Homework 7

8.7. Prove the properties of the Operator Norm

1. Prove $\|X\| \leq \|X\|\|Y\|$

$$\|XY\| = \sup_{\alpha \neq 0} \frac{\|XY|\alpha\rangle\|}{\||\alpha\rangle\|} = \frac{\|XY|\alpha\rangle\|}{\|Y|\alpha\rangle\|}\frac{\|Y|\alpha\rangle\|}{\||\alpha\rangle\|}$$

If we consider $\|Y|\alpha\rangle\| = \beta$ then,

$$\|XY\| = \sup_{\alpha \neq 0} \frac{\|X|\beta\rangle\|}{\||\beta\rangle\|}\frac{\|Y|\alpha\rangle\|}{\||\alpha\rangle\|}$$

$$\leq \sup_{\beta \neq 0} \frac{\|X|\beta\rangle\|}{\||\beta\rangle\|} \sup_{\alpha \neq 0} \frac{\|Y|\alpha\rangle\|}{\||\alpha\rangle\|}$$

$$\leq \|X\|\|Y\|$$

2. Prove $\|X^\dagger\| = \|X\|$

Let λ be the Eigen value such that $\|A\alpha\| = \lambda\|\alpha\|$

$$\|A\| = \sup_{\alpha \neq 0} \frac{\|A|\alpha\rangle\|}{\||\alpha\rangle\|} = \sup_{\alpha \neq 0} \frac{\lambda\|\alpha\|}{\||\alpha\rangle\|}$$

We know that, $XX^\dagger|\alpha\rangle = X^\dagger X|\alpha\rangle = \lambda\alpha$

$$= \sup_{\alpha \neq 0} \frac{\lambda\|\alpha\|}{\||\alpha\rangle\|} = \sup_{\alpha \neq 0} \frac{\|A^\dagger|\alpha\rangle\|}{\||\alpha\rangle\|} = \|A^\dagger\|$$

$$\Rightarrow \|X^\dagger\| = \|X\|$$

3. Prove $\|X \otimes Y\| = \|X\|\|Y\|$

We know that,
$\|X \otimes I|\alpha\rangle\| \leq \|X\|\|\alpha\|$
$\|I \otimes Y|\alpha\rangle\| \leq \|Y\|\|\alpha\|$

If we combine above equations, we get,
$\|X \otimes Y\| \leq \|X \otimes I\|\|I \otimes Y\|$
$= \|X\|\|Y\|$

4. Prove $\|U\| = 1$

$$\|U\| = \sup_{\alpha \neq 0} \frac{\|U|\alpha\rangle\|}{\||\alpha\rangle\|}$$

We know that, Unitary operator preserves the Gemoetry of the vector.
Its inner product will not change,
$\Rightarrow \|U|\alpha\rangle\| = \sqrt{\langle\alpha \mid \alpha\rangle}$
$\||\alpha\rangle\| = \sqrt{\langle\alpha \mid \alpha\rangle}$

$$\|U\| = \frac{\sqrt{\langle\alpha \mid \alpha\rangle}}{\sqrt{\langle\alpha \mid \alpha\rangle}} = 1$$

8.8.a If $\tilde{U}$ approximates U with precision δ, then $\tilde{U}^{-1}$ approximates U^{-1} with the same precision δ

1. Let precision be δ, then $\|\tilde{U} - U\| \leq \delta$.

2. Use ancillas $\tilde{U}^{-1}, U^{-1}$ on the left and right side respectively,

$$\delta \geq \|\tilde{U}^{-1}\|\|\tilde{U} - U\|\|U^{-1}\|$$

$$\geq \|\tilde{U}^{-1}\tilde{U}U^{-1} - \tilde{U}^{-1}UU^{-1}\|$$

3. We know that, $\tilde{U}^{-1}\tilde{U} = UU^{-1} = I$

$$\Rightarrow \delta \geq \|U^{-1} - \tilde{U}^{-1}\|$$

4. From above equation, we can say that $\tilde{U}^{-1}$ approximates U^{-1} with the precision δ

8.8.b If unitary operators $\tilde{U}_k$ approximate unitary operators $U_k (1 \leq k \leq L)$ with precision δ_k, then $\tilde{U}_L \ldots \tilde{U}_1$ approximate $U_L \ldots U_1$ with precision $\sum_k \delta_k$

1. If we consider $k = 2$, the approximate realization can be written as,

$$\|\tilde{U}_1\tilde{U}_2 - U_1U_2\| = \|\tilde{U}_1\tilde{U}_2 - U_1U_2 + \tilde{U}_1U_2 - U_2\tilde{U}_1\|$$

$$= \|\tilde{U}_1(\tilde{U}_2 - U_2) + (\tilde{U}_1 - U_1)U_2\|$$

By using Triangle inequality,
$\leq \|\tilde{U}_1(\tilde{U}_2 - U_2)\| + \|(\tilde{U}_1 - U_1)U_2\|$

$$\leq \|\tilde{U}_1\|\|(\tilde{U}_2 - U_2)\| + \|(\tilde{U}_1 - U_1)\|\|U_2\|$$

2. As our operators are unitary, we can say $\|\tilde{U}_1\| = \|U_2\| = 1$

$$\|\tilde{U}_1\tilde{U}_2 - U_1U_2\| = \|(\tilde{U}_2 - U_2)\| + \|(\tilde{U}_1 - U_1)\| = \delta_2 + \delta_1$$

3. By above equation, we can say that $\tilde{U}_L \dots \tilde{U}_1$ approximate $U_L \dots U_1$ with precision $\sum_k \delta_k$

1.

After K iterations of G in Grover's algorithm, we obtained

$$G^K|\psi(1,1)\rangle = |\psi(\tfrac{1}{\sqrt{a}}\sin((2K+1)\theta), \tfrac{1}{\sqrt{b}}(\cos((2K+1)\theta))\rangle$$

1 what is the probability of measuring state $|A\rangle$

we know that

$$|\psi(\alpha,\beta)\rangle = \alpha|A\rangle + \beta|B\rangle \quad -①$$

where $|A\rangle := \frac{1}{\sqrt{2^n}}|s\rangle$; $|B\rangle = \frac{1}{\sqrt{2^n}}\sum|x\rangle$

$a := \langle A|A\rangle = \frac{1}{2^n}$ $\quad b := \langle B|B\rangle = 1-a \quad -②$

$|\psi(1,1)\rangle = |A\rangle + |B\rangle$ from -①

$$G^K|\psi(1,1)\rangle = |\psi(\tfrac{1}{\sqrt{a}}\sin((2K+1)\theta), \tfrac{1}{\sqrt{b}}\cos((2K+1)\theta))\rangle$$

$$= \frac{1}{\sqrt{a}}\sin((2K+1)\theta)|A\rangle + \frac{1}{\sqrt{b}}\cos((2K+1)\theta)|B\rangle \quad \text{- from ①}$$

we know that, if $|v\rangle = d_x|x\rangle + d_y|y\rangle$
the probability of measuring $|x\rangle$ in the state is $|d_x|^2$
the Probability of measuring $|y\rangle$ in the state v is $|d_y|$

Similarly, Probability of measuring $|A\rangle$ in $G^K|\psi(1,1)\rangle$ is

$$= \left|\frac{1}{\sqrt{a}}\sin((2K+1)\theta)\right|^2 = \frac{1}{a}(\sin(2K+1)\theta)^2$$

$$\geq \sin((2K+1)\theta)^2$$

If $K = \pi/4\theta$, $\sin((2K+1)\theta)^2 = \sin(\frac{\pi}{2}+\theta)^2 = (\cos\theta)^2$

we know that

if $a = 1/2^n$ $\sin\theta = \sqrt{a}$; $\cos\theta = \sqrt{b}$

Probability of measuring $|A\rangle$ $\geq (\cos\theta)^2$

$\geq b$

$\geq (1-a)$ (from ②)

2.

(2) validate

$$\begin{bmatrix} b-a & 2b \\ -2a & b-a \end{bmatrix} = \frac{1}{2\sqrt{ab}} \begin{bmatrix} i\sqrt{b} & -i\sqrt{b} \\ \sqrt{a} & \sqrt{a} \end{bmatrix} \begin{bmatrix} \bar{d}^2 & 0 \\ 0 & d^2 \end{bmatrix} \begin{bmatrix} -i\sqrt{a} & \sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{bmatrix}$$

$$d = e^{i\theta} = \sqrt{b} + i\sqrt{a}$$

$$\bar{d} = \sqrt{b} - i\sqrt{a}$$

$$\begin{bmatrix} i\sqrt{b} & -i\sqrt{b} \\ \sqrt{a} & \sqrt{a} \end{bmatrix} \begin{bmatrix} \bar{d}^2 & 0 \\ 0 & d^2 \end{bmatrix} = \begin{array}{cc} ① & ② \\ \left[\begin{array}{c} i\sqrt{b}\,\bar{d}^2 \\ \sqrt{a}\,\bar{d}^2 \end{array}\right. & \left.\begin{array}{c} -i\sqrt{b}\,d^2 \\ \sqrt{a}\,d^2 \end{array}\right] \\ ③ & ④ \end{array}$$

① $i\sqrt{b}\,(b-a-2i\sqrt{ab}) = (ib\sqrt{b} - ia\sqrt{b} + 2b\sqrt{a})$

② $-i\sqrt{b}\,(b-a+2i\sqrt{ab}) = (-ib\sqrt{b} + ia\sqrt{b} + 2b\sqrt{a})$

③ $\sqrt{a}\,(b-a-2i\sqrt{ab}) = (b\sqrt{a} - a\sqrt{a} - 2ia\sqrt{b})$

④ $\sqrt{a}\,(b-a+2i\sqrt{ab}) = (b\sqrt{a} - a\sqrt{a} + 2ia\sqrt{b})$

$$\begin{bmatrix} ① & ② \\ ③ & ④ \end{bmatrix} \begin{bmatrix} -i\sqrt{a} & \sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{bmatrix} = \begin{array}{cc} Ⓟ & Ⓠ \\ \left[\begin{array}{c} -i\sqrt{a}① + i\sqrt{a}② \\ -i\sqrt{a}③ + i\sqrt{a}④ \end{array}\right. & \left.\begin{array}{c} \sqrt{b}\,(① + ②) \\ \sqrt{b}\,(③ + ④) \end{array}\right] \\ Ⓡ & Ⓢ \end{array}$$

$$P = 2\sqrt{ab}\,(b-a)$$

$$Q = 4b\sqrt{ab}$$

$$R = -4a\sqrt{ab}$$

$$S = 2\sqrt{ab}\,(b-a)$$

$$= \frac{1}{\sqrt{2ab}} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \frac{1}{2\sqrt{ab}} \begin{bmatrix} 2\sqrt{ab}\,(b-a) & 4b\sqrt{ab} \\ -4a\sqrt{ab} & 2\sqrt{ab}\,(b-a) \end{bmatrix}$$

$$= \begin{bmatrix} b-a & 2b \\ -2a & b-a \end{bmatrix}$$