# Homework 9

**1.1.** What is the dimension of $dim(\mathbb{A} + \mathbb{B})$ ?

**1.** We have the following definition

$\mathbb{A}$ and $\mathbb{B}$ are subspaces and $\mathbb{A} + \mathbb{B} = \{|a\rangle + |b\rangle, |a\rangle \in \mathbb{A}, |b\rangle \in \mathbb{B}\}$

We know that, the dimension of a vector space is the number of vectors in its basis.
Let A and B are the basis for $\mathbb{A}$ and $\mathbb{B}$
We know that basis vectors A and B are orthonormal basis, so they are linearly independent.
We know that basis vector formed by $A + B$ span the subspaces $\mathbb{A} + \mathbb{B}$.

**2.** From above points we can say,
$A + B$ is basis for the vector spaces formed by $\mathbb{A} + \mathbb{B}$
$\Rightarrow dim(\mathbb{A} + \mathbb{B}) = dim(\mathbb{A}) + dim(\mathbb{B})$

**1.2.** Show that $\mathcal{P}(|v\rangle, \mathbb{A} + \mathbb{B}) = \mathcal{P}(|v\rangle, \mathbb{A}) + \mathcal{P}(|v\rangle, \mathbb{B})$

**1.** If $|v\rangle$ is the state of the input vector, then quantum probability of finding the system in state x is, $\mathcal{P}(|v\rangle, x) = \langle v \mid x\rangle\langle x \mid v\rangle = \langle v \mid \Pi_x \mid v\rangle$, where $\Pi_x$ is the orthogonal projection operator on to the subspace spanned by $|x\rangle$
$\Pi_x = \sum_j |x_j\rangle\langle x_j|$

**2.** By above definition, the quantum probability of finding a system in state $\mathbb{A} + \mathbb{B}$ is
$\mathcal{P}(|v\rangle, \mathbb{A} + \mathbb{B}) = \langle v \mid \Pi_{\mathbb{A}+\mathbb{B}} \mid v\rangle$

**3.** From above two points, we know that $\Pi_{\mathbb{A}+\mathbb{B}} = \sum_j |e_j\rangle\langle e_j|$ where $e_j \in A + B$
We know that basis $|e_j\rangle$ is formed by $|a\rangle + |b\rangle$, where $|a\rangle \in \mathbb{A}, |b\rangle \in \mathbb{B}$
$\Rightarrow \Pi_{\mathbb{A}+\mathbb{B}} = \sum_j |e_j\rangle\langle e_j| = \sum_n |a\rangle\langle a| + \sum_m |b\rangle\langle b|$
$\Rightarrow \Pi_{\mathbb{A}+\mathbb{B}} = \Pi_{\mathbb{A}} + \Pi_{\mathbb{B}}$

**4.** By point 3, we can rewrite the definition in point 2 as,
$\mathcal{P}(|v\rangle, \mathbb{A} + \mathbb{B}) = \langle v \mid \Pi_{\mathbb{A}+\mathbb{B}} \mid v\rangle = \langle v \mid \Pi_{\mathbb{A}} + \Pi_{\mathbb{B}} \mid v\rangle$

**5.** Applying the projection operator $\Pi$ on a state vector is a linear transformation and as it is a vector space homomorphishm, we can say,

$\langle v \mid \Pi_{\mathbb{A}} + \Pi_{\mathbb{B}} \mid v\rangle = \langle v \mid \Pi_{\mathbb{A}} \mid v\rangle + \langle v \mid \Pi_{\mathbb{B}} \mid v\rangle$

$\Rightarrow \mathcal{P}(|v\rangle, \mathbb{A} + \mathbb{B}) = \mathcal{P}(|v\rangle, \mathbb{A}) + \mathcal{P}(|v\rangle, \mathbb{B})$

**1.3.** Show that $\Pi_{\mathbb{A}}\Pi_{\mathbb{B}} = \Pi_{\mathbb{B}}\Pi_{\mathbb{A}}$

**1.** We know that,

$\Pi_{\mathbb{A}} = \sum_i |a_i\rangle\langle a_i|$
$\Pi_{\mathbb{B}} = \sum_j |b_j\rangle\langle b_j|$

**2.** $\Pi_{\mathbb{A}}\Pi_{\mathbb{B}}$ will be of the form,

$(|a_1\rangle\langle a_1| + \cdots + |a_i\rangle\langle a_i|).(|b_1\rangle\langle b_1| + \cdots + |b_j\rangle\langle b_j|)$

$\Rightarrow \sum_{i,j} |a_i\rangle\langle a_i||b_j\rangle\langle b_j|)$

$\Rightarrow \sum_{i,j} |a_i\rangle\langle a_i \mid b_j\rangle|b_j\rangle$

But we know that $\mathbb{A}, \mathbb{B}$ are orthogonal $\Rightarrow \langle a_i \mid b_j\rangle = 0$

$\Pi_{\mathbb{A}}\Pi_{\mathbb{B}} = \sum_{i,j} |a_i\rangle\langle a_i \mid b_j\rangle|b_j\rangle = 0$

**3.** Similarly, $\Pi_{\mathbb{B}}\Pi_{\mathbb{A}} = \sum_{i,j} |b_j\rangle\langle b_j \mid a_i\rangle|a_i\rangle = 0$

**4.** From points 2,3 we can say,

$\Pi_{\mathbb{A}}\Pi_{\mathbb{B}} = \Pi_{\mathbb{B}}\Pi_{\mathbb{A}} = 0$

**2.1.** Show that $\Pi_{\mathbb{A}\otimes\mathbb{B}} = \Pi_{\mathbb{A}} \otimes \Pi_{\mathbb{B}}$

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Show that $\Pi_{A\otimes B} = \Pi_A \otimes \Pi_B$

$A = \{a_1, a_2 \dots a_i\}$ $\quad B = \{b_1, b_2 \dots b_j\}$

$|A\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \end{bmatrix}$ $\quad \langle A| = [a_1\ a_2 \cdots a_i]$

$|B\rangle = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \end{bmatrix}$ $\quad \langle B| = [b_1\ b_2 \dots b_j]$, say $\langle B| = B^\dagger$

we know that,

$\Pi_{A\otimes B} = \sum |A\otimes B\rangle\langle A\otimes B|$

$= \sum |AB\rangle\langle AB|$

$= \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_i B \end{bmatrix} [a_1B^\dagger\ a_2B^\dagger \cdots a_iB^\dagger]$

$= \begin{bmatrix} a_1^2 BB^\dagger & a_1a_2BB^\dagger & \cdots \\ \vdots & a_2^2 BB^\dagger & \cdots \\ \vdots & \vdots & a_i^2 BB^\dagger \end{bmatrix}$

$= \begin{bmatrix} a_1^2 & a_1a_2 & \cdots \\ \vdots & a_2^2 & \cdots \\ & \vdots & a_i^2 \end{bmatrix} \otimes BB^\dagger$ — (1)

$\Pi_A \otimes \Pi_B$

$\Pi_A = \sum_i |a_i\rangle\langle a_i|$

$\Pi_B = \sum_j |b_j\rangle\langle b_j|$

$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \end{bmatrix} [a_1\ a_2 \cdots a_i] \otimes BB^\dagger$

$= \begin{bmatrix} a_1^2 & a_1a_2 & \cdots \\ \vdots & a_2^2 & \cdots \\ & & a_i^2 \end{bmatrix} \otimes BB^\dagger$ — (2)

From (1) & (2)

we can say that

$\Pi_{A\otimes B} = \Pi_A \otimes \Pi_B$

**2.2.** Show that $\mathcal{P}(\rho \otimes \mathcal{T}, \mathbb{A} \otimes \mathbb{B}) = \mathcal{P}(\rho, \mathbb{A})\mathcal{P}(\mathcal{T}, \mathbb{B})$

**1.** Given a quantumn state $|\alpha\rangle$, the density matrix of $|\alpha\rangle$ is its outer product $|\alpha\rangle\langle\alpha|$

**2.** We know that the probability of observing a state to be in 'm', in terms of a density matrix is of the form,

$$\mathcal{P}(|\alpha\rangle, m) = \sum_k \langle \alpha_k \mid \Pi_m \mid \alpha_k \rangle$$

$$= Tr(\sum_k \langle \alpha_k \mid \Pi_m \mid \alpha_k \rangle)$$

$$= Tr(\sum_k |\alpha_k\rangle\langle\alpha_k|\Pi_m)$$, as Trace is not affected by cyclic permutation.

$$= Tr(\rho\Pi_m)$$, where $\rho$ is the density matrix

**3.** Using above notation, we can say that,

$$\mathcal{P}(\rho \otimes \mathcal{T}, \mathbb{A} \otimes \mathbb{B}) = Tr((\rho \otimes \mathcal{T})\Pi_{\mathbb{A}\otimes\mathbb{B}})$$

$$= Tr((\rho \otimes \mathcal{T})\Pi_{\mathbb{A}} \otimes \Pi_{\mathbb{B}})$$, from problem 2.1.

**4.** We know that for a linear transformation, $(p \otimes q)(|r\rangle \otimes |s\rangle) = (p|r\rangle) \otimes (r|s\rangle)$

**5.** Result from point 3 can be written as,

$$\mathcal{P}(\rho \otimes \mathcal{T}, \mathbb{A} \otimes \mathbb{B}) = Tr(\rho\Pi_{\mathbb{A}} \otimes \mathcal{T}\Pi_{\mathbb{B}})$$

$$= Tr(\rho\Pi_{\mathbb{A}})Tr(\mathcal{T}\Pi_{\mathbb{B}})$$

$$= \mathcal{P}(\rho, \mathbb{A})\mathcal{P}(\mathcal{T}, \mathbb{B})$$