Homework 6

6.5.a. Let $H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Write the matrix of the operator $H[2]$ acting on the space $B^{\otimes 3}$

1. We have qubit space of 3 and Hardman operator on subset of 2 qubits, given by below formula,

$$X[p] = I_{B^{\otimes(p-1)}} \otimes I_{B^{\otimes(n-p)}}$$

2. In our case $n = 3, p = 2$, we get, $H[2] = I_{B^{\otimes(1)}} \otimes H \otimes I_{B^{\otimes(1)}}$

$$H[2] = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \otimes \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\end{bmatrix}$$

6.5.b.

Let U be an arbitrary two-qubit operator with matrix elements $u_{jk} = \langle j|u|k\rangle$, where $j,k \in \{00, 01, 10, 11\}$. Write matrix for $U[3,1]$.

$$\text{matrix} = \text{elements of } \langle j|U|k\rangle = \sum_{jk} U_{jk} |j\rangle\langle k|$$

U = 2-qubit operator = $2^2 \times 2^2$ matrix

$$U = \left[\begin{array}{cc|cc} U_{0000} & U_{0001} & U_{0010} & U_{0011} \\ U_{0100} & U_{0101} & U_{0110} & U_{0111} \\ \hline U_{1000} & U_{1001} & U_{1010} & U_{1011} \\ U_{1100} & U_{1101} & U_{1110} & U_{1111} \end{array}\right] \qquad \begin{bmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{bmatrix}$$

Decompose matrix U into 1-qubit matrices

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \overbrace{\begin{bmatrix} U_{0000} & U_{0001} \\ U_{0100} & U_{0101} \end{bmatrix}}^{V_{00}} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \overbrace{\begin{bmatrix} U_{0010} & U_{0011} \\ U_{0110} & U_{0111} \end{bmatrix}}^{V_{01}}$$
$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \underbrace{\begin{bmatrix} U_{1000} & U_{1001} \\ U_{1100} & U_{1101} \end{bmatrix}}_{V_{10}} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \underbrace{\begin{bmatrix} U_{1010} & U_{1011} \\ U_{1110} & U_{1111} \end{bmatrix}}_{V_{11}}$$

Apply U on the subset of qubits $[3,1]$

$$U[3,1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}[3] \cdot V_{00}[1] + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}[3] \cdot V_{01}[1]$$
$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}[3] \cdot V_{10}[1] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}[3] \cdot V_{11}[1]$$

we know that $X[P] = I_B^{\otimes(P-1)} \otimes X \otimes I_B^{\otimes(n-P)}$

n = n-qubit space

P = qubit

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}[3] = I_{B^{\otimes}(2)} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_0 = I_{4\times 4} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$n=3,\ P=3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V_{00}[1] = I_{B^{\otimes}(0)} \otimes V_{00} \otimes I_{B^{\otimes}2} = \begin{bmatrix} U_{0000} & U_{0001} \\ U_{0100} & U_{0101} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$n=3;\ P=1$

$$= \begin{bmatrix} U_{0000} & 0 & 0 & 0 & U_{0001} & 0 & 0 & 0 \\ 0 & U_{0000} & 0 & 0 & 0 & U_{0001} & 0 & 0 \\ 0 & 0 & U_{0000} & 0 & 0 & 0 & U_{0001} & 0 \\ 0 & 0 & 0 & U_{0000} & 0 & 0 & 0 & U_{0001} \\ U_{0100} & 0 & 0 & 0 & U_{0101} & 0 & 0 & 0 \\ 0 & U_{0100} & 0 & 0 & 0 & U_{0101} & 0 & 0 \\ 0 & 0 & U_{0100} & 0 & 0 & 0 & U_{0101} & 0 \\ 0 & 0 & 0 & U_{0100} & 0 & 0 & 0 & U_{0101} \end{bmatrix}$$

$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [3] \right) \left(V_{00} [1] \right)$ – matrix multiplication

$$= \begin{bmatrix} U_{0000} & 0 & 0 & 0 & U_{0001} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{0000} & 0 & 0 & 0 & U_{0001} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_{0100} & 0 & 0 & 0 & U_{0101} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{0100} & 0 & 0 & 0 & U_{0101} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} [3] \right) \left(V_{01} [1] \right) = \begin{bmatrix} 0 & U_{0010} & 0 & 0 & 0 & U_{0011} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{0010} & 0 & 0 & 0 & U_{0011} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_{0110} & 0 & 0 & 0 & U_{0111} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{0110} & 0 & 0 & 0 & U_{0111} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V_{01} = \begin{bmatrix} U_{0010} & U_{0011} \\ U_{0110} & U_{0111} \end{bmatrix}$$

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} [3] \right) \left(V_{10} [1] \right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_{1000} & 0 & 0 & 0 & U_{1001} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{1000} & 0 & 0 & 0 & U_{1001} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_{1100} & 0 & 0 & 0 & U_{1101} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{1100} & 0 & 0 & 0 & U_{1101} & 0 \end{bmatrix}$$

$$V_{10} = \begin{bmatrix} U_{1000} & U_{1001} \\ U_{1100} & U_{1101} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}[3])(V_{11}[1]) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_{1010} & 0 & 0 & 0 & U_{1011} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{1010} & 0 & 0 & 0 & U_{1011} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_{1110} & 0 & 0 & 0 & U_{1111} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{1110} & 0 & 0 & 0 & U_{1111} \end{bmatrix}$$

$$V_{11} = \begin{bmatrix} U_{1010} & U_{1011} \\ U_{1110} & U_{1111} \end{bmatrix}$$

$$U[3,1] = \begin{bmatrix} U_{0000} & U_{0010} & 0 & 0 & U_{0001} & U_{0011} & 0 & 0 \\ U_{1000} & U_{1010} & 0 & 0 & U_{1001} & U_{1011} & 0 & 0 \\ 0 & 0 & U_{0000} & U_{0010} & 0 & 0 & U_{0001} & U_{0011} \\ 0 & 0 & U_{1000} & U_{1010} & 0 & 0 & U_{1001} & U_{1011} \\ U_{0100} & U_{0110} & 0 & 0 & U_{0101} & U_{0111} & 0 & 0 \\ U_{1100} & U_{1110} & 0 & 0 & U_{1101} & U_{1111} & 0 & 0 \\ 0 & 0 & U_{0100} & U_{0110} & 0 & 0 & U_{0101} & U_{0111} \\ 0 & 0 & U_{1100} & U_{1110} & 0 & 0 & U_{1101} & U_{1111} \end{bmatrix}$$

1. Prove that the inner product and the tensor product commute: $\langle \alpha \otimes \beta \mid \gamma \otimes \delta \rangle = \langle \alpha \mid \gamma \rangle \langle \beta \mid \delta \rangle$

1. We know that $|\varepsilon\rangle = \langle \varepsilon^\dagger|$, where $\dagger = Conjugate\ transpose$. Let,

$$|\alpha\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix},\ |\beta\rangle = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},\ |\gamma\rangle = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix},\ |\delta\rangle = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

$$\langle \alpha^\dagger| = \begin{bmatrix} \alpha_1^\dagger & \alpha_2^\dagger \end{bmatrix},\ \langle \beta^\dagger| = \begin{bmatrix} \beta_1^\dagger & \beta_2^\dagger \end{bmatrix},\ \langle \gamma^\dagger| = \begin{bmatrix} \gamma_1^\dagger & \gamma_2^\dagger \end{bmatrix},\ \langle \delta^\dagger| = \begin{bmatrix} \delta_1^\dagger & \delta_2^\dagger \end{bmatrix}$$

2. Using above vectors, we can define tensors as below,

$$|\alpha \otimes \beta\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \otimes \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \text{ and } \langle \alpha \otimes \beta| = \begin{bmatrix} \alpha_1^\dagger & \alpha_2^\dagger \end{bmatrix} \otimes \begin{bmatrix} \beta_1^\dagger & \beta_2^\dagger \end{bmatrix} = \begin{bmatrix} \alpha_1^\dagger\beta_1^\dagger & \alpha_1^\dagger\beta_2^\dagger & \alpha_2^\dagger\beta_1^\dagger & \alpha_2^\dagger\beta_2^\dagger \end{bmatrix}$$

$$|\gamma \otimes \delta\rangle = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \otimes \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \gamma_1\delta_1 \\ \gamma_1\delta_2 \\ \gamma_2\delta_1 \\ \gamma_2\delta_2 \end{bmatrix}$$

$$\langle \alpha \otimes \beta \mid \gamma \otimes \delta \rangle = \alpha_1^\dagger\beta_1^\dagger\gamma_1\delta_1 + \alpha_1^\dagger\beta_2^\dagger\gamma_1\delta_2 + \alpha_2^\dagger\beta_1^\dagger\gamma_2\delta_1 + \alpha_2^\dagger\beta_2^\dagger\gamma_2\delta_2$$

3. We will derive inner product as below,

$$\langle \alpha \mid \gamma \rangle = \begin{bmatrix} \alpha_1^\dagger & \alpha_2^\dagger \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \alpha_1^\dagger\gamma_1 + \alpha_2^\dagger\gamma_2 \text{ and } \langle \beta \mid \delta \rangle = \begin{bmatrix} \beta_1^\dagger & \beta_2^\dagger \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \beta_1^\dagger\delta_1 + \beta_2^\dagger\delta_2$$

$$\langle \alpha \mid \gamma \rangle \langle \beta \mid \delta \rangle = [\alpha_1^\dagger\gamma_1 + \alpha_2^\dagger\gamma_2][\beta_1^\dagger\delta_1 + \beta_2^\dagger\delta_2] = \alpha_1^\dagger\beta_1^\dagger\gamma_1\delta_1 + \alpha_1^\dagger\beta_2^\dagger\gamma_1\delta_2 + \alpha_2^\dagger\beta_1^\dagger\gamma_2\delta_1 + \alpha_2^\dagger\beta_2^\dagger\gamma_2\delta_2$$

4. From the results of point 2 and 3, we can say that,

$$\langle \alpha \otimes \beta \mid \gamma \otimes \delta \rangle = \langle \alpha \mid \gamma \rangle \langle \beta \mid \delta \rangle$$

2. An $n - ary$ function