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JULIA SETS AND QUASI-STABLE ORBITS IN THE COMPLEX PLANE

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Abstract—The dynamic behavior and attractors of quasi-stable orbits of 0 for the iteration $z_{n+1} \leftarrow z_n^2 + c$ are explored. Relationships to their associated Julia sets are also discussed in this tutorial.

1. INTRODUCTION

Various graphical aspects of the Mandelbrot set (M-set) have been reported extensively since its initial development[1-4]. The primary graphical techniques have involved escape time algorithms that show, quite beautifully, the complexity in boundary regions of the M-set. Each pixel in such displays is a representation of the outcome of a specific orbit in the complex plane, and has a unique Julia set associated with it. The dynamics of such orbits, which are presented in this tutorial, can reveal details of its Julia set and also the attractors that are contained within it.

2. ITERATED MAPS

The generation of M-set images is determined by the *orbits of* 0 (initial $z = 0$), of the iteration:

$$z_{n+1} = z_n^2 + c. \qquad (1)$$

If the progress of the iteration is plotted in the complex plane (for a particular value of c) it forms a two-dimensional *iterated map*. In general, the paths of such maps either diverge, and head to infinity, or remain bounded to a point or set of points[5]. All values of c whose orbits are bounded are members the M-set and have associated Julia sets that are connected[2]. A connected Julia set forms the boundary between the basins of attraction of the stable orbits and the attraction to infinity[6].

Orbits that are bounded generally fall into two categories: either they are *superstable* or *asymptotically-stable*. In the superstable case the orbit is exactly periodic. In the latter case the orbit approaches, asymptotically, single (or multiple) point attractors as the iteration proceeds. It has been shown that some periodic orbits can form fractal shapes, or strange attractors, in the complex plane within their associated Julia set[7].

The *escape time*, or dwell, of unstable orbits is the precise number of iterations for which the magnitude of Z becomes greater than 2. The final trajectory, which heads off to infinity, can be considered an orbit's *path to infinity*. For certain c values, the orbit is strongly attracted to infinity and has a direct path to infinity; the antithesis of asymptotically stable orbits that collapse directly to attractive points in the complex plane.

Other unstable orbits escape to infinity after relatively long escape times. Such *quasi-stable* orbits tend to be equally attracted to regions within the complex plane and to infinity. As a consequence, the iterated map of such an orbit fills in a *boundary region* between the attraction to infinity and the central attractor(s). Since the associated Julia set *is* the boundary between the central attractors and infinity, as stated, this boundary region of the iterated map is also the Julia set. For unstable orbits, the associated Julia set is, in fact, a Cantor set[2] and is therefore not closed. The Cantor sets of quasi-stable orbits look very much like the closed Julia sets, but what the iterated maps show is that these sets have narrow breaks through which the orbit's path eventually escapes.

An example of a quasi-stable orbit of 0 is shown in Fig. 1 for a particular value of c. In Fig. 1a the magnitude of z vs. iteration number is plotted. It can be seen from the plot that the orbit has some semblance of periodicity but eventually escapes after 650 iterations. Figure 1b shows the iterated map of this orbit in the complex plane and points out the main features associated with such quasi-stable orbits. The central attractor is the location in the complex plane that the orbit tends to gravitate towards. Since the orbit is quasi-stable it is equally attracted to infinity and the orbit tends to skirt a boundary region during its iterated lifetime before escaping. In Fig. 1c the associated Julia set is superimposed on the iterated map. The Julia sets for these particular displays, unless otherwise stated, were computed by the inverse iteration method[4] and plotted on the same scale as the iterated maps. It can be seen that the Julia set opens up along broken cusps or folds. These cusps act as escape routes from the central attractor to the attraction at infinity as observed by the orbit's path to infinity.

3. STEPWISE ITERATED MAPS

Another effective way of displaying iterated maps is to vary the real part, or the imaginary part, of the value of c in small steps, and plot the first 100, or so, iterations for each successive orbit, all on the same *stepwise iterated map*. By overlaying orbits in this manner the progression of the attractive regions in the complex plane can be seen directly; as the value for c is moved about the M-set the location of the orbit's attractor(s) change correspondingly. Sudden changes of attractors in this manner is analogous to a phenomena known as *crises*[8], which is a description of the transient na-

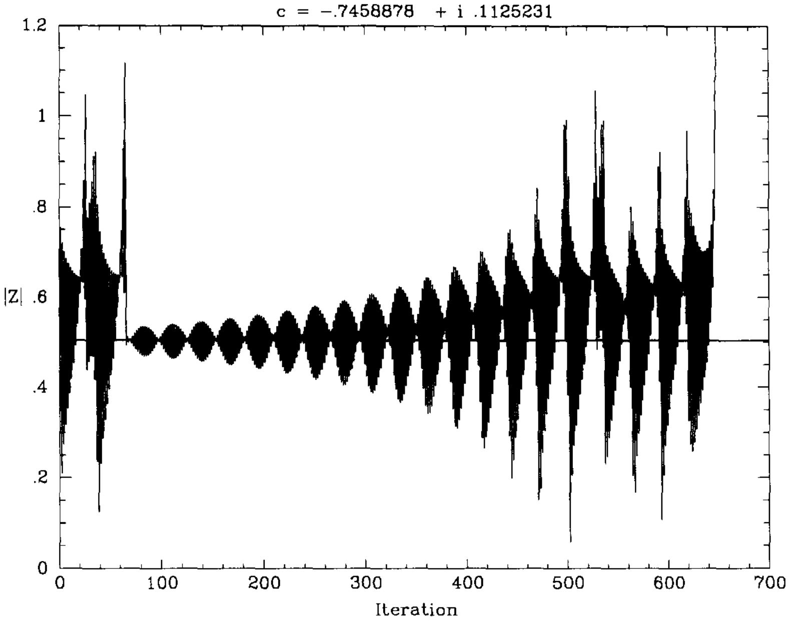


Fig. 1a. Plot of the iteration, Eq. 1, for a particular c value, showing the quasi-stable nature of the orbit before it finally becomes unstable and heads off to infinity.

ture of chaotic attractors in iterated maps. As the attractive regions change, the orbits undergo corresponding changes in behavior. It is apparent from such iterated maps that orbits about attractors that are on the verge of such a crises tend to be quasi-stable.

These maps also show an apparent coherence between the paths about the attractors as they change; the view of a single orbit may show little structure but when plotted with a series of orbits a pattern emerges; see for example Figs. 5a and 6c.

4. ORBITS AS DYNAMICAL SYSTEMS

The orbits generated by Eq. 1 can be viewed as dynamical systems, with each successive iteration representing an increment of time and the path of a particular orbit representing a trajectory in two-dimensional space. The dynamics of some quasi-stable orbits can exhibit quite complex behavior if the value of c is chosen carefully. As the path of an orbit unfolds, oscillations of outward spirals, slow dances around strange attractors, and chaotic pulsations can be ob-

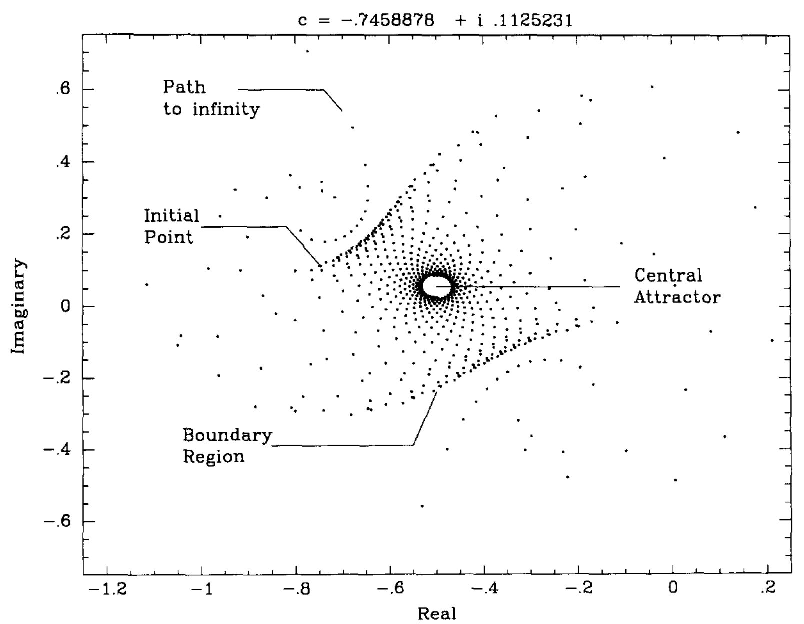


Fig. 1b. Iterated map of the same c value used in Fig. 1a, showing some of the orbit's characteristic features.

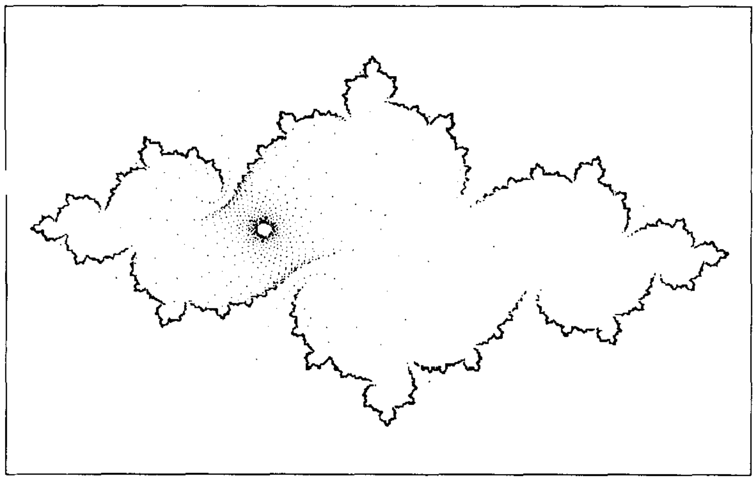


Fig. 1c. Overlay of the associated Julia set with the iterated map of Fig. 1b. Note how the escape route of the orbit occurs through the broken cusps in the Julia set.

served. The orbits take on almost life-like behavior at times. There are also display tricks that can be used to enhance the images. For instance, by changing the color (or shade) of the points plotted every 1,000 iterations, or so, surprisingly coherent patterns can be brought out. This also allows one to follow dense patterns more easily. This is how the grey-shade patterns were generated in some of the figures. Also, clearing the screen during an orbit or slowing down the iteration process can reveal other interesting aspects of the iterated map.

5. GRAPHICS AND OBSERVATIONS

A sampling of such iterated maps is now discussed. Refer to Fig. 7 as to where in the M-set the c values for a particular figure is located.

Figure 2a is a particularly interesting orbit that begins

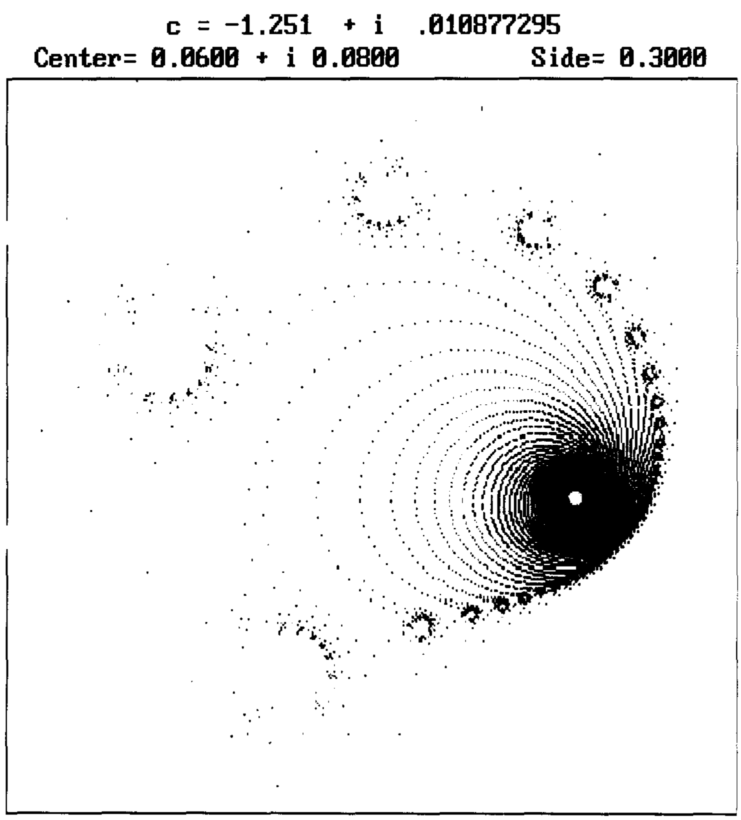


Fig. 2a. Magnified view of quasi-stable 0 orbit.

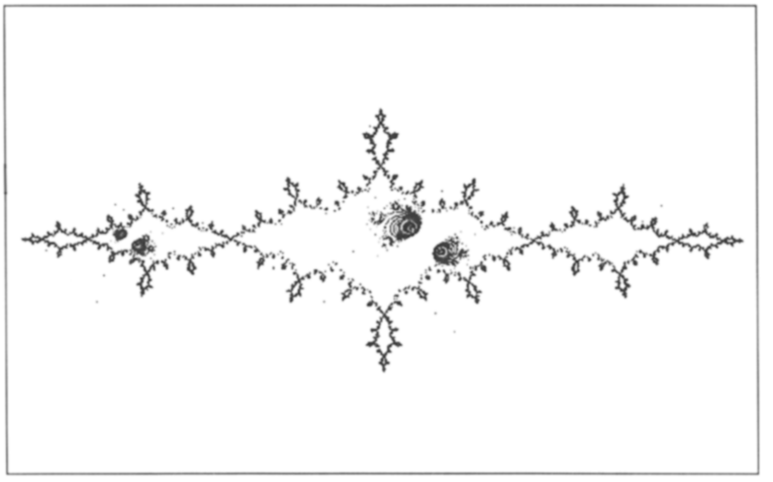


Fig. 2b. Expanded view of Fig. 2a showing the extent of the associated Julia set and the entire iterated map which is period 4.

by filling the outer boundary regions, that act as mini-attractors, but is soon pulled into the central region where the orbit spirals outward before escaping to infinity. Figure 2b shows the orbit and its associated Julia set; note that there are, in fact, four regions in the complex plane (period 4) that the iteration actually maps out.

In Fig. 2c a portion of the associated *filled in* Julia set[3] is displayed. Notice that the filled in Julia set shows clearly how the quasi-stable orbit is bound within it before the trajectory escapes to infinity. The boxed section represents the display area shown in Fig. 2a.

Figure 3a is an excellent example of a "long-lived" quasi-stable orbit. The pattern is formed by a series of jumps toward the central attractor followed by outward spirals that slowly fill the boundary seen in the display. The random (chaotic) points seen outside the strange attractor are formed by pulses that occur after an out-

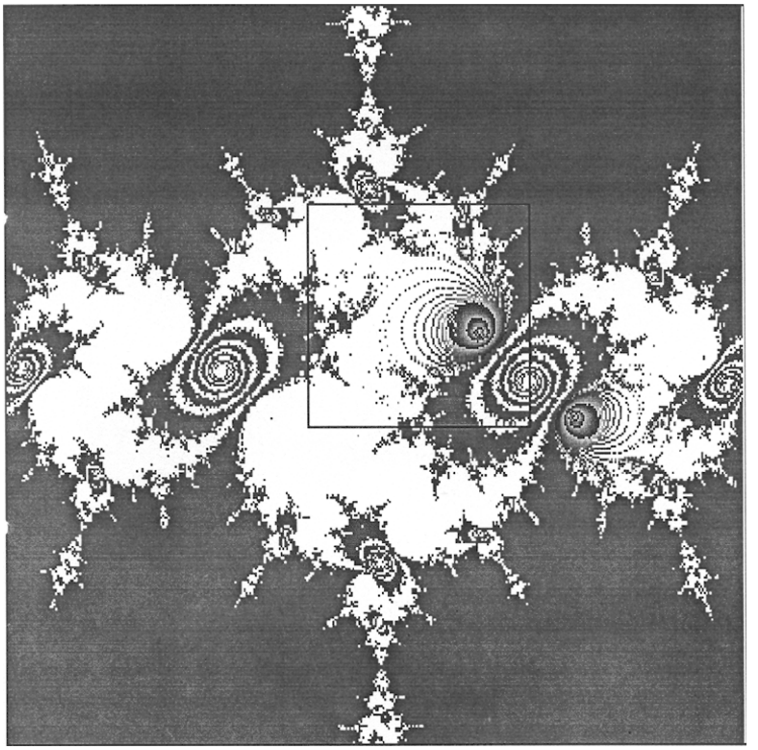


Fig. 2c. Display of the filled in Julia set of Fig. 2b, showing how it bounds the quasi-stable orbit. The boxed area is the region displayed in Fig. 2a.

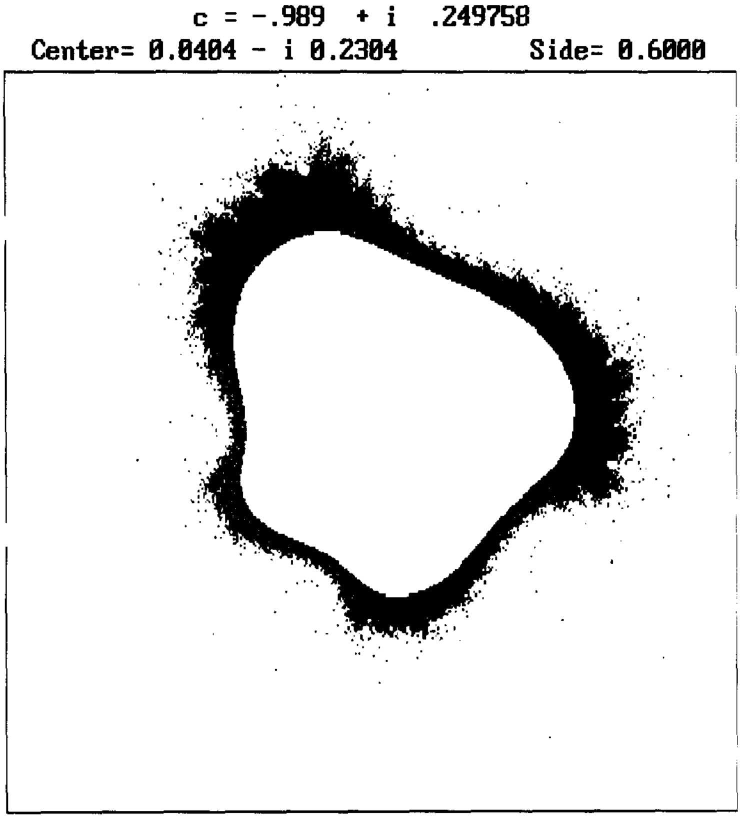


Fig. 3a. Example of a quasi-stable orbit with a very large time of escape. In this case, over 2 million iterations occur before escape.

ward spiral oscillation fills the attractor region. Although these scattered points seem to be outside the boundary region, they are somehow connected because the iteration recovers and returns to the boundary region and continues its peculiar dance again until, finally, this quasi-stable orbit heads off to infinity after over 2 *million iterations.* In Fig. 3b the same map is shown with an expanded display scale and Julia set outline. Note that there are actually two structures formed in the complex plane; the smaller one is near the initial point (c value) and the larger shape is displayed in Fig. 3a.

The iterated map shown in Figure 4a forms three boundary regions about a central attractor. The inner spiral-type structure was formed by a jump from the three boundary regions in toward the central attractor

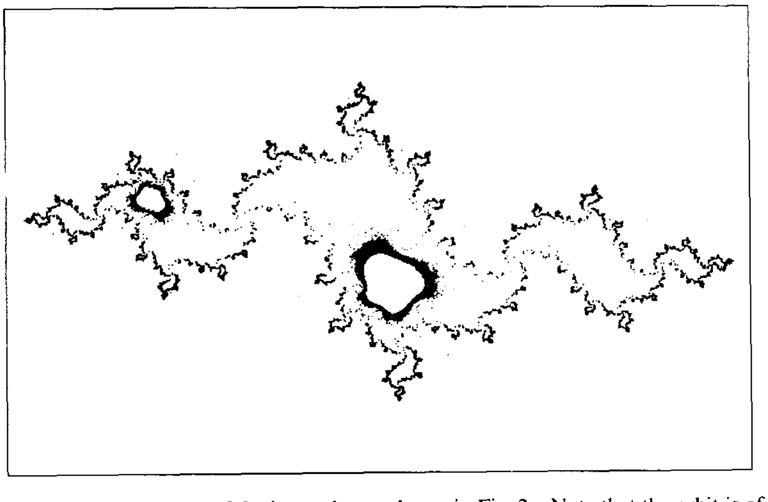


Fig. 3b. The associated Julia set of the iterated map shown in Fig. 2a. Note that the orbit is of period 2.

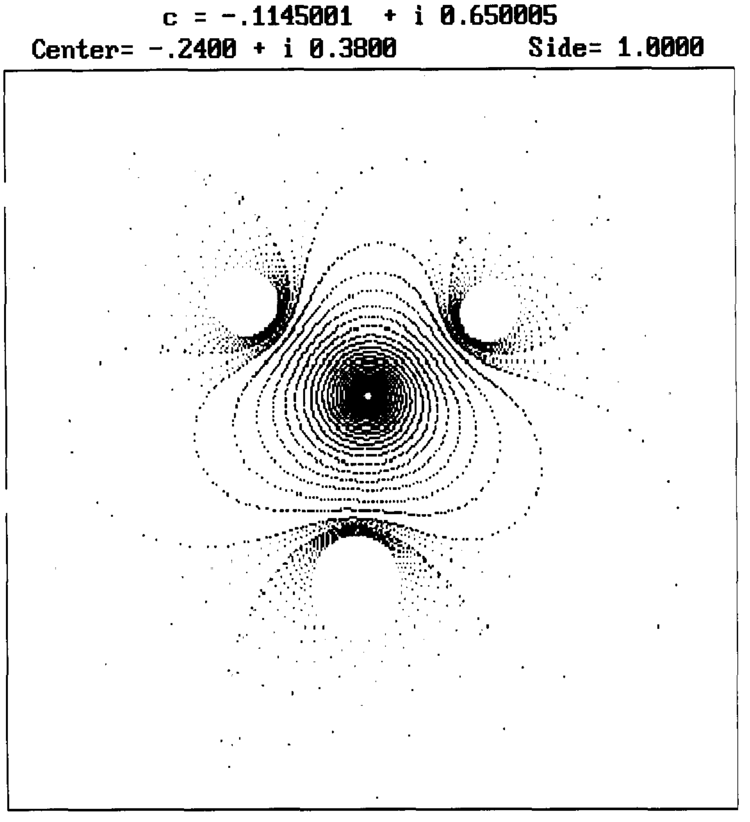


Fig. 4a. Iterated map showing an orbit influenced equally by three outer regions of attraction and a central attractor.

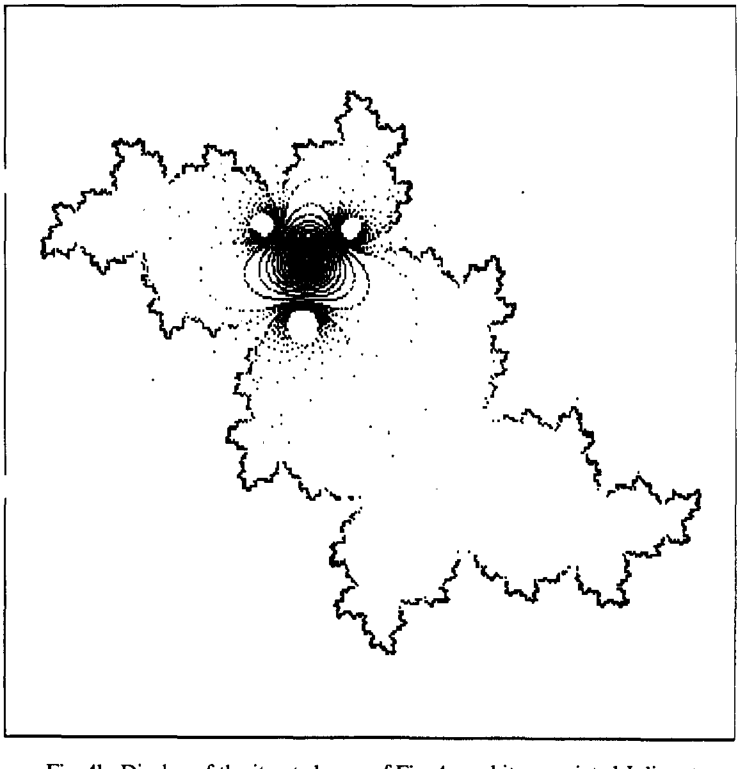


Fig. 4b. Display of the iterated map of Fig. 4a and its associated Julia set.

and then the orbit spiraled outward toward the attraction at infinity. It can be seen from Fig. 4b how the orbit skirts the Julia set region before heading off to infinity through the open cusps that pierce inward toward the central attractor.

Figure 5a is an example of a stepwise iterated map. In this case, the imaginary part of c is incremented while the real part is held constant; the track of these steps within the M-set is shown in Fig. 7. A sample of individual orbits of 0 along this track and their associated Julia sets are also shown in Figs. 5b–e. As the steps progress, the attractive regions evolve from a single point into three separate attractive regions. Notice that the spirals in Figs. 5c–e, which are asymptotically stable orbits, change from three to one and finally two arms respectively. Also note that Fig. 4a is very near the point where the attractive regions change from one region to three, which is quite apparent in the behavior of its orbit.

Figure 6a is another example of a quasi-stable orbit. As seen in Fig. 6b, the orbit forms along the inside edge of its associated Julia set. In Fig. 6c, another example of a stepwise iterated map was made by choosing the same real value as the orbit shown in Fig. 6a and incrementing in small steps about its imaginary part. As the steps progress: at first the individual orbits are bound by 19 individual regions of attraction and then, slowly, the iterations begin to collapse toward the central attractor producing the spiral arms shown. Notice that the dynamics of the quasi-stable orbit in Fig. 6a is a consequence of the dual attraction between these outer 19 attractors and the central attractor. Note that, since the imaginary part is incremented only .002 units, the track of the stepwise map cannot be resolved in Fig. 7.

6. CONCLUSION

Stable orbits are bound by the basins of attraction within their associated Julia set. It is the opening up of this set into a Cantor set that allows the orbit to be simultaneously attracted to both the inner regions and infinity. This dual attraction yields orbits having complex dynamics that convey the outline of the set, as observed by the iterated maps shown.

The inner attractive region also contributes to the complexity of the orbit's dynamics due to the unstable nature of its attractors. The stepwise iterated maps demonstrate the instability of these attractors about certain regions in the complex plane, whereas individual quasi-stable iterated maps can yield information about the strange attractors themselves [7, 8, 9].

It is the M-set, of course, that points the way to the location of these quasi-stable orbits; the c values chosen for these displays occur along the fractal shorelines of the Mandelbrot seas and rivers.

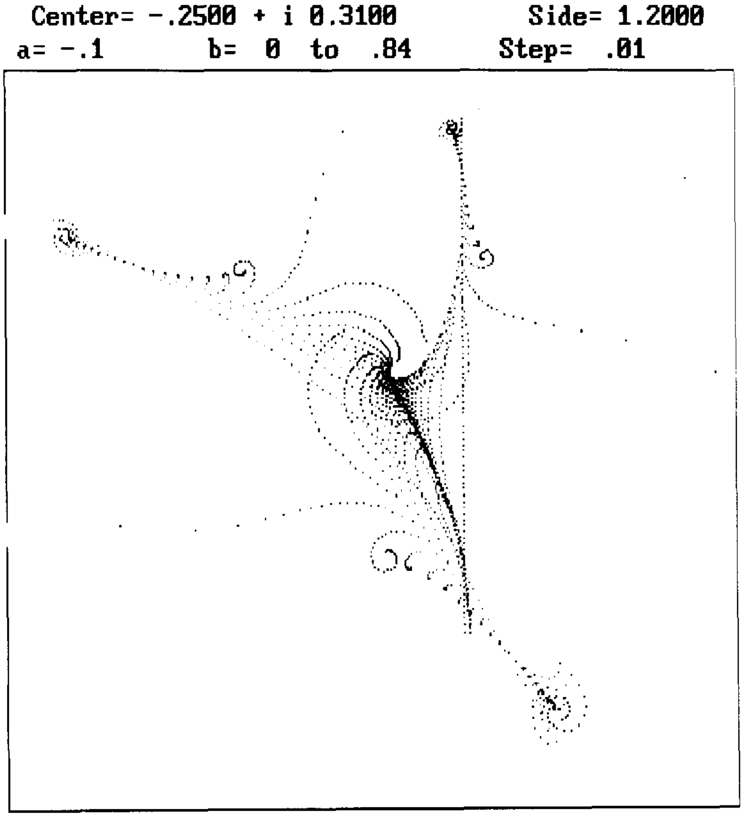
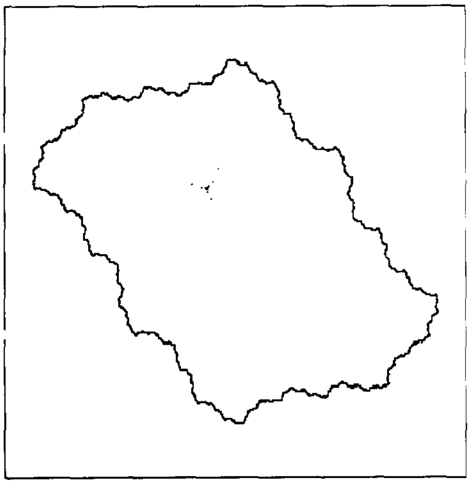
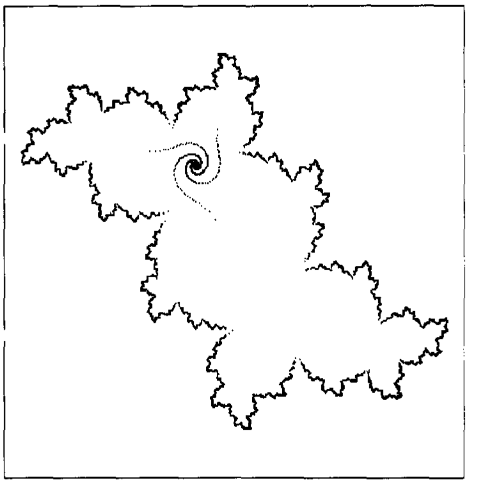


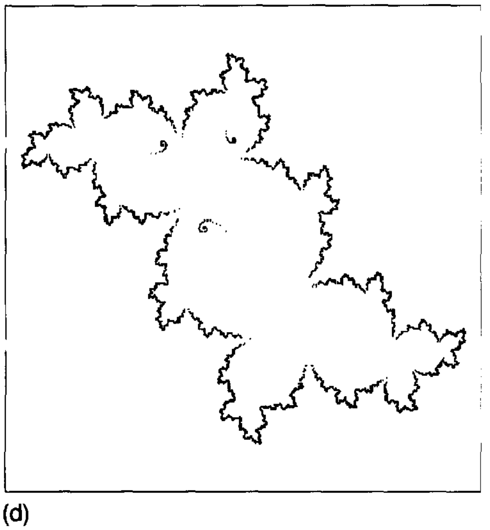
Fig. 5a. An example of a stepwise iterated map, the track of which is shown as a dotted line in Fig. 7.



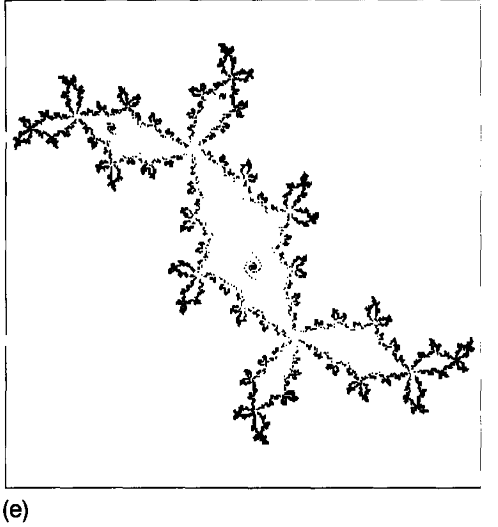
(b)



(c)



(d)



(e)

Fig. 5b, c, d, e. A sequence of individual orbits (and their associated Julia sets) along the track of the stepwise map shown in Fig. 5a.

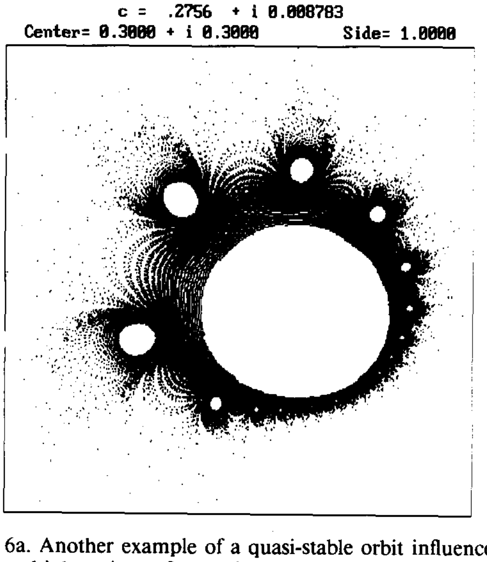


Fig. 6a. Another example of a quasi-stable orbit influenced by multiple regions of attraction about a central attractor.

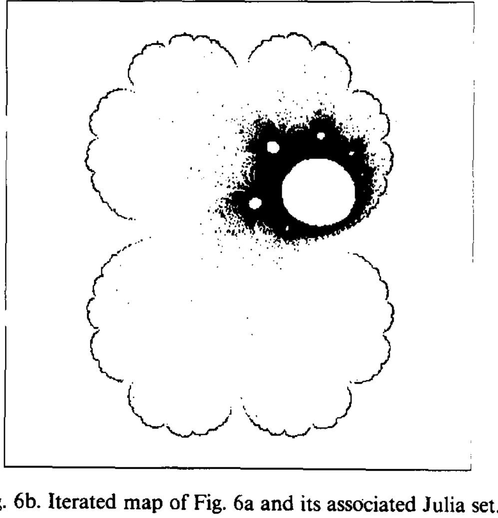


Fig. 6b. Iterated map of Fig. 6a and its associated Julia set.

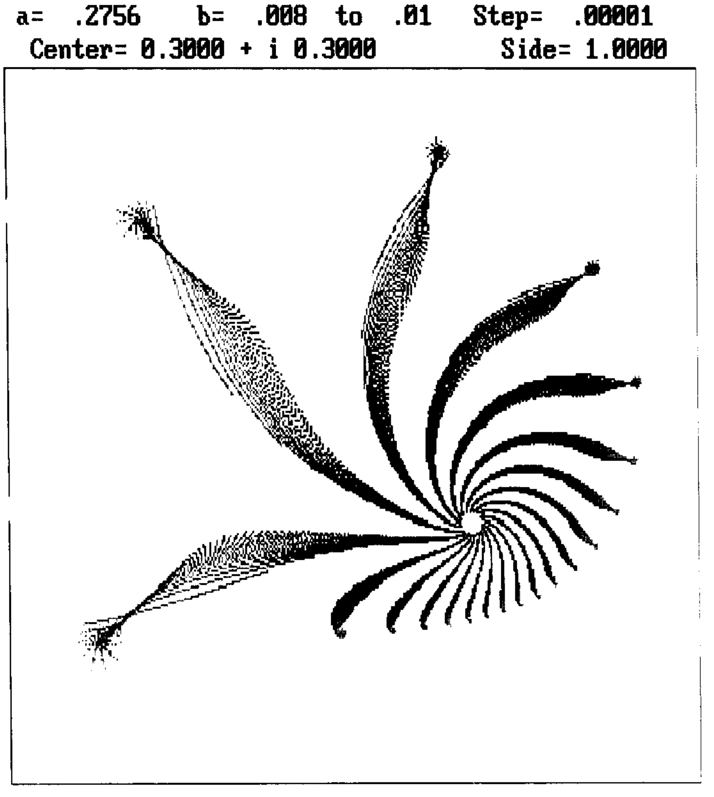


Fig. 6c. Stepwise iterated map about the c value chosen for the 0 orbit shown in Fig. 6a.

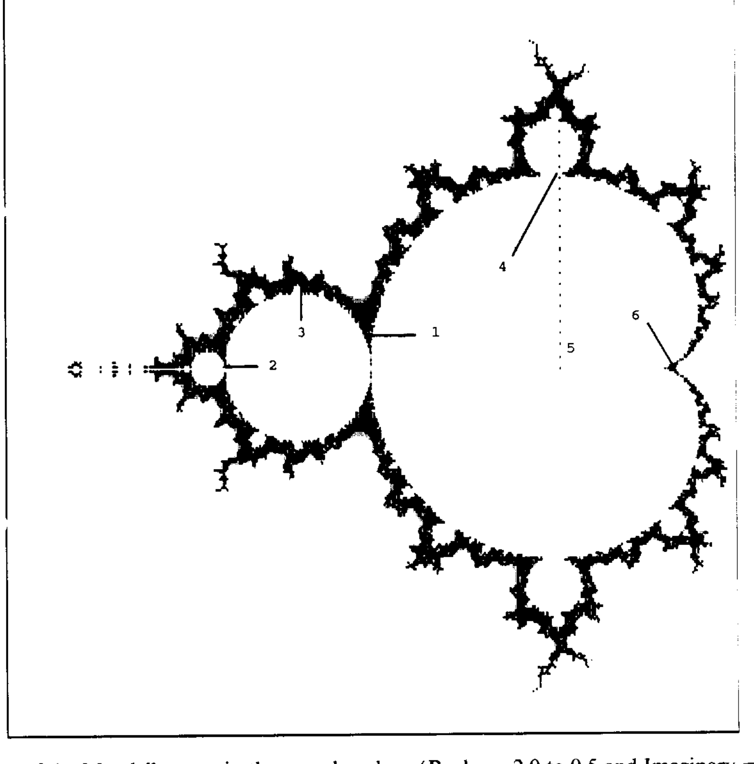


Fig. 7. Display of the Mandelbrot set in the complex plane (Real = -2.0 to 0.5 and Imaginary = -1.25 to 1.25), showing the location of the c values chosen for various iterated maps shown in the corresponding figure numbers. The vertical dotted line is the track of the orbits displayed in Fig. 5.

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APPENDIX A

All maps were computed with programs written with the mathcoprocessor version of Microsoft Quick Basic ver. 3.0. Double precision variables were used with all calculations. Different computer languages and compilers may, of course, produce slightly varied results due to more, or less, computational precision.

APPENDIX B

Input and display parameters for iterated maps shown in figures.

			Display Window				
			Center				
Fig.	Real	Imaginary	x	y	x side	y side	time of escape
1a	−0.7458878	0.1125231	—	—	—	—	650
1b	−0.7458878	0.1125231	−0.5	0.0	1.5	1.5	650
1c	−0.7458878	0.1125231	0.0	0.3	3.2	2.0	650
2a	−1.251	.010877295	0.06	0.08	0.3	0.3	70,123
2b	−1.251	.010877295	0.0	0.0	3.5	3.5	70,123
2c	−1.251	.010877295	0.0	0.0	1.0	1.0	70,123
3a	−0.989	0.249758	0.0404	0.2304	0.6	0.6	2,211,697
3b	−0.989	0.249758	0	0	3.4	3.4	2,211,697
4a	−0.1145001	0.650005	−0.24	0.38	1.0	1.0	7,587
4b	−0.1145001	0.650005	0.0	0.0	2.8	2.8	7,587
5a	−0.1	0.0 to 0.84 (Step = .01)	−0.25	0.31	1.2	1.2	*150
5b	−0.1	0.41	0.0	0.0	2.7	2.7	—
5c	−0.1	0.64	0.0	0.0	2.7	2.7	—
5d	−0.1	0.66	0.0	0.0	2.7	2.7	—
5e	−0.1	0.83	0.0	0.0	2.7	2.7	—
6a	0.2756	0.008783	0.3	0.3	1.0	1.0	327,648
6b	0.2756	0.008783	0.0	0.0	2.5	2.5	327,648
6c	0.2756	0.008 to 0.01 (Step = .00001)	0.3	0.3	1.0	1.0	*300

* Maximum number of iterations displayed per step.