QUANTUM ALGORITHMS
HOMEWORK 5 SELECTED SOLUTIONS

PROF. MATTHEW MOORE

AP 1. Let $\mathbb{V}$ and $\mathbb{S}$ be vector spaces over $\mathbb{C}$ with bases $\mathcal{B}_{\mathbb{V}}$ and $\mathcal{B}_{\mathbb{S}}$, respectively. Define

$$\mathbb{V} \times \mathbb{S} = \big\{(v, s) \mid v \in V \text{ and } s \in S\big\}$$

and recognize it as a vector space by *coordinate-wise* interpretation of the vector space axioms. That is,

$$\begin{aligned} &(v_1, s_1) + (v_2, s_2) = (v_1 + v_2, s_1 + s_2) && \text{for } v_1, v_2 \in V \text{ and } s_1, s_2 \in S, \\ &\lambda \cdot (v_1, s_1) = (\lambda \cdot v_1, \lambda \cdot s_1) && \text{for } v_1 \in V, s_1 \in S, \text{ and } \lambda \in \mathbb{C} \text{ a scalar.} \end{aligned}$$

If $R : \mathbb{A} \to \mathbb{V}$ and $T : \mathbb{A} \to \mathbb{S}$ are linear functions, then we can define a linear function $(R \times T) : \mathbb{A} \to \mathbb{V} \times \mathbb{S}$ by

$$(R \times T)a = \big(Ra, Ta\big) \qquad \text{for } a \in A.$$

(i) Let

$$\mathcal{C} = \big\{(b_v, b_s) \mid b_v \in \mathcal{B}_{\mathbb{V}} \text{ and } b_s \in \mathcal{B}_{\mathbb{S}}\big\}.$$

Show that $\mathcal{C}$ is not linearly independent.

(ii) Prove that

$$\mathcal{B}_{\mathbb{V}\times\mathbb{S}} = \big\{(b_v, 0), (0, b_s) \mid b_v \in \mathcal{B}_{\mathbb{V}} \text{ and } b_s \in \mathcal{B}_{\mathbb{S}}\big\}$$

is a basis for $\mathbb{V} \times \mathbb{S}$. What is the dimension of $\mathbb{V} \times \mathbb{S}$?

(iii) Let $R : \mathbb{A} \to \mathbb{V}$ and $T : \mathbb{A} \to \mathbb{S}$ be linear functions. Suppose that $\mathbb{A}$, $\mathbb{V}$, and $\mathbb{S}$ have ordered bases

$$\mathcal{B}_{\mathbb{A}} = \big\{a_1, a_2\big\}, \qquad \mathcal{B}_{\mathbb{V}} = \big\{v_1, v_2, v_3\big\}, \qquad \mathcal{B}_{\mathbb{S}} = \{s_1, s_2\},$$

and that the matrix representations of R and T relative to these bases are

$$(R)_{\mathcal{B}_{\mathbb{A}}\to\mathcal{B}_{\mathbb{V}}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad \text{and} \qquad (T)_{\mathcal{B}_{\mathbb{A}}\to\mathcal{B}_{\mathbb{S}}} = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}.$$

Using the lexicographic order for the basis $\mathcal{B}_{\mathbb{V}\times\mathbb{S}}$ (i.e. ordering by $\mathcal{B}_{\mathbb{V}}$ first, and then $\mathcal{B}_{\mathbb{S}}$), find the matrix representation for $(R \times T)$ (that is, find $(R \times T)_{\mathcal{B}_{\mathbb{A}}\to\mathcal{B}_{\mathbb{V}\times\mathbb{S}}}$).

Solution:

(i): Choose distinct elements $v_1, v_2 \in \mathcal{B}_{\mathbb{V}}$ and $s_1, s_2 \in \mathcal{B}_{\mathbb{S}}$. We have

$$(v_1, s_1) + (v_2, s_2) - (v_1, s_2) - (v_2, s_1) = (0, 0),$$

so $\mathcal{C}$ cannot be linearly independent.

Date: March 11, 2021.

(ii): *Proof.* Let $(\alpha,\beta) \in \mathbb{V} \times \mathbb{S}$. It follows that $\alpha \in \mathbb{V}$ and $\beta \in \mathbb{S}$. Since $\mathcal{B}_{\mathbb{V}}$ and $\mathcal{B}_{\mathbb{S}}$ are bases for $\mathbb{V}$ and $\mathbb{S}$, respectively, there are scalars λ_v, μ_s such that

$$\alpha = \sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v \qquad\qquad \beta = \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s.$$

Therefore

$$\begin{aligned}(\alpha,\beta) &= \left(\sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v, \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s\right) = \left(\sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v, 0\right) + \left(0, \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s\right) \\ &= \sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v (b_v, 0) + \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s (0, b_s) = \sum_{\substack{b_v \in \mathcal{B}_{\mathbb{V}} \\ b_s \in \mathcal{B}_{\mathbb{S}}}} \lambda_v (b_v, 0) + \mu_s (0, b_s).\end{aligned}$$

It follows that $\mathbb{C}\text{-span}(\mathcal{B}_{\mathbb{V}\times\mathbb{S}}) = \mathbb{V} \times \mathbb{S}$. It remains to show linear independence. Following the equation above backwards, if

$$0 = (0,0) = \sum_{\substack{b_v \in \mathcal{B}_{\mathbb{V}} \\ b_s \in \mathcal{B}_{\mathbb{S}}}} \lambda_v (b_v, 0) + \mu_s (0, b_s) = \left(\sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v, \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s\right),$$

then both the sums are equal to 0, and therefore all the coefficients are 0 since $\mathcal{B}_{\mathbb{V}}$ and $\mathcal{B}_{\mathbb{S}}$ are linearly independent. □

(iii): The basis described in the problem is

$$\mathcal{B}_{\mathbb{V}\times\mathbb{S}} = \big\{(v_1,0),(v_2,0),(v_3,0),(0,s_1),(0,s_1)\big\}.$$

From the definition of $\times$ for linear functions and the definitions of R and T, we have that

$$\begin{aligned}(R\times T)a_1 &= (Ra_1, Ta_1) = \big(v_1 + 3v_2 + 5v_3, -s_1 + 3s_2\big) \\ &= (v_1,0) + 3(v_2,0) + 5(v_3,0) - (0,s_1) + 3(0,s_2) \qquad\qquad \text{and} \\ (R\times T)a_2 &= (Ra_2, Ta_2) = \big(2v_1 + 4v_2 + 6v_3, 2s_1 - 2s_2\big) \\ &= 2(v_1,0) + 4(v_2,0) + 6(v_3,0) + 2(0,s_1) - 2(0,s_2).\end{aligned}$$

Therefore, the first column of $(R\times T)_{\mathcal{B}_{\mathbb{A}}\to\mathcal{B}_{\mathbb{V}\times\mathbb{S}}}$ is $(1,3,5,-1,3)^T$ and the second column is $(2,4,6,2,-2)^T$:

$$(R\times T)_{\mathcal{B}_{\mathbb{A}}\to\mathcal{B}_{\mathbb{V}\times\mathbb{S}}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ -1 & 2 \\ 3 & -2 \end{pmatrix}.$$

AP 2. Let $T : \mathbb{A} \to \mathbb{B}$ be a linear transformation between vector spaces with ordered bases

$$\mathcal{B}_{\mathbb{A}} = \{\, |1\rangle, |2\rangle, |3\rangle \,\} \qquad\qquad \mathcal{B}_{\mathbb{B}} = \{\, |1\rangle, |2\rangle \,\}.$$

Suppose that T has matrix with respect to these bases

$$T = \begin{pmatrix} 9 & 6 & -3 \\ -4 & -8 & 8 \end{pmatrix}.$$

(i) Show that the matrix for T can be written

$$T = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} a_{ij} \, |i\rangle \langle j|$$

(note that $|1\rangle \in \mathcal{B}_{\mathbb{A}}$ is a 3-dimensional vector, while $|1\rangle \in \mathcal{B}_{\mathbb{B}}$ is a 2-dimensional vector).

(ii) Show that for fixed $|i\rangle \in \mathcal{B}_{\mathbb{A}}$ and $|j\rangle \in \mathcal{B}_{\mathbb{B}}$

$$\big(|j\rangle\langle i|\big)|v\rangle = \langle i \mid v\rangle |j\rangle$$

for all $\langle v| \in \mathbb{A}$. From this, prove that $|j\rangle\langle i|$ defines a linear transformation from $\mathbb{A} \to \mathbb{B}$.

(iii) Suppose that

$$R = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} b_{ij} |i\rangle\langle j|$$

for $b_{ij} \in \mathbb{C}$. Use the previous part to prove that R is a linear transformation from $\mathbb{A} \to \mathbb{B}$.

Solution:

(i): The linear transformation T is uniquely characterized by its action on $\mathcal{B}_{\mathbb{A}}$,

$$\begin{aligned} T|1\rangle &= 9|1\rangle - 4|2\rangle, \\ T|2\rangle &= 6|1\rangle - 8|2\rangle, \\ T|2\rangle &= -3|1\rangle + 8|2\rangle \end{aligned}$$

(these equalities come from the matrix representation of T given in the problem). Therefore, if S is any other linear transformation acting the same way on $\mathcal{B}_{\mathbb{A}}$ then $S = T$. Let

$$S = \big(9|1\rangle\langle 1| - 4|2\rangle\langle 1|\big)\big(6|1\rangle\langle 2| - 8|2\rangle\langle 2|\big)\big(-3|1\rangle\langle 3| + 8|2\rangle\langle 3|\big).$$

S is itself a linear combination of linear transformations (by items (ii) and (iii) below), and is thus a linear transformation. Showing that S acts on the basis vectors $\mathcal{B}_{\mathbb{A}}$ in the same manner as T is a straightforward calculation. It follows that $S = T$.

(ii): *Proof.* Regarded as a matrix, the object $|j\rangle\langle i|$ has dimensions $\dim(\mathbb{B}) \times \dim(\mathbb{A})$, so $(|j\rangle\langle i|)|v\rangle$ is a defined quantity. Matrix multiplication is the same as function composition, so it is associative. Therefore

$$\big(|j\rangle\langle i|\big)|v\rangle = \langle j|\big(|i\rangle\langle v|\big) = \langle j|\langle i \mid v\rangle = \langle i \mid v\rangle\langle j|.$$

(the last equality follows from $\langle i \mid v\rangle \in \mathbb{C}$ being a scalar). □

(iii): *Proof.* R is a linear combination of linear transformations (by item (ii) above). It is therefore sufficient to show that if S and T are linear transformations and $\lambda, \mu \in \mathbb{C}$, then is $\lambda S + \mu T$ also a linear transformation. We have

$$(\lambda S + \mu T)0 = \lambda S0 + \mu T0 = \lambda 0 + \mu 0 = 0 + 0 = 0,$$

$$\begin{aligned} (\lambda S + \mu T)(|\alpha\rangle + |\beta\rangle) &= \lambda S(|\alpha\rangle + |\beta\rangle) + \mu T(|\alpha\rangle + |\beta\rangle) \\ &= \lambda S|\alpha\rangle + \lambda S|\beta\rangle + \mu T|\alpha\rangle + \mu T|\beta\rangle \\ &= \lambda S|\alpha\rangle + \mu T|\alpha\rangle + \lambda S|\beta\rangle + \mu T|\beta\rangle \\ &= (\lambda S + \mu T)|\alpha\rangle + (\lambda S + \mu T)|\beta\rangle, \end{aligned}$$

$$(\lambda S + \mu T)(\nu|\beta\rangle) = \lambda S(\nu|\beta\rangle) + \mu T(\nu|\beta\rangle) = \nu\lambda S|\beta\rangle + \nu\mu T|\beta\rangle = \nu(\lambda S + \mu T)|\beta\rangle.$$

Hence $(\lambda S + \mu T)$ is a linear transformation. □

AP 3. Let $\mathbb{V}$ and $\mathbb{S}$ be vector spaces over $\mathbb{C}$ with bases $B_{\mathbb{V}}$ and $B_{\mathbb{S}}$, respectively.

(i) Prove that

$$B_{\mathbb{V}\otimes\mathbb{S}} = \big\{ b_v \otimes b_s \mid b_v \in B_{\mathbb{V}} \text{ and } b_s \in B_{\mathbb{S}} \big\}$$

is a basis of $\mathbb{V} \otimes \mathbb{S}$. What is the dimension of $\mathbb{V} \otimes \mathbb{S}$?

(ii) Let $R : \mathbb{V} \to \mathbb{A}$ and $T : \mathbb{S} \to \mathbb{B}$ be linear functions. Suppose that $\mathbb{A}$, $\mathbb{V}$, and $\mathbb{S}$ have ordered bases the same as in the previous question and that $\mathbb{B}$ has ordered basis $B_{\mathbb{B}} = \{b_1, b_2\}$ and that the matrix representations of R and T relative to these bases are

$$(R)_{B_{\mathbb{V}} \to B_{\mathbb{A}}} = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -2 & -1 \end{bmatrix} \qquad \text{and} \qquad (T)_{B_{\mathbb{S}} \to B_{\mathbb{B}}} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Using the lexicographic order for the basis $B_{\mathbb{V}\otimes\mathbb{S}}$, find the matrix representation for $(R \otimes T)$ (that is, find $(R \times T)_{B_{\mathbb{V}\otimes\mathbb{S}} \to B_{\mathbb{A}\otimes\mathbb{B}}}$). [*Hint: Kronecker product.*]

Solution:

(i): *Proof.* We being by showing that $\mathbb{V} \otimes \mathbb{S} = \mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}})$. Let $\alpha \in V$ and $\beta \in S$. It follows that α and β can be decomposed into linear combinations of their respective bases,

$$\alpha = \sum_{b_v \in B_{\mathbb{V}}} \lambda_v b_v, \qquad \beta = \sum_{b_s \in B_{\mathbb{S}}} \mu_s b_s, \qquad \text{for } \lambda_v, \mu_s \in \mathbb{C}.$$

Using the bilinearity of the tensor, we have

$$\begin{aligned}\alpha \otimes \beta &= \left(\sum_{b_v \in B_{\mathbb{V}}} \lambda_v b_v \right) \otimes \left(\sum_{b_s \in B_{\mathbb{S}}} \mu_s b_s \right) = \sum_{b_v \in B_{\mathbb{V}}} \lambda_v \left(b_v \otimes \sum_{b_s \in B_{\mathbb{S}}} \mu_s b_s \right) \\ &= \sum_{b_v \in B_{\mathbb{V}}} \lambda_v \sum_{b_s \in B_{\mathbb{S}}} \mu_s (b_v \otimes b_s) = \sum_{\substack{b_v \in B_{\mathbb{V}} \\ b_s \in B_{\mathbb{S}}}} \lambda_v \mu_s (b_v \otimes b_s).\end{aligned}$$

It follows that

$$\{\alpha \otimes \beta \mid \alpha \in \mathbb{V}, \beta \in \mathbb{S}\} \subseteq \mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}}).$$

Applying $\mathbb{C}$-span to both sides yields

$$\mathbb{V} \otimes \mathbb{S} = \mathbb{C}\text{-span}\{\alpha \otimes \beta \mid \alpha \in \mathbb{V}, \beta \in \mathbb{S}\} = \mathbb{C}\text{-span}\left(\mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}})\right) = \mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}}).$$

Therefore $B_{\mathbb{V}\otimes\mathbb{S}}$ spans $\mathbb{V} \otimes \mathbb{S}$.

Next we show that $B_{\mathbb{V}\otimes\mathbb{S}}$ is linearly independent. Simple tensors $\alpha \otimes \beta$ and $\alpha' \otimes \beta'$ are equal if and only if there is $\lambda \in \mathbb{C}$ such that $\alpha = \lambda\alpha$ and $\lambda\beta = \beta'$. From this, we obtain the equalities

$$0 = \alpha \otimes 0 = 0 \otimes \beta \qquad \text{for all } \alpha \in \mathbb{V}, \beta \in \mathbb{S}$$

(the left-most 0 is the zero vector in $\mathbb{V} \otimes \mathbb{S}$). Suppose that we have a linear combination of basis vectors that is equal to 0. Bilinearity gives us

$$0 = \sum_{\substack{b_v \in B_{\mathbb{V}} \\ b_s \in B_{\mathbb{S}}}} \lambda_{vs} (b_v \otimes b_s) = \sum_{b_v \in B_{\mathbb{V}}} \sum_{b_s \in B_{\mathbb{S}}} \lambda_{vs} (b_v \otimes b_s) = \sum_{b_v \in B_{\mathbb{V}}} \left(b_v \otimes \sum_{b_s \in B_{\mathbb{S}}} \lambda_{vs} b_s \right),$$

so for *fixed* $b_v \in B_{\mathbb{V}}$ we have

$$0 = \sum_{b_s \in B_{\mathbb{S}}} \lambda_{vs} b_s.$$

Since $B_{\mathbb{S}}$ is linearly independent, this implies that for fixed b_v we have $\lambda_{vs} = 0$ for all $b_s \in B_{\mathbb{S}}$. Doing this for all $b_v \in B_{\mathbb{V}}$ yields $\lambda_{vs} = 0$ for all $b_v \in B_{\mathbb{V}}$ and $b_s \in B_{\mathbb{S}}$, as desired. □

(ii): Ordered as described, the bases we are looking at are

$$\begin{aligned} B_{\mathbb{V}\otimes\mathbb{S}} &= \{v_1 \otimes s_1, v_1 \otimes s_2, v_2 \otimes s_1, v_2 \otimes s_2, v_3 \otimes s_1, v_3 \otimes s_2\} \qquad \text{and} \\ B_{\mathbb{A}\otimes\mathbb{B}} &= \{a_1 \otimes b_1, a_1 \otimes b_2, a_2 \otimes b_1, a_2 \otimes b_2\}. \end{aligned}$$

From the definition of $R \otimes T$, we have that $(R \otimes T)(v \otimes s) = (Rv) \otimes (Ts)$. It follows from this and bilinearity that

$$\begin{aligned}
(R \otimes T)(v_1 \otimes s_1) &= (Rv_1) \otimes (Ts_1) = \big(-a_1 + 3a_2 \big) \otimes \big(-2b_1 + b_2 \big) \\
&= 2(a_1 \otimes b_1) - (a_1 \otimes b_2) - 6(a_2 \otimes b_1) + 3(a_2 \otimes b_2), \\
&\big(\text{ so column 1 of } (R \times T)_{B_{\mathbb{V} \otimes \mathbb{S}} \to B_{\mathbb{A} \otimes \mathbb{B}}} \text{ is } (2, -1, -6, 3)^T \ \big) \\
(R \otimes T)(v_1 \otimes s_2) &= (Rv_1) \otimes (Ts_2) = \big(-a_1 + 3a_2 \big) \otimes \big(b_1 - 2b_2 \big) \\
&= -(a_1 \otimes b_1) + 2(a_1 \otimes b_2) + 3(a_2 \otimes b_1) - 6(a_2 \otimes b_2), \\
&\big(\text{ so column 2 of } (R \times T)_{B_{\mathbb{V} \otimes \mathbb{S}} \to B_{\mathbb{A} \otimes \mathbb{B}}} \text{ is } (-1, 2, 3, -6)^T \ \big) \\
(R \otimes T)(v_2 \otimes s_1) &= (Rv_2) \otimes (Ts_1) = \big(2a_1 - 2a_2 \big) \otimes \big(-2b_1 + b_2 \big) \\
&= -4(a_1 \otimes b_1) + 2(a_1 \otimes b_2) + 4(a_2 \otimes b_1) - 2(a_2 \otimes b_2), \\
&\big(\text{ so column 3 of } (R \times T)_{B_{\mathbb{V} \otimes \mathbb{S}} \to B_{\mathbb{A} \otimes \mathbb{B}}} \text{ is } (-4, 2, 4, -2)^T \ \big) \\
(R \otimes T)(v_2 \otimes s_2) &= (Rv_2) \otimes (Ts_2) = \big(2a_1 - 2a_2 \big) \otimes \big(b_1 - 2b_2 \big) \\
&= 2(a_1 \otimes b_1) - 4(a_1 \otimes b_2) - 2(a_2 \otimes b_1) + 4(a_2 \otimes b_2), \\
&\big(\text{ so column 4 of } (R \times T)_{B_{\mathbb{V} \otimes \mathbb{S}} \to B_{\mathbb{A} \otimes \mathbb{B}}} \text{ is } (2, -4, -2, 4)^T \ \big) \\
(R \otimes T)(v_3 \otimes s_1) &= (Rv_3) \otimes (Ts_1) = \big(-a_1 - a_2 \big) \otimes \big(-2b_1 + b_2 \big) \\
&= 2(a_1 \otimes b_1) - (a_1 \otimes b_2) + 2(a_2 \otimes b_1) - (a_2 \otimes b_2), \\
&\big(\text{ so column 5 of } (R \times T)_{B_{\mathbb{V} \otimes \mathbb{S}} \to B_{\mathbb{A} \otimes \mathbb{B}}} \text{ is } (2, -1, 2, -1)^T \ \big) \\
(R \otimes T)(v_3 \otimes s_3) &= (Rv_3) \otimes (Ts_3) = \big(-a_1 - a_2 \big) \otimes \big(b_1 - 2b_2 \big) \\
&= -(a_1 \otimes b_1) + 2(a_1 \otimes b_2) - (a_2 \otimes b_1) + 2(a_2 \otimes b_2). \\
&\big(\text{ so column 6 of } (R \times T)_{B_{\mathbb{V} \otimes \mathbb{S}} \to B_{\mathbb{A} \otimes \mathbb{B}}} \text{ is } (-1, 2, -1, 2)^T \ \big).
\end{aligned}$$

From all this we have

$$(R \otimes T)_{B_{\mathbb{V} \otimes \mathbb{S}} \to B_{\mathbb{A} \otimes \mathbb{B}}} = \begin{pmatrix} 2 & -1 & -4 & 2 & 2 & -1 \\ -1 & 2 & 2 & -4 & -1 & 2 \\ -6 & 3 & 4 & -2 & 2 & -1 \\ 3 & -6 & -2 & 4 & -1 & 2 \end{pmatrix},$$

which is the Kronecker product of $(R)_{B_{\mathbb{V}} \to B_{\mathbb{A}}}$ and $(T)_{B_{\mathbb{S}} \to B_{\mathbb{B}}}$!

6.2. Consider two linear maps, $A : \mathcal{L} \to \mathcal{L}'$ and $B : \mathcal{M} \to \mathcal{M}'$. Prove that there is a unique linear map $C = A \otimes B : \mathcal{L} \otimes \mathcal{M} \to \mathcal{L}' \otimes \mathcal{M}'$ such that $C(u \otimes v) = A(u) \otimes B(v)$ for any $u \in \mathcal{L}$, $v \in \mathcal{M}$.

Solution:

Proof. Elements of $\mathcal{L} \otimes \mathcal{M}$ are linear combinations of simple tensors, $u \otimes v$. Define a *function* C on the simple tensors of $\mathcal{L} \otimes \mathcal{M}$ by

$$C(u \otimes v) \coloneqq A(u) \otimes B(v).$$

Extend C to a function on all of $\mathcal{L} \otimes \mathcal{M}$ by linearity.

One thing we must check is that when we "defined" C, we didn't accidentally write an inconsistent definition that depends on the exact representation of $u \otimes v$. An example of this would be if we defined a function f on the fractions to be $f(a/b) = a \cdot b$. This function is not well-defined since $f(1/2) = 2$ and $f(2/4) = 8$, but $1/2 = 2/4$ and $2 \neq 8$. This property is called being "well-defined". Logically, we want to prove that if $u \otimes v = u' \otimes v'$ then $C(u \otimes v) = C(u' \otimes v')$.

Claim. *C is well-defined.*

Proof of claim. Suppose that $u, u' \in \mathcal{L}$, $v, v' \in \mathcal{M}$, and $u \otimes v = u' \otimes v'$. For simple tensors, the only way this is possible is if there is a scalar $\lambda \in \mathbb{C}$ such that $u = \lambda u'$ and $\lambda v = v'$. We have

$$\begin{aligned} C(u \otimes v) &= A(u) \otimes B(v) = A(\lambda u') \otimes B((1/\lambda)v') = \big(\lambda A(u')\big) \otimes \big((1/\lambda)B(v')\big) \\ &= \lambda(1/\lambda)\big(A(u') \otimes B(v')\big) = A(u') \otimes B(v') = C(u' \otimes v'). \end{aligned}$$

Therefore C is well-defined. ○

Next, we claim that C is a linear transformation. This follows immediately from the definition since we extended C by linearity. Finally, we must prove that C is unique. Since there is only one way to extend by linearity, C is entirely determined by its action on the simple tensors. It follows that there is just a single linear transformation that acts like C on simple tensors, so C is unique. □