QUANTUM ALGORITHMS
EXAM 1

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DUE: 2020-04-09

Instructions

- Solutions must be typed. Submit solutions by email.
- Solutions will be graded based on correctness, quality, and presentation. Turn in something that you are proud of.
- You may make use of any non-human assistance — any book, the web (but do not ask for help online), etc. Solutions must be self-contained.
- You may ask me questions about the problems.
- **You must submit a "draft" of your solution no later than 04-02.**

We will consider a generalization of Grover's algorithm. Suppose that we are given the following.

- $\mathcal{A}$, a quantum circuit.
- Basis vector $|\texttt{start}\rangle$ and quantum state $|\texttt{end}\rangle$ (i.e. norm 1) with $\mathcal{A}\,|\texttt{start}\rangle = |\texttt{end}\rangle$.
- $|\texttt{end}\rangle = |A\rangle + |B\rangle$ with $\langle A \mid B\rangle = 0$, $\langle A \mid A\rangle = a$, and $\langle B \mid B\rangle = b = 1 - a$.

Let us consider $|A\rangle$ as consisting of a superposition of "correct" outcomes of algorithm $\mathcal{A}$. Upon measuring $|\texttt{end}\rangle$, the probability of observing $|A\rangle$ is a.

We assume that we have a basis $(\psi_i)_{i\in I}$ such that $I = A \cup B$ and a function $\chi : I \to \{0, 1\}$ such that $\chi(A) = 1$ and $\chi(B) = 0$. Define

$$|\Psi(\alpha, \beta)\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$$

and note that $|\Psi(1, 1)\rangle = |\texttt{end}\rangle$. Define $\mathcal{G} = \mathcal{A} \circ D_{\texttt{s}} \circ \mathcal{A}^\dagger \circ D_A$ where D_A and $D_{\texttt{s}}$ are reflection operators (use the phase shift $e^{i\theta}$ for both).

Give a careful analysis of the circuit $\mathcal{G} \circ \mathcal{A}$, addressing all of the points below.

- Draw the circuits for both reflection operators and also write out the operators for them (e.g. $D_0 = (e^{i\theta} - 1)\,|0\rangle\,\langle 0| + I$ as in the usual Grover's algorithm).
- Calculate $\mathcal{G}\,|\Psi(1, 1)\rangle$ and write your answer in the form $|\Psi(x, y)\rangle$ for some x, y (you should specify their values).
- Suppose that $\mathcal{A}$ works with probability $1/4$ (i.e. $a = 1/4$). Show that taking $\theta = \pi$ (as in the usual Grover's circuit) makes $\mathcal{G} \circ \mathcal{A}$ acting on $|\texttt{start}\rangle$ exact.
- Show that when $\mathcal{A}$ works with probability $1/2$ there is a choice of θ so that $\mathcal{G} \circ \mathcal{A}$ acting on $|\texttt{start}\rangle$ is exact. What is the value of $e^{i\theta}$ in this case?
- Show that when $\mathcal{A}$ works with probability in the interval $[1/4, 1]$ there is a choice of θ so that $\mathcal{G} \circ \mathcal{A}$ acting on $|\texttt{start}\rangle$ is exact. Give a formula to determine it given a.
- When $\mathcal{A}$ works with probability in the interval $(0, 1/4)$, is there a choice of θ that makes $\mathcal{G} \circ \mathcal{A}$ acting on $|\texttt{start}\rangle$ exact? Fully justify your answer.

(When a circuit $\mathcal{B}$ gives the correct answer with probability 1, then $\mathcal{B}$ is said to be *exact*.)