Chapter 5 - Polar Coordinates

1 Understand and use polar coordinates

2 Convert between polar and Cartesian coordinates

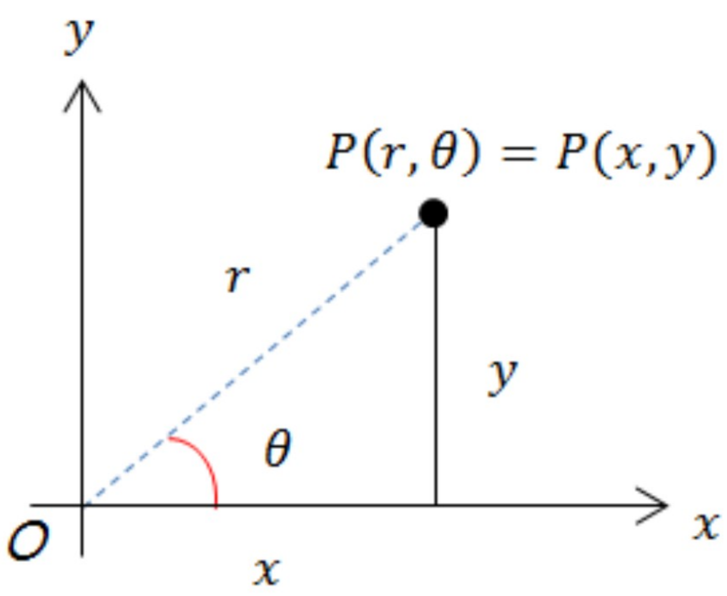
3 Sketch curves with *r* given as a function of θ

4 Find the area enclosed by a polar curve

5 Find tangents parallel to, or at right angles to, the initial line

What are polar coordinates?

You've actually encountered polar coordinates already via complex numbers.
Recall that you could define complex numbers either in Cartesian form, or in '**polar form**' using the distance from the origin and anticlockwise angle from the positive x-axis.



Converting to/from polar coordinates

$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ ← I advise always to draw/imagine a diagram rather than use this

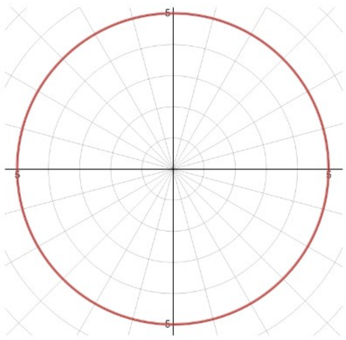
Cartesian	Polar
$(0,2)$	
	$(3,\pi)$
$(1,1)$	
$(-5,12)$	
	$\left(6,-\frac{\pi}{6}\right)$

Polar equation → Cartesian equation

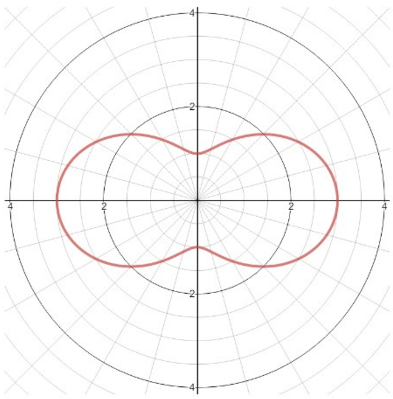
Tip: Use
$r^2 = x^2 + y^2$
$x = r\cos\theta$
$y = r\sin\theta$

Find Cartesian equations for the following:

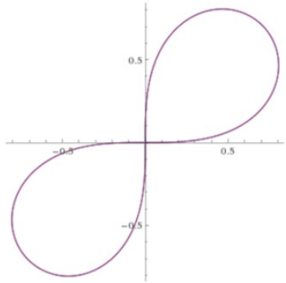
$\boldsymbol{r = 5}$



$\boldsymbol{r = 2 + \cos 2\theta}$

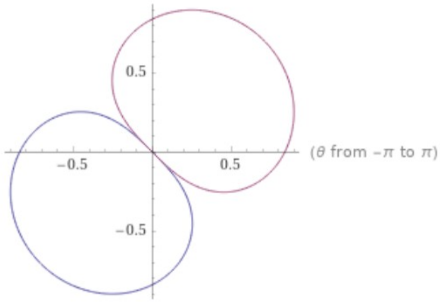


$$r^2 = \sin 2\theta$$

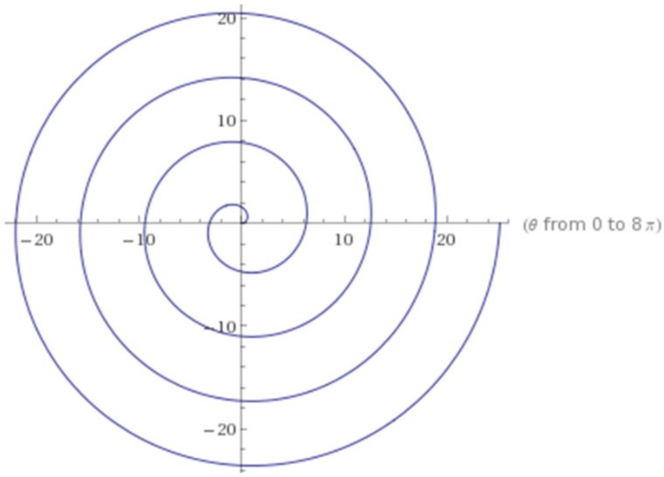


Tip: Use
$r^2 = x^2 + y^2$
$x = r\cos\theta$
$y = r\sin\theta$

$$r^2 = \sin(\theta + \frac{\pi}{4})$$



Why is polar form useful?



Think why...

This spiral pattern has the very simple polar equation $\boldsymbol{r = \theta}$.
In Cartesian form:

$$\sqrt{x^2 + y^2} = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

which is terrible to work with.

Cartesian equation → Polar equation

Converting to polar is easier, but the harder part is often finding how to simplify the expression. Know your double angle formulae!

Tip: Use
$r^2 = x^2 + y^2$
$x = r\cos\theta$
$y = r\sin\theta$

Find polar equations for the following:

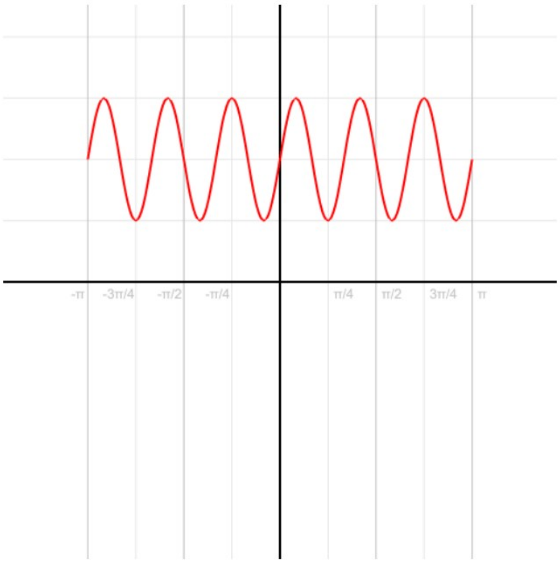
$$\boldsymbol{y^2 = 4x}$$

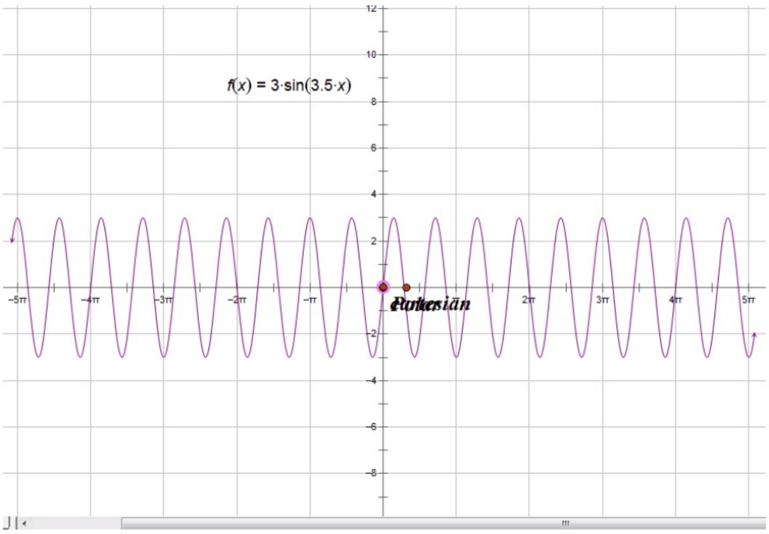
Tip: Polar equations usually start
$r =$ or $r^2 =$

$$\boldsymbol{x^2 - y^2 = 5}$$

$$\boldsymbol{y\sqrt{3} = x + 4}$$

Sketching Curves of Polar Equations





How would you sketch each of the following?

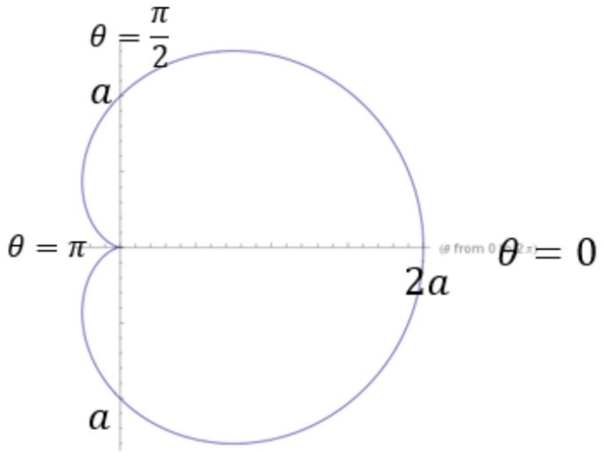
$r = a$	$\theta = \alpha$	$r = a\theta$

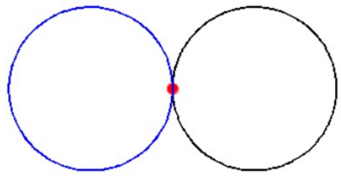
Sketching using tables of values

$$r = a(1 + \cos\theta)$$

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\boldsymbol{r}$	$2a$	a	0	a	$2a$

Note: technically polar angles are between $-\pi$ and π, but 0 to 2π keeps things simple, and it of course doesn't matter when generating values.



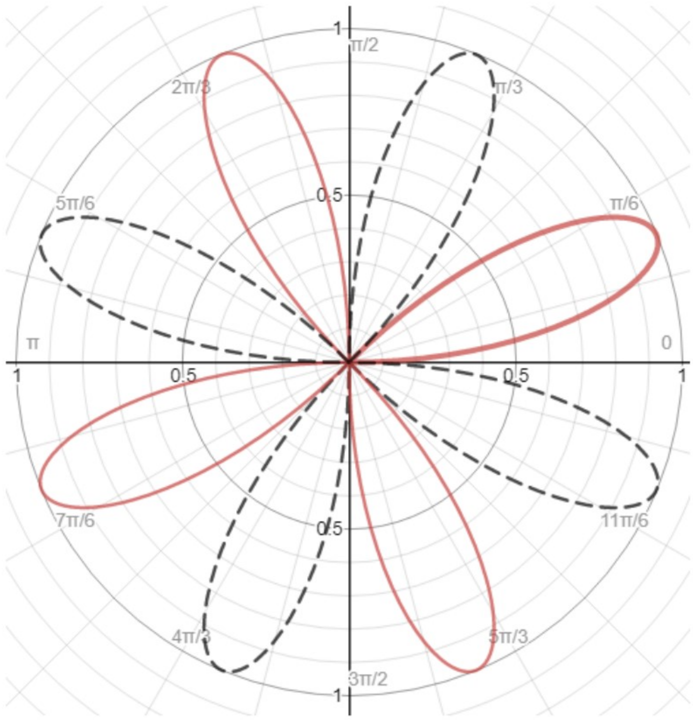


This is a **cardioid**

Important Note: It's <u>usually</u> possible to have negative r (where we'd end up on the opposite side of the origin). However Core Pure Yr2 assumes that we only sketch parts of curves where $r \geq 0$.

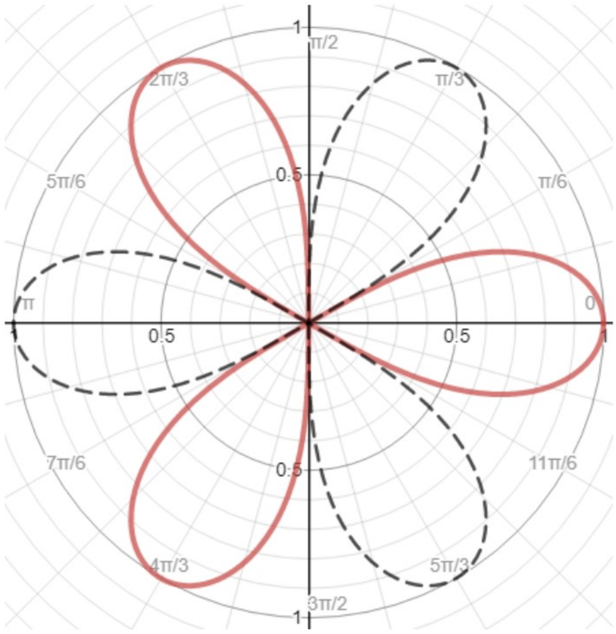
$$r = \sin 4\theta$$

If $r \geq 0$ then $0 \leq 4\theta \leq \pi$ or…



https://www.desmos.com/calculator/dco6wnw5vd

$$r^2 = a^2 \cos 3\theta$$



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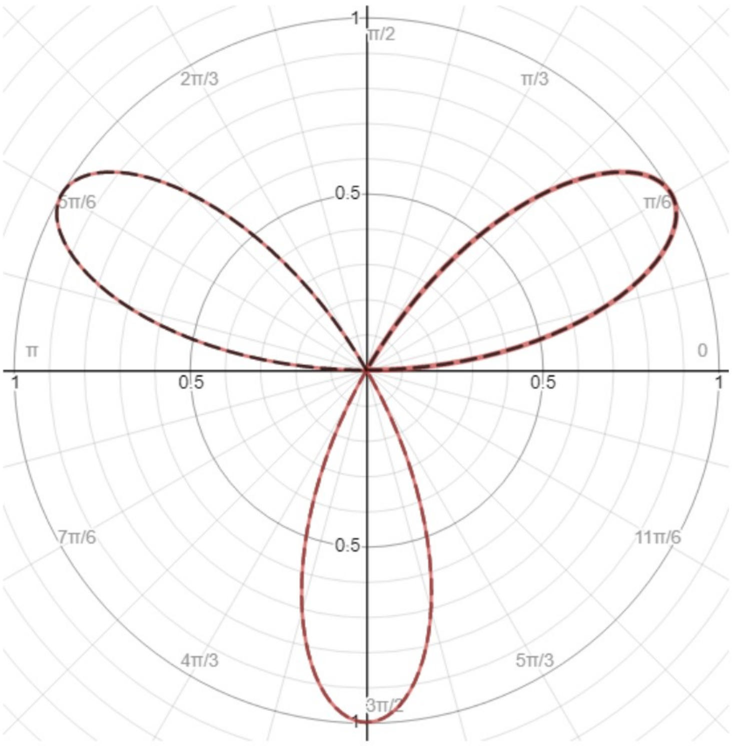
$$r = \sin 3\theta$$



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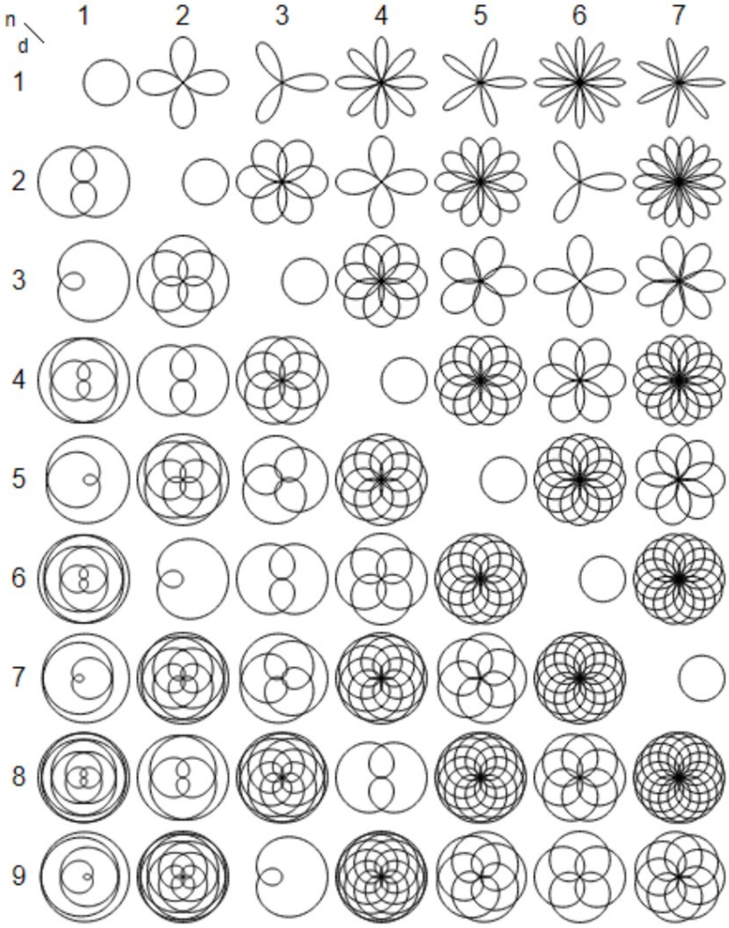
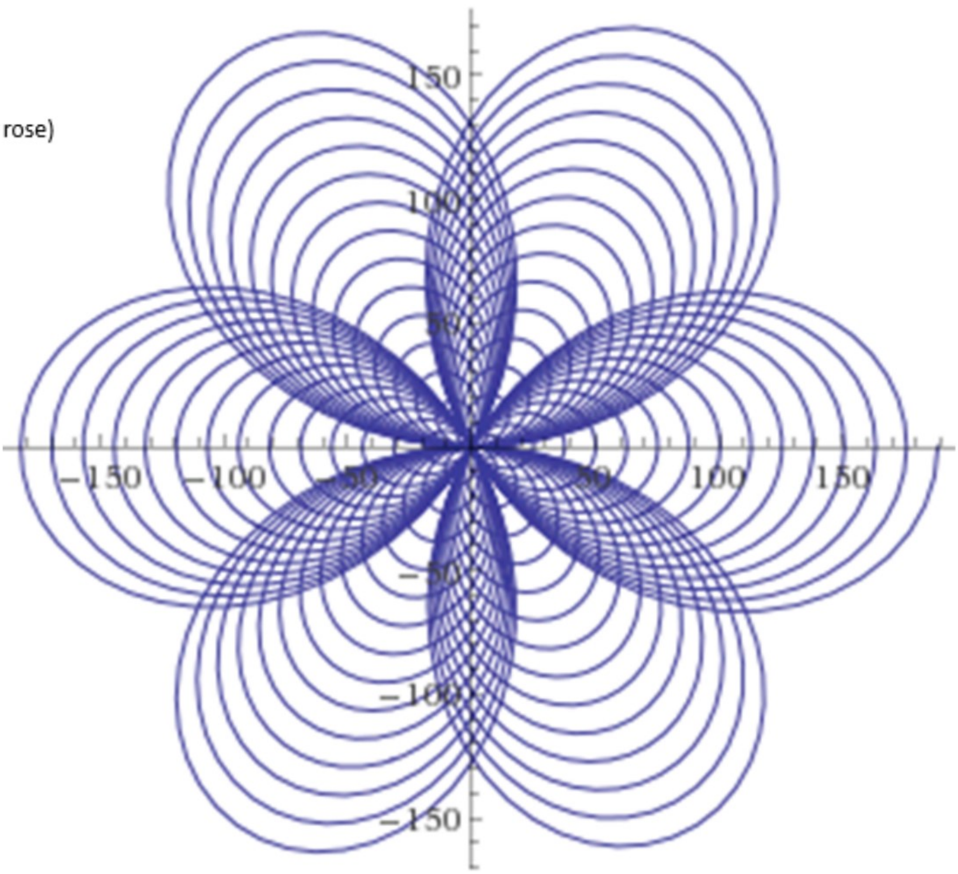
Rose curves defined by $r = \cos k\theta$, for various values of k=n/d.



$$\boxed{\boldsymbol{r = \theta \cos\left(\frac{3}{2}\theta\right)}}$$

(This is a spiral combined with a polar rose)

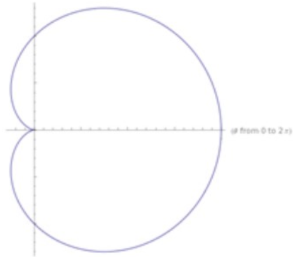


Egg vs Dimple

$$r = a(p + q\cos\theta)$$

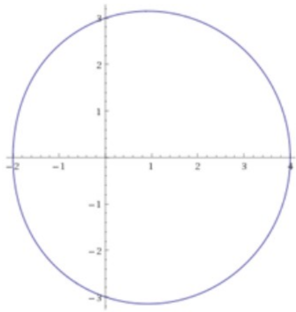
If we require that $r \geq 0$ then the curve is defined for all θ if:

$\boldsymbol{p \geq q}$



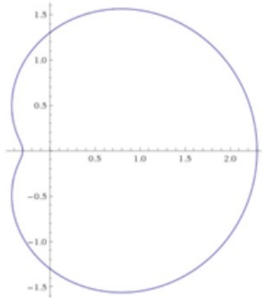
Case 1

When $p = q$ we get a cardioid (where the curve reaches the origin when $\theta = \pi$



Case 2

When $p \geq 2q$ we get an egg/oval shape
(and if $q = 0$, a circle centred at the origin).



Case 3

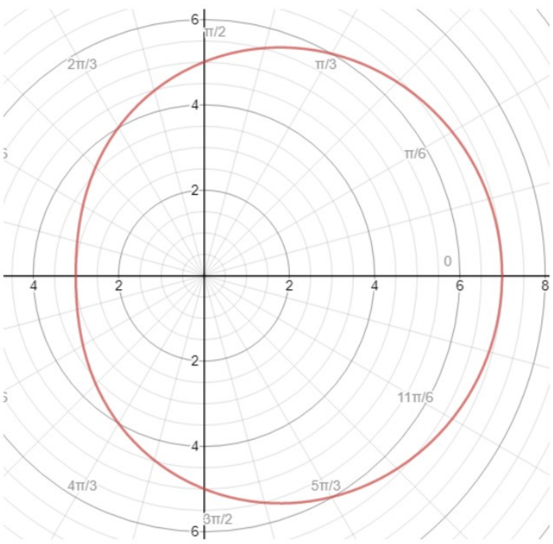
When $q < p < 2q$ we get dimple shape
(as with a cardioid, although here the curve will never be at the origin because $r > 0$ and not equal to 0).

(We will see why we get the 'egg' vs 'dimple' later.)

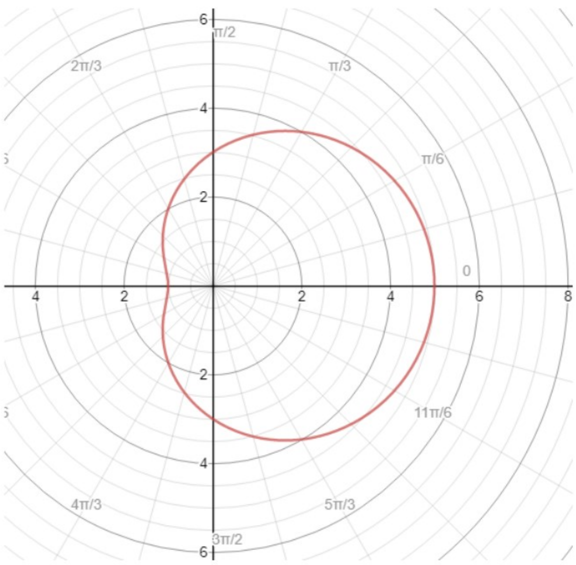
$$r = a(p + q\cos\theta)$$

Egg if $p \geq 2q$

Sketch $r = a(5 + 2\cos\theta)$

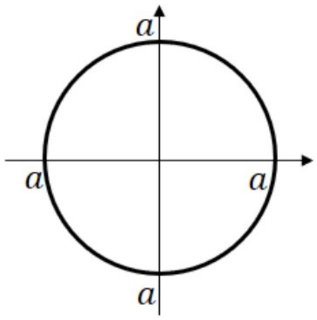


Sketch $r = a(3 + 2\cos\theta)$

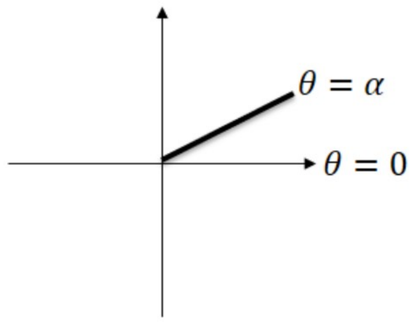


Summary so far

$$r = a$$

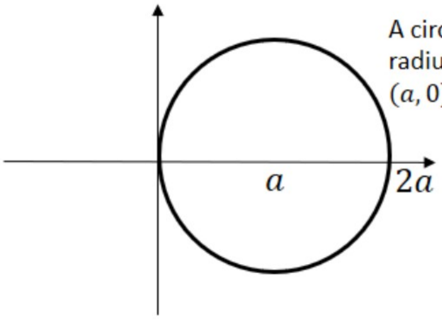


$$\theta = \alpha$$



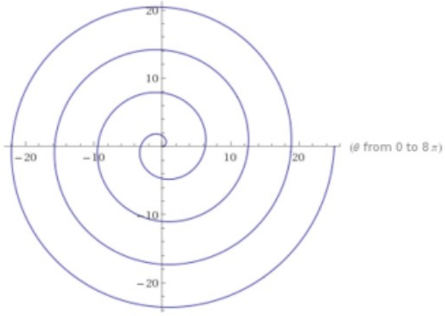
You can prove these by converting equation to Cartesian.

$$r = 2a\cos\theta$$



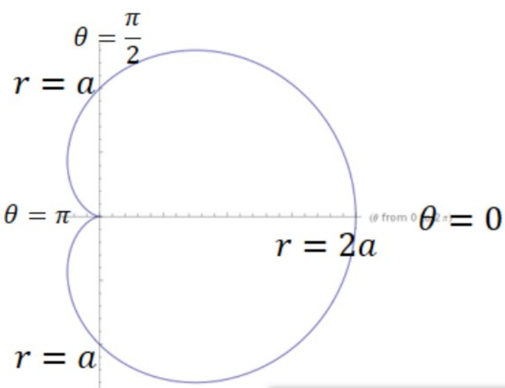
A circle of radius a centre $(a, 0)$.

$$r = k\theta$$

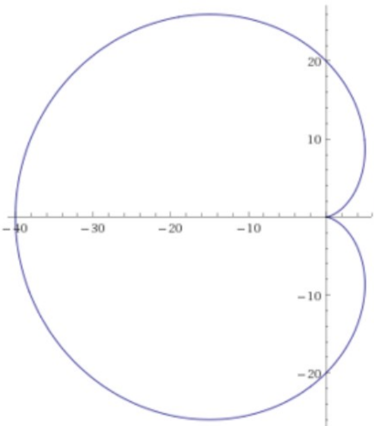


$$r = a(1 + \cos\theta)$$

(special name: cardioid)

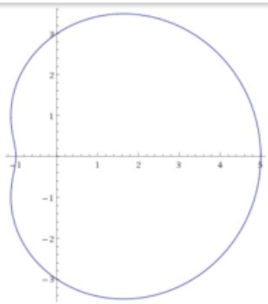


$$r = a(1 - \cos\theta)$$



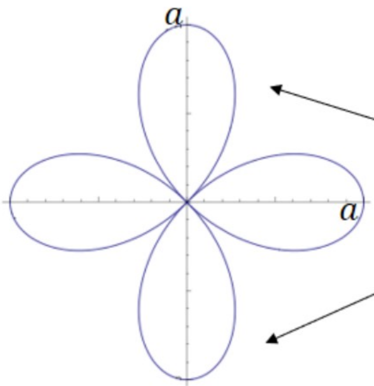
Think about it: now when $\theta = 0$, $1 - cos\theta = 0$ so we start at the origin. And when $\theta = \pi$, r will be at its maximum.

$$r = a(3 + 2\cos\theta)$$



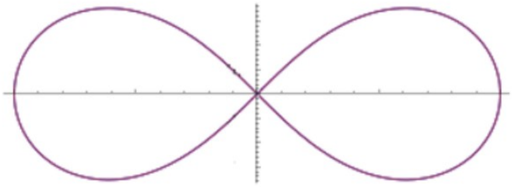
$p < 2q$ therefore dimpled.

$$r = a\cos 2\theta$$



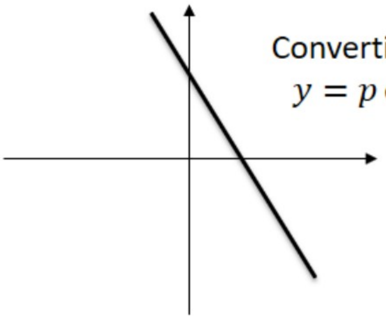
However because CP2 requires that $r \geq 0$, we won't see top and bottom petals.

$$r^2 = a^2\cos 2\theta$$



However because the LHS is squared $\therefore$ positive, it forces the RHS to be positive, so regardless of whether we restrict $r > 0$, those other two petals won't be there.

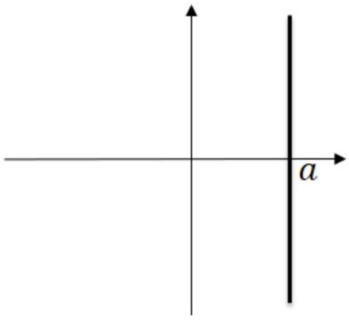
$$r = p\sec(\alpha - \theta)$$



Converting to Cartesian:

$$y = p\,\mathrm{cosec}\,\alpha - \cot\alpha\ x$$

$$r = a\sec(\theta)$$



$x = r\cos\theta$

$\therefore r = x\sec\theta$

$\therefore x\sec\theta = a\sec\theta$

$\therefore x = a$

Ex 5B Q1-2

Connections to Argand Diagrams

(a) Show on an Argand diagram the locus of points given by the values of z satisfying

$|z - 3 - 4i| = 5$

(b) Show that this locus of points can be represented by the polar curve $r = 6\cos\theta + 8\sin\theta$

Ex 5B Q3-4

Integration

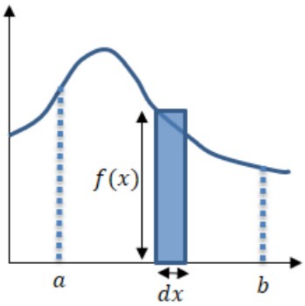
When integrating normal Cartesian 2D areas, we know we're summing a bunch of infinitely thin rectangles:

Area of each rectangle:

$$= f(x)\,dx$$

Adding them all for area:

$$\int_a^b f(x)\,dx$$



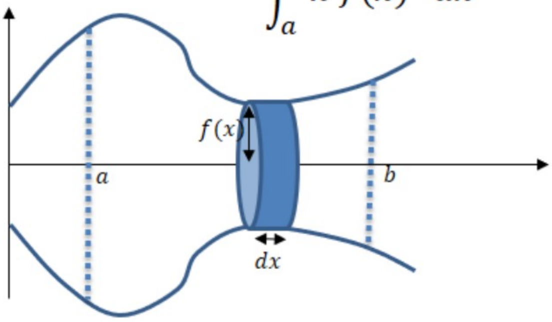
Similarly in Core Pure Years 1 and 2, we could get a volume of revolution by summing the volumes of infinitely thin cylinders:

Volume of each cylinder

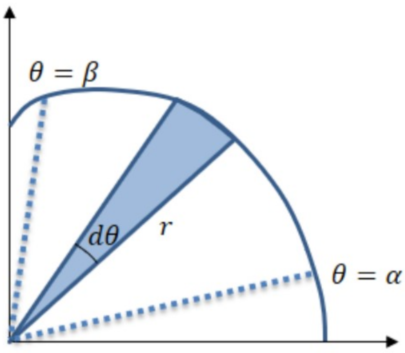
$$= \pi\, f(x)^2\,dx$$

Adding them all for total volume:

$$\int_a^b \pi\, f(x)^2\,dx$$



Can we apply to same principle to find the area bound between a polar curve and two half lines $\theta = \alpha$ and $\theta = \beta$?



Area of each sector:

$$= \frac{1}{2} r^2 d\theta$$

Adding them all for total area:

$$\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

$$\boxed{A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta}$$

Integration Recap

$$\int \cos\theta \, d\theta$$

$$\int \sin 3\theta \, d\theta$$

$$\int (\sin 3\theta - 2\cos 5\theta) d\theta$$

Rearranged Double Angle: You should memorise:
$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$\int \cos^2\theta \, d\theta$$

$$\int \sin^2\theta \, d\theta$$

$$\int \sin^2 5\theta \, d\theta$$

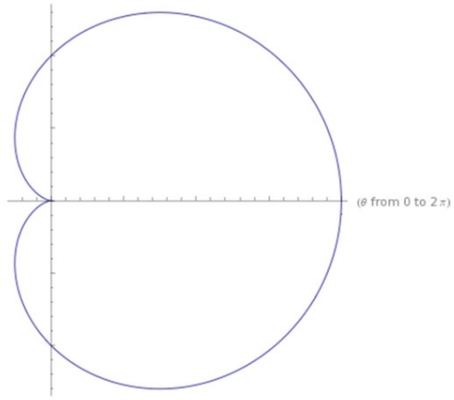
$$\int (1 + \cos 2\theta)^2 \, d\theta$$

$$\int (1 + \sin 3\theta)^2 \, d\theta$$

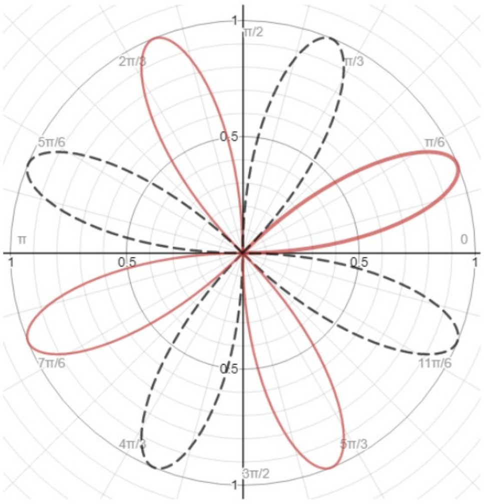
$$\int (2 + 3\cos 5\theta)^2 \, d\theta$$

Areas enclosed by polar curves

Find the area enclosed by the cardioid with equation
$r = a(1 + \cos\theta)$



Find the area of one loop of the polar rose $r = a\sin 4\theta$



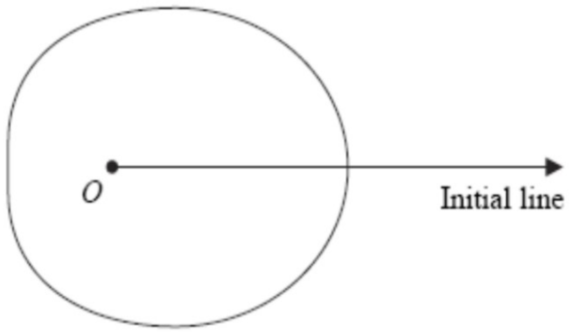


Figure 1 shows a sketch of the curve with polar equation

$$r = a + 3\cos\theta, \qquad a > 0, \qquad 0 \le \theta < 2\pi$$

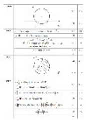
The area enclosed by the curve is $\frac{107}{2}\pi$.

Find the value of a.

(8)

Ex 5C Q1-4

Areas on Argand Diagrams



6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathrm{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

The set of points A is defined by

$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \left\{z : |z - 4 - 3\mathrm{i}| \leqslant 5\right\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

Ex 5C Q7-8

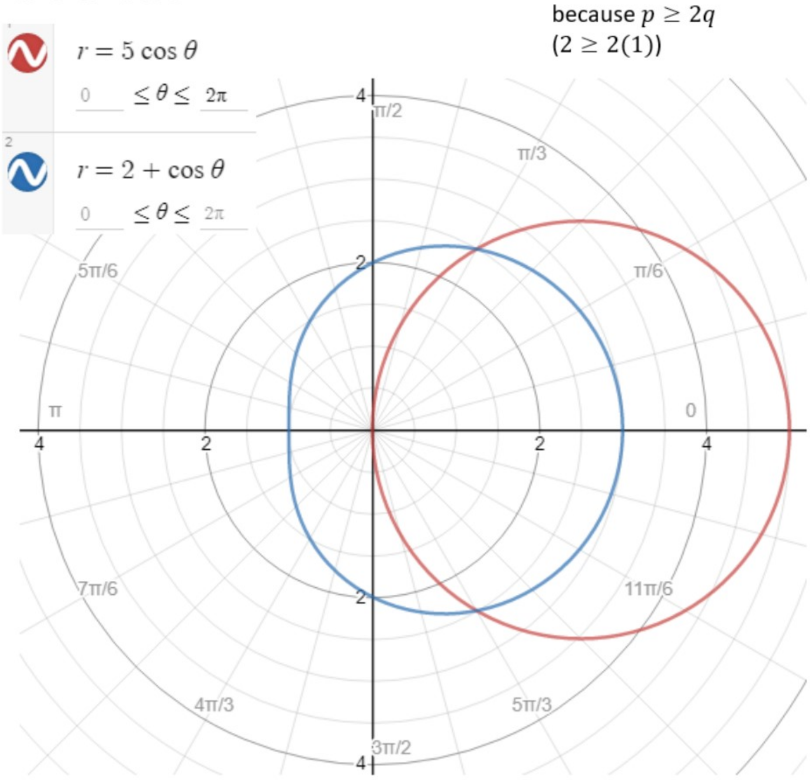
Intersecting/Complicated Areas

No dimple because $p \geq 2q$
$(2 \geq 2(1))$

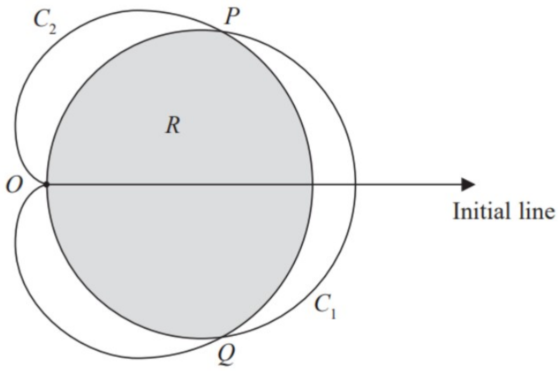
(a) The diagram shows the curves with equations $r = 2 + \cos\theta$ and $r = 5\cos\theta$

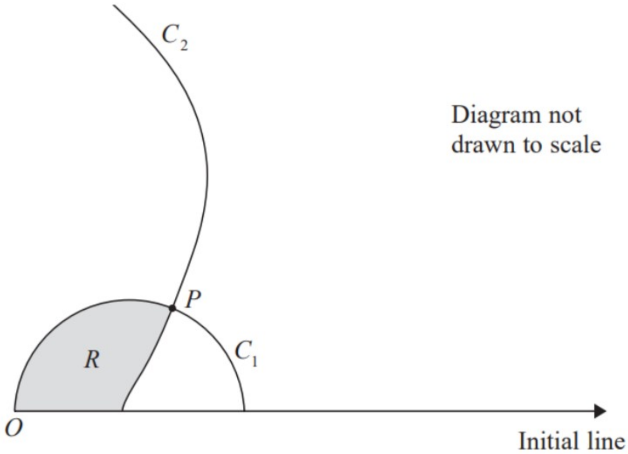
(b) Find the polar coordinates of the points of intersection of these two curves.

(c) Find the exact value of the area of the finite region bound between the two curves.



What strategy would you use to find the shaded area?





What strategy would you use to find the shaded area?

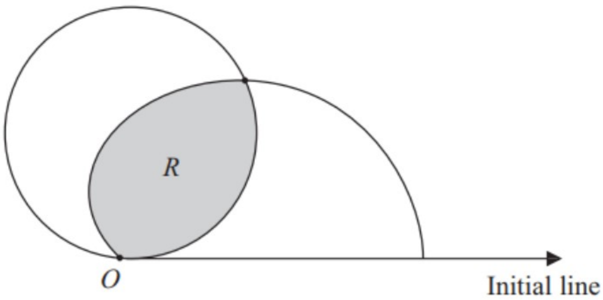


Figure 1

Figure 1 shows the two curves given by the polar equations

$$r = \sqrt{3}\sin\theta, \quad 0 \leqslant \theta \leqslant \pi$$
$$r = 1 + \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$$

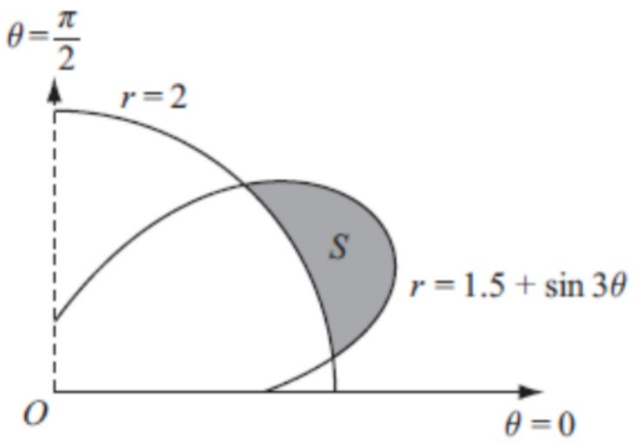
Figure 1 shows the curves given by the polar equations

$$r = 2, \qquad 0 \le \theta < \frac{\pi}{2}$$

$$r = 1.5 + \sin 3\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$

(a) Find the coordinates of the points where the curves intersect. **(3)**

(b) The region S for which $r > 2$ and $r < 1.5 + \sin 3\theta$ is shown. Find, by integration, area of S giving your answer in the form $a\pi + b\sqrt{3}$ where a and b are simplified fractions. **(7)**



Ex 5C Q5-6

What strategy would you use to find the shaded area?

$r = 7.5 + 1.5\cos 6\theta$

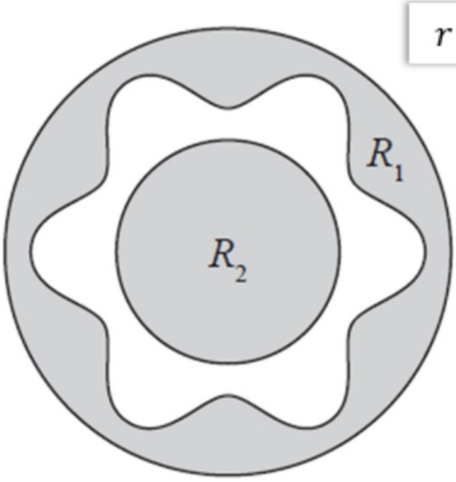
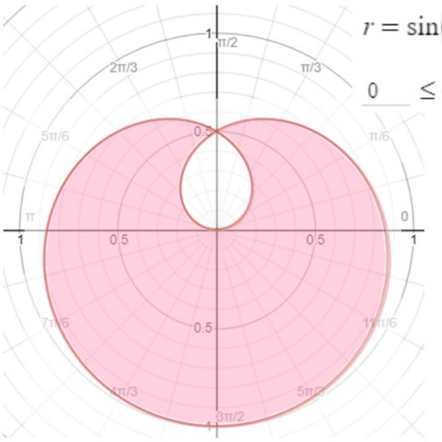


Figure 1

What strategy would you use to find the shaded area?

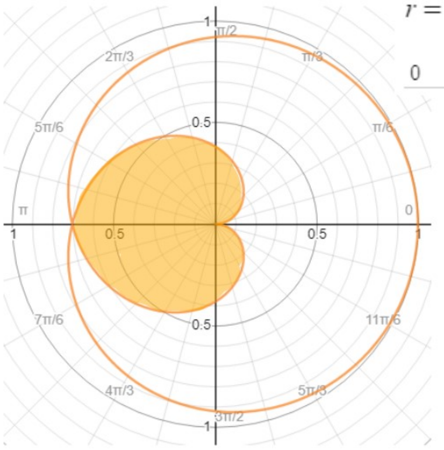
$r = \sin\left(\frac{\theta}{3}\right)$

$0 \leq \theta \leq 3\pi$

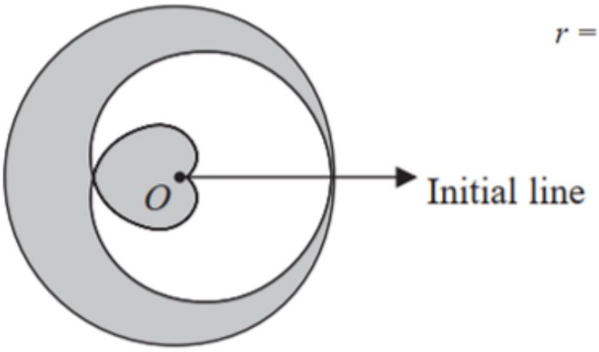


$r = \sin\left(\frac{\theta}{4}\right)$

$0 \leq \theta \leq 4\pi$



What strategy would you use to find the shaded area?

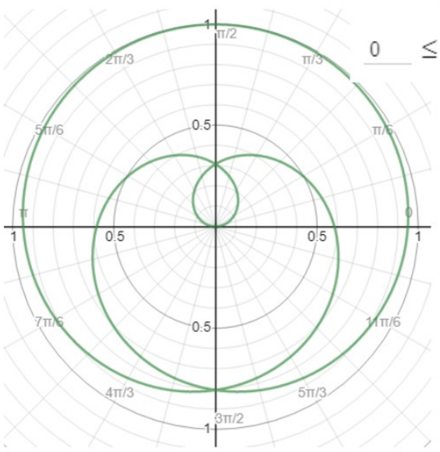


$$r = \sin\left(\frac{\theta}{6}\right) \qquad 0 \leqslant \theta \leqslant 6\pi$$

Shade part of these and come up with your own strategy

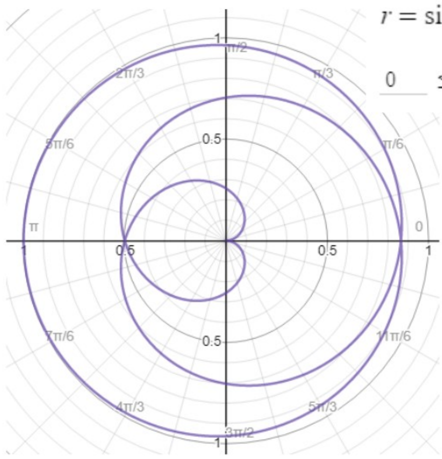
$$r = \sin\left(\frac{\theta}{5}\right)$$

$0 \leq \theta \leq 5\pi$

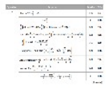


$$r = \sin\left(\frac{\theta}{6}\right)$$

$0 \leq \theta \leq 6\pi$



Ex 5C Q9-10
Mixed Ex, Q9, 12, 13



4.

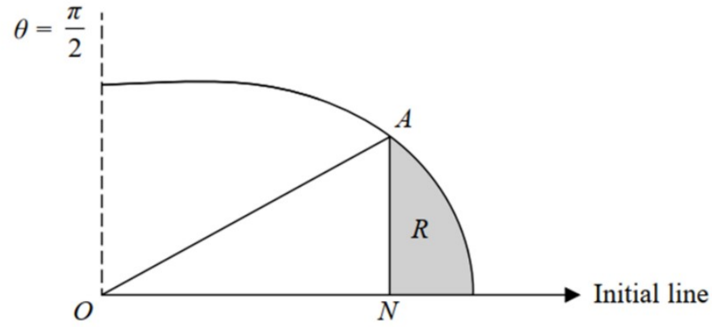


Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 4 + \cos 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{9}{2}$

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

Find the exact area of the shaded region R, giving your answer in the form $p\pi + q\sqrt{3}$ where p and q are rational numbers to be found.

(9)

3.

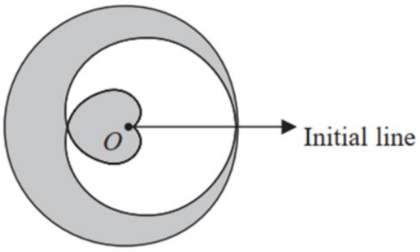


Figure 1

Figure 1 shows a sketch for the design of a logo. The logo is defined by the polar curve with equation

$$r = \sin\left(\frac{\theta}{6}\right) \qquad 0 \leqslant \theta \leqslant 6\pi$$

The inner closed section and outer closed section of the curve, shown shaded in Figure 1, are to be coloured the same colour. The remaining section is to be left clear.

(a) Use algebraic integration to find the area of the coloured sections of the logo.

(6)

A copy of this logo is to be painted on a white wall of a building such that the total width of the logo is 12 m.

Tins of coloured paint with an advertised minimum coverage area of 30 m^2 are to be used to paint the coloured sections of the logo onto the wall. Given that two coats of paint will be needed,

(b) find the minimum number of tins of this paint that should be bought to ensure that the coloured sections of the logo can be painted onto the wall.

(4)

Figure 1 shows the design for a new type of security wheel nut for a car.
The inner circle has a radius of 5 mm and the outer circle has a radius of 10 mm.
The curve, C, between the two circles, is modelled by the polar equation

$$r = 7.5 + 1.5\cos 6\theta \qquad 0 \leqslant \theta < 2\pi$$

where r is measured in millimetres.

The regions R_1 and R_2 are shown shaded in Figure 1 and both regions must be coated in a special paint.

The region R_1 is enclosed between the outer circle and C.

The region R_2 is enclosed by the inner circle.

Find the area that must be coated in the special paint, according to the model.

Give your answer in cm² to 2 decimal places.

[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(7)

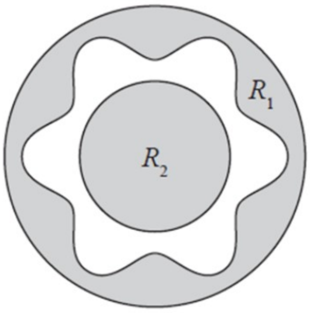


Figure 1

Tangents and Normals

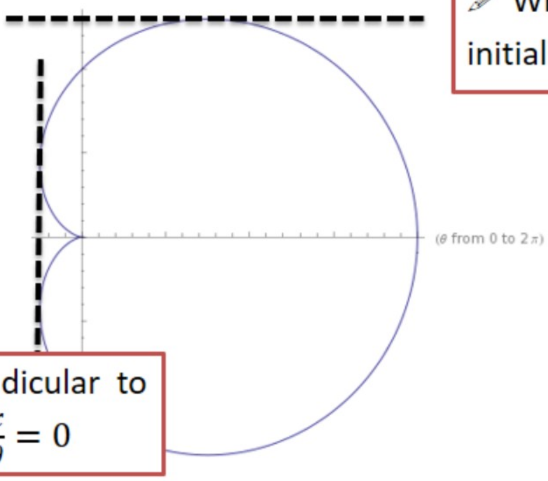
Remember how you found the gradient given equations in parametric form?

e.g. $x = \cos\theta\,,\ y = \sin\theta$

$$\frac{dy}{dx} = \frac{\left(\boldsymbol{\frac{dy}{d\theta}}\right)}{\left(\boldsymbol{\frac{dx}{d\theta}}\right)} = \frac{\boldsymbol{\cos\theta}}{\boldsymbol{-\sin\theta}} = \boldsymbol{-\cot\theta}$$

We know that in the polar world:

$$x = r\cos\theta \qquad y = r\sin\theta$$



When parallel to the initial line, $\frac{dy}{d\theta} = 0$

When perpendicular to the initial line, $\frac{dx}{d\theta} = 0$

$$\frac{dy}{dx} = \frac{\left(\boldsymbol{\frac{dy}{d\theta}}\right)}{\left(\boldsymbol{\frac{dx}{d\theta}}\right)}$$

Find the coordinates of the points on $r = a(1 + \cos\theta)$ where the tangents are parallel to the initial line $\theta = 0$.

Since we're about to find $\frac{dy}{d\theta}$, start with $y = r\sin\theta$
and ensure expression is only in terms of θ.

The curve C has polar equation

$$r = 1 + 2\cos\theta\,, \qquad 0 \le \theta \le \frac{\pi}{2}$$

At the point P on C, the tangent to C is parallel to the initial line.
Given that O is the pole, find the exact length of the line OP.

Find the equations and the points of contact of the tangents to the curve

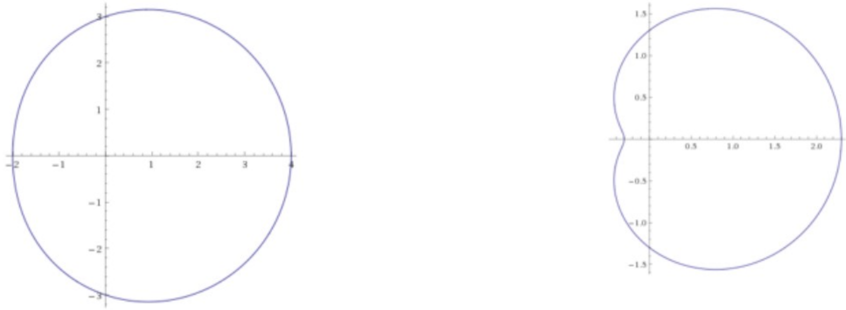
$$r = a \sin 2\theta, \qquad 0 \le \theta \le \frac{\pi}{2}$$

that are (a) parallel to the initial line and (b) perpendicular to the initial line.

Ex 5D

Proof of dimple vs egg

Prove that for $r = p + q\cos\theta$ we have a 'dimple' if $p < 2q$.



What's the difference in terms of number of tangents perpendicular to the initial line?

The first has 2 tangents perpendicular to the initial line. The second has 3!

$$x = r\cos\theta = (p + q\cos\theta)\cos\theta = p\cos\theta + q\cos^2\theta$$
$$\frac{dx}{d\theta} = 0 \Rightarrow \ -p\sin\theta - 2q\sin\theta\cos\theta = 0$$
$$-\sin\theta\,(p + 2q\cos\theta) = 0$$
$$\sin\theta = 0 \ \ or \cos\theta = -\frac{p}{2q}$$

The first corresponds to the left and right-most tangents (where $\theta = 0$ and $\theta = \pi$).
If $p > 2q$ then $\cos\theta < -1$ which has no solutions, so there will be no other tangent.
If $p = 2q$ then $\cos\theta = -1$ which gives us $\theta = \pi$, i.e. there is no extra solution.