First-Order Circuits

Dr. Albert T. L. Lee

Department of Electrical and Electronic Engineering
Faculty of Engineering
The University of Hong Kong

Topics

- ✓ Source-free RC/RL circuit
- ✓ Step response of RC/RL circuit
- ✓ First-Order Op Amp Circuits

First-Order Circuits

- We have considered the three passive elements (resistors, capacitors, and inductors) and one active element (op amp) individually, we are prepared to consider circuits that contain various combinations of two or three of the passive elements.

- We will carry out the analysis of *RC* or *RL* circuits by applying Kirchhoff's Laws.

- Applying Kirchhoff's Law to pure *resistive circuits* generates *algebraic equations.*

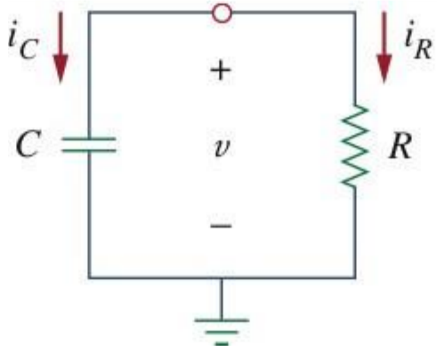
- Applying Kirchhoff's Law to *RC and RL circuits* produces *differential equations.*

- The differential equations resulting from analyzing *RC* and *RL* circuits are of the first order.

We assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors.

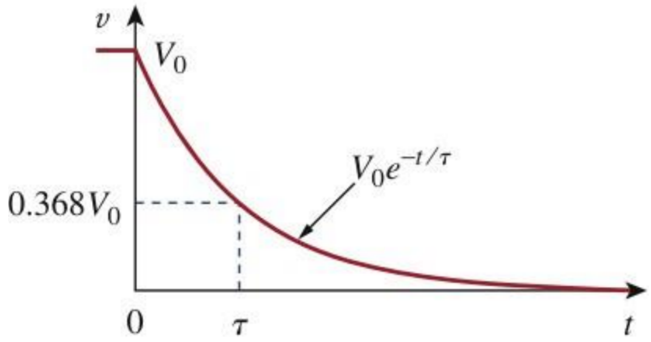
Source-free RC Circuit

- Consider a parallel combination of **a resistor and an initially charged capacitor**.
- The energy stored in the capacitor is released to the resistor.
- Of course, after a very long time, all the energy in the capacitor would be released and dissipated in the resistor. At that time, the voltage across the capacitor would be zero and the current in the circuit would also be zero.
- But, how would the circuit behave during discharge?
- Since the capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is $v(0) = V_o$, with the corresponding energy stored $w(0) = (1/2)\times C\times V_o^2$.
- Applying KCL at the top node yields $i_C + i_R = 0$.
- By definition, $i_C = Cdv/dt$ and $i_R = v/R$.
- Thus, $\dfrac{dv}{dt} + \dfrac{v}{RC} = 0$. This is the first-order differential equation!



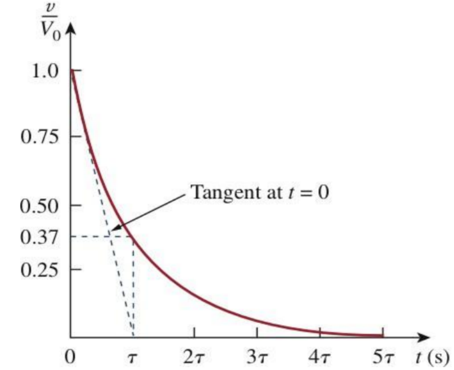
Source-free RC Circuit

- To solve it, we rearrange the terms as $\frac{dv}{v} = -\frac{1}{RC}dt$
- Integrating both sides, we get (A is the integration constant)
- Thus, $\ln\frac{v}{A} = -\frac{t}{RC}$ => $\ln v = -\frac{t}{RC} + \ln A$
- Taking powers of e produces $v(t) = Ae^{-t/RC}$
- With the initial condition $v(0) = V_o$, we have $A = V_o$
- Hence, the final solution is $v(t) = V_o e^{-t/RC}$
- The voltage response for the RC circuit is an exponential decaying function, starting from its initial value V_o
- The rapidity with which the voltage decreases is expressed in terms of the time constant, denoted by $\tau = RC$
- In terms of the time constant, we have $\boxed{v(t) = V_o e^{-t/\tau}}$
- Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the <u>natural response</u> of the circuit.



Source-free RC Circuit

- At $t = 5\tau$, $v(t = 5\tau)/V_o = 0.00674 < 1\%$. It is customary to assume that the capacitor is fully discharged after 5 time constants

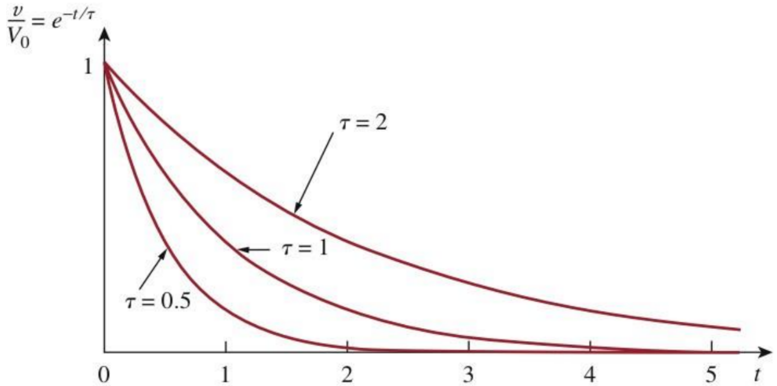


- Another meaning of time constant:
  Draw a tangent at $t = 0$, τ is the intercept at time axis
- Smaller the time constant (smaller resistance to the current), the faster the response (discharge)
- Since

$$i_R(t) = \frac{v(t)}{R} = \frac{V_o}{R} e^{-t/\tau}$$

  The power dissipated in the resistor is

$$p(t) = v(t) i_R(t) = \frac{V_o^2}{R} e^{-2t/\tau}$$

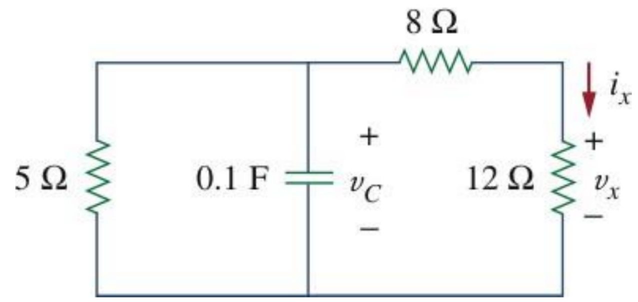


- The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p(t')dt' = \int_0^t \frac{V_o^2}{R} e^{-2t'/\tau} dt' = -\frac{\tau V_o^2}{2R} e^{-2t'/\tau}\Big|_0^t = \frac{1}{2} C V_o^2 (1 - e^{-2t/\tau})$$

- As $t \to \infty$, $w_R(\infty) \to (1/2)CV_o^2$. That is, all the energy initially stored in the capacitor is dissipated in the resistor.

Example (1): Source-free RC Circuit

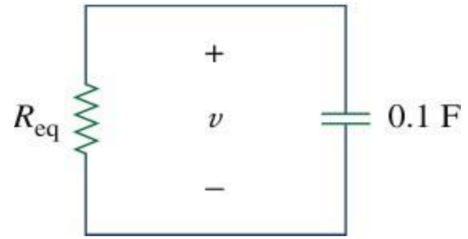


Question: Let $v_C(0) = 15$ V. Find v_C, v_x and i_x for $t > 0$.

Solution:

- First, we first need to make the circuit in the original figure conform with the standard source-free RC circuit.
- The 8 Ω and 12 Ω are in series and they are in parallel with the 5 Ω resistor.
- The combined resistance is $R_{eq} = (8+12)(5) / (8+12+5) = 4\ \Omega$.
- The time constant $\tau = R_{eq}C = 4\ (0.1) = 0.4$ s.
- Thus, $v_C(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.4} = 15e^{-2.5t}$
- To find v_x, we can use the voltage division formula, i.e.,

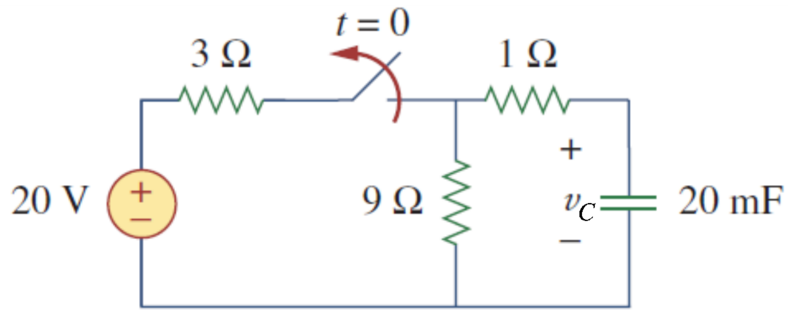
$$v_x(t) = \frac{12}{12+8} v_C(t) = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\,\mathrm{V}$$



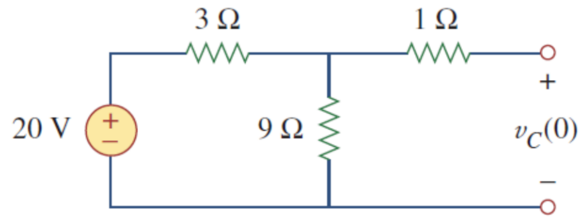
- Finally, $i_x = v_x / 12 = 0.75e^{-2.5t}$ A

Example (2): Source-free RC Circuit

Question: The switch in the circuit has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

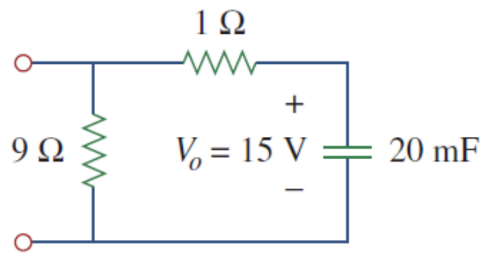


$$v_C(t) = \frac{9}{9+3}(20) = 15\text{ V}, \qquad t < 0$$



Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor before and immediately after the switch is open at $t = 0$ will be the same. That is, $v_C(0) = 15$ V.

For t > 0, the switch is opened, and we have the RC circuit.



$\tau = R_{eq}C = (9 + 1) \times 20 \times 10^{-3} = 0.2$ s

$v_C(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2} = 15e^{-5t}$ V

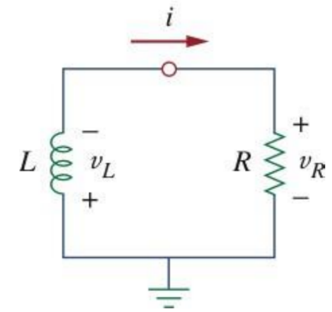
The initial energy stored in the capacitor is:
$w_C(0) = \frac{1}{2}Cv^2_C(0) = 2.25$ J

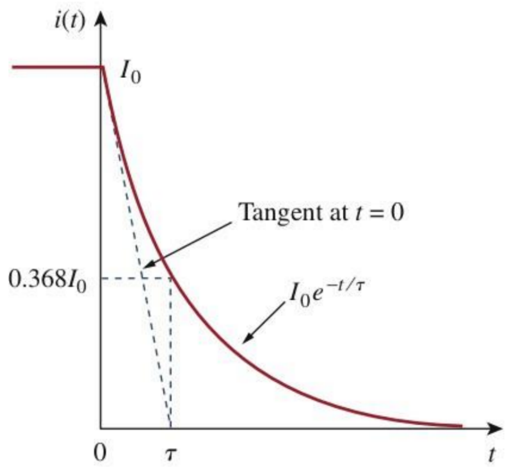
Example (2): Source-free RL Circuit

- Consider the parallel connection of a resistor and an inductor.
- We want to determine the response for $i(t)$.
- We select the inductor current as the response to be determined in order to take advantage of the idea that the inductor current cannot change instantaneously.
- At $t = 0$, we assume that the inductor has an initial current $i(0) = I_o$.
- Applying KVL around the loop, we have $v_L + v_R = 0$.
- But $v_L = L\,di/dt$ and $v_R = iR$. Thus, by rearranging the terms and integrating gives

$$\int_{I_o}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt' \quad \Rightarrow \quad \ln i \Big|_{I_o}^{i(t)} = -\frac{Rt'}{L}\Big|_0^t \quad \Rightarrow \quad \ln \frac{i(t)}{I_o} = -\frac{Rt}{L}$$

- Taking the powers of e, and define $\tau = L/R$, we have $\boxed{i(t) = I_o e^{-t/\tau}}$
- The current response is an exponential decay of the initial current.





Source-free RL Circuit

- Since $v_R(t) = iR = I_o R e^{-t/\tau}$, the power dissipated in the resistor is:

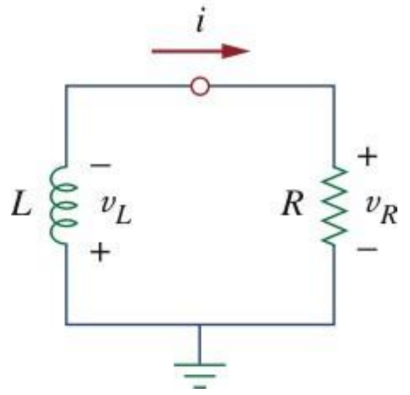
$$p(t) = v_R(t) i(t) = I_o^2 R \cdot e^{-2t/\tau}$$

- The energy absorbed by the resistor is:

$$w_R(t) = \int_0^t p(t')dt' = \int_0^t I_o^2 R \cdot e^{-2t'/\tau} dt' = -\frac{1}{2}\tau I_o^2 R \cdot e^{-2t'/\tau}\Big|_0^t$$

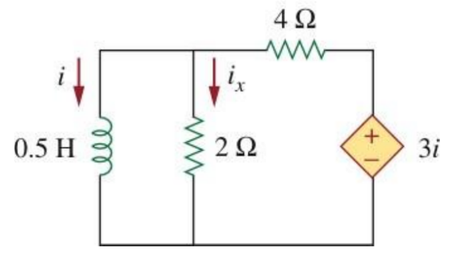
$$= \frac{1}{2} L I_o^2 (1 - e^{-2t/\tau})$$

- As $t \to \infty$, $w_R(\infty) \to (1/2)LI_o^2$. That is, the initial energy stored in the inductor is eventually dissipated in the resistor.



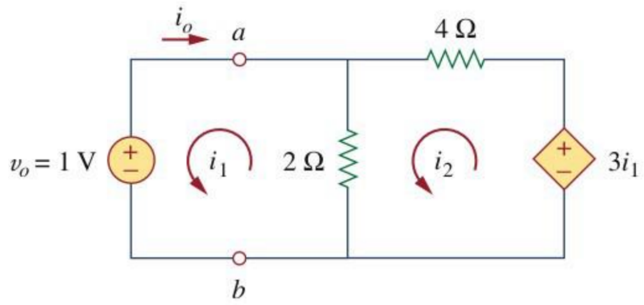
Example (3): Source-free RL Circuit

Question: Assuming that $i(0)$ = 10 A, calculate $i(t)$ and $i_x(t)$ in the circuit.



Solution:

- We have to obtain the equivalent resistance at the inductor terminal and then apply $i(t) = I_o e^{-t/\tau}$
- To find the equivalent resistance, we disconnect the inductor and insert a voltage source of 1 V at the inductor terminal.



- Applying KVL to the two loops results in

$$\begin{cases} 2(i_1 - i_2) + 1 = 0 \\ 4i_2 + 2(i_2 - i_1) - 3i_1 = 0 \end{cases}$$

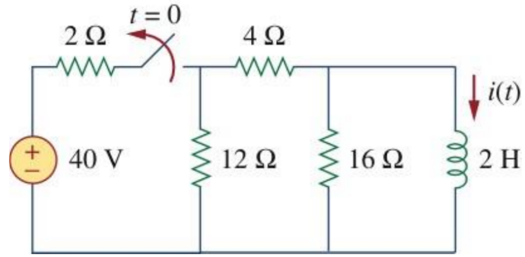
- Simplifying and solving the equations gives

$$i_1 = -3\text{A}, \quad i_o = -i_1 = 3\text{A}$$

- Hence $R_{eq} = v_o/i_o$ = 1/3 Ω.
- The time constant is $\tau = L/R_{eq}$ = 0.5/(1/3) = 3/2 s
- Thus, the current through the inductor is $i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}$ A
- The voltage across the inductor is $v = L\,di/dt = (-10/3)e^{-(2/3)t}$ V
- Since the inductor and the 2Ω resistor are in parallel, $i_x = v/2 = -1.6667e^{-(2/3)t}$ A

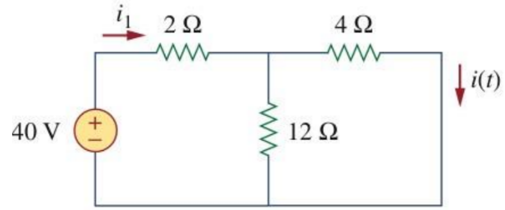
Example (4): Source-free RL Circuit

Question: The switch in the circuit has been closed for a long time.
At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.

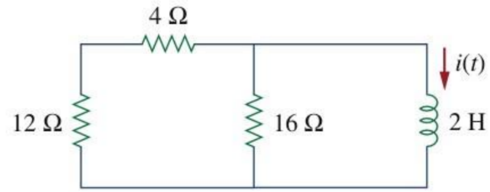


Solution:

- For $t < 0$, the switch is closed, and the inductor act as a short circuit.
  The 16Ω resistor is short-circuited, the resulting circuit is shown in Fig (a).
- In Fig (a), $i_1 = 40 / (4 // 12 + 2) = 8$ A
- By current division, $i = \dfrac{12}{12+4} i_1 = 6 \text{ A}$
- Since the current through an inductor cannot change instantaneously, the current before and immediately after the switch opens is $i(0) = 6$ A.
- When $t > 0$, the switch is open and the voltage source is disconnected, we now have a source-free RL circuit, as shown in Fig. (b).
- Combining the resistors, we have $R_{eq} = (12 + 4) // 16 = 8\ \Omega$.
- The time constant is $\tau = L/R_{eq} = 2/8 = 1/4$ s.
- Thus, $i(t) = i(0)e^{-t/\tau} = 6e^{-4t} \text{A}$



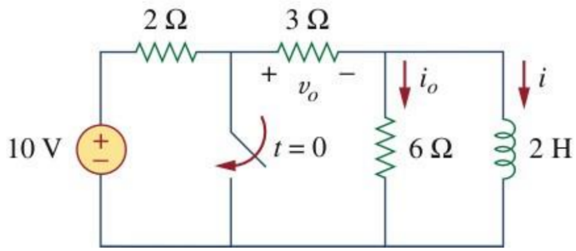
(a)



(b)

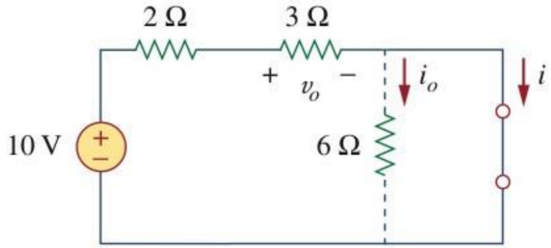
Example (5): Source-free RL Circuit

Question: In the circuit shown, find i_o, v_o and i for all time, assuming that the switch was open for a long time.



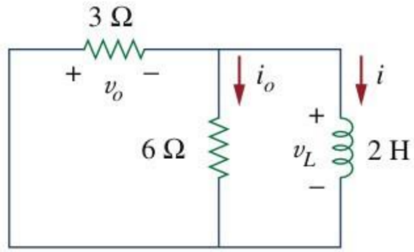
Solution:

- For $t < 0$, the switch is open. Since the inductor acts like a short circuit in DC, the 6 Ω resistor is short-circuited, so we have the circuit shown in Fig. (a).
- Hence, $i_o = 0$ and $i(t) = \dfrac{10}{2+3} = 2\text{A}$; $\quad v_o(t) = 3i(t) = 6\text{V} \qquad$ for $t < 0$



(a)

- For $t > 0$, the switch is closed. So, the voltage source is short-circuited. Now we have a source-free RL circuit, as shown in Fig. (b).
- At the inductor terminals, $R_{eq} = 3 \,//\, 6 = 2\ \Omega$. Hence, the time constant is: $\tau = L/R_{eq} = 2/2 = 1$ s
- Therefore, $i(t) = i(0)e^{-t/\tau} = 2e^{-t}$ A



(b)

- Since the inductor is in parallel with the 6 Ω and 3 Ω resistors,

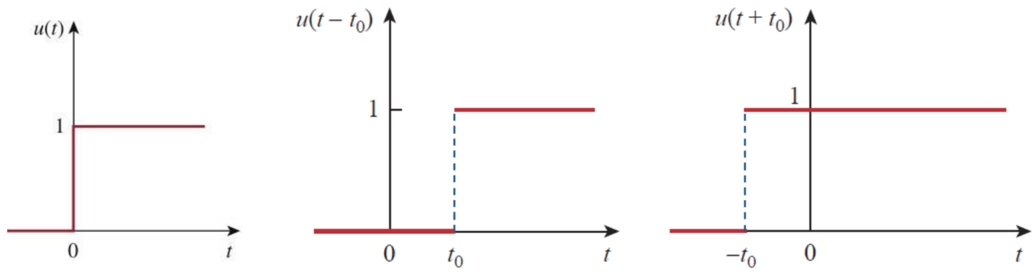
$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\text{V} \qquad \text{for } t > 0$$

$$i_o(t) = v_L/6 = -(2/3)e^{-t}\text{A} \qquad \text{for } t > 0$$

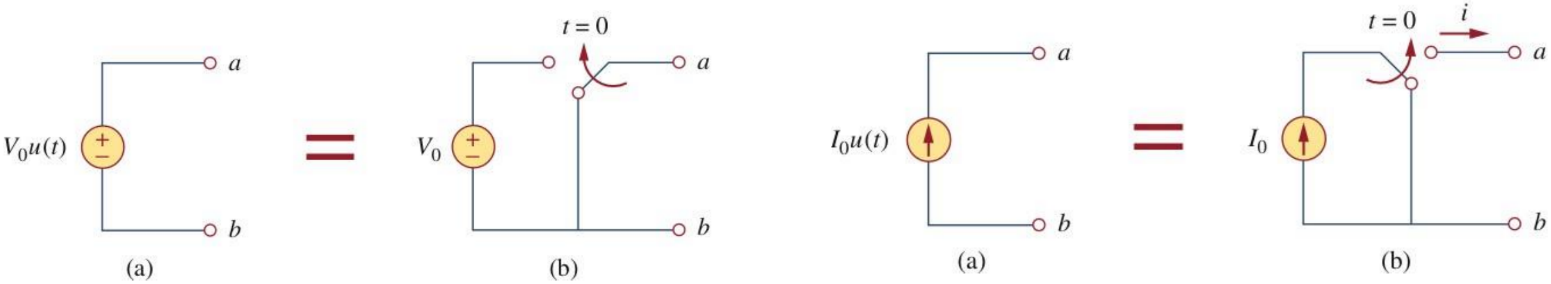
Step Response of an RC Circuit

- Unit step function:

$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$



- When a dc source is suddenly applied, the voltage or current source can be modeled as a step function

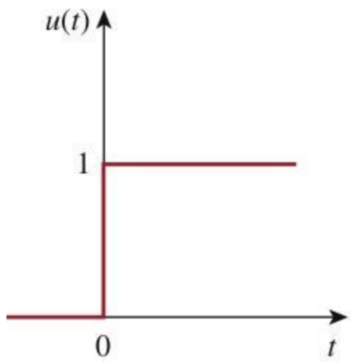


- The response of a circuit to such a sudden application of a dc voltage or current source is called a step response.

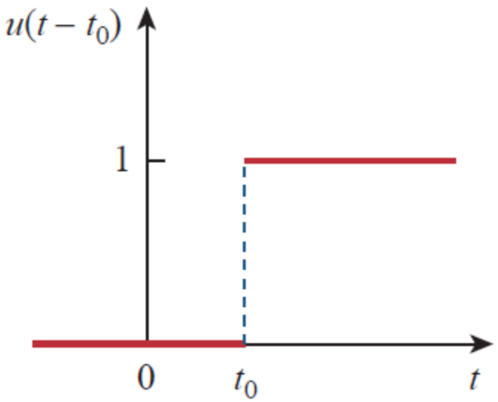
Step Response of an RC Circuit

- Unit step function:

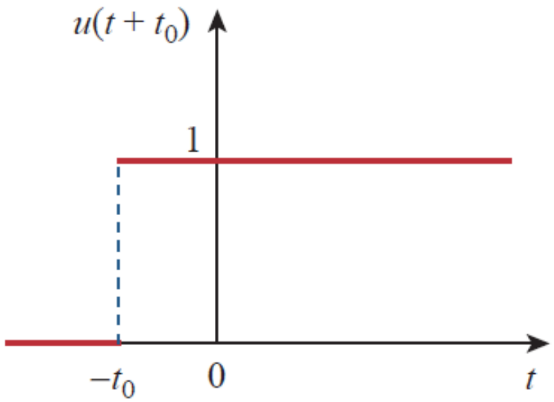
$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$



Singularity functions are functions that either are discontinuous or have discontinuous derivatives.



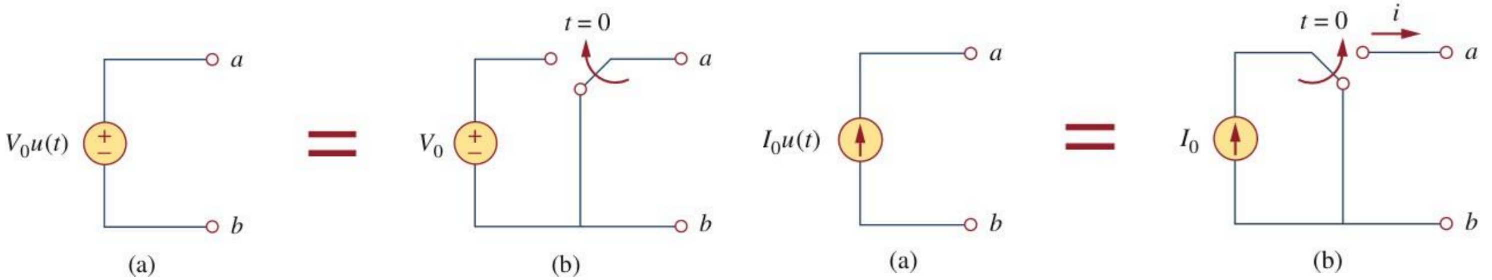
The unit step function delayed by t_0.



The unit step advanced by t_0.

Step Response of an RC Circuit

- Unit step function:
- When a dc source is suddenly applied, the voltage or current source can be modeled as a step function
- The response of a circuit to such a sudden application of a dc voltage or current source is called step response



Step Response of an RC Circuit

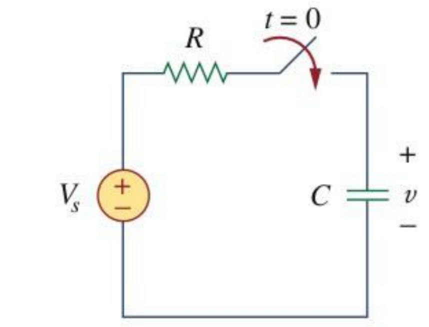
- Consider the *RC* circuit shown in Fig. (a) which can be replaced by the circuit in Fig. (b).
- We select the capacitor voltage as the circuit response to be determined.
- Assume the initial voltage of the capacitor is V_o.
- Since the voltage of a capacitor cannot change instantaneously, $v(0^-) = v(0^+) = V_o$, where $v(0^-)$ is the voltage across the capacitor just before switching, and $v(0^+)$ is its voltage immediately after switching
- Applying KCL, we have (for $t > 0$)

$$C\frac{dv}{dt} = \frac{V_s - v}{R} \qquad \Rightarrow \qquad \frac{dv}{v - V_s} = -\frac{dt}{RC}$$

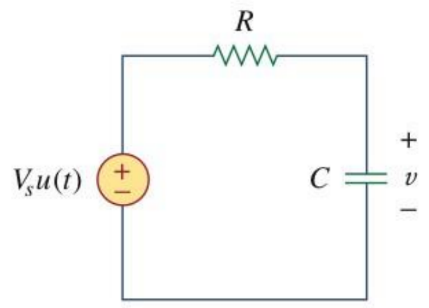
- Integrating both sides and introducing the initial conditions, we have

$$\ln(v' - V_s)\Big|_{V_o}^{v(t)} = -\frac{t'}{RC}\Big|_0^t \qquad \Rightarrow \qquad \ln\frac{v(t) - V_s}{V_o - V_s} = -\frac{t}{RC}$$

- Taking the exponential of both sides, we have $\boxed{v(t) = V_s + (V_o - V_s)e^{-t/\tau}}$



(a)

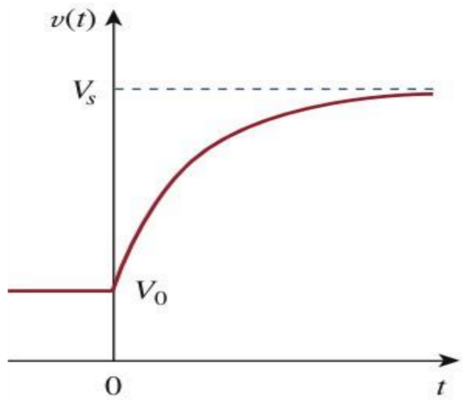


(b)

Step Response of an RC Circuit

- Combining with the capacitor voltage for $t < 0$, we have

$$v(t) = \begin{cases} V_o & t < 0 \\ V_s + (V_o - V_s)e^{-t/\tau} & t > 0 \end{cases}$$



- Special case: When $V_o = 0$, $v(t) = V_s(1-e^{-t/\tau})\, u(t)$
- The complete circuit response can be decomposed in two ways:

1. $$\underbrace{v(t)}_{\text{complete response}} = \underbrace{V_o e^{-t/\tau}}_{\text{natural response}} + \underbrace{V_s(1-e^{-t/\tau})}_{\text{forced response}}$$

   - The natural response is due to the initial stored charge, it is the same as the source-free RC circuit response.
   - The forced response is due to the application of external source, it is the same as the above special case when $V_o = 0$.

2. $$\underbrace{v(t)}_{\text{complete response}} = \underbrace{(V_o - V_s)e^{-t/\tau}}_{\text{transient response}} + \underbrace{V_s}_{\text{steady state response}}$$

   - The transient response is temporary.
   - The steady-state response is the response as $t \to \infty$.

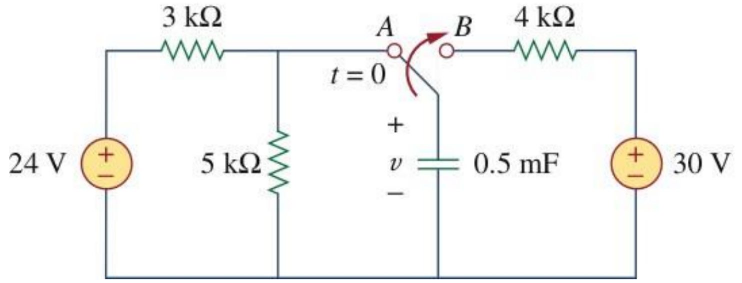
Step Response of an RC Circuit

- Source-free *RC* response: $v(t) = V_o e^{-t/\tau}$
- *RC* circuit step response: $v(t) = V_s + (V_o - V_s)e^{-t/\tau}$
- Notice that for source-free *RC* circuit, it is equivalent to V_s=0 in step response
- Therefore, both cases can be summarized using the response

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}}$$

where $v(0)$ is the initial voltage at t=0^+ and $v(\infty)$ is the final or steady-state value.

Example: The switch has been in position *A* for a long time. At t = 0, the switch moves to *B*. Determine $v(t)$ for t > 0 and calculate its values at t = 1 s and 4 s.



- We have to find $v(0)$, $v(\infty)$ and τ.
- For t < 0, the switch is at position *A*. The capacitor acts like an open circuit in DC, v is the same as the voltage across the 5 kΩ resistor:

$$v(0^-) = \frac{5}{5+3}(24) = 15\text{V}$$

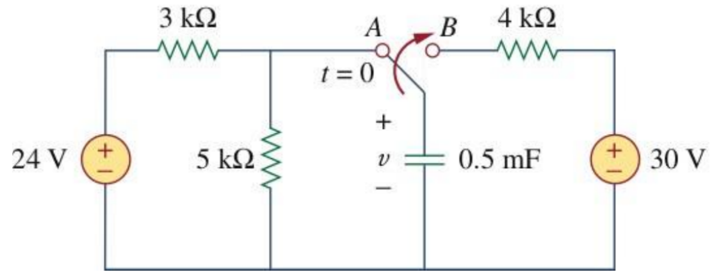
- Since the capacitor voltage cannot change instantaneously, $v(0) = v(0^+) = v(0^-)$ = 15 V.

Step Response of an RC Circuit

- For $t > 0$, the switch is in position B, the circuit becomes an simple RC circuit with step input
- The time constant is $\tau = RC = 4\times10^3 \times 0.5\times10^{-3} = 2$ s
- Also notice that $v(\infty) = 30$ V
- Thus, $v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$

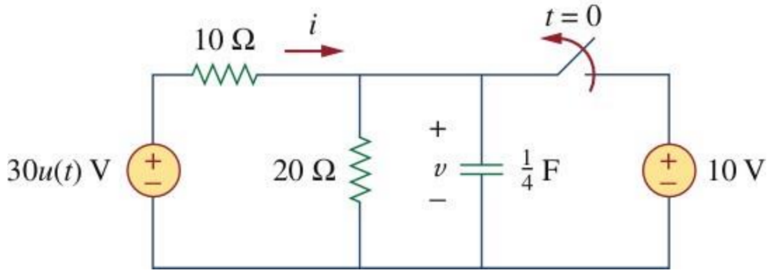
$$= 30 + (15 - 30)e^{-t/2}$$

$$= 30 - 15e^{-0.5t}$$



- $v(1) = 30 - 15e^{-0.5} = 20.9$ V, $v(4) = 30 - 15e^{-2} = 27.97$ V

Example: In the figure shown, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.



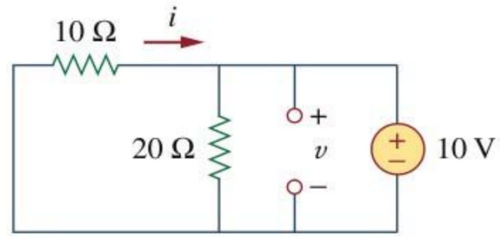
- The resistor current i can be discontinuous at $t = 0$, while the capacitor voltage cannot. Hence, it is better to find v and then obtain i from v.
- We have to find $v(0)$, $v(\infty)$ and τ.
- For $t < 0$, the switch is closed and the 30 V source has no effect (it can be replaced by a short circuit). Furthermore, the capacitor acts like an open circuit since the switch has been closed for a long time.

Step Response of an RC Circuit

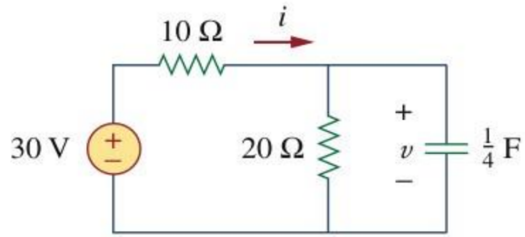
- So, the circuit becomes that in Fig. (a).
- From this circuit we obtain $v = 10$ V, $I = -v/10 = -1$ A.
- Since the capacitor voltage cannot change instantaneously $v(0) = v(0^-) = 10$ V.
- For $t > 0$, the switch is opened and the 10 V source is disconnected from the circuit. The 30V source is now operative, so the circuit becomes that in Fig. (b).
- After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again. By voltage division,

$$v(\infty) = \frac{20}{20+10}(30) = 20\text{V}$$

- To obtain the time constant, we have to find out the Thevenin equivalent circuit at the capacitor terminals
- The Thevenin resistance at the capacitor terminals is: $R_{TH} = 10 \| 20 = \frac{20}{3}\Omega$
- The time constant is: $\tau = R_{TH}C = \frac{20}{3}\cdot\frac{1}{4} = \frac{5}{3}s$



(a)



(b)

Step Response of an RC Circuit

- Thus, $v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$

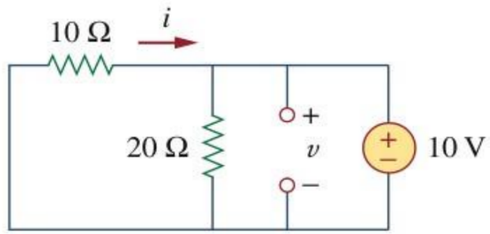
$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\text{V}$$

- To obtain i, we notice from Fig. (b) that $i = (30 - v) / 10 = (1 + e^{-0.6t})$ A

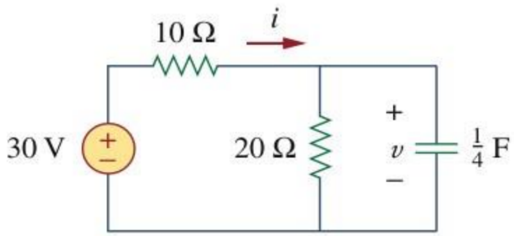
- Notice that the capacitor voltage is continuous while the resistor current is not.

$$v(t) = \begin{cases} 10\text{V} & t < 0 \\ (20 - 10e^{-0.6t})\text{V} & t \geq 0 \end{cases}$$

$$i(t) = \begin{cases} -1\text{A} & t < 0 \\ (1 + e^{-0.6t})\text{A} & t > 0 \end{cases}$$



(a)



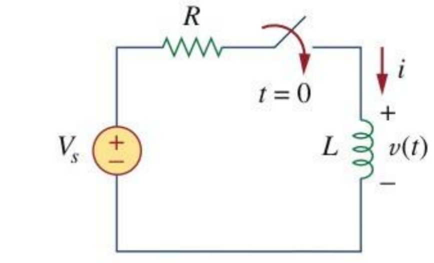
(b)

Step Response of an RL Circuit

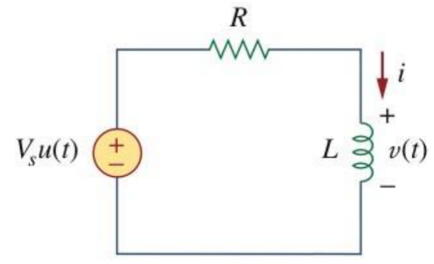
- Consider the *RL* circuit in Fig. (a), which may be replaced by the circuit in Fig. (b).
- Our goal is to find the inductor current *i* as the circuit response.
- The response can be obtained by applying the Kirchhoff's laws as done in the *RC* circuit.
- We can also find the circuit response by finding the transient response i_t and steady-state response i_{ss}.
- We know that the transient response is always a decaying exponential, i.e., $i_t = Ae^{-t/\tau}$, where $\tau = L/R$ and A is a constant to be determined.
- The steady-state current is $i_{ss} = V_s / R$.
- Therefore, the complete response is in the form:

$$i(t) = Ae^{-t/\tau} + V_s / R \qquad (1)$$

- Now, we need to determine the constant *A* from the initial value of *i*.



(a)

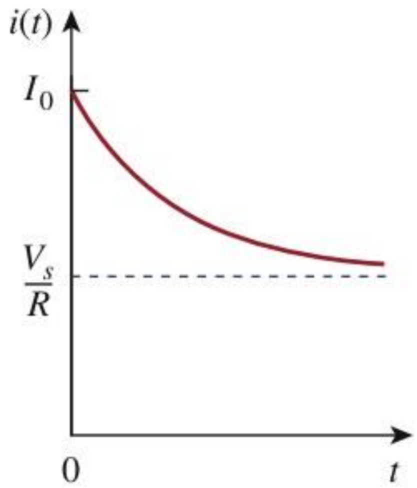


(b)

Step Response of an RL Circuit

- Assume I_o be the initial current through the inductor, which may come from a source other than V_s.
- Since the current through the inductor cannot change instantaneously, $I(0^+) = I(0^-) = I_o$.
- Thus, at $t = 0$, equation (1) (from previous slide) becomes $I_o = A + V_s/R$ => $A = I_o - V_s/R$.
- Substituting the result of A back into (1) gives

$$i(t) = \frac{V_s}{R} + \left(I_o - \frac{V_s}{R} \right) e^{-t/\tau}$$



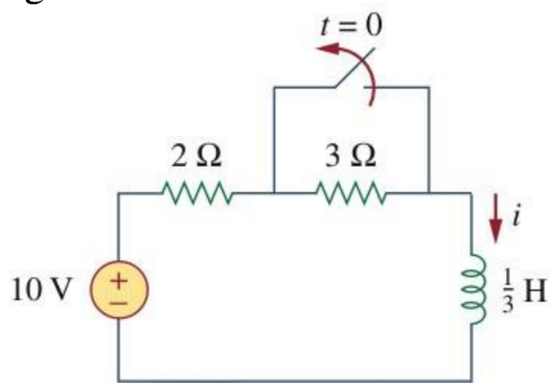
- The above response and the source-free *RL* circuit response can be summarized by the following equation:

$$\boxed{i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}}$$

where $i(0)$ and $i(\infty)$ are the initial and final values of i, respectively.

Example (6):Step Response of an RL Circuit

Question: Find $i(t)$ in the circuit for $t > 0$. Assume that the switch has been closed for a long time.



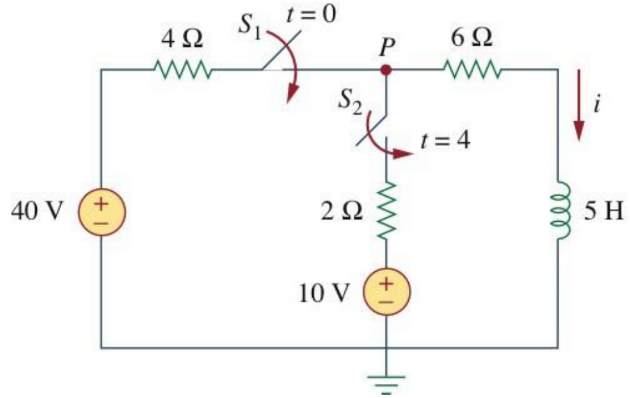
Solution:

- We have to find $i(0)$, $i(\infty)$ and τ.
- For $t < 0$, the 3-Ω resistor is short-circuited and the inductor acts like a short circuit.
- The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is $i(0^-) = 10/2 = 5$ A.
- Since the inductor current cannot change instantaneously, $i(0) = i(0^+) = i(0^-) = 5$ A.
- When $t > 0$, the 2Ω and 3Ω are in series so that $i(\infty) = 10/(2+3) = 2$ A.
- The time constant is $\tau = L/R = (1/3)/(2 + 3) = 1/15$ s.
- Thus,

$$
\begin{aligned}
i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\
&= 2 + (5-2)e^{-15t} \\
&= 2 + 3e^{-15t} \qquad \text{for } t > 0
\end{aligned}
$$

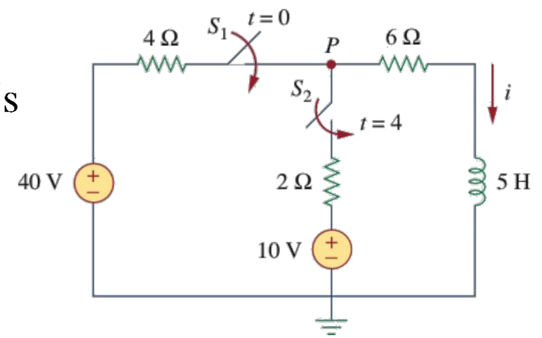
Example (7):Step Response of an RL Circuit

Question: At t = 0, switch 1 (S_1) is closed and switch 2 (S_2) is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for t = 2 s and t = 5 s.



- We need to consider three time intervals
  - $t < 0$ s
  - $0 \le t \le 4$ s
  - $t > 4$ s
- For $t < 0$, both switches S_1 and S_2 are open so that $i = 0$.
- Since the inductor current cannot change instantly, $i(0^-) = i(0) = i(0^+) = 0$.
- For $0 \le t \le 4$, S_1 is closed so that the 4-Ω and 6-Ω resistors are in series (remember, at this time S_2 is still open).
- Hence, assuming for now that S_1 is closed for a very long time, $i(\infty) = 40/(4 + 6) = 4$ A
- The equivalent resistance for the present RL circuit is 4 + 6 = 10 Ω. So, $\tau = L/R = 5/10 = 0.5$s
- Therefore,

$$
\begin{aligned}
i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\
&= 4 + (0 - 4)e^{-2t} \\
&= 4(1 - e^{-2t})\text{A} \qquad \text{for } 0 \le t \le 4
\end{aligned}
$$



Example (7): Step Response of an RL Circuit

- For t > 4s, S_2 is also closed. The 10-V voltage source is connected and the circuit changes. This sudden change does *not* affect the inductor current because it cannot change instantly.
- Thus the initial current is: $i(4) = i(4^-) = 4(1 - e^{-8}) \approx 4\text{A}$
- To find $i(\infty)$, we have to apply KCL at node P.
- Let v be the voltage at node P, using KCL, we have

$$\frac{40-v}{4} + \frac{10-v}{2} = \frac{v}{6} \quad => \quad v = \frac{180}{11}\text{V}$$

- Therefore, $i(\infty) = v/6 = 30/11 = 2.727$ A
- To find the time constant, we need to find the Thevenin resistance at the inductor terminals:

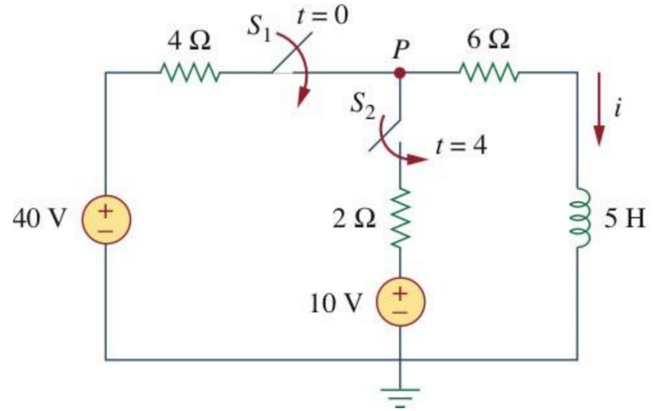
$$R_{TH} = 4 \| 2 + 6 = \frac{4 \times 2}{4+2} + 6 = \frac{22}{3}\Omega$$

- The time constant is $\tau = L/R_{TH}$ =5/(22/3) = 15/22 s
- Hence, $i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}$

$$= 2.727 + (4 - 2.727)e^{-1.4667(t-4)}$$

$$= 2.727 + 1.273e^{-1.4667(t-4)} \qquad \text{for } t \geq 4$$

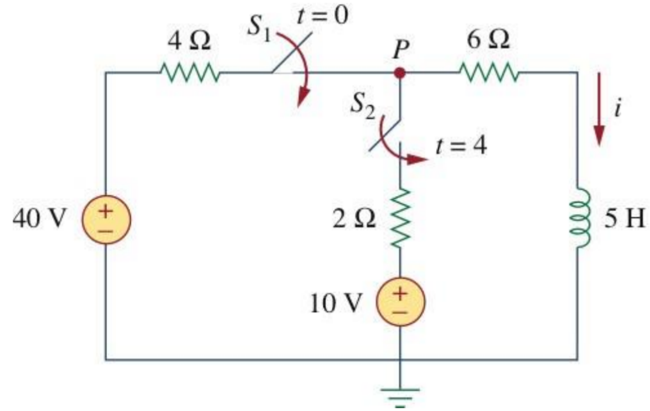
- We need (t–4) in the exponential because of the time delay.



Example (7): Step Response of an RL Circuit

Putting all together,

$$i(t) = \begin{cases} 0 & \text{for } t < 0 \\ 4(1 - e^{-2t}) & \text{for } 0 \le t \le 4 \\ 2.727 + 1.273e^{-1.4667(t-4)} & \text{for } t \ge 4 \end{cases}$$



At $t = 2$ s, $i(2) = 4(1 - e^{-4}) = 3.93$ A

At $t = 5$ s, $i(5) = 2.727 + 1.273e^{-1.4667} = 3.02$ A

First-Order Op Amp Circuits

- An op amp circuit containing a storage element will exhibit first-order behaviour.
- As usual, we analyse op amp circuits using nodal analysis. Sometimes, the Thevenin equivalent circuit is used to reduce the op amp circuit to one that we can easily handle.
- RC types of op amp circuits, depending on the location of the capacitor with respect to the op amp, i.e., the capacitor can be located in the input, the output, or the feedback loop.

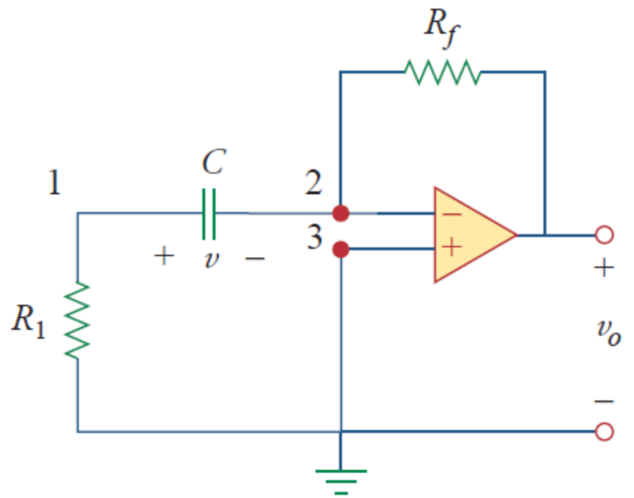
Example: For the op amp circuit, find v_o for $t > 0$, given that $v(0) = 3$.

If v_1 is the voltage at node 1, at that node, KCL gives

$$\frac{0 - v_1}{R_1} = C\frac{dv}{dt}$$

$$\frac{dv}{dt} + \frac{v}{CR_1} = 0$$

$$v(t) = V_0 e^{-t/\tau}, \qquad \tau = R_1 C$$



First-Order Op Amp Circuits

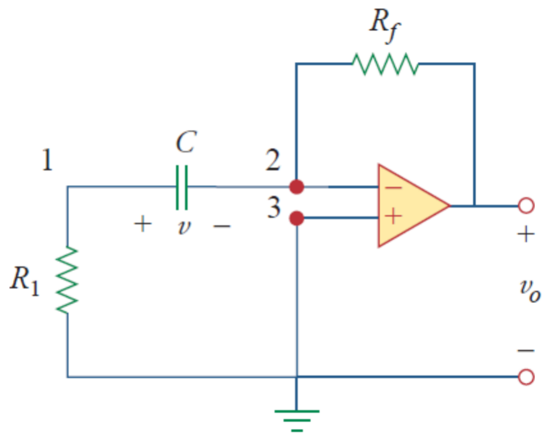
But $v(0) = 3 = V_o$ and $\tau = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1$

Hence, $v(t) = 3e^{-10t}$

By applying KCL at node 2, we have

$$C\frac{dv}{dt} = \frac{0-v_o}{R_f}$$

$$v_o = -R_f C\frac{dv}{dt} = -80 \times 10^3 \times 5 \times 10^{-6}(-30e^{-10t}) = 12e^{-10t}V$$



First-Order Op Amp Circuits

METHOD 2: Since $v(0^+) = v(0^-) = 3\text{V}$, we apply KCL at node 2 in the circuit of Fig. 7.55(b) to obtain

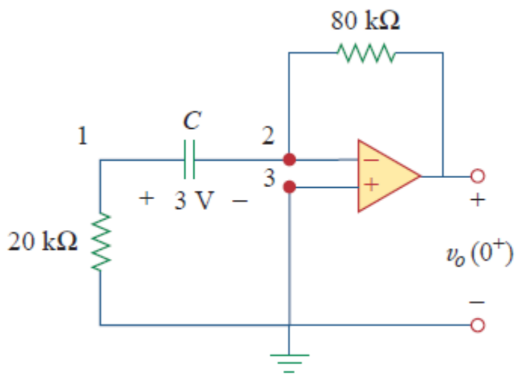
$$\frac{3}{20{,}000} + \frac{0 - v_o(0^+)}{80{,}000} = 0 \qquad v_o(0^+) = 12 \text{ V}$$

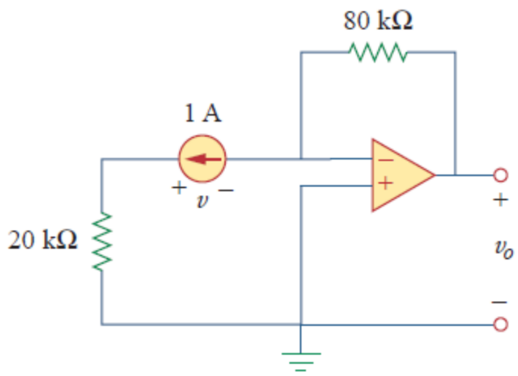
To find τ we need the equivalent resistance across the capacitor terminals. If we remove the capacitor and replace it by a 1-A current source

$$20{,}000(1) - v = 0 \qquad \Rightarrow \qquad v = 20 \text{ kV}$$

$$R_{\text{eq}} = \frac{v}{1} = 20 \text{ k}\Omega \qquad \tau = R_{\text{eq}}C = 0.1$$

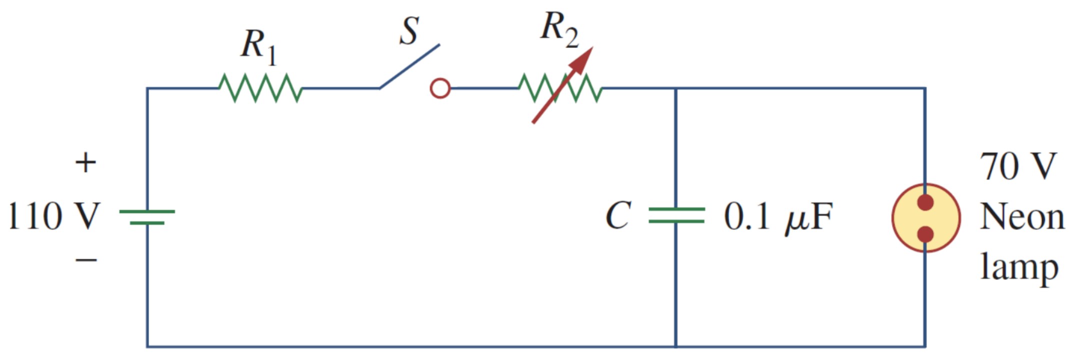
$$v_o(t) = v_o(\infty) + [v_o(0) - v_o(\infty)]e^{-t/\tau}$$
$$= 0 + (12 - 0)e^{-10t} = 12e^{-10t} \text{ V}, \qquad t > 0$$





Applications: Delay Circuits

- An RC circuit can be used to provide various time delays.
- Basically, it consists of an RC circuit with the capacitor connected in parallel with a neon lamp. The voltage source can provide enough voltage to fire the lamp. When the switch is closed, the capacitor voltage increases gradually toward 110 V at a rate determined by the circuit's time constant, $(R_1 + R_2)C$.
- The lamp will act as an open circuit and not emit light until the voltage across it exceeds a particular level, say 70 V. When the voltage level is reached, the lamp fires (goes on), and the capacitor discharges through it. Due to the low resistance of the lamp when on, the capacitor voltage drops fast and the lamp turns off. The lamp acts again as an open circuit and the capacitor recharges.



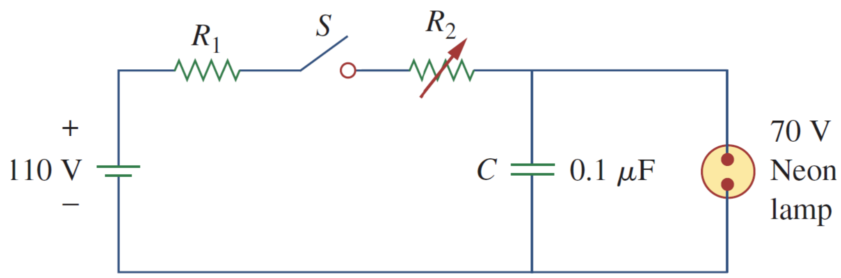
Other example: The warning blinkers commonly found on road construction sites.

Example (8): Delay Circuits

Question: Assume that $R_1 = 1.5$ MΩ, $0 < R_2 < 2.5$ MΩ.

(a) Calculate the extreme limits of the time constant of the circuit.

(b) How long does it take for the lamp to glow for the first time after the switch is closed ? Let R_2 assume its largest value.



Solution:

(a) The smallest value for R_2 is 0 Ω and the corresponding time constant is:

$$\tau = (R_1 + R_2)C = (1.5 \times 10^6 + 0) \times 0.1 \times 10^{-6} = 0.15 \text{ s}$$

The largest value for R_2 is 2.5 MΩ and the corresponding time constant is:

$$\tau = (R_1 + R_2)C = (1.5 + 2.5) \times 10^6 \times 0.1 \times 10^{-6} = 0.4 \text{ s}$$

(b) Assuming that the capacitor is initially uncharged, $v_c(0) = 0$, $v_c(\infty) = 110$.

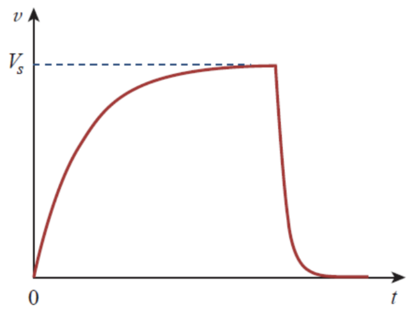
$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau} = 110[1 - e^{-t/\tau}]$$

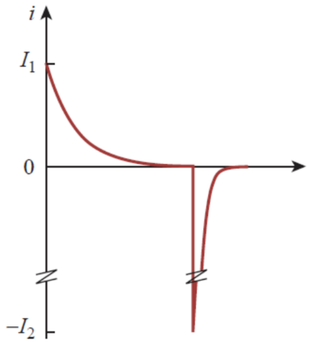
$$70 = 110[1 - e^{-t_0/\tau}] \quad \Rightarrow \quad \frac{7}{11} = 1 - e^{-t_0/\tau}$$

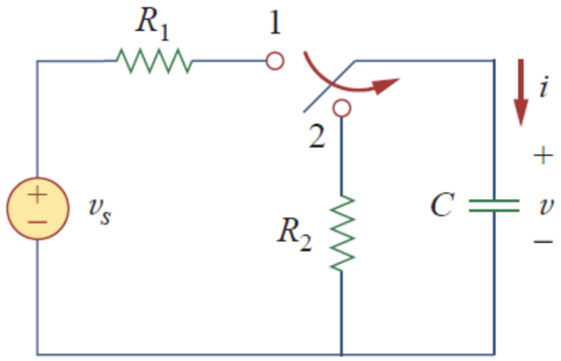
Hence, $t_0 = \tau \ln\frac{11}{4} = 0.4 \ln 2.75 = 0.4046 \text{ s}$

Applications: Photoflash Unit

- An electronic flash unit provides a common example of an *RC* circuit. This application exploits the ability of the capacitor to oppose any abrupt change in voltage.
- It consists essentially of a high-voltage dc supply, a current-limiting large resistor and a capacitor C in parallel with the flash lamp of low resistance When the switch is in position 1, the capacitor charges slowly due to the large time constant.
- The capacitor voltage rises gradually from zero to V_s while its current decreases gradually from to zero. The charging time is approximately five times the time constant.
- With the switch in position 2, the capacitor voltage is discharged. The low resistance of the lamp permits a high discharge current with peak in a short duration.







Example (9): Electronic Flashgun

Question: An electronic flashgun has a current-limiting resistor and electrolytic capacitor charged to 240 V. If the lamp resistance is find: (a) the peak charging current, (b) the time required for the capacitor to fully charge, (c) the peak discharging current, (d) the total energy stored in the capacitor, and (e) the average power dissipated by the lamp.

(a) The peak charging current is $I_1 = \frac{V_s}{R_1} = \frac{240}{6 \times 10^3} = 40 \text{ mA}$

(b) $t_{\text{charge}} = 5R_1C = 5 \times 6 \times 10^3 \times 2000 \times 10^{-6} = 60 \text{ s} = 1 \text{ minute}$

(c) The peak discharging current is $I_2 = \frac{V_s}{R_2} = \frac{240}{12} = 20 \text{ A}$

(d) The energy stored is $W = \frac{1}{2}CV_s^2 = \frac{1}{2} \times 2000 \times 10^{-6} \times 240^2 = 57.6 \text{ J}$

(e) The energy stored in the capacitor is dissipated across the lamp during the discharging period.

$$t_{\text{discharge}} = 5R_2C = 5 \times 12 \times 2000 \times 10^{-6} = 0.12 \text{ s}$$

$$p = \frac{W}{t_{\text{discharge}}} = \frac{57.6}{0.12} = 480 \text{ watts}$$