Finals

2. Show that $[D] * [E] = [D \otimes E]$

2.1. We have below information,

$$D|v_1\rangle = |w_1\rangle + |w_2\rangle + |w_3\rangle \Rightarrow (D|v_1\rangle)_w = [1\ 1\ 1]^T$$
$$D|v_2\rangle = 2|w_2\rangle - |w_3\rangle \Rightarrow (D|v_2\rangle)_w = [0\ 2\ -1]^T$$

$$E|x_1\rangle = |y_1\rangle - |y_2\rangle \Rightarrow (E|x_1\rangle)_y = [1\ -1]^T$$
$$E|x_2\rangle = 2|y_2\rangle \Rightarrow (E|x_2\rangle)_y = [0\ 2]^T$$
$$E|x_3\rangle = |y_1\rangle + |y_2\rangle \Rightarrow (E|x_3\rangle)_y = [1\ 1]^T$$

2.2. We know that,

$$D : V \to W = [(D|v_1\rangle)_w \quad (D|v_2\rangle)_w]$$
$$E : X \to Y = [(E|x_1\rangle)_y \quad (E|x_2\rangle)_y \quad (E|x_3\rangle)_y]$$

$$D_{V\to W} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} E_{X\to Y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

2.3. $[D] * [E] = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1E & 0E \\ 1E & 2E \\ 1E & -1E \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 2 \\ -1 & 2 & 1 & -2 & 4 & 2 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 1 & -2 & -1 \end{bmatrix}$$

2.4. We know that $D \otimes E$ is the operator works on $V \otimes X$, $W \otimes Y$
$\Rightarrow (D \otimes E) : V \otimes X \to W \otimes Y$ with dimension $= dim(D \otimes E)$
$\Rightarrow dim(D \otimes E) = dim(V \otimes X) \times dim(W \otimes Y)$
$= (2 * 3) \times (3 * 2) = 6 \times 6$

Suppose $[D \otimes E] = (D \otimes E)|V \otimes X\rangle = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{bmatrix}$, then

$C_1 = (D \otimes E)(v_1 \otimes x_1) = (Dv_1) \otimes (Ex_1) = (w_1 + w_2 + w_3) \otimes (y_1 - y_2)$
$= 1(w_1 \otimes y_1) - 1(w_1 \otimes y_2) + 1(w_2 \otimes y_1) - 1(w_2 \otimes y_2) + 1(w_3 \otimes y_1) - 1(w_3 \otimes y_2)$
$= [1\ -1\ 1\ -1\ 1\ -1]^T$

$C_2 = (D \otimes E)(v_1 \otimes x_2) = (Dv_1) \otimes (Ex_2) = (w_1 + w_2 + w_3) \otimes (2y_2)$
$= 0(w_1 \otimes y_1) + 2(w_1 \otimes y_2) + 0(w_2 \otimes y_1) + 2(w_2 \otimes y_2) + 0(w_3 \otimes y_1) + 2(w_3 \otimes y_2)$
$= [0\ 2\ 0\ 2\ 0\ 2]^T$

$C_3 = (D \otimes E)(v_1 \otimes x_3) = (Dv_1) \otimes (Ex_3) = (w_1 + w_2 + w_3) \otimes (y_1 + y_2)$
$= 1(w_1 \otimes y_1) + 1(w_1 \otimes y_2) + 1(w_2 \otimes y_1) + 1(w_2 \otimes y_2) + 1(w_3 \otimes y_1) + 1(w_3 \otimes y_2)$
$= [1\ 1\ 1\ 1\ 1\ 1]^T$

$C_4 = (D \otimes E)(v_2 \otimes x_1) = (Dv_2) \otimes (Ex_1) = (2w_2 - w_3) \otimes (y_1 - y_2)$
$= 0(w_1 \otimes y_1) - 0(w_1 \otimes y_2) + 2(w_2 \otimes y_1) - 2(w_2 \otimes y_2) - 1(w_3 \otimes y_1) + 1(w_3 \otimes y_2)$
$= [0\ 0\ 2\ \ -2\ \ -1\ 1]^T$

$C_5 = (D \otimes E)(v_2 \otimes x_2) = (Dv_2) \otimes (Ex_2) = (2w_2 - w_3) \otimes (2y_2)$
$= 0(w_1 \otimes y_1) - 0(w_1 \otimes y_2) + 0(w_2 \otimes y_1) + 4(w_2 \otimes y_2) - 0(w_3 \otimes y_1) - 2(w_3 \otimes y_2)$
$= [0\ 0\ 0\ 4\ 0\ \ -2]^T$

$C_6 = (D \otimes E)(v_2 \otimes x_3) = (Dv_2) \otimes (Ex_3) = (2w_2 - w_3) \otimes (y_1 + y_2)$
$= 0(w_1 \otimes y_1) - 0(w_1 \otimes y_2) + 2(w_2 \otimes y_1) + 2(w_2 \otimes y_2) - 1(w_3 \otimes y_1) - 1(w_3 \otimes y_2)$
$= [0\ 0\ 2\ \ -2\ \ -1\ \ -1]^T$

$$\Rightarrow [D \otimes E] = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 2 \\ -1 & 2 & 1 & -2 & 4 & 2 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 1 & -2 & -1 \end{bmatrix}$$

2.5. From points 3 and 4,
$[D] * [E] = [D \otimes E]$

1. Show that $\mathcal{F}_{\mathbb{G}} = \bigotimes_{i=1}^{k} \mathcal{F}_{\mathbb{Z}/m_i\mathbb{Z}}$

1.1. We have a finite abelian group $\mathbb{G} = \Pi_{i=1}^{k}\mathbb{Z}/m_i\mathbb{Z}$
$\Rightarrow \mathbb{G} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$ where, $|G| = p_1 p_2 \ldots p_k$

We can denote the elments of the group $g \in \mathbb{G}$ as a tuple of k elements
$g = (g_1, \ldots, g_k)$ where $g_j \in \mathbb{Z}_{p_j}$

1.2. To define the Fourier transform, we consider the characters of G.
Let $\mathcal{X} : \mathbb{G} \to \mathbb{C}^*$ be a character.
Let $\beta_1 = (1, 0, \ldots, 0) \in \mathbb{G}, \beta_2 = (0, 1, \ldots, 0) \in \mathbb{G}, \ldots, \beta_k = (0, 0, \ldots, 1) \in \mathbb{G}$
Then for any element $g = (g_1, \ldots, g_k)$ we have
$\mathcal{X}(g) = \mathcal{X}(\sum_{j=1}^{k} g_j\beta_j) = \Pi_{j=1}^{k}\mathcal{X}(\beta_j)^{g_j}$

1.3. From above point we know that, $\mathcal{X}(\beta_j)$ is determined by the values of β_j
We also know that the set of characters forms orthogonal basis,
$\mathcal{X}(\beta_j)^{g_j} = \mathcal{X}(1)^{g_j}$
$\Rightarrow \mathcal{X}(\beta_j)$ can be dertermined by N-th root of unity
$\mathcal{X}(\beta_j) = \omega_{p_k}^{h_j}$ for some interger h_j
$\Rightarrow$ a given character from $\mathcal{X}$ can be determined by the tuple $h = (h_1, \ldots, h_k)$
$\mathcal{X}_g(h) = \Pi_{j=1}^{k}\omega_{p_k}^{g_j h_j}$

We know that $g = (g_1, \ldots, g_k)$
$\Rightarrow \sum_{g \in \mathbb{G}} \mathcal{X}_g(h) = \sum_{g_1 \in \mathbb{Z}_1} \cdots \sum_{g_k \in \mathbb{Z}_k} \Pi_{j=1}^{k}\omega_{p_k}^{g_j h_j}$
$\Rightarrow \sum_{g \in \mathbb{G}} \mathcal{X}_g(h) = (\sum_{g_1 \in \mathbb{Z}_1} \omega_{p_1}^{g_1 h_1}) \ldots (\sum_{g_k \in \mathbb{Z}_k} \omega_{p_k}^{g_k h_k})$

1.4. We know that Quantum fourier transform for a given group $\mathbb{G}$ is defined as,
$\mathcal{F}_{\mathbb{G}} = \frac{1}{\sqrt{|\mathbb{G}|}} \sum_{g,h \in G} \mu(g,h)|g\rangle\langle h|$
where $\mu(g,h) = \Pi_{i=1}^{k}\omega_{m_i}^{g_i h_i}$, $\omega_{m_i} = exp(i2\pi/m_i)$

When the group $\mathbb{G}$ is a finite abelian group $\mathbb{G} = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$, then
Quantum fourier transform $\mathcal{F}_{\mathbb{Z}/m_i\mathbb{Z}}$,
$\mathcal{F}_{\mathbb{Z}/m_i\mathbb{Z}} = \frac{1}{\sqrt{|\mathbb{G}|}} \sum_{g,h \in G} \mathcal{X}(g,h)|g_1, \ldots, g_k\rangle\langle h_1, \ldots, h_k|$

from above points, we can rewrite the QFT for finite abeilian groups as,

$$\mathcal{F}_{\mathbb{Z}/m_i\mathbb{Z}} = \frac{1}{\sqrt{|p_1 p_2 \ldots p_k|}}(\sum_{g_1 \in \mathbb{Z}_1} \omega_{p_1}^{g_1 h_1}) \ldots (\sum_{g_k \in \mathbb{Z}_k} \omega_{p_k}^{g_k h_k}))|g_1\rangle \otimes |g_2\rangle \cdots \otimes |g_k\rangle\langle h_1| \otimes \langle h_2| \cdots \otimes \langle h_k|$$

$$\mathcal{F}_{\mathbb{Z}/m_i\mathbb{Z}} = (\frac{1}{\sqrt{|p_1|}} \sum_{g_1 \in \mathbb{Z}_1} \omega_{p_1}^{g_1 h_1})|g_1\rangle\langle h_1|) \otimes (\frac{1}{\sqrt{|p_2|}} \sum_{g_2 \in \mathbb{Z}} \omega_{p_2}^{g_2 h_2})|g_2\rangle\langle h_2|) \otimes \cdots \otimes (\frac{1}{\sqrt{|p_k|}} \sum_{g_k \in \mathbb{Z}_k} \omega_{p_k}^{g_k h_k})|g_k\rangle\langle h_k|)$$

1.5. Thus, from above points, we can write

$$\mathcal{F}_{\mathbb{Z}/m_i\mathbb{Z}} = \mathcal{F}_{\mathbb{Z}/m_1\mathbb{Z}} \otimes \mathcal{F}_{\mathbb{Z}/m_2\mathbb{Z}} \otimes \cdots \otimes \mathcal{F}_{\mathbb{Z}/m_k\mathbb{Z}}$$

$$\Rightarrow \mathcal{F}_{\mathbb{G}} = \bigotimes_{i=1}^{k} \mathcal{F}_{\mathbb{Z}/m_i\mathbb{Z}}$$

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"I pledge on my honor that I have neither given nor received unauthorized aid on this assignment."

Signature: Madhu Peduri