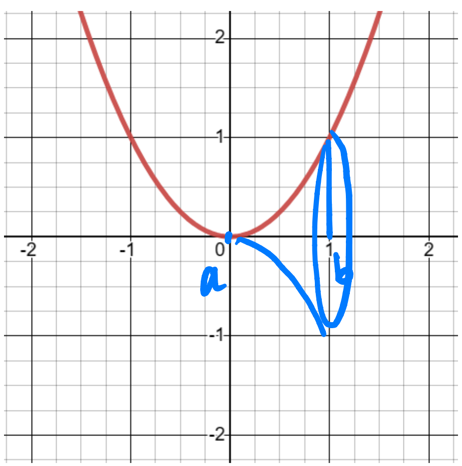
CP2: Chapter 4, Volumes of Revolution (Year 2)

This is effectively the same content as Year 1, except that you can (and should!) expect the integration to be much more challenging.

Make sure you've covered Integration from normal maths, and know that in the exam, you can also be tested on all integration from Further Maths, too.

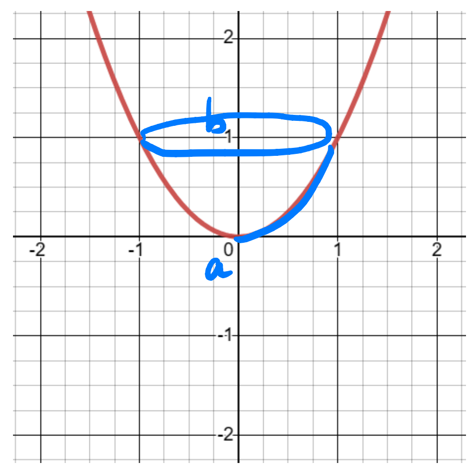
$$\boxed{V = \pi \int_a^b y^2 \, dx}$$

about x-axis



$$\boxed{V = \pi \int_a^b x^2 \, dy}$$

about y-axis



About the x-axis

The region R is bounded by the curve with equation $y = \sin 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$. Find the volume of the solid formed when region R is rotated through 2π radians about the x-axis.

$$V = \pi \int_0^{\pi/2} y^2 \, dx$$

$$= \pi \int_0^{\pi/2} \sin^2 2x \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{2}}$$

$$= \pi \left(\frac{1}{2} \times \frac{\pi}{2} - \frac{1}{8}\sin 2\pi\right) = \pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{4} \text{ units}^3$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

MEMORISE THESE

Ex 4A

About the y-axis

The diagram shows the curve with equation $y = 4\ln x - 1$. The finite region R, shown in the diagram, is bounded by the curve, the x-axis, the y-axis and the line $y = 4$. Region R is rotated by 2π radians about the y-axis. Use integration to show that the exact value of the volume of the solid generated is $2\pi\sqrt{e}(e^2 - 1)$.

$$V = \pi \int_0^4 x^2 \, dy$$

$$y = 4\ln x - 1$$

$$\frac{y+1}{4} = \ln x$$

$$x = e^{\frac{y+1}{4}}$$

$$x^2 = \left(e^{\frac{y+1}{4}}\right)^2$$

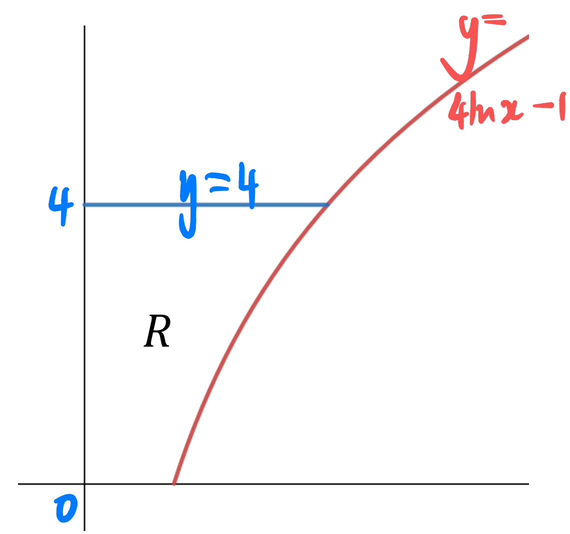
$$x^2 = e^{\frac{y+1}{2}}$$

$$V = \pi \int_0^4 e^{\frac{y+1}{2}} \, dy$$

$$V = \pi \int_0^4 e^{\frac{1}{2}y + \frac{1}{2}} \, dy$$

$$V = \pi \left[2e^{\frac{1}{2}y + \frac{1}{2}} \right]_0^4$$

$$V = \pi \left(2e^{5/2} - 2e^{1/2} \right)$$



Ex 4B

$$V = 2\pi\left(e^{5/2} - e^{1/2}\right)$$

$$V = 2\pi e^{1/2}\left(e^{2} - 1\right)$$

$$V = \underline{2\pi\sqrt{e}\left(e^{2} - 1\right)} \quad \text{units}^3$$

Parametric Curves

You can also be expected to find volumes of revolutions for curves defined parametrically.

$y = f(t)$

$$V = \pi \int_a^b y^2 \, dx$$

$$V = \pi \int_a^b x^2 \, dy$$

$$V = \pi \int_p^q y^2 \frac{dx}{dt} \, dt$$

$$V = \pi \int_p^q x^2 \frac{dy}{dt} \, dt$$

We multiply by $\frac{dx}{dt}$ or $\frac{dy}{dt}$ so that we can integrate with respect to t

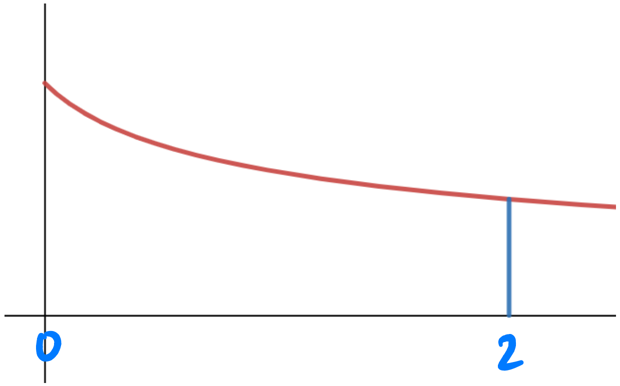
Be careful – the limits must match what the function is being integrated with respect to!

The curve C has parametric equations $x = t(1+t), y = \frac{1}{1+t}, t \geq 0$.

The region R is bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.

Find the exact volume of the solid formed when R is rotated 2π radians about the x-axis.

$$V = \pi \int y^2 \, dx = \pi \int y^2 \frac{dx}{dt} \, dt$$



Limits

$x = 0$

$0 = t(1+t)$

$t = 0$, $\cancel{t = -1}$

$x = 2$

$2 = t(1+t)$

$2 = t + t^2$

$0 = t^2 + t - 2$

$0 = (t+2)(t-1)$

$\cancel{t = -2}$, $t = 1$

$$V = \pi \int_0^1 y^2 \frac{dx}{dt} \, dt$$

$$y^2 = \frac{1}{(1+t)^2}$$

$x = t(1+t)$

$x = t + t^2$

$$\frac{dx}{dt} = 1 + 2t$$

Ex 4C

$$V = \pi \int_0^1 \frac{1}{(1+t)^2} \times (1+2t) \, dt$$

$$= \pi \int_0^1 \frac{1+2t}{(1+t)^2} \, dt$$

$$= \pi \int_0^1 \left(\frac{2}{1+t} - \frac{1}{(1+t)^2} \right) dt$$

$$= \pi \int_0^1 \left(\frac{2}{1+t} - (1+t)^{-2} \right) dt$$

$$= \pi \left[2\ln|1+t| + (1+t)^{-1} \right]_0^1$$

$$= \pi \left(2\ln 2 + \frac{1}{2} - 2\ln 1 - 1 \right)$$

$$= \pi \left(2\ln 2 - \frac{1}{2} \right) \text{ units}^3$$

$$\frac{1+2t}{(1+t)^2} = \frac{A}{(1+t)} + \frac{B}{(1+t)^2}$$

$$1+2t = A(1+t) + B$$

compare t// $A = 2$

constants// $1 = A + B$

$$1 = 2 + B$$

$$B = -1$$

Modelling

The diagram shows a model of a goldfish bowl. The cross-section of the model is described by the curve with parametric equations $x = 2\sin t, y = 2\cos t + 2$, $\frac{\pi}{6} \leq t \leq \frac{11\pi}{6}$, where the units of x and y are in cm.

The goldfish bowl is formed by rotating this curve about the y-axis to form a solid of revolution.

a) Find the volume of water required to fill the model to a height of 3 cm

The real goldfish bowl has a maximum diameter of 48 cm.

b) Find the volume of water required to fill the real goldfish bowl to the corresponding height.

$x = 2\sin t$ $\qquad y = 2\cos t + 2$

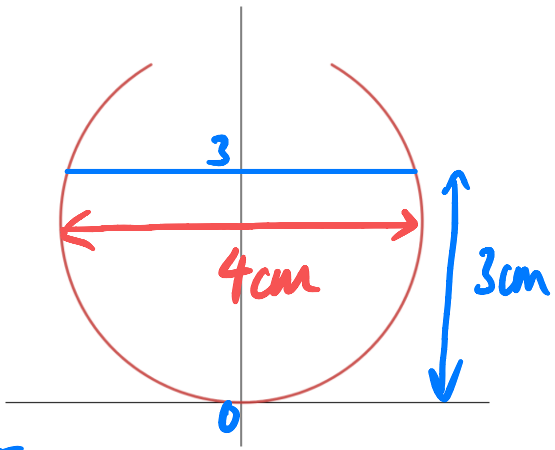
$V = \pi \int x^2 \frac{dy}{dt}\, dt$

Limits $\quad y = 0$

$0 = 2\cos t + 2$

$-\frac{2}{2} = \cos t$

$\cos t = -1 \qquad t = \pi$



$\int_{\pi}^{\frac{5\pi}{3}}$

Ex 4D

$y = 3$

$3 = 2\cos t + 2$

$\frac{1}{2} = \cos t$

$t = \frac{\pi}{3}, \frac{5\pi}{3}$ we could use either

$x^2 = 4\sin^2 t$

$y = 2\cos t + 2$

$\frac{dy}{dt} = -2\sin t$

$V = \pi \int_{\pi}^{\frac{5\pi}{3}} -8\sin^3 t \, dt$

$= -8\pi \int_{\pi}^{\frac{5\pi}{3}} \sin^3 t \, dt$

$= -8\pi \int_{\pi}^{\frac{5\pi}{3}} (\sin t - \sin t \cos^2 t) \, dt$

$\sin^3 t = \sin t \sin^2 t$

$= \sin t (1 - \cos^2 t)$

$= \sin t - \sin t \cos^2 t$ ✓ ✓

$$= -8\pi\left[-\cos t + \frac{1}{3}\cos^3 t\right]_{\pi}^{\frac{5\pi}{3}}$$

$\cos\frac{5\pi}{3} = \frac{1}{2} = \cos\frac{\pi}{3}$

$\cos\pi = -1$

$$= -8\pi\left(-\cos\frac{5\pi}{3} + \frac{1}{3}\cos^3\frac{5\pi}{3} + \cos\pi - \frac{1}{3}\cos^3\pi\right)$$

$$= -8\pi\left(-\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 - 1 - \frac{1}{3}(-1)^3\right)$$

$$= -8\pi\left(-\frac{1}{2} + \frac{1}{24} - 1 + \frac{1}{3}\right)$$

$$= -8\pi\left(-\frac{9}{8}\right)$$

$cm^3 = ml$

$$= \underline{\underline{9\pi}}$$

b) Real bowl has a diameter 12 times wider

S.F $= 12$ Volume S.F. $= 12^3$

Volume $= 9\pi \times 12^3 = 48858\ cm^3$

$= 48900\ cm^3 = 48900\ ml$

$= 48 \cdot 9$ litres