Your notes

Edexcel A Level Further Maths: Decision Maths 1

Simplex Algorithm

Contents

Simplex Algorithm – Slack Variables & Initial Tableau

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Introduction to the Simplex Algorithm

What is the simplex algorithm?

- The **simplex algorithm** is an alternative to the graphical method for solving **linear programming** problems
  - It is particularly useful when there are **more than 2 decision variables** as these cannot be drawn graphically (Not very easily at least!)
  - Essentially the simplex algorithm works by considering each **vertex** of the **basic feasible region** in turn until an **optimal solution** is found
- Initially the **simplex algorithm** can **only** be applied to an LP problem that has a **basic feasible solution**
  - For problems where the basic feasible **region** contains the origin, the basic feasible **solution** is the **origin**
    - In practical situations though, this is often not realistic
      - For example, the number of chairs and tables made by a furniture manufacturer trying to maximise their profit
      - if they make no chairs and no tables, they won't make any profit!
      - but the origin would satisfy all constraints if it is in the feasible region
  - For problems where the basic feasible region does not contain the **origin** the algorithm is adapted such that a **basic feasible solution** is found first
    - then the simplex algorithm can be applied as usual

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Slack Variables

What are slack variables?

- In the first instance, the **simplex algorithm** deals with **constraints** (inequalities) involving ≤ (excluding the **non-negativity** constraints)
  - **Slack variables** are used to turn **inequalities** involving ≤ into **equations**
- A **slack** variable, as the name implies, takes up the spare (slack) that a function falls short on in "less than or equal to" constraints
  - A **slack** variable will be **added** to (the left hand side of) each **constraint** so will be **non-negative**
- The **number** of slack variables required in a problem will be the **same** as the number of (non-negativity) **constraints** involving ≤

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Worked example

The following linear programming problem is to be solved using the simplex algorithm.

Maximise

$$P = 8x + 5y + 7z$$

subject to

$$2x + 3y \leq 10$$
$$x + 2y + 5z \leq 60$$
$$5y + 3z \leq 40$$
$$x,\ y,\ z \geq 0$$

Use slack variables to write the constraints (except the non-negativity constraint) of the linear programming problem as equations.

Use s_1 to 'use up the slack' in the inequality $2x + 3y \leq 10$

$$2x + 3y + s_1 = 10$$

use s_2 for $x + 2y + 5z \leq 60$

$$x + 2y + 5x + s_2 = 60$$

and s_3 for $5y + 3z \leq 40$

$$5y + 3z + s_3 = 40$$

The constraints as equations using slack variables are

$$\mathbf{2x + 3y + s_1 = 10}$$
$$\mathbf{x + 2y + 5x + s_2 = 60}$$
$$\mathbf{5y + 3z + s_3 = 40}$$

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Initial Tableau

What is the initial tableau?

- The simplex algorithm is performed by creating a series of matrices, or **tables,** showing the values of each **decision variable**, each **slack variable** and the **objective function** until an optimal solution is found
- The first of these tables is called the **initial tableau**
- To set up the initial tableau, the equations derived from the inequalities (constraints) are needed along with a rearrangement of the objective function such that zero is on one side
  - e.g. $P = 8x + 5y + 7z$ becomes $P - 8x - 5y - 7z = 0$
- The **initial tableau** is created by
  - columns represent the **basic variable** (b.v.), the **decision variables** ($x,\ y,\ z$), the **slack variables** $(s_1,\ s_2,\ s_3)$ and the **value** (the RHS of each equation)
  - rows are completed with the coefficients of each **equation/constraint**
    - the **last** row is the (rearranged) **objective function**
  - the basic variable column is completed **last**
    - the basic variable for each row is the variable that has a 1 in that row but 0's in all other rows of that column
    - the last row is always the **objective** row

e.g. Constructing the initial tableau for the objective function

$$P - 8x - 5y - 7z = 0$$

and equations (constraints)

$$2x + 3y + s_1 = 10$$
$$x + 2y + 5x + s_2 = 60$$
$$5y + 3z + s_3 = 40$$

COLUMN 1 ABBREVIATED TO C1

EACH ROW IS THE COEFFICIENTS OF THE VARIABLES, WITH THE RHS BEING THE 'VALUE'

THE BASIC VARIABLE (B. V.) COLUMN IS COMPLETED LAST BY LOOKING FOR THE COLUMN ONLY WITH A 1 AND 0'S IN IT

B.V.	x	y	z	s_1	s_2	s_3	VALUE
s_1	2	3	0	1	0	0	10
s_2	1	2	5	0	1	0	60
s_3	0	5	3	0	0	1	40
P	−8	−5	−7	0	0	0	0

ROW 4 ABBREVIATED TO R4

CELL R2C3

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Simplex Algorithm

How do I apply the simplex algorithm?

- Once the **initial tableau** is set up the first iteration can begin
- **STEP 1**
  For each row (except the objective row), find the θ**-value** by **dividing** the '**Value**' by the **entry** in the **pivot column**
  - Deduce the **pivot column** by looking for the **most negative** entry in the **objective row** (excluding 'Value')
- **STEP 2**
  Find the **pivot element** (often just called the pivot) by finding where the **pivot column** and **pivot row** meet
  - Deduce the **pivot row** by selecting the row with the **least positive** θ**-value**
- **STEP 3**
  Use a series of **row operations** to update the tableau
  - manipulate the **pivot row** so that the **pivot element** is **1**
  - for each 'old row' (i.e. each row from the previous tableau) use a multiple of the 'new pivot row' to make each entry in the pivot column **zero,** including the **objective row**
- **STEP 4**
  Complete the updated tableau by completing the header column of **basic variables** (b.v.)
  - to do this look for the **column** with **one 1** and **all other** entries **0**

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Worked example

The initial tableau for using the simplex algorithm is given below. Apply the first iteration of the simplex algorithm.

b.v.	x	y	z	s_1	s_2	s_3	Value
s_1	2	3	0	1	0	0	10
s_2	1	2	5	0	1	0	60
s_3	0	5	3	0	0	1	40
P	-8	-5	-7	0	0	0	0

- **STEP 1**
  The most negative entry in the objective row is -8, so column 1 (C1) is the pivot column
  The θ-values for each row are

  R1: $\theta_1 = 10 \div 2 = 5$

  R2: $\theta_2 = 60 \div 1 = 60$

  R3: - n/a since division by zero is undefined

- **STEP 2**
  The least positive θ-value is $\theta_1 = 5$ so the pivot row is row 1 (R1)
  The pivot element is R1C1

  Pivot is 2

- **STEP 3**
  Use the row operation $0.5'\text{R}1'$ to make the pivot element 1
  (The apostrophes indicate 'old row 1' - i.e. the row from the previous tableau.)
  For every other row use a row operation that makes the entry in the pivot column 0
  (e.g. the objective row (R4) will be 'old row 4 plus 8 lots of new row 1' ($'\text{R}4' + 8\text{R}1$))

- **STEP 4**
  Complete the b.v. column by looking for a column with one 1 and all others 0

b.v.	x	y	z	s_1	s_2	s_3	Value	Row Op.
x	1	1.5	0	0.5	0	0	5	$0.5'\text{R}1'$
s_2	0	0.5	5	-0.5	1	0	55	$'\text{R}2' - \text{R}1$
s_3	0	5	3	0	0	1	40	$'\text{R}3'$
P	0	7	-7	4	0	0	40	$'\text{R}4' + 8\text{R}1$

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How do I know when the simplex algorithm is complete?

- The simplex algorithm is **compete** when there are **no negative** entries in the **objective row** (excluding 'Value')
  - This tableau is called the **final tableau**
  - The **optimal solution** can be found from the final tableau

How do I find the optimal solution from the final tableau?

- The **basic variable** (**b.v.**) and **value** column indicate the optimal **solution**
  - The **variables** in the **basic value column** take the **value** in their **value column**
  - The **objective function** will then have a **maximum value** of the entry in its **value column**
  - All other variables are **non-basic** and take the value of 0, regardless of values in the table

Your notes

Worked example

Apply one more iteration of the simplex algorithm to the tableau below and hence find the optimal solution to the linear programming problem it represents.

b.v.	x	y	z	s_1	s_2	s_3	Value
x	1	1.5	0	0.5	0	0	5
s_2	0	0.5	5	-0.5	1	0	55
s_3	0	5	3	0	0	1	40
P	0	7	-7	4	0	0	40

First find the pivot column; -7 in C3 is the most negative entry in the objective row

Find the θ-values for each row

$$\theta_1 = \text{n/a}$$
$$\theta_2 = 55 \div 5 = 11$$
$$\theta_3 = 40 \div 3 = 13.333...$$

11 is the least positive so the pivot is 5 (R2C3)

Use row operations to make the pivot 1 and other entries in its column 0
Complete the b.v. column

b.v.	x	y	z	s_1	s_2	s_3	Value	Row Op.
x	1	1.5	0	0.5	0	0	5	$'\text{R1}'$
s_2	0	0.1	1	-0.1	0.2	0	11	$0.2'\text{R2}'$
s_3	0	4.7	0	0.3	-0.6	1	7	$'\text{R3}' - 3\text{R2}$
P	0	7.7	0	3.3	1.4	0	117	$'\text{R4}' + 7\text{R2}$

There are no negative entries in the objective row so the algorithm is complete and the optimal solution can be read from the final tableau

Basic variables (in header column) take the values from their 'Value' column
Non-basic variable take the value zero
The maximum value of the objective function is the 'Value' in its row

$$x = 5,\ z = 11,\ s_3 = 7$$

$$y = 0,\ s_1 = 0,\ s_2 = 0$$

P **has a maximum value of 117**

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Two-stage Simplex Method

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Introduction to Two-Stage Simplex Method

What is the two-stage simplex method?

- The **simplex algorithm** solves **maximisation** linear programming problems where all (non-negativity) constraints involve ≤
- The **two-stage simplex method** adapts the simplex algorithm to solve linear programming problems
  - involving **constraints** with ≥
  - where the objective function is to be **minimised**
  - In both cases the simplex algorithm cannot be used directly
    - it is based on the basic feasible **region** containing the basic feasible **solution** of the **origin**
- The two-stage simplex method is two applications of the simplex algorithm
  - The **first stage** of the two-stage simplex method finds a **basic feasible solution** (if one exists)
  - If a basic feasible solution is found the **second stage** applies the simplex algorithm as usual

Surplus & Artificial Variables

What are surplus variables?

- **Surplus variables** are used to turn **inequalities** involving ≥ into **equations**
  - A **surplus** variable, as the name implies, takes up the excess (surplus) that a function has in "greater than or equal to" constraints
  - A **surplus** variable will be **subtracted** from (the left hand side of) each **constraint** so will be **non-negative**
- The **number** of surplus variables required in a problem will be the **same** as the number of (non-negativity) constraints that contain **≥**
  - Any inequalities involving **≤** will still require slack variables
- Whenever a surplus variable is introduced, an **artificial variable** will also be needed

What are artificial variables?

- **Artificial variables** are used to ensure each constraint with a **surplus variable** contains a **basic variable**
  - A **surplus** variable is **subtracted,** so will have a **coefficient** of **-1**
    - **basic** variables have a **coefficient** of **1**
  - Artificial variables will **always** start with a coefficient of 1

Examiner Tip

- You need to be able to recognise when the simplex algorithm **cannot** (directly) be used to solve a linear programming problem
  - This is due to the presence of constraints involving ≥ (or a mixture of ≤ and ≥)

Your notes

Your notes

Worked example

The following linear programming problem is to be solved using the two-stage simplex method.

Maximise

$$P = 2x + 4y + 3z$$

subject to

$$\begin{aligned} x + y + z &\geq 20 \\ 2x - y + 2z &\geq 25 \\ 2x + 3y + 4z &\leq 80 \\ x, y, z &\geq 0 \end{aligned}$$

Use slack, surplus and artificial variables as necessary to write the constraints (except the non-negativity constraint) of the linear programming problem as equations.

Use s_1 and a_1 as surplus and artificial variables in the first constraint as it involves ≥

$$x + y + z - s_1 + a_1 = 20$$

The second constraint involves ≥ too so will also need a surplus and artificial variable

$$2x - y + 2z - s_2 + a_2 = 25$$

The third constraint involves ≤ so requires a slack variable

$$2x + 3y + 4z + s_3 = 80$$

The constraints as equations using slack, surplus and artificial variables are

$$\begin{aligned} x + y + z - s_1 + a_1 &= 20 \\ 2x - y + 2z - s_2 + a_2 &= 25 \\ 2x + 3y + 4z + s_3 &= 80 \end{aligned}$$

How do I set up the initial tableau for the two-stage simplex method?

- To set up the initial tableau, the equations derived from the inequalities (constraints) are needed along with
  - a rearrangement of the objective function such that zero is on one side
    - e.g. $P = 2x + 4y + 3z$ becomes $P - 2x - 4y - 3z = 0$

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- a **new objective function,** I. which is the **negative sum** of the **artificial variables**
  - e.g. where two artificial variables a_1 and a_2 have been introduced, $I = -\left(a_1 + a_2\right)$
  - In general, $I = -\left(a_1 + a_2 + \ldots + a_n\right)$ for n artificial variables
  - artificial variables should be written in terms of the **decision** and **surplus** variables before being substituted into I
  - I is then rearranged such that all variables are on one side and a constant is on the other
- The initial tableau can then be constructed with two objective rows
  - one for P, one for I

Your notes

Worked example

Construct the initial tableau for the two-stage simplex method for the objective function

$$P - 2x - 4y - 3z = 0$$

and equations (constraints)

$$x + y + z - s_1 + a_1 = 20$$
$$2x - y + 2z - s_2 + a_2 = 25$$
$$2x + 3y + 4z + s_3 = 80$$

Introduce I (P is already in the required form)

$$I = -(a_1 + a_2)$$

Rewrite a_1 and a_2 in terms of x, y, z, s_1 and s_2 (s_3 is a slack variable)

$$a_1 = 20 - x - y - z + s_1$$
$$a_2 = 25 - 2x + y - 2z + s_2$$

Substitute into I

$$I = -(20 - x - y - z + s_1 + 25 - 2x + y - 2z + s_2)$$
$$I = -(45 - 3x - 3z + s_1 + s_2)$$

Rearrange so all variables are on the same side as (positive) I and a constant on the other

$$I - 3x - 3z + s_1 + s_2 = -45$$

Set up the initial tableau two objective rows for P and I

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value
a_1	1	1	1	-1	0	0	1	0	20
a_2	2	-1	2	0	-1	0	0	1	25
s_3	2	3	4	0	0	1	0	0	80
P	-2	-4	-3	0	0	0	0	0	0

I	-3	0	3	1	1	0	0	0	-45

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Maximising Problems

How do I apply the first-stage of the two-stage simplex method to maximising problems?

- In the **first-stage**, I will be **maximised**
  - the **maximum** I can be is **0**
  - this is when **each artificial variable equals 0**
    - artificial variables cannot be negative
- At the end of the **first-stage**
  - if $I \neq 0$ then a basic feasible region does **not** exist and the linear programming problem **cannot** be solved
    - there will be no need (or point) in progressing to the second-stage
  - if $I = 0$ then a basic feasible region does exist
    - the objective row I and the artificial variable columns can be removed from the tableau before commencing the second stage

Your notes

Your notes

Worked example

The initial tableau for using the two-stage simplex method is given below.
Apply the first-stage of the two-stage simplex method to maximise I.
Give your final answer as the tableau with I, a_1 and a_2 removed.

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value
a_1	1	1	1	-1	0	0	1	0	20
a_2	2	-1	2	0	-1	0	0	1	25
s_3	2	3	4	0	0	1	0	0	80
P	-2	-4	-3	0	0	0	0	0	0
I	-3	0	3	1	1	0	0	0	-45

Apply the simplex algorithm to maximise I

There is a negative entry in the (I) objective row
-3 is the most negative

C1 is the pivot column, the θ-values are

$$\theta_1 = 20 \div 1 = 20$$
$$\theta_2 = 25 \div 2 = 12.5$$
$$\theta_3 = 80 \div 2 = 40$$

θ_2 is the least positive so R2 is the pivot row
The pivot element is in cell R2C1 (highlighted in tableau above)

Pivot is 2

Apply the appropriate row operations, starting with the pivot row, R2

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	Row Op.
a_1	0	1.5	0	-1	0.5	0	1	-0.5	7.5	$'\text{R1}' - \text{R2}$
x	1	-0.5	1	0	-0.5	0	0	0.5	12.5	$0.5'\text{R2}'$
s_3	0	4	2	0	1	1	0	-1	55	$'\text{R3}' - 2\text{R2}$
P	0	-5	-1	0	-1	0	0	1	25	$0.5'\text{R4}' + 2\text{R2}$

I	0	-1.5	0	1	-0.5	0	0	1.5	-7.5	$'R5' + 3R2$

There is a negative entry in the (I) objective row
-1.5 is the most negative

C2 is the pivot column, the θ-values are

$$\theta_1 = 7.5 \div 1.5 = 5$$
$$\theta_2 = 12.5 \div (-0.5) = -25$$
$$\theta_3 = 55 \div 4 = 13.75$$

θ_1 is the least positive so R1 is the pivot row
The pivot element is in cell R1C2

Pivot is 1.5

Apply the appropriate row operations, starting with the pivot row, R1

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	Row Op.
y	0	1	0	-2/3	1/3	0	2/3	-1/3	5	$1.5'R1'$
x	1	0	1	-1/3	-1/3	0	1/3	1/3	15	$'R2' + 0.5R1$
s_3	0	0	2	8/3	-1/3	1	-8/3	1/3	35	$'R3' - 4R1$
P	0	0	-1	-10/3	2/3	0	10/3	-2/3	50	$'R4' + 5R1$
I	0	0	0	0	0	0	1	1	0	$'R5' + 1.5R1$

There are no negative values in the (I) objective row so I has been maximised and the first-stage of the method is complete
Reading from the tableau, we see that neither a_1 nor a_2 are basic variables so both have the value 0, and the value of I is 0

The final answer is required as the tableau with I, a_1 and a_2 removed

b.v.	x	y	z	s_1	s_2	s_3	Value
y	**0**	**1**	**0**	**-2/3**	**1/3**	**0**	**5**
x	**1**	**0**	**1**	**-1/3**	**-1/3**	**0**	**15**
s_3	**0**	**0**	**2**	**8/3**	**-1/3**	**1**	**35**

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P	0	0	-1	-10/3	2/3	0	50

Your notes

How do I apply the second-stage of the two-stage simplex method to maximising problems?

- After the first-stage, the value of I should be zero
  - If it is not then a **feasible region does not** exist and the linear programming problem **cannot** be solved
- In the **second-stage,** the simplex algorithm is applied as usual to maximise the objective function P

Your notes

Worked example

After the first-stage of the two-stage simplex method has been applied to a linear programming problem the following tableau is formed. Apply the second-stage of the two-stage simplex method to find the optimal solution.

b.v.	x	y	z	s_1	s_2	s_3	Value
y	0	1	0	-2/3	1/3	0	5
x	1	0	1	-1/3	-1/3	0	15
s_3	0	0	2	8/3	-1/3	1	35
P	0	0	-1	-10/3	2/3	0	50

The second-stage of the two stage simplex method is to apply the simplex algorithm

There is a negative entry in the objective row
-10/3 is the most negative

C4 is the pivot column, the θ-values are

$$\theta_1 = 5 \div (-2/3) = -7.5$$
$$\theta_2 = 15 \div (-1/3) = -45$$
$$\theta_3 = 35 \div 8/3 = 13.125$$

θ_3 is the least positive so R3 is the pivot row
The pivot element is in cell R3C4 (highlighted in tableau above)

Pivot is 8/3

Apply the appropriate row operations, starting with the pivot row, R3

b.v.	x	y	z	s_1	s_2	s_3	Value	Row Op.
y	0	1	0.5	0	0.25	0.25	13.75	$'\text{R1}' + 2/3\ \text{R3}$
x	1	0	1.25	0	-0.375	0.125	18.375	$'\text{R2}' + 1/3\ \text{R3}$
s_1	0	0	0.75	1	-0.125	0.375	13.126	$8/3\ '\text{R3}'$
P	0	0	1.5	0	0.25	1.25	93.75	$'\text{R4}' + 10/3\ \text{R1}$

There are no negative values in the objective row so the algorithm is complete and the optimal solution can be read from the final tableau

Your notes

x, y, s_1 are basic variables, s_2, s_3 are non-basic

$$x = 18.375,\ y = 13.75,\ s_1 = 13.126$$

$$z = s_2 = s_3 = 0$$

P has a maximum value of 93.75

Minimising Problems

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How do I apply the two-stage simplex method to minimising problems?

- Minimising the objective function is the same as maximising the negative of the objective function
- So in **minimisation problems,** we **maximise** the **negative** of the **objective function**
  - e.g. Minimising $C = 4x + 2y$ is the same as maximising $P = -C = -(4x + 2y) = -4x - 2y$
- The two-stage simplex method for minimisation problems is otherwise the same as for maximisation problems
  - The **first-stage maximises** the (new) objective function, I
    - where I is the **negative sum** of the **artificial variables**
    - $I = -\left(a_1 + a_2 + \ldots + a_n\right)$
  - At the end of the first-stage, if $I = 0$ a feasible region exists
    - the tableau can have the row for I and the columns for the artificial variable removed
    - the second-stage of applying the simplex algorithm will then maximise P
- For minimisation problems, the **maximum** P will need interpreting as the **minimum** for C
  - C is often used as 'costs' require minimising (P is often for profit)

Your notes

Worked example

Solve the following linear programming problem using the two-stage simplex method.

Minimise

$$C = 4x + 2y$$

subject to

$$\begin{aligned} x + 8y &\geq 9 \\ x + y &\leq 9 \\ 6x - y &\geq 5 \\ x, y &\geq 0 \end{aligned}$$

Minimising C is the same as maximising P, where

$$P = -C = -4x - 2y$$

P will need rewriting in the correct form for the initial tableau

$$P + 4x + 2y = 0$$

Use slack, surplus and artificial variables to rewrite the constraints as equations

$$\begin{aligned} x + 8y - s_1 + a_1 &= 9 \\ x + y + s_2 &= 9 \\ 6x - y - s_3 + a_2 &= 5 \end{aligned}$$

Introduce, find and write I in the correct format for the initial tableau

$$\text{Let } I = -(a_1 + a_2)$$

$$I = -(9 - x - 8y + s_1 + 5 - 6x + y + s_3)$$

$$I = -(14 - 7x - 7y + s_1 + s_3)$$

$$I - 7x - 7y + s_1 + s_3 = -14$$

Construct the initial tableau

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value

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a_1	1	8	-1	0	0	1	0	9
s_2	1	1	0	1	0	0	0	9
a_2	6	-1	0	0	-1	0	1	5
P	4	2	0	0	0	0	0	0
I	-7	-7	1	0	1	0	0	-14

Apply the simplex algorithm to maximise I

There is a negative entry in the (I) objective row
-7 is the most negative (either can be chosen, we're choosing the first)

C1 is the pivot column, the θ-values are

$$\theta_1 = 9 \div 1 = 9$$
$$\theta_2 = 9 \div 1 = 9$$
$$\theta_3 = 5 \div 6 = 0.833...$$

θ_3 is the least positive so R3 is the pivot row
The pivot element is in cell R3C1

Pivot is 6

Apply the appropriate row operations, starting with the pivot row, R3

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value	Row Op.
a_1	0	49/6	-1	0	1/6	1	-1/6	49/6	$'R1' - R3$
s_2	0	7/6	0	1	1/6	0	-1/6	49/6	$'R2' - R3$
x	1	-1/6	0	0	-1/6	0	1/6	5/6	$1/6\,'R3'$
P	0	8/3	0	0	2/3	0	-2/3	-10/3	$'R4' - 4R3$
I	0	-49/6	1	0	-1/6	0	-7/6	-49/6	$'R5' + 7R3$

There is a negative entry in the (I) objective row
-49/6 is the most negative

C2 is the pivot column, the θ-values are

Your notes

$$\theta_1 = 49/6 \div 49/6 = 1$$
$$\theta_2 = 49/6 \div 7/6 = 7$$
$$\theta_3 = 5/6 \div (-1/6) = -5$$

θ_1 is the least positive so R1 is the pivot row
The pivot element is in cell R1C2

Pivot is 49/6

Apply the appropriate row operations, starting with the pivot row, R1

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value	Row Op.
y	0	1	-6/49	0	1/49	6/49	-1/49	1	$6/49\ \text{'R1'}$
s_2	0	0	1/7	1	1/7	-1/7	-1/7	7	$\text{'R2'} - 7/6\ \text{R1}$
x	1	0	-1/49	0	-8/49	1/49	8/49	1	$\text{'R3'} + 1/6\ \text{R1}$
P	0	0	16/49	0	30/49	-16/49	-30/49	-6	$\text{'R4'} - 8/3\ \text{R1}$
I	0	0	0	0	0	1	1	0	$\text{'R5'} + 49/6\ \text{R1}$

There are no negative entries in the (I) objective row so the first-stage is complete, $a_1 = a_2 = I = 0$

Remove a_1, a_2, I from the tableau to complete the first-stage

b.v.	x	y	s_1	s_2	s_3	Value
y	0	1	-6/49	0	1/49	1
s_2	0	0	1/7	1	1/7	7
x	1	0	-1/49	0	-8/49	1
P	0	0	16/49	0	30/49	-6

There are no negative values in the objective row so the method is complete (no second-stage required) and the optimal solution can be read from the final tableau

x, y, s_2 are basic variables, s_1, s_3 are non-basic

Change from maximising P to minimising C

$$C_{min} = -P_{max} = -(-6) = 6$$

The solution to the linear programming problem is

$$x = 1,\ y = 1$$

with C minimised at $C = 6$

$$(s_1 = s_3 = 0,\ s_2 = 7)$$

Your notes

Your notes

Big-M Method

Big-M Method

What is the Big-M method?

- The **Big-M method** is an adaption of the **simplex algorithm**
  - It is an alternative to the two-stage simplex method
- The **Big-M** method can be used to
  - solve problems involving constraints that contain ≥
  - solve minimisation (as well as maximisation) linear programming problems
- The Big-M method has the **advantage** of not requiring two stages
  - Each tableau requires just one objective row
- The big-M method has the **disadvantage** that some of the algebra can get awkward to track and follow
- M is an arbitrarily large positive number
  - This is so expressions such as $1 - M$ are definitely negative and $M - 12$ will definitely be positive
  - M is never actually assigned a value, nor would it need to be calculated

How do I rewrite the constraints and objective function for the Big-M method?

- **STEP 1**
  Use slack, surplus and artificial variables to convert the constraints of a linear programming problem into equations
- **STEP 2**
  Rearrange each constraint containing an **artificial variable** such that the artificial variable is the subject
- **STEP 3**
  Subtract MA from the **objective function,** where
  - A is the sum of the artificial variables ($a_1 + a_2 + \ldots$)
  - M is an arbitrarily large, positive number

Your notes

Worked example

The linear programming problem formulated below is to be solved using the Big-M method.

Maximise

$$P = 3x + 2y$$

subject to

$$\begin{aligned} x - y &\leq 4 \\ x + 2y &\leq 16 \\ 2x + 3y &\geq 18 \\ 2x - y &\geq 0 \\ x,\ y &\geq 0 \end{aligned}$$

Rewrite the constraints and objective function such that the initial tableau for the Big-M method can be produced.
(You do not need to produce the initial tableau.)

- **STEP 1**
  $x - y \leq 4$ requires a slack variable

  $$x - y + s_1 = 4$$

  $x + 2y \leq 16$ requires a slack variable

  $$x + 2y + s_2 = 16$$

  $2x + 3y \geq 18$ requires a surplus and an artificial variable

  $$2x + 3y - s_3 + a_1 = 18$$

  $2x - y \geq 0$ requires a surplus and an artificial variable

  $$2x - y - s_4 + a_2 = 2$$

  $$(s_1,\ s_2,\ s_3,\ s_4,\ a_1,\ a_2 \geq 0)$$

- **STEP 2**
  Rearrange each constraint containing an artificial variable

Your notes

$$a_1 = 18 - 2x - 3y + s_3$$
$$a_2 = 2 - 2x + y + s_4$$

- STEP 3
  Subtract MA from P (find A first)
  $$A = 18 - 2x - 3y + s_3 + 2 - 2x + y + s_4$$
  $$A = 20 - 4x - 2y + s_3 + s_4$$

  and so

  $$P = 3x + 2y - AM$$
  $$P = 3x + 2y - M(20 - 4x - 2y + s_3 + s_4)$$
  $$\boldsymbol{P - (4M + 3)x - (2M + 2)y + Ms_3 + Ms_4 = -20M}$$

How do I apply the Big-M method?

- Once the constraints and objective function and rewritten the initial tableau for the Big-M method can be produced
  - (Some of) the entries in the objective line will be algebraic - in terms of M
- Apply the **simplex algorithm** as usual to solve the problem
  - A tableau is optimal when there are no negative entries in the objective row
    - Big-M makes negative entries easy to spot!
  - The row operations for the objective row are a little harder as they involve adding or subtracting algebraic terms that are in terms of M
  - As previously, use apostrophes to indicate 'old rows' in the row operations column

Examiner Tip

- If a question doesn't specify, it is a good idea to write down the values of **all** variables as the final answer
  - the decision variables, and any **slack, surplus** and **artificial** variables
  - only the basic variables take their values from the final tableau
  - non-basic variables **always** have the value zero
- Remember to also state the **objective function** value, stating whether it is a maximum or minimum

Worked example

A maximisation linear programming problem has been formulated so it is ready to be solved using the Big-M adaption of the simplex algorithm.

$$x - y + s_1 = 4$$
$$x + 2y + s_2 = 16$$
$$2x + 3y - s_3 + a_1 = 18$$
$$2x - y - s_4 + a_2 = 2$$
$$P - (4M+3)x - (2M+2)y + Ms_3 + Ms_4 = -20M$$

Form the initial tableau and apply the simplex algorithm to find the optimal solution to the problem.

The initial tableau is formed from the rearranged constraints and objective function

b.v.	x	y	s_1	s_2	s_3	s_4	a_1	a_2	Value
s_1	1	-1	1	0	0	0	0	0	4
s_2	1	2	0	1	0	0	0	0	16
a_1	2	3	0	0	-1	0	1	0	18
a_2	2	-1	0	0	0	-1	0	1	2
P	$-(4M+3)$	$-(2M+2)$	0	0	M	M	0	0	$-20M$

There is at least one negative entry in the objective line so the tableau is not yet optimal
Apply an iteration of the simplex algorithm
$-(4M+3)$ is the most negative

C1 is the pivot column; the θ-values are

$$\theta_1 = 4 \div 1 = 4$$
$$\theta_2 = 16 \div 1 = 16$$
$$\theta_3 = 18 \div 2 = 9$$
$$\theta_4 = 2 \div 2 = 1$$

Your notes

θ_4 is the least positive so R4 is the pivot row

The pivot element is in cell R4C1

Pivot is 2

The pivot (R4C1) needs to be changed to 1 through a row operation
Every other entry in C1 should be changed to 0 through row operations

b.v.	x	y	s_1	s_2	s_3	s_4	a_1	a_2	Value	Row Op.
s_1	0	-0.5	1	0	0	0.5	0	-0.5	3	$'R1' - R4$
s_2	0	2.5	0	1	0	0.5	0	-0.5	15	$'R2' - R4$
a_1	0	4	0	0	-1	1	1	-1	16	$'R3' - 2R4$
x	1	-0.5	0	0	0	-0.5	0	0.5	1	$0.5'R4'$
P	0	$-(4M+3.5)$	0	0	M	$-(M+1.5)$	0	$2M+1.5$	$3-16M$	$'R5' + (4M+3)R4$

There is a negative entry in the objective row, so apply a second iteration
$-(4M+3.5)$ is the most negative

C2 is the pivot column; the θ-values are

$$\theta_1 = 3 \div (-0.5) = -6$$
$$\theta_2 = 15 \div 2.5 = 6$$
$$\theta_3 = 16 \div 4 = 4$$
$$\theta_4 = 1 \div (-0.5) = -2$$

θ_3 is the least positive so R3 is the pivot row

The pivot element is in cell R3C2

Pivot is 4

Apply the appropriate row operations, starting with the pivot row, R3

b.v.	x	y	s_1	s_2	s_3	s_4	a_1	a_2	Value	Row Op.
s_1	0	0	1	0	-0.125	0.625	0.125	-0.625	5	'R1' + 0.5R3
s_2	0	0	0	1	0.625	-0.125	-0.625	0.125	5	'R2' - 2.5R3
y	0	1	0	0	-0.25	0.25	0.25	-0.25	4	0.25'R3'

Your notes

Your notes

x	1	0	0	0	-0.125	-0.375	0.125	0.375	3	'R4' + 0.5R3
P	0	0	0	0	-0.875	-0.625	$M+0.875$	$M+0.625$	17	$'\text{R5}'+(4M+3.5)\text{R3}$

There is a negative entry in the objective row, so apply a third iteration
-0.875 is the most negative

C5 is the pivot column; the θ-values are

$$\theta_1 = 5 \div (-0.125) = -40$$
$$\theta_2 = 5 \div 0.625 = 8$$
$$\theta_3 = 4 \div (-0.25) = -16$$
$$\theta_4 = 3 \div (-0.125) = -24$$

θ_2 is the least positive so R2 is the pivot row
The pivot element is in cell R2C5

Pivot is 0.625

Apply the appropriate row operations, starting with the pivot row, R2

b.v.	x	y	s_1	s_2	s_3	s_4	a_1	a_2	Value	Row Op.
s_1	0	0	1	0.2	0	0.6	0	-0.6	6	$'\text{R1}'+0.125\text{R2}$
s_3	0	0	0	1.6	1	-0.2	-1	0.2	8	$1.6'\text{R2}'$
y	0	1	0	0.4	0	0.2	0	-0.2	6	$'\text{R3}'+0.25\text{R2}$
x	1	0	0	0.2	0	-0.4	0	0.4	4	$'\text{R4}'+0.125\text{R2}$
P	0	0	0	1.4	0	-0.8	M	$M+0.8$	24	$'\text{R5}'+0.875\text{R2}$

There is a negative entry in the objective row, so apply a fourth iteration
-0.8 is the most negative

C6 is the pivot column; the θ-values are

$$\theta_1 = 6 \div 0.6 = 10$$
$$\theta_2 = 8 \div (-0.2) = -40$$
$$\theta_3 = 6 \div 0.2 = 30$$
$$\theta_4 = 4 \div (-0.4) = -10$$

Your notes

θ_1 is the least positive so R1 is the pivot row

The pivot element is in cell R1C6

Pivot is 0.6

Apply the appropriate row operations, starting with the pivot row, R1

b.v.	x	y	s_1	s_2	s_3	s_4	a_1	a_2	Value	Row Op.
s_4	0	0	5/3	1/3	0	1	0	-1	10	$5/3\text{'R1'}$
s_3	0	0	1/3	5/3	1	0	-1	0	10	$\text{'R2'} + 0.2\text{R1}$
y	0	1	-1/3	1/3	0	0	0	0	4	$\text{'R3'} - 0.2\text{R1}$
x	1	0	2/3	1/3	0	0	0	0	8	$\text{'R4'} + 0.4\text{R1}$
P	0	0	4/3	5/3	0	0	M	M	32	$\text{'R5'} + 0.8\text{R1}$

There are no negative entries in the objective row so the tableau is optimal

x, y, s_3, s_4 are the basic variables and so their values are read from the table

s_1, s_2, a_1, a_2 are the non-basic variables and so their values are all zero

$P = 3x + 2y$ **is maximised when** $x = 8$ **and** $y = 4$ **at** $P = 32$

$s_3 = 10,\ s_4 = 10$

$s_1 = s_2 = a_1 = a_2 = 0$

How do I use the Big-M method for minimisation problems?

- To use the Big-M method for minimising the objective function
  - Introduce a new **objective function,** Q, say, such that
    $Q = -P$
  - After finding P in the required form for tableau entry, write Q in the same way
  - Use Q as the objective row in the simplex algorithm
- Maximising Q is the same as minimising P
- Make sure to interpret the final tableau correctly in light of this adaption!
  - i.e. $P_{min} = -Q_{max}$

How do I know when the Big-M method is complete or optimal?

- A Big-M method tableau provides an **optimal solution** when there are no negative entries in the objective row

- Remember that M is an (arbitrarily) **large positive** number
  - For example, $M-7$ will be positive, $7-M$ would be negative
- Questions may ask for an interpretation of a tableau after any iteration of the Big-M method

Your notes