Circuit Theorems

Dr. Albert T. L. Lee

Department of Electrical and Electronic Engineering
Faculty of Engineering
The University of Hong Kong

Circuit Theorems

I. Linear and Superposition

II. Source Transformation

III. Thevenin’s Theorem

IV. Norton’s Theorem

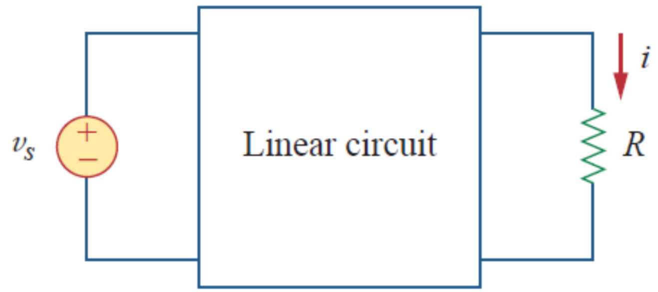
Introduction

- In the last session, nodal and mesh methods are used to analyze circuits with 3 or 4 unknown node voltages or mesh currents.
- For circuits with increased complexity, these approaches are more difficult to use as they involve tedious calculations.
- Engineers have developed some principles and theorems to simplify the circuit analysis, e.g. linearity, superposition, source transformation, Thevenin's and Norton's theorems.
- Linearity is the property of an element describing a linear relationship between cause and effect. For example, Ohm's law states that $v = iR$, for a constant R (resistance). It shows a linear relationship between voltage (v) and current (i) for a resistor.

Linearity

- A mathematical function $f(x)$ is linear if $f(k_1x_1+ k_2x_2)= k_1f(x_1) + k_2f(x_2)$
- e.g., $f(x) = 2x$ is linear since $f(k_1x_1+ k_2x_2) = 2(k_1x_1+ k_2x_2) = k_1(2x_1)+ k_2(2x_2) = k_1f(x_1) + k_2f(x_2)$
- Physical meaning of linearity: If input is doubled, output will be doubled.
- A circuit element is linear if its *v-i* relationship $v=f(i)$ satisfies $f(k_1i_1+ k_2i_2)= k_1f(i_1) + k_2f(i_2)$.
- e.g., for a resistor, the *v-i* relationship (i.e., Ohm's law) is $v = iR$, which satisfies the linear property. Therefore, **resistor is a linear circuit element.**
- Capacitor and inductor (to be discussed later) are also linear elements.
- A linear circuit is one whose output is linearly related to its input.

Linearity (cont'd)



In the above circuit diagram, the linear circuit is represented by a black box. Assume it contains no independent sources.

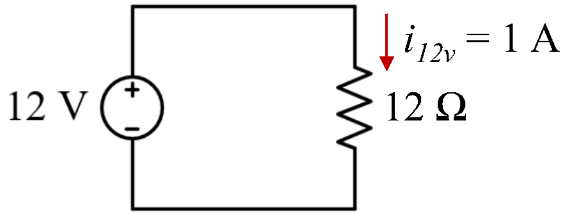
Now, the linear circuit is excited by a voltage source v_s, where $v_s = 10$ V. The current i across the resistor (R) is measured to be 2 A.

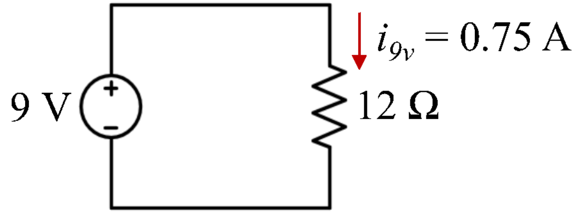
According to the linearity principle, $v_s = 1$ V produces $i = 0.2$ A across R. Conversely, $i = 10$ mA corresponds to $v_s = 50$ mV.

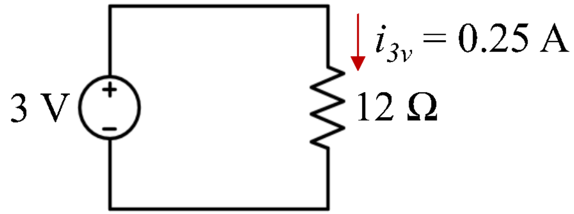
In general, a linear circuit consists of only linear elements and linear dependent sources.

Simple Illustration of Linearity

$f(x+y) = f(x) + f(y)$





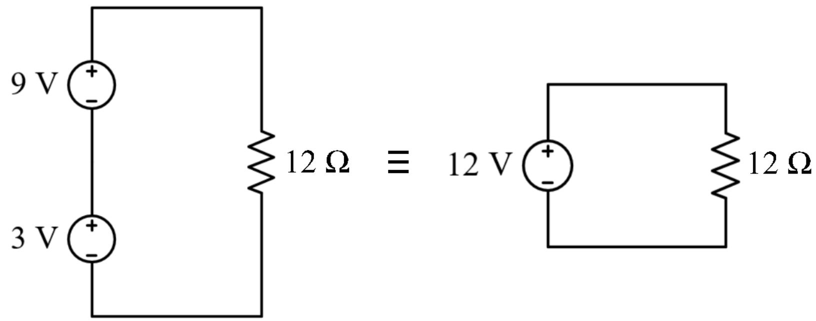


$f(12) = v/R = 12/12 = 1\ A$

$f(9) = v/R = 9/12 = 0.75\ A$

$f(3) = v/R = 3/12 = 0.25\ A$

Hence, $f(12) = f(9+3) = f(9) + f(3) = 1\ A$



Example (1): Linearity

Question: In Fig. 1, find I_o when (i) $v_s = 12$ V and (ii) $v_s = 24$ V.

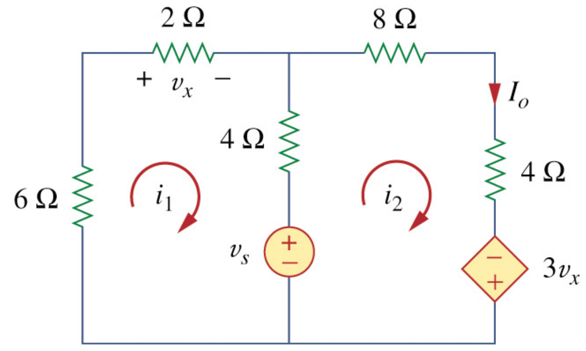


Fig. 1

Ans:

By applying KVL to the two loops, we have

$$12i_i - 4i_2 + v_s = 0 \qquad (1)$$

$$-4i_i + 16i_2 - 3v_x - v_s = 0 \qquad (2)$$

Since $v_x = 2i_1$, equation (2) can therefore be written as:

$$-10i_i + 16i_2 - v_s = 0 \qquad (3)$$

By solving (1) and (3), we have $i_2 = \dfrac{v_s}{76}$ (4)

Since $I_o = i_2$, $v_s = 12$ V => $I_o = 12/76 \approx 0.158$ A

Similarly, when $v_s = 24$ V, $I_o = 24/76 \approx 0.316$ A

Hence, when v_s is doubled, I_o is also doubled.

Ramification of Linearity

- As seen in the previous example, every voltage (or current) is proportional to the independent source voltage (v_s) because the circuit is linear.
- If a circuit contains two or more independent sources, one way to determine the value of a specific variable (either voltage or current) is to use nodal or mesh analysis, as discussed in the previous session titled "Methods of Analysis".
- Another way is to determine the contribution of each independent source to the variable and then add them up. This is known as **superposition**.

Superposition

- The superposition principle states that the voltage (or current) across an element in a **linear circuit** is **the algebraic sum** of the voltages (or current) across that element due to each independent source acting alone (turning off the rest of the sources).
- When applying the principle of superposition, consider <u>one independent source at a time</u> while the other **voltage sources** are replaced by **short circuit** and all other **current sources** are replaced by **open circuit**.
- Note that the <u>dependent sources are left intact</u>.

Example (2): Superposition

Question: In Fig. 2(a), find i_o by using the superposition theorem.

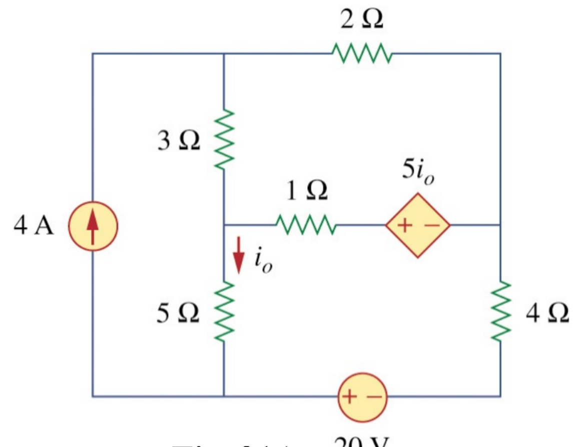


Fig. 2(a)

Ans:

Let $i_o = i_o' + i_o''$, where i_o' and i_o'' are due to the 4 A current source and 20 V voltage source, respectively. The dependent source is left intact.

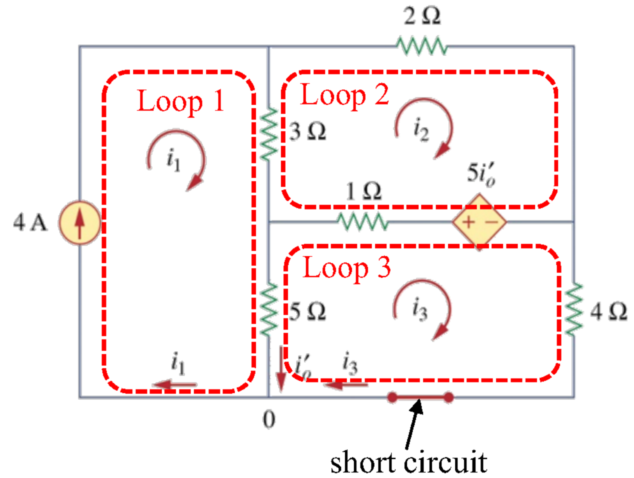
First, to obtain i_o', we turn off the 20 V source (i.e., replace it by a short circuit).

We apply mesh analysis in order to obtain i_o'

For loop 1, $i_1 = 4$ A (1)

For loop 2, $-3i_1 + 6i_2 - i_3 - 5i_o' = 0$ (2)

For loop 3, $-5i_1 - i_2 + 10i_3 + 5i_o' = 0$ (3)



It seems that we only have three equations with four unknowns!

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Example (2): Superposition (cont'd)

But, if we take a closer look at node 0, we can obtain another equation, i.e.,

$i_3 = i_1 - i_o' = 4 - i_o'$ (4)

Now, by substituting (1) and (4) into (2) and (3), the following two simultaneous equations can be written.

$$\begin{cases} 3i_2 - 2i_o' = 8 \\ i_2 + 5i_o' = 20 \end{cases}$$

By solving these equations, we have $i_o' = 52/17 \approx 3.06$ A

Second, to obtain i_o'', we turn off the 4 A current source (i.e., replace it by an open circuit), as shown in Fig. 2(b).

For loop 4, KVL gives $6i_4 - i_5 - 5i_o'' = 0$ (5)

For loop 5, KVL gives $-i_4 + 10i_5 - 20 + 5i_o'' = 0$ (6)

But, note that $i_o'' = -i_5$ (7)

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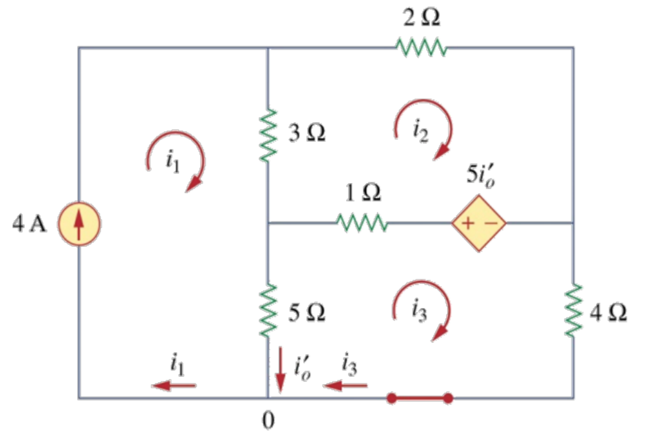


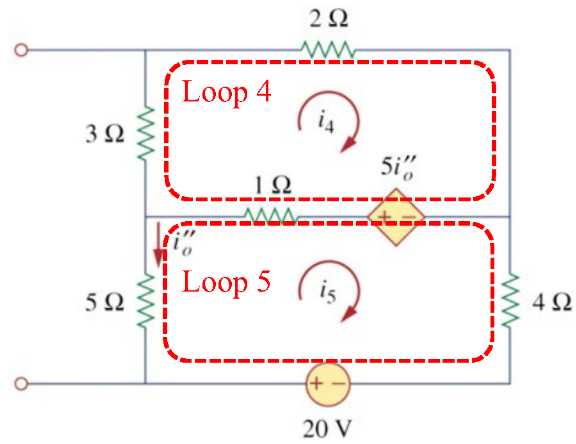


Fig. 2(b)

Example (2): Superposition (cont'd)

Equation (5), (6) and (7) are repeated here for convenience.

$$6i_4 - i_5 - 5i_o'' = 0 \qquad (5)$$

$$-i_4 + 10i_5 - 20 + 5i_o'' = 0 \qquad (6)$$

$$i_o'' = -i_5 \qquad (7)$$

By substituting (7) into (5) and (6), we have

$$\begin{cases} 6i_4 - 4i_o'' = 0 \\ i_4 + 5i_o'' = -20 \end{cases}$$

By solving these equations, $i_o'' = -60/17 \approx -3.53$ A

Therefore, $i_o = i_o' + i_o'' = 3.06 - 3.53 = -0.47$ A

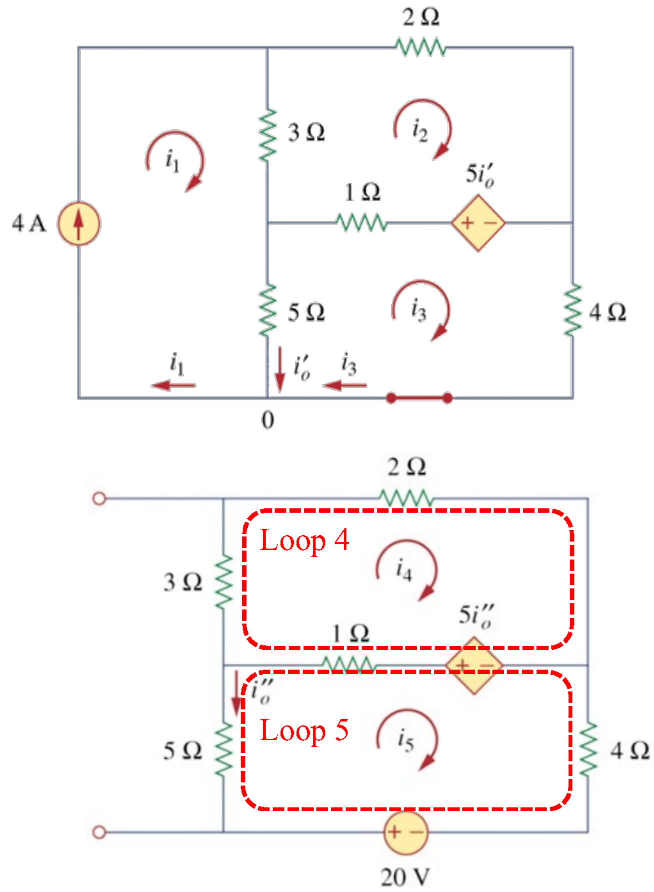
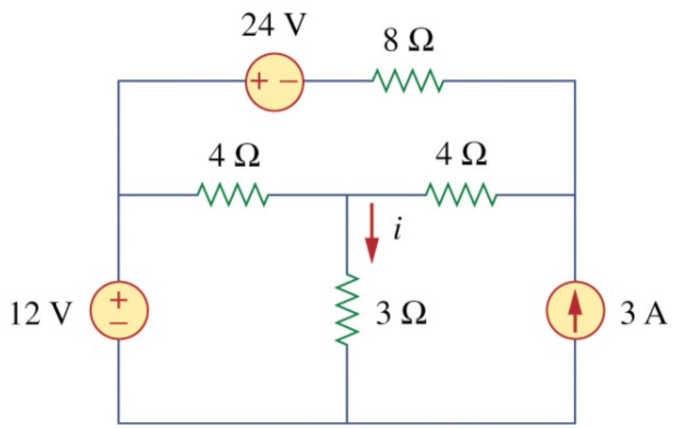


Fig. 2(b)

Example (3): Superposition

Question: Use the superposition theorem to find the current *i*, as shown in the following circuit.



Ans:

Since there are three independent sources, we can decompose the current i into three components, namely, $i = i_1 + i_2 + i_3$, where i_1, i_2 and i_3 are due to the 12 V, 24 V, and 3 A sources, respectively.

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Example (3): Superposition (cont'd)

- To obtain i_1, consider the circuit shown in Fig. 3(a). The value of i_1 can be easily determined by combining the resistors. So, $i_1 = 2$ A.
- To obtain i_2, consider the circuit in Fig. 3(b). By using mesh analysis, $i_2 = -1$ A.
- To obtain i_3, consider the circuit in Fig. 3(c). By using nodal analysis, $i_3 = 1$ A.
- Hence, $i = i_1 + i_2 + i_3 = 2 - 1 + 1 = \mathbf{2}$ **A**.

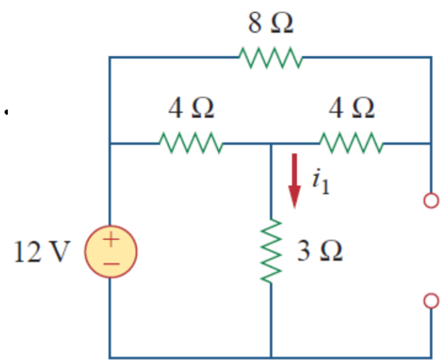
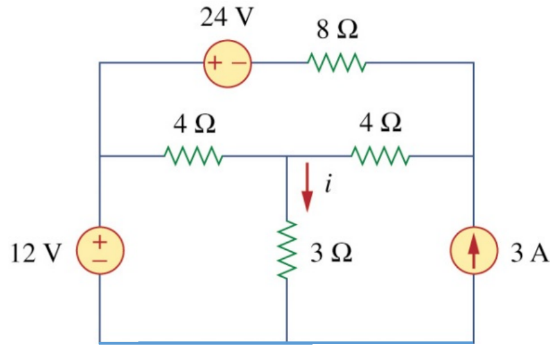


Fig. 3(a)



Original circuit

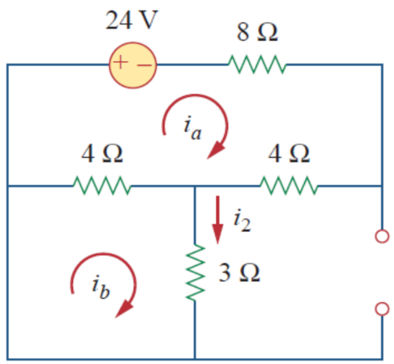


Fig. 3(b)

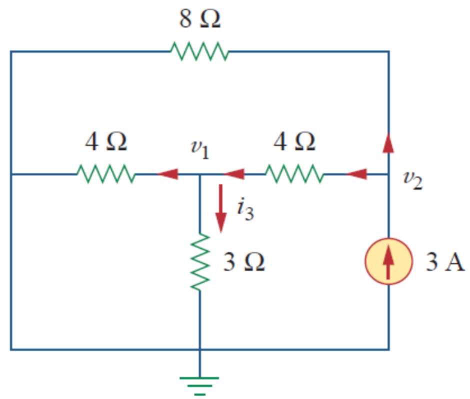


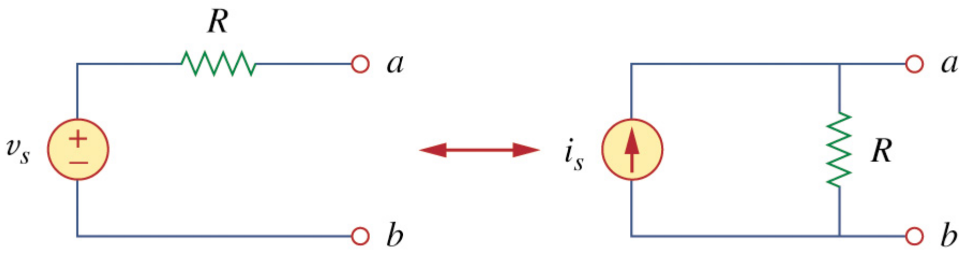
Fig. 3(c)

Summary of Superposition

- Superposition is based on linearity.
- For this reason, it is not applicable to power calculation because the power absorbed by a resistor depends on the square of the voltage (or current).
- If the power value is needed, the voltage (or current) across the circuit element must be calculated first by using superposition.
- Superposition helps reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and current sources by open circuits.

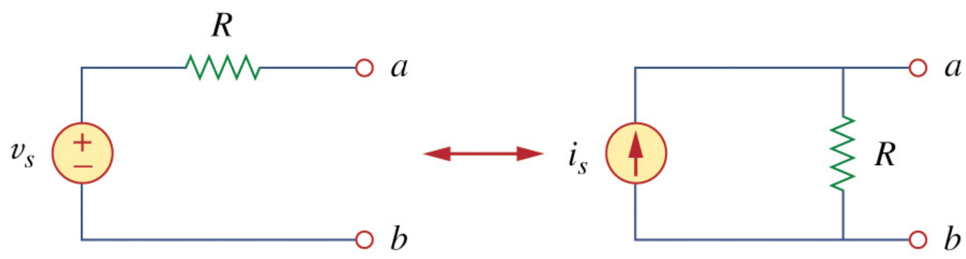
Source Transformation

- Previously, we learned that series-parallel combination and wye-delta transformation helps simplify circuits. Superposition is another method of solving circuit problems.
- Source transformation is another tool for simplifying circuits.
- A source transformation is the process of replacing a voltage source v_s in series with a resistor R by a current source i_s in parallel with a resistor R, or vice versa.



- The two circuits are equivalent if they have the *same* voltage-current relation at terminal *a-b*.
  - If the sources are turned off, the equivalent resistance at terminal *a-b* in both circuits is R.
  - If *a-b* are short-circuited, the short-circuit current flowing from a to b is $i_{sc} = v_s/R$ in the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the right-hand side. In order for the two circuits to be equivalent, we need $v_s/R = i_s$.

Source Transformation (cont'd)



- If *a-b* are not connected, the open-circuit voltage for the left-hand circuit is v_s whereas that for the right-hand circuit is $i_s R$. Therefore, v_s/R = is would make the two circuits the same.

- The arrow of the current source is directed toward the positive terminal of the voltage source.

- Source transformation is *not* possible when $R = 0$, which is the case with an ideal voltage source. In practice, non-ideal voltage source means $R \neq 0$. Likewise, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source.

- Source transformation also applied to dependent sources.

Example (4): Source Transformation

Question: Use source transformation to find v_o in Fig. 4(a).

1. Transform the current and voltages to obtain the circuit shown in Fig. 4(b).
2. Combining the 4 Ω and 2 Ω in series and transforming the 12 V voltage source gives us the circuit in Fig. 4(c).
3. Then we combine the 3 Ω and 6 Ω in parallel, which can be represented by an equivalent resistance of 2 Ω.
4. We can also combine the 2A and 4 A current sources, which is equivalent to the 2 A current source, as shown in Fig. 4(d).
5. From Fig. 4(d), the current division can be applied as follows.

$$i = \frac{2}{2+8}(2) = 0.4 \text{ A}$$

6. Finally, $v_o = 8i = 8(0.4) = 3.2$ V

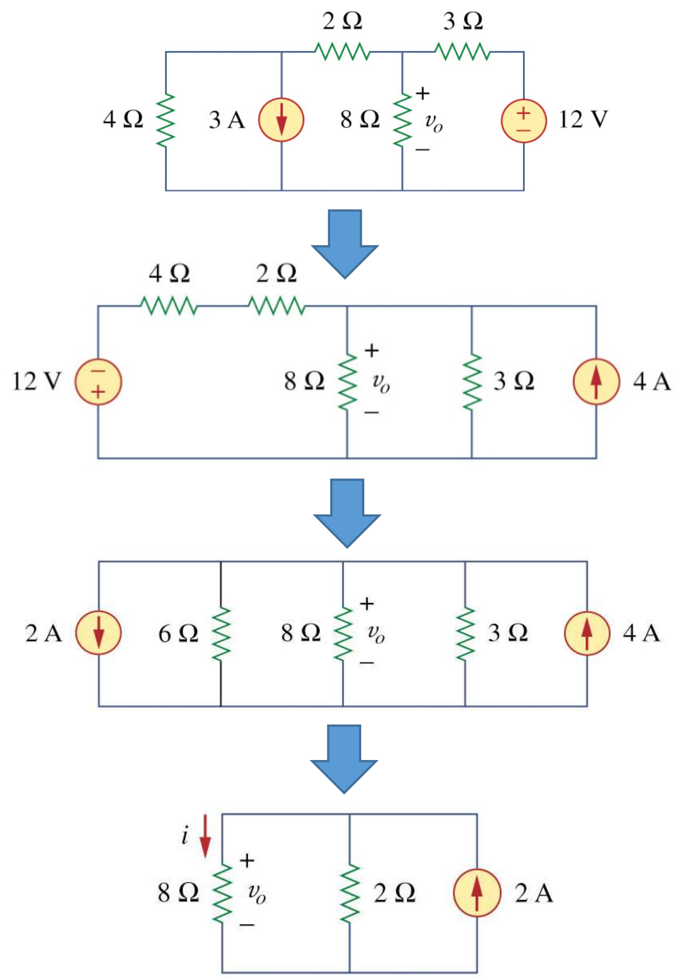


Fig. 4(a)

Fig. 4(b)

Fig. 4(c)

Fig. 4(d)

Example (5): Source Transformation

Question: Find v_x in Fig. 5(a) by employing source transformation.

1. Transform the dependent source and the 6 V independent source. Hence, the circuit in Fig. 5(b) is obtained.
2. The 18 V voltage source is *not* transformed since it is not connected in series with any resistor.
3. The two 2 Ω resistors in parallel combine to give a 1 Ω resistor.
4. Since 1 Ω resistor is in parallel with the 3A current source, they can be transformed into a 3 V voltage source, as shown in Fig. 5(c).
5. By applying KVL around the loop 1 in Fig. 5(c), we have

$$-v_x + 4i + v_x + 18 = 0 \Rightarrow i = -4.5 \text{ A}$$

6. Finally, $v_x = 3 - i(1) = 3 - (-4.5) = 3 + 4.5 = 7.5$ V

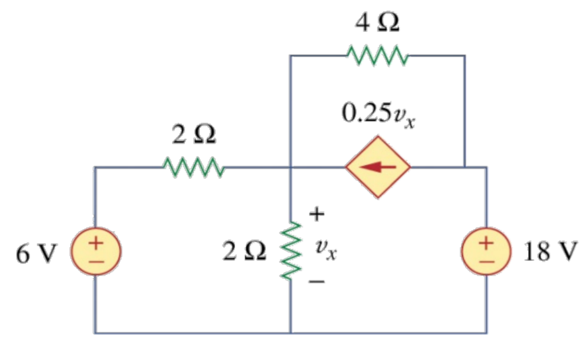


Fig. 5(a)

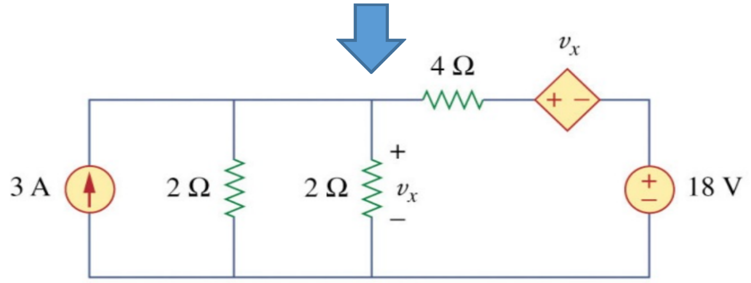


Fig. 5(b)

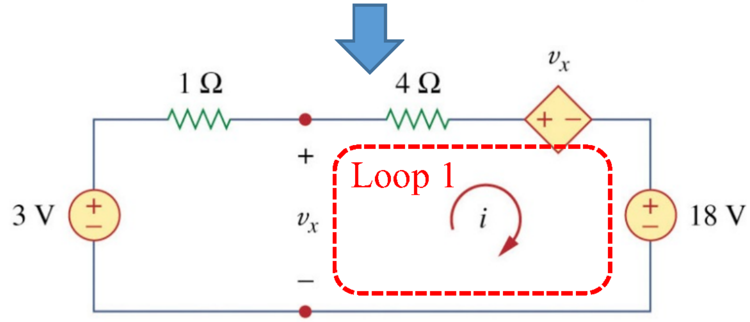
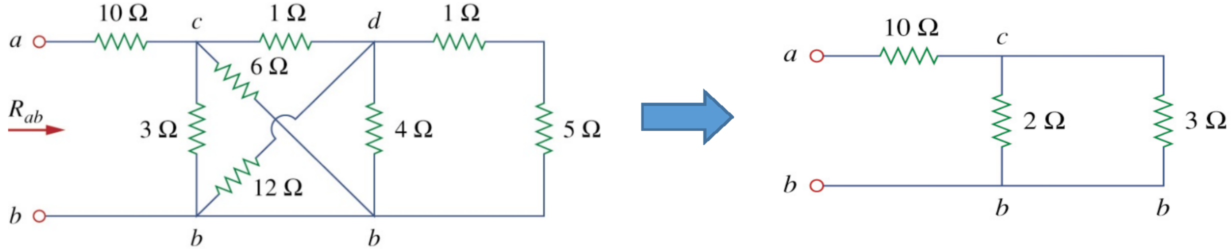


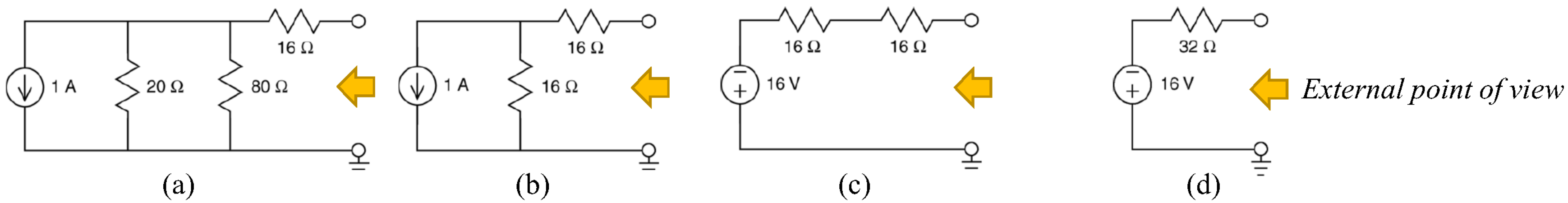
Fig. 5(c)

Circuit Equivalence

- Two circuits are equivalent if they have the same *i*-*v* characteristic at a specified pair of terminals.
- Equivalent circuits for resistive networks are obtained from reductions of parallel and series connections.

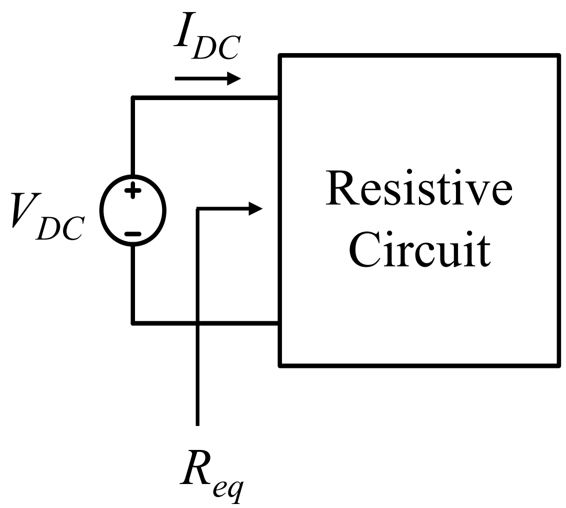


Circuits with source:



- From an external point of view, the circuits (a) to (d) are equivalent as they share identical *i*-*v* characteristics.
- (d) is said to be the equivalent circuit of (a), although the internal circuit variables are now unavailable.
- How do we obtain the equivalent circuit ?

Determination of Equivalent Resistance



If a DC input voltage (V_{DC}) is applied to a resistive circuit (as shown above) and an input current (I_{DC}) is measured, the equivalent resistance of this circuit is given by:

$$R_{eq} = \frac{V_{DC}}{I_{DC}}$$

Thevenin’s Theorem

- It often occurs in practice that a particular element in a circuit is variable (usually called the load) while the remaining elements are fixed.
- A typical example of this is the household outlet.
- Each time the variable element is changed, the entire circuit has to analyzed all over again.
- To avoid this problem, Thevenin’s theorem provides a technique by which the fixed part of the circuit can be replaced by an equivalent circuit.
- Thevenin’s theorem states that a linear two-terminal circuit [see Fig. (a)] can be replaced by an equivalent circuit consisting of a voltage source V_{TH} in series with a resistor R_{TH} [see Fig. (b)].
- But, how to find V_{TH} and R_{TH} ?

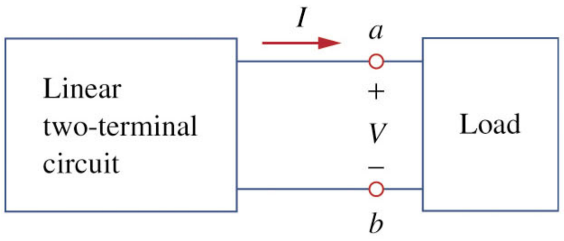


Fig. (a)

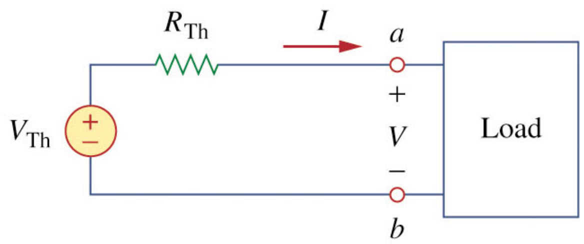
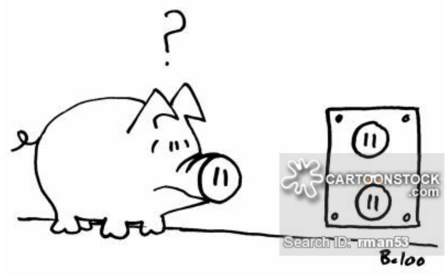


Fig. (b)



Thevenin's Theorem (cont'd)

- Suppose the terminals a-b are made open-circuited (by removing the load). No current flows and hence, the open-circuit voltage across the terminals a-b [see Fig. (a) in previous slide] must be equal to V_{TH} [see Fig. (b) in previous slide], i.e., $V_{TH} = v_{oc}$.

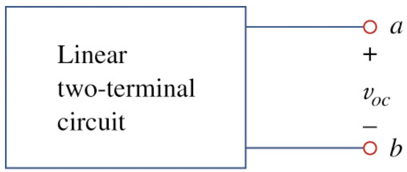
- Again, with the load remains disconnected and terminals a-b open-circuited, we turn off all independent sources, the equivalent resistance at the terminals a-b must be equal to R_{TH}. i.e., $R_{TH} = R_{in}$.

- Summary: V_{TH} is the open-circuit voltage at the terminals and R_{TH} is the output or equivalent resistance at the terminals when the independent sources are turned off.

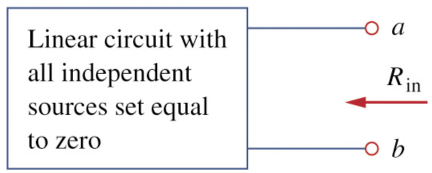
- Now, the question becomes: How to find v_{oc} and R_{in} ?

- Finding v_{oc} is relatively straightforward.

- To find R_{in} , we need to consider TWO cases.



$V_{Th} = v_{oc}$



$R_{Th} = R_{in}$

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Thevenin's Theorem (cont'd)

- Case 1: If the network has no dependent sources, we **turn off all independent sources**. $R_{\mathrm{TH}} = R_{in}$ can be found readily by combining resistances.

- Case 2: If the network has dependent sources, we turn off all independent sources, but *not* the dependent sources. A voltage source v_o (or current sources i_o) is applied across terminals a-b to determine the resulting current i_o (or resulting voltage v_o). Then, $R_{\mathrm{TH}} = R_{in} = v_o/i_o$.

- In either case, we may assume any values of v_o or $i_{o.}$ Notice that case 1 is a special case of case 2.

Example (6): Thevenin's Theorem

Question: Find the Thevenin equivalent circuit of the circuit [see Fig. 6(a)], to the left of terminal a-b. Then find the current through the load resistor (R_L) when (i) $R_L = 6\ \Omega$; (ii) $R_L = 16\ \Omega$; (iii) $R_L = 36\ \Omega$.

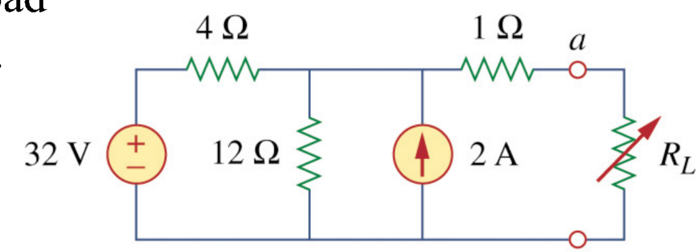

Fig. 6(a)

- We find R_{TH} by turning off the 32 V voltage source (replacing it by a short circuit) and the 2A current source (replacing it with an open circuit). So, the original circuit in Fig. 6(a) becomes that in Fig. 6(b).

- Since there is no dependent sources, R_{TH} can be obtained by combining the resistances, i.e.,

$$R_{TH} = 4 \| 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$

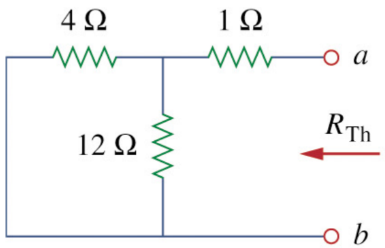

Fig. 6(b)

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Example (6): Thevenin's Theorem (cont'd)

- To find V_{TH}, consider the circuit in Fig. 6(c).
- Apply the nodal analysis at node marked with V_{TH}

$$\frac{32 - V_{TH}}{4} + 2 = \frac{V_{TH}}{12}$$

- Solving the above equation gives $V_{TH} = 30$ V.
- The Thevenin equivalent circuit is therefore shown in Fig. 6(d).
- $I_L = V_{TH}/(R_{TH} + R_L) = 30/(4 + R_L)$
- When $R_L = 6$, $I_L = 30/(4 + 6) = 3$ A
- When $R_L = 16$, $I_L = 30/(4 + 16) = 1.5$ A
- When $R_L = 36$, $I_L = 30/(4 + 36) = 0.75$ A

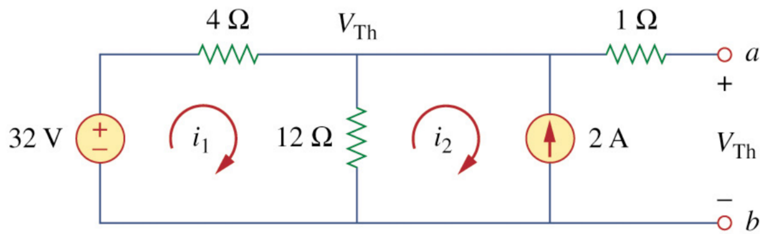


Fig. 6(c)

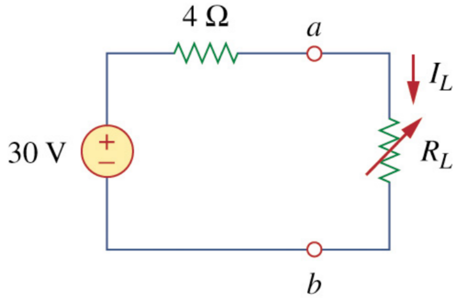


Fig. 6(d)

Example (7): Thevenin's Theorem

Question: Find the Thevenin equivalent at terminal a-b of the circuit shown in Fig. 7(a).

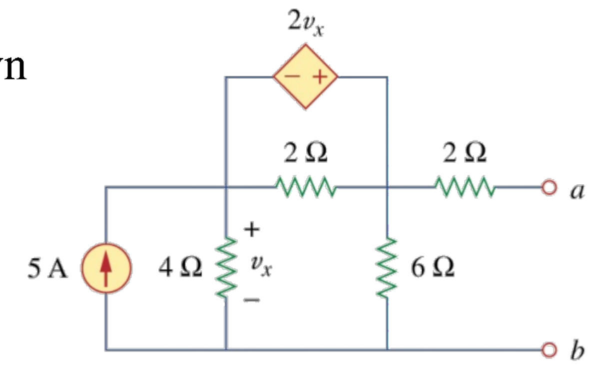


Fig. 7(a)

- This circuit has one dependent source ($2v_x$).
- To find R_{TH}, we set the independent current source equal to zero. Leave the dependent source unchanged.
- Because of the presence of the dependent source, the method of combining resistances will *not* work.
- Instead, we excite the network with a voltage source v_o , connected across terminal a-b [see Fig. 7(b)]. We may set v_o = 1 V for easy computation.
- Our goal is to find the current i_o .
- Apply mesh analysis to loop 1 in Fig. 7(b), we have

$$-2v_x + 2(i_1 - i_2) = 0 \qquad (1)$$

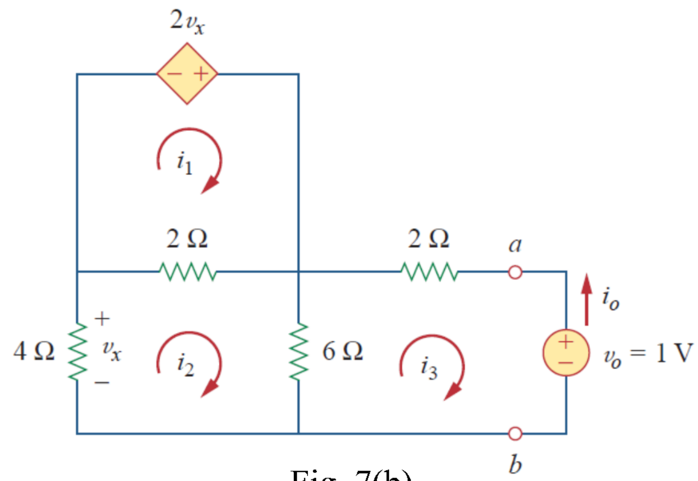


Fig. 7(b)

(Continued on next slide)

Example (7): Thevenin's Theorem (cont'd)

- But, $v_x = -4i_2$. Putting this back into equation (1) in previous slide gives

$$i_1 = -3i_2$$

- Apply KVL to loop 2 and loop 3 yields the following two equations.

$$4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0$$
$$6(i_3 - i_2) + 2i_3 + 1 = 0$$

- Solving these equations gives $i_3 = -1/6$ A. Since $i_o = -i_3$, $i_o = 1/6$ A.
- Hence $R_{TH} = v_o / i_o = 6\ \Omega$
- To get V_{TH}, find v_{oc} in Fig. 7(c). By applying mesh analysis, we have

$$\begin{cases} i_1 = 5 \\ 4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0 \\ -2v_x + 2(i_3 - i_2) = 0 \end{cases}$$

- Putting $v_x = 4(i_1 - i_2)$ and solving the above set of equations leads to $i_2 = 10/3$ A.
- Hence, $V_{TH} = v_{oc} = 6i_2 = 20$ V.

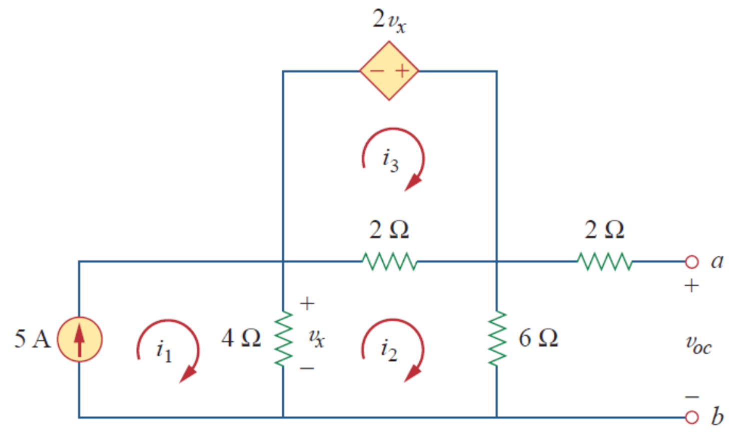
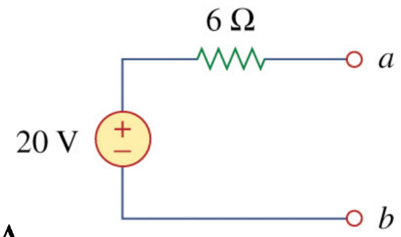


Fig. 7(c)



Norton's Theorem

- Norton equivalent and Thevenin equivalent are very closely related.
- In fact, by source transformation, we should have $I_N = V_{\mathrm{TH}}/R_{\mathrm{TH}}$.
- If we put $I_N = i_{sc}$, $V_{\mathrm{TH}} = v_{oc}$ into the above expression, we have $R_{\mathrm{TH}} = v_{oc}/i_{sc}$. This gives us the third way to find R_{TH} .
- Summary:
  - To determine the Thevenin or Norton equivalent circuit requires two out of the following three quantities (compute the two that take the least effort):

    (1) v_{oc} across terminals a-b;

    (2) i_{sc} at terminals a-b;

    (3) R_{in} at terminals a-b when all independent sources are turned off.
  - If Thevenin equivalent is needed, given v_{oc} and i_{sc}, $R_{\mathrm{TH}} = v_{oc}/i_{sc}$ can be used to find the equivalent resistance.
  - If Norton equivalent is needed, given v_{oc} and R_{in} , $I_N = i_{sc}$ can be found from $R_{in} = v_{oc}/i_{sc}$.

Example (8): Norton's Theorem

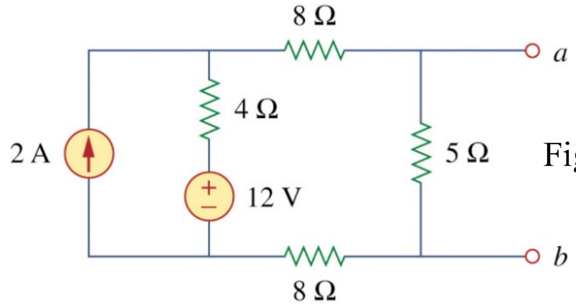


Fig. 8(a)

Question: Find the Norton equivalent circuit at terminal a-b of the circuit shown in Fig. 8(a).

- Find R_{TH} in Thevenin equivalent.
- Set all independent sources equal to zero, which produces the circuit in Fig. 8(b), from which the equivalent resistance is obtained as follows.

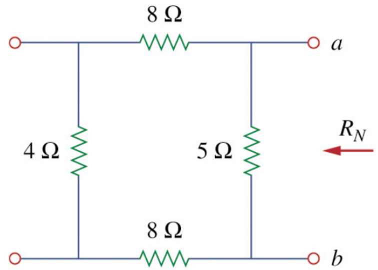


Fig. 8(b)

$$R_N = 5 \,||\, (8+4+8) = 5 \,||\, 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

- To find I_N, we short-circuit terminals a and b [see Fig. 8(c)].
- Ignore the 5Ω since it has been short-circuited.
- Applying mesh analysis, we obtain
- $i_1 = 2\text{A}, \quad 20i_2 - 4i_1 - 12 = 0$
- Solving these equations, we obtain $i_2 = 1\text{ A} = I_N$
- The Norton equivalent is shown in Fig. 8(d).

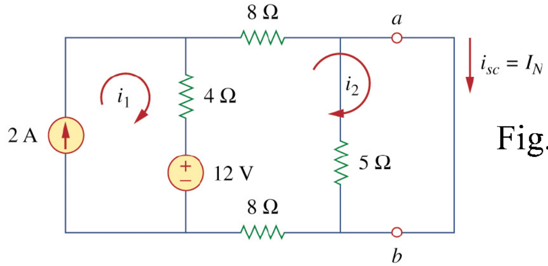


Fig. 8(c)

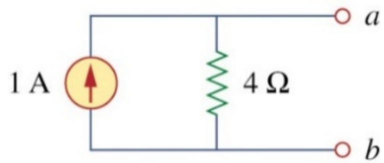


Fig. 8(d)

Example (8): Norton's Theorem (con'd)

- Alternatively, we may determine I_N from V_{TH}/R_{TH} .
- V_{TH} is the open-circuit voltage as shown in Fig. 8(e).
- Using mesh analysis, we have

$$i_3 = 2\text{ A},\quad 25i_4 - 4i_3 - 12 = 0 \quad \Rightarrow \quad i_4 = 0.8\text{ A}$$

- $v_{oc} = V_{TH} = 5i_4 = 4\text{V}$
- Hence, $I_N = V_{TH}/R_{TH} = 4/4 = 1\text{ A}$

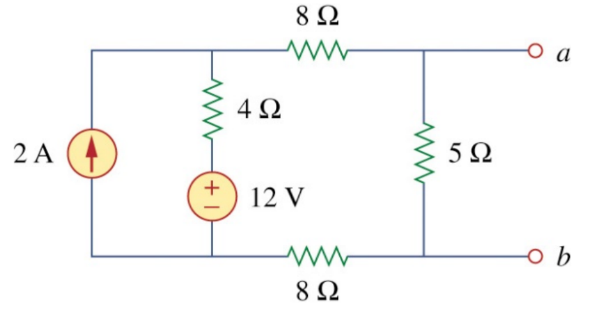


Fig. 8(a)

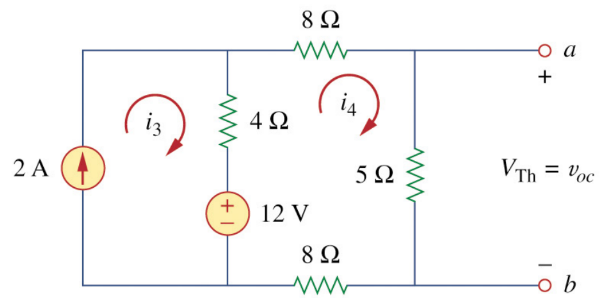
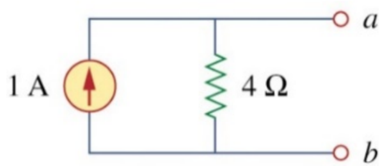
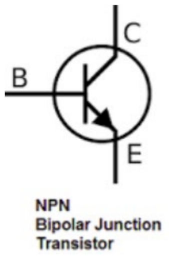
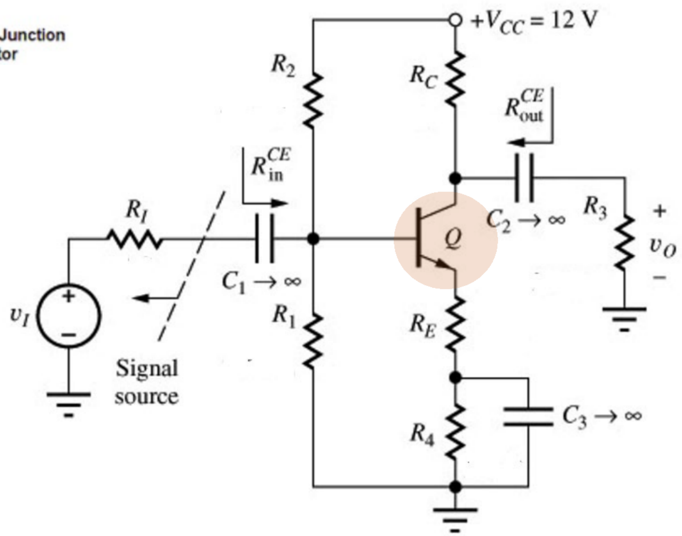


Fig. 8(e)



Why do we want to find equivalent circuits ?

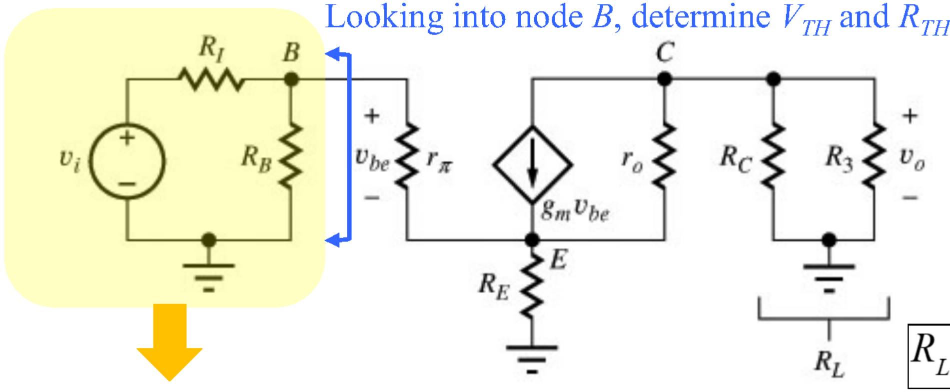




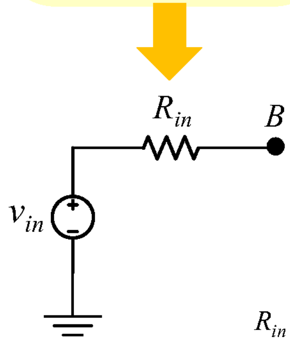
BJT common-emitter (CE) amplifier

Small-signal equivalent of the CE amplifier

Looking into node B, determine V_{TH} and R_{TH}



$$R_L = R_C \| R_3$$



By Thevenin Theorem,

$V_{TH} = vi \times [R_B / (R_I + R_B)] \Rightarrow v_{in} = V_{TH}$

$R_{TH} = R_I \,//\, R_B \Rightarrow R_{in} = R_{TH}$