The Complex Logistic Map

S.E. Jorás[1]
UDESC - Joinville

C. Manchein[2]
UFPR

ABSTRACT

Many systems in Classical Mechanics can be easily managed if one extends the configuration space to allow for complex values of the (generalized) coordinates — for instance, the complex-angular-momentum approach to scattering of light. This work is a first step towards the implementation of such procedure in the study of chaotic systems.

For the sake of simplicity, we have chosen the real Logistic Map,

$$x_{n+1} = r\, x_n(1 - x_n) \quad ,$$

the usual textbook example of a chaotic map. In spite of its mathematical simplicity, it shares many charateristics with more complicated maps and dynamical systems, such as bifurcations and periodicity windows. In this work we make the usual extension to complex variables, namely $x_n \to z_n$, obtaining a nonlinear bidimensional map.

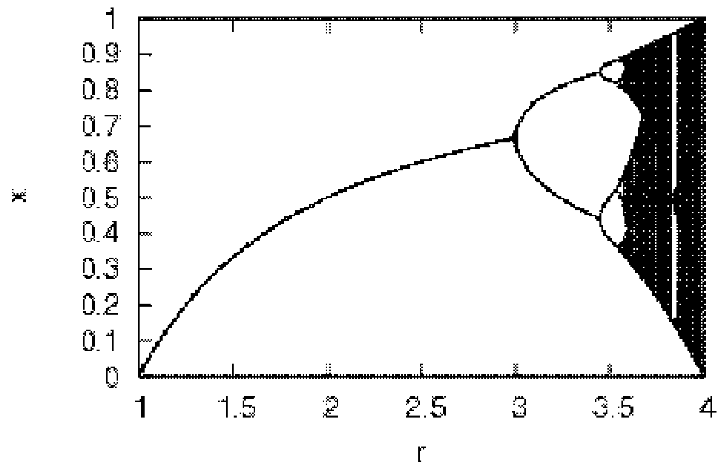
[1] joras@joinville.udesc.br
[2] cmanchein@fisica.ufpr.br

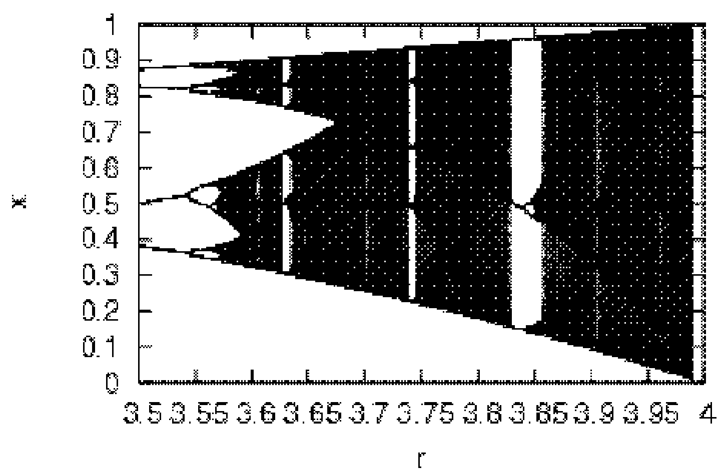
1 INTRODUCTION

The logistic map is a widely used textbook example of a chaotic system. It is defined by the map

$$x_{n+1} = r\, x_n(1 - x_n) \quad . \tag{1}$$

In spite of its mathematical simplicity, it shares many important characterisctics with more complcated maps, such as self-similarity and periodicity windows.





2 THE COMPLEXIFICATION

Complex solutions have been used in different fields of physics, from optics [1] to finite temperature field theory [2] and cosmology [3].

The complexification should not be taken too literally. One can see it as just a mathematical artifact which simplifies the calculations.

By making

$$x_n \longrightarrow z_n = x_n + i\, y_n \qquad , \tag{2}$$

we transform the Logistic map into a pair of coupled maps:

$$\begin{cases} x_{n+1} \equiv f(x_n, y_n) = r(x_n - x_n^2 + y_n^2) \\ \\ y_{n+1} \equiv g(x_n, y_n) = r\, y_n(1 - 2x_n) \end{cases} \tag{3}$$

representing the Real and Imaginary parts, respectively.

The fixed point $y = 0$ turns stable whenever a periodicity window opens up. That can be easily seen upon linearization at $(x, 0)$: the eigenvalues in both directions are the same function of the control parameter r.

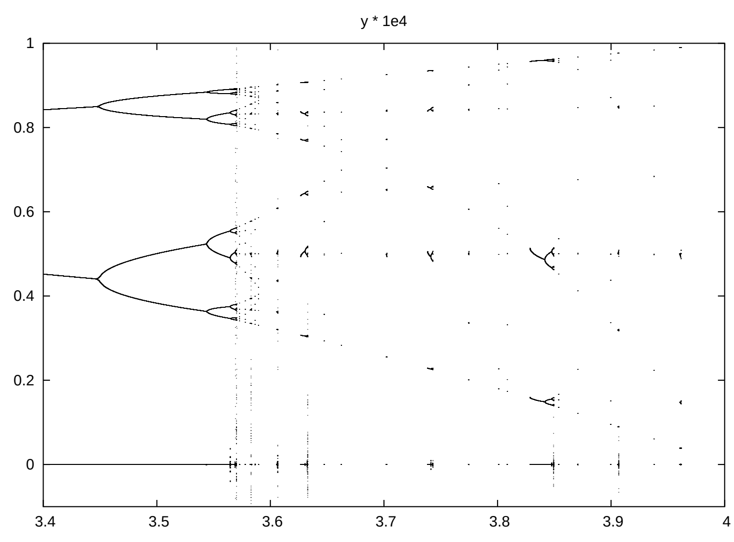


Figure 1: Real and Imaginary parts of the complex logistic map. The imaginary part was multiplied by 10^4.

3 Averages and Oscillations

There has been a large interest [4, 5, 6] in the behaviour of map average

$$\bar{x} \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n \tag{4}$$

just before a tangent bifurcation, i.e, when a periodicity window opens up.

In the particular subject of the logistic map, Cavalcante it et al [4, 5] numerically found oscillations in the average $\bar{x}(r)$ just before the bifurcation.

The behaviour could be fitted by the expression

$$\bar{x}(r) = \bar{x}_o + A\{1 - \exp[-\xi(r_c - r)^\nu]\}\{1 + B\sin[\Omega(r_c - r)^\nu + \psi]\}. \tag{5}$$

At the period-7 one, they have found [4] $A = 0.0029$, $\xi = 616$, $\nu = 0.5$, $B = 0.026$, $\Omega = 0.44$ and $\psi = 2.7$.

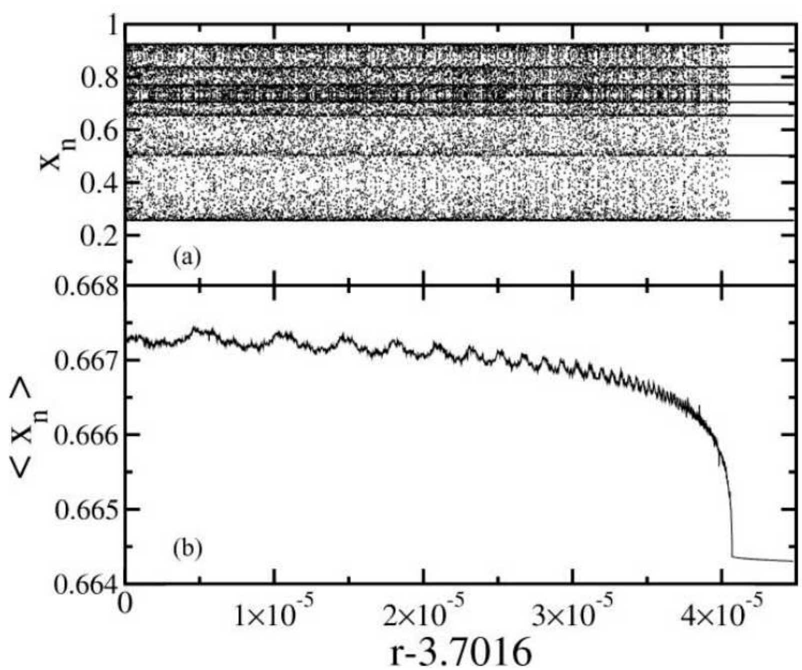


Figure 2: Figure from Ref. [4].

4 PERIOD-3 WINDOW

Of course, one can find the same behaviour close to a period-3 window, which can be fitted accordingly:

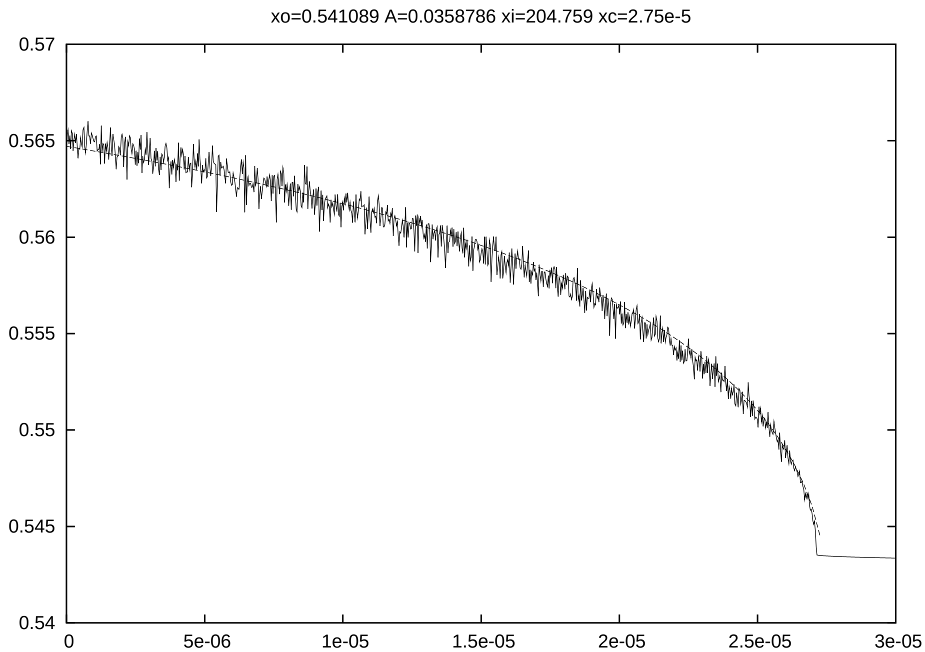


Figure 3: Analogoulsy to the previous figure, but at the period-3 window.

For the remaining of this poster we will restrict ourselves to the period-3 window. We believe the same arguments can be applied close to any tangent bifurcation, but it can be more easily seen at this particular choice, due to the low number (3, of course) of periodic stable solutions.

4.1 FIXED IMAGINARY POINTS

The fixed points of a map $x_{n+1} = f_r(x_n)$ can be defined by

$$x_{FP} = f(x_{FP}) \qquad , \tag{6}$$

i.e, are the roots of

$$g(x_{FP}) \equiv f(x_{FP}) - x_{FP} = 0 \qquad . \tag{7}$$

For the problem we have in mind, namely the period-3 window in the logistic map, the function $g(x)$ is a 8th-order polynomial, with real coefficients. Such an expression has always 8 roots, not all of them real. Actually, such a reasoning works for every periodic window in the logistic map. That is why our argument, as follows, should also hold at other tangent bifurcations.

We propose to study the periodic solutions before the bifurcation corresponding to the openning of the periodic window itself. What is the contribuition, if any, of the complex solutions of the equation $g(z) = 0$ to the average $\bar{x}(r)$?

4.2 CATASTROPHE THEORY

The beahaviour of the function $g(z)$ or rather of its squared modulus $h(z) \equiv |g(z)|^2$ (recall we are now dealing with complex variables) around a tangent bifurcation can be modeled by the corresponding catastrophe, the fold:

$$h(z) = \frac{1}{3}z^3 + u\,z \qquad . \tag{8}$$

In order to reproduce the results, we must introduce a scale

$$z \longrightarrow A\,z \tag{9}$$

and a shift, since the imaginary roots do **not** collapse at $z = 0$:

$$z \longrightarrow z - z_o \qquad . \tag{10}$$

which yields

$$h(z) \longrightarrow h(z) = \frac{A}{3}z[(z - z_o)^2 + u] \tag{11}$$

where we have redefined $h(z)$ and $u \rightarrow u/3$ to avoid cluttering the notation.

Since we know the solutions are imaginary if $r < r_c$ — when the imaginary solutions collapse into real ones and the periodicity window opens up — we will further write

$$h(z) = \frac{A}{3}(z - z_o)\left[(z - z_o)^2 + b^2(r_c - r)\right] \qquad , \tag{12}$$

where b is a Real number.

Actually, since that happens around every pair of complex solutions, we should write

$$h_j(z) = \frac{A_j}{3}(z - z_j)\left[(z - z_j)^2 + b_j^2(r_c - r)\right] \qquad , \tag{13}$$

which describes the behaviour of $h(z)$ close to each root.

4.3 SUMMING UP THE SOLUTIONS

The roots are then:

$$z_1 \quad , \quad z_1 \pm b_1(r - r_c)^{1/2} \tag{14}$$
$$z_2 \quad , \quad z_2 \pm b_2(r - r_c)^{1/2} \tag{15}$$
$$z_3 \quad , \quad z_3 \pm b_3(r - r_c)^{1/2} \tag{16}$$

If we take only the positive signal, the average is then

$$\begin{aligned} M &= \frac{1}{6}\left[2z_1 + 2z_2 + 2z_3 + b_1(r - r_c)^{1/2} + b_2(r - r_c)^{1/2} + b_3(r - r_c)^{1/2}\right] \\ &= \frac{2}{6} z_T \left[1 + \frac{b_T}{2z_T}(r - r_c)^{1/2}\right] \end{aligned} \tag{17}$$

where

$$b_T \equiv b_1 + b_2 + b_3 \tag{18}$$
$$z_T \equiv z_1 + z_2 + z_3 \tag{19}$$

If we further argue that Eq. (17) is just an e xpansion close to $r = r_c$, we can suppose that the "correct" expression is

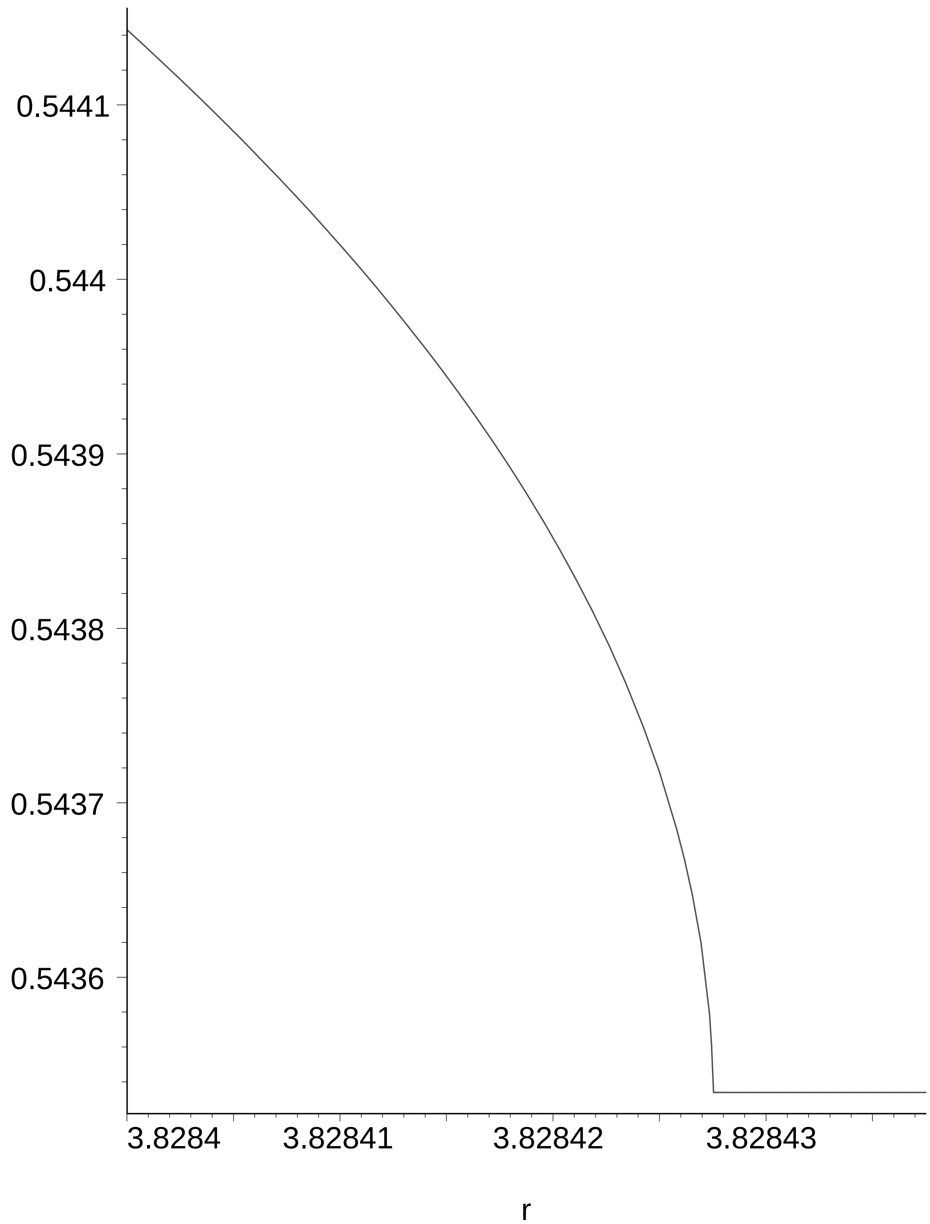
$$M = \frac{1}{3} z_T \exp\left[\frac{b_T}{2z_T}(r - r_c)^{1/2}\right] \tag{20}$$

Indeed, a crude estimative taking the real roots leads to

$$b_T \approx 0.35 \tag{21}$$

$$z_T \approx 1.629 \tag{22}$$

and to the plot



5 NEXT STEPS

- Taking only the positive imaginary part solutions can be justified in some finite temperature field theories [7], but it does need more support here.
- We believe the scale can be adjusted by a fine determination of the b's.
- Ondulations can be found if we further assume $b = b(r)$

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