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Determination of Mandelbrot Set's Hyperbolic Component Centres

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Abstract—In this work we present a very fast and parsimonious method to calculate the centre coordinates of hyperbolic components in the Mandelbrot set. The method we use constitutes an extension for the complex domain of the one developed by Myrberg for the real map $x \mapsto x^2 - p$, in which, given the symbolic sequence of a superstable orbit, the parameter value originating such a superstable orbit is worked out. We show that, when dealing with complex domain sequences, some of the solutions obtained correspond to the centres of the Mandelbrot set's hyperbolic components, while some others do not exist. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

According to Hao and Zheng [1], historically, symbolic dynamics appeared in topological dynamical systems theory during the 30's. Being the only rigorous way to describe the chaotic movement in dynamical systems, its abstract mathematical form is hard to understand and to apply to the study of concrete models. However, in the case of unimodal maps, an applied symbolic dynamics was developed, constituting a valuable tool. The most important study about symbolic dynamics belongs to Metropolis, Stein and Stein [2] (MSS). In a previous work [3], we showed a sketch of the hyperbolic components generation in one dimensional quadratic maps using symbolic dynamics and the MSS algorithm.

Let us consider the real Mandelbrot map $x_{n+1} = x_n^2 + c$. To represent the dynamics of a given orbit, we do not record the exact value of each iterate x_i, but consider simply if it falls to the left (L), to the right (R) or on the critical point (C). Thus, for instance, the symbolic sequence of the period-2 superstable orbit is CLCLCL... When restricted to a single period, it is written simply as CL.

Myrberg [4] proposed a straightforward method which leads to the parameter value when the corresponding symbolic sequence is known. To find, for instance, the c-parameter value of the period-3 superstable orbit with symbolic sequence CLR, it is enough to solve the equation $c = -\sqrt{-c + \sqrt{-c}}$, which can be easily accomplished by iterating (note that the solution is the fixed point of the map $c_{n+1} = -\sqrt{-c_n + \sqrt{-c_n}}$).

In general, the c value of the real Mandelbrot map's parameter originating a real superstable orbit of a given symbolic sequence CLRXX ... X (the first three letters are always CLR for superstable orbits on the real axis) can be obtained by Myrberg's formula [4]:

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$$c = -\sqrt{-c+\sqrt{-c\pm\sqrt{-c\pm\ldots\pm\sqrt{-c}}}}, \tag{1}$$

where the symbol $\pm$ stands for a $+$ sign or a $-$ sign, according to the expression of the orbit's symbolic dynamics, assigning the $+$ sign to the letter R and the $-$ sign to the letter L. The first sign before the square root is always $-$ and corresponds to the first L in the orbit's symbolic dynamics, whereas the second sign is always $+$, corresponding to the first R (CLRXX.. X).

This simple but efficient method, put forward by Myrberg and used later by Kaplan [5], Zheng and Hao [6], Schroeder [7], and others, allows us to obtain, quickly and parsimoniously, the parameter value of an orbit of any period, n, for the real Mandelbrot map. For example, the parameter value $c = -1.78187000761051\ldots$, corresponding to the period-24 orbit of symbolic sequence $\mathrm{CLR^2LRL^2R^2\overline{LR^2}^4LR}$, where $\mathrm{\overline{LR^2}^4 = LR^2LR^2LR^2LR^2}$ can easily be found by applying this method [8].

In this work, we pretend to prove that this method also works for the complex plane extension, i.e., it will be shown that, when applying the method devised by Myrberg for finding the parameter value originating orbits in the complex plane, the centres of the hyperbolic components [9] of the Mandelbrot set are obtained.

2. EXTENSION OF MYRBERG'S METHOD TO THE COMPLEX PLANE

Let us consider now the complex Mandelbrot map

$$z_{n+1} = p_c(z) = z_n^2 + c, \tag{2}$$

where $z,c \in \mathbb{C}$. As is well known, the Mandelbrot set, M, can be defined as the set of c-parameter values for which the orbit of the critical point, $z_0 = 0$, remains bounded, i.e., it does not tend to infinity as n tends to infinity:

$$M = \{c\in\mathbb{C}\ |p_c^n(0)| \not\to \infty \text{ when } n\to\infty\}, \tag{3}$$

where $p_c^n(z)$ represents the n-th iteration of function (2), dependant on the c-parameter.

The real solutions considered in the introduction correspond to the centres of cardioids or disks (hyperbolic components) on the Mandelbrot set's antenna (on the real axis), but what happens when we obtain pairs of complex conjugate solutions after applying Myrberg's method to sequences of letters not corresponding to real orbits? Let us consider, for example, the orbit of symbolic sequence CLL. In this case, we obtain two complex conjugate solutions. When carefully examining these solutions, it is easy to realise that they correspond to the centres of the two biggest disks attached to the main cardioid of the Mandelbrot set (see Fig. 1). Therefore, solving Myrberg's formula for different combinations of signs (different symbolic sequences), we normally obtain pairs of complex conjugate solutions which correspond to centres of cardioids or disks on the set's periphery, although in some cases we cannot obtain a solution because Myrberg's formula does not converge.

In this context, and only as a very simple and useful operative tool, we shall develop an informal way of assigning complex sequences using the location of the orbit relative to the imaginary axis because it serves to our purpose of calculating the centres of the hyperbolic components. Thus, we shall assign the letter L to the iterates falling to the left of the imaginary axis and the R, to the iterates falling to the right. C will represent the map's critical point, $z = 0+0i$. It is important to note that we are aware that sequences of complex numbers do not have a natural order and thus we do not pretend a rigorous description of orbits nor an extension of the kneading theory to the complex case (see the beautiful external rays theory by Douady and Hubbard [10] or [11]), but a useful operative way of describing orbits in order to find

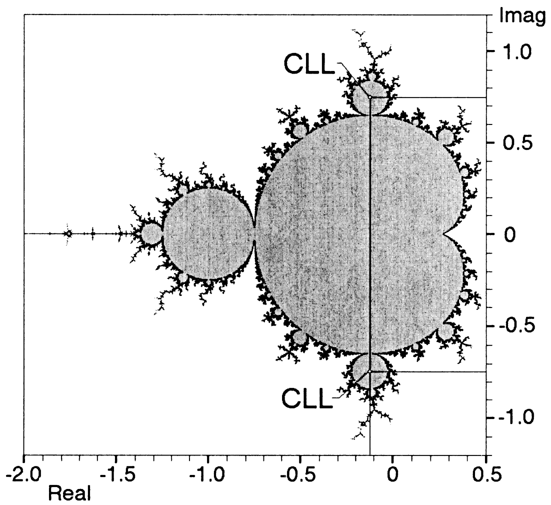


Fig. 1. Mandelbrot set representation, with the disks of complex symbolic sequence CLL whose centres are at $c = -0.1225611669\ldots \pm 0.7448617667\ldots i$.

Table 1. Superstable orbits corresponding to the complex symbolic sequence CLL

c	$-0.1225611669.. + 0.7448617667..\,i$	$-0.1225611669.. - 0.7448617667..\,i$
z_0	$0+0i$	$0+0i$
z_1	$-0.1225611669.. + 0.7448617667..\,i$	$-0.1225611669.. - 0.7448617667..\,i$
z_2	$-0.66235897876.. + 0.562279512088..\,i$	$-0.66235897876.. - 0.562279512088..\,i$

hyperbolic component centres, as will be seen later in this section. Another point to notice is that, with this notation, there always exist two symmetric orbits with respect to the real axis for each symbolic sequence, although it represents no inconvenience for our goal. For example, the symbolic sequence of the superstable period-3 orbits corresponding to the two disks tangent to the main cardioid of the Mandelbrot set (Fig. 1) is CLL for both disks. Hence, a single symbolic sequence gives rise to a pair of complex conjugate centres. In Tab. 1 are listed the values of these disks' centres, along with the orbit followed by the critical point. Therefore, when talking about symbolic sequences in the complex case, we always mean two symmetric complex orbits with respect to the real axis.

By simple combinatory, the maximum possible number of symbolic sequences of period n is 2^{n-1} (for example, for $n = 5$, there exist 16 possible symbolic sequences), although not all of them will exist, as will be seen next. Provided that the first two letters following the C do not need to be L and R, respectively, as in the real case, all the possible $\pm$ signs in Myrberg's formula

$$c_{n+1} = \pm\sqrt{-c_n \pm \sqrt{-c_n \pm \ldots \pm \sqrt{-c_n}}}, \tag{4}$$

should be considered. We call eqn (4) the Myrberg map. As shown in [12], the number of period-n real orbits and the total number of period-n orbits in unimodal maps, both real and complex, can be calculated from a formula devised by Weiss and Rogers [13] and by Lutzky [14]. These

Table 2. Number of real orbits, R_n, and total number of orbits (both real and complex), C_n, depending on the period n

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C_n	1	3	6	15	27	63	120	252	495	1023	2010	4095	8127	16365	32640
R_n	1	1	2	3	5	9	16	28	51	93	170	315	585	1091	2048

results are summarised in Tab. 2, which we have worked out until period 16. R_n is the number of real orbits, and C_n is the total number of orbits, both real and complex. According to Tab. 2, for $n = 5$ there exist 15 superstable orbits, from which 3 are real and consequently 12 must be complex (6 complex conjugated solutions). This means that out of the 16 possible letter combinations to make up the symbolic sequences, only 3 sequences correspond to real orbits (CLR^3, CLR^2L, $CLRL^2$), while 6 sequences correspond to 6 pairs of symmetrical orbits (CLRLR, CL^4, CL^2RL, CL^2R^2, CR^2LR, CR^3L). The remaining other 7 sequences do not exist (CL^3R, CRL^3, CRL^2R, CRLRL, $CRLR^2$, CR^2L^2, CR^4).

A method to work out the Mandelbrot set's hyperbolic component centre coordinates of a given period, consists of calculating the c-values for which the origin (the map's critical point) belongs to the period-n cycles in question: from eqn (2), the values of the centres looked for, $\bar{c}$, can be found as the zeroes of the polynomials

$$p_c^n(0) \equiv p_c^{n-1}(\bar{c}) = 0. \tag{5}$$

As is easily observed, for high periods this method becomes highly costly, requiring a long computing time as n increases. The simple algebraic method developed by Stephenson [12, 15, 16] of constructing exact analytical solutions to the parts of the boundaries of the Mandelbrot set in the form of polynomial maps of the unit circle, becomes almost impracticable for periods $n \geqslant 10$, because the size of the numerical coefficients and the number of terms in the various polynomials soon become too large to handle.

In this discouraging context, we show that the complex solutions obtained from eqn (4), when fed with the symbolic sequences corresponding to superstable points in the interior of whatever hyperbolic components and considering that $c \in \mathbb{C}$ in eqn (4), correspond to the centres of those cardioids and disks, on the Mandelbrot set's periphery. In this way, the extension of Myrberg's formula to the complex plane allows us to easily find the centres of the hyperbolic components of the Mandelbrot set.

Thus, when applying Myrberg's method with $c \in \mathbb{C}$ to a given sequence, where we do not know in advance whether it exists or not, one of three cases takes place:

a) The method converges to a real solution: this means that the orbit with such symbolic sequence exists and is real. The value to which it converged is the centre of the disk or cardioid centred at the Mandelbrot set's antenna.
b) The method converges to a complex solution: this means that we have found one of the two complex conjugated centres, and therefore there are two complex symmetric orbits with such symbolic sequence.
c) The method does not converge to a value, real nor complex. In this case, we do not know if this is due to a non-existent orbit, or to the failure of the method to converge. In the former situation, we know of no method yet that can tell whether a given complex symbolic sequence exists or not. In the next section, we shall see how to overcome the difficulty of a non-converging but existing sequence.

3. EXCEPTIONAL POINTS

In those cases when Myrberg's method does not converge to a solution, we found a way to transform the iterative eqn (4) into an alternative one which does converge. For example, for the period-5 complex symbolic sequence CR^3L, corresponding to a disk centre on the periphery of the Mandelbrot set, the dynamical system:

$$c_{n+1} = f(c_n) = +\sqrt{-c_n+\sqrt{-c_n+\sqrt{-c_n-\sqrt{-c_n}}}} \tag{6}$$

does not converge to the value looked for, $\bar{c} = 0.3795135881\ldots \pm 0.3349323056\ldots\, i$, i.e., $\bar{c}$ is an unstable fixed point of the map (6). As it is well known, the stability of a fixed point is determined by its multiplier, $\lambda_{\bar{c}}$, defined as the absolute value of the derivative of the map's function evaluated at that fixed point, $\bar{c}$.

In system (6), the multiplier, $\lambda_{\bar{c}} = 1.15670052\ldots$, is greater than unity if evaluated at the point $\bar{c}$, corresponding to the disk's centre. Therefore, for this map, $\bar{c}$ represents a repelling fixed point. The way to solve this difficulty consists of studying an equivalent dynamical system that does present an attracting fixed point at $\bar{c}$. Squaring both members in system (6):

$$c^2 = -c+\sqrt{-c+\sqrt{-c-\sqrt{-c}}}$$

and reordering:

$$c(c+1) = +\sqrt{-c+\sqrt{-c-\sqrt{-c}}}$$

we obtain the new dynamical system

$$c_{n+1} = g(c_n) = \frac{+\sqrt{-c_n+\sqrt{-c_n-\sqrt{-c_n}}}}{c_n+1}$$

whose multiplier evaluated at the disk's centre, $\lambda'_{\bar{c}} = 0.331021177\ldots$, is now smaller than unity, thus leading to the desired attracting fixed point $\bar{c}$.

We call this strategy of looking for a new dynamical system whose multiplier is smaller than unity *first order deflation*. If, after applying the first order deflation, the system does not converge to the solution, it can be applied again. For example, let the period-7 complex sequence CR^5L, correspond to a disk centre. The dynamical system used to find the centre is

$$c_{n+1} = +\sqrt{-c_n+\sqrt{-c_n+\sqrt{-c_n+\sqrt{-c_n+\sqrt{-c_n-\sqrt{-c_n}}}}}}, \tag{7}$$

that does not converge to a value. Squaring and reordering (7) leads to:

$$c^2+c = +\sqrt{-c+\sqrt{-c+\sqrt{-c+\sqrt{-c-\sqrt{-c}}}}},$$

whose first order deflation does not converge either to a value. If we square and reorder again, we obtain:

$$c(c^3+2c^2+c+1) = +\sqrt{-c+\sqrt{-c+\sqrt{-c-\sqrt{-c}}}},$$

leading to the new dynamical system

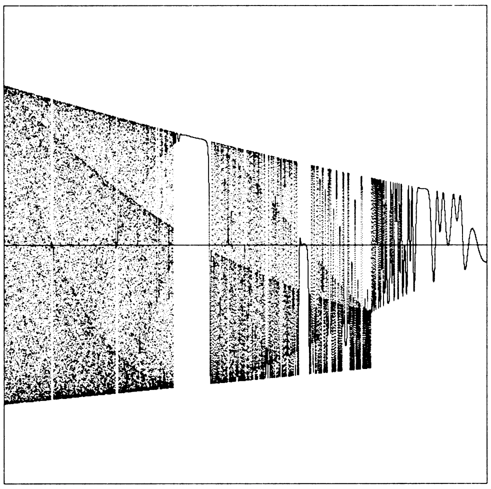


Fig. 2. Real Mandelbrot map. Critical polynomial for $n = 20$.

$$c_{n+1} = \frac{+\sqrt{-c_n+\sqrt{-c_n+\sqrt{-c_n-\sqrt{-c_n}}}}}{c_n^3+2c_n^2+c_n+1},$$

with the same fixed point as (7), but now converging to the disk's centre, $\bar{c} = 0.3760086818\ldots \pm 0.147493712\ldots i$. This procedure is called *second order deflation*.

Deflation can be repeated over and over, so as to obtain a converging formulation of (4). However, the deflation procedure is also limited, as will be discussed in the next section.

4. LIMITATIONS OF THE DEFLATION METHOD

It was shown how the method devised by Myrberg can be used not only for real values of the map's parameter, but for complex values as well. In the latter situation, in some cases when the dynamical system (4) does not converge to a value, it must be transformed into another one that does converge. This method, although working well in most cases, does not always work. Thus, for period $n = 7$ (with 27 complex conjugate solutions and 9 real solutions), an equivalent formulation of dynamical system (4), converging for the symbolic sequence CRL^2RL^2, was not found, although the method works correctly for the rest of the sequences. For $n = 8$ (with 52 complex conjugate solutions and 16 real solutions), the method does not converge when trying to obtain 7 centres, although it succeeds in finding the remaining 61 centres. The finding of alternative ways of formulating the dynamical system (4), so that it does converge in all cases, remains an open problem.

As a consequence, we deal with a method that, even though not infallible in all complex cases, constitutes a valuable helping tool when working with high periods. Up to now, no other method is known to algebraically find the hyperbolic components' centres, other than the resolution of eqn (5) for $c \in \mathbb{C}$ or Stephenson's method [12, 15, 16]. Unfortunately, for a relatively small period, such as $n = 20$, we find that the grade of eqn (5) is 524,288. To illustrate the problem's complexity, in Fig. 2 the polynomial corresponding to eqn (5) has been plotted. As can be appreciated, the number of real solutions (zero-crossings) is huge, exactly 26,214, whereas the total number of

Table 3. Performance of complex Myrberg's method.

Period	# of symbolic sequences	# of sequences solved	Rate of convergence
2	1	1	1.00
3	2	2	1.00
4	4	4	1.00
5	9	9	1.00
6	16	16	1.00
7	36	35	0.97
8	68	61	0.90
9	140	123	0.88
10	273	239	0.88
11	558	482	0.86
12	1090	941	0.86
13	2205	1873	0.85
14	4356	3674	0.84
15	8728	7262	0.83
16	17344	14176	0.82

complex conjugated solutions is 248,778. Even in the case that all solutions could be found, each of them should be checked until the c-value corresponding to the given symbolic sequence is found. Myrberg's method, however, offers the possibility of finding the solution fast and parsimoniously.

Let us see it with some examples. Firstly, given the period-20 symbolic sequence $CL^4RL^9RL^4$, when applying Myrberg's formula, we easily reach the solution $c = -0.5403140239\ldots \pm 0.6121975424\ldots i$, that corresponds to a cardioid centre on the periphery of the set.

Secondly, for the period-20 symbolic sequence $CL^2R^2L^2RL^3RL^3R^2L^2R$, we obtain directly the solution $c = -0.1680237817\ldots \pm 1.041706480\ldots i$, corresponding to another cardioid centre as well.

Thirdly, for the period-20 symbolic sequence $CLRL^4RL^4RL^2RLRL^2$, we also obtain directly the solution $c = -1.624274421\ldots \pm 0.002163607760\ldots i$, corresponding to a disk centre.

However, Myrberg's method does not converge when trying the period-20 symbolic sequence $CR^2L^2R^2LR^3LR^3L^2R^2L$. But when applying the first order deflation, then the solution $c = 0.3634622757\ldots \pm 0.5893619612\ldots i$ is quickly reached, that corresponds to a cardioid centre.

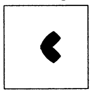
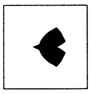
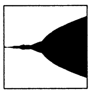
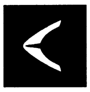
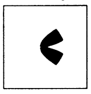
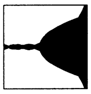
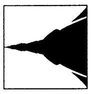
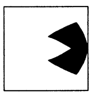
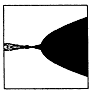
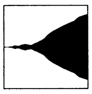
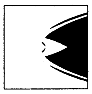
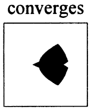
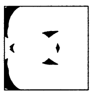
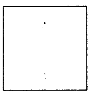
As a last example, let us consider the period-30 symbolic sequence, $CL^4RL^{14}RL^9$. If the polynomial resolution method is used, we find that the grade of eqn (5) for $n = 30$ becomes 536,870,912, absolutely intractable. Likewise, it would be very costly to explore each of the solutions to check which of the symbolic sequences they originate coincides with the given one. On the contrary, if that sequence is tried with Myrberg's method, the solution $c = -0.5513177379\ldots \pm 0.6273074955\ldots i$, that corresponds to a disk centre, is reached without problems.

The overall performance of the extension of Myrberg's method is summed up in Tab. 3 as the rate of convergence when searching for the centres (superstable points). The number of symbolic sequences is calculated as $(C_n - R_n)/2 + R_n$. It has to be noted that this number is not simply C_n, since in the case of complex orbits there are always two orbits with the same symbolic sequence.

5. REVIEW OF RESULTS

In Tab. 4 we present the results obtained when the Myrberg method was applied through period-7 symbolic sequences. The asterisks in the 'Symbolic Sequence' column inform of the order of deflation when applied: one * means first order deflation, two *'s mean second order

Table 4. Results of the application of complex Myrberg's method through period-7 symbolic sequences. * indicates first order deflation. ** indicates second order deflation.

Symbolic Sequence	Centre Coordinates	Basin of attraction
C	0.0	converges
CL	−1.0	converges
CLR	−1.75487766	converges
CLL	−0.12256116 ± 0.74486176i	converges
CLRR	−1.94079980	converges
CLRL	−1.31070264	converges
CLLR	−0.15652016 ± 1.03224719i	converges
CRRL	0.28227139 ± 0.53006061i	
CLRRR	−1.98542425	converges
CLRRL	−1.86078252	converges
CLRLR	−1.25636793 ± 0.38032096i	converges
CLRLL	−1.62541372	converges
CLLRR	−0.19804209 ± 1.10026953i	converges
CLLRL	−0.04421235 ± 0.98658097i	converges
CLLLL	−0.50434017 ± 0.56276576i	converges
CRRLR	0.35925922 ± 0.64251373i	
CRRRL*	0.37951358 ± 0.33493230i	
CLRRRR	−1.99637613	converges
CLRRRL	−1.96677321	converges
CLRRLR	−1.77289290	converges
CLRRLL	−1.90728009	converges
CLRLRR	−1.28408492 ± 0.42726889i	converges
CLRLRL	−1.13800066 ± 0.24033240i	
CLRLLL	−1.47601464	converges
CLLRRR	−0.21752674 ± 1.11445426i	converges
CLLRRL	−0.16359826 ± 1.09778064i	converges
CLLRLR	−0.01557038 ± 1.02049736i	converges
CLLRLL	−0.11341865 ± 0.86056947i	converges
CLLLLR	−0.59689164 ± 0.66298074i	converges
CRRLLL	0.39653457 ± 0.60418181i	
CRRRLR*	0.44332563 ± 0.37296241i	
CRRRRL*	0.38900684 ± 0.21585065i	
CRRLRR*	0.35989273 ± 0.68476202i	
CLRRRRR	−1.99909568	converges
CLRRRRL	−1.99181417	converges
CLRRRLR	−1.95370589	converges
CLRRRLL	−1.97717958	converges
CLRRLRR	−1.76926167 ± 0.05691950i	converges
CLRRLRL	−1.83231520	converges
CLRRLLR	−1.92714770	converges
CLRRLLL	−1.88480357	converges
CLRLRRR	−1.29255806 ± 0.43819881i	converges
CLRLRRL	−1.26228728 ± 0.40810432i	converges
CLRLRLL	−1.25273588 ± 0.34247064i	converges
CLRLLRL	−1.67406609	converges
CLRLLLR	−1.40844648 ± 0.13617199i	converges
CLRLLLL	−1.57488914	converges
CLLRRRR	−0.22491595 ± 1.11626015i	converges
CLLRRRL	−0.20728383 ± 1.11748077i	converges
CLLRRLR	−0.15751605 ± 1.10900651i	converges
CLLRRLL	−0.17457822 ± 1.07142767i	converges
CLLRLRR	−0.01423348 ± 1.03291477i	converges
CLLRLRL	−0.00693568 ± 1.00360386i	
CLLRLLR	−0.27210246 ± 0.84236469i	converges
CLLRLLL	−0.12749997 ± 0.98746090i	converges
CLLLRLR	−1.02819385 ± 0.36137651i	converges
CLLLLRR	−0.62353248 ± 0.68106441i	converges
CLLLLRL	−0.53082780 ± 0.66828872i	converges
CLLLLLL	−0.62243629 ± 0.42487843i	converges
CRRRRRL**	0.37600868 ± 0.14474937i	converges
CRRRRLR*	0.43237619 ± 0.22675990i	converges
CRRRLRR*	0.45277450 ± 0.39617012i	
CRRRLLL*	0.45682328 ± 0.34775870i	
CRRLRLL*	0.37689324 ± 0.67856869i	
CRRLLRL	0.38653917 ± 0.56932471i	converges
CRRLLLR	0.41291602 ± 0.61480676i	
CRLLRRR*	0.35248253 ± 0.69833723i	
CRLLRLL*	0.12119278 ± 0.61061169i	
CRLLLLR*	0.01489546 ± 0.84814876i	converges

deflation. The plots in the 'Basin of attraction' column correspond to a graphical representation of the convergence of the Myrberg's formula. Initial points converging in the side-4 square centred at (0, 0) to the centre of the superstable orbit are depicted in black. When the formula converges for any point in the interior of the side-4 square, then the word 'converges' appears.

6. CONCLUSIONS

We used and extended Myrberg's method to find the c-parameter value of superstable complex orbits, when only the symbolic sequence is known. Although the method presented in this paper does not converge in all complex cases, it is the only computationally reasonable alternative known to tackle the calculation of centres corresponding to hyperbolic components from a given symbolic sequence.

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