Top 10 Hardest Questions

Core Pure 1 and 2, Edexcel

2019-2022 Papers

(no AS papers featured)

DO NOT WRITE IN THIS AREA

#10

5. (a)

$$y = \tan^{-1} x$$

Assuming the derivative of $\tan x$, prove that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + x^2}$$

(3)

$$\mathrm{f}(x) = x \tan^{-1} 4x$$

(b) Show that

$$\int \mathrm{f}(x)\,\mathrm{d}x = Ax^2 \tan^{-1} 4x + Bx + C \tan^{-1} 4x + k$$

where k is an arbitrary constant and A, B and C are constants to be determined.

(5)

(c) Hence find, in exact form, the mean value of $\mathrm{f}(x)$ over the interval $\left[0, \frac{\sqrt{3}}{4}\right]$

(2)

DO NOT WRITE IN THIS AREA

5a) $y = \tan^{-1}x$

$\tan y = x$

$1 + \tan^2 y = \sec^2 y$

$1 + x^2 = \sec^2 y$

$\sec^2 y = \frac{dx}{dy}$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

5b) $f(x) = x\tan^{-1}4x$

$\int f(x)\,dx = \int x\tan^{-1}4x\,dx$

$= \frac{1}{2}x^2\tan^{-1}4x - \int \frac{1}{2}x^2 \times \frac{4}{1+16x^2}\,dx$

Int. by Parts

$u = \tan^{-1}4x \qquad v = \frac{1}{2}x^2$

$u' = \frac{4}{1+(4x)^2} \qquad v' = x$

$= \frac{4}{1+16x^2}$

$$= \frac{1}{2}x^2\tan^{-1}4x - \int \frac{2x^2}{1+16x^2}\,dx$$

Alg. Division

$$\begin{array}{r|l} & \frac{1}{8} \\ 16x^2+1 & 2x^2 \\ & 2x^2+\frac{1}{8} \\ \hline & -\frac{1}{8} \end{array}$$

$$= \frac{1}{2}x^2\tan^{-1}4x - \int \left(\frac{1}{8} - \frac{1/8}{1+16x^2}\right)dx$$

$$= \frac{1}{2}x^2\tan^{-1}4x - \int \frac{1}{8}\,dx + \frac{1}{8}\int \frac{1}{1+16x^2}\,dx$$

formula booklet

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}$$

$$a^2 = \frac{1}{16}, \quad a = \frac{1}{4}$$

$$= \frac{1}{2}x^2\tan^{-1}4x - \frac{1}{8}x + \frac{1}{8}\times\frac{1}{16}\int \frac{1}{\frac{1}{16}+x^2}\,dx$$

$$= \frac{1}{2}x^2\tan^{-1}4x - \frac{1}{8}x + \frac{1}{128}\times 4\arctan 4x + k$$

$$= \frac{1}{2}x^2\tan^{-1}4x - \frac{1}{8}x + \frac{1}{32}\tan^{-1}4x + k$$

$$A = \frac{1}{2}, \quad B = -\frac{1}{8}, \quad C = \frac{1}{32}$$

c) $$\frac{1}{\sqrt{3}/4}\left[\frac{1}{2}x^2\tan^{-1}4x - \frac{1}{8}x + \frac{1}{32}\tan^{-1}4x\right]_0^{\sqrt{3}/4}$$

$$= \frac{4\sqrt{3}}{3}\left(\frac{1}{2}\times\frac{3}{16}\times\tan^{-1}\sqrt{3} - \frac{1}{8}\times\frac{\sqrt{3}}{4} + \frac{1}{32}\tan^{-1}\sqrt{3}\right)$$

DO NOT WRITE IN THIS AREA

$$= \frac{4\sqrt{3}}{3}\left(\frac{3}{32} \times \frac{\pi}{3} - \frac{\sqrt{3}}{32} + \frac{1}{32} \times \frac{\pi}{3}\right)$$

$$= \frac{4\sqrt{3}}{3}\left(\frac{\pi}{24} - \frac{\sqrt{3}}{32}\right)$$

$$= \frac{\sqrt{3}}{18}\pi - \frac{1}{8}$$

DO NOT WRITE IN THIS AREA

#10

Question	Scheme	Marks
5(a)	$y=\tan^{-1}x \Rightarrow \tan y = x \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y}=\sec^2 y$ $y=\tan^{-1}x \Rightarrow \tan y = x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}\sec^2 y = 1$	M1
	$\frac{\mathrm{d}x}{\mathrm{d}y}=1+\tan^2 y$ or $\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+\tan^2 y\right)=1$	M1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$ *	A1*
		(3)
(b)	$\frac{\mathrm{d}\left(\tan^{-1}4x\right)}{\mathrm{d}x}=\frac{4}{1+16x^2}$	B1
	$\int x\tan^{-1}4x\,\mathrm{d}x=\alpha x^2\tan^{-1}4x-\int \alpha x^2 \times '\frac{4}{1+16x^2}'\,\mathrm{d}x$	M1
	$\int x\tan^{-1}4x\,\mathrm{d}x=\frac{x^2}{2}\tan^{-1}4x-\int \frac{x^2}{2}\times\frac{4}{1+16x^2}\,\mathrm{d}x$	A1
	$=\ldots-\frac{1}{8}\int\frac{16x^2+1-1}{1+16x^2}\,\mathrm{d}x=\ldots-\frac{1}{8}\int\left(1-\frac{1}{1+16x^2}\right)\mathrm{d}x$ or let $4x=\tan u \Rightarrow \frac{1}{8}\int\frac{\tan^2 u}{1+\tan^2 u}\times\frac{1}{4}\sec^2 u\,\mathrm{d}u$ $\Rightarrow \frac{1}{32}\int\tan^2 u\,\mathrm{d}u=\frac{1}{32}\int\sec^2 u-\ u\,\mathrm{d}u$	M1
	$=\frac{x^2}{2}\tan^{-1}4x-\frac{1}{8}x+\frac{1}{32}\tan^{-1}4x+k$	A1
		(5)
(c)	Mean value $=\left(\frac{1}{\frac{\sqrt{3}}{4}-0}\right)\left[\frac{x^2}{2}\tan^{-1}4x-\frac{1}{8}x+\frac{1}{32}\tan^{-1}4x\right]_0^{\frac{\sqrt{3}}{4}}$ $=\frac{4}{\sqrt{3}}\left(\left(\frac{3}{32}\times\frac{\pi}{3}-\frac{1}{8}\times\frac{\sqrt{3}}{4}+\frac{1}{32}\times\frac{\pi}{3}\right)-0\right)$	M1
	$=\frac{\sqrt{3}}{72}\left(4\pi-3\sqrt{3}\right)$ or $\frac{\sqrt{3}}{18}\pi-\frac{1}{8}$ oe	A1
		(2)

#9

2. In an Argand diagram, the points A and B are represented by the complex numbers $-3 + 2\text{i}$ and $5 - 4\text{i}$ respectively. The points A and B are the end points of a diameter of a circle C.

(a) Find the equation of C, giving your answer in the form

$$|z - a| = b \qquad a \in \mathbb{C},\ b \in \mathbb{R}$$

(3)

The circle D, with equation $|z - 2 - 3\text{i}| = 2$, intersects C at the points representing the complex numbers z_1 and z_2

(b) Find the complex numbers z_1 and z_2

(6)

a) Centre is midpoint of A and B

$A(-3,2)$

$B(5,-4)$

$$a = \frac{-3+2i+5-4i}{2} = 1 - i$$

Radius, b, is half the distance between A and B

$$b = \frac{1}{2}\sqrt{8^2+6^2} = \frac{1}{2}\times 10 = 5$$

DO NOT WRITE IN THIS AREA

Hence $|z-(1-i)|=5$

b) C $(x-1)^2+(y+1)^2=25$

$x^2-2x+1+y^2+2y+1=25$

$x^2-2x+y^2+2y=23$ (1)

D $(x-2)^2+(y-3)^2=4$

$x^2-4x+4+y^2-6y+9=4$

$x^2-4x+y^2-6y=-9$ (2)

(1) − (2)

$2x+8y=32$

$x+4y=16$

$x=16-4y=4(4-y)$

Sub into (1)

$x^2-2x+y^2+2y=23$

$16(4-y)^2-2\times4(4-y)+y^2+2y=23$

$16(16-8y+y^2)-8(4-y)+y^2+2y=23$

$256-128y+16y^2-32+8y+y^2+2y=23$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

$$17y^2 - 118y + 201 = 0$$

$$(y-3)(17y-67) = 0$$

$y = 3$

$x = 4(4-3)$

$= 4$

$$\underline{\underline{z_1 = 4 + 3i}}$$

$y = \frac{67}{17}$

$x = 4\left(4 - \frac{67}{17}\right)$

$= \frac{4}{17}$

$$\underline{\underline{z_2 = \frac{4}{17} + \frac{67}{17}i}}$$

#9

2(a)	Centre of circle C is $(1, -1)$	B1
	$r=\sqrt{(5-1)^2+(-4+1)^2}=5$ or $r=\sqrt{(-3-1)^2+(2+1)^2}=5$ or $r=\frac{1}{2}\sqrt{(-3-5)^2+(2+4)^2}=5$	M1
	$\lvert z-1+\mathrm{i}\rvert=5$ or $\lvert z-(1-\mathrm{i})\rvert=5$	A1
		(3)
(b)	$(x-1)^2+(y+1)^2=25,\quad (x-2)^2+(y-3)^2=4$ $x^2-2x+1+y^2+2y+1=25$ $x^2-4x+4+y^2-6y+9=4$ $\Rightarrow 2x+8y=32$	M1
	$(16-4y)^2-4(16-4y)+4+y^2-6y+9=4$ or $x^2-4x+4+\left(\frac{16-x}{4}\right)^2-6\left(\frac{16-x}{4}\right)+9=4$	M1
	$17y^2-118y+201=0$ or $17x^2-72x+16=0$	A1
	$17y^2-118y+201=0\Rightarrow(17y-67)(y-3)=0\Rightarrow y=\frac{67}{17},3$ or $17x^2-72x+16=0\Rightarrow(17x-4)(x-4)=0\Rightarrow x=\frac{4}{17},4$	M1
	$y=\frac{67}{17},3\Rightarrow x=\frac{4}{17},4$ or $x=\frac{4}{17},4\Rightarrow y=\frac{67}{17},3$	M1
	$4+3\mathrm{i},\ \frac{4}{17}+\frac{67}{17}\mathrm{i}$	A1
		(6)

DO NOT WRITE IN THIS AREA

#8

7. The plane Π has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Show that vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ is perpendicular to Π.

(2)

(b) Hence find a Cartesian equation of Π.

(2)

The line l has equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

where t is a scalar parameter.

The point A lies on l.

Given that the shortest distance between A and Π is $2\sqrt{29}$

(c) determine the possible coordinates of A.

(4)

DO NOT WRITE IN THIS AREA

a) $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -2+6-4=0$

$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 4+0-4=0$

Hence $2\underline{i} + 3\underline{j} - 4\underline{k}$ is perp to both vectors on the plane, and so is therefore perp. to Π.

b) $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

$\underline{r} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

$2x + 3y - 4z = 6 + 9 - 8$

$2x + 3y - 4z = 7$

c) Shortest dist. between point and plane

A is $\begin{pmatrix} 4+t \\ -5+6t \\ 2-3t \end{pmatrix}$

Formula booklet

For (α, β, γ) and $n_1 x + n_2 y + n_3 z + d = 0$

$$\text{dist} = \frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

$\alpha = 4+t$ $\quad n_1 = 2$ $\quad d = -7$

$\beta = 6t-5$ $\quad n_2 = 3$

$\gamma = 2-3t$ $\quad n_3 = -4$

$$2\sqrt{29} = \frac{|2(4+t) + 3(6t-5) - 4(2-3t) - 7|}{\sqrt{2^2 + 3^2 + 4^2}}$$

$$58 = |8 + 2t + 18t - 15 - 8 + 12t - 7|$$

$$58 = |32t - 22|$$

So $32t - 22 = 58$ or $-(32t-22) = 58$

$t = \underline{2.5}$ $\qquad 32t - 22 = -58$

$t = \underline{-1.125}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

If $t=2.5$

$A(6.5, 10, -5.5)$

If $t=-1.125$

$A(2.875, -11.75, 5.375)$

DO NOT WRITE IN THIS AREA

#8

7(a)	$\begin{pmatrix}-1\\2\\1\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=-2+6-4=0$ and $\begin{pmatrix}2\\0\\1\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=4+0-4=0$ Alt: $\begin{pmatrix}-1\\2\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}2\times1-1\times0\\-(-1\times1-1\times2)\\-1\times0-2\times2\end{pmatrix}=\ldots$	M1
	As $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ is perpendicular to both direction vectors (two non-parallel vectors) of Π then it must be perpendicular to Π	A1
		(2)
(b)	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}3\\3\\2\end{pmatrix}.\begin{pmatrix}2\\3\\-4\end{pmatrix}\Rightarrow\ldots$	M1
	$2x+3y-4z=7$	A1
		(2)
(c)	$\dfrac{\lvert 2(4+t)+3(-5+6t)-4(2-3t)-7\rvert}{\sqrt{2^2+3^2+(-4)^2}}=2\sqrt{29}\Rightarrow t=\ldots$	M1
	$t=-\dfrac{9}{8}$ and $t=\dfrac{5}{2}$	A1
	$\mathbf{r}=\begin{pmatrix}4\\-5\\2\end{pmatrix}-\dfrac{9}{8}\begin{pmatrix}1\\6\\-3\end{pmatrix}=\ldots$ or $\mathbf{r}=\begin{pmatrix}4\\-5\\2\end{pmatrix}+\dfrac{5}{2}\begin{pmatrix}1\\6\\-3\end{pmatrix}=\ldots$	M1
	$\left(\dfrac{23}{8},-\dfrac{47}{4},\dfrac{43}{8}\right)$ and $\left(\dfrac{13}{2},10,-\dfrac{11}{2}\right)$	A1
		(4)

#7

6. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2i$.

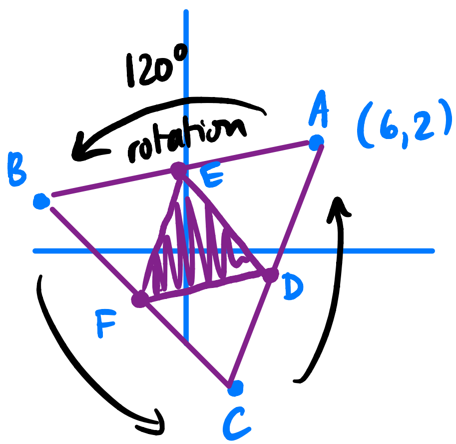
(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + iy$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF.

(3)



a) rotation 120°

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

We will rotate (6,2)

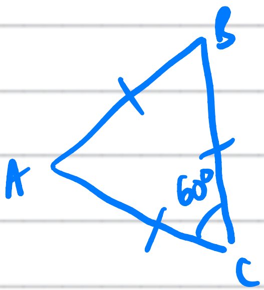
$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -3-\sqrt{3} \\ -1+3\sqrt{3} \end{pmatrix}$$

So B is $(-3-\sqrt{3}) + (-1+3\sqrt{3})i$

$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^2 \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -3+\sqrt{3} \\ -1-3\sqrt{3} \end{pmatrix}$$

So C is $(-3+\sqrt{3})+(-1-3\sqrt{3})i$

b) Area $DEF = \frac{1}{4}$ Area ABC



$AB = \sqrt{(6-(-3-\sqrt{3}))^2+(2-(-1+3\sqrt{3}))^2}$

$= \sqrt{(9-\sqrt{3})^2+(3+3\sqrt{3})^2}$

$= \sqrt{120}$ $\qquad AB = BC = AC$

Area $DEF = \frac{1}{4} \times \frac{1}{2} ab\sin C = \frac{1}{8}\sqrt{120}\sqrt{120}\sin 60$

$= \underline{\underline{\frac{15\sqrt{3}}{2}}}$

#7

6(a)	Examples: $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2\mathrm{i})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\mathrm{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\mathrm{e}^{\mathrm{i}\arctan\left(\frac{2}{6}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$	M1
	$\left(-3-\sqrt{3}\right)$ **or** $\left(3\sqrt{3}-1\right)\mathrm{i}$	A1
	$\left(-3-\sqrt{3}\right)+\left(3\sqrt{3}-1\right)\mathrm{i}$	A1
	Examples: $\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2\mathrm{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\mathrm{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\mathrm{e}^{\mathrm{i}\arctan\left(\frac{2}{6}\right)}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$	M1
	$\left(-3+\sqrt{3}\right)$ **or** $\left(-3\sqrt{3}-1\right)\mathrm{i}$	A1
	$\left(-3+\sqrt{3}\right)+\left(-3\sqrt{3}-1\right)\mathrm{i}$	A1
		(6)
(b) Way 1	Area $ABC = 3\times\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$ or Area $AOB = \frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$	M1
	Area $DEF = \frac{1}{4}ABC$ or $\frac{3}{4}AOB$	dM1
	$=\frac{3}{8}\times 40\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1
		(3)

(b) Way 2	$D\left(\frac{3-\sqrt{3}}{2},\frac{3\sqrt{3}+1}{2}\right)$ $OD=\sqrt{\left(\frac{3-\sqrt{3}}{2}\right)^2+\left(\frac{3\sqrt{3}+1}{2}\right)^2}=\sqrt{10}$ Area $DOF=\frac{1}{2}\sqrt{10}\sqrt{10}\sin 120°$	M1	2.1
	Area $DEF = 3DOF$	dM1	3.1a
	$=3\times\frac{1}{2}\times\sqrt{10}\sqrt{10}\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 3	$AB=\sqrt{\left(9+\sqrt{3}\right)^2+\left(3-3\sqrt{3}\right)^2}=\sqrt{120}$ Area $ABC=\frac{1}{2}\sqrt{120}\sqrt{120}\sin 60°\left(=30\sqrt{3}\right)$	M1	2.1
	Area $DEF=\frac{1}{4}ABC$	dM1	3.1a
	$=\frac{1}{4}\times 30\sqrt{3}=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 4	$D\left(\frac{3-\sqrt{3}}{2},\frac{3\sqrt{3}+1}{2}\right)$, $E(-3,-1)$, $F\left(\frac{3+\sqrt{3}}{2},\frac{-3\sqrt{3}+1}{2}\right)$ $DE=\sqrt{\left(\frac{3-\sqrt{3}}{2}+3\right)^2+\left(\frac{3\sqrt{3}+1}{2}+1\right)^2}\left(=\sqrt{30}\right)$ Area $DEF=\frac{1}{2}\sqrt{30}\sqrt{30}\sin 60°$	M1 dM1	2.1 3.1a
	$=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 5	Area $ABC=\frac{1}{2}\begin{vmatrix}6 & -3-\sqrt{3} & \sqrt{3}-3 & 6\\ 2 & 3\sqrt{3}-1 & -3\sqrt{3}-1 & 2\end{vmatrix}=30\sqrt{3}$	M1	2.1
	Area $DEF=\frac{1}{4}ABC$	dM1	3.1a
	$=\frac{1}{4}\times 30\sqrt{3}=\frac{15\sqrt{3}}{2}$	A1	1.1b

#6

6. The curve C has equation

$$r = a(p + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where a and p are positive constants and $p > 2$

There are exactly four points on C where the tangent is perpendicular to the initial line.

(a) Show that the range of possible values for p is

$$2 < p < 4$$

(5)

(b) Sketch the curve with equation

$$r = a(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi \qquad \text{where } a > 0$$

(1)

John digs a hole in his garden in order to make a pond.

The pond has a uniform horizontal cross section that is modelled by the curve with equation

$$r = 20(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where r is measured in centimetres.

The depth of the pond is 90 centimetres.

Water flows through a hosepipe into the pond at a rate of 50 litres per minute.

Given that the pond is initially empty,

(c) determine how long it will take to completely fill the pond with water using the hosepipe, according to the model. Give your answer to the nearest minute.

(7)

(d) State a limitation of the model.

(1)

a) $r = a(p + 2\cos\theta)$

perp to initial line if $\frac{dx}{d\theta} = 0$

$x = r\cos\theta$

$x = a(p + 2\cos\theta)\cos\theta$

Product Rule

$u = a(p + 2\cos\theta)$ $\quad v = \cos\theta$

$u' = -2a\sin\theta$ $\quad v' = -\sin\theta$

$\frac{dx}{d\theta} = -a(p + 2\cos\theta)\sin\theta - 2a\sin\theta\cos\theta$

$0 = -\cancel{a}p\sin\theta - 2\cancel{a}\cos\theta\sin\theta - 2\cancel{a}\sin\theta\cos\theta$

$0 = \sin\theta(p + 4\cos\theta)$

$\sin\theta = 0$

gives 2 solutions,

$\theta = 0, \pi$

So $p + 4\cos\theta = 0$

must give 2 more solutions

$-\frac{p}{4} = \cos\theta$

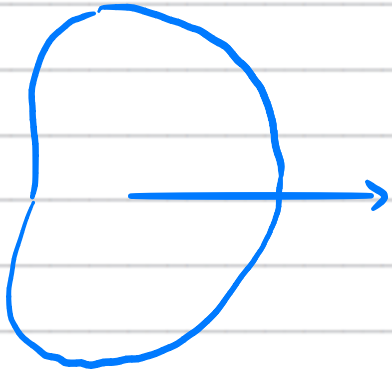
As $p > 2$, p is positive, hence

$-\frac{p}{4} > -1$ for 2 solutions to exist

$-p > -4$

$p < 4$ Hence $\underline{2 < p < 4}$

b)



$r = 20(3+2\cos\theta)$

c) $\text{Area} = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = 2 \times \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$= \int_0^{\pi} 400(9 + 12\cos\theta + 4\cos^2\theta) \, d\theta$

$= 400\int_0^{\pi} \left(9 + 12\cos\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\right) d\theta$

$= 400\int_0^{\pi} (11 + 12\cos\theta + 2\cos 2\theta) \, d\theta$

$= 400\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{\pi}$

$= 400(11\pi + 0 + 0 - 0 - 0 - 0) = 4400\pi \text{ cm}^2$

So $V = 90 \times 4400\pi$

$= 396000\pi \text{ cm}^3$ $\quad$ $1000 \text{ cm}^3 = 1$ litre

$= 396\pi$ litres

time $= \dfrac{396\pi}{50} = 24.881\ldots$ minutes

$= 25$ minutes.

d) The curve equation is unlikely to be a perfect match for the pond's shape.

#6

6(a)

$x = r\cos\theta = a(p + 2\cos\theta)\cos\theta$

Leading to $\frac{dx}{d\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta(p + 2\cos\theta)$

or $\frac{dx}{d\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta$

or

$x = a(p\cos\theta + 2\cos^2\theta) = a(\cos 2\theta + p\cos\theta + 1)$

leading to $\frac{dx}{d\theta} = \alpha\sin 2\theta + \beta\sin\theta$

$\frac{dx}{d\theta} = a[-2\sin\theta\cos\theta - \sin\theta(p + 2\cos\theta)]$

or

$\frac{dx}{d\theta} = -4a\sin\theta\cos\theta - ap\sin\theta$ or $\frac{dx}{d\theta} = -2a\sin 2\theta - ap\sin\theta$

$a[-2\sin\theta\cos\theta - \sin\theta(p + 2\cos\theta)] = 0$

$\pm a(4\sin\theta\cos\theta + p\sin\theta) = 0$

$a\sin\theta(4\cos\theta + p) = 0$

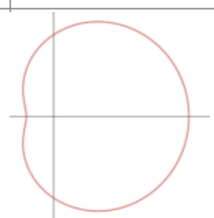
Either $\sin\theta = 0$ or $\cos\theta = -\frac{p}{4}$

$\sin\theta = 0$ implies 2 solutions (tangents which are perpendicular to the initial line) e.g. $\theta = 0, \pi$

Therefore two solutions to $\cos\theta = -\frac{p}{4}$ are required

$-\frac{p}{4} > -1 \Rightarrow p < 4$ as p is a positive constant $2 < p < 4$ *

(b)



Correct shape and position.
Condone cusp

(c)

Area =

$2 \times \frac{1}{2}\int_0^{\pi}[20(3 + 2\cos\theta)]^2\, d\theta = 400\int_0^{\pi}(9 + 12\cos\theta + 4\cos^2\theta)\, d\theta$

or $= \int_0^{\pi}(3600 + 4800\cos\theta + 1600\cos^2\theta)\, d\theta$

or

$\frac{1}{2}\int_0^{2\pi}[20(3 + 2\cos\theta)]^2\, d\theta = 200\int_0^{2\pi}(9 + 12\cos\theta + 4\cos^2\theta)\, d\theta$

or $= \int_0^{2\pi}(1800 + 2400\cos\theta + 800\cos^2\theta)\, d\theta$

	$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta \Rightarrow$ $A = ...\int(9 + 12\cos\theta + 2 + 2\cos 2\theta)\, d\theta = \alpha\theta \pm \beta\sin\theta \pm \lambda\sin 2\theta$	M1
	$= 400[11\theta + 12\sin\theta + \sin 2\theta]$ or $= 200[11\theta + 12\sin\theta + \sin 2\theta]$	A1
	Using limits $\theta = 0$ and $\theta = \pi$ or $\theta = 0$ and $\theta = 2\pi$ as appropriate and subtracts the correct way round provided there is an attempt at integration $= 400[11\pi - 0] = 4400\pi = 13823.0\,(\text{cm}^2)$ or $= 200[11(2\pi) - 0] = 4400\pi = 13823.0\,(\text{cm}^2)$	M1
	Volume $=$ area $\times 90 = 396\,000\pi = 1\,244\,070.691\,(\text{cm}^3)$	M1
	time $= \frac{1\,244\,070.691}{50\,000} = ...$ or volume $= 1244$ litres therefore time $= \frac{1244}{50} = ...$	M1
	25 (minutes)	A1
		(7)
(d)	For example Polar equation is not likely to be accurate. Some comment that the sides will not be smooth and draws an appropriate conclusion. The hole may not be uniform depth The pond may leak/ ground may absorb some water	B1

#5

4. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3, \quad \sum_{r=1}^{n} r^2 \quad \text{and} \quad \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2 - 1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

a) odd numbers are $2r-1$ for $r=1,2,3\ldots$

$$\sum_{r=1}^{n}(2r-1)^3 = \sum_{r=1}^{n}(8r^3 - 12r^2 + 6r - 1)$$

$$= 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

DO NOT WRITE IN THIS AREA

$$= \frac{8}{4}n^2(n+1)^2 - \frac{12}{6}n(n+1)(2n+1) + \frac{6}{2}n(n+1) - n$$

$$= n\left(2n(n+1)^2 - 2(n+1)(2n+1) + 3(n+1) - 1\right)$$

$$= n\left(2n(n^2+2n+1) - 2(2n^2+3n+1) + 3n + 3 - 1\right)$$

$$= n\left(2n^3 + 4n^2 + 2n - 4n^2 - 6n - 2 + 3n + 3 - 1\right)$$

$$= n(2n^3 - n)$$

$$= \underline{n^2(2n^2-1)}$$

b) Doesn't have to start at 1 !

$$\sum_{r=n}^{n+9}(2r-1)^3 = \sum_{r=1}^{n+9}(2r-1)^3 - \sum_{r=1}^{n-1}(2r-1)^3$$

$$99\,800 = (n+9)^2\left(2(n+9)^2-1\right) - (n-1)^2\left(2(n-1)^2-1\right)$$

$$99\,800 = \underline{(n^2+18n+81)(2n^2+36n+161)} - \underline{(n^2-2n+1)(2n^2-4n+1)}$$

	$2n^2$	$36n$	161
n^2	$2n^4$	$36n^3$	$161n^2$
$18n$	$36n^3$	$648n^2$	$2898n$
81	$162n^2$	$2916n$	13041

$-$

	$2n^2$	$-4n$	1
n^2	$2n^4$	$-4n^3$	n^2
$-2n$	$-4n^3$	$8n^2$	$-2n$
1	$2n^2$	$-4n$	1

$$99800 = 2n^4 - 2n^4 + 72n^3 + 8n^3 + 971n^2 - 11n^2 + 5814n + 6n + 13041 - 1$$

$$0 = 80n^3 + 960n^2 + 5820n - 86760$$

$$n = 6$$

So smallest is $2 \times 6 - 1 = \underline{\underline{11}}$

#5

4(a)	A complete attempt to find the sum of the cubes of the first n odd numbers using three of the standard summation formulae. Attempts to find $\sum(2r+1)^3$ or $\sum(2r-1)^3$ by expanding and using summation formulae	M1
	$\sum_{r=1}^{n}(2r-1)^3=\sum_{r=1}^{n}(8r^3-12r^2+6r-1)=8\sum_{r=1}^{n}r^3-12\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$ or $\sum_{r=0}^{n-1}(2r+1)^3=\sum_{r=0}^{n-1}(8r^3+12r^2+6r+1)=8\sum_{r=0}^{n-1}r^3+12\sum_{r=0}^{n-1}r^2+6\sum_{r=0}^{n-1}r+\sum_{r=0}^{n-1}1$	M1
	$=8\frac{n^2}{4}(n+1)^2-12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)-n$ or $=8\frac{(n-1)^2}{4}(n)^2+12\frac{(n-1)}{6}(n)(2n-1)+6\frac{(n-1)}{2}(n)+n$	M1 A1
	Multiplies out to achieve a correct intermediate line for example $n\ n+1\ \ 2n^2-2n+1\ \ -n=2n^4-2n^3+n^2+2n^3-2n^2+n-n$ $2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$ leading to $=n^2(2n^2-1)$ cso *	A1 *
		(5)

(b)	$\sum_{r=n}^{n+9}(2r-1)^3=\sum_{r=1}^{n+9}(2r-1)^3-\sum_{r=1}^{n-1}(2r-1)^3$ $=(n+9)^2(2(n+9)^2-1)-(n-1)^2(2(n-1)^2-1)=99800$ or $\sum_{r=n+1}^{n+10}(2r-1)^3=\sum_{r=1}^{n+10}(2r-1)^3-\sum_{r=1}^{n}(2r-1)^3$ $=(n+10)^2(2(n+10)^2-1)-(n)^2(2n^2-1)=99800$ or $\sum_{r=n-9}^{n}(2r-1)^3=\sum_{r=1}^{n}(2r-1)^3-\sum_{r=1}^{n-10}(2r-1)^3$ $=(n)^2(2(n)^2-1)-(n-10)^2(2(n-10)^2-1)=99800$	M1
	$80n^3+960n^2+5820n-86760=0$ or $80n^3+1200n^2+7980n-79900=0$ or $80n^3-1200n^2+7980n-119700=0$	A1
	Solves cubic equation	dM1
	Achieves $n=6$ and the smallest number as 11 or Achieves $n=5$ and the smallest number as 11 or Achieves $n=15$ and the smallest number as 11	A1
		(4)

#4

9. (a) Use a hyperbolic substitution and calculus to show that

$$\int \frac{x^2}{\sqrt{x^2 - 1}}\,\mathrm{d}x = \frac{1}{2}\left[x\sqrt{x^2 - 1} + \operatorname{arcosh} x\right] + k$$

where k is an arbitrary constant.

(6)

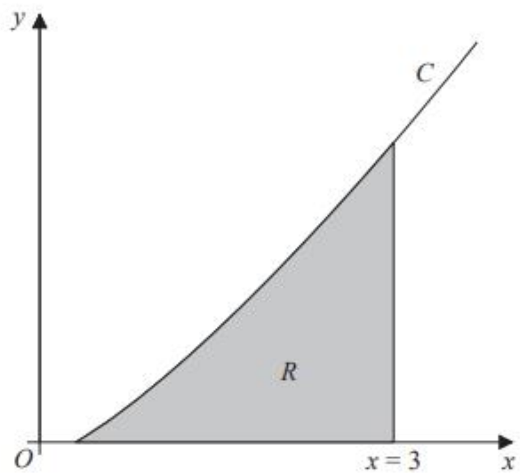


Figure 1

Figure 1 shows a sketch of part of the curve C with equation

$$y = \frac{4}{15}x \operatorname{arcosh} x \qquad x \geqslant 1$$

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Using algebraic integration and the result from part (a), show that the area of R is given by

$$\frac{1}{15}\left[17\ln\left(3 + 2\sqrt{2}\right) - 6\sqrt{2}\right]$$

(5)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx$$

a) $x = \cosh u$ $\qquad x^2 = \cosh^2 u$

$\frac{dx}{du} = \sinh u$ $\qquad \sqrt{x^2-1} = \sqrt{\cosh^2 u - 1}$

$dx = \sinh u \, du$ $\qquad = \sinh u$

$\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u$

$\cosh^2 u = \frac{1}{2} + \frac{1}{2}\cosh 2u$

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \int \frac{\cosh^2 u}{\cancel{\sinh u}} \times \cancel{\sinh u}\,du$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cosh 2u\right) du$$

$$= \frac{1}{2}u + \frac{1}{4}\sinh 2u + k$$

$$= \frac{1}{2}\operatorname{arcosh} x + \frac{1}{4} \times 2\sinh u \cosh u + k$$

$$= \frac{1}{2}\operatorname{arcosh} x + \frac{1}{2}x\sqrt{x^2-1} + k$$

$$= \frac{1}{2}\left(x\sqrt{x^2-1} + \operatorname{arcosh} x\right) + k$$

b) $\int_1^3 \frac{4}{15} x \operatorname{arcosh} x \, dx = \frac{4}{15}\int_1^3 x \operatorname{arcosh} x \, dx$

$= \frac{4}{15}\left(\left[\frac{1}{2}x^2 \operatorname{arcosh} x\right]_1^3 - \frac{1}{2}\int_1^3 \frac{x^2}{\sqrt{x^2-1}}\,dx\right)$

Int by parts

$u = \operatorname{arcosh} x$ $\qquad v = \frac{1}{2}x^2$

$u' = \frac{1}{\sqrt{x^2-1}}$ $\qquad v' = x$

BICEN MATHS

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

$$= \frac{4}{15}\left(\left[\frac{1}{2}x^2 \operatorname{arcosh} x\right]_1^3 - \frac{1}{2}\times\frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh} x\right]_1^3\right)$$

Formula booklet

$\operatorname{arcosh} x = \ln(x+\sqrt{x^2-1})$

$\operatorname{arcosh} 3 = \ln(3+\sqrt{8})$

$= \ln(3+2\sqrt{2})$

$$= \frac{4}{15}\left(\frac{9}{2}\operatorname{arcosh} 3 - \cancel{\frac{1}{2}\operatorname{arcosh} 1} - \frac{1}{4}\left(3\sqrt{8} + \operatorname{arcosh} 3 - 0 - \cancel{\operatorname{arcosh} 1}\right)\right)$$

$$= \frac{4}{15}\left(\frac{9}{2}\ln(3+2\sqrt{2}) - \frac{1}{4}\left(6\sqrt{2} + \ln(3+2\sqrt{2})\right)\right)$$

$$= \frac{4}{15}\left(\frac{9}{2}\ln(3+2\sqrt{2}) - \frac{1}{4}\ln(3+2\sqrt{2}) - \frac{3\sqrt{2}}{2}\right)$$

$$= \frac{4}{15}\left(\frac{17}{4}\ln(3+2\sqrt{2}) - \frac{3\sqrt{2}}{2}\right)$$

$$= \frac{17}{15}\ln(3+2\sqrt{2}) - \frac{2\sqrt{2}}{5}$$

$$= \frac{1}{15}\left(17\ln(3+2\sqrt{2}) - 6\sqrt{2}\right)$$

BICEN MATHS

#4

9(a)	$\int \frac{x^2}{\sqrt{x^2-1}}\,dx \to \int f(u)\,du$ Uses the substitution $x=\cosh u$ fully to achieve an integral in terms of u only, including replacing the dx	M1
	$\int \frac{\cosh^2 u}{\sqrt{\cosh^2 u-1}}\sinh u\,(du)$	A1
	Uses correct identities $\cosh^2 u-1=\sinh^2 u$ and $\cosh 2u=2\cosh^2 u-1$ to achieve an integral of the form $A\int(\cosh 2u \pm 1)\,du \quad A>0$	M1
	Integrates to achieve $A\left(\pm\frac{1}{2}\sinh 2u \pm u\right)(+c) \quad A>0$	M1
	Uses the identity $\sinh 2u=2\sinh u\cosh u$ and $\cosh^2 u-1=\sinh^2 u$ $\to \sinh 2u=2x\sqrt{x^2-1}$	M1
	$\frac{1}{2}\left[x\sqrt{x^2-1}+\text{ar}\cosh x\right]+k$ * cso	A1*
		(6)
(b)	Uses integration by parts the correct way around to achieve $\int \frac{4}{15}x\,\text{arcosh}x\,dx = Px^2\text{arcosh}x - Q\int\frac{x^2}{\sqrt{x^2-1}}\,dx$	M1
	$=\frac{4}{15}\left(\frac{1}{2}x^2\text{arcosh}x-\frac{1}{2}\int\frac{x^2}{\sqrt{x^2-1}}\,dx\right)$	A1
	$=\frac{4}{15}\left(\frac{1}{2}x^2\text{arcosh}x-\frac{1}{2}\left(\frac{1}{2}\left[x\sqrt{x^2-1}+\text{arcosh}x\right]\right)\right)$	B1ft
	Uses the limits $x=1$ and $x=3$ the correct way around and subtracts $=\frac{4}{15}\left(\frac{1}{2}(3)^2\text{arcosh}3-\frac{1}{2}\left(\frac{1}{2}\left[3\sqrt{(3)^2-1}+\text{arcosh}3\right]\right)\right)-\frac{4}{15}(0)$	dM1
	$=\frac{4}{15}\left(\frac{9}{2}\ln\left(3+\sqrt{8}\right)-\frac{3\sqrt{8}}{4}-\frac{1}{4}\ln\left(3+\sqrt{8}\right)\right)$ $=\frac{1}{15}\left[17\ln\left(3+2\sqrt{2}\right)-6\sqrt{2}\right]$*	A1*
		(5)

#3

9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \ldots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

(2)

a) $a = 1$ $\quad S_\infty = \frac{a}{1-r} = \underline{\underline{\frac{1}{1-z}}}$

$r = z$

DO NOT WRITE IN THIS AREA

b) i) $z = \frac{1}{2}(\cos\theta + i\sin\theta) = \frac{1}{2}e^{i\theta}$

Hence $1 + z + z^2 + \dots = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \dots$

and $\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots = \text{Im}\left(1 + \frac{1}{2}e^{i\theta} + \dots\right) = \text{Im}\left(\frac{1}{1-z}\right)$

$$\frac{1}{\left(1 - \frac{1}{2}e^{i\theta}\right)} \times \frac{\left(1 - \frac{1}{2}e^{-i\theta}\right)}{\left(1 - \frac{1}{2}e^{-i\theta}\right)} = \frac{1 - \frac{1}{2}e^{-i\theta}}{1 + \frac{1}{4} - \frac{1}{2}e^{i\theta} - \frac{1}{2}e^{-i\theta}} = \frac{1 - \frac{1}{2}e^{-i\theta}}{\frac{5}{4} - \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

$e^{i\theta} + e^{-i\theta} = \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}$

$= 2\cos\theta$

$$= \frac{1 - \frac{1}{2}e^{-i\theta}}{\frac{5}{4} - \frac{1}{2}\left(2\cos\theta\right)}$$

$$= \frac{2 - e^{-i\theta}}{\frac{5}{2} - 2\cos\theta}$$

$$= \frac{2 - (\cos\theta - i\sin\theta)}{\frac{5}{2} - 2\cos\theta}$$

$$= \frac{2-\cos\theta}{\frac{5}{2}-2\cos\theta} + \frac{i\sin\theta}{\frac{5}{2}-2\cos\theta}$$

Hence $1+\frac{1}{2}\sin 2\theta+\frac{1}{4}\sin 3\theta \ldots = \mathrm{Im}\left(\frac{1}{1-z}\right) = \frac{\sin\theta}{\frac{5}{2}-2\cos\theta}$

$$= \underline{\underline{\frac{2\sin\theta}{5-4\cos\theta}}}$$

b)ii) Purely imaginary if $\mathrm{Re}\left(\frac{1}{1-z}\right)=0$

i.e. if $\frac{2-\cos\theta}{\frac{5}{2}-2\cos\theta} = 0$

$$2-\cos\theta = 0$$

$$\cos\theta = 2$$

But $|\cos\theta| \le 1$ for all θ, hence $1+z+z^2+\ldots$ cannot ever be purely imaginary as it will always have a real part.

#3

9(a)	$\frac{1}{1-z}$	B1
		(1)
(b)(i)	$1+z+z^2+z^3+\ldots$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+\ldots$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\mathrm{i}\sin 3\theta)+\ldots$	M1
	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}$ or $\frac{1}{1-z}=\frac{2}{2-(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{2-(\cos\theta-\mathrm{i}\sin\theta)}{2-(\cos\theta-\mathrm{i}\sin\theta)}$	M1
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\ldots\right\}=\frac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\ldots\right\}=\frac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2=1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2=4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+\ldots=\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*
	Alternative $1+z+z^2+z^3+\ldots$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+\ldots$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\mathrm{i}\sin 3\theta)+\ldots$	M1

	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}\times\frac{1-\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}}{1-\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}}$	M1
	$\frac{1-\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}}{1-\frac{1}{4}\mathrm{e}^{\mathrm{i}\theta}-\frac{1}{4}\mathrm{e}^{-\mathrm{i}\theta}+\frac{1}{4}}=\frac{4-2\mathrm{e}^{-\mathrm{i}\theta}}{5-2\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)}=\frac{4-2(\cos\theta-\mathrm{i}\sin\theta)}{5-2(2\cos\theta)}$	M1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+\ldots=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*
		(5)
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}=0\Rightarrow\cos\theta=2$	M1
	As $(-1\le)\cos\theta\le 1$ therefore there is no solution to $\cos\theta=2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta}=\frac{1}{2}+\frac{3}{2(5-4\cos\theta)}$	M1
	$-1\le\cos\theta\le 1$ therefore $\frac{1}{6}<\frac{3}{2(5-4\cos\theta)}<\frac{3}{2}$ so sum must contain real part	A1
	Alternative 2 $\frac{1}{1-z}=k\mathrm{i}\Rightarrow z=1+\frac{\mathrm{i}}{k}$	M1
	mod $z>1$ contradiction hence cannot be purely imaginary	A1
		(2)

#2

8. Two different colours of paint are being mixed together in a container.

The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.

Initially the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.

The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second
- adding blue paint to the container at a rate of 1 litre per second
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let r litres be the amount of red paint in the container at time t seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2 - \frac{r}{\alpha}$$

where α is a positive constant to be determined.

(2)

(b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model. Give your answer to the nearest second.

(6)

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red paint and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer.

(1)

DO NOT WRITE IN THIS AREA

a) 2 litres of red arrive per second, hence $\frac{dr}{dt} = 2$ for paint in.

The proportion of paint leaving is $\frac{3}{30}$

i.e. $\frac{1}{10}$ of the liquid leaves every second

so $\frac{1}{10}$ of the red paint is leaving i.e. $-\frac{1}{10}r = -\frac{r}{10}$

So overall $\frac{dr}{dt} = 2 - \frac{r}{10}$ $\quad\quad$ $\underline{\underline{\alpha = 10}}$

b) $\frac{dr}{dt} + 0.1r = 2$ $\quad\quad$ $\frac{dy}{dx} + P(x)y = Q(x)$

I.F. $e^{\int 0.1\, dt} = e^{0.1t}$

$e^{0.1t}\frac{dr}{dt} + 0.1e^{0.1t}r = 2e^{0.1t}$

$$\frac{d}{dt}\left(re^{0.1t}\right) = 2e^{0.1t}$$

$$re^{0.1t} = \int 2e^{0.1t}\,dt$$

$$re^{0.1t} = 20e^{0.1t} + c$$

$t=0,\ r=10$ $\quad 10 = 20 + c$

$$c = -10$$

Hence $\quad re^{0.1t} = 20e^{0.1t} - 10$

Equal red and blue means $r=15$ (total is always 30)

$$15e^{0.1t} = 20e^{0.1t} - 10$$

$$10 = 5e^{0.1t}$$

$$2 = e^{0.1t}$$

$$\ln 2 = 0.1t$$

$$10\ln 2 = t$$

$$t = 6.93\ldots = 7 \text{ seconds}$$

c) model: 7 secs. $\quad \frac{7}{9} = 77.8\%$ (1dp) so model

reality: 9 secs. $\quad$ underestimated by 22.2%,

this is very large difference, so model not good.

#2

8(a)	Volume of paint = 30 litres therefore Rate of paint out $= 3 \times \frac{r}{30}$ litres per second		M1
	$\frac{\mathrm{d}r}{\mathrm{d}t} = 2 - \frac{r}{10}$		A1
			(2)
(b)	Rearranges $\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{r}{10} = 2$ and attempts integrating factor IF $= \mathrm{e}^{\int \frac{1}{10}\mathrm{d}t} = \ldots$	Separates the variables $\int \frac{1}{20-r}\,\mathrm{d}r = \frac{1}{10}\mathrm{d}t$ $\Rightarrow \ldots$	M1
	$r\mathrm{e}^{\frac{t}{10}} = \int 2\mathrm{e}^{\frac{t}{10}}\,\mathrm{d}t \Rightarrow r\mathrm{e}^{\frac{t}{10}} = \lambda \mathrm{e}^{\frac{t}{10}}(+c)$	Integrates to the form $\lambda \ln(20-r) = \frac{1}{10}t(+c)$	M1
	$r\mathrm{e}^{\frac{t}{10}} = 20\mathrm{e}^{\frac{t}{10}} + c$	$-\ln(20-r) = \frac{1}{10}t + c$	A1ft
	$t = 0, r = 10 \Rightarrow c = \ldots$		M1
	$r = \frac{20\mathrm{e}^{\frac{t}{10}} - 10}{\mathrm{e}^{\frac{t}{10}}} = 15$ rearranges to achieve $\mathrm{e}^{\frac{t}{10}} = \alpha$ and solves to find a value for t or $r = 20 - 10\mathrm{e}^{-\frac{t}{10}} = 15$ rearranges to achieve $\mathrm{e}^{-\frac{t}{10}} = \beta$ and solves to find a value for t	$-\ln(20-15) = \frac{1}{10}t - \ln 10$ Leading to a value for t	M1
	$t =$ awrt 7 seconds		A1
			(6)
(c)	The model predicts 7 seconds but it actually takes 9 seconds so (over) 2 seconds out (over 20%), therefore it is not a good model		B1ft
			(1)

DO NOT WRITE IN THIS AREA

#1

2.

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix **A** represents the linear transformation M.

Prove that, for the linear transformation M, there are no invariant lines.

(5)

Invariant if $y = mx + c$

$$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \text{ gives } Y = mX + c$$

$4x - 2(mx+c) = X$

$4x - 2mx - 2c = X$

$(4-2m)x - 2c = X$

$5x + 3(mx+c) = Y$

$5x + 3mx + 3c = Y$

$(5+3m)x + 3c = Y$

$Y = mX + c$

$(5+3m)x + 3c = m(4-2m)x - 2mc + c$

$(5+3m - 4m + 2m^2)x + 3c + 2mc - c = 0$

$(2m^2 - m + 5)x + 2c + 2mc = 0$

Prove this is never = to 0.

i.e. $2m^2 - m + 5 \neq 0$

Discriminant $b^2 - 4ac = 1 - 4 \times 2 \times 5$
$= -39$

As $b^2 - 4ac < 0$, there are no solutions, so the equation can never $= 0$, hence no invariant lines.

Alt. method, also accepted:

$$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{invariant if } Y = mX$$

$4x - 2mx = X \qquad 5x + 3mx = Y$

$Y = mX$

$5x + 3mx = m(4x - 2mx)$

$5x + 3mx = 4mx - 2m^2x$

$x(2m^2 - m + 5) = 0$

$b^2 - 4ac = 1 - 4 \times 2 \times 5 = \underline{\underline{-39}}$ etc.

#1

2	$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix}=\begin{pmatrix} X \\ mX+c \end{pmatrix}$ leading to an equation in x, m, c and X		M1
	$4x-2(mx+c)=X$ and $5x+3(mx+c)=mX+c$		A1
	$5x+3(mx+c)=m\left(4x-2(mx+c)\right)+c$ leading to $5+3m=4m-2m^2$	$(3c=-2mc+c)$	M1
	$2m^2-m+5=0 \Rightarrow b^2-4ac=(-1)^2-4(2)(5)=\ldots$	Solves $3c=-2mc+c \Rightarrow m=\ldots$	dM1
	Correct expression for the discriminant $=\{-39\}<0$ therefore there are no invariant lines.	$m=-1$ and shows a contradiction in $5+3m=4m-2m^2$ therefore there are no invariant lines.	A1
	Alternative		
	$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix}=\begin{pmatrix} X \\ mX \end{pmatrix}$ leading to an equation in x, m and X		M1
	$4x-2(mx)=X$ and $5x+3(mx)=mX$		A1
	$5x+3(mx)=m\left(4x-2(mx)\right)$ leading to $5+3m=4m-2m^2$		M1
	$2m^2-m+5=0 \Rightarrow b^2-4ac=(-1)^2-4(2)(5)=\ldots$		dM1
	Correct expression for the discriminant $=\{-39\}<0$ therefore there are no invariant lines that pass through the origin no invariant lines.		A1
			(5)

DO NOT WRITE IN THIS AREA

Number	Question	Year	Paper	Mark	Total	Percentage
1	2	2021	CP2	0.63	5	12.6%
2	8	2021	CP1	1.34	9	14.9%
3	9	2021	CP2	1.34	8	16.8%
4	9	2021	CP1	1.85	11	16.8%
5	4	2021	CP2	2.52	9	28.0%
6	6	2021	CP2	4.04	14	28.9%
7	6	2019	CP2	2.67	9	29.7%
8	7	2021	CP1	2.43	8	30.4%
9	2	2020	CP2	3.18	9	35.3%
10	5	2020	CP2	3.66	10	36.6%
11	3	2021	CP2	2.43	6	40.5%
12	2	2021	CP1	2.9	7	41.4%
13	6	2021	CP1	5.01	12	41.8%
14	9	2022	CP2	3.37	8	42.1%
15	7	2020	CP1	4.84	11	44.0%
16	8	2021	CP2	4.84	11	44.0%
17	3	2021	CP1	2.67	6	44.5%
18	6	2020	CP2	6.25	14	44.6%
19	5	2021	CP1	3.15	7	45.0%
20	1	2021	CP1	2.72	6	45.3%
21	2	2019	CP1	3.18	7	45.4%
22	5	2021	CP2	3.65	8	45.6%
23	4	2019	CP1	2.29	5	45.8%
24	7	2020	CP2	5.05	11	45.9%
25	1	2021	CP2	2.38	5	47.6%
26	5	2019	CP1	6.3	13	48.5%
27	3	2020	CP2	6.96	14	49.7%
28	7	2021	CP2	4.6	9	51.1%
29	4	2020	CP1	4.61	9	51.2%
30	5	2020	CP1	8.73	17	51.4%
31	4	2019	CP2	4.23	8	52.9%
32	4	2022	CP2	3.18	6	53.0%
33	4	2021	CP1	4.92	9	54.7%
34	5	2019	CP2	6.8	12	56.7%
35	4	2020	CP2	5.67	10	56.7%
36	8	2019	CP2	6.31	11	57.4%
37	2	2020	CP1	4.04	7	57.7%
38	6	2020	CP1	7.03	12	58.6%
39	7	2019	CP1	4.11	7	58.7%
40	3	2019	CP2	3.61	6	60.2%
41	1	2020	CP2	4.24	7	60.6%
42	3	2020	CP1	5.54	9	61.6%
43	10	2022	CP1	8.68	14	62.0%
44	7	2019	CP2	7.02	11	63.8%
45	8	2019	CP1	11.59	18	64.4%
46	7	2022	CP2	6.65	10	66.5%
47	9	2022	CP1	4.04	6	67.3%
48	1	2019	CP2	6.95	10	69.5%
49	2	2019	CP2	5.81	8	72.6%
50	8	2022	CP1	8.86	12	73.8%
51	3	2022	CP1	4.46	6	74.3%
52	6	2019	CP1	4.47	6	74.5%
53	2	2022	CP1	3	4	75.0%
54	8	2022	CP2	9.77	13	75.2%
55	4	2022	CP1	5.38	7	76.9%
56	6	2022	CP1	5.38	7	76.9%
57	3	2022	CP2	8.92	11	81.1%
58	7	2022	CP1	5.68	7	81.1%
59	3	2019	CP1	8.18	10	81.8%
60	5	2022	CP2	4.91	6	81.8%
61	2	2022	CP2	6.6	8	82.5%
62	1	2020	CP1	8.39	10	83.9%
63	5	2022	CP1	5.08	6	84.7%
64	6	2022	CP2	8.59	10	85.9%
65	1	2022	CP1	5.28	6	88.0%
66	1	2022	CP2	2.69	3	89.7%
67	1	2019	CP1	8.25	9	91.7%