Capacitors and Inductors

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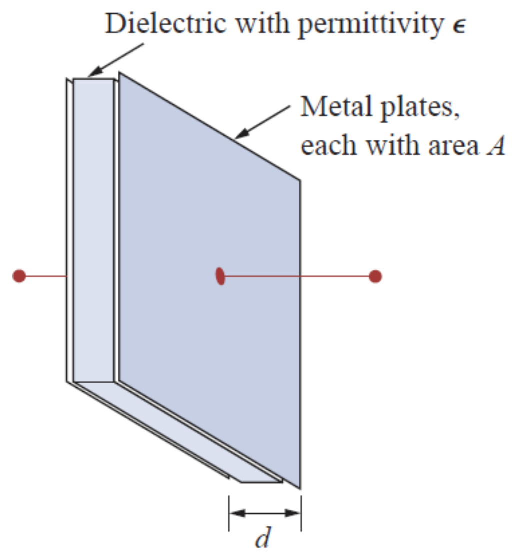
Topics

- Properties of Capacitor
- Series and parallel capacitors
- Properties of inductor
- Series and parallel inductors
- Op amp with capacitor

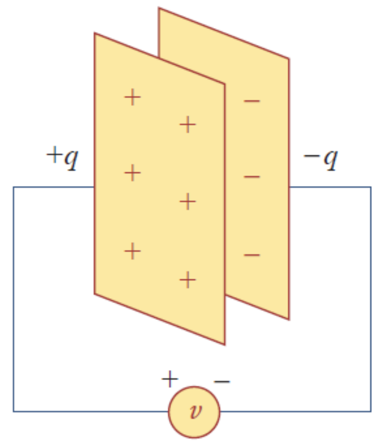
Capacitors and Inductors

- So far, we have focused only on resistive circuits.
- In this chapter, we will learn two other important passive linear circuit elements: capacitor and inductor.
- Unlike resistors, capacitors and inductors do *not* dissipate energy. Instead, they **store and release energy**.

Capacitors

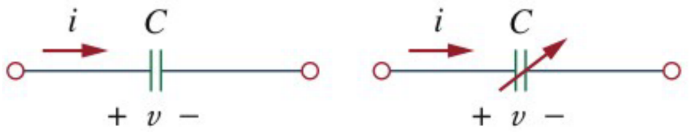


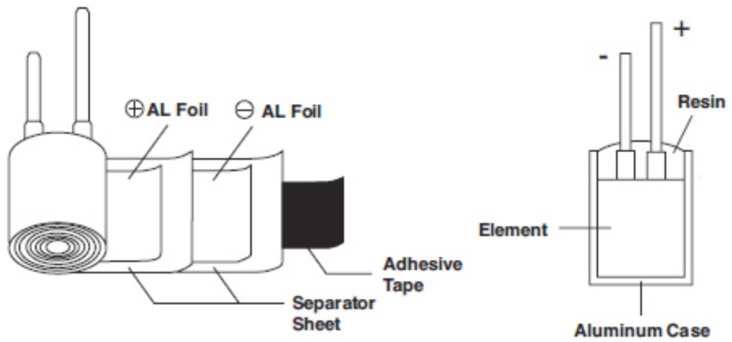
- A capacitor stores energy in its electric field.
- Typically, a capacitor is constructed as two conducting plates separated by an insulator as shown on the right.
- When a voltage source v is connected to the capacitor, the source deposits positive charges $+q$ on one plate and negative charge $-q$ on the other plate.
- The amount of charge stored is directly proportional to the voltage v, i.e., $\boxed{q = Cv}$, where the proportional constant C is called the capacitance.
- The unit of capacitance is the farad (F).

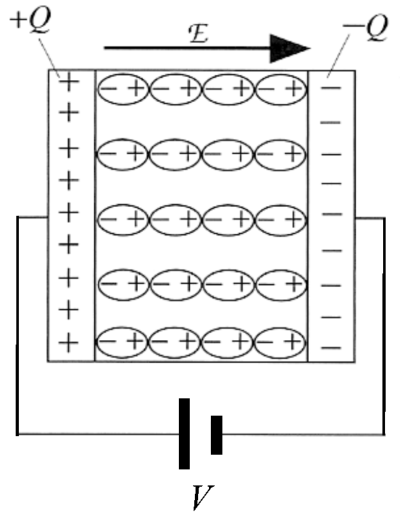


Capacitors (cont'd)

- The typical values of commercially-available capacitors are in the range of pF and μF.
- Note that 1 farad = 1 coulomb / volt.
- The symbols of capacitors are shown in the following figure. Note that the current reference direction is going into the positive terminal of the capacitor.

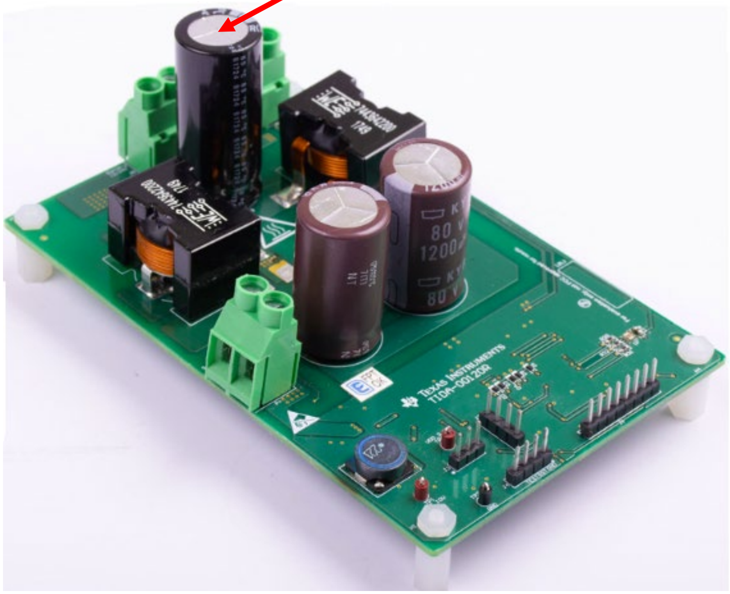


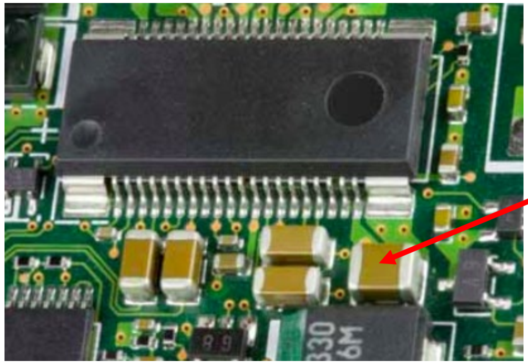




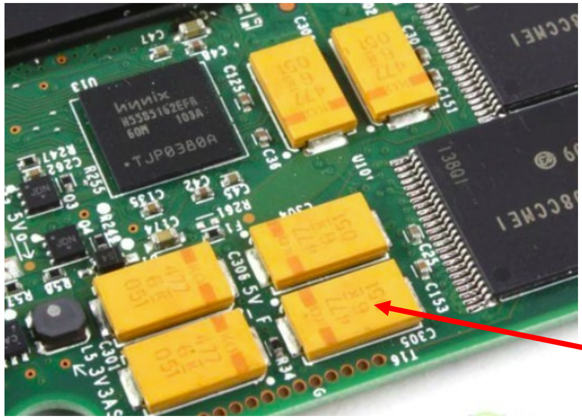
Examples of Capacitor

Electrolytic capacitors (E-cap)





Ceramic SMD capacitor



Tantalum capacitor

Capacitor Equations

- To obtain the current-voltage relationship, take the derivative on both sides of $q = Cv$ with respect to t. This gives

$$\boxed{i = C\frac{dv}{dt}}$$

- Integrating both sides of the above equation gives the voltage-current relationship.

$$\boxed{v(t) = \frac{1}{C}\int_{t_o}^{t} i(t')dt' + v(t_o)}$$

- The instantaneous power delivered to the capacitor is: $\boxed{p = vi = Cv\frac{dv}{dt}}$

Capacitor Equations (cont'd)

- The energy stored in the capacitor is therefore given by:

$$\boxed{w(t) = \int_{-\infty}^{t} p(t')dt' = C\int_{-\infty}^{t} v(t')\frac{dv(t')}{dt'}dt' = C\int_{-\infty}^{t} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}}$$

- Assume that the capacitor is uncharged at t = –∞, we have $\boxed{w(t) = \frac{1}{2}Cv(t)^2}$

- Key properties of a capacitor:
  - A capacitor is an open circuit to DC after it is fully charged.
  - The voltage on a capacitor cannot change abruptly (a sudden voltage change requires an infinite current).
  - An ideal capacitor does *not* dissipate energy.

Example (1): Capacitors

Question: (a) Calculate the charge stored in a 3 pF capacitor with 20 V across it; and

(b) Find the energy stored in the same capacitor.

Answer: (a) Since $q = Cv$, $q = 3 \times 10^{-12} \times 20 = 60$ pC

(b) The energy stored is: $w = \frac{1}{2}Cv^2 = \frac{1}{2}\left(3\times10^{-12}\right)\left(20\right)^2 = 6\times10^{-10} = 600 \text{ pJ}$

Example (2): Capacitors

Question: The voltage across a 5 μF capacitor is given by $v(t) = 10\cos(6000t)$.
Calculate the current through it.

Answer: By definition, $i(t) = C\dfrac{dv(t)}{dt} = \left(5\times 10^{-6}\right)\dfrac{d}{dt}\left(10\times\cos 6000t\right)$

$$= -5\times 10^{-6}\times 10\times 6000\sin 6000t$$

$$= -0.3\sin 6000t \text{ A}$$

Example (3): Capacitors

Question: Determine the voltage across a 2 μF capacitor if the current through it is $i(\mathrm{t}) = 6e^{-3000t}$ mA. Assume the capacitor voltage is zero at $t = 0$.

Answer:

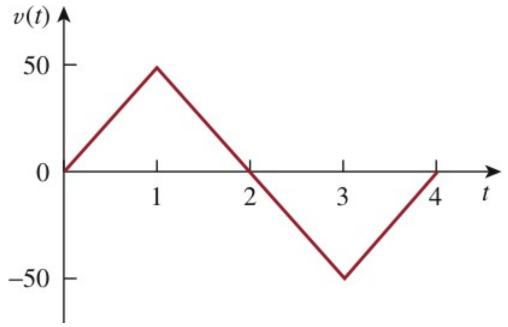
- We can use this equation, i.e., $v(t) = \frac{1}{C}\int_{t_o}^{t} i(t')dt' + v(t_o)$ (1)
- Set $t_o = 0$ and substituting the given expression of $i(t)$ into (1), we have

$$v(t) = \frac{1}{2\times 10^{-6}}\int_0^t 6e^{-3000t'}dt'\cdot 10^{-3} = \frac{3\times 10^3}{-3000}e^{-3000t'}\Big|_0^t = (1-e^{-3000t})\mathrm{V}$$

Example (4): Capacitors

Question: Determine the current through a 200 μF capacitor. The voltage across this capacitor can be described mathematically as

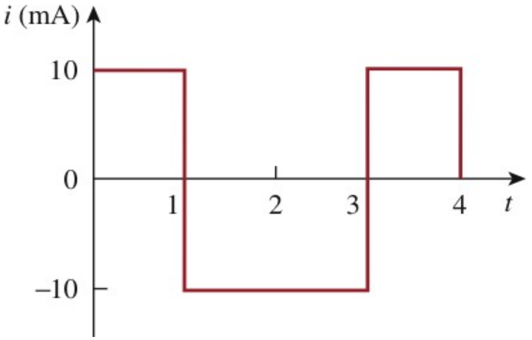
$$v(t)=\begin{cases}50t\text{V} & 0<t<1\\ 100-50t\text{V} & 1<t<3\\ -200+50t\text{V} & 3<t<4\\ 0 & \text{otherwise}\end{cases}$$



Answer:

Since $i = C dv/dt$ and C = 200 μF, we take the derivative of $v(t)$ to obtain $i(t)$ as follows.

$$i(t)=200\times10^{-6}\times\begin{cases}50 & 0<t<1\\ -50 & 1<t<3\\ 50 & 3<t<4\\ 0 & \text{otherwise}\end{cases}=\begin{cases}10\text{mA} & 0<t<1\\ -10\text{mA} & 1<t<3\\ 10\text{mA} & 3<t<4\\ 0 & \text{otherwise}\end{cases}$$



Example (5): Capacitors

Question: Obtain the energy stored in each capacitor in Fig. (a) under DC conditions.

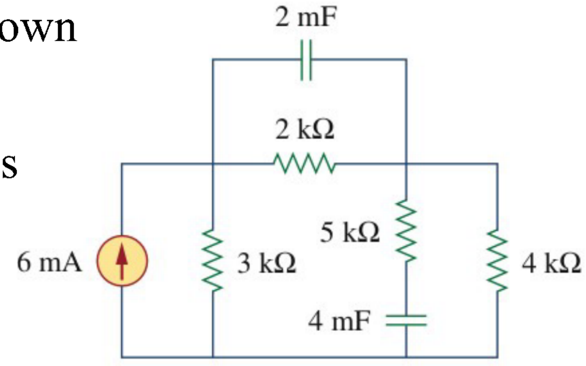
- Under DC conditions, we replace each capacitor with an open circuit, as shown in Fig. (b).
- The current through the series combination of the 2 kΩ and 4 kΩ resistors is obtained by current division as $i=\frac{3}{3+2+4}(6\text{mA})=2\text{mA}$
- The voltages v_1 and v_2 across the capacitors can be written as

  $v_1=2000i=4\text{V},\ \ v_2=4000i=8\text{V}$

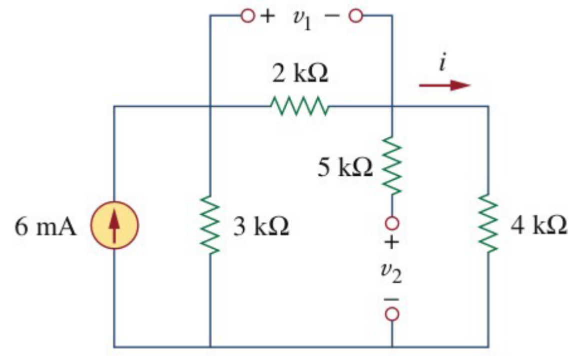
- The energies stored in the capacitors can be obtained as

$$w_1=\frac{1}{2}C_1v_1^2=\frac{1}{2}(2\times10^{-3})(4)^2=16\text{mJ}$$

$$w_2=\frac{1}{2}C_2v_2^2=\frac{1}{2}(4\times10^{-3})(8)^2=128\text{mJ}$$



(a)



(b)

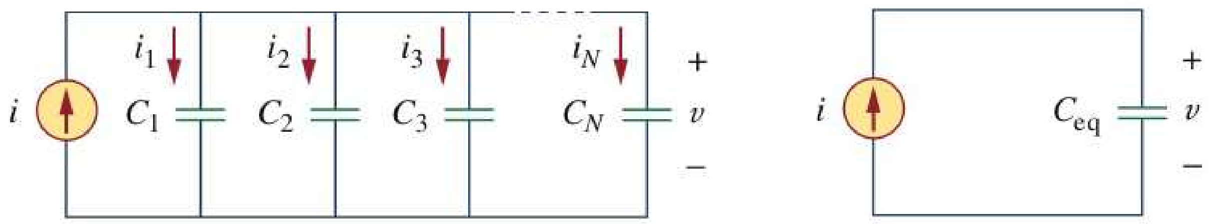
Parallel-connected Capacitors

- We know from resistive circuits that the series-parallel combination is a powerful tool for reducing circuits.
- This technique can be extended to series-parallel connections of capacitors.
- Firstly, consider the case of N capacitors connected in parallel.
  - Since the capacitors have the same voltage *v* across them, KCL is applied to obtain the total current i, where $i = i_1 + i_2 + \ldots + i_N$. But, since $i_k = C_k\, dv/dt$, we have

$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \ldots + C_N \frac{dv}{dt} = \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

  - Hence, the equivalent capacitance of N parallel capacitors is the sum of the individual capacitances, i.e.,

$$\boxed{C_{eq} = C_1 + C_2 + \ldots + C_N}$$



Series-connected Capacitors

- Consider N capacitors connected in series.
  - Apply KCL to the loop in Fig. (a), we have

    $v = v_1 + v_2 + \ldots + v_N$

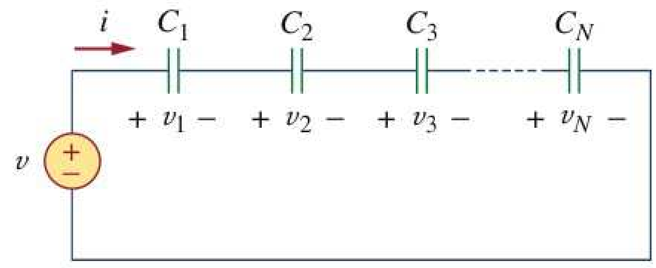
  - Notice that the same current i flows through all the capacitors and

    $$v_k = \frac{1}{C_k}\int_{t_o}^{t} i(t)dt + v_k(t_o)$$

  - Therefore, $$v = \frac{1}{C_1}\int_{t_o}^{t} i(t)dt + v_1(t_o) + \frac{1}{C_2}\int_{t_o}^{t} i(t)dt + v_2(t_o) + \ldots + \frac{1}{C_N}\int_{t_o}^{t} i(t)dt + v_N(t_o)$$

    $$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_N}\right)\int_{t_o}^{t} i(t)dt + \left[v_1(t_o) + v_2(t_o) + \ldots + v_N(t_o)\right]$$

    $$\triangleq \frac{1}{C_{eq}}\int_{t_o}^{t} i(t)dt + v(t_o)$$



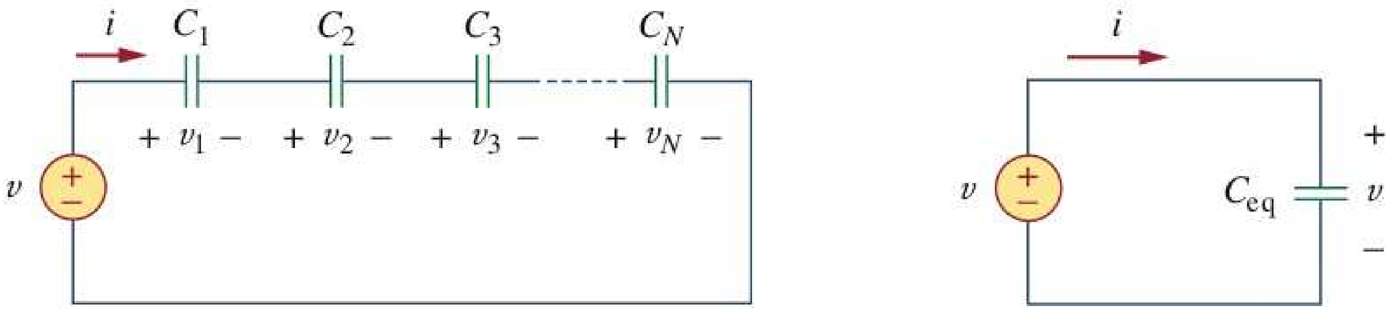
(a)

Series-connected Capacitors (cont'd)

- From the equation in the previous slide, we know that the equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

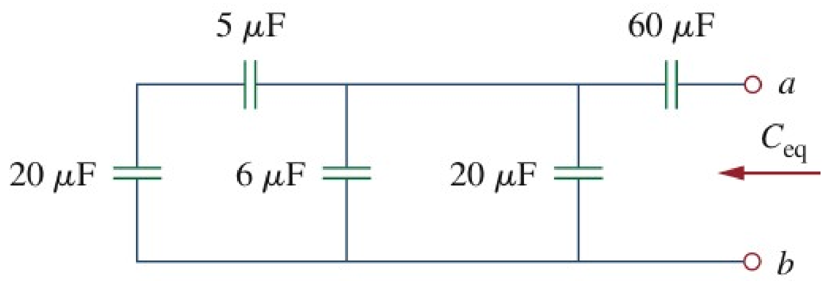
$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_N}}$$

In particular, when $N = 2$ (i.e., two capacitors in series), $C_{eq} = \dfrac{C_1 C_2}{C_1 + C_2}$.



Example (6): Capacitors

Question: Find the equivalent capacitance between terminals a and b in the following circuit.

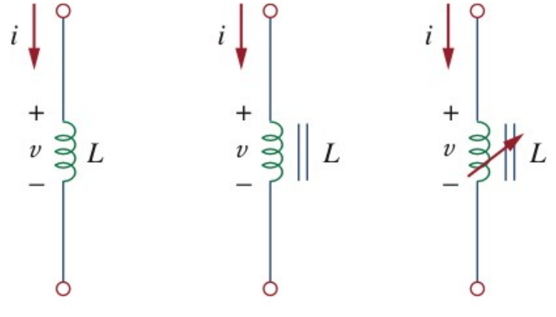


- The 20 μF and 5 μF capacitors are in series. So, the equivalent capacitance is: $\dfrac{20 \times 5}{20+5} = 4\ \mu\text{F}$
- The 4 μF capacitor is in parallel with the 6 μF and 20 μF capacitors. Hence, their combined capacitance is:

  $4 + 6 + 20 = 30\ \mu\text{F}$
- The 30 μF capacitor is in series with the 60 μF capacitor. Hence, the equivalent capacitance between terminals a and b (C_{eq}) is given by:

  $C_{eq} = (30 \times 60)/(30 + 60) = 20\ \mu\text{F}$

Inductors



- An inductor stores energy in its magnetic field.
- The voltage-current relationship is given by: $\boxed{v = L\dfrac{di}{dt}}$

  where L is the inductance and the unit is Henry (H).
- Note that the current reference direction is going into the positive side of the voltage polarity.
- The current-voltage relationship is obtained from integrating both sides of above voltage-current relationship, i.e., $di = (1/L)vdt$. Hence, we have

$$\boxed{i(t) = \frac{1}{L}\int_{t_o}^{t} v(t')dt' + i(t_o)}$$

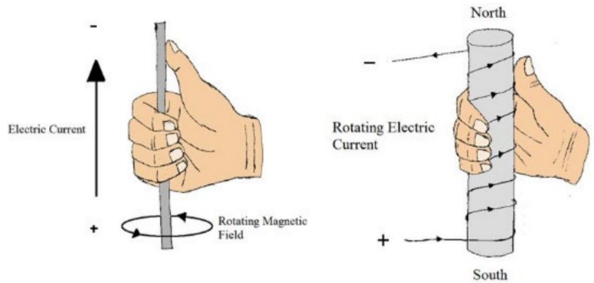
Inductors (cont'd)

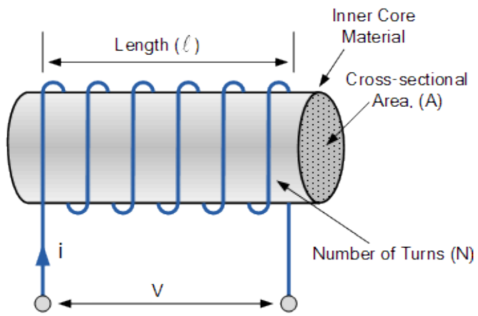
- The power delivered to the inductor is: $p = vi = L(di/dt)i$
- The energy stored is:

$$w(t) = \int_{-\infty}^{t} p(t')dt' = \int_{-\infty}^{t} \left(L\frac{di(t')}{dt'}\right)i(t')dt' = L\int_{-\infty}^{t} i(t')di(t') = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty)$$

- If we assume $i(-\infty) = 0$, then $\boxed{w(t) = \frac{1}{2}Li^2(t)}$
- Key properties of inductor:
  - An inductor acts like a short circuit to DC.
  - The current through an inductor cannot change instantaneously.
  - Ideal inductor does not dissipate energy.

Right Hand Rule





Example (7): Inductors

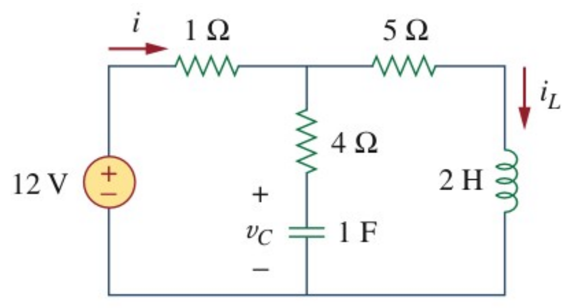
Question: The current through a 0.1 H inductor is $i(t) = 10te^{-5t}$ A. Determine the voltage across the inductor and the energy stored in it.

- Since $v = L\,di/dt$ and $L = 0.1$ H, $v = 0.1\dfrac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1-5t)$ V
- The energy stored is:

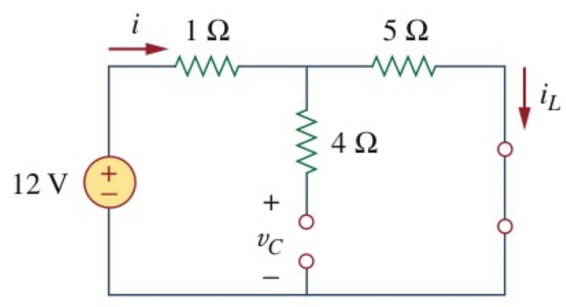
$$w(t) = \frac{1}{2}Li(t)^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t} \text{ J}$$

Example (8): RLC circuit

Question: In the circuit shown in Fig. (a), assume DC condition, find (a) i, v_c, and I_L; (b) the energy stored in the capacitor and inductor.



(a)



(b)

- Under DC condition, the capacitor acts like an open circuit whereas the inductor acts like a short circuit, as shown in Fig. (b).
- (a) From Fig. (b), it is evident that $i = i_L = 12 / (1+5) = 2$ A
  The voltage $v_c = 5i_L = 10$ V
- (b) Let w_c and w_L be the energy stored in the capacitor and the inductor, respectively.

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10)^2 = 50 \text{ J}$$

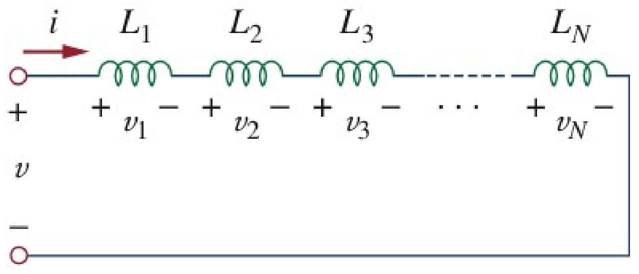
$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2)^2 = 4 \text{ J}$$

Series-connected Inductors

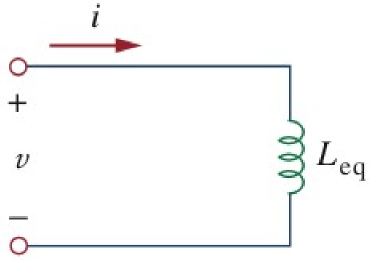
- Consider a series connection of N inductors, as depicted in Fig. (a).
- The same current (i) flows through all inductors.
- Apply KVL to the loop, $v = v_1 + v_2 + \ldots + v_N$.
- Substituting $v_k = L_k di/dt$ yields the following expression.

$$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \ldots + L_N \frac{di}{dt} = (L_1 + L_2 + \ldots + L_N) \frac{di}{dt}$$

- Therefore, $\boxed{L_{eq} = L_1 + L_2 + \ldots + L_N}$



(a)



(b)

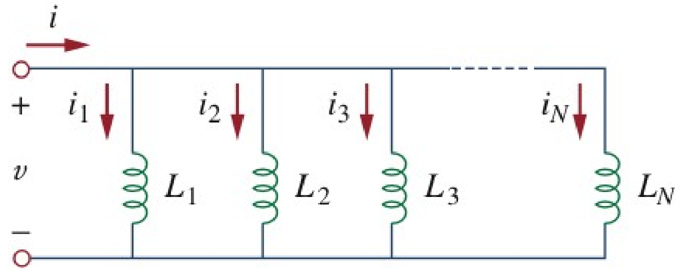
Parallel-connected Inductors

- Consider N inductors connected in parallel, as depicted in Fig. (a).
- The same voltage (v) is applied across each of the N inductors.
- By applying KCL, $i = i_1 + i_2 + \ldots + i_N$.
- Since $i_k = \frac{1}{L_k}\int_{t_o}^{t} v dt + i_k(t_o)$, we have

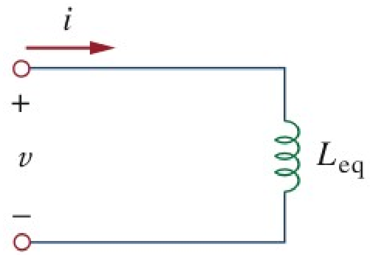
$$i = \frac{1}{L_1}\int_{t_o}^{t} v dt + i_1(t_o) + \frac{1}{L_2}\int_{t_o}^{t} v dt + i_2(t_o) + \ldots + \frac{1}{L_N}\int_{t_o}^{t} v dt + i_N(t_o)$$

$$= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_o}^{t} v dt + \left(\sum_{k=1}^{N} i_k(t_o)\right)$$

- Therefore, $\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}$
- In particular, when $N = 2$, $L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$



(a)



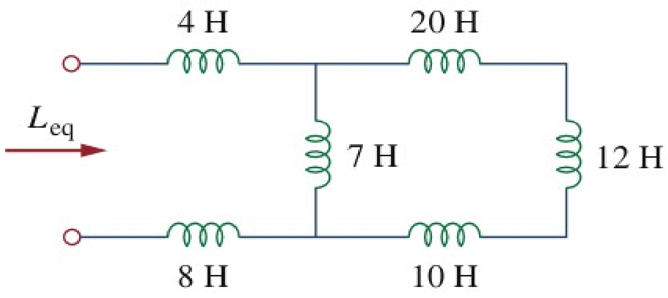
(b)

Summary

Relation	**Resistor (R)**	**Capacitor (C)**	**Inductor (L)**
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

Example (9): Inductors

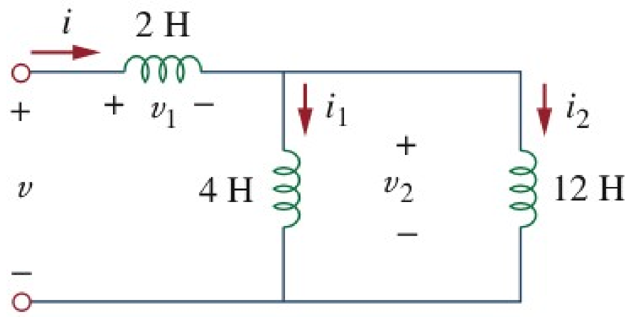
Question: Find the equivalent inductance of the following circuit.



- The 20 H, 12 H and 10 H inductors are connected in series. Thus, their equivalent inductance is 42 H.
- The 42 H inductor is now in parallel with the 7 H inductor. Therefore, they can be combined to give (42)(7) / (42 + 7) = 6 H.
- Since the 6 H inductor is in series with the 4 H and 8 H inductors, the equivalent inductance (L_{eq}) is given by the sum of these three inductances, i.e., $L_{eq} = 4 + 6 + 8 = 18$ H.

Example (10): Inductors

Question: Consider the circuit shown in Fig. (a). Given $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.



(a)

- (a) From $i(t) = 4(2 - e^{-10t})$, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$, $i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5$ mA.
- (b) $L_{eq} = 2 + (4 \parallel 12) = 2 + 3 = 5$ H.
  Thus, $v(t) = L_{eq}\, di/dt = 5(4)(-1)(-10)e^{-10t} = 200e^{-10t}$ mV
  $v_1(t) = 2\, di/dt = 2(4)(-1)(-10)e^{-10t} = 80e^{-10t}$ mV
  $v_2(t) = v(t) - v_1(t) = 120e^{-10t}$ mV
- (c)

$$i_1(t) = \frac{1}{4}\int_0^t v_2(t')dt' + i_1(0) = \frac{120}{4}\int_0^t e^{-10t'}dt' + 5\text{mA} = -3e^{-10t'}\Big|_0^t + 5\text{mA} = 8 - 3e^{-10t}\text{mA}$$

$$i_2(t) = \frac{1}{12}\int_0^t v_2(t')dt' + i_2(0) = \frac{120}{12}\int_0^t e^{-10t'}dt' - 1\text{mA} = -e^{-10t'}\Big|_0^t - 1\text{mA} = -e^{-10t}\text{mA}$$

Op Amp with Capacitor

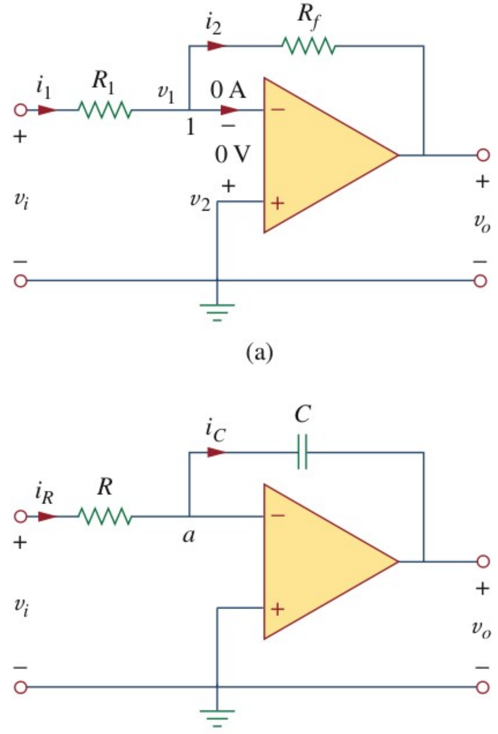
- By replacing the feedback resistor (R_f) in the familiar inverting amplifier in Fig. (a) with a capacitor, an integrator can be obtained, as shown in Fig. (b).
- At node a in Fig. (b), we have $i_R = i_C$.
- But, $i_R = v_i / R$ and $i_C = -C dv_o/dt$
- By equating i_R and i_C, we have

$$\frac{v_i}{R} = -C\frac{dv_o}{dt} \quad => \quad dv_o = -\frac{1}{RC}v_i dt$$

- By integrating both sides of the above equation gives:

$$v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_i(t')dt'$$

- If the capacitor is not charged at $t = 0$, we have $\boxed{v_o(t) = -\frac{1}{RC}\int_0^t v_i(t')dt'}$

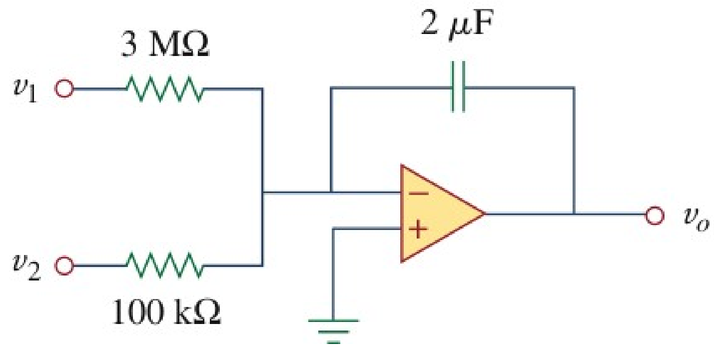


(a)

(b)

Example (11): Op Amp with Capacitor

Question: If $v_1(t) = 10\cos 2t$ mV and $v_2(t) = 0.5t$ mV, find $v_o(t)$ in the following op amp circuit. Assume the voltage across the capacitor is initially zero.



This is a summing integrator whose output voltage can be expressed as

$$v_o = -\frac{1}{R_1C}\int_0^t v_1(t')dt' - \frac{1}{R_2C}\int_0^t v_2(t')dt'$$

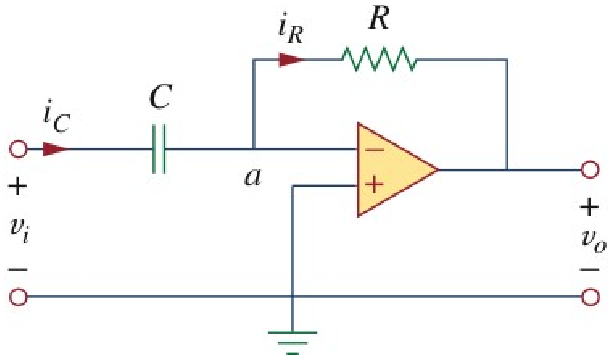
$$= -\frac{1}{3\times10^6\times2\times10^{-6}}\int_0^t 10\cos 2t'dt' - \frac{1}{100\times10^3\times2\times10^{-6}}\int_0^t 0.5t'dt'$$

$$= -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0.2}\frac{0.5t^2}{2}$$

$$= -0.833\sin 2t - 1.25t^2 \text{mV}$$

Differentiator

- If the input resistor in the inverting amplifier is replaced by a capacitor, as shown in the following op amp circuit, we have a differentiator.



- Apply KCL at node a, we have $i_C = i_R$.
- But, $i_C = C dv_i/dt$ and $i_R = -v_o / R$
- By equating i_C and i_R , we have

$$\boxed{v_o = -RC\frac{dv_i}{dt}}$$

- Although inductors can theoretically be used with op amp, they are seldom used in practice because they tend to be more bulky and expensive.

Example (12): Op Amp with Capacitor

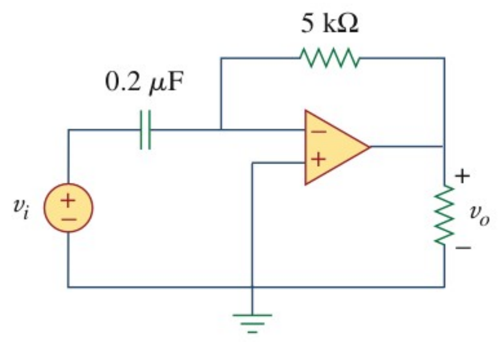
Question: Sketch the output voltage (v_o) for the circuit in Fig. (a), given the input voltage in Fig. (b). Assume $v_o = 0$ at $t = 0$.

- Observe that this is a differentiator with RC = $5 \times 10^3 \times 0.2 \times 10^{-6} = 10^{-3}$ s.
- From Fig. (b), for $0 < t < 4$ ms, the input voltage (v_i) can be expressed as

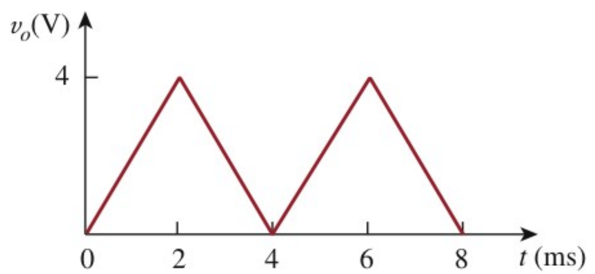
$$v_i = \begin{cases} 2000t & 0 < t < 2 \text{ ms} \\ 8 - 2000t & 2 < t < 4 \text{ ms} \end{cases}$$

- The output voltage can be obtained as

$$v_o = -RC\frac{dv_i}{dt} = \begin{cases} -2\text{V} & 0 < t < 2 \text{ ms} \\ 2\text{V} & 2 < t < 4 \text{ ms} \end{cases}$$



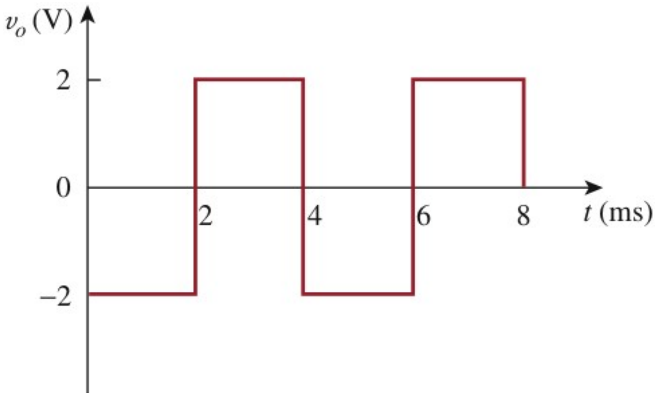
(a)

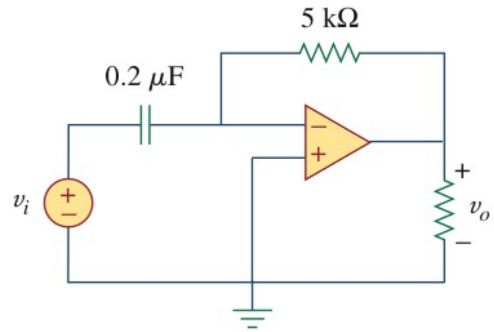


(b)

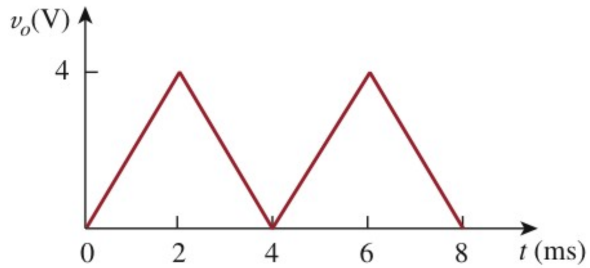
Example (12): Op Amp with Capacitor
(Cont'd)

- From Fig. (b), for $4 < t < 8$ ms, the output voltage (v_o) is the same as that for $0 < t < 4$ ms.
- Thus, the output voltage is sketched below.





(a)



(b)

Analog Computer

- Op amps were originally developed for electronic analog computers.
- Analog computers can be programmed to solve mathematical models of mechanical or electrical systems. This models are often expressed in terms of differential equations.
- Solving simple differential equations using the analog computer requires cascading three types of op amp circuits, namely, integrator circuits, summing amplifiers, and inverting (or non-inverting) amplifiers for negative or positive scaling.

Suppose we would like to find the solution $x(t)$ of the following equation.

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \qquad t > 0$$

where a, b, and c are constants, and $f(t)$ is an arbitrary forcing function.

Analog Computer (cont'd)

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \qquad t > 0$$

where a, b, and c are constants, and $f(t)$ is an arbitrary forcing function.

The solution can be obtained by first solving the highest-order derivative term. That is,

$$\frac{d^2x}{dt^2} = \frac{f(t)}{a} - \frac{b}{a}\frac{dx}{dt} - \frac{c}{a}x$$

Now, to obtain dx/dt, the d^2x/dt^2 term is integrated.

Likewise, to obtain x, the dx/dt *term* is integrated.

Hence, the analog computer is implemented by connecting the necessary summers, inverters, and integrators.

Example (13): Analog Computer

Question: Design an analog computer circuit to solve the following differential equation.

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = 10\sin 4t, \qquad t > 0$$

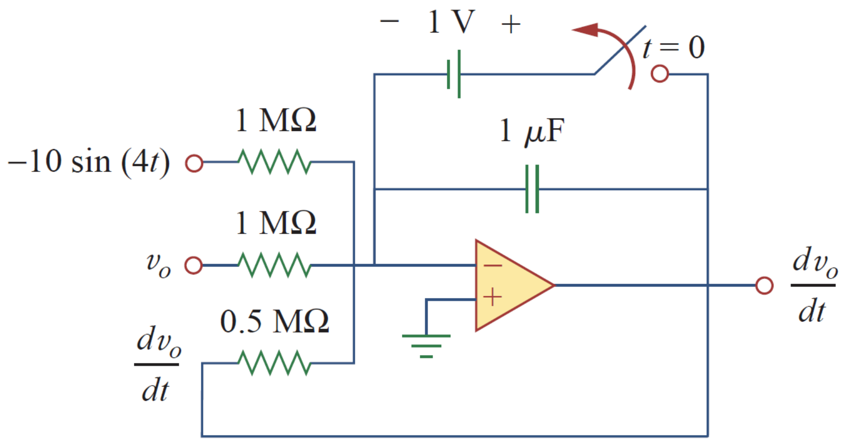
Subject to $v_o(0) = -4$ and $v_o'(t) = 1$, where the prime refers to the time derivative.

Solution:

1. We implement equation using the **summing integrator**

$$\frac{dv_o}{dt} = -\int_0^t \left(-10\sin 4t + 2\frac{dv_o}{dt} + v_o\right)dt + v_o'(0)$$

$$-\frac{1}{RC}\int_0^t v_o\,dt \quad RC=1$$

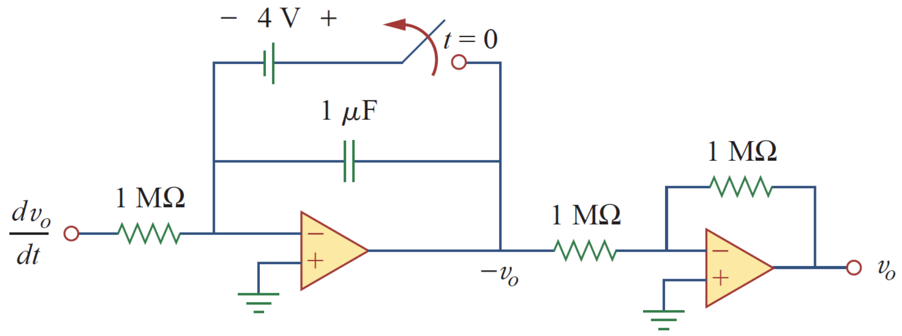


Example (13): Analog Computer (cont'd)

2. The next step is to obtain v_o by integrating dv_o/dt and inverting the result.

$$v_o = -\int_0^t \left(-\frac{dv_o}{dt}\right) dt + v(0)$$

$$-\frac{1}{RC}\int_0^t v_o\, dt \qquad RC=1$$



Example (13): Analog Computer (cont'd)

$$\frac{d^2 v_o}{dt^2} + 2\frac{dv_o}{dt} + v_o = 10 \sin 4t, \qquad t > 0$$