Additional questions for chapter 4

1. *A stock price is currently \$ 100. Over the next two six-month periods it is expected to go up by 10% or go down by 10%. The risk-free interest rate is 8% per annum with continuous compounding.*

   (i) What is the value of a one-year European call option with a strike price of \$ 100.

   (ii) What is the value of a one-year European put option with a strike price of \$ 100.

   (iii) Verify that the European call and the European put satisfy put-call parity.

   Solution:
   Parameters are $u = 0.1, d = -0.1, 1 + r = e^{0.5\times0.08}$. So the risk-neutral probability is $p^* = 0.7$. After evaluation of the options at the terminal nodes we use the risk-neutral valuation to get (i)

   $$\pi_C(0) = e^{-2(0.5\times0.08)} \left[0.7^2 \times 21 + 2 \times 0.7(1 - 0.7) \times 0 + (1 - 0.7)^2 \times 0\right] = 9.61$$

   and (ii)

   $$\pi_P(0) = e^{-2(0.5\times0.08)} \left[0.7^2 \times 0 + 2 \times 0.7(1 - 0.7) \times 1 + (1 - 0.7)^2 \times 19\right] = 1.92$$

   (iii) For put-call parity one has to verify $S - \pi_C + \pi_P = Ke^{-r}$, here :

   $$100 - 9.61 + 1.92 = 100e^{-0.08}.$$

2. *Assume a standard 3-period CRR binomial model. The price of the stock is currently \$100. The risk-free interest rate with continuous compounding is 6% per annum. Over the next three 4 month periods, the stock is expected to go up by 8% or go down by 7% in each period.*

   (a) *What is the value of a one-year European call with strike price \$103?*

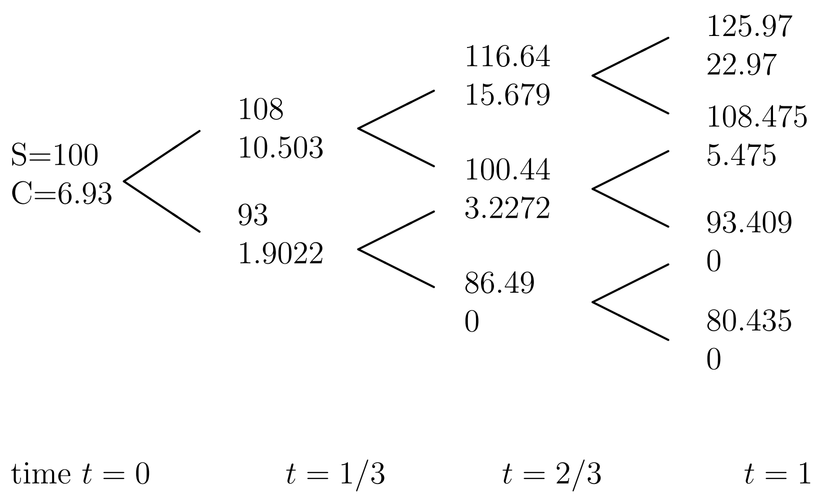
   (b) *What is the value of a one-year European put with strike price \$103?*

   (c) *Verify the Put-Call parity for the European call and the European put.*

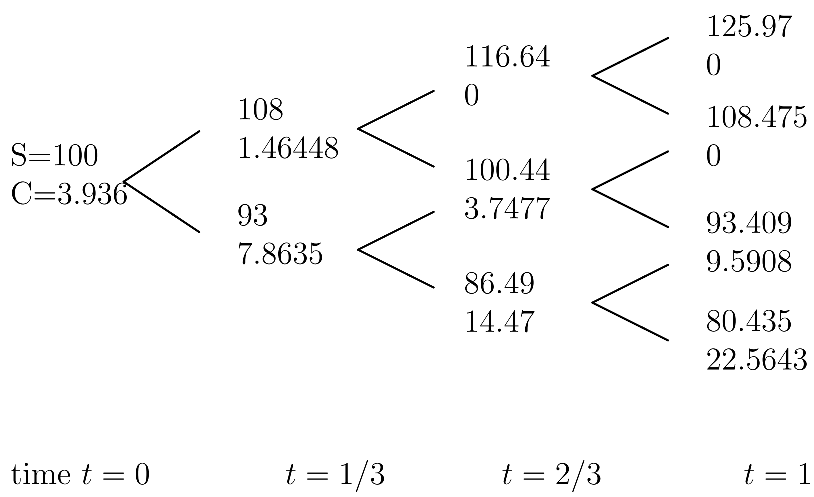
Solution:
We first calculate the Martingale probability in the tree. We get

$$p = \frac{r-d}{u-d} = \frac{e^{0.06/3} - 1 + 0.07}{0.08 + 0.07} = 0.6013423$$

(a) The tree for the call option looks as follows:



(b) The tree for the put option is:



(c) The Put-Call parity holds:

$$C - P = 6.9342 - 3.936 = 2.9982 = 100 - 103e^{-0.06} = 100 - 97.0017 = S - Ke^{-rT}.$$

3. *Consider a 3-period Cox-Ross-Rubinstein model. The annual interest rate is* $r = 0.05$ *(discrete),* $u = 0.1$ *and* $d = -0.1$*. The initial price of the stock is* $S(0) = 100$*. The time horizon is* $T = 3$ *years.*

   (a) *Calculate the risk-neutral probability and the stock prices at each node in the binomial tree (correct up to 2 decimal places after the decimal point).*

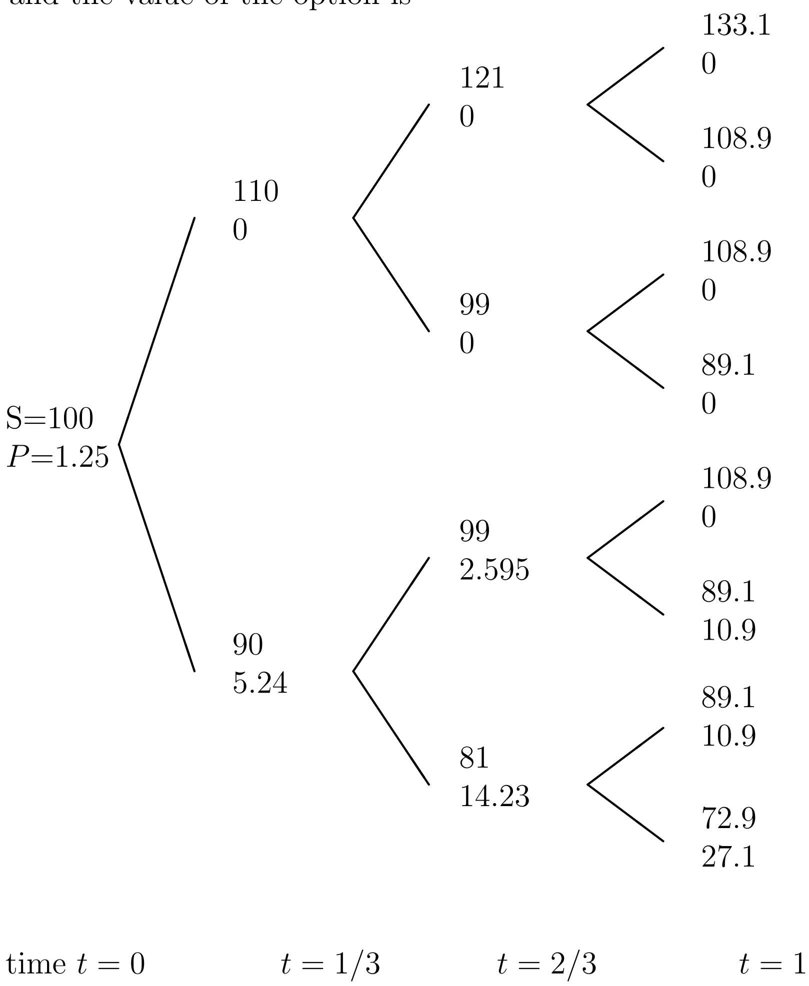
   (b) *Calculate the value of the European option with payoff*

$$P(T) = \begin{cases} \sup\limits_{0 \le t \le T} S_t - S_T & S_t < 110 \ \forall t \\ 0 & \textit{otherwise} \end{cases}$$

   (c) *Find a replicating portfolio for the above option for the first trading period.*

Solution:

(a) For the risk-neutral probability we get $p = \frac{r-d}{u-d} = \frac{3}{4}$. The tree with the stock prices and the value of the option is



(b) The replicating portfolio can be found by solving the equations

$$1.05 \cdot \varphi_1 + 110 \cdot \varphi_2 = 0$$
$$1.05 \cdot \varphi_1 + 90 \cdot \varphi_2 = 5.24$$

As solution we get $\varphi_1 = 27.45$ and $\varphi_2 = -0.262$.

4. *Construct a three period binomial tree using the parameters* $r = 0.1$ *(discrete, per period),* $u = 0.15$, $d = -0.05$ *and* $S_0 = 100$.

   (a) *Find the price of a European Put* P *with strike 105 and maturity date* $T = 3$.

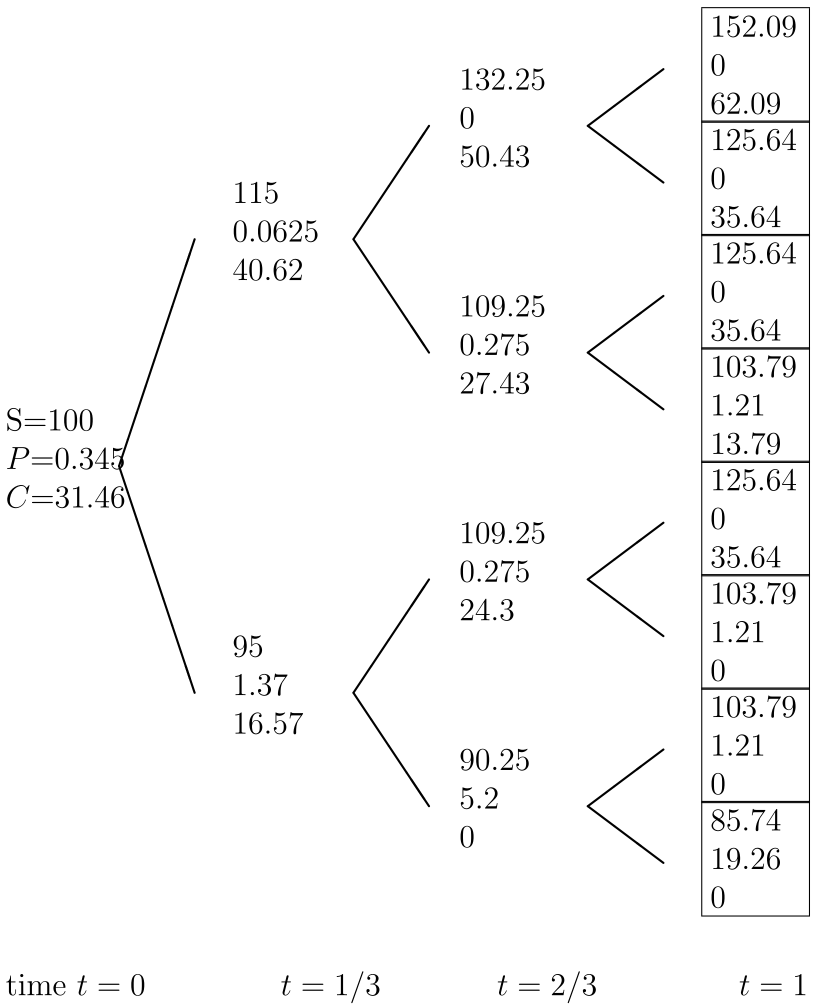
   (b) *Find the price of the knock in Call option* C *with knock in level* $H = 110$, *strike* $K = 90$ *and maturity date* $T = 3$, *i.e.*

$$C = \begin{cases} (S(T) - 90)^+ & \exists t : S_t > H = 110 \\ 0 & S_t \leq H = 110\ \forall t. \end{cases}$$

Solution:
The risk neutral probability is

$$p = \frac{r-d}{u-d} = \frac{0.1 + 0.05}{0.15 + 0.05} = \frac{3}{4}$$

We first set up a tree with the stock price movements, then compute the values of the two options:



5. *Assume a 3-period Cox-Ross-Rubinstein model. The annual interest rate with continuous compounding is $r = 0.06$. The volatility of the stock is $\sigma = 0.2$ with a price of $S(0) = 100$. Furthermore, there exists an American Put with maturity date $T = 1$ und strike $K = 90$.*

   (a) *Calculate the risk-neutral probability and the stock prices at each node in the binomial tree (correct up to 2 decimal places after the decimal point).*

   (b) *Calculate the value of the American Put for all nodes in the tree.*

   (c) *What is the optimal stopping time? Justify your answer.*

Solution:

(a) The parameter-values are

$$\Delta = \frac{1}{3}, \quad 1+r_d = e^{r\Delta} = 1.0202, \quad 1+u = e^{\sigma\sqrt{\Delta}} = 1.1224, \quad 1+d = e^{-\sigma\sqrt{\Delta}} = 0.8909.$$

For the risk-neutral probability we get

$$p^* = \frac{r_d - d}{u - d} = 0.5584.$$

The tree with the stock prices is

time $t = 0$	$t = 1/3$	$t = 2/3$	$t = 1$
			141.40
		125.98	
	112.24		112.24
S=100		89.09	
	89.09		89.09
		79.38	
			70.72

(b) The prices for the american Put are

time $t = 0$	$t = 1/3$	$t = 2/3$	$t = 1$
			0
		0	
	0.17		0
P=2.176		0.394	
	4.812		0.91
		$\max\{10.62, 8.84\}$	
			19.28

(c) Let $\Omega = \{u, d\}^3$. The optimal exercise date is

$$\tau(\omega) = \begin{cases} n = 2 & \omega \in \{ddu, ddd\} \\ n = 3 & \text{otherwise.} \end{cases}$$

For $\omega \in \{(ddu), (ddd)\}$, we have $\frac{1}{1+r_d}\mathbb{E}[p^* f_{32} + (1 - p^*) f_{33}] < (K - S_0(1 + d)^2)^+$. Here f_{ij} denotes the price of the claim in period i with j-down movements.

6. *Assume that we have a three period CRR model with initial stock price $S = \$150$, interest rate $r = 0.05$ and volatility $\sigma = 0.2$.*

   (a) *What is the value of an American Put with strike \$150, which matures in 6 months?*

   (b) *What is the value of an American Call with strike \$150, which matures in 6 months?*

   (c) *Verify that the following inequalities hold:*

$$S - K \leq C_A - P_A \leq S - Ke^{-rT}$$

Solution:
The martingale probability is $p = 0.5308$ with $u = 0.085$, $d = -0.0784$ and $r = 0.0084$.

(a), (b) For the American Put and Call we get:

			191.63 0 41.63
		176.61 0 27.85	
	162.76 2.55 17.78		162.76 0 12.76
S=150 P_A=7.57 C_A=11.01		150 5.47 6.72	
	138.24 13.40 3.54		138.24 11.76 0
		127.40 22.60 0	
			117.41 32.59 0
time $t = 0$	$t = 1/6$	$t = 2/6$	$t = 0.5$

(c) We have $0 \leq 11.01 - 7.57 = 3.44 \leq 3.7035 = 150 - 150 \cdot e^{-0.025}$

7. *Show that a security market is arbitrage-free with respect to Φ iff it is arbitrage-free with respect to Φ_a. Here Φ is the set of all self-financing trading strategies and Φ_a is the set of all admissible strategies, that means all $\varphi \in \Phi$ with $V_\varphi(t) \geq 0 \quad t = 0, \ldots, T$.*

Solution:
First note, if $\varphi \in \Phi_a$ is an arbitrage strategy, then it is by definition of Φ_a also a strategy in Φ. We now have to show that if we have an arbitrage strategy $\varphi \in \Phi$, then there exists an arbitrage strategy $\psi \in \Phi_a$.

Assume that $\varphi \in \Phi$ is an arbitrage strategy. Then we have $V_\varphi(0) = 0$, $P(V_\varphi(T) \geq 0) = 1$ and $P(V_\varphi(T) > 0) > 0$. We have to distinguish between two cases:

Case 1: $V_\varphi(t) \geq 0 \quad t = 0, \ldots, T$. Then $\varphi \in \Phi_a$ and we found the admissible arbitrage strategy.

Case 2: $\exists t^*, A \in \mathcal{F}_{t^*}$ with $V_\varphi(t^*, \omega) < 0 \; \forall \omega \in A$ and $V_\varphi(t) \geq 0 \; t > t^*$. Then define a new strategy ψ. Set $\psi(u, \omega) = 0 \; \forall \omega \in A^c \; \forall u$. Furthermore $\psi(u, \omega) = 0 \; \omega \in A$ and $u \leq t^*$. For the remaining possibilities set

$$\psi_0(u, \omega) = \varphi_0(u, \omega) - \frac{V_\varphi(t^*, \omega)}{S_0(t^*, \omega)} \quad \forall \omega \in A \quad u > t^*$$

and

$$\psi_i(u, \omega) = \varphi_i(u, \omega) \quad \forall \omega \in A \quad i = 1, \ldots, d \quad u > t^*$$

We have to show that this strategy is self-financing and admissible. For $\omega \in A^c$ we clearly have no problem. There is nothing to show for $\omega \in A, u \leq t^*$. ψ is also clearly self-financing for $u > t^* + 1$ as it just replicates the other strategy there. We have to show that $\psi(t^*)S(t^*) = \psi(t^* + 1)S(t^*)$. For $\omega \in A$ we have

$$\psi_0(t^* + 1)S_0(t^*) = \varphi_0(t^* + 1)S_0(t^*) - V_\varphi(t^*) \quad \text{and} \quad \psi_i(t^* + 1) = \varphi_i(t^* + 1)$$

Thus we get

$$\psi(t^*+1)S(t^*) = \mathbb{1}_A(\varphi(t^*+1)S(t^*) - V_\varphi(t^*)) = \mathbb{1}_A(\varphi(t^*)S(t^*) - V_\varphi(t^*)) = 0 = \psi(t^*)S(t^*)$$

It remains to show that ψ is admissible and an arbitrage opportunity. We get

$$V_\psi(t) = 0 \quad t \leq t^*$$

and

$$V_\psi(t) = \mathbb{1}_A \left(\varphi(t)S(t) - V_\varphi(t^*) \frac{S_0(t)}{S_0(t^*)} \right) = \mathbb{1}_A \left(V_\varphi(t) - V_\varphi(t^*) \frac{S_0(t)}{S_0(t^*)} \right) \geq 0$$

and > 0 on A for $t = T$ because $V_\varphi(t^*) < 0$. We also have that $V_\psi(t) = 0 \; \forall t \leq t^*$. Therefore ψ is admissible and an arbitrage opportunity.

8. *(a) State the Black-Scholes formula for an European Call and Put. (Hint: The Put-Call parity $C - P = S - Ke^{-r(T-t)}$ might be useful)*

   (b) Replicate the European straddle with payoff $D(T) = |S(T) - K|$ using standard European options.

   (c) What is the Black-Scholes price of the straddle?

   (d) What is the Δ of the straddle? How much does the value of the straddle approximately change if the stock price changes from S_t to $S_t + \varepsilon$? (Hint: The Δ of the Call is $N(d_1)$)

Solution:

(a) The Black-Scholes formula for an European Call and Put is

$$\begin{aligned} C(t) &= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \\ P(t) &= Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

(b) We can replicate the straddle $D(T) = |S(T) - K|$ by buying one call and one put, both with strike K.

(c) The Black-Scholes price of the straddle is

$$\begin{aligned} D(t) = C(t) + P(t) &= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) + Ke^{-r(T_t)}N(-d_2) - S(t)N(-d_1) = \\ &= S(t)(2N(d_1) - 1) - Ke^{-r(T-t)}(2N(d_2) - 1). \end{aligned}$$

(d) The Delta of the straddle is

$$\Delta_D = \Delta_C + \Delta_P = N(d_1) - N(-d_1) = 2N(d_1) - 1.$$

When the stock price changes from S_t to $S_t + \varepsilon$, then the price of the straddle changes about $\varepsilon(2N(d_1) - 1)$.

9. *Consider a financial market in which the Black-Scholes formula for a European call option holds. The risk-free interest rate (cont. compounding) is r. The underlying stock has value S with volatility σ. For a European call with strike K and maturity T, show that the following relations hold:*

$$\begin{aligned}
\Delta = \frac{\partial C}{\partial S} &= N(d_1) \\
\Gamma = \frac{\partial C}{\partial S^2} &= \frac{N'(d_1)}{S\sigma\sqrt{T-t}} \\
\Theta = \frac{\partial C}{\partial t} &= -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) \\
\rho = \frac{\partial C}{\partial r} &= K(T-t)e^{-r(T-t)}N(d_2) \\
\nu = \frac{\partial C}{\partial \sigma} &= SN'(d_1)\sqrt{T-t}
\end{aligned}$$

Show that the call satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0.$$

Solution:
We first show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$:

$$\begin{aligned}
SN'(d_1) - Ke^{-r(T-t)}N'(d_2) &= \frac{1}{\sqrt{2\pi}}\left(Se^{-d_1^2/2} - Ke^{-r(T-t)}e^{-d_2^2/2}\right) = \\
&= \frac{1}{\sqrt{2\pi}}\left(Se^{-d_1^2/2} - Ke^{-r(T-t)}e^{-d_1^2/2+d_1\sigma\sqrt{T-t}-\sigma^2(T-t)/2}\right) = \\
&= \frac{1}{\sqrt{2\pi}}\left(Se^{-d_1^2/2} - Se^{-d_1^2/2}\frac{K}{S}e^{-r(T-t)}e^{d_1\sigma\sqrt{T-t}-\sigma^2(T-t)/2}\right) = \\
&= \frac{1}{\sqrt{2\pi}}\left(Se^{-d_1^2/2} - Se^{-d_1^2/2}\frac{K}{S}e^{-r(T-t)}e^{\log(S/K)+(r+\sigma^2/2)(T-t)-\sigma^2(T-t)/2}\right) = \\
&= \frac{1}{\sqrt{2\pi}}\left(Se^{-d_1^2/2} - Se^{-d_1^2/2}\right) = 0
\end{aligned}$$

Now we calculate the Greeks:

(a)

$$\begin{aligned}
\Delta = \frac{\partial C}{\partial S} &= N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} = \\
&= N(d_1) + SN'(d_1)\left(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\right) = \\
&= N(d_1)
\end{aligned}$$

(b)

$$\Gamma = \frac{\partial C}{\partial S^2} = \frac{\partial \Delta}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

(c)

$$\begin{aligned}
\Theta = \frac{\partial C}{\partial t} &= SN'(d_1)\frac{\partial d_1}{\partial t} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t} - Kre^{-r(T-t)}N(d_2) = \\
&= SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right) - Kre^{-r(T-t)}N(d_2) = \\
&= -SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)
\end{aligned}$$

(d)

$$\begin{aligned}
\rho = \frac{\partial C}{\partial r} &= SN'(d_1)\frac{\partial d_1}{\partial r} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r} + K(T-t)e^{-r(T-t)}N(d_2) = \\
&= SN'(d_1)\left(\frac{\partial d_1}{\partial r} - \frac{\partial d_2}{\partial r}\right) + K(T-t)e^{-r(T-t)}N(d_2) = \\
&= K(T-t)e^{-r(T-t)}N(d_2)
\end{aligned}$$

(e)

$$\begin{aligned}
\nu = \frac{\partial C}{\partial \sigma} &= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma} = \\
&= SN'(d_1)\left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right) = \\
&= SN'(d_1)\sqrt{T-t}.
\end{aligned}$$

The partial differential equation holds because:

$$\begin{aligned}
&\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = \\
&= -SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) + \\
&+ rSN(d_1) + \\
&+ \frac{1}{2}\sigma^2 S^2 \frac{N'(d_1)}{S\sigma\sqrt{T-t}} + \\
&+ rC = \\
&= r(SN(d_1) - Ke^{-r(T-t)}N(d_2) - C) = 0.
\end{aligned}$$

10. *Prove the following limit relations used in the proof of Proposition 3.5.1, assuming that* $k_n \to \infty$ $(n \to \infty)$*:*

$$\lim_{n\to\infty} \hat{p}_n = \frac{1}{2}, \quad \lim_{n\to\infty} k_n(1-2\hat{p}_n)\sqrt{\Delta_n} = -T\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)$$

Solution:
We have the following definitions for the variables:

$$\begin{aligned}
\Delta_n &= \frac{T}{k_n} \\
u_n &= e^{\sigma\sqrt{\Delta_n}} - 1 \\
d_n &= e^{-\sigma\sqrt{\Delta_n}} - 1 \\
r_n &= e^{r\Delta_n} - 1 \\
p_n^* &= \frac{r_n - d_n}{u_n - d_n} \\
\hat{p}_n &= p_n^* \frac{1+u_n}{1+r_n}
\end{aligned}$$

Then, we get for the first limit relation:

$$\lim_{n\to\infty} \hat{p}_n = \lim_{n\to\infty} p_n^* \underbrace{\lim_{n\to\infty} \frac{1+u_n}{1+r_n}}_{\to 1 (n\to\infty)} = \lim_{n\to\infty} \frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}} = \frac{1}{2}.$$

In order to show the last equality, it suffices to show that

$$\lim_{x\to 0_+} \frac{e^{rx^2} - e^{-\sigma x}}{e^{\sigma} x - e^{-\sigma x}} = \frac{1}{2}.$$

as $\sqrt{\Delta_n} \to 0_+ (n\to\infty)$.
By L'Hospital we get

$$\lim_{x\to 0_+} \frac{e^{rx^2} - e^{-\sigma x}}{e^{\sigma} x - e^{-\sigma x}} == \lim_{x\to 0_+} \frac{2xre^{rx^2} + \sigma e^{-\sigma x}}{\sigma e^{\sigma x} + \sigma e^{-\sigma x}} = \frac{1}{2}.$$

For the second limit relation we get:

$$\begin{aligned}
\lim_{n\to\infty} k_n(1-2\hat{p}_n)\sqrt{\Delta_n} &= \lim_{n\to\infty} \sqrt{Tk_n}\left(1 - 2\frac{e^{r\Delta_n} - e^{-\sigma\sqrt{\Delta_n}}}{e^{\sigma\sqrt{\Delta_n}} - e^{-\sigma\sqrt{\Delta_n}}} \frac{e^{-r\Delta_n}}{e^{-\sigma\sqrt{\Delta_n}}}\right) = \\
&= \lim_{n\to\infty} \sqrt{T} \frac{1 - e^{-2\sigma\sqrt{\Delta_n}} - 2 + 2e^{-\sigma\sqrt{\Delta_n} - r\Delta_n}}{\frac{1}{\sqrt{k_n}}\left(1 - e^{-2\sigma\sqrt{\Delta_n}}\right)} \\
&= \sqrt{T}\left(-\sqrt{T}\left(\frac{\sigma}{2}+\frac{r}{\sigma}\right)\right) = -T\left(\frac{\sigma}{2}+\frac{r}{\sigma}\right).
\end{aligned}$$

For the second to last equation it suffices to show that:

$$\lim_{x\to 0_+} \frac{-e^{-2\sigma x} - 1 + 2e^{-\sigma x - rx^2}}{\frac{x}{\sqrt{T}}\left(1 - e^{-2\sigma x}\right)} = -\sqrt{T}\left(\frac{\sigma}{2}+\frac{r}{\sigma}\right)$$

as $\sqrt{\Delta_n} \to 0_+ (n \to \infty)$.

We are using L'Hospital twice and get:

$$\begin{aligned}
&\lim_{x \to 0_+} \frac{-e^{-2\sigma x} - 1 + 2e^{-\sigma x - rx^2}}{\frac{x}{\sqrt{T}}\left(1 - e^{-2\sigma x}\right)} = \\
&= \lim_{x \to 0_+} \sqrt{T} \frac{2\sigma e^{-2\sigma x} - 2(\sigma + 2rx)e^{-\sigma x - rx^2}}{(1 - e^{-2\sigma x}) + 2x\sigma e^{-2\sigma x}} = \\
&= \lim_{x \to 0_+} \sqrt{T} \frac{-4\sigma^2 e^{-2\sigma x} + 2(\sigma + 2rx)^2 e^{-\sigma x - rx^2} - 4re^{-\sigma x - rx^2}}{2\sigma e^{-2\sigma x} + 2\sigma e^{-2\sigma x} - 4x\sigma^2 e^{-2\sigma x}} = \\
&= \sqrt{T} \frac{-4\sigma^2 + 2\sigma^2 - 4r}{4\sigma} = -\sqrt{T}\left(\frac{\sigma}{2} + \frac{r}{\sigma}\right).
\end{aligned}$$

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