Homework 10

1.1. Prove $A = A_0 \cup A_1$ and $A_0 \cap A_1 = \phi$?

1. We know that $\mathbb{Z}^n = 0, 1, 2 \ldots n$ and
$\mathbb{Z}_2^n = 0, 1, 0, 1 \ldots l$ is the cyclic group of order 2, where l = 0,1 if n = even,odd

2. We have $A \leq \mathbb{Z}_2^n$ and $g \in \mathbb{Z}_2^n$ and
$A_0 = \{a \in A | a.g = 0\}, A_1 = \{a \in A, g \neq 0 | a.g = 1\}$, where A_0, A_1 are right cosets of A

3. If e is the identity, then $e \in A \Rightarrow g \in A.g \; \forall \; g \in \mathbb{Z}_2^n$
We can say that, all right cosets $A.g$ are non empty
For a given 'g' A_0 is the coset having all zeroes of subgroup A and
For a given 'g' A_1 is the coset having all ones of subgroup A and
$\Rightarrow A = A_0 \cup A_1$

4. Suppose cosets are not disjoint $A_0 \cap A_1 \neq \phi$
Let $g \in A_0 \cap A_1$
Then there is $h, k \in A$ and such that $g = ph = qk$ where $p \in A_0 \; and \; q \in A_1$
$\Rightarrow p = qkh^{-1} \in qA$ and $q = phk^{-1} \in pA$
Let $ph^{`} \in pA$
$ph^{`} = qkh^{-1}h^{`}$
$\Rightarrow ph^{`}$ is of the form qA $\Rightarrow ph^{`} \in qA$
Thus $pA \subset qA$ and by symmetry $qA \subset pA$
$\Rightarrow pA = qA \Rightarrow A_0 = A_1$

By our assumption $A_0 \cap A_1 \neq \phi$ resulted in cosets being $A_0 = A_1$
$\Rightarrow$ cosets A_0, A_1 are disjoint
$\Rightarrow A_0 \cap A_1 = \phi$

1.2. Prove that $a + A_0 = A_1$ and $a + A_1 = A_0 \Rightarrow |A_0| = |A_1|$ if $A_1 \neq \phi$

1. We know that $\mathbb{Z}_2^n$ consists of two cosets,
A_0, the even numbers and A_1, the odd numbers

2. We can write,
$A_0 = \{\ldots, -4, -2, 0, 2, 4, \ldots\} = 0 + 2\mathbb{Z}$
$A_1 = \{\ldots, -3, -1, 1, 3, 5, \ldots\} = 1 + 2\mathbb{Z}$

3. Say $a = 1 \in A_1$,
$a + A_0 = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$
$a + A_1 = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
Above cosets remain same for any value $a \in A_1$

4. From above points, we can say that,
$a + A_0 = A_1$ and the number of elements in A_0, A_1 are equal.
$a + A_1 = A_0$ and the number of elements in A_0, A_1 are equal.
$\Rightarrow |A_0| = |A_1|$ where $|A_1| \neq \phi$

1.3. Prove that $\sum_{a\in A}(-1)^{a\cdot g} = \begin{cases} |A| & \text{if } a\cdot g = 0 \text{ for all } a \in A, \\ 0 & \text{otherwise.} \end{cases}$

1. We know that, $A = A_0 \cup A_1 \Rightarrow |A| = |A_0| + |A_1|$
$\sum_{a\in A}(-1)$ is the summation of $(-1) = (-1 + -1 + \cdots + |A|)$

2. $\sum_{a\in A}(-1)^{a\cdot g} = \sum_{a\in A} 1$, if $a.g = 0$
$\Rightarrow \sum_{a\in A} 1 = (1 + 1 + \cdots + |A|) = |A|$

3. We know that subgroup A has equal number of odd and even numbers
$|A_0| = |A_1|$
$\Rightarrow \sum_{a\in A}(-1)^{a\cdot g} = 0$

3. Prove that A is maximal if and only if $A^{\perp}$ is minimal

1. Let $P \in A^{\perp} and a \in A$
$\Rightarrow P.a = 0$

2. Say we have another subgroup $A \leq B$
Let $Q \in B^{\perp}$ and $b \in B$
$\Rightarrow Q.b = 0$
$\Rightarrow Q.a = 0$ as $A \leq B \Rightarrow Q \in A^{\perp}$
$\Rightarrow B^{\perp} \leq A^{\perp}$

3. Using point 2, by symmetry, we can say
If $B \leq A$, then $A^{\perp} \leq B^{\perp}$
$\Rightarrow$ If A is maximal if and only if $A^{\perp}$ is minimal