Homework 5

1.1 Show that $C - span(C) = V \times S$ but that C is not always a basis for $V \times S$.

1. We have coordinates of the vectors as $V = (v_1, v_2), S = (s_1, s_2)$. We can write bases as $b_v = \{b_{v1}, b_{v2}\}, b_s = \{b_{s1}, b_{s2}\}$ respectively.

2. We have coordinates of the vector space $V \times S = \{v_1 + v_2, s_1 + s_2\}$. Now we have the set $C = \{b_v, b_s\}$ and C-span(C) will have the dinstinct combinations which are both linearly dependent and independent. If we consider a set where $b_{v1} = -b_{s1}$ *and* $b_{v2} = -b_{s2}$, then such combination would make the vector space $V \times S = 0$. As we can have linearly dependent combinations, C-Span(C) cannot be basis for vector space $V \times S$ always.

1.2 Prove that $B_{v\times s} = \{(b_v, 0), (0, b_s) \mid b_v \in B_v \; and \; b_s \in B_s\}$ is a basis for $V \times S$. What is the dimension of $V \times S$.

1. A set, b , is the basis of a vector space, V, if b spans V and is linearly independent.

2. If $B_{v\times s}$ is the basis of $V \times S$ then,
$V \times S = \sum \lambda_v(b_v + 0), \sum \lambda_s(0 + b_s)$
$V \times S = \sum \lambda_v b_v, \sum \lambda_s b_s$
$V \times S = \sum \lambda_v b_v, \sum \lambda_s b_s$
$V \times S = (v, s) = C - span(B_{v\times s})$

3. A set of vectors are linearly independent if they are not dependent, that is, one of them cannot be derived by addition or scalar multiplication of others.
$\sum \lambda_v(b_v + 0) = \sum \lambda_s(0 + b_s)$ only if $\lambda_v = \lambda_s = 0$
This states that $B_{v\times s}$ is linearly independent.

4. From above points, we can say $B_{v\times s}$ is the basis for the vector space $V \times S$. Size of the basis is equal to the dimension of the vector space.
$dim(V \times S) = 2$

1.3 Find the matrix representation for $(R \times T)_{B_A \to B_{v\times s}}$

1. We have a linear mapping $R : A \to V$ where matrix R transform the ordered base $B_A \to B_V$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 + 2a_2 \\ 3a_1 + 4a_2 \\ 5a_1 + 6a_2 \end{bmatrix}$$

2. We have a linear mapping $T : A \to S$ where matrix R transform the ordered base $B_A \to B_S$

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_1 + 2a_2 \\ 3a_1 - 2a_2 \end{bmatrix}$$

3. We have a linear mapping $R \times T : A \to (V \times S)$ where matrix $(R \times T)$ transform the ordered base $B_A \to B_{V\times S}$

$B_{V\times S}$ is the lexicographic order of B_V *and* $B_S = \{v_1, v_2, v_3, s_1, s_2\}$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} a_1 + 2a_2 \\ 3a_1 + 4a_2 \\ 5a_1 + 6a_2 \\ -a_1 + 2a_2 \\ 3a_1 - 2a_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ -1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

4. From above point we can derive the matrix $(R \times T)_{B_A \to B_{V\times S}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ -1 & 2 \\ 3 & -2 \end{bmatrix}$

2.1 Let $T : A \to B$ is a linear transformation between vector spaces with ordered bases $B_A = \{|1\rangle, |2\rangle, |3\rangle\}$ and $B_B = \{|1\rangle, |2\rangle\}$

1. Let $i \in B_B$ *and* $j \in B_A$, then a matrix representation of T is obtained by applying T to every vector in the basis of A and expressing the result as a linear combination of basis vectors of B.

$$T(|j\rangle) = \sum_{j=1}^{2} a_{ij}|i\rangle$$

2. Using above notation, we can write the vectors of A in terms of B as below

$T(|1\rangle) = 9|1\rangle - 4|2\rangle$
$T(|2\rangle) = 6|1\rangle - 8|2\rangle$
$T(|3\rangle) = -3|1\rangle + 8|2\rangle$

3. From the below summation a_{ij} are entries of $m \times n$ matrix of of our linear transformation function A.

$$T = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}|i\rangle\langle j| \text{ where } m = 3,\ n = 2$$

4. We write the equations in point 2 as per the summation in point 3, we get below

$= (9|1\rangle - 4|2\rangle)\langle 1| + (6|1\rangle - 8|2\rangle)\langle 2| + (-3|1\rangle + 8|2\rangle)\langle 3|$
$= 9|1\rangle\langle 1| - 4|2\rangle\langle 1| + 6|1\rangle\langle 2| - 8|2\rangle\langle 2| - 3|1\rangle\langle 3| + 8|2\rangle\langle 3|$

5. If we gather the a_{ij} from above equation in the order of $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$, we get our matrix which is equal to the given one.

$$\begin{bmatrix} 9 & 6 & -3 \\ -4 & -8 & 8 \end{bmatrix}$$

2.2 From the equation in point 2.1.3, we can verify that the matrix is the product of T by $|v_j$ is equal to $T(v_j)$. This is because of the property that matrix multiplication is associative.

$(|j\rangle\langle i|)|v_k\rangle = \langle i|v_k|j\rangle$

2.3 By definition of a linear transformation, which was used in 2.1.3, a function $R : A \to B$ which can be represented as a matrix R of the below form is a linear transformation.

$R = \sum_{i\in B_B}^{m} \sum_{j\in B_A}^{n} b_{ij}|i\rangle\langle j|$ where $m = dim(A),\ n = dim(B)$

3.1 $B_{v\otimes s} = \{b_v \otimes b_s\} \mid b_v \in B_v \ and\ b_s \in B_S$

1. Let v_i be i^{th} vector in the basis b_v and similarly s_j be a j^{th} vector in the basis b_s.

2. Using Kronecker product, we can write different linear combinations of $b_v \otimes b_s$ as below

$$\begin{bmatrix} a_1 v_1[\sum b_1 s_j] \\ a_2 v_2[\sum b_2 s_j] \\ \dots \\ a_i v_i[\sum b_j s_j] \end{bmatrix}$$

3. Above matrix can be represented as below summation and spans the Vector space $V \otimes S$

$\sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j v_i s_j$ where m = dim(V) and n = dim(S)

4. Basis b_v and b_s are orthonormal basis and can be treated as linearly independent.
As $B_V \otimes B_S$ spans $V \otimes S$ and linearly independent, we can consider that as its basis.
Dimension of $V \otimes S = dim(B_V) \times dim(B_S)$

3.2 $R \otimes T = (R)_{B_V \to B_A} \otimes (T)_{B_S \to B_B}$

1. We have $(R)_{B_V \to B_A} = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -2 & -1 \end{bmatrix}$ and $(T)_{B_S \to B_B} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

2. Using Kronecker's product, we get below matrix form

$$\begin{bmatrix} -1\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} & 2\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} & -1\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \\ 3\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} & -2\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} & -1\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -4 & 2 & 2 & -1 \\ -1 & 2 & 2 & -4 & -1 & 2 \\ -6 & 3 & 4 & -2 & 2 & -1 \\ 3 & -6 & -2 & 4 & -1 & 2 \end{bmatrix}$$