Your notes

Edexcel A Level Further Maths: Core Pure

3.2 Series

Contents

3.2.1 Sums of Integers, Squares & Cubes

Your notes

Sums of Integers, Squares & Cubes

How can we use sigma notation?

- When writing the sum of a series you can use sigma notation
  - The series $u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$ can instead be written as $\sum_{r=1}^{r=n} u_r$ or $\sum_{r=1}^{n} u_r$
  - This means "the sum of all the terms from u_1 to u_n for the sequence described by u_r"
  - $\sum_{r=1}^{3} 2r + 3$ would mean $(2(1) + 3) + (2(2) + 3) + (2(3) + 3) = 21$
- Using the following relations, summations can be grouped together (or ungrouped) to make some calculations easier:

$$\sum_{r=p}^{q} a\mathrm{f}(r) + b = \sum_{r=p}^{q} a\mathrm{f}(r) + \sum_{r=p}^{q} b = a\sum_{r=p}^{q} \mathrm{f}(r) + \sum_{r=p}^{q} b$$

$$\sum_{r=p}^{q} a\mathrm{f}(r) + b\mathrm{g}(r) = \sum_{r=p}^{q} a\mathrm{f}(r) + \sum_{r=p}^{q} b\mathrm{g}(r) = a\sum_{r=p}^{q} \mathrm{f}(r) + b\sum_{r=p}^{q} \mathrm{g}(r)$$

- *a* and *b* are constants, and $\mathrm{f}(r)$ and $\mathrm{g}(r)$ are a functions of *r*
- Note that the top and bottom summation limits (p and q) are the same for all the sums
  - This is important – if the top and bottom limits don't all match then the relation is no longer valid!
- This can be very useful when $\mathrm{f}(r)$ or $\mathrm{g}(r) = r,\ r^2$ or r^3, as $\sum b$ and $\sum r$ (or r^2 or r^3) are straightforward to find using formulae
- A useful result to remember is that $\sum_{r=1}^{n} a = a + a + a + \ldots(n \text{ times}) = n \times a$

What are the formulae for finding sums of integers, squares, and cubes?

- There are several useful formulae for summing integers, square numbers, and cube numbers
- The sum of the first n natural numbers is given by
  - $\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$
    - This formula **is not** given in the formula book

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- The sum of the first n square numbers is given by
  - $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
    - This formula **is** given in the formula book
- The sum of the first n cube numbers is given by
  - $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$
    - This formula **is** given in the formula book
    - Notice that this is equal to the formula for the sum of the first n natural numbers, squared
- Using the relations given above, a more complicated summation can often be broken down into sums of constants, natural numbers, squares, and cubes
  - For example, $\sum 2r^3 - 3r^2 - 8r - 3 = 2\sum r^3 - 3\sum r^2 - 8\sum r - \sum 3$

Examiner Tip

- You can find summations using sigma notation on most advanced scientific calculators or graphics calculators – you can use this to check your answers
- Bear in mind, however, that the question will normally require you to show your full working

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Worked example

Find $\sum_{r=1}^{7}(2r+1)(r-3)(r+1)$

Expand $\sum_{r=1}^{7}(2r+1)(r-3)(r+1) = \sum_{r=1}^{7} 2r^3 - 3r^2 - 8r - 3$

Rewrite as several simpler sums, factoring out constants

$$2\sum_{r=1}^{7} r^3 - 3\sum_{r=1}^{7} r^2 - 8\sum_{r=1}^{7} r - \sum_{r=1}^{7} 3$$

Could even write the last term as $3\sum_{r=1}^{7} 1$

Use the following standard formulae

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

$$2\left(\frac{1}{4}(7^2)(8)^2\right) - 3\left(\frac{1}{6}(7)(8)(15)\right) - 8\left(\frac{1}{2}(7)(8)\right) - (7\times 3)$$

$$\boxed{= 903}$$

3.2.2 Method of Differences

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Method of Differences

What is the Method of Differences?

- The **Method of Differences** is a way of turning longer and more complicated sums into shorter and simpler ones
- Sometimes when summing series, you will notice that many of the terms (or parts of the terms) simply "cancel out" or eliminate each other
  - This can turn a very long series summation, into a much simpler shorter one
- In the case of $\sum_{r=1}^{n}(f(r)-f(r+1))$
  - $f(1)-f(2)$
  - $f(2)-f(3)$
  - $f(3)-f(4)$
  - $f(4)-f(5)$
  - (and so on, until...)
  - $f(n-1)-f(n)$
    - This is the penultimate term
  - $f(n)-f(n+1)$
    - This is the last term
- You can see that when these are summed, most of the terms will cancel out
  - This leaves just $f(1)-f(n+1)$
  - We can say that $\sum_{r=1}^{n}(f(r)-f(r+1))=f(1)-f(n+1)$

How can I use partial fractions along with the method of differences?

- You will often need to use partial fractions to change the general term into a sum of two or three terms, rather than a single fraction
  - For example, $\frac{6}{(r+1)(r+3)}$ can be rewritten as $\frac{3}{r+1}-\frac{3}{r+3}$
- This may lead to a more interesting pattern of cancellations than was seen for $f(r)-f(r+1)$
- For example, $\sum_{r=1}^{n}\frac{3}{r+1}-\frac{3}{r+3}$ can be written as $\sum_{r=1}^{n}f(r)-f(r+2)$, where $f(r)=\frac{3}{r+1}$, and the terms can then be listed as:
  - $f(1)-f(3)$
  - $f(2)-f(4)$

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    - $f(3) - f(5)$
    - $f(4) - f(6)$
    - $f(5) - f(7)$
    - (and so on, until...)
    - $f(n-1) - f(n+1)$
    - $f(n) - f(n+2)$
  - When these are summed, it will just leave $f(1) + f(2) - f(n+1) - f(n+2)$
  - You can then evaluate this expression with $f(r) = \dfrac{3}{r+1}$ to get to your final answer
- It is helpful to use $f(r)$ notation to spot the pattern, rather than substituting $r = 1,\ r = 2,\ \ldots$ into the expression every time, especially with more complicated expressions
  - You need to consider carefully which term to make $f(r)$ and then how the other terms in the expression relate to it
  - If this is difficult for a particular expression, it may be more straightforward to substitute $r = 1,\ r = 2,\ \ldots$ into each term in the series and spot any patterns that way
  - This is essentially writing the series out in full until you spot which terms will cancel
  - In your working, however, you should still write out the last two or three terms in terms of $n,\ n-1,\ n-2$, etc.

How can I use the method of differences for series with expressions containing more than two terms?

- The general term of the series may have more than two terms, which can sometimes make spotting which terms will cancel more challenging
  - For example, $\displaystyle\sum_{r=1}^{n} \frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2}$
  - This can be written as $2f(r) - 3f(r+1) + f(r+2)$ where $f(r) = \dfrac{1}{r}$
  - Writing out the first five terms and the last three terms we get:
    - $2f(1) - 3f(2) \mathbf{+ f(3)}$
    - $2f(2) \mathbf{- 3f(3)} + f(4)$
    - $\mathbf{2f(3)} - 3f(4) + f(5)$
    - $2f(4) - 3f(5) + f(6)$
    - $2f(5) - 3f(6) + f(7)$
    - (and so on until...)
    - $2f(n-2) - 3f(n-1) + f(n)$
    - $2f(n-1) - 3(fn) + f(n+1)$
    - $2f(n) - 3f(n+1) + f(n+2)$
  - In this case, look at the diagonals starting at the top right

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- We have $f(3)$, $-3f(3)$ and $2f(3)$ which sum to 0
  - This pattern repeats for the other diagonals
- We will eventually be left with only
$2f(1) - 3f(2) + 2f(2) + f(n+1) - 3f(n+1) + f(n+2) = 2f(1) - f(2) - 2f(n+1) + f(n+2)$
- Evaluating this with $f(r) = \frac{1}{r}$ results in an answer of $\frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}$

What other uses are there for the method of differences?

- Method of differences can also be used to prove the formulae for the sum of r, r^2 and r^3
  - For example, this can be used to prove that the result for the sum of squares is indeed $\frac{1}{6}n(n+1)(2n+1)$
    - By expanding brackets it can be shown that $(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$, and then the sum of both sides of that equation from $r = 1$ to $r = n$ can be considered
    - $\sum_{r=1}^{n}((2r+1)^3 - (2r-1)^3) = \sum_{r=1}^{n}(24r^2 + 2)$
    - The left-hand side can be found in terms of n using method of differences, and the right-hand side can be rearranged to give $2n + 24\sum_{r=1}^{n} r^2$
    - That equation can then be rearranged to give an expression for $\sum_{r=1}^{n} r^2$
    - When these proofs have appeared previously in exams, they have tended to be structured to help you work through the steps
- You may have to use your algebraic method of differences result to find a numerical answer, usually in the last part of a question
  - The question will often ask you to evaluate the sum starting from $r = 20$ (or some other arbitrary value) rather than from $r = 1$
  - To help with this, remember that: $\sum_{r=p}^{q} u_r = \sum_{r=1}^{q} u_r - \sum_{r=1}^{p-1} u_r$
  - You may also find it helpful to recall that for constants a and b:
  $\sum_{r=p}^{q}(au_r + b) = a\sum_{r=p}^{q} u_r + \sum_{r=p}^{q} b$

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Examiner Tip

- Mark schemes often specify how many terms from the start and end of the series should be written down – it is usually two or three, so always write down the first three and last three terms
- Don't be afraid to write out more terms than this to make sure you spot the pattern, and can easily decide which terms will cancel and which will not
- Check your algebraic answer by substituting in numbers to make sure it works; you can use your calculator to find summations in sigma notation

Your notes

Worked example

(a) Express $\frac{2}{r(r+2)}$ in partial fractions.

Use standard method for partial fractions

$$\frac{2}{r(r+2)} = \frac{A}{r} - \frac{B}{r+2}$$

$$2 = A(r+2) - B(r)$$

let $r=-2$: $\quad 2 = 0 + 2B \Rightarrow B = 1$

let $r=0$: $\quad 2 = 2A - 0 \Rightarrow A = 1$

$$\boxed{\frac{2}{r(r+2)} = \frac{1}{r} - \frac{1}{r+2}}$$

(b) Hence show that $\sum_{r=1}^{n} \frac{2}{r(r+2)} = \frac{3n^2+5n}{2(n+1)(n+2)}$ using the method of differences.

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Using result from (a) we are finding

$$\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+2} = \sum_{r=1}^{n} f(r) - f(r+2) \quad \text{where} \quad f(r) = \frac{1}{r}$$

Investigate the pattern of the summation for increasing values of r from 1 to n

$$f(1) - \cancel{f(3)}$$
$$f(2) - \cancel{f(4)}$$
$$\cancel{f(3)} - \cancel{f(5)}$$
$$\cancel{f(4)} - \cancel{f(6)}$$
$$\cancel{f(5)} - \cancel{f(7)}$$
$$\vdots$$
$$\cancel{f(n-1)} - f(n+1)$$
$$\cancel{f(n)} - f(n+2)$$

Most of the terms eliminate eachother, except $f(1)$, $f(2)$, $-f(n+1)$, $-f(n+2)$

Reduces to $f(1) + f(2) - f(n+1) - f(n+2)$

Evaluate, recalling that $f(r) = \frac{1}{r}$

$$\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

Now need to rearrange to desired form

$$\frac{3}{2} - \frac{n+2}{(n+1)(n+2)} - \frac{n+1}{(n+1)(n+2)}$$

$$\frac{3(n+1)(n+2)}{2(n+1)(n+2)} - \frac{n+2}{2(n+1)(n+2)} - \frac{n+1}{2(n+1)(n+2)}$$

$$\boxed{\frac{3n^2+5n}{2(n+1)(n+2)}}$$