QUANTUM ALGORITHMS
HOMEWORK 5 SELECTED SOLUTIONS

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AP 1. Let $\mathbb{V}$ and $\mathbb{S}$ be vector spaces over $\mathbb{C}$ with bases $\mathcal{B}_{\mathbb{V}}$ and $\mathcal{B}_{\mathbb{S}}$, respectively. Define

$$\mathbb{V} \times \mathbb{S} = \big\{(v, s) \mid v \in V \text{ and } s \in S\big\}$$

and recognize it as a vector space by *coordinate-wise* interpretation of the vector space axioms. That is,

$$\begin{aligned} &(v_1, s_1) + (v_2, s_2) = (v_1 + v_2, s_1 + s_2) && \text{for } v_1, v_2 \in V \text{ and } s_1, s_2 \in S, \\ &\lambda \cdot (v_1, s_1) = (\lambda \cdot v_1, \lambda \cdot s_1) && \text{for } v_1 \in V, s_1 \in S, \text{ and } \lambda \in \mathbb{C} \text{ a scalar.} \end{aligned}$$

If $R : \mathbb{A} \to \mathbb{V}$ and $T : \mathbb{A} \to \mathbb{S}$ are linear functions, then we can define a linear function $(R \times T) : \mathbb{A} \to \mathbb{V} \times \mathbb{S}$ by

$$(R \times T)a = \big(Ra, Ta\big) \qquad \text{for } a \in A.$$

(i) Let

$$\mathcal{C} = \big\{(b_v, b_s) \mid b_v \in \mathcal{B}_{\mathbb{V}} \text{ and } b_s \in \mathcal{B}_{\mathbb{S}}\big\}.$$

Show that $\mathbb{C}\text{-span}(\mathcal{C}) = \mathbb{V} \times \mathbb{S}$ but that $\mathcal{C}$ is *not always* a basis for $\mathbb{V} \times \mathbb{S}$.

(ii) Prove that

$$\mathcal{B}_{\mathbb{V}\times\mathbb{S}} = \big\{(b_v, 0), (0, b_s) \mid b_v \in \mathcal{B}_{\mathbb{V}} \text{ and } b_s \in \mathcal{B}_{\mathbb{S}}\big\}$$

is a basis for $\mathbb{V} \times \mathbb{S}$. What is the dimension of $\mathbb{V} \times \mathbb{S}$?

(iii) Let $R : \mathbb{A} \to \mathbb{V}$ and $T : \mathbb{A} \to \mathbb{S}$ be linear functions. Suppose that $\mathbb{A}$, $\mathbb{V}$, and $\mathbb{S}$ have ordered bases

$$\mathcal{B}_{\mathbb{A}} = \big\{a_1, a_2\big\}, \qquad \mathcal{B}_{\mathbb{V}} = \big\{v_1, v_2, v_3\big\}, \qquad \mathcal{B}_{\mathbb{S}} = \{s_1, s_2\},$$

and that the matrix representations of R and T relative to these bases are

$$(R)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{V}}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \text{and} \qquad (T)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{S}}} = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}.$$

Using the lexicographic order for the basis $\mathcal{B}_{\mathbb{V}\times\mathbb{S}}$ (i.e. ordering by $\mathcal{B}_{\mathbb{V}}$ first, and then $\mathcal{B}_{\mathbb{S}}$), find the matrix representation for $(R \times T)$ (that is, find $(R \times T)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{V}\times\mathbb{S}}}$).

Solution:

(i): Choose distinct elements $v_1, v_2 \in \mathcal{B}_{\mathbb{V}}$ and $s_1, s_2 \in \mathcal{B}_{\mathbb{S}}$. We have

$$(v_1, s_1) + (v_2, s_2) - (v_1, s_2) - (v_2, s_1) = (0, 0),$$

so $\mathcal{C}$ cannot be linearly independent and hence cannot be a basis.

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(ii): *Proof.* Let $(\alpha,\beta) \in \mathbb{V} \times \mathbb{S}$. It follows that $\alpha \in \mathbb{V}$ and $\beta \in \mathbb{S}$. Since $\mathcal{B}_{\mathbb{V}}$ and $\mathcal{B}_{\mathbb{S}}$ are bases for $\mathbb{V}$ and $\mathbb{S}$, respectively, there are scalars λ_v, μ_s such that

$$\alpha = \sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v \qquad\qquad \beta = \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s.$$

Therefore

$$\begin{aligned}(\alpha,\beta) &= \left(\sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v, \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s\right) = \left(\sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v, 0\right) + \left(0, \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s\right) \\ &= \sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v (b_v, 0) + \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s (0, b_s) = \sum_{\substack{b_v \in \mathcal{B}_{\mathbb{V}} \\ b_s \in \mathcal{B}_{\mathbb{S}}}} \lambda_v (b_v, 0) + \mu_s (0, b_s).\end{aligned}$$

It follows that $\mathbb{C}\text{-span}(\mathcal{B}_{\mathbb{V}\times\mathbb{S}}) = \mathbb{V} \times \mathbb{S}$. It remains to show linear independence. Following the equation above backwards, if

$$0 = (0,0) = \sum_{\substack{b_v \in \mathcal{B}_{\mathbb{V}} \\ b_s \in \mathcal{B}_{\mathbb{S}}}} \lambda_v (b_v, 0) + \mu_s (0, b_s) = \left(\sum_{b_v \in \mathcal{B}_{\mathbb{V}}} \lambda_v b_v, \sum_{b_s \in \mathcal{B}_{\mathbb{S}}} \mu_s b_s\right),$$

then both the sums are equal to 0, and therefore all the coefficients are 0 since $\mathcal{B}_{\mathbb{V}}$ and $\mathcal{B}_{\mathbb{S}}$ are linearly independent. □

(iii): The basis described in the problem is

$$\mathcal{B}_{\mathbb{V}\times\mathbb{S}} = \big\{(v_1,0),(v_2,0),(v_3,0),(0,s_1),(0,s_1)\big\}.$$

From the definition of $\times$ for linear functions and the definitions of R and T, we have that

$$\begin{aligned}(R \times T)a_1 &= (Ra_1, Ta_1) = \big(v_1 + 3v_2 + 5v_3, -s_1 + 3s_2\big) \\ &= (v_1,0) + 3(v_2,0) + 5(v_3,0) - (0,s_1) + 3(0,s_2) \qquad\qquad \text{and} \\ (R \times T)a_2 &= (Ra_2, Ta_2) = \big(2v_1 + 4v_2 + 6v_3, 2s_1 - 2s_2\big) \\ &= 2(v_1,0) + 4(v_2,0) + 6(v_3,0) + 2(0,s_1) - 2(0,s_2).\end{aligned}$$

Therefore, the first column of $(R \times T)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{V}\times\mathbb{S}}}$ is $(1,3,5,-1,3)^T$ and the second column is $(2,4,6,2,-2)^T$:

$$(R \times T)_{\mathcal{B}_{\mathbb{A}} \to \mathcal{B}_{\mathbb{V}\times\mathbb{S}}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ -1 & 2 \\ 3 & -2 \end{pmatrix}.$$

AP 2. Let $T : \mathbb{A} \to \mathbb{B}$ be a linear transformation between vector spaces with ordered bases

$$\mathcal{B}_{\mathbb{A}} = \{\, |1\rangle, |2\rangle, |3\rangle \,\} \qquad\qquad \mathcal{B}_{\mathbb{B}} = \{\, |1\rangle, |2\rangle \,\}.$$

Suppose that T has matrix with respect to these bases

$$T = \begin{pmatrix} 9 & 6 & -3 \\ -4 & -8 & 8 \end{pmatrix}.$$

(i) Show that the matrix for T can be written

$$T = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} a_{ij} \, |i\rangle \langle j|$$

(note that $|1\rangle \in \mathcal{B}_{\mathbb{A}}$ is a 3-dimensional vector, while $|1\rangle \in \mathcal{B}_{\mathbb{B}}$ is a 2-dimensional vector).

(ii) Show that for fixed $|i\rangle \in \mathcal{B}_{\mathbb{A}}$ and $|j\rangle \in \mathcal{B}_{\mathbb{B}}$

$$\big(|j\rangle \langle i| \big) |v\rangle = \langle i \mid v\rangle |j\rangle$$

for all $\langle v| \in \mathbb{A}$. From this, prove that $|j\rangle \langle i|$ defines a linear transformation from $\mathbb{A} \to \mathbb{B}$.

(iii) Suppose that

$$R = \sum_{\substack{|j\rangle \in \mathcal{B}_{\mathbb{A}} \\ |i\rangle \in \mathcal{B}_{\mathbb{B}}}} b_{ij} |i\rangle \langle j|$$

for $b_{ij} \in \mathbb{C}$. Use the previous part to prove that R is a linear transformation from $\mathbb{A} \to \mathbb{B}$.

Solution:

(i): The linear transformation T is uniquely characterized by its action on $\mathcal{B}_{\mathbb{A}}$,

$$\begin{aligned} T|1\rangle &= 9|1\rangle - 4|2\rangle, \\ T|2\rangle &= 6|1\rangle - 8|2\rangle, \\ T|2\rangle &= -3|1\rangle + 8|2\rangle \end{aligned}$$

(these equalities come from the matrix representation of T given in the problem). Therefore, if S is any other linear transformation acting the same way on $\mathcal{B}_{\mathbb{A}}$ then $S = T$. Let

$$S = \big(9|1\rangle\langle 1| - 4|2\rangle\langle 1|\big)\big(6|1\rangle\langle 2| - 8|2\rangle\langle 2|\big)\big(-3|1\rangle\langle 3| + 8|2\rangle\langle 3|\big).$$

S is itself a linear combination of linear transformations (by items (ii) and (iii) below), and is thus a linear transformation. Showing that S acts on the basis vectors $\mathcal{B}_{\mathbb{A}}$ in the same manner as T is a straightforward calculation. It follows that $S = T$.

(ii): *Proof.* Regarded as a matrix, the object $|j\rangle \langle i|$ has dimensions $\dim(\mathbb{B}) \times \dim(\mathbb{A})$, so $(|j\rangle \langle i|) |v\rangle$ is a defined quantity. Matrix multiplication is the same as function composition, so it is associative. Therefore

$$\big(|j\rangle \langle i| \big) |v\rangle = \langle j| \big(|i\rangle \langle v| \big) = \langle j| \langle i \mid v\rangle = \langle i \mid v\rangle \langle j|.$$

(the last equality follows from $\langle i \mid v\rangle \in \mathbb{C}$ being a scalar). □

(iii): *Proof.* R is a linear combination of linear transformations (by item (ii) above). It is therefore sufficient to show that if S and T are linear transformations and $\lambda, \mu \in \mathbb{C}$, then is $\lambda S + \mu T$ also a linear transformation. We have

$$(\lambda S + \mu T)0 = \lambda S0 + \mu T0 = \lambda 0 + \mu 0 = 0 + 0 = 0,$$

$$\begin{aligned} (\lambda S + \mu T)(|\alpha\rangle + |\beta\rangle) &= \lambda S(|\alpha\rangle + |\beta\rangle) + \mu T(|\alpha\rangle + |\beta\rangle) \\ &= \lambda S|\alpha\rangle + \lambda S|\beta\rangle + \mu T|\alpha\rangle + \mu T|\beta\rangle \\ &= \lambda S|\alpha\rangle + \mu T|\alpha\rangle + \lambda S|\beta\rangle + \mu T|\beta\rangle \\ &= (\lambda S + \mu T)|\alpha\rangle + (\lambda S + \mu T)|\beta\rangle, \end{aligned}$$

$$(\lambda S + \mu T)(\nu|\beta\rangle) = \lambda S(\nu|\beta\rangle) + \mu T(\nu|\beta\rangle) = \nu\lambda S|\beta\rangle + \nu\mu T|\beta\rangle = \nu(\lambda S + \mu T)|\beta\rangle.$$

Hence $(\lambda S + \mu T)$ is a linear transformation. □