Homework 8

(1) Reflection operator on the vector $|a\rangle$

$$D_a|x\rangle = e^{i\theta}|x\rangle \quad \text{if } \langle a|x\rangle \neq 0$$
$$= |x\rangle \quad \text{if } \langle a|x\rangle = 0$$

We saw in Grover's Algorithm,

If $P_0 = (e^{i\theta}-1)|0^n\rangle\langle 0^n\rangle + I$, then

$$D_0|x\rangle = e^{i\theta}|0^n\rangle \quad \text{if } x = |0^n\rangle$$
$$|0^n\rangle \quad \text{else}$$

In similar lines, if

$$D_A = \frac{(e^{i\theta}-1)}{a}|A\rangle\langle A| + I \quad \text{where } \langle A|A\rangle = a \neq 0$$

then $D_A|A\rangle = \left(\frac{(e^{i\theta}-1)}{a}|A\rangle\langle A\rangle + I\right)|A\rangle$

$$= \frac{e^{i\theta}}{a}|A\rangle\langle A|A\rangle - \frac{1}{a}|A\rangle\langle A|A\rangle + |A\rangle$$

$$= e^{i\theta}|A\rangle$$

Say $D_A|B\rangle = \left(\left(\frac{e^{i\theta}-1)}{a}|A\rangle\langle A\rangle + I\right)|B\rangle = \frac{(e^{i\theta}-1)}{a}|A\rangle\langle A|B\rangle + |B\rangle$

$$= |B\rangle \qquad (\langle A|B\rangle = 0)$$

So we can say operator for D_A can be written as

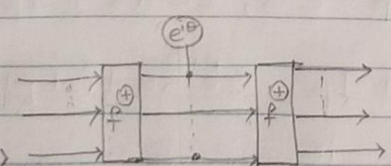
$$\underline{D_A} = \frac{(e^{i\theta}-1)}{a}|A\rangle\langle A| + I$$

Similarly for $|start\rangle$ which is a basis vector

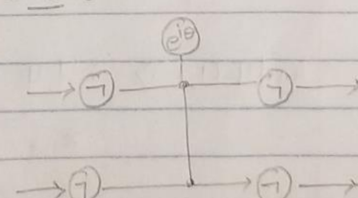
$$\underline{D_s} = (e^{i\theta}-1)|start\rangle\langle start| + I$$

<u>Circuit</u>

D_A $|0\rangle$



<u>Circuit</u>



(2) We know that $\langle A|A\rangle = a$

$\langle B|B\rangle = b$

$$G = A \circ D_s \circ A^{+}$$

$$G|\psi(\alpha,\beta)\rangle = -A \circ D_s \circ A^{+} \circ D_A |\psi(1,1)\rangle$$

We know that $D_A|A\rangle = e^{i\theta}|A\rangle$

$$\Rightarrow D_A|\psi(\alpha,\beta)\rangle = \psi(e^{i\theta}\alpha, \beta) \quad - (1)$$

$$A = A^{-1} = A^{+}$$

$$\Rightarrow \quad -A \circ D_s \circ A^{+}$$

$$= -2|\psi(1,1)\rangle\langle\psi(1,1)| + I \quad - (2)$$

From (1) & (2)

$$G|\psi(\alpha,\beta)\rangle = (2|\psi(1,1)\rangle\langle\psi(1,1)| - I)|\psi(e^{i\theta}\alpha,\beta)\rangle$$

Now we calculate

$$\langle\psi(1,1)|\psi(e^{i\theta}\alpha, \beta)\rangle$$

$$= (\langle A| + \langle B|)(e^{i\theta}\alpha|A\rangle + \beta|B\rangle)$$

$$= e^{i\theta}a\alpha + b\beta \quad (\because \langle A|A\rangle = a;\ \langle B|B\rangle = b) \quad - (3)$$

$$G|\psi(\alpha,\beta)\rangle = 2|\psi(1,1)\rangle(e^{i\theta}a\alpha + b\beta) - \psi(e^{i\theta}\alpha, \beta)\rangle$$

$$= |\psi(2e^{i\theta}a\alpha + 2b\beta - e^{i\theta}\alpha,\ 2e^{i\theta}a\alpha + 2b\beta - \beta)\rangle$$

$$= |\psi((2a-1)e^{i\theta}\alpha + 2b\beta,\ 2e^{i\theta}a\alpha + (2b-1)\beta)\rangle$$

$$G|\psi(1,1)\rangle = |\psi((2a-1)e^{i\theta} + \beta,\ 2e^{i\theta}a + (2b-1))\rangle$$

$$x = (2a-1)e^{i\theta} + \beta$$

$$y = 2e^{i\theta}a + (2b-1)$$