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Continuous Interpolation of the Complex Discrete Map $z \rightarrow \lambda z(1-z)$, and Related Topics

On the dynamics of iterated maps, **IX.**

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Abstract

One striking aspect of the orderliness of chaos is that many of its geometric aspects are governed by fractals, and that many of its physical aspects are governed by its fractal geometry. The present work reports several observations concerning the dynamics of a continuous interpolate, forward and backward, of the quadratic map of the complex plane. In the difficult limit case $|\lambda| = 1$, the dynamics is known to have rich structures that depend on whether Arg $\lambda/2\pi$ is rational or is a Siegel number. This paper establishes that these rich structures have counterparts for $|\lambda| < 1$. The observations concern an intrinsic tiling that covers the interior of a J-set and rules the Schröder interpolation of the forward dynamics, its intrinsic inverse, and the periodic or chaotic limit properties of the intrinsic inverse.

1. Introduction

One striking aspect of the orderliness of chaos is that many of its geometric aspects are governed by fractals, and that many of its physical aspects are governed by its fractal geometry. The present work reports several observations on the fractal geometry of the dynamics of iterated maps.

This paper is number IX in a series that stretches over the years [1], but if the results needed from previous Parts are restated. The iteration of rational maps of the complex plane was studied actively circa 1918. Since 1980, special interest attaches to families of rational maps parameterized by complex numbers. Blanchard [2] gives a nice survey with an extensive bibliography. Two fractal sets play an essential role here: The *J-set* (Julia 1918, Fatou 1919) in the z-plane, and the boundary of the *M-set* (Mandelbrot 1980 [1, Part I, 3]) in the parameter space. A map's J-set is the closure of the unstable fixed points and fixed cycles, and the M-set is the set of parameter values for which the J-set is connected. To simplify the statements while enhancing the credit due to P. Fatou, the complement of the J-set is called *F-set* in [2]; here, the maximal open components of the F-set will be called *F-components.*

To simplify this paper, the discussion will be limited to the inexhaustible quadratic map written in the form $z \rightarrow f(z) = \lambda z - \lambda z^2$; it reduces to $z \rightarrow f^*(z) = z^2 - \mu$, with $\mu = \lambda^2/4 - \lambda/2$, by a linear change of z. F-components are defined for parameter values in the M^0-set (a subset of M), i.e., the set of parameter values such that the map $f(z)$ has a finite limit cycle (plus the usual limit point at infinity). The advantage of the form $f(z)$ is [1, Parts I or II, 3, p. 188 and 189] that the disc $|\lambda| < 1$ belongs to M^0, and that for $|\lambda| < 1$ the limit point is $z = 0$ and the multiplier at the limit point is λ itself.

This paper reports on new structures that apply for $|\lambda| < 1$, and are dominated by the argument $\theta = \text{Arg}\,\lambda$. It is known that when $|\lambda| = 1$ the arithmetic properties of $\theta/2\pi$ determine such structures as bifurcation and the "petal" [2, p. 101], or Siegel discs. It is shown in this paper that generalizations to the above structures are already present when $|\lambda| < 1$.

These structures were observed thanks to the fractal geometric intuition attained while illustrating (apparently, for the first time) the behavior of the solutions of the Schröder equation, and deducing an intrinsic tiling of the F-components.

The main observation is a "universality" result: The topological structure of the intrinsic tiling depends solely on θ and not on $|\lambda|$. Hence, this topological structure can be directly inferred from the bifurcation/petal or Siegel disc structure that corresponds to the same θ in the limit case $|\lambda| = 1$.

A corollary is that the fractal dimension of the J-set varies smoothly as $|\lambda| \rightarrow 1$ while θ is fixed. This may account intuitively for certain theorems I have heard sketched by N. Sibony.

These and related observations touch upon a topic that has aroused a little interest for a long time, and is mentioned in the title because of its attractiveness. The iterates $z_0, z_1 = f(z_0) \ldots, z_k = f(z_{k-1}) = f_k(z_0)$, which form a sequence with an integer index k, can be embedded intrinsically into a sequence $z_t = f_t(z_0)$, where t is real, by solving the "Schröder equation". When z_0 is in the fundamental tile of the intrinsic tiling, the continuous time can, moreover, be inverted, so that $z_t = f_t(z_0)$ becomes defined for positive *and* negative reals. The forward motion is very plain when $|\lambda| < 1$, since there is a stable attractor point, but the arithmetic properties of $\theta/2\pi$ are essential to the dynamics of the backward motion.

It cannot be excluded that some observations reported here are known, but were reviewed as non-intuitive, hence had remained obscure. While I have been spared such rediscoveries, thus far, they have been all-too-common in the study of iteration!

2. The Schröder function

General values of $\lambda \in M^0$ will be considered in Section 8, but elsewhere we suppose for simplicity that $\lambda \neq 0$ but $|\lambda| < 1$. The J-set is then a Jordan curve whose interior – which is the bounded F-component – is the domain of attraction of $z_t = 0$. To each point in this interior, Schröder long ago (circa 1870) attached the function $\sigma(z) = \lim_{k \rightarrow \infty} \lambda^{-k} f_k(z)$. This function satisfies the Schröder functional equation $\sigma[f(z)] = \lambda\sigma(z)$.

The Figures give, for various values of λ, the approximate maps of the function $\sigma(z)$, in the form of a fan of curves of constant argument, and also, in some cases, of curves of constant modulus. To construct them, cut slices of pie fanning from z_t, alternatingly colored in black and in white, and separated by equidistant half-lines. Then draw an annulus around $z_t = 0$, whose outer radius is some small $\rho^* > 0$. When $f_k(z_0)$ first falls within the disc $|f_k(z_0)| < \rho^*$, check whether $\lambda^{-k} f_k(z_0)$ falls on a black or a white point, and color z_0 accordingly. This yields black strips whose boundaries approximate the isolines of the argument of $\theta(z)$. To cut these black strips by white pieces of curves that outline approximate isolines of $|\sigma(z)|$, do not color z_0 if the first $f_k(z_0)$ of modulus $< \rho^*$ is of modulus $> \rho^*(1-\epsilon)$, with a suitable small $\epsilon > 0$. As $\rho^* \to 0$, the approximation improves, and the spurious pattern near $z = 0$ ebbs away.

3. The intrinsic tiling

It is clear by inspection of the Figures that the Schröder function defines an intrinsic tiling of the bounded F-component. The tiles' boundaries are smooth curves, except for a countable number of 90° kinks. The tile that contains z_f is to be called the *fundamental tile*, and every other tile is a pre-image of this fundamental tile.

(There is a resemblance with the hyperbolic tiling developed by H. Poincaré and F. Klein for Kleinian groups of fractional linear maps of the complex plane.)

4. Schröder interpolation of $\lambda z(1-z)$ to continuous time, in the fundamental tile

In terms of the Schröder function $\sigma(z)$, the interpolation of $f(z)$ in the fundamental tile simply amounts to using a logarithmic spiral to interpolate a sequence of points known to lie along this spiral. In terms of σ, this intrinsic interpolation is continuous and invertible, both forward, between z_0 and the forward attractor $z_f = 0$, and backwards, between z_0 and a backward attractor B, which is a subset of the J-set to be discussed in Section 6. In terms of z, however, the interpolation is not continuous, except when λ is real and positive. Otherwise, this interpolation is continuous in the disc around $z_f = 0$, in which the leaves of the fundamental domain are attached to each other. In the remainder of the fundamental tile, where the leaves are split, the interpolation is discontinuous.

(In addition to the above "fundamental", one can define "harmonic" and "subharmonic" interpolations. Given an integer h, the harmonic interpolation is the map of the curve $\sigma(z_t) = (\lambda^{1/h})^t \sigma(z_0)$, and the subharmonic interpolation is the map of the curve $\sigma(z_t) = (\lambda^h)^t \sigma(z_0)$. The harmonics contain all the points $\sigma(z_k) = \lambda^k \sigma(z_0)$, but the subharmonics do not.)

(An alternative interpolation, in terms of the "Poincaré function" was advanced by S. Lattès in 1917; Dubuc [4] compares the two interpolations.)

5. Dependence of the intrinsic tiling upon λ

A striking observation is that the intrinsic tiling topology only depends on $\theta = \operatorname{Arg}\lambda$ and not on $|\lambda|$, and is a sharply discontinuous function of θ (Figs. 1 and 2). This is a noteworthy result, because the J-set that wraps up the tiling varies continuously with $|\lambda|$ and θ, as long as $\lambda \in M^0$.

Interesting consequences follow, concerning the intrinsic interpolation of $f(z)$ as function of time and of λ. The forward motion from a given z_0 is continuous in θ and $|\lambda|$. The backward motion is continuous in $|\lambda|$; its dependence on θ becomes discontinuous after a long enough time, but for a short time it may be continuous.

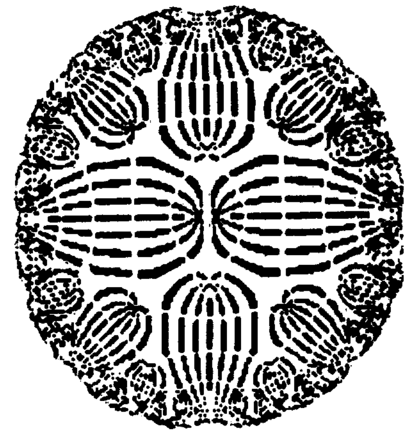

Fig. 1. Intrinsic tiling for λ real $\in\,]0, 1[$.

6. The backward attractor B, intersection of the J-set with the boundary of the fundamental tile, and the leaves of the fundamental tile

The set B is of course invariant but unstable under the action of $f(z)$. It will be called the fundamental invariant set of J.

The rational case. If $\theta/2\pi = m/n$, the set B contains n points, each of them being invariant under the action of $f_n(z)$. The direct map $f(z)$ has a fixed point as attractor, hence a cycle of size 1. However, the intrinsic determination of the multivalues inverse map f^{-1}, identified as lying in the fundamental tile, has a limit cycle of size n, namely B.

To each point of B corresponds an interval of values of $\operatorname{Arg}[\sigma(z)]$ of width $2\pi/n$, which reduces to an interval of values of $\operatorname{Arg} z$ near $z = 0$. Each interval defines a "leaf" in the fundamental tile. The leaves are attached to each other in the neighborhood of $z_f = 0$, but they split as one moves away from z_f, and are separate in the neighborhood of the J-set. Each leaf should be viewed as made of two half-leaves, separated by a "rib" that bisects the leaf in terms of $\operatorname{Arg}[\sigma(z)]$.

It is known that the J-set can be mapped intrinsically upon a circle, hence each point of the J-set can be given an intrinsic argument φ. Each point of the B-set has an argument φ and an interval of arguments $\operatorname{Arg}[\sigma(z)]$. The action of $f(z)$ replaces φ by 2φ, and replaces $\operatorname{Arg}[\sigma(z)]$ by $\operatorname{Arg}[\sigma(z)] + \operatorname{Arg}\lambda$. The mismatch between these two operations is the reason why the study of iteration is rife with fractals.

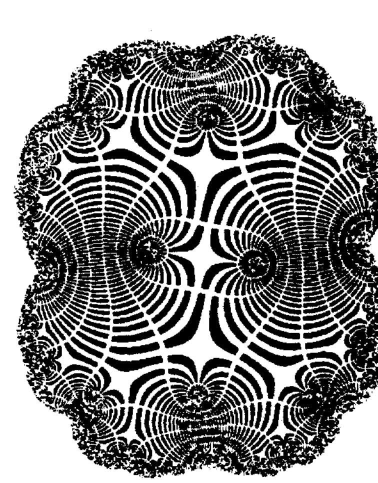

Fig. 2. Intrinsic tiling for a λ near theinterval $]0, 1[$. A small change in λ, compared to Fig. 1, changes the tiling drastically.

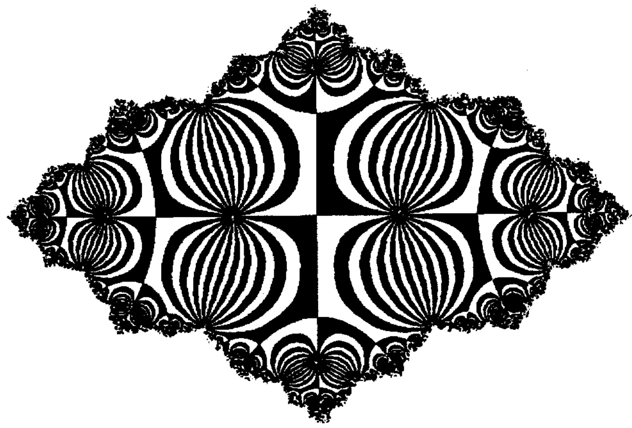

Fig. 3. Intrinsic tiling for λ real $\in]-1, 0[$.

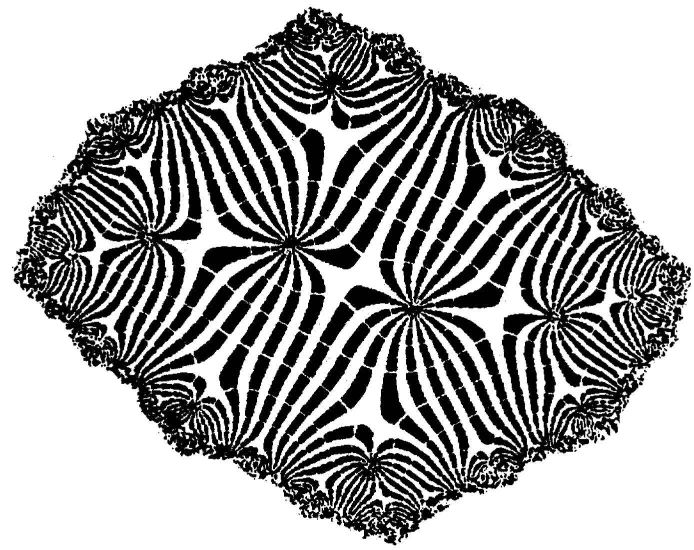

Fig. 4. Intrinsic tiling for $\operatorname{Arg}\lambda/2\pi = m/n$, with $n = 5$, and $|\lambda|$ well below 1.

Observation. The φ arguments of the points of the B-set are identical to the φ arguments that the point $z_f = 0$ takes in the limit case when λ is replaced by the parameter $\lambda' = \lambda/|\lambda|$ and bifurcation occurs. In other words, the φ arguments of the n points of B only depend upon λ via the value of θ.

On the circle, when the point of argument φ is invariant under the map $\theta \to 2^k\theta$, the ratio $\varphi/2\pi$ is represented by a sequence of binary digits that is periodic with the period 2^k. There are $2^k/k$ cycles, many of them being reducible to each other. There is a rule for determining the special invariant unstable set that maps upon B.

The irrational case. If $\theta/2\pi$ is irrational, B is a Cantor set (fractal dust). In order to become convinced that such invariant Cantor dusts exist, it suffices to take on the circle a point for which $\varphi/2\pi$ is an infinite sequence of binary digits such that 0 is never followed by 0. The points of the form 2φ obviously share this property. (The fractal dimension satisfies $2^D = 1.618$.)

When the B-set is a Cantor dust, each half-leaf is associated with a trema of B (an open interval in the complement of Γ). However, the leaves of the fundamental tile no longer split into two half-leaves. Stated alternatively: one of the half-leaves is degenerate and the rib runs along the side of the leaf that is not pointed towards the midpoint of a trema.

7. Observations relative to the limit $|\lambda| \to 1$, taken radially

In the limit $|\lambda| = 1$, Julia and Siegel had shown it is import-

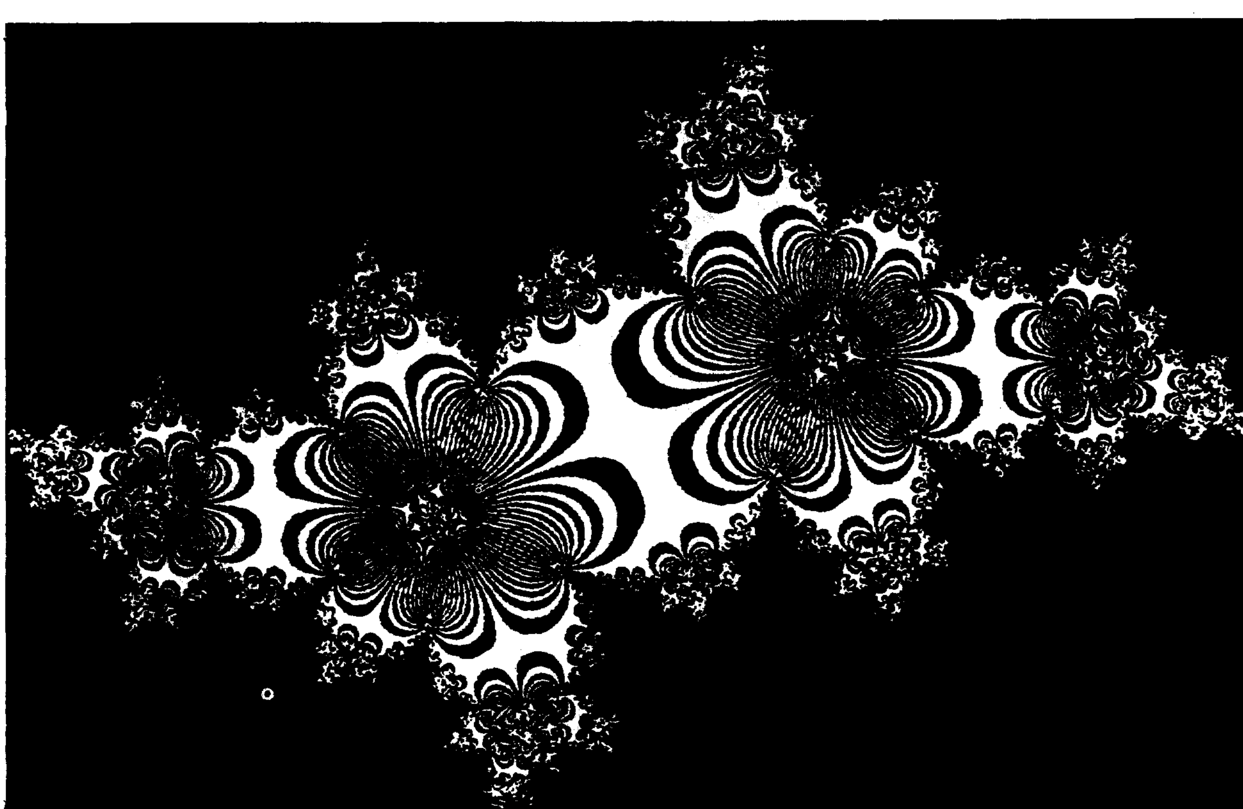

Fig. 5. Intrinsic tiling for the same $\operatorname{Arg}\lambda$ as in Fig. 4, and $|\lambda|$ approaching 1. The leaves split in an overall radial direction.

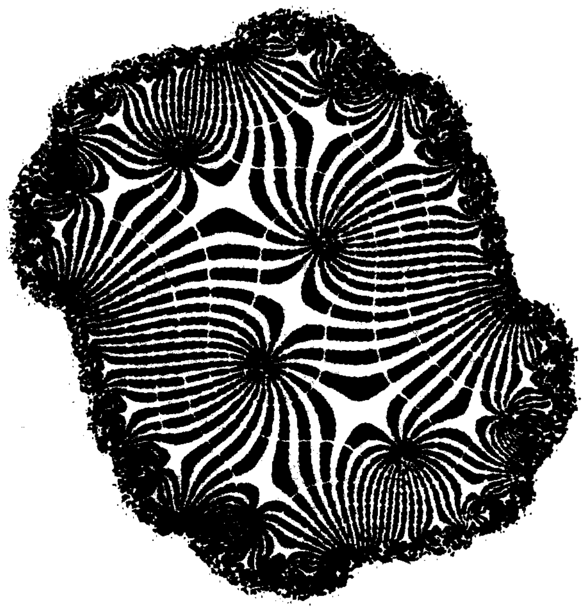

Fig. 6. Intrinsic tiling for Arg $\lambda/1\pi$ the golden mean, a Siegel number, and $|\lambda|$ well below 1.

ant whether the number $\theta/2\pi$ is (a) rational or (b) irrational and a Siegel number, or (c) irrational but not a Siegel number. The rational cases, $\theta/2\pi = m/n$, correspond to $\lambda \in M^0$, and the topology of the J-set is determined mostly by n. When $|\lambda| = 1$ and $\theta/2\pi$ is irrational and a "Siegel number", the J-set is made up of the boundary of a "Siegel disc" and of this disc's pre-images under $f(z)$. The novelty reported in the present paper is that the arithmetic nature of $\theta/2\pi$ is already important for $|\lambda| < 1$. To link the properties relative to $|\lambda| = 1$ and $|\lambda| < 1$, let $|\lambda| \to 1$ radially, that is, with invariant θ. The J-set acquires double points, as the points of the B-set move either toward or around $z_f = 0$.

When $\theta/2\pi$ is the rational m/n, (Figs. 4 and 5) the n points in the B-set all converge to $z_f = 0$. Each leaf splits, along its rib, into its two halves. Thus, each tile becomes split into n pieces, each of them made of two half-leaves, and all having equal widths in terms of Arg $[\sigma(z)]$. The fundamental tile becomes identified with the petal [2, p. 101].

When $\theta/2\pi$ is a Siegel irrational, (Figs. 6 and 7) the endpoints of each trema in the B-set converge circumferentially to the same point – which depends on the trema. As a result, the tiles keep their identity as $|\lambda| \to 1$. The fundamental domain tends

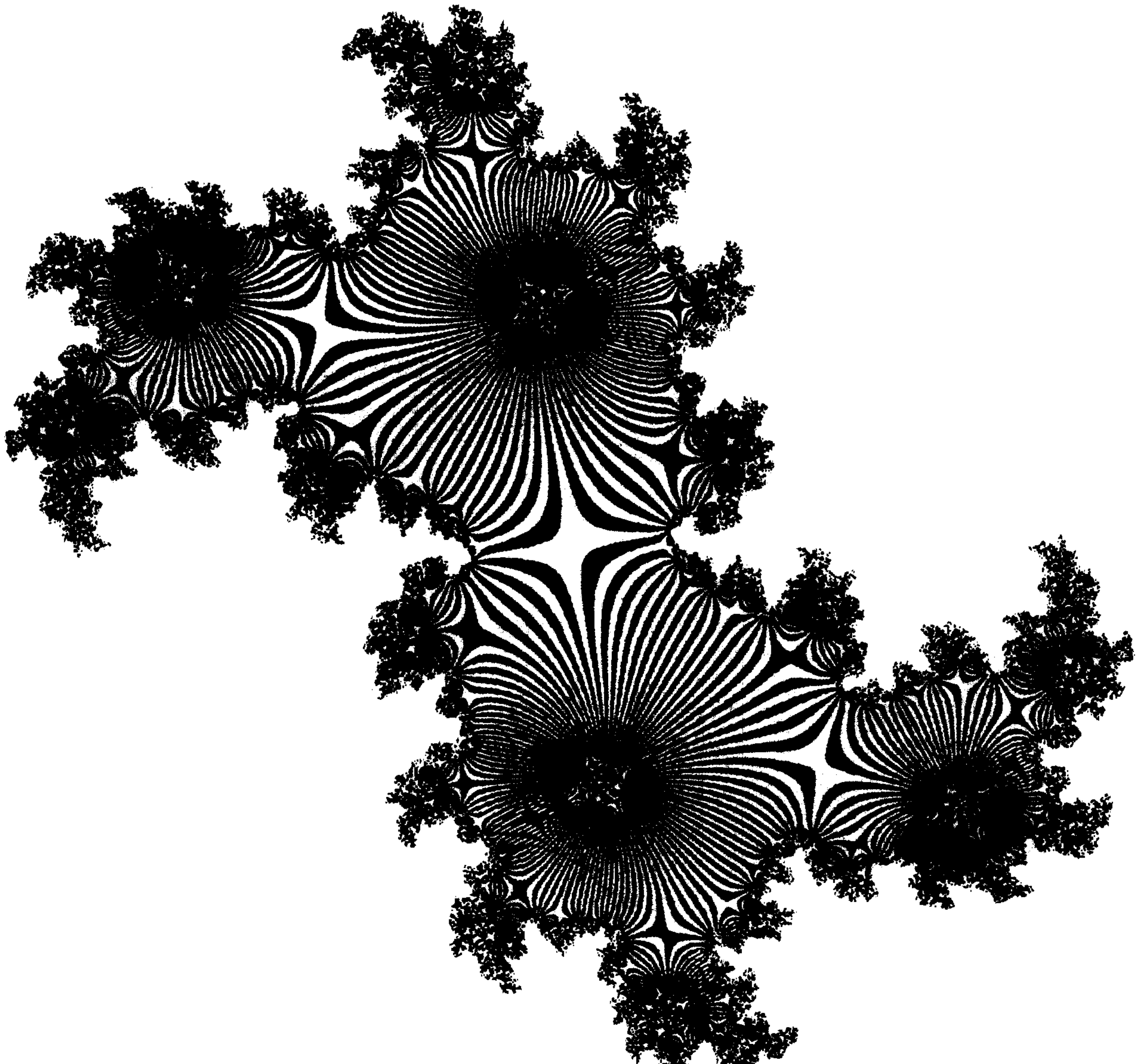

Fig. 7. Intrinsic tiling for the same Arg λ as in Fig. 6, and $|\lambda|$ approaching 1. The leaves do not split, rather are on the way to coalescing into a Siegel disc.

to become increasingly separated from the remainder of the interior of the J-set. At the limit $|\lambda| = 1$, it becomes a Siegel disc, surrounded by a fractal Jordan curve that is part of the J-set.

The remaining case, when $\theta/2\pi$ is a non-Siegel irrational, is known to be hard. A conjectural scenario is that every point in the B-set again converges to $z_f = 0$, and the fundamental domain becomes split into a denumerable infinity of leaves (≡ half-leaves) of unequal widths in terms of Arg $[\sigma(z)]$.

Thus, attainment of $|\lambda| = 1$ has different effects upon the intrinsic tiling and upon its J-set wrapping. The tiling depends discontinuously upon θ, both when $|\lambda| < 1$ and when $|\lambda| = 1$. The wrapping's dependence upon θ is continuous when $|\lambda| < 1$, but discontinuous when $|\lambda| = 1$.

The shape of the J-set changing smoothly as $|\lambda| \to 1$ radially, the fractal dimension $D(\lambda)$ of the J-set converges smoothly to a limit. But for $|\lambda| = 1$, the shape of J varies discontinuously with θ. Therefore, the radial limit of $D(\lambda)$ seems to be an extremely unsmooth function of θ.

8. Case when $\lambda \in M^0$ but $|\lambda| > 1$ and $|\lambda - 2| > 1$

For such λ's, there is a limit cycle of size $N(\lambda)$ and the multiplier is some $\Lambda(\lambda)$ satisfying $|\Lambda| < 1$. To each λ with $|\Lambda| \neq 0$ one can associate a parameter value λ' that lies in the same atom of M^0 and satisfies $|\Lambda(\lambda')| = 1$ and Arg $[\Lambda(\lambda')] =$ Arg $[\Lambda(\lambda)]$. The arithmetic properties of Arg $[\Lambda(\lambda')]/2\pi$ determine the tiling structure for the parameter value λ, and the limit behavior of the intrinsic inverse of $f_N(z)$.

9. The superstable case

For $\lambda = 0$, the argument θ is not defined. This is a superstable parameter value, the nucleus of an atom in the M-set. In that case, the Schröder equation is replaced by the Böttscher equation, and each F-component – including the exterior of the J-set – is a single tile. The discussion is reserved for a later occasion.

Acknowledgement

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