QUANTUM ALGORITHMS
HOMEWORK 11 ADDITIONAL PROBLEMS

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1. Let $t \in \mathbb{N}$.

(i) Prove that
$$x^t - 1 = (x-1)\sum_{k=0}^{t-1} x^k.$$

(ii) Prove that $x = e^{2\pi i(m/t)}$ is a solution to $x^t - 1$ for $m \in \mathbb{Z}$.

(iii) Let $m \in \mathbb{Z}$ with $0 \leq m < t$. Use the previous parts to prove that
$$\sum_{k=0}^{t-1} e^{2\pi i(km/t)} = \begin{cases} t & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the n-qubit Quantum Fourier Transform ($\mathcal{QFT}_n$) is characterized by its action on basis vector $|x\rangle$,

$$\begin{aligned}
&\mathcal{QFT}_n |x\rangle \\
&= 2^{-n/2}\Big(|0\rangle + e^{2\pi i[0.x_n]} |1\rangle \Big) \otimes \Big(|0\rangle + e^{2\pi i[0.x_{n-1}x_n]} |1\rangle \Big) \otimes \cdots \otimes \Big(|0\rangle + e^{2\pi i[0.x_1x_2\ldots x_n]} |1\rangle \Big) \\
&= 2^{-n/2} \bigotimes_{k=1}^{n} |0\rangle + e^{2\pi i[x]/2^k} |1\rangle,
\end{aligned}$$

where $[0.\cdots]$ represents the number with binary decimal representation $0.\cdots$ and likewise $[x]$ represents the number with binary representation $x \in \{0,1\}^n$.

2. **(i)** Explicitly calculate $\mathcal{QFT}_n |0^n\rangle$.

(ii) Explicitly calculate $\mathcal{QFT}_n |1^n\rangle$.

3. Show that
$$\mathcal{QFT}_n |x\rangle = 2^{-n/2} \sum_{y \in \{0,1\}^n} e^{2\pi i[x][y]/2^n} |y\rangle,$$
where $[x]$ represents the number with binary representation $x \in \{0,1\}^n$ (and so $[x][y]$ is the product of x and y, regarded as binary numbers).

Problems continue on the next page.

4. Use the previous problem to prove that

$$\mathcal{QFT}_n^\dagger \left|x\right\rangle = 2^{-n/2} \sum_{y \in \{0,1\}^n} e^{-2\pi i [x][y]/2^n} \left|y\right\rangle$$

for basis vector $\left|x\right\rangle \in \{0,1\}$ defines the inverse of $\mathcal{QFT}_n$.

Hint 1: Show that $\mathcal{QFT}_n \circ \mathcal{QFT}_n^\dagger \left|x\right\rangle = \mathcal{QFT}_n^\dagger \circ \mathcal{QFT}_n \left|x\right\rangle = \left|x\right\rangle$.

Hint 2: You may find this identity useful

$$\sum_{k=0}^{2^n-1} e^{2\pi i\, k\ell/2^n} = 0 \qquad \text{if} \qquad \ell \neq 0.$$

5. **(i)** Write the matrix for $\mathcal{QFT}_3$ in the standard computational basis (i.e. $\left|x\right\rangle$ for $x \in \{0,1\}^n$). Use the notation $\omega_n = e^{2\pi i/n}$. [*Hint: this is an* 8×8 *matrix.*]

(ii) Write the matrix for $\mathcal{QFT}_3^\dagger$ in the standard computational basis (i.e. $\left|x\right\rangle$ for $x \in \{0,1\}^n$).