QUANTUM ALGORITHMS
HOMEWORK 11 SELECTED SOLUTIONS

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AP 1. **(i)** Explicitly calculate $\mathcal{QFT}_n |0^n\rangle$.

(ii) Explicitly calculate $\mathcal{QFT}_n |1^n\rangle$.

Solution: We have

$$\begin{aligned}
\mathcal{QFT}_n |0^n\rangle &= 2^{-n/2} \bigotimes_{k=1}^{n} |0\rangle + e^{2\pi i(0)/2^k} |1\rangle = 2^{-n/2} \bigotimes_{k=1}^{n} |0\rangle + |1\rangle = 2^{-n/2} \big(|0\rangle + |1\rangle \big)^{\otimes n} \\
&= 2^{-n/2} \sum_{x \in \{0,1\}^n} |x\rangle , \\
\mathcal{QFT}_n |1^n\rangle &= 2^{-n/2} \bigotimes_{k=1}^{n} |0\rangle + e^{2\pi i[1^n]/2^k} |1\rangle = 2^{-n/2} \bigotimes_{k=1}^{n} |0\rangle + e^{2\pi i(2^n-1)/2^k} |1\rangle \\
&= 2^{-n/2} \bigotimes_{k=1}^{n} |0\rangle + e^{-2\pi i/2^k} |1\rangle = 2^{-n/2} \sum_{x_1,\ldots,x_n \in \{0,1\}} \left(\prod_{k=1}^{n} e^{-2\pi i x_k/2^k} \right) |x_1 \cdots x_n\rangle \\
&= 2^{-n/2} \sum_{x_1,\ldots,x_n \in \{0,1\}} e^{-2\pi i \sum x_k/2^k} |x_1 \cdots x_n\rangle \\
&= 2^{-n/2} \sum_{x \in \{0,1\}^n} e^{-2\pi i[x]/2^n} |x\rangle .
\end{aligned}$$

AP 2. Show that

$$\mathcal{QFT}_n |x\rangle = 2^{-n/2} \sum_{y \in \{0,1\}^n} e^{2\pi i[x][y]/2^n} |y\rangle ,$$

where $[x]$ represents the number with binary representation $x \in \{0,1\}^n$ (and so $[x][y]$ is the product of x and y, regarded as binary numbers).

Solution:

Proof. We have

$$\begin{aligned}
\mathcal{QFT}_n |x\rangle &= 2^{-n/2} \bigotimes_{k=1}^{n} |0\rangle + e^{2\pi i[x]/2^k} |1\rangle = 2^{-n/2} \bigotimes_{k=1}^{n} \sum_{y_k \in \{0,1\}} e^{2\pi i y_k [x]/2^k} |y_k\rangle \\
&= 2^{-n/2} \sum_{y_1,\ldots,y_k \in \{0,1\}} e^{2\pi i[x] \sum y_k/2^k} |y_1 \cdots y_n\rangle = 2^{-n/2} \sum_{y \in \{0,1\}^n} e^{2\pi i[x][y]/2^n} |y\rangle . \square
\end{aligned}$$

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AP 3. Use the previous problem to prove that

$$\mathcal{QFT}_n^\dagger |x\rangle = 2^{-n/2} \sum_{y\in\{0,1\}^n} e^{-2\pi i[x][y]/2^n} |y\rangle$$

for basis vector $|x\rangle \in \{0,1\}$ defines the inverse of $\mathcal{QFT}_n$.

Hint 1: Show that $\mathcal{QFT}_n \circ \mathcal{QFT}_n^\dagger |x\rangle = \mathcal{QFT}_n^\dagger \circ \mathcal{QFT}_n |x\rangle = |x\rangle$.

Hint 2: You may find this identity useful

$$\sum_{k=0}^{2^n-1} e^{2\pi i\, k\ell/2^n} = 0 \qquad \text{if} \qquad \ell \neq 0.$$

Solution:

Proof. We have

$$\begin{aligned}
\mathcal{QFT}_n \circ \mathcal{QFT}_n^\dagger |x\rangle &= \mathcal{QFT}_n\Big(2^{-n/2} \sum_{y\in\{0,1\}^n} e^{-2\pi i[x][y]/2^n} |y\rangle\Big)\\
&= 2^{-n/2} \sum_{y\in\{0,1\}^n} e^{-2\pi i[x][y]/2^n}\ \mathcal{QFT}_n |y\rangle\\
&= 2^{-n/2} \sum_{y\in\{0,1\}^n} e^{-2\pi i[x][y]/2^n} \Big(2^{-n/2} \sum_{z\in\{0,1\}^n} e^{2\pi i[y][z]/2^n} |z\rangle\Big)\\
&= 2^{-n} \sum_{z\in\{0,1\}^n} \Big(\sum_{y\in\{0,1\}^n} e^{2\pi i[y]\big([x]-[z]\big)/2^n}\Big) |z\rangle .
\end{aligned}$$

Using the identity in Hint 2, the inner sum simplifies,

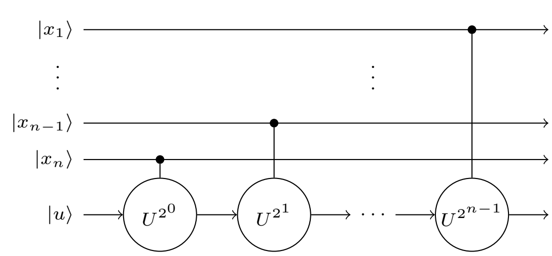
$$\sum_{y\in\{0,1\}^n} e^{2\pi i[y]\big([x]-[z]\big)/2^n} = \left\{ \begin{array}{cl} 0 & \text{if } x \neq z \\ \displaystyle\sum_{y\in\{0,1\}^n} 1 & \text{if } x = z \end{array} \right\} = \begin{cases} 0 & \text{if } x \neq z, \\ 2^n & \text{if } x = z. \end{cases}$$

Therefore the summand is nonzero only when $z = x$, in which case it is $2^n |x\rangle$. Thus,

$$2^{-n} \sum_{z\in\{0,1\}^n} \Big(\sum_{y\in\{0,1\}^n} e^{2\pi i[y]\big([x]-[z]\big)/2^n}\Big) |z\rangle = 2^{-n}2^n |x\rangle = |x\rangle ,$$

as desired. The calculation showing $\mathcal{QFT}_n^\dagger \circ \mathcal{QFT}_n |x\rangle = |x\rangle$ is quite similar. ☐

AP 5. Let $\mathcal{P}$ represent the portion of the eigenvalue approximation circuit shown below.



We consider the circuit for arbitrary unitary m-dimensional U, $|u\rangle \in \mathfrak{B}^m$, and $x \in \{0,1\}^n$ (the eigenvalue estimation circuit took $x = 0^n$ and $|u\rangle$ to be an eigenvector).

Show that $\mathcal{P}|x,u\rangle = |x\rangle \otimes U^{[x]} |u\rangle$, where $[x]$ is the number with binary representation x and $U^{[x]}$ is matrix exponentiation.

Solution:

Proof. Let us consider a single rail, $|x_k\rangle$. We have

$$|x_k\rangle \otimes |u\rangle \to \left\{\begin{matrix} |0\rangle \otimes |u\rangle & \text{if } x_k = 0 \\ |1\rangle \otimes U^{2^k}|u\rangle & \text{if } x_k = 1 \end{matrix}\right\} = |x_k\rangle \otimes U^{x_k 2^k}|u\rangle\,.$$

Applying this to the whole input vector $|x_1 \cdots x_n\rangle$, we have

$$\mathcal{P}\,|x,u\rangle = |x\rangle \otimes \left(\prod_{k=1}^{n} U^{x_k 2^k}\right)|u\rangle = |x\rangle \otimes U^{\sum x_k 2^k}|u\rangle = |x\rangle \otimes U^{[x]}|u\rangle\,,$$

as claimed. □