QUANTUM ALGORITHMS
HOMEWORK 7 SELECTED SOLUTIONS

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8.7. Prove the properties (8.10)–(8.13) of the operator norm.

Solution: The properties in question are

$$\|XY\| \leq \|X\|\|Y\|, \tag{8.10}$$

$$\|X^\dagger\| = \|X\|, \tag{8.11}$$

$$\|X \otimes Y\| = \|X\|\|Y\|, \tag{8.12}$$

$$\|U\| = 1 \qquad \text{if } U \text{ is unitary.} \tag{8.13}$$

We will prove each of these in turn.

Proof. We have

$$\begin{aligned}
\|XY\| &= \sup_{|\xi\rangle \neq 0} \frac{\|XY\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \left(\frac{\|XY\,|\xi\rangle\,\|}{\|Y\,|\xi\rangle\,\|}\right)\left(\frac{\|Y\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right) \\
&\leq \left(\sup_{|\xi\rangle \neq 0} \frac{\|XY\,|\xi\rangle\,\|}{\|Y\,|\xi\rangle\,\|}\right)\left(\sup_{|\xi\rangle \neq 0} \frac{\|Y\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right) \leq \left(\sup_{|\zeta\rangle \neq 0} \frac{\|X\,|\zeta\rangle\,\|}{\|\,|\zeta\rangle\,\|}\right)\left(\sup_{|\xi\rangle \neq 0} \frac{\|Y\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right) \\
&= \|X\|\|Y\|,
\end{aligned}$$

proving (8.10).

Next, we begin by noting that $\|\,\langle\xi|\,\| = \|\,|\xi\rangle\,\|$. Therefore

$$\|X^\dagger\| = \sup_{|\xi\rangle \neq 0} \frac{\|X^\dagger\,|\xi\rangle\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \frac{\|\,\left|X^\dagger\xi\right\rangle\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \frac{\|\,\langle\xi X|\,\|}{\|\,|\xi\rangle\,\|} = \sup_{|\xi\rangle \neq 0} \frac{\|\,\langle\xi|\,X\|}{\|\,|\xi\rangle\,\|} = \|X\|,$$

proving (8.11).

Next, we have

$$\|\,|\alpha\rangle \otimes |\beta\rangle\,\| = \sqrt{\langle\alpha\otimes\beta \mid \alpha\otimes\beta\rangle} = \sqrt{\langle\alpha \mid \alpha\rangle\,\langle\beta \mid \beta\rangle} = \|\alpha\|\|\beta\|.$$

It follows that

$$\begin{aligned}
\|X \otimes Y\| &= \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \frac{\|(X\otimes Y)(|\xi\rangle \otimes |\zeta\rangle)\|}{\|\,|\xi\rangle \otimes |\zeta\rangle\,\|} = \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \frac{\|\,|X\xi\rangle \otimes |Y\zeta\rangle\,\|}{\|\,|\xi\rangle \otimes |\zeta\rangle\,\|} \\
&= \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \frac{\|\,|X\xi\rangle\,\|\|\,|Y\zeta\rangle\,\|}{\|\,|\xi\rangle\,\|\|\,|\zeta\rangle\,\|} = \sup_{|\xi\rangle\otimes|\zeta\rangle \neq 0} \left(\frac{\|\,|X\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right)\left(\frac{\|\,|Y\zeta\rangle\,\|}{\|\,|\zeta\rangle\,\|}\right) \\
&= \left(\sup_{|\xi\rangle \neq 0} \frac{\|\,|X\xi\rangle\,\|}{\|\,|\xi\rangle\,\|}\right)\left(\sup_{|\zeta\rangle \neq 0} \frac{\|\,|Y\zeta\rangle\,\|}{\|\,|\zeta\rangle\,\|}\right) = \|X\|\|Y\|,
\end{aligned}$$

proving (8.12).

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Finally, if U is unitary then

$$\begin{aligned}\|U\| &= \sup_{|\xi\rangle\neq 0} \frac{\|U|\xi\rangle\|}{\||\xi\rangle\|} = \sup_{|\xi\rangle\neq 0} \frac{\sqrt{\langle U\xi \mid U\xi\rangle}}{\||\xi\rangle\|} = \sup_{|\xi\rangle\neq 0} \frac{\sqrt{\langle \xi \mid \xi\rangle}}{\||\xi\rangle\|} = \sup_{|\xi\rangle\neq 0} \frac{\||\xi\rangle\|}{\||\xi\rangle\|} \\ &= \sup_{|\xi\rangle\neq 0} 1 = 1,\end{aligned}$$

proving (8.13). □

8.8. Prove the two basic properties of approximation with ancillas:

a) If $\tilde{U}$ approximates U with precision δ, then $\tilde{U}^{-1}$ approximates U^{-1} with the same precision δ.

b) If unitary operators $\tilde{U}_k$ approximate unitary operators U_k $(1 \leq k \leq L)$ with precision δ_k, then $\tilde{U}_L \cdots \tilde{U}_1$ approximates $U_L \cdots U_1$ with precision $\sum_k \delta_k$.

Solution: Before we begin, we build some tools. Define $V : \mathbb{B}^{\otimes n} \to \mathbb{B}^{\otimes N}$ by $V|\xi\rangle = |\xi\rangle \otimes |0^{N-n}\rangle$. The statement "$\tilde{U}$ approximates U with ancillas with precision δ" is equivalent to

$$\|\tilde{U}V - VU\| \leq \delta.$$

We are now ready to do the proofs.

a): *Proof.* Note that $\tilde{U}$ is always taken to be *unitary*. Using the fact that $\|T\| = 1$ and $\|T\|\|X\| = \|TX\|$ for unitary T, we have

$$\begin{aligned}\|\tilde{U}^{-1}V - VU^{-1}\| &= \|\tilde{U}\|\|\tilde{U}^{-1}V - VU^{-1}\|\|U\| = \|\tilde{U}(\tilde{U}^{-1}V - VU^{-1})U\| \\ &= \|\tilde{U}\tilde{U}^{-1}VU - \tilde{U}VU^{-1}U\| = \|VU - \tilde{U}V\| = \|\tilde{U}V - VU\| \leq \delta.\end{aligned}$$

Hence $\tilde{U}^{-1}$ approximates U^{-1} with precision δ. □

b): *Proof.* We will proceed by induction on L. The base case of $L = 1$ is included in the assumptions, so it certainly holds. Assume now that the claim holds for a product of $L-1$ matrices. Let $\tilde{W} = \tilde{U}_2 \cdots \tilde{U}_L$ and $W = U_2 \cdots U_L$. These are products of $L-1$ matrices, so by the inductive hypothesis we have that $\tilde{W}$ approximates W with ancillas with precision $\sum_{k=2}^{L} \delta_k$. We now have

$$\begin{aligned}\|\tilde{U}_1 \cdots \tilde{U}_L V - VU_1 \cdots U_L\| &= \|\tilde{U}_1\tilde{W}V - VU_1W\| \\ &= \|\tilde{U}_1(\tilde{W}V - VW) + (\tilde{U}_1V - VU_1)W\| \\ &\leq \|\tilde{U}_1(\tilde{W}V - VW)\| + \|(\tilde{U}_1V - VU_1)W\| \\ &= \|\tilde{U}_1\|\|\tilde{W}V - VW\| + \|\tilde{U}_1V - VU_1\|\|W\| \\ &= \|\tilde{W}V - VW\| + \|\tilde{U}_1V - VU_1\| \\ &\leq \sum_{k=2}^{L} \delta_k + \delta_1 = \sum_{k=1}^{L} \delta_k\end{aligned}$$

(we use the triangle inequality on lines $2-3$). This completes the induction and proves the claim. □

AP 1. After k iterations of $\mathcal{G}$ in Grover's algorithm, we obtained

$$\mathcal{G}^k|\Psi(1,1)\rangle = \Big|\Psi\Big(a^{-1/2}\sin\big((2k+1)\theta\big),\ b^{-1/2}\cos\big((2k+1)\theta\big)\Big)\Big\rangle$$

where θ is such that $\sin(\theta) = \sqrt{a}$. Show that when $k = \lfloor \pi/(4\theta) \rfloor$, upon measuring this state the probability of observing a state in $|A\rangle$ is $\geq 1-a$.

Solution: Given a state vectors $|\alpha\rangle$ and $|\beta\rangle$, the probability of observing $|\alpha\rangle$ to be among the vectors in $|\beta\rangle$ is $|\langle\alpha \mid \beta\rangle|^2$. The vector $|A\rangle$ is not a state vector since it doesn't have norm 1, but we can normalize it. Therefore, the probability of measuring a state $|\alpha\rangle$ and observing it to be among the vectors in $|A\rangle$ is

$$\left|\frac{1}{\| \, |A\rangle \, \|} \langle A \mid \alpha\rangle\right|^2.$$

For the state vector given in the problem, this is

$$\begin{aligned}
&\frac{1}{a}\left|\Big\langle A \mid \Psi\Big(a^{-1/2}\sin\big((2k+1)\theta\big),\ b^{-1/2}\cos\big((2k+1)\theta\big)\Big)\Big\rangle\right|^2 \\
&\quad = \frac{1}{a}\left|\Big\langle A \mid a^{-1/2}\sin\big((2k+1)\theta\big)\,|A\rangle + b^{-1/2}\cos\big((2k+1)\theta\big)\,|B\rangle\Big\rangle\right|^2 \\
&\quad = \frac{1}{a}\left|a^{-1/2}\sin\big((2k+1)\theta\big)\,\langle A \mid A\rangle\right|^2 = \frac{1}{a}\left|a^{-1/2}\sin\big((2k+1)\theta\big)a\right|^2 \\
&\quad = \sin\big((2k+1)\theta\big)^2.
\end{aligned}$$

Thus the probability of observing a correct answer is $\sin((2k+1)\theta)^2$.

Let $k = \lfloor \pi/(4\theta) \rfloor$ and recall that θ is such that $\sin(\theta) = \sqrt{a}$. As a function in x, $\sin((2x+1)\theta)$ is decreasing near $x = \pi/(4\theta)$. Using this (at the marked inequality), we have

$$\sin\big((2k+1)\theta\big) \overset{*}{\geq} \sin\big((\pi/(2\theta)+1)\theta\big) = \sin\big(\pi/2+\theta\big) = \cos(\theta) = \sqrt{b} = \sqrt{1-a}.$$

Hence the probability of measuring a correct answer is $\sin((2k+1)\theta)^2 \geq 1-a$.

AP 3. Recall that the Fibonacci sequence $(f_i)_{i\in\mathbb{N}}$ is defined

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+1} = f_n + f_{n-1}.$$

(i) Show that

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) Use the same technique that we used to find a closed form of the recurrence in Grover's algorithm to find a closed form for the Fibonacci sequence.

Hint: $f_n = (1/\sqrt{5})(\varphi^n - \psi^n)$ *where* $\varphi = (1/2)(1+\sqrt{5})$ *is the golden ratio and* $\psi = (1/2)(1-\sqrt{5})$ *is its conjugate.*

Solution: Let F be the matrix mentioned in the problem. We begin by diagonalizing F. The eigenvalues are solutions to

$$\begin{aligned}
\det(F - xI) &= \det\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right) = \det\begin{pmatrix} 1-x & 1 \\ 1 & -x \end{pmatrix} \\
&= (1-x)(-x) - 1 = x^2 - x - 1,
\end{aligned}$$

so

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \psi = \frac{1-\sqrt{5}}{2}$$

are the eigenvalues. Note that from $x^2 - x - 1 = 0$, we obtain $x^2 = x+1$ and $x(x-1) = 1$. Hence $\varphi^2 = \varphi + 1$, $\varphi^{-1} = \varphi - 1$, and $(\varphi - 1)^{-1} = \varphi$. Similar identities hold for ψ. We also have $\psi = 1 - \varphi$.

Next, we find the eigenvectors by computing the nullspace of $F - xI$ for $x = \varphi$ and $x = \psi$. For $x = \varphi$, we row reduce $F - \varphi I$,

$$\begin{pmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{pmatrix} \sim \begin{pmatrix} 1 & -\varphi \\ 0 & 0 \end{pmatrix},$$

yielding the eigenvector $(\varphi, 1)^T$. Similarly, for $x = \psi$ we row reduce $F - \psi I$ to obtain the eigenvector $(\psi, 1)^T$. It follows that

$$F = \begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \begin{pmatrix} 1 & \psi \\ -1 & \varphi \end{pmatrix}.$$

Finally, we have

$$\begin{aligned} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix} \begin{pmatrix} 1 & \psi \\ -1 & \varphi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi & \psi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^n \\ -\psi^n \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi^{n+1} - \psi^{n+1} \\ \varphi^n - \psi^n \end{pmatrix}. \end{aligned}$$

Therefore $f_n = (1/\sqrt{5})(\varphi^n - \psi^n)$.