Subspaces

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I did not get to make it to subspaces today in class, so I decided to make this study sheet for you guys to briefly discuss Sub Spaces.

1 Introduction

We all know what Vector Spaces are (ie. $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, etc) and we also know that they have many properties. A few of the most important are that Vector Spaces are *closed* both under addition and scalar multiplication. What does that mean? Being closed under addition means that if we took any vectors x_1 and x_2 and added them together, their sum would also be in that vector space.

ex.

Take $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Both vectors belong to $\mathbb{R}^3$. Their sum, which is $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$ is also a member of $\mathbb{R}^3$.

Being closed under scalar multiplication means that vectors in a vector space, when multiplied by a scalar (any real number), it still belongs to the same vector space.

ex. Consider $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$. If I multiply the vector by a scalar, say, 10, I will get $10\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 40 \\ 30 \end{pmatrix}$. Which is still in $\mathbb{R}^3$

There are two very important notions of a Vector Space, and will end up being very important in defining a Sub Space.

2 Subspaces

Now we are ready to define what a subspace is. Strictly speaking, A Subspace is a Vector Space included in another larger Vector Space. Therefore, all properties of a Vector Space, such as being closed under addition and scalar multiplication still hold true when applied to the Subspace.

ex. We all know $\mathbb{R}^3$ is a Vector Space. It satisfies all the properties including being closed under addition and scalar multiplication. Consider the set of all vectors $S = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ such at x and y are real numbers. This is also a Vector Space because all the conditions of a Vector Space are satisfied, including the important conditions of being closed under addition and scalar multiplication.

ex.

Consider the vector $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$ Which are both contained in S. If we add them together we get $\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$, which is is still in S. We can also multiply each one by a scalar, say $\frac{1}{2}$ and get $\begin{pmatrix} \frac{1}{2} \\ 2 \\ \frac{3}{2} \end{pmatrix}$, and $\begin{pmatrix} \frac{5}{2} \\ 1 \\ 0 \end{pmatrix}$, which are both still in S.

So we see that S is a Vector Space, but it is important to notice that all of S is contained in $\mathbb{R}^3$. By this, I mean any vector in S can also be found in $\mathbb{R}^3$. Therefore, S is a *SUBSPACE* of $\mathbb{R}^3$.

Other examples of Sub Spaces:

- The line defined by the equation $y = 2x$, also defined by the vector definition $\begin{pmatrix} t \\ 2t \end{pmatrix}$ is a subspace of $\mathbb{R}^2$
- The plane $z = -2x$, otherwise known as $\begin{pmatrix} t \\ 0 \\ -2t \end{pmatrix}$ is a subspace of $\mathbb{R}^3$
- In fact, in general, the plane $ax + by + cz = 0$ is a subspace of $\mathbb{R}^3$ if $abc \neq 0$. This one is tricky, try it out. Test whether or not any arbitrary vectors x_1, and x_s are closed under addition and scalar multiplication.

2.1 Subspace Test

Given a space, and asked whether or not it is a Sub Space of another Vector Space, there is a very simple test you can preform to answer this question. There are only two things to show:

The Subspace Test To test whether or not S is a subspace of some Vector Space $\mathbb{R}^n$ you must check two things:

1. if s_1 and s_2 are vectors in S, their sum must also be in S
2. if s is a vector in S and k is a scalar, ks must also be in S

In other words, to test if a set is a subspace of a Vector Space, you only need to check if it closed under addition and scalar multiplication. Easy!

ex.
Test whether or not the plane $2x + 4y + 3z = 0$ is a subspace of $\mathbb{R}^3$.

To test if the plane is a subspace, we will take arbitrary points $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, both of which lie on the plane, and we will check both points of the subspace test.

1. Closed under addition: Consider $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$. We will test if the point also lies in the plane. We will take our original polynomial, $2x + 4y + 3z = 0$, and substitute x with $x_1 + x_2$, y with $y_1 + y_2$, and z with $z_1 + z_2$ and get

$$2(x_1 + x_2) + 4(y_1 + y_2) + 3(z_1 + z_2) = 0$$

From here we can distribute and get:

$$2x_1 + 2x_2 + 4y_1 + 4y_2 + 3z_1 + 3z_2 = 0$$

which we can reorganize to get

$$(2x_1 + 4y_1 + 3z_1) + (2x_2 + 4y_2 + 3y_2) = 0$$

We also know that both $2x_1 + 4y_1 + xz_1$ and $2x_2 + 4y_2 + 3z_2$ are both 0, so the equation becomes $0 + 0 = 0$, which proves that the plane is closed under addition.

2. Closed under scalar multiplication: Consider the point $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ on the plane. Also consider the scalar k. If we multiply $k\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} kx_1 \\ ky_1 \\ kz_1 \end{pmatrix}$ we need to check if this is also in the plane. To do this, we will plug in the point into the original plane. So we have

$$2kx_1 + 4ky_1 + 3kz_1 = 0$$

which factors into

$$k(2x_1 + 4y_1 + 3z_1) = 0$$

And because we know $2x_1 + 4y_1 + 3z_1 = 0$ we obtain $0 = 0$. So the plane is closed under scalar multiplication. Phew! We proved both parts of the Subspace test so we have proved that the plane defined by the equation $2x_1 + 4y_1 + 3z_1 = 0$ is a subspace of $\mathbb{R}^3$.