QUANTUM ALGORITHMS
HOMEWORK 6 SELECTED SOLUTIONS

PROF. MATTHEW MOORE

6.2. Consider two linear maps, $A : \mathcal{L} \to \mathcal{L}'$ and $B : \mathcal{M} \to \mathcal{M}'$. Prove that there is a unique linear map $C = A \otimes B : \mathcal{L} \otimes \mathcal{M} \to \mathcal{L}' \otimes \mathcal{M}'$ such that $C(u \otimes v) = A(u) \otimes B(v)$ for any $u \in \mathcal{L}$, $v \in \mathcal{M}$.

Solution:

Proof. Elements of $\mathcal{L} \otimes \mathcal{M}$ are linear combinations of simple tensors, $u \otimes v$. Define a *function* C on the simple tensors of $\mathcal{L} \otimes \mathcal{M}$ by

$$C(u \otimes v) \coloneqq A(u) \otimes B(v).$$

Extend C to a function on all of $\mathcal{L} \otimes \mathcal{M}$ by linearity.

One thing we must check is that when we "defined" C, we didn't accidentally write an inconsistent definition that depends on the exact representation of $u \otimes v$. An example of this would be if we defined a function f on the fractions to be $f(a/b) = a \cdot b$. This function is not well-defined since $f(1/2) = 2$ and $f(2/4) = 8$, but $1/2 = 2/4$ and $2 \neq 8$. This property is call being "well-defined". Logically, we want to prove that if $u \otimes v = u' \otimes v'$ then $C(u \otimes v) = C(u' \otimes v')$.

Claim. *C is well-defined.*

Proof of claim. Suppose that $u, u' \in \mathcal{L}$, $v, v' \in \mathcal{M}$, and $u \otimes v = u' \otimes v'$. For simple tensors, the only way this is possible is if there is a scalar $\lambda \in \mathbb{C}$ such that $u = \lambda u'$ and $\lambda v = v'$. We have

$$\begin{aligned} C(u \otimes v) = A(u) \otimes B(v) &= A(\lambda u') \otimes B((1/\lambda)v') = \big(\lambda A(u')\big) \otimes \big((1/\lambda)B(v')\big) \\ &= \lambda(1/\lambda)\big(A(u') \otimes B(v')\big) = A(u') \otimes B(v') = C(u' \otimes v'). \end{aligned}$$

Therefore C is well-defined. ◦

Next, we claim that C is a linear transformation. This follows immediately from the definition since we extended C by linearity. Finally, we must prove that C is unique. Since there is only one way to extend by linearity, C is entirely determined by its action on the simple tensors. It follows that there is just a single linear transformation that acts like C on simple tensors, so C is unique. □

6.5. a) Let $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Write the matrix of the operator $H[2]$ acting on the space $\mathfrak{B}^{\otimes 3}$.

b) Let U be an arbitrary two-qubit operator with matrix elements $u_{jk} = \langle j| U |k\rangle$, where $j, k \in \{00, 01, 10, 11\}$. Write the matrix for $U[3, 1]$.

Date: March 4, 2020.

Solution:

a): H is a one-qubit operator, so

$$H[2] \coloneqq I_{\mathfrak{B}^{\otimes 1}} \otimes H \otimes I_{\mathfrak{B}^{\otimes 1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} & 0 \cdot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} & 1 \cdot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

b): We partition the matrix described in the problem into 2×2 matrices:

$$U = \begin{pmatrix} u_{00|00} & u_{00|01} & u_{00|10} & u_{00|11} \\ u_{01|00} & u_{01|01} & u_{01|10} & u_{01|11} \\ u_{10|00} & u_{10|01} & u_{10|10} & u_{10|11} \\ u_{11|00} & u_{11|01} & u_{11|10} & u_{11|11} \end{pmatrix} = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}.$$

It follows that we have a representation of U as the sum of tensor produces of 2×2 matrices,

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes U_{00} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes U_{01} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes U_{10} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_{11},$$

so we have

$$U[3,1] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [3]\, U_{00}[1] + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} [3]\, U_{01}[1] + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [3]\, U_{10}[1] + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} [3]\, U_{11}[1].$$

As in the previous part, we have

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [3]\, U_{00}[1] &= \left(I_{\mathfrak{B}^{\otimes 2}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \left(U_{00} \otimes I_{\mathfrak{B}^{\otimes 2}} \right) \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&\qquad \cdot \begin{pmatrix}
u_{00|00} & 0 & 0 & 0 & u_{00|01} & 0 & 0 & 0 \\
0 & u_{00|00} & 0 & 0 & 0 & u_{00|01} & 0 & 0 \\
0 & 0 & u_{00|00} & 0 & 0 & 0 & u_{00|01} & 0 \\
0 & 0 & 0 & u_{00|00} & 0 & 0 & 0 & u_{00|01} \\
u_{01|00} & 0 & 0 & 0 & u_{01|01} & 0 & 0 & 0 \\
0 & u_{01|00} & 0 & 0 & 0 & u_{01|01} & 0 & 0 \\
0 & 0 & u_{01|00} & 0 & 0 & 0 & u_{01|01} & 0 \\
0 & 0 & 0 & u_{01|00} & 0 & 0 & 0 & u_{01|01}
\end{pmatrix} \\
&= \begin{pmatrix}
u_{00|00} & 0 & 0 & 0 & u_{00|01} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_{00|00} & 0 & 0 & 0 & u_{00|01} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
u_{01|00} & 0 & 0 & 0 & u_{01|01} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_{01|00} & 0 & 0 & 0 & u_{01|01} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\end{aligned}$$

Similarly,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} [3]\ U_{01}[1] = \begin{pmatrix} 0 & u_{00|10} & 0 & 0 & 0 & u_{00|11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{00|10} & 0 & 0 & 0 & u_{00|11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_{01|10} & 0 & 0 & 0 & u_{01|11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{01|10} & 0 & 0 & 0 & u_{01|11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} [3]\ U_{10}[1] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_{10|00} & 0 & 0 & 0 & u_{10|01} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{10|00} & 0 & 0 & 0 & u_{10|01} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_{11|00} & 0 & 0 & 0 & u_{11|01} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{11|00} & 0 & 0 & 0 & u_{11|01} & 0 \end{pmatrix}, \qquad \text{and}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} [3]\ U_{11}[1] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_{10|10} & 0 & 0 & 0 & u_{10|11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{10|10} & 0 & 0 & 0 & u_{10|11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_{11|10} & 0 & 0 & 0 & u_{11|11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{11|10} & 0 & 0 & 0 & u_{11|11} \end{pmatrix}.$$

Adding these four matrices together gives us

$$U[3,1] = \begin{pmatrix} u_{00|00} & u_{00|10} & 0 & 0 & u_{00|01} & u_{00|11} & 0 & 0 \\ u_{10|00} & u_{10|10} & 0 & 0 & u_{10|01} & u_{10|11} & 0 & 0 \\ 0 & 0 & u_{00|00} & u_{00|10} & 0 & 0 & u_{00|01} & u_{00|11} \\ 0 & 0 & u_{10|00} & u_{10|10} & 0 & 0 & u_{10|01} & u_{10|11} \\ u_{01|00} & u_{01|10} & 0 & 0 & u_{01|01} & u_{01|11} & 0 & 0 \\ u_{11|00} & u_{11|10} & 0 & 0 & u_{11|01} & u_{11|11} & 0 & 0 \\ 0 & 0 & u_{01|00} & u_{01|10} & 0 & 0 & u_{01|01} & u_{01|11} \\ 0 & 0 & u_{11|00} & u_{11|10} & 0 & 0 & u_{11|01} & u_{11|11} \end{pmatrix}$$

AP 1. Let $\mathbb{V}$ and $\mathbb{S}$ be vector spaces over $\mathbb{C}$ with bases $B_{\mathbb{V}}$ and $B_{\mathbb{S}}$, respectively.

(i) Prove that

$$B_{\mathbb{V}\otimes\mathbb{S}} = \{b_v \otimes b_s \mid b_v \in B_{\mathbb{V}} \text{ and } b_s \in B_{\mathbb{S}}\}$$

is a basis of $\mathbb{V}\otimes\mathbb{S}$. What is the dimension of $\mathbb{V}\otimes\mathbb{S}$?

(ii) Let $R : \mathbb{V} \to \mathbb{A}$ and $T : \mathbb{S} \to \mathbb{B}$ be linear functions. Suppose that $\mathbb{A}$, $\mathbb{V}$, and $\mathbb{S}$ have ordered bases the same as in the previous question and that $\mathbb{B}$ has ordered basis $B_{\mathbb{B}} = \{b_1, b_2\}$

and that the matrix representations of R and T relative to these bases are

$$(R)_{B_{\mathbb{V}}\to B_{\mathbb{A}}} = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -2 & -1 \end{bmatrix} \qquad \text{and} \qquad (T)_{B_{\mathbb{S}}\to B_{\mathbb{B}}} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Using the lexicographic order for the basis $B_{\mathbb{V}\otimes\mathbb{S}}$, find the matrix representation for $(R\otimes T)$ (that is, find $(R\times T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}}$). [*Hint: Kronecker product.*]

Solution:

(i): *Proof.* We being by showing that $\mathbb{V}\otimes\mathbb{S} = \mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}})$. Let $\alpha\in V$ and $\beta\in S$. It follows that α and β can be decomposed into linear combinations of their respective bases,

$$\alpha = \sum_{b_v\in B_{\mathbb{V}}} \lambda_v b_v, \qquad \beta = \sum_{b_s\in B_{\mathbb{S}}} \mu_s b_s, \qquad \text{for } \lambda_v,\mu_s\in\mathbb{C}.$$

Using the bilinearity of the tensor, we have

$$\begin{aligned}\alpha\otimes\beta &= \left(\sum_{b_v\in B_{\mathbb{V}}} \lambda_v b_v\right)\otimes\left(\sum_{b_s\in B_{\mathbb{S}}} \mu_s b_s\right) = \sum_{b_v\in B_{\mathbb{V}}} \lambda_v\left(b_v\otimes\sum_{b_s\in B_{\mathbb{S}}} \mu_s b_s\right)\\ &= \sum_{b_v\in B_{\mathbb{V}}} \lambda_v \sum_{b_s\in B_{\mathbb{S}}} \mu_s(b_v\otimes b_s) = \sum_{\substack{b_v\in B_{\mathbb{V}}\\ b_s\in B_{\mathbb{S}}}} \lambda_v\mu_s(b_v\otimes b_s).\end{aligned}$$

It follows that

$$\big\{\alpha\otimes\beta \mid \alpha\in\mathbb{V},\beta\in\mathbb{S}\big\}\subseteq \mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}}).$$

Applying $\mathbb{C}$-span to both sides yields

$$\mathbb{V}\otimes\mathbb{S} = \mathbb{C}\text{-span}\big\{\alpha\otimes\beta \mid \alpha\in\mathbb{V},\beta\in\mathbb{S}\big\} = \mathbb{C}\text{-span}\big(\mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}})\big) = \mathbb{C}\text{-span}(B_{\mathbb{V}\otimes\mathbb{S}}).$$

Therefore $B_{\mathbb{V}\otimes\mathbb{S}}$ spans $\mathbb{V}\otimes\mathbb{S}$.

Next we show that $B_{\mathbb{V}\otimes\mathbb{S}}$ is linearly independent. Simple tensors $\alpha\otimes\beta$ and $\alpha'\otimes\beta'$ are equal if and only if there is $\lambda\in\mathbb{C}$ such that $\alpha=\lambda\alpha$ and $\lambda\beta=\beta'$. From this, we obtain the equalities

$$0 = \alpha\otimes 0 = 0\otimes\beta \qquad \text{for all } \alpha\in\mathbb{V},\beta\in\mathbb{S}$$

(the left-most 0 is the zero vector in $\mathbb{V}\otimes\mathbb{S}$). Suppose that we have a linear combination of basis vectors that is equal to 0. Bilinearity gives us

$$0 = \sum_{\substack{b_v\in B_{\mathbb{V}}\\ b_s\in B_{\mathbb{S}}}} \lambda_{vs}(b_v\otimes b_s) = \sum_{b_v\in B_{\mathbb{V}}}\sum_{b_s\in B_{\mathbb{S}}} \lambda_{vs}(b_v\otimes b_s) = \sum_{b_v\in B_{\mathbb{V}}}\left(b_v\otimes\sum_{b_s\in B_{\mathbb{S}}} \lambda_{vs} b_s\right),$$

so for *fixed* $b_v\in B_{\mathbb{V}}$ we have

$$0 = \sum_{b_s\in B_{\mathbb{S}}} \lambda_{vs} b_s.$$

Since $B_{\mathbb{S}}$ is linearly independent, this implies that for fixed b_v we have $\lambda_{vs}=0$ for all $b_s\in B_{\mathbb{S}}$. Doing this for all $b_v\in B_{\mathbb{V}}$ yields $\lambda_{vs}=0$ for all $b_v\in B_{\mathbb{V}}$ and $b_s\in B_{\mathbb{S}}$, as desired. □

(ii): Ordered as described, the bases we are looking at are

$$\begin{aligned}B_{\mathbb{V}\otimes\mathbb{S}} &= \Big\{v_1\otimes s_1, v_1\otimes s_2, v_2\otimes s_1, v_2\otimes s_2, v_3\otimes s_1, v_3\otimes s_2\Big\} \qquad \text{and}\\ B_{\mathbb{A}\otimes\mathbb{B}} &= \Big\{a_1\otimes b_1, a_1\otimes b_2, a_2\otimes b_1, a_2\otimes b_2\Big\}.\end{aligned}$$

From the definition of $R\otimes T$, we have that $(R\otimes T)(v\otimes s) = (Rv)\otimes(Ts)$. It follows from this and bilinearity that

$$\begin{aligned}(R\otimes T)(v_1\otimes s_1) &= (Rv_1)\otimes(Ts_1) = \big(-a_1+3a_2\big)\otimes\big(-2b_1+b_2\big)\\ &= 2(a_1\otimes b_1) - (a_1\otimes b_2) - 6(a_2\otimes b_1) + 3(a_2\otimes b_2),\end{aligned}$$

$$\big(\text{ so column 1 of } (R\times T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}} \text{ is } (2,-1,-6,3)^T \ \big)$$

$$\begin{aligned}(R\otimes T)(v_1\otimes s_2) &= (Rv_1)\otimes(Ts_2) = \big(-a_1+3a_2\big)\otimes\big(b_1-2b_2\big)\\ &= -(a_1\otimes b_1)+2(a_1\otimes b_2)+3(a_2\otimes b_1)-6(a_2\otimes b_2),\end{aligned}$$

$$\big(\text{ so column 2 of } (R\times T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}} \text{ is } (-1,2,3,-6)^T \ \big)$$

$$\begin{aligned}(R\otimes T)(v_2\otimes s_1) &= (Rv_2)\otimes(Ts_1) = \big(2a_1-2a_2\big)\otimes\big(-2b_1+b_2\big)\\ &= -4(a_1\otimes b_1)+2(a_1\otimes b_2)+4(a_2\otimes b_1)-2(a_2\otimes b_2),\end{aligned}$$

$$\big(\text{ so column 3 of } (R\times T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}} \text{ is } (-4,2,4,-2)^T \ \big)$$

$$\begin{aligned}(R\otimes T)(v_2\otimes s_2) &= (Rv_2)\otimes(Ts_2) = \big(2a_1-2a_2\big)\otimes\big(b_1-2b_2\big)\\ &= 2(a_1\otimes b_1)-4(a_1\otimes b_2)-2(a_2\otimes b_1)+4(a_2\otimes b_2),\end{aligned}$$

$$\big(\text{ so column 4 of } (R\times T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}} \text{ is } (2,-4,-2,4)^T \ \big)$$

$$\begin{aligned}(R\otimes T)(v_3\otimes s_1) &= (Rv_3)\otimes(Ts_1) = \big(-a_1-a_2\big)\otimes\big(-2b_1+b_2\big)\\ &= 2(a_1\otimes b_1)-(a_1\otimes b_2)+2(a_2\otimes b_1)-(a_2\otimes b_2),\end{aligned}$$

$$\big(\text{ so column 5 of } (R\times T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}} \text{ is } (2,-1,2,-1)^T \ \big)$$

$$\begin{aligned}(R\otimes T)(v_3\otimes s_3) &= (Rv_3)\otimes(Ts_3) = \big(-a_1-a_2\big)\otimes\big(b_1-2b_2\big)\\ &= -(a_1\otimes b_1)+2(a_1\otimes b_2)-(a_2\otimes b_1)+2(a_2\otimes b_2).\end{aligned}$$

$$\big(\text{ so column 6 of } (R\times T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}} \text{ is } (-1,2,-1,2)^T \ \big).$$

From all this we have

$$(R\otimes T)_{B_{\mathbb{V}\otimes\mathbb{S}}\to B_{\mathbb{A}\otimes\mathbb{B}}} = \begin{pmatrix} 2 & -1 & -4 & 2 & 2 & -1\\ -1 & 2 & 2 & -4 & -1 & 2\\ -6 & 3 & 4 & -2 & 2 & -1\\ 3 & -6 & -2 & 4 & -1 & 2\end{pmatrix},$$

which is the Kronecker product of $(R)_{B_{\mathbb{V}}\to B_{\mathbb{A}}}$ and $(T)_{B_{\mathbb{S}}\to B_{\mathbb{B}}}$!

AP 2. Prove that the inner product and the tensor product commute:

$$\langle \alpha\otimes\beta \mid \gamma\otimes\delta\rangle = \langle\alpha\mid\gamma\rangle\langle\beta\mid\delta\rangle\,.$$

This is asserted on page 57 of the textbook.

Solution:

Proof. In order for the inner product to be defined, α and γ must be elements of the same vector space, say $\mathbb{A}$. Likewise β and δ must be elements of the same vector space, say $\mathbb{B}$. Let $\mathbb{A}$ and $\mathbb{B}$ have ordered basis

$$\{\tau_1,\ldots,\tau_n\} \qquad\qquad \{\sigma_1,\ldots,\sigma_m\}$$

respectively. We may furthermore assume that these bases are *orthonormal* (we either assume this, or define the inner product in terms of them so that they are). It follows that each of the vectors α, β, γ, δ have decompositions in terms of their respective bases, say

$$\alpha = \sum_{i=1}^n a_i\tau_i, \qquad \gamma = \sum_{i=1}^n c_i\tau_i, \qquad \beta = \sum_{i=1}^m b_i\sigma_i, \qquad \delta = \sum_{i=1}^m d_i\sigma_i.$$

for $a_i, c_i, b_i, d_i \in \mathbb{C}$. It follows that

$$\langle \alpha \mid \gamma \rangle = \sum_{i=1}^{n} a_i^* c_i \qquad \langle \beta \mid \delta \rangle = \sum_{i=1}^{m} b_i^* d_i$$

and hence

$$\langle \alpha \mid \gamma \rangle \langle \beta \mid \delta \rangle = \left(\sum_{i=1}^{n} a_i^* c_i \right) \left(\sum_{i=1}^{m} b_i^* d_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i^* c_i b_j^* d_j.$$

We now examine the tensors. From above and from bilinearity, we have

$$\alpha \otimes \beta = \left(\sum_{i=1}^{n} a_i \tau_i \right) \otimes \left(\sum_{i=1}^{m} b_i \sigma_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_i \tau_i \otimes \sigma_j \qquad \text{and}$$

$$\gamma \otimes \delta = \left(\sum_{i=1}^{n} c_i \tau_i \right) \otimes \left(\sum_{i=1}^{m} d_i \sigma_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_i \tau_i \otimes \sigma_j.$$

Using the bilinearity of the inner product, this yields

$$\begin{aligned} \langle \alpha \otimes \beta \mid \gamma \otimes \delta \rangle &= \left\langle \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_i \tau_i \otimes \sigma_j \mid \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_i \tau_i \otimes \sigma_j \right\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{\ell=1}^{m} (a_i b_i)^* c_i d_i \langle \tau_i \otimes \sigma_j \mid \tau_k \otimes \sigma_\ell \rangle . \end{aligned}$$

Using the orthonormality of the bases, we have

$$\langle \tau_i \otimes \sigma_j \mid \tau_k \otimes \sigma_\ell \rangle = \begin{cases} 1 & \text{if } i = k \text{ and } j = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the four-sum above reduces

$$\begin{aligned} \langle \alpha \otimes \beta \mid \gamma \otimes \delta \rangle &= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{\ell=1}^{m} (a_i b_j)^* c_k d_\ell \langle \tau_i \otimes \sigma_j \mid \tau_k \otimes \sigma_\ell \rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i^* b_j^* c_i d_j. \end{aligned}$$

This is equal to $\langle \alpha \mid \gamma \rangle \langle \beta \mid \delta \rangle$ as calculated in the previous paragraph, as claimed. $\square$