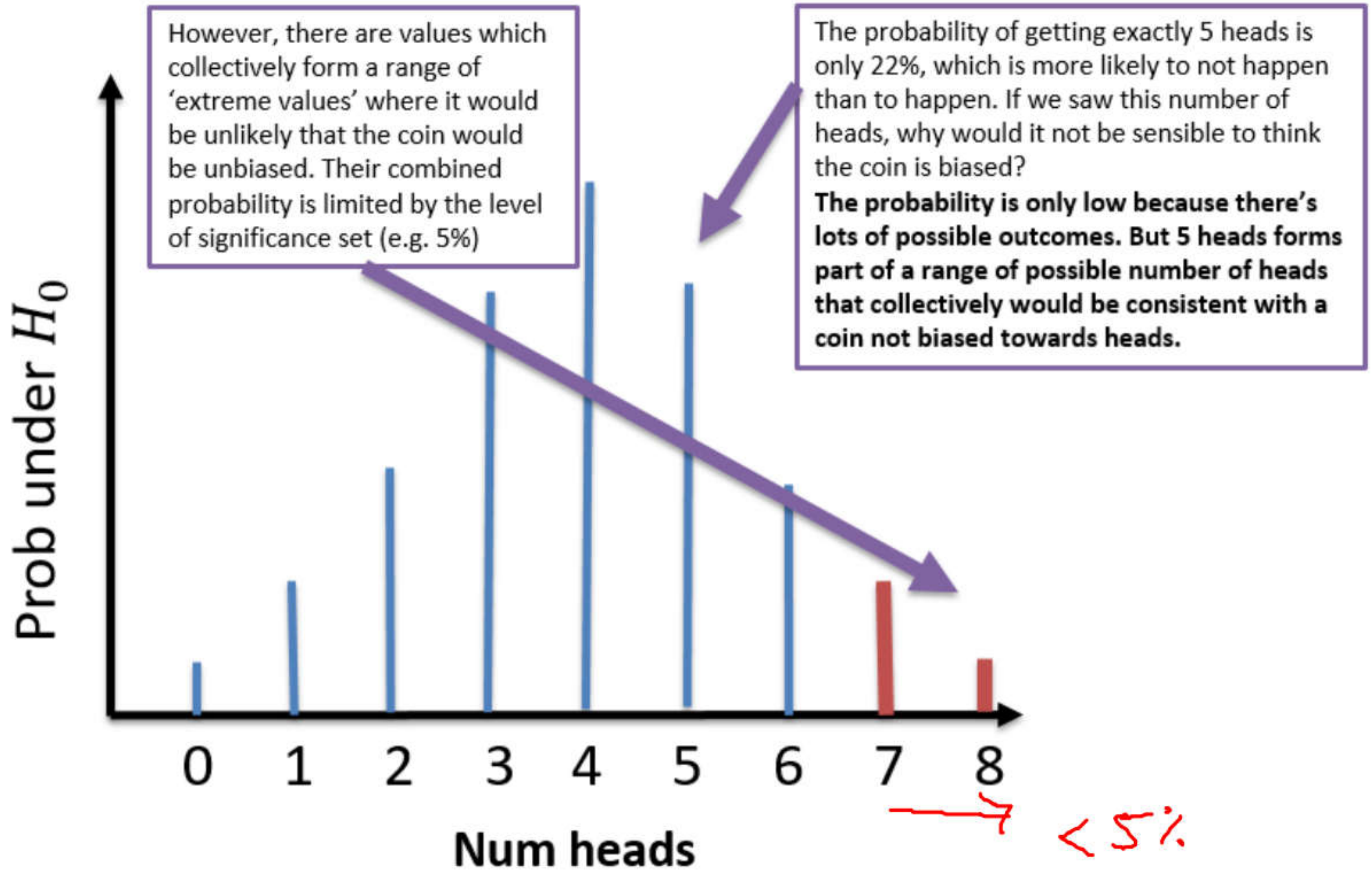
Critical Regions and Values

John wants to see whether a coin is unbiased or whether **it is biased towards coming down heads**. He tosses the coin 8 times and counts the number of times X, it lands head uppermost. **What values would lead to John's hypothesis being rejected?**

As before, we're interested how likely a given outcome is likely to happen 'just by chance' under the null hypothesis (i.e. when the coin is not biased).



John wants to see whether a coin is unbiased or whether **it is biased towards coming down heads**. He tosses the coin 8 times and counts the number of times X, it lands head uppermost. **What values would lead to John's hypothesis being rejected**, if the significance level was 5%?

null hypothesis

$H_0 \quad p = 0.5$

What's the probability that we would see **6 heads**, or an **even more extreme value**? Is this sufficiently unlikely to support John's claim that the coin is biased?

$$P(X \geq 6) = 1 - P(X \leq 5)$$
$$= 1 - 0.8555$$
$$= 0.1445 \quad 14\%$$

What's the probability that we would see **7 heads**, or an **even more extreme value**?

$$P(X \geq 7) = 1 - P(X \leq 6) = 1 - 0.9648$$
$$= 0.0352$$
$$3.5\%$$

C.D.F. Binomial table: $p = 0.5, n = 8$

x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
3	0.3633
4	0.6367
5	0.8555
6	0.9648
7	0.9961

7

✎ The value(s) on the boundary of the critical region are called **critical value(s)**.

✎ The **critical region** is the range of values of the test statistic that would lead to you rejecting H_0

$X \geq 7$

we conclude that there is evidence to reject H_0, i.e. the coin is biased.

✎ The **actual significance level** is the actual probability of being in the critical region. $= 0.0352$

Determine the critical region when we throw a coin where we're trying to establish if there's the specified bias, given the specified number of throws, when the level of significance is 5%. Here p is the probability of landing on heads (i.e. success is H on coin)

Coin thrown 5 times. Trying to establish if biased towards heads. $H_1: p > 0.5$

$p = 0.5, n = 5$

x	$P(X \le x)$
0	0.0312
1	0.1875
2	0.5000
3	0.8125
4	0.9688

$P(X \le 4) = 0.9688$

$P(X \geq 5) = 1 - 0.9688$

$= 0.0312$

$\cancel{X \geq 5}$ $X = 5$

Coin thrown 10 times. Trying to establish if biased towards heads. $p > 0.5$

$p = 0.5, n = 10$

x	$P(X \le x)$
0	0.0010
1	0.0107
2	0.0547
...	...
7	0.9453
8	0.9893
9	0.9990

$X \geq 9$

Coin thrown 10 times. Trying to establish if biased towards tails. $p < 0.5$

$p = 0.5, n = 10$

x	$P(X \le x)$
0	0.0010
1	0.0107
2	0.0547
...	...
7	0.9453
8	0.9893
9	0.9990

$X \leq 1$

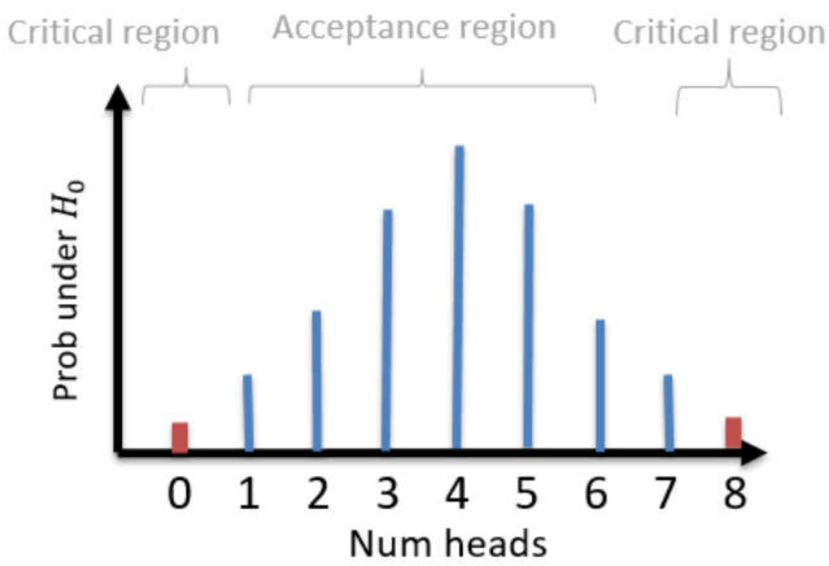
Tip: At the positive tail, use the value AFTER the first that exceeds 95% (100 - 5).

At the negative tail, we just use the first value that goes under the significance level.

Two-tailed test

Suppose I threw a coin 8 times and was now interested in how ~~may~~ many heads would suggest it was a **biased coin** (i.e. either way!). How do we work out the critical values now, with 5% significance?

We split the 5% so there's 2.5% at either tail, then proceed as normal:



Critical region at positive tail:
Look at closest value above 0.975 (then go one above):

$X = 8$

Critical region at negative tail:
Look at closest value below 0.025.

$X = 0$

C.D.F. Binomial table: $p = 0.5, n = 8$	
x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
…	…
6	0.9648
7	0.9961
8	1

A random variable X has binomial distribution $B(40, p)$. A single observation is used to test $H_0: p = 0.25$ against $H_1: p \neq 0.25$. → 1% at each end

The ≠ indicates bias either way, i.e. two-tailed.

a) Using the 2% level of significance, find the critical region of this test. The probability in each tail should be as close as possible to 0.01.

This means you find the closest to 0.01 (even if slightly above) rather than the closest under 0.01

b) Write down the actual significance level of the test.

a) $X \leq 3$ $\quad P(X \leq 3) = 0.0047$

$X \geq 17$ $\quad P(X \geq 17) = 1 - 0.9884$

$= 0.0116$

b) $0.0047 + 0.0116 = 0.0164$

C.D.F. Binomial table: $p = 0.25, n = 40$

x	$P(X \leq x)$
2	0.0010
3	0.0047
4	0.0160
5	0.0433
16	0.9884
17	0.9953
18	0.9983
19	0.9994

Doing a full one-tailed hypothesis test

John tosses a coin 8 times and it comes up heads 6 times. He claims the coin is **biased towards heads**. With a significance level of 5%, test his claim.

Let

X be the number of heads.

$X \sim B(8, 0.5)$

$B(8, p)$ p is the probability of Heads

$H_0: p = 0.5$

$H_1: p > 0.5$

Assume H_0 is true,

$P(X \geq 6) = 0.1445$

$0.1445 > 0.05$

So, there is no evidence to reject H_0.

The coin is a fair coin.

STEP 1: Define test statistic X (stating its distribution), and the parameter p.

STEP 2: Write null and alternative hypotheses.

STEP 3: Determine probability of observed test statistic (or 'more extreme'), assuming null hypothesis.
i.e. Determine probability we'd see this outcome just by chance.

STEP 4: Two-part conclusion:
1. Do we reject H_0 or not?
2. Put in context of original problem.

C.D.F. Binomial table:
$p = 0.5, n = 8$

x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
3	0.3633
4	0.6367
5	0.8555
6	0.9648
7	0.9961

Alternative method using critical regions

We can also find the critical region and see if the test statistic lies within it.

John tosses a coin 8 times and it comes up heads 6 times. He claims the coin is **biased towards heads**. With a significance level of 5%, test his claim.

X is number of heads.
p is probability of heads.
$X \sim B(8, p)$

$H_0: p = 0.5$
$H_1: p > 0.5$

Critical region $X \geq 7$
Because 6 is not in the crit. region, we do not reject Ho, and so the coin is not biased towards heads.

STEP 1: Define test statistic X (stating its distribution), and the parameter p.

STEP 2: Write null and alternative hypotheses.

STEP 3 (Alternative): Determine critical region.

STEP 4: Two-part conclusion:
1. Do we reject H_0 or not?
2. Put in context of original problem.

C.D.F. Binomial table: $p = 0.5, n = 8$

x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
3	0.3633
4	0.6367
5	0.8555
6	0.9648
7	0.9961

The standard treatment for a particular disease has a $\frac{2}{5}$ probability of success. A certain doctor has undertake research in this area and has produced a new drug which has been successful with 11 out of 20 patients. The doctor claims the new drug represents an improvement on the standard treatment.
Test, at the 5% significance level, the claim made by the doctor.

STEP 1: Define test statistic X (stating its distribution), and the parameter p.

X is the number of patients successfully treated.

p is the probability of successful treatment.

$X \sim B(20, p)$

STEP 2: Write null and alternative hypotheses.

$H_0: p = 0.4$

$H_1: p > 0.4$

STEP 3: Determine probability of observed test statistic (or 'more extreme'), assuming null hypothesis.

Assume H_0, $X \sim B(20, 0.4)$

$$P(X \geq 11) = 1 - P(X \leq 10)$$
$$= 1 - 0.8725$$
$$= 0.1275 > 0.05$$

STEP 4: Two-part conclusion:
1. Do we reject H_0 or not?
2. Put <u>in context of original problem</u>.

So, there is not enough evidence to reject H_0.
The doctor's claim is not supported, same level of success.

Your Turn

Edexcel S2 Jan 2011 Q2

$\to p = 0.2$

A student takes a multiple choice test. The test is made up of 10 questions each with 5 possible answers. The student gets 4 questions correct. Her teacher claims she was guessing the answers. Using a one tailed test, at the 5% level of significance, test whether or not there is evidence to reject the teacher's claim.
State your hypotheses clearly. **(6)**

$10 \times 0.2 = 2$ questions

4 right!

X is the number of correct questions
p is the probability of getting it right.

$H_0: p = 0.2$

$H_1: p > 0.2$

Assume H_0, $X \sim B(10, 0.2)$ $\to$ $1 - P(X \leq 3)$

$P(X \geq 4) = 0.1209 > 0.05$

Not enough evidence to reject H_0,
so teacher's claim is upheld, student was guessing

Crit. reg. is
$X \geq 5$

Her mark is 4,
so not in region.
No evidence to reject H_0.

Ex 7C

Note the mark for stating distribution of X under null hypothesis.

$H_0 : p = 0.2 \quad H_1 : p > 0.2$	B1
Under H_0, $X \sim \text{Bin}(10,0.2)$	B1
$P(X \geq 4) = 1 - P(X \leq 3)$ OR $P(X \leq 4) = 0.9672$	M1
$= 1 - 0.8791$ $\quad P(X \geq 5) = 0.0328$	
$= 0.1209$ $\quad$ CR $X \geq 5$	A1
$0.1209 > 0.05$. Insufficient evidence to reject H_0 so teacher's claim is supported.	M1A1ft
	[6]

Note two-mark conclusion.

Two-Tailed Tests

We have already seen that if we're interest in bias 'either way', we have two tails, and therefore have to split the critical region by **halving the significance level at each end.**

NV NV V

2.5% at each end.

Over a long period of time it has been found that in Enrico's restaurant the ratio of non-veg to veg meals is 2 to 1. In Manuel's restaurant in a random sample of 10 people ordering meals, 1 ordered a vegetarian meal. Using a 5% level of significance, test whether or not the proportion of people eating veg meals in Manuel's restaurant is different to that in Enrico's restaurant.

X is the number of veg meals ordered.
p is the probability of ordering veg meal.

$X \sim B(10, p)$

$\cancel{P(X \geq 1) = 0.98265...}$

$H_0: p = \frac{1}{3}$

$H_1: p \neq \frac{1}{3}$

Assume H_0, $X \sim B(10, \frac{1}{3})$

$P(X \leq 1) = 0.1040 > 0.025$

not enough evidence to reject H_0,

So, proportion of veg in both restaurants is the same.

Edexcel S2 Jan 2006 Q7a

A teacher thinks that 20% of the pupils in a school read the Deano comic regularly.

He chooses 20 pupils at random and finds 9 of them read the Deano.

(*a*) (i) Test, at the 5% level of significance, whether or not there is evidence that the percentage of pupils that read the Deano is different from 20%. State your hypotheses clearly.

(ii) State all the possible numbers of pupils that read the Deano from a sample of size 20 that will make the test in part (*a*)(i) significant at the 5% level. **(9)**

expect

20% of 20 = 4 comic readers

X is the number of students reading Deano.

p is the probability a student reads Deano.

$X \sim B(20, p)$

$H_0: p = 0.2$

$H_1: p \neq 0.2$

Assume H_0, $X \sim B(20, 0.2)$

$P(X \geq 9) = 1 - P(X \leq 8)$

$= 1 - 0.9900$

$= 0.01$ < 0.025

so, evidence to reject H_0, the % of readers is not 20%.

ii) Critical values/regions

$X = 0$

$X \geq 9$

$P(X=0) = 0.0115$

$P(X \geq 9) = 1 - 0.99$

$= 0.01$

actual significance

$0.01 + 0.0115$

$= 0.0215$

Ex 7D

(a)(i)	$H_0 : p = 0.2, H_1 : p \neq 0.2$	$p =$	B1B1
	$P(X \geq 9) = 1 - P(X \leq 8)$ or attempt critical value/region		M1
	$= 1 - 0.9900 = 0.01$ CR $X \geq 9$		
	$0.01 < 0.025$ or $9 \geq 9$ or $0.99 > 0.975$ or $0.02 < 0.05$ or lies in interval with correct interval stated.		A1
	Evidence that the percentage of pupils that read Deano is not 20%		A1
(ii)	$X \sim \text{Bin}(20, 0.2)$	may be implied or seen in (i) or (ii)	B1
	So 0 or [9,20] make test significant.	0,9,between "their 9" and 20	B1B1B1
			(9)