Homework 11

1.1. Prove that $x^t - 1 = (x-1)\sum_{k=0}^{t-1} x^k$

1. Consider the polynomial $1 + x + \cdots + x^{t-1}$

$$\Rightarrow (1 + x + \cdots + x^{t-1})\frac{(x-1)}{(x-1)}$$

$$\Rightarrow \frac{(x + x^2 + \cdots + x^{t-1} + x^t - 1 - x - x^2 - \cdots - x^{t-1})}{(x-1)}$$

$$\Rightarrow 1 + x + \cdots + x^{t-1} = \frac{(x^t - 1)}{(x-1)}$$

$$\Rightarrow \sum_{k=0}^{t-1} x^k = \frac{(x^t - 1)}{(x-1)}$$

2. $x^t - 1 = (x-1)\sum_{k=0}^{t-1} x^k$

1.2. Prove that $x = e^{2\Pi i(m/t)}$ is the solution to $x^t - 1$ for $m \in \mathbb{Z}$

1. Substitute $e^{2\Pi i(m/t)}$ in the equation $x^t - 1$

$$\Rightarrow x^t - 1 = (e^{2\Pi i(m/t)})^t - 1 = e^{2\Pi i\frac{mt}{t}} - 1 = (e^{i2\Pi})^m - 1$$

2. We know that,
$e^{i\theta} = cos(\theta) + isin(\theta)$ and
$(cos(\theta) + isin(\theta))^m = cos(m\theta) + isin(m\theta)$

$$\Rightarrow x^t - 1 = (e^{i2\Pi})^m - 1 = cos(2\Pi m) + isin(2\Pi m) - 1$$

3. if $m \in \mathbb{Z}$, then
$cos(2\Pi m) = 1$ and $sin(2\Pi m) = 0$

$$\Rightarrow x^t - 1 = cos(2\Pi m) + isin(2\Pi m) - 1 = 1 + 0 - 1 = 0$$

4. Thus $e^{2\Pi i(m/t)}$ is the root for the equation $x^t - 1$ for $m \in \mathbb{Z}$

1.3. Prove that $\sum_{k=0}^{t-1} e^{2\Pi i(km/t)} = \begin{cases} t & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$

1. When $m = 0$, $e^{2\Pi i(m/t)} = 1$

$\sum_{k=0}^{t-1} e^{2\Pi i(km/t)} = 1 + 1 + \cdots + t - 1,$

$\Rightarrow \sum_{k=0}^{t-1} e^{2\Pi i(km/t)} = t$

2. When $m \neq 0$ and $0 \leq m \leq t$,

$\sum_{k=0}^{t-1} e^{2\Pi i(km/t)} = \sum_{k=0}^{t-1} (e^{2\Pi i(m/t)})^k$

$\Rightarrow \sum_{k=0}^{t-1} (e^{2\Pi i(m/t)})^k = \frac{(e^{2\Pi i(m/t)})^t - 1}{e^{2\Pi i(m/t)} - 1}$, using 1.1

3. From 1.2, we know that, $e^{i2\Pi m} = 1$

Thus for $m \neq 0$ and $0 \leq m \leq t$,

$\Rightarrow \sum_{k=0}^{t-1} (e^{2\Pi i(m/t)})^k = \frac{e^{i2\Pi m} - 1}{e^{2\Pi i(m/t)} - 1} = \frac{1 - 1}{e^{2\Pi i(m/t)} - 1} = 0$

2. By given definition, $QFT|x\rangle = \frac{1}{\sqrt{2^n}} \bigotimes_{k=1}^{n} (|0\rangle + e^{\frac{i2\Pi[x]}{2^k}} |1\rangle)$

1. We rearrage the definition of QFT using summation and product as below,

$$QFT = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} [\prod_{k=1}^{n} e^{\frac{i2\Pi[x]}{2^k}}] |j_1 j_2 \ldots j_n\rangle$$

In above equation, we can rewrite the product component as below

$\prod_{k=1}^{n} e^{\frac{i2\Pi[x]}{2^k}} = e^{i2\Pi\alpha}$ where $\alpha = \sum_{l=1}^{n} \frac{j_l}{2^l}$

2.1. If we perform QFT on $|0^n\rangle$, then

$\alpha = 0$, because, $j_l = 0$

$$\Rightarrow \prod_{k=1}^{n} e^{\frac{i2\Pi[x]}{2^k}} = e^{i2\Pi\alpha} = 1$$

$$\Rightarrow QFT|0^n\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j_1 j_2 \ldots j_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{j\in\{0,1\}^n} |j\rangle$$

2.2. If we perform QFT on $|1^n\rangle$, then

$$\alpha = \sum_{l=1}^{n} \frac{j_l}{2^l} = \frac{[1 + 2 + 2^2 + \cdots + 2^{k-1}]}{2^k} = \frac{[j]}{2^k}$$

$$\Rightarrow QFT|1^n\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{\dfrac{i2\Pi[j]}{2^k}} |j_1 j_2 \ldots j_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{j \in \{0,1\}^n} e^{\dfrac{i2\Pi[j]}{2^k}} |j\rangle$$

3. We have $QFT|x\rangle = \dfrac{1}{\sqrt{2^n}} \bigotimes_{k=1}^{n} (|0\rangle + e^{\frac{i2\Pi[x]}{2^k}} |1\rangle)$

1. We rearrage the definition of QFT using summation and product as below,

$$QFT|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} [\prod_{k=1}^{n} e^{\frac{i2\Pi[x]}{2^k}}] |y_1 y_2 \ldots y_n\rangle$$

In above equation, we can rewrite the product component as below
$\prod_{k=1}^{n} e^{\frac{i2\Pi[x]}{2^k}} = e^{i2\Pi[x]\alpha}$ where $\alpha = \sum_{l=1}^{n} \frac{y_l}{2^l}$

2. If we consider, $\alpha = \sum_{l=1}^{n} \frac{y_l}{2^l} = [y]$ where $[x], [y]$ are binary decimal representations.

$$\Rightarrow QFT|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{\frac{i2\Pi[x][y]}{2^k}} |y_1 y_2 \ldots y_n\rangle$$

$$\Rightarrow QFT|x\rangle = 2^{-n/2} \sum_{y \in \{0,1\}^n} e^{\dfrac{i2\Pi[x][y]}{2^n}} |y\rangle$$

4.

4. show that $QFT_n^{\dagger}|x\rangle = 2^{-n/2}\sum_{y\in\{0,1\}^n} e^{\frac{-i2\pi[x][y]}{2^n}}|y\rangle$

① Compute

$$QFT_n \circ QFT_n^{\dagger}|x\rangle = QFT_n\left[2^{-n/2}\sum_{y\in\{0,1\}^n} e^{\frac{-i2\pi[x][y]}{2^n}}|y\rangle\right]$$

$$= 2^{-n/2}\sum_{y\in\{0,1\}^n} e^{\frac{-i2\pi[x][y]}{2^n}} QFT_n|y\rangle$$

$$= 2^{-n/2}\sum_{y\in\{0,1\}^n} e^{-i2\pi[x][y]/2^n}\left(2^{-n/2}\sum_{z\in\{0,1\}^n} e^{i2\pi[y][z]/2^n}|z\rangle\right)$$

$$= 2^{-n}\sum_{z\in\{0,1\}^n}\sum_{y\in\{0,1\}^n} e^{-i2\pi[y]([x]-[z])/2^n}|z\rangle$$

From problem 1 & Hint 2,

$$\sum_{y\in\{0,1\}^n} e^{-i2\pi[y]([x]-[z])/2^n} = \begin{cases} 2^n & x = z \\ 0 & x \neq z \end{cases}$$

$$= 2^{-n}\sum_{z\in\{0,1\}^n} 2^n|z\rangle = |x\rangle$$

② Compute

$$QFT^{\dagger} \circ QFT|x\rangle = QFT_n^{\dagger}\left[2^{-n/2}\sum_{y\in\{0,1\}^n} e^{i2\pi[x][y]/2^n}|y\rangle\right]$$

$$= 2^{-n/2}\sum_{y\in\{0,1\}^n} e^{i2\pi[x][y]/2^n} QFT^{\dagger}|y\rangle$$

$$= 2^{-n/2}\sum_{y\in\{0,1\}^n} e^{i2\pi[x][y]/2^n}\left(2^{-n/2}\sum_{z\in\{0,1\}^n} e^{-i2\pi[y][z]/2^n}|z\rangle\right)$$

$$= 2^{-n} \sum_{z \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} e^{i 2\pi ([x]-[z])[y] / 2^n} |z\rangle$$

we know that

$$\sum_{y \in \{0,1\}^n} e^{i 2\pi [y]([x]-[z]) / 2^n} = \begin{cases} 2^n & \text{if } x = z \\ 0 & \text{if } x \neq z \end{cases}$$

$$= 2^{-n} \sum_{z \in \{0,1\}^n} 2^n |z\rangle = \sum_{z \in \{0,1\}^n} |z\rangle = |x\rangle$$

From (1) & (2) we can prove the property

$$QFT \circ QFT^{\dagger} |x\rangle = QFT^{\dagger} \circ QFT |x\rangle = |x\rangle$$

by using

$$QFT^{\dagger} = 2^{-n/2} \sum_{y \in \{0,1\}^n} e^{-i 2\pi [x][y] / 2^n} |y\rangle$$

5.

5. Find QFT_3, $QFT_3^{\dagger}$

$QFT_3 =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} \cdot \frac{1}{\sqrt{8}}$$

$$\omega = i\pi/4$$

$$= \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & (1+i)/\sqrt{2} & i & \frac{(-1+i)}{\sqrt{2}} & -1 & \frac{(-1-i)}{\sqrt{2}} & -i & \frac{(1-i)}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & (-1+i)/\sqrt{2} & -i & \frac{(1+i)}{\sqrt{2}} & -1 & \frac{(1-i)}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & (-1-i)/\sqrt{2} & i & \frac{(1-i)}{\sqrt{2}} & -1 & \frac{(1+i)}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{(1-i)}{\sqrt{2}} & -i & \frac{(-1-i)}{\sqrt{2}} & -1 & \frac{(-1+i)}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \end{bmatrix}$$

$QFT_3^{\dagger} =$

$$\frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} & 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} \\ 1 & -i & 1 & i & -1 & -i & 1 & i \\ 1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} & 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} & 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} \\ 1 & i & 1 & -i & -1 & i & 1 & -i \\ 1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} & 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} \end{bmatrix}$$