Algebraic Expressions: Index Laws and Surds

base

$$3^5$$

exponent or power or index (plural: indices)

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{mn}$$
$$(ab)^n = a^n b^n$$

Simplify $(a^3)^2 \times 2a^2$

Simplify $(4x^3y)^3$

Simplify $2x^2(3+5x) - x(4-x^2)$

Simplify $\frac{x^3-2x}{3x^2}$

Pro Tip: A common student error is to get the sign wrong of $+x^3$

Pro Tip: While $\frac{a+b}{c}$ can be split into $\frac{a}{c}+\frac{b}{c}$, a common student error is to think that $\frac{a}{b+c} = \frac{a}{b}+\frac{a}{c}$

Your Turn

1 Simplify $\left(\frac{2a^5}{a^2}\right)^2 \times 3a$

2 Simplify $\frac{2x+x^5}{4x^3}$

3 Expand and simplify
$$2x(3-x^2) - 4x^3(3-x)$$

4 Simplify $2^x \times 3^x$

Note: This is using $(ab)^n = a^n b^n$ law backwards.

Exercise 1A

1 Simplify these expressions:

a $x^3 \times x^4$ **b** $2x^3 \times 3x^2$ **c** $\frac{k^3}{k^2}$

d $\frac{4p^3}{2p}$ **e** $\frac{3x^3}{3x^2}$ **f** $(y^2)^5$

g $10x^5 \div 2x^3$ **h** $(p^3)^2 \div p^4$ **i** $(2a^3)^2 \div 2a^3$

j $8p^4 \div 4p^3$ **k** $2a^4 \times 3a^5$ **l** $\frac{21a^3b^7}{7ab^4}$

m $9x^2 \times 3(x^2)^3$ **n** $3x^3 \times 2x^2 \times 4x^6$ **o** $7a^4 \times (3a^4)^2$

p $(4y^3)^3 \div 2y^3$ **q** $2a^3 \div 3a^2 \times 6a^5$ **r** $3a^4 \times 2a^5 \times a^3$

3 Simplify these fractions:

a $\frac{6x^4 + 10x^6}{2x}$ **b** $\frac{3x^5 - x^7}{x}$ **c** $\frac{2x^4 - 4x^2}{4x}$

d $\frac{8x^3 + 5x}{2x}$ **e** $\frac{7x^7 + 5x^2}{5x}$ **f** $\frac{9x^5 - 5x^3}{3x}$

Exercise 1A

Answers

1 **a** x^7 **b** $6x^5$ **c** k **d** $2p^2$
e x **f** y^{10} **g** $5x^2$ **h** p^2
i $2a^3$ **j** $2p$ **k** $6a^9$ **l** $3a^2b^3$
m $27x^8$ **n** $24x^{11}$ **o** $63a^{12}$ **p** $32y^6$
q $4a^6$ **r** $6a^{12}$

3 **a** $3x^3 + 5x^5$ **b** $3x^4 - x^6$ **c** $\frac{x^3}{2} - x$
d $4x^2 + \frac{5}{2}$ **e** $\frac{7x^6}{5} + x$ **f** $3x^4 - \frac{5x^2}{3}$

Extension

1 *[MAT 2006 1A]*

Which of the following numbers is largest?

- $\left(\left(2^3\right)^2\right)^3$
- $\left(2^3\right)^{\left(2^3\right)}$
- $2^{\left(\left(3^2\right)^3\right)}$
- $2^{\left(3^{\left(2^3\right)}\right)}$

2 *[MAT 2012 1B]*

Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers.

Then N will definitely be a square number whenever:

- k is even;
- $k + n$ is odd;
- k is odd but $m + n$ is even;
- $k + n$ is even.

Negative and Fractional Indices

$$a^0 = 1$$
$$a^{\frac{1}{m}} = \sqrt[m]{a}$$
$$a^{\frac{n}{m}} = (\sqrt[m]{a})^n$$
$$a^{-m} = \frac{1}{a^m}$$

Pro Note: $\sqrt{9}$ only means the <u>positive</u> square root of 9, i.e. 3 not -3. Otherwise, what would be the point of the $\pm$ in the quadratic formula before the $\sqrt{b^2 - 4ac}$?

Prove that $x^{\frac{1}{2}} = \sqrt{x}$

Evaluate $27^{-\frac{1}{3}}$

Evaluate $32^{\frac{2}{5}}$

Simplify $\left(\frac{1}{9}x^6y\right)^{\frac{1}{2}}$

Evaluate $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

If $b = \frac{1}{9}a^2$, determine $3b^{-2}$ in the form kb^n where k, n are constants.

Exercise 1D

1 Simplify:

a $x^3 \div x^{-2}$ **b** $x^5 \div x^7$ **c** $x^{\frac{3}{2}} \times x^{\frac{5}{2}}$

d $(x^2)^{\frac{3}{2}}$ **e** $(x^3)^{\frac{5}{3}}$ **f** $3x^{0.5} \times 4x^{-0.5}$

g $9x^{\frac{2}{3}} \div 3x^{\frac{1}{6}}$ **h** $5x^{\frac{7}{5}} \div x^{\frac{2}{5}}$ **i** $3x^4 \times 2x^{-5}$

j $\sqrt{x} \times \sqrt[3]{x}$ **k** $(\sqrt{x})^3 \times (\sqrt[3]{x})^4$ **l** $\dfrac{(\sqrt[3]{x})^2}{\sqrt{x}}$

2 Evaluate:

a $25^{\frac{1}{2}}$ **b** $81^{\frac{3}{2}}$ **c** $27^{\frac{1}{3}}$

d 4^{-2} **e** $9^{-\frac{1}{2}}$ **f** $(-5)^{-3}$

g $\left(\frac{3}{4}\right)^0$ **h** $1296^{\frac{3}{4}}$ **i** $\left(\frac{25}{16}\right)^{\frac{3}{2}}$

j $\left(\frac{27}{8}\right)^{\frac{2}{3}}$ **k** $\left(\frac{6}{5}\right)^{-1}$ **l** $\left(\frac{343}{512}\right)^{-\frac{2}{3}}$

3 Simplify:

a $(64x^{10})^{\frac{1}{2}}$ **b** $\dfrac{5x^3 - 2x^2}{x^5}$ **c** $(125x^{12})^{\frac{1}{3}}$ **d** $\dfrac{x + 4x^3}{x^3}$

e $\dfrac{2x + x^2}{x^4}$ **f** $\left(\frac{4}{9}x^4\right)^{\frac{3}{2}}$ **g** $\dfrac{9x^2 - 15x^5}{3x^3}$ **h** $\dfrac{5x + 3x^2}{15x^3}$

Answers

Exercise 1D

1 **a** x^5 **b** x^{-2} **c** x^4 **d** x^3
e x^5 **f** $12x^0 = 12$ **g** $3x^{\frac{1}{2}}$ **h** $5x$
i $6x^{-1}$ **j** $x^{\frac{5}{6}}$ **k** $x^{\frac{17}{6}}$ **l** $x^{\frac{1}{6}}$

2 **a** 5 **b** 729 **c** 3 **d** $\frac{1}{16}$
e $\frac{1}{3}$ **f** $\frac{-1}{125}$ **g** 1 **h** 216
i $\frac{125}{64}$ **j** $\frac{9}{4}$ **k** $\frac{5}{6}$ **l** $\frac{64}{49}$

3 **a** $8x^5$ **b** $\frac{5}{x^2} - \frac{2}{x^3}$ **c** $5x^4$
d $\frac{1}{x^2} + 4$ **e** $\frac{2}{x^3} + \frac{1}{x^2}$ **f** $\frac{8}{27}x^6$
g $\frac{3}{x} - 5x^2$ **h** $\frac{1}{3x^2} + \frac{1}{5x}$

Extension

[MAT 2007 1A]

Let r and s be integers. Then

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

is an integer if

- $r + s \leq 0$
- $s \leq 0$
- $r \leq 0$
- $r \geq s$

Hint:

Surds

Recap:

A surd is a root of a number that does not simplify to a rational number.

Laws:

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$
$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

Note: A *rational* number is any which can be expressed as $\frac{a}{b}$ where a, b are integers. $\frac{2}{3}$ and $\frac{4}{1} = 4$ are rational numbers, but π and $\sqrt{2}$ are not.

$\sqrt{3} \times 2$

$\sqrt{12} + \sqrt{27}$

$3\sqrt{5} \times 2\sqrt{5}$

$(\sqrt{8} + 1)(\sqrt{2} - 3)$

$\sqrt{8}$

Exercise 1E

1 Do not use your calculator for this exercise. Simplify:

a $\sqrt{28}$ **b** $\sqrt{72}$ **c** $\sqrt{50}$

d $\sqrt{32}$ **e** $\sqrt{90}$ **f** $\frac{\sqrt{12}}{2}$

g $\frac{\sqrt{27}}{3}$ **h** $\sqrt{20}+\sqrt{80}$ **i** $\sqrt{200}+\sqrt{18}-\sqrt{72}$

j $\sqrt{175}+\sqrt{63}+2\sqrt{28}$ **k** $\sqrt{28}-2\sqrt{63}+\sqrt{7}$ **l** $\sqrt{80}-2\sqrt{20}+3\sqrt{45}$

m $3\sqrt{80}-2\sqrt{20}+5\sqrt{45}$ **n** $\frac{\sqrt{44}}{\sqrt{11}}$ **o** $\sqrt{12}+3\sqrt{48}+\sqrt{75}$

2 Expand and simplify if possible:

a $\sqrt{3}(2+\sqrt{3})$ **b** $\sqrt{5}(3-\sqrt{3})$ **c** $\sqrt{2}(4-\sqrt{5})$

d $(2-\sqrt{2})(3+\sqrt{5})$ **e** $(2-\sqrt{3})(3-\sqrt{7})$ **f** $(4+\sqrt{5})(2+\sqrt{5})$

g $(5-\sqrt{3})(1-\sqrt{3})$ **h** $(4+\sqrt{3})(2-\sqrt{3})$ **i** $(7-\sqrt{11})(2+\sqrt{11})$

3 Simplify $\sqrt{75}-\sqrt{12}$ giving your answer in the form $a\sqrt{3}$, where a is an integer.

Exercise 1E

Answers

1 **a** $2\sqrt{7}$ **b** $6\sqrt{2}$ **c** $5\sqrt{2}$ **d** $4\sqrt{2}$
e $3\sqrt{10}$ **f** $\sqrt{3}$ **g** $\sqrt{3}$ **h** $6\sqrt{5}$
i $7\sqrt{2}$ **j** $12\sqrt{7}$ **k** $-3\sqrt{7}$ **l** $9\sqrt{5}$
m $23\sqrt{5}$ **n** 2 **o** $19\sqrt{3}$

2 **a** $2\sqrt{3}+3$ **b** $3\sqrt{5}-\sqrt{15}$
c $4\sqrt{2}-\sqrt{10}$ **d** $6+2\sqrt{5}-3\sqrt{2}-\sqrt{10}$
e $6-2\sqrt{7}-3\sqrt{3}+\sqrt{21}$ **f** $13+6\sqrt{5}$
g $8-6\sqrt{3}$ **h** $5-2\sqrt{3}$
i $3+5\sqrt{11}$

3 $3\sqrt{3}$

Extension

[SMC 2014 Q24] Which of the following is smallest?

- $10-3\sqrt{11}$
- $8-3\sqrt{7}$
- $5-2\sqrt{6}$
- $9-4\sqrt{5}$
- $7-4\sqrt{3}$

Hint:

Rationalising the Denominator

Here's a surd.
What could we multiply it by such that it's no longer an irrational number?

$$\sqrt{5}$$

In this fraction, the denominator is irrational.
'**Rationalising the denominator**' means making the denominator a rational number.
What could we multiply this fraction by to both rationalise the denominator, but <u>leave the value of the fraction unchanged?</u>

$$\frac{1}{\sqrt{2}}$$

Side Note: There's two reasons why we might want to do this:
1. For aesthetic reasons, it makes more sense to say "half of root 2" rather than "one root two-th of 1". It's nice to divide by something whole!
2. It makes it easier for us to add expressions involving surds.

$$\frac{3}{\sqrt{2}} =$$

$$\frac{6}{\sqrt{3}} =$$

$$\frac{7}{\sqrt{7}} =$$

$$\frac{15}{\sqrt{5}} + \sqrt{5} =$$

Test Your Understanding:

$$\frac{12}{\sqrt{3}} =$$

$$\frac{2}{\sqrt{6}} =$$

$$\frac{4\sqrt{2}}{\sqrt{8}} =$$

More Complex Denominators

$$\frac{1}{\sqrt{2}+1}$$

We basically use the same expression but with the sign reversed (this is known as the *conjugate*). That way, we obtain the difference of two squares. Since $(a+b)(a-b)=a^2-b^2$, any surds will be squared and thus we'll end up with no surds in the denominator.

$$\frac{3}{\sqrt{6}-2}$$

You can explicitly expand out $(\sqrt{6}-2)(\sqrt{6}+2)$ in the denominator, but remember that $(a-b)(a+b)=a^2-b^2$ so we can mentally obtain $6-4=2$
Just remember: 'difference of two squares'!

$$\frac{4}{\sqrt{3}+1}$$

$$\frac{3\sqrt{2}+4}{5\sqrt{2}-7}$$

Your Turn

Rationalise the denominator and simplify
$$\frac{4}{\sqrt{5}-2}$$

Rationalise the denominator and simplify
$$\frac{2\sqrt{3}-1}{3\sqrt{3}+1}$$

AQA IGCSE FM June 2013 Paper 1

Solve $y\left(\sqrt{3}-1\right)=8$
Give your answer in the form $a+b\sqrt{3}$ where a and b are integers.

1 Rationalise the denominator and simplify the following:

a) $\frac{1}{\sqrt{5}+2}=$

b) $\frac{\sqrt{3}}{\sqrt{3}-1}=$

c) $\frac{\sqrt{5}+1}{\sqrt{5}-2}=$

d) $\frac{2\sqrt{3}-1}{3\sqrt{3}+4}=$

e) $\frac{5\sqrt{5}-2}{2\sqrt{5}-3}=$

2 Expand and simplify:
$$\left(\sqrt{5}+3\right)\left(\sqrt{5}-2\right)\left(\sqrt{5}+1\right)=$$

3 Rationalise the denominator, giving your answer in the form $a+b\sqrt{3}$.
$$\frac{3\sqrt{3}+7}{3\sqrt{3}-5}=$$

4 Solve $x\left(4-\sqrt{6}\right)=10$ giving your answer in the form $a+b\sqrt{6}$.

5 Solve $y\left(1+\sqrt{2}\right)-\sqrt{2}=3$
$$y=\frac{3+\sqrt{2}}{1+\sqrt{2}}=$$

Simplify:

6 $\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}+\sqrt{a}}=$