Appendix A

Vectors

In this appendix, we give a summary of the properties of vectors which are used in the text.

A.1 Definitions and Elementary Properties

A *vector* $\boldsymbol{a}$ is an entity specified by a magnitude, written a or $|\boldsymbol{a}|$, and a direction in space. It is to be contrasted with a *scalar*, which is specified by a magnitude alone. The vector $\boldsymbol{a}$ may be represented geometrically by an arrow of length a drawn from any point in the appropriate direction. In particular, the position of a point P with respect to a given origin O may be specified by the *position vector* $\boldsymbol{r}$ drawn from O to P as in Fig. A.1.

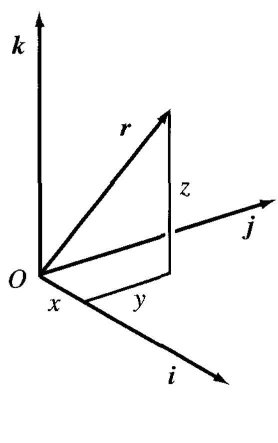

Fig. A.1

Any vector can be specified, with respect to a given set of Cartesian axes, by three components. If x, y, z are the Cartesian co-ordinates of P, with O as origin, then we write $\boldsymbol{r} = (x, y, z)$, and say that x, y, z are the *components* of $\boldsymbol{r}$. (See Fig. A.1.) We often speak of P as 'the point $\boldsymbol{r}$'. When P coincides with O, its position vector is the *zero vector* $\mathbf{0} = (0, 0, 0)$ of length 0 and indeterminate direction. For a general vector, we write $\boldsymbol{a} = (a_x, a_y, a_z)$, where a_x, a_y, a_z are its components.

The product of a vector $\boldsymbol{a}$ and a scalar c is $c\boldsymbol{a} = (ca_x, ca_y, ca_z)$. If $c > 0$, it is a vector in the same direction as $\boldsymbol{a}$, and of length ca; if $c < 0$, it is in the opposite direction, and of length $|c|a$. In particular, if $c = 1/a$, we obtain the *unit vector* in the direction of $\boldsymbol{a}$, $\hat{\boldsymbol{a}} = \boldsymbol{a}/a$.

Addition of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ may be defined geometrically by drawing one vector from the head of the other, as in Fig. A.2. (This is the 'parallelogram law' for addition of forces — or vectors in general.) Subtraction

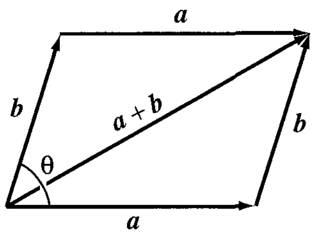

Fig. A.2

is defined similarly by Fig. A.3. In terms of components,

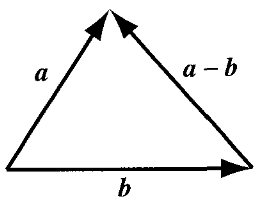

Fig. A.3

$$\boldsymbol{a} + \boldsymbol{b} = (a_x + b_x, a_y + b_y, a_z + b_z),$$
$$\boldsymbol{a} - \boldsymbol{b} = (a_x - b_x, a_y - b_y, a_z - b_z).$$

It is often useful to introduce three unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, pointing in the directions of the x-, y-, z-axes, respectively. They form what is known as an *orthonormal triad* — a set of three mutually perpendicular vectors of unit length. It is clear from Fig. A.1 that any vector $\boldsymbol{r}$ can be written as a sum of three vectors along the three axes,

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}. \tag{A.1}$$

Mathematically, any set of three quantities may be grouped together and regarded as the components of a vector. It is important to realize, however, that when we say that some physical quantity is a vector we mean more than just that it needs three numbers to specify it. What we mean is that these three numbers must transform in the correct way under a change of axes.

For example, consider a new set of axes $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ related to $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ by a rotation through an angle φ about the z-axis (see fig. A.4):

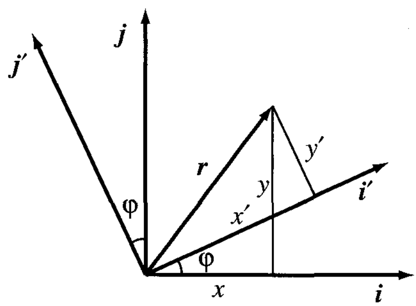

Fig. A.4

$$\begin{aligned} \boldsymbol{i}' &= \boldsymbol{i}\cos\varphi + \boldsymbol{j}\sin\varphi, \\ \boldsymbol{j}' &= -\boldsymbol{i}\sin\varphi + \boldsymbol{j}\cos\varphi, \\ \boldsymbol{k}' &= \boldsymbol{k}. \end{aligned} \tag{A.2}$$

The co-ordinates x', y', z' of P with respect to the new axes are defined by

$$\boldsymbol{r} = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}'.$$

Substituting (A.2) and comparing with (A.1), we see that $x = x'\cos\varphi - y'\sin\varphi$, *etc*, or equivalently $x' = x\cos\varphi + y\sin\varphi$, *etc*. Physically, then, a vector $\boldsymbol{a}$ is an object represented with respect to any set of axes by three components (a_x, a_y, a_z) *which transform under rotations in the same way as* (x, y, z), *i.e.*, in matrix notation,

$$\begin{bmatrix} a'_x \\ a'_y \\ a'_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}. \tag{A.3}$$

A.2 The Scalar Product

If θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then by elementary trigonometry the length of their sum is given by

$$|\boldsymbol{a} + \boldsymbol{b}|^2 = a^2 + b^2 + 2ab\cos\theta. \tag{A.4}$$

It is useful to define their *scalar product* $\boldsymbol{a} \cdot \boldsymbol{b}$ (‘$\boldsymbol{a}$ *dot* $\boldsymbol{b}$’) as

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta. \tag{A.5}$$

Note that this is equal to the length of $\boldsymbol{a}$ multiplied by the projection of $\boldsymbol{b}$ on $\boldsymbol{a}$, or *vice versa*. (See Fig. A.5.)

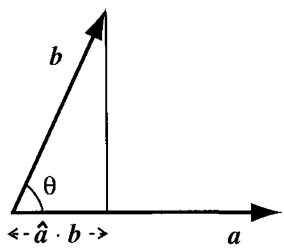

Fig. A.5

In particular, the *square of* $\boldsymbol{a}$ is

$$\boldsymbol{a}^2 = \boldsymbol{a} \cdot \boldsymbol{a} = a^2.$$

Thus we can write (A.4) as

$$(\boldsymbol{a} + \boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a} \cdot \boldsymbol{b},$$

and, similarly, the square of the difference is

$$(\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{a} \cdot \boldsymbol{b}.$$

All the ordinary rules of algebra are valid for the sums and scalar products of vectors, save one. (For example, the commutative law of addition, $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ is obvious from Fig. A.2, and the other laws can be deduced from appropriate figures.) The one exception is the following: for two scalars, $ab = 0$ implies that either $a = 0$ or $b = 0$ (or, of course, both), but we can find two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ for which $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. In fact, this is the case if $\theta = \pi/2$, that is, if the vectors are orthogonal:

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0 \qquad \text{if} \qquad \boldsymbol{a} \perp \boldsymbol{b}.$$

The scalar products of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1, \qquad \boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = 0.$$

Thus, taking the scalar product of each in turn with (A.1), we find

$$\boldsymbol{i} \cdot \boldsymbol{r} = x, \qquad \boldsymbol{j} \cdot \boldsymbol{r} = y, \qquad \boldsymbol{k} \cdot \boldsymbol{r} = z.$$

These relations express the fact that the components of $\boldsymbol{r}$ are equal to its projections on the three co-ordinate axes.

More generally, if we take the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we find

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z, \tag{A.6}$$

and in particular,

$$\boldsymbol{r}^2 = r^2 = x^2 + y^2 + z^2. \tag{A.7}$$

A.3 The Vector Product

Any two non-parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ drawn from O define a unique axis through O perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$. It is useful to define the *vector product* $\boldsymbol{a} \wedge \boldsymbol{b}$ (‘$\boldsymbol{a}$ *cross* $\boldsymbol{b}$’, sometimes also written $\boldsymbol{a} \times \boldsymbol{b}$) to be a vector along this axis whose magnitude is the area of the parallelogram with edges $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$|\boldsymbol{a} \wedge \boldsymbol{b}| = ab \sin\theta. \tag{A.8}$$

(See Fig. A.6.) To distinguish between the two opposite directions along the axis, we introduce a convention: the direction of $\boldsymbol{a} \wedge \boldsymbol{b}$ is that in which a right-hand screw would move when turned from $\boldsymbol{a}$ to $\boldsymbol{b}$.

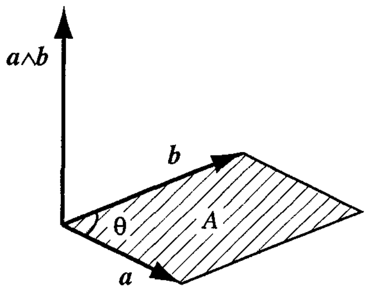

Fig. A.6

A vector whose sense is merely conventional, and would be reversed by changing from a right-hand to a left-hand convention, is called an *axial* vector, as opposed to an ordinary, or *polar*, vector. For example, velocity and force are polar vectors, but angular velocity is an axial vector (see §5.1). The vector product of two polar vectors is thus an axial vector.

The vector product has one very important, but unfamiliar, property. If we interchange $\boldsymbol{a}$ and $\boldsymbol{b}$, we reverse the sign of the vector product:

$$\boldsymbol{b} \wedge \boldsymbol{a} = -\boldsymbol{a} \wedge \boldsymbol{b}. \tag{A.9}$$

It is essential to remember this fact when manipulating any expression involving vector products. In particular, the vector product of a vector with itself is the zero vector,

$$\boldsymbol{a} \wedge \boldsymbol{a} = \mathbf{0}.$$

More generally, $\boldsymbol{a} \wedge \boldsymbol{b}$ vanishes if $\theta = 0$ or π:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \mathbf{0} \qquad \text{if} \qquad \boldsymbol{a} \parallel \boldsymbol{b}.$$

If we choose our co-ordinate axes to be right-handed, then the vector products of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are

$$\begin{aligned} &\boldsymbol{i} \wedge \boldsymbol{i} = \boldsymbol{j} \wedge \boldsymbol{j} = \boldsymbol{k} \wedge \boldsymbol{k} = \mathbf{0}, \\ &\boldsymbol{i} \wedge \boldsymbol{j} = \boldsymbol{k}, \qquad \boldsymbol{j} \wedge \boldsymbol{i} = -\boldsymbol{k}, \\ &\boldsymbol{j} \wedge \boldsymbol{k} = \boldsymbol{i}, \qquad \boldsymbol{k} \wedge \boldsymbol{j} = -\boldsymbol{i}, \\ &\boldsymbol{k} \wedge \boldsymbol{i} = \boldsymbol{j}, \qquad \boldsymbol{i} \wedge \boldsymbol{k} = -\boldsymbol{j}. \end{aligned} \tag{A.10}$$

Thus, when we form the vector product of two arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we obtain

$$\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{i}(a_y b_z - a_z b_y) + \boldsymbol{j}(a_z b_x - a_x b_z) + \boldsymbol{k}(a_x b_y - a_y b_x).$$

This relation may conveniently be expressed in the form of a determinant

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{A.11}$$

From any three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, we can form the *scalar triple product* $(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c}$. Geometrically, it represents the volume V of the parallelepiped with adjacent edges $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (see Fig. A.7). For, if φ is the angle between $\boldsymbol{c}$

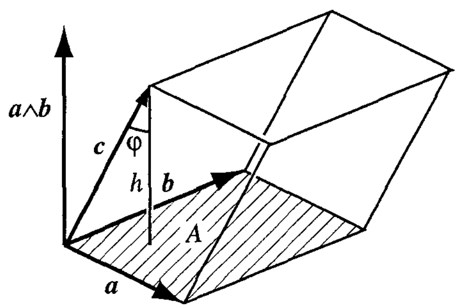

Fig. A.7

and $\boldsymbol{a} \wedge \boldsymbol{b}$, then

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c} = |\boldsymbol{a} \wedge \boldsymbol{b}| c \cos \varphi = Ah = V,$$

where A is the area of the base, and $h = c \cos \varphi$ is the height. The volume is reckoned positive if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ form a right-handed triad, and negative if they form a left-handed triad. For example, $(\boldsymbol{i} \wedge \boldsymbol{j}) \cdot \boldsymbol{k} = 1$, but $(\boldsymbol{i} \wedge \boldsymbol{k}) \cdot \boldsymbol{j} = -1$.

In terms of components, we can evaluate the scalar triple product by taking the scalar product of $\boldsymbol{c}$ with (A.11). We find

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{A.12}$$

Either from this formula, or from its geometric interpretation, we see that the scalar triple product is unchanged by any cyclic permutation of

$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, but changes sign if any pair is interchanged:

$$\begin{aligned}(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c} &= (\boldsymbol{b} \wedge \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \wedge \boldsymbol{a}) \cdot \boldsymbol{b} \\ &= -(\boldsymbol{b} \wedge \boldsymbol{a}) \cdot \boldsymbol{c} = -(\boldsymbol{c} \wedge \boldsymbol{b}) \cdot \boldsymbol{a} = -(\boldsymbol{a} \wedge \boldsymbol{c}) \cdot \boldsymbol{b}.\end{aligned} \tag{A.13}$$

Moreover, we may interchange the dot and the cross:

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c}). \tag{A.14}$$

(For this reason, a more symmetrical notation, $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$, is sometimes used.)

Note that the scalar triple product vanishes if any two vectors are equal, or parallel. More generally, it vanishes if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are coplanar.

From three vectors we can also form the *vector triple product* $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$. Since this vector is perpendicular to $\boldsymbol{a} \wedge \boldsymbol{b}$, it must lie in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$, and must therefore be a linear combination of these two vectors. It is not hard to show by writing out the components, that

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}. \tag{A.15}$$

Similarly,

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}. \tag{A.16}$$

Note that these vectors are unequal, so that we cannot omit the brackets in a vector triple product. It is useful to note that in both of these formulae the term with the positive sign is the middle vector times the scalar product of the other two.

A.4 Differentiation and Integration of Vectors

We are often concerned with vectors which are functions of some scalar parameter, for example the position vector of a particle as a function of time, $\boldsymbol{r}(t)$. The vector distance travelled by the particle in a short time interval Δt is

$$\Delta \boldsymbol{r} = \boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t).$$

(See Fig. A.8.) The velocity, or derivative of $\boldsymbol{r}$ with respect to t, is defined just as for scalars, as the limit of a ratio,

$$\dot{\boldsymbol{r}} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t}. \tag{A.17}$$

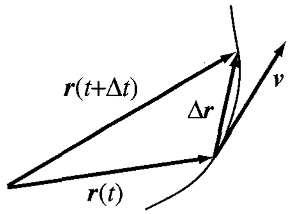

Fig. A.8

In the limit, the direction of this vector is that of the tangent to the path of the particle, and its magnitude is the speed in the usual sense. In terms of co-ordinates,

$$\dot{\boldsymbol{r}} = (\dot{x}, \dot{y}, \dot{z}).$$

Derivatives of other vectors are defined similarly. In particular, we can differentiate again to form the acceleration vector $\ddot{\boldsymbol{r}} = \mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$.

It is easy to show that all the usual rules for differentiating sums and products apply also to vectors. For example,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a} \wedge \boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} \wedge \boldsymbol{b} + \boldsymbol{a} \wedge \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t},$$

though in this particular case one must be careful to preserve the order of the two factors, because of the antisymmetry of the vector product.

Note that the derivative of the magnitude of $\boldsymbol{r}$, $\mathrm{d}r/\mathrm{d}t$, is *not* the same thing as the magnitude of the derivative, $|\mathrm{d}\boldsymbol{r}/\mathrm{d}t|$. For example, if the particle is moving in a circle, r is constant, so that $\dot{r} = 0$, but clearly $|\dot{\boldsymbol{r}}|$ is not zero. In fact, applying the rule for differentiating a scalar product to $\boldsymbol{r}^2$, we obtain

$$2r\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}(r^2) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}^2) = 2\boldsymbol{r} \cdot \dot{\boldsymbol{r}},$$

which may also be written

$$\dot{r} = \hat{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}. \tag{A.18}$$

Thus the rate of change of the distance r from the origin is equal to the radial component of the velocity vector.

We can also define the integral of a vector. If $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, then we also write

$$\boldsymbol{r} = \int \boldsymbol{v}\,\mathrm{d}t,$$

and say that $\boldsymbol{r}$ is the *integral* of $\boldsymbol{v}$. If we are given $\boldsymbol{v}(t)$ as a function of time, and the initial value of $\boldsymbol{r}$, $\boldsymbol{r}(t_0)$, then the position at any later time is given by the definite integral

$$\boldsymbol{r}(t_1) = \boldsymbol{r}(t_0) + \int_{t_0}^{t_1} \boldsymbol{v}(t)\,\mathrm{d}t. \tag{A.19}$$

This is equivalent to three scalar equations for the components, for example,

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} v_x(t)\,\mathrm{d}t.$$

One can show, exactly as for scalars, that the integral in (A.19) may be expressed as the limit of a sum.

A.5 Gradient, Divergence and Curl

There are many quantities in physics which are functions of position in space; for example, temperature, gravitational potential, or electric field. Such quantities are known as *fields.* A *scalar field* is a scalar function $\phi(x, y, z)$ of position in space; a *vector field* is a vector function $\boldsymbol{A}(x, y, z)$. We can also indicate the position in space by the position vector $\boldsymbol{r}$ and write $\phi(\boldsymbol{r})$ or $\boldsymbol{A}(\boldsymbol{r})$.

Now let us consider the three partial derivatives of a scalar field, $\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z$. They form the components of a vector field, known as the *gradient* of ϕ, and written $\operatorname{grad}\phi$, or $\boldsymbol{\nabla}\phi$ ('*del* ϕ', or occasionally '*nabla* ϕ'). To show that they really are the components of a *vector*, we have to show that it is defined in a manner which is independent of the choice of axes. We note that if $\boldsymbol{r}$ and $\boldsymbol{r} + \mathrm{d}\boldsymbol{r}$ are two neighbouring points, then the difference between the values of ϕ at these points is

$$\mathrm{d}\phi = \phi(\boldsymbol{r} + \mathrm{d}\boldsymbol{r}) - \phi(\boldsymbol{r}) = \frac{\partial\phi}{\partial x}\,\mathrm{d}x + \frac{\partial\phi}{\partial y}\,\mathrm{d}y + \frac{\partial\phi}{\partial z}\,\mathrm{d}z = \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{\nabla}\phi. \tag{A.20}$$

Now, if the distance $|\mathrm{d}\boldsymbol{r}|$ is fixed, then this scalar product takes on its maximum value when $\mathrm{d}\boldsymbol{r}$ is in the direction of $\boldsymbol{\nabla}\phi$. Hence we conclude that the direction of $\boldsymbol{\nabla}\phi$ is the direction in which ϕ increases most rapidly.

Moreover, its magnitude is the rate of increase of ϕ with distance in this direction. (This is the reason for the name 'gradient'.) Clearly, therefore, we could *define* $\boldsymbol{\nabla}\phi$ by these properties, which are independent of any choice of axes.

We are often interested in the value of the scalar field ϕ evaluated at the position of a moving particle, $\phi(\boldsymbol{r}(t))$. From (A.20) it follows that the rate of change of ϕ is

$$\frac{\mathrm{d}\phi(\boldsymbol{r}(t))}{\mathrm{d}t} = \dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}\phi. \tag{A.21}$$

The symbol $\boldsymbol{\nabla}$ may be regarded as a vector which is also a differential operator (like $\mathrm{d}/\mathrm{d}x$), given by

$$\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}.$$

We can also apply it to a vector field $\boldsymbol{A}$. The *divergence* of $\boldsymbol{A}$ is defined to be the scalar field

$$\operatorname{div}\boldsymbol{A} = \boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{A.22}$$

and the *curl* of $\boldsymbol{A}$ to be the vector field

$$\operatorname{curl}\boldsymbol{A} = \boldsymbol{\nabla}\wedge\boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{A.23}$$

This latter expression is an abbreviation for the expanded form

$$\boldsymbol{\nabla}\wedge\boldsymbol{A} = \boldsymbol{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$

(Instead of $\operatorname{curl}\boldsymbol{A}$, the alternative notation $\operatorname{rot}\boldsymbol{A}$ is sometimes used, particularly in non-English-speaking countries.)

To understand the physical significance of these operations, it is helpful to think of the velocity field in a fluid: $\boldsymbol{v}(\boldsymbol{r})$ is the fluid velocity at the point $\boldsymbol{r}$.

Let us consider a small volume of fluid, $\delta V = \delta x\,\delta y\,\delta z$, and try to find its rate of change as it moves with the fluid. Consider first the length δx. To a first approximation, over a short time interval $\mathrm{d}t$, the velocity components in the y and z directions are irrelevant; the length δx changes because the x components of velocity, v_x, at its two ends are slightly different, by an amount $(\partial v_x/\partial x)\,\delta x$. Thus in a time $\mathrm{d}t$, the change in δx is $\mathrm{d}\,\delta x = (\partial v_x/\partial x)\,\delta x\,\mathrm{d}t$, whence

$$\frac{\mathrm{d}\,\delta x}{\mathrm{d}t} = \frac{\partial v_x}{\partial x}\,\delta x.$$

Taking account of similar changes in δy and δz, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x\,\delta y\,\delta z) = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta x\,\delta y\,\delta z,$$

or, equivalently,

$$\frac{\mathrm{d}\,\delta V}{\mathrm{d}t} = (\boldsymbol{\nabla}\cdot\boldsymbol{v})\,\delta V. \tag{A.24}$$

Thus $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ represents the proportional rate of increase of volume: positive $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ means expansion, negative $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ compression. In particular, if the fluid is *incompressible*, then $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$.

It is possible to show in a similar way that a non-zero $\boldsymbol{\nabla}\wedge\boldsymbol{v}$ means that locally the fluid is rotating. This vector, called the *vorticity*, represents the local angular velocity of rotation (times 2; see Problem 10).

The rule for differentiating products can also be applied to expressions involving $\boldsymbol{\nabla}$. For example, $\boldsymbol{\nabla}\cdot(\boldsymbol{A}\wedge\boldsymbol{B})$ is a sum of two terms, in one of which $\boldsymbol{\nabla}$ acts on $\boldsymbol{A}$ only and in the other on $\boldsymbol{B}$ only. The gradient of a product of scalar fields can be written

$$\boldsymbol{\nabla}(\phi\psi) = \psi\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\psi,$$

and similarly

$$\boldsymbol{\nabla}\cdot(\phi\boldsymbol{A}) = \boldsymbol{A}\cdot\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\cdot\boldsymbol{A}.$$

But, when vector products are involved, we have to remember that the order of the factors as a product of vectors cannot be changed without affecting the signs. Thus we have

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\wedge\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\wedge\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\wedge\boldsymbol{B}),$$

and, similarly,

$$\boldsymbol{\nabla}\wedge(\phi\boldsymbol{A}) = \phi(\boldsymbol{\nabla}\wedge\boldsymbol{A}) - \boldsymbol{A}\wedge(\boldsymbol{\nabla}\phi).$$

We may apply the vector differential operator $\boldsymbol{\nabla}$ twice. The divergence of the gradient of a scalar field ϕ is called the *Laplacian* of ϕ,

$$\boldsymbol{\nabla}^2\phi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{A.25}$$

Some operations always give zero. Just as $\boldsymbol{a}\wedge\boldsymbol{a} = \mathbf{0}$, we find that the curl of a gradient vanishes,

$$\boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\phi = \mathbf{0}. \tag{A.26}$$

For example, its z component is

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = 0.$$

Similarly, one can show that the divergence of a curl vanishes:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\wedge\boldsymbol{A}) = 0. \tag{A.27}$$

An important identity, analogous to the expansion of the vector triple product (A.16), gives the curl of a curl,

$$\boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A}, \tag{A.28}$$

where of course

$$\boldsymbol{\nabla}^2\boldsymbol{A} = \frac{\partial^2\boldsymbol{A}}{\partial x^2} + \frac{\partial^2\boldsymbol{A}}{\partial y^2} + \frac{\partial^2\boldsymbol{A}}{\partial z^2}.$$

It may easily be proved by inserting the expressions in terms of components.

A.6 Integral Theorems

There are three important theorems for vectors which are generalizations of the fundamental theorem of the calculus,

$$\int_{x_0}^{x_1}\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f(x_1) - f(x_0).$$

First, consider a curve C in space, running from $\boldsymbol{r}_0$ to $\boldsymbol{r}_1$ (see Fig. A.9). Let the directed element of length along C be $\mathrm{d}\boldsymbol{r}$. If ϕ is a scalar field, then according to (A.20), the change in ϕ along this element of length is

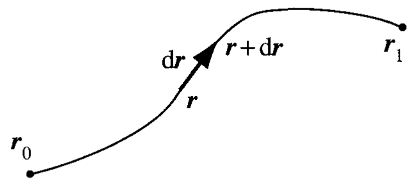

Fig. A.9

$d\phi = d\boldsymbol{r} \cdot \boldsymbol{\nabla}\phi$. Thus, integrating from $\boldsymbol{r}_0$ to $\boldsymbol{r}_1$, we obtain the first of the integral theorems,

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} d\boldsymbol{r} \cdot \boldsymbol{\nabla}\phi = \phi(\boldsymbol{r}_1) - \phi(\boldsymbol{r}_0). \tag{A.29}$$

The integral on the left is called the *line integral* of $\boldsymbol{\nabla}\phi$ along C. (Note that, as here, it is often more convenient to place the differential symbol $d\boldsymbol{r}$ to the *left* of the integrand.)

This theorem may be used to relate the potential energy function $V(\boldsymbol{r})$ for a conservative force to the work done in going from some fixed point $\boldsymbol{r}_0$, where V is chosen to vanish, to $\boldsymbol{r}$. Thus, if $F = -\boldsymbol{\nabla}V$, then

$$V(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{r} \cdot \boldsymbol{F}. \tag{A.30}$$

When $\boldsymbol{F}$ is conservative, this integral depends only on its end-points, and not on the path C chosen between them. Conversely, if this condition is satisfied, we can define V by (A.30), and the force must be conservative. The condition that two line integrals of the form (A.30) should be equal whenever their end-points coincide may be restated by saying the the line integral round any *closed* path should vanish. Physically, this means that no work is done in taking a particle round a loop which returns to its starting point. The integral round a closed loop is usually denoted by the symbol $\oint_C$. Thus we require

$$\oint_C d\boldsymbol{r} \cdot \boldsymbol{F} = 0, \tag{A.31}$$

for all closed loops C.

This condition may be simplified by using the second of the integral theorems — *Stokes' theorem.* Consider a curved surface S, bounded by the closed curve C. If one side of S is chosen to be the 'positive' side, then the positive direction round C may be defined by the right-hand-screw convention (see Fig. A.10). Take a small element of the surface, of area $\mathrm{d}S$, and let

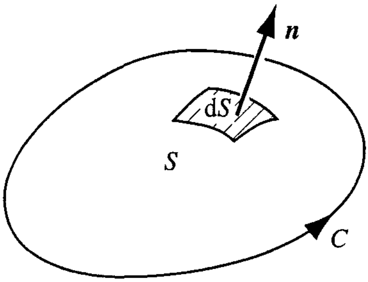

Fig. A.10

$\boldsymbol{n}$ be a unit vector normal to the element, and directed towards its positive side. Then the *directed* element of area is defined to be $\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S$. Stokes' theorem states that if $\boldsymbol{A}$ is any vector field, then

$$\iint_S \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{\nabla} \wedge \boldsymbol{A}) = \oint_C \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{A}. \tag{A.32}$$

The application of this theorem to (A.31) is immediate. If the line integral round C is required to vanish for all closed curves C, then the surface integral must vanish for all surfaces S. But this is only possible if the integrand vanishes identically. So the condition for a force to be conservative is

$$\boldsymbol{\nabla} \wedge \boldsymbol{F} = \boldsymbol{0}. \tag{A.33}$$

We shall not prove Stokes' theorem. However, it is easy to verify for a small rectangular surface. (The proof proceeds by splitting up the surface into small sub-regions.) Suppose S is a rectangle in the xy-plane, of area $\mathrm{d}x\,\mathrm{d}y$. Then $\mathrm{d}\boldsymbol{S} = \boldsymbol{k}\,\mathrm{d}x\,\mathrm{d}y$, so the surface integral is

$$\boldsymbol{k} \cdot (\boldsymbol{\nabla} \wedge \boldsymbol{A})\,\mathrm{d}x\,\mathrm{d}y = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y. \tag{A.34}$$

The line integral consists of four terms, one from each edge. The two terms arising from the edges parallel to the x-axis involve the x component of $\boldsymbol{A}$ evaluated for different values of y. They therefore contribute

$$A_x(y)\,\mathrm{d}x - A_x(y+\mathrm{d}y)\,\mathrm{d}x = -\frac{\partial A_x}{\partial y}\,\mathrm{d}x\,\mathrm{d}y.$$

Similarly, the other pair of edges yields the first term of (A.34).

One can also find a necessary and sufficient condition for a field $\boldsymbol{B}(\boldsymbol{r})$ to have the form of a curl,

$$\boldsymbol{B} = \boldsymbol{\nabla} \wedge \boldsymbol{A}.$$

By (A.27), such a field must satisfy

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{A.35}$$

The proof that this is also a sufficient condition (which we shall not give in detail) follows much the same lines as before. One can show that it is sufficient that the surface integral of $\boldsymbol{B}$ over any *closed* surface should vanish:

$$\iint_S \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B} = 0, \qquad (S \text{ closed})$$

and then use the third of the integral theorems, *Gauss' theorem.* This states that if V is a volume in space bounded by the closed surface S, then for any vector field $\boldsymbol{B}$,

$$\iiint_V \mathrm{d}V\, \boldsymbol{\nabla} \cdot \boldsymbol{B} = \iint_S \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B}, \tag{A.36}$$

where $\mathrm{d}V$ denotes the volume element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, and the positive side of S is taken to be the outside.

It is again easy to verify Gauss' theorem for a small rectangular volume, $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. The volume integral is

$$\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{A.37}$$

The surface integral consists of six terms, one for each face. Consider the faces parallel to the xy-plane, with directed surface elements $\boldsymbol{k}\,\mathrm{d}x\,\mathrm{d}y$ and $-\boldsymbol{k}\,\mathrm{d}x\,\mathrm{d}y$. Their contributions involve $\boldsymbol{k} \cdot \boldsymbol{B} = B_z$, evaluated for different values of z. Thus they contribute

$$B_z(z + \mathrm{d}z)\,\mathrm{d}x\,\mathrm{d}y - B_z(z)\,\mathrm{d}x\,\mathrm{d}y = \frac{\partial B_z}{\partial z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Similarly, the other terms of (A.37) come from the other pairs of faces.

A.7 Electromagnetic Potentials

An important application of these theorems is to the electromagnetic field.

The basic equations of electromagnetic theory are *Maxwell's equations*. In the absence of dielectric or magnetic media, they may be expressed in terms of two fields, the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$. There is one pair of homogeneous equations,

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{A.38}$$

and a second pair involving also the electric charge density ρ and current density $\boldsymbol{j}$,

$$\mu_0^{-1} \boldsymbol{\nabla} \wedge \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j}, \qquad \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \tag{A.39}$$

in which μ_0 and ϵ_0 are universal constants.

The second equation in (A.38) is just the condition (A.35). It follows that there must exist a *vector potential* $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla} \wedge \boldsymbol{A}. \tag{A.40}$$

Then, substituting in the first of the equations (A.38), we find that $\boldsymbol{\nabla} \wedge (\boldsymbol{E} + \partial \boldsymbol{A}/\partial t) = \boldsymbol{0}$. It follows that there must exist a *scalar potential* ϕ such that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{A.41}$$

These potentials are not unique. If Λ is any scalar field, then the potentials

$$\phi' = \phi + \frac{\partial \Lambda}{\partial t}, \qquad \boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{\nabla}\Lambda \tag{A.42}$$

define the same fields $\boldsymbol{E}$ and $\boldsymbol{B}$ as do ϕ and $\boldsymbol{A}$. This is called a *gauge transformation*. We may eliminate this arbitrariness by imposing an extra condition, for example the *radiation gauge* (or *Coulomb gauge*) condition

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \tag{A.43}$$

In the static case, where all the fields are time-*in*dependent, Maxwell's equations separate into a pair of electrostatic equations, and a magnetostatic pair. Then ϕ becomes the ordinary electrostatic potential, satisfying

Poisson's equation (6.48). The vector potential, by (A.39) and (A.40) satisfies

$$\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{A}) = \mu_0 \boldsymbol{j}.$$

Using (A.28), and imposing the radiation gauge condition (A.43), we find

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}. \tag{A.44}$$

This is the analogue of Poisson's equation. The solution is of the same form as (6.15), namely

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3\boldsymbol{r}'. \tag{A.45}$$

Thus, given any static distribution of charges and currents, we may calculate the potentials ϕ and $\boldsymbol{A}$, and hence the fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

A.8 Curvilinear Co-ordinates

Another use of the integral theorems is to provide expressions for the gradient, divergence and curl in terms of curvilinear co-ordinates.

Consider a set of orthogonal curvilinear co-ordinates (see §3.5) q_1, q_2, q_3. Let us denote the elements of length along the three co-ordinate curves by $h_1\,\mathrm{d}q_1, h_2\,\mathrm{d}q_2, h_3\,\mathrm{d}q_3$. For example, in cylindrical polars

$$h_\rho = 1, \qquad h_\varphi = \rho, \qquad h_z = 1, \tag{A.46}$$

while in spherical polars

$$h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r\sin\theta. \tag{A.47}$$

Now consider a scalar field ψ, and two neighbouring points (q_1, q_2, q_3) and $(q_1 + \mathrm{d}q_1, q_2, q_3)$. Then the difference between the values of ψ at these points is

$$\frac{\partial\psi}{\partial q_1}\,\mathrm{d}q_1 = \mathrm{d}\psi = \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{\nabla}\psi = h_1\,\mathrm{d}q_1\,(\boldsymbol{\nabla}\psi)_1,$$

where $(\boldsymbol{\nabla}\psi)_1$ is the component of $\boldsymbol{\nabla}\psi$ in the direction of increasing q_1. Hence we find

$$(\boldsymbol{\nabla}\psi)_1 = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1}, \tag{A.48}$$

with similar expressions for the other components. Thus in cylindrical and spherical polars, we have

$$\nabla\psi = \left(\frac{\partial\psi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\psi}{\partial\varphi}, \frac{\partial\psi}{\partial z}\right), \tag{A.49}$$

and

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right). \tag{A.50}$$

To find an expression for the divergence, we use Gauss' theorem, applied to a small volume bounded by the co-ordinate surfaces. The volume integral is

$$(\nabla\cdot\boldsymbol{A})\, h_1\,\mathrm{d}q_1\, h_2\,\mathrm{d}q_2\, h_3\,\mathrm{d}q_3.$$

In the surface integral, the terms arising from the faces which are surfaces of constant q_3 are of the form $A_3h_1\,\mathrm{d}q_1\,h_2\,\mathrm{d}q_2$, evaluated for two different values of q_3. They therefore contribute

$$\frac{\partial}{\partial q_3}(h_1h_2A_3)\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3.$$

Adding the terms from all three pairs of faces, and comparing with the volume integral, we obtain

$$\nabla\cdot\boldsymbol{A} = \frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3A_1)}{\partial q_1} + \frac{\partial(h_3h_1A_2)}{\partial q_2} + \frac{\partial(h_1h_2A_3)}{\partial q_3}\right). \tag{A.51}$$

In particular, in cylindrical and spherical polars,

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}, \tag{A.52}$$

and

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}. \tag{A.53}$$

To find the curl, we use Stokes' theorem in a similar way. Let us consider a small element of a surface $q_3 =$ constant, bounded by curves of constant q_1 and of q_2. Then the surface integral is

$$(\nabla\wedge\boldsymbol{A})_3\, h_1\,\mathrm{d}q_1\, h_2\,\mathrm{d}q_2.$$

In the line integral around the boundary, the two edges of constant q_2 involve $A_1 h_1\, dq_1$ evaluated for different values of q_2, and so contribute

$$-\frac{\partial}{\partial q_2}(h_1 A_1)\, dq_1\, dq_2.$$

Hence, adding the contribution from the other pair of edges, we obtain

$$(\boldsymbol{\nabla} \wedge \boldsymbol{A})_3 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 A_2)}{\partial q_1} - \frac{\partial(h_1 A_1)}{\partial q_2}\right), \tag{A.54}$$

with similar expressions for the other components. Thus, in particular, in cylindrical and spherical polars

$$\boldsymbol{\nabla} \wedge \boldsymbol{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}, \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}, \frac{1}{\rho}\left[\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right]\right), \tag{A.55}$$

and

$$\begin{aligned}\boldsymbol{\nabla} \wedge \boldsymbol{A} = \Bigg(&\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi}\right],\\ &\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{1}{r}\frac{\partial(r A_\varphi)}{\partial r}, \frac{1}{r}\left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\Bigg).\end{aligned} \tag{A.56}$$

Finally, combining the expressions for the divergence and gradient, we can find the Laplacian of a scalar field. It is

$$\begin{aligned}\boldsymbol{\nabla}^2\psi = \frac{1}{h_1 h_2 h_3}\Bigg[&\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial q_2}\right)\\ &+ \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\Bigg].\end{aligned} \tag{A.57}$$

In cylindrical polars,

$$\boldsymbol{\nabla}^2\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}, \tag{A.58}$$

and, in spherical polars,

$$\boldsymbol{\nabla}^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}. \tag{A.59}$$

A.9 Tensors

Scalars and vectors are the first two members of a family of objects known collectively as *tensors*, and described by $1, 3, 9, 27, \ldots$ components. Scalars and vectors are called tensors of *valence* 0 and *valence* 1, respectively. (Sometimes the word *rank* is used instead of 'valence', but there is then a possibility of confusion with a different usage of the same word in matrix theory.)

In this section, we shall be concerned with the next member of the family, the tensors of valence 2, often called *dyadics*. We shall use the word 'tensor' in this restricted sense, to mean a tensor of valence 2.

Tensors occur most frequently when one vector $\boldsymbol{b}$ is given as a linear function of another vector $\boldsymbol{a}$, according to the matrix equation

$$\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}. \tag{A.60}$$

An example is the relation (9.17) between the angular momentum $\boldsymbol{J}$ and angular velocity $\boldsymbol{\omega}$ of a rigid body.

The nine elements of the 3×3 matrix in (A.60) are the components of a tensor, which we shall denote by the sans-serif capital T. By an obvious extension of the dot product notation for the scalar product of two vectors, we may write (A.60) as

$$\boldsymbol{b} = \mathsf{T} \cdot \boldsymbol{a}. \tag{A.61}$$

For example, (9.17) may be written $\boldsymbol{J} = \mathsf{I} \cdot \boldsymbol{\omega}$, where I is the *inertia tensor.*

We can go on to form the scalar product of (A.61) with another vector, $\boldsymbol{c}$, obtaining a scalar, $\boldsymbol{c} \cdot \mathsf{T} \cdot \boldsymbol{a}$. Note that in general this is not the same as $\boldsymbol{a} \cdot \mathsf{T} \cdot \boldsymbol{c}$. In fact,

$$\boldsymbol{a} \cdot \mathsf{T} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \tilde{\mathsf{T}} \cdot \boldsymbol{a}, \tag{A.62}$$

where $\tilde{\mathsf{T}}$ is the *transposed* tensor of T, obtained by reflecting in the leading diagonal, *e.g.*, $\tilde{T}_{xy} = T_{yx}$.

The tensor T is called *symmetric* if $\tilde{\mathsf{T}} = \mathsf{T}$, *i.e.*, if $T_{ji} = T_{ij}$ for all i, j. It is *antisymmetric* if $\tilde{\mathsf{T}} = -\mathsf{T}$, or $T_{ji} = -T_{ij}$ for all i, j.

An interesting example of an antisymmetric tensor is provided by the relation (5.2) giving the velocity $\boldsymbol{v}$ as a function of position $\boldsymbol{r}$ in a body rotating with angular velocity $\boldsymbol{\omega}$. It is a linear relation and so may be written in the form (A.60), specifically as

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

There is an important special tensor,

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

called the *unit tensor* or *identity tensor*, with the property that $\mathbf{1} \cdot \boldsymbol{a} = \boldsymbol{a}$ for all vectors $\boldsymbol{a}$.

From any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we can form a tensor T by multiplying their elements together (without adding), *i.e.*, $T_{ij} = a_i b_j$. This is the *tensor product* (or *dyadic product* or *outer product*) of $\boldsymbol{a}$ and $\boldsymbol{b}$, written $\mathsf{T} = \boldsymbol{ab}$, with no dot or cross. Note that $\mathsf{T}\cdot\boldsymbol{c} = (\boldsymbol{ab})\cdot\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$, so the brackets are in fact unnecessary. In matrix notation, $\boldsymbol{ab}$ is the product of the column vector $\boldsymbol{a}$ and the row vector $\boldsymbol{b}$, while the scalar product (or *inner product*) $\boldsymbol{a} \cdot \boldsymbol{b}$ is the row $\boldsymbol{a}$ times the column $\boldsymbol{b}$.

We can deduce the correct transformation law of a tensor under a rotation of axes: its components transform just like the products of components of two vectors. If we symbolize (A.2) formally as $\boldsymbol{a}' = \mathsf{R}\cdot\boldsymbol{a}$, then the correct transformation law of a tensor is $\mathsf{T}' = \mathsf{R}\cdot\mathsf{T}\cdot\tilde{\mathsf{R}}$. (This denotes a product of three 3×3 matrices.)

The use of the tensor product allows us to write some old results in a new way. For example, for any vector $\boldsymbol{a}$,

$$\mathbf{1}\cdot\boldsymbol{a} = \boldsymbol{a} = \boldsymbol{i}(\boldsymbol{i}\cdot\boldsymbol{a}) + \boldsymbol{j}(\boldsymbol{j}\cdot\boldsymbol{a}) + \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{a}) = (\boldsymbol{ii}+\boldsymbol{jj}+\boldsymbol{kk})\cdot\boldsymbol{a},$$

whence

$$\boldsymbol{ii}+\boldsymbol{jj}+\boldsymbol{kk} = \mathbf{1}, \tag{A.63}$$

as may easily be verified by writing out the components.

Similarly, we may write the relation (9.16) between angular momentum and angular velocity in the form

$$\boldsymbol{J} = \sum m(r^2\boldsymbol{\omega} - \boldsymbol{rr}\cdot\boldsymbol{\omega}) = \mathsf{I}\cdot\boldsymbol{\omega},$$

where the inertia tensor I is given explicitly by

$$\mathsf{I} = \sum m(r^2\mathbf{1} - \boldsymbol{rr}).$$

Note the difference between the unit tensor $\mathbf{1}$ and the inertia tensor I. It is easy to check that the nine components of this equation reproduce the relations (9.15).

Note that if $\mathsf{T} = \boldsymbol{ab}$, then $\tilde{\mathsf{T}} = \boldsymbol{ba}$, whence in particular the inertia tensor I is symmetric.

A.10 Eigenvalues; Diagonalization of a Symmetric Tensor

In this section, we discuss a theorem that has very wide applicability.

Let T be a symmetric tensor. A vector $\boldsymbol{a}$ is called an *eigenvector* of T, with *eigenvalue* λ, if

$$\mathsf{T} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}, \tag{A.64}$$

or, equivalently $(\mathsf{T} - \lambda \mathbf{1}) \cdot \boldsymbol{a} = \mathbf{0}$. (Compare (11.17), which is also an eigenvalue equation.) The condition for the existence of a non-trivial solution is that the determinant of the coefficients vanishes,

$$\det(\mathsf{T} - \lambda \mathbf{1}) = \begin{vmatrix} T_{xx} - \lambda & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} - \lambda & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} - \lambda \end{vmatrix} = 0.$$

This is a cubic equation for λ. Its three roots are either all real, or else one real and one complex conjugate pair. However, for a symmetric tensor T with real elements the latter possibility can be ruled out.

To see this, suppose that λ is a complex eigenvalue, and let $\boldsymbol{a}$ be the corresponding eigenvector, whose components may also be complex. Now, taking the complex conjugate of $\mathsf{T} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}$, we obtain $\mathsf{T} \cdot \boldsymbol{a}^* = \lambda^* \boldsymbol{a}^*$, where λ^* denotes the complex conjugate of λ, and $\boldsymbol{a}^* = (a_x^*, a_y^*, a_z^*)$. Multiplying these two equations by $\boldsymbol{a}^*$ and $\boldsymbol{a}$ respectively, we obtain

$$\boldsymbol{a}^* \cdot \mathsf{T} \cdot \boldsymbol{a} = \lambda \boldsymbol{a}^* \cdot \boldsymbol{a}, \qquad \text{and} \qquad \boldsymbol{a} \cdot \mathsf{T} \cdot \boldsymbol{a}^* = \lambda^* \boldsymbol{a} \cdot \boldsymbol{a}^*.$$

But since T is symmetric, the left-hand sides of these equations are equal, by (A.62). Hence the right-hand sides must be equal too. Since $\boldsymbol{a}^* \cdot \boldsymbol{a} = |a_x|^2 + |a_y|^2 + |a_z|^2 = \boldsymbol{a} \cdot \boldsymbol{a}^*$, this means that $\lambda^* = \lambda$, *i.e.*, λ must be real.

Thus we have shown that there are three real eigenvalues, say $\lambda_1, \lambda_2, \lambda_3$, and three corresponding real eigenvectors, $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$. (We consider the case where two eigenvalues are equal below.) Next, we show that the eigenvectors are orthogonal. For, if

$$\mathsf{T} \cdot \boldsymbol{a}_1 = \lambda_1 \boldsymbol{a}_1, \qquad \mathsf{T} \cdot \boldsymbol{a}_2 = \lambda_2 \boldsymbol{a}_2,$$

then, multiplying the first equation by $\boldsymbol{a}_2$ and the second by $\boldsymbol{a}_1$, and again using the symmetry of $\mathbf{T}$, we obtain

$$\lambda_1 \boldsymbol{a}_2 \cdot \boldsymbol{a}_1 = \lambda_2 \boldsymbol{a}_1 \cdot \boldsymbol{a}_2.$$

Thus if $\lambda_1 \neq \lambda_2$, then $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = 0$.

If all three eigenvalues are distinct, then the three eigenvectors are orthogonal. Moreover, it is clear that if $\boldsymbol{a}$ is an eigenvector, then so is any multiple of $\boldsymbol{a}$, so that we may choose to normalize it, defining $\boldsymbol{e}_1 = \boldsymbol{a}_1/a_1$. Then the three normalized eigenvectors form an orthonormal triad, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. If we choose these as axes, then $\mathbf{T}$ must take the diagonal form

$$\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{A.65}$$

For, since $\boldsymbol{e}_1 = (1, 0, 0)$, $\mathbf{T} \cdot \boldsymbol{e}_1$ is simply the first column of $\mathbf{T}$, and this must be $\lambda_1 \boldsymbol{e}_1 = (\lambda_1, 0, 0)$. Similarly for the other columns.

This relationship between $\mathbf{T}$ and the eigenvectors may also be expressed, using the tensor-product notation, in a co-ordinate-independent form, namely

$$\mathbf{T} = \lambda_1 \boldsymbol{e}_1 \boldsymbol{e}_1 + \lambda_2 \boldsymbol{e}_2 \boldsymbol{e}_2 + \lambda_3 \boldsymbol{e}_3 \boldsymbol{e}_3. \tag{A.66}$$

Finally, we have to show that these results still hold if two or three eigenvalues coincide. The simplest way to do this is to add a small quantity ϵ to one of the diagonal components of $\mathbf{T}$, to make the eigenvalues slightly different. So long as $\epsilon \neq 0$, the tensor must have three orthonormal eigenvectors. By continuity, this must still be true in the limit $\epsilon \to 0$. (The symmetry of $\mathbf{T}$ is important here, because without the consequent orthogonality of eigenvectors we could not exclude the possibility that two eigenvectors that are distinct for $\epsilon \neq 0$ have the same limit as $\epsilon \to 0$. Indeed, this *does* happen for non-symmetric tensors, as will be seen in a different context in Appendix C.)

We have shown, therefore, that any symmetric tensor may be diagonalized by a suitable choice of axes. This was the result we used for the inertia tensor in Chapter 9. In that case, the eigenvectors are the principal axes, and the eigenvalues the principal moments of inertia. The procedure for finding normal co-ordinates for an oscillating system, discussed in §11.2, is essentially the same. In that case, it is the potential energy function that is brought to 'diagonal' form. Eigenvalue equations also appear in the analysis of dynamical systems in Chapter 13 and in many other branches of physics, in particular playing a big role in quantum mechanics.

Problems

1. Given $\boldsymbol{a} = (3, -1, 2), \boldsymbol{b} = (0, 1, 1)$ and $\boldsymbol{c} = (2, 2, -1)$, find:
   (a) $\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{a} \cdot \boldsymbol{c}$ and $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$;
   (b) $\boldsymbol{a} \wedge \boldsymbol{b}$, $\boldsymbol{a} \wedge \boldsymbol{c}$ and $\boldsymbol{a} \wedge (\boldsymbol{b} + \boldsymbol{c})$;
   (c) $(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c}$ and $(\boldsymbol{a} \wedge \boldsymbol{c}) \cdot \boldsymbol{b}$;
   (d) $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ and $(\boldsymbol{a} \wedge \boldsymbol{c}) \wedge \boldsymbol{b}$;
   (e) $(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$ and $(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$.
2. Find the angles between the vectors $\boldsymbol{a} \wedge \boldsymbol{b}$ and $\boldsymbol{c}$, and between $\boldsymbol{a} \wedge \boldsymbol{c}$ and $\boldsymbol{b}$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are as in Problem 1.
3. Show that $\boldsymbol{c} = (a\boldsymbol{b} + b\boldsymbol{a})/(a + b)$ bisects the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two vectors.
4. Find $\boldsymbol{\nabla}\phi$ if $\phi = x^3 - xyz$. Verify that $\boldsymbol{\nabla} \wedge \boldsymbol{\nabla}\phi = \boldsymbol{0}$, and evaluate $\boldsymbol{\nabla}^2\phi$.
5. (a) Find the gradients of $u = x + y^2/x$ and $v = y + x^2/y$, and show that they are always orthogonal.
   (b) Describe the contour curves of u and v in the xy-plane. What does (a) tell you about these curves?
6. Draw appropriate figures to give geometric proofs for the following laws of vector algebra:
$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c});$$
$$\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b};$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}.$$
   (Note that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ need not be coplanar.)
7. Show that $(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot (\boldsymbol{c} \wedge \boldsymbol{d}) = \boldsymbol{a} \cdot \boldsymbol{c}\,\boldsymbol{b} \cdot \boldsymbol{d} - \boldsymbol{a} \cdot \boldsymbol{d}\,\boldsymbol{b} \cdot \boldsymbol{c}$. Hence show that $(\boldsymbol{a} \wedge \boldsymbol{b})^2 = \boldsymbol{a}^2\boldsymbol{b}^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.
8. Express $\boldsymbol{\nabla} \wedge (\boldsymbol{a} \wedge \boldsymbol{b})$ in terms of scalar products.
9. If the vector field $\boldsymbol{v}(\boldsymbol{r})$ is defined by $\boldsymbol{v} = \omega\boldsymbol{k} \wedge \boldsymbol{r}$, verify that $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, and evaluate the vorticity $\boldsymbol{\nabla} \wedge \boldsymbol{v}$.
10. *Show that, if u and v are scalar fields, the maxima and minima of u on the surface $v = 0$ are points where $\boldsymbol{\nabla}u = \lambda\boldsymbol{\nabla}v$ for some value of λ. Interpret this equation geometrically. (*Hint*: On $v = 0$ only two coordinates can vary independently. Thus δz for example can usually be expressed in terms of δx and δy. We require that δu should vanish for

all infinitesimal variations satisfying this constraint.) Show that this problem is equivalent to finding the *unrestricted* maxima and minima of the function $w(\boldsymbol{r}, \lambda) = u - \lambda v$ as a function of the *four* independent variables x, y, z and λ. Here λ is called a *Lagrange multiplier.* What is the role of the equation $\partial w/\partial\lambda = 0$?

11. *Evaluate the components of $\boldsymbol{\nabla}^2 \boldsymbol{A}$ in cylindrical polar co-ordinates by using the identity (A.28). Show that they are *not* the same as the scalar Laplacians of the components of $\boldsymbol{A}$.
12. *Find the radiation-gauge vector potential at large distances from a circular loop of radius a carrying an electric current I. [*Hint*: Consider first a point $(x, 0, z)$, and expand the integrand in powers of a/r, keeping only the linear term. Then express your answer in spherical polars.] Hence find the magnetic field — the field of a *magnetic dipole.* Express the results in terms of the *magnetic moment* $\boldsymbol{\mu}$, a vector normal to the loop, of magnitude $\mu = \pi a^2 I$.
13. *Calculate the vector potential due to a short segment of wire of directed length $\mathrm{d}\boldsymbol{s}$, carrying a current I, placed at the origin. Evaluate the corresponding magnetic field. Find the force on another segment, of length $\mathrm{d}\boldsymbol{s}'$, carrying current I', at $\boldsymbol{r}$. (To compute the force, treat the current element as a collection of moving charges.) Show that this force does not satisfy Newton's third law. (To preserve the law of conservation of momentum, one must assume that, while this force is acting, some momentum is transferred to the electromagnetic field.)
14. *Given $u = \cos\theta$ and $v = \ln r$, evaluate $\boldsymbol{A} = u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u$. Find the divergence and curl of $\boldsymbol{A}$, and verify that $\boldsymbol{\nabla}\cdot\boldsymbol{A} = u\boldsymbol{\nabla}^2 v - v\boldsymbol{\nabla}^2 u$ and that $\boldsymbol{\nabla}\wedge\boldsymbol{A} = 2\boldsymbol{\nabla}u\wedge\boldsymbol{\nabla}v$.
15. *Show that the rotation which takes the axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ into $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ may be specified by $\boldsymbol{r} \to \boldsymbol{r}' = \mathsf{R}\cdot\boldsymbol{r}$, where the tensor R is $\mathsf{R} = \boldsymbol{i}'\boldsymbol{i} + \boldsymbol{j}'\boldsymbol{j} + \boldsymbol{k}'\boldsymbol{k}$. Write down the matrix of components of R if the rotation is through an angle θ about the y-axis. What is the tensor corresponding to the rotation which takes $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ back into $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$? Show that $\tilde{\mathsf{R}}\cdot\mathsf{R} = \mathbf{1}$. (Such tensors are said to be *orthogonal.*)
16. *The *trace* of a tensor T is the sum of its diagonal elements, $\mathrm{tr}(\mathsf{T}) = \sum_i T_{ii}$. Show that the trace is equal to the sum of the eigenvalues, and that the determinant $\det(\mathsf{T})$ is equal to the product of the eigenvalues.
17. *The *double dot* product of two tensors is defined as $\mathsf{S}{:}\mathsf{T} = \mathrm{tr}(\mathsf{S}\cdot\mathsf{T}) = \sum_i\sum_j S_{ij}T_{ji}$. Evaluate $\mathbf{1} : \mathbf{1}$ and $\mathbf{1} : \boldsymbol{rr}$. Show that

$$(3\boldsymbol{r}'\boldsymbol{r}' - r'^2\mathbf{1}) : (\boldsymbol{rr} - \tfrac{1}{3}r^2\mathbf{1}) = 3(\boldsymbol{r}'\cdot\boldsymbol{r})^2 - r'^2 r^2.$$

Hence show that the expansion (6.19) of the potential may be written

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(q\frac{1}{r} + \boldsymbol{d}\cdot\frac{\boldsymbol{r}}{r^3} + \tfrac{1}{2}\mathbf{Q} : \frac{\boldsymbol{rr} - \frac{1}{3}r^2\mathbf{1}}{r^5} + \cdots\right),$$

and write down an expression for the *quadrupole tensor* $\mathbf{Q}$. Show that $\mathrm{tr}(\mathbf{Q}) = 0$, and that in the axially symmetric case it has diagonal elements $-\frac{1}{2}Q, -\frac{1}{2}Q, Q$, where Q is the quadrupole moment defined in Chapter 6. Show also that the gravitational quadrupole tensor is related to the inertia tensor $\mathbf{I}$ by $\mathbf{Q} = \mathrm{tr}(\mathbf{I})\mathbf{1} - 3\mathbf{I}$.

18. *In an elastic solid in equilibrium, the force across a small area may have both a normal component (of compression or tension) and transverse components (shearing stress). Denote the ith component of force per unit area across an area with normal in the jth direction by T_{ij}. These are the components of the *stress tensor* $\mathbf{T}$. By considering the equilibrium of a small volume, show that the force across area A with normal in the direction of the unit vector $\boldsymbol{n}$ is $\boldsymbol{F} = \mathbf{T}\cdot\boldsymbol{n}A$. Show also by considering the equilibrium of a small rectangular volume that $\mathbf{T}$ is symmetric. What physical significance attaches to its eigenvectors?

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Appendix B

Conics

Conic sections, or *conics* for short, are most simply defined as curves in a plane whose equation in Cartesian co-ordinates is quadratic in x and y. The name derives from the fact that they can be obtained by making a plane section through a circular cone. They turn up in several physical applications, particularly in the theory of orbits under an inverse square law force. It may be useful to gather together the relevant mathematical information.

B.1 Cartesian Form

The most general conic would have an equation of the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where $A, B, \ldots, F$ are real constants, but by choosing the axes appropriately we can reduce this to a simpler form.

First, we look at the quadratic part, $Ax^2 + 2Bxy + Cy^2$. It is always possible by rotating the axes to eliminate the constant B. This is another example of the diagonalization process described in §11.3 and §A.10. The quadratic part of the equation is then reduced to a sum of squares, $A'x'^2 + C'y'^2$. We then forget about the original co-ordinates, and drop the primes. The nature of the curve is largely determined by the ratio A/C of the new constants.

Let us assume for the moment that A and C are both non-zero (we will come back later to the special case where that isn't true). Then we can choose to shift the origin (adding constants to x and y, *e.g.*, $x' = x + D/A$) so as to remove D and E. If F is also non-zero, we can move it to the other side of the equation, and divide by $-F$, to get the standard form of the equation,

$$Ax^2 + Cy^2 = 1. \tag{B.1}$$

$F = 0$ is a degenerate case: if A and C have the same sign, the only solution is $x = y = 0$; if they are of opposite sign, the equation factorizes, and so represents a pair of straight lines, $y = \pm\sqrt{-A/C}\,x$.

We cannot allow both A and C in (B.1) to be negative; the equation would then have no solutions at all. So we can distinguish two cases:

1. Both A and C are positive. Without loss of generality we can assume that $A \leq C$. (If $A > C$, we simply interchange the x and y axes.) Defining new positive constants a and b by $A = 1/a^2$ and $C = 1/b^2$, we finally arrive at the canonical form of the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{B.2}$$

This is the equation of an *ellipse* (see Fig. B.1). Here $a \geq b$; a is the *semi-major axis* and b is the *semi-minor axis*. (In the special case

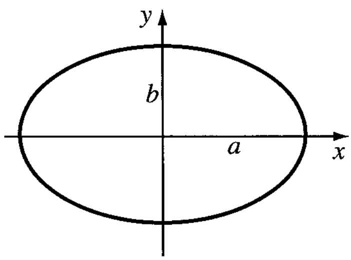


Fig. B.1

$a = b$, we have a *circle* of radius a.)

2. A and C have opposite signs. Again, without loss of generality, we can assume that $A > 0$ and $C < 0$. So, defining $A = 1/a^2$ and $C = -1/b^2$, we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{B.3}$$

the equation of a *hyperbola* (see Fig. B.2); a and b are still called the *semi-major axis* and the *semi-minor axis* respectively, although it is no longer necessarily true that a is the larger. Note that this curve has two

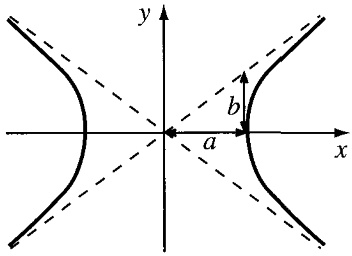


Fig. B.2

separate branches on opposite sides of the origin. At large distances, it asymptotically approaches the two straight lines $y = \pm(b/a)x$ shown on the figure.

We still have to consider the special case where one of the constants A and C vanishes. (They cannot both vanish, otherwise we have simply a linear equation, representing a straight line.) Without loss of generality, we may assume that $A = 0$ and $C \neq 0$. As before, we can shift the origin in the y direction to eliminate E. On the other hand, D cannot be zero (otherwise x doesn't appear at all in the equation). This time, we can choose the origin

Fig. B.3

in the x direction to make $F = 0$ (by setting $x' = x + F/2D$). Finally, defining $l = -D/C$, we arrive at the canonical form

$$y^2 = 2lx, \tag{B.4}$$

which is the equation of a *parabola* (see Fig. B.3).

Areas are easy to compute using the Cartesian form of the equation. For example, we can solve (B.2) for y and integrate to find the area of the ellipse; the result, which generalizes the familiar πr^2 for a circle, is πab. (In fact, the ellipse may be thought of as a circle of radius a which has been squashed uniformly in the y direction in the ratio b/a.)

B.2 Polar Form

When we are looking for orbits under the action of a central force, it is usually convenient to use polar co-ordinates. The form of the equation that emerged from the discussion in §4.4 was

$$r(e\cos\theta \pm 1) = l, \tag{B.5}$$

where e and l are constants satisfying $e \geq 0$, $l > 0$ (the upper and lower signs refer to the attractive and repulsive cases, respectively).

It is interesting to see how this form is related to the Cartesian form above. If we rearrange (B.5) and square it, we obtain for both signs the equation

$$x^2 + y^2 = (l - ex)^2. \tag{B.6}$$

This can easily be put into one of the canonical forms above; which one depends on the value of e.

I. If $e < 1$, we can 'complete the square' in (B.6) and write it as

$$(1 - e^2)x^2 + 2elx + \frac{e^2 l^2}{1 - e^2} + y^2 = \frac{l^2}{1 - e^2}. \tag{B.7}$$

Dividing by $l^2/(1 - e^2)$, this reduces almost to the form (B.2), where

$$a = \frac{l}{1 - e^2}, \qquad b = \frac{l}{\sqrt{1 - e^2}}. \tag{B.8}$$

The only difference is that the origin is not at the centre of the ellipse: (B.7) is equivalent to

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{B.9}$$

an ellipse with centre at $(-ae, 0)$.

II. If $e > 1$, we complete the square in the same way and divide by $l^2/(e^2 - 1)$, obtaining

$$\frac{(x - ae)^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{B.10}$$

where now

$$a = \frac{l}{e^2 - 1}, \qquad b = \frac{l}{\sqrt{e^2 - 1}}. \tag{B.11}$$

This is a hyperbola with centre at $(ae, 0)$. The left-hand branch, intersecting the x-axis at $(ae - a, 0)$, corresponds to an orbit under an attractive inverse square law force, while the right-hand one, meeting it at $(ae + a, 0)$, corresponds to the repulsive case.

III. Finally, if $e = 1$, the equation can be written

$$y^2 = l^2 - 2lx, \tag{B.12}$$

which is a parabola with its apex at $(l/2, 0)$, and oriented in the opposite direction to (B.4).

In all these cases, the position of the origin is one *focus* of the conic. In cases I and II there is a second focus symmetrically placed on the other side of the centre; for the parabola, the second focus is at infinity. (The plural of *focus* is *foci*.) The reason for the name is an intriguing geometric property (see Problem 2): if we have a perfect mirror in the shape of an ellipse light from a source at one focus will converge to the second focus. Similarly, a source at the focus of a parabolic mirror generates a parallel beam, which makes parabolic mirrors ideal for certain applications. For a hyperbolic mirror with a source at one focus, the reflected light will appear to come from a virtual image at the second focus.

Problems

1. The equation (B.2) of an ellipse can be written in parametric form as $x = a\cos\psi$, $y = b\sin\psi$. Show [using the identity $b^2 = (1 - e^2)a^2$)] that the distances between the point labelled ψ and the two foci, $(\pm ae, 0)$, are $a(1 \mp e\cos\psi)$, and hence that the sum of the two distances is a constant. (This result provides a commonly used method of drawing an ellipse, by tying a string between two pegs at the foci, stretching it round a pencil, and drawing a curve while keeping the string taut.)

2. *Using the parametrization of the previous question, show that the slope of the curve is given by $\mathrm{d}y/\mathrm{d}x = -(b/a)\cot\psi$. Hence show that the angles between the curve and the two lines joining it to the foci are equal. (One way is to find the scalar products between the unit vector tangent to the curve and the unit vectors from the two foci. This result provides a proof of the focussing property: light from one focus converges to the other.)
[The results stated in Problems 1, 2 imply that all radiation originating at one focus of an ellipse at a particular time is then reflected to the other focus with the same time of arrival — a consequence with many applications, both peaceful and otherwise.]

Appendix C

Phase Plane Analysis near Critical Points

In this appendix we give a summary of the types of behaviour exhibited by a general autonomous dynamical system near critical points in the phase plane ($n = 2$), as indicated in §13.3.

C.1 Linear Systems and their Classification

We saw in §13.3 that, in the local expansion near a critical point (x_0, y_0), the key to the local behaviour and to the stability of the equilibrium at the critical point is, normally, the behaviour of the linear system

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = M \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{C.1}$$

which is obtained from (13.14) by neglecting higher-order terms in the expansion. The 2×2 Jacobian matrix M [a tensor of valence 2 (§A.10)] has constant entries, which are found as derivatives of the functions $F(x, y), G(x, y)$ evaluated at the critical points (x_0, y_0), as in (13.11), (13.13).

Consider

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, d are real constants. For the critical point itself, at which $\xi = 0, \eta = 0$, to be an *isolated critical point* it is necessary that the determinant of M is non-zero. That is to say $ad - bc \neq 0$ and M then has an inverse. If this condition is not satisfied, so that M is singular, then there is at least a *critical line* through $\xi = 0, \eta = 0$, rather than just the single point; we do not consider this case further here.

If we seek a solution to (C.1) in the form

$$\boldsymbol{\xi}(t) \equiv \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} \mathrm{e}^{\lambda t} \equiv \boldsymbol{\xi}_0 \mathrm{e}^{\lambda t}, \tag{C.2}$$

then we require

$$M\boldsymbol{\xi}_0 = \lambda \boldsymbol{\xi}_0 \tag{C.3}$$

and this is an eigenvalue/eigenvector problem. (See §A.10, although M may not now be symmetric.)

Here the eigenvalues λ_1, λ_2 satisfy the quadratic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0,$$

that is

$$\lambda^2 - (\mathrm{tr} M)\lambda + (\det M) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2) = 0, \tag{C.4}$$

so that $\lambda_1 + \lambda_2 = \mathrm{tr} M$, the *trace* of M and $\lambda_1 \lambda_2 = \det M$, the *determinant* of M (see Appendix A, Problem 16). The eigenvalues λ_1, λ_2 lead to corresponding eigenvectors $\boldsymbol{\xi}_{01}, \boldsymbol{\xi}_{02}$ respectively, in principle, but, since the matrix M is not necessarily symmetric we have eigenvalues which may not be real and a set of eigenvectors which may not be orthogonal or even complete.

There are various cases depending on the nature of the eigenvalues and we can consider separately the cases $\lambda_1 \neq \lambda_2$ and $\lambda_1 = \lambda_2$.

1. $\lambda_1 \neq \lambda_2$. In this case, because of the linearity of the system, we can write

$$\boldsymbol{\xi}(t) = c_1 \boldsymbol{\xi}_{01} \mathrm{e}^{\lambda_1 t} + c_2 \boldsymbol{\xi}_{02} \mathrm{e}^{\lambda_2 t}, \tag{C.5}$$

with c_1, c_2 constants. The vectors $\boldsymbol{\xi}_{01}, \boldsymbol{\xi}_{02}$ are independent in this case and any vector can be expressed as a linear combination of them. In particular $\boldsymbol{\xi}_0 \equiv \boldsymbol{\xi}(0)$ leads to the unique values of c_1, c_2 corresponding to given initial conditions. In this situation we can carry out a linear change of variables

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = S \begin{bmatrix} \bar{\xi} \\ \bar{\eta} \end{bmatrix},$$

similar to the change to normal co-ordinates in §11.4, in such a way that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\bar{\xi}\\ \bar{\eta}\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\bar{\xi}\\ \bar{\eta}\end{bmatrix}. \tag{C.6}$$

In this *similarity transformation* the 2×2 matrix $S=[\boldsymbol{\xi}_{01}\,\vdots\,\boldsymbol{\xi}_{02}]$ and the diagonal matrix in (C.6) above then takes the form $S^{-1}MS$. It should be noted here that $\lambda_i, \boldsymbol{\xi}_{0i}, c_i$ $(i=1,2)$ could be complex, but even then (C.5) is the formal expression of the solution for $\boldsymbol{\xi}$. In the case when M is symmetric then the eigenvalues λ_1, λ_2 are real and the eigenvectors $\boldsymbol{\xi}_{01}, \boldsymbol{\xi}_{02}$ are orthogonal. If the eigenvectors are normalized to have unit length then $S^{-1}\equiv\tilde{S}$, *i.e.* S is a *rotation matrix*.

2. $\lambda_1=\lambda_2\,(\equiv\lambda)$. In this case, λ is necessarily real and we may find that the matrix reduction to diagonal form indicated above may, or may not, be possible:

   (a) If we *can* find two distinct eigenvectors corresponding to λ then the above machinery will go through trivially, since the matrix $M=\lambda I$ in this case, where I is the unit matrix, and *all* non-zero vectors are eigenvectors!

   (b) If there are *not* two distinct eigenvectors corresponding to λ then the best that can be done by a linear transformation is to reduce the system to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\bar{\xi}\\ \bar{\eta}\end{bmatrix}=\begin{bmatrix}\lambda & 0\\ 1 & \lambda\end{bmatrix}\begin{bmatrix}\bar{\xi}\\ \bar{\eta}\end{bmatrix}, \tag{C.7}$$

   since the diagonal form is not now achievable. The system (C.7) has the solution

$$\begin{bmatrix}\bar{\xi}\\ \bar{\eta}\end{bmatrix}=c_1\begin{bmatrix}0\\ 1\end{bmatrix}\mathrm{e}^{\lambda t}+c_2\begin{bmatrix}1\\ t\end{bmatrix}\mathrm{e}^{\lambda t}, \tag{C.8}$$

   with c_1, c_2 constants.

Depending on the eigenvalues λ_1, λ_2 there are then various possible cases to consider. These are listed below together with sketches of typical patterns of local trajectories. The orientation and sense of rotation in these patterns depends on the system concerned. However, in each case the directions of the arrows indicate evolution with time t along the trajectories.

Case 1

λ_1, λ_2 real, unequal, *same sign* $\Longrightarrow$ (*improper*) *node*, *e.g. negative* sign (Fig. C.1).

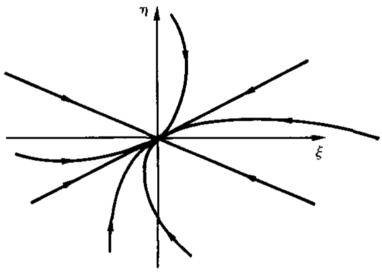


Fig. C.1

All trajectories except for one pair approach the critical point tangent to the same line. The critical point is *asymptotically stable.*

If the sign of λ_1, λ_2 is *positive* then the local structure is similar to that above, but with the sense of the arrows reversed. The critical point is then *unstable*.

Case 2

λ_1, λ_2 real, equal or unequal magnitude, *opposite sign* $\Longrightarrow$ *saddle* (or *hyperbolic point*) (Fig. C.2). This type of critical point is always *unstable*.

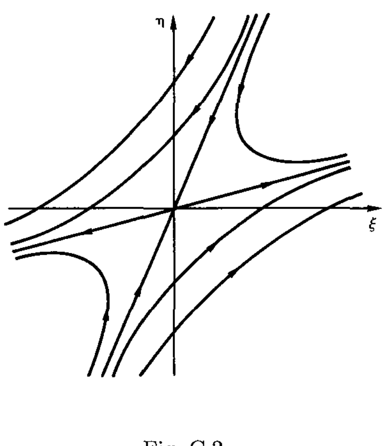


Fig. C.2

Case 3

$\lambda_1 = \lambda_2 = \lambda$ (necessarily real).

1. When $M = \lambda I$ we have a *(proper) node*, *e.g.* λ *negative* (Fig. C.3). This critical point is *asymptotically stable*.

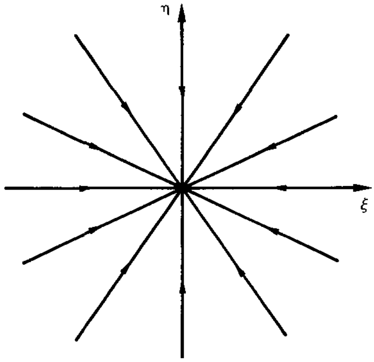


Fig. C.3

If λ is *positive* then the local structure is similar to that above, but with the sense of the arrows reversed. The critical point is then *unstable*.
(A *proper node* is sometimes called a *star*, *focus*, *source* or *sink* as appropriate.)

2. When M may *not* be diagonalized, so that there is only a single eigenvector corresponding to λ, we have an *improper* (or *inflected*) *node*, *e.g.* λ *negative* (Fig. C.4). This critical point is *asymptotically stable*.

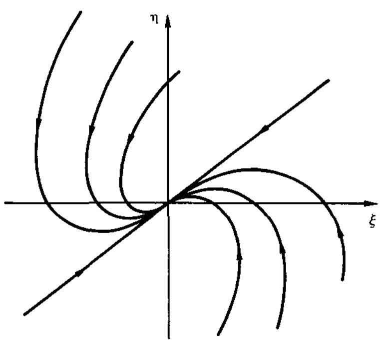


Fig. C.4

If λ is *positive* then the local structure is similar to that above, but with the sense of the arrows reversed. The critical point is then *unstable*.

Case 4

$\lambda_{1,2}$ a complex conjugate pair $\mu \pm \mathrm{i}\nu$, with $\mu \neq 0 \implies$ *spiral* (*point*), *e.g.* μ *negative* (Fig. C.5).

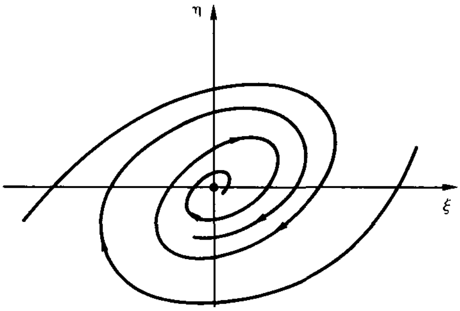


Fig. C.5

This critical point is *asymptotically stable*.

If μ is *positive* then the local structure is similar to that above, but with the sense of the arrows reversed. The critical point is then *unstable*.

(A *spiral* is sometimes called a *spiral source* or *spiral sink* as appropriate.)

Case 5

$\lambda_{1,2}$ a pure imaginary conjugate pair $\pm\mathrm{i}\nu \implies$ *centre* (or *elliptic point*) (Fig. C.6).

The sense of the arrows may be different, but this type of critical point is always *stable*.

We can solve equation (C.4) for the eigenvalues λ_1, λ_2 in terms of $\mathrm{tr}M$ and $\det M$ obtaining

$$\lambda_{1,2} = \tfrac{1}{2}(\mathrm{tr}M \pm \sqrt{\Delta}), \tag{C.9}$$

where the discriminant $\Delta = (\mathrm{tr}M)^2 - 4(\det M)$. We may then represent the types of behaviour in the phase plane near a critical point schematically.

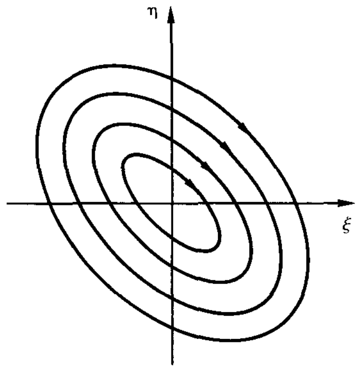


Fig. C.6

(See Fig. C.7.) Note that along the line $\det M = 0$ the critical point is not isolated.

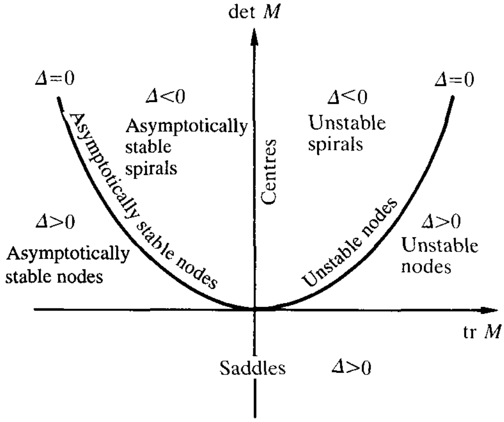


Fig. C.7

C.2 Almost Linear Systems

We have seen that the local analysis near a typical critical point in the phase plane leads to (13.14) and this equation differs from the linear system (C.1) in that it includes some higher-order terms. For the almost linear system (13.14) the classification of the corresponding linear system (C.1) determines the local phase portrait and the type of stability in almost every case. Small changes produced by the higher-order terms are evidently going to be crucial, if at all, only in the particular cases:

Case 3

The equal real eigenvalues λ, λ (*node*) *could* split to give $\lambda \pm \epsilon$ (*node*) or $\lambda \pm \mathrm{i}\epsilon$ (*spiral*), where ϵ is small. However, the *stability* of the critical point would still be just that predicted by the linear system analysis.

Case 5

The pure imaginary conjugate pair of eigenvalues $\pm\mathrm{i}\nu$ (*centre*) *could* become $\pm\mathrm{i}(\nu+\epsilon)$ (*centre*) or $\epsilon \pm \mathrm{i}\nu$ (*spiral*), where ϵ is small. Naturally a centre would still indicate that the critical point is *stable*. However, the spiral would be crucially dependent for its stability on the sign of the new real part ϵ of the eigenvalue pair. If $\epsilon > 0$ then the critical point is *unstable*, whereas if $\epsilon < 0$ then the critical point is *asymptotically stable*.

So, for the systems we are considering, it is only when the exactly linear analysis of §C.1 predicts that a critical point is a centre that we need to be suspicious of the predictions of the exactly linear analysis. Whether (13.14) has a *true centre* or an *unstable* or *asymptotically stable spiral* has to be resolved by a closer scrutiny of the particular system in hand.

It is the case, in fact, that the trajectories near a critical point in the phase plane have a topological equivalence in the *linear* and *almost linear* systems except when there is a zero eigenvalue (*i.e.* the critical point is not isolated) or when the eigenvalues are pure imaginary (*i.e.* a centre) — this is guaranteed by a *theorem* due to *Hartman and Grobman*. For example, we can examine the system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -y + x^2,\end{aligned} \tag{C.10}$$

which has only one critical point (at the origin $x = 0, y = 0$). For the linear system, in the expansion about the origin we have

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so that the eigenvalues are $\lambda_1 = 1, \lambda_2 = -1$ with eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

respectively. The trajectories near the origin, which is a *saddle*, are indicated in Fig. C.8(a). For the exact nonlinear system (C.10) we can write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} + x, \tag{C.11}$$

so that

$$y = \frac{x^2}{3} + \frac{c}{x}, \qquad \text{with } c \text{ constant,}$$

together with a second solution $x = 0$ (for all y).

The exact family of trajectories near the origin is indicated in Fig. C.8(b). It should be noted that the trajectories which go directly into and directly out of the critical point O (respectively the stable and unstable manifolds) correspond directly at and near O for the exactly linear and almost linear systems — a general result usually known as the *stable manifold theorem.*

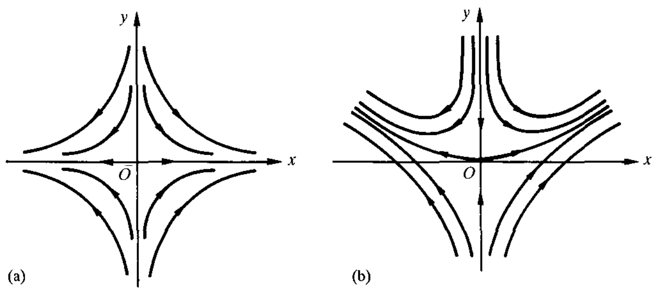


Fig. C.8

C.3 Systems of Third (and Higher) Order

As we indicated in §13.6, higher-order systems can be analyzed in a similar fashion to that carried out in §13.3 and earlier in this appendix. That is to say the critical points are found and local analysis effected about each of them by linearization. The resulting eigenvalue/eigenvector problems determine the local stability of the critical points and the *local* phase portrait structures normally determine the *global* phase portrait for the complete system.

For a third-order system, *e.g.* the Lorenz system of §13.6, the matrix M for a particular critical point is 3×3 [again a tensor of valence 2 (§A.9,10)], so that the three eigenvalues satisfy a cubic equation. This implies that at least one of the eigenvalues must be real, with the others either both real or a complex conjugate pair. We only note here that, if all the eigenvalues are negative real or have negative real part, then the critical point is asymptotically stable. Even if only one of the eigenvalues is positive or if the complex conjugate pair has positive real part, then the critical point is unstable.

Problems

1. Find the critical points of the following systems and classify them according to their local linear approximations:

   (a) $\dot{x} = -3x + y,\ \dot{y} = 4x - 2y;$
   (b) $\dot{x} = 3x + y,\ \dot{y} = 2x + 2y;$
   (c) $\dot{x} = -6x + 2xy - 8,\ \dot{y} = y^2 - x^2;$
   (d) $\dot{x} = -2x - y + 2,\ \dot{y} = xy;$
   (e) $\dot{x} = 4 - 4x^2 - y^2,\ \dot{y} = 3xy;$
   (f) $\dot{x} = \sin y,\ \dot{y} = x + x^3;$
   (g) $\dot{x} = y,\ \dot{y} = \left[\frac{\omega^2 - \alpha - y^2}{1 + x^2}\right] x$ in the cases $\omega^2 < \alpha$ and $\omega^2 > \alpha$.

2. For the nonlinear oscillator equation $\ddot{x} + x = x^3$, write $\dot{x} = y$ and show that there are two saddle points and one centre in the linear approximation about the critical points in the (x, y) phase plane. Integrate the equations of the system to obtain an 'energy' equation and use this to show that

   (a) the centre is a *true* centre for the full system;
   (b) the equation of the separatrices through the saddles is

$$2y^2 = x^2(x^2 - 2) + 1.$$

Appendix D

Discrete Dynamical Systems — Maps

In this appendix we consider discrete dynamical systems in which a space is effectively mapped onto itself repeatedly. We recognized in Chapter 13 that for some systems it is appropriate and useful to observe at discrete time intervals, which are not necessarily equal — this is, for example, often the case for biological systems.

Also we saw in §14.2 the concept of a Poincaré return map, where the evolution of a system through its dynamics induces a map of a Poincaré section onto itself. Examining properties of maps, in their own right, will give insight into mechanisms of chaotic breakdown in continuous systems as well.

D.1 One-dimensional Maps

We consider a map given by

$$x_{n+1} = F(x_n), \tag{D.1}$$

for $n = 0, 1, 2, \ldots$ and with F a known function, and consider possible behaviours of x_n for suitable initial values x_0, as we *iterate* to find successively $x_1 = F(x_0)$, $x_2 = F(x_1) \equiv F\big(F(x_0)\big) \equiv F^{(2)}(x_0)$, etc. We can expect to find any *fixed points* X as solutions of

$$X = F(X). \tag{D.2}$$

To examine the stability of the fixed point X, we may write $x_n + \epsilon_n = X$, for each n, and, when ϵ_n is small, we can expand $F(x_n)$ in (D.1) in the form

$$F(x_n) = F(X - \epsilon_n) = F(X) - \epsilon_n F'(X) + \tfrac{1}{2}\epsilon_n^2 F''(X) + \ldots, \tag{D.3}$$

where $F'(X) = [\mathrm{d}F(x)/\mathrm{d}x]_{x=X}$, etc.

A fixed point X is asymptotically stable (and therefore an attractor) if $|F'(X)| < 1$. We may consider different cases (where $\epsilon_n \to 0$):

- $0 < |F'(X)| < 1 \implies \epsilon_{n+1} \simeq F'(X)\epsilon_n$ as $n \to \infty$, and we have *first-order convergence*.
- $F'(X) = 0,\ F''(X) \neq 0 \implies \epsilon_{n+1} \simeq -\frac{1}{2}F''(X)\epsilon_n^2$ as $n \to \infty$, and we have *second-order convergence*.

While this sequence may be continued, the key criterion is that stated above for $|F'(X)|$. We note that the case $|F'(X)| > 1$ leads to instability of X, and that the case $|F'(X)| = 1$ depends more specifically on the function $F(X)$.

A familiar example of what is normally second-order convergence is the *Newton–Raphson* iteration process to find roots of a single equation $f(x) = 0$. Here $x_{n+1} = x_n - f(x_n)/f'(x_n)$ and each root has a *basin of attraction*, so that we can find all the roots by judicious choices of x_0.

The very simplest map is the linear map:

$$x_{n+1} = rx_n, \tag{D.4}$$

with r constant (and, say, non-negative), and it is evident that $x_n = r^n x_0$ in this case. Here there are various behaviours depending on r:

- $0 \leq r < 1$: $x_n \to 0$ for all x_0 [asymptotic stability of $X = 0$].
- $r = 1$: $x_n = x_0$ for all x_0 [steady state].
- $r > 1$: $x_n = x_0 \exp(n \ln r)$ [exponential growth].

If this were a biological model, of *e.g.* a seasonal breeding population x_n, then the rate constant r is crucial in determining the fate of any initial population x_0.

The logistic map

A simple nonlinear map, derived from (D.4), is

$$x_{n+1} = rx_n - sx_n^2, \tag{D.5}$$

which is called the *logistic map* and has apparent similarity with the logistic differential equation (13.3). In a biological context the rate constant r, quantifying the ability of the population to reproduce, is balanced by the parameter s, which quantifies the effect of overcrowding. This model formed the centrepiece of what has become a very influential paper — 'Simple mathematical models with very complicated dynamics', May, *Nature*, **261**, 459–467, 1976.

A simple scaling $\bar{x}_n = sx_n/r$ leads to $\bar{x}_{n+1} = r\bar{x}_n(1 - \bar{x}_n)$, and it is evident that the overbar may then be dropped, in order to find the map

$$x_{n+1} = rx_n(1 - x_n), \tag{D.6}$$

which is the *logistic map in standard form*, with r the single key parameter.

Naturally the primary physical/biological interest is in the case where the x interval $[0, 1]$ is mapped by (D.6) onto $[0, 1]$, which requires $0 \leq r \leq 4$ — for other applications this r restriction might well be absent.

Despite the apparent similarity with the continuous system (13.3) the maps (D.5), (D.6) have some very different and complex properties.

We see immediately that there are two fixed points of (D.6) — at $X = 0$, $X = 1 - 1/r$ — and, in each case, $F'(X) = r(1 - 2X)$.

It is helpful for maps to consider [see Fig. D.1(a)] a plot of $y = x$ and $y = F(x)$ so that by tracing the vertical and horizontal lines between the two we can follow the sequence $x_0 \to x_1 \to x_2 \to \ldots$, and thus see whether or not it might converge.

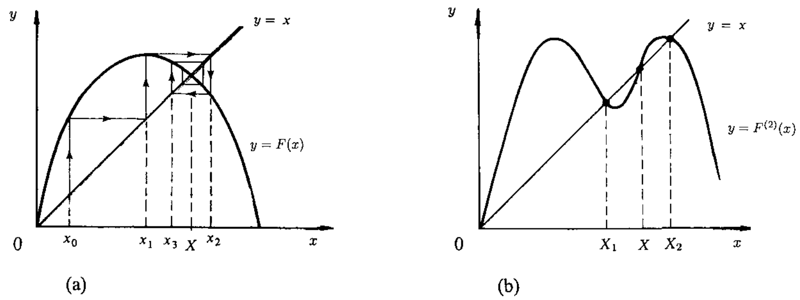


Fig. D.1

The behaviours of the map (D.6) for different values of r can be summarized as follows:

- $0 \leq r < 1$: $X = 0$ is asymptotically stable and $X = 1 - 1/r$ is unstable. [$X = 0$ is a *point attractor* — if the corresponding *linear* model population cannot sustain itself then overcrowding makes it worse!]
- $1 < r < 3$: $X = 0$ is unstable and $X = 1 - 1/r$ is asymptotically stable — see Fig. D.1(a) for example. [$X = 1 - 1/r$ is a *point attractor* —

exponential growth is stabilized by overcrowding, in very similar fashion to the behaviour of the logistic differential equation (13.3).]

- $3 < r \leq 4$: $X = 0$ and $X = 1 - 1/r$ are now both unstable. As r increases successive '*period-doubling bifurcations*' occur as asymptotic stability is exchanged between lower- and higher-order cycles (termed a *supercritical flip bifurcation*):

  * $3 < r < 1 + \sqrt{6} = 3.449\,48\cdots$: $x_n \to$ an asymptotically stable 2-cycle (or period-2 solution). If we examine $x_{n+2} = F(x_{n+1}) = F^{(2)}(x_n)$, then we obtain an equation of degree 4 for the fixed points of this iterated mapping. Of course $X = 0$, $X = 1 - 1/r$ are two of the roots of this equation. The nontrivial solutions X_1, X_2, such that $F(X_1) = X_2$, $F(X_2) = X_1$ [see Fig. D.1(b)], are roots of the quadratic $r^2X^2 - r(r+1)X + (r+1) = 0$. For asymptotic stability it is necessary (see Problem 1) that $|4 + 2r - r^2| < 1$ or equivalently, for positive r, $3 < r < 1 + \sqrt{6}$.
  * $3.449\,48\cdots < r < 3.544\,09\cdots$: $x_n \to$ an asymptotically stable 4-cycle found from an equation of degree 16; four more roots are the trivial solutions $0, 1 - 1/r, X_1, X_2$ (the other roots, when real, give 4-cycles arising through a different process).
  * $3.544\,09\cdots < r < 3.564\,40\cdots$: $x_n \to$ an asymptotically stable 8-cycle, and so on (each time in a shorter interval in r) ... until
  * $r = 3.569\,94\cdots$: Accumulation point of 2^∞-cycle.
  * $3.569\,94\cdots < r \leq 4$: For some values of r there are asymptotically stable cycles of different lengths, but for others the x_n values range seemingly over a whole continuous interval, in an apparently random fashion. An intriguing fact is that odd-period cycles only appear for $r > 3.678\,57\cdots$.

The logistic map attractors are shown in Fig. D.2. The numbers along the top are the cycle periods.

It should be noted that this figure has an approximate self-similarity at higher magnification, in that the *period-doubling cascade* is broadly repeated as other-period asymptotically stable cycles become unstable — *e.g.* the period-3 cycle in Fig. D.2.

That the period-doubling proceeds broadly in geometric fashion in the limit was discovered by Feigenbaum in 1975. Here the r-intervals Δ_i between bifurcations and the measures d_i of width in X of successive cycles

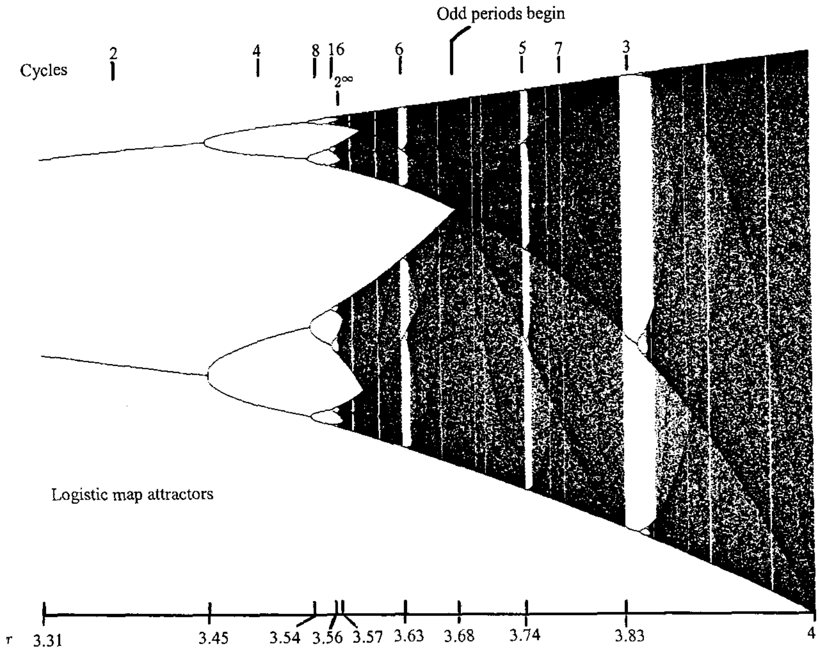


Fig. D.2

are such that (see Figs. D.2, D.3)

$$\begin{aligned} \Delta_i/\Delta_{i+1} &\to \delta = 4.669\,201\,6\ldots, \\ d_i/d_{i+1} &\to \alpha = 2.502\,907\,8\ldots, \end{aligned} \tag{D.7}$$

in each case in the limit as $i \to \infty$.

Feigenbaum noted that essentially all 'humped' mapping functions $F(x)$ lead to intersections of $y = x$ with $y = F(x)$, $y = F^{(2)}(x)$, $y = F^{(4)}(x), \ldots$ which are similar up to a rescaling. In a process of '*renormalization*' the precise form of $F(x)$ is lost and in 1976 Feigenbaum discovered a universal function $g(x)$, which is self-reproducing under such rescaling and iteration, so describing this universal property:

$$g(x) = -\alpha g\left[g\left(\frac{-x}{\alpha}\right)\right] \tag{D.8}$$

(see Feigenbaum, *Journal of Statistical Physics*, **21**, 669–706, 1979).

The universality leads to δ, α as universal constants and δ is usually identified as the *Feigenbaum number*.

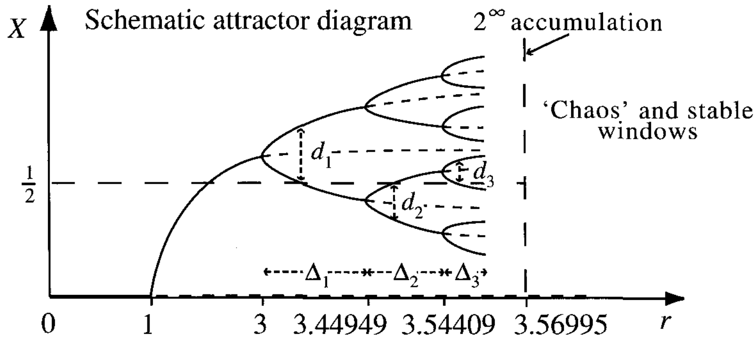


Fig. D.3

As to the onset of cycles with other periods, there is a theorem, due to *Sarkovskii* (1964), which is as follows:

If a mapping function $F(x)$ has a point x_p which is cyclic of order p, then it must also have a point x_q of period q, for every q which precedes p in the sequence:

$$
\begin{array}{ccccc}
1 \Leftarrow 2 \Leftarrow 4 \Leftarrow & \cdots & \Leftarrow 2^n \Leftarrow & \cdots & \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots \Leftarrow 2^m.9 & \Leftarrow 2^m.7 & \Leftarrow 2^m.5 & \Leftarrow 2^m.3 & \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots \Leftarrow 2^2.9 & \Leftarrow 2^2.7 & \Leftarrow 2^2.5 & \Leftarrow 2^2.3 & \\
\cdots \Leftarrow 2.9 & \Leftarrow 2.7 & \Leftarrow 2.5 & \Leftarrow 2.3 & \\
\cdots \Leftarrow 9 & \Leftarrow 7 & \Leftarrow 5 & \Leftarrow 3. &
\end{array}
$$

For example, the existence of a 3-cycle implies the existence of cycles of all the other periods!

Naturally the odd periods cannot arise from period-doubling, but do so *via* a rather different process which may be examined by similar methods to those employed above.

This theorem can be proved using a continuity/intermediate value theorem argument. However it says nothing about stability of these cycles, or the ranges of r (in our logistic example) for which they may be observed. The vast majority of these cycles are unstable when all are present, and it is this which leads to a brief summary statement in the form 'Period 3 implies chaos', as essentially random behaviour of iterates x_n occurs (Li and Yorke, *American Mathematical Monthly*, **82**, 985–992, 1975, in which, incidentally, the term *chaos* was first introduced!).

What can be said about values of r in the logistic map which lead to distributions of iterates, rather than to asymptotically stable cycles?

Analytically this is a tough problem. However, it happens that there are two positive values of r for which a formal exact solution of (D.6) is known — $r = 2$ and $r = 4$.

The former ($r = 2$) allows $(1 - 2x_{n+1}) = (1 - 2x_n)^2$ leading to $x_n = \frac{1}{2}[1 - (1 - 2x_0)^{2^n}]$. This is not especially interesting, since $x_n \to \frac{1}{2}$ as we should expect.

However the latter possibility ($r = 4$) allows us to substitute $x_n = \frac{1}{2}[1 - \cos(2\pi\theta_n)] \equiv \sin^2(\pi\theta_n)$ and this leads us to $\theta_{n+1} = 2\theta_n$ and then $\theta_n = 2^n\theta_0$. Here θ is evidently periodic, with period 1, in that the same x is generated by θ and by $\theta + 1$. Thus we may write any θ_0 we choose in a binary representation just using negative powers of 2 — for example, $\theta_0 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{64} + \cdots = 0.101\,101\ldots$. Then $\theta_1 = 0.011\,01\ldots$, $\theta_2 = 0.110\,1\ldots$, etc, since the integer part may be cancelled at each stage on account of the periodic property.

For almost all choices of θ_0 then the θ_n will be uniformly distributed on the interval $[0, 1]$, since each digit in the binary expansion of θ_0 could be chosen with equal likelihood to be a 0 or a 1. As a consequence of θ being uniformly distributed, then x is not. Indeed x has a probability distribution given by $P(x) = 1/[\pi\sqrt{x(1-x)}]$ (see Figs. D.4, D.2).

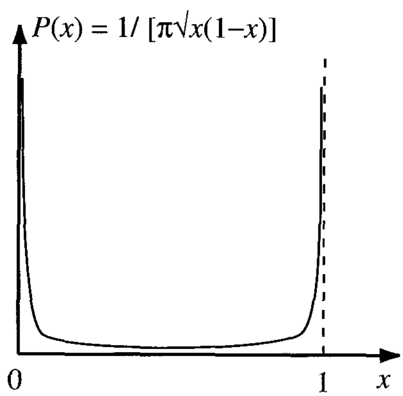


Fig. D.4

We note that $P(x)$ itself *is* the attractor here for $r = 4$, in the sense that for almost all choices of x_0 the distribution of x_n will approach $P(x)$ for large n. It is called an *invariant probability distribution.*

For other values of r there are theoretical results for the corresponding *probability distribution attractors* (via an equation for the distribution function — Perron–Frobenius), but no slick result like that above for $r = 4$.

The distribution attractors are also characterized by an exponential divergence of iterates, leading to the sensitivity to initial conditions characteristic of chaos. If we choose to examine x_0 and $x_0 + \epsilon_0$, with ϵ_0 very small, then $\epsilon_n \simeq \epsilon_0 \exp(\lambda n)$ on average, and we have divergence or convergence of iterates according as $\lambda > 0$ or $\lambda < 0$. Here λ is a *Lyapunov exponent* (see §13.7).

Since we have $F^{(n)}(x_0 + \epsilon_0) - F^{(n)}(x_0) \simeq \epsilon_0 e^{\lambda n}$, then we have

$$\begin{aligned}\lambda &\simeq \lim_{n\to\infty}\left\{\frac{1}{n}\ln\left(\left|\frac{d}{dx}F^{(n)}(x)\right|\right)\right\}\\ &= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln|F'(x_i)|\right\}.\end{aligned} \tag{D.9}$$

Values of λ can be found by numerical computation. In measuring λ empirically in a particular practical case, we allow n to become large enough so that our estimate of λ can settle down to a steady value. We also average over various different x_0, in order to avoid an atypical result through a single unfortunate choice. For the logistic map Fig. D.5 shows a plot of λ as r varies.

Here $F'(x) = r(1 - 2x)$, so that we expect:

- $0 < r < 1$: $\lambda = \ln r$,
- $1 < r < 3$: $\lambda = \ln|2 - r|$ (leading to an infinite spike when $r = 2$),
- $3 < r < 1 + \sqrt{6}$: $\lambda = \frac{1}{2}\ln|r(1 - 2X_1)| + \frac{1}{2}\ln|r(1 - 2X_2)|$ [leading to an infinite spike when $X_1 = \frac{1}{2}$ (requiring $r = 1 + \sqrt{5} = 3.236$)],
- $r = 4$: since $P(x) = 1/[\pi\sqrt{x(1-x)}]$ so we have

$$\lambda = \int_0^1 (\ln|F'(x)|)P(x)\,dx = \int_0^1 \frac{\ln|4(1-2x)|\,dx}{\pi\sqrt{x(1-x)}} = \ln 2 = 0.693\,147\ldots.$$

In any event, the regions of r for which there are positive values of λ indicate the sensitivity to initial conditions at these r values. While there are other measures of the complexity/disorder of the iterates in these cases —

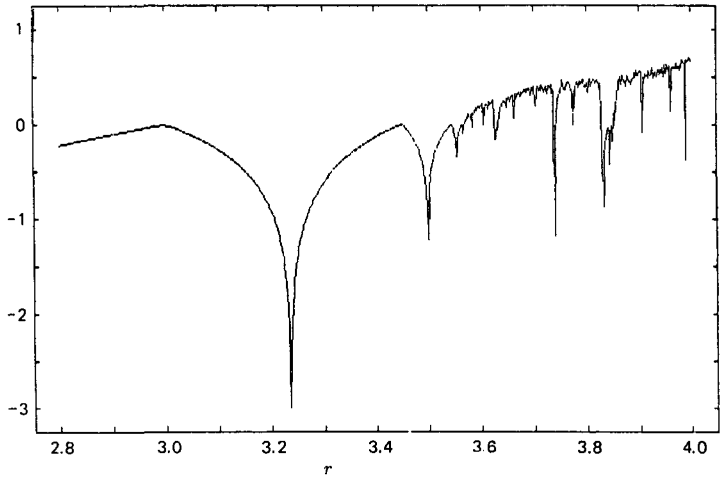


Fig. D.5 [Baker and Gollub (*Chaotic Dynamics*, 2nd ed., Cambridge University Press, 1996)]

e.g. the system 'entropy' — these will not be pursued further here. However, see Problem 4 for another (and simpler) example of a chaotic map.

The logistic map (D.6), together with other similar maps, involves *stretching* and *folding*, so that states diverging exponentially are still broadly confined to a bounded region. The loss of information about initial conditions, as the iteration process proceeds in a chaotic regime, is associated with the *non-invertibility* of the mapping function $F(x)$, *i.e.* while x_{n+1} is uniquely determined from x_n, each x_n can come from 2 possible x_{n-1}, 4 possible x_{n-2}, etc, and eventually from 2^n possible x_0. Hence system memory of initial conditions becomes blurred!

Many continuous systems — for example the Lorenz system of §13.6 — exhibit a similar period-doubling in their dynamics.

D.2 Two-dimensional Maps

For two-dimensional maps any new universality has proved harder to find! However, there are interesting phenomena. We may write our generic map in the form

$$\begin{aligned} x_{n+1} &= F(x_n, y_n), \\ y_{n+1} &= G(x_n, y_n), \end{aligned} \tag{D.10}$$

for integer n and with F, G known functions. Again we may seek fixed points (X, Y) as solutions of

$$\begin{aligned} X &= F(X,Y), \\ Y &= G(X,Y). \end{aligned} \tag{D.11}$$

By a Taylor expansion, near X, Y and similar to that carried out in §13.3 and in Appendix C, we obtain

$$\begin{pmatrix} x_{n+1} - X \\ y_{n+1} - Y \end{pmatrix} \simeq M \begin{pmatrix} x_n - X \\ y_n - Y \end{pmatrix}, \text{ with } M = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}_{(x,y)=(X,Y)} . \tag{D.12}$$

It now follows that $(x_n, y_n) \to (X, Y)$ if and only if the eigenvalues of the matrix M all have modulus less than 1; this is needed to force $M^n \to$ the zero matrix as $n \to \infty$.

When instability sets in as parameters are changed, we then typically have a 2-cycle to examine with $(X_1, Y_1) \rightleftarrows (X_2, Y_2)$ and where these points are the solutions of

$$\begin{aligned} X &= F[F(X,Y), G(X,Y)], \\ Y &= G[F(X,Y), G(X,Y)], \end{aligned} \tag{D.13}$$

other than the fixed points of (D.10) found earlier and which satisfy (D.11). The asymptotic stability of this 2-cycle depends on the eigenvalues of the matrix product

$$M_1 M_2 = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}_{(x,y)=(X_1,Y_1)} \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}_{(x,y)=(X_2,Y_2)} , \tag{D.14}$$

and so on.

The Hénon map

Probably the most celebrated example is the Hénon map (Hénon, *Communications in Mathematical Physics*, **50**, 69–77, 1976):

$$\begin{aligned} x_{n+1} &= 1 - a x_n^2 + y_n, \\ y_{n+1} &= b x_n, \end{aligned} \tag{D.15}$$

with a, b real parameters, and where the normal physical interest is in $|b| \leq 1$. This map was constructed to exhibit some behaviours similar to those of the Lorenz system of §13.6. Geometrically the map may be considered to be a composition of three separate simple maps — an area-preserving fold, a contraction $|b|$ in the x direction, an area-preserving rotation.

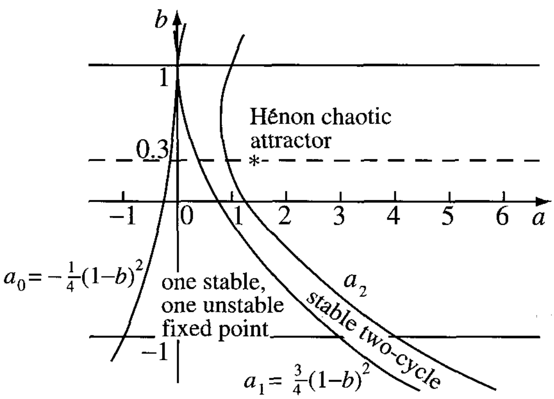
There are two fixed points given by

$$\begin{aligned} X &= X_\pm \equiv [-(1-b) \pm \sqrt{(1-b)^2 + 4a}]/2a, \\ Y &= Y_\pm \ \equiv bX_\pm, \end{aligned} \tag{D.16}$$

and these are real and distinct if and only if $a > a_0 = -\frac{1}{4}(1-b)^2$. One of the fixed points is then always unstable and the other is asymptotically stable if $a < a_1 = \frac{3}{4}(1-b)^2$ (see Problem 7). For $a > a_1$, both fixed points are unstable and we get period-doubling to a 2-cycle, 4-cycle, etc. The two-cycle stability is determined through (D.14) by the eigenvalues of the matrix product

$$M_1 M_2 = \begin{pmatrix} 4a^2 X_1 X_2 + b & -2aX_1 \\ -2abX_2 & b \end{pmatrix}.$$

This leads to asymptotic stability of the 2-cycle only when $a_1 < a < a_2$ with $a_2 = (1-b)^2 + \frac{1}{4}(1+b)^2$. The period-doubling cascade then continues (see Fig. D.6).



b	a_0	a_1	$a(2^\infty)$	$a(*)$
0	−0.25	0.75	−1.40	
0.3	−0.1225	0.3675	−1.06	1.40

b=0 corresponds to the one-dimensional logistic map (see Problem 2)

Fig. D.6

Beyond the cascade, and embedded among other-period cycles, there are (a, b) values where the attractor is very complex (see, *e.g.*, Fig. D.7, where $a = 1.4, b = 0.3$).

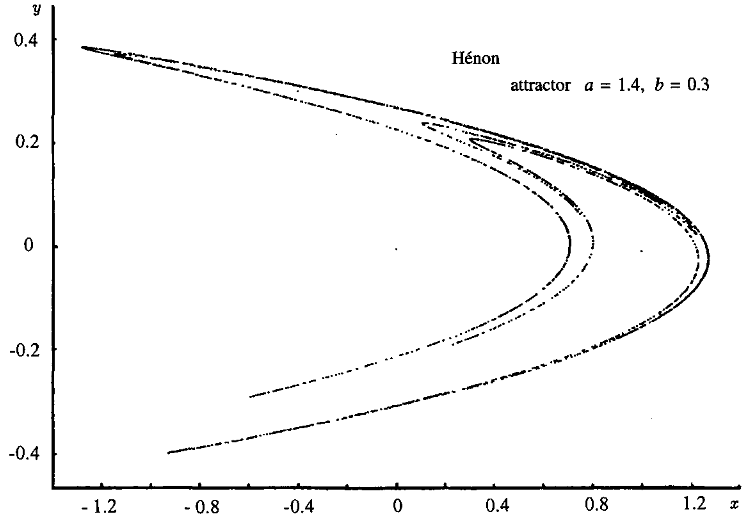


Fig. D.7

Here there is *stretching* along the strands of the attractor and *squeezing* across them — with associated Lyapunov exponents which are respectively positive (along) and negative (across) (see Problem 8).

This chaotic attractor — termed '*strange*' — has some similar features to the Lorenz attractor of Fig. 13.20. It also has a fractal character in that it has a broad similarity in features and relative scale at all magnifications.

The repetition of stretching, squeezing and folding onto the original region is characteristic of a technical construct — the Smale 'horseshoe' (1960) — which is now known to be a trademark of chaotic systems.

Smale 'stretched' the unit square in one co-ordinate direction and 'squeezed' it in the other direction, then placing the resulting strip (with one 'fold') over the original square — with inevitable overlap. Infinite repetition of this sequence of operations leads to the identification of an *attractor* consisting of all the points which remain within the original square indefinitely. This attractor has great topological complexity and, while it *guarantees* the 'sensitivity to initial conditions' of chaos, *proving* its existence for a particular system may be very tough.

A formal demonstration of *chaotic* dynamics for the Hénon map is contained in a paper by Benedicks and Carleson (*Annals of Mathematics*, **133**, 73–169, 1991).

In passing (and for reference in §D.3) we note that a map which preserves area would have $|\det M| = 1$ in (D.12) and then the stretching and squeezing exactly compensate each other (see Problem 11). For the Hénon map this is the case only when $|b| = 1$. Equal-area maps are of special interest since the Poincaré return map for a section through a Hamiltonian system (see §14.2) has the equal-area property. Some consequences are explored briefly in §D.3.

D.3 Twist Maps and Torus Breakdown

In §14.1 we noted that each trajectory of an integrable Hamiltonian system with n degrees of freedom is confined to the surface of an n-torus in the $2n$-dimensional phase space. The n-tori corresponding to the range of initial conditions are 'nested' in the phase space.

In §14.2 we introduced the concept of a Poincaré surface of section, as a slice through the dynamical structure. For $n = 2$ degrees of freedom this section of the nested torus structure is a continuum of closed curves, each one of which is intersected by one of its own torus trajectories in a sequence of points (see Fig. 14.4).

If we make use of action/angle variables, as described in §14.3 and particularly in the polar form of Fig. 14.5(c), then the closed curves of the Poincaré section can be taken as concentric circles. The intersection points of a trajectory with its own particular one of these circles (of radius r) will necessarily be twisted successively around the origin (the centre of the circle) through an angle $2\pi\alpha$. Here α is the *rotation number*, which is the ratio of normal frequencies characterizing the particular torus concerned — it therefore depends on r.

We can then introduce the notion of a *twist map*, T [Moser (1973)]; in polars:

$$\begin{pmatrix} r_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} r_n \\ \theta_n + 2\pi\alpha(r_n) \end{pmatrix} \equiv T\left[\begin{pmatrix} r_n \\ \theta_n \end{pmatrix}\right]. \tag{D.17}$$

When a perturbation Hamiltonian is introduced, as in §14.6, we can model the modified situation using a *perturbed twist map*, T_ϵ:

$$\begin{pmatrix} r_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} r_n + \epsilon f(r_n, \theta_n) \\ \theta_n + 2\pi\alpha(r_n) + \epsilon g(r_n, \theta_n) \end{pmatrix} \equiv T_\epsilon \left[\begin{pmatrix} r_n \\ \theta_n \end{pmatrix} \right], \qquad \text{(D.18)}$$

with ϵ small and positive and with f, g known smooth functions.

We can now ask what happens to a circle of a particular radius, which is mapped to itself (with a twist) by T, as the perturbation quantified by ϵ in T_ϵ grows from zero — *i.e.* as the *integrable system* T becomes a *near-integrable system* T_ϵ, in such a way that *area is still preserved.*

In fact the answer to this question depends on the rotation number α corresponding to our particular circle chosen.

If α is an *irrational* number then, for ϵ small enough, the circle undergoes some distortion (perturbation) but is certainly not destroyed. This is also true for the corresponding torus in the full phase space. This result is in accord with the KAM theory referred to in §14.6.

As we shall see, it is circles for (D.17), and their tori, corresponding to *rational* α, which break down under perturbation, leading to sensitivity to initial conditions and chaos.

The rational α are 'scanty, but dense' among the real numbers (see Problem 12) and these α correspond to resonances in the system. As in the discussion of the problem of small denominators in §14.6, the sensitivity to initial conditions is strongest for the rational $\alpha = k/s$ with small values of s. As the perturbation grows with ϵ, the breakdown associated with each such rational α broadens, so that progressive overlap occurs, leading eventually to complete breakdown of the torus structure.

Let us examine the twist map (D.17) for three neighbouring circles C_-, C, C_+ corresponding respectively to $\alpha_- < k/s$, $\alpha = k/s$, $\alpha_+ > k/s$ (say), with k, s positive integers and with α_-, α_+ irrational. (We have here chosen to take α to be an increasing function of radius r.)

Then applying the twist map T successively s times we find that the effect of $T^{(s)}$ (see Fig. D.8) is to map the circles to themselves, with C invariant and with C_-, C_+ twisted 'rigidly' clockwise, anticlockwise respectively — this is so since $2\pi s\alpha_- < 2\pi k$ and $2\pi s\alpha_+ > 2\pi k$. All points of C are fixed under the iterated mapping $T^{(s)}$.

Since α_-, α_+ are irrational the circles C_-, C_+ are only mildly distorted when we apply the map T_ϵ successively s times instead — *i.e.* when we iterate to consider the effect of $T_\epsilon^{(s)}$. However, the inner and outer circles C_-, C_+ are still twisted clockwise, anticlockwise respectively — see Fig. D.9.

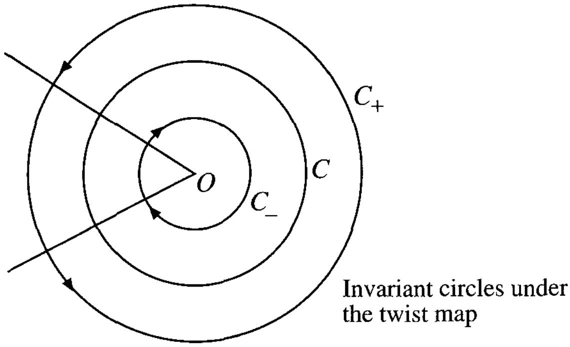


Fig. D.8

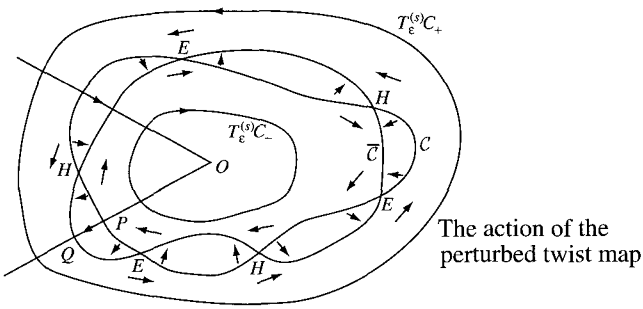


Fig. D.9

Moving outwards radially from O in any direction we can, by continuity, find a point on that radius arm which undergoes *no* net twist when the iterated map $T_\epsilon^{(s)}$ is applied — P is mapped only *radially* (to Q) by this process. By considering all possible radius arms we construct the closed curve $\mathcal{C}$ consisting of all such points P which are mapped only radially by $T_\epsilon^{(s)}$ — to $\bar{\mathcal{C}}$.

For the map T_ϵ to model a section of a Hamiltonian dynamical system (as, of course, does T itself) the map T_ϵ — and hence $T_\epsilon^{(s)}$ — must be one of equal-area, as reflected in §14.2.

Since the areas contained within the curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ must be *equal*, there is in general an even number of intersections of these curves, which correspond, of course, to fixed points of the iterated map $T_\epsilon^{(s)}$. These points are all that remains fixed from the original invariant circle C of the unperturbed iterated map $T^{(s)}$. We now note that the fixed points are of alternating type as we move around $\mathcal{C}$ (or $\bar{\mathcal{C}}$) — see Fig. D.10, where the iterated mapping sense of flow is indicated.

(a) Elliptic point (E) stable (b) Hyperbolic point (H) unstable

Fig. D.10

There are evidently in all $2ns$ such fixed points of the iterated map $T_\epsilon^{(s)}$, where n is a positive integer (usually 1).

The statement of existence of these fixed points, of their multiplicity and their alternating stability is a result known as the Poincaré–Birkhoff Theorem (1927).

It turns out that the elliptic fixed points E are themselves surrounded (at higher scales) by elliptic and hyperbolic fixed points corresponding to even higher-order frequency resonances.

For the hyperbolic fixed points H the unstable and stable manifolds (*q.v.* also in §C.2) for neighbouring such points in the same family cross to form what are called *homoclinic intersections*. The resulting instability at all scales leads inevitably to sensitivity to initial conditions, in that trajectories of the system have to twist and turn, forming a *homoclinic tangle*, in order not to *self*-intersect, while maintaining the equal-area property of the return map. This results in the stretching, squeezing and folding associated with the *Smale horseshoe* referred to in §D.2 and this forces a dense interweaving of ordered and chaotic motions within Hamiltonian systems (Fig. D.11). Recognition by Poincaré of the existence of such tangles was the first mathematical realization of the presence of what we now call chaos.

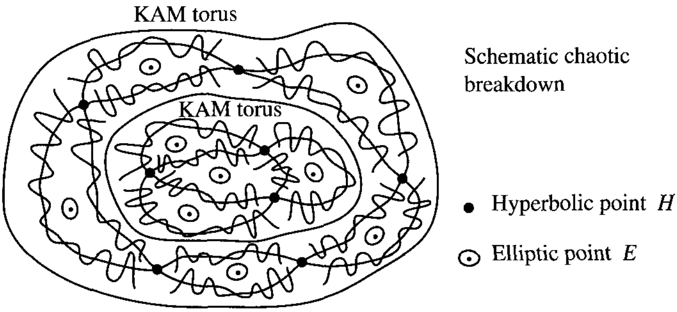


Fig. D.11

An example of a map in which some of the breakdown to chaos in a Poincaré section is apparent is that for the oval billiard map (Fig. 14.13).

For further exploration of breakdown to chaos under parameter change, see Problem 13.

Problems

1. For the logistic map (D.6) with r non-negative:

   (a) Show that there is a point attractor for $0 \leq r < 3$.
   (b) Show that there is a two-cycle attractor for $3 < r < 1 + \sqrt{6}$.
   (c) Show that a, b, s can be found such that y_n satisfies the logistic map with parameter s ($\neq r$) and $x_n = a + by_n$.
   (d) Hence determine the principal period-doubling bifurcation points for the logistic map on the range $-2 \leq r \leq 4$.

2. Show that the map $y_{n+1} = 1 - ay_n^2$ is just the logistic map (D.6) for x_n with x_n, y_n related linearly by $y_n = \alpha + \beta x_n$ and $a = \frac{1}{4}r(r-2)$. [This is an example of the fact that all quadratic maps $y_{n+1} = A + By_n + Cy_n^2$ are essentially just the logistic map, in that (D.6) can be obtained by a suitably chosen linear relation between y_n and x_n.]
3. *Allowing r and x_n to be complex in the logistic map (D.6), find regions of the complex r plane for which the map has (a) a point attractor, (b) a 2-cycle attractor. (c) Sketch the corresponding regions when these results are expressed in terms of the complex a plane for the y_n map of Problem 2.

4. The tent map

$$x_{n+1} = \begin{cases} 2x_n, & (0 \le x_n \le \frac{1}{2}) \\ 2(1-x_n), & (\frac{1}{2} < x_n \le 1) \end{cases}$$

has unstable fixed points. Show that this map exhibits extreme sensitivity to initial conditions, in that an uncertainty ϵ_0 in x_0 is rapidly magnified. Estimate the number of iterations after which the range of uncertainty in the iterates is the complete interval $[0, 1]$.

5. For the cubic map $x_{n+1} = ax_n - x_n^3$, where a is real, show that, when $|a| < 1$, there is an asymptotically stable fixed point $X = 0$ and that, when $1 < a < 2$ there are two such fixed points at $X = \pm\sqrt{a-1}$. What happens when a becomes > 2?

6. Explore the one-dimensional maps of Problems 1–5 using a programmable calculator or (better) a computer. Further interesting examples are $x_{n+1} = \exp[a(1-x_n)]$, $x_{n+1} = a\sin x_n$.

7. For the Hénon map (D.15) show that:

   (a) when $-\frac{1}{4}(1-b)^2 < a < \frac{3}{4}(1-b)^2$ there are two real fixed points, one of which is asymptotically stable,

   (b) *when $\frac{3}{4}(1-b)^2 < a < (1-b)^2 + \frac{1}{4}(1+b)^2$ there is an asymptotically stable 2-cycle.

   [*Hint*: Since $M = \begin{pmatrix} -2aX & 1 \\ b & 0 \end{pmatrix}$ the eigenvalues of M are λ_1, λ_2 with $\lambda_1\lambda_2 = -b$, $\lambda_1 + \lambda_2 = -2aX$. To determine stability, it is useful to consider a sketch of the function $f(\lambda) \equiv b/\lambda - \lambda$ and look for points where $f(\lambda) = 2aX$ in order to find (a, b) such that both the eigenvalues satisfy $|\lambda_i| < 1$.]

8. *Show that a small circle of radius ϵ centred at any (X, Y) becomes a small ellipse under a single iteration of the Hénon map (D.15). Explain how the semi-axes of the ellipse are related to Lyapunov exponents λ_1, λ_2 and show that $\lambda_1 + \lambda_2 = \ln|b|$ with $\lambda_1 > 0 > \lambda_2$, implying simultaneous 'stretch' and 'squeeze'.

9. *The Lozi map is (D.10) with $F(x, y) = 1 + y - a|x|$, $G(x, y) = bx$, where a, b are real parameters.

   (a) When $|b| < 1$ and $|a| < 1 - b$, show that there is one asymptotically stable fixed point.

   (b) Find the 2-cycle when $|b| < 1$ and $a > 1 - b$ and determine its stability.

10. Explore the two-dimensional maps of Problems 7–9 using a computer.

11. By calculating Lyapunov exponents examine sensitivity to initial conditions of the equal-area maps of the unit square ($0 \le x, y \le 1$):
    (a) Arnold's cat map $x_{n+1} = x_n + y_n$, $y_{n+1} = x_n + 2y_n$ (each modulo 1).
    (b) The baker's transformation

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, \frac{1}{2}y_n) & (0 \le x_n < \frac{1}{2}), \\ (2x_n - 1, \frac{1}{2}(y_n + 1)) & (\frac{1}{2} \le x_n \le 1). \end{cases}$$

12. Consider a system of two degrees of freedom with two natural frequencies ω_1, ω_2 in the light of the discussion of periodicity and degeneracy in §§14.1,14.6 and the real ratio ω_1/ω_2. Show that for numbers on the real line:
    (a) between any two irrationals we can certainly find a rational;
    (b) between any two rationals we can certainly find an irrational.

    (Note that, despite (a), there are vastly more irrationals than rationals — unlike the former the latter are 'countable', so that the rationals are 'scanty, but dense'. This emphasizes that for most systems periodicity (closure) is relatively rare, although it is still the case that for any irrational ω_1/ω_2 there are rationals arbitrarily close by.)
13. *As an exercise on near-integrable systems explore the '*standard* map' (Chirikov–Taylor) analytically/computationally:

$$I_{n+1} = I_n + K \sin \phi_n,$$
$$\phi_{n+1} = \phi_n + I_{n+1}.$$

    (*Note*: This equal-area map models the twist around a Poincaré section by the dynamics (as in §14.2 and in more detail in §D.3), when the integrable case ($K = 0$) is expressed in action/angle variables (I, ϕ). As K increases there appear resonance zones, periodic orbits and bands of chaos as tori I = constant undergo progressive breakdown. See Chirikov, *Physics Reports*, **52**, 263–379, 1979.)

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Answers to Problems

CHAPTER 1

1. $m_A/m_B = 3$; $3\boldsymbol{v}/4$.
2. $m_1/m_2 = r_2/r_1$.
3. $2m\ddot{\boldsymbol{r}} = \boldsymbol{F} = \boldsymbol{F}_{21} + \boldsymbol{F}_{31}$; $\boldsymbol{r} = (m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3)/(m_2 + m_3)$.
4. $0.12\,\mathrm{m}$.
5. $\boldsymbol{r}'_{ij} = \boldsymbol{r}_{ij}$, $\boldsymbol{p}'_i = \boldsymbol{p}_i - m_i\boldsymbol{v}$, $\boldsymbol{F}'_{ij} = \boldsymbol{F}_{ij}$.
6. $400\,\mathrm{N}$, $300\,\mathrm{N}$.
7. $\arcsin 0.135 = 7.76°$ E of N; $60.6\,\mathrm{min}$; $130\,\mathrm{km}$, $8.62°$ W of S.
8. $1.2\,\mathrm{m}$; $(\pm 0.4, 0.6, -1.2)$, $(0, -0.9, -1.2)$; $7\,\mathrm{N}$, $7\,\mathrm{N}$, $10\,\mathrm{N}$.
10. $(2.2 \times 10^{-3})° = 7.9''$.

CHAPTER 2

1. (x in m, t in s) $x = -3\cos 2t + 4\sin 2t = 5\cos(2t - 2.214) = \mathrm{Re}[(-3 - 4\mathrm{i})\mathrm{e}^{2\mathrm{i}t}]$; $t = 0.322\,\mathrm{s}$, $1.107\,\mathrm{s}$.
2. $z = -(mg/k)(1 - \cos\omega t)$, $\omega = \sqrt{k/m}$.
3. $0.447\,\mathrm{s}$, $14.2\,\mathrm{mm}$.
4. $15.7°\,\mathrm{s}^{-1}$, $\theta = 5°\cos\pi t - 8.66°\sin\pi t = 10°\cos(\pi t + \pi/3)$.
5. $V = -GMm/x$; $\sqrt{2GM(R^{-1} - a^{-1})}$; $8\,\mathrm{km\,s^{-1}}$.
6. $V = \frac{1}{4}cx^4$; $\sqrt{c/2m}\,a^2$; $x = \pm a$.
7. $V = \frac{1}{2}kx^2 - c\ln x$; $x = \sqrt{c/k}$; $\omega = \sqrt{2k/m}$.
8. $F = -mk$, $F = mk$; oscillation; $2\sqrt{2a/k}$.
9. $F = kx$, $|x| < a$; $F = 0$, $|x| > a$; oscillation between two turning points if $k < 0$ and $E < 0$, 1 turning point if $k > 0$ and $E < \frac{1}{2}ka^2$, otherwise no turning points.
10. Earlier by $(2a/v) - (2/\omega)\arctan(\omega a/v)$.
11. (a) no turning points, (b) 1 turning point, (c) 1 or 2 turning points.

12. $x \doteq -a$; $2\pi\sqrt{2ma^3/c}$; (a) $|v| < \sqrt{c/ma}$, (b) $v < -\sqrt{c/ma}$ or $\sqrt{c/ma} < v < \sqrt{2c/ma}$, (c) $v > \sqrt{2c/ma}$.
13. $z = (g/\gamma^2)(1 - \mathrm{e}^{-\gamma t}) - gt/\gamma$; $\dot{z} \to -g/\gamma$.
14. 8.05 s, 202 m.
15. $\sqrt{gk}\,\arctan(\sqrt{k/g}\,u)$, $(1/2k)\ln(1 + ku^2/g)$.
16. $\sqrt{g/k}$, $(gk)^{-1/2}\ln(\mathrm{e}^{kh} - \sqrt{\mathrm{e}^{2kh} - 1})$.
17. $\pm\sqrt{2(g/l)(\cos\theta - \cos\theta_0)}$; $2\pi\sqrt{l/g}$; $\theta = \theta_0\cos(\sqrt{g/l}\,t)$.
18. For $\theta = \pi - \alpha$, $\ddot{\alpha} = (g/l)\alpha$; 0.95 s; $2\pi\,\mathrm{s}^{-1}$.
19. $\sqrt{c/m}(a^2 - x^2)$; $x = a\tanh(\sqrt{c/m}\,at)$.
20. $x = a$; π/ω; $\sqrt{a^2 + v^2/4\omega^2} \pm v/2\omega$.
21. $z = (mg/k)[-1 + (1 + \gamma t)\mathrm{e}^{-\gamma t}]$, $\gamma = \sqrt{k/m}$; 16 mm.
22. 1.006 s; 5.33°, 1.17°.
23. $x = (v/\omega)\mathrm{e}^{-\gamma t}\sin\omega t \to vt\mathrm{e}^{-\gamma t}$.
25. $\omega_1 = \sqrt{\omega_0^2 + \gamma^2} \pm \gamma$.
26. $\bar{E} = \frac{1}{4}ma_1^2(\omega_1^2 + \omega_0^2)$, $W = 2\pi m\gamma\omega_1 a_1^2$.
27. 3; final velocities: $-6\,\mathrm{m\,s^{-1}}$, $7\,\mathrm{m\,s^{-1}}$, $10\,\mathrm{m\,s^{-1}}$; $T = 364.5\,\mathrm{J}$.
28. $v_n = e^n\sqrt{2gh}$.
29. $a_n = c/m\omega^2 n(1 + n^2)$.
31. $\tau = 1.017 \times 2\pi\sqrt{l/g}$.
32. $G(t) = (\mathrm{e}^{-\gamma_- t} - \mathrm{e}^{-\gamma_+ t})/m(\gamma_+ - \gamma_-)$, $t > 0$;
$x = \frac{c}{m}\left[\frac{1}{\gamma_+ - \gamma_-}\left(\frac{1}{\gamma_-^2}\mathrm{e}^{-\gamma_- t} - \frac{1}{\gamma_+^2}\mathrm{e}^{-\gamma_+ t}\right) - \frac{2\gamma}{\omega_0^4} + \frac{t}{\omega_0^2}\right]$.

CHAPTER 3

1. (a) $V = -\frac{1}{2}ax^2 - ayz - bxy^2 - \frac{1}{3}bz^3$, (c) $V = -ar^2\sin\theta\sin\varphi$, (f) $V = -\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{r})^2$.
2. $\frac{1}{2}a + b$.
3. (i) 0, (ii) $\frac{1}{2}a$.
4. (a) πa^2, $\boldsymbol{F}$ not conservative; (b) 0, $\boldsymbol{F}$ may be conservative.
5. $\boldsymbol{F} = c[3(\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{r} - r^2\boldsymbol{k}]/r^5$; $F_r = 2c\cos\theta/r^3$, $F_\theta = c\sin\theta/r^3$, $F_\varphi = 0$.
6. 382 m, 883 m; 30°; 17.7 s, 10.2 s.
7. $z = x\tan\alpha - gx^2/2v^2\cos^2\alpha$; $\alpha = \pi/4 + \beta/2$.
8. $z = wx/u - gx^2/2u^2 - \gamma gx^3/3u^3$; 42.3°, 823 m.
9. 6.89 km, 7.35 km, 7.18 km.
10. 4ω; $m\omega^3 l^4/r^3$; $\Delta T = \frac{3}{2}m\omega^2 l^2$.
11. $v/2$, $v^2 = 4ka^2/3m$; $4ka/3m$, $-5ka/6m$.
12. $\ddot{\theta} = 2F/ma - (g/a)\sin\theta$, $\dot{\theta}^2 = 4F\theta/ma - (2g/a)(1 - \cos\theta)$; $F_0 = 0.362\,mg$.

13. $\ddot{\theta} = (1 - \sin\theta)g/3a$, $\dot{\theta}^2 = (\theta - 1 + \cos\theta)2g/3a$; $F = mg(1 + 2\sin\theta)/6$; $\theta = 7\pi/6$; thereafter the two bodies move independently until string tautens.
14. $\dot{u} = (M + m)g\sin\alpha/(M + m\sin^2\alpha)$, $\dot{v} = mg\sin\alpha\cos\alpha/(M + m\sin^2\alpha)$; $\alpha = \arcsin(2/3) = 41.8°$.
15. $z = c^{-2}\sin^2\theta$, $x = c^{-2}(\theta - \frac{1}{2}\sin 2\theta)$.
17. $\cot\theta = \cot\theta_0\cos(\varphi - \varphi_0)$, ($\theta_0, \varphi_0$ constants).
18. $\frac{m}{4}\left[\frac{\xi+\eta}{\xi}\ddot{\xi} - \frac{1}{2}\eta\left(\frac{\dot{\xi}}{\xi} - \frac{\dot{\eta}}{\eta}\right)^2\right] = F_\xi$, $\frac{m}{4}\left[\frac{\xi+\eta}{\eta}\ddot{\eta} - \frac{1}{2}\xi\left(\frac{\dot{\xi}}{\xi} - \frac{\dot{\eta}}{\eta}\right)^2\right] = F_\eta$.
19. (a) and (b): $r = k/(c + g)$, $\theta = 0$, unstable; (a) only: $r = k/(c - g)$, $\theta = \pi$, stable.
20. $T = \frac{1}{2}m\sum_i h_i^2\dot{q}_i^2$, $p_i = mh_i^2\dot{q}_i$, $\boldsymbol{e}_i \cdot \boldsymbol{p} = mh_i\dot{q}_i$.
21. $\ddot{\boldsymbol{r}} = (\ddot{\rho} - \rho\dot{\varphi}^2)\boldsymbol{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\boldsymbol{e}_\varphi + \ddot{z}\boldsymbol{k} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2)\boldsymbol{e}_\theta + (r\sin\theta\,\ddot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi} + 2\dot{r}\sin\theta\,\dot{\varphi})\boldsymbol{e}_\varphi$.
23. $\partial\boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta, \partial\boldsymbol{e}_\theta/\partial\theta = -\boldsymbol{e}_r, \partial\boldsymbol{e}_r/\partial\varphi = \boldsymbol{e}_\varphi\sin\theta, \partial\boldsymbol{e}_\theta/\partial\varphi = \boldsymbol{e}_\varphi\cos\theta$, $\partial\boldsymbol{e}_\varphi/\partial\varphi = -(\boldsymbol{e}_r\sin\theta + \boldsymbol{e}_\theta\cos\theta)$, others zero.
24. $mc^2[(\cosh^2\lambda - \cos^2\theta)\ddot{\lambda} + \frac{1}{2}\sinh 2\lambda(\dot{\lambda}^2 - \dot{\theta}^2) + \sin 2\theta\,\dot{\lambda}\dot{\theta}] = F_\lambda$, $mc^2[(\cosh^2\lambda - \cos^2\theta)\ddot{\theta} - \frac{1}{2}\sin 2\theta(\dot{\lambda}^2 - \dot{\theta}^2) + \sinh 2\lambda\,\dot{\lambda}\dot{\theta}] = F_\theta$.
25. $y = \lambda + a\cosh[(x - b)/a]$, a, b, λ constants.
26. A circle, $x^2 + y^2 - 2by = a^2$, b constant.

CHAPTER 4

1. 4.22×10^4 km.
2. 1.61×10^5 km, 0.176 AU $= 2.64 \times 10^7$ km.
3. 11.9 yrs, 13.1 km s^{-1}.
4. 5.46 yrs.
5. 38.6 km s^{-1}, 7.4 km s^{-1}.
6. 1.62 m s^{-2}, 2.38 km s^{-1}; 25.8 m s^{-2}, 60.2 km s^{-1}.
7. 84.4 min, 108 min, 173 min.
8. $1.0 \times 10^{11}\,M_{\rm S}$ (assuming that the mass distribution is spherical — this is the mass inside the radius of the Sun's orbit).
9. $\sqrt{k/m}\,a$; $r^2 = \frac{1}{2}a^2(3 + 2\cos\alpha \pm \sqrt{5 + 4\cos\alpha})$; $r = 2a, a$; $r = a, 0$.
10. $U = J^2/2mr^2 + \frac{1}{2}k(r - a)^2$; $\omega = \sqrt{1/2}\,\omega_0$; $\omega' = \sqrt{5/2}\,\omega_0$; 2.24 radial oscillations per orbit.
11. $(2/3\pi)$ yrs = 77.5 days.
12. Ratio is 1.013, 2.33, 134.
13. $1/r^2 = mE/J^2 + \sqrt{(mE/J^2)^2 - mk/J^2}\cos 2(\theta - \theta_0)$.

14. Hyperbola with origin at the centre.
15. 7.77 days.
16. $8.8\,\mathrm{km\,s^{-1}}$, $5.7\,\mathrm{km\,s^{-1}}$; $97°$ ahead of Earth; $82°$ ahead of Jupiter.
17. $4.26R_{\mathrm{E}}$, $18.4°$; $38.3\,\mathrm{km\,s^{-1}}$, 4.87 yrs.
19. $GMm/2a, -GMm/a$.
20. $x = a(e - \cosh\psi), y = b\sinh\psi$; $r = a(e\cosh\psi - 1)$, $t = (abm/J)(e\sinh\psi - \psi)$.
21. $5.7\,\mathrm{km\,s^{-1}}$, opposite to Jupiter's orbital motion; $5.7\,\mathrm{km\,s^{-1}}$; $3.9 \times 10^6\,\mathrm{km} = 56R_{\mathrm{J}}$, $23R_{\mathrm{J}}$.
22. $14.3\,\mathrm{km\,s^{-1}}$ at $23.5°$ to Jupiter's orbital direction; 9.2 AU, 16.2 yrs; 3.6 AU.
23. $14.3\,\mathrm{km\,s^{-1}}$, in plane normal to Jupiter's orbit, at $23.5°$ to orbital direction; 7.8 AU, 16.2 yrs; 2.5 AU.
24. $\cos\theta = (1 - l/R)/\sqrt{1 - l/a}$; $60°$, $6.45\,\mathrm{km\,s^{-1}}$.
25. With $n^2 = |1 + mk/J^2|$, $b^2 = J^2/2m|E|$:
    $J^2 + mk > 0, E > 0 : r\cos n(\theta - \theta_0) = b$;
    $J^2 + mk = 0, E > 0 : r(\theta - \theta_0) = \pm b$;
    $J^2 + mk < 0, E > 0 : r\sinh n(\theta - \theta_0) = \pm b$;
    $J^2 + mk < 0, E = 0 : re^{\pm n\theta} = r_0$;
    $J^2 + mk < 0, E < 0 : r\cosh n(\theta - \theta_0) = b$.
26. $\mathrm{d}\sigma/\mathrm{d}\Omega = k\pi^2(\pi - \theta)/mv^2\theta^2(2\pi - \theta)^2\sin\theta$.
27. $\omega = \sqrt{(-ka^2 - c)/ma^5}$, $\omega' = \sqrt{(-ka^2 + c)/ma^5}$.
28. 0.123 m, 2.44 m.
29. $1.13 \times 10^{-11}\,\mathrm{m}$, $8.1 \times 10^3\,\mathrm{s^{-1}}$.
30. $\dot{r} = eJ\sin\theta/ml$, $\ddot{r} = eJ^2\cos\theta/m^2lr^2$, rad. accel. $= -J^2/m^2lr^2$.

CHAPTER 5

1. $2.2\,\mathrm{m\,s^{-2}} = 0.086\,g_{\mathrm{J}}$, $5.1 \times 10^{-3}\,\mathrm{m\,s^{-2}} = 1.9 \times 10^{-5}\,g_{\mathrm{S}}$.
2. $15.3\,\mathrm{s^{-1}}$.
3. 0.20 mm, 78 mm.
4. $465\,\mathrm{m\,s^{-1}}$; (a) $542\,\mathrm{m\,s^{-1}}$, (b) $187\,\mathrm{m\,s^{-1}}$, (c) $743\,\mathrm{m\,s^{-1}}$.
5. (a) 99.88 t wt, (b) 100.29 t wt, (c) 99.46 t wt.
6. $2.53 \times 10^{-3}\,\mathrm{N}$ to south, $1.46 \times 10^{-3}\,\mathrm{N}$ up.
7. $0.013\,\mathrm{mbar\,km^{-1}}$.
8. $0.155°$.
9. 12.0 s; 100 kg wt, 20 kg wt; decreasing weight and a Coriolis force of 79 N.
10. 47.4 mm.

11. $\boldsymbol{F} = m\ddot{\boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v} \wedge \boldsymbol{B})$ with $\boldsymbol{E} = E\boldsymbol{k}$, $\boldsymbol{B} = B\boldsymbol{k}$;
$x = (mv/qB)\sin(qBt/m), y = (mv/qB)[\cos(qBt/m) - 1]$,
$z = qEt^2/2m$; $z = (2mE/qa^2B^2)y^2$; depends only on m/q.
12. $l = \pi mv/qB$; $E = 2 \times 10^6\,\mathrm{V\,m^{-1}}$, $l = 0.089\,\mathrm{m}$.
13. $\sim 10^5\,\mathrm{T}$, $1.76 \times 10^{11}\,\mathrm{s^{-1}}$.
14. $5.1 \times 10^{16}\,\mathrm{s^{-1}}$, $-3.3 \times 10^{16}\,\mathrm{s^{-1}}$.
15. 124 m.
18. $m\ddot{\boldsymbol{r}} = \boldsymbol{F} - m\boldsymbol{a}$.
19. $\begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.
20. $T = \frac{1}{2}m(\dot{\boldsymbol{r}} + \boldsymbol{\omega} \wedge \boldsymbol{r})^2$.

CHAPTER 6

1. $\phi = (q/2\pi\epsilon_0 a^2)(\sqrt{a^2+z^2} - |z|)$, $\boldsymbol{E} = (q\boldsymbol{k}/2\pi\epsilon_0 a^2)\left(\frac{z}{|z|} - \frac{z}{\sqrt{a^2+z^2}}\right)$;
$\boldsymbol{E} \to \boldsymbol{k}(\sigma/2\epsilon_0)(z/|z|)$.
2. $Q = -\frac{1}{2}qa^2$; for $\theta = 0$, $\phi \approx (q/4\pi\epsilon_0)(1/z - a^2/4z^3)$.
3. When $\boldsymbol{d}$ is in same direction as $\boldsymbol{E}$.
4. $\boldsymbol{E} = (3\boldsymbol{d}\cdot\boldsymbol{r}\boldsymbol{r} - r^2\boldsymbol{d})/4\pi\epsilon_0 r^5$, $V = (r^2\boldsymbol{d}\cdot\boldsymbol{d}' - 3\boldsymbol{d}\cdot\boldsymbol{r}\,\boldsymbol{d}'\cdot\boldsymbol{r})/4\pi\epsilon_0 r^5$;
(a) $\boldsymbol{F} = -\boldsymbol{F}' = -6\boldsymbol{k}(dd'/4\pi\epsilon_0 r^4)$, $\boldsymbol{G} = \boldsymbol{G}' = \boldsymbol{0}$;
(b) $\boldsymbol{F} = -\boldsymbol{F}' = 3\boldsymbol{k}(dd'/4\pi\epsilon_0 r^4)$, $\boldsymbol{G} = \boldsymbol{G}' = \boldsymbol{0}$;
(c) $\boldsymbol{F} = -\boldsymbol{F}' = 3\boldsymbol{i}(dd'/4\pi\epsilon_0 r^4)$, $\boldsymbol{G} = -\boldsymbol{j}(dd'/4\pi\epsilon_0 r^3)$, $\boldsymbol{G}' = 2\boldsymbol{G}$;
(d) $\boldsymbol{F} = \boldsymbol{F}' = \boldsymbol{0}$, $\boldsymbol{G} = -\boldsymbol{G}' = -\boldsymbol{k}(dd'/4\pi\epsilon_0 r^3)$.
5. $V = \frac{1}{2}\sum_{i\neq j}(q_i q_j/4\pi\epsilon_0 r_{ij})$.
6. $\frac{3}{5}(q^2/4\pi\epsilon_0 a)$; $\frac{1}{2}(q^2/4\pi\epsilon_0 a)$.
7. $4.5 \times 10^5\,\mathrm{C}$, $0.28\,\mathrm{J\,m^{-2}}$.
8. $8\pi\sigma_0 a^4/5$, $2\pi\sigma_0^2 a^3/25\epsilon_0$.
9. $3qa^2(x^2 - y^2)/4\pi\epsilon_0 r^5$,
$\boldsymbol{E} = (3qa^2/4\pi\epsilon_0 r^7)\big((3x^2 - 7y^2 - 2z^2)x, (7x^2 - 3y^2 + 2z^2)y, 5(x^2 - y^2)z\big)$.
10. $-Gm/r + (Gma^2/r^3)(3\cos^2\theta - 1)$;
$-Gm/a + (Gm/4a^3)(2z^2 - x^2 - y^2)$, $\boldsymbol{g} = (Gm/2a^3)(x, y, -2z)$.
11. $-6Gm/r + (7Gma^4/4r^5)[3 - 5(x^4 + y^4 + z^4)/r^4]$.
12. $-\sqrt{(8\pi G\rho_0 a^3/3)(r^{-1} - a^{-1})}$; 6.7×10^6 yrs; 14.9 mins, 29.5 mins.
13. $4.0 \times 10^{40}\,\mathrm{J}$.
14. 1/9.5.
15. 3.0.
16. 79 m; 78 R_{E}.
17. $(M_{\mathrm{E}}/M_{\mathrm{M}})^2(R_{\mathrm{M}}/R_{\mathrm{E}})^4 = 35$; 2.8 km.

18. 12.5 m.
19. $(8\pi/5)\rho_0 d_0 r^4$; 1.13.
22. $q\mu^2 e^{-\mu r}/4\pi\epsilon_0 r$; $-q$.
23. $2\pi G\rho^2 R^2/3 = 1.7 \times 10^{11}\,\mathrm{Pa}{=}1.7\,\mathrm{Mbar}$.
24. $\Phi = -k\rho$; $a = \sqrt{\pi k/4G}$.
25. $-1.43 \times 10^{-6}\,\mathrm{s}^{-1}$; 51 days.
26. 17.9 yrs (should be 18.6 yrs).

CHAPTER 7

1. 258 days.
2. 7.4×10^5 km from centre of Sun, *i.e.*, just outside the Sun; 0.28°.
3. $0.00125\,M_0$; $m_1 \geq 0.00125\,M_0$.
4. $z_1 = l + m_1 vt/M - \frac{1}{2}gt^2 + (m_2 v/M\omega)\sin\omega t$,
   $z_2 = m_1 vt/M - \frac{1}{2}gt^2 - (m_1 v/M\omega)\sin\omega t$, with $\omega = \sqrt{k/\mu}$ and $v < l\omega$.
5. $m_1/m_2 = 1$.
6. 12; 0.071.
7. $62.7°, 55.0°, 640\,\mathrm{keV}$.
8. $T_1^* = m_2 Q/M, T_2^* = m_1 Q/M$; 3.2 MeV, 0.8 MeV.
9. $\ln 10^6/\ln 2 \approx 20$.
10. 90°; 45°, 45°.
11. $2.41b$; $(0.65v, 0.15v, 0), (0.35v, -0.15v, 0)$.
12. $T^*/T = m_2/M$; $\to 1$ or 0.
13. 3×10^{-6}, +450 km, +2.4 min.
15. $a^2 \cos\theta(1/\sin^4\theta + 1/\cos^4\theta)$, where $a = e^2/2\pi\epsilon_0 mv^2$. (The second term comes from recoiling target particles.)
16. $1.8 \times 10^3\,\mathrm{s}^{-1}$, same for both.
17. $2m\ddot{\boldsymbol{R}} = \boldsymbol{0}$, $\frac{1}{2}m\ddot{\boldsymbol{r}} = q\boldsymbol{E} - (q^2/4\pi\epsilon_0 r^3)\boldsymbol{r}$; $z = 2qE/m\omega^2$.

CHAPTER 8

1. $0.99\,\mathrm{km\,s^{-1}}$, 164 kg.
2. $(2.44 + 1.48 =)\, 3.91\,\mathrm{km\,s^{-1}}$, 143 kg.
3. $4.74\,\mathrm{km\,s^{-1}}$.
4. 3 stages, 1.48×10^5 kg.
5. $14.2\,\mathrm{km\,s^{-1}}$, $2.06\,\mathrm{km\,s^{-1}}$, 2.8×10^6 kg.
6. $\frac{1}{2}M_0 u^2(1 - e^{-v/u})$.
7. 44.6 km, 33.9 km.
8. $10.3\,\mathrm{km\,s^{-1}}$, $(3.07 - 0.07 =)\ 3.0\,\mathrm{km\,s^{-1}}$; 6.13 t.
9. If $\boldsymbol{u}_1 = (v, 0)$: $(-1, 0)v/5$, $(3, \pm\sqrt{3})v/5$, $(|\boldsymbol{v}_2| = 2\sqrt{3}v/5)$.

10. $-\rho A v^2, A = \pi r^2$; because scattering is isotropic.
11. $\delta a = -2(I/m)\sqrt{(1+e)a^3/GM(1-e)}$.
12. $\mathrm{d}a/\mathrm{d}t = \mathrm{d}l/\mathrm{d}t = -2\rho A v a/m$.
13. $-20.9\,\mathrm{s}$, $-16.9\,\mathrm{km}$; $-3.37\,\mathrm{s}$, $-2.78\,\mathrm{km}$.
14. (a) 6.2 h, 1.85 d; (b) 3.15 d, 40.4 d.

CHAPTER 9

1. $4\sqrt{2}a/3, (3\sqrt{2}g/4a)^{1/2}$.
2. 64 r.p.m., 5.3×10^{-6} J from work done by insect; dissipated to heat.
3. (a) $E, \boldsymbol{P}$; (b) $\boldsymbol{J}$ about leading edge; (c) E; $3v/8a, 5/8$; $[16(\sqrt{2}-1)ga/3]^{1/2}$.
4. (a) 1.011 s, (b) 1.031 s.
5. $4a/3$; $3bX/4Ma^2$, $3bX/4a$; $b = 4a/3$.
6. (a) $3Mg\cos\varphi$; (b) $(Mg/8)(-9\sin 2\varphi, 11 + 9\cos 2\varphi)$, $(3Mg/2)(-\sin 2\varphi, 1+\cos 2\varphi)$.
7. $9 \times 10^{-6}\,\mathrm{kg\,m^2}$, $16 \times 10^{-6}\,\mathrm{kg\,m^2}$, $25 \times 10^{-6}\,\mathrm{kg\,m^2}$; $(1.08, 1.44, 0) \times 10^{-4}\,\mathrm{kg\,m^2\,s^{-1}}$; 6.3×10^{-3} N.
8. (a) $(8, 8, 2)Ma^2/3$; (b) $(11, 11, 2)Ma^2/3$.
9. $2M(a^5 - b^5)/5(a^3 - b^3)$.
10. 25.6 s, 1.097×10^3 J.
11. 60°.
12. $I_1 = I_2 = 3M(a^2 + 4h^2)/20, I_3 = 3Ma^2/10$; 1/2; $Z = 3h/4$, $I_1^* = I_2^* = 51Ma^2/320, I_3^* = I_3$.
13. 1.55 s.
14. 112 s.
15. $0.244\,\mathrm{s^{-1}}$.
17. 8.83 m.
18. (a) 2.64 Hz ($\Omega = 16.6\,\mathrm{s^{-1}}$); (b) 3.44 Hz ($\Omega = 21.6\,\mathrm{s^{-1}}$).
20. $2.50 \times 10^{-12}\,\mathrm{s^{-1}} = 16.3''\,\mathrm{yr^{-1}}$.
21. 22.3 s.

CHAPTER 10

1. $\pm g/4$.
2. $\sqrt{4mgl/(M+2m)a^2}$.
3. $Mmg/(M+2m)$.
4. $M^2mg/k(M+2m)^2$.
6. $g/7, 3g/7, -5g/7$.
7. $24mg/7, 12mg/7$.

8. $62.62\,\mathrm{s}^{-1}$, $4.347\,\mathrm{s}^{-1}$ (*cf.* $4.065\,\mathrm{s}^{-1}$); $371.5\,\mathrm{s}^{-1}$ (3548 r.p.m.).
9. $(M + m\sin^2\theta)l\ddot{\theta} + ml\dot{\theta}^2\cos\theta\sin\theta + (M+m)g\sin\theta = 0$; $1.40\,\mathrm{s}^{-1}$.
10. $I_1\ddot{\varphi} = I_3\omega_3\Omega\sin\lambda\cos\varphi - I_1\Omega^2\sin^2\lambda\sin\varphi\cos\varphi$; $(I_1/I_3)\Omega\sin\lambda$; east and west.
11. I_1, I_3 are replaced by $I_1^* < I_1, I_3^* = I_3$; large Ω is bigger, small Ω is slightly smaller.
12. $\arcsin(1/\sqrt{3}) = 35.3°$.
14. (a) as at $t = 0$ except that for $l/2 - ct < x < l/2 + ct$, $y = a - 2act/l$; (b) $y = 0$; (c) $y(x, l/c) = -y(x, 0)$.
15. $\ddot{x} - 2\omega\dot{y} - \omega^2 x = -GM_1(x + a_1)/r_1^3 - GM_2(x - a_2)/r_2^3$,
    $\ddot{y} + 2\omega\dot{x} - \omega^2 y = -GM_1 y/r_1^3 - GM_2 y/r_2^3$, with
    $r_1^2 = (x + a_1)^2 + y^2, r_2^2 = (x - a_2)^2 + y^2, \omega^2 = GM/a^3$.

CHAPTER 11

1. $x = a(\cos\omega_1 t\cos\omega_2 t + \frac{1}{2}\sqrt{2}\sin\omega_1 t\sin\omega_2 t)$,
   $y = a(2\cos\omega_1 t\cos\omega_2 t + \frac{3}{2}\sqrt{2}\sin\omega_1 t\sin\omega_2 t)$,
   where $\omega_{1,2} = \frac{1}{2}(\omega_+ \pm \omega_-)$ and $\omega_\pm = \sqrt{(2 \pm \sqrt{2})g/l}$.
2. $\omega^2 = g/l, g/l + k/M + k/m$; $A_X/A_x = 1, -m/M$; $2Ma/(M+m)$; no.
3. $x_0 = 2mg/k, y_0 = 3mg/k$; $\omega^2 = (3 \pm \sqrt{5})k/2m$.
4. $\omega^2 = \omega_0^2, \omega_0^2 + \omega_{\mathrm{s}}^2, \omega_0^2 + 3\omega_{\mathrm{s}}^2$;
   $A_x : A_y : A_z = 1:1:1,\ 1:0:-1,\ 1:-2:1$.
5. $2a/3, a$.
6. (a) $\omega^2 = k/m, 3k/m$; (b) $\omega^2 = (1 - a/l)k/m, 3(1 - a/l)k/m$.
7. $\omega^2 = (M + m)g/ma, g/2a$.
8. $\theta = (\varphi_0/10)(\cos 2\pi t - \cos 3\pi t)$ (t in s); $\varphi_0/5, t = 1\,\mathrm{s}$.
9. 1.0025 s, 0.099 75 s; 0.5025 mm.
10. $(0.401\sin 6.27t - 0.0399\sin 63.0t)$ mm, (t in s).
11. $\phi = (q/4\pi\epsilon_0)[14/a + (4x^2 + y^2)/2a^3]$; $\omega_1^2 = q^2/4\pi\epsilon_0 a^3 m$, $\omega_2^2 = 4\omega_1^2$.
12. $\omega^2 = g/l, g/l, 3g/l$.
13. $A_{1,2} = (F/\sqrt{2m})(\omega_{1,2}^2 - \omega^2 + 2\mathrm{i}\gamma_{1,2}\omega)$, with $\omega_1^2 = \omega_0^2$, $\omega_2^2 = \omega_0^2 + 2\omega_{\mathrm{s}}^2$,
    $\gamma_1 = \alpha/2m$, $\gamma_2 = (\alpha + 2\beta)/2m$; $\alpha > \sqrt{3k/\omega_0}$.
14. $q_{2r} = 0$, $q_{2r+1} = (-1)^r 4\sqrt{2}la/\pi^2(2r+1)^2$.

CHAPTER 12

1. $\omega^2 = g\cos\alpha/r\sin^2\alpha$; $\arcsin(1/\sqrt{3}) = 35.3°$.
2. $\Omega^2 = \omega^2 - g^2/l^2\omega^2$.
3. $H = (p_x - p_y)^2/6m + p_y^2/2m + \frac{1}{2}ky^2 - mgy$;
   $y = mg(1 - \cos\omega t)/k$, $x = x_0 - y/4$, $\omega^2 = 4k/3m$.

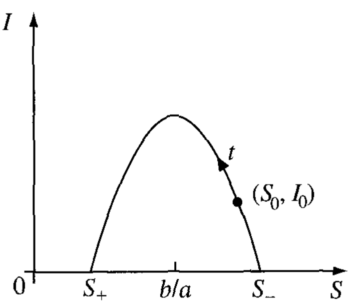
4. $J^2 = p_\theta^2 + p_\varphi^2/\sin^2\theta$.
5. $H = \dfrac{p_\theta^2}{2ml^2} + \dfrac{(lp_x - p_\theta\cos\theta)^2}{2(M + m\sin^2\theta)l^2} + mgl(1-\cos\theta)$.
6. $H = \dfrac{p_X^2 + p_Y^2}{2M} + \dfrac{p_\theta^2}{2(I_1^* + MR^2\sin^2\theta)} + \dfrac{(p_\varphi - p_\psi\cos\theta)^2}{2I_1^*\sin^2\theta}$
$+\dfrac{p_\psi^2}{2I_3^*} + MgR\cos\theta$; $\omega_{3,\min}^2 = 4I_1^*MgR/I_3^2$, reduced by a factor I_1^*/I_1.
7. $(p_\varphi - p_\psi z)^2 - 2I_1(1-z^2)(E - Mgrz - p_\psi^2/2I_3) = 0$.
8. $H = (\boldsymbol{p} - q\boldsymbol{A})^2/2m + q\phi$.
9. $H = \dfrac{p_\rho^2}{2m} + \dfrac{p_\varphi^2}{2m\rho^2} - \dfrac{qq'}{4\pi\epsilon_0\rho} + \dfrac{q^2B^2\rho^2}{2m}$.
10. $\sqrt{q'm/\pi\epsilon_0 qB^2}$; $\sqrt{3}\omega_{\rm L}$.
11. $0 < b < a$, $a^2/4b$.
12. $M_1/M_2 > \frac{1}{2}(25 + \sqrt{621}) = 24.96$.
13. $\Omega^2 = -p^2 = \frac{1}{2}\omega^2\{1 \pm \sqrt{1 - 27M_1M_2/(M_1+M_2)^2}\}$; 11.90 yrs, 147.4 yrs.
14. $\dot\varphi = \dfrac{J}{m\rho^2}$; $\dfrac{\partial U}{\partial\rho} = -\dfrac{J^2}{m\rho^3} - \dfrac{qJB_z}{m\rho}$, $\dfrac{\partial U}{\partial z} = \dfrac{qJB_\rho}{m\rho}$.

CHAPTER 13

1. (a) $\bar f = k^2/4\sigma$; equilibria $x_\pm = (k \pm \sqrt{k^2 - 4f\sigma})/2\sigma$, x_+ is asymptotically stable. (b) $\dot x \le \bar f - f < 0 \implies t_0 \le k/\sigma(f - \bar f)$. In practice, $\bar f$ is not precisely known or constant.
2. Trajectories $\frac{1}{2}\dot\theta^2 - (g/l)\cos\theta =$ constant ($= g/l$ on separatrices).
4. $\dot x = y$, $\dot y = -\omega_0^2 x - \mu y$. Critical point $(0,0)$ which is (i) *asymptotically stable spiral*, (ii) (*improper*) *asymptotically stable node*, (iii) (*inflected*) *asymptotically stable node.*
5. $(0,0)$ and $(1,0)$ are *asymptotically stable nodes*; $(\frac{1}{2},0)$ is a *saddle.* [*Note* that the nodes are local minima of $U(x,y)$.]
6. $f(x) = -c\ln x + dx$ has a single minimum and $f \to +\infty$ when $x \to 0^+$ and when $x \to +\infty$.
7. $(0,0)$ has $\lambda_{1,2} = 1, \frac{1}{2}$ with eigenvectors $(1,0),(0,1)$: *unstable node*; $(0,2)$ has $\lambda_{1,2} = -1, -\frac{1}{2}$ with $(1,3),(0,1)$: *asymptotically stable node*; $(1,0)$ has $\lambda_{1,2} = -1, -\frac{1}{4}$ with $(1,0),(1,-\frac{3}{4})$: *asymptotically stable node*; $(\frac{1}{2},\frac{1}{2})$ has $\lambda_{1,2} = (-5 \pm \sqrt{57})/16$ with $(-\frac{1}{2}, \frac{1}{2} + \lambda_i)$: *saddle.* Nearly all initial conditions lead to extinction for one or other species in Fig. 13.9 $\implies$ no stable coexistence. Second set of parameters $\implies$ $(0,0)$: *unstable node*; $(0,\frac{3}{2})$ and $(\frac{1}{2},0)$: *saddles*; $(\frac{1}{4},1)$:

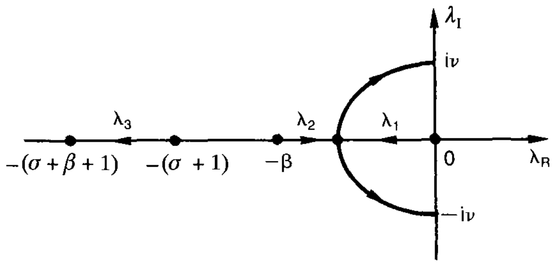
asymptotically stable node; in this case there is asymptotically stable coexistence.

8. $M = \begin{bmatrix} -c_1 & a_2 \\ a_1 & -c_2 \end{bmatrix} \implies \lambda^2 + (c_1 + c_2)\lambda + (c_1c_2 - a_1a_2) = 0$. Then $c_1c_2 > a_1a_2 \implies \lambda_1, \lambda_2$ real, negative: *asymptotically stable node*, coexistence in first quadrant; and $c_1c_2 < a_1a_2 \implies \lambda_1, \lambda_2$ real, opposite sign: *saddle* in third quadrant and trajectories run away $\to +\infty$.
9. Critical points at $(0,0)$, $\left[\frac{\mu(R-1)}{(\mu R+a)}, \frac{a(R-1)}{R(a+\mu)}\right]$. $R < 1$: *asymptotically stable node, saddle.* $R > 1$: *saddle, asymptotically stable node.* Disease maintains itself only when $R > 1$.
10. At $(0,0)$, $\lambda_{1,2} = (\epsilon \pm \sqrt{\epsilon^2 - 4})/2$; *unstable spiral* ($\epsilon < 2$) or *node* ($\epsilon \geq 2$).
11. At $(0,0)$, $\lambda_{1,2} = \alpha\beta \pm \mathrm{i}$, so we have an *asymptotically stable spiral* ($\beta < 0$), *unstable spiral* ($\beta > 0$) or *centre* ($\beta = 0$). In polars, $\dot{\theta} = 1$ and $\dot{r} = \alpha r(\beta - r^2)$; for $\beta > 0$, $r \to \sqrt{\beta}$ as t and $\theta \to \infty$; when $\beta < 0$, $r \to 0$.
12. (a) $J_1\dot{J}_1 + J_2\dot{J}_2 + J_3\dot{J}_3 = 0$; then integrate to get $|\boldsymbol{J}| =$ constant. (b) $(\pm J, 0, 0), (0, 0, \pm J)$ are *centres* on the sphere; $(0, \pm J, 0)$ are *saddles.* (c) $\dot{J}_3 = 0 \implies J_3 = I_3\Omega$. $J_{1,2}$ satisfy $\ddot{J}_i + [(I_1 - I_3)/I_1]^2\Omega^2 J_i = 0$. (d) Critical point $(0,0,0)$, with $\lambda_{1,2,3} = -|\mu|/I_{1,2,3}$ and eigenvectors $(1,0,0), (0,1,0), (0,0,1)$. As $\omega \to 0$, $\boldsymbol{\omega} \to$ rotation about axis with the largest moment of inertia, I_3.
13. (a) $\mathrm{d}N/\mathrm{d}t \equiv 0$ and $\mathrm{d}I/\mathrm{d}S = \dot{I}/\dot{S} = -1 + b/aS$. (b) $\mathrm{d}I/\mathrm{d}S = 0$ when $S = b/a$ and $\mathrm{d}^2I/\mathrm{d}S^2 = -b/aS^2 < 0$, so maximum. $I = 0$ at $S = S_-, S_+$. (c) $S_0 = b/a + \delta, I_0 = \epsilon$. Write $S = b/a + \xi$ and expand $\implies$ trajectory passes through $(b/a - \delta, \epsilon)$. Since S_0 can be made arbitrarily close to S_-, so S_+ is arbitrarily close to $b/a - \delta$ and these susceptibles escape infection.



14. For a critical point (X,Y,Z), the matrix $M = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - Z & -1 & -X \\ Y & X & -\beta \end{bmatrix}$.

(a) $\lambda_1 = -\beta < 0$, $\lambda_{2,3} = -\frac{1}{2}(\sigma+1) \pm \frac{1}{2}\sqrt{(\sigma+1)^2 - 4(1-\rho)\sigma}$, so that $\lambda_{2,3}$ are real and are both negative only when $0 < \rho < 1$.
(c) $\rho = 1 \implies$ cubic becomes $\lambda(\lambda+\beta)(\lambda+\sigma+1) = 0$. If the cubic has the form $(\lambda+\mu)(\lambda^2+\nu^2) = 0$ then $\mu = \sigma+\beta+1$, $\nu^2 = \beta(\sigma+\rho)$, $\mu\nu^2 = 2\sigma\beta(\rho-1) \implies$ result for ρ_{crit}. (d) Write $\rho = 1+\epsilon$ (with σ, β



constant, ϵ small) and find changes in eigenvalues by putting $\lambda = \lambda_i + \xi(\epsilon)$ and performing a linear analysis of the cubic in (b).
(e) Evidently $RHS = 0$ is an ellipsoid and the distance of (x,y,z) from $\bar{\boldsymbol{O}} \equiv (0,0,\rho+\sigma)$ decreases where $RHS < 0$, *i.e.*, outside the ellipsoid and so *a fortiori* outside any sphere centred at $\bar{\boldsymbol{O}}$ which contains it.

15. (b) For a critical point (X_1, X_2, Y), the matrix
$$M = \begin{bmatrix} -\mu & Y & X_2 \\ Y - A & -\mu & X_1 \\ -X_2 & -X_1 & 0 \end{bmatrix}.$$
(c) $\boldsymbol{\nabla} \cdot (\dot{\boldsymbol{x}}) = -2\mu < 0$. (d) $X_1^2 + X_2^2 + \bar{Y}^2 = C$, where $C > 0$, is an ellipsoid (oblate). Trajectories move towards smaller C values whenever $(X_1, X_2, \bar{Y})$ is below the paraboloid $\bar{Y} = \mu(X_1^2 + X_2^2)/\sqrt{2}$.

16. $\Delta x = \Delta x_0 \cos\omega t + (\Delta y_0/\omega)\sin\omega t$, $\Delta y = -\omega\,\Delta x_0 \sin\omega t + \Delta y_0 \cos\omega t$. Each is bounded and so is $\sqrt{(\Delta x)^2 + (\Delta y)^2}$.

17. Trajectories in the xy-plane are parabolic arcs in $x \geq 0$. The higher the energy the longer the period of oscillation (between successive bounces). Between bounces, $\Delta x = \Delta x_0 + \Delta y_0 t$, $\Delta y = \Delta y_0$, so that $\sqrt{(\Delta x)^2 + (\Delta y)^2} \sim \kappa t$ for t increasing, and bounces don't affect this result.

18. Evidently, we have $(1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, *etc.* in successive rows, *i.e.*, $(\frac{1}{2})^n \times$ (row of Pascal's triangle). When $n = 16$, we have

$(\frac{1}{2})^{16}$[1,16,120,560,1820,4368,8008,11440, 12870,11440,8008,4368, 1820,560,120,16,1].

CHAPTER 14

1. Return trajectories have $-|k|/R \le E < 0$.
$t_0 = \sqrt{2h^3m/|k|}\left[\pi/2 - \arcsin\left(\sqrt{R/h}\right) + \sqrt{(R/h)(1-R/h)}\right] \to \sqrt{h^3m/2|k|}\,\pi$ as $R/h \to 0$, *i.e.*, Kepler's third law for a flat ellipse, major axis h in this limit.
2. Explicitly, $[F, H] = 0$ in each case. In (c), the vector $\boldsymbol{A}$ may be considered using Cartesians or polars. (Then the orientation of the orbit may be specified using $\alpha = \arctan(A_y/A_x)$ and $|\boldsymbol{A}|^2 = 1 - (2J^2|E|/mk^2)$, consistent with (4.30).)
3. H $(\equiv E)$ and p_θ are constants of the motion. Steady $r = r_0$ when $p_\theta = mr_0^2\Omega_0$ with $\Omega_0 = \pm\sqrt{g/r_0}$. Small oscillations $r = r_0 + \Delta \implies$ SHM for Δ with frequency $\sqrt{3g/2r_0}$.
4. $\omega = \sqrt{g\cos\alpha/r_0\sin^2\alpha}$, rotation rate at r_0. Small oscillations $r = r_0 + \Delta \implies$ SHM for Δ with frequency $\varpi \equiv \sqrt{3}(\omega\sin\alpha)$. Closure for rational $\sqrt{3}\sin\alpha$.
5. Effective potential $p_\theta^2/2mr^2 - |k|/r \implies$ minimum $-k^2m/2p_\theta^2$ when $r = r_0 = p_\theta^2/m|k|$, circle; (r, p_r) curves closed (around $r = r_0$) when $E < 0$, corresponding to (r,θ) ellipses; $E \ge 0$ curves stretch to $r \to \infty$, (r,θ) hyperbolae, parabola. We always have closure for this system — given E, Poincaré section is a single point on an (r, p_r) curve, a different point for each choice of θ section.
6. $H \equiv E$ here. Action $I = 2E\sqrt{l/g} \implies E = \frac{1}{2}\sqrt{g/l}\,I$, and frequency $\omega = \partial H/\partial I = \frac{1}{2}\sqrt{g/l}$.
7. Oscillation between $x = \pm[l + \sqrt{2E/k}]$. Action $I = (2l/\pi)\sqrt{2mE} + E/\Omega \implies \sqrt{E} = \sqrt{I\Omega + \beta^2} - \beta$. $\phi(x) = (\partial/\partial I)\int^x p\,\mathrm{d}x$ with $\phi = \omega(I)t +$ constant, $\omega = \Omega\sqrt{E}/(\sqrt{E} + \beta)$, $T = 2\pi/\omega$. Small $E \implies T \to \infty$ as $E^{-1/2}$; large $E \implies T \to 2\pi/\Omega$.
8. $I = \frac{1}{2}\sqrt{mk}[E/k - \sqrt{\lambda/k}] \implies E(I)$; $\phi(q) = (\partial/\partial I)\int^q p\,\mathrm{d}q \equiv \omega t + \beta$, with $\omega = \partial E/\partial I = 2\sqrt{k/m}$. Evaluating $\phi(q)$ gives $q(t) = \left[\frac{E}{k} + \frac{\sqrt{E^2-\lambda k}}{k}\sin(\omega t + \beta)\right]^{1/2}$. (*Note.* For the isotropic oscillator there are two radial oscillations for each complete elliptical trajectory.)

9. Action $I_1 = (\sqrt{-2mE}/\pi)\int_{r_1}^{r_2}\sqrt{(r_2-r)(r-r_1)}(\mathrm{d}r/r)$, where $r_1+r_2=-|k|/E$ and $r_1r_2=-I_2^2/2mE \implies$ result. So $E(I_1,I_2)$ and $\omega_1=\partial E/\partial I_1 \equiv \omega_2$ (see (14.18)).
10. Consider the lines from a bounce point to the foci and the angles they make with the trajectory just before and just after the bounce. $\Lambda = [(x+ae)p_y - p_x y][(x-ae)p_y - p_x y]$. Change to λ,θ variables; H from Chapter 3, Problem 24 using $p_\lambda = mc^2(\cosh^2\lambda - \cos^2\theta)\dot{\lambda}$, $p_\theta = mc^2(\cosh^2\lambda-\cos^2\theta)\dot{\theta}$. Turning value for λ is when $p_\lambda = 0 \implies$ tangency condition.
11. (a) Motion within curve $(E/mg\mu r) = 1-(1/\mu)\cos\theta$ (ellipse). (b) $\mu=1$: bounding curve $E=2mgr\sin^2(\theta/2)$; $\mu<1$: curve as in (a) — hyperbola for $E\neq 0$, two straight lines for $E=0$.
12. Hamilton's equations: $\dot{x}=p_x+\omega y$, $\dot{p}_x=\omega p_y-\omega^2x+\partial U/\partial x$, $\dot{y}=p_y-\omega x$, $\dot{p}_y=-\omega p_x-\omega^2 y+\partial U/\partial y$ and $H=\frac{1}{2}(p_x+\omega y)^2+\frac{1}{2}(p_y-\omega x)^2-U$. System autonomous, $H=$ constant; $U\geq C$ for possible motion and for the Earth/Moon system the critical C is that corresponding to the 'equilibrium' point between Earth and Moon.
13. Action $I=E/\omega$, so $E\propto\omega\propto l^{-1/2}$. Maximum sideways displacement $= l\theta_{\max}\propto l^{1/4}$. Maximum acceleration $|\omega^2 l\theta_{\max}|\propto l^{-3/4}$.
14. Action $I\propto L\sqrt{H}$ and $H\propto v^2 \implies v\propto 1/L$. Temperature $\propto 1/L^2 \implies$ pressure $\propto 1/L^5 \implies$ pressure $\propto$ (density)$^{5/3}$.
15. Action $I=(2m/3\pi)\sqrt{2gq_0^3\sin\alpha}$ and $E=mgq_0\sin\alpha \implies$ result. Frequency $\omega=2\pi\sqrt{g\sin\alpha/8q_0}$. Evidently, I = constant $\implies$ $E\propto(\sin\alpha)^{2/3}$, $q_0\propto(\sin\alpha)^{-1/3}$, period $2\pi/\omega\propto(\sin\alpha)^{-2/3}$. So $E_2/E_1=0.69$, $q_{02}/q_{01}=1.20$, $\omega_1/\omega_2=1.44$.
16. Evidently $E\propto k^2$ and $\tau\propto k^{-2}$. Since $\tau\propto\sqrt{a^3/|k|}$ (see (4.31)), $a\propto 1/|k| \implies k$ decreases, a increases. Since eccentricity $e=\sqrt{1+2EI_2^2/mk^2}$ (see (4.30)), e remains constant.

APPENDIX A

1. (a) $1,2,3$; (b) $(-3,-3,3),(-3,7,8),(-6,4,11)$; (c) $-15,15$; (d) $(-3,3,0),(-1,3,-3)$; (e) as (d).
2. $164.2°, 16.2°$.
4. $(3x^2-yz,-xz,-xy)$; $6x$.
5. (a) $(1-y^2/x^2, 2y/x, 0), (2x/y, 1-x^2/y^2, 0)$; (b) circles passing through the origin with centres on x- or y-axes, respectively, intersecting at right angles.

8. $\boldsymbol{a}(\boldsymbol{\nabla}\cdot\boldsymbol{b})+(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a}-\boldsymbol{b}(\boldsymbol{\nabla}\cdot\boldsymbol{a})-(\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b}$.
9. $2\omega\boldsymbol{k}$.
11. $(\boldsymbol{\nabla}^2\boldsymbol{A})_\rho=\boldsymbol{\nabla}^2(A_\rho)-\dfrac{1}{\rho^2}\left(A_\rho+2\dfrac{\partial A_\varphi}{\partial\varphi}\right)$,
$(\boldsymbol{\nabla}^2\boldsymbol{A})_\varphi=\boldsymbol{\nabla}^2(A_\varphi)-\dfrac{1}{\rho^2}\left(A_\varphi-2\dfrac{\partial A_\rho}{\partial\varphi}\right)$, $(\boldsymbol{\nabla}^2\boldsymbol{A})_z=\boldsymbol{\nabla}^2(A_z)$.
12. $A_r=A_\theta=0,\ A_\varphi=\mu_0\mu\sin\theta/4\pi r^2$;
$B_r=\mu_0\mu\cos\theta/2\pi r^3,\ B_\theta=\mu_0\mu\sin\theta/4\pi r^3,\ B_\varphi=0$;
$\boldsymbol{A}=(\mu_0/4\pi r^3)\boldsymbol{\mu}\wedge\boldsymbol{r}$; $\boldsymbol{B}=(\mu_0/4\pi r^5)(3\boldsymbol{\mu}\cdot\boldsymbol{rr}-r^2\boldsymbol{\mu})$.
13. $\boldsymbol{B}=(\mu_0 I/4\pi r^3)\mathrm{d}\boldsymbol{s}\wedge\boldsymbol{r}$; $\boldsymbol{F}=(\mu_0 II'/4\pi r^3)\mathrm{d}\boldsymbol{s}'\wedge(\mathrm{d}\boldsymbol{s}\wedge\boldsymbol{r})$;
$\boldsymbol{F}+\boldsymbol{F}'=(\mu_0 II'/4\pi r^3)\boldsymbol{r}\wedge(\mathrm{d}\boldsymbol{s}\wedge\mathrm{d}\boldsymbol{s}')\neq\boldsymbol{0}$.
14. $A_r=r^{-1}\cos\theta,\ A_\theta=r^{-1}\ln r\sin\theta,\ A_\varphi=0$; $r^{-2}(1+2\ln r)\cos\theta$;
$(0,0,2r^{-2}\sin\theta)$.
15. $\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$.
17. $\mathsf{Q}=\iiint\rho(\boldsymbol{r}')(3\boldsymbol{r}'\boldsymbol{r}'-r'^2\mathbf{1})\mathrm{d}^3\boldsymbol{r}'$.

APPENDIX B

1. Distances $r_{1,2}$ are given by
$r_{1,2}^2=(a\cos\psi\mp ae)^2+(b\sin\psi)^2=a^2(1\mp e\cos\psi)^2$.
2. Tangent vector is $\boldsymbol{t}=(-a\sin\psi,b\cos\psi)$, unit vectors from two foci are $\boldsymbol{n}_{1,2}=(1\mp e\cos\psi)^{-1}(\cos\psi\mp e,(b/a)\sin\psi)$, scalar products are $\boldsymbol{t}\cdot\boldsymbol{n}_{1,2}=\pm ae\sin\psi$.

APPENDIX C

1. (a) $(0,0)$, $\lambda_{1,2}=-\frac{5}{2}\pm\frac{1}{2}\sqrt{17}$, eigenvectors $(1,3+\lambda_i)$, *asymptotically stable node.*
   (b) $(0,0)$, $\lambda_{1,2}=4,1$, eigenvectors $(1,\lambda_i-3)$, *unstable node.*
   (c) $(-1,-1)$, $\lambda_{1,2}=-5\pm\sqrt{5}$, eigenvectors $(-2,\lambda_i+8)$, *asymptotically stable node*;
   $(4,4)$, $\lambda_{1,2}=5\pm\mathrm{i}\sqrt{55}$, *unstable spiral.*
   (d) $(0,2)$, $\lambda_{1,2}=-1\pm\mathrm{i}$, *asymptotically stable spiral*;
   $(1,0)$ $\lambda_{1,2}=1,-2$, eigenvectors $(-1,\lambda_i+2)$, *saddle* ($\therefore$ unstable).
   (e) $(0,2)$, $\lambda_{1,2}=\pm\mathrm{i}2\sqrt{6}$, *centre* ($\therefore$ stable);
   $(0,-2)$, $\lambda_{1,2}=\pm\mathrm{i}2\sqrt{6}$, *centre* ($\therefore$ stable);

$(1,0)$, $\lambda_{1,2} = -8, 3$, eigenvectors $(1,0), (0,1)$, resp., *saddle* ($\therefore$ unstable);
$(-1,0)$, $\lambda_{1,2} = 8, -3$, eigenvectors $(1,0), (0,1)$, resp., *saddle* ($\therefore$ unstable).

(f) $(0, n\pi)$, n odd: $\lambda_{1,2} = \pm \mathrm{i}$, *centres* ($\therefore$ stable);
n even: $\lambda_{1,2} = \pm 1$, eigenvectors $(1,1), (1,-1)$, resp., *saddles* ($\therefore$ unstable).

(g) $(0,0)$, $\lambda_{1,2} = \pm\sqrt{\omega^2 - \alpha}$, so $\omega^2 > \alpha$, *saddle* ($\therefore$ unstable) and $\omega^2 < \alpha$, *centre* ($\therefore$ stable).

(*Note*: In each case, consider local sketches near the critical points and how these build into the global phase portrait.)

2. $\dot{x} = y$, $\dot{y} = x^3 - x \implies (0,0)$, *centre*, $(\pm 1, 0)$, *saddles*;
(a) $\mathrm{d}y/\mathrm{d}x = (x^3 - x)/y \implies \frac{1}{2}y^2 = \frac{1}{4}x^4 - \frac{1}{2}x^2 + c$, *i.e.*, $y = \pm\sqrt{\frac{1}{2}x^4 - x^2 + 2c}$; symmetry about x axis rules out spirals;
(b) separatrices given by $c = \frac{1}{4}$.

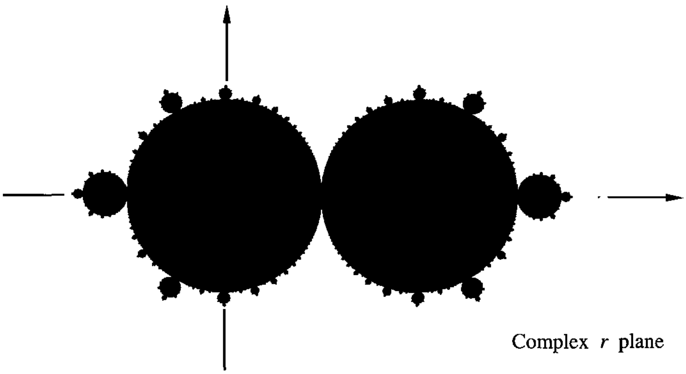
APPENDIX D

1. (b) $\frac{\mathrm{d}}{\mathrm{d}x}F^{(2)}(x) = F'(x)F'(F(x)) \implies$ asymptotically stable when $|F'(X_1)F'(X_2)| < 1 \implies |-r^2 + 2r + 4| < 1 \implies 3 < r < 1 + \sqrt{6}$ (if $r \geq 0$).
(c) $a = 1 - 1/r, b = 2/r - 1, s = 2 - r$.
(d) Applying (c) to (a), (b), etc, yields asymptotically stable points/cycles:

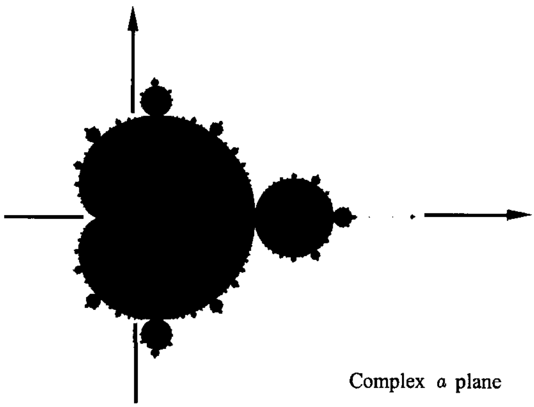
$s = 2 - r$	Y	r	X
$0 \to 1$	0	$2 \to 1$	$1 - 1/r$
$1 \to 3$	$1 - 1/s$	$1 \to -1$	0
$3 \to 1 + \sqrt{6}$	$(Y_1(s), Y_2(s))$	$-1 \to 1 - \sqrt{6}$	$(X_1(r), X_2(r))$
3.57	2^∞ accum.	-1.57	2^∞ accum.
4	range limit	-2	range limit

2. $r = 1 \pm \sqrt{1 + 4a}$, $\beta = -2\alpha = r/a$.
3. (a) $X = 0$ asymptotically stable in circle $|r| < 1$, $X = 1 - 1/r$ asymptotically stable in circle $|2 - r| < 1$.
(b) asymptotically stable 2-cycle when $|-r^2 + 2r + 4| < 1$, or $|r - 1 - \sqrt{5}|.|r - 1 + \sqrt{5}| < 1$ (two disjoint ovals).

Note: If we continue in this way, adding the regions of the complex r plane in which there are cycles of any finite length we arrive at the figure shown:

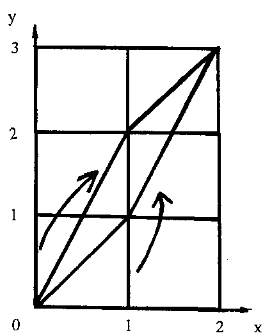


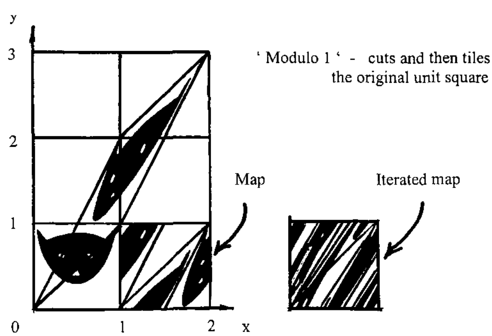
(c) In the complex a plane, the circles become a heart-shaped region with a cusp at $a = -\frac{1}{4}$; the 2-cycles yield a circle $|a-1| = \frac{1}{4}$. The entire set yields the Mandelbrot 'signature' (see Peitgen and Richter, *The Beauty of Fractals*, Springer, 1986):



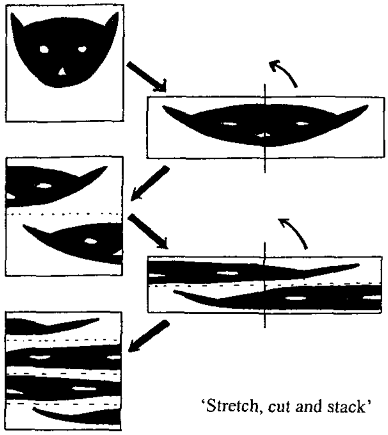
4. Fixed points at $X = 0$, $X = \frac{2}{3}$ have $|F'(X)| = 2 \implies$ instability. Uncertainty $\epsilon_n = \epsilon_0 2^n \geq 1$ when $n \geq \ln(1/\epsilon_0)/\ln 2$. *Note*: $\epsilon_n = \epsilon_0 2^n \implies$ Lyapunov exponent $\lambda = \ln 2 > 0$.
5. $X = 0$ asymptotically stable if $|a| < 1$. $X = \pm\sqrt{a-1}$ asymptotically stable when $|3-2a| < 1$, or $1 < a < 2$. At $a = 2$, there is period-doubling on each branch, followed by a period-doubling cascade.

7. (a) Both $|\lambda_1|, |\lambda_2| < 1 \implies |b| < 1$ and also $|2aX| < 1 - b$. (X_-, Y_-) is always unstable; (X_+, Y_+) is asymptotically stable if $a < \frac{3}{4}(1-b)^2$.
(b) Nontrivial 2-cycle $\implies X_i$ roots of
$a^2X^2 - a(1-b)X + (1-b)^2 - a = 0$, with $Y = b(1 - aX^2)/(1-b)$; real roots if $a > \frac{3}{4}(1-b)^2$. Eigenvalues λ of M_1M_2 satisfy
$\lambda^2 - (4a^2X_1X_2 + 2b)\lambda + b^2 = 0 \implies$ for asymptotic stability $|b| < 1$ and $|4(1-b)^2 - 4a + 2b| < 1 + b^2 \implies a < (1-b)^2 + \frac{1}{4}(1+b)^2$ [using an argument similar to that in the Hint].
8. $x_n = X + \xi$, $y_n = Y + \eta \implies x_{n+1} = 1 - aX^2 + Y + \bar{\xi}$, $y_{n+1} = bX + \bar{\eta}$, with $\bar{\xi} = -2aX\xi + \eta$, $\bar{\eta} = b\xi$. Circle $\xi^2 + \eta^2 = \epsilon^2$ becomes
$[\bar{\xi} + (2aX/b)\bar{\eta}]^2 + \bar{\eta}^2/b^2 = \epsilon^2$, ellipse with semi-axes $\epsilon/\sqrt{\mu_1}$, $\epsilon/\sqrt{\mu_2}$ where $\mu_1\mu_2 = 1/b^2$, $\mu_1 + \mu_2 = 1 + (1 + 4a^2X^2)/b^2$. Area of ellipse $= \pi|b|\epsilon^2 \implies$ area reduction. $(1-\mu_1)(1-\mu_2) < 0 \implies$
$0 < \mu_1 < 1 < \mu_2$ (say) $\implies$ Lyapunov exponents $\lambda_1 > 0 > \lambda_2$.
9. (a) Fixed points $P_1 = (1 + a - b)^{-1}(1, b)$ if $a > b - 1$,
$P_2 = (1 - a - b)^{-1}(1, b)$ if $a > 1 - b$. If $|b| < 1$ and $|a| < 1 - b$ only P_1 exists, $\lambda - b/\lambda = -a \implies P_1$ asymptotically stable.
(b) For nontrivial 2-cycle $Q_1 = (X_1, Y_1) \rightleftarrows Q_2 = (X_2, Y_2)$, X_1, X_2 must be of opposite sign. Hence $X_{1,2} = (1 - b \mp a)/[a^2 + (1-b)^2]$,
$Y_{1,2} = bX_{2,1}$.
11. (a) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \xi \\ Y + \eta \end{pmatrix} \implies \begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} \implies$ eigenvalues
$\mu_{1,2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{5} \implies$ Lyapunov exponents $\lambda_{1,2} = \ln \mu_{1,2} = \pm \ln 2.618 = \pm 0.9624$ — stretch and squeeze. See diagram:



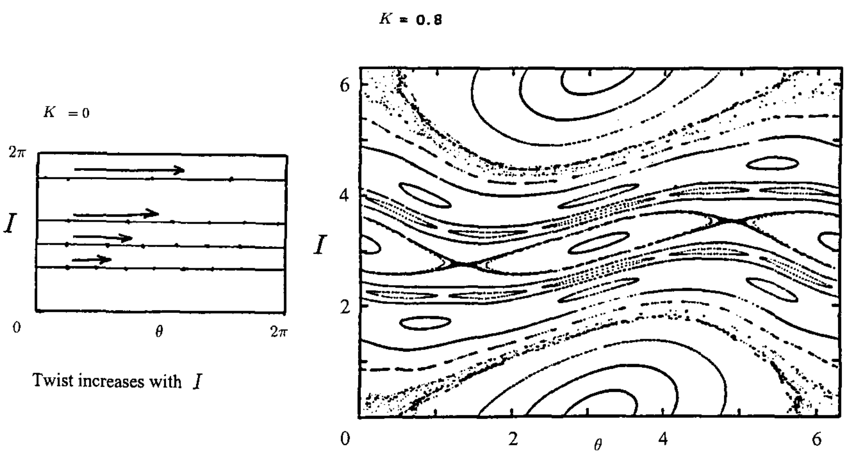


(b) eigenvalues $\mu_1 = 2$, $\mu_2 = \frac{1}{2} \implies \lambda_{1,2} = \pm \ln 2 = \pm 0.6931$. See diagram:



'Stretch, cut and stack'

12. (a) α, β distinct irrationals with $\beta - \alpha > \epsilon > 0$, then choose integer N such that $\epsilon > 1/N$; mesh integer multiples of $1/N$, at least one is between α and β. (b) Since $\sqrt{2}$ is irrational, then *e.g.*, $(1 - 1/\sqrt{2})(p_1/q_1) + (1/\sqrt{2})(p_2/q_2)$ is irrational and lies between the rationals p_1/q_1 and p_2/q_2.
13. We may take both θ *and* I to be 2π-periodic here. See diagram:



[from Reinhardt and Dana, *Proc. Roy. Soc.* **413**, 157–170, 1987]

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Note: Page numbers in italics refer to items in the problems.