Recurrence Relations - a **VERY** popular exam topic

$$u_n = 2n^2 + 3$$

This is an example of a position-to-term sequence, because each term is based on the position n.

$$u_{n+1} = 2u_n + 4$$
$$u_1 = 3$$

3, 10, 24, 52

We need the first term because the recurrence relation alone is not enough to know what number the sequence starts at.

But a term might be defined based on previous terms.
If u_n refers to the current term, u_{n+1} refers to the next term.
So the example in words says "the next term is twice the previous term + 4"

This is known as a term-to-term sequence, or more formally as a **recurrence relation**, as the sequence 'recursively' refers to itself.

Important Note: With recurrence relation questions, the **the sequence will likely not be arithmetic nor geometric**. So your previous u_n and S_n formulae do not apply.

Edexcel C1 May 2013 (R)

6. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = (x_n)^2 - kx_n, \qquad n \geq 1,$$

where k is a constant.

(*a*) Find an expression for x_2 in terms of k. **(1)**

$x_2 = x_1^2 - kx_1 = 1 - k$

(*b*) Show that $x_3 = 1 - 3k + 2k^2$. **(2)**

$x_3 = x_2^2 - kx_2$

$= (1-k)^2 - k(1-k)$

$= 1 - 2k + k^2 - k + k^2$

$x_3 = 1 - 3k + 2k^2$

Given also that $x_3 = 1$,

(*c*) calculate the value of k. **(3)**

$1 = 1 - 3k + 2k^2$

$0 = 2k^2 - 3k$

$0 = k(2k-3)$

$k = 0$ or $k = \frac{3}{2}$

(*d*) Hence find the value of $\sum_{n=1}^{100} x_n$. **(3)**

$x_1 = 1$

$x_2 = 1 - \frac{3}{2} = -\frac{1}{2}$

$x_3 = 1$

$\sum_{n=1}^{100} x_n = (1 + -\frac{1}{2}) + (1 + -\frac{1}{2}) + (1 + -\frac{1}{2}) + \ldots + (1 + -\frac{1}{2})$

$= \frac{1}{2} \times 50 = \frac{50}{2} = 25$

6. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = (x_n)^2 - kx_n, \qquad n \geq 1,$$

where k is a constant.

(*a*) Find an expression for x_2 in terms of k.

$$x_2 = (x_1)^2 - kx_1 = 1 - k$$

(1)

(*b*) Show that $x_3 = 1 - 3k + 2k^2$.

$$\begin{aligned} x_3 &= (x_2)^2 - kx_2 \\ &= (1-k)^2 - k(1-k) \\ &= 1 - 3k + 2k^2 \end{aligned}$$

(2)

Given also that $x_3 = 1$,

(*c*) calculate the value of k.

$$k = \frac{3}{2}$$

(3)

(*d*) Hence find the value of $\sum_{n=1}^{100} x_n$.

$$= 1 + \left(-\frac{1}{2}\right) + 1 + \left(-\frac{1}{2}\right) + \cdots$$

$$= 50 \times \left(1 - \frac{1}{2}\right) = 25$$

(3)

3. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 3$$

$$a_{n+1} = \frac{a_n - 3}{a_n - 2}, \qquad n \in \mathbb{N}$$

(a) Find $\displaystyle\sum_{r=1}^{100} a_r$

(3)

(b) Hence find $\displaystyle\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r$

(1)

$a_1 = 3$

$a_2 = \dfrac{3-3}{3-2} = \dfrac{0}{1} = 0$

$a_3 = \dfrac{0-3}{0-2} = \dfrac{3}{2}$

$a_4 = \dfrac{\frac{3}{2} - 3}{\frac{3}{2} - 2} = 3$

$a_5 = 0$

$a_6 = \dfrac{3}{2}$

$$\sum_{r=1}^{100} a_r = \left(3 + 0 + \frac{3}{2}\right) + \left(3 + 0 + \frac{3}{2}\right) + \ldots + 3$$

$$= 33\left(3 + 0 + \frac{3}{2}\right) + 3$$

$$= \underline{\underline{151.5}}$$

b) $\displaystyle\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r = 151.5 + 33\left(3 + \frac{3}{2}\right)$

$$= \underline{\underline{300}}$$

Question	Scheme	Marks	AOs
3 (a)	$a_1 = 3$, $a_2 = 0$, $a_3 = 1.5$, $a_4 = 3$	M1	1.1b
	$\sum_{r=1}^{100} a_r = 33(4.5) + 3$	M1	2.2a
	$= 151.5$	A1	1.1b
		(3)	
(b)	$\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r = (2)(151.5) - 3 = 300$	B1ft	2.2a
		(1)	
			(4 marks)

15. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_{n+1} = k - \frac{3k}{a_n} \qquad n \in \mathbb{Z}^+$$

where k is a constant.

a b c a b c a b c

The sequence is periodic of order 3

Given that $a_2 = 2$

(a) show that $k^2 + k - 12 = 0$

(3)

Given that $a_1 \neq a_2$

(b) find the value of $\sum_{r=1}^{121} a_r$

(4)

$a_2 = 2$

$a_3 = k - \frac{3k}{2}$

$= -\frac{k}{2}$

$a_4 = k - \frac{3k}{-\frac{k}{2}}$

$= k - \frac{3}{-\frac{1}{2}}$

$= k + 6$

$a_5 = k - \frac{3k}{k+6} = 2$

$2 = k - \frac{3k}{k+6}$

$2(k+6) = k(k+6) - 3k$

$2k + 12 = k^2 + 6k - 3k$

$0 = k^2 + k - 12$

$k = 3 \quad k = -4$

If $k = 3$ ✓

$a_2 = 2 \quad a_3 = -\frac{3}{2} \quad a_4 = 9 \quad a_5 = 2$

If $k = -4$

$a_2 = 2 \quad a_3 = -\frac{-4}{2} = 2 \quad a_4 = 2 \quad a_5 = 2$

$k \neq -4$

$a_1 \neq a_2$ if $k = -4$

$$\sum_{r=1}^{121} a_r = \left(9 + 2 - \frac{3}{2}\right) + \left(9 + 2 - \frac{3}{2}\right) + 9 \ldots + 9$$

$$= 40\left(9 + 2 - \frac{3}{2}\right) + 9$$

$$= \underline{\underline{389}}$$

Question	Scheme	Marks	AOs
15	$a_{n+1} = k - \dfrac{3k}{a_n},\ n \in \mathbb{Z}^+;$ k is a constant Sequence $a_1, a_2, a_3, \ldots$ where $a_2 = 2$ is periodic of order 3		
(a)	$a_3 = k - \dfrac{3k}{2} = -\dfrac{1}{2}k\,;$ $\quad a_4 = k - \dfrac{3k}{(-\frac{1}{2}k)} = k+6$	M1	1.1b
	$\{a_5 = a_2 \Rightarrow\}\ a_5 = k - \dfrac{3k}{k+6} = 2$	M1	3.1a
	$\Rightarrow k(k+6) - 3k = 2(k+6) \Rightarrow k^2 + 6k - 3k = 2k + 12$ $\Rightarrow k^2 + k - 12 = 0$ *	A1*	2.1
		(3)	
(b)	$(k+4)(k-3) = 0 \Rightarrow k = -4, 3$	M1	3.1a
	$k = 3;\ \{a_2 = 2,\}\ a_3 = -\dfrac{3}{2},\ a_4 = 9$ $\{k = -4;\ \{a_2 = 2,\}\ a_3 = 2\ \{\Rightarrow a_4 = 2,\ a_1 = 2\,;$ so reject as $a_1 = a_2\}\}$	A1	1.1b
	Note: $k = 3;\ a_1 = 9, a_2 = 2,\ a_3 = -\dfrac{3}{2},\ a_4 = 9$, etc.		
	$\displaystyle\sum_{r=1}^{121} a_r = 40\left(2 - \frac{3}{2} + 9\right) + 9$	M1	2.2a
	$= 40(9.5) + 9 = 380 + 9 = 389$	A1	1.1b
		(4)	
			(7 marks)

8. (i) Find the value of

$$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$$

(3)

(ii) Show that

$$\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) = 2$$

(3)

8i) $\displaystyle\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r = 20 \times \left(\frac{1}{2}\right)^4 + 20 \times \left(\frac{1}{2}\right)^5 + 20 \times \left(\frac{1}{2}\right)^6$

$= \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \ldots$

$= \dfrac{5/4}{1-\frac{1}{2}} = \underline{\underline{\frac{5}{2}}}$

$S_\infty = \dfrac{a}{1-r}$

$a = \frac{5}{4}$

$r = \frac{1}{2}$

8ii) $\displaystyle\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) = \log_5 \frac{3}{2} + \log_5 \frac{4}{3} + \log_5 \frac{5}{4} + \ldots + \log_5 \frac{49}{48} + \log_5 \frac{50}{49}$

$$= \log_5 \left(\frac{\cancel{3} \times \cancel{4} \times \cancel{5} \times \cdots \times \cancel{49} \times 50}{2 \times \cancel{3} \times \cancel{4} \times \cdots \times \cancel{48} \times \cancel{49}} \right) = \log_5 \left(\frac{50}{2} \right)$$

$$= \log_5 25$$

$$= \underline{\underline{2}}$$

Question 8 (Total 6 marks)

Part	Working or answer an examiner might expect to see	Mark	Notes
(i)	$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$ $= \sum_{r=1}^{\infty} 20 \times \left(\frac{1}{2}\right)^r - (10 + 5 + 2.5)$	M1	This mark is given for a method to find the sum to infinity of a GP
	$= \frac{10}{1-\frac{1}{2}} - (10 + 5 + 2.5)$	M1	This mark is given for a method to use a correct sum formula with a correct first term
	$= 2.5$	A1	This mark is given for a correct value for the sum
(ii)	$\sum_{n=1}^{48} \log_5 \left(\frac{n+2}{n+1}\right)$ $= \log_5 \frac{50}{49} + \log_5 \frac{49}{48} + \ldots + \log_5 \frac{4}{3} + \log_5 \frac{3}{2}$	M1	This mark is given for writing out at least four terms of the sum, including the first two and the last two
	$= \log_5 \frac{3 \times 4 \times \ldots \times 48 \times 49 \times 50}{2 \times 3 \times 4 \times \ldots \times 48 \times 49} = \log_5 \frac{50}{2}$	M1	This mark is given for using the rules of logs and cancelling terms
	$= 2$	A1	This mark is given for a full proof to show the expression is equal to 2 as required

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4. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = a x_n + 5, \qquad n \geq 1,$$

where a is a constant.

(a) Write down an expression for x_2 in terms of a.

(1)

(b) Show that $x_3 = a^2 + 5a + 5$.

(2)

Given that $x_3 = 41$

(c) find the possible values of a.

(3)

a) $x_2 = ax_1 + 5 = \underline{\underline{a+5}}$

b) $x_3 = ax_2 + 5 = a(a+5) + 5 = a^2 + 5a + 8$

$41 = a^2 + 5a + 5$

$0 = a^2 + 5a - 36$ $\qquad a = 4 \quad \text{or} \quad a = -9$

Ex 3G

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4. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = a x_n + 5, \qquad n \geq 1,$$

where a is a constant.

(*a*) Write down an expression for x_2 in terms of a.

$$\boxed{x_2 = a + 5}$$

(1)

(*b*) Show that $x_3 = a^2 + 5a + 5$.

$$\boxed{x_3 = a(a+5) + 5 = \cdots}$$

(2)

Given that $x_3 = 41$

(*c*) find the possible values of a.

$$a^2 + 5a + 5 = 41$$
$$a^2 + 5a - 36 = 0$$
$$(a+9)(a-4) = 0$$
$$a = -9 \text{ or } 4$$

(3)