Chapter 6: Probability Distributions

1 :: General Probability Distributions

"Given that $P(X = x) = \frac{k}{x}$, find the value of k."

2 :: Binomial Distribution

"I toss an unfair coin, with probability heads of 0.6, 10 times. What's the probability I see 5 heads?"

3 :: Cumulative Binomial Probabilities

"I toss an unfair coin, with probability heads of 0.6, 10 times. What's the probability I see at most 3 heads?"

Probability distributions

You are already familiar with the concept of **variable** in statistics: a collection of values (e.g. favourite colour of students in the room):

$\boldsymbol{x}$	red	green	blue	orange
$\boldsymbol{P(X = x)}$	0.3	0.4	0.1	0.2

If each is assigned a probability of occurring, it becomes a **random variable**.

> A **random variable X represents a single experiment/trial**. It consists of outcomes with a probability for each.

i.e. X is a random variable (capital letter), but x is a particular outcome.

$$P(X = x)$$

"The probability that... | ...the outcome of the random variable X... | ...was the specific outcome x"

A shorthand for $P(X = x)$ is $p(x)$ (note the lowercase p).
It's like saying "the probability that the outcome of my coin throw was heads" ($P(X = heads)$) vs "the probability of heads" ($p(heads)$)

Probability Distributions vs Probability Functions

There are two ways to write the mapping from outcomes to probabilities:

The "{" means we have a 'piecewise function'. This just simply means we choose the function from a list depending on the input.

As a function:

$$p(x) = \begin{cases} 0.1x, & x = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

e.g. if $x = 2$, then the probability is $0.1 \times 2 = 0.2$

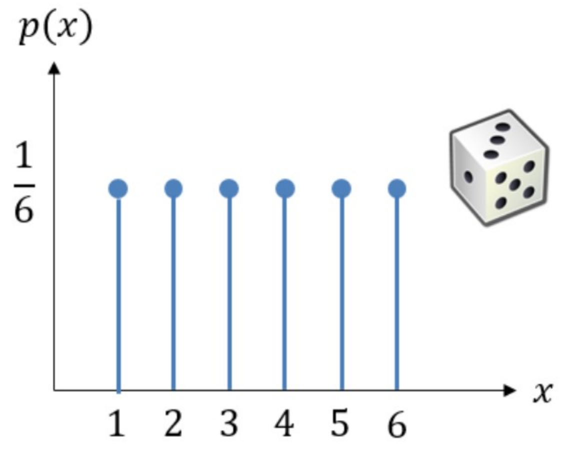
As a table:

$\boldsymbol{x}$	1	2	3	4
$\boldsymbol{p(x)}$				

The random variable X represents the **number of tails when three coins are tossed**. Write down the probability distribution as a table and as a function.

Discrete Uniform Distribution

We can also represent a probability distribution graphically:



✏ The throw of a die is an example of a **discrete uniform distribution** because the probability of each outcome is the same.

x	1	2	3	4	5	6
$P(X = x)$						

Probabilities add to 1

1. A discrete random variable X has the probability function

$$P(X = x) = \begin{cases} k(1-x)^2 & x = -1,\ 0,\ 1 \text{ and } 2 \\ 0 & \text{otherwise.} \end{cases}$$

(*a*) Show that $k = \frac{1}{6}$.

(3)

x	-1	0	1	2
$p(x)$				

Probability of a Range

x	2	3	4	5
$p(x)$	0.1	0.3	0.2	0.4

Determine:

$P(X > 3) =$

$P(2 \leq X < 4) =$

$P(2X + 1 \geq 6) =$

Your Turn

A biased four-sided dice with faces numbered 1, 2, 3 and 4 is rolled. The number on the top-most face is modelled as a random variable X.

Given that $P(X = x) = \frac{k}{x}$

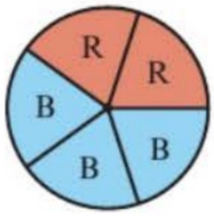
a) Find the value of k

b) Give the probability distribution of X in table form

c) Find the probability that i) $X > 2$ ii) $1 < X \leq 3$ iii) $X > 4$

Trickier problems

This spinner is spun until it lands on red or has been spun four times in total.
Find the probability distribution of the random variable S, the number of times the spinner is spun.



Ex 6A Q11
Mixed Ex 6 Q6

5. A biased spinner can only land on one of the numbers 1, 2, 3 or 4. The random variable X represents the number that the spinner lands on after a single spin and $P(X = r) = P(X = r + 2)$ for $r = 1, 2$

Given that $P(X = 2) = 0.35$

(a) find the complete probability distribution of X.

(2)

~~Ambroh spins the spinner 60 times~~.

~~(b) Find the probability that more than half of the spins land on the number 4~~
~~Give your answer to 3 significant figures.~~

(3)

The random variable $Y = \dfrac{12}{X}$

(c) Find $P(Y - X \leqslant 4)$

(3)

A spinner is designed so that the score S is given by the following probability distribution.

s	0	1	2	4	5
$P(S = s)$	p	0.25	0.25	0.20	0.20

(*a*) Find the value of p.

(2)

Tom and Jess play a game with this spinner. The spinner is spun repeatedly and S counters are awarded on the outcome of each spin. If S is even then Tom receives the counters and if S is odd then Jess receives them. The first player to collect 10 or more counters is the winner.

(*e*) Find the probability that Jess wins after 2 spins.

(2)

(*f*) Find the probability that Tom wins after exactly 3 spins.

(4)

(*g*) Find the probability that Jess wins after exactly 3 spins.

(3)

Extra Questions - exam style

The discrete random variable X has probability distribution given by

x	-1	0	1	2	3
$P(X=x)$	$\frac{1}{5}$	a	$\frac{1}{10}$	a	$\frac{1}{5}$

where a is a constant.

(*a*) Find the value of a.

(2)

(*b*) The random variable $Y = 6 - 2X$.

Calculate $P(X \geq Y)$.

(3)

The discrete random variable X has probability function

$$
P(X = x) = \begin{cases} k(2-x) & x = 0, 1, 2 \\ k(x-2) & x = 3 \\ 0 & \text{otherwise} \end{cases}
$$

where k is a positive constant.

(a) Show that $k = 0.25$

(2)

Two independent observations X_1 and X_2 are made of X.

(b) Show that $P(X_1 + X_2 = 5) = 0$

(1)

(c) Find the complete probability function for $X_1 + X_2$.

(3)

(d) Find $P(1.3 \leqslant X_1 + X_2 \leqslant 3.2)$

(2)

(Total 8 marks)

The discrete random variable X has the probability distribution

x	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$4k$

(*a*) Show that $k = 0.1$

(1)

Two independent observations X_1 and X_2 are made of X.

(*e*) Show that $P(X_1 + X_2 = 4) = 0.1$

(2)

(*f*) Complete the probability distribution table for $X_1 + X_2$.

(2)

y	2	3	4	5	6	7	8
$P(X_1 + X_2 = y)$	0.01	0.04	0.10		0.25	0.24	

(g) Find $P(1.5 < X_1 + X_2 \leq 3.5)$

(2)

A fair blue die has faces numbered 1, 1, 3, 3, 5 and 5. The random variable B represents the score when the blue die is rolled.

(*a*) Write down the probability distribution for B.

(2)

(*b*) State the name of this probability distribution.

(1)

A second die is red and the random variable R represents the score when the red die is rolled.

The probability distribution of R is

r	2	4	6
$\mathrm{P}(R = r)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

Tom invites Avisha to play a game with these dice.

Tom spins a fair coin with one side labelled 2 and the other side labelled 5. When Avisha sees the number showing on the coin she then chooses one of the dice and rolls it. If the number showing on the die is **greater** than the number showing on the coin, Avisha wins, otherwise Tom wins.

Avisha chooses the die which gives her the best chance of winning each time Tom spins the coin.

(*f*) Find the probability that Avisha wins the game, stating clearly which die she should use in each case.

(4)

Q3	(a)	$2a+\frac{2}{5}+\frac{1}{10}=1$ (or equivalent)	M1	
		$a=\frac{1}{4}$ or 0.25	A1	(2)
	(b)	$\mathrm{E}(X)=\underline{1}$	B1	(1)
	(c)	$\mathrm{E}(X^2)=1\times\frac{1}{5}+1\times\frac{1}{10}+4\times\frac{1}{4}+9\times\frac{1}{5}$ (= 3.1)	M1	
		$\mathrm{Var}(X)=3.1-1^2$, $= \underline{2.1}$ or $\frac{21}{10}$ oe	M1 A1	(3)
	(d)	$\mathrm{Var}(Y)=(-2)^2\,\mathrm{Var}(X)$, $= \underline{8.4}$ or $\frac{42}{5}$ oe	M1 A1	(2)
	(e)	$X\geq Y$ when $X=3$ or 2, so probability $=\text{"}\frac{1}{4}\text{"}+\frac{1}{5}$	M1 A1ft	
		$=\frac{9}{20}$ oe	A1	(3)
				Total 11

4a	$2k+k+0+k=1$	M1	2.1	4th
	$\Rightarrow 4k=1$, so $k=0.25$ (answer given).	A1*	1.1b	Calculate probabilities from discrete distributions.
		(2)		
4b	$\mathrm{P}(X_1+X_2=5)=\mathrm{P}(X_1=3 \text{ and } X_2=2)+\mathrm{P}(X_1=2 \text{ and } X_2=3)$ $=0+0=0$ (answer given).	B1*	2.4	4th Calculate probabilities from discrete distributions.
		(1)		
4c	see table below	M1 A1 A1	2.5 1.1b 1.1b	4th Calculate probabilities from discrete distributions.
		(3)		

x_1+x_2	0	1	2	3	4	5	6
$\mathrm{P}(X_1+X_2)$	0.25	0.25	0.0625	0.25	0.125	0	0.0625

6. (a)	$k+2k+3k+4k=1$ or $10k=1$ $k=0.1$ (*) [allow verification with a comment e.g. "so $k=0.1$"]	B1cso	(1)
(b)	$\mathrm{E}(X)=1\times0.1+2\times0.2+3\times0.3+4\times0.4=3$	M1 A1	(2)
(c)	$\mathrm{E}(X^2)=1\times0.1+4\times0.2+9\times0.3+16\times0.4=10$	M1 A1	(2)
(d)	$\mathrm{Var}(X)=10-9(=1)$ $\mathrm{Var}(2-5X)=5^2\mathrm{Var}(X)=25$	M1 M1 A1	(3)
(e)	$\mathrm{P}(1,3)+\mathrm{P}(2,2)=2\times0.1\times0.3+0.2\times0.2=0.1$ (*)	M1 A1cso	(2)
(f)	see table below	B1 B1	(2)
(g)	$\mathrm{P}(2)+\mathrm{P}(3)=0.05$	M1A1	(2)
			[14]

X_1+X_2	2	3	4	5	6	7	8
p	0.01	0.04	0.1	**0.2**	0.25	0.24	**0.16**

6. (a)

b	1	3	5
$\mathrm{P}(B=b)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Also allow b values 1,1,3,3,5,5 and probabilities all $\frac{1}{6}$ — B1 B1 (2)

(b)	Discrete Uniform {distribution}	B1	(1)
(c)	[E(B) =] 3 (by symmetry)	B1	(1)
(d)	$[\mathrm{E}(R)=]\ 2\times\frac{2}{3}+4\times\frac{1}{6}+6\times\frac{1}{6}$ $=\underline{3}$	M1 A1	(2)
(e)	$[\mathrm{E}(R^2)=]\ 2^2\times\frac{2}{3}+4^2\times\frac{1}{6}+6^2\times\frac{1}{6}$ $\left[=\frac{34}{3}\right]$ $[\mathrm{Var}(R)=]\frac{34}{3}-3^2=\frac{7}{3}$ (or any exact equivalent. NB 2.33 is A0)	M1 dM1, A1	(3)
(f)	Coin lands on **2**, choose **blue** die; coin lands on **5** choose **red** die $\mathrm{P(Avisha\ wins)}=\frac{1}{2}\times\left(\frac{1}{3}+\frac{1}{3}\right)+\frac{1}{2}\times\frac{1}{6}$ $=\frac{5}{12}$ (allow awrt 0.417)	B2/1/0 M1 A1	(4)

Probability experiments with multiple trials

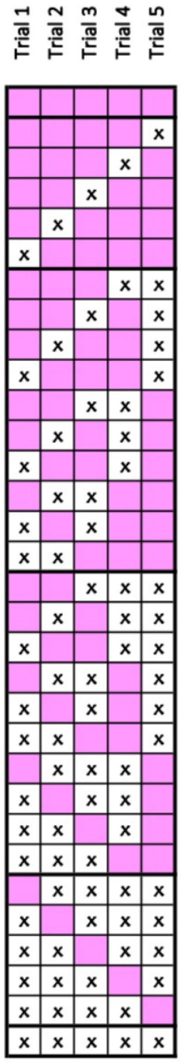
Suppose the probability that I successfully win a game is 0.25.
If I play the game 3 times, what is the probability that I win exactly once?

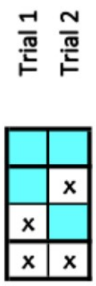
Trial 1	Trial 2	Trial 3
		x
	x	
x		
	x	x
x		x
x	x	
x	x	x

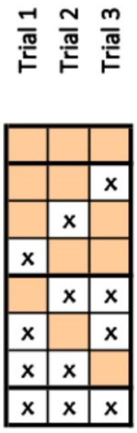
Suppose the probability that I successfully win a game is 0.3.
If I play the game 4 times, what is the probability that I win exactly three times?

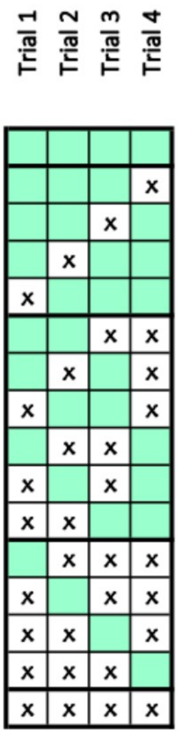
Trial 1	Trial 2	Trial 3	Trial 4
			x
		x	
	x		
x			
		x	x
	x		x
x			x
	x	x	
x		x	
x	x		
	x	x	x
x		x	x
x	x		x
x	x	x	
x	x	x	x

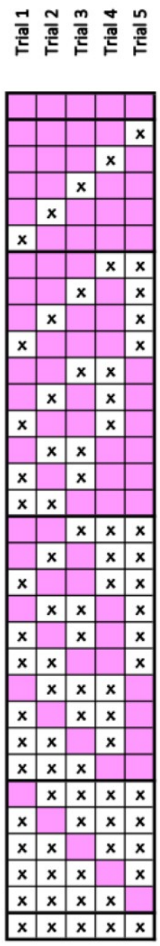
Suppose the probability that I successfully win a game is 0.4.
If I play the game 5 times, what is the probability that I win exactly twice?

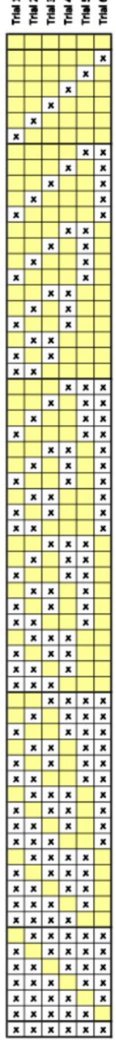












What's the pattern?
How can we predict the number of different ways of achieving a certain number of successes?

The Binomial Distribution

				1					
			1		1				
		1		2		1			
	1		3		3		1		
1		4		6		4		1	

0^{th} row

1^{st} row

2^{nd} row

3^{rd} row

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Suppose I was playing a game 15 times.
How many different ways are there of me winning exactly 5 times?

Given that the probability of me winning is 0.36, what is the probability that I win exactly 5 of the 15 games?

The Binomial Distribution - definition

✎ You can model a random variable X with a binomial distribution $B(n, p)$ if

- there are a fixed number of trials, n,
- there are two possible outcomes: 'success' and 'failure',
- there is a fixed probability of success, p
- the trials are independent of each other

If $X \sim B(n, p)$ then:

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$$

In the example below, 'success' was 'left handed'.

r is the number of successes out of n.

"$\sim$" means "has the distribution"

On a table of 8 family members, 6 people are left handed.

a) Suggest a suitable model for a random variable X: the number of left-handed people in a group of 8, where the probability of being left-handed is 0.1.
b) Find the probability 6 people are left handed.
c) Suggest why the chosen model may not have been appropriate.

The random variable $X \sim B\left(12, \frac{1}{6}\right)$. Find:

a) $P(X = 2)$

b) $P(X = 9)$

c) $P(X \leq 1)$

A company claims that a quarter of the bolts sent to them are faulty. To test this claim the number of faulty bolts in a random sample of 40 is recorded.

(a) Give two reasons why a binomial distribution may be a suitable model for the number of faulty bolts in the sample.

(b) Find the probability that exactly 10 of them are faulty.

Records kept in a hospital show that 3 out of every 10 patients who visit the accident and emergency department have to wait more than 30 minutes. Find, to 3 decimal places, the probability that of the first 12 patients who come to the accident and emergency department:

a) None have to wait longer than 30 minutes

b) Exactly 4 have to wait more than 30 minutes

Your Turn

1 $X \sim B(6, 0.2)$

What is $P(X = 2)$?

What is $P(X \geq 5)$?

2 I have a bag of 2 red and 8 white balls. X represents the number of red balls I chose after 5 selections (with replacement).

a How is X distributed?

b Determine the probability that I chose 3 red balls.

Cumulative Probabilities

Often we wish to find the probability of a range of values.
For a Binomial distribution, this was relatively easy if the range was narrow, e.g. $P(X \leq 1) = P(X = 0) + P(X = 1)$, but would be much more computationally expensive if we wanted say $P(X \leq 6)$.

If $X \sim B(10{,}0.3)$, find $P(X \leq 6)$.

How to calculate on your calculator:

Press Menu then 'Distributions'.
Choose "Binomial CD" (the C stands for 'Cumulative').
Choose 'Variable'.

$$x = 6$$
$$N = 10$$
$$p = 0.3$$

Pressing = gives the desired value.

Using tables

Look up $n = 10$ and the column $p = 0.3$.
Then look up the row $x = 6$.
The value should be 0.9894.

Note: The Classwiz calculator *ALWAYS* does $X \leq N$ rather than $X < N\ or\ X > N\ or\ X \geq N$.
The graphics calculator can choose upper and lower limits, but are always inclusive, i.e. $M \leq X \leq N$

The random variable $X \sim B(20{,}0.4)$. Find:

$P(X \leq 7) =$

$P(X < 6) =$

$P(X \geq 15) =$

The random variable $M \sim B(25, 0.25)$.
Find:

$P(M > 20) =$

$P(6 < M \leq 10) =$

$P(M = 6) =$

Write the following in terms of cumulative probabilities, e.g. $P(X < 7) = P(X \leq 6)$

$P(X < 5) =$

$P(X \geq 7) =$

$P(X > 7) =$

$P(10 \leq X < 20) =$

$P(1 \leq X < 8) =$

$P(10 \leq X \leq 20) =$

$P(X = 100) =$

$P(20 < X < 30) =$

"at least 30" =

"greater than 30" =

Ex 6C
Q1-6

In a computer game, you have 20 attempts to try and knock some monkeys off a tree branch.
The probability of knocking a monkey off a tree branch is 0.3.
Determine the probability that someone:

a) Knocks less than 6 monkeys off the branch.
b) Knocks at least 9 monkeys off the branch.

The game gives you a prize (one banana) if the you knock at least 9 monkeys off the branch.
A student plays the game 5 times.
c) Calculate the probability that they win at least 4 bananas.

Mixed Ex 6 Q15

Dealing with Probability Ranges

A spinner is designed so that probability it lands on red is 0.3. Jane has 12 spins.

a) Find the probability that Jane obtains at least 5 reds.

Jane decides to use this spinner for a class competition. She wants the probability of winning a prize to be < 0.05. Each member of the class will have 12 spins and the number of reds will be recorded.

b) Find how many reds are needed to win the prize.

At Morpeth University, students have 20 exams at the end of the year. All students pass each individual exam with probability 0.45. Students are only allowed to continue into the next year if they pass some minimum of exams out of the 20.
What do the university administrators set this minimum number such that the probability of continuing to next year is at least 90%?

Ex 6C
Q7-10