# QUANTUM ALGORITHMS
# HOMEWORK 5 ADDITIONAL PROBLEMS

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**1.** Let $\mathbb{V}$ and $\mathbb{S}$ be vector spaces over $\mathbb{C}$ with bases $\mathcal{B}_{\mathbb{V}}$ and $\mathcal{B}_{\mathbb{S}}$, respectively. Define

$$\mathbb{V} \times \mathbb{S} = \big\{(v, s) \mid v \in V \text{ and } s \in S\big\}$$

and recognize it as a vector space by *coordinate-wise* interpretation of the vector space axioms. That is,

$$\begin{aligned} (v_1, s_1) + (v_2, s_2) &= (v_1 + v_2, s_1 + s_2) && \text{for } v_1, v_2 \in V \text{ and } s_1, s_2 \in S, \\ \lambda \cdot (v_1, s_1) &= (\lambda \cdot v_1, \lambda \cdot s_1) && \text{for } v_1 \in V, s_1 \in S, \text{ and } \lambda \in \mathbb{C} \text{ a scalar.} \end{aligned}$$

If $R : \mathbb{A} \to \mathbb{V}$ and $T : \mathbb{A} \to \mathbb{S}$ are linear functions, then we can define a linear function $(R \times T) : \mathbb{A} \to \mathbb{V} \times \mathbb{S}$ by

$$(R \times T)a = \big(Ra, Ta\big) \qquad \text{for } a \in A.$$

**(i)** Let

$$\mathcal{C} = \big\{(b_v, b_s) \mid b_v \in \mathcal{B}_{\mathbb{V}} \text{ and } b_s \in \mathcal{B}_{\mathbb{S}}\big\}.$$

Show that $\mathbb{C}$-span$(\mathcal{C}) = \mathbb{V} \times \mathbb{S}$ but that $\mathcal{C}$ is *not always* a basis for $\mathbb{V} \times \mathbb{S}$.

**(ii)** Prove that

$$\mathcal{B}_{\mathbb{V}\times\mathbb{S}} = \big\{(b_v, 0), (0, b_s) \mid b_v \in \mathcal{B}_{\mathbb{V}} \text{ and } b_s \in \mathcal{B}_{\mathbb{S}}\big\}$$

is a basis for $\mathbb{V} \times \mathbb{S}$. What is the dimension of $\mathbb{V} \times \mathbb{S}$?

**(iii)** Let $R : \mathbb{A} \to \mathbb{V}$ and $T : \mathbb{A} \to \mathbb{S}$ be linear functions. Suppose that $\mathbb{A}$, $\mathbb{V}$, and $\mathbb{S}$ have ordered bases

$$\mathcal{B}_{\mathbb{A}} = \big\{a_1, a_2\big\}, \qquad \mathcal{B}_{\mathbb{V}} = \big\{v_1, v_2, v_3\big\}, \qquad \mathcal{B}_{\mathbb{S}} = \{s_1, s_2\},$$

and that the matrix representations of $R$ and $T$ relative to these bases are

$$(R)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{V}}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \text{and} \qquad (T)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{S}}} = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}.$$

Using the lexicographic order for the basis $\mathcal{B}_{\mathbb{V}\times\mathbb{S}}$ (i.e. ordering by $\mathcal{B}_{\mathbb{V}}$ first, and then $\mathcal{B}_{\mathbb{S}}$), find the matrix representation for $(R \times T)$ (that is, find $(R \times T)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{V}\times\mathbb{S}}}$).

**Problems continue on the next page.**

**2.** Let $T : \mathbb{A} \to \mathbb{B}$ be a linear transformation between vector spaces with ordered bases

$$\mathcal{B}_{\mathbb{A}} = \{\, |1\rangle, |2\rangle, |3\rangle \,\} \qquad \mathcal{B}_{\mathbb{B}} = \{\, |1\rangle, |2\rangle \,\}.$$

Suppose that $T$ has matrix with respect to these bases

$$T = \begin{pmatrix} 9 & 6 & -3 \\ -4 & -8 & 8 \end{pmatrix}.$$

**(i)** Show that the matrix for $T$ can be written

$$T = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} a_{ij} \, |i\rangle \langle j|$$

(note that $|1\rangle \in \mathcal{B}_{\mathbb{A}}$ is a 3-dimensional vector, while $|1\rangle \in \mathcal{B}_{\mathbb{B}}$ is a 2-dimensional vector).

**(ii)** Show that for fixed $|i\rangle \in \mathcal{B}_{\mathbb{A}}$ and $|j\rangle \in \mathcal{B}_{\mathbb{B}}$

$$\big( |j\rangle \langle i| \big) |v\rangle = \langle i \mid v\rangle \, |j\rangle$$

for all $\langle v| \in \mathbb{A}$. From this, prove that $|j\rangle \langle i|$ defines a linear transformation from $\mathbb{A} \to \mathbb{B}$.

**(iii)** Suppose that

$$R = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} b_{ij} \, |i\rangle \langle j|$$

for $b_{ij} \in \mathbb{C}$. Use the previous part to prove that $R$ is a linear transformation from $\mathbb{A} \to \mathbb{B}$.

**3.** Let $\mathbb{V}$ and $\mathbb{S}$ be vector spaces over $\mathbb{C}$ with bases $B_{\mathbb{V}}$ and $B_{\mathbb{S}}$, respectively.

**(i)** Prove that

$$B_{\mathbb{V}\otimes\mathbb{S}} = \big\{ b_v \otimes b_s \mid b_v \in B_{\mathbb{V}} \text{ and } b_s \in B_{\mathbb{S}} \big\}$$

is a basis of $\mathbb{V} \otimes \mathbb{S}$. What is the dimension of $\mathbb{V} \otimes \mathbb{S}$?

**(ii)** Let $R : \mathbb{V} \to \mathbb{A}$ and $T : \mathbb{S} \to \mathbb{B}$ be linear functions. Suppose that $\mathbb{A}$, $\mathbb{V}$, and $\mathbb{S}$ have ordered bases the same as in the previous question and that $\mathbb{B}$ has ordered basis $B_{\mathbb{B}} = \big\{ b_1, b_2 \big\}$ and that the matrix representations of $R$ and $T$ relative to these bases are

$$(R)_{B_{\mathbb{V}} \to B_{\mathbb{A}}} = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -2 & -1 \end{bmatrix} \qquad \text{and} \qquad (T)_{B_{\mathbb{S}} \to B_{\mathbb{B}}} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Using the lexicographic order for the basis $B_{\mathbb{V}\otimes\mathbb{S}}$, find the matrix representation for $(R \otimes T)$ (that is, find $(R \times T)_{B_{\mathbb{V}\otimes\mathbb{S}} \to B_{\mathbb{A}\otimes\mathbb{B}}}$). [*Hint: Kronecker product.*]