QUANTUM ALGORITHMS
HOMEWORK 7 ADDITIONAL PROBLEMS

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1. After k iterations of $\mathcal{G}$ in Grover's algorithm, we obtained

$$\mathcal{G}^k \left|\Psi(1,1)\right\rangle = \left|\Psi\Big(a^{-1/2}\sin\big((2k+1)\theta\big),\ b^{-1/2}\cos\big((2k+1)\theta\big)\Big)\right\rangle$$

where θ is such that $\sin(\theta) = \sqrt{a}$. Show that when $k = \lfloor \pi/(4\theta) \rfloor$, upon measuring this state the probability of observing a state in $|A\rangle$ is $\geq 1 - a$.

2. We solved the recurrence in Grover's algorithm by diagonalizing a matrix,

$$\begin{pmatrix} b-a & 2b \\ -2a & b-a \end{pmatrix} = \frac{1}{\sqrt{a}} \begin{pmatrix} i\sqrt{b} & -i\sqrt{b} \\ \sqrt{a} & \sqrt{a} \end{pmatrix} \cdot \begin{pmatrix} \overline{\lambda}^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} \cdot \frac{1}{2\sqrt{b}} \begin{pmatrix} -i\sqrt{a} & \sqrt{b} \\ i\sqrt{a} & \sqrt{b} \end{pmatrix}$$

where $\lambda = e^{i\theta} = \sqrt{b} + i\sqrt{a}$ (so $\sin(\theta) = \sqrt{a}$), $\overline{\lambda}$ is the conjugate of λ, and $b = 1 - a$ with $a \in [0,1]$.

Verify that the matrices multiply as claimed in the above equation.

3. Recall that the Fibonacci sequence $(f_i)_{i\in\mathbb{N}}$ is defined

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+1} = f_n + f_{n-1}.$$

(i) Show that

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) Use the same technique that we used to find a closed form of the recurrence in Grover's algorithm to find a closed form for the Fibonacci sequence.

Hint: $f_n = (1/\sqrt{5})(\varphi^n - \psi^n)$ *where* $\varphi = (1/2)(1+\sqrt{5})$ *is the golden ratio and* $\psi = (1/2)(1-\sqrt{5})$ *is its conjugate.*