QUANTUM ALGORITHMS
HOMEWORK 9 SELECTED SOLUTIONS

PROF. MATTHEW MOORE

AP 1. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are orthogonal to each other.

(i) What is $\dim(\mathbb{A}+\mathbb{B})$?

(ii) Show that $\mathcal{P}(|v\rangle, \mathbb{A}+\mathbb{B}) = \mathcal{P}(|v\rangle, \mathbb{A}) + \mathcal{P}(|v\rangle, \mathbb{B})$.

(iii) Show that $\Pi_{\mathbb{A}}\Pi_{\mathbb{B}} = \Pi_{\mathbb{B}}\Pi_{\mathbb{A}}$.

Solution: **(i)** Let $\mathcal{A}$ and $\mathcal{B}$ be orthonormal bases for $\mathbb{A}$ and $\mathbb{B}$, respectively. It follows that $\mathcal{A}\cup\mathcal{B}$ is a spanning set of $\mathbb{A}+\mathbb{B}$. Furthermore, since $\mathbb{A}$ and $\mathbb{B}$ are orthogonal, $\mathcal{A}$ and $\mathcal{B}$ are orthogonal. It follows that $\mathcal{A}\cup\mathcal{B}$ is a set of orthogonal vectors, and any set of orthogonal vectors not containing the zero vector $\vec{0}$ is linearly independent. Thus $\dim(\mathbb{A}+\mathbb{B}) = \dim(\mathbb{A}) + \dim(\mathbb{B})$.

(ii): *Proof.* Let $\mathcal{A} = \{|a_1\rangle, \ldots, |a_n\rangle\}$ and $\mathcal{B} = \{|b_1\rangle, \ldots, |b_m\rangle\}$ be orthonormal bases for $\mathbb{A}$ and $\mathbb{B}$, respectively. From the previous part, we have that $\mathcal{A}\cup\mathcal{B}$ is an orthonormal basis for $\mathbb{A}+\mathbb{B}$. Define

$$|c_i\rangle = \begin{cases} a_i & \text{if } 1 \le i \le n, \\ b_{i-n} & \text{if } n < i \le n+m \end{cases}$$

so that $\mathcal{A}\cup\mathcal{B} = \{|c_1\rangle, \ldots, |c_{n+m}\rangle\}$. We have

$$\begin{aligned} \Pi_{\mathbb{A}+\mathbb{B}} &= \sum_{i=1}^{n+m} |c_i\rangle\langle c_i| = \sum_{i=1}^{n} |c_i\rangle\langle c_i| + \sum_{i=n+1}^{n+m} |c_i\rangle\langle c_i| \\ &= \sum_{i=1}^{n} |a_i\rangle\langle a_i| + \sum_{i=1}^{m} |b_i\rangle\langle b_i| = \Pi_{\mathbb{A}} + \Pi_{\mathbb{B}}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{P}(|v\rangle, \mathbb{A}+\mathbb{B}) &= \langle v|\Pi_{A+\mathbb{B}}|v\rangle = \langle v|\left(\Pi_{\mathbb{A}} + \Pi_{\mathbb{B}}\right)|v\rangle \\ &= \langle v|\Pi_{\mathbb{A}}|v\rangle + \langle v|\Pi_{\mathbb{B}}|v\rangle = \mathcal{P}(|v\rangle, \mathbb{A}) + \mathcal{P}(|v\rangle, \mathbb{B}). \end{aligned}$$

□

(iii): *Proof.* Using the same definitions for $\mathcal{A}$ and $\mathcal{B}$ as in the previous part, we have

$$\begin{aligned} \Pi_{\mathbb{A}}\Pi_{\mathbb{B}} &= \left(\sum_{i=1}^{n} |a_i\rangle\langle a_i|\right)\left(\sum_{j=1}^{m} |b_j\rangle\langle b_j|\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} |a_i\rangle\langle a_i \mid b_j\rangle\langle b_j| \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} |a_i\rangle(0)\langle b_j| = |\vec{0}\rangle\langle\vec{0}|. \end{aligned}$$

Reversing the order also gives $\Pi_{\mathbb{B}}\Pi_{\mathbb{A}} = |\vec{0}\rangle\langle\vec{0}|$. □

Date: March 31, 2020.

AP 2. Suppose that $\mathbb{A} \leq \mathbb{V}$ and $\mathbb{B} \leq \mathbb{W}$ are two subspaces.

(i) Prove that $\Pi_{\mathbb{A}\otimes\mathbb{B}} = \Pi_{\mathbb{A}} \otimes \Pi_{\mathbb{B}}$.

(ii) Let ρ and τ be density matrices. Prove that $\mathcal{P}(\rho\otimes\tau, \mathbb{A}\otimes\mathbb{B}) = \mathcal{P}(\rho, \mathbb{A})\mathcal{P}(\tau, \mathbb{B})$. You may use the fact that $\mathrm{Tr}(X \otimes Y) = \mathrm{Tr}(X)\,\mathrm{Tr}(Y)$.

Solution:

(i): *Proof.* Let $\mathcal{A} = \{|a_1\rangle, \ldots, |a_n\rangle\}$ and $\mathcal{B} = \{|b_1\rangle, \ldots, |b_m\rangle\}$ be orthonormal bases for A and $\mathbb{B}$, respectively. It follows that

$$\mathcal{C} = \left\{ |a_i, b_j\rangle \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m \right\}$$

is an orthonormal basis for $\mathbb{A} \otimes \mathbb{B}$. Therefore

$$\begin{aligned}
\Pi_{A\otimes\mathbb{B}} &= \sum_{i,j} |a_i, b_j\rangle \langle a_i, b_j| = \sum_{i,j} \big(|a_i\rangle \otimes |b_j\rangle \big)\big(\langle a_i| \otimes \langle b_j| \big) \\
&= \sum_{i,j} |a_i\rangle \langle a_i| \otimes |b_j\rangle \langle b_j| = \left(\sum_i |a_i\rangle \langle a_i| \right) \otimes \left(\sum_j |b_j\rangle \langle b_j| \right) \\
&= \Pi_{\mathbb{A}} \otimes \Pi_{\mathbb{B}}.
\end{aligned}$$

□

(ii): *Proof.* We have

$$\begin{aligned}
\mathcal{P}(\rho\otimes\tau, \mathbb{A}\otimes\mathbb{B}) &= \mathrm{Tr}\left((\rho\otimes\tau)\Pi_{A\otimes\mathbb{B}} \right) = \mathrm{Tr}\left((\rho\otimes\tau)(\Pi_{\mathbb{A}} \otimes \Pi_{\mathbb{B}}) \right) \\
&= \mathrm{Tr}\left((\rho\Pi_{\mathbb{A}}) \otimes (\tau\Pi_{\mathbb{B}}) \right) = \mathrm{Tr}\left(\rho\Pi_{\mathbb{A}} \right) \mathrm{Tr}\left(\tau\Pi_{\mathbb{B}} \right) \\
&= \mathcal{P}(\rho, \mathbb{A})\mathcal{P}(\tau, \mathbb{B})
\end{aligned}$$

(we use the result of the previous part on line 1). □