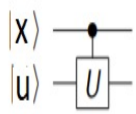
Homework 12

1. Prove that $\mathcal{P}|x, u\rangle = |x\rangle \otimes U^{[x]}|u\rangle$

1. Consider the single, k, unit of the circuit



2. This is a controlled gate operation on vector $|x\rangle$ with operator U

$$\wedge(u)|i, \alpha\rangle = \begin{cases} |i\rangle \otimes u|\alpha\rangle & \text{if } i = 1, \\ |i, \alpha\rangle = |i\rangle \otimes |\alpha\rangle & \text{otherwise.} \end{cases}$$

3. We can write the output of k^{th} unit as,

$$\mathcal{P}|x_k, u\rangle = \begin{cases} |1\rangle \otimes u^{2^k}|u\rangle & \text{if } x = 1, \\ |0\rangle \otimes |u\rangle & \text{if } x = 0. \end{cases}$$

$$\mathcal{P}|x_k, u\rangle = |x_k\rangle \otimes U^{2^k x_k}|u\rangle$$

4. We can write the output for the full circuit as below,

$$\mathcal{P}|x, u\rangle = (|x_0\rangle \otimes U^{2^{n-1}x_0}|u\rangle) \otimes (|x_1\rangle \otimes U^{2^{n-2}x_1}|u\rangle) \otimes \cdots \otimes (|x_{n-1}\rangle \otimes U^{2^0 x_{n-1}}|u\rangle)$$

$$= |x_0\rangle \otimes |x_1\rangle \otimes \cdots \otimes |x_{n-1}\rangle \otimes U^{2^{n-1}x_0}.U^{2^{n-2}x_1}.\dots.U^{2^0 x_{n-1}}.|u\rangle$$

$$= |x_0\rangle \otimes |x_1\rangle \otimes \cdots \otimes |x_{n-1}\rangle \otimes e^{i2\Pi 2^{n-1}x_0}.e^{i2\Pi 2^{n-2}x_1}.\dots.e^{i2\Pi 2^0 x_{n-1}}.|u\rangle$$

$$= |x_0\rangle \otimes |x_1\rangle \otimes \cdots \otimes |x_{n-1}\rangle \otimes e^{i2\Pi[2^{n-1}x_0 + 2^{n-2}x_1 + 2^0 x_{n-1}]}.|u\rangle$$

$= |x\rangle \otimes U^{[x]} \otimes |u\rangle$, where $[x]$ is the number with binary representation x

2. Prove that $\langle\alpha \mid \beta\rangle = 0$ and $|\alpha\rangle, |\beta\rangle$ are orthogonal

1. We have U as the unitary operator for the eigenvectors $|\alpha\rangle, |\beta\rangle$ with eigenvalues λ, μ

$\Rightarrow U|\alpha\rangle = \lambda|\alpha\rangle \Rightarrow \langle\alpha|U^\dagger = \langle\alpha|\lambda$

Similarly, $\Rightarrow U|\beta\rangle = \mu|\beta\rangle \Rightarrow \langle\beta|U^\dagger = \langle\beta|\mu$

2. From above equations, we can say that

$\langle\alpha|UU^\dagger|\beta\rangle = \langle\alpha|\lambda\mu|\beta\rangle$

As U is a unitary operator, $U.U^\dagger = 1$
$\Rightarrow \langle\alpha \mid \beta\rangle = \lambda\mu\langle\alpha \mid \beta\rangle$

3. If $\langle\alpha \mid \beta\rangle \neq 0$, then
$\lambda = \mu = 1$ or $\mu = \bar{\lambda}$, where $\lambda, \mu \in \mathbb{C}$ of the form $e^{i2\Pi\frac{k}{t}}$
But this is against the given condition $\lambda \neq \mu$

4. So, if $\lambda \neq \mu$, then,
$\langle\alpha \mid \beta\rangle = \lambda\mu\langle\alpha \mid \beta\rangle = 0$
$\Rightarrow |\alpha\rangle, |\beta\rangle$ are orthogonal

3. Show that $\hat{l_\oplus}(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \Leftrightarrow |\psi\rangle = |0\rangle$ or $|\psi\rangle = |1\rangle$

1. Suppose, $\hat{l_\oplus}$ is the quantumn clone operator,

$\Rightarrow \hat{l_\oplus}|0\rangle|0\rangle = |0\rangle|0\rangle$ and $\hat{l_\oplus}|1\rangle|0\rangle = |1\rangle|1\rangle$

2. Let us consider the qubit in the state $|\psi\rangle = \dfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

if we apply the clone operator on ψ, by linearity,

$$\hat{l_\oplus}|\psi\rangle|0\rangle = \hat{l_\oplus}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$$
$$= \frac{1}{\sqrt{2}}(\hat{l_\oplus}|0\rangle|0\rangle + \hat{l_\oplus}|1\rangle|0\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$
$$\neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \neq \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle)$$

3. Thus, cloning operator $\hat{l_\oplus}$ work if $|01\rangle = |10\rangle = 0$.
$\Rightarrow$ only if $|\psi\rangle = |0\rangle$ or $|1\rangle$

4. Show that U clones $|\varphi\rangle$ and $|\psi\rangle$ if and only if $|\varphi\rangle = |\psi\rangle$ or $\langle\varphi \mid \psi\rangle = 0$

1. Given cloning is a unitary transformation, which should preserve the geometry of the vectors.
We use inner product to verify. Inner product should be same (preserves geometry)
for given equations,

$U(|\varphi\rangle \otimes |0^n\rangle) = |\varphi\rangle \otimes |\varphi\rangle$, $U(|\psi\rangle \otimes |0^n\rangle) = |\psi\rangle \otimes |\psi\rangle$

2. $(\langle\varphi| \otimes \langle 0^n|)U^\dagger U(|\varphi\rangle \otimes |0^n\rangle)$

$= (\langle\varphi| \otimes \langle 0^n|)(|\varphi\rangle \otimes |0^n\rangle) = \langle\varphi \mid \psi\rangle\langle 0^n \mid 0^n\rangle - 1$

3. $(\langle\varphi| \otimes \langle\varphi|)(|\psi\rangle \otimes |\psi\rangle)$

$= \langle\varphi \mid \psi\rangle\langle\varphi \mid \psi\rangle - 2$

4. Equation 1 and 2 are equal,

$\langle\varphi \mid \psi\rangle\langle 0^n \mid 0^n\rangle = \langle\varphi \mid \psi\rangle\langle\varphi \mid \psi\rangle$

if and only if
$\Rightarrow \langle\varphi \mid \psi\rangle = 0$
$\Rightarrow |\varphi\rangle|\psi\rangle$ are orthogonal

or
$\Rightarrow \langle\varphi \mid \psi\rangle = 1$
$\Rightarrow |\varphi\rangle = |\psi\rangle$

5. Show that, Quantum cloning operators do not work for all pair of states

1. Let us consider the qubit in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
if we apply the clone operator on ψ, by linearity,
$\hat{l_\oplus}(|\psi\rangle \otimes |0\rangle) = \hat{l_\oplus}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$
$= \frac{1}{\sqrt{2}}(\hat{l_\oplus}|0\rangle|0\rangle + \hat{l_\oplus}|1\rangle|0\rangle)$
$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

$|\psi\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle)$

2. $\Rightarrow \hat{l_\oplus}(|\psi\rangle \otimes |0\rangle) \neq |\psi\rangle \otimes |\psi\rangle$ Cloning failed on this pair of state.