Chapter 4: Graphs and Transformations

1a:: Cubic Graphs

Sketch the graph with equation:
$$y = x(x-3)^2$$

1b:: Quartic Graphs

Sketch the graph with equation:
$$y = (x-1)^2(x+1)^2$$

1c:: Reciprocal Graphs

Sketch the graph with equation $y = -\frac{3}{x^2}$

2:: Points of Intersection

Sketch the curves $y = \frac{4}{x^2}$ and $y = x^2(x-1)$ on the same axes. Using your sketch, state, with a reason, the number of real solutions to the equation $x^4(x-1) - 4 = 0$.

3:: Graph Transformations

If $f(x) = x^2(x+1)$, sketch the graph of $y = f(x+a)$, indicating any intercepts with the axes.

Polynomial Graphs

We have previously seen that a **polynomial** expression is of the form:
$$a + bx + cx^2 + dx^3 + ex^3 + \cdots$$
where $a, b, c, d, e, \ldots$ are constants (which could be 0).

The **order** of a polynomial is its highest power.

Order	Name
0	(e.g. "4")
1	(e.g. "$2x-1$")
2	(e.g. "x^2+3")
3	
4	
5	

These are covered in Chapter 5.

Chapter 2 explored the graphs for these.

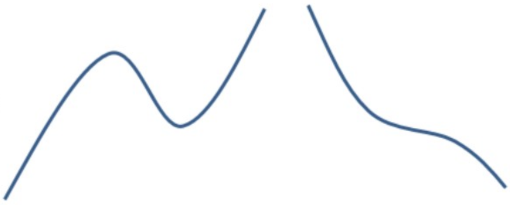
We will cover these now.

While these are technically beyond the A Level syllabus, we will look at how to sketch polynomials in general.

Shapes of Polynomial Graphs

Order:

2 

3 

4 

What property connects the order of the polynomial and the shape?

In Chapter 2 how did we tell what way up a quadratic is, and why does this work?

e.g. If $y = 2x^2 + 3$, try a large positive value like $x = 1000$. We can see we'd get a large positive y value. Thus as $x \to \infty, y \to \infty$

Equation	If $a > 0$	Resulting Shape	If $a < 0$	Resulting Shape
$y = ax^2 + bx + c$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$			
$y = ax^3 + bx^2 + cx + d$				
$y = ax^4 + bx^3 + cx^2 + dx + e$				
$y = ax^5 + bx^4 + \cdots$				

If $a > 0$, what therefore can we say about the shape if:

- **The order is odd**: It goes uphill (from left to right)
- **The order is even**: The tails go upwards.

(And we have the opposite if $a < 0$)

Sketching Cubics

Sketch the curve with equation
$$y = (x-2)(1-x)(1+x)$$

Features you must consider:

Shape?

Roots?

y-intercept?

Sketch the curve with equation
$$y = x^2(x-1)$$

Shape?

Roots?

y-intercept?

If there is a repeated root, it does not cut through the axis, but just touches it and 'bounces' back

Sketch the curve with equation

$$y = (2 - x)(x + 1)^2$$

Sketch the curve with equation

$$y = (x - 4)^3$$

If there is a triple repeated root, there is a point of inflection

A point of inflection is where the curve goes from 'convex' to 'concave' (or vice versa), i.e. curves in one direction before and curves in another direction after. You might have encountered these terms in Physics.

Cubics with Limited Roots

Sketch the curve with equation
$$y = (x+1)(x^2+x+1)$$

We don't have enough information to determine the exact shape.

Your Turn

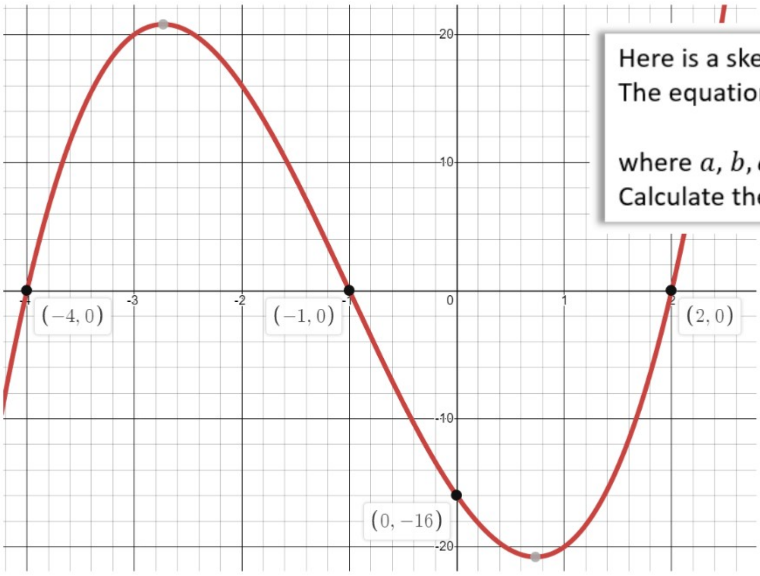
1 Sketch the curve with equation
$$y = x(x-3)^2$$

2 Sketch the curve with equation
$$y = -(x+2)^3$$

3 

A curve has this shape, touches the x axis at 3 and crosses the x axis at -2. Give a suitable equation for this graph.

Finding the equation yourself



Here is a sketch of the curve C with equation $y = f(x)$.
The equation of the curve C can be written in the form.

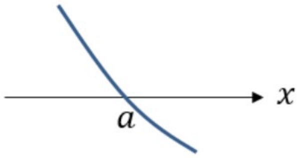
$$y = ax^3 + bx^2 + cx + d$$

where a, b, c and d are integers.
Calculate the values of a, b, c, d.

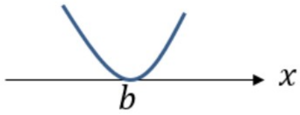
Summary

If we sketched $y = (x-a)(x-b)^2(x-c)^3$ what happens on the x-axis at:

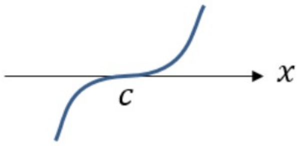
$x = a$: The line **crosses** the axis.



$x = b$: The line **touches** the axis.



$x = c$: **Point of inflection** on the axis.



Ex4A

Quartics

This feels very similar to sketching cubics.
Recall that if the x^4 term is positive, the 'tails' both go upwards, otherwise downwards.

Sketch the curve with equation
$y = x(x+1)(x-2)(x-3)$

Sketch the curve with equation

$$y = (x-2)^2(x+1)(3-x)$$

Sketch the curve with equation

$$y = (x+1)(x-1)^3$$

Sketch the curve with equation
$$y = (x - 2)^4$$

Your Turn

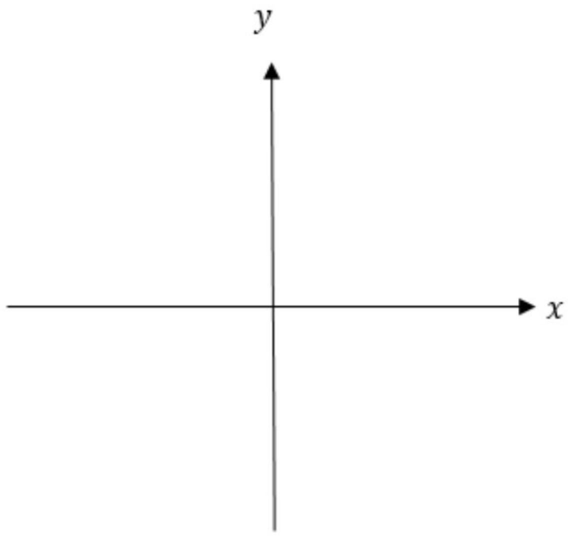
Sketch the curve with equation
$$y = x^2(x + 1)(x - 1)$$

Sketch the curve with equation
$$y = -(x + 1)(x - 3)^3$$

Reciprocal Graphs

Sketch $y = \frac{1}{x}$

x						
y						



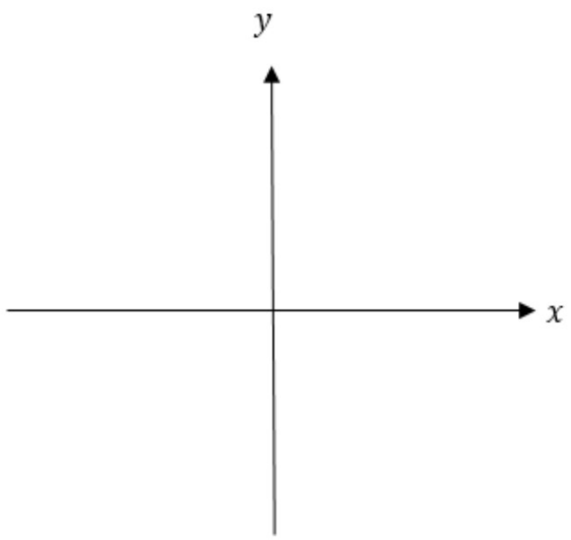
✏ An asymptote is a line which the graph approaches but never reaches.

Asymptotes of $y = \frac{a}{x}$:

$y = 0$,

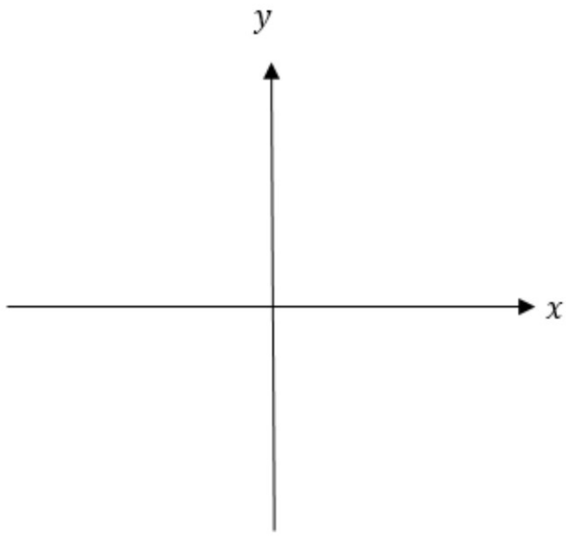
$x = 0$

Sketch $y = \frac{k}{x}$ where $k > 0$

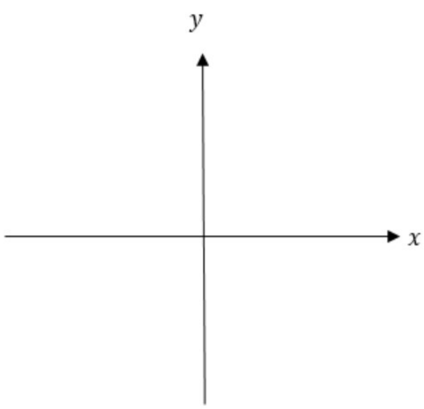


Sketch $y = -\frac{1}{x}$

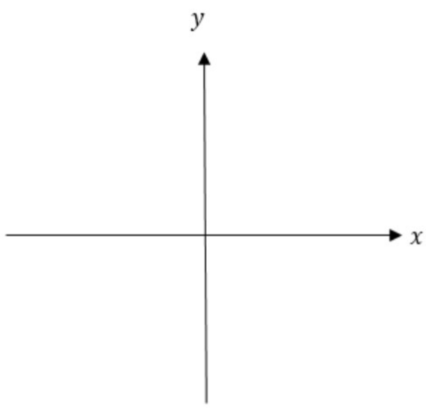
x						
y						



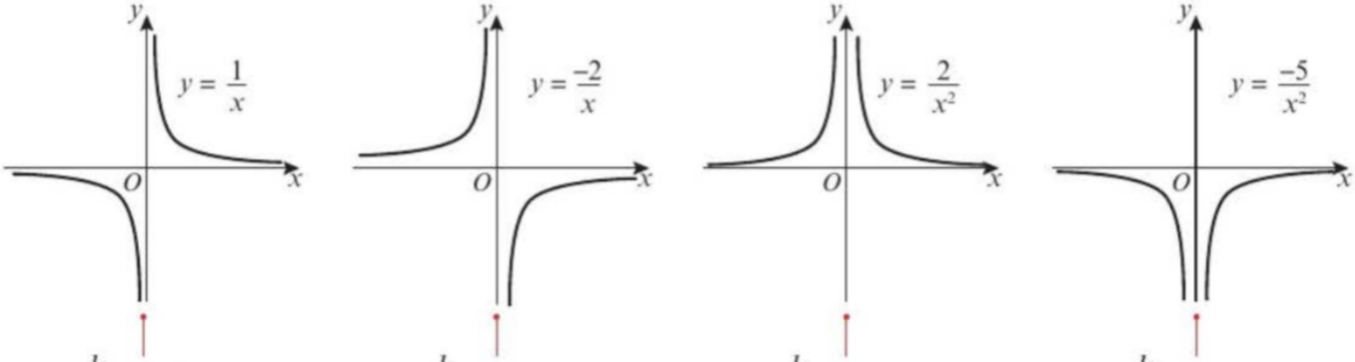
Sketch $y = \frac{1}{x^2}$



Sketch $y = -\frac{1}{x^2}$

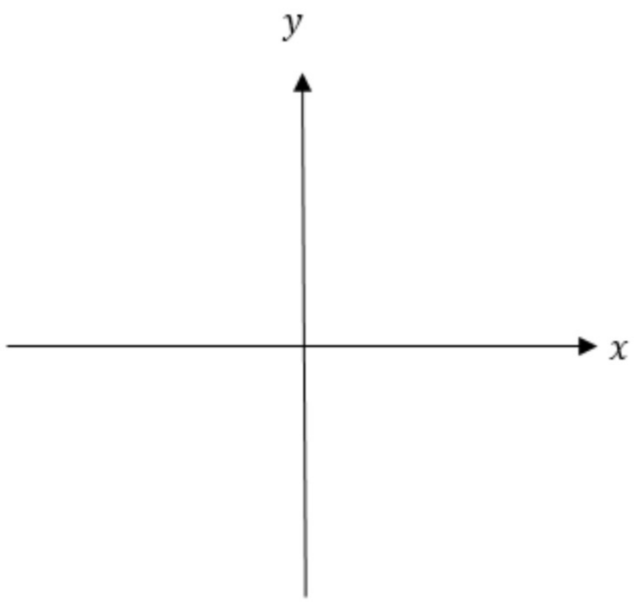


Summary



$y = \frac{k}{x}$ with $k > 0$. $y = \frac{k}{x}$ with $k < 0$. $y = \frac{k}{x^2}$ with $k > 0$. $y = \frac{k}{x^2}$ with $k < 0$.

On the same axes, sketch $y = \frac{1}{x}$ and $y = \frac{3}{x}$



Points of Intersection

If $y = f(x)$ and $y = g(x)$, then the x values of the points of intersection can be found when $f(x) = g(x)$.

On the same diagram sketch the curves with equations $y = x(x-3)$ and $y = x^2(1-x)$. Find the coordinates of their points of intersection.

On the same diagram sketch the curves with equations $y = \frac{4}{x^2}$ and $y = x^2(x-3)$
State, giving a reason, the number of real solutions to the equation $\frac{4}{x^2} - x^2(x-3) = 0$

Further example involving unknown constants

On the same diagram sketch the curves with equations $y = x^2(3x - a)$ and $y = \frac{b}{x}$, where a, b are positive constants. State, giving a reason, the number of real solutions to the equation $x^2(3x-a) - \frac{b}{x} = 0$

On the same diagram sketch the curves with equations $y = x(x-4)$ and $y = x(x-2)^2$, and hence find the coordinates of any points of intersection.

Hint: Remember you can use the discriminant to reason about the number of solutions of a quadratic.

Ex4D

Translation of Functions

Suppose $f(x) = x^2$

Sketch $y = f(x)$:

Then $f(x+2) = (x+2)^2$

Sketch $y = f(x+2)$

Then $f(x) + 2 = x^2 + 2$

Sketch $y = f(x) + 2$

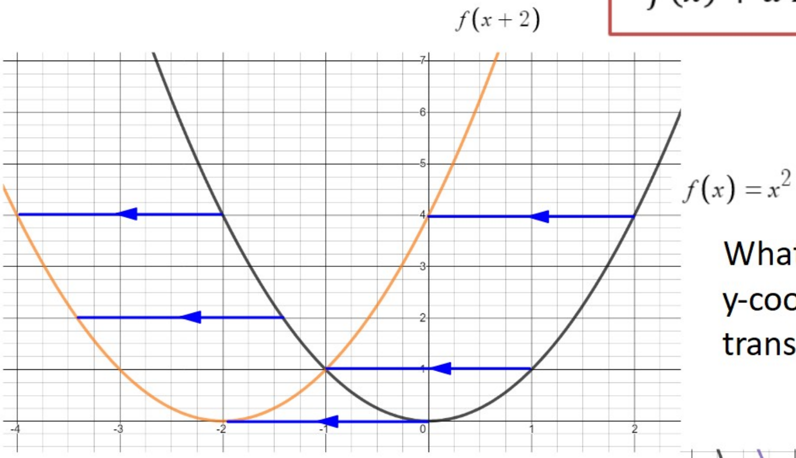
What do you notice about the relationship between the graphs of $y = f(x)$ and $y = f(x+2)$?

What do you notice about the relationship between the graphs of $y = f(x)$ and $y = f(x) + 2$?

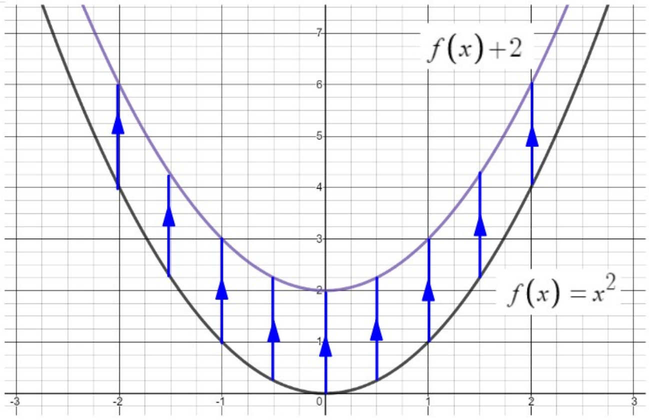
Desmos Demonstration

$f(x+a)$ is a translation of $f(x)$ by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$

$f(x)+a$ is a translation of $f(x)$ by $\begin{pmatrix} 0 \\ a \end{pmatrix}$



What has happened to the y-coordinates when f(x) is transformed to f(x+2)?



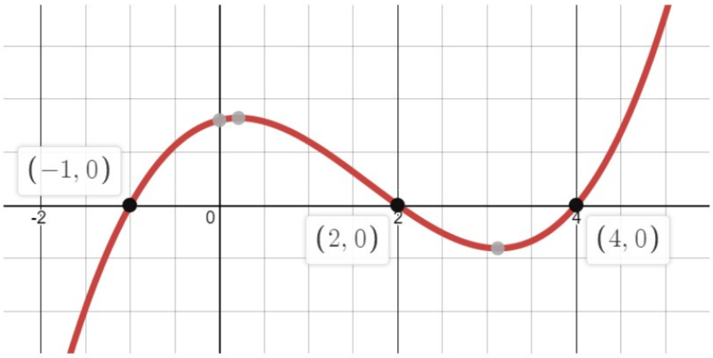
What has happened to the y-coordinates when f(x) is transformed to f(x)+2?

Sketch the graph of $y=\frac{2}{x}+1$, ensuring you indicate any intercepts with the axes.

Sketch $y=\frac{2}{x+1}$

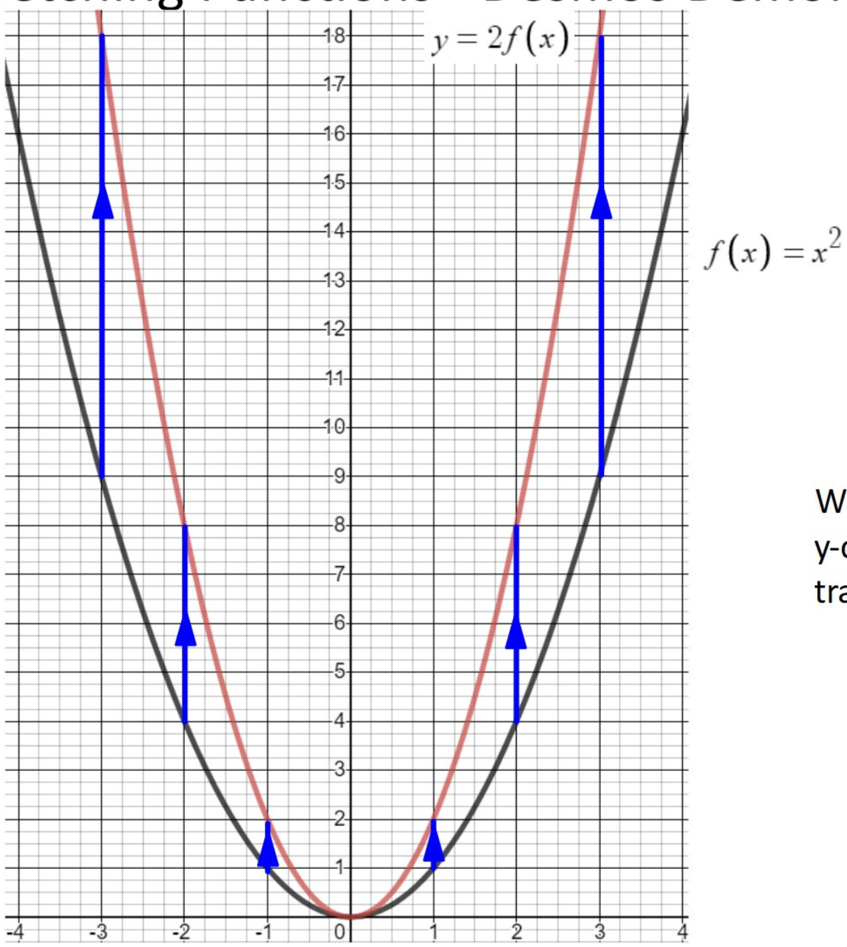
Sketch $y=\frac{4}{x^2}+3$

The graph below shows $y = f(x)$
Given that $y = f(x + a)$ passes through the origin, state the possible values of a

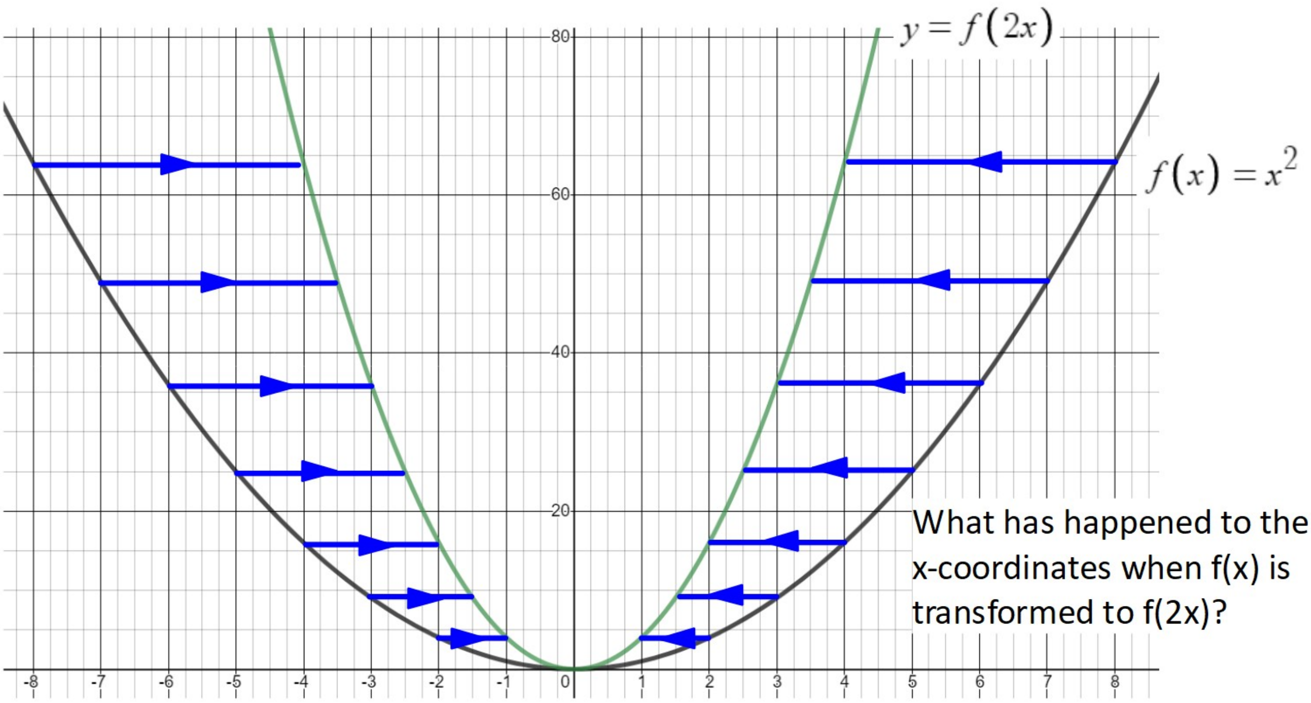


Sketch $y = x(x + 2)$. On the same axes, sketch $y = (x - a)(x - a + 2)$, where $a > 2$.

Stretching Functions - Desmos Demonstration



What has happened to the y-coordinates when f(x) is transformed to 2f(x)?

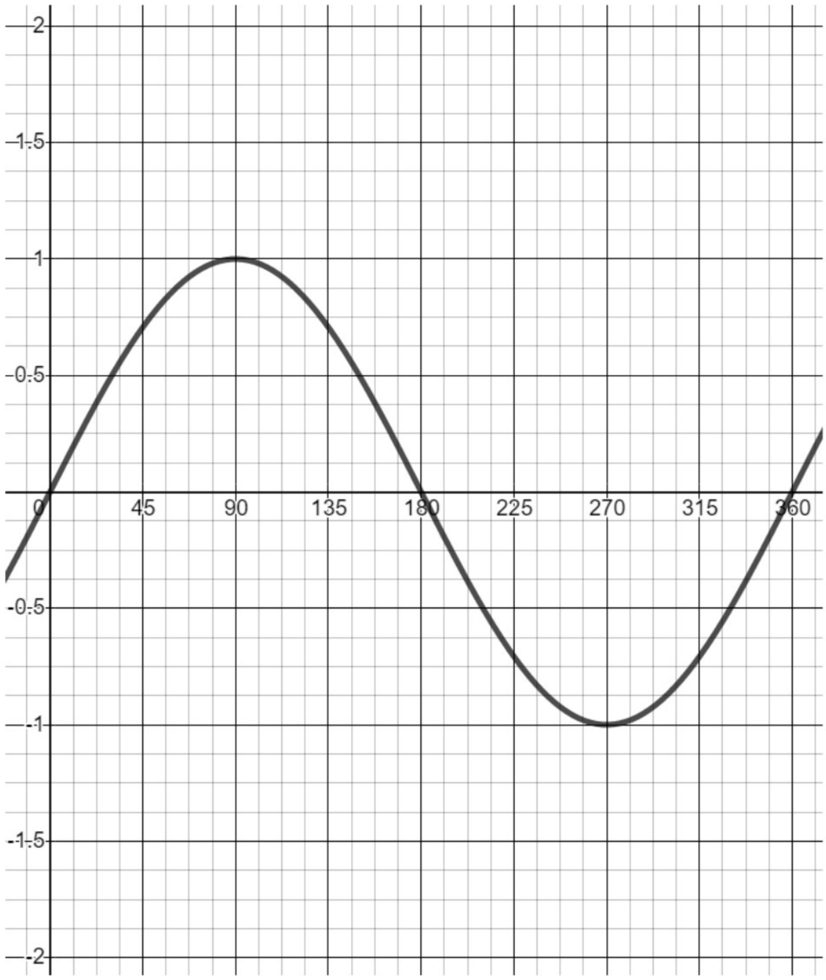


What has happened to the x-coordinates when f(x) is transformed to f(2x)?

$af(x)$ is a stretch of $f(x)$ by a factor of a in the $y-$direction

$f(ax)$ is a stretch of $f(x)$ by a factor of $\frac{1}{a}$ in the $x-$direction

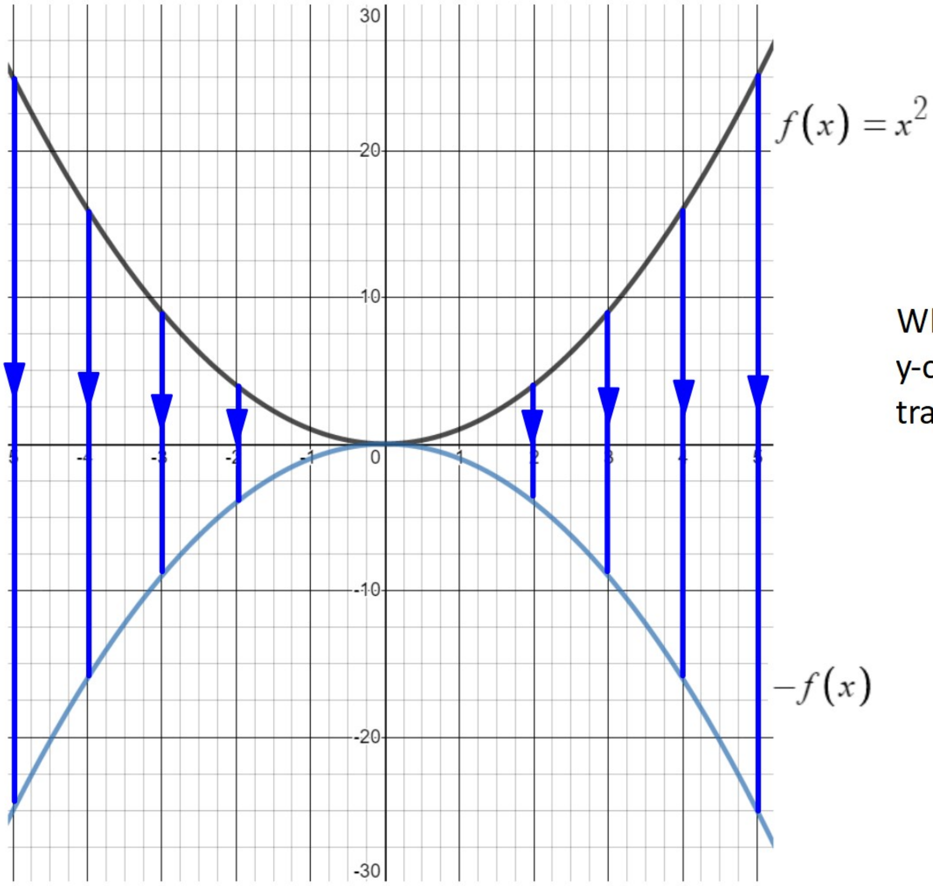
The curve below is $y = \sin x$
Sketch $y = 2\sin x$



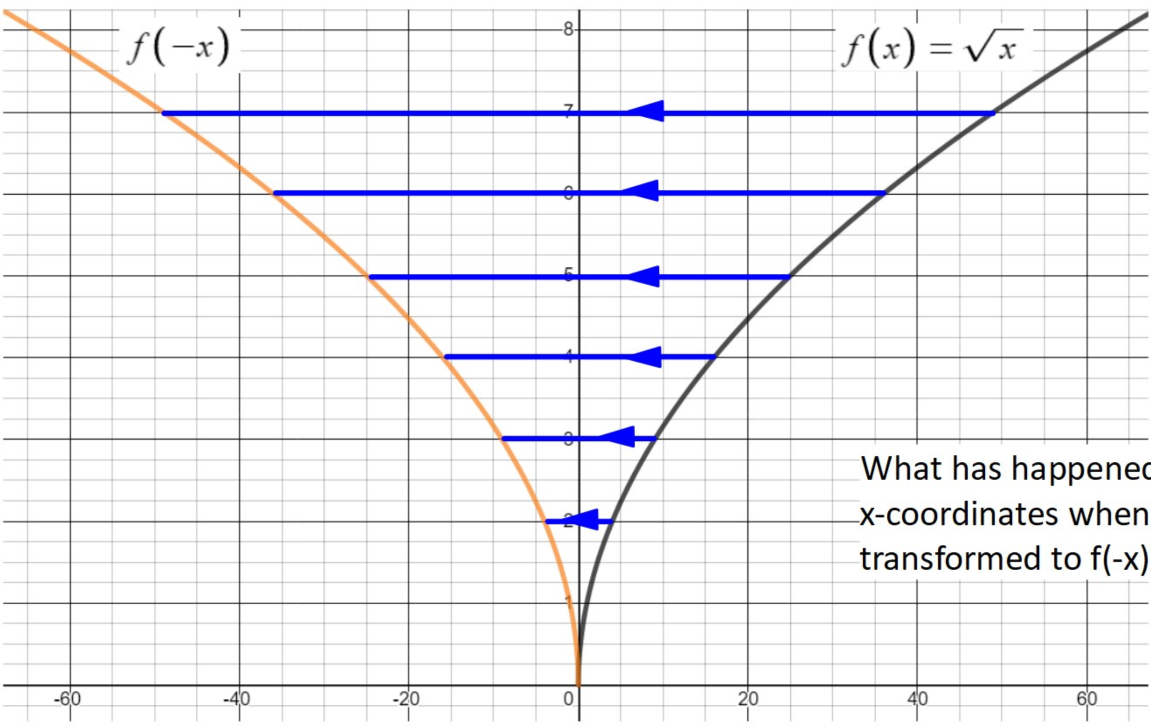
Sketch $y = x^2(x-4)$. On the same axes, sketch the graph with equation
$$y = (2x)^2(2x-4)$$

If $y = (x+1)(x-2)$, sketch $y = f(x)$ and $y = f(\frac{x}{3})$ on the same axes.

Flipping (reflecting) Functions - Desmos Demonstration



What has happened to the y-coordinates when f(x) is transformed to -f(x)?



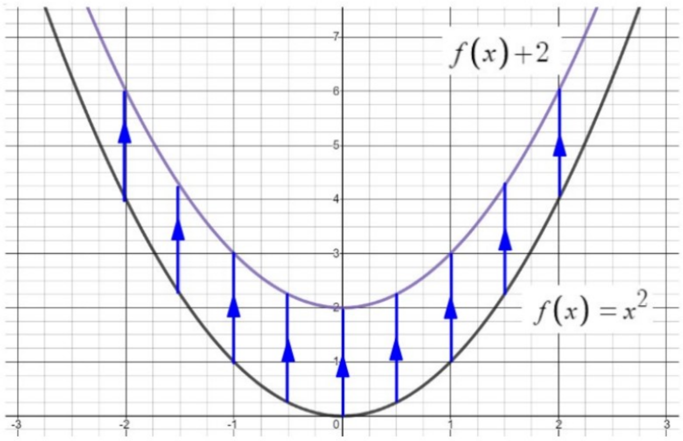
What has happened to the x-coordinates when f(x) is transformed to f(-x)?

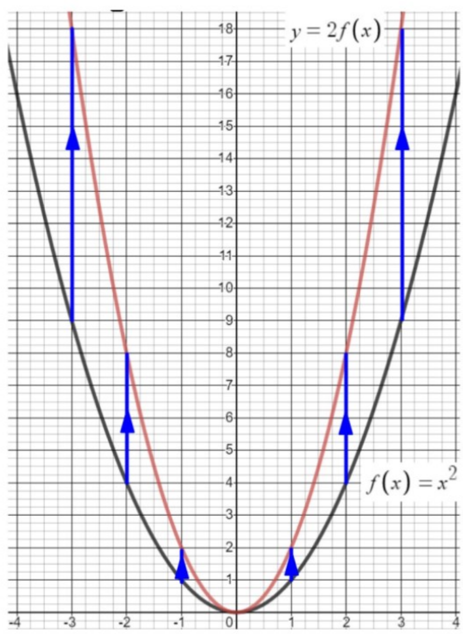
$-f(x)$ is $f(x)$ flipped in the y − direction

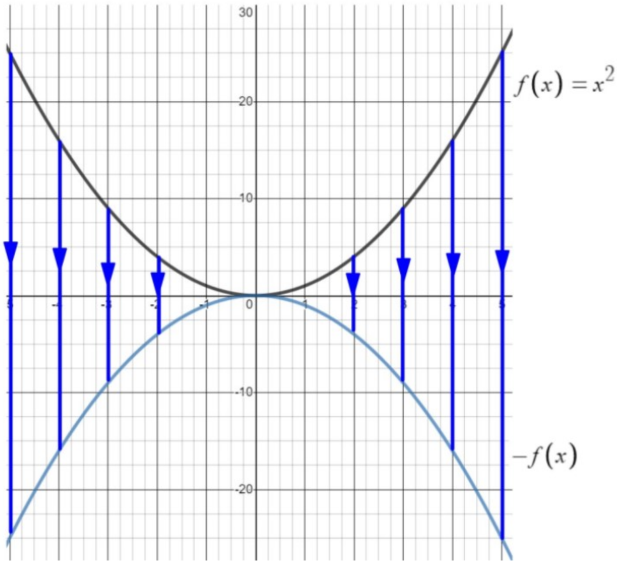
$f(-x)$ is $f(x)$ flipped in the x − direction

If $y = x(x+2)$, sketch $y = f(x)$ and $y = -f(x)$ on the same axes.

Sketch $y = x(x+3)(x+5)$.
On the same axes, sketch $y = -x(3-x)(5-x)$





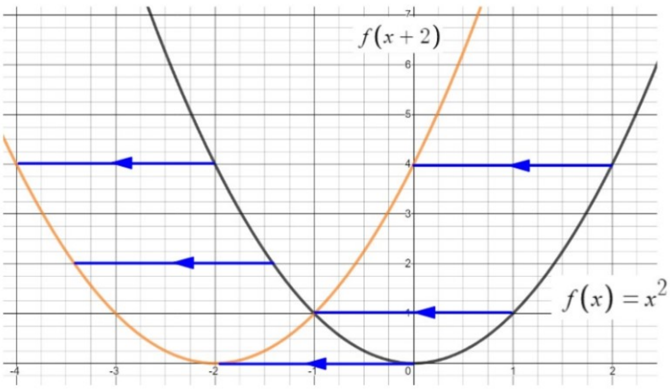


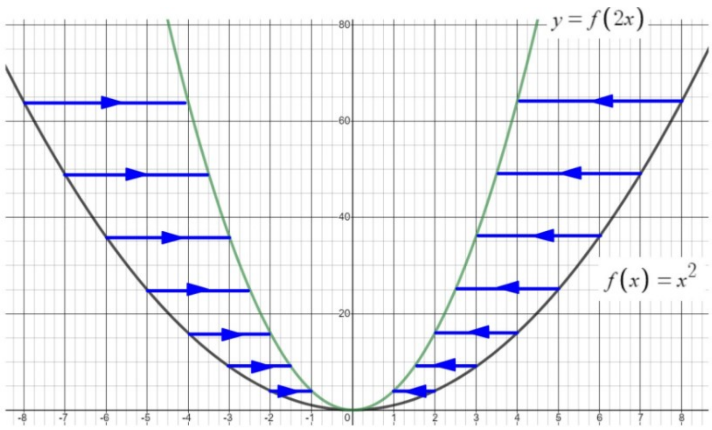
These are all affecting the **<u>y-coordinates</u>**.

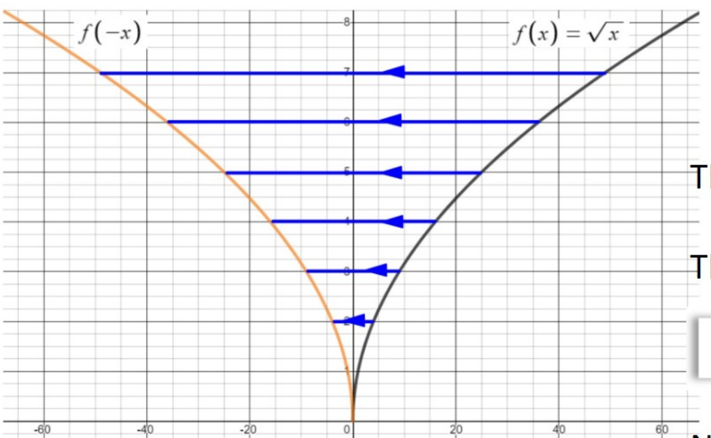
They are the transformations

$f(x)+a$	$af(x)$	$-f(x)$

Notice how the function is changed **<u>outside its brackets</u>**







These are all affecting the **<u>x-coordinates</u>**.

They are the transformations

$f(x+a)$	$f(ax)$	$f(-x)$

Notice how the function is changed **<u>inside its brackets</u>**

Summary of Transformations

$f(x) + a$

$f(x + a)$

$af(x)$

$f(ax)$

$-f(x)$

$f(-x)$

Effect of transformation on specific points

Sometimes you will not be given the original function, but will be given a sketch with specific points and features you need to transform.
Where would each of these points end up?

$\boldsymbol{y = f(x)}$	Effect on the coordinate	$\mathbf{(4, 3)}$	$\mathbf{(1, 0)}$	$\mathbf{(6, -4)}$
$y = f(x + 1)$				
$y = f(2x)$				
$y = 3f(x)$				
$y = f(x) - 1$				
$y = f\left(\frac{x}{4}\right)$				
$y = f(-x)$				
$y = -f(x)$				

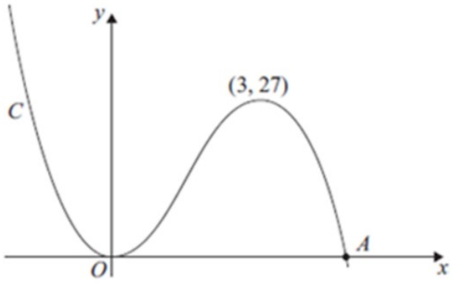


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^2(9 - 2x).$$

There is a minimum at the origin, a maximum at the point (3, 27) and C cuts the x-axis at the point A.

(*a*) Write down the coordinates of the point A.

(1)

(*b*) On separate diagrams sketch the curve with equation

(i) $y = \mathrm{f}(x + 3)$,

(ii) $y = \mathrm{f}(3x)$.

On each sketch you should indicate clearly the coordinates of the maximum point and any points where the curves cross or meet the coordinate axes.

(6)

The curve with equation $y = \mathrm{f}(x) + k$, where k is a constant, has a maximum point at (3, 10).

(*c*) Write down the value of k.

(1)

Ex4G

13. (a) Factorise completely $x^3 + 10x^2 + 25x$

(2)

(b) Sketch the curve with equation

$$y = x^3 + 10x^2 + 25x$$

showing the coordinates of the points at which the curve cuts or touches the x-axis.

(2)

The point with coordinates $(-3, 0)$ lies on the curve with equation

$$y = (x + a)^3 + 10(x + a)^2 + 25(x + a)$$

where a is a constant.

(c) Find the two possible values of a.

(3)