QUANTUM ALGORITHMS
HOMEWORK 8 ADDITIONAL PROBLEMS

PROF. MATTHEW MOORE

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We will consider a generalization of Grover's algorithm. Suppose that we are given the following.

- $\mathcal{A}$, a quantum circuit.
- Basis vector $|\texttt{start}\rangle$ and quantum state $|\texttt{end}\rangle$ (i.e. norm 1) with $\mathcal{A}\,|\texttt{start}\rangle = |\texttt{end}\rangle$.
- $|\texttt{end}\rangle = |A\rangle + |B\rangle$ with $\langle A \mid B\rangle = 0$, $\langle A \mid A\rangle = a$, and $\langle B \mid B\rangle = b = 1 - a$.

Let us consider $|A\rangle$ as consisting of a superposition of "correct" outcomes of algorithm $\mathcal{A}$. Upon measuring $|\texttt{end}\rangle$, the probability of observing $|A\rangle$ is a.

We assume that we have a basis $(\psi_i)_{i\in I}$ such that $I = A \cup B$ and a function $\chi : I \to \{0, 1\}$ such that $\chi(A) = 1$ and $\chi(B) = 0$. Define

$$|\Psi(\alpha, \beta)\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$$

and note that $|\Psi(1, 1)\rangle = |\texttt{end}\rangle$. Define $\mathcal{G} = \mathcal{A} \circ D_{\texttt{s}} \circ \mathcal{A}^\dagger \circ D_A$ where D_A is a reflection operator for $|A\rangle$ and $D_{\texttt{s}}$ is a reflection operator for $|\texttt{start}\rangle$ (use the phase shift $e^{i\theta}$ for both).

1. Draw the circuits for both reflection operators and also write out the operators for them (e.g. $D_0 = (e^{i\theta} - 1)\,|0\rangle\,\langle 0| + I$ as in the usual Grover's algorithm).

2. Calculate $\mathcal{G}\,|\Psi(1, 1)\rangle$ and write your answer in the form $|\Psi(x, y)\rangle$ for some x, y (you should specify their values).

3. Suppose that $\mathcal{A}$ works with probability 1/4 (i.e. $a = 1/4$). Show that taking $\theta = \pi$ (as in the usual Grover's circuit) makes $\mathcal{G} \circ \mathcal{A}$ acting on $|\texttt{start}\rangle$ exact (i.e. it produces a correct answer with probability 1).

4. Show that when $\mathcal{A}$ works with probability 1/2 there is a choice of θ so that $\mathcal{G} \circ \mathcal{A}$ acting on $|\texttt{start}\rangle$ is exact. What is the value of $e^{i\theta}$ in this case?