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Bifurcations and averages in the logistic map

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Abstract. Bifurcations in the logistic map are studied numerically by inspection of the average of a long iterated series. The tangent bifurcations, at the entrance of the periodic windows, and the crisis, when chaotic bands merge, show distinct power law scalings for these statistical properties.

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Among the one-dimensional maps, the logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

is probably the one most frequently used in applications (Ott 1993, and references therein) and its dynamical bifurcations have been extensively explored. It is well known, as depicted in figure 1(a), that as r varies from 0 to 4 this map iterated with initial condition $x_0 \in (0,1)$ shows cascades of period-doubling bifurcations, chaos, tangent bifurcations back into periodic windows, further period-doubling cascades to chaos with chaotic bands merging crisis until $r = 4$ where the map becomes conjugate to the tent map.

The period-doubling cascades in quadratic maps have universal scaling law for their Lyapunov exponents near criticality (Huberman and Rudnick 1980, Claiborne Johnston and Hilborn 1988). The tangent bifurcations were also characterized by scaling laws for the duration of the intermittent nearly periodic iterations, as r approaches the threshold value to enter the periodic window (Bergé *et al.* 1984). In recent work, Jacobs *et al.* (1997) have addressed the scaling laws for the chaotic transients in one parameter one-dimensional maps (Ott 1993, chapter 5). All these, and many other authors, study the scaling laws for the transient time or the Lyapunov exponents near bifurcations, associated with trajectories on the attractors of the system.

Herein we present a numerical study of the bifurcations in the logistic map as manifested in the average value of x_n:

$$\bar{x} = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=0}^{N-1} x_n\right]$$

Such a quantity is the simplest statistical property that one can inspect in a time series. Yet, as can be seen in figure 1(b), this average does contain clear evidence of

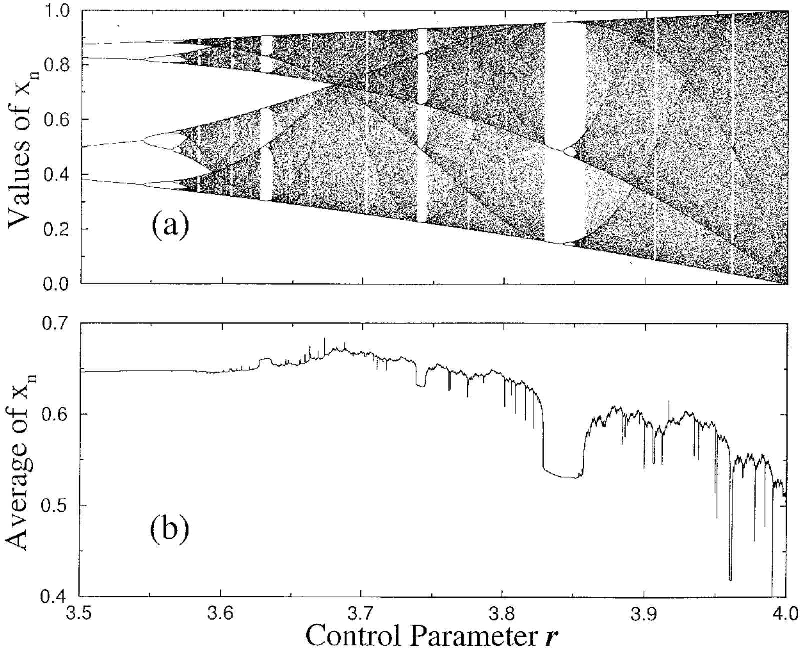

Figure 1. Bifurcation diagrams of the logistic map. (*a*) The usual diagram showing 100 values of x_n obtained after 10^7 iterations. The steps on r were $\delta r = 2 \times 10^{-4}$ and at each value the initial condition was $x_0 = 0.5$. (*b*) Bifurcation diagram showing the average of 10^7 iterations. The initial condition and the steps on the control parameter are as in (*a*).

the bifurcations occurring in the dynamical system. Scaling exponents for this average (and also for the statistical variance) have been identified at the tangent bifurcations and at the chaotic bands merging crisis.

The numerical averages were done by adding 10^7 iterates of x_n, taking $x_0 = 0.5$ as initial value at each step of the control parameter r. Tests with nearby different initial values did not give observable difference on the averages. A double precision algorithm running in a 32 bits microcomputer calculated at least 1200 steps of r. Segments of the average with steps $\delta r \leqslant 5 \times 10^{-4}$ were calculated at regions of bifurcations in the map. The cascade of period-doubling leading to chaos at $r_\infty = 3.569946\ldots$ has an average behavior as shown in figure 2. No manifestation of the bifurcation is obtained on $\bar{x}$. The same is observed crossing $r_0 = 3.856805\ldots$ when the three chaotic bands merge in a single band. On the other hand, the averages at tangent bifurcations entering the periodic windows clearly show the bifurcations, as seen in figures 4 and 5, for periods 3, 5 and 7, respectively. Transients in the calculations, whose length increase as one approaches the critical values of r (Huberman and Rudnick 1980, Claiborne Johnston and Hilborn 1988, Jacobs *et al.* 1997) had to be longer than 10^4 iterates to affect the numerical average results within the third decimal place. Rounding up in the iteration value by the computer (Sauer *et al.* 1997) seems to have prevented the occurrence of averages on unstable periodic orbits. This must also be the reason for robustness of the calculations with respect to initial condition. However, narrow windows of stable periodicity, which are known to

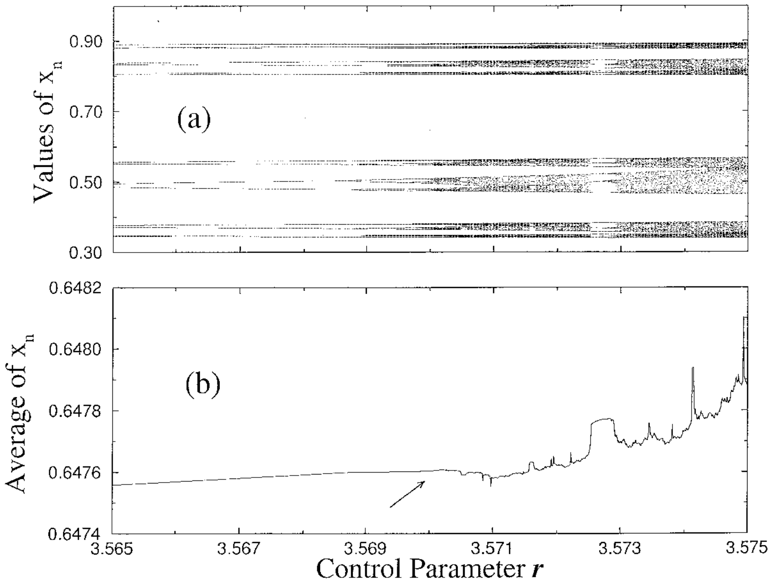

Figure 2. Bifurcation diagrams showing a blow-up of the convergence of the period-doubling cascade at $r_\infty = 3.569946\ldots$. (*a*) One hundred values of x_n, saved after 10^7 iterations initiated at $x_0 = 0.5$ and calculated at steps of $\delta r = 10^{-5}$. (*b*) The average of 10^7 iterates of the map. Despite the existence of an infinite number of periodic orbits such a bifurcation, pointed by the arrow, has no significant effect on the average. Beyond the cascade bifurcation one sees further bifurcations, similar (reversed) to those of figure 1(*b*). This is evidence of the self-similarity of the map.

cover the 0–4 interval in a dense manner (Ott 1993: 42), do appear when their width in r is within the numerical precision.

In searching for critical exponents, attempts were made to fit the chaotic numerical averages to the expression

$$\bar{x}(r) = \bar{x}_i + A_i\{1 - \exp[-\xi_i(r_i^c - r)^{\nu_i}]\}$$

where $\bar{x}_i$ is the average at the onset of the periodic window i which begins at r_i^c. These values were introduced in the equation with six decimal places as read from the numerical results. The constants A_i, ξ_i and the expected critical exponent ν_i were best fitted over the numerical data. The heavy line shown in figure 6 corresponds to a fit with $\nu_3 \simeq 0.47$. Similar fittings gave $\nu_5 \simeq 0.38$ and $\nu_7 \simeq 0.40$, for windows 5 and 7, respectively. Using data intervals closer to the bifurcation gives different ν_i, compatible with the value 0.5 but one cannot definitively conclude that the averages approaching tangent bifurcations follow a polynomial critical exponent dependence, $(r_i^c - r)^{\nu_i}$. The tangent bifurcations into periodic windows of even periods 6, 10 and so on, before the bands merge at $r_0 = 3.674\ldots$ have the same behavior as their respective counterpart with periods 3, 5, etc. The chaotic crisis of bands merging after the period 3 window is shown in figure 7. There, the best fit curve for the average gave a value $\nu \simeq 0.5$.

Another property of the averages numerically investigated was their maxima and minima, as a function of r. Unable to find analytical predictions for these extrema in

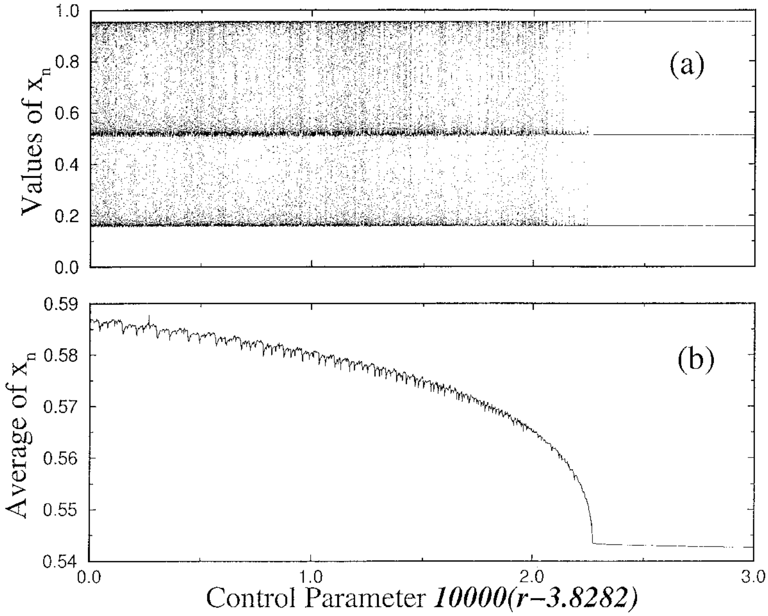

Figure 3. Bifurcation diagrams at the tangent bifurcation entering the period 3 window at $r_3^c = 3.82841\ldots$. (*a*) Plot of 100 values of x_n, saved after 10^7 iterations initiated at $x_0 = 0.5$ and calculated at steps of $\delta r = 2.5 \times 10^{-7}$. (*b*) The average of 10^7 iterates of the map as in 3(*a*).

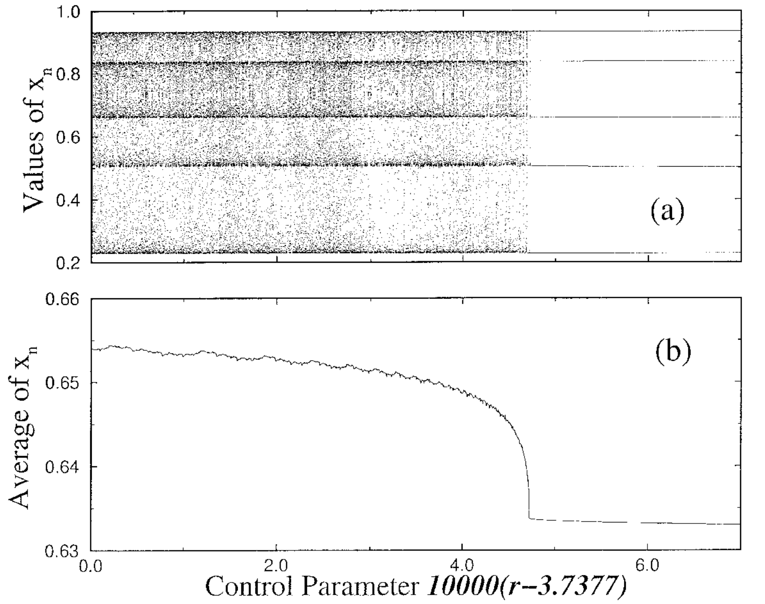

Figure 4. Bifurcation diagrams at the tangent bifurcation entering the period 5 window at $r_5^c = 3.73817\ldots$. (*a*) Plot of 100 values of x_n, saved after 10^7 iterations initiated at $x_0 = 0.5$ and calculated at steps of $\delta r = 6 \times 10^{-7}$. (*b*) The average of 10^7 iterates of the map as in (*a*).

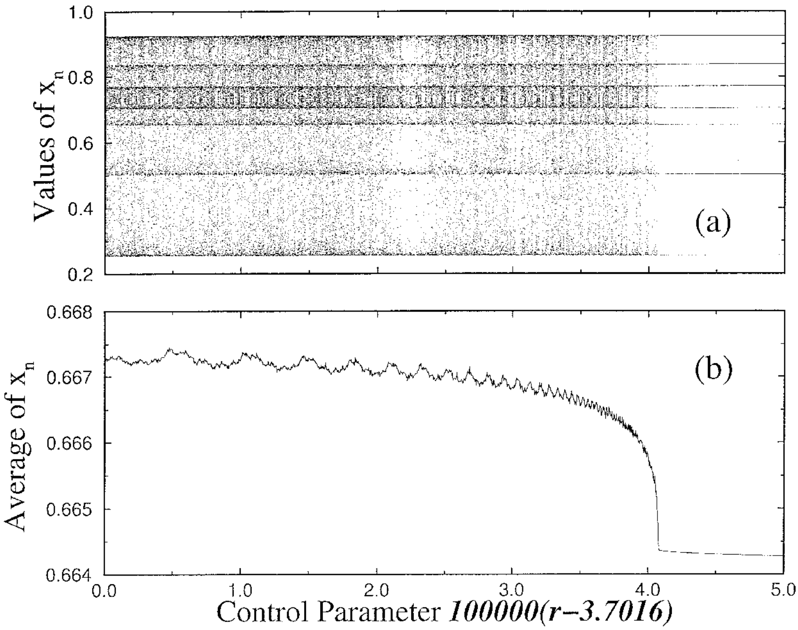

Figure 5. Bifurcation diagrams at the tangent bifurcation entering the period 7 window at $r_3^c = 3.70164\ldots$. (*a*) Plot of 100 values of x_n, saved after 10^7 iterations initiated at $x_0 = 0.5$ and calculated at steps of $\delta r = 4 \times 10^{-8}$. (*b*) The average of 10^7 iterates of the map done as in (*a*).

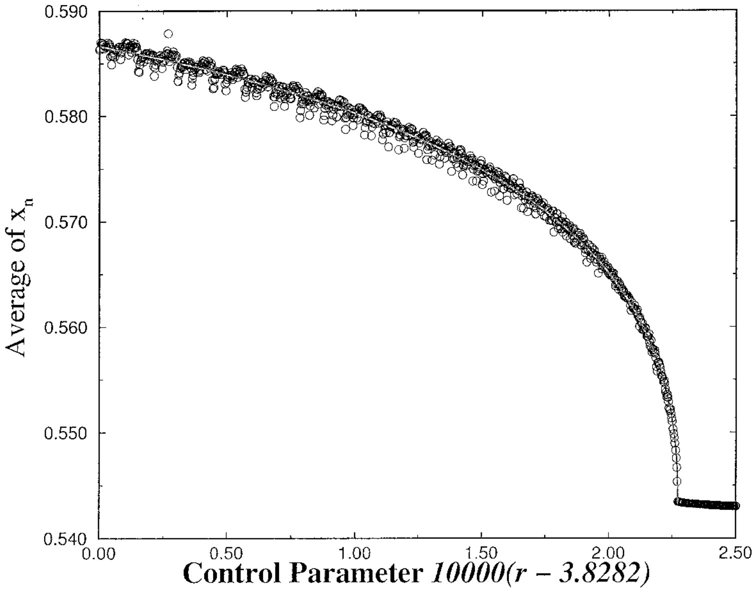

Figure 6. Fitting of the bifurcation diagram of the average at the tangent bifurcation entering the period 3 window at $r_3^c = 3.82841\ldots$. The data are the same as in figure 3(*b*). The heavy line was best fitted with equation (3) of the text and corresponds to a dependence $\bar{x} \sim (r_i^c - r)^{\nu_i}$ with $\nu_3 \simeq 0.47$.

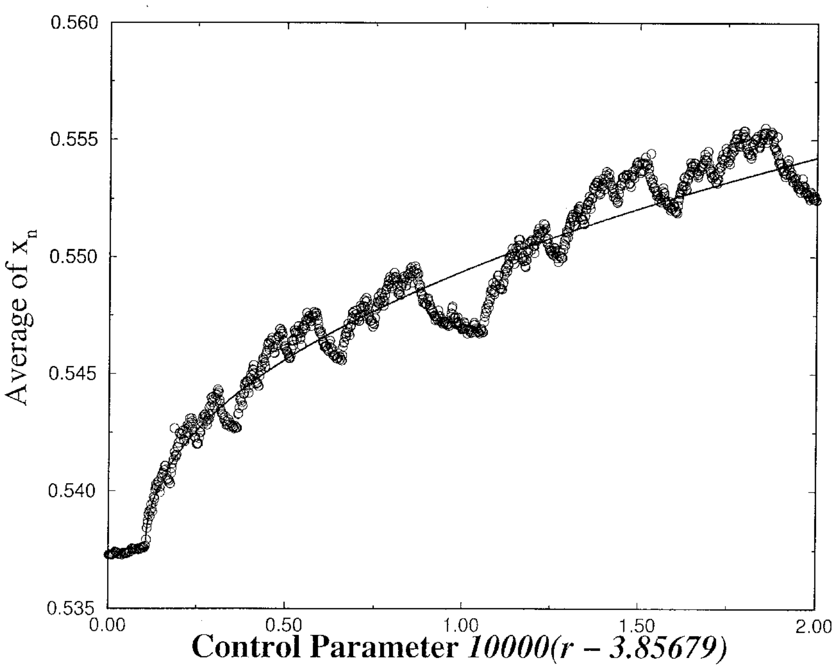

Figure 7. Bifurcation diagram at the crisis of chaotic bands merging after the period 3 window at $r_3^c = 3.82841\ldots$. The averages of 10^7 iterations, each one initiated at $x_0 = 0.5$, were calculated at steps of $\delta r = 1.75 \times 10^{-7}$. The heavy line is a fitting of equation (3) that gave $\nu_c \simeq 0.497\ldots$.

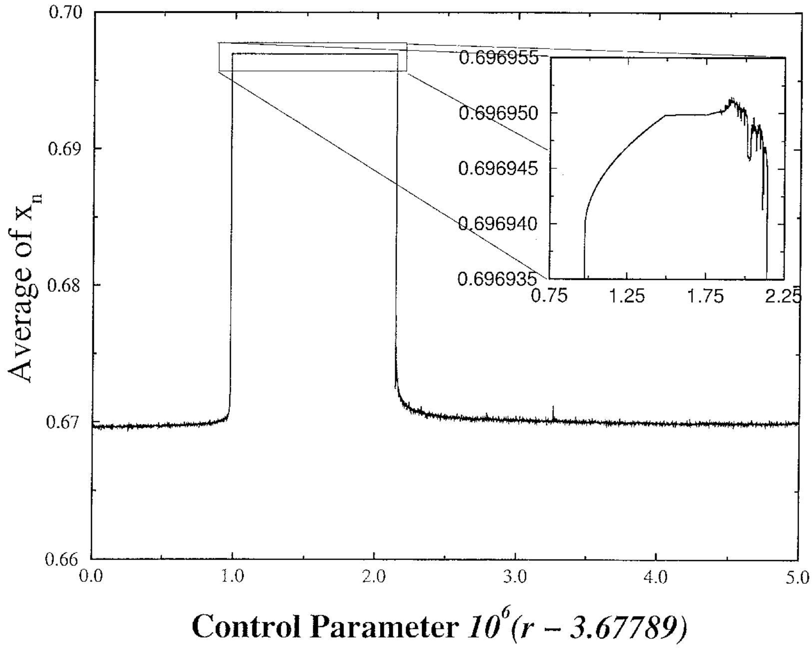

Figure 8. Bifurcation diagrams of the average around its probable maximum value. The average was calculated over 10^7 iterations, each one initiated at $x_0 = 0.5$ with steps $\delta r = 5 \times 10^{-10}$.

the literature we scanned the range $3 < r < 4$. The maximum of $\bar{x}$ found was $0.696951\ldots$, within a periodic window beginning at $r = 3.677891\ldots$, as shown in figure 8. The end of this window, re-entering chaos as r increases, reproduces figure 1, with the expected self-similarity of the map. In principle one should be able to find the value of the maximum average with as many decimal places as wished. Analogous results are valid for the window of minimum average, found as $\bar{x}_{\min} = 0.28262\ldots$, within a window centred at $r = 3.997585\ldots$. The variance of the iterated logistic map also shows the bifurcations appearing in the averages. One must expect that the observed properties of the average have their explanation on the properties of the natural measure of the map. Such study is beyond the scope of the numerical experiment reported here.

To conclude we mention our original motivation and the relevance of this study for characterization of experimental chaotic systems. When very fast chaotic pulsations occur in a real system one may inspect the average of a dynamical variable using a slow signal detector, sensitive to the average. Such procedure is common in measuring the average power of periodically pulsed lasers and electronic devices. It was used in a preliminary experiment on chaotic lasers by Oliveira-Neto and collaborators (Oliveira-Neto *et al.* 1996). It is our intention to continue studying the applications of the averages to identify bifurcations and associated dynamical properties of chaotic attractors in real electronic circuits and lasers (Cavalcante and Rios Leite 1999).

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