Lecture Notes | 2024

Topology Class Notes

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Introduction

These lecture notes have been transcribed by me (Patrick Mullen). The environment was set up by Soham Chatterjee, sohamc@cmi.ac.in, and can be found here

This course was taught in W24 at the University of Michigan by Pierre-Louis Blayac.

Contents

Chapter 1

Introduction

1.1 The Standard Topology on Euclidean Space

Topology, from the greek *topos*, meaning "place" or "locality", and *logos* meaning "study", can be thought of as the study of shape. More specifically, the study of how geometric objects behave under continuous deformations.

There are a variety of different (equivalent) approaches to topology, including but not limited open sets, neighborhoods, metrics, convergence of sequences, and continuity of functions. All of the preceding are discussed in this course, but we will rely heavily on the concept of open sets. Before getting to the subject, we review some important fundamentals.

Firstly, we denote the set of real numbers as $\mathbb{R}$, and the set of d-tuples as $\mathbb{R}^d$. The latter of these sets is sometimes referred to as "Euclidean d-space".

Definition 1.1.1: The (Standard) Inner Product and (Standard) Norm

Let $x = (x_1, ..., x_d), y = (y_1, ..., y_d) \in \mathbb{R}$. The standard inner product is a map $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(x, y) = \langle x, y \rangle = x_1 y_1 + ... x_d y_d$$

We then define the (standard) norm as a function $g : \mathbb{R}^d \to \mathbb{R}$ as

$$g(x) = ||x|| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + ... x_d^2}$$

Theorem 1.1.1

For all $x, x', y, y' \in \mathbb{R}^d$ the following properties regarding the inner product hold

1. Bilinearity:

$$\langle \lambda x + \lambda' x', \mu y + \mu' y' \rangle = \lambda\mu\langle x, y \rangle + \lambda'\mu\langle x' y \rangle + \lambda\mu'\langle x, y' \rangle + \lambda'\mu'\langle x', y' \rangle$$

2. Symmetry:

$$\langle x, y \rangle = \langle y, x \rangle$$

3. Positivity:

$$\langle x, x \rangle \geq 0 \text{ with equality if and only if } x = 0$$

4. Cauchy Schwarz Inequality:

$$|\langle x, y \rangle| \leq ||x||\,||y||, \text{ with equality if and only if } x \parallel y$$

5. Triangle Inequality:

$$||x + y|| \leq ||x|| + ||y||, \text{ with equality if and only } y = 0 \text{ or } \exists a \geq 0 \text{ s.t. } x = ay$$

Proof: TODO: prove the above statements from definition □

Definition 1.1.2: The (Standard) Metric on $\mathbb{R}^d$

The (standard) metric on $\mathbb{R}^d$ is a map $d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R}$

$$d(x, y) = ||x - y||$$

Theorem 1.1.2

For all $x, y, z \in \mathbb{R}$, the following properties regarding the (standard) metric hold:

1. Positivity:
$$d(x, y) \geq 0 \text{ with equality if and only if } x = y$$
2. Symmetry:
$$d(x, y) = d(y, x)$$
3. Triangle Inequality
$$d(x, z) \leq d(x, y) + d(y, z)$$

Proof: TODO: prove above statements from definition. □

Definition 1.1.3: (Standard) Open Ball and Closed Ball

Let $x \geq 0$ and $R \geq 0$. Then the (standard) open ball is the set

$$B(x, R) = \{y \in \mathbb{R}^d | d(x, y) < R\}$$

and the (standard) close ball is the set

$$\overline{B}(x, R) = \{y \in \mathbb{R}^d | d(x, y) \leq R\}$$

In $\mathbb{R}$, $B(x, R)$ is just the open interval $(x - R, x + R)$. In $\mathbb{R}^2$, $B(x, R)$ is the interior of the circle centered at x of radius R. In $\mathbb{R}^3$, $B(x, R)$ is the the interior of the sphere centered at x of radius R. The closed counterparts of these sets include the boundaries of the described sets.

Definition 1.1.4: Neighborhoods in $\mathbb{R}^d$

Let $U \subset \mathbb{R}^d$ and $x \in U$. Then U is a **neighborhood** of x if it contains a nonempty ball centered at x, i.e. if

$$\exists \epsilon > 0, B(x, \epsilon) \subset U$$

Additionally, U is a (standard) open ball of $\mathbb{R}^d$ if it is a neighborhood of all its points, i.e.

$$\forall y \in U, \exists \epsilon > 0, B(x, \epsilon) \subset U$$

Note:-

1. and $\mathbb{R}$ are open.
2. Open balls are open.

Proof. TODO: prove the statements in the note □

Theorem 1.1.3

1. Let $(V_\alpha)_{\alpha \in I}$ be a (possibly infinite) family of open subsets of $\mathbb{R}^d$. Then $\bigcup_{\alpha \in I} V_\alpha$ is open.

2. Let $V_1, ..., V_n \in \mathbb{R}^d$ be open. Then $V_1 \cap ... \cap V_n$ is open.

Proof. TODO: prove the above statements $\square$

Example 1.1.1

The set $U = \{(x_1, ..., x_d) \in \mathbb{R}^d | x_d > 0\}$ is an open subset of $\mathbb{R}^d$.

Proof: Let $x \in U$ and set $\epsilon = x_d$. We then want to show that $B(x, \epsilon) \subset U$. Let $y \in B(x, \epsilon)$. Then $|x_d - y_d| \leq ||x - y|| = \sqrt{|x_1 - y_1|^2 + ... + |x_d - y_d^2|}$ $\square$

Note:-

The collection of open subsets of $\mathbb{R}^d$ is called the Standard Topology on $\mathbb{R}^d$.

Definition 1.1.5: Convergence of Sequences in $\mathbb{R}^d$

Let $(x_n)_{n\geq 0}$ be a sequence in $\mathbb{R}^d$. We say $x_n \to_x$ or $\lim\limits_{n\to\infty} x_n = x$ if

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, d(x_n, x) < \epsilon$$

Note that $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.

Example 1.1.2

$\lim\limits_{n\to\infty} \frac{1}{n} = 0$
TODO: Proof

Theorem 1.1.4

Let $x \in \mathbb{R}$ and $(x_n)_{n\geq 0}$ a sequence in $\mathbb{R}^d$. Then

$$\lim_{n\to\infty} x_n = x \iff \forall V \text{ nbhd of } x, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in V$$

Proof: ($\implies$): Assume $\lim_{n\to\infty} x_n = x$. Let V be a neighborhood of x. Then, by the definition of neighborhood, $\exists \epsilon > 0$ such that $B(x, \epsilon) \subset V$. Then, by the definition of convergence, $\exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x_n \in B(x, \epsilon)$. Then, since $B(x, \epsilon) \subset V$, we have $\forall n > n_0, x_n \in V$. Then, since V is arbitrary, $\forall V$ a neighborhood of $x, \exists n_0 \in \mathbb{R}, \forall n > n_0, x_n \in V$. Therefore,

$$\lim_{n\to\infty} x_n = x \implies \forall V \text{ nbhd of } x, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in V$$

($\impliedby$) Assume $\forall V$ nbhd of $x, \exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x_n \in V$. Let $\epsilon > 0$. We know that $B(x, \epsilon)$ is a neighborhood of x. Then, by our assumption, $\exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x \in B(x, \epsilon)$. Therefore, by the definition of convergence, $\lim_{n\to\infty} x_n = x$. Therefore,

$$\lim_{n\to\infty} x_n = x \impliedby \forall V \text{ nbhd of } x, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in V$$

$\square$

Definition 1.1.6: Continuity of Functions from $\mathbb{R}^d$ to $\mathbb{R}^k$

A map $f : \mathbb{R}^d \to \mathbb{R}^k$ is **continuous at** $x \in \mathbb{R}^d$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in B(x, \delta), d(f(x), f(y)) < \epsilon$$

Note that $d(f(x), f(y)) < \epsilon$ is equivalent to $f(y) \in B(f(y), \epsilon)$, so f is continuous if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in B(x, \delta), f(y) \in B(f(y), \epsilon)$$

Another, equivalent definition is that f is continuous if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } f(B(x, \delta)) \subset B(f(x), \epsilon)$$

We say that f is **continuous** if it is continuous at every point.

Example 1.1.3

1. Let $v \in \mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^d$ defined by the rule $x \mapsto x + v$ is continuous on $\mathbb{R}^d$.
   Let $x \in \mathbb{R}^d$ and let $\epsilon > 0$. Set $\delta = \epsilon$. We then want to show that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Let $y \in B(x, \delta)$. Then

   $$d(f(y), f(x)) = ||f(y) - f(x)|| = ||y + v - (x + v)|| = ||y - x|| < \delta = \epsilon$$

   Therefore, $f(y) \in B(f(x), \epsilon)$.
2. TODO: fill in remaining examples

Theorem 1.1.5

Let $f : \mathbb{R}^d \to \mathbb{R}^k$. Then the following statements hold

1. f is continuous at $x \in \mathbb{R}^d \iff \forall V$ neighborhood of $f(x)$, $f^{-1}(V)$ is a neighborhood of x.
2. f is continuous on $\mathbb{R}^d \iff \forall U \subset \mathbb{R}^d$ open, $f^{-1}(U) \subset \mathbb{R}^k$ is open.
3. f is continuous at $x \in \mathbb{R}^d \iff \forall (x_n)_{n \geq 0}$ in $\mathbb{R}^d$ converging to x, $f(x_n) \to f(x)$ as $n \to \infty$.

Proof: TODO: complete proof of above statements □

Definition 1.1.7: Open Subset

Let $X \subset \mathbb{R}^d$. A subset $U \subset X$ is **open in** X if $\forall x \in U, \exists \epsilon > 0$ such that $B(x, \epsilon) \cap X \subset U$.

A Topology of X, then, is a collection of subsets of X that are open in X. This definition will be expanded and made rigorous later in the course.

Example 1.1.4 (Torus)

TODO: insert torus example here (preferable with illustration)

Definition 1.1.8: Homeomorphism

Let $X, Y \subset \mathbb{R}^d$ be subsets. Then X, Y are **homeomorphic** if $\exists f : X \to Y$ a bijection such that

$$\forall U \subset X, U \text{ is open in } X \iff f(U) \text{ open in } Y$$

The function f is called a **homeomorphism**.

1.2 General Topologies, Open Sets, and Neighborhoods

In the previous section, we defined things in terms of $\mathbb{R}^d$, the standard metric, and the standard inner product.

Definition 1.2.1: Topology

Let X be a set. A **topology** on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X called **open subsets** such that

1. $\emptyset, X \in \mathcal{T}$
2. Stability under unions. For any $(U_\alpha)_{\alpha \in I}$ of elements of $\mathcal{T}$, $\bigcup_{\alpha \in I} U_\alpha$ is also an element of $\mathcal{T}$.
3. Stability under finite intersection. For any $U_1, ..., U_n \in \mathcal{T}$, we have $U_1 \cap ... \cap U_n \in \mathcal{T}$.

The pair (X, d) is called a **topological space**.

Example 1.2.1

Let $X = \mathbb{R}^d$.

1. The collection of standard open sets is a topology (the standard topology) as discussed in the previous section.
2. $\{\emptyset, \mathbb{R}^d\}$ is a topology on $\mathbb{R}^d$.
3. $\mathcal{P}(\mathbb{R}^d)$ is a topology

If $\mathcal{T}_S$ is the standard topology, then

$$\{\emptyset, \mathbb{R}^d\} \subset \mathcal{T}_S \subset \mathcal{P}(\mathbb{R}^d)$$

Definition 1.2.2: Coarse and Fine

Let X be a set, and $\mathcal{T}_1, \mathcal{T}_2$ be two topologies. If $\mathcal{T}_1 \subset \mathcal{T}_2$, we say $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, and $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$

.

Note:-

Given X a set, the set $\{\emptyset, X\}$ is called the **coarse topology** (it is the coarsest topology). The powerset of X, $\mathcal{P}(X)$, is called the discrete topology (it is the finest).

Example 1.2.2

Consider $X = \{0, 1, 2\}$. Then $\mathcal{T} = \{\emptyset, X, \{0\}, \{1, 2\}\}$ is a topology on X. It's easy to check that it's stable under union, stable under finite intersection, and contains both $\emptyset$ and X.
TODO: input illustration
There exist many different topologies on X. The number of unique topologies is bounded above by the cardinality of $\mathcal{P}(\mathcal{P}(X))$, which in this case is 2^{2^3} or 256. Another example of such a topology is $\mathcal{T}_2 = \{\emptyset, X, \{0\}\}$. A non-example of a topology is $\{\emptyset, X, \{0\}, \{1\}, \{2\}\}$, since it is not stable under unions.

Definition 1.2.3: Neighborhood

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is a neighborhood of $x \in X$ if $\exists U \in X$ open such that $x \in U \subset A$.

Theorem 1.2.1

Let $(X, \mathcal{T})$ be a topological space. Then $\forall x \in X, \mathcal{N}(x)$ is defined as the collection of all neighborhoods of x. That is

$$\mathcal{N}(x) = \{N \subset X | N \text{ is a neighborhood of } x\}$$

Then, $\forall x \in X$ the following statements hold:

1. $X \in \mathcal{N}(x)$
2. $\forall N \in \mathcal{N}(x), x \in N$
3. $\forall N \in \mathcal{N}(x), \forall A \subset X$, if $N \subset A$, then $A \subset \mathcal{N}(x)$
4. $\forall N_1, ..., N_k \in \mathcal{N}(x), N_1 \cap ... \cap N_k \in \mathcal{N}(x)$
5. $\forall N \in \mathcal{N}(x), \exists N' \in \mathcal{N}(x)$ such that $\forall y \in N', N \in \mathcal{N}(y)$

Moreover, $\forall U \subset X, U$ is open if and only if U is a neighborhood of all its points.

Proof: TODO: Insert proof here □

1.3 Sequences in Topological Spaces

While the sequences we study are not remarkably different than those encountered in the typical analysis course, we will generalize the concept slightly, especially in regards to the definition of convergence of sequences.

Definition 1.3.1: Convergence and Continuity

Let $(X, \mathcal{T})$ be a topological space, $(x_n)_n$ in X and $x \in X$. We say that $x_n \to x$ if

$$\forall N \text{ a neighborhood }, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in N$$

Example 1.3.1

Let $X = \{0, 1, 2\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}, \{0, 1\}\}$. Then $x_n \to 1$ if and only if $\exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x_n =$ 0 or 1. In particular $0 \to 1$.

Theorem 1.3.1

Let $(X, \mathcal{T})$ be a topological space. Let $(x_n)_{n \geq 0}$ be a sequence in X that is constant after some time: $\exists n_0 \geq 0$ integer such that $x_n = x_{n_0} \forall n \geq n_0$. Then $x_n \to x_{n_0}$ as $n \to \infty$.

Proof: Let N be a neighborhood of x_{n_0}. Then $\forall n \geq n_0, x_n = x_{n_0}$, so $x_n \to x_{n_0}$. □

Definition 1.3.2: Continuity

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. A map $f : X \to Y$ is called continuous if

$$\forall U \subset Y \text{ open}, f^{-1}(U) \subset X \text{ is open}$$

Theorem 1.3.2

$f : \mathbb{R}^d \to \mathbb{R}^k$ is continuous if and only if $\forall x \in \mathbb{R}^d, \forall \epsilon > 0, \exists \delta < 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$.

Proof: ($\implies$) Assume that $f : \mathbb{R}^d \to \mathbb{R}^k$ is continuous. Let $x \in \mathbb{R}^d$ and $\epsilon > 0$. We know $B(f(x), \epsilon)$. We know $B(f(x), \epsilon)$ is open, so $f^{-1}(B(x, \epsilon))$ is open by continuity. Then, since $x \in f^{-1}(B(x, \epsilon)), \exists \delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$. Then, $f(B(x, \delta)) \subset B(f(x), \epsilon)$.

($\impliedby$) Assume $\forall x \in \mathbb{R}^d, \forall \epsilon > 0, \exists \delta < 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Let $U \subset \mathbb{R}^d$ be open. Then, for f to be continuous we want to show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$. We have that $f(x) \in U$ is open, so $\exists > 0$, such that $B(f(x), \epsilon) \subset U$. By assumption, $\exists \delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Then $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon)) \subset f^{-1}(U)$. Therefore, $f^{-1}(U)$ is open. Therefore, $\forall U \in \mathbb{R}^k$ open, $f^{-1}(U) \subset \mathbb{R}^d$ is open.

Therefore, $f : \mathbb{R}^d \to \mathbb{R}^k$ is continuous if and only if $\forall x \in \mathbb{R}^d, \forall \epsilon > 0, \exists \delta < 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. $\square$

Example 1.3.2

Let $X = \{-1, 0, 1\}$ and $\mathcal{T} = \{\emptyset, X, \{-1\}, \{1\}, \{-1, 1\}\}$ be a topology on X.
Then $f : \mathbb{R} \to X$ defined by the rule

$$x \mapsto \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 1 \end{cases}$$

Then f is continuous, because

- $f^{-1}(\emptyset) = \emptyset$
- $f^{-1}(X) = \mathbb{R}$
- $f^{-1}(\{-1\}) = (-\infty, 0)$
- $f^{-1}(\{1\}) = (0, \infty)$
- $f^{-1}(\{-1, 1\}) = (-\infty, 0) \cup (0, \infty)$

are all open.

1.4 Bases of Topologies

Definition 1.4.1: The Subspace Topology

Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ be a subset. The subspace topology on Y is

$$\mathcal{T}_{|Y} = \{U \cap Y | U \in \mathcal{T}\}$$

Proof that the subspace topology is, in fact, a topology

Proof:

1. $\emptyset = \emptyset \cap Y \in \mathcal{T}_{|Y}$ and $Y = X \cap Y \subset \mathcal{T}_{|Y}$, so $\mathcal{T}_{|Y}$ satisfies the first condition for being a topology.
2. Let $(V_\alpha)_{\alpha \in I}$ be a family of sets in $\mathcal{T}_{|Y}$. Then, by the definition of the subspace topology, $\forall \alpha \in I, \exists U_\alpha \in \mathcal{T}$ such that $V_\alpha = U_\alpha \cap Y$. Then
$$\bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} (U_\alpha \cap Y) = \bigcup_{\alpha \in I} (U_\alpha) \cap Y$$
By the definition of a topology, $\bigcup_{\alpha \in I} (U_\alpha) \in \mathcal{T}$, so $\bigcup_{\alpha \in I} (V_\alpha) \in \mathcal{T}_{|Y}$. Therefore, $\mathcal{T}_{|Y}$ is stable under unions, satisfying the second condition of being a topology.

3. Let $V_1, ..., V_n \in \mathcal{T}_{|Y}$. Then, $\forall i = 1, ..., n, \exists U_i \in \mathcal{T}$ such that $V_i = U_i \cap Y$. Then

$$V_1 \cap ... \cap V_n = (U_1 \cap Y) \cap ... \cap (U_n \cap Y) = (U_1 \cap ... \cap U_n) \cap Y$$

Then, clearly, $U_1 \cap ... \cap U_n \in \mathcal{T}$, so $V_1 \cap ... \cap V_n \in \mathcal{T}_{|Y}$. Thus, $\mathcal{T}$ is stable under finite intersection, satisfying the third condition of being a topology.
Therefore, $\mathcal{T}_{|Y}$ is a topology. □

Example 1.4.1

1. $\overline{B}(0,1), S(0,1) = \overline{B}(0,1) \setminus B(0,1) = \{x \in \mathbb{R}^d | \, ||x|| = 1\}$
2. Torus (same definition as before). TODO: insert definition and illustration
3. Cantor Set. TODO: insert definition

Theorem 1.4.1 Moore's Law
Let X be a set and $(\mathcal{T}_\alpha)_{\alpha \in I}$ be a family of topologies on X. Then

$$\mathcal{T} = \bigcap_{\alpha \in I} \mathcal{T}_\alpha \subset \mathcal{P}(X)$$

is also a topology and we have

$$U \in \mathcal{T} \iff U \text{ is open } \forall \mathcal{T}_\alpha$$

Proof: TODO: insert proof of Moore's Law here. □

dfnTopology Generated by a SubsetLet X be a set and $A \subset \mathcal{P}(X)$. Then the **topology generated by** A is the intersection of all topologies containing A. Then A is called a subbasis for this topology.

Example 1.4.2
The standard topology on $\mathbb{R}^d$ is generated by the set of open balls. (exercise)
TODO: Prove

Definition 1.4.2

Let X be a set. A basis of X is a $\mathcal{B} \subset \mathcal{P}(X)$ such that

1. $\bigcup_{B \in \mathcal{B}} B = X$. In words, $\mathcal{B}$ *covers* X.
2. $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Moreover, $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ if $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{B}$ generates $\mathcal{T}$.

Example 1.4.3
Open balls in $\mathbb{R}^d$ form a basis for the standard topology.

Proof: The first condition is self evident. To check that the second condition from the definition of a basis holds:
Let $x_1, x_2 \in \mathbb{R}^d, r_1, r_2 \geq 0$. Let $x_3 \in B(x_1, r_1) \cap B(x_2, r_2)$. Set $r_3 = \min(r_1 - d(x_1, x_3), r_2 - d(x_2, x_3))$.
Then $B(x_3, r_3) \subset B(x_1, r_1) \cap B(x_2, r_2)$. Therefore, $\mathcal{B}$ is a basis for $\mathcal{T}$. □

Note:-
Note that any topology satisfies the conditions for being a basis.

Example 1.4.4
On $\mathbb{R}, \mathcal{B} = \{[x, y) | x < y \in \mathbb{R}\}$ is a basis.

Proof:

1. $\mathbb{R} = \bigcup_{i \in \mathbb{R}} [t, t+1)$

2. Let $B_1 = [x_1, y_1)$ and $B_2 = [x_2, y_2) \subset \mathbb{R}$. Let $x \in B_1 \cap B_2$. Then we have

$$\max(x_1, x_2) \leq x < \min(y_1, y_2)$$

so take $x_3 = \max(x_1, x_2)$, and $y_3 = \min(y_1, y_2)$. Then

$$x_3 \leq x < y_3$$

Let $B_3 = [x_3, y_3)$. Then $x \in B_3 \subset B_1 \cap B_2$.

Therefore, $\mathcal{B}$ is a basis of $\mathbb{R}$. □

Theorem 1.4.2
Let X be a set, and $\mathcal{T} \subset \mathcal{P}(x)$. Then $\mathcal{T}$ is a topology if and only if the following hold:

1. $\emptyset, X \in \mathcal{T}$
2. $\mathcal{T}$ is stable under union
3. $\forall U, V \in \mathcal{T}, U \cap V \in \mathcal{T}$

To prove the above theorem, the forward direction is tribial. For the reverse direction, the first two conditions are also just assumed, and the proofs of those statements are trivial. For the third condition, we simply need to induct on n to demonstrate stability under finite intersection. Providing a rigorous proof is left as an exercise to the reader.

Theorem 1.4.3
Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U$$

Proof: ($\implies$) Assume that $\mathcal{B}$ is a basis for some $\mathcal{T}$.Let $\mathcal{T}'$ be the collection of all $U \subset X$ such that ($\forall x \in U, \exists B \in \mathcal{B}$ such that $x \in B \subset U$). Clearly, $\mathcal{B} \subset \mathcal{T}'$. We then want to show that $\mathcal{T}'$ is a topology.

1. $\emptyset \in \mathcal{T}'$ and $X \in \mathcal{T}'$ since $X = \bigcup_{B \in \mathcal{B}} B$
2. Let $(U_\alpha)_(\alpha \in I)$ be a family in $\mathcal{T}'$. Let $x \in U = \bigcup_{\alpha \in I} U_\alpha$.

   TODO: finish inputting proof □

Corollary 1.4.1
Let X be a set, and $A \subset \mathcal{P}(X)$. Let $\mathcal{T}$ be the topology generated by A. Then

$$\mathcal{B} = \{A_1 \cap ... \cap A_n | A_i \in A\} \cup \{\emptyset, X\}$$

is a basis for $\mathcal{T}$. Therefore, $U \subset X$ is $\mathcal{T}$-open if and only if it is $\emptyset$ or X, or if it is a union of finite intersections of elements of A.

Proof:

1. $X \in \mathcal{B}$

TODO: complete proof □

1.5 The Product Topology

Definition 1.5.1: The Product Topology

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. The **product topology** on $X_1 \times X_2$ is the topology generated by

$$\{U_1 \times U_2 | U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}$$

Theorem 1.5.1

Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and let $\mathcal{B}_1, \mathcal{B}_2$ be bases of X_1, X_2 respectively. Then

$$\mathcal{B} = \{B_1 \times B_2 | B_i \in \mathcal{B}_i\}$$

is a basis for the product topology on $X = X_1 \times X_2$. In particular, $\{U_1 \times U_2 | U_i \in \mathcal{T}_i\}$

Proof:

1. $X = X_1 \times X_2 = (\bigcup_{B_1 \in \mathcal{B}_1} B_1) \times (\bigcup_{B_2 \in \mathcal{B}_2} B_2) = \bigcup_{(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2} B_1 \times B_2 = \bigcup_{B \in \mathcal{B}} B$
2. TODO: finish proof

$\square$

Example 1.5.1

The product topology $\mathbb{R} \times \mathbb{R}$ is the standard topology. Open balls are not products of open sets of $\mathbb{R}$.

Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$. Consider the set

$$\mathcal{T} = \{U \subset X | \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$$

Under what conditions on $\mathcal{B}$ is $\mathcal{T}$ a topology?

For the first condition, we have $\emptyset \in \mathcal{T}$, since $\emptyset$ satisfies the condition. Then, for $X \in \mathcal{T}$, we need $\mathcal{B}$ to cover X.

The second condition is satisfied without further specification, since any union of sets in $\mathcal{T}$ inherently contains the elements of B that caused the sets to be in $\mathcal{T}$ in the first place.

For the third condition, we only need $\forall U, V \in \mathcal{T}, U \cap V \mathcal{T}$. For this to hold for this particular $\mathcal{T}$, we consider that $x \in U \cap V$ implies that $\exists B_1, B_2$ such that $x \in B_1 \subset U$ and $x \in B_2 \subset V$. From there, we need the second axiom of the basis for $\mathcal{T}$ to be a topology.

Theorem 1.5.2

Let X be a set, $\mathcal{B}$ be a basis, and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then $U \subset X$ is $\mathcal{T}$-open if $\forall x \in U, \exists B \in \mathcal{B}$ such that $x \in B \subset U$.
Additionally, $N \subset X$ is a neighborhood of x if $\exists B \in \mathcal{B}$ such that $x \in B \subset N$.

Note:-

Let X_1, X_2 be topological spaces. Then $N \subset X_1 \times X_2$ is a neighborhood of $X_1 \times X_2$ for the product topology if and only if $\exists U_1 \subset X_1, U_2 \subset X_2$ both open such that $x \in U_1 \times U_2 \subset N$.

Definition 1.5.2: Cartesian Product

Let $(X_\alpha)_{\alpha \in I}$ be a family of sets. Then $\prod_{\alpha \in I} X_\alpha$ is the set of lists $(x_\alpha)_{\alpha \in I} = x$ such that for all $\alpha \in I$, $x_\alpha \in X_\alpha$ (the α-coordinate of x).

Example 1.5.2

If $I = \{1, 2\}$, then $\prod_{\alpha \in I} X_\alpha = X_1 \times X_2$. If $X_\alpha = X \forall \alpha \in I$, then $\prod_{\alpha \in I} X_\alpha = X^I$ is identified with the set of functions $f : I \to X$.

Definition 1.5.3: The Box Topology

Let $((X_\alpha, \mathcal{T}_\alpha))_{\alpha\in I}$ be a family of topological spaces. The **Box Topology** on $\prod_{\alpha\in I} X_\alpha$ is the topology generated by the basis

$$\mathcal{B} = \{\prod_{\alpha\in I} U_\alpha | U_\alpha \subset X_\alpha \text{ open } \forall \alpha \in I\}$$

Corollary 1.5.1

Let $I = \mathbb{N}$ and $X_\alpha = \mathbb{R}, \forall \alpha \in I$. So

$$\prod_{\alpha\in I} X_\alpha = \mathbb{R}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{R}\}$$

Then no sequence of positive functions can converge to $f : \mathbb{N} \to \mathbb{R}$ defined by $k \mapsto 0$. We want to find a neighborhood N of f such that $f_n \notin N, \forall n \geq 0$.
Set

$$N = (-f_0(0), f_0(0)) \times (-f_1(1), f_1(1)) \times ...(-f_k(k), f_k(k)) \times ...$$

Then $f_0(0) \notin \mathbb{N}$, but $f_k(k) \in \mathbb{N}$.

Definition 1.5.4: General Product Topology

Let $((X_\alpha, \mathcal{T}_\alpha))_{\alpha\in I}$ be a family of topological spaces. The **product topology** on $\prod_{\alpha\in I} X_\alpha$ is the topology generated by the basis

$$\mathcal{B} = \{\prod_{\alpha\in I} U_\alpha | \forall \alpha \in I, U_\alpha \subset \mathcal{T}_\alpha \text{ and } \{\alpha \in I | U_\alpha \neq X_\alpha\} \text{ is finite}\}$$

Theorem 1.5.3

Let $(X_\alpha)_{\alpha\in I}$ be a family of topological spaces. Let $\alpha \in I$. Set $\pi_\alpha : \prod_{\beta\in I} X_\beta \to X_\alpha$, called the **projection map**. Then π_α is continuous for the product topology.

Proof: Let $\alpha \in X_\alpha$ be open. Then, to demonstrate continuity, we need to show that $\pi_\alpha^{-1}(U_\alpha) \subset X = \prod_{\beta\in I} X_\beta$ is open. Then,

$$\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta\in I} U_\beta \text{ where } B_\beta = X_\alpha \text{ if } \beta \neq \alpha$$

$$\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta\in I\setminus\{\alpha\}} X_\beta \times U_\beta$$

□

Note:-

The product topology is generated by

$$\mathcal{A} = \{\pi_\alpha^{-1}(U_\alpha) | \alpha \in I, U_\alpha \subset X_\alpha \text{ is open}\} \subset \mathcal{B}$$

Indeed, let $B = \prod_{\alpha\in I} U_\alpha \in \mathcal{B}$. Then $J = \{\alpha \in I | U_\alpha \neq X_\alpha\}$. Then $B = \bigcap_{\alpha\in J} \pi_\alpha^{-1}(U_\alpha)$ the finite intersection of $\mathcal{A}$.

Theorem 1.5.4

Let $(X_\alpha)_{\alpha\in I}$ be a family of topological spaces. Let $(x_n)_{n\geq 0}$ be a sequence in $\prod_{\alpha\in I} X_\alpha$. Then $\forall n \geq 0, \pi_\alpha(x_n)$ is the α-coordinate of x_n. Then

$$x_n \to x \iff \forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x) \text{ in } X_\alpha$$

Proof: ($\implies$) Assume $x_n \to x$. Let $\alpha \in I$. Then we want to show that $\pi_\alpha(x_n) \to \pi_\alpha(x)$. Let $N \subset X_\alpha$ be a neighborhood of $\pi_\alpha(x)$. Then, since π_α is continuous, we have that $\pi_\alpha^{-1}(N)$ is a neighborhood of x. Since $x_n \to x$, $\exists n_0$ such that $\forall n > n_0, x_n \in \pi_\alpha^{-1}(N)$, by definition. Then, $\forall n > n_0, \pi_\alpha(x_n) \in N$. Therefore, $\forall N$ neighborhood of $\pi_\alpha(x), \exists n_0$ such that $\forall n > n_0, \pi_\alpha(x_n) \in N$. Thus, $\pi_\alpha(x_n) \to \pi_\alpha(x)$. Since $\alpha \in I$ is arbitrary, we have $\forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x)$ in X_α. Therefore,

$$x_n \to x \implies \forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x) \text{ in } X_\alpha$$

($\impliedby$) Now assume $\forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x)$ in X_α. Let N be a neighborhood of $x \in X$. Then $\forall \alpha \in I, \exists U_\alpha \subset X_\alpha$ an open set such that $J = \{\alpha \in I | U_\alpha \neq X_\alpha\}$ is finite and $x \in \prod_{\alpha \in I} U_\alpha \subset N$. Then, since $\pi_\alpha(x_n) \to \pi_\alpha(x)$, we have that $\forall \alpha \in J, \exists n_\alpha$ such that $\forall n > n_\alpha, \pi_\alpha(x_n) \in U_\alpha$. Set $n_0 = \max\{n_\alpha | \alpha \in J\}$, which is well defined since J is finite. Then $\forall n > n_0, x_n \in \bigcap_{\alpha \in J} \pi_\alpha^{-1}(U_\alpha) = \prod_{\alpha \in I} U_\alpha \subset N$. So $x_n \to x$.

Therefore,

$$x_n \to x \iff \forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x) \text{ in } X_\alpha$$

□

1.6 Metric Spaces

Definition 1.6.1: Metric Space

Let X be a set. A **metric** on X is a map $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$

1. d is symmetric:
$$d(x, y) = d(y, x)$$
2. d is positive definite
$$d(x, y) \geq 0 \text{ with equality if and only if } x = y$$
3. d satisfies the triangle Inequality
$$d(x, z) \leq d(x, y) + d(y, z)$$

Then the pair (X, d) is called the **metric space**.

Example 1.6.1 (Some examples)

The Euclidean metric, or L^2 metric on $\mathbb{R}^k$

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + ... + |x_k - y_k|^2}$$

L^1-metric:

$$d(x, y) = |x_1 - y_1| + ... + |x_k - y_k|$$

L^∞-metric:

$$d(x, y) = \max(|x_1 - y_1|, ..., |x_k - y_k|)$$

Definition 1.6.2: The Metric Topology

Let (X, d) be a metric space. Then, $\forall x \in X, r \geq 0$ we define the open ball as

$$B_d(x, r) = \{y \in X | d(x, y) < r\}$$

The **metric topology** induced by d is the topoogy generated by the set of open balls, which form a basis.
Then, $N \subset X$ is a neighborhood of x is $\exists \epsilon > 0$ such that $B_d(x, \epsilon) \subset N$.
A topological space is **metrizable** if $\exists d$ a metric which induces it.

Theorem 1.6.1

Let X be a set and d, d' two metrics with corresponding metric topologies $\mathcal{T}, \mathcal{T}'$. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ if and only if

$$\forall x \in X, r > 0, \exists \epsilon > 0 \text{ such that } B_d(x, \epsilon) \subset B_{d'}(x, r)$$

Proof: ($\Longrightarrow$) Assume $\mathcal{T}$ is finer than $\mathcal{T}'$. Let $x \in X$ and $r > 0$. Then $B_{d'}(x, r)$ is $\mathcal{T}'$-open and hence is $\mathcal{T}$-open since $\mathcal{T}' \subset \mathcal{T}$. Then $B_{d'}(x, r)$ is a $\mathcal{T}$ neighborhood of x, so $\exists \epsilon > 0$ such that $B_d(x, \epsilon) \subset B_{d'}(x, r)$.

($\Longleftarrow$) Assume $\forall x \in X, r > 0, \exists \epsilon > 0$ such that $B_d(x, \epsilon) \subset B_{d'}(x, \epsilon)$. Let $U \in \mathcal{T}'$. We then want to show that $U \in \mathcal{T}$. Take $x \in U$. As U is a $\mathcal{T}'$-neighborhood of x, we have that $\exists \epsilon > 0$ such that $B_{d'}(x, \epsilon) \subset U$. By assumption, $\exists \delta > 0$ such that $B_d(x, \delta) \subset B_{d'}(x, \epsilon) \subset U$. Therefore, U is a $\mathcal{T}$-neighborhood of x, so $U \subset \mathcal{T}$. ☐

Example 1.6.2

The metrics d, d_1, and d_∞ all induce the same topology. For instance
TODO: Complete example

Theorem 1.6.2

Let (X, d_X) be a metric space, and Y a topological space. A sequence $(x_n)_{n \geq 0} \subset X$ converges to $x \in X$ if $\forall \epsilon > 0 \exists n_0$ such that $x_n \in B(x, \epsilon) \forall n > n_0$. Then

$$f : X \to Y \text{ is continuous } \iff \forall x \in X, \forall N \subset Y \text{ neighborhood of } f(x), \exists \epsilon > 0 \text{ such that } f(B(x, \epsilon)) \subset N$$

Theorem 1.6.3

Let X, Y be topological spaces, and let $f : X \to Y$ be a mapping.

1. If f is continuous then $\forall (x_n)_{n \geq 0} \subset X$ converging to x, we have $f(x_n) \to f(x)$.
2. Suppose X is metrizable. If $x_n \to x$ implies $f(x_n) \to f(x)$, then f is continuous.

Proof:

1. Assume f is continuous. Let $x_n \to x \in X$. Then, we want to show that $f(x_n) \to f(x)$. Let $N \subset Y$ be a neighborhood of $f(x)$. Then, since f is continuous, $f^{-1}(N)$ is a neighborhood of x ($\exists f(x) \in U \subset N, f^{-1}(U)$ is open and a subset of $f^{-1}(N)$). Since $x_n \to x, \exists n_0$ such that $\forall n > n_0, x_n \in f^{-1}(N)$. Then $f(x_n) \in N$.
2. Let X be metrizable, and assume $x_n \to x$ implies $f(x_n) \to f(x)$. Let d be a metric on X inducing the topology. Let $x \in X$ and $N \subset Y$ be a neighborhood of $f(x)$. Suppose for contradiction that $\forall \epsilon > 0, f(B(x, \epsilon)) \not\subset N$. In particular, $\forall n \geq 1, \exists x_n \in B(x, \frac{1}{n})$ such that $f(x_n) \notin N$. Now apply our supposition ot $\epsilon = \frac{1}{n}$. Then we can check that $f(x_n) \to f(x)$ and $x_n \to x$ contradicts the statement that $\exists n \geq 1, \exists x_n \in B(x, \frac{1}{n})$ such that $f(x_n) \notin N$.

☐

Definition 1.6.3: Homeomorphism

Let X, Y be a topological space. A homeomorphism from X to Y is a bijection $f : X \to Y$ such that $U \subset X$ is open if and only if $f(U) \subset Y$.
X and Y are homeomorphic if $\exists$ a homeomorphism.

Corollary 1.6.1 (of Theorem 1.6.3)

Let X, Y be topological spaces. Then $f : X \to Y$ is a homeomorphism if and only if f is a bijection such that $x_n \to x$ if and only if $f(x_n) \to f(x)$.

Definition 1.6.4: Boundary Metric of d

Let (X, d) be a metric space. Then $\overline{d}(x, y) = \min(d(x, y), 1) \leq 1$ is a metric that defines the same topology.

Proof: Symmetry and positive definitness are trivial.
To prove the triangle inequality:
Let $x, y, z \in X$. Case 1: assume $d(x, y) \geq 1$ of $d(y, z) \geq 1$. Then $\overline{d}(x, y) + \overline{d}(y, z) \geq 1 \leq \overline{d}(x, z)$.
Case 2: $d(x, y) = \overline{d}(x, y)$ and $d(y, z) = \overline{d}(y, z)$. Then

$$\overline{d}(x, y) + \overline{d}(y, z) = d(x, y) + d(y, z) \geq d(x, z) \geq \overline{d}(x, z)$$

Therefore, $\overline{d}$ satisfies the triangle inequality. $\square$

Definition 1.6.5: Diameter

Let (X, d) be a metric space, and $A \subset X$. Then we define diameter as

$$\operatorname{diam}_d(A) = \sup\{d(x, y) | x, y \in A\}$$

Example 1.6.3
$\operatorname{diam}_{\mathbb{R}}((0, 1) \cup \{2\}) = 2$

Theorem 1.6.4
Let $((X_\alpha, d_\alpha))_{\alpha \in I}$ be a family of metric spaces. Suppose $\exists c > 0$ such that $\forall \alpha \in I, \operatorname{diam}_{d_\alpha} X_\alpha \leq c$. Then

$$d((x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}) = \sup\{d_\alpha(x_\alpha, y_\alpha) | \alpha \in I\}$$

is well defined and is a metric (where $((x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}) \in \prod_{\alpha \in I} X_\alpha)$. Moreover, the metric topology is finer than the product topology. We have equality if and only if $\forall \epsilon > 0, I_\epsilon = \{\alpha \in I | \operatorname{diam} X_\alpha \geq \epsilon\}$ is finite.

Proof:

1. First, we must prove that d is a metric. Proving that d is symmetric and positive definite is trivial, so we must demonstrate that d satisfies the triangle inequality.

   Let $(x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}, (z_\alpha)_{\alpha \in I}$ be sequences such that $x_\alpha, y_\alpha, z_\alpha \in X_\alpha, \forall \alpha \in I$. Then we want to show that

   $$d((x_\alpha)_{\alpha \in I}, (z_\alpha)_{\alpha \in I}) + d((z_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}) \leq d((x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I})$$

   For convenience, let $x = (x_\alpha)_{\alpha \in I}, y = (y_\alpha)_{\alpha \in I}$, and $z = (z_\alpha)_{\alpha \in I}$. We have that $\forall \alpha \in I$,

   $$d(x, z) + d(z, y) \geq d_\alpha(x_\alpha, z_\alpha) + d_\alpha(z_\alpha, y_\alpha) \geq d_\alpha(x_\alpha, y_\alpha)$$

   So $d(x, z) + d(z, y)$ is an upper bound of $\{d_\alpha(x_\alpha, y_\alpha | \alpha \in I)\}$. Then $d(x, y)$ is the least upper bound of this set, so

   $$d(x, y) \leq d(x, z) + d(z, y)$$

   Therefore d is a metric and is well defined.

2. Next, to demonstrate the "moreover.." statement, we must demonstrate that the metric topology, $\mathcal{T}$, is finer than the product topology, $\mathcal{T}'$. That is, we must show that $\mathcal{T}' \subset \mathcal{T}$.

   We have that a basis for $\mathcal{T}$ is the set $\mathcal{B} = \{\text{open balls}\}$, and a basis for $\mathcal{T}'$ is $\mathcal{B}' = \{\prod_{\alpha \in I} B(x_\alpha, r_\alpha) | r_\alpha = \infty$ for all but finitely many $\alpha\}$. Let $B' = \prod_{\alpha \in I} B(x_\alpha, r_\alpha) \in \mathcal{B}'$. Let $(y_\alpha)_{\alpha \in I} \in \mathcal{B}'$. Set $\epsilon = \inf\{r_\alpha - d_\alpha(x_\alpha, y_\alpha)\}$. Note that this is necessarily greater than 0, and equal to ∞ for "most" α's. Then

   $$B((y_\alpha)_{\alpha \in I}) = \prod_{\alpha \in I} B(y_\alpha, \epsilon) \subset B(x_\alpha, r_\alpha) \subset B'$$

   So B' is a neighborhood of $(y_\alpha)_{\alpha \in I}$. Therefore $B' \in \mathcal{T}$, so $B' \in \mathcal{T}'$. Hence, the metric topology $\mathcal{T}$ is finer than $\mathcal{T}'$.

3. Suppose $I_\epsilon = \{\alpha \in I | \text{diam} X_\alpha \geq \epsilon\}$ is finite $\forall \epsilon > 0$. We want to show that $\mathcal{T} \subset \mathcal{T}'$. Let $x = (x_\alpha)_{\alpha \in I} \in X$ and $r > 0$. Then, let $y \in B(x, r)$. Set $\epsilon = r - d(x, y)$. Then $B(y, \epsilon) \subset B(x, r)$. Then we want to find N, a $\mathcal{T}'$-neighborhood of y contained in $B(y, \epsilon)$. So, $\forall \alpha \in I_{\epsilon/2}$, set $\epsilon_\alpha = \frac{\epsilon}{2}$. Then, $\forall \alpha \in I \setminus I_{\epsilon/2}$, set $\epsilon_\alpha = \infty$. Then $\prod_{\alpha \in I} B(y_\alpha, \epsilon) \subset B(y, \epsilon)$, and $\prod_{\alpha \in I} B(y_\alpha, \epsilon)$ is a $\mathcal{T}'$-neighborhood y. Indeed, if $(z_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} B(y_\alpha, \epsilon_\alpha)$. Then $\forall \alpha \in I$, if $\alpha \in I_\epsilon$, then $d_\alpha(y_\alpha, z_\alpha) \leq \epsilon/2$. If $\alpha \in I \setminus I_\epsilon, d_\alpha(y_\alpha, z_\alpha) \leq \epsilon/2$ as $\text{diam} X_\alpha \leq \epsilon/2$.

4. TODO: finish proof later

□

Theorem 1.6.5

Let (X, d) be a metric space. Then

$$x_n \to x \text{ in the metric topology} \iff d(x_n, x) \to 0 \text{ in the standard topology of } \mathbb{R}$$

Example 1.6.4

In $\mathbb{R}^{\mathbb{Z}_{\geq 0}} = \{f : \mathbb{Z}_{\geq 1} \to \mathbb{R}\} = \{(x_n)_{n \geq 1} \text{ sequence in } \mathbb{R}\}$. Then

$$d((x_n)_{n\geq 1}, (y_n)_{n \geq 1}) = \sup\{\min(|x_n - y_n|, 1) | n \geq 1\}$$

is a metric. It does not induce the product topology, but rather the **uniform topology**.
This works for $\mathbb{R}^I$, where I is any set. For instance, $\mathbb{R}^\mathbb{R} = \{f : \mathbb{R} \to \mathbb{R}\}$. Then

$$d(f, g) = \sup\{\min(|f(x) - g(x)|, 1) | x \in \mathbb{R}\}$$

However, $d'((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) = \sup\{\min(|x_n - y_n|, \frac{1}{n}) | n \geq 1\}$ induces the product topology.

Note:-

In the space of functions from $\mathbb{R} \to \mathbb{R}$, the ball of radius ϵ centered at a function f, $B(f, \epsilon)$, is the set of functions whose graph is contained in the "ϵ-tubular neighborhood" of f.

Chapter 2

Metric Space

2.1 Definition

Definition 2.1.1: Metric Space X

A set X with a function d $X \times X \to \mathbb{R}_{\geq 0}$ such that

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Notice that there is no homogeneity condition, and it does ot make sense as we don't have a field. In fact there is no notion of addition. But the condition **??** of norm has to be satisfied by this distance. Also we don't have a translational condition i.e. distance between x, y and distance between $x + v, y + v$ has to be same. Hence

Note:-

A metric space need not be a vector space. So it doesn't need a zero, or a notion of addition or scalar multiplication.

If I take a metric space and take any subset of it. And those three conditions of distance functions are still satisfied.

Note:-

Any subset of metric space is a metric space under the same distance function.

2.2 Open and Closed Ball and Set

Definition 2.2.1: Open Ball and Closed Ball in a Metric Space

An open ball of radius r with center $c \in X$ in a metric space X is

$$B_r(c) = \{x \in X \mid d(c, x) < r\}$$

and a closed ball is

$$\overline{B_r(c)} = \{x \in X \mid d(c, x) \leq r\}$$

Definition 2.2.2: Open Set and Closed Ball in a Metric Space

An open set in a metric space X is one of the form of union of some open balls and a closed set in a metric space X is one of the form of $X\backslash$some open sets

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Example 2.2.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup\limits_{x\in X} B_r(x)$ (Any $r>0$ will do)
- $B_r(x)$ is open

Closed Set:
- $X,\ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Question 1

Is the set $x-$axis$\backslash\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Now this works well for points which are above or below the $x-$axis. But for origin no matter how small the ball we take it willl have points from $x-$axis. Hence the set is not a closed set.

Question 2

Any continuous path in $\mathbb{R}^2$ is closed where path $= f:[0,1]\to\mathbb{R}^2$

Solution: This is true. To be proved later.
Analogous to: For continuous function $f:[0,1]\to\mathbb{R}$, the image is a closed interval

Question 3

If i take $X = x-$axis $\cup$ $y-$axis then is it open

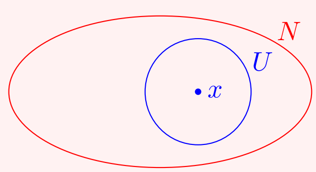
Solution: Yes because here the space is only the union of those two axis. So any ball would be like a cross or line but it just as the metric space given to us. [It is open for this metric space but not open in $\mathbb{R}^2$]

Note:-

If $S\subset X$, then S itself has a collection of open sets of S by containing S as a metric space.

Definition 2.2.3: Neighborhood

For a point x in metric space X, a neighborhood of x is a set N such that $x\in$an open set $U\subset N$

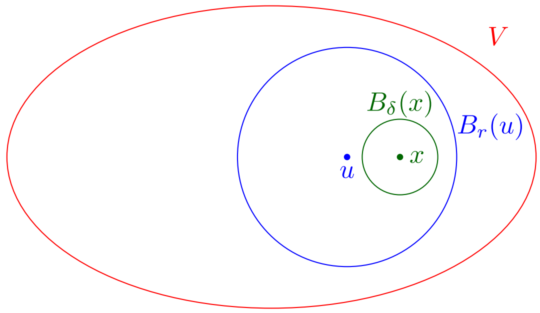


If N itself is open. then we say that N is an open neighborhood of x

Theorem 2.2.1

If $x \in$ open set V then $\exists\, \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leq d(u, x) + d(x, y) < d + \delta < r$$

$\square$

Note:-

V is open $\iff \bigcup_{x \in V} B_r(x)$ (where r depends on x)

Theorem 2.2.2

Let X be a metric space.

1. Union of open sets is open
2. Intersection of two open sets is open

Analogues to these as we are just taking complement of the open sets

1′. Arbitrary intersection of closed sets is closed

2′. Finite union of closed sets is closed.

Proof: 1. Let $\{V_\alpha\}_{\alpha \in I}$ be a collection of open sets where I is an index set. We want ti show $\bigcup_{\alpha \in I} V_\alpha$ is open in X. Since each V_α is open $V_\alpha = \bigcup_{\beta \in J_\alpha} B_{r_\beta}(c_\beta)$ Then

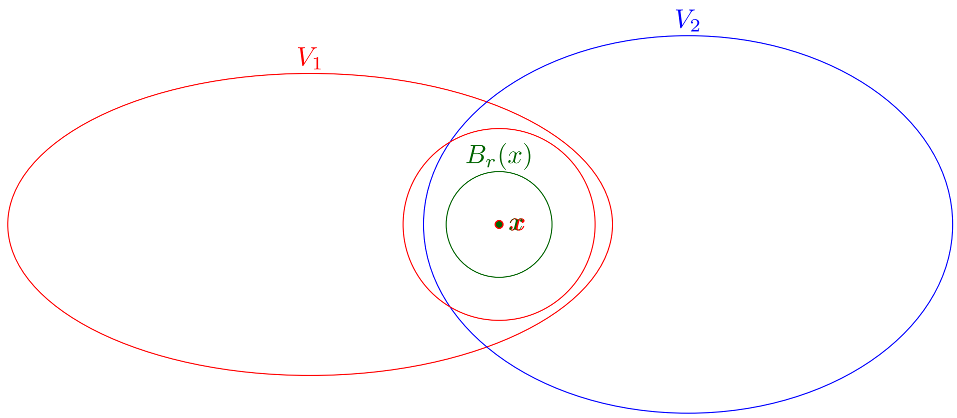
$$\begin{aligned}\bigcup_{\alpha \in I} V_\alpha &= \bigcup_{\alpha \in I} \bigcup_{\beta \in J_\alpha} B_{r_\beta}(c_\beta) \\ &= \bigcup_{\beta \in \sqcup J_\alpha} B_{r_\beta}(c_\beta)\end{aligned}$$

which is still a union of balls

$\square$

2. The statement implies intersection of finite number of open sets is open. We can prove this by induction.

We will do by showing that for each $x \in V_1 \cap V_2$ $\exists\, r > 0$ s.t. $B_r(x) \subset V_1 \cap V_2$



As $x \in V_1$ $\exists$ r_1 such that $x \in B_{r_1}(x) \subset V_1$. Similarly $x \in V_2$ $\exists$ r_2 such that $x \in B_{r_2}(x) \subset V_2$. Take $r = \min\{r_1, r_2\}$. Thus we have $x \in B_r(x) \subset V_1 \cap V_2$
The second part for closed sets are left as exercise $\square$

2.3 Topological Space

Definition 2.3.1: Topological Space

A topological space is a set X together with a collection of subsets of X (i.e. a subset of the power set of X) that is closed under taking arbitrary unions and finite intersections. This collection is called a topology on X

Note:-

Union means $\bigcup\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \exists\ \alpha \text{ s.t. } x \in S_\alpha\}$
Intersection means $\bigcap\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \forall \alpha,\ x \in S_\alpha\}$

Question 4

Suppose i have a topological space X under given some topology. Is the entire set open ? And that the empty set is open ?

Solution: If $I = \phi$, $\bigcup\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \exists\ \alpha \in I \text{ s.t. } x \in S_\alpha\}$ gives ϕ and $\bigcap\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \forall \alpha \in I,\ x \in S_\alpha\}$ gives X because $\forall\ \alpha \in I$ condition is vacuously true for each $x \in X$.

Note:-

Intersection of empty families are not defined in set theory. This brings a very important point. In a set theory you have to have a universe. (Set theory have to avoid paradoxes, Russel Paradox) At the beginning you construct a large enough universe and you taking subsets only from that universe. Notice all subsets we are considering here are subsets of X and here we defined how we union and intersection mean. Though it still this asks what our axioms of set theory. So you can change the part of the definition of topological space like this "...with a collection of subsets of X including the empty set and the whole space..."

(If you don't like this as it is)

Note:-

If S is a subset of metric space X, then S is itself a metric space and as such open/closed sets as subsets of metric space

Question 5

Is there any connection between being open in X and being open in S (Similar question for closed)

Solution: Let $x \in S$. Now, Ball of radius r in $S = S\cap$ Ball of radius r in X. Therefore

$$\begin{aligned}\text{Open Set in } S &= \bigcup \text{ Balls in } S\\ &= \bigcup (\text{Balls in } X \cap S)\\ &= \left(\bigcup \text{ Balls in } X\right) \cap S\\ &= \text{Open set } X \cap S\end{aligned}$$

Part 2 is left as exercise

Corollary 2.3.1

If $S \subset X$ is open in X then a subset T of S is open in $S \iff T$ is open in X

Corollary 2.3.2

If $S \subset X$ is closed in X then a subset T of S is closed in $S \iff T$ is closed in X

Definition 2.3.2: Subspace of a Topological Space X

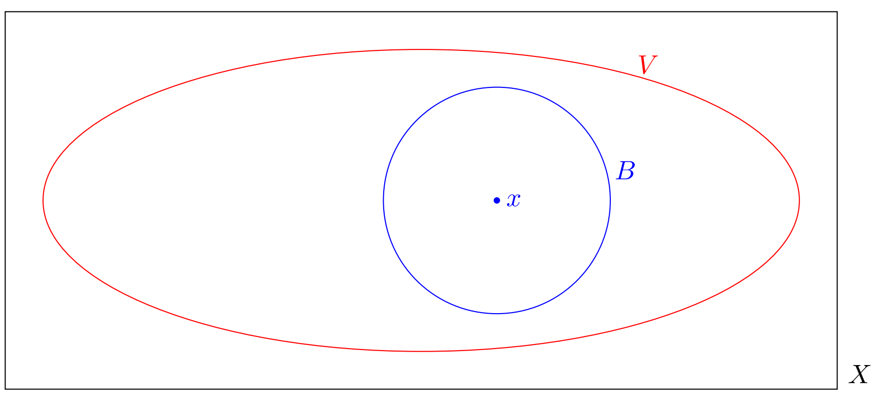
For any subset S of a topological space X, the collection $S \cap U$, U open in X is called a subspace.

Question 6

Prove that subspace of a metric space X defines a topology on X

Wrong Concept 2.1

If $x \in$ open V then there exists $r > 0$ such that $x \in B_r(x) \subset V$



Idea: Why not we take $r = \inf\{\text{distance from } x \text{ to boundary of ball } B\}$.

Now we first have to ensure $r > 0$. Suppose that's true.

Then we have to define boundary. What is boundary, We can give a reasonable definition (Boundary has already a definition but we don't know that for now). Let boundary of $B = \{x \in X \mid d(c, x) = \delta\}$ Now this definition is not proper for our purpose.Because if we take union of all balls in V then we will have lots of points as boundary but part of them should not be considered as boundary. Even if we take this definition.

Then the big question comes/ We are taking a infimum of a certain set of real numbers. The very first question arises is whether this set is nonempty. For example if we take B to be the metric space it self we have no boundary.

Questions which come thorough this.

- Is there a meaningful way to define boundary
- Can we modify the idea

Chapter 3

Continuity in Metric Space

3.1 Limit Point and Closure

Definition 3.1.1: Limit Point

$S \subset X$ is a metric space. We say that $x \in X$ is a limit point of S if $\exists$ a sequence $\{s_n\}$ with all $s_n \in S \setminus \{x\}$ such that $s_n \to x$ (each s_n is different from x)

Theorem 3.1.1

x is a limit point of $S \iff$ every neighborhood of x in X contains a point of S other than S.

Proof: **If Part:**

Let x be a limit point of S. Therefore take a sequence $\{s_n\}$ in $S \setminus \{x\}$ with $s_n \to x$.

To prove what we want it is enough to show that $B_r(x) \cap S$ contains a point other than x. As $s_n \to x$, $\exists$ N s.t. $\forall\ n > N$ $d(x, s_n) < r$ i.e. $s_n \in B_r(x)$. In particular $s_n]inB_r(x) \cap (S \setminus \{x\})$

Only If Part:

We need to produce a sequence $\{s_n\} \in S \setminus \{x\}$ with $\lim s_n = x$. Take $s_n \in B_{\frac{1}{n}}(x) \cap (S \setminus \{x\})$ See that $\lim\limits_{n\to\infty} s_n = x$. This is essentially because $\frac{1}{n} \to 0$.

Complete the rest of the proof.

$\square$

Definition 3.1.2: Closure

Given a topological space X and $S \subset X$, the closure of the set S is $\overline{S}$ the smallest closed set containing S.

Theorem 3.1.2

$$\begin{aligned}
\overline{S} &= \text{Smallest closed set of } X \text{ containing } S = \text{A} \\
&= S \cup (\text{limit points of } S) = \text{B} \\
&= \{x \in X \mid x = \lim_{n\to\infty} \text{ for some sequence } \{s_n\} \text{ in } S\} = \text{C} \\
&= \{x \in X \mid \text{Every neighborhood of } x \text{ intersects } S\} = \text{D}
\end{aligned}$$

Proof: ***$A \subset D$***

$$\begin{aligned}
A^c &= \bigcup (\text{All open set } V \text{ s.t. } V \cap S = \phi) \\
D^c &= \{x \in X \mid \exists \text{ open neighborhood of } x, B \text{ s.t. } B \cap S = \phi\}
\end{aligned}$$

Clearly for all $x \in D^c$, $x \in A^c$. Hence $D^c \subset A^c \implies A \subset D$

D* $\subset$ *B

Take $x \in D$. Suppose $x \notin S$. Now any neighborhood of x intersects S in a point hence it has to be a different point from x since $x \notin S$. Therefore x is a limit point of S. $D \subset B$

B* $\subset$ *C

If $x \in S$ then take a sequence

$\square$

Question 7

What does it mean to be smallest closed set containing the set S here ?

Solution: $\bigcap$ All closed sets containing S is automatically closed and hence the smallest closed set containing S.

Proof: For proof of Theorem 3.1.2 notice A,B,C,D all contains S (obvious).

Note:-

We don't need to show B,C,D are closed. We can also take the sets element wise and show each set is a subset of the other. This may simplify our way of proof. (exercise)

Now see A and D completely deal with topology. A is about closed sets and D is about open sets. So A and D close to each other. Now by the 3.1.1 we have equivalence of C and D. So we can prove like this

$$A \iff D \iff B \,\&\, C$$

Left as exercise $\square$

Note:-

For these kind of proofs instead of looking for the most efficient way try to find a path that allows you to go from anywhere to anywhere

3.2 Continuity

Definition 3.2.1: Continuity

$f : X \to Y$ function between metric spaces is continuous at $a \in X$ if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ s.t.

$$\begin{array}{ccc} d(x,a) < \delta & \implies & d(f(a), f(x)) < \epsilon \\ \Updownarrow & & \Updownarrow \\ x \in B_\delta(x) & \implies & f(x) \in B_\epsilon(f(a)) \end{array}$$

That means f^{-1}(Any ball around $f(a)$) $\supset$ Ball around a.

So $f : X \to Y$ is continuous at all points $\iff$ f^{-1}(Any ball intersecting the range) $\supset$ A ball

Note:-

We can not say f^{-1}(Any ball) because because we need a ball that contains a point in the range

Theorem 3.2.1

f is continuous $\iff$ f^{-1}(Any open set in Y) is open in X

Proof: **If Part:-**

It is enough to show f^{-1}(Any ball) is open on X because f^{-1} preserves unions $f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha \left(f^{-1}(V_\alpha)\right)$

Let B is any open set (as its conceptually simpler to take open set here instead of a ball) in Y. Let $a \in f^{-1}(B)$. Hence we can say $f(a) \in B$. Since B is an open set we can say there is a ball $B_\epsilon(f(a)) \subset B$. Since f is continuous $\exists\ \delta$ such that $f(x) \in B_\epsilon(f(a))$ whenever $x \in B_\delta(a)$. Now $f^{-1}(B) \supset f^{-1}\Big(B_\epsilon(f(a))\Big) \supset B_\delta(a)$ Hence $f^{-1}(B)$ is open.

Only If Part:-

Lets prove continuity ar $a \in X$. We are given that $f^{-1}\Big(B_\epsilon(f(a))\Big)$ is open and obviously contains a. Therefore $f^{-1}\Big(B_\epsilon\big(f(a)\big)\Big)$ contains a ball around a. Take $\delta =$ Radius of the ball. □

Question 8

For a metric space X, show that $\overline{S} = \{x \in X \mid \lim\limits_{n\to\infty} s_n = x\}$ for some sequence $\{s_n\}$ in S.

Question 9

For a function $f : X \to Y$ between metric spaces, show that the followings are equivalent.

1. f is continuous
2. f^{-1}(Open Set) is open
3. f^{-1}(Closed Set) is closed
4. $f(\overline{S}) = \overline{f(S)}$
5. $x_n \to x \implies f(x_n) \to f(x)$

One or more of the above are wrong so check if they are true and if not then find the true statement.

Solution: 4 is wrong. How to correct and rest is left as exercise

Question 10

For $f : X \to Y$ any set map

(i) f^{-1} preserves unions, intersections, complements

(ii) Is there any condition on f under which f possesses the property above ?

Example 3.2.1 (Continuous Function)

1. Any constant function.
2. $X \xrightarrow{f} Y \xrightarrow{g} Z$ f, g continuous $\implies g \cdot f$ is continuous
3. Is $S \subset X$ then $S \xrightarrow{\text{Inclusion}} X$ is continuous
4. Projection $\underset{(x_1,x_2,\cdots,x_n)\mapsto x_i}{\mathbb{R}^n \to \mathbb{R}}$

   More generally for example $\underset{(x,y,z)\mapsto(x,x,y,y)}{\mathbb{R}^3 \to \mathbb{R}^4}$
5. Map from metric space to euclidean space.

$$\left.\begin{aligned} &X \to \mathbb{R}^n \\ &x \mapsto (f_1(x), f_2(x), \cdots, f_n(x)) \end{aligned}\right\} \underset{\text{each } f_i \text{is continuous}}{\overset{f \text{ is continuous}}{\iff}}$$

6. $\mathbb{R}\times\mathbb{R}\to\mathbb{R}:\ (x,y)\mapsto x\pm y, xy$ are continuous.

   We need to prove $x_n\to x$ and $y_n\to y$ in $\mathbb{R} \implies \begin{cases} x_n\pm y_n\to x\pm y \\ x_ny_n\to xy\end{cases}$

   $\mathbb{R}\setminus\{0\}\to\mathbb{R}:\ x\mapsto\frac{1}{x}$ is continuous

7. sum and product of two continuous real valued function on X are continuous

$$f,g:X\xrightarrow{f,g}\mathbb{R}\text{ continuous}\implies \underset{x}{X}\ \underset{\longmapsto}{\xrightarrow{f,g}}\ \underset{(f(x),g(x))}{\mathbb{R}\times\mathbb{R}}\xrightarrow{+}\mathbb{R}$$

$$f:X\to\mathbb{R}\implies\frac{1}{f}:\underbrace{X\setminus f^{-1}(0)}_{\text{open set in }X}\to\mathbb{R}\text{ is continuous}$$

   $\{0\}$ is closed in $\mathbb{R}$, so $f^{-1}(0)$ is closed in X by continuity of f

   Therefore any polynomial in continuous real valued functions on X is continuous.

8. **Special Case:**

   - $\mathbb{R}^n\xrightarrow{T}\mathbb{R}^m$ linear map is continuous where $(x_1,x_2,\cdots,x_n)\longmapsto(a_{11}x_1+\cdots+a_{1n}x_n, a_{21}x_1+\cdots+a_{2n}x_n,\cdots,a_{m1}x_1+\cdots+a_{mn}x_n)$

$$\text{Matrix of }T=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}$$

   - $\underset{A}{M_{n\times n}(\mathbb{R})}\ \underset{\longmapsto}{\to}\ \underset{det(A)}{\mathbb{R}}$ is continuous

     $\frac{1}{\det}:GL_n(\mathbb{R})\to\mathbb{R}$

     Here $M_{n\times n}$ is a vector space of dimension n^2 in which $GL_n(\mathbb{R})=\{A\mid\det(A)\neq 0\}$ is an open set.

   - $GL_n(\mathbb{R})\underset{A\longmapsto A^{-1}}{\to}GL_n(\mathbb{R})$ is continuous.

9. Any norm (f) on $\mathbb{R}^n$ is uniformly continuous w.r.t usual topology on $\mathbb{R}^n$ i.e. $f:\mathbb{R}^n\to\mathbb{R}$ is continuous w.r.t usual norms ($\|\cdot\|=p0$norm for $p=1,2,\infty$) on $\mathbb{R}^n(\|\cdot\|)$ and $\mathbb{R}(|\cdot|)$

Theorem 3.2.2

Any norm (f) on $\mathbb{R}^n$ is uniformly continuous w.r.t usual topology on $\mathbb{R}^n$ i.e. $\forall\ \epsilon>0\ \forall\ x,y\in\mathbb{R}^n\ \exists\ \delta>0$ s.t. $\|x-y\|<\delta\implies|f(x)-f(y)|<\epsilon$

Proof:

$$\left.\begin{array}{r}f(x)\le f(y)+f(x-y)\\ \|\\ f(y)\le f(x)+f(y-x)\end{array}\right\}|f(x)-f(y)|\le f(x-y)$$

Let $x=\sum x_ie_i$ and $y=\sum y_ie_i$ where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$.

$$f(x-y)=f\left(\sum(x_i-y_i)e_i\right)\le\sum f\left((x_i-y_i)e_i\right)=|x_i-y_i|f(e_i)$$

Notice $\sum|x_i-y_i|=\|x-y\|_1$. Let $M=\max\{f(e_i)\}$ Then

$$|f(x)-f(y)|\le f(x-y)\le M\|x-y\|_1$$

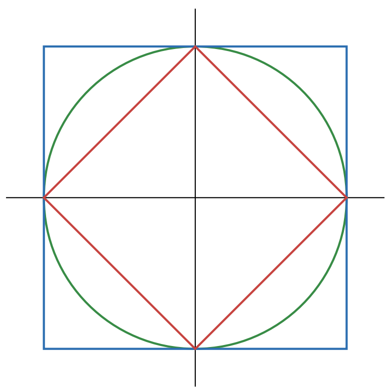
Thus $\|x-y\|<\frac{\epsilon}{M}\implies|f(x)-f(y)|<\epsilon$ $\square$

Chapter 4

Equivalence of Norms

We back to Normed Linear Space for a little while.

In $\mathbb{R}^n$, $u = (u_1, u_2, \cdots, u_n)$ where each $u_i \in \mathbb{R}$. we have $p-$norm: $\|u\|_p = \left(\sum_i |u_i|^p\right)^{\frac{1}{p}}$ where $1 \le p \le \infty$.
Balls in $\mathbb{R}^2$ w.r.t. $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$.



Observe: A set V in $\mathbb{R}^2$ is

$$\begin{aligned}
\text{open w.r.t. } \|\cdot\|_1 \quad &\iff \quad V = \bigcup_{u \in V} \text{Box in } V \text{ centered box} \\
\text{open w.r.t. } \|\cdot\|_2 \quad &\iff \quad V = \bigcup_{u \in V} \text{Diamond in } V \text{ centered box} \\
\text{open w.r.t. } \|\cdot\|_\infty \quad &\iff \quad V = \bigcup_{u \in V} \text{Circle in } V \text{ centered box}
\end{aligned}$$

Definition 4.1: Equivalence of Norms

Suppose $\|\cdot\|$, $\|\cdot\|'$ are two norms in vector space V, We say that the two norms are equivalent if there are constants $\alpha, \beta > 0$ s.t.

$$\alpha\|x\|' \le \|x\| \le \beta\|x\|'$$

Example 4.0.1 (Norm Equivalence)

1. $\boldsymbol{p = \infty}$ **and** $\boldsymbol{p = 1}$

   $\|x\|_\infty = \max\{|x_i| \mid 1 \le i \le n\} \le \|x\|_1 = \sum_i |x_i|$

   $\|x\|_\infty \ge$ each $|x_i| \implies n\|x\|_\infty \ge \|x\|_1$

   Hence

   $$\|x\|_\infty| \le \|x\|_1 \le n\|x\|_\infty \text{ and } \frac{1}{n}\|x\|_1 \le \|x\|_\infty \le \|x\|_1$$

2. $\boldsymbol{p = \infty}$ **and** $\boldsymbol{p = 2}$
$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

Theorem 4.1
All norms on a finite dimensional vector space are equivalent

Proof: Proved in Theorem 5.2.7 □

Theorem 4.2
Suppose $\|\cdot\|$ and $\|\cdot\|'$ are equivalent on a vector space V. Then

(i) $\{x_n\} \to x$ w.r.t. $\|\cdot\| \iff \{x_n\} \to x$ w.r.t $\|\cdot\|'$

(ii) $S \subset V$ is open w.r.t $\|\cdot\| \iff S$ is open w.r.t $\|\cdot\|'$

Proof: For both proofs if we just prove one direction the we are done actually since we can just replace the words to prove for opposite direction,

(i) **If Part:-**

Since $\|\cdot\|, \|\cdot\|'$ are equivalent we have $\exists\ \alpha, \beta$ such that $\alpha\|x\|' \leq \|x\| \leq \beta\|x\|'$. So if we show $\alpha \left\|x_n - x\right\| < \|x_n - x\| < \alpha\epsilon$ we are done.

Let $\{x_n\} \to x$ w.r.t $\|\cdot\|$ i.e. $\forall\ \epsilon > 0\ \exists\ N$ s.t. $\forall\ n > N\ \ \|x_n - x\| < \alpha\epsilon$. Hence we have $\alpha \left\|x_n - x\right\|' < \alpha\epsilon$. Hence $\forall\ \epsilon > 0\ \exists\ N$ such that $\forall\ n > N\ \ \left\|x_n - x\right\|' < \epsilon$

□

(ii) **Only If Part:-**

V is open w.r.t $\|\cdot\| \iff \bigcup\limits_{x \in V} B_r(x)$ and V' is open w.r.t $\|\cdot\|' \iff \bigcup\limits_{x \in V} B'_r(x)$

Now we have
$$B_r(x) = \{y \mid \|y - x\| < r\} \text{ and } B'_r(x) = \{y \mid \|y - x\|' < s\}$$

Hence by equivalence of the norms for any v
$$\alpha\|v\|' \leq \|v\| \leq \beta\|v\|'$$

Since $\|v\| < r$ we have
$$\|v\|' \leq \frac{r}{\beta} \implies B'_{\frac{r}{\beta}}(x) \subset B_r(x)$$

□

Corollary 4.1
$p = 1$ and $p = \infty$ on $\mathbb{R}^n$ (and $\mathbb{C}^n$) give the same topology as $p = 2$ norm

Corollary 4.2
Let x_m be a square in $\mathbb{R}^n$. $\overline{x_m} = (x_{m_1}, x_{m_2}, \cdots, x_{m_n})$. Then $\{\overline{x_m}\} \to x = (x_1, x_2, \cdots, x_n)$ w.r.t $\|\cdot\|_2 \iff \{x_{m_i}\} \to x_i$ in $\mathbb{R}$ for each i.

Note:-

We can check this w.r.t $\|\cdot\|_\infty$

$$\begin{aligned}
\overline{x_m} \to \overline{x} \text{ w.r.t } \|\cdot\|_\infty \quad &\iff \quad \forall\ \epsilon > 0\ \exists\ N \text{ s.t. } \forall\ m > N\ \max\{|x_{m_i} - x_i| \mid 1 \le i \le n\} \\
&\iff \quad \text{each } |x_{m_i} - x_i| < \epsilon\ \forall\ i \\
&\iff \quad \lim_{n\to\infty} x_{m_i} = x_i\ \forall\ i
\end{aligned}$$

Chapter 5

Compactness

5.1 Sequentially Compact

Definition 5.1.1: Sequentially Compact

Let (X, d) be a metric space. X is called sequentially compact if every sequence in X has a convergent subsequence. (Often applied to a subset S of X)

Note:-

For S to be sequentially compact the limit of subsequence must be in S

Definition 5.1.2: Boundedness

A subset S of (X, d) is bounded if $S \subset B_r(x)$ for some $x \in X$ and $r > 0$

Note:-

Boundedness depends on the metric but if two metrics are "equivalent" analogous to norms)

Theorem 5.1.1

A subset K of $\mathbb{R}^n$ is sequentially compact $\iff$ K is closed and bounded

Proof: Proof in steps

1. **A closed interval $[a, b]$ in $\mathbb{R}$ is sequentially compact**

   Proof: Given a sequence $x_1, x_2, \cdots$ in $\mathbb{R}$ in $[a, b]$ we can extract a monotonic subsequence as follows:

   We call x_i to be a peak if $x_i > x_j \ \forall\ j > i$. Now there are two cases. If number of peaks is infinite then the next peak comes after the previous one so smaller than the previous one. So its a strictly decreasing sequence. If number of peaks are finite then at some point we cant find a peak with this property that means no matter which term i peak there is at least one term after that which is greater than or equal to that term. y_1 =a term after the last peak. and y_{i+1} =a term after y_i such that $y_{i+1} \geq y_i$. Hence $y_1, y_2, \cdots$ is a weakly increasing sequence.

   When $\{x_n\}$ contained in $[a, b]$ by boundedness of the monotonic subsequence, it converges to its sup/inf and the limit is in $[a, b]$ $\square$

2. **$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is sequentially compact (w.r.t $p-$norm for $p = 1, 2, \infty$. Later for any norm)**

   Proof: Recall a sequence $\{x_m\} \to x$ in $\mathbb{R}^n \iff$ The sequence converges in each coordinate i.e. $x_{m_i} \to x_i$

   Take a sequence in the given box. Extract a subsequence whose entries in 1st slot converge (necessarily to x_i in $[a_1, b_1]$ by step 1 From this sequence, extract a further subsequence whose entries in second slot converge to $x_2 \in [a_2, b_2]$. Continue □

3. **Every closed subset of a sequentially compact set is sequentially compact**

   Proof: Exercise □

   This will show each closed and bounded subset of the Euclidean Space $\mathbb{R}^n$ is sequentially compact. (because such a set will be contained in a box)

4. **If K is sequentially compact then K is closed and bounded**

   Proof: If K is not closed then some limit point x of K will not be in K. Then there is a sequence $\{y_m\}$ in K converges to $x \notin K$ violating sequential compactness of K.

   If K is not bounded take $\{x_m\} \in K$ with $\|x_m\| \geq n$ then $\{x_m\}$ can not be convergent □

□

Note:-
Step 4 works for any metric space. Then we need to have a ball instead of norm

Theorem 5.1.2
If K is a sequentially compact of a metric space X, then K is closed and bounded

Proof: Same argument as step 4 use x_m such that $d(x_m, x) \geq m$ □

Question 11

If K is closed and bounded in $(X, d) \implies K$ is sequentially compact

Solution: No. Any counter-example. Define a metric on real number which induces same topology as the normal topology in such a way that there is a closed and bounded set that is not compact.

Question 12

1. If V, W are normed linear spaces can we define a norm on $V \times W$?
2. If V, W are metric spaces can we define a metric on $V \times W$?
3. If V, W are topological spaces can we define a topology on $V \times W$?

5.2 Open Cover and Compactness

Definition 5.2.1: Open Cover

Let $\{V_\alpha\}_{\alpha \in I}$ be a family of subsets of metric space X we say that $\{V_\alpha\}_{\alpha \in I}$ is a cover of X if $\bigcup_\alpha V_\alpha = X$ and we say that $\{V_\alpha\}_{\alpha \in I}$ is an open cover if each V_α is open (in X)

Definition 5.2.2: Compact

X is called compact if each open cover of X has a finite subcover i.e. $\{V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}\} \subset \{V_\alpha\}_{\alpha \in I}$ with $V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n} = X$

Note:-

1. This definition makes sense for any topological space X.

   If X is a metric space then it is a fact that X is compact $\iff$ X is sequentially compact. This is not true for general topological spaces. Both implications fail.

2. Reformulation of compactness for subset K of X in terms of open sets of X

$$
\begin{array}{lcl}
K \text{ is compact} & \iff & \text{Every cover of } K \text{ bt open sets of } K \text{ has a finite subcover.} \\
\multicolumn{3}{c}{\text{As open sets of } K \text{ are precisely (open sets of } X) \cap K. \text{ We have the following}} \\
K \text{ is compact} & \iff & \text{For any family } \{V_\alpha \cap K\}_{\alpha \in I} \text{ where } V_\alpha \text{ are open in } X \text{ whose union is } K, \\
 & & \text{there is a finite subcover.} \\
 & \iff & \text{For any family } \{V_\alpha \cap K\}_{\alpha \in I} \text{ of open sets in } X \text{ such that } \bigcup\limits_{\alpha \in I} V_\alpha \supset K, \\
 & & \text{there must be a finite subfamily } V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n} \text{ with } \bigcup\limits_{i=1}^{n} V_{\alpha_i} \supset K
\end{array}
$$

If i take this definition of compactness of a subset K of metric space X then K is compact as subset of $X \iff K$ is compact as a subset of it itself

Theorem 5.2.1 Haine Borel Theorem

$$K \subset \mathbb{R}^n \text{ is compact} \quad \iff \quad K \text{ is closed and boundeded}$$

(w.r.t $p = 1, 2$ or ∞ norm as they are equivalent.)

Proof: **Only If Part:-**

Proof in steps

1. Closed interval $[a, b]$ is compact in $\mathbb{R}$. ***Proof:*** Theorem 5.2.4
2. Closed box $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact in $\mathbb{R}^n$. ***Proof:*** Theorem 5.2.6
3. A closed subset of a compact set is compact. ***Proof:*** Theorem 5.2.3

These steps would give the backward direction of Haine Borel Theorem i.e. suppose K is closed and bounded in $\mathbb{R}^n \implies K \in [-M.M]^n \implies$ compact by 2

If Part:-

Bounded: First we have to show that K is compact $\implies K$ is bounded an i.e. $K \subset B_r(x)$ in (X, d) for some $x \in X$, $r > 0$

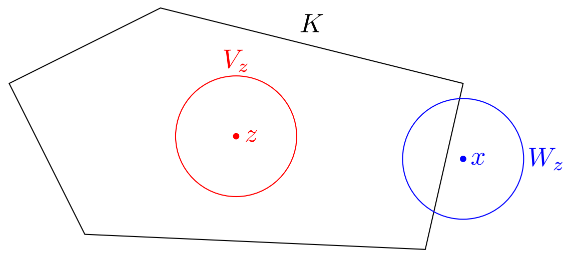
Consider open cover $\{B_n(x)\}_{n \in \mathbb{Z}^+}$ of X and hence of K. This must have a finite subcover $B_{n_1}(x), B_{n_2}(x), \cdots, B_{n_k}(x)$. Take $r = \max\{n_1, n_2, \cdots, n_k\}$ Hence

$$K \text{ is compact} \implies K \text{ is closed}$$

Closed: We will show that $X \setminus K$ is open. Pick $x \notin K$. Enough to construct an open neighborhood $U_x \ni x$ such that $U_x \cap x = \phi$

Take $z \in K$. Let $c = d(x, z)$ then

$$\underset{\displaystyle W_z}{\underset{\|}{B_{\frac{c}{3}}(x)}} \quad \cap \quad \underset{\displaystyle V_z}{\underset{\|}{B_{\frac{c}{3}}(z)}} = \phi \quad \text{by triangle inequality}$$



Now $\bigcup\limits_{z\in K} V_z \supset K$. So $\{V_z\}$ is an open cover of K. By compactness we have $V_{z_1} \cup V_{z_2} \cup \cdots \cup V_{z_n} \supset K$. As $W_z \cap V_z = \phi \ \forall z \in K$. We have $\underbrace{(W_{z_1} \cup W_{z_2} \cup \cdots \cup W_{z_n})}_{\substack{\text{Finite intersection of} \\ \text{open neighborhoods of } x, \\ \text{so call this } U_x}} \cap K = \phi$ ☐

Key fact that made this work: For $x \neq z$ in X, we could find open neighborhoods of V and W (of x and z respectively) such that $V \cap W = \phi$. Topological spaces that satisfy this property are called Housdorff.

What we proved is the following

Theorem 5.2.2

For a Housdorff Topological space X any compact subset K is closed and bounded

Theorem 5.2.3 Haine Borel Theorem - If Part: Step 3

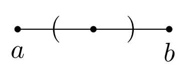
C is a closed subset of compact set $X \implies C$ is compact.

Proof: Take any open cover $\{V_\alpha\}_{\alpha\in I}$ of C by open sets in X i.e $\bigcup\limits_{\alpha} V_\alpha \supset C$. Now $\{V_\alpha\}_{\alpha\in I} \cup \{X \setminus C\}$ is an open cover of x. We have a finite subcover by compactness of X. The same subcover (after dropping $X \setminus C$ if necessary) works for C. ☐

Wrong Concept 5.1: Closed interval $[a, b]$ is compact in $\mathbb{R}$

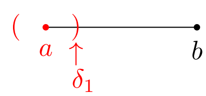
Suppose $\{V_\alpha\}_{\alpha\in I}$ is an open cover of $[a, b]$ by open sets in $\mathbb{R}$.

Hence every one of the points in the interval is covered by one of the V_α. Hence there is some interval contained in the V_α



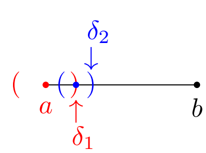
So i could just ignore the V_α and say for each point in the interval we can get an open interval that is part of a V_α. So how can i find a subcover. I could simply travel from one end to the other.

So i start with a so a must be contained in some open interval



Not only that i have covered up a small segment of the closed interval, upto a point, $a+\delta_1$. Say $[a, a+\delta_1) \subset V_1$.

Let $a + \delta_1$ is contained in some open interval which is contained in V_2 upto the point $a + \delta_2$



Now continue.

What is wrong with this ?

We could have smaller and smaller intervals. For example length of first interval can be $\frac{1}{3}$, length of second interval can be $\frac{1}{9}$, length of third interval can be $\frac{1}{27}$ and so on. So its a geometric progression and it will sum less than 1. So i just may not get there in finite number of steps.

Question 13

Suppose X is a topological space that is compact and 5.2 (Take x ro be a compact metric space if you like). Prove that given disjoint compact subsets K and L, there are disjoint open sets U and V with $K \subset U$ and $L \subset V$(First do it for $K =$ single point)

In the above exercise we could have replaced the word compact with another word which is closed because X is given to be compact so any closed set will be compact and in a Housdorff space compact subset is also closed.

Note:-

Cauchy Sequence in Metric space need not converge. For example $(0, 1)$ and take the sequence $\frac{1}{n}$. It wants to converge to 0 but 0 is not there.

Theorem 5.2.4 Haine Borel Theorem - If Part: Step 1

$[0, 1]$ is compact in $\mathbb{R}$

Proof: Let $\{V_\alpha\}_{\alpha\in I}$ be a family of open sets in $\mathbb{R}$ covering $[0, 1]$.

Let $S = \{a \in [0, 1] \mid [0, a]$ can be covered by a finite number of V_α's$\}$. Our goal is to prove $1 \in S$.

Let $0 \leq x < y \leq 1$. So $[0, x] \subset [0, y]$. This $y \in S \implies x \in S$ i.e $x \notin S \implies y \notin S$. Now S is nonempty because $0 \in S$ and S is bounded. Let $u =$ lub of S. Clearly $0 \leq u \leq 1$. Hence it is enough to show $u = 1$ and $u \in S$.

$0 \in$ some open set V_α. Hence $\exists\ \epsilon > 0$ $B_\epsilon(0) \subset V_\alpha$. Hence $\forall$ point $x \in [0, \epsilon)$ $x \in S$

For $a \in [0, u)$, $a \in S$ (otherwise a itself would be an upper bound for S). As $\{V_\alpha\}_{\alpha\in I}$ cover $[0, 1]$, $u \in V_\beta$. So $\exists\ \epsilon > 0$ such that $(u - \epsilon, u + \epsilon) \subset V_\beta$ As $u - \epsilon \in S$ we have $V_{\alpha_1} \sup V_{\alpha_2} \sup \cdots V_{\alpha_k} \supset [0, u - \epsilon]$ Then $V_{\alpha_\beta} \cup V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots V_{\alpha_k} \supset \left[0, u + \frac{\epsilon}{2}\right]$. So $u = 1$ because otherwise some $u + \delta \in S$ contradicting that u is an upper bound. □

Question 14

Can the strategy from the last time be made to work ti actually extract a finite subcover of a given cover.

Theorem 5.2.5

Suppose $X \xrightarrow{f} Y$ continuous and $K \subset X$ is compact. Then $f(K)$ is compact

Proof: Let $\{V_\alpha\}_{\alpha\in I}$ be an open cover of $f(k)$ by open sets V_α of Y. So

$$\bigcup_\alpha V_\alpha \supset f(K) \implies f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha) \supset f^{-1}(f(K)) \supset K$$

Thus $\left\{f^{-1}(V_\alpha)\right\}_{\alpha\in I}$ is an open (because of continuity Theorem 3.2.1) cover of K.

Extract a finite subcover

$$\begin{aligned} & f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \cdots f^{-1}(V_{\alpha_m}) \supset K \\ \implies & f\left(f^{-1}(V_{\alpha_2}) \cup \cdots f^{-1}(V_{\alpha_m})\right) \supset f(K) \\ \implies & \bigcup_{i=1}^{m} f\left(f^{-1}(V_{\alpha_i})\right) \supset f(K) \end{aligned}$$

As $V_{\alpha_i} \supset f\left(f^{-1}\left(V_{\alpha_i}\right)\right)$ we have $\bigcup\limits_{i=1}^{m} V_{\alpha_i} \supset f(K)$ ☐

Question 15

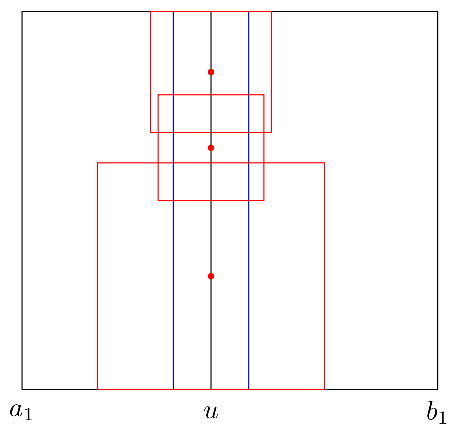
f(Sequentially compact K) is sequentially compact

Theorem 5.2.6 Haine Borel Theorem - If Part: Step ②

$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact in $\mathbb{R}^n$

Proof: Induction on n. $n = 1$ we already proved in Theorem 5.2.4.Let $\mathcal{F} = \{V_\alpha\}_{\alpha \in I}$ be a cover of K by open sets in $\mathbb{R}^n$. Fix $u \in [a_1, b_1]$ and consider $\{u\} \times \underbrace{[a_2, b_2] \times \cdots \times [a_n, b_n]}_{\substack{=C \text{ is compact} \\ \text{by induction on } n}}$ Hence $\{u\} \times C$ is compact because $\mathbb{R}^{n-1} \to \mathbb{R}^n$ which maps $(y_2, y \cdots, y_n) \mapsto (u, y_2, \cdots, y_n)$ or $f(C) = \{u\} \times C$ is continuous.

For each $p = (u, y_2, \cdots, y_n)$ in $\{u\} \times C$ pick an open neighborhood $V_p \in \mathcal{F}$. Hence $V_p \supset (x - \epsilon, x + \epsilon) \times \underbrace{(y_2 - \epsilon, y_2 + \epsilon) \times \cdots \times (y_n - \epsilon, y_n + \epsilon)}_{W_p}$ for some $\epsilon = \epsilon_p$ depending on p



By compactness of $\{u\} \times C$, extract a finite subcover of the cover $\{W_p\}$. Hence $W_{p_1} \cup W_{p_2} \cup \times \cup W_{p_k} \supset \{u\} \times C$. Since its a union of open sets we have in fact $W_{p_1} \cup W_{p_2} \cup \times \cup W_{p_k} \supset (u - \epsilon, u + \epsilon) \times C$ where $\epsilon = \min\{\epsilon_{p_1}, \epsilon_{p_2}, \cdots, \epsilon_{p_k}\}$. Let $\mathcal{F}_u = \{V_{p_1}, V_{p_2} < \cdots, V_{p_k}\}$. So

$$V_{p_1} \cup V_{p_2} \cup \cdots \cup V_{p_k} \supset W_{p_1} \cup W_{p_2} \cup \cdots \cup W_{p_k} \supset (u - \epsilon, u + \epsilon) \times C$$

i.e. this finite subcover $\mathcal{F}_u$ cover not just the slice but a tube around it.

Now as u varies in $[a_1, b_1]$, $(u - \epsilon_u, u + \epsilon_u)$ gives an open cover. Extract a finite subcover $(u_1 - \epsilon_{u_1}, u_1 + \epsilon_{u_1}), (u_2 - \epsilon_{u_2}, u_2 + \epsilon_{u_2}), \cdots, (u_l + \epsilon_{u_l}, u_l + \epsilon_{u_l})$. Then $\mathcal{F}_{u_1} \cup \mathcal{F}_{u_2} \cup \cdots \cup \mathcal{F}_{u_l}$ is a finite subcover of $[a_1, b_1] \times C = K$ ☐

Question 16

Why the map $\mathbb{R}^{n-1} \to \mathbb{R}^n$ which maps $(y_2, y \cdots, y_n) \mapsto (u, y_2, \cdots, y_n)$ or $f(C) = \{u\} \times C$ is continuous ?

Question 17

X, Y are topological spaces. $K \subset X$ and $Y \subset Y$ are compact subsets. Then $K \times L$ is compact subset of $X \times Y$ where Open sets of $X \times Y$ are $\bigcup$(Open set of X)$\times$(Open set in Y)

Theorem 5.2.7

All norms on $\mathbb{R}^n$ are equivalent

Proof: Enough to show any norm $f \sim \|\cdot\|$

$$\text{i.e } \alpha\|u\| \leq f(u) \leq \beta\|u\| \forall\ u$$
$$\text{i.e } \alpha \leq \frac{f(u)}{\|u\|} \leq \beta\ \forall\ u \forall\ u \neq 0$$

Note that $\frac{f(x)}{\|x\|} = f\left(\frac{x}{\|x\|}\right) = f(u)$ where $u = \frac{x}{\|x\|}$, so $\|u\| = 1$. Hence it is enough to show that

$$\alpha \leq f(u) \leq \beta$$

for any u with $\|u\| = 1$

Let $S = \{u \mid \|u\| = 1\}$ is the unit sphere in $\mathbb{R}^n$, which is closed and bounded

$$\begin{aligned} S \text{ is closed and bounded} \quad &\implies \quad S \text{ is compact} \\ &\implies \quad f(S) \text{ is compact in } \mathbb{R} \\ &\implies \quad f(S) \text{ is closed and bounded in } \mathbb{R} \\ &\implies \quad f(S) \text{ has largest element in } \beta \text{ and smallest} \\ &\qquad\quad \text{element } \alpha \text{ such that } \alpha \leq f(S) \leq \beta \end{aligned}$$

□