QUANTUM ALGORITHMS
HOMEWORK 7 SELECTED SOLUTIONS

PROF. MATTHEW MOORE

8.7. Prove the properties (8.10)–(8.13) of the operator norm.

Solution: The properties in question are

$$\|XY\| \leq \|X\|\|Y\|, \tag{8.10}$$

$$\|X^\dagger\| = \|X\|, \tag{8.11}$$

$$\|X \otimes Y\| = \|X\|\|Y\|, \tag{8.12}$$

$$\|U\| = 1 \qquad \text{if } U \text{ is unitary.} \tag{8.13}$$

We will prove each of these in turn.

Proof. We have

$$\begin{aligned}
\|XY\| &= \sup_{|\xi\rangle \neq 0} \frac{\|XY\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \left(\frac{\|XY\,|\xi\rangle\,\|}{\|Y\,|\xi\rangle\,\|}\right)\left(\frac{\|Y\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right) \\
&\leq \left(\sup_{|\xi\rangle \neq 0} \frac{\|XY\,|\xi\rangle\,\|}{\|Y\,|\xi\rangle\,\|}\right)\left(\sup_{|\xi\rangle \neq 0} \frac{\|Y\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right) \leq \left(\sup_{|\zeta\rangle \neq 0} \frac{\|X\,|\zeta\rangle\,\|}{\|\,|\zeta\rangle\,\|}\right)\left(\sup_{|\xi\rangle \neq 0} \frac{\|Y\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right) \\
&= \|X\|\|Y\|,
\end{aligned}$$

proving (8.10).

Next, we begin by noting that $\|\,\langle\xi|\,\| = \|\,|\xi\rangle\,\|$. Therefore

$$\|X^\dagger\| = \sup_{|\xi\rangle \neq 0} \frac{\|X^\dagger\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \frac{\|\,\left|X^\dagger\xi\right\rangle\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \frac{\|\,\langle\xi X|\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \frac{\|\,\langle\xi|\,X\|}{\|\,|\xi\rangle\,\|} = \|X\|,$$

proving (8.11).

Next, we have

$$\|\,|\alpha\rangle \otimes |\beta\rangle\,\| = \sqrt{\langle\alpha\otimes\beta \mid \alpha\otimes\beta\rangle} = \sqrt{\langle\alpha \mid \alpha\rangle\,\langle\beta \mid \beta\rangle} = \|\alpha\|\|\beta\|.$$

It follows that

$$\begin{aligned}
\|X \otimes Y\| &= \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \frac{\|(X\otimes Y)(|\xi\rangle \otimes |\zeta\rangle)\|}{\|\,|\xi\rangle \otimes |\zeta\rangle\,\|} = \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \frac{\|\,|X\xi\rangle \otimes |Y\zeta\rangle\,\|}{\|\,|\xi\rangle \otimes |\zeta\rangle\,\|} \\
&= \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \frac{\|\,|X\xi\rangle\,\|\|\,|Y\zeta\rangle\,\|}{\|\,|\xi\rangle\,\|\|\,|\zeta\rangle\,\|} = \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \left(\frac{\|\,|X\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right)\left(\frac{\|\,|Y\zeta\rangle\,\|}{\|\,|\zeta\rangle\,\|}\right) \\
&= \left(\sup_{|\xi\rangle \neq 0} \frac{\|\,|X\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right)\left(\sup_{|\zeta\rangle \neq 0} \frac{\|\,|Y\zeta\rangle\,\|}{\|\,|\zeta\rangle\,\|}\right) = \|X\|\|Y\|,
\end{aligned}$$

proving (8.12).

Date: April 1, 2020.

Finally, if U is unitary then

$$\begin{aligned}\|U\| &= \sup_{|\xi\rangle\neq 0} \frac{\|U|\xi\rangle\|}{\||\xi\rangle\|} = \sup_{|\xi\rangle\neq 0} \frac{\sqrt{\langle U\xi \mid U\xi\rangle}}{\||\xi\rangle\|} = \sup_{|\xi\rangle\neq 0} \frac{\sqrt{\langle \xi \mid \xi\rangle}}{\||\xi\rangle\|} = \sup_{|\xi\rangle\neq 0} \frac{\||\xi\rangle\|}{\||\xi\rangle\|} \\ &= \sup_{|\xi\rangle\neq 0} 1 = 1,\end{aligned}$$

proving (8.13). □

8.8. Prove the two basic properties of approximation with ancillas:

a) If $\tilde{U}$ approximates U with precision δ, then $\tilde{U}^{-1}$ approximates U^{-1} with the same precision δ.

b) If unitary operators $\tilde{U}_k$ approximate unitary operators U_k $(1 \leq k \leq L)$ with precision δ_k, then $\tilde{U}_L \cdots \tilde{U}_1$ approximates $U_L \cdots U_1$ with precision $\sum_k \delta_k$.

Solution: Before we begin, we build some tools. Define $V : \mathbb{B}^{\otimes n} \to \mathbb{B}^{\otimes N}$ by $V|\xi\rangle = |\xi\rangle \otimes |0^{N-n}\rangle$. The statement "$\tilde{U}$ approximates U with ancillas with precision δ" is equivalent to

$$\|\tilde{U}V - VU\| \leq \delta.$$

We are now ready to do the proofs.

a): *Proof.* Note that $\tilde{U}$ is always taken to be *unitary*. Using the fact that $\|T\| = 1$ and $\|T\|\|X\| = \|TX\|$ for unitary T, we have

$$\begin{aligned}\|\tilde{U}^{-1}V - VU^{-1}\| &= \|\tilde{U}\|\|\tilde{U}^{-1}V - VU^{-1}\|\|U\| = \|\tilde{U}(\tilde{U}^{-1}V - VU^{-1})U\| \\ &= \|\tilde{U}\tilde{U}^{-1}VU - \tilde{U}VU^{-1}U\| = \|VU - \tilde{U}V\| = \|\tilde{U}V - VU\| \leq \delta.\end{aligned}$$

Hence $\tilde{U}^{-1}$ approximates U^{-1} with precision δ. □

b): *Proof.* We will proceed by induction on L. The base case of $L = 1$ is included in the assumptions, so it certainly holds. Assume now that the claim holds for a product of $L - 1$ matrices. Let $\tilde{W} = \tilde{U}_2 \cdots \tilde{U}_L$ and $W = U_2 \cdots U_L$. These are products of $L - 1$ matrices, so by the inductive hypothesis we have that $\tilde{W}$ approximates W with ancillas with precision $\sum_{k=2}^{L} \delta_k$. We now have

$$\begin{aligned}\|\tilde{U}_1 \cdots \tilde{U}_L V - VU_1 \cdots U_L\| &= \|\tilde{U}_1\tilde{W}V - VU_1W\| \\ &= \|\tilde{U}_1(\tilde{W}V - VW) + (\tilde{U}_1V - VU_1)W\| \\ &\leq \|\tilde{U}_1(\tilde{W}V - VW)\| + \|(\tilde{U}_1V - VU_1)W\| \\ &= \|\tilde{U}_1\|\|\tilde{W}V - VW\| + \|\tilde{U}_1V - VU_1\|\|W\| \\ &= \|\tilde{W}V - VW\| + \|\tilde{U}_1V - VU_1\| \\ &\leq \sum_{k=2}^{L} \delta_k + \delta_1 = \sum_{k=1}^{L} \delta_k\end{aligned}$$

(we use the triangle inequality on lines 2 – 3). This completes the induction and proves the claim. □