# QUANTUM ALGORITHMS
# HOMEWORK 12 ADDITIONAL PROBLEMS

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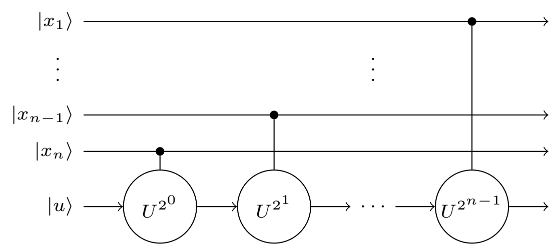
**1.** Let $\mathcal{P}$ represent the portion of the eigenvalue approximation circuit shown below.




We consider the circuit for arbitrary unitary $m$-dimensional $U$, $|u\rangle \in \mathfrak{B}^m$, and $x \in \{0,1\}^n$ (the eigenvalue estimation circuit took $x = 0^n$ and $|u\rangle$ to be an eigenvector).

Show that $\mathcal{P}\,|x, u\rangle = |x\rangle \otimes U^{[x]}\,|u\rangle$, where $[x]$ is the number with binary representation $x$ and $U^{[x]}$ is matrix exponentiation.

**2.** Let $U$ be a unitary operator and suppose that $|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors with respective eigenvalues $\lambda, \mu \in \mathbb{C}$. Prove that if $\lambda \neq \mu$ then $\langle \alpha \mid \beta \rangle = 0$ (i.e. $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal).

**3.** Let $\iota : \{0,1\} \to \{0,1\}$ be the identity function, defined by $\iota(x) = x$. The function $\iota_\oplus(x,y) = (x, x \oplus y)$ has the property that

$$\iota_\oplus(x, 0) = (x, x),$$

meaning that it *clones* the bit $x$ in the first register to the second register.

Let $|\psi\rangle \in \mathfrak{B}$ be an arbitrary 1-qubit quantum state. Show that

$$\widehat{\iota_\oplus}(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \qquad \Leftrightarrow \qquad |\psi\rangle = |0\rangle \text{ or } |\psi\rangle = |1\rangle\,.$$

That is, the quantum operator $\widehat{\iota_\oplus}$ corresponding to the classical 1-bit cloning operator $\iota_\oplus$ fails to clone $|\psi\rangle$ unless $|\psi\rangle$ is in a state corresponding to a classical bit.

**4.** Let $U$ be a $(2n)$-qubit operator that *clones* two $n$-qubit quantum states, $|\varphi\rangle, |\psi\rangle \in \mathfrak{B}^{\otimes n}$, meaning

$$U(|\varphi\rangle \otimes |0^n\rangle) = |\varphi\rangle \otimes |\varphi\rangle \qquad \text{and} \qquad U(|\psi\rangle \otimes |0^n\rangle) = |\psi\rangle \otimes |\psi\rangle\,.$$

Prove that $U$ clones $|\varphi\rangle$ and $|\psi\rangle$ if and only if $|\varphi\rangle = |\psi\rangle$ or $\langle \varphi \mid \psi \rangle = 0$. [*Hint: take the inner product of the two equations.*]

**5.** Use the previous question to prove that a there are no quantum cloning operators that work for all pairs of states.