QUANTUM ALGORITHMS
HOMEWORK 8 SELECTED SOLUTIONS

PROF. MATTHEW MOORE

AP 1. After k iterations of $\mathcal{G}$ in Grover's algorithm, we obtained

$$\mathcal{G}^k |\Psi(1,1)\rangle = \Big|\Psi\Big(a^{-1/2}\sin\big((2k+1)\theta\big),\ b^{-1/2}\cos\big((2k+1)\theta\big)\Big)\Big\rangle$$

where θ is such that $\sin(\theta) = \sqrt{a}$. Show that when $k = \lfloor \pi/(4\theta) \rfloor$, upon measuring this state the probability of observing a state in $|A\rangle$ is $\geq 1 - a$.

Solution: Given a state vectors $|\alpha\rangle$ and $|\beta\rangle$, the probability of observing $|\alpha\rangle$ to be among the vectors in $|\beta\rangle$ is $|\langle \alpha \mid \beta \rangle|^2$. The vector $|A\rangle$ is not a state vector since it doesn't have norm 1, but we can normalize it. Therefore, the probability of measuring a state $|\alpha\rangle$ and observing it to be among the vectors in $|A\rangle$ is

$$\left| \frac{1}{\| \, |A\rangle \, \|} \langle A \mid \alpha \rangle \right|^2.$$

For the state vector given in the problem, this is

$$\begin{aligned}
&\frac{1}{a}\left|\Big\langle A \mid \Psi\Big(a^{-1/2}\sin\big((2k+1)\theta\big),\ b^{-1/2}\cos\big((2k+1)\theta\big)\Big)\Big\rangle\right|^2 \\
&\quad = \frac{1}{a}\left|\Big\langle A \mid a^{-1/2}\sin\big((2k+1)\theta\big)|A\rangle + b^{-1/2}\cos\big((2k+1)\theta\big)|B\rangle\Big\rangle\right|^2 \\
&\quad = \frac{1}{a}\left|a^{-1/2}\sin\big((2k+1)\theta\big)\langle A \mid A\rangle\right|^2 = \frac{1}{a}\left|a^{-1/2}\sin\big((2k+1)\theta\big)a\right|^2 \\
&\quad = \sin\big((2k+1)\theta\big)^2.
\end{aligned}$$

Thus the probability of observing a correct answer is $\sin((2k+1)\theta)^2$.

Let $k = \lfloor \pi/(4\theta) \rfloor$ and recall that θ is such that $\sin(\theta) = \sqrt{a}$. As a function in x, $\sin((2x+1)\theta)$ is decreasing near $x = \pi/(4\theta)$. Using this (at the marked inequality), we have

$$\sin\big((2k+1)\theta\big) \overset{*}{\geq} \sin\big((\pi/(2\theta)+1)\theta\big) = \sin\big(\pi/2+\theta\big) = \cos(\theta) = \sqrt{b} = \sqrt{1-a}.$$

Hence the probability of measuring a correct answer is $\sin((2k+1)\theta)^2 \geq 1 - a$.

AP 3. Recall that the Fibonacci sequence $(f_i)_{i\in\mathbb{N}}$ is defined

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+1} = f_n + f_{n-1}.$$

(i) Show that

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) Use the same technique that we used to find a closed form of the recurrence in Grover's algorithm to find a closed form for the Fibonacci sequence.

Date: April 2, 2020.

Hint: $f_n = (1/\sqrt{5})(\varphi^n - \psi^n)$ *where* $\varphi = (1/2)(1+\sqrt{5})$ *is the golden ratio and* $\psi = (1/2)(1-\sqrt{5})$ *is its conjugate.*

Solution: Let F be the matrix mentioned in the problem. We begin by diagonalizing F. The eigenvalues are solutions to

$$\begin{aligned}\det(F - xI) &= \det\left(\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}\right) = \det\begin{pmatrix}1-x & 1\\ 1 & -x\end{pmatrix}\\ &= (1-x)(-x) - 1 = x^2 - x - 1,\end{aligned}$$

so

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \psi = \frac{1-\sqrt{5}}{2}$$

are the eigenvalues. Note that from $x^2 - x - 1 = 0$, we obtain $x^2 = x + 1$ and $x(x-1) = 1$. Hence $\varphi^2 = \varphi + 1$, $\varphi^{-1} = \varphi - 1$, and $(\varphi - 1)^{-1} = \varphi$. Similar identities hold for ψ. We also have $\psi = 1 - \varphi$.

Next, we find the eigenvectors by computing the nullspace of $F - xI$ for $x = \varphi$ and $x = \psi$. For $x = \varphi$, we row reduce $F - \varphi I$,

$$\begin{pmatrix}1-\varphi & 1\\ 1 & -\varphi\end{pmatrix} \sim \begin{pmatrix}1 & -\varphi\\ 0 & 0\end{pmatrix},$$

yielding the eigenvector $(\varphi, 1)^T$. Similarly, for $x = \psi$ we row reduce $F - \psi I$ to obtain the eigenvector $(\psi, 1)^T$. It follows that

$$F = \begin{pmatrix}\varphi & \psi\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi & 0\\ 0 & \psi\end{pmatrix}\begin{pmatrix}\varphi & \psi\\ 1 & 1\end{pmatrix}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & \psi\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi & 0\\ 0 & \psi\end{pmatrix}\begin{pmatrix}1 & \psi\\ -1 & \varphi\end{pmatrix}.$$

Finally, we have

$$\begin{aligned}\begin{pmatrix}f_{n+1}\\ f_n\end{pmatrix} &= \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & \psi\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n & 0\\ 0 & \psi^n\end{pmatrix}\begin{pmatrix}1 & \psi\\ -1 & \varphi\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & \psi\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n & 0\\ 0 & \psi^n\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & \psi\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n\\ -\psi^n\end{pmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n+1} - \psi^{n+1}\\ \varphi^n - \psi^n\end{pmatrix}.\end{aligned}$$

Therefore $f_n = (1/\sqrt{5})(\varphi^n - \psi^n)$.