Quantum Algorithms: Quiz 7 2021-03-25

Name: ______________________________

1. Let U be a unitary matrix. Using the definition of the operator norm, show that $\|U\| = 1$.

Solution: Recall that unitary matrices are defined to be those matrices which preserve the bracket. It follows that for a vector $|\alpha\rangle$,

$$\|\alpha\| = \sqrt{\langle \alpha \mid \alpha \rangle} = \sqrt{\langle \alpha | \, U^\dagger U \, | \alpha \rangle} = \sqrt{\big(U \, |\alpha\rangle\,\big)^\dagger \big(U \, |\alpha\rangle\,\big)} = \big\| U \, |\alpha\rangle \, \big\|.$$

In particular, if $\|\alpha\| = 1$ then $\big\| U \, |\alpha\rangle \, \big\| = 1$. From the definition of the operator norm, we have

$$\|U\| = \sup_{\|\alpha\|=1} \big\| U \, |\alpha\rangle \, \| = \sup_{\|\alpha\|=1} \|\alpha\| = \sup_{\|\alpha\|=1} 1 = 1.$$