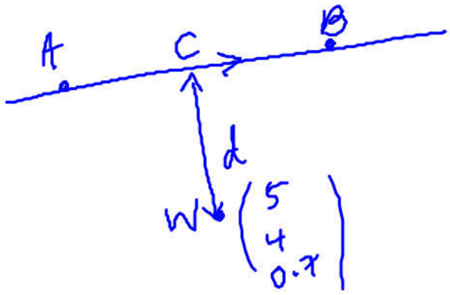
9 A birdwatcher is located on a hilltop. Relative to a fixed origin O, the position vector of the birdwatcher is $\begin{pmatrix} 5 \\ 4 \\ 0.7 \end{pmatrix}$ km. The birdwatcher is able to spot any bird that flies within 500 m of her position. A kestrel flies from point A to point B, where points A and B have position vectors $\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ km and $\begin{pmatrix} 12 \\ 0 \\ 1.2 \end{pmatrix}$ km respectively. The kestrel is modelled as flying in a straight line.

0.5km

a Use the model to determine whether the birdwatcher is able to spot the kestrel. **(7 marks)**

b Give one criticism of the model. **(1 mark)**



$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 12 \\ 0 \\ 1.2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \\ 1.2 \end{pmatrix}$$

$$l \quad \underline{r} = \begin{pmatrix} 12 + 9\lambda \\ 0 - 5\lambda \\ 1.2 + 1.2\lambda \end{pmatrix}$$

$$\vec{WC} = \underline{c} - \underline{w} = \begin{pmatrix} 7 + 9\lambda \\ -4 - 5\lambda \\ 0.5 + 1.2\lambda \end{pmatrix} \qquad \vec{WC} \cdot \vec{AB} = 0$$

$$63 + 81\lambda + 20 + 25\lambda + 0.6 + 1.44\lambda = 0$$

$$107.44\lambda = -83.6$$

$$\lambda = \frac{-1045}{1343}$$

$$\vec{WC} = \begin{pmatrix} -2.978 \times 10^{-3} \\ -\frac{147}{1343} \\ -\frac{1165}{2686} \end{pmatrix}$$

$WC = 0.447$ km $\qquad$ Yes $WC < 0.5$ km

Shortest distance between a point and a plane

The perpendicular distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$ is $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$.

Find the perpendicular distance from the point with coordinates $(3,2,-1)$ to the plane with equation $2x - 3y + z = 5$.

$(3, 2, -1)$ $\quad 2x - 3y + z - 5 = 0$

$\alpha \quad \beta \quad \gamma \qquad n_1 = 2 \quad n_2 = -3 \quad n_3 = 1 \quad d = -5$

$$d = \frac{|2\times 3 - 3\times 2 + 1\times -1 - 5|}{\sqrt{2^2 + 3^2 + 1^2}}$$

$$d = \frac{6}{\sqrt{14}} = \frac{3\sqrt{14}}{7}$$

[June 2013 Q8(R)] The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(*a*) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1.

(3)

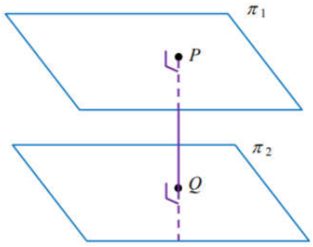
Distance between

(α, β, γ) and $n_1x + n_2y + n_3z + d = 0$

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Shortest distance between two parallel planes

- find *any* point on the plane
- use the formula for shortest distance between point and plane



Example: Find the distance between the parallel planes

π_1: $\boxed{2x - 6y + 3z = 9}$ and π_2: $\underset{n_1}{2}x - \underset{n_2}{6}y + \underset{n_3}{3}z = 5$ $d = -5$

$x = 0$
$y = 0$
$z = 3$

$(\underset{\alpha}{0}, \underset{\beta}{0}, \underset{\gamma}{3})$ ← shortest dist

$$\text{shortest dist} = \frac{|3 \times 3 - 5|}{\sqrt{2^2 + 6^2 + 3^2}} = \frac{4}{\sqrt{49}} = \frac{4}{7}$$

$$\left|\frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1{}^2 + n_2{}^2 + n_3{}^2}}\right|$$

Ex 9F Q5, 6, 10

5 Find the shortest distance between the parallel planes.

~~**a** $\mathbf{r}.(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 55$ and $\mathbf{r}.(6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) = 22$~~

$$\alpha \quad \beta \quad \gamma$$

b $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + \mathbf{k}) + \mu(8\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = \boxed{14\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}} + \lambda(3\mathbf{j} + \mathbf{k}) + \mu(8\mathbf{i} - 9\mathbf{j} - \mathbf{k})$

Find a vector perp to both $\begin{pmatrix} 8 \\ 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ Let $n = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} 8 \\ 3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8x + 3y + 3z = 0$$

and $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4x + z = 0.$

Let $z = -4$ $\quad 8 + 3y - 12 = 0$ $\quad \underline{n} = \begin{pmatrix} 1 \\ 4/3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix}$

$x = 1$ $\quad 3y = 4$

$y = \frac{4}{3}$

$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

$\underline{r} \cdot \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix} = 9 + 16 - 12 = 13$

Hence Π_1 is

$3x + 4y - 12z = 13$

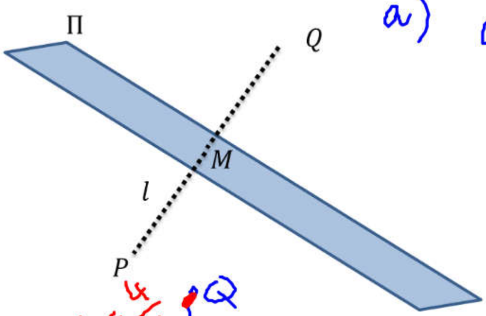
$3x + 4y - 12z - 13 = 0$

$$\boxed{\begin{array}{l} \alpha = 14 \;\; \beta = 2 \;\; \gamma = 2 \\ n_1 = 3 \;\; n_2 = 4 \;\; n_3 = -12 \;\; d = -13 \end{array}}$$

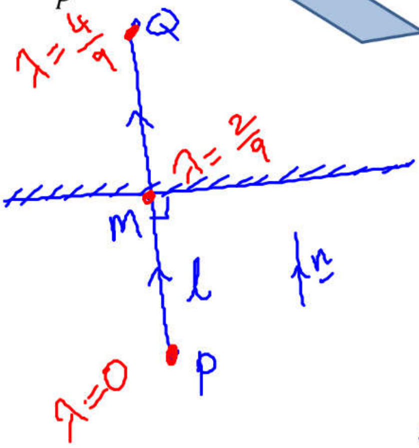
So $\text{dist} = \dfrac{|14 \times 3 + 2 \times 4 + 2 \times -12 - 13|}{\sqrt{3^2 + 4^2 + 12^2}} = \dfrac{13}{13} = 1$

Reflecting a point in a plane

The plane Π has equation $r \cdot (\overset{n_1}{i} + \overset{n_2}{2j} + \overset{n_3}{2k}) = \overset{-d}{5}$. The point P has coordinates $(\overset{\alpha}{1},\overset{\beta}{3},\overset{\gamma}{-2})$.
(a) Find the shortest distance between P and Π.
The point Q is the reflection of the point P in Π.
(b) Find the coordinates of point Q.



a) $$\text{dist} = \frac{|1\times1 + 3\times2 - 2\times2 - 5|}{\sqrt{1^2+2^2+2^2}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$



Reflection $\longrightarrow$ intersection of a line and plane.

$$l: \; \underline{r} = \begin{pmatrix} 1+\lambda \\ 3+2\lambda \\ -2+2\lambda \end{pmatrix} \qquad \Pi: \; \underline{r}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} = 5$$

$$\begin{pmatrix} 1+\lambda \\ 3+2\lambda \\ -2+2\lambda \end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} = 5 \qquad 1+\lambda+6+4\lambda-4+4\lambda = 5$$

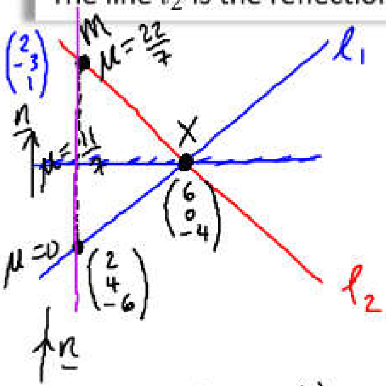
$$9\lambda = 2$$

$$\lambda = \frac{2}{9}$$

$$\lambda = \frac{4}{9} \quad \underline{q} = \begin{pmatrix} 1+4/9 \\ 3+8/9 \\ -2+8/9 \end{pmatrix} = \begin{pmatrix} 13/9 \\ 35/9 \\ -10/9 \end{pmatrix} \qquad \underline{\underline{Q\left(\frac{13}{9}, \frac{35}{9}, -\frac{10}{9}\right)}}$$

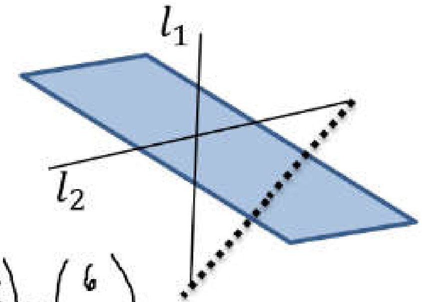
Reflecting a line in a plane

The line l_1 has equation $\frac{x-2}{2} = \frac{y-4}{-2} = \frac{z+6}{1}$. The plane Π has equation $2x - 3y + z = 8$.
The line l_2 is the reflection of line l_1 in the plane Π. Find a vector equation of the line l_2.



- find the intersection
- reflect a point on l_1 to l_2
- find the equation through these 2 points.

The key here is that we need to reflect two points on the line through the plane, then find the equation of the line through these new points.



$l_1 \quad r: \begin{pmatrix} 2+2\lambda \\ 4-2\lambda \\ -6+\lambda \end{pmatrix} \qquad \Pi: \ \underline{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 8$

$4 + 4\lambda - 12 + 6\lambda - 6 + \lambda = 8$

$11\lambda = 22$

$\lambda = 2$

$\underline{x} = \begin{pmatrix} 2+2\times2 \\ 4-2\times2 \\ -6+2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix}$

Equation of line perp to plane through a point on l_1 (the pink line).

$\underline{r} = \begin{pmatrix} 2+2\mu \\ 4-3\mu \\ -6+\mu \end{pmatrix} \qquad \underline{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 8$

Ex 9F Q7, 8, 12

$4 + 4\mu - 12 + 9\mu - 6 + \mu = 8$

$14\mu = 22$

$\mu = \frac{11}{7}$

$\underline{m} = \begin{pmatrix} 2 + \frac{44}{7} \\ 4 - \frac{66}{7} \\ -6 + \frac{22}{7} \end{pmatrix} = \begin{pmatrix} 58/7 \\ -38/7 \\ -20/7 \end{pmatrix}$

Our l_2 passes through M and X

$\overrightarrow{MX} = \underline{x} - \underline{m}$

$= \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 58/7 \\ -38/7 \\ -20/7 \end{pmatrix}$

$\overrightarrow{MX} = \begin{pmatrix} -16/7 \\ 38/7 \\ -8/7 \end{pmatrix}$

because $\overrightarrow{MX}$ is direction, I can simplify, so direction is $\begin{pmatrix} -16 \\ 38 \\ -8 \end{pmatrix}$ or $\begin{pmatrix} -8 \\ 19 \\ -4 \end{pmatrix}$

So $l_2 \quad \underline{r} = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix} + t\begin{pmatrix} -8 \\ 19 \\ -4 \end{pmatrix}$