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Physica A 295 (2001) 291–296

PHYSICA A

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Power law periodicity in the tangent bifurcations of the logistic map

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Abstract

Numerical studies were carried out for the average of the logistic map on the tangent bifurcations from chaos into periodic windows. A critical exponent of $\frac{1}{2}$ is found on the average amplitude as one approaches the transition. Additionally, the averages oscillate with a period that decreases with the same exponent. This Power Law Periodicity is related to the reinjection mechanism of the map. The undulations appear at control parameter values much earlier than the values where the critical exponent of the bifurcation shows significant changes in the average amplitude.

PACS: 42.50.Tj; 42.55.Em; 05.45.+b; 42.65.Sf

Keywords: Dynamical bifurcations; Chaos; Logistic map

1. Introduction

Critical behavior of nonlinear systems have large application interest on catastrophe prediction. The log periodicity in systems with discrete scale invariance, associated to complex critical indices, have been introduced by Sornette [1], and their application to study earthquake seismological data is an active research subject [2,3]. The evidence of log periodicity in the topological entropy of bidimensional maps has also been studied [4]. Here, we give the results of numerical studies on the logistic map, where instead of log periodicity one encounters Power Law Periodicity (PLP) on the oscillation of a property of the dynamical system near bifurcation.

Statistical properties of chaotic attractors have been studied for decades [5]. Lorenz has used averages in the quadratic map to emphasize his well known proposition on the

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PII: S0378-4371(01)00090-5

unpredictability of the weather on long time scales [6]. In recent works, the properties of the average power in an experiment with a chaotic laser [7] and the properties of the average in bifurcations of the logistic map [8] and model equations for a chaotic laser [9] have been shown to give signatures of the bifurcations from and into chaotic behavior.

This work presents numerical results for the dynamical bifurcations of the logistic map. Bifurcations diagrams which are usually exhibited by giving the peak of the pulses are given here by the average of the iterates. The properties of the average throughout the intermittence bifurcations are the main results reported.

2. The bifurcations in the logistic map

The logistic map has been extensively studied as a prototype of one-dimensional maps with chaos [5]. The iterates of

$$x_{n+1} = rx_n(1 - x_n)\,, \tag{1}$$

for $0 \leqslant r \leqslant 4$ can be found in most textbooks on chaos. Period doubling and saddle-node bifurcations are the abundant bifurcations occurring as the control parameter r increases towards 4. At the saddle-node bifurcations the system enters a periodic window, following an intermittent chaotic transition. The iterates of this map on the near-periodic channel were studied in detail by Pomeau and Maneville [5].

The bifurcation that occurs at a critical parameter $r_{c(7)} = 3.7016407\ldots$ corresponds to a period 7 window, as shown in Fig. 1(a). It consists of the transition from sequences of near-periodic iterations with bursts of visits to the wide range of the attractor. The number of iterations in the channel of near-periodicity grows as $(r_{c(7)} - r)^{-1/2}$ [5].

3. Averages and Bifurcations

The average of a dynamical variable has been pointed out as relevant in the early days of studies on chaotic maps [6]. Such property may manifest the signature of the bifurcation as shown in experiments [7] and model calculations [9] with lasers and in the logistic map [8]. The bifurcations were studied numerically by taking a typical initial condition x_0 and calculating x_n which was used to obtain the average

$$\bar{x} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x_n\,. \tag{2}$$

The tangent bifurcation in the average near the period 7 window at $r = 3.7016407\ldots$ is shown in Fig. 1(b). Here the average of 20 averages of 10^7 iterates, initiated at x_0 randomly chosen within $(\frac{1}{2} \pm 0.1)$, were calculated. A critical exponent $v_i = 1/2$ results as one fits the average to the expression [8]

$$\bar{x}(r) = \bar{x}_i + A_i \left\{1 - \exp\left[-\xi_i(r_{c(i)} - r)^{v_i}\right]\right\}\,. \tag{3}$$

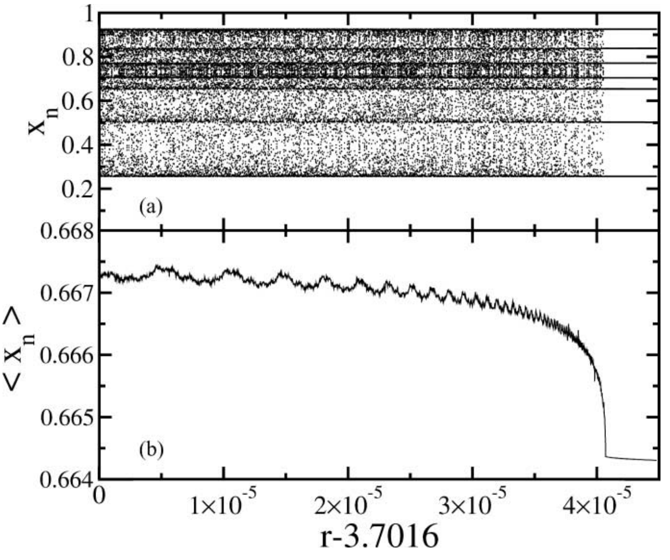

Fig. 1. Bifurcation diagram calculated for the logistic map. (a) the iterates of x_n and (b) the average $\bar{x}$, near the largest period 7 window, whose critical parameter is $r_{c(7)} = 3.7016407\ldots$.

The tangent bifurcations into periods 3 and 5 windows have the same exponent but a detailed study of higher periods was not pursued. In addition to the power law on the amplitude of the average one clearly observes undulations with increasing frequency as one approaches $r_{c(7)}$. The numerically observed frequency variation, as a function of $r_{c(7)} - r$, could not be accounted for by a logarithmic frequency dependence. The variation is much faster and a power dependence $(r_{c(7)} - r)^{-v_7}$ was searched for.

To obtain the fitting of the undulations the expression of Eq. (3) was extended to include a power law periodic term as

$$\bar{x}(r) = \bar{x}_i + A_i\{1 - \exp[-\xi_i(r_{c(i)} - r)^{v_i}]\}\left\{1 + B_i \sin\left[\Omega_i/(r_{c(i)} - r)^{v_i} + \Psi_i\right]\right\} . \tag{4}$$

Fig. 2(a) shows the graphic of Eq. (4) that best fitted the average period 7 window bifurcation. The choice $v_7 = \frac{1}{2}$ was made by inspection of the frequency dependence on the numerical results. The values obtained for the coefficients fitted are: $A_7 = 0.0029$, $\xi_7 = 616$, $B_7 = 0.026$, $\Omega_7 = 0.44$, and $\Psi_7 = 2.7$. The superposition of the fitting over the numerical average is shown in Fig. 2(b).

To verify possible numerical artifacts in the calculations, the averages have also been calculated with different initial conditions and number of iterates. A single initial condition, $x_0 = \frac{1}{2}$, averaging 2×10^8 iterates gave results similar to Fig. 1(b) but some narrow downgoing spikes appear. These are expected from the fact that the occurrence of periodic windows (very narrow) is dense within $0 < r < 4$ [5]. The Lyapunov exponents were also calculated in the same range of r and with the same

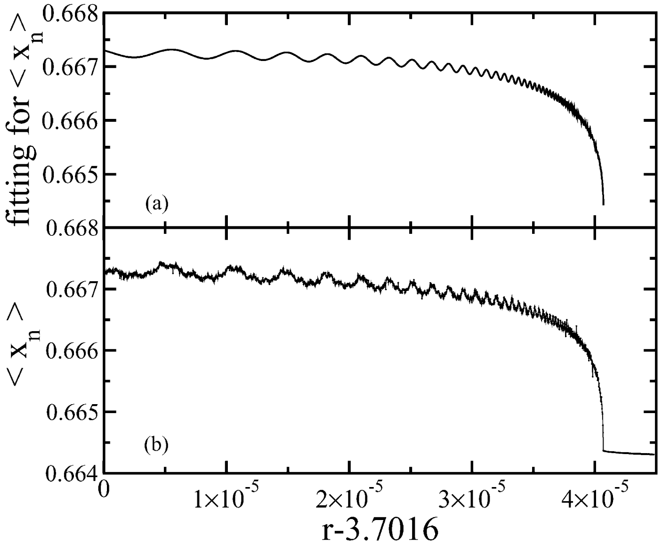

Fig. 2. Average of the iterates of the logistic map near the period 7 window. (a) Using Eq. (4) with the parameters extracted from Fig. 1(b) and (b) superposition of the numerical average (Fig. 1(b)) and its fitted expression (Fig. 2(a)).

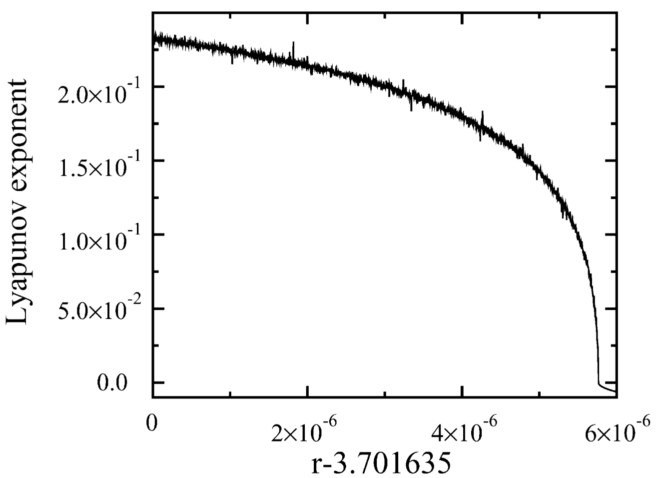

Fig. 3. Bifurcation diagram for the Lyapunov exponent near the largest period 7 window. A critical exponent $v_{c(7)} = 1/2$ fits the positive values, before the bifurcation.

conditions used for Fig. 1. As shown in Fig. 3 the critical exponent v_7 for the power dependence on $r_{c(7)} - r$ is $\frac{1}{2}$ but no undulations are visible. Another quantity obtained numerically was the number of iterates in the near-periodic channels. Fig. 4 gives the number of iterates in the channel near $\frac{1}{2}$. The number grows as $(r_{c(7)} - r)^{-1/2}$ as predicted many years ago by Pomeau and Maneville for this type I intermittence [5]. Again no undulations are manifested in this number. These observations imply that the

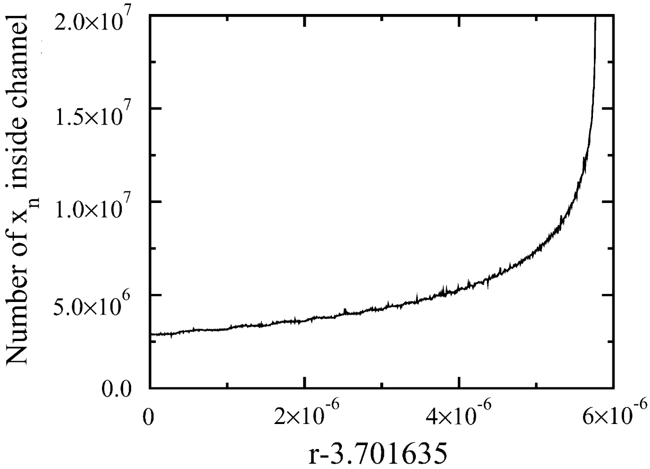

Fig. 4. Number of points iterated within the channel of the period 7 window as function of the control parameter. As opposed to the average, no undulations occurs for this number. A curve fitting is consistent with $(r_{c(7)} - r)^{-1/2}$ dependence, near the bifurcation.

undulations in the average are originated in the mechanism of reinjection of the iterates in the channel of near-periodic behaviour. The undulations are therefore related to the global nature of the nonlinear map [10]. [1]

4. Conclusions

Bifurcation diagrams were calculated numerically for the logistic map. The tangent bifurcations from chaos into periodic pulsation show a characteristic average power dependence [9], with a critical exponent $\frac{1}{2}$. Furthermore, an evidence for Power Law Periodicity in the critical behavior was found in the average of the iterates.

The undulations, whose amplitude correspond to 5% of the average value, appear at control parameter values $r - r_c$ at least one order of magnitude greater than the values that give significant change in the average approaching the bifurcation. This fact may be most relevant to use the undulations to predict bifurcations. The origin of such undulations, that cannot be attributed to a discrete scale invariance of the map [1] is under investigation. They appear to be caused by the global properties of the nonlinear map.

[1] Preliminary studies of a map with a tangent bifurcation leading to type I intermittence bifurcation from periodic to quasi-periodic regimes indicate that the number of iterates in the near periodic channel has the universal dependence $(r_c - r)^{-1/2}$ [5], while the average of the iterates is completely different for different maps.

Acknowledgements

This work was partially supported by Brazilian Agencies: Conselho Nacional de Pesquisa e Desenvolvimento (CNPq), Financiadora de Estudos e Projetos (FINEP).

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