Find the speed and direction of each sphere after the collision.



Your notes

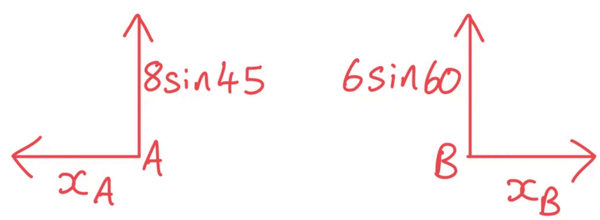
Split the initial velocities into their components

Before



The components in the direction of the common tangent are unaffected by the impulse

After



Conservation of momentum

$(\rightarrow)\quad 2\times 8\cos 45 + 4\times(-6\cos 60) = 2(-x_A) + 4x_B$

$① \quad 8\sqrt{2} - 12 = 4x_B - 2x_A$

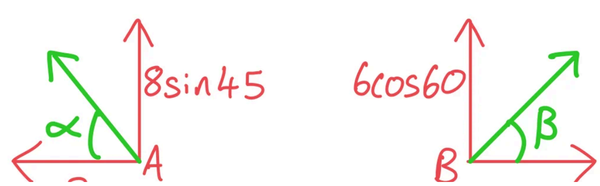
$e = \dfrac{\text{separation}}{\text{approach}} \qquad \dfrac{1}{2} = \dfrac{x_A + x_B}{8\cos 45 + 6\cos 60}$

$② \quad x_A + x_B = 2\sqrt{2} + \dfrac{3}{2}$

Solving ① and ② simultaneously

$x_A = 3$

$x_B = 2\sqrt{2} - \dfrac{3}{2}$



Your notes

3

$2\sqrt{2} - \frac{3}{2}$

$$\text{Speed} = \sqrt{(8\sin 45)^2 + 3^2}$$
$$= 6.40 \text{ ms}^{-1}$$
$$\alpha = \tan^{-1}\left(\frac{8\sin 45}{3}\right)$$
$$= 62.1^\circ$$

or 118° from positive x-axis

$$S = \sqrt{(6\cos 60)^2 + \left(2\sqrt{2} - \frac{3}{2}\right)^2}$$
$$= 5.36 \text{ ms}^{-1}$$
$$\beta = \tan^{-1}\left(\frac{6\cos 60}{2\sqrt{2} - \frac{3}{2}}\right)$$

$= 75.7^\circ$ from positive x-axis

Sphere A: 6.40 ms^{-1}
at 118° from positive x-axis

Sphere B: 5.36 ms^{-1}
at 75.7° from positive x-axis

Your notes

Problem Solving with Oblique Collisions

Energy in 2D Collisions

How do I find the kinetic energy loss from a collision?

- A common question, usually as a follow up to a collisions question, is to find the kinetic energy lost as a result of the impact
- If $e = 1$ kinetic energy is conserved in the collision
  - If $e < 1$ there will be a decrease in the total kinetic energy of the two particles
- Recall that kinetic energy can be calculated using $K.E. = \frac{1}{2}mv^2$
- As the velocity is squared, its sign does not affect the kinetic energy
  - Hence, v^2 here is effectively the speed squared
- When dealing with motion in two dimensions, the velocity may be described in two components, e.g. $\boldsymbol{v} = (3\mathbf{i} - 4\mathbf{j})\ \mathrm{ms}^{-1}$
  - To use this with $K.E. = \frac{1}{2}mv^2$, the magnitude of the vector must be found, using Pythagoras
    - In this case $|v| = \sqrt{3^2 + 4^2} = 5\ \mathrm{ms}^{-1}$
    - If $m = 7\ \mathrm{kg}$ then $K.E. = \frac{1}{2} \times 7 \times 5^2 = 87.5\ \mathrm{J}$
- To find the loss in kinetic energy due to a collision, find the difference between the kinetic energy before the collision, and the kinetic energy after the collision
  - The question may ask to find the loss in kinetic energy for one particular particle,
    - or it could ask to find the total loss in kinetic energy for both particles

Examiner Tip

- When finding the speed from two components, your working will look like this: $v = \sqrt{x^2 + y^2}$
  - When finding kinetic energy, the speed is squared, so your working may look like this: $\frac{1}{2}m\left(\sqrt{x^2 + y^2}\right)^2$
  - It can be quicker to not do the square rooting part if you know you are only using it to find v^2
  - e.g. $\frac{1}{2}m(x^2 + y^2)$
- You can also use the scalar product to find the kinetic energy:
  - $\frac{1}{2}m(\boldsymbol{v} \cdot \boldsymbol{v})$ where $\boldsymbol{v}$ is the vector form of the velocity

Your notes

Your notes

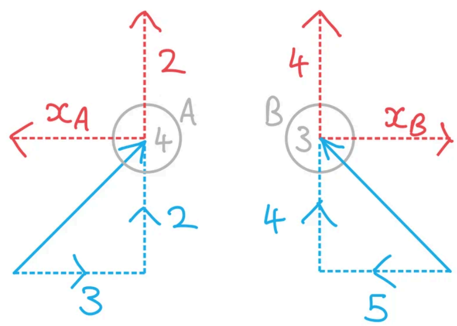
Worked example

A smooth sphere *A* of mass 4 kg is moving on a smooth horizontal surface with velocity $(3\mathbf{i}+2\mathbf{j})\ \mathrm{ms}^{-1}$. Another smooth sphere *B* of mass 3 kg and the same radius as *A* is moving on the same surface with velocity $(-5\mathbf{i}+4\mathbf{j})\ \mathrm{ms}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{i}$. The coefficient of restitution between the spheres is $\frac{3}{4}$.

Find the kinetic energy lost in the impact in total.

Your notes

Draw a diagram showing components before and after



Conservation of momentum

$$(\rightarrow) \quad 4 \times 3 + 3 \times (-5) = 4 \times (-x_A) + 3 \times x_B$$

$$① \quad -3 = 3x_B - 4x_A$$

$$e = \frac{\text{separation}}{\text{approach}} \qquad \frac{3}{4} = \frac{x_A + x_B}{3+5}$$

$$② \quad 6 = x_A + x_B$$

Solving ① and ② $\quad x_A = 3 \quad x_B = 3$

Velocities before the collision $\quad U_A = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad U_B = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

Velocities after the collision $\quad V_A = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad V_B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Kinetic Energy $= \frac{1}{2}mv^2$ where v^2 is $(\text{speed})^2$

Speed is the magnitude of the velocity vector

K.E. before $\quad \frac{1}{2} \times 4 \times (3^2 + 2^2) = 26\text{ J}$

$$\frac{1}{2} \times 3 \times (4^2 + 5^2) = \underline{61.5\text{ J}}$$

$$87.5\text{ J}$$

Your notes

K.E. after $\frac{1}{2} \times 4 \times (3^2 + 2^2) = 26\text{ J}$

$\frac{1}{2} \times 3 \times (3^2 + 4^2) = \underline{37.5\text{ J}}$

63.5 J

Loss in K.E. $= 87.5 - 63.5$

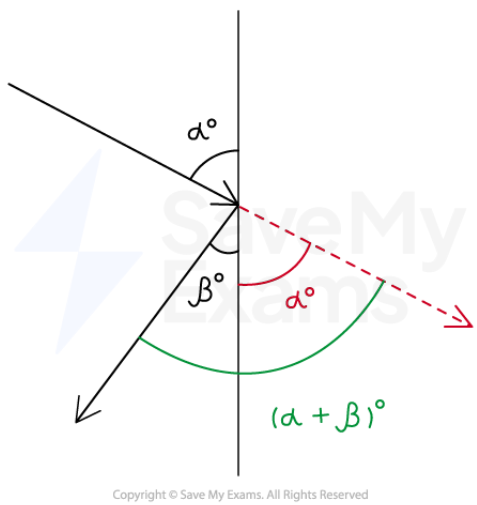
$\boxed{24\text{ J}}$

Your notes

Angles of Deflection

How do I find the angle of deflection after collision with a surface?

- Once the speed and direction of a sphere after a collision have been calculated, a common follow up question is to find the angle of deflection
- The angle of deflection is the angle by which the path of the object has changed from its original trajectory
- To find the angle of deflection it is helpful to sketch a new diagram, with only the angles marked on it
  - The diagram below shows a particle which collides with the surface at angle $\alpha°$ and leaves the surface at angle $\beta°$
- Draw a dashed line showing the continuing path of the object, if it had not collided with the surface
- Use vertically opposite angles to mark the other angle which is equal to $\alpha°$
- It can now be seen that the angle of deflection, the angle by which the path of the object has turned,
  - is equal to $(\alpha + \beta)°$

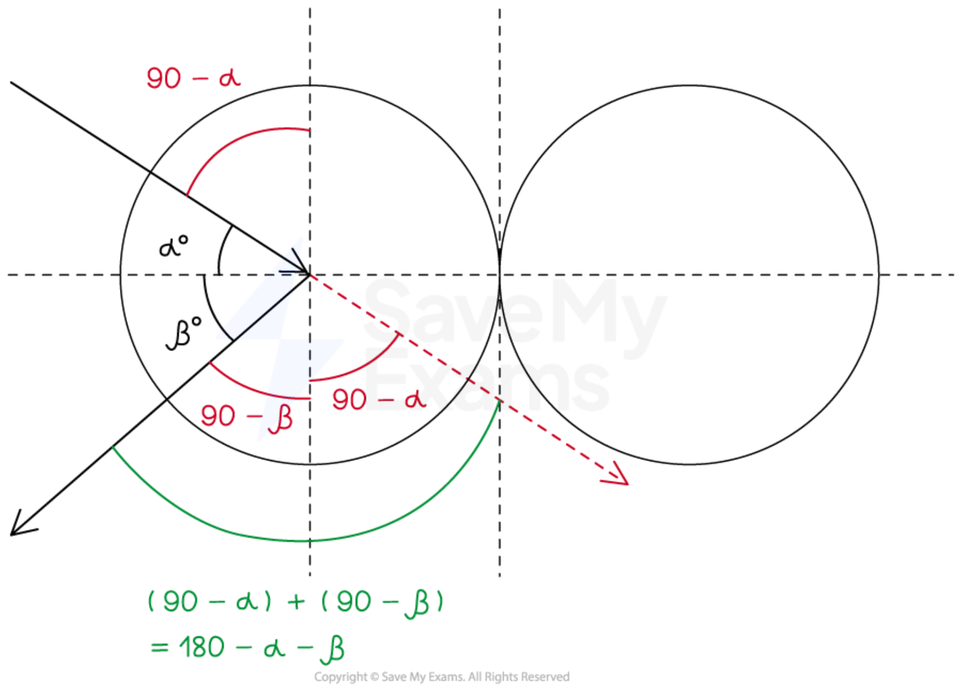


How do I find the angle of deflection after a collision between two spheres?

- Exactly the same concept described for finding the angle of deflection after colliding with a surface, applies when two spheres collide
  - The "surface" when two spheres collide, is the common tangent line between the two spheres
- The angle of deflection is still the angle by which the path of the object has changed from its original trajectory
- The main difference with spheres, is that the angles may be marked in varying ways on the diagram depending on the problem
- To deal with this draw a diagram clearly showing:
  - the velocity of the sphere before and after the collision,
  - a dashed line showing the continuing path of the object, if it had not collided with the other sphere,
  - any angles that were given or have been calculated,

Your notes

- and the common tangent line of the spheres
- Remembering that the angle of deflection is the angle between the continued original path of the sphere, and its new path, find any missing angles on the diagram using:
  - vertically opposite angles are equal,
  - angles on a straight line sum to 180°,
  - and angles in a right angle sum to 90°



- In the above diagram, a sphere collides at an angle $\alpha°$ above the horizontal, and after the collision has a direction which forms angle $\beta°$ below the horizontal
  - By drawing the common tangent line, the angle $90 - \alpha$ can be filled in, adjacent to α
  - Similarly, $90 - \beta$ can be filled in, adjacent to β
  - By using vertically opposite angles, another $90 - \alpha$ angle can be marked on the diagram
  - This then shows (in green) that the angle of deflection, the angle by which the path of the object has turned, is equal to $(90 - \alpha) + (90 - \beta)$
  - This simplifies to $180 - \alpha - \beta$
- Note that this answer does not always apply, as it depends which angles are given and marked on the diagram

How do I use the scalar (dot) product to find the angle of deflection?

- The scalar product (or dot product) can also be used to find the angle of deflection
- We can use the property: $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$
- If $\boldsymbol{a}$ is the vector describing the velocity of the sphere before the collision,

Your notes

- and $\boldsymbol{b}$ is the vector describing the velocity of the sphere after the collision,
- then θ will be the angle of deflection
- This can then be rearranged to $\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}\right)$

Examiner Tip

- Drawing a separate diagram focusing on only the angles can be helpful, without the algebra and masses etc that you may have used earlier in the question
  - You should include the velocities both before and after on the same diagram to do this

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Worked example

Two small smooth spheres *A* and *B* collide when the line joining their centres is parallel to $\mathbf{i}$. Before the collision the velocity of *A* is $(3\mathbf{i}+2\mathbf{j})\ \mathrm{ms}^{-1}$ and the velocity of *B* is $(-4\mathbf{i}+5\mathbf{j})\ \mathrm{ms}^{-1}$. After the collision the velocity of A is $(-5.4\mathbf{i}+2\mathbf{j})\ \mathrm{ms}^{-1}$ and the velocity of B is $(1.6\mathbf{i}+5\mathbf{j})\ \mathrm{ms}^{-1}$.

Find the angle of deflection of sphere *A*.

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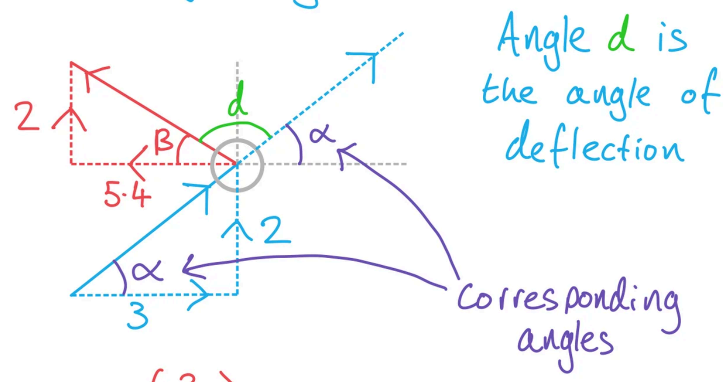
Draw a diagram for sphere A

Show the continued trajectory



Find angle α $\quad \tan^{-1}\left(\frac{2}{3}\right) = 33.690...^\circ$

Find angle β $\quad \tan^{-1}\left(\frac{2}{5.4}\right) = 20.323...^\circ$

Angles on a straight line sum to 180°

$$\alpha + \beta + d = 180$$

$$33.690... + 20.323... + d = 180$$

$$\boxed{\text{Angle of deflection} = 126^\circ \ (3 \text{ s.f.})}$$

Alternatively using scalar product

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta$$

$$\underline{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} -5.4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -5.4 \\ 2 \end{pmatrix} = \sqrt{3^2 + 2^2}\,\sqrt{5.4^2 + 2^2}\,\cos\theta$$

$$-16.2 + 4 = \sqrt{13}\,\sqrt{33.16}\,\cos\theta$$

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$$\cos\theta = \frac{-12.2}{\sqrt{13}\sqrt{33.16}}$$

$$\theta = \cos^{-1}\left(\frac{-12.2}{\sqrt{13}\sqrt{33.16}}\right)$$

$$\boxed{\theta = 126^\circ \ (3 \text{ s.f.})}$$

Your notes

Problem Solving with Oblique Collisions

Tips for problem solving with oblique collisions

- Drawing a large, clear diagram will always help and prevent working from becoming squashed together
  - Establish a routine of how you lay out your diagrams and working, so you can do it routinely and quickly in an exam
- Adding an arrow, or pair of arrows, to show the direction of the impulse will help remind you which components will be affected
- Fill in the things you can work out straight away
  - For example the components of the velocity perpendicular to the impulse are unaffected, so you can usually fill this in immediately
- Use sensible labels for velocities and components of velocities
  - e.g. x_A for the unknown horizontal component of the velocity of sphere *A*, y_A for an unknown vertical component, and v_A for the unknown final speed $\left(\sqrt{x_A{}^2 + y_A{}^2}\right)$ of sphere *A*.
- If you are not sure how to find a piece of missing information, and have filled in everything you can work out by inspection, follow the usual processes to form equations which may help you
  - Apply conservation of linear momentum in the direction parallel to the impulse
    - $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$
  - Apply Newton's law of restitution in the direction of the impulse
    - $e = \dfrac{\text{Speed of separation of the objects}}{\text{Speed of approach of the objects}}$
  - This will often lead to a pair of simultaneous equations which can be solved
  - Be careful with positive and negative signs when forming these equations, the signs of the velocities will depend on how they are modelled in the diagram
    - e.g. If the arrows are drawn pointing away from each other in the "after" diagram, the speed of separation will be $x_A + x_B$
    - If the arrows were drawn both pointing to the right, the speed of separation would be $x_B - x_A$
- The equation $I = m(v - u)$, where I is the impulse, can also sometimes be useful with 2D collisions
  - It can be used with vectors for $\mathbf{I}$, $\mathbf{v}$, and $\mathbf{u}$:
    - $\mathbf{I} = m(\mathbf{v} - \mathbf{u})$
  - This is particularly helpful when a question does not describe the direction of a wall, as it can be used to find the direction of the impulse, which is always perpendicular to the wall
- Make use of the scalar (dot) product formulae for collisions with a surface
  - $\mathbf{v} \cdot \mathbf{P} = -e\,\mathbf{u} \cdot \mathbf{P}$ where $\mathbf{P}$ is the direction perpendicular to the surface
  - $\mathbf{v} \cdot \mathbf{W} = \mathbf{u} \cdot \mathbf{W}$ where $\mathbf{W}$ is the direction of the surface (or wall)
    - Remember that the direction of the wall and the direction of the impulse will be perpendicular, e.g. $(3\mathbf{i} + 1\mathbf{j})$ and $(-1\mathbf{i} + 3\mathbf{j})$
  - These formulae are most useful when a surface is not in a "nice" direction

    - e.g. when it is not parallel to the *x* or *y* axis when the sphere is moving in the *xy*-plane
- You may need to consider kinetic energy before and after the collision for one or both of the objects using,
  - $K.E. = \frac{1}{2}mv^2$

Your notes

Examiner Tip

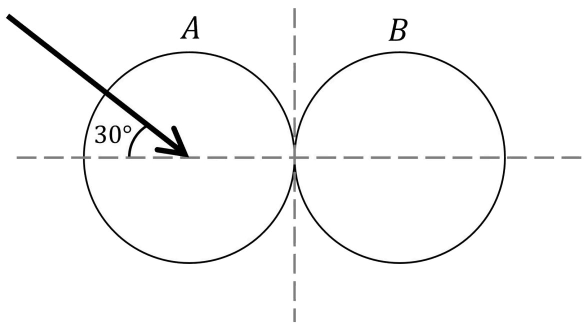
- Sometimes questions will be entirely algebraic,
  - in this case apply the same procedures and methods as above,
  - and form equations using conservation of linear momentum and Newton's law of restitution
  - Label any angles on the diagram clearly and use simple angle geometry to help you
    - e.g. Vertically opposite angles are equal, angles in a right angle sum to 90°, and angles on a straight line sum to 180°
- Sometimes you may need to use trigonometric identities from your pure maths knowledge to help simplify or rearrange to reach a required expression

Your notes

Worked example

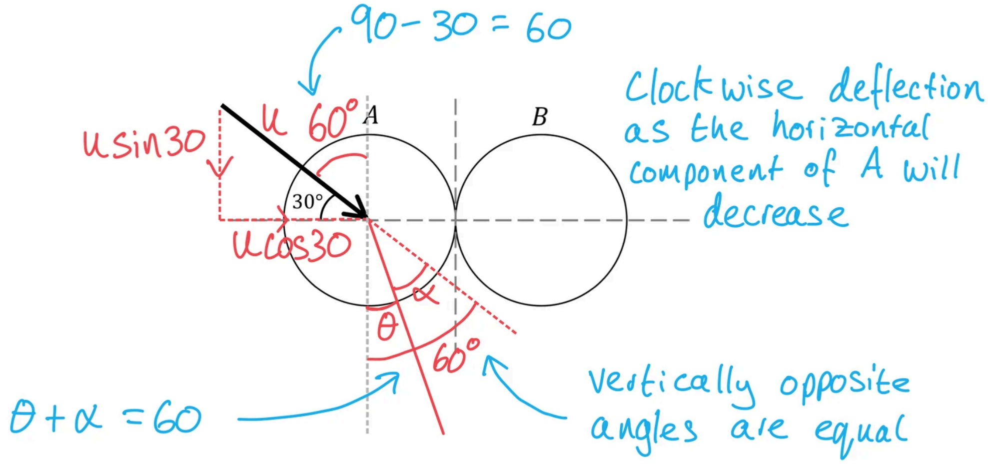
A smooth uniform sphere *A* collides with an identical sphere *B* which is at rest. When the spheres collide *A* is moving such that it forms an angle of 30° to the line joining the centres of the spheres, as shown in the diagram. The coefficient of restitution between the two spheres is e. Sphere *A* is deflected by angle α as a result of the collision, where $0° < \alpha < 60°$.

Show that $\tan \alpha = \dfrac{\sqrt{3}(1+e)}{5-3e}$.

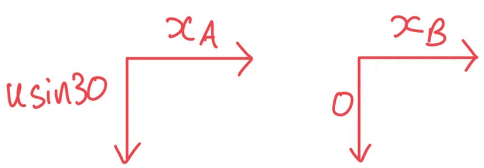


Your notes

Draw a diagram showing the angle of deflection



Velocities after the collision



Model x_A as to the right, as $0° < \alpha < 60°$

Conservation of momentum

$$(\rightarrow) \quad mu\cos 30 + 0 = mx_A + mx_B$$

$$① \quad \frac{\sqrt{3}}{2}u = x_A + x_B$$

$e = \dfrac{\text{separation}}{\text{approach}}$

$$e = \frac{x_B - x_A}{u\cos 30}$$

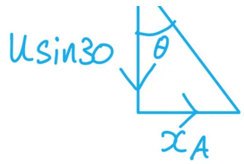
$$② \quad \frac{\sqrt{3}}{2}ue = x_B - x_A$$

$$① - ② \quad \frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{2}ue = 2x_A$$

$$x_A = \frac{\sqrt{3}}{4}u(1-e)$$

Velocity of A after collision

$\frac{\sqrt{3}}{4}u(1-e)$



$$\tan\theta = \frac{x_A}{u\sin 30} = \frac{\text{[illegible]}}{\frac{1}{2}u}$$

$$\tan\theta = \frac{\sqrt{3}}{2}(1-e)$$

From the first diagram, $\theta + \alpha = 60$

$$\alpha = 60 - \theta \qquad \tan\alpha = \tan(60-\theta)$$

Expand using $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\tan\alpha = \frac{\tan 60 - \tan\theta}{1 + \tan 60 \tan\theta}$$

$$\tan\alpha = \frac{\sqrt{3} - \frac{\sqrt{3}}{2}(1-e)}{1 + \sqrt{3}\left(\frac{\sqrt{3}}{2}(1-e)\right)}$$

$$\tan\alpha = \frac{2\sqrt{3} - \sqrt{3} + \sqrt{3}e}{2 + 3 - 3e}$$

$$\boxed{\tan\alpha = \frac{\sqrt{3}(1+e)}{5-3e}}$$

Your notes