Your Turn

Edexcel C4 Jan 2012 Q4

Given that $y = 2$ at $x = \dfrac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}. \qquad \textbf{(5)}$$

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}$$

$$y\frac{dy}{dx} = \frac{3}{\cos^2 x}$$

$$\int y\,dy = \int \frac{3}{\cos^2 x}\,dx$$

$$\int y\,dy = \int 3\sec^2 x\,dx$$

$$\frac{1}{2}y^2 = 3\tan x + c$$

$$y = 2,\ x = \frac{\pi}{4}$$

$$2 = 3 + c$$

$$c = -1$$

$$\frac{1}{2}y^2 = 3\tan x - 1$$

$$y = \sqrt{6\tan x - 2}$$

11b)
11a) $\int \frac{3x+4}{x}\,dx = \int \left(\frac{3x}{x} + \frac{4}{x}\right) dx$

$= \int \left(3 + \frac{4}{x}\right) dx$

$= 3x + 4\ln|x| + c$.

11b). $\underline{y=16}, \underline{x=1}$ $\quad \frac{dy}{dx} = \frac{3x\sqrt{y} + 4\sqrt{y}}{x}$. $\quad y=f(x)$.

$\frac{dy}{dx} = \sqrt{y}\,\frac{(3x+4)}{x}$

$\int \frac{1}{\sqrt{y}}\,dy = \int \frac{3x+4}{x}\,dx$

$2y^{1/2} = 3x + 4\ln|x| + c$

$8 = 3 + c$

$c = 5$

$y = \left(\frac{1}{2}(3x + 4\ln|x| + c)\right)^2$

$y = \left(\frac{1}{2}(3x + 4\ln|x| + 5)\right)^2$

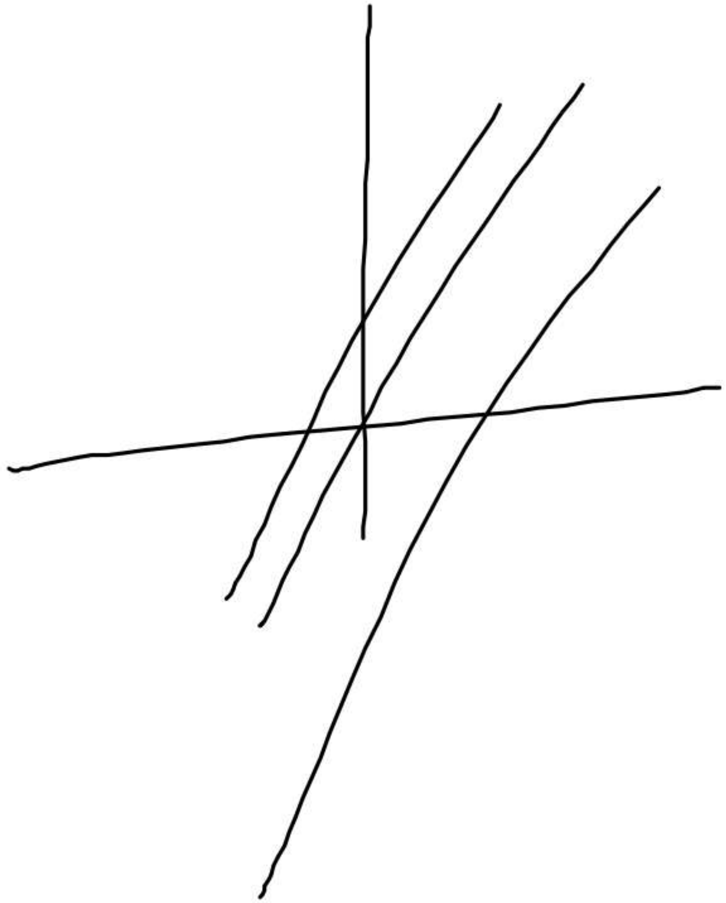
$$\frac{dy}{dx} = 2$$

$$\boxed{y = 2x + c}$$

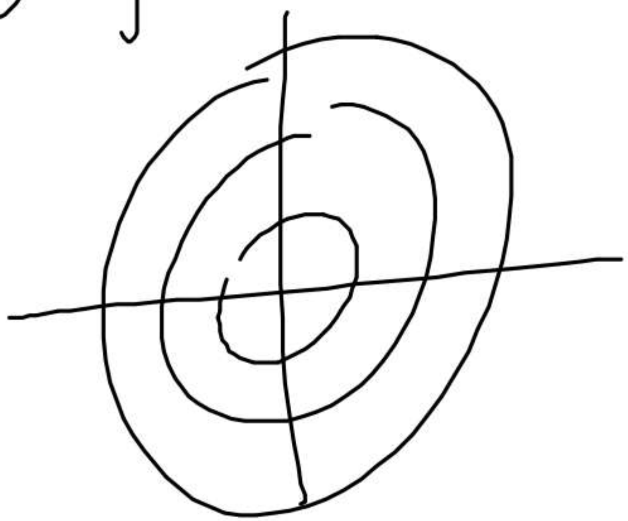
$y = 2x$

$y = 2x + 1$

$y = 2x - 7$



$x^2 + y^2 = c^2$



Forming differential equations

Q The rate of increase of a rabbit population (with population P, where time is t) is **proportional to** the current population. Form a differential equation, and find its general solution.

$$\frac{dP}{dt} = kP$$

$$\frac{1}{P}\frac{dP}{dt} = k$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = kt + c$$

$$P = e^{kt+c}$$

$$P = Ae^{kt}$$

$$e^{kt+c} = e^{kt} \times e^{c}$$
$$= Ae^{kt}$$

$$A = e^{c}$$

$t=0,\ P=300$

$$300 = Ae^{0}$$
$$300 = A.$$

$t=3,\ P=900$

Water in a manufacturing plant is held in a large cylindrical tank of diameter 20m. Water flows out of the bottom of the tank through a tap at a rate proportional to the cube root of the volume.

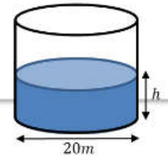
(a) Show that t minutes after the tap is opened, $\frac{dh}{dt} = -k\sqrt[3]{h}$ for some constant k.

(b) Show that the general solution of this differential equation may be written $h = (P - Qt)^{\frac{3}{2}}$, where P and Q are constants. $t=0,\ h=27$ $t=10,\ h=8$

Initially the height of the water is 27m. 10 minutes later, the height is 8m.

(c) Find the values of the constants P and Q.

(d) Find the time in minutes when the water is at a depth of 1m. $t=?\ ,\ h=1$



a)

$$\frac{dV}{dt} \propto -\sqrt[3]{V}$$

$$\frac{dV}{dt} = -c\sqrt[3]{V}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$V = \pi \times r^2 \times h$$
$$V = 100\pi h$$
$$\frac{dV}{dh} = 100\pi$$

$$\frac{dh}{dt} = \frac{1}{100\pi} \times -c\sqrt[3]{V}$$

$$\frac{dh}{dt} = \frac{1}{100\pi} \times -c\sqrt[3]{100\pi h}$$

$$\frac{dh}{dt} = -\frac{c\sqrt[3]{100\pi}}{100\pi}\sqrt[3]{h} \qquad k = \frac{c\sqrt[3]{100\pi}}{100\pi}$$

$$\frac{dh}{dt} = -k\sqrt[3]{h} \qquad \frac{dh}{dt} = -kh^{1/3}$$

Ex 11K
Odd Questions

b)

$$h^{-1/3}\frac{dh}{dt} = -k$$

$$\int h^{-1/3}\,dh = \int -k\,dt$$

$$\frac{3}{2}h^{2/3} = -kt + a$$

$$h^{2/3} = -\frac{2kt}{3} + \frac{2a}{3}$$

$$h = \left(\frac{2a}{3} - \frac{2k}{3}t\right)^{3/2}$$

$$h = (P - Qt)^{3/2} \qquad P = \frac{2a}{3} \qquad Q = -\frac{2k}{3}$$

c) $t=0,\ h=27$

$$27 = (P - 0)^{3/2}$$
$$27^{2/3} = P$$
$$P = 9$$

$t=10,\ h=8$

$$8 = (9 - Q\times 10)^{3/2}$$
$$8^{2/3} = 9 - 10Q$$
$$4 = 9 - 10Q$$
$$10Q = 5$$
$$Q = \frac{1}{2}$$

$$h = \left(9 - \frac{1}{2}t\right)^{3/2}$$

d) $h=1,\ t=?$

$$1 = \left(9 - \frac{1}{2}t\right)^{3/2}$$
$$1^{2/3} = 9 - \frac{1}{2}t$$
$$1 = 9 - \frac{1}{2}t$$
$$\frac{1}{2}t = 8$$

$t = 16$ minutes.

Edexcel C4 June 2005 Q8

Liquid is pouring into a container at a constant rate of 20 $\text{cm}^3\ \text{s}^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

(*a*) Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 20 - kV,$$

where k is a positive constant. **(2)**

The container is initially empty.

(*b*) By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k. **(6)**

Given also that $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$ when $t = 5$,

(*c*) find the volume of liquid in the container at 10 s after the start. **(5)**

Jan 08 C4

Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 $cm^3 s^{-1}$ and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(*a*) Show that at time *t* seconds, the height *h* cm of liquid in the cylinder satisfies the differential equation

$$\frac{dh}{dt} = 0.4 - k\sqrt{h},$$

where *k* is a positive constant.

(3)

$400 = c\sqrt{25}$

$c = 80$

$V = 4000h$

$\frac{dV}{dh} = 4000$

$\frac{dV}{dt} = 1600 - c\sqrt{h}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dh}{dt} = \frac{1}{4000}\left(1600 - c\sqrt{h}\right) = 0.4 - \frac{c}{4000}\sqrt{h}$

$k = \frac{c}{4000}$ $\quad k = \frac{80}{4000}$

$\frac{dh}{dt} = 0.4 - k\sqrt{h}$

When $h = 25$, water is leaking out of the hole at 400 $cm^3 s^{-1}$.

(*b*) Show that $k = 0.02$.

(1)

part (b) in red

$\frac{dV}{dt} = 1200, \quad h = 25$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$1200 = 4000 \times \left(0.4 - k\sqrt{25}\right)$

$0.3 = 0.4 - 5k$

$5k = 0.1$

$k = 0.02$

(*c*) Separate the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h}$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh.$$

(2)

Using the substitution $h = (20 - x)^2$, or otherwise,

(*d*) find the exact value of $\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh$.

(6)

(*e*) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$\boldsymbol{\sin x}$	Standard result	$-\cos x$	No
$\boldsymbol{\cos x}$	Standard result	$\sin x$	No
$\boldsymbol{\tan x}$	In formula booklet, but use $\int \frac{\sin x}{\cos x} dx$ which is of the form $\int \frac{kf'(x)}{f(x)} dx$	$\ln\lvert\sec x\rvert$	Yes
$\boldsymbol{\sin^2 x}$	For both $\sin^2 x$ and $\cos^2 x$ use identities for $\cos 2x$ $$\cos 2x = 1 - 2\sin^2 x$$ $$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$	$\frac{1}{2}x - \frac{1}{4}\sin 2x$	No
$\boldsymbol{\cos^2 x}$	$$\cos 2x = 2\cos^2 x - 1$$ $$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$	$\frac{1}{2}x + \frac{1}{4}\sin 2x$	No
$\boldsymbol{\tan^2 x}$	$$1 + \tan^2 x \equiv \sec^2 x$$ $$\tan^2 x \equiv \sec^2 x - 1$$	$\tan x - x$	No
$\boldsymbol{cosec\ x}$	Would use substitution $u = cosec\ x + \cot x$, but too hard for exam.	$-\ln\lvert cosec\ x + \cot x\rvert$	Yes
$\boldsymbol{\sec x}$	Would use substitution $u = \sec x + \tan x$, but too hard for exam.	$\ln\lvert\sec x + \tan x\rvert$	Yes
$\boldsymbol{\cot x}$	$\int \frac{\cos x}{\sin x} dx$ which is of the form $\int \frac{f'(x)}{f(x)} dx$	$\ln\lvert sin\ x\rvert$	Yes

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)	Formula booklet?
$\boldsymbol{cosec^2 x}$	By observation.	$-\boldsymbol{\cot x}$	No!
$\boldsymbol{sec^2 x}$	By observation.	$\tan x$	Yes (but memorise)
$\boldsymbol{cot^2 x}$	$1+\cot^2 x \equiv cosec^2 x$	$-\cot x - x$	No
$\boldsymbol{sin\ 2x\ cos\ 2x}$	For any product of sin and cos with same coefficient of x, use double angle. $\sin 2x \cos 2x \equiv \frac{1}{2}\sin 4x$	$-\frac{1}{8}\cos 4x$	No
$\boldsymbol{\frac{1}{x}}$		$\ln x$	No
$\boldsymbol{\ln x}$	Use IBP, where $u = \ln x, \frac{dv}{dx} = \ln x$	$x \ln x - x$	No
$\boldsymbol{\frac{x}{x+1}}$	Use algebraic division. $\frac{x}{x+1} \equiv 1 - \frac{1}{x+1}$	$x - \ln\|x+1\|$	
$\boldsymbol{\frac{1}{x(x+1)}}$	Use partial fractions.	$\ln\|x\| - \ln\|x+1\|$	

$f(x)$	How to deal with it	$\int f(x)dx$ (+constant)
$\frac{4x}{x^2+1}$	Reverse chain rule. Of form $\int \frac{kf'(x)}{f(x)}$	$2\ln\lvert x^2+1\rvert$
$\frac{x}{(x^2+1)^2}$	Power around denominator so NOT of form $\int \frac{kf'(x)}{f(x)}$. Rewrite as product. $x(x^2+1)^{-2}$ Reverse chain rule (i.e. "Consider $y = (x^2+1)^{-1}$" and differentiate)	$-\frac{1}{2}(x^2+1)^{-1}$
e^{2x+1} $\frac{1}{1-3x}$	For any function where 'inner function' is linear expression, divide by coefficient of x	$\frac{1}{2}e^{2x+1}$ $-\frac{1}{3}\ln\lvert 1-3x\rvert$
$x\sqrt{2x+1}$	IBP or use sensible substitution. $u = 2x+1$ or even better, $u^2 = 2x+1$.	$\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$
$\sin^5 x\ cos\ x$	Reverse chain rule.	$\frac{1}{6}\sin^6 x$

A Whole Load of Integration

This is it; where all the integration you've seen comes together. You need to find the following integrals without any clue as to how to do them! You could use 'guess and check', partial fractions, parts, substitution or more than one of the above!

1 $\int \cos(3x-1)dx$

2 $\int e^{1-x}dx$

3 $\int \frac{2x+1}{\left(x^2+x-1\right)^2}dx$

4 $\int \cos 2x dx$

5 $\int \ln 2x dx$

6 $\int \frac{x}{\left(x^2-1\right)^3}dx$

7 $\int \sqrt{2x-3}dx$

8 $\int \frac{4x-1}{(x-1)^2(x+2)}dx$

9 $\int x^3 \ln x dx$

10 $\int \frac{5}{2x^2-7x+3}dx$

11 $\int (x+1)e^{x^2+2x}dx$

12 $\int \frac{\sin x - \cos x}{\sin x + \cos x}dx$

13 $\int x^2 \sin 2x dx$

14 $\int \sin^3 2x dx$

1) $\frac{1}{3}\sin(3x-1) + c$
2) $-e^{1-x} + c$
3) $\frac{-1}{x^2+x-1} + c$
4) $\frac{1}{2}\sin(2x) + c$
5) $x\ln(2x) - x + c$
6) $\frac{-1}{4}\left(x^2-1\right)^{-2} + c$
7) $\frac{1}{3}(2x-3)^{3/2} + c$
8) $-\ln|x+2| + \ln|x-1| - (x-1)^{-1} + c$
9) $\frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + c$
10) $-\ln|2x-1| + \ln|x-3| + c$
11) $\frac{1}{2}e^{x^2+2x} + c$
12) $-\ln|\sin x + \cos x| + c$
13) $\frac{1}{4}\cos 2x - \frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + c$
14)