微积分计算公式

WRITTEN & REVIEWED BY MINQI PAN, ZHIJIE ZHOU

(1) 直坐标面元：$\mathrm{d}S = \mathrm{d}x\mathrm{d}y$
(2) 极坐标面元：$\mathrm{d}S = r\,\mathrm{d}r\,\mathrm{d}\theta$

(3) 球坐标面元：$\mathrm{d}S_\rho = \rho^2 \sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta$，微小变动：$\varphi$至$\varphi+\mathrm{d}\varphi$，$\theta$至$\theta+\mathrm{d}\theta$，径向距离$\rho$不变，面元处在球皮上
(4) 球坐标面元：$\mathrm{d}S_\varphi = \rho\,\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta$，微小变动：$\rho$至$\rho+\mathrm{d}\rho$，$\theta$至$\theta+\mathrm{d}\theta$，天顶角$\varphi$不变，面元处在一个顶点为原点的圆锥面上
(5) 球坐标面元：$\mathrm{d}S_\theta = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi$，微小变动：$\rho$至$\rho+\mathrm{d}\rho$，$\varphi$至$\varphi+\mathrm{d}\varphi$，方位角$\theta$不变，面元处在一个竖直半平面上
(6) 球坐标体元：$\mathrm{d}V = \rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$

(7) 柱坐标面元：$\mathrm{d}S_\rho = \rho\,\mathrm{d}\theta\,\mathrm{d}z$，情形：$\rho$不变，面元处在一个竖直圆柱面上
(8) 柱坐标面元：$\mathrm{d}S_\theta = \mathrm{d}\rho\,\mathrm{d}z$，情形：$\theta$不变，面元处在一个竖直半平面上
(9) 柱坐标面元：$\mathrm{d}S_z = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi$，情形：$z$不变，面元处在一个水平平面上
(10) 柱坐标体元：$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z$

(11) 极坐标换元公式：$x = r\cos\theta$，$y = r\sin\theta$，雅可比：r
(12) 球坐标换元公式：$x = \rho\,\sin\varphi\,\cos\theta$，$y = \rho\,\sin\varphi\,\sin\theta$，$z = \rho\,\cos\varphi$，雅可比：$\rho^2\,\sin\varphi$
(13) 柱坐标换元公式：$x = \rho\cos\theta$，$y = \rho\sin\theta$，$z = z$，雅可比：ρ
(14) 广义极坐标换元：$x = ra\cos\theta$，$y = rb\sin\theta$，雅可比：rab
(15) 广义球坐标换元：$x = a\rho\,\sin\varphi\,\cos\theta$，$y = b\rho\,\sin\varphi\,\sin\theta$，$z = c\rho\,\cos\varphi$，雅可比：$abc\rho^2\,\sin\varphi$

(16) $f(x)$旋转生成体的侧面积：$2\pi\int_a^b f(x)\sqrt{1+\left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right]^2}\,\mathrm{d}x$，体积：$\pi\int_a^b [f(x)]^2\mathrm{d}x$

(17) 极坐标表示的弧$r(\theta)$从$\theta=a$到$\theta=b$间的弧长：$\int_a^b\sqrt{[r(\theta)]^2+\left[\frac{\mathrm{d}r(\theta)}{\mathrm{d}\theta}\right]^2}\mathrm{d}\theta$，夹在$\theta=a$和$\theta=b$之间部分的面积：$\frac{1}{2}\int_a^b[r(\theta)]^2\,\mathrm{d}\theta$

(18) 曲线$\mathbf{r}(t)$在$\mathbf{r}(t_0)$处的切线方程：$\mathbf{r}(t) = \mathbf{r}(t_0) + \mathbf{r}'(t_0)(t-t_0)$
(19) 曲线$\mathbf{r}(t)$的线元：$\mathrm{d}l = |\mathbf{r}'(t)|\mathrm{d}t$
(20) 曲线$\mathbf{r}(t)$从$t=a$到$t=b$段的弧长：$\int_a^b|\mathbf{r}'(t)|\,\mathrm{d}t$

(21) 曲面$\mathbf{r}(u,v)$的法向量：$\mathbf{n} = \frac{\partial\mathbf{r}}{\partial u}(u,v)\times\frac{\partial\mathbf{r}}{\partial v}(u,v)$
(22) 曲面$\mathbf{r}(u,v)$过$\mathbf{r_0}$点的切平面方程：$(\mathbf{r}-\mathbf{r_0})\cdot\mathbf{n} = 0$，其中$\mathbf{n}$为$\mathbf{r_0}$处法向量
(23) 曲面$\mathbf{r}(u,v)$的面元：$\mathrm{d}S = |\mathbf{n}|\mathrm{d}u\mathrm{d}v$，其中$\mathbf{n}$为面元处的法向量（定义如上上条，不可随意数乘）
(24) 为计算方便，定义$E = \mathbf{r}_u\cdot\mathbf{r}_u, F = \mathbf{r}_u\cdot\mathbf{r}_v, G = \mathbf{r}_v\cdot\mathbf{r}_v$，则曲面$\mathbf{r}(u,v)$的面元：$\mathrm{d}S = \sqrt{EF-G^2}dudv$

(25) 特别地，当$\mathbf{r}(x,y) = (x,y,f(x,y))$，曲面$\mathbf{r}(x,y)$的法向量：$\mathbf{n} = (-f_x,-f_y,1)$
(26) 特别地，当$\mathbf{r}(x,y) = (x,y,f(x,y))$，曲面$\mathbf{r}(x,y)$的面元：$\mathrm{d}S = \sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\mathrm{d}x\mathrm{d}y$
(27) 特别地，当$\mathbf{r}(x,y) = (x,y,f(x,y))$，曲面$\mathbf{r}(x,y)$的面积：$\iint_D\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\mathrm{d}x\mathrm{d}y$

(28) 直坐标系线元：$\mathrm{d}r^2 = \mathrm{d}x^2+\mathrm{d}y^2$
(29) 极坐标系线元：$\mathrm{d}r^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^{\ 2}$
(30) 球坐标系线元：$\mathrm{d}r^2 = \mathrm{d}\rho^2+\rho^2\mathrm{d}\varphi^2+\rho^2\sin^2\varphi\,\mathrm{d}\theta^2 \quad\Rightarrow\quad \mathrm{d}\mathbf{r} = \mathrm{d}\rho\,\hat{\boldsymbol{\rho}}+\rho\,\mathrm{d}\varphi\,\hat{\boldsymbol{\varphi}}+\rho\sin\varphi\mathrm{d}\theta\,\hat{\boldsymbol{\theta}}$
(31) 柱坐标系线元：$\mathrm{d}r^2 = \mathrm{d}\rho^2+\rho^2\mathrm{d}\theta^2+dz^2 \quad\Rightarrow\quad \mathrm{d}\mathbf{r} = \mathrm{d}\rho\,\hat{\boldsymbol{\rho}}+\rho\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}}+\mathrm{d}z\,\hat{\mathbf{z}}$

(32) 直坐标系nabla算子：$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}$
(33) 柱坐标系nabla算子：$\nabla = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}+\hat{\boldsymbol{\theta}}\frac{1}{\rho}\frac{\partial}{\partial\theta}+\hat{\mathbf{z}}\frac{\partial}{\partial z}$,
(34) 直坐标系Laplace算子：$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \nabla\cdot\nabla = \nabla^2$
(35) 柱坐标系Laplace算子：$\Delta f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial z^2}$.
(36) 球坐标系Laplace算子：$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$

(37) 牛顿－莱布尼茨公式：$\varphi(\mathbf{b})-\varphi(\mathbf{a}) = \int_a^b\nabla\varphi(\mathbf{r})\cdot\mathrm{d}\mathbf{r}$
(38) 斯托克斯公式：$\oint_{\partial S}\mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_S\nabla\times\mathbf{F}\cdot\mathrm{d}\mathbf{S}$
(39) 高斯公式：$\oiint_{\partial\Omega}\mathbf{A}\cdot\mathrm{d}\mathbf{S} = \iiint_\Omega\nabla\cdot\mathbf{A}dv$

(40) $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$
(41) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$
(42) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$
(43) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})\,(\mathbf{B}\cdot\mathbf{D})-(\mathbf{B}\cdot\mathbf{C})\,(\mathbf{A}\cdot\mathbf{D})$
(44) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\,\mathbf{C}-(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\,\mathbf{D}$

(45) $\mathrm{curl}\,(\mathrm{grad}\,f) = \nabla\times(\nabla f) = 0$
(46) $\mathrm{div}\,(\mathrm{curl}\,\vec{v}) = \nabla\cdot\nabla\times\vec{v} = 0$
(47) $\mathrm{div}\,(\mathrm{grad}\,f) = \nabla\cdot(\nabla f) = \nabla^2 f = \Delta f$
(48) $\nabla\times\nabla\times\vec{v} = \nabla(\nabla\cdot\vec{v})-\nabla^2\vec{v}$
(49) $\nabla(\nabla\cdot\vec{v}) = \nabla\cdot(\nabla\otimes\vec{v})$
(50) $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}$
(51) $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A}+\nabla\times\mathbf{B}$
(52) $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$
(53) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$
(54) $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$
(55) $\mathbf{A}\times(\nabla\times\mathbf{B}) = \nabla_B\,(\mathbf{A}\cdot\mathbf{B})-(\mathbf{A}\cdot\nabla)\,\mathbf{B}$,
(56) $\nabla\cdot(\psi\mathbf{A}) = \mathbf{A}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{A}$
(57) $\nabla\times(\psi\mathbf{A}) = \psi\nabla\times\mathbf{A}+\nabla\psi\times\mathbf{A}$
(58) $\nabla(\psi\,\phi) = \phi\,\nabla\psi+\psi\,\nabla\phi$

(59) $\mathrm{arctanh}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$
(60) $\mathrm{arcsinh}(x) = \ln(x+\sqrt{x^2+1})$

(61) $\int\frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x)$
(62) $\int\frac{1}{1-x^2}\,\mathrm{d}x = \mathrm{arctanh}(x)$
(63) $\int\frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \mathrm{arcsinh}(x)$
(64) $\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin(x)$
(65) $\int\sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\,x\sqrt{1+x^2}+\frac{1}{2}\,\mathrm{arcsinh}\,(x)$
(66) $\int\sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\,x\sqrt{1-x^2}+\frac{1}{2}\,\arcsin\,(x)$
(67) $\int\tan x\,\mathrm{d}x = -\ln\,(\cos\,(x))$
(68) $\int\cot x\,\mathrm{d}x = \ln\,(\sin\,(x))$
(69) $\int\frac{1}{\cos^2(x)}\,\mathrm{d}x = \tan x$
(70) $\int\frac{1}{\sin^2(x)}\,\mathrm{d}x = -\cot x$