marica.pelanti@ensta-paris.fr

Introduction to CFD
MF206 ENSTA Paris
2020-2021

Final Assignment

Finite Volume Methods for the 1D Shallow Water Equations

Instructions

- **Due date: Wednesday, April 7, 2021 (23:59)**
- Upload your electronic report on Moodle by the due date. **Do not send me a copy by e-mail.**
- You can write your work in English or in French.
- Include your codes in your report or relevant portions.
- Be sure to provide a complete description of your plots (variable plotted, time at which it is plotted, method used, number of cells used). Recall that you can print .png figures in Matlab by using the command `print -dpng figure.png`.
- You have to work independently.

1 1D Shallow Water equations

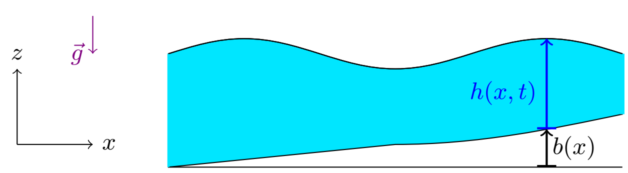

Figure 1: Shallow water flow configuration.

The shallow water equations (or Saint-Venant equations) model a free surface incompressible inviscid flow under the assumption $\frac{H}{L} \ll 1$, where H is a characteristic flow height and L a characteristic horizontal length. The equations are obtained by depth-averaging the Navier–Stokes equations under the considered hypotheses. Here we consider a one-dimensional flow over a topography $b(x)$ (see Figure 1). The 1D shallow water equations can be written in the following conservative form:

$$\frac{\partial U}{\partial t} + \frac{\partial \mathcal{F}(U)}{\partial x} = \Psi\,, \tag{1a}$$

where

$$U = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \mathcal{F}(U) = \begin{bmatrix} hu \\ hu^2 + g\frac{h^2}{2} \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 \\ -gh\frac{\partial b}{\partial x} \end{bmatrix}. \tag{1b}$$

Here $h(x,t)$ is the flow depth, $u(x,t)$ the flow velocity in the x direction, $b = b(x)$ denotes the bottom topography, and g is the gravity constant. Note that the free surface level is $h + b$.

2 Homogeneous system

To begin with, we consider the case of flat bottom, $b(x) = 0$.

(a) Consider the quasi-linear form of system (1)

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0. \tag{2}$$

Write the Jacobian matrix $A(U) = \frac{\partial \mathcal{F}(U)}{\partial U}$ $(A_{kj} = \frac{\partial \mathcal{F}_k}{\partial U_j})$. Show that the eigenvalues of this matrix are

$$\lambda_1(U) = u - \sqrt{gh} \quad \text{and} \quad \lambda_2(U) = u + \sqrt{gh}. \tag{3}$$

The corresponding eigenvectors are (you do not have to show this however):

$$r_1(U) = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix} \quad \text{and} \quad r_2(U) = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}. \tag{4}$$

Assuming $h > 0$ the eigenvalues are real and distinct and the eigenvectors are linearly independent, hence the system of equations is strictly hyperbolic. We can solve it numerically by using the techniques for hyperbolic conservation laws. Note that the addition of a non-zero bottom topography source term $b(x)$ does not change the nature of the equations. Show that the two characteristic fields are genuinely nonlinear, that is $\nabla\lambda_k(U) \cdot r_k(U) \neq 0$, $k = 1, 2$, $\forall U$ in the set of states U with $h > 0$. Consider now a Riemann problem for the shallow water equations with left and right states U_ℓ, U_r, respectively, assuming $h_\ell, h_r > 0$ (no dry states $h = 0$). What is the possible wave structure of the solution of this Riemann problem, knowing that the two characteristic fields are genuinely nonlinear?

2.1 Dam-break problem

We wish to simulate a dam-break problem, whose initial conditions correspond to a Riemann problem. An analytical solution can be derived. A dam at $\bar{x}$ initially separates two uniform regions, which have different water heights, with higher height in our test in the left region. The initial velocity is assumed equal to zero in both regions. At time $t > 0$ the dam is abruptly broken. This generates a shock wave (hydraulic jump) propagating to the right and a rarefaction wave propagating to the left. Specifically, we consider here the following initial condition, in terms of the variables $W = (h, u)$:

$$W(x,0) = \begin{cases} W_\ell = (3,0) & \text{if } x \leq \bar{x}\,, \\ W_r = (1,0) & \text{if } x > \bar{x}\,. \end{cases} \tag{5}$$

The flow variables are nondimensionalized. The problem is considered over the space interval $[-4, 4]$ and the dam location is $\bar{x} = 0$. The gravity constant is set to $g = 1$. We will consider free flow (transmissive) boundary conditions both on the left and on the right of the considered interval. Figure 2 shows the exact solution of the problem at $t = 1.2$.

2.2 Discretization

We consider a uniform discretization of the physical space interval $[-4, 4]$ with N_C cells. The ith cell, $i \in \mathbb{Z}$, is centered at x_i and covers the interval $[x_{i-1/2}, x_{i+1/2}]$. We denote with Δx the grid spacing $\Delta x = x_{i+1} - x_i = x_{i+1/2} - x_{i-1/2}$, see Figure 3. Moreover, two additional ghost cells are considered outside the physical domain, to the left and to the right of the interval limits, hence the total number of cells is $N_C + 4$. The ghost cells are used to impose boundary conditions. We numerically approximate the hyperbolic system of conservation laws (1) with $b(x) = 0$ by using three-point finite volume conservative schemes of the form:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}[F(U_i^n, U_{i+1}^n) - F(U_{i-1}^n, U_i^n)]\,, \tag{6}$$

where Δt is the time step and $n \in \mathbb{N}$ denotes the time level. The numerical solution U_i^n represents an approximation of the average of the true solution at t^n over the ith cell interval $[x_{i-1/2}, x_{i+1/2}]$.

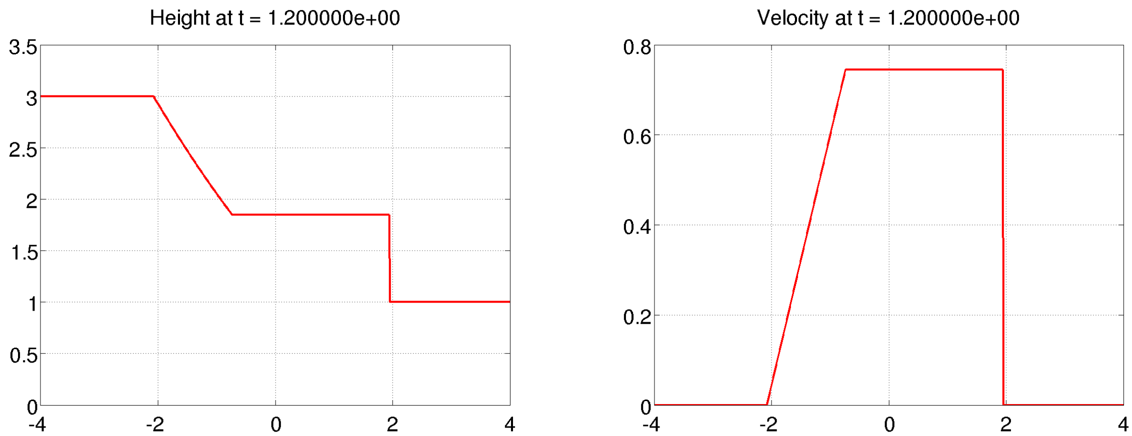

Figure 2: Exact solution of the dam-break problem (5) at $t = 1.2$. Left: flow height h; Right: velocity u.

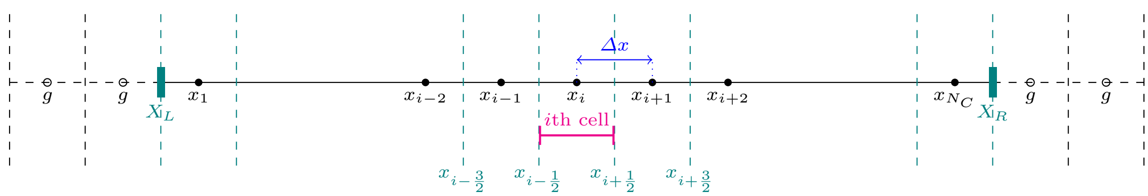

Figure 3: Uniform Finite Volume grid with N_C cells over an interval $[X_L, X_R]$. Two additional ghost cells are considered on the left and on the right of the interval limits.

(b) Solve the given dam-break problem by employing the Rusanov method and the Roe method, which have the conservative form (6). The numerical fluxes of these methods are defined in the Appendix. For the implementation you can use the Matlab script `numlessfin_sw_templ21.m`, which already defines the finite volume grid and the initial and boundary conditions, and the Matlab function `fluxswRSn_templ.m` to have an idea on how you can define the numerical flux. The Matlab script `numlessfin_sw_templ21.m` also calls a function `fswrpex.m` that computes the exact solution for this problem at a given time (if the parameter `iexact = 1`). If you run this script you will see the exact solution as plotted in Figure 2.

Run simulations with the two methods until a final time $t_\mathrm{f} = 1.2$ by taking $N_C = 200$ and $N_C = 1000$ grid cells and a fixed time step Δt, with ratio $\frac{\Delta t}{\Delta x} = 0.4$. Compare in the same plots the water height h and the velocity u computed by the two methods and the exact solution. Comment your results.

(c) Verify that the physically conserved quantities h (height) and hu (momentum) are conserved at the discrete level by the finite volume scheme (you can choose Rusanov or Roe flux to check this). To this aim, consider the (vectorial) integral form of the system of conservation laws (1) with no source term Ψ over the considered space interval $[x_1, x_2] = [-4, 4]$ and the time interval $[0, t_\mathrm{f}]$:

$$\int_{x_1}^{x_2} U(x, t_\mathrm{f}) dx = \int_{x_1}^{x_2} U(x, 0) dx + \int_0^{t_\mathrm{f}} \mathcal{F}(U(x_1, t)) dt - \int_0^{t_\mathrm{f}} \mathcal{F}(U(x_2, t)) dt \,. \tag{7}$$

This gives an expression of the amount of $U = [h, hu]^\mathrm{T}$ over the interval $[x_1, x_2]$ at time t_f in terms of the amount of U at time $t = 0$ and the total flux at each boundary during this time period. Compute the exact values of the integrals $\int_{x_1}^{x_2} h(x, t_\mathrm{f}) dx$ and $\int_{x_1}^{x_2} (hu)(x, t_\mathrm{f}) dx$ by using (7) and check that the integrals calculated with the discrete values of the numerical solution ($\Delta x \sum_{i=1}^{N_C} h_i$ and $\Delta x \sum_{i=1}^{N_C} (hu)_i$) agree with the exact ones.

3 System with topography source term

We now consider the case of variable bottom topography, $b(x) \neq 0$.

(a) Show that the stationary states at rest ($u = 0$, $\frac{\partial(\cdot)}{\partial t} = 0$) are defined by the condition

$$h + b = \text{constant}, \tag{8}$$

which expresses the fact that the water level is horizontal (so-called "lake at rest" condition).

One difficulty in the numerical solution of the shallow water equations with bottom topography is the need to guarantee the preservation of the condition (8) at the discrete level, that is we need to ensure that if at time t^n we have $u_i = 0$, $h_i + b_i = \text{constant}\ \forall i$, then these conditions are maintained at time level t^{n+1}. Numerical schemes able to guarantee this property are called *well-balanced*. Here we use the well-balanced Hydrostatic Reconstruction scheme described in [1, 2]. This Finite Volume scheme has the form:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left[\tilde{F}_\ell(U_i^n, U_{i+1}^n, b_i, b_{i+1}) - \tilde{F}_r(U_{i-1}^n, U_i^n, b_{i-1}, b_i)\right], \tag{9}$$

where b_i are discrete values of the bottom topography b associated to the cell $[x_{i-1/2}, x_{i+1/2}]$ centered in x_i, as the variables U_i, and the numerical fluxes are defined as:

$$\tilde{F}_\ell(U_L, U_R, b_L, b_R) = F(U_L^*, U_R^*) + \begin{bmatrix} 0 \\ \frac{1}{2}g(h_L^2 - {h_L^*}^2) \end{bmatrix}, \tag{10a}$$

$$\tilde{F}_r(U_L, U_R, b_L, b_R) = F(U_L^*, U_R^*) + \begin{bmatrix} 0 \\ \frac{1}{2}g(h_R^2 - {h_R^*}^2) \end{bmatrix}. \tag{10b}$$

Above $F(U_L^*, U_R^*)$ is a consistent numerical flux for the system with no bottom source term ($b(x) = 0$), that is $F(U, U) = \mathcal{F}(U)$, as for instance the Rusanov and Roe numerical fluxes in the Appendix. The so-called reconstructed states U_L^* and U_R^* are defined as

$$U_L^* = (h_L^*, h_L^* u_L) \quad \text{and} \quad U_R^* = (h_R^*, h_R^* u_R), \tag{10c}$$

where

$$h_L^* = (h_L + b_L - b^*)_+ \quad \text{and} \quad h_R^* = (h_R + b_R - b^*)_+ \tag{10d}$$

where $b^* = \max(b_L, b_R)$, and $(\cdot)_+$ denotes the positive part, $(\phi)_+ = \max(0, \phi)$.

(b) Show that the scheme defined above preserves the stationary states with zero velocity $u = 0$ (Hint: assume $u_L = u_R = 0$, $h_L + b_L = h_R + b_R$, and show that $U_L^* = U_R^*$, then $\tilde{F}_\ell = \mathcal{F}(U_L)$, $\tilde{F}_r = \mathcal{F}(U_R)$).

(c) Implement the scheme (9)-(10) by using the Rusanov numerical flux for the flux $F(U_L^*, U_R^*)$. Apply the scheme to the problem specified hereafter (taken from [3]) over the interval $[0, 1]$ with bottom topography defined by $b(x) = 0.25(\cos(\pi(x - 0.5)/0.1) + 1)$ if $|x - 0.5| < 0.1$, and $b(x) = 0$ otherwise. The initial condition consists of an initial perturbation of a steady condition defined by $h + b = 1$, $u = 0$. A perturbation of the water height $h(x, 0) = 1 + \varepsilon$ is taken for $0.1 < x < 0.2$, with $\epsilon = 0.2$. The flow is initially at rest, $u(x, 0) = 0$. Use free flow (transmissive) boundary conditions on both sides of the spatial interval. Compute the solution with $N_C = 200$ grid cells and fixed Δt with $\frac{\Delta t}{\Delta x} = 0.8$. Plot results for the water height at times $t = 0.1$, $t = 0.4$, $t = 0.7$, and $t = 2$. At $t = 0.1$ you should observe that the initial water bump splits into two waves going in opposite directions. The left-going wave soon exits from the left edge of the interval. At $t = 2$ you should observe approximately steady conditions. Then, run a simulation with $N_C = 1000$ grid cells and plot results at $t = 0.7$. Figure (4) is taken from [3], and it shows the initial condition of the problem and the solution at $t = 0.7$. For the implementation, you can use the given Matlab script `numlessfin_swb_templ21.m`. This script already implements the initial and boundary conditions. Finally, in [3] you can see the numerical difficulties occurring if you solve a problem of this type with a scheme that is not well-balanced.

(d) Now use the scheme with the Rusanov flux that you implemented for the previous problem to solve the oscillating lake problem with dry regions proposed in [1]. In this test we consider a lake over a topography defined by $b(x) = 0.5(1 - 0.5\cos(\pi(x - 0.5)/0.5) + 1))$ over the interval $[0, 1]$. The lake is initially at rest but with a sinusoidal perturbation of the free surface $\eta(x) = 0.04\sin((x - 0.5)/.25)$ centered at $(0, 0.4)$. See the initial condition in Figure 5. Then the flow oscillates. To set the initial condition just set `iproblb = 2` in the given Matlab script `numlessfin_swb_templ21.m`. Run a simulation until a final time $t_f = 19.8$, at which the flow height reaches approximately a maximum level at the left shore. Use $N_C = 200$ and $\frac{\Delta t}{\Delta x} = 0.8$. Note that in this problem we have dry regions with $h = 0$ (vacuum regions). For most applications of the shallow water equations it is important to be able to simulate problems with dry regions. The Rusanov method can handle data with vacuum since it is able to preserve the non-negativity of the flow height at the discrete level. In contrast, Roe's scheme may fail in problems involving dry states, since it might compute negative values of the flow height.

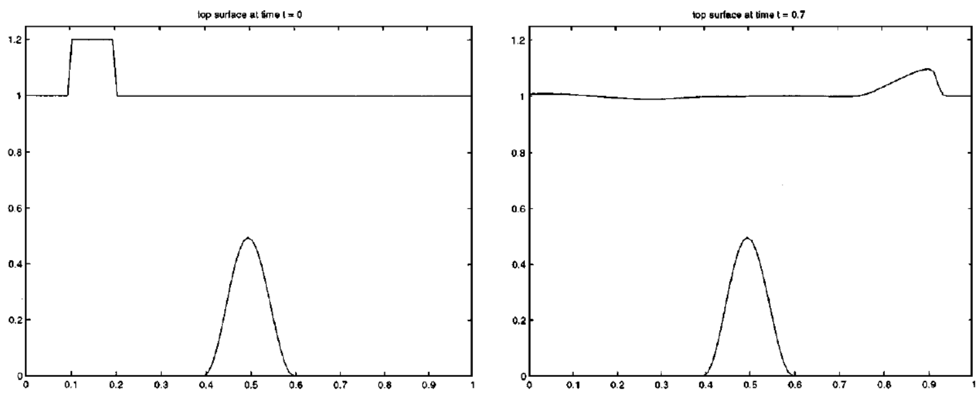

Figure 4: Figure from [3]. Initial condition and solution at $t = 0.7$ for the problem described in Section 3.

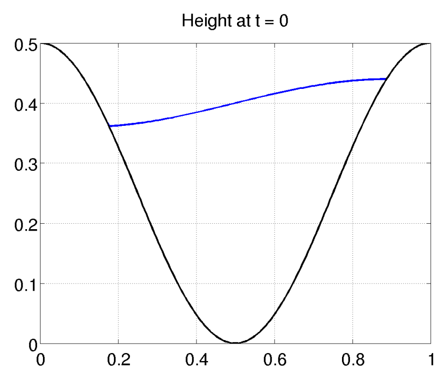

Figure 5: Initial condition of the oscillating lake problem [1].

A Rusanov scheme

The numerical flux of the Rusanov scheme is:

$$F(U_i, U_{i+1}) = \frac{1}{2}[\mathcal{F}(U_i) + \mathcal{F}(U_{i+1}) - S_{i+1/2}(U_{i+1} - U_i)]\,, \tag{11a}$$

where $S_{i+1/2}$ denotes the local maximum absolute value of characteristic speeds, hence for the shallow water equations:

$$S_{i+1/2} = \max(|u_{i+1} + \sqrt{gh_{i+1}}|, |u_{i+1} - \sqrt{gh_{i+1}}|, |u_i + \sqrt{gh_i}|, |u_i - \sqrt{gh_i}|). \tag{11b}$$

B Roe scheme

The numerical flux of the Roe's scheme is:

$$F^{\mathrm{Roe}}(U_i, U_{i+1}) = \mathcal{F}(U_i) + \sum_{k=1}^{2} (\hat{\lambda}_k^- \alpha_k \hat{r}_k)_{i+1/2} \,, \tag{12}$$

or equivalently

$$F^{\mathrm{Roe}}(U_i, U_{i+1}) = \mathcal{F}(U_{i+1}) - \sum_{k=1}^{2} (\hat{\lambda}_k^+ \alpha_k \hat{r}_k)_{i+1/2} \,, \tag{13}$$

where we have used the notation $\lambda^- \equiv \min(\lambda, 0)$ (negative part of λ) and $\lambda^+ \equiv \max(\lambda, 0)$ (positive part of λ). Note that we can also obtain a centered expression of the flux by taking $\frac{1}{2}$ of the sum of the two expressions (12) and (13):

$$F^{\mathrm{Roe}}(U_i, U_{i+1}) = \frac{1}{2}(\mathcal{F}(U_i) + \mathcal{F}(U_{i+1})) - \frac{1}{2} \sum_{k=1}^{2} (|\hat{\lambda}_k| \alpha_k \hat{r}_k)_{i+1/2} \,. \tag{14}$$

Above, $(\hat{\lambda}_k)_{i+1/2}$ and $(\hat{r}_k)_{i+1/2}$, $k = 1, 2$, are the eigenvalues and the eigenvectors of the Roe matrix $\hat{A}_{i+1/2} = \hat{A}_{i+1/2}(U_i, U_{i+1})$ (associated to the interface $i + 1/2$), and $(\alpha_k)_{i+1/2}$ are the coefficients of the projection $U_{i+1} - U_i = \sum_{k=1}^{2} (\alpha_k \hat{r}_k)_{i+1/2}$. For the shallow water equations the Roe matrix is defined as $\hat{A} = A(\hat{U}_{\mathrm{Roe}})$, $\hat{U}_{\mathrm{Roe}} = \hat{U}_{\mathrm{Roe}}(\hat{h}, \hat{u})$, with, for the interface $i + 1/2$,

$$\hat{h}_{i+1/2} = \frac{1}{2}(h_i + h_{i+1}) \quad \text{and} \quad \hat{u}_{i+1/2} = \frac{\sqrt{h_i}\, u_i + \sqrt{h_{i+1}}\, u_{i+1}}{\sqrt{h_i} + \sqrt{h_{i+1}}} \,. \tag{15}$$

The Roe eigenvalues and Roe eigenvectors are (omitting here the subscript $i + 1/2$)

$$\hat{\lambda}_1 = \hat{u} - \sqrt{g\hat{h}} \quad \text{and} \quad \hat{\lambda}_2 = \hat{u} + \sqrt{g\hat{h}}, \tag{16}$$

$$\hat{r}_1 = \begin{bmatrix} 1 \\ \hat{u} - \sqrt{g\hat{h}} \end{bmatrix} \quad \text{and} \quad \hat{r}_2 = \begin{bmatrix} 1 \\ \hat{u} + \sqrt{g\hat{h}} \end{bmatrix}. \tag{17}$$

The coefficients $\alpha_{i+1/2} = ([\alpha_1\ \alpha_2]^{\mathrm{T}})_{i+1/2}$ can be obtained by $\alpha_{i+1/2} = \hat{R}^{-1}_{i+1/2}(U_{i+1} - U_i)$, where $\hat{R}$ is the matrix whose columns are the Roe eigenvectors, $\hat{R} = [\hat{r}_1\ \hat{r}_2]$. We can easily obtain these coefficients α_k analytically:

$$\alpha_1 = \tfrac{1}{2\sqrt{g\hat{h}}}((\hat{u} + \sqrt{g\hat{h}})\mathit{\Delta} U^{(1)} - \mathit{\Delta} U^{(2)}), \quad \alpha_2 = \tfrac{1}{2\sqrt{g\hat{h}}}(-(\hat{u} - \sqrt{g\hat{h}})\mathit{\Delta} U^{(1)} + \mathit{\Delta} U^{(2)}), \tag{18}$$

where $\mathit{\Delta} U^{(\xi)}$ is the ξth component of the vector $\mathit{\Delta} U = U_{i+1} - U_i$ (and the quantities α_k, $\hat{h}$, $\hat{u}$ above correspond to the interface $i + 1/2$).

References

[1] E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein, and B. Perthame. A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows. *SIAM J. Sci. Comput.*, 25(6):2050–2065, 2004.

[2] F. Bouchut. *Nonlinear stability of finite volume methods for hyperbolic conservation laws, and well-balanced schemes for sources.* Birkhäuser-Verlag, 2004.

[3] R. J. LeVeque. Balancing source terms and flux gradients in high-resolution Godunov methods: The quasi-steady wave-propagation algorithm. *J. Comput. Phys.*, 146:346–365, 1998.