Second-Order Circuits

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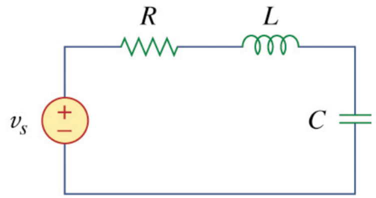
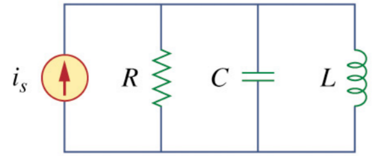
Second-Order Circuits

- In the previous lecture, we consider circuits with a single storage element (a capacitor or an inductor).
- Such circuits are first-order because the differential equations describing them are first-order.
- Here, we will consider circuits containing two storage elements.
- These are known as second-order circuits because their responses are described by differential equations that contain second derivatives.
- Typical examples of second-order circuits are *RLC* circuits as shown in (a) and (b).
- Other examples are *RL* and *RC* circuits, as shown in (c) and (d).
- Second-order circuit may have two storage elements of different type or the same type (provided that elements of the same type *cannot* be represented by an equivalent single element).



(a)



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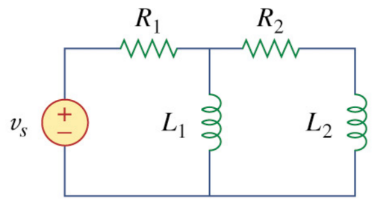
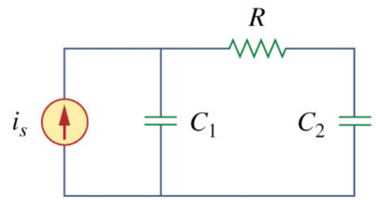
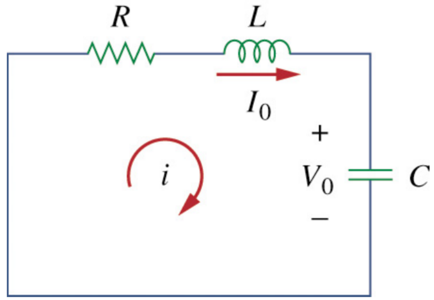


(c)



(d)

Source-free Series RLC Circuit



- Consider the series *RLC* circuit as shown in the figure on the right.
- The circuit is excited by the energy initially stored in the capacitor and the inductor.
- Apply KVL around the loop, we have $Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t')dt' + V_o(-\infty) = 0$
- To eliminate the integral, we differentiate w.r.t. t and rearranging terms, i.e.,

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} = 0 \qquad (1)$$

- This is a second-order differential equation.
- We need to find $i(t)$ that satisfies (1) above.
- Before we continue, let's look at how to solve a second-order differential equation.

Methods for solving second-order differential equation

A second order linear homogeneous ordinary differential equation with *constant coefficients* can be expressed as

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

First, write down the characteristic equation as

$$a_2 s^2 + a_1 s + a_0 = 0$$

Then, solve the characteristic equation to obtain s_1 and s_2.

Three possible cases for s_1 and s_2 :

Solution of characteristic equation	General solution
$s_1, s_2 \in \mathrm{R}, \ s_1 \neq s_2$	$y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
$s_1, s_2 \in \mathrm{R}, \ s_1 = s_2 = s$	$y(t) = (A_1 + A_2 t) e^{st}$
$s_1 = a + bj, \ s_2 = a - bj$, where $a, b \in \mathrm{R}$	$y(t) = (A_1 \cos(bt) + A_2 \sin(bt)) e^{at}$

where A_1 and A_2 are coefficients to be determined from the initial conditions.

Source-free Series RLC Circuit

- Now, we continue to solve the differential equation for the series *RLC* circuit
- From the differential equation $\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} = 0$
- The characteristic equation is $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
- The two roots of the above quadratic equation are

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\,, \qquad s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

- A more compact way of expressing the roots is by letting $\alpha = R/(2L)$ and

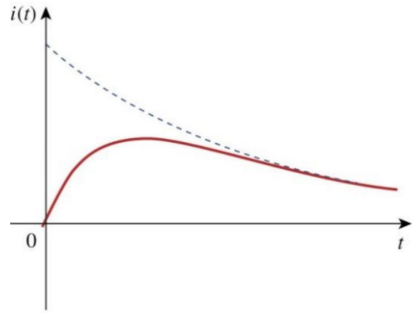
$$\omega_o = 1/\sqrt{LC} \qquad \boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}\,, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}}$$

- ω_o is known as the undamped natural frequency; α is the damping factor
- Depending on the values of s_1 and s_2, we have three cases:

Source-free Series RLC Circuit

Overdamped case ($\alpha > \omega_o$). $\alpha > \omega_o$ is equivalent to $C > 4L/R^2$. When this happens, both s_1 and s_2 are real and negative, and $s_1 \neq s_2$. The response is represented by the following form.

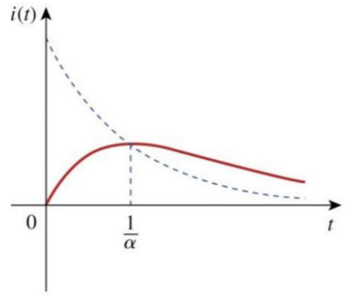
$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$



(a)

Critically damped case ($\alpha = \omega_o$). $\alpha = \omega_o$ is equivalent to $C = 4L/R^2$. When this happens, both $s_1 = s_2 = -\alpha = -R/(2L)$. The response is represented by the following form.

$$\boxed{i(t) = (A_1 + A_2 t)e^{-\alpha t}}$$



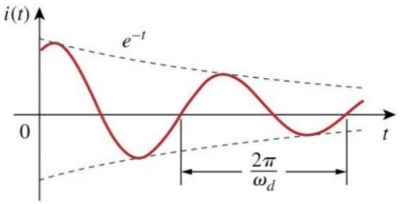
(b)

Underdamped case ($\alpha < \omega_o$). In this case, $C < 4L/R^2$. The roots s_1 and s_2 may be written as

$$\boxed{\begin{aligned} s_1 &= -\alpha + \sqrt{-(\omega_o^2 - \alpha^2)} = -\alpha + j\omega_d \\ s_2 &= -\alpha - \sqrt{-(\omega_o^2 - \alpha^2)} = -\alpha - j\omega_d \end{aligned}}$$

where $j \triangleq \sqrt{-1}$ and $\omega_d \triangleq \sqrt{\omega_o^2 - \alpha^2}$. The response is represented by the following form.

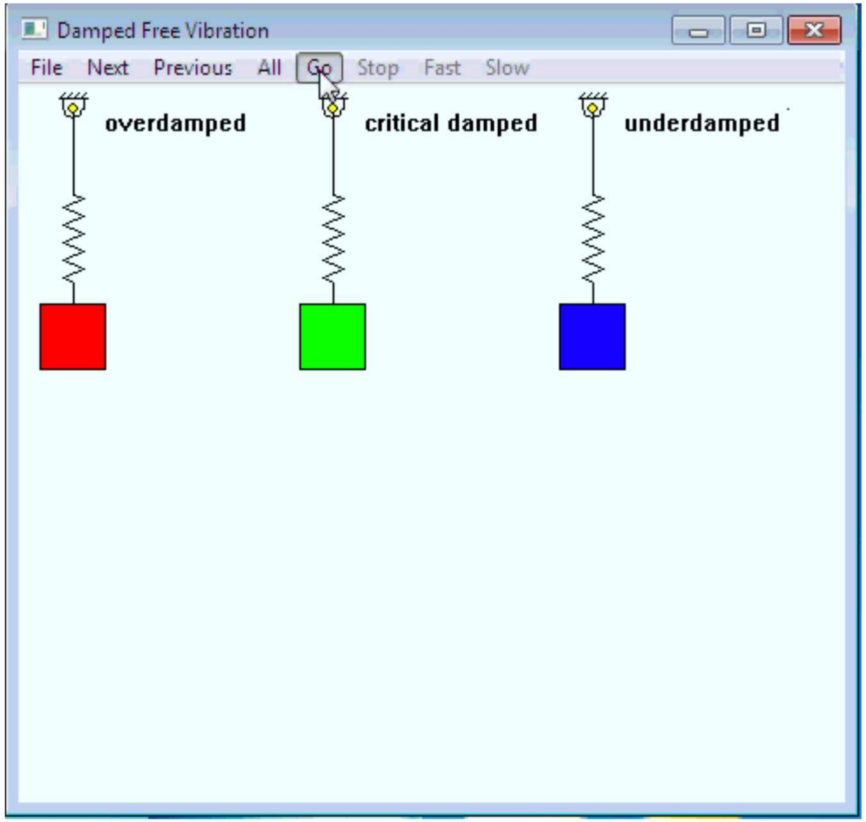
$$\boxed{i(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}}$$

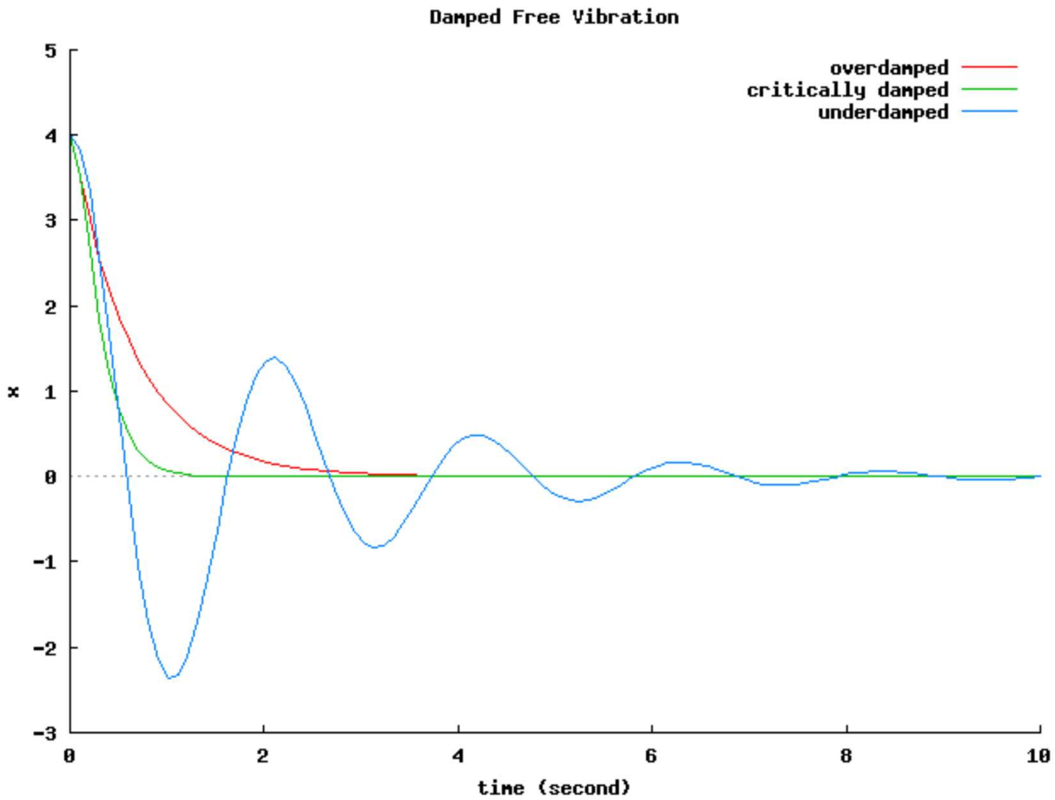


(c)

Source-free Series RLC Circuit (cont'd)

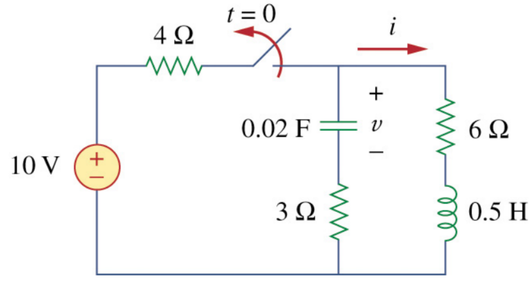
- Once $i(t)$ is found, other circuit quantities such as individual element voltages can easily be found.
- The behavior of the above network is captured by the idea of damping, which is the gradual loss of the initial stored energy.
- The damping effect is due to the presence of resistance R.
- Special case: If $R = 0$, then $\alpha = 0$, and we have an LC circuit. Since $\alpha < \omega_o$ in this case, the response is not only undamped but also oscillatory. The circuit is said to be loss-less because of the absence of the dissipating or damping element (R).
- By adjusting the value of R, the response may be made undamped, overdamped, critically damped or underdamped.
- **Oscillatory response** is possible due to the presence of the two types of storage elements, allowing the energy to flow back and forth between them.
- In general, it is difficult to tell from the waveforms the difference between the overdamped and critically damped responses.
- With the same initial conditions, critically damped case decays the fastest, while the overdamped case takes the longest time to settle.





Source: https://www.softintegration.com/docs/ch/qanimate/examples/vibration/

Example (1): Second-Order Circuits



Question: Find $i(t)$ in the circuit shown. Assume that the circuit has reached steady state at $t = 0^-$.

- For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a short circuit. The equivalent circuit is shown in Fig. (a)
- Thus, at $t = 0^-$, $i(0^-) = 10 / (4 + 6) = 1$ A and $v(0^-) = 6i(0^-) = 6$ V
- For $t > 0$, the switch is opened and the voltage source is disconnected. With the 6 Ω and 3 Ω resistors combined, the equivalent circuit is shown in Fig. (b), which is a source-free series *RLC* circuit.
- s_1 and s_2 are calculated as:

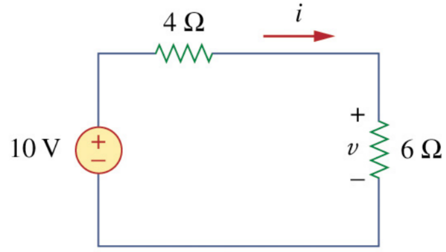
$$\alpha = R/2L = 9/2(0.5) = 9, \qquad \omega_o = 1/\sqrt{LC} = 1/\sqrt{(0.5)(0.02)} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -9 \pm \sqrt{81 - 100} = -9 \pm j4.359$$

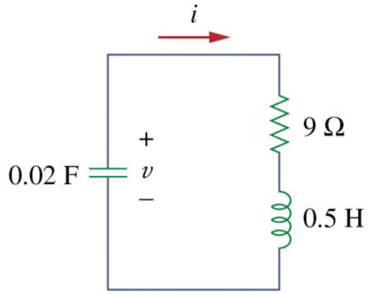
- Hence, the response is underdamped and is expressed in the following form.

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \qquad (1)$$

- We now obtain A_1 and A_2 using the **initial conditions**.



(a)



(b)

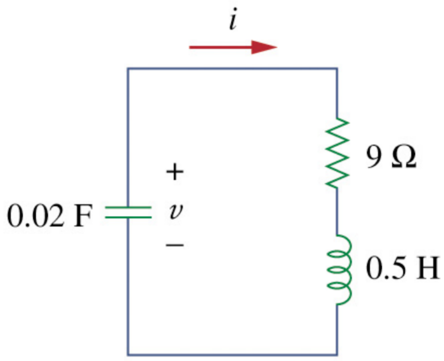
Example (1): Second-Order Circuits (cont'd)

- Since there are two unknowns, we need **two** initial conditions.
- Since the inductor current cannot change instantly, we have the **first initial condition** $i(0) = 1 = A_1$.
- Apply the KVL to the circuit in Fig. (b), we have $-v(t) + 9i(t) + L\frac{di(t)}{dt} = 0$
- At $t = 0$, the equation reduces to $$\frac{di(0)}{dt} = \frac{1}{L}(v(0) - 9i(0)) = \frac{1}{0.5}(6 - 9(1)) = -6\text{A}/s$$
- This forms the second initial condition
- Taking the derivative of $i(t)$ in eq. (1) in the previous slide, we get

$$\frac{di(t)}{dt} = -9e^{-9t}(A_1 \cos 4.395t + A_2 \sin 4.395t) + e^{-9t}(4.395)(-A_1 \sin 4.395t + A_2 \cos 4.395t)$$

- Put $t = 0$, we have $\frac{di(0)}{dt} = -9A_1 + 4.395A_2$
- With $A_1 = 1$ and the **second initial condition**, we obtain $-6 = -9 + 4.359A_2$ or $A_2 = 0.6882$
- Substituting the values of A_1 and A_2 into (1) yields the complete solution

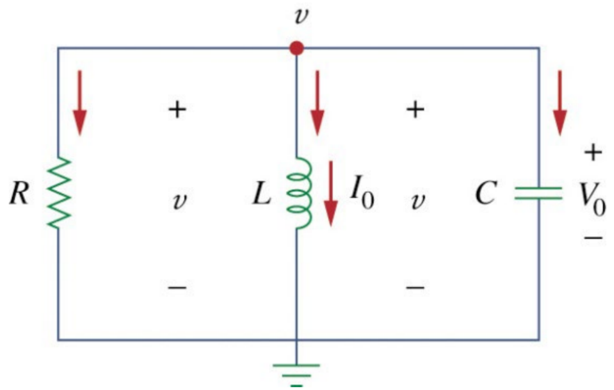
$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t)\text{A}$$



(b)

Source-free Parallel RLC Circuit

- Consider the parallel *RLC* circuit.
- Assume the initial inductor current is I_o and the initial capacitor voltage is V_o.
- Applying the KCL at the top node



$$\frac{v(t)}{R}+\frac{1}{L}\int_{-\infty}^{t} v(t')dt'+C\frac{dv(t)}{dt}=0$$

- Taking derivative with respect to t and dividing by C results in

$$\frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=0$$

- The characteristic equation is $s^2+\frac{1}{RC}s+\frac{1}{LC}=0$
- The roots of the characteristic equation are $s_{1,2}=-\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$ $\Leftrightarrow$ $s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_o^2}$
- where $\alpha=1/(2RC)$, $\omega_o=1/\sqrt{LC}$

Source-free Parallel RLC Circuit

Three possible solutions:

Overdamped case ($\alpha > \omega_o$). This occurs when $L > 4R^2C$ and the roots of the characteristic equation are real and distinct. The form of the response is:

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

Critically damped case ($\alpha = \omega_o$). This occurs when $L = 4R^2C$ and the roots of the characteristic equation are real and equal. The form of the response is:

$$\boxed{v(t) = (A_1 + A_2 t) e^{-\alpha t}}$$

Underdamped case ($\alpha < \omega_o$). This occurs when $L < 4R^2C$ and the roots of the characteristic equation are complex and may be expressed as

$s_{1,2} = -\alpha \pm j\omega_d$ where $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$. The form of the response is

$$\boxed{v(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t) e^{-\alpha t}}$$

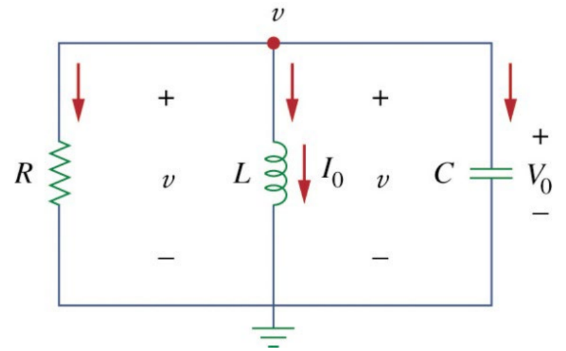
The constants A_1 and A_2 in each case can be determined from the initial conditions.

Example (2): Source-free Parallel RLC Circuit

Question: For the parallel RLC circuit, find $v(t)$ for $t > 0$.

Assume $v(0) = V_o = 5$ V, $i_L(0) = I_o = 0$, $L = 1$ H, and $C = 10$ mF.

Consider three cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$



- Case 1. If $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times 1.923\times 10\times 10^{-3}} = 26$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 10\times 10^{-3}}} = 10$$

- Since $\alpha > \omega_o$, the response is overdamped. The roots of the characteristic equation are: $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -2, -50$
- The corresponding form of response is: $v(t) = A_1 e^{-2t} + A_2 e^{-50t}$
- We now apply the initial conditions to get A_1 and A_2. The first initial condition is $v(0) = 5$ V.
- To obtain the second initial condition, we apply KCL at the top node,

$$\frac{v(t)}{R} + i_L(t) + C\frac{dv(t)}{dt} = 0 \qquad \Rightarrow \qquad \frac{dv(0)}{dt} = -\frac{v(0) + i_L(0)R}{RC} = -260$$

Example (2): Source-free Parallel RLC Circuit (cont'd)

- Since $v(0) = A_1 + A_2$,

$$\frac{dv(t)}{dt} = -2A_1e^{-2t} - 50A_2e^{-50t} \quad \Rightarrow \quad \frac{dv(0)}{dt} = -2A_1 - 50A_2$$

- Therefore, $\begin{cases} A_1 + A_2 = 5 \\ -2A_1 - 50A_2 = -260 \end{cases}$

- Solving the equations, we obtain $A_1 = -0.2083$ and $A_2 = 5.208$.
- The complete response is thus $v(t) = -0.2083e^{-2t} + 5.208e^{-50t}$

- Case 2. When $R = 5\ \Omega$, $\alpha = \frac{1}{2RC} = \frac{1}{2\times5\times10\times10^{-3}} = 10$ and ω_o remains the same.

- Since $\alpha = \omega_o$ in this case, the response is critically damped and $s_1 = s_2 = -10$ and the form of response is $v(t) = (A_1 + A_2t)e^{-10t}$

- To get A_1 and A_2, we apply the initial conditions $v(0) = 5$. Hence, $\frac{dv(0)}{dt} = -\frac{v(0) + i_L(0)R}{RC} = -\frac{5+0}{5\times10\times10^{-3}} = -100$

Example (2): Source-free Parallel RLC Circuit (cont'd)

- By differentiating $v(t) = (A_1 + A_2 t)e^{-10t}$ with respect to t, we have $\frac{dv(t)}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$
- Therefore, the two initial conditions translate to

$$\begin{cases} v(0) = A_1 = 5 \\ \dfrac{dv(0)}{dt} = -10A_1 + A_2 = -100 \end{cases} \quad \Rightarrow \quad A_1 = 5,\ A_2 = -50$$

- The complete response is $v(t) = (5 - 50t)e^{-10t}\text{V}$
- Case 3. When $R = 6.25\ \Omega$, $\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$ while ω_o= 10 remains the same.
- As $\alpha < \omega_o$ in this case, the response is underdamped. The roots of the characteristic equation are

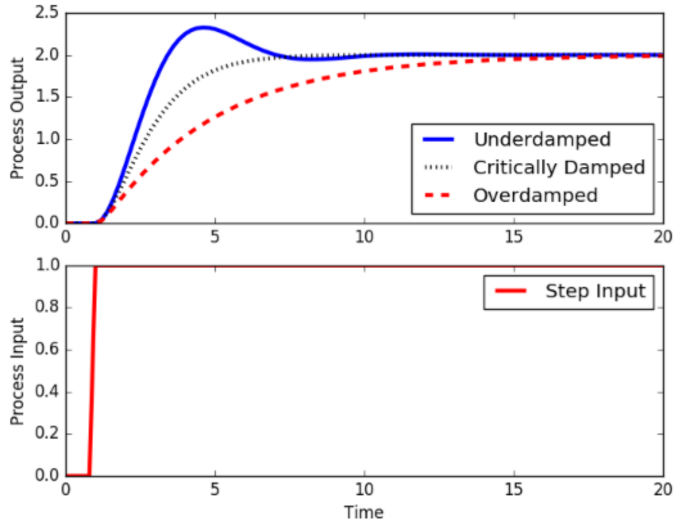
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} = -8 \pm j6$$

- Hence, the form of response is $v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$
- The initial conditions are $v(0) = 5$ and

$$\frac{dv(0)}{dt} = -\frac{v(0) + i_L(0)R}{RC} = -\frac{5+0}{6.25 \times 10 \times 10^{-3}} = -80$$

Example (2): Source-free Parallel RLC Circuit (cont'd)

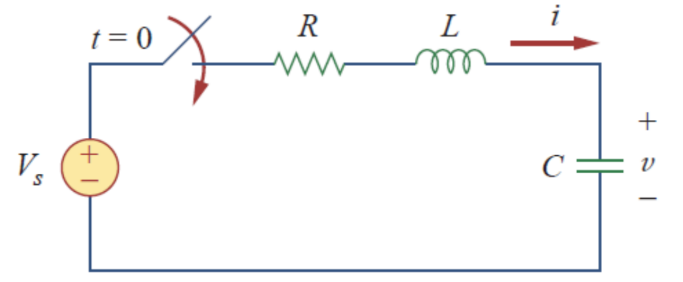
- Differentiating the response gives $\dfrac{dv(t)}{dt} = (-8A_1\cos 6t - 8A_2\sin 6t - 6A_1\sin 6t + 6A_2\cos 6t)e^{-8t}$.
- At $t = 0$, $-80 = -8A_1 + 6A_2$.
- Because $v(0)=A_1=5$, we have $A_2 = -6.667$.
- The complete response is $v(t) = (5\cos 6t - 6.667\sin 6t)e^{-8t}$
- Notice that in parallel *RLC* circuit, increasing the value of *R*, the degree of damping decreases (Yet, in series *RLC* circuit, increasing the value of *R*, the degree of damping increases).



Source: apmonitor.com

Step Response of a Series RLC Circuit

- The step response is obtained by the sudden application of a dc source.
- Consider the series RLC circuit. Apply KVL around the loop for t > 0.



$$i = C\frac{dv}{dt} \qquad L\frac{di}{dt} + Ri + v = V_s$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

The solution has two components: (1) Transient response; (2) Steady-state response.

$$v(t) = v_t(t) + v_{ss}(t)$$

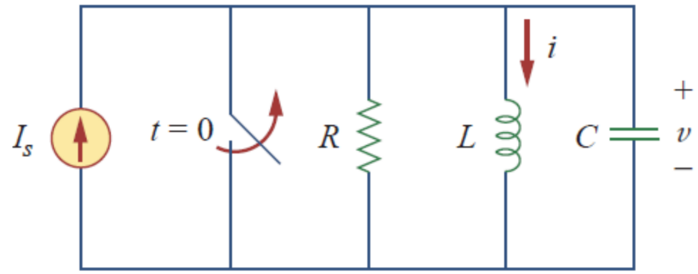
The final value of the capacitor voltage is the same as the source voltage.

$$v_{ss}(t) = v(\infty) = V_s$$

$$\boxed{\begin{aligned} v(t) &= V_s + A_1e^{s_1t} + A_2e^{s_2t} && \text{(Overdamped)} \\ v(t) &= V_s + (A_1 + A_2t)e^{-\alpha t} && \text{(Critically damped)} \\ v(t) &= V_s + (A_1\cos\omega_d t + A_2\sin\omega_d t)e^{-\alpha t} && \text{(Underdamped)} \end{aligned}}$$

Step Response of a Parallel RLC Circuit

- Consider the parallel RLC circuit.
- We want to find *i* due to a sudden application of a dc current.
- Applying KCL at the top node for t > 0, we can write:



$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \qquad v = L\frac{di}{dt}$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

The complete solution consists of the transient response and the steady-state response

$$i(t) = i_t(t) + i_{ss}(t)$$

$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

$$i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$i(t) = I_s + (A_1 \cos\omega_d t + A_2 \sin\omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$