Numerical Methods

In the GCSE9-1 syllabus you covered 'iteration', which allowed you to find successfully better approximations to the solutions of an equation. We'll revisit this, but also see a more powerful method for approximating solutions.

1:: Locating Roots

What it means to find the root of an equation and when we can be sure a root lies in a stated range.

"Show that $f(x) = x^3 - 4x^2 + 3x + 1$ has a root between $x = 1.4$ and $x = 1.5$".

2:: Using iteration to approximate roots to $f(x) = 0$

[Jan 2010] 2. $\quad \mathrm{f}(x) = x^3 + 2x^2 - 3x - 11$

(*a*) Show that $\mathrm{f}(x) = 0$ can be rearranged as **(2)**

$$x = \sqrt{\left(\frac{3x+11}{x+2}\right)}, \quad x \neq -2.$$

The equation $\mathrm{f}(x) = 0$ has one positive root α.

The iterative formula $x_{n+1} = \sqrt{\left(\frac{3x_n+11}{x_n+2}\right)}$ is used to find an approximation to α.

(*b*) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2, x_3 and x_4. **(3)**

3:: The Newton-Raphson Method

A numerical method that tends to converge to (i.e. approach) the root faster, by **following the tangent of the graph**.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Why do we need numerical methods?

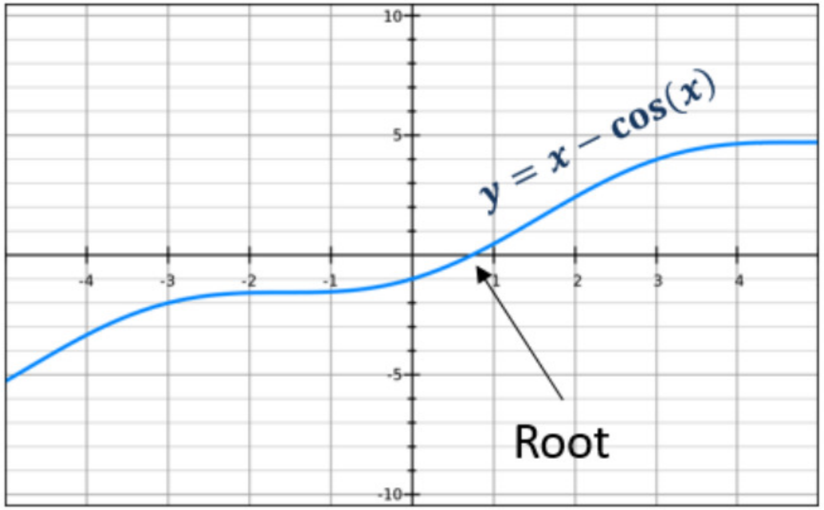
Finding the root of a function $f(x)$ is to **solve the equation $f(x) = 0$**
(i.e. the inputs such that the output of the function is 0)

However, for some functions, the 'exact' root is either complicated and difficult to calculate:

$$x^3 + 2x^2 - 3x + 4 = 0 \quad \Rightarrow \quad x = \frac{1}{3}\left(-2 - \frac{13}{\sqrt[3]{89 - 6\sqrt{159}}} - \sqrt[3]{89 - 6\sqrt{159}}\right)$$

or there's no 'algebraic' expression at all! (involving roots, logs, sin, cos, etc.)

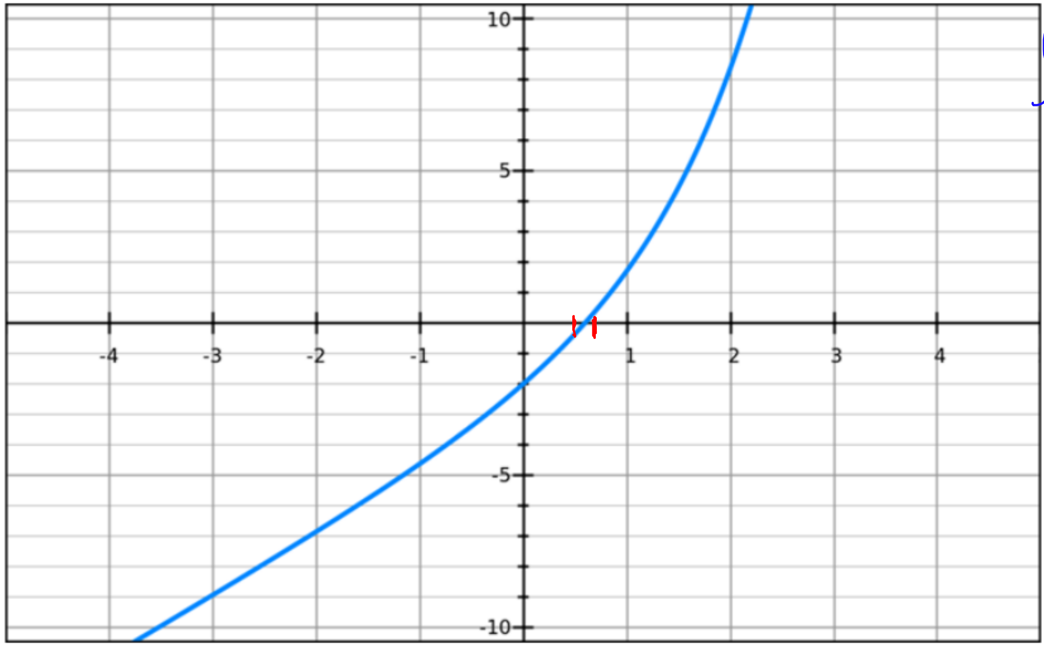
$$x - \cos(x) = 0 \quad \Rightarrow \quad$$ **Exact solution not expressible**



But there are a variety of 'numerical methods' which get progressively better solutions to an equation in the form $f(x) = 0$.
You have already seen 'iteration' at GCSE as one such method.

Proving a solution lies in a range

Show that $f(x) = e^x + 2x - 3$ has a root between $x = 0.5$ and $x = 0.6$



Substitute in $x=0.5$ and $x=0.6$ into $f(x)$.

$f(0.5) = e^{0.5} + 2\times 0.5 - 3$

$= -0.35127...$ negative

$f(0.6) = e^{0.6} + 2\times 0.6 - 3$

$= 0.02211...$ positive

Because there is a change in sign and $f(x)$ is continuous there is a root for $f(x)$ between 0.5 and 0.6.

Exam Tip: In the mark scheme they're looking for:

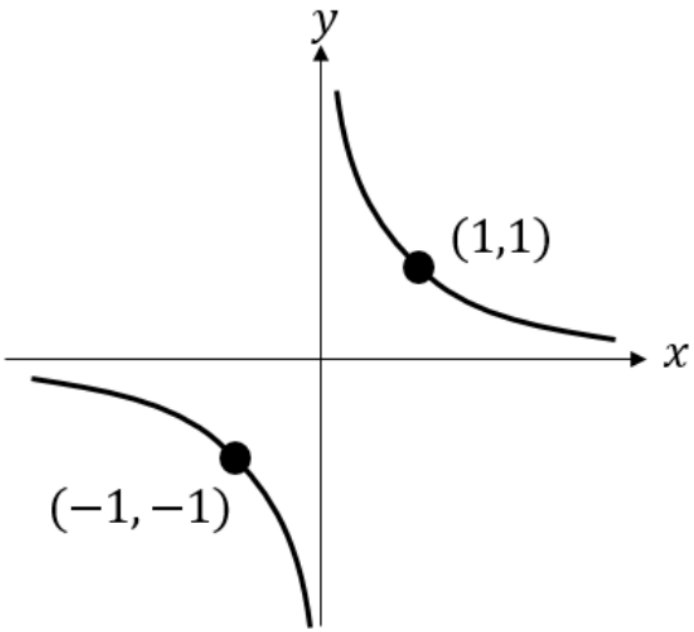
1. Finding the function output for the two values.
2. Referring to a 'change in sign'.
3. Commenting that f(x) is continuous

…why only if the function is continuous?

A function is continuous if you can trace it without taking your pen off the page.

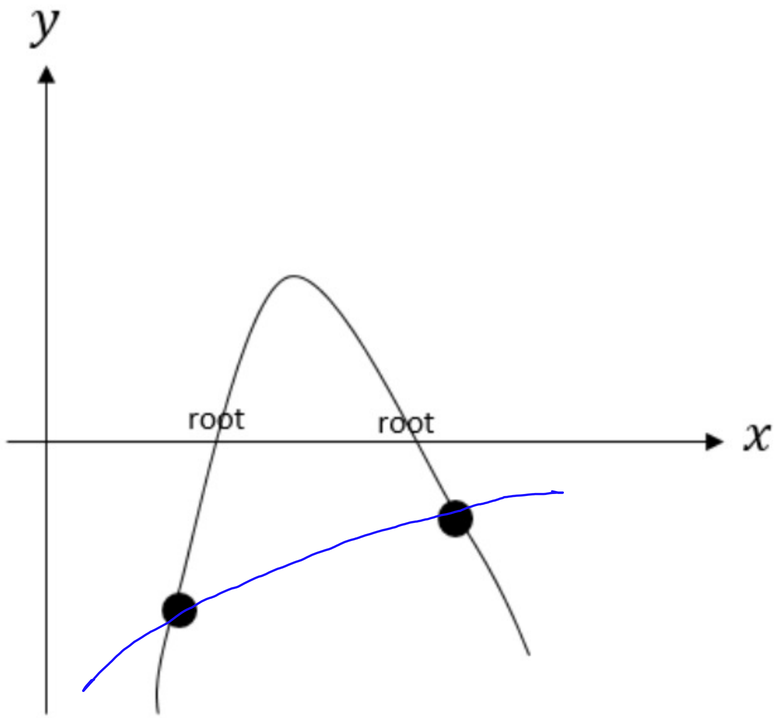
When $f(x) = \frac{1}{x}$, then $f(-1) = -1$ and $f(1) = 1$. There is a change in sign therefore $f(x)$ has a root in the range [-1, 1]

Why is this incorrect?



A function is **continuous** if the line **does not 'jump'**. A root is only guaranteed with a sign change if the function is continuous, as otherwise the line can skip past 0 (in this case due to a vertical asymptote.

No sign change doesn't *always* mean there isn't a root



Beware! Just because there isn't a sign change, doesn't mean there's no root in that interval.

The sign change method fails to detect a root if there were an **even number of roots** in that interval.

Proving a solution to a given accuracy

Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

The root of $g(x) = 0$ is α.

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places. **(3)**

$[2.3065, 2.3075]$

$g(2.3065) = e^{2.3065-1} + 2.3065 - 6$

$= -2.75... \times 10^{-4} < 0$

$g(2.3075) = 4.419.. \times 10^{-3} > 0$

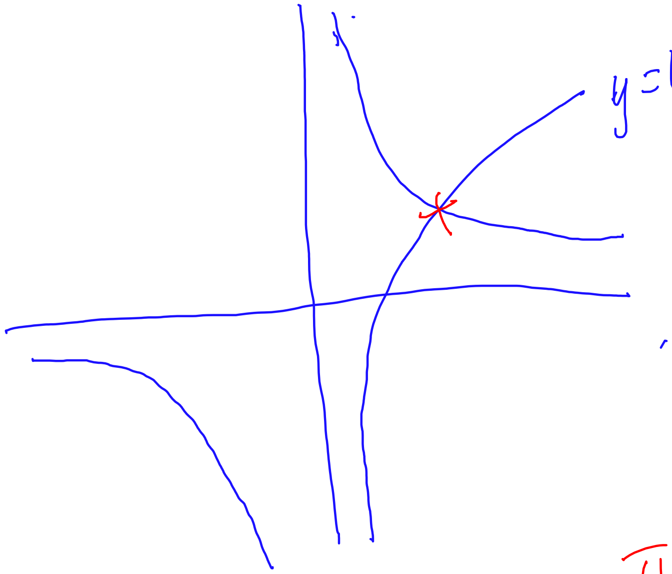
There is a change in sign, and $g(x)$ is continuous, so α is in the interval $[2.3065, 2.3075]$, ~~49999~~ and hence

$\alpha = 2.307$ to 3dp.

(a) Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Explain how your diagrams shows that the function $y = \ln(x) - \frac{1}{x}$ has only one root. $=0$

(b) Show that this root lies in the interval $1.7 < x < 1.8$

(c) Given that the root of $f(x)$ is α, show that $\alpha = 1.763$ correct to 3 decimal places.



$y = \ln x - \frac{1}{x}$

The root is $\ln x - \frac{1}{x} = 0$

$\ln x = \frac{1}{x}$

(intersection of $y = \ln x$, $y = \frac{1}{x}$

The function only has one root because the graphs only intersect once,

b) $f(x) = \ln x - \frac{1}{x}$

$f(1.7) = \ln 1.7 - \frac{1}{1.7}$

$= -0.657 < 0$

$f(1.8) = 0.03223 > 0$

There is a change in ~~sig~~ sign and $f(x)$ is continuous so the root lies in interval $1.7 < x < 1.8$

c) $f(1.7625) = -6.4 \times 10^{-4}$

$f(1.7635) = 2.46... \times 10^{-4}$

Change in sign, continuous function,

$\alpha = 1.763$ correct to 3 d p.

Ex 10A

Using iteration to approximate a root

To solve $f(x) = 0$ by an iterative method, rearrange into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$

We'll see why it works later.

Edexcel C3 Jan 2013

$$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

Tip: The difficulty is that there's multiple choices of x to isolate on one side of the equation. Therefore use the target equation to give clues for how to rearrange.

$$0 = e^{x-1} + x - 6$$
$$6 - x = e^{x-1}$$
$$\ln(6-x) = x - 1$$
$$x = \ln(6-x) + 1$$

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(*b*) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

x_0, x_1, x_2 represent successively better approximations of the root, where x_0 is the starting value.

$$x_1 = \ln(6 - x_0) + 1$$
$$= \ln(6 - 2) + 1$$
$$= 2.3863 \text{ (4dp)}$$

$$x_2 = \ln(6 - x_1) + 1$$
$$x_2 = 2.2847$$
$$x_3 = 2.3125$$
$$x_4 = 2.3050$$

converging to the root

Calculator Tip: Initially type x_0 (i.e. 2) onto your calculator.
Now just type:

$$\ln(6 - ANS) + 1$$

And then spam your $=$ key to get successive iterations.

Exam Tip: Show the substitution for x_1 to ensure you get the method mark. But then just write the final value for x_2 and thereafter, as the remaining marks will be 'accuracy' ones.

If the x_n values get closer and closer together the iterations **converge to the root**, ~~so iteration has failed.~~ The iteration is **convergent**.

If the x_n values get further and further apart the iterations **diverge**, so iteration has failed. The iteration is **divergent**.

If they bounce back and forth between values, we say the iteration **oscillates** or is **periodic** or is **non-convergent**.

Your Turn

Edexcel C3 June 2012 Q2

$$f(x) = x^3 + 3x^2 + 4x - 12$$

(*a*) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{4(3-x)}{(3+x)}\right)}, \quad x \neq -3.$$ **(3)**

$$\boxed{x^2 = \frac{12-4x}{3+x} \qquad 3x^2 + x^3 = 12 - 4x}$$

a) $0 = x^3 + 3x^2 + 4x - 12$

$12 - 4x = x^3 + 3x^2$

$12 - 4x = x^2(x+3)$

$\sqrt{\frac{4(3-x)}{(3+x)}} = x$

The equation $x^3 + 3x^2 + 4x - 12 = 0$ has a single root which is between 1 and 2.

(*b*) Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{4(3-\text{ANS})}{(3+\text{ANS})}\right)}, \quad n \geq 0,$$

with $x_0 = 1$ to find, to 2 decimal places, the value of x_1, x_2 and x_3. **(3)**

b) $x_1 = \sqrt{\frac{4(3-1)}{(3+1)}}$

$= \sqrt{2}$

$= 1.41$

$1 =$

$\sqrt{\frac{4(3-\text{ANS})}{(3+\text{ANS})}}$

$x_2 = 1.26$

$x_3 = 1.31$

The root of $f(x) = 0$ is α.

(*c*) By choosing a suitable interval, prove that $\alpha = 1.272$ to 3 decimal places. **(3)**

$f(1.2715) = 1.2715^3 + 3 \times 1.2715^2 + 4 \times 1.2715 - 12 = -8.21 \times 10^{-3} < 0$

$f(1.2725) = 8.27\ldots \times 10^{-3} > 0$ Change in sign, $f(x)$ is continuous, so, root is 1.272 to 3 dp.

Why does this method work?

Rearrange $x^2 - x - 1 = 0$ to make 3 different iterative formulae

$$x^2 - x - 1 = 0$$

Way 1

$$\boxed{x = x^2 - 1}$$

Way 2

$$x^2 = x + 1$$

$$\boxed{x = \sqrt{x+1}}$$

Way 3

$$x^2 - x - 1 = 0$$

$$x(x-1) - 1 = 0$$

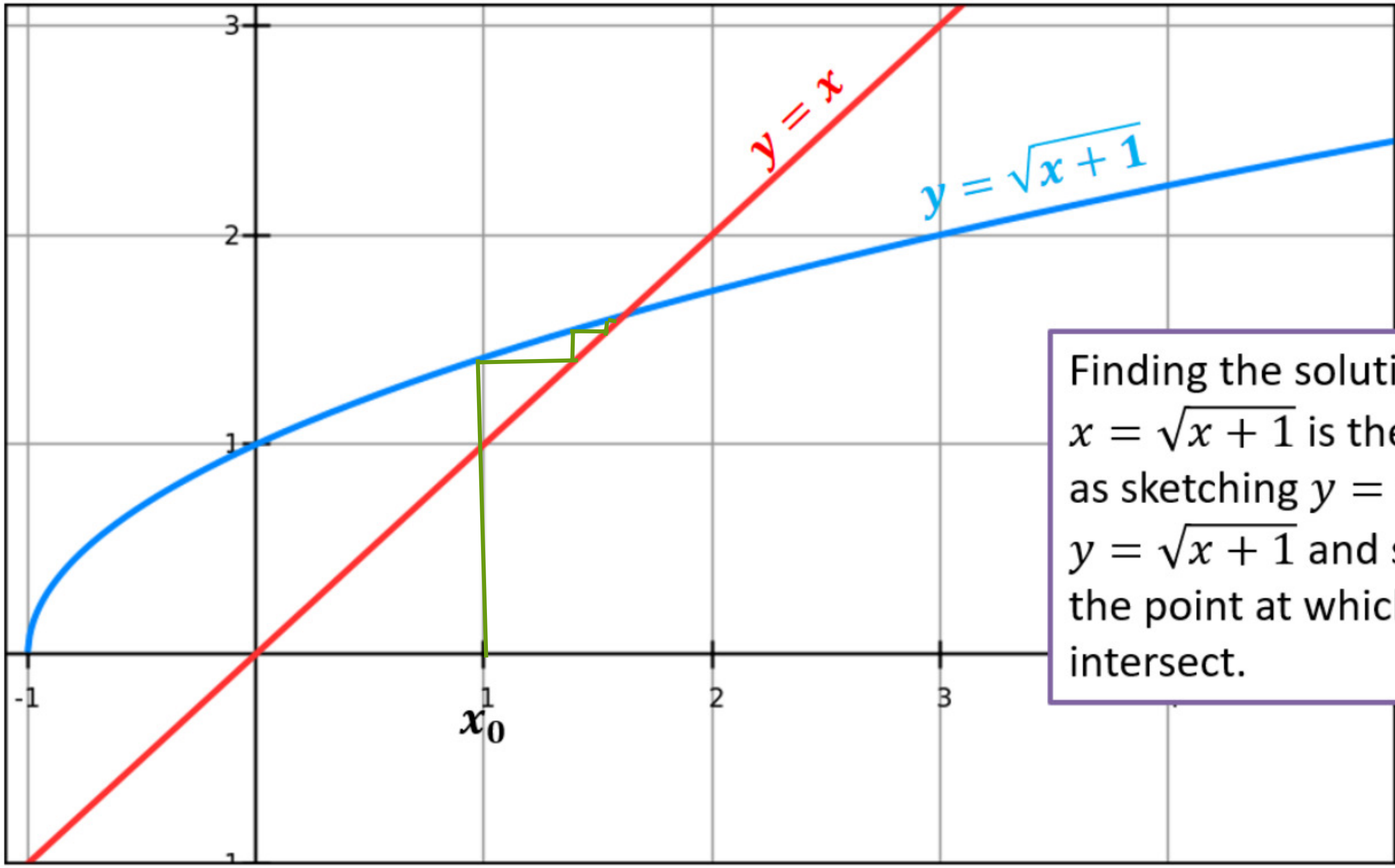
$$x(x-1) = 1$$

$$\boxed{x = \frac{1}{x-1}}$$

Staircase diagrams

Solve $x^2 - x - 1 = 0$

Recall we put in the form $x = g(x)$: in this case $x = \sqrt{x+1}$ is one possible rearrangement.
We can then use the recurrence $x_{n+1} = \sqrt{x_n + 1}$. Why does **this** recurrence work? (and not others?)



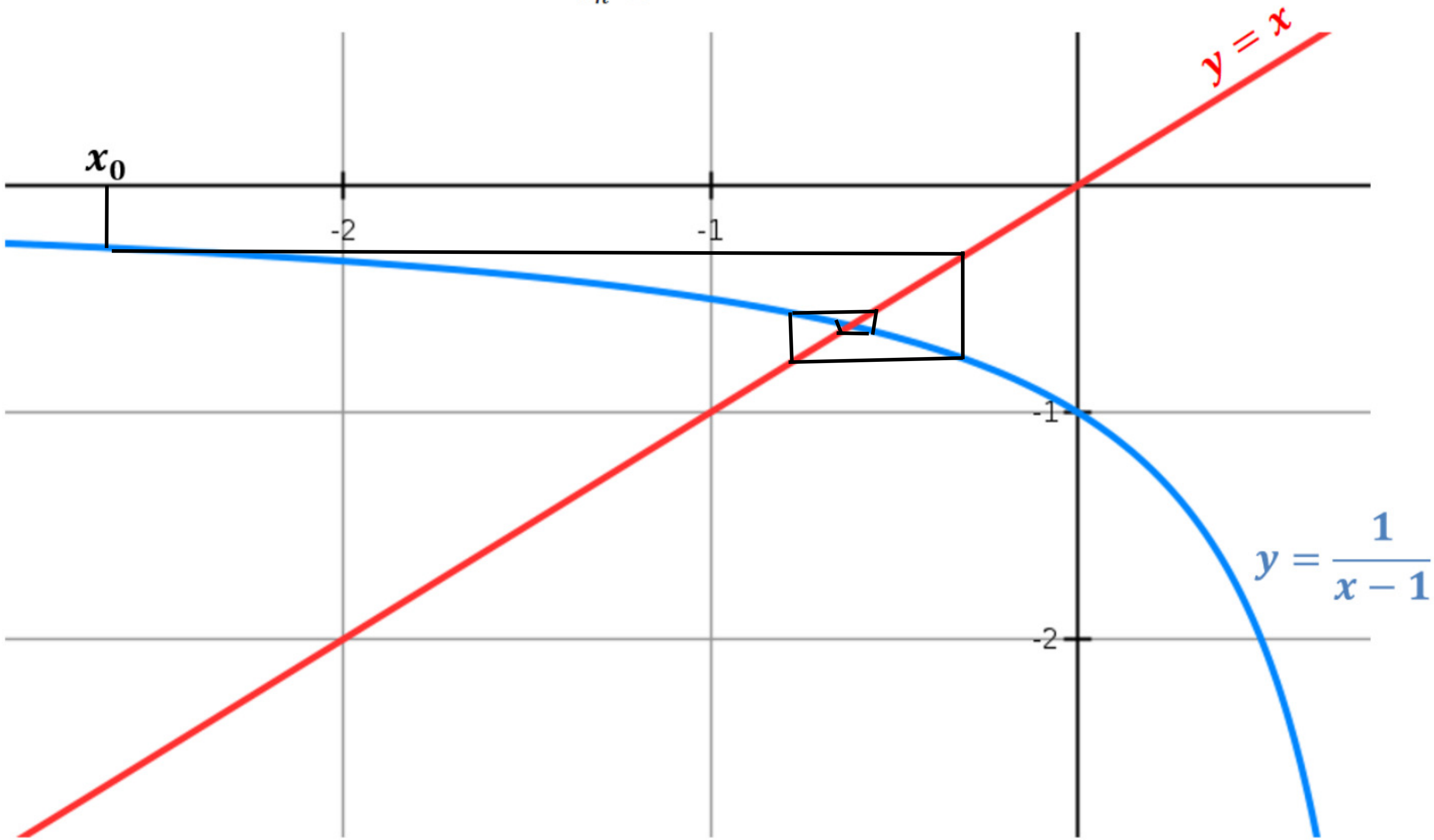
Finding the solution to $x = \sqrt{x+1}$ is the same as sketching $y = x$ and $y = \sqrt{x+1}$ and seeing the point at which they intersect.

Cobweb diagrams

Solve $x^2 - x - 1 = 0$

We could also have rearranged differently to $x = \frac{1}{x-1}$

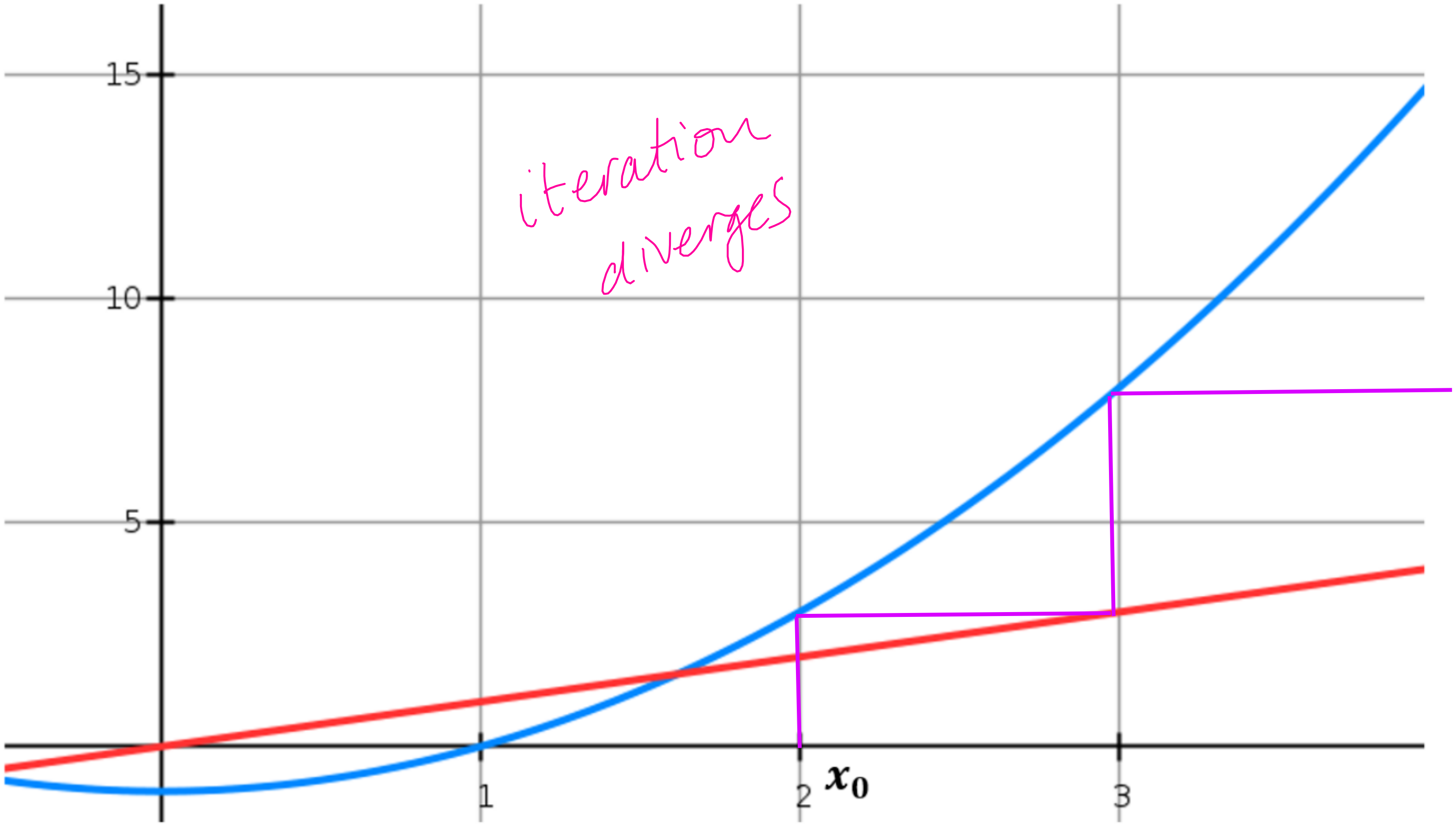
Therefore we use the recurrence $x_{n+1} = \frac{1}{x_n - 1}$. What happens this time?



And when iteration fails…

Solve $x^2 - x - 1 = 0$

But again, we could have rearranged differently! $x = x^2 - 1$
Therefore we use the recurrence $x_{n+1} = x_n^2 - 1$. What happens this time?

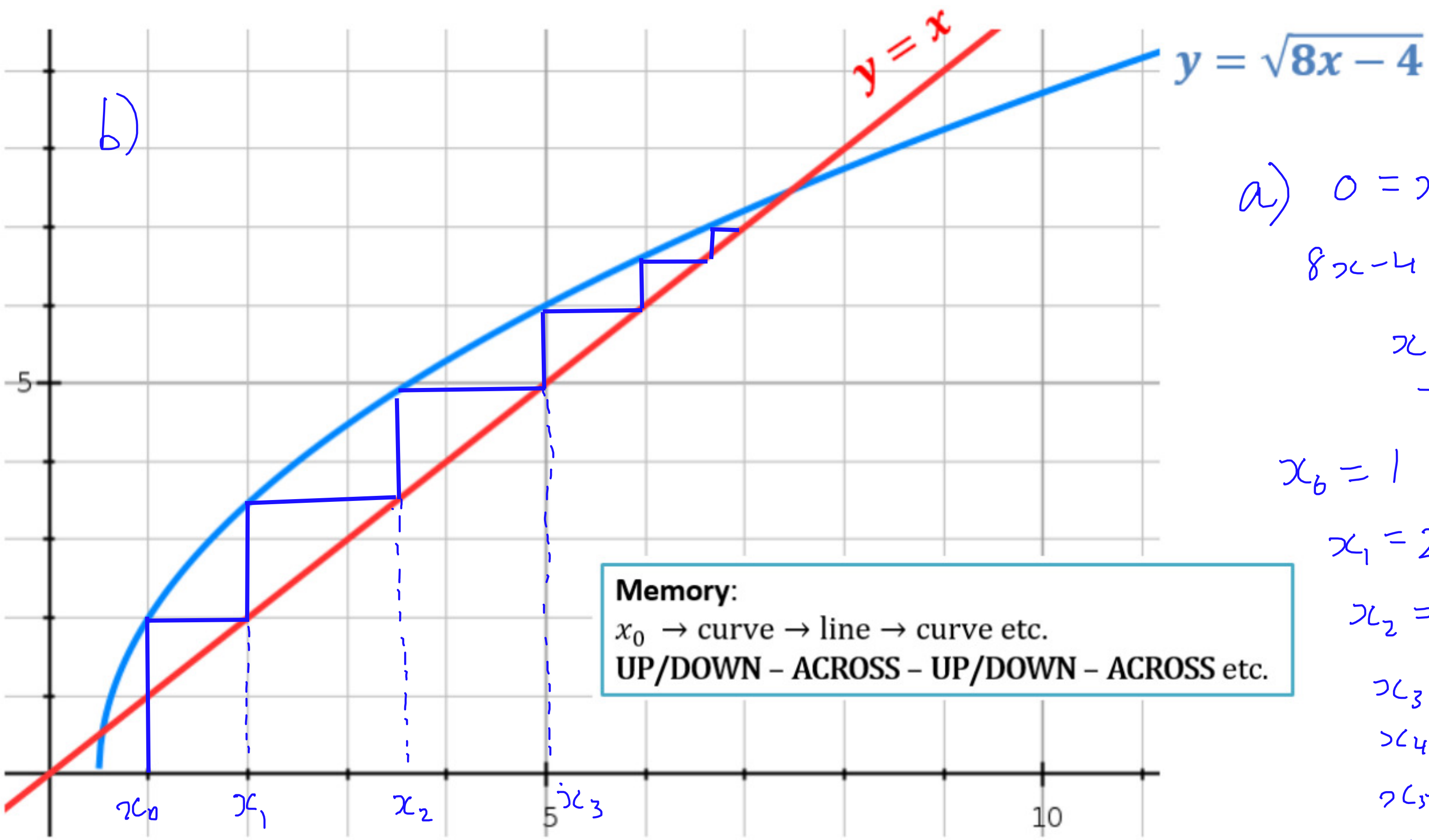


Your Turn

$$f(x) = x^2 - 8x + 4$$

(a) Show that the root of the equation $f(x) = 0$ can be written as $x = \sqrt{8x - 4}$

(b) Using the iterative formula $x_{n+1} = \sqrt{8x_n - 4}$, and starting with $x_0 = 1$, draw a staircase diagram, indicating x_0, x_1, x_2 on your x-axis, as well as the root α.



a) $0 = x^2 - 8x + 4$

$8x - 4 = x^2$

$x = \sqrt{8x - 4}$

$x_0 = 1$

$x_1 = 2$

$x_2 = 3.46$

$x_3 = 4.87$

$x_4 =$

x_5

$\alpha = 7.46$

Ex 10B

4. The curve with equation $y = 2\ln(8 - x)$ meets the line $y = x$ at a single point, $x = \alpha$.

(a) Show that $3 < \alpha < 4$

(2)

a) $y = 2\ln(8-x)$

$y = x$

$2\ln(8-x) = x$

$f(x) = 2\ln(8-x) - x$

$f(3) = 2\ln(8-3) - 3 = 0.2188$

$f(4) = -0.227$

Change in sign, $f(x)$ is continuous, so $3 < \alpha < 4$.

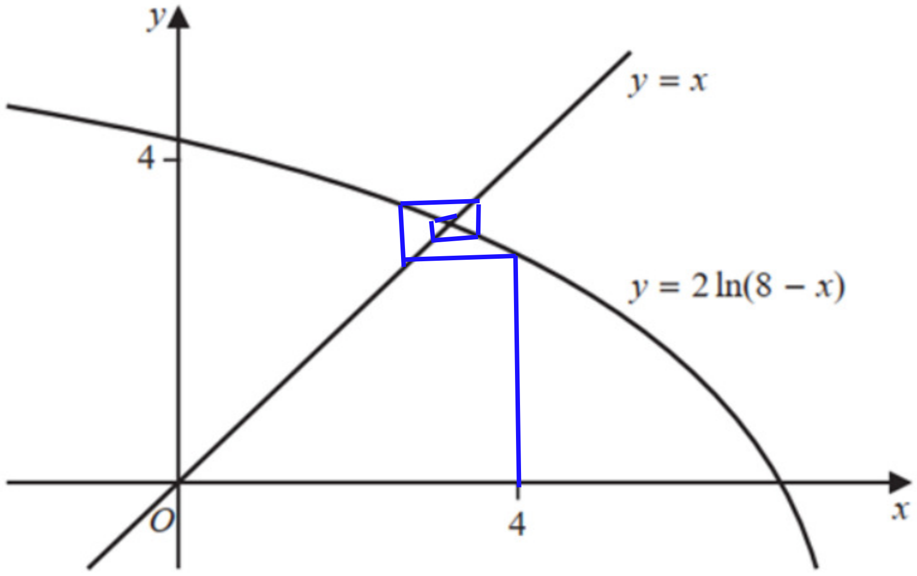


Figure 2

Figure 2 shows the graph of $y = 2\ln(8 - x)$ and the graph of $y = x$.

A student uses the iteration formula

$$x_{n+1} = 2\ln(8 - x_n), \quad n \in \mathbb{N}$$

in an attempt to find an approximation for α.

Using the graph and starting with $x_1 = 4$

staircase diagram

(b) determine whether or not this iteration formula can be used to find an approximation for α, justifying your answer.

(2)

Yes it can be used. The diagram shows the cobweb spirals inwards, converging to the root α.

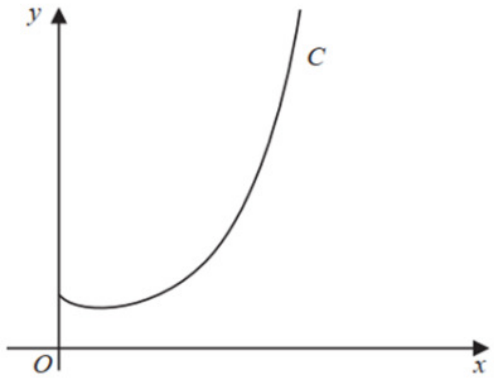


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x,\ x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{\,1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

a) $y = x^x$

$u = x \quad v = \ln x$

$u' = 1 \quad v' = \frac{1}{x}$

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = y(1 + \ln x)$

$\frac{dy}{dx} = 0 \qquad y(1+\ln x) = 0$

$y = 0$ or $1 + \ln x = 0$

$x^x = 0$

$\ln x = -1$

$x = e^{-1}$

$\underline{x = 0.3679}$ (4dp)

b) $y = 2$

$2 = x^x$

$f(x) = x^x - 2$

$f(1.5) = 1.5^{1.5} - 2 = -0.162\ldots < 0$

$f(1.6) = 1.6^{1.6} - 2 = 0.121 > 0$

So, change in sign, $f(x)$ is continuous,

So $\underline{1.5 < \alpha < 1.6}$