Chapter 8: Binomial Expansion

1:: Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

2:: Factorial Notation

Given that $\binom{8}{3} = \frac{8!}{3!a!}$, find the value of a.

4:: Using expansions for estimation

Use your expansion to estimate the value of 1.05^{10} to 5 decimal places.

3:: Binomial Expansion

Find the first 4 terms in the binomial expansion of $(4+5x)^{10}$, giving terms in ascending powers of x.

Looking for patterns in expanding binomials

a) Expand $(a+b)^0$ $\quad 1$

b) Expand $(a+b)^1$ $\quad 1a + 1b$

c) Expand $(a+b)^2$ $\quad 1a^2 + 2ab + 1b^2$

d) Expand $(a+b)^3$ $\quad 1a^3 + 3a^2b + 3ab^2 + 1b^3$

e) Expand $(a+b)^4$ $\quad 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

What do you notice about:

The coefficients: They follow Pascal's triangle (we'll explore on next slide).

The powers of a and b: Power of a decreases each time (starting at the power)
Power of b increases each time (starting at 0)

Pascal’s Triangle

The second number of each row tells us what row we should use for an expansion.

So if we were expanding $(2+x)^4$, the power is 4, so we use this row.

In Pascal’s Triangle, each term (except for the 1s) is the sum of the two terms above.

Tip: I recommend memorising each row up to what you see here.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 1 1

We’ll see later WHY each row gives us the coefficients in an expansion of $(a+b)^n$

Find the expansion of $(2+3x)^4$

Coefficient					
$2\downarrow$					
$3x\uparrow$					

Find the expansion of $(3 - 2x)^3$

Coefficient				
$3 \downarrow$				
$-2x \uparrow$				

Getting a single term in the expansion

The coefficient of x^2 in the expansion of $(2 - cx)^5$ is 720.
Find the possible value(s) of the constant c.

Edexcel C2

(*a*) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$(2 + kx)^7$$

where k is a constant. Give each term in its simplest form. **(4)**

Given that the coefficient of x^2 is 6 times the coefficient of x,

(*b*) find the value of k. **(2)**

Ex 8A

Factorial and Choose Function

$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
said "n factorial", is the number of ways of arranging n objects in a line.

Suppose you had three letters, A, B and C, and wanted to arrange them in a line to form a 'word', e.g. ACB or BAC.

$$^nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

said "n choose r", is the number of ways of 'choosing' r things from n, such that the order in our selection does not matter.
These are also known as **binomial coefficients**.

For example, if you a football team captain and need to choose 4 people from amongst 10 in your class, there are $\binom{10}{4} = \frac{10!}{4!6!} = 210$ possible selections.

(Note: the $\binom{10}{4}$ notation is preferable to $^{10}C_4$)

Use the nCr button on your calculator (your calculator input should display "10C4")

Calculate the value of the following. You may use the factorial button, but not the nCr button.

$$^nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

a) $5!$

b) $\binom{5}{3}$

c) $0!$

d) $\binom{20}{1}$

e) $\binom{20}{0}$

f) $\binom{20}{2}$

g) $\binom{20}{18}$

Why do we care?

If the power in the binomial expansion is large, e.g. $(x+3)^{20}$, it is no longer practical to go this far down Pascal's triangle. We can instead use the choose function to get numbers from anywhere within the triangle. We'll practise doing this after the next exercise.

1	0th row	$\binom{0}{0}$
1 1	1st row	$\binom{1}{0}$ $\binom{1}{1}$
1 2 1	2nd row	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
1 3 3 1	3rd row	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
1 4 6 4 1		$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

Ex 8B

Calculate the value of the following.

$${}^nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\binom{n}{0}$

$\binom{n}{1}$

$\binom{n}{2}$

$\binom{n}{3}$

$\binom{n}{4}$

The Binomial Coefficients – for all values of n

nth row $\binom{n}{0}$ $\binom{n}{1}$ $\binom{n}{2}$ $\binom{n}{3}$ $\binom{n}{4}$ etc.

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2!}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$$

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4!}$$

Why are Pascal's and Binomial Coefficients linked?

In the previous section we learnt about the 'choose' function and how this related to Pascal's Triangle.

1 5 10 10 5 1

$$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$$

Why do rows of Pascal's Triangle give us the coefficients in a Binomial Expansion?

One possible selection of terms from each bracket.

Consider: $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$
Each term of the expansion involves picking one term from each bracket.
How many times will a^3b^2 appear in the expansion?
To get a^3b^2 we must have chosen 3 a's from the 5 brackets (the rest b's).
That's $\binom{5}{3}$ ways, giving us $\binom{5}{3}a^3b^2$ in the expansion of $(a+b)^5$.

The binomial expansion:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{r} a^{n-r}b^r + \cdots + b^n$$

Find the first 4 terms in the expansion of $(3x+1)^{10}$, in ascending powers of x.

Find the first 3 terms in the expansion of $\left(2-\frac{1}{3}x\right)^7$, in ascending powers of x.

Find the first 4 terms in the expansion of $(3x - 2y)^5$, in ascending powers of x.

Ex 8C

Getting a single term in the expansion

In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r} a^{n-r} b^r$

Expression	Power of x in term wanted.	Term in expansion
$(a + x)^{10}$	3	
$(2x - 1)^{75}$	50	
$(3 - x)^{12}$	7	
$(3x + 4)^{16}$	3	

The coefficient of x^4 in the expansion of $(1 + qx)^{10}$ is 3360.
Find the possible value(s) of the constant q.

In the expansion of $(1 + ax)^{10}$, where a is a non-zero constant the coefficient of x^3 is double the coefficient of x^2.
Find the value of a.

Estimating Powers

Edexcel C2 Jan 2012 Q3

(*a*) Find the first 4 terms of the binomial expansion, in ascending powers of x, of

$$\left(1+\frac{x}{4}\right)^8,$$

giving each term in its simplest form.

(4)

(*b*) Use your expansion to estimate the value of $(1.025)^8$, giving your answer to 4 decimal places.

(3)

Edexcel C2 Jan 2008 Q3

(*a*) Find the first 4 terms of the expansion of $\left(1+\frac{x}{2}\right)^{10}$ in ascending powers of x, giving each term in its simplest form.

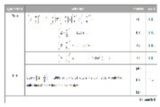
(4)

(*b*) Use your expansion to estimate the value of $(1.005)^{10}$, giving your answer to 5 decimal places.

(3)

Ex 8E

Exam Questions



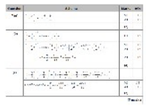
7. (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of $\left(2-\frac{x}{2}\right)^7$, giving each term in its simplest form.

(4)

(b) Explain how you would use your expansion to give an estimate for the value of 1.995^7

(1)

7. (*a*) Expand $\left(1+\dfrac{3}{x}\right)^2$, simplifying each term.



(2)

(*b*) Use the binomial expansion to find, in ascending powers of x, the first four terms in the expansion of

$$\left(1+\frac{3}{4}x\right)^6,$$

simplifying each term.

(4)

(*c*) Hence find the coefficient of x in the expansion of

$$\left(1+\frac{3}{x}\right)^2\left(1+\frac{3}{4}x\right)^6.$$

(2)

(Total for Question 7 is 8 marks)

11. (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$\left(2 - \frac{x}{16}\right)^9$$

giving each term in its simplest form.

(4)

$$f(x) = (a + bx)\left(2 - \frac{x}{16}\right)^9, \text{ where } a \text{ and } b \text{ are constants}$$

Given that the first two terms, in ascending powers of x, in the series expansion of $f(x)$ are 128 and $36x$,

(b) find the value of a,

(2)

(c) find the value of b.

(2)