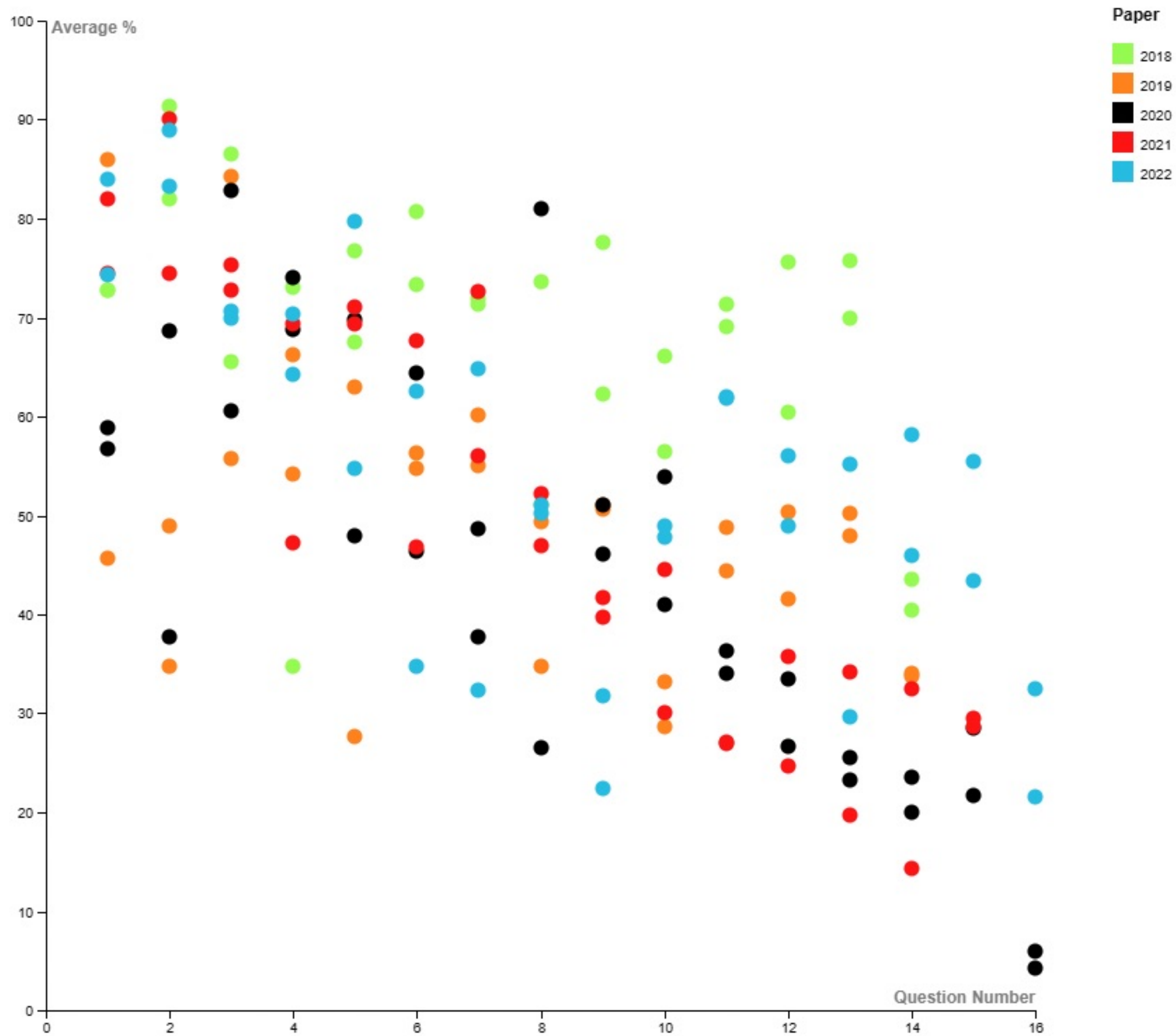

Average %
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#10: DIFFERENTIAL EQUATIONS

2020 PAPER 1

23.6%

14. A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{dr}{dt} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

a) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$\frac{dV}{dt} = -c$

$= \frac{1}{4\pi r^2} \times -c$

$= -\frac{c}{4\pi} \times \frac{1}{r^2}$ Let $k = \frac{c}{4\pi}$

$= -\frac{k}{r^2}$

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Question 14 continued

b) $\frac{dr}{dt} = -\frac{k}{r^2}$

$\leftarrow r^2$

$r^2\, dr = -k\, dt$

$\int r^2\, dr = \int -k\, dt$

$\frac{1}{3}r^3 = -kt + c$

When $t=0$, $r=40$ $\longrightarrow$ $\frac{1}{3}(40)^3 = c$

$c = \frac{64000}{3}$

$t=5$, $r=20$ $\longrightarrow$ $\frac{1}{3}(20)^3 = -5k + \frac{64000}{3}$

$k = \frac{11200}{3}$

Hence $\frac{1}{3}r^3 = -\frac{11200}{3}t + \frac{64000}{3}$

$r^3 = 64000 - 11200t$

c) $r > 0$, hence $64000 - 11200t > 0$

$\frac{64000}{11200} > t$

$\frac{40}{7} > t$

Hence $0 \le t < \frac{40}{7}$

#9: DIFFERENTIATION

2020 PAPER 2

23.3%

13. The function g is defined by

$$g(x) = \frac{3\ln(x) - 7}{\ln(x) - 2} \qquad x > 0 \qquad x \neq k$$

where k is a constant.

(a) Deduce the value of k.

(1)

(b) Prove that

$$g'(x) > 0$$

for all values of x in the domain of g.

(3)

(c) Find the range of values of a for which

$$g(a) > 0$$

(2)

a) $\ln k - 2 = 0$

$\ln k = 2$

$k = e^2$

b) Quotient Rule

$u = 3\ln x - 7$ $\qquad$ $v = \ln x - 2$

$u' = \frac{3}{x}$ $\qquad$ $v' = \frac{1}{x}$

$g'(x) = \frac{vu' - uv'}{v^2}$

$$g'(x) = \frac{\frac{3}{x}(\ln x - 2) - \frac{1}{x}(3\ln x - 7)}{(\ln x - 2)^2}$$

$$= \frac{\cancel{\frac{3}{x}\ln x} - \frac{6}{x} - \cancel{\frac{3}{x}\ln x} + \frac{7}{x}}{(\ln x - 2)^2}$$

$$= \frac{\frac{1}{x}}{(\ln x - 2)^2}$$

As $x > 0$, $\frac{1}{x} > 0$, and $\ln x - 2$ is squared, it is also ≥ 0,

hence $g'(x) > 0$

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Question 13 continued

c) $g(a) > 0$

$\frac{3\ln a - 7}{\ln a - 2} > 0$

$3\ln a - 7 > 0$

$\ln a > \frac{7}{3}$

$a > e^{7/3}$

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Turn over

9.

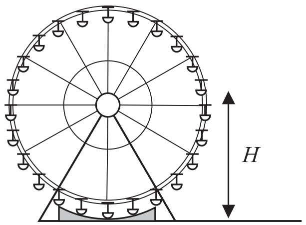


Figure 4

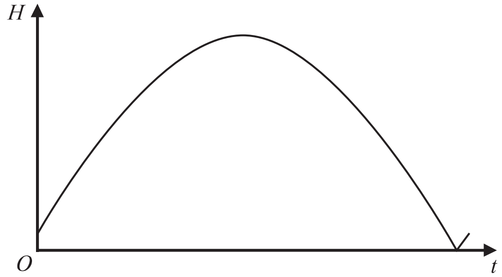


Figure 5

Figure 4 shows a sketch of a Ferris wheel.

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A\sin(bt + \alpha)°|$$

where A, b and α are constants.

Figure 5 shows a sketch of the graph of H against t, for one revolution of the wheel.

Given that

- the maximum height of the passenger above the ground is 50 m
- the passenger is 1 m above the ground when the wheel starts turning
- the wheel takes 720 seconds to complete one revolution

(a) find a complete equation for the model, giving the exact value of A, the exact value of b and the value of α to 3 significant figures.

(4)

(b) Explain why an equation of the form

$$H = |A\sin(bt + \alpha)°| + d$$

where d is a positive constant, would be a more appropriate model.

(1)

#8: MODELLING WITH TRIGONOMETRY

2022 PAPER 2
22.4%

Question 9 continued

shifts α to the left
$f(x+\alpha)$

a) $H = |A\sin(bt+\alpha)|$

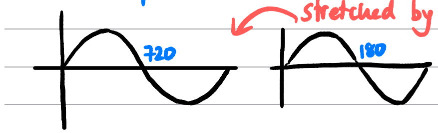
scales the amplitude
$af(x)$

stretches by factor $\frac{1}{b}$
$f(bx)$

$A = 50$ (max height)

720 for one revolution

stretched by SF 4



Hence $b = \frac{1}{4}$

When $t=0$, $H=1$

$1 = |50\sin(\frac{1}{4}t+\alpha)|$

$1 = 50\sin\alpha$

$\alpha = \underline{1.15}$ (3sf)

Hence $H = |50\sin(\frac{1}{4}b + 1.15)|$

b) In this second model the passenger now does not touch the ground which is more realistic

(Total for Question 9 is 5 marks)

#7: IMPLICIT DIFFERENTIATION, POINTS OF INFLECTION

2020 PAPER 1

21.7%

15. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27}$

(3)

Product Rule

a) $x^2 \tan y = 9$

$u = x^2 \qquad v = \tan y$

$u' = 2x \qquad v' = \sec^2 y \frac{dy}{dx}$

$x^2 \sec^2 y \frac{dy}{dx} + 2x \tan y = 0$

$\frac{dy}{dx} = \frac{-2x \tan y}{x^2 \sec^2 y}$

$\tan y = \frac{9}{x^2}$

$1 + \tan^2 y = \sec^2 y$

$1 + \left(\frac{9}{x^2}\right)^2 = \sec^2 y$

$1 + \frac{81}{x^4} = \sec^2 y$

$\frac{dy}{dx} = \frac{-2x \times \frac{9}{x^2}}{x^2\left(1 + \frac{81}{x^4}\right)}$

$\frac{dy}{dx} = \frac{-\frac{18}{x}}{x^2 + \frac{81}{x^2}} \qquad \times x^2$

$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$

Question 15 continued

b) Quotient Rule

$u = -18x \qquad v = x^4 + 81$

$u' = -18 \qquad v' = 4x^3$

$\frac{d^2y}{dx^2} = \frac{-18(x^4+81) + 18x(4x^3)}{(x^4+81)^2}$

$= \frac{-18x^4 - 1458 + 72x^4}{(x^4+81)^2}$

$\frac{d^2y}{dx^2} = \frac{54x^4 - 1458}{(x^4+81)^2}$

Point of inflection if: $\frac{d^2y}{dx^2} = 0$ and there is a change of sign of $\frac{d^2y}{dx^2}$ either side of that solution

$\frac{d^2y}{dx^2} = 0$

$54x^4 - 1458 = 0$

$x^4 = 27$

$x = \sqrt[4]{27}$

When $x < \sqrt[4]{27}$, $\frac{d^2y}{dx^2} < 0$ and when $x > \sqrt[4]{27}$, $\frac{d^2y}{dx^2} > 0$

hence there is a point of inflection when $x = \sqrt[4]{27}$

#6: PARAMETRIC INTEGRATION

2022 PAPER 1 21.6%

Advance Info.

16.

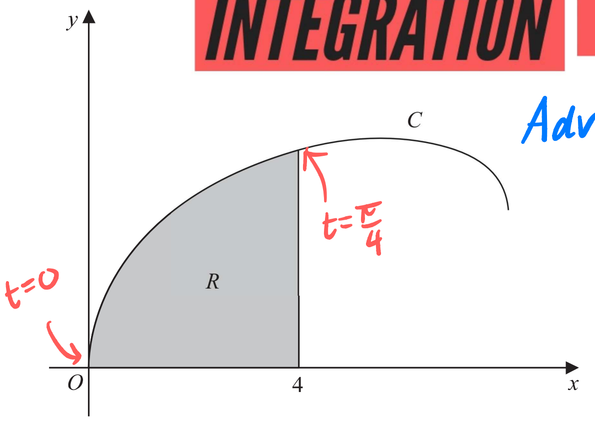


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a \left(8 - 8\cos 4t + 48\sin^2 t\cos t\right)\mathrm{d}t$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

a) Find a

$x = 4$

$4 = 8\sin^2 t$

$\frac{1}{2} = \sin^2 t$

$\pm\frac{\sqrt{2}}{2} = \sin t \qquad$ (But $0 \leq t \leq \frac{\pi}{2}$, so $\sin t = \frac{\sqrt{2}}{2}$)

$t = \frac{\pi}{4}$

Hence $a = \frac{\pi}{4}$

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Question 16 continued

$x = 8(\sin t)^2$

$\frac{dx}{dt} = 16\sin t\cos t$

$R = \int_0^{\pi/4} y\frac{dx}{dt}\,dt = \int_0^{\pi/4}(2\sin 2t + 3\sin t)\,16\sin t\cos t\,dt$

$= \int_0^{\frac{\pi}{4}}(32\sin 2t\sin t\cos t + 48\sin^2 t\cos t)\,dt$

$\sin 2t = 2\sin t\cos t$

$= \int_0^{\frac{\pi}{4}}(16\sin 2t\sin 2t + 48\sin^2 t\cos t)\,dt$

$\cos 2t = 1 - 2\sin^2 t$

$2\sin^2 t = 1 - \cos 2t$

$= \int_0^{\pi/4}(8 \times 2\sin^2 2t + 48\sin^2 t\cos t)\,dt$

$= \int_0^{\pi/4}(8(1 - \cos 4t) + 48\sin^2 t\cos t)\,dt$

$= \int_0^{\pi/4}(8 - 8\cos 4t + 48\sin^2 t\cos t)\,dt$

b) $R = \left[8t - 2\sin 4t + 16\sin^3 t\right]_0^{\frac{\pi}{4}}$

$= 8\left(\frac{\pi}{4}\right) - 2\sin\pi + 16\sin^3\left(\frac{\pi}{4}\right) - 0 - 0 - 0$

$= 2\pi - 0 + 16\left(\frac{\sqrt{2}}{2}\right)^3$

$= \underline{\underline{2\pi + 4\sqrt{2}}}$

#5: CIRCLES

2020 PAPER 2
20.0%

14. A circle C with radius r

- lies only in the 1st quadrant
- touches the x-axis and touches the y-axis

The line l has equation $2x + y = 12$

(a) Show that the x coordinates of the points of intersection of l with C satisfy

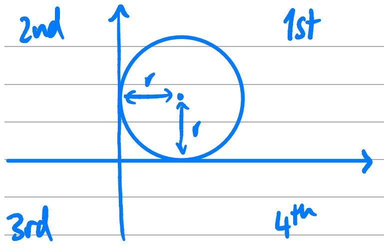
$$5x^2 + (2r - 48)x + (r^2 - 24r + 144) = 0$$

(3)

Given also that l is a tangent to C,

(b) find the two possible values of r, giving your answers as fully simplified surds.

(4)



Centre (r, r)

$(x-r)^2 + (y-r)^2 = r^2$

Intersection with $2x+y=12$

$y = 12-2x$

$(x-r)^2 + (12-2x-r)^2 = r^2$

	12	$-2x$	$-r$
12	144	$-24x$	$-12r$
$-2x$	$-24x$	$+4x^2$	$+2xr$
$-r$	$-12r$	$+2xr$	$+r^2$

$x^2 - 2xr + r^2 + 144 - 48x - 24r + 4x^2 + 4xr + r^2 = r^2$

$5x^2 + 2xr - 48x + r^2 - 24r + 144 = 0$

$5x^2 + (2r-48)x + (r^2 - 24r + 144) = 0$

b) tangent, $b^2 - 4ac = 0$

$a = 5,\ b = 2r - 48,\ c = r^2 - 24r + 144$

$(2r-48)^2 - 4(5)(r^2 - 24r + 144) = 0$

$4r^2 - 192r + 2304 - 20r^2 + 480r - 2880 = 0$

$-16r^2 + 288r - 576 = 0$

$r = 9 + 3\sqrt{5},\ r = 9 - 3\sqrt{5}$

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Question 14 continued

Turn over

#4: PARAMETRIC EQUATIONS

2021 PAPER 1

19.7%

13. A curve C has parametric equations

$$x = \frac{t^2+5}{t^2+1} \qquad y = \frac{4t}{t^2+1} \qquad t \in \mathbb{R}$$

Show that all points on C satisfy

$$(x-3)^2 + y^2 = 4$$

(3)

$$x - 3 = \frac{t^2+5}{t^2+1} - \frac{3(t^2+1)}{t^2+1}$$

$$= \frac{t^2+5-3t^2-3}{t^2+1}$$

$$= \frac{2-2t^2}{t^2+1}$$

$(2-2t^2)^2 =$

$$(x-3)^2 = \frac{4-8t^2+4t^4}{(t^2+1)^2}$$

$$y^2 = \frac{16t^2}{(t^2+1)^2}$$

Hence $(x-3)^2 + y^2 = \dfrac{4-8t^2+4t^4}{(t^2+1)^2} + \dfrac{16t^2}{(t^2+1)^2}$

$$= \frac{4t^4+8t^2+4}{(t^2+1)^2}$$

$$= \frac{4(t^4+2t^2+1)}{(t^2+1)^2} = \frac{4\cancel{(t^2+1)^2}}{\cancel{(t^2+1)^2}}$$

$$= 4$$

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P 6 8 7 3 1 A 0 4 2 5 2

Question 13 continued

(Total for Question 13 is 3 marks)

P 6 8 7 3 1 A 0 4 3 5 2

Turn over

#3: DIFFERENTIAL EQUATIONS

2021 PAPER 2

14.3%

14.

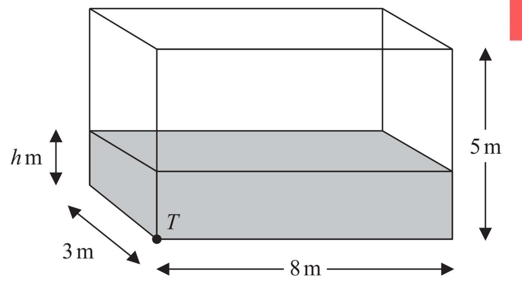

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

$\frac{dV}{dt} = 0.48$

$\frac{dV}{dt} = 0.48 - 0.1h$

$\frac{dV}{dt} = -0.1h$

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

Question 14 continued

$V = 8 \times 3 \times h$

$V = 24h$

$\frac{dV}{dh} = 24$

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$

$\frac{dh}{dt} = \frac{1}{24}(0.48 - 0.1h)$

$24\frac{dh}{dt} = 0.48 - 0.1h$

$\times 50$

$1200\frac{dh}{dt} = 24 - 5h$

b) $\frac{1200}{24-5h}\frac{dh}{dt} = 1$

$\int \frac{1200}{24-5h}\,dh = \int 1\,dt$

$-240\ln|24-5h| = t + c$

When $t=0$, $h=2$

$-240\ln 14 = c$

Hence $-240\ln|24-5h| = t - 240\ln 14$

$240(\ln 14 - \ln|24-5h|) = t$

$\ln\left(\frac{14}{24-5h}\right) = \frac{t}{240}$

$\frac{14}{24-5h} = e^{t/240}$

$\frac{24-5h}{14} = e^{-t/240}$

$24 - 14e^{-t/240} = 5h$

$h = 4.8 - 2.8e^{-t/240}$

c) As $t \to \infty$

$-2.8e^{-t/240} \to 0$

So $h \to 4.8$

This is less than 5m so never fills up.

16. Use algebra to prove that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3

(4)

#2: PROOF

2020 PAPER 2

6.0%

Cannot exhaust using evens and odds...

instead :

24 31 17

$3k$ • multiples of 3

$3k+1$ • one more than a multiple of 3

$3k-1$ • one less than a multiple of 3

So if k is a positive integer...

$(3k)^2 = 9k^2$
$= 3(3k^2)$ so is a multiple of 3

$(3k+1)^2 = 9k^2 + 6k + 1$
$= 3(3k^2 + 2k) + 1$ so is one more than a multiple of 3

$(3k-1)^2 = 9k^2 - 6k + 1$
$= 3(3k^2 - 2k) + 1$ so is one more than a multiple of 3

Hence we have proven the statement.

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Question 16 continued

16. Prove by contradiction that there are no positive integers p and q such that

$$4p^2 - q^2 = 25$$

(4)

#1: PROOF BY CONTRADICTION

2020 PAPER 1

4.3%

Assume, for contradiction, that there are positive integers p and q such that

$$4p^2 - q^2 = 25$$

D.O.T.S.

$$(2p+q)(2p-q) = 25$$

$5 \times 5 = 25$
$25 \times 1 = 25$ are the only integer factor pairs for 25

Hence $2p+q = 5$
$2p-q = 5$
$p = 2.5, \quad q = 0$
Neither p and q are positive integers

or $2p+q = 25$
$2p-q = 1$
$p = 6.5, \quad q = 12$
p is not an integer.

Hence we have contradicted our assumption, and so there are no positive integers p and q such that $4p^2 - q^2 = 25$

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Question 16 continued