QUANTUM ALGORITHMS
HOMEWORK 13 SELECTED SOLUTIONS

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AP 1. Let $\iota : \{0,1\} \to \{0,1\}$ be the identity function, defined by $\iota(x) = x$. The function $\iota_\oplus(x,y) = (x, x \oplus y)$ has the property that

$$\iota_\oplus(x,0) = (x,x),$$

meaning that it *clones* the bit x in the first register to the second register.

Let $|\psi\rangle \in \mathfrak{B}$ be an arbitrary 1-qubit quantum state. Show that

$$\widehat{\iota_\oplus}(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \qquad \Leftrightarrow \qquad |\psi\rangle = |0\rangle \text{ or } |\psi\rangle = |1\rangle\,.$$

That is, the quantum operator $\widehat{\iota_\oplus}$ corresponding to the classical 1-bit cloning operator $\iota_\oplus$ fails to clone $|\psi\rangle$ unless $|\psi\rangle$ is in a state corresponding to a classical bit.

Solution: Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. If $\widehat{\iota_\oplus}$ clones $|\psi\rangle$ then

$$\begin{aligned}\widehat{\iota_\oplus}\big(\,|\psi\rangle \otimes |0\rangle\,\big) &= |\psi\rangle \otimes |\psi\rangle = \big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big) \otimes \big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big)\\ &= \alpha^2\,|00\rangle + \alpha\beta\,|01\rangle + \alpha\beta\,|10\rangle + \beta^2\,|11\rangle\,.\end{aligned}$$

Evaluating directly yields

$$\begin{aligned}\widehat{\iota_\oplus}\big(\,|\psi\rangle \otimes |0\rangle\,\big) &= \widehat{\iota_\oplus}\big((\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |0\rangle\,\big) = \alpha\,\widehat{\iota_\oplus}\big(\,|0\rangle \otimes |0\rangle\,\big) + \beta\,\widehat{\iota_\oplus}\big(\,|1\rangle \otimes |0\rangle\,\big)\\ &= \alpha\,|00\rangle + \beta\,|11\rangle\,.\end{aligned}$$

It follows that $\widehat{\iota_\oplus}$ clones ψ if and only if

$$\alpha^2\,|00\rangle + \alpha\beta\,|01\rangle + \alpha\beta\,|10\rangle + \beta^2\,|11\rangle = \alpha\,|00\rangle + \beta\,|11\rangle\,.$$

Vectors are equal if and only if their basis coefficients are equal, so this is true if and only if

$$\alpha = \alpha^2, \qquad \beta = \beta^2, \qquad \alpha\beta = 0.$$

$\alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$. Combined with $|\alpha|^2 + |\beta|^2 = 1$, this gives two solutions: $(\alpha,\beta) = (1,0)$ or $(\alpha,\beta) = (0,1)$. These correspond to $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$, respectively.

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AP 2. Let U be a $(2n)$-qubit operator that *clones* two n-qubit quantum states, $|\varphi\rangle, |\psi\rangle \in \mathfrak{B}^{\otimes n}$, meaning

$$U(|\varphi\rangle \otimes |0^n\rangle) = |\varphi\rangle \otimes |\varphi\rangle \qquad \text{and} \qquad U(|\psi\rangle \otimes |0^n\rangle) = |\psi\rangle \otimes |\psi\rangle .$$

Prove that U clones $|\varphi\rangle$ and $|\psi\rangle$ if and only if $|\varphi\rangle = |\psi\rangle$ or $\langle \varphi \mid \psi \rangle = 0$. [*Hint: take the inner product of the two equations.*]

Solution:

Proof. We will take the inner product of the left and right hand sides of the equations, respectively, and then equate them. We begin with the left hand sides. Using the fact that U is unitary (and hence preseves inner product) and $\otimes$ distributes over the inner product, we have

$$\begin{aligned}\Big\langle \Big(U(|\varphi\rangle \otimes |0^n\rangle)\Big) \mid \Big(U(|\psi\rangle \otimes |0^n\rangle)\Big) \Big\rangle &= \big(\, \langle\varphi| \otimes \langle 0^n| \,\big)\big(\, |\psi\rangle \otimes |0^n\rangle \,\big) \\ &= \langle \varphi \mid \psi\rangle \langle 0^n \mid 0^n\rangle = \langle \varphi \mid \psi\rangle .\end{aligned}$$

Taking the inner product of the left hand sides, we have

$$\big(\, \langle\varphi| \otimes \langle\varphi| \,\big)\big(\, \langle\psi| \otimes \langle\psi| \,\big) = \langle \varphi \mid \psi\rangle \langle \varphi \mid \psi\rangle = \langle \varphi \mid \psi\rangle^2$$

Equating these two yields $\langle \varphi \mid \psi\rangle = \langle \varphi \mid \psi\rangle^2$. The only solutions to $x^2 - x = 0$ in $\mathbb{C}$ are $x = 0$ and $x = 1$, so we have that $\langle \varphi \mid \psi\rangle = 0$ or $\langle \varphi \mid \psi\rangle = 1$. The first implies that $|\varphi\rangle$ and $|\psi\rangle$ are orthogonal, while the second implies that $|\varphi\rangle = |\psi\rangle$ since $\| \, |\varphi\rangle \, \| = \| \, |\psi\rangle \, \| = 1$. All of the logic in this proof can be reversed, so the same proof works for the converse. □

AP 3. Use the previous question to prove that a there are no quantum cloning operators that work for all pairs of states.

Solution:

Proof. Let $|\varphi\rangle, |\psi\rangle \in \mathfrak{B}^{\otimes n}$ be a pair of non-equal non-orthogonal states. For instance,

$$|\varphi\rangle = |0^n > \qquad \text{and} \qquad |\psi\rangle = \frac{1}{\sqrt{2}}\big(\, |0^n\rangle + |1^n\rangle \,\big).$$

Suppose that U clones both $|\varphi\rangle$ and $|\psi\rangle$. By the previous problem this implies that $|\varphi\rangle = |\psi\rangle$ or $\langle \varphi \mid \psi\rangle = 0$. Neither of these are true, so it must be that no such U exists. □