QUANTUM ALGORITHMS
HOMEWORK 9 ADDITIONAL PROBLEMS

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Definition. Let $\mathbb{V}$ be a vector space and let $\mathbb{A}, \mathbb{B} \leq \mathbb{V}$ be subspaces.

- We say that $\mathbb{A}$ is *orthogonal* to $\mathbb{B}$ if for every $|a\rangle \in A$ and $|b\rangle \in B$ we have $\langle a \mid b\rangle = 0$.
- Define the *sum* of $\mathbb{A}$ and $\mathbb{B}$ to be $\mathbb{A} + \mathbb{B} = \Big\{ |a\rangle + |b\rangle \mid |a\rangle \in A, |b\rangle \in B \Big\}$.

1. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are orthogonal to each other.

(i) What is $\dim(\mathbb{A} + \mathbb{B})$?

(ii) Show that $\mathcal{P}(|v\rangle, \mathbb{A} + \mathbb{B}) = \mathcal{P}(|v\rangle, \mathbb{A}) + \mathcal{P}(|v\rangle, \mathbb{B})$.

(iii) Show that $\Pi_{\mathbb{A}}\Pi_{\mathbb{B}} = \Pi_{\mathbb{B}}\Pi_{\mathbb{A}}$.

2. Suppose that $\mathbb{A} \leq \mathbb{V}$ and $\mathbb{B} \leq \mathbb{W}$ are two subspaces.

(i) Prove that $\Pi_{\mathbb{A}\otimes\mathbb{B}} = \Pi_{\mathbb{A}} \otimes \Pi_{\mathbb{B}}$.

(ii) Let ρ and τ be density matrices. Prove that $\mathcal{P}(\rho \otimes \tau, \mathbb{A} \otimes \mathbb{B}) = \mathcal{P}(\rho, \mathbb{A})\mathcal{P}(\tau, \mathbb{B})$. You may use the fact that $\operatorname{Tr}(X \otimes Y) = \operatorname{Tr}(X)\operatorname{Tr}(Y)$.