Lecture Notes | 2024

Topology Class Notes

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Introduction

These lecture notes have been transcribed by me (Patrick Mullen). The environment was set up by Soham Chatterjee, sohamc@cmi.ac.in, and can be found here

This course was taught in W24 at the University of Michigan by Pierre-Louis Blayac.

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Chapter 1

Introduction

1.1 The Standard Topology on Euclidean Space

Topology, from the greek *topos*, meaning "place" or "locality", and *logos* meaning "study", can be thought of as the study of shape. More specifically, the study of how geometric objects behave under continuous deformations.

There are a variety of different (equivalent) approaches to topology, including but not limited open sets, neighborhoods, metrics, convergence of sequences, and continuity of functions. All of the preceding are discussed in this course, but we will rely heavily on the concept of open sets. Before getting to the subject, we review some important fundamentals.

Firstly, we denote the set of real numbers as $\mathbb{R}$, and the set of d-tuples as $\mathbb{R}^d$. The latter of these sets is sometimes referred to as "Euclidean d-space".

Definition 1.1.1: The (Standard) Inner Product and (Standard) Norm

Let $x = (x_1, ..., x_d), y = (y_1, ..., y_d) \in \mathbb{R}$. The standard inner product is a map $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(x, y) = \langle x, y\rangle = x_1y_1 + ...x_dy_d$$

We then define the (standard) norm as a function $g : \mathbb{R}^d \to \mathbb{R}$ as

$$g(x) = ||x|| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + ...x_d^2}$$

Theorem 1.1.1

For all $x, x', y, y' \in \mathbb{R}^d$ the following properties regarding the inner product hold

1. Bilinearity:
$$\langle \lambda x + \lambda' x', \mu y + \mu' y'\rangle = \lambda\mu\langle x, y\rangle + \lambda'\mu\langle x' y\rangle + \lambda\mu'\langle x, y'\rangle + \lambda'\mu'\langle x', y'\rangle$$
2. Symmetry:
$$\langle x, y\rangle = \langle y, x\rangle$$
3. Positivity:
$$\langle x, x\rangle \geq 0 \text{ with equality if and only if } x = 0$$
4. Cauchy Schwarz Inequality:
$$|\langle x, y\rangle| \leq ||x||\,||y||, \text{ with equality if and only if } x \parallel y$$
5. Triangle Inequality:
$$||x + y|| \leq ||x|| + ||y||, \text{ with equality if and only } y = 0 \text{ or } \exists a \geq 0 \text{ s.t. } x = ay$$

Proof: TODO: prove the above statements from definition ☐

Definition 1.1.2: The (Standard) Metric on $\mathbb{R}^d$

The (standard) metric on $\mathbb{R}^d$ is a map $d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $\forall x, y \in \mathbb{R}$

$$d(x, y) = ||x - y||$$

Theorem 1.1.2

For all $x, y, z \in \mathbb{R}$, the following properties regarding the (standard) metric hold:

1. Positivity:
$$d(x, y) \geq 0 \text{ with equality if and only if } x = y$$
2. Symmetry:
$$d(x, y) = d(y, x)$$
3. Triangle Inequality
$$d(x, z) \leq d(x, y) + d(y, z)$$

Proof: TODO: prove above statements from definition. ☐

Definition 1.1.3: (Standard) Open Ball and Closed Ball

Let $x \geq 0$ and $R \geq 0$. Then the (standard) open ball is the set

$$B(x, R) = \{y \in \mathbb{R}^d | d(x, y) < R\}$$

and the (standard) close ball is the set

$$\overline{B}(x, R) = \{y \in \mathbb{R}^d | d(x, y) \leq R\}$$

In $\mathbb{R}$, $B(x, R)$ is just the open interval $(x - R, x + R)$. In $\mathbb{R}^2$, $B(x, R)$ is the interior of the circle centered at x of radius R. In $\mathbb{R}^3$, $B(x, R)$ is the the interior of the sphere centered at x of radius R. The closed counterparts of these sets include the boundaries of the described sets.

Definition 1.1.4: Neighborhoods in $\mathbb{R}^d$

Let $U \subset \mathbb{R}^d$ and $x \in U$. Then U is a **neighborhood** of x if it contains a nonempty ball centered at x, i.e. if

$$\exists \epsilon > 0, B(x, \epsilon) \subset U$$

Additionally, U is a (standard) open ball of $\mathbb{R}^d$ if it is a neighborhood of all its points, i.e.

$$\forall y \in U, \exists \epsilon > 0, B(x, \epsilon) \subset U$$

Note:-

1. and $\mathbb{R}$ are open.
2. Open balls are open.

Proof. TODO: prove the statements in the note ☐

Theorem 1.1.3

1. Let $(V_\alpha)_{\alpha \in I}$ be a (possibly infinite) family of open subsets of $\mathbb{R}^d$. Then $\bigcup_{\alpha \in I} V_\alpha$ is open.

2. Let $V_1, ..., V_n \in \mathbb{R}^d$ be open. Then $V_1 \cap ... \cap V_n$ is open.

Proof. TODO: prove the above statements □

Example 1.1.1

The set $U = \{(x_1, ..., x_d) \in \mathbb{R}^d | x_d > 0\}$ is an open subset of $\mathbb{R}^d$.

Proof: Let $x \in U$ and set $\epsilon = x_d$. We then want to show that $B(x, \epsilon) \subset U$. Let $y \in B(x, \epsilon)$. Then $|x_d - y_d| \leq ||x - y|| = \sqrt{|x_1 - y_1|^2 + ... + |x_d - y_d^2|}$ □

Note:-

The collection of open subsets of $\mathbb{R}^d$ is called the Standard Topology on $\mathbb{R}^d$.

Definition 1.1.5: Convergence of Sequences in $\mathbb{R}^d$

Let $(x_n)_{n \geq 0}$ be a sequence in $\mathbb{R}^d$. We say $x_n \rightarrow_x$ or $\lim\limits_{n \to \infty} x_n = x$ if

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, d(x_n, x) < \epsilon$$

Note that $d(x_n, x) < \epsilon$ is equivalent to $x_n \in B(x, \epsilon)$.

Example 1.1.2

$\lim\limits_{n \to \infty} \frac{1}{n} = 0$
TODO: Proof

Theorem 1.1.4

Let $x \in \mathbb{R}$ and $(x_n)_{n \geq 0}$ a sequence in $\mathbb{R}^d$. Then

$$\lim_{n \to \infty} x_n = x \iff \forall V \text{ nbhd of } x, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in V$$

Proof: ($\implies$): Assume $\lim_{n \to \infty} x_n = x$. Let V be a neighborhood of x. Then, by the definition of neighborhood, $\exists \epsilon > 0$ such that $B(x, \epsilon) \subset V$. Then, by the definition of convergence, $\exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x_n \in B(x, \epsilon)$. Then, since $B(x, \epsilon) \subset V$, we have $\forall n > n_0, x_n \in V$. Then, since V is arbitrary, $\forall V$ a neighborhood of $x, \exists n_0 \in \mathbb{R}, \forall n > n_0, x_n \in V$. Therefore,

$$\lim_{n \to \infty} x_n = x \implies \forall V \text{ nbhd of } x, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in V$$

($\impliedby$) Assume $\forall V$ nbhd of $x, \exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x_n \in V$. Let $\epsilon > 0$. We know that $B(x, \epsilon)$ is a neighborhood of x. Then, by our assumption, $\exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x \in B(x, \epsilon)$. Therefore, by the definition of convergence, $\lim_{n \to \infty} x_n = x$. Therefore,

$$\lim_{n \to \infty} x_n = x \impliedby \forall V \text{ nbhd of } x, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in V$$

□

Definition 1.1.6: Continuity of Functions from $\mathbb{R}^d$ to $\mathbb{R}^k$

A map $f : \mathbb{R}^d \to \mathbb{R}^k$ is **continuous at** $x \in \mathbb{R}^d$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in B(x, \delta), d(f(x), f(y)) < \epsilon$$

Note that $d(f(x), f(y)) < \epsilon$ is equivalent to $f(y) \in B(f(y), \epsilon)$, so f is continuous if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in B(x, \delta), f(y) \in B(f(y), \epsilon)$$

Another, equivalent definition is that f is continuous if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } f(B(x, \delta)) \subset B(f(x), \epsilon)$$

We say that f is **continuous** if it is continuous at every point.

Example 1.1.3

1. Let $v \in \mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^d$ defined by the rule $x \mapsto x + v$ is continuous on $\mathbb{R}^d$. Let $x \in \mathbb{R}^d$ and let $\epsilon > 0$. Set $\delta = \epsilon$. We then want to show that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Let $y \in B(x, \delta)$. Then

   $$d(f(y), f(x)) = ||f(y) - f(x)|| = ||y + v - (x + v)|| = ||y - x|| < \delta = \epsilon$$

   Therefore, $f(y) \in B(f(x), \epsilon)$.

2. TODO: fill in remaining examples

Theorem 1.1.5

Let $f : \mathbb{R}^d \to \mathbb{R}^k$. Then the following statements hold

1. f is continuous at $x \in \mathbb{R}^d \iff \forall V$ neighborhood of $f(x), f^{-1}(V)$ is a neighborhood of x.
2. f is continuous on $\mathbb{R}^d \iff \forall U \subset \mathbb{R}^d$ open, $f^{-1}(U) \subset \mathbb{R}^k$ is open.
3. f is continuous at $x \in \mathbb{R}^d \iff \forall (x_n)_{n\geq 0}$ in $\mathbb{R}^d$ converging to x, $f(x_n) \to f(x)$ as $n \to \infty$.

Proof: TODO: complete proof of above statements □

Definition 1.1.7: Open Subset

Let $X \subset \mathbb{R}^d$. A subset $U \subset X$ is **open in** X if $\forall x \in U, \exists \epsilon > 0$ such that $B(x, \epsilon) \cap X \subset U$.

A Topology of X, then, is a collection of subsets of X that are open in X. This definition will be expanded and made rigorous later in the course.

Example 1.1.4 (Torus)

TODO: insert torus example here (preferable with illustration)

Definition 1.1.8: Homeomorphism

Let $X, Y \subset \mathbb{R}^d$ be subsets. Then X, Y are **homeomorphic** if $\exists f : X \to Y$ a bijection such that

$$\forall U \subset X, U \text{ is open in } X \iff f(U) \text{ open in } Y$$

The function f is called a **homeomorphism**.

1.2 General Topologies, Open Sets, and Neighborhoods

In the previous section, we defined things in terms of $\mathbb{R}^d$, the standard metric, and the standard inner product.

Definition 1.2.1: Topology

Let X be a set. A **topology** on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X called **open subsets** such that

1. $\emptyset, X \in \mathcal{T}$
2. Stability under unions. For any $(U_\alpha)_{\alpha \in I}$ of elements of $\mathcal{T}$, $\bigcup_{\alpha \in I} U_\alpha$ is also an element of $\mathcal{T}$.
3. Stability under finite intersection. For any $U_1, ..., U_n \in \mathcal{T}$, we have $U_1 \cap ... \cap U_n \in \mathcal{T}$.

The pair (X, d) is called a **topological space**.

Example 1.2.1

Let $X = \mathbb{R}^d$.

1. The collection of standard open sets is a topology (the standard topology) as discussed in the previous section.
2. $\{\emptyset, \mathbb{R}^d\}$ is a topology on $\mathbb{R}^d$.
3. $\mathcal{P}(\mathbb{R}^d)$ is a topology

If $\mathcal{T}_S$ is the standard topology, then

$$\{\emptyset, \mathbb{R}^d\} \subset \mathcal{T}_S \subset \mathcal{P}(\mathbb{R}^d)$$

Definition 1.2.2: Coarse and Fine

Let X be a set, and $\mathcal{T}_1, \mathcal{T}_2$ be two topologies. If $\mathcal{T}_1 \subset \mathcal{T}_2$, we say $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, and $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$

.

Note:-

Given X a set, the set $\{\emptyset, X\}$ is called the **coarse topology** (it is the coarsest topology). The powerset of X, $\mathcal{P}(X)$, is called the discrete topology (it is the finest).

Example 1.2.2

Consider $X = \{0, 1, 2\}$. Then $\mathcal{T} = \{\emptyset, X, \{0\}, \{1, 2\}\}$ is a topology on X. It's easy to check that it's stable under union, stable under finite intersection, and contains both $\emptyset$ and X.
TODO: input illustration
There exist many different topologies on X. The number of unique topologies is bounded above by the cardinality of $\mathcal{P}(\mathcal{P}(X))$, which in this case is 2^{2^3} or 256. Another example of such a topology is $\mathcal{T}_2 = \{\emptyset, X, \{0\}\}$. A non-example of a topology is $\{\emptyset, X, \{0\}, \{1\}, \{2\}\}$, since it is not stable under unions.

Definition 1.2.3: Neighborhood

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is a neighborhood of $x \in X$ if $\exists U \in X$ open such that $x \in U \subset A$.

Theorem 1.2.1

Let $(X, \mathcal{T})$ be a topological space. Then $\forall x \in X, \mathcal{N}(x)$ is defined as the collection of all neighborhoods of x. That is

$$\mathcal{N}(x) = \{N \subset X | N \text{ is a neighborhood of } x\}$$

Then, $\forall x \in X$ the following statements hold:

1. $X \in \mathcal{N}(x)$
2. $\forall N \in \mathcal{N}(x), x \in N$
3. $\forall N \in \mathcal{N}(x), \forall A \subset X$, if $N \subset A$, then $A \subset \mathcal{N}(x)$
4. $\forall N_1, ..., N_k \in \mathcal{N}(x), N_1 \cap ... \cap N_k \in \mathcal{N}(x)$
5. $\forall N \in \mathcal{N}(x), \exists N' \in \mathcal{N}(x)$ such that $\forall y \in N', N \in \mathcal{N}(y)$

Moreover, $\forall U \subset X, U$ is open if and only if U is a neighborhood of all its points.

Proof: TODO: Insert proof here □

1.3 Sequences in Topological Spaces

While the sequences we study are not remarkably different than those encountered in the typical analysis course, we will generalize the concept slightly, especially in regards to the definition of convergence of sequences.

Definition 1.3.1: Convergence and Continuity

Let $(X, \mathcal{T})$ be a topological space, $(x_n)_n$ in X and $x \in X$. We say that $x_n \to x$ if

$$\forall N \text{ a neighborhood }, \exists n_0 \in \mathbb{R} \text{ such that } \forall n > n_0, x_n \in N$$

Example 1.3.1

Let $X = \{0, 1, 2\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}, \{0, 1\}\}$. Then $x_n \to 1$ if and only if $\exists n_0 \in \mathbb{R}$ such that $\forall n > n_0, x_n =$ 0 or 1. In particular $0 \to 1$.

Theorem 1.3.1

Let $(X, \mathcal{T})$ be a topological space. Let $(x_n)_{n \geq 0}$ be a sequence in X that is constant after some time: $\exists n_0 \geq 0$ integer such that $x_n = x_{n_0} \forall n \geq n_0$. Then $x_n \to x_{n_0}$ as $n \to \infty$.

Proof: Let N be a neighborhood of x_{n_0}. Then $\forall n \geq n_0, x_n = x_{n_0}$, so $x_n \to x_{n_0}$. □

Definition 1.3.2: Continuity

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. A map $f : X \to Y$ is called continuous if

$$\forall U \subset Y \text{ open}, f^{-1}(U) \subset X \text{ is open}$$

Theorem 1.3.2
$f : \mathbb{R}^d \to \mathbb{R}^k$ is continuous if and only if $\forall x \in \mathbb{R}^d, \forall \epsilon > 0, \exists \delta < 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$.

Proof: ($\implies$) Assume that $f : \mathbb{R}^d \to \mathbb{R}^k$ is continuous. Let $x \in \mathbb{R}^d$ and $\epsilon > 0$. We know $B(f(x),\epsilon)$. We know $B(f(x),\epsilon)$ is open, so $f^{-1}(B(x,\epsilon))$ is open by continuity. Then, since $x \in f^{-1}(B(x,\epsilon)), \exists \delta > 0$ such that $B(x,\delta) \subset f^{-1}(B(f(x),\epsilon))$. Then, $f(B(x,\delta)) \subset B(f(x),\epsilon)$.
($\impliedby$) Assume $\forall x \in \mathbb{R}^d, \forall \epsilon > 0, \exists \delta < 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$. Let $U \subset \mathbb{R}^d$ be open. Then, for f to be continuous we want to show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$. We have that $f(x) \in U$ is open, so $\exists > 0$, such that $B(f(x),\epsilon) \subset U$. By assumption, $\exists \delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$. Then $B(x,\delta) \subset f^{-1}(B(f(x),\epsilon)) \subset f^{-1}(U)$. Therefore, $f^{-1}(U)$ is open. Therefore, $\forall U \in \mathbb{R}^k$ open, $f^{-1}(U) \subset \mathbb{R}^d$ is open.

Therefore, $f : \mathbb{R}^d \to \mathbb{R}^k$ is continuous if and only if $\forall x \in \mathbb{R}^d, \forall \epsilon > 0, \exists \delta < 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$. $\square$

Example 1.3.2
Let $X = \{-1,0,1\}$ and $\mathcal{T} = \{\emptyset, X, \{-1\}, \{1\}, \{-1,1\}\}$ be a topology on X.
Then $f : \mathbb{R} \to X$ defined by the rule

$$x \mapsto \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 1 \end{cases}$$

Then f is continuous, because

- $f^{-1}(\emptyset) = \emptyset$
- $f^{-1}(X) = \mathbb{R}$
- $f^{-1}(\{-1\}) = (-\infty, 0)$
- $f^{-1}(\{1\}) = (0, \infty)$
- $f^{-1}(\{-1,1\}) = (-\infty,0) \cup (0,\infty)$

are all open.

1.4 Bases of Topologies

Definition 1.4.1: The Subspace Topology

Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ be a subset. The subspace topology on Y is

$$\mathcal{T}_{|Y} = \{U \cap Y | U \in \mathcal{T}\}$$

Proof that the subspace topology is, in fact, a topology

Proof:

1. $\emptyset = \emptyset \cap Y \in \mathcal{T}_{|Y}$ and $Y = X \cap Y \subset \mathcal{T}_{|Y}$, so $\mathcal{T}_{|Y}$ satisfies the first condition for being a topology.
2. Let $(V_\alpha)_{\alpha \in I}$ be a family of sets in $\mathcal{T}_{|Y}$. Then, by the definition of the subspace topology, $\forall \alpha \in I, \exists U_\alpha \in \mathcal{T}$ such that $V_\alpha = U_\alpha \cap Y$. Then
$$\bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} (U_\alpha \cap Y) = \bigcup_{\alpha \in I} (U_\alpha) \cap Y$$
By the definition of a topology, $\bigcup_{\alpha \in I}(U_\alpha) \in \mathcal{T}$, so $\bigcup_{\alpha \in I}(V_\alpha) \in \mathcal{T}_{|Y}$. Therefore, $\mathcal{T}_{|Y}$ is stable under unions, satisfying the second condition of being a topology.

3. Let $V_1, ..., V_n \in \mathcal{T}_{|Y}$. Then, $\forall i = 1, ..., n, \exists U_i \in \mathcal{T}$ such that $V_i = U_i \cap Y$. Then

$$V_1 \cap ... \cap V_n = (U_1 \cap Y) \cap ... \cap (U_n \cap Y) = (U_1 \cap ... \cap U_n) \cap Y$$

Then, clearly, $U_1 \cap ... \cap U_n \in \mathcal{T}$, so $V_1 \cap ... \cap V_n \in \mathcal{T}_{|Y}$. Thus, $\mathcal{T}$ is stable under finite intersection, satisfying the third condition of being a topology.
Therefore, $\mathcal{T}_{|Y}$ is a topology. □

Example 1.4.1

1. $\overline{B}(0,1), S(0,1) = \overline{B}(0,1) \setminus B(0,1) = \{x \in \mathbb{R}^d | \, ||x|| = 1\}$
2. Torus (same definition as before). TODO: insert definition and illustration
3. Cantor Set. TODO: insert definition

Theorem 1.4.1 Moore's Law
Let X be a set and $(\mathcal{T}_\alpha)_{\alpha \in I}$ be a family of topologies on X. Then

$$\mathcal{T} = \bigcap_{\alpha \in I} \mathcal{T}_\alpha \subset \mathcal{P}(X)$$

is also a topology and we have

$$U \in \mathcal{T} \iff U \text{ is open } \forall \mathcal{T}_\alpha$$

Proof: TODO: insert proof of Moore's Law here. □

dfnTopology Generated by a SubsetLet X be a set and $A \subset \mathcal{P}(X)$. Then the **topology generated by** A is the intersection of all topologies containing A. Then A is called a subbasis for this topology.

Example 1.4.2
The standard topology on $\mathbb{R}^d$ is generated by the set of open balls. (exercise)
TODO: Prove

Definition 1.4.2

Let X be a set. A basis of X is a $\mathcal{B} \subset \mathcal{P}(X)$ such that

1. $\bigcup_{B \in \mathcal{B}} B = X$. In words, $\mathcal{B}$ *covers* X.
2. $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Moreover, $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ if $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{B}$ generates $\mathcal{T}$.

Example 1.4.3
Open balls in $\mathbb{R}^d$ form a basis for the standard topology.

Proof: The first condition is self evident. To check that the second condition from the definition of a basis holds:
Let $x_1, x_2 \in \mathbb{R}^d, r_1, r_2 \geq 0$. Let $x_3 \in B(x_1, r_1) \cap B(x_2, r_2)$. Set $r_3 = \min(r_1 - d(x_1, x_3), r_2 - d(x_2, x_3))$. Then $B(x_3, r_3) \subset B(x_1, r_1) \cap B(x_2, r_2)$. Therefore, $\mathcal{B}$ is a basis for $\mathcal{T}$. □

Note:-
Note that any topology satisfies the conditions for being a basis.

Example 1.4.4
On $\mathbb{R}, \mathcal{B} = \{[x,y)|x < y \in \mathbb{R}\}$ is a basis.

Proof:

1. $\mathbb{R} = \bigcup_{i \in \mathbb{R}} [t, t+1)$
2. Let $B_1 = [x_1, y_1)$ and $B_2 = [x_2, y_2) \subset \mathbb{R}$. Let $x \in B_1 \cap B_2$. Then we have

$$\max(x_1, x_2) \leq x < \min(y_1, y_2)$$

so take $x_3 = \max(x_1, x_2)$, and $y_3 = \min(y_1, y_2)$. Then

$$x_3 \leq x < y_3$$

Let $B_3 = [x_3, y_3)$. Then $x \in B_3 \subset B_1 \cap B_2$.

Therefore, $\mathcal{B}$ is a basis of $\mathbb{R}$. □

Theorem 1.4.2
Let X be a set, and $\mathcal{T} \subset \mathcal{P}(x)$. Then $\mathcal{T}$ is a topology if and only if the following hold:

1. $\emptyset, X \in \mathcal{T}$
2. $\mathcal{T}$ is stable under union
3. $\forall U, V \in \mathcal{T}, U \cap V \in \mathcal{T}$

To prove the above theorem, the forward direction is tribial. For the reverse direction, the first two conditions are also just assumed, and the proofs of those statements are trivial. For the third condition, we simply need to induct on n to demonstrate stability under finite intersection. Providing a rigorous proof is left as an exercise to the reader.

Theorem 1.4.3
Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U$$

Proof: ($\implies$) Assume that $\mathcal{B}$ is a basis for some $\mathcal{T}$.Let $\mathcal{T}'$ be the collection of all $U \subset X$ such that ($\forall x \in U, \exists B \in \mathcal{B}$ such that $x \in B \subset U$). Clearly, $\mathcal{B} \subset \mathcal{T}'$. We then want to show that $\mathcal{T}'$ is a topology.

1. $\emptyset \in \mathcal{T}'$ and $X \in \mathcal{T}'$ since $X = \bigcup_{B \in \mathcal{B}} B$
2. Let $(U_\alpha)_(\alpha \in I)$ be a family in $\mathcal{T}'$. Let $x \in U = \bigcup_{\alpha \in I} U_\alpha$.

   TODO: finish inputting proof □

Corollary 1.4.1
Let X be a set, and $A \subset \mathcal{P}(X)$. Let $\mathcal{T}$ be the topology generated by A. Then

$$\mathcal{B} = \{A_1 \cap ... \cap A_n | A_i \in A\} \cup \{\emptyset, X\}$$

is a basis for $\mathcal{T}$. Therefore, $U \subset X$ is $\mathcal{T}$-open if and only if it is $\emptyset$ or X, or if it is a union of finite intersections of elements of A.

Proof:

1. $X \in \mathcal{B}$

TODO: complete proof □

1.5 The Product Topology

Definition 1.5.1: The Product Topology

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. The **product topology** on $X_1 \times X_2$ is the topology generated by

$$\{U_1 \times U_2 | U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}$$

Theorem 1.5.1

Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and let $\mathcal{B}_1, \mathcal{B}_2$ be bases of X_1, X_2 respectively. Then

$$\mathcal{B} = \{B_1 \times B_2 | B_i \in \mathcal{B}_i\}$$

is a basis for the product topology on $X = X_1 \times X_2$. In particular, $\{U_1 \times U_2 | U_i \in \mathcal{T}_i\}$

Proof:

1. $X = X_1 \times X_2 = (\bigcup_{B_1 \in \mathcal{B}_1} B_1) \times (\bigcup_{B_2 \in \mathcal{B}_2} B_2) = \bigcup_{(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2} B_1 \times B_2 = \bigcup_{B \in \mathcal{B}} B$
2. TODO: finish proof

□

Example 1.5.1

The product topology $\mathbb{R} \times \mathbb{R}$ is the standard topology. Open balls are not products of open sets of $\mathbb{R}$.

Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$. Consider the set

$$\mathcal{T} = \{U \subset X | \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$$

Under what conditions on $\mathcal{B}$ is $\mathcal{T}$ a topology?

For the first condition, we have $\emptyset \in \mathcal{T}$, since $\emptyset$ satisfies the condition. Then, for $X \in \mathcal{T}$, we need $\mathcal{B}$ to cover X.

The second condition is satisfied without further specification, since any union of sets in $\mathcal{T}$ inherently contains the elements of B that caused the sets to be in $\mathcal{T}$ in the first place.

For the third condition, we only need $\forall U, V \in \mathcal{T}, U \cap V \mathcal{T}$. For this to hold for this particular $\mathcal{T}$, we consider that $x \in U \cap V$ implies that $\exists B_1, B_2$ such that $x \in B_1 \subset U$ and $x \in B_2 \subset V$. From there, we need the second axiom of the basis for $\mathcal{T}$ to be a topology.

Theorem 1.5.2

Let X be a set, $\mathcal{B}$ be a basis, and $\mathcal{T}$ the topology generated by $\mathcal{B}$. Then $U \subset X$ is $\mathcal{T}$-open if $\forall x \in U, \exists B \in \mathcal{B}$ such that $x \in B \subset U$.
Additionally, $N \subset X$ is a neighborhood of x if $\exists B \in \mathcal{B}$ such that $x \in B \subset N$.

Note:-

Let X_1, X_2 be topological spaces. Then $N \subset X_1 \times X_2$ is a neighborhood of $X_1 \times X_2$ for the product topology if and only if $\exists U_1 \subset X_1, U_2 \subset X_2$ both open such that $x \in U_1 \times U_2 \subset N$.

Definition 1.5.2: Cartesian Product

Let $(X_\alpha)_{\alpha \in I}$ be a family of sets. Then $\prod_{\alpha \in I} X_\alpha$ is the set of lists $(x_\alpha)_{\alpha \in I} = x$ such that for all $\alpha \in I$, $x_\alpha \in X_\alpha$ (the α-coordinate of x).

Example 1.5.2

If $I = \{1, 2\}$, then $\prod_{\alpha \in I} X_\alpha = X_1 \times X_2$. If $X_\alpha = X \forall \alpha \in I$, then $\prod_{\alpha \in I} X_\alpha = X^I$ is identified with the set of functions $f : I \to X$.

Definition 1.5.3: The Box Topology

Let $((X_\alpha, \mathcal{T}_\alpha))_{\alpha \in I}$ be a family of topological spaces. The **Box Topology** on $\prod_{\alpha \in I} X_\alpha$ is the topology generated by the basis

$$\mathcal{B} = \{\prod_{\alpha \in I} U_\alpha | U_\alpha \subset X_\alpha \text{ open } \forall \alpha \in I\}$$

Corollary 1.5.1

Let $I = \mathbb{N}$ and $X_\alpha = \mathbb{R}, \forall \alpha \in I$. So

$$\prod_{\alpha \in I} X_\alpha = \mathbb{R}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{R}\}$$

Then no sequence of positive functions can converge to $f : \mathbb{N} \to \mathbb{R}$ defined by $k \mapsto 0$. We want to find a neighborhood N of f such that $f_n \notin N, \forall n \geq 0$.
Set

$$N = (-f_0(0), f_0(0)) \times (-f_1(1), f_1(1)) \times ...(-f_k(k), f_k(k)) \times ...$$

Then $f_0(0) \notin \mathbb{N}$, but $f_k(k) \in \mathbb{N}$.

Definition 1.5.4: General Product Topology

Let $((X_\alpha, \mathcal{T}_\alpha))_{\alpha \in I}$ be a family of topological spaces. The **product topology** on $\prod_{\alpha \in I} X_\alpha$ is the topology generated by the basis

$$\mathcal{B} = \{\prod_{\alpha \in I} U_\alpha | \forall \alpha \in I, U_\alpha \subset \mathcal{T}_\alpha \text{ and } \{\alpha \in I | U_\alpha \neq X_\alpha\} \text{ is finite}\}$$

Theorem 1.5.3

Let $(X_\alpha)_{\alpha \in I}$ be a family of topological spaces. Let $\alpha \in I$. Set $\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha$, called the **projection map**. Then π_α is continuous for the product topology.

Proof: Let $\alpha \in X_\alpha$ be open. Then, to demonstrate continuity, we need to show that $\pi_\alpha^{-1}(U_\alpha) \subset X = \prod_{\beta \in I} X_\beta$ is open. Then,

$$\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta \in I} U_\beta \text{ where } B_\beta = X_\alpha \text{ if } \beta \neq \alpha$$

$$\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta \in I \setminus \{\alpha\}} X_\beta \times U_\beta$$

□

Note:-

The product topology is generated by

$$\mathcal{A} = \{\pi_\alpha^{-1}(U_\alpha) | \alpha \in I, U_\alpha \subset X_\alpha \text{ is open}\} \subset \mathcal{B}$$

Indeed, let $B = \prod_{\alpha \in I} U_\alpha \in \mathcal{B}$. Then $J = \{\alpha \in I | U_\alpha \neq X_\alpha\}$. Then $B = \bigcap_{\alpha \in J} \pi_\alpha^{-1}(U_\alpha)$ the finite intersection of $\mathcal{A}$.

Theorem 1.5.4

Let $(X_\alpha)_{\alpha \in I}$ be a family of topological spaces. Let $(x_n)_{n \geq 0}$ be a sequence in $\prod_{\alpha \in I} X_\alpha$. Then $\forall n \geq 0, \pi_\alpha(x_n)$ is the α-coordinate of x_n. Then

$$x_n \to x \iff \forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x) \text{ in } X_\alpha$$

Proof: ($\Longrightarrow$) Assume $x_n \to x$. Let $\alpha \in I$. Then we want to show that $\pi_\alpha(x_n) \to \pi_\alpha(x)$. Let $N \subset X_\alpha$ be a neighborhood of $\pi_\alpha(x)$. Then, since π_α is continuous, we have that $\pi_\alpha^{-1}(N)$ is a neighborhood of x. Since $x_n \to x$, $\exists n_0$ such that $\forall n > n_0, x_n \in \pi_\alpha^{-1}(N)$, by definition. Then, $\forall n > n_0, \pi_\alpha(x_n) \in N$. Therefore, $\forall N$ neighborhood of $\pi_\alpha(x), \exists n_0$ such that $\forall n > n_0, \pi_\alpha(x_n) \in N$. Thus, $\pi_\alpha(x_n) \to \pi_\alpha(x)$. Since $\alpha \in I$ is arbitrary, we have $\forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x)$ in X_α. Therefore,

$$x_n \to x \implies \forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x) \text{ in } X_\alpha$$

($\Longleftarrow$) Now assume $\forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x)$ in X_α. Let N be a neighborhood of $x \in X$. Then $\forall \alpha \in I, \exists U_\alpha \subset X_\alpha$ an open set such that $J = \{\alpha \in I | U_\alpha \neq X_\alpha\}$ is finite and $x \in \prod_{\alpha \in I} U_\alpha \subset N$. Then, since $\pi_\alpha(x_n) \to \pi_\alpha(x)$, we have that $\forall \alpha \in J, \exists n_\alpha$ such that $\forall n > n_\alpha, \pi_\alpha(x_n) \in U_\alpha$. Set $n_0 = \max\{n_\alpha | \alpha \in J\}$, which is well defined since J is finite. Then $\forall n > n_0, x_n \in \bigcap_{\alpha \in J} \pi_\alpha^{-1}(U_\alpha) = \prod_{\alpha \in I} U_\alpha \subset N$. So $x_n \to x$.

Therefore,

$$x_n \to x \iff \forall \alpha \in I, \pi_\alpha(x_n) \to \pi_\alpha(x) \text{ in } X_\alpha$$

□

1.6 Metric Spaces

Definition 1.6.1: Metric Space

Let X be a set. A **metric** on X is a map $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$

1. d is symmetric:
$$d(x, y) = d(y, x)$$
2. d is positive definite
$$d(x, y) \geq 0 \text{ with equality if and only if } x = y$$
3. d satisfies the triangle Inequality
$$d(x, z) \leq d(x, y) + d(y, z)$$

Then the pair (X, d) is called the **metric space**.

Example 1.6.1 (Some examples)

The Euclidean metric, or L^2 metric on $\mathbb{R}^k$

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + ... + |x_k - y_k|^2}$$

L^1-metric:

$$d(x, y) = |x_1 - y_1| + ... + |x_k - y_k|$$

L^∞-metric:

$$d(x, y) = \max(|x_1 - y_1|, ..., |x_k - y_k|)$$

Definition 1.6.2: The Metric Topology

Let (X, d) be a metric space. Then, $\forall x \in X, r \geq 0$ we define the open ball as

$$B_d(x, r) = \{y \in X | d(x, y) < r\}$$

The **metric topology** induced by d is the topoogy generated by the set of open balls, which form a basis.
Then, $N \subset X$ is a neighborhood of x is $\exists \epsilon > 0$ such that $B_d(x, \epsilon) \subset N$.
A topological space is **metrizable** if $\exists d$ a metric which induces it.

Theorem 1.6.1

Let X be a set and d, d' two metrics with corresponding metric topologies $\mathcal{T}, \mathcal{T}'$. Then $\mathcal{T}$ is finer than $\mathcal{T}'$ if and only if

$$\forall x \in X, r > 0, \exists \epsilon > 0 \text{ such that } B_d(x, \epsilon) \subset B_{d'}(x, r)$$

Proof: ($\Longrightarrow$) Assume $\mathcal{T}$ is finer than $\mathcal{T}'$. Let $x \in X$ and $r > 0$. Then $B_{d'}(x, r)$ is $\mathcal{T}'$-open and hence is $\mathcal{T}$-open since $\mathcal{T}' \subset \mathcal{T}$. Then $B_{d'}(x, r)$ is a $\mathcal{T}$ neighborhood of x, so $\exists \epsilon > 0$ such that $B_d(x, \epsilon) \subset B_{d'}(x, r)$.

($\Longleftarrow$) Assume $\forall x \in X, r > 0, \exists \epsilon > 0$ such that $B_d(x, \epsilon) \subset B_{d'}(x, \epsilon)$. Let $U \in \mathcal{T}'$. We then want to show that $U \in \mathcal{T}$. Take $x \in U$. As U is a $\mathcal{T}'$-neighborhood of x, we have that $\exists \epsilon > 0$ such that $B_{d'}(x, \epsilon) \subset U$. By assumption, $\exists \delta > 0$ such that $B_d(x, \delta) \subset B_{d'}(x, \epsilon) \subset U$. Therefore, U is a $\mathcal{T}$-neighborhood of x, so $U \subset \mathcal{T}$. □

Example 1.6.2

The metrics d, d_1, and d_∞ all induce the same topology. For instance
TODO: Complete example

Theorem 1.6.2

Let (X, d_X) be a metric space, and Y a topological space. A sequence $(x_n)_{n \geq 0} \subset X$ converges to $x \in X$ if $\forall \epsilon > 0 \exists n_0$ such that $x_n \in B(x, \epsilon) \forall n > n_0$. Then

$$f : X \to Y \text{ is continuous } \iff \forall x \in X, \forall N \subset Y \text{ neighborhood of } f(x), \exists \epsilon > 0 \text{ such that } f(B(x, \epsilon)) \subset N$$

Theorem 1.6.3

Let X, Y be topological spaces, and let $f : X \to Y$ be a mapping.

1. If f is continuous then $\forall (x_n)_{n \geq 0} \subset X$ converging to x, we have $f(x_n) \to f(x)$.
2. Suppose X is metrizable. If $x_n \to x$ implies $f(x_n) \to f(x)$, then f is continuous.

Proof:

1. Assume f is continuous. Let $x_n \to x \in X$. Then, we want to show that $f(x_n) \to f(x)$. Let $N \subset Y$ be a neighborhood of $f(x)$. Then, since f is continuous, $f^{-1}(N)$ is a neighborhood of x ($\exists f(x) \in U \subset N, f^{-1}(U)$ is open and a subset of $f^{-1}(N)$). Since $x_n \to x, \exists n_0$ such that $\forall n > n_0, x_n \in f^{-1}(N)$. Then $f(x_n) \in N$.

2. Let X be metrizable, and assume $x_n \to x$ implies $f(x_n) \to f(x)$. Let d be a metric on X inducing the topology. Let $x \in X$ and $N \subset Y$ be a neighborhood of $f(x)$. Suppose for contradiction that $\forall \epsilon > 0, f(B(x, \epsilon)) \not\subset N$. In particular, $\forall n \geq 1, \exists x_n \in B(x, \frac{1}{n})$ such that $f(x_n) \notin N$. Now apply our supposition ot $\epsilon = \frac{1}{n}$. Then we can check that $f(x_n) \to f(x)$ and $x_n \to x$ contradicts the statement that $\exists n \geq 1, \exists x_n \in B(x, \frac{1}{n})$ such that $f(x_n) \notin N$.

□

Definition 1.6.3: Homeomorphism

Let X, Y be a topological space. A homeomorphism from X to Y is a bijection $f : X \to Y$ such that $U \subset X$ is open if and only if $f(U) \subset Y$.
X and Y are homeomorphic if $\exists$ a homeomorphism.

Corollary 1.6.1 (of Theorem 1.6.3)

Let X, Y be topological spaces. Then $f : X \to Y$ is a homeomorphism if and only if f is a bijection such that $x_n \to x$ if and only if $f(x_n) \to f(x)$.

Definition 1.6.4: Boundary Metric of d

Let (X, d) be a metric space. Then $\overline{d}(x, y) = \min(d(x, y), 1) \leq 1$ is a metric that defines the same topology.

Proof: Symmetry and positive definitness are trivial.
To prove the triangle inequality:
Let $x, y, z \in X$. Case 1: assume $d(x, y) \geq 1$ of $d(y, z) \geq 1$. Then $\overline{d}(x, y) + \overline{d}(y, z) \geq 1 \leq \overline{d}(x, z)$.
Case 2: $d(x, y) = \overline{d}(x, y)$ and $d(y, z) = \overline{d}(y, z)$. Then

$$\overline{d}(x, y) + \overline{d}(y, z) = d(x, y) + d(y, z) \geq d(x, z) \geq \overline{d}(x, z)$$

Therefore, $\overline{d}$ satisfies the triangle inequality. □

Definition 1.6.5: Diameter

Let (X, d) be a metric space, and $A \subset X$. Then we define diameter as

$$\mathrm{diam}_d(A) = \sup\{d(x, y) | x, y \in A\}$$

Example 1.6.3
$\mathrm{diam}_{\mathbb{R}}((0, 1) \cup \{2\}) = 2$

Theorem 1.6.4
Let $((X_\alpha, d_\alpha))_{\alpha \in I}$ be a family of metric spaces. Suppose $\exists c > 0$ such that $\forall \alpha \in I, \mathrm{diam}_{d_\alpha} X_\alpha \leq c$. Then

$$d((x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}) = \sup\{d_\alpha(x_\alpha, y_\alpha) | \alpha \in I\}$$

is well defined and is a metric (where $((x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}) \in \prod_{\alpha \in I} X_\alpha$). Moreover, the metric topology is finer than the product topology. We have equality if and only if $\forall \epsilon > 0, I_\epsilon = \{\alpha \in I | \mathrm{diam} X_\alpha \geq \epsilon\}$ is finite.

Proof:

1. First, we must prove that d is a metric. Proving that d is symmetric and positive definite is trivial, so we must demonstrate that d satisfies the triangle inequality.

   Let $(x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}, (z_\alpha)_{\alpha \in I}$ be sequences such that $x_\alpha, y_\alpha, z_\alpha \in X_\alpha, \forall \alpha \in I$. Then we want to show that

   $$d((x_\alpha)_{\alpha \in I}, (z_\alpha)_{\alpha \in I}) + d((z_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}) \leq d((x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I})$$

   For convenience, let $x = (x_\alpha)_{\alpha \in I}, y = (y_\alpha)_{\alpha \in I}$, and $z = (z_\alpha)_{\alpha \in I}$. We have that $\forall \alpha \in I$,

   $$d(x, z) + d(z, y) \geq d_\alpha(x_\alpha, z_\alpha) + d_\alpha(z_\alpha, y_\alpha) \geq d_\alpha(x_\alpha, y_\alpha)$$

   So $d(x, z) + d(z, y)$ is an upper bound of $\{d_\alpha(x_\alpha, y_\alpha | \alpha \in I)\}$. Then $d(x, y)$ is the least upper bound of this set, so

   $$d(x, y) \leq d(x, z) + d(z, y)$$

   Therefore d is a metric and is well defined.

2. Next, to demonstrate the "moreover.." statement, we must demonstrate that the metric topology, $\mathcal{T}$, is finer than the product topology, $\mathcal{T}'$. That is, we must show that $\mathcal{T}' \subset \mathcal{T}$.

   We have that a basis for $\mathcal{T}$ is the set $\mathcal{B} = \{\text{open balls}\}$, and a basis for $\mathcal{T}'$ is $\mathcal{B}' = \{\prod_{\alpha \in I} B(x_\alpha, r_\alpha) | r_\alpha = \infty \text{ for all but finitely many } \alpha\}$. Let $B' = \prod_{\alpha \in I} B(x_\alpha, r_\alpha) \in \mathcal{B}'$. Let $(y_\alpha)_{\alpha \in I} \in \mathcal{B}'$. Set $\epsilon = \inf\{r_\alpha - d_\alpha(x_\alpha, y_\alpha)\}$. Note that this is necessarily greater than 0, and equal to ∞ for "most" α's. Then

   $$B((y_\alpha)_{\alpha \in I}) = \prod_{\alpha \in I} B(y_\alpha, \epsilon) \subset B(x_\alpha, r_\alpha) \subset B'$$

   So B' is a neighborhood of $(y_\alpha)_{\alpha \in I}$. Therefore $B' \in \mathcal{T}$, so $B' \in \mathcal{T}'$. Hence, the metric topology $\mathcal{T}$ is finer than $\mathcal{T}'$.

3. Suppose $I_\epsilon = \{\alpha \in I | \text{diam} X_\alpha \geq \epsilon\}$ is finite $\forall \epsilon > 0$. We want to show that $\mathcal{T} \subset \mathcal{T}'$. Let $x = (x_\alpha)_{\alpha \in I} \in X$ and $r > 0$. Then, let $y \in B(x, r)$. Set $\epsilon = r - d(x, y)$. Then $B(y, \epsilon) \subset B(x, r)$. Then we want to find N, a $\mathcal{T}'$-neighborhood of y contained in $B(y, \epsilon)$. So, $\forall \alpha \in I_{\epsilon/2}$, set $\epsilon_\alpha = \frac{\epsilon}{2}$. Then, $\forall \alpha \in I \setminus I_{\epsilon/2}$, set $\epsilon_\alpha = \infty$. Then $\prod_{\alpha \in I} B(y_\alpha, \epsilon) \subset B(y, \epsilon)$, and $\prod_{\alpha \in I} B(y_\alpha, \epsilon)$ is a $\mathcal{T}'$-neighborhood y. Indeed, if $(z_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} B(y_\alpha, \epsilon_\alpha)$. Then $\forall \alpha \in I$, if $\alpha \in I_\epsilon$, then $d_\alpha(y_\alpha, z_\alpha) \leq \epsilon/2$. If $\alpha \in I \setminus I_\epsilon, d_\alpha(y_\alpha, z_\alpha) \leq \epsilon/2$ as $\text{diam} X_\alpha \leq \epsilon/2$.
4. TODO: finish proof later

$\square$

Theorem 1.6.5

Let (X, d) be a metric space. Then

$$x_n \to x \text{ in the metric topology } \iff d(x_n, x) \to 0 \text{ in the standard topology of } \mathbb{R}$$

Example 1.6.4

In $\mathbb{R}^{\mathbb{Z}_{\geq 0}} = \{f : \mathbb{Z}_{\geq 1} \to \mathbb{R}\} = \{(x_n)_{n \geq 1} \text{ sequence in } \mathbb{R}\}$. Then

$$d((x_n)_{n \geq 1}, (y_n)_{n \geq 1}) = \sup\{\min(|x_n - y_n|, 1) | n \geq 1\}$$

is a metric. It does not induce the product topology, but rather the **uniform topology**.
This works for $\mathbb{R}^I$, where I is any set. For instance, $\mathbb{R}^\mathbb{R} = \{f : \mathbb{R} \to \mathbb{R}\}$. Then

$$d(f, g) = \sup\{\min(|f(x) - g(x)|, 1) | x \in \mathbb{R}\}$$

However, $d'((x_n)_{n \geq 1}, (y_n)_{n \geq 1}) = \sup\{\min(|x_n - y_n|, \frac{1}{n}) | n \geq 1\}$ induces the product topology.

Note:-

In the space of functions from $\mathbb{R} \to \mathbb{R}$, the ball of radius ϵ centered at a function f, $B(f, \epsilon)$, is the set of functions whose graph is contained in the "ϵ-tubular neighborhood" of f.

1.7 The Quotient Topology

Definition 1.7.1: Inclusion Map

Let X be a topological space, and $Y \subset X$. We call $i : Y \to X$ defined by the rule $y \mapsto y$. Then the subspace topology on $Y = \{U \cap Y | U \subset X\}$ where $U \cap Y = i^{-1}(U)$. So the subspace topology is $\{i^{-1}(U) | U \subset X \text{ is open}\}$. This is the coarsest topology that makes i continuous.

Definition 1.7.2

Let X be a topological space, Y a set, and $f : X \to Y$ a map. Then $\mathcal{T} = \{U \cap Y | f^{-1}(U) \subset X \text{ is open}\}$ is a topology called the **Quotient Topology**. It is the finest topology that makes f continuous.

Proof: Let X be a topological space and $\mathcal{T} = \{U \cap Y | f^{-1}(U) \subset X \text{ is open}\}$. Then we want to demonstrate that $\mathcal{T}$ is indeed a topology.

1. $f^{-1}(U) = \emptyset$, which is open, so $\emptyset \in \mathcal{T}$. Then $f^{-1}(Y) = X$, which is open. Therefore, $Y \in \mathcal{T}$.
2. Let $(U_\alpha)_{\alpha \in I}$ be a family of sets in $\mathcal{T}$. Then

$$f^{-1}(\bigcup_{\alpha \in I} U_\alpha) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha)$$

By definition of $\mathcal{T}$, $f^{-1}(U_\alpha)$ is open in X for all $\alpha \in I$. Therefore, it's union is open since openness is stable under union. So $f^{-1}(\bigcup_{\alpha \in I} U_\alpha)$ is open in X. Then, by the definition of $\mathcal{T}$, we have $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$. Thus, $\mathcal{T}$ is stable under union.

3. Let $U, V \in \mathcal{T}$. We want to demonstrate that $U \cap V \in \mathcal{T}$. We have that

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

Note that $f^{-1}(U)$ and $f^{-1}(V)$ are both open in X, so their intersection is also open in X. Therefore, by definition of $\mathcal{T}$, we have $U \cap V \in \mathcal{T}$. Thus, $\mathcal{T}$ is stable under finite intersection.

$\mathcal{T}$ is therefore a topology. $\square$

Example 1.7.1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x, y) \mapsto x$. Then the quotient of the standard topology on $\mathbb{R}^2$ under f is the standard topology on $\mathbb{R}$. The proof is left as an exercise.

Definition 1.7.3: Equivalence Relation

Let X be a set. A **relation** is a map $\sim: X \times X \to \{\text{True, False}\}$, where True indicates that x and y are in relation, and False indicates that x and y are not in relation.
The map $\sim$ is called an **equivalence relation** if it satisfies the following conditions $\forall x, y, z \in X$:

1. **Symmetry:** $x \sim y \iff y \sim x$
2. **Reflexivity:** $x \sim x$
3. **Transitivity:** $(x \sim y \text{ and } y \sim z) \implies x \sim z$

If $\sim$ is an equivalence relation, the **equivalence class** of $x \in X$ is $\overline{x} = \{y \in X | y \sim x\}$.
The **quotient set** is $X/\sim = \{\overline{x} | x \in X\} \subset \mathcal{P}(X)$ the set of equivalence classes.
The **quotient map** $f : X \to X/\sim$ defined by $x \mapsto \overline{x}$ is surjective.

Example 1.7.2

Let X, Y be sets, and $f : X \to Y$ be a surjective map. Let $x \sim x'$ if and only if $f(x) = f(x')$. Then $\sim$ is an equivalence relation. Moreover, $\overline{f} : X/\sim \to Y$ defined by $\overline{x} \mapsto f(x)$ is well-defined and bijective.

Example 1.7.3

Let $X \subset [0,1]^2 \subset \mathbb{R}^2$ have the subspace topology. Then we want to "identify" $(0, y)$ with $(1, y)$ and $(x, 0)$ with $(x, 1)$. Set $(x, y) \sim (x', y')$ if and only if

$$\begin{cases} x = x' \text{ and } y = y' \\ x = x' \text{ and } \begin{cases} y = 0 \text{ and } y' = 1 \\ y = 1 \text{ and } y' = 0 \end{cases} \\ y = y' \text{ and } \begin{cases} x = 0 \text{ and } x' = 1 \\ x = 1 \text{ and } x' = 0 \end{cases} \end{cases}$$

Check that it is an equivalence relation as an exercise.

1.8 Closed Sets and Limit Points

Definition 1.8.1: Closed Set

Let X be a topological space. A subset $C \subset X$ is **closed** if its complement, $X \setminus C$, is open.

Example 1.8.1

In $\mathbb{R}$, $\{x\}$ is closed (since $\mathbb{R} \setminus C$ is open). Also, $[a, b]$ is closed, as $(-\infty, a) \cup (b, \infty)$ is open. The sets (a, b) and $[a, b)$ are not closed.

Theorem 1.8.1 Properties of Closed Sets

Let X be a topological space. Then

1. $\emptyset, X$ are closed.
2. Closed is stable under intersection.
3. Closed is stable under finite unions.

Proof: 1. TODO: Later

□

Theorem 1.8.2

Let X, Y be a topological space, and let $f : X \to Y$ be a mapping. Then f is continuous if and only if $\forall C \subset Y, f^{-1}(C)$ is closed in X

Proof: TODO: finish later □

Example 1.8.2

In $\mathbb{R}^d$, let $f : \mathbb{R}^d \to \mathbb{R}$ be a map. Let $x \in \mathbb{R}^d$, and define f by the rule $y \mapsto d(x, y)$ is continuous. The proof is left as an exercise

Example 1.8.3

$\{(x, y) \in \mathbb{R}^2 | x + y = 1\} \subset \mathbb{R}^2$ is closed.

Theorem 1.8.3

Let X be a topological space and $C \subset X$. Then

1. if C is closed, then $\forall (x_n)_{n \geq 0} \subset C$ such that $x_n \to x, x \in C$. In this case, we say C is "closed under sequential limits.
2. Suppose X is metrizable. If C is stable under limit then it is closed.

Proof: 1. Let $C \subset X$ be closed, and let $(x_n)_{n \geq 0} \subset C$ such that $x_n \to x$ for some $x \in X$. Suppose for contradiction that $x \in X \setminus C$. Since C is closed $X \setminus C$ is open, so it is a neighborhood of x. Since $x_n \to x, \exists n_0$ such that $\forall n \geq n_0, x_n \in X \setminus C$. However, this contradicts the fact that $x_n \in C, \forall n \geq 0$. Therefore, $x \in C$, so C is closed under sequential limits.

2. Let d be a metric inducing a topology on X. To attempt a proof by contrapositive, suppose C is not closed, and therefore $X \setminus C$ is not open. Therefore, $\exists x \in X \setminus C$ such that $X \setminus C$ is not a neighborhood of x. Then, $\forall \epsilon > 0, B(x, \epsilon) \not\subset X \setminus C$. Then, $\forall n \geq 1, \exists x_n \in B(x, \frac{1}{n})$ such that $x_n X \setminus C$. Therefore, $x_n \in C$. Then $d(x_n, x) < \frac{1}{n}$, so $x_n \to x \notin C$. Therefore, C is not stable under sequential limits.

□

Definition 1.8.2: Closure, Interior, Boundary, and Limit Point

Let X be a topological space and $A \subset X$.

1. The **closure** of A, denoted $\overline{A}$, is defined as the intersection of all closed sets containing A. This is equivalent to the smallest closed set containing A, or the set $\{x \in X | \text{any neighborhood meets } A\}$.
2. The **interior** of A, denoted $\mathring{A}$ or $\mathrm{Int}(A)$, is defined as the union of all open sets contained in A. This is equivalent to the biggest set contained in A, or the set $\{x \in A | x \text{ has a neighborhood contained in } A\}$.
3. The **boundary** of A, denoted ∂A, is defined as $\overline{A} \setminus \mathring{A} = \overline{A} \cap (X \setminus \mathring{A})$. Note that ∂A is closed, and that $\mathring{A} \sqcup \partial A$.
4. Let $x \in X$. Then x is **accumulation** or **limit point** if all x meets $A \setminus \{x\}$.

Proof of the equivalence of the definition of interior with the listed set

Proof: Let $U = \{x \in A | \exists N \subset A \text{ a neighborhood of } x\}$. We want to show that $U = \mathring{A}$.

($\subset$) To demonstrate that $U \subset \mathring{A}$, we just need to show that U is open. Let $x \in U$. Then $\exists N$ an open neighborhood of x such that $N \subset A$. Then $\exists V \subset X$ open such that $x \in V \subset N \subset A$. Then $V \subset U$. Indeed $\forall y \in V, y \in V \subset A$, so $y \in U$, hence U is a neighborhood of x, so U is open. Therefore, $U \subset \mathring{A}$.

($\supset$) Let $x \in \mathring{A}$. Then $x \in \mathring{A} \subset A$. Note that $\mathring{A}$ is a neighborhood of x, so $x \in U$. Therefore, $\mathring{A} \subset U$ and $\mathring{A} = U$. $\square$

Theorem 1.8.4

Let X be a topological space, and $A \subset X$. Then $\overline{X \setminus A} = X \setminus \mathring{A}$ and $\mathrm{Int}(X \setminus A) = X \setminus \overline{A}$.

Proof: Let $\mathcal{T} = \{C \subset X | C \text{ is closed and contains } X \setminus A\}$. Then $C \in \mathcal{T}$ if and only if $X \setminus C$ open and contained in A. Let $\mathcal{U} = \{U \subset X | U \text{ is open abd contained in } A\}$. Then

$$\overline{X \setminus A} = \bigcap_{C \in \mathcal{T}} C = \bigcap_{U \in \mathcal{U}} X \setminus U = X \setminus (\bigcup_{U \in \mathcal{U}} U) = X \setminus \mathrm{Int}(A)$$

$\square$

Theorem 1.8.5

Let X be a topological space and $A \subset X$.Then

1. $\overline{A} \supset \{x \in X | \exists (x_n)_{n \geq 0} \subset A, \text{ such that } x_n \to x\}$
2. Suppose X is metrizable, then $\bar{A} = \{x \in X | \exists (x_n)_{n \geq 0} \subset A, \text{ such that } x_n \to x\}$

Example 1.8.4

Let X be a topological space and $Y \subset X$ with subspace topology. Then

$$\{\text{closed subsets of } Y\} = \{C \cap Y | C \subset X \text{ closed}\}$$

Indeed, if $C \subset X$ closed, then $Y \setminus (C \cap Y) = (X \setminus C) \cap Y$ open in Y. If $C' \subset Y$ is closed, then $Y \setminus C' \subset Y$ is open. Then, $\exists U \subset X$ such that $Y \setminus C' = U \cap Y$. Then

$$C' = Y \setminus (U \cap Y) = (X \setminus U) \cap Y$$

Note that $X \setminus U$ is closed in X.

Theorem 1.8.6

Let $A \subset Y$ be a subset. Then $\overline{A}^Y$ is the closure of A in Y, and $\overline{A}^Y = \overline{A}^X \cap Y$.

$\overline{A}^X \cap Y$ is closed and contains A, so $\overline{A}^X \cap Y \supset \overline{A}^Y$. Conversely, $\overline{A}^Y = C \cap Y$ for some $C \subset X$ closed. Then $A \subset C$, so $\overline{A}^X \subset C$, and $\overline{A}^X \cap Y \subset C \cap Y$. So $\overline{A}^Y = \overline{A}^X \cap Y$.

Corollary 1.8.1

Let X_1, X_2 be two topological spaces. Let $C_1 \subset X_1$ and $C_2 \subset X_2$ be closed. Then $C_1 \times C_2$ is closed in $X_1 \times X_2$. Indeed,

$$(X_1 \times X_2) \setminus (C_1 \times C_2) = (X_1 \times (X_2 \setminus C_2)) \cup ((X_1 \setminus C_1) \times C_2)$$

Note that $X_1 \setminus C_1, X_2 \setminus C_2$ are open since C_1, C_2 are closed. Then, $X_1 \times (X_2 \setminus C_2)$ and $(X_1 \setminus C_1) \times C_2$ are both open, so their union is open.

Proof: Indeed,

$$(X_1 \times X_2) \setminus (C_1 \times C_2) = (X_1 \times (X_2 \setminus C_2)) \cup ((X_1 \setminus C_1) \times C_2)$$

Note that $X_1 \setminus C_1, X_2 \setminus C_2$ are open since C_1, C_2 are closed. Then, $X_1 \times (X_2 \setminus C_2)$ and $(X_1 \setminus C_1) \times C_2$ are both open, so their union is open. $\square$

Note:-

Let $A_1 \subset X_1$ and $A_2 \subset X_2$. Then $\overline{A_1 \times A_2} = \overline{A}_1 \times \overline{A}_2$, which is closed and contains $A_1 \times A_2$.

Proof: We wat to show that $\overline{A}_1 \times \overline{A}_2 \subset \overline{A_1 \times A_2}$.
Let $(x_1, x_2) \in \overline{A}_1 \times \overline{A}_2$. Let N be a neighborhood of (x_1, x_2) in $X_1 \times X_2$. Then, $\exists U_1 \subset X_1$ neighborhood of x_1 and $\exists U_2 \subset X_2$ neighborhood of x_2 such that $U_1 \times U_2 \subset N$.
Since, for $i \in \{1, 2\}, \exists y_i \in U_i \cap A_i$,we have that $(y_1, y_2) \in (U_1 \times U_2) \cap (A_1 \times A_2) \subset N \cap (A_1 \times A_2)$. Therefore, $(x_1, x_2) \in \overline{A_1 \times A_2}$ $\square$

Note:-

For A, B two sets,

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof: Let A, B be two sets. $(\subset)$ Since closed-ness is stable under finite union, we have that $\overline{A} \subset \overline{B}$ is closed and contains $A \cup B$. Therefore, since the closeure of a set is the smallest closed set containing that set, we have that

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}$$

$(\supset)$ Clearly $A \subset A \cup B$, so $\overline{A} \subset \overline{A \cup B}$. Similarly, $B \subset A \cup B$, so $\overline{B} \subset \overline{A \cup B}$. Therefore,

$$\overline{A \cup B} \supset \overline{A} \cup \overline{B}$$

Thus,

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

$\square$

Definition 1.8.3: Hausdorff Space

Let X be a topological space. Then X is **<u>Hausdorff</u>** if $\forall x \neq y \in X, \exists U$ a neighborhood of x and $\exists V$ a neighborhood of y such that $U \cap V = \emptyset$.

Most topological spaces we will look at will be Hausdorff with the exception of some finite ones.

Example 1.8.5

1. Metric spaces (X, d) are Hausdorff. For $x \neq y \in X$, we only have to look at $B(x, \frac{d(x,y)}{2}) \cap B(y, \frac{d(x,y)}{2}) = \emptyset$ to see this.

2. $X = \{0,1,2\}, \mathcal{T}\{\emptyset, X, \{0\}, \{1,2\}\}$ is **not** Hausdorff.

3. If X is Hausdorff and $Y \subset X$ is a subspace, then Y is Hausdorff. The proof is left as an exercise.

4. Let X_1, X_2 be two Hausdorff spaces. Then $X_1 \times X_2$ is Hausdorff. Let $(x_1, x_2) \neq (y_1, y_2) \in X_1 \times X_2$. If $x_1 \neq y_1$, then $\exists U_1, V_1$ respective neighborhoods of x, y such that $U_1 \times V_1 = \emptyset$. Then $(U_1 \times X_2) \cap (V_1 \times X_2) = \emptyset$. The case where $x_1 = y_1$ but $x_2 \neq y_2$ is identical.

Theorem 1.8.7

Let X be a Hausdorff topological space. If $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$

Proof: Assume for contradiction that $x \neq y$. Then $\exists U, V$ respective neighborhoods of x, y such that $U \cap V \neq 0$. Since $x_n \to x, \exists n_0$ such that $\forall n > n_0, x_n \in U$, and since $x_n \to y, \exists k_0$ such that $\forall n > k_0, x_n \in V$. Then, $\forall n > \max(n_0, k_0), x_n \in U \cap V$, which is a contradiction. □

Theorem 1.8.8

Let X be a Hausdorff topological space and let $A = \{x_1, ..., x_n\} \subset X$ be a finite subset. Then A is closed.

Proof: TODO: transcribe proof later □

Note:-

Consider the set of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ and take $\mathcal{T} = \{U \subset \mathbb{N} | \mathbb{N} \setminus U \text{ is finite}\}$. Then every finite set is closed, but $(\mathbb{N}, \mathcal{T})$ is not Hausdorff. The proof is left as an exercise.

Definition 1.8.4: Continuous at a Point

Let X, Y be topological spaces, take $x \in X$, and let $f : X \to Y$ be a map. Then f is **continuous at x** if $\forall N$ neighborhood of $f(x)$, $f^{-1}(N)$ is a neighborhood of x.

Theorem 1.8.9

Let X, Y be topological spaces. Then $f : X \to Y$ is continuous if and only if it is continuous at every point

Proof: TODO: transcribe and complete proof □

Example 1.8.6

Let X, Y, Z be topological spaces.

1. Constant maps are continuous.

2. Let $A \subset X$. Then the inclusion map $A \hookrightarrow X$ is continuous.

3. Let $f : X \to Y$ and $g : Y \to Z$ be maps such that f is continuous at $x \in X$ and g is continuous at $f(x) \in Y$. Then $g \circ f : X \to Z$ is continuous at x.

4. Let $A \subset X$ be a subspace and $f : X \to Y$ be a continuous map. Then $f_{|A} : A \to Y$ defined by $a \mapsto f(a)$ is continuous.

TODO: finish inserting examples later

Chapter 2

Connectedness

2.1 Paths

Definition 2.1.1: Path

Let X be a topological space, and take $x, y \in X$. A **path** from x to y is a continuous map $p : [a,b] \to X$, where $a \leq b \in \mathbb{R}$, such that $p(a) = x$ and $p(b) = y$.

- if $p : [a,b] \to X$ and $q : [b,c] \to X$ are paths such that $p(b) = q(b)$, the **concatenation** is $h : [a,c] \to X$ defined by

  $$\begin{cases} t \mapsto p(t) \text{ if } t < b \\ t \mapsto q(t) \text{ if } t \geq b \end{cases}$$

  h is well defined.

Note:-

If $p : [a,b] \to X$ is a path from x to y, then $q : [0,1] \to X$ defined by $t \mapsto p((1-t)a + tb)$ is also a path from x to y.

Theorem 2.1.1 The Pasting Lemma

Let X, Y be topological spaces, and let $A, B \subset X$ be closed such that $X = A \cup B$. Let $f : A \to Y$ and $g : B \to Y$ be continuous such that $f(x) = g(x) \forall x \in A \cap B$. Take $h : X \to Y$ to be the function defined by

$$h(x) = \begin{cases} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B \end{cases}$$

Then h is well-defined and continuous.

Note:-

Let X be a topological space and $C \subset X$ a closed subspace. A subset $A \subset C$ is closed in C if and only if A is closed in X.

Proof: $(\implies)$ Let $A \subset C$ be closed in C. Since A is closed in $C, \exists D \subset X$ such that $A = D \cap C$, wher D, and since closed-ness is stable under intersection, $D \cap C$ is closed in X. $(\impliedby)$ Let $A \subset C$ be closed in X. Then $A = A \cap C$, so A is closed in C. □

Then, for the proof of the pasting lemma:

Proof: Let $C \subset Y$ be closed. We want to show that $h^{-1}(C) \subset X$ be closed. Note that

$$h^{-1}(C) = (h^{-1}(C) \cap A) \cup (h^{-1}(C) \cap B) = f^{-1}(C) \cup g^{-1}(C)$$

Where $f^{-1}(C) \subset A$ and $g^{-1}(C) \subset B$ are closed in X by the previous fact. Therefore, $h^{-1}(C)$ is closed in X as closed sets are stable under finite union. □

Corollary 2.1.1

Concatenations of paths are paths.

2.2 Connectedness

Definition 2.2.1

Let X be a topological space and $x, y \in X$. Then

- x and y are **path-connected in X** if $\exists$ a path in X from x to y.
- X is path-connected if any two points are path connected
- Note that path-connectedness in X is an equivalence relation. The equivalence class of $x \in X$ is the path-connected component of x.

Proof: Proof of equivalence relation

1. **Reflexivity**: take the constant path.
2. **Symmetry**: If $p : [0,1] \to X$ connects x to y, then $q : [0,1] \to X$ defined by $t \mapsto p(1-t)$ connects y to x.
3. **Transitivity**: If $p[0,1] \to X$ is a path from x to y, and $q[0,1] \to X$ is a path from y to z, then the concatenation of p and q is a path from x to z.

□

Example 2.2.1

In $\mathbb{R}^k$, balls $B(x,r)$ are path-connected. In fact, $B(x,r)$ is **convex**, $\forall y, z \in B(x,r), \forall t \in [0,1], (1-t)y + tz \in B(x,r)$.

Theorem 2.2.1

Let X be a topological space and $p : [0,1] \to X$ a path from x to y. Let $U \subset X$ be an open neighborhood of x such that $y \notin U$. Then $\exists t \in [0,1]$ such that $p(t) \in \partial U$ the boundary of U.

Proof: TODO: insert proof here □

Definition 2.2.2: Separation

Let X be a topological space. A **separation** is a pair (U, V) of disjoint non-empty open sets that cover X. i.e. it is a pair $(U, X \setminus U)$ where U is a non-empty proper clopen (closed and open) subset.

Definition 2.2.3: Connected

X is connected if there is no separation.

Theorem 2.2.2

If X is path connected, then X is connected.

Proof: TODO: insert proof here □

Definition 2.2.4: Interval

Let $I \subset \mathbb{R}$. I is an interval if $I = \{a\}, [a,b], (a,b), [a,b), (-\infty, b), (-\infty, \infty), \emptyset$, etc.

Theorem 2.2.3

Let $A \subset \mathbb{R}$. The following are equivalent:

1. A is an interval
2. A is convex
3. A is path-connected
4. A is connected

Theorem 2.2.4

Let X, Y be a topological space and let $f : X \to Y$ be continuous. Then

1. if X is connected, then $f(X)$ is connected.
2. if X is path-connected, then $f(X)$ is path-connected.

Proof: TODO: insert proof here □

Proof: TODO: insert proof here □

Theorem 2.2.5 The Intermediate Value Theorem

Let X be a connected topological space and $f : X \to \mathbb{R}$ be continuous. Then $f(X) \subset \mathbb{R}$ is an interval, i.e. $\forall x, y \in X, \forall t \in \mathbb{R}$ if $f(x) \leq t \leq f(y)$ then $\exists z \in X$ such that $f(z) = t$.

Proof: TODO: insert proof here □

Example 2.2.2

$Y = \{(t, \sin(\frac{1}{t}))|t > 0\} \cup \{0\} \times [-1, 1]$ is connected, but not path connected.

Proof: TODO: Insert proof here (from lecture on 2/12) □

Theorem 2.2.6

Let X be a topological space and $A \subset X$ be connected. Then $\overline{A}$ is connected. In particular, if A is dense in X (i.e. $\overline{A} = X$) then X is connected.

Proof: TODO: insert proof here □

Theorem 2.2.7

Let X be a topological space and take $x, y \in X$. Suppose $p : [0, 1] \to X$ is a path from x to y and $\exists U$ open such that $x \in U$ but $y \notin U$. Then $\exists t$ such that $p(t) \in \partial U$.

Theorem 2.2.8

Let X, Y be $\begin{cases} \text{connected} \\ \text{path-connected} \end{cases}$. Then $X \times Y$ is $\begin{cases} \text{connected} \\ \text{path-connected} \end{cases}$.

Proof: TODO: insert proof here □

Theorem 2.2.9

Let X be a topological space, and let $(A_\alpha)_{\alpha\in I}$ be a collection of (path-)connected subspaces such that $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$. Then $\bigcup_{\alpha\in I} A_\alpha \neq \emptyset$ is (path-)connected.

Proof: TODO: insert proof here $\square$

Definition 2.2.5: Connected Elements

Let X be a topological space. Then $x, y \in X$ are **connected in** X if $\exists A \subset X$ a connected subspace such that $x, y \in A$. Note that this is an equivalence relation. The **connected component** of x is its equivalence class.

Proof: TODO: insert proof here $\square$

Note:-

If $\sim$ is an equivalence relation on X, then the equivalence classes form a partition of X. X is the disjoint union of all equivalence classes. If two equivalence classes intersect, then they are equal.

Theorem 2.2.10

1. If x, y are path connected in X, then x, y are connected in X.
2. The path connected component of x is contained in the connected complement of x.
3. The connected component of x is the union of all connected subspaces containing x, or equivalently, the largest connected subspace containing x.

Definition 2.2.6

Let X be a topological space and let $x \in X$. X is **locally (path) connected at** x if $\forall N$ open neighborhood of x, $\exists N' \subset N$ which is (path) connected. X is **locally connected** if it is locally connected at every point.

Note:-

A topological space being locally connected at x does not mean that there exists a connected open neighborhood. Similarly connectedness does not imply local connectedness.

Note:-

If X is locally connected, then any open $U \subset X$ is locally connected.

Theorem 2.2.11

Let X be a topological space and $A \subset X$ be a connected subspace. Then $\overline{A}$ is connected.

Corollary 2.2.1

If X is a topological space, then any connected component is closed.
Note that **path** connected components are not necessarily closed.

Proof: TODO: insert proof here $\square$

Theorem 2.2.12 Locally Connected

Let X be a locally (path) connected subspace. Then the (path) connected components are clopen.

Proof: TODO: insert proof here $\square$

Theorem 2.2.13

Let X be a locally path connected space. Then X is locally connected, and $\forall x \in X$, the path connected component of x is equal to the connected component of x.

Chapter 3

Compactness

3.1 Definition

Definition 3.1.1: Cover

Let X be a topological space and $Y \subset X$ a subspace. A collection $\mathcal{A} \subset \mathcal{P}(X)$ of subspace of X **covers** Y if $Y \subset \bigcup_{A \in \alpha} A$. $\mathcal{A}$ can be replaced by $(A_\alpha)_{\alpha \in I}$ and $\bigcup_{A \in \mathcal{A}} A$ by $\bigcup_{\alpha \in I} A_\alpha$. α is then called a **cover** of Y. α is an **open cover** if every $A \in \mathcal{A}$ is open.

Example 3.1.1

TODO: insert illustration here

Definition 3.1.2: Compact

Let X be a topological space. We say X is **compact** if every open cover $\mathcal{U} \subset \mathcal{P}(X)$ admits a finite subcover, i.e. $\exists U_1, ..., U_n \in \mathcal{U}$ such that $X \subset U_1 \cup ... \cup U_n$.

Example 3.1.2

Any finite space is compact. $\mathbb{R}^k$ is **not compact**. In fact, any subspace $X \subset \mathbb{R}^k$ is not compact. $\mathcal{U} = \{B(x,1) : x \in X\}$ is an open cover of X but does not admit a finite subcover. Otherwise, if $X \subset B(x_1,1) \cup ... \cup B(x_n,1)$, then X is bounded.

Note:-

Let X be a topological space and $Y \subset X$ a subspace. Then Y is compact if and only if every open cover of Y in X admits a finite subcover.

Proof: TODO: insert proof here □

Theorem 3.1.1

$[0,1]$ is compact.

Proof: TODO: insert proof here □

Theorem 3.1.2

Let X, Y be a topological space and $f : X \to Y$ be continuous. If X is compact, then $f(X)$ is compact.

Corollary 3.1.1
$[a,b] \subset \mathbb{R}$ is compact $\forall a \leq b \in \mathbb{R}$.

Theorem 3.1.3
Let X, Y be compact spaces. Then $X \times Y$ is compact.

Corollary 3.1.2
Any parallelepiped $[a_1, b_1] \times ... \times [a_k, b_k] \subset \mathbb{R}^k$ is compact.

Theorem 3.1.4
Let X, Y be topological spaces with X compact. Let $y \in Y$ and $U \subset X \times Y$ an open set containing $X \times \{y\}$. Then $\exists N$ a neighborhood of y such that $X \times N \subset U$.

Proof: TODO: insert proofs of the previous statements. □

Theorem 3.1.5
Let X be a compact space and $Y \subset X$ a closed subset. Then Y is compact.

Corollary 3.1.3
Any closed, bounded $A \subset \mathbb{R}^k$ is compact.

Proof: TODO: insert proofs of the previous statements. □

Theorem 3.1.6
Let X be a Hausdorff space. Then $Y \subset X$ being compact implies that Y is closed.

Corollary 3.1.4
$X \subset \mathbb{R}^k$ is compact if and only if it is closed and bounded.

Theorem 3.1.7 Extremal Value Theorem
Let X be a compact space and $f : X \to \mathbb{R}$ be continuous. Then $\exists a, b \in X$ such that $\forall x \in X$

$$f(a) \leq f(x) \leq f(b)$$

Note that we can just use $\min\{f(y) : y \in X\}$ for $f(a)$ and $\max\{f(y) : y \in X\}$ for $f(b)$.

Claim 3.1.1
Let $A \subset \mathbb{R}$ be bounded from above. Then $\sup A \in \overline{A}$ ($\exists (x_n)_n \subset A$ such that $x_n \to \sup A$). In particular, if A is closed, then $\sup A \in \overline{A}$.

Proof: By definition, $\sup A$ is the least upper bound of A, so $\forall n \geq 1, \sup A - \frac{1}{n}$ is not an upper bound. Then, $\exists x_n \in A$ such that $x_n > \sup A - \frac{1}{n}$. Moreover, $x_n \leq \sup A$. Then $d(x_n, \sup A) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore $x_n \to \sup A$. □

For the proof of the Extremal Value Theorem, we use the previous claim:

Proof: TODO: insert proof of Extremal Value Theorem here. □

Note:-

Let X be a topological space, and take $\mathcal{A} \subset \mathcal{P}(X)$ a collection of subsets of X. Let $\mathcal{B} = \{X \setminus A | A \in \mathcal{A}\}$. Note that

1. $\mathcal{A}$ is made of open sets if and only if $\mathcal{B}$ is made of closed sets
2. $\mathcal{A} = \{X \setminus B | B \in \mathcal{B}\}$
3. $\mathcal{A}$ covers X if and only if $\bigcap\limits_{B \in \mathcal{B}} B = \bigcap\limits_{A \in \mathcal{A}} X \setminus A = X \setminus \bigcup\limits_{A \in \mathcal{A}} A$ is empty.

Theorem 3.1.8

Let X be a topological space. Then X is compact if and only if for any $\mathcal{C} \subset \mathcal{P}(X)$ collection of closed sets such that $\bigcap_{C \in \mathcal{C}} C = \emptyset, \exists C_1, ..., C_n \in \mathcal{C}$ such that $C_1 \cap ... \cap C_n = \emptyset$.
$\mathcal{C} \subset \mathcal{P}(X)$ of closed sets satisfies the finite intersection property: $\forall C_1, ..., C_n \in \mathcal{C}, C_1 \cap ... \cap C_n = \emptyset$ if and only if we have $\cap_{C \in \mathcal{C}} C = \emptyset$.

The second aspect of the above theorem is somewhat trivial.

Corollary 3.1.5

Let X be a topological space and $(C_n)_{n \geq 1}$ be a non-increasing ($C_{n+1} \subset C_n$) sequence of $\begin{cases} \text{closed} \\ \text{compact} \end{cases}$ subspace. Then $\bigcap\limits_{n \geq 1} C_n \neq \emptyset$. If X is Hausdorff, then $\bigcap\limits_{n \geq 1} C$ is closed and compact.

3.2 Sequential Compactness

Definition 3.2.1: Subsequence

Let $(x_n)_{n \geq 1}$ be a sequence in a set X. A subsequence of $(x_n)_{n \geq 1}$ is a sequence of the form $(x_{n_k})_{k \geq 1}$ where $(n_k)_{k \geq 1}$ is an increasing sequence of positive integers (called an **extraction**).

Example 3.2.1

$\left(\frac{1}{2^k}\right)_{k \geq 1}$ is a subsequence of $\left(\frac{1}{n}\right)_{n \geq 1}$. Just take $n_k = 2^k$.

Note:-

Let $(n_k)_{k \geq 1}$ be an extraction. Then $n_k > n_{k-1}$, so $n_k \geq n_{k-1} + 1 \geq n_{k-2} + 2 \geq ... \geq n_1 + k - 1 \geq k \geq 1$.

Definition 3.2.2

Let X be a topological space. Let $(x_n)_{n \geq 1} \subset X$. An **accumulation point** of $(x_n)_{n \geq 1}$ is a limit of a converging subsequence of $(x_n)_{n \geq 1}$.

Example 3.2.2

The accumulation points of $((-1)^n)_{n \geq 1}$ are -1 and 1.
TODO: finish this example and warning

Claim 3.2.1
Let (X, d) be a metric space, and let $(x_n)_{n\geq 1} \subset X$ be a sequence. Then the set of accumulation points is

$$\bigcap_{n_0 \geq 1} \overline{\{x_n : n \geq n_0\}}$$

Theorem 3.2.1
Let (X, d) be a compact metric space. Then any sequence admits a converging subsequence.

Theorem 3.2.2
Let (X, d) be a metric space. Then X is compact if and only if X is sequentially compact.

Theorem 3.2.3 Lebesgue Number Lemma
Let (X, d) be sequentially compact metric space. Let $\mathcal{U} \subset \mathcal{P}(X)$ be an open cover. Then, $\exists \delta > 0$ such that $\forall x \in X, \exists U \in \mathcal{U}$ such that $B(x, \delta) \subset U$

TODO: insert proof of Lebesgue Nummer Lemma
TODO: insert proof of previous theorem, using Lemma

Definition 3.2.3: Uniform Continuity

Let (X, d_X), (Y, d_Y) be metric spaces. Let $f : X \to Y$ is **uniformly continuous** if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x_0 \in X, \forall x \in X, if d_x(x_0, x) < \delta$ then $d_Y(f(x_0), f(x)) < \epsilon$.
Recall continuity means $\forall x_0 \in X, \forall \epsilon > 0, \exists \delta > 0$ such that $x \in X$ if $d_X(x, x_0) < \delta$ then $d_Y(f(x), f(x_0)) < \epsilon$.
In the definition continuity, ϵ and δ depend on x_0. In the definition of uniform continuity, ϵ and δ do not.

Example 3.2.3
$f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x$ is uniformly continuous (take $\delta = \epsilon$). $g : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$.

Theorem 3.2.4
Let $(X, d_X), (Y, d_Y)$ be metric spaces, and let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

Proof: TODO: insert proof here □

3.3 Local Compactness

Definition 3.3.1: Local Compactness

Let X be a topological space and take $x \in X$.

- X is **locally compact at** x if $\exists N$ a compact neighborhood of x.
- X is **locally compact** if it is locally compact at every point.

Warning: this time, we don't require that "every neighborhood contains a compact neighborhood".

Example 3.3.1

1. Compactness implies local compactness/
2. $\mathbb{R}^k$ is locally compact (any closed ball).
3. $\mathbb{R}$ is not locally compact at 0.

TODO: insert proofs of examples here

Theorem 3.3.1

Let X be a locally compact Hausdorff space. Then $\forall x \in X$, any neighborhood contains a compact neighborhood. Ths implies that open subsets of X are locally compact.

TODO: insert proof of theorem here (from lecture on 3/4)

Theorem 3.3.2

Let X be a locally compact Hausdorff space such that $\infty \notin X$. Then there is a unique topology $\mathcal{T}_{\overline{X}} \subset \mathcal{P}(\overline{X})$ on $\overline{X} = X \cup \{\infty\}$ such that

1. $\forall U \subset X, U \in \mathcal{T}_{\overline{X}}$ if and only if $U \in \mathcal{T}_X$. Note that this implies $\mathcal{T}_X \subset \mathcal{T}_{\overline{X}}$.
2. $\overline{X}$ is a compact Hausdorff space.

Corollary 3.3.1

Locally compact Hausdorff spaces are exactly open subspaces of compact Hausdorff spaces

Example 3.3.2

$\mathbb{R}^d \hookrightarrow \mathbb{S}^d = \{v \in \mathbb{R}^{d+1} : ||v|| = 1\}$
TODO: complete example later

Proof: TODO: insert proof of theorem here □

Theorem 3.3.3 Tychonoff Theorem

Any product of compact spaces is compact.

The Tychonoff Theorem is admitted without proof

Example 3.3.3

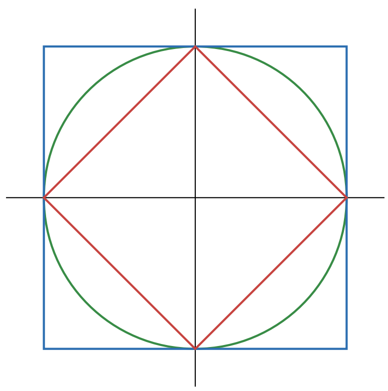
$[0, 1]^{\mathbb{R}}$ is compact

Chapter 4

Equivalence of Norms

We back to Normed Linear Space for a little while.

In $\mathbb{R}^n$, $u = (u_1, u_2, \cdots, u_n)$ where each $u_i \in \mathbb{R}$. we have $p-$norm: $\|u\|_p = \left(\sum_i |u_i|^p\right)^{\frac{1}{p}}$ where $1 \le p \le \infty$.
Balls in $\mathbb{R}^2$ w.r.t. $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$.



Observe: A set V in $\mathbb{R}^2$ is

$$\begin{aligned}
&\text{open w.r.t. } \|\cdot\|_1 &&\iff && V = \bigcup_{u \in V} \text{Box in } V \text{ centered box}\\
&\text{open w.r.t. } \|\cdot\|_2 &&\iff && V = \bigcup_{u \in V} \text{Diamond in } V \text{ centered box}\\
&\text{open w.r.t. } \|\cdot\|_\infty &&\iff && V = \bigcup_{u \in V} \text{Circle in } V \text{ centered box}
\end{aligned}$$

Definition 4.1: Equivalence of Norms

Suppose $\|\cdot\|$, $\|\cdot\|'$ are two norms in vector space V, We say that the two norms are equivalent if there are constants $\alpha, \beta > 0$ s.t.

$$\alpha\|x\|' \le \|x\| \le \beta\|x\|'$$

Example 4.0.1 (Norm Equivalence)

1. $\boldsymbol{p = \infty}$ **and** $\boldsymbol{p = 1}$

   $\|x\|_\infty = \max\{|x_i| \mid 1 \le i \le n\} \le \|x\|_1 = \sum_i |x_i|$

   $\|x\|_\infty \ge \text{each } |x_i| \implies n\|x\|_\infty \ge \|x\|_1$

   Hence

   $$\|x\|_\infty| \le \|x\|_1 \le n\|x\|_\infty \text{ and } \frac{1}{n}\|x\|_1 \le \|x\|_\infty \le \|x\|_1$$

2. $\boldsymbol{p = \infty}$ **and** $\boldsymbol{p = 2}$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

Theorem 4.1

All norms on a finite dimensional vector space are equivalent

Proof: Proved in Theorem 5.2.7 □

Theorem 4.2

Suppose $\|\cdot\|$ and $\|\cdot\|'$ are equivalent on a vector space V. Then

(i) $\{x_n\} \to x$ w.r.t. $\|\cdot\| \iff \{x_n\} \to x$ w.r.t $\|\cdot\|'$

(ii) $S \subset V$ is open w.r.t $\|\cdot\| \iff S$ is open w.r.t $\|\cdot\|'$

Proof: For both proofs if we just prove one direction the we are done actually since we can just replace the words to prove for opposite direction,

(i) **If Part:-**

Since $\|\cdot\|, \|\cdot\|'$ are equivalent we have $\exists\ \alpha, \beta$ such that $\alpha\|x\|' \leq \|x\| \leq \beta\|x\|'$. So if we show $\alpha\left\|x_n - x\right\| < \|x_n - x\| < \alpha\epsilon$ we are done.

Let $\{x_n\} \to x$ w.r.t $\|\cdot\|$ i.e. $\forall\ \epsilon > 0\ \exists\ N$ s.t. $\forall\ n > N\ \ \|x_n - x\| < \alpha\epsilon$. Hence we have $\alpha\left\|x_n - x\right\|' < \alpha\epsilon$. Hence $\forall\ \epsilon > 0\ \exists\ N$ such that $\forall\ n > N\ \ \left\|x_n - x\right\|' < \epsilon$

□

(ii) **Only If Part:-**

V is open w.r.t $\|\cdot\| \iff \bigcup\limits_{x \in V} B_r(x)$ and V' is open w.r.t $\|\cdot\|' \iff \bigcup\limits_{x \in V} B'_r(x)$

Now we have

$$B_r(x) = \{y \mid \|y - x\| < r\} \text{ and } B'_r(x) = \{y \mid \|y - x\|' < s\}$$

Hence by equivalence of the norms for any v

$$\alpha\|v\|' \leq \|v\| \leq \beta\|v\|'$$

Since $\|v\| < r$ we have

$$\|v\|' \leq \frac{r}{\beta} \implies B'_{\frac{r}{\beta}}(x) \subset B_r(x)$$

□

Corollary 4.1

$p = 1$ and $p = \infty$ on $\mathbb{R}^n$ (and $\mathbb{C}^n$) give the same topology as $p = 2$ norm

Corollary 4.2

Let x_m be a square in $\mathbb{R}^n$. $\overline{x_m} = (x_{m_1}, x_{m_2}, \cdots, x_{m_n})$. Then $\{\overline{x_m}\} \to x = (x_1, x_2, \cdots, x_n)$ w.r.t $\|\cdot\|_2 \iff \{x_{m_i}\} \to x_i$ in $\mathbb{R}$ for each i.

Note:-

We can check this w.r.t $\|\cdot\|_\infty$

$$\begin{aligned}
\overline{x_m} \to \overline{x} \text{ w.r.t } \|\cdot\|_\infty &\iff \forall\ \epsilon > 0\ \exists\ N \text{ s.t. } \forall\ m > N\ \max\{|x_{m_i} - x_i| \mid 1 \le i \le n\} \\
&\iff \text{each } |x_{m_i} - x_i| < \epsilon\ \forall\ i \\
&\iff \lim_{n\to\infty} x_{m_i} = x_i\ \forall\ i
\end{aligned}$$

Chapter 5

Compactness

5.1 Sequentially Compact

Definition 5.1.1: Sequentially Compact

Let (X, d) be a metric space. X is called sequentially compact if every sequence in X has a convergent subsequence. (Often applied to a subset S of X)

Note:-

For S to be sequentially compact the limit of subsequence must be in S

Definition 5.1.2: Boundedness

A subset S of (X, d) is bounded if $S \subset B_r(x)$ for some $x \in X$ and $r > 0$

Note:-

Boundedness depends on the metric but if two metrics are "equivalent" analogous to norms)

Theorem 5.1.1

A subset K of $\mathbb{R}^n$ is sequentially compact $\iff$ K is closed and bounded

Proof: Proof in steps

1. **A closed interval $[a, b]$ in $\mathbb{R}$ is sequentially compact**

   Proof: Given a sequence $x_1, x_2, \cdots$ in $\mathbb{R}$ in $[a, b]$ we can extract a monotonic subsequence as follows:

   We call x_i to be a peak if $x_i > x_j \; \forall \; j > i$. Now there are two cases. If number of peaks is infinite then the next peak comes after the previous one so smaller than the previous one. So its a strictly decreasing sequence. If number of peaks are finite then at some point we cant find a peak with this property that means no matter which term i peak there is at least one term after that which is greater than or equal to that term. y_1 =a term after the last peak. and y_{i+1} =a term after y_i such that $y_{i+1} \geq y_i$. Hence $y_1, y_2, \cdots$ is a weakly increasing sequence.

   When $\{x_n\}$ contained in $[a, b]$ by boundedness of the monotonic subsequence, it converges to its sup/inf and the limit is in $[a, b]$ □

2. **$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is sequentially compact (w.r.t p−norm for $p = 1, 2, \infty$. Later for any norm)**

   Proof: Recall a sequence $\{x_m\} \to x$ in $\mathbb{R}^n$ $\iff$ The sequence converges in each coordinate i.e. $x_{m_i} \to x_i$

   Take a sequence in the given box. Extract a subsequence whose entries in 1st slot converge (necessarily to x_i in $[a_1, b_1]$ by step 1 From this sequence, extract a further subsequence whose entries in second slot converge to $x_2 \in [a_2, b_2]$. Continue □

3. **Every closed subset of a sequentially compact set is sequentially compact**

   Proof: Exercise □

   This will show each closed and bounded subset of the Euclidean Space $\mathbb{R}^n$ is sequentially compact. (because such a set will be contained in a box)

4. **If K is sequentially compact then K is closed and bounded**

   Proof: If K is not closed then some limit point x of K will not be in K. Then there is a sequence $\{y_m\}$ in K converges to $x \notin K$ violating sequential compactness of K.

   If K is not bounded take $\{x_m\} \in K$ with $\|x_m\| \geq n$ then $\{x_m\}$ can not be convergent □

□

Note:-

Step 4 works for any metric space. Then we need to have a ball instead of norm

Theorem 5.1.2

If K is a sequentially compact of a metric space X, then K is closed and bounded

Proof: Same argument as step 4 use x_m such that $d(x_m, x) \geq m$ □

Question 1

If K is closed and bounded in (X, d) $\implies$ K is sequentially compact

Solution: No. Any counter-example. Define a metric on real number which induces same topology as the normal topology in such a way that there is a closed and bounded set that is not compact.

Question 2

1. If V, W are normed linear spaces can we define a norm on $V \times W$?
2. If V, W are metric spaces can we define a metric on $V \times W$?
3. If V, W are topological spaces can we define a topology on $V \times W$?

5.2 Open Cover and Compactness

Definition 5.2.1: Open Cover

Let $\{V_\alpha\}_{\alpha \in I}$ be a family of subsets of metric space X we say that $\{V_\alpha\}_{\alpha \in I}$ is a cover of X if $\bigcup\limits_{\alpha} V_\alpha = X$ and we say that $\{V_\alpha\}_{\alpha \in I}$ is an open cover if each V_α is open (in X)

Definition 5.2.2: Compact

X is called compact if each open cover of X has a finite subcover i.e. $\{V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}\} \subset \{V_\alpha\}_{\alpha \in I}$ with $V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n} = X$

Note:-

1. This definition makes sense for any topological space X.

   If X is a metric space then it is a fact that X is compact $\iff$ X is sequentially compact. This is not true for general topological spaces. Both implications fail.

2. Reformulation of compactness for subset K of X in terms of open sets of X

   K is compact $\iff$ Every cover of K bt open sets of K has a finite subcover.
   As open sets of K are precisely (open sets of X) $\cap K$. We have the following
   K is compact $\iff$ For any family $\{V_\alpha \cap K\}_{\alpha\in I}$ where V_α are open in X whose union is K, there is a finite subcover.
   $\iff$ For any family $\{V_\alpha \cap K\}_{\alpha\in I}$ of open sets in X such that $\bigcup\limits_{\alpha\in I} V_\alpha \supset K$, there must be a finite subfamily $V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}$ with $\bigcup\limits_{i=1}^{n} V_{\alpha_i} \supset K$

   If i take this definition of compactness of a subset K of metric space X then K is compact as subset of X $\iff$ K is compact as a subset of it itself

Theorem 5.2.1 Haine Borel Theorem

$$K \subset \mathbb{R}^n \text{ is compact} \quad \iff \quad K \text{ is closed and boundeded}$$

(w.r.t $p = 1, 2$ or ∞ norm as they are equivalent.)

Proof: **Only If Part:-**

Proof in steps

1. Closed interval $[a, b]$ is compact in $\mathbb{R}$. ***Proof:*** Theorem 5.2.4
2. Closed box $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact in $\mathbb{R}^n$. ***Proof:*** Theorem 5.2.6
3. A closed subset of a compact set is compact. ***Proof:*** Theorem 5.2.3

These steps would give the backward direction of Haine Borel Theorem i.e. suppose K is closed and bounded in $\mathbb{R}^n \implies K \in [-M.M]^n \implies$ compact by 2

If Part:-

Bounded: First we have to show that K is compact $\implies$ K is bounded an i.e. $K \subset B_r(x)$ in (X, d) for some $x \in X$, $r > 0$

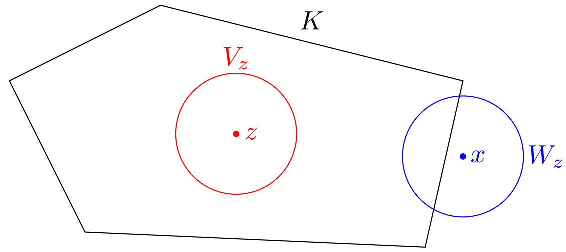
Consider open cover $\{B_n(x)\}_{n\in\mathbb{Z}^+}$ of X and hence of K. This must have a finite subcover $B_{n_1}(x), B_{n_2}(x), \cdots, B_{n_k}(x)$. Take $r = \max\{n_1, n_2, \cdots, n_k\}$ Hence

$$K \text{ is compact} \implies K \text{ is closed}$$

Closed: We will show that $X \setminus K$ is open. Pick $x \notin K$. Enough to construct an open neighborhood $U_x \ni x$ such that $U_x \cap x = \phi$

Take $z \in K$. Let $c = d(x, z)$ then

$$\underset{\substack{\parallel \\ W_z}}{B_{\frac{c}{3}}(x)} \quad \cap \quad \underset{\substack{\parallel \\ V_z}}{B_{\frac{c}{3}}(z)} = \phi \quad \text{by triangle inequality}$$



Now $\bigcup\limits_{z\in K} V_z \supset K$. So $\{V_z\}$ is an open cover of K. By compactness we have $V_{z_1} \cup V_{z_2} \cup \cdots \cup V_{z_n} \supset K$. As $W_z \cap V_z = \phi \ \forall z \in K$. We have $\underbrace{(W_{z_1} \cup W_{z_2} \cup \cdots \cup W_{z_n})}_{\substack{\text{Finite intersection of} \\ \text{open neighborhoods of } x, \\ \text{so call this } U_x}} \cap K = \phi$ $\square$

Key fact that made this work: For $x \neq z$ in X, we could find open neighborhoods of V and W (of x and z respectively) such that $V \cap W = \phi$. Topological spaces that satisfy this property are called Housdorff.

What we proved is the following

Theorem 5.2.2

For a Housdorff Topological space X any compact subset K is closed and bounded

Theorem 5.2.3 Haine Borel Theorem - If Part: Step 3

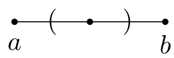
C is a closed subset of compact set $X \implies C$ is compact.

Proof: Take any open cover $\{V_\alpha\}_{\alpha \in I}$ of C by open sets in X i.e $\bigcup\limits_{\alpha} V_\alpha \supset C$. Now $\{V_\alpha\}_{\alpha\in I} \cup \{X \setminus C\}$ is an open cover of x. We have a finite subcover by compactness of X. The same subcover (after dropping $X \setminus C$ if necessary) works for C. $\square$

Wrong Concept 5.1: Closed interval $[a, b]$ is compact in $\mathbb{R}$

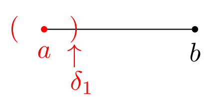
Suppose $\{V_\alpha\}_{\alpha \in I}$ is an open cover of $[a, b]$ by open sets in $\mathbb{R}$.

Hence every one of the points in the interval is covered by one of the V_α. Hence there is some interval contained in the V_α



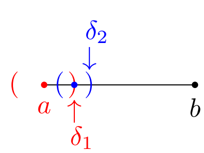
So i could just ignore the V_α and say for each point in the interval we can get an open interval that is part of a V_α. So how can i find a subcover. I could simply travel from one end to the other.

So i start with a so a must be contained in some open interval



Not only that i have covered up a small segment of the closed interval, upto a point, $a+\delta_1$. Say $[a, a+\delta_1) \subset V_1$.

Let $a + \delta_1$ is contained in some open interval which is contained in V_2 upto the point $a + \delta_2$



Now continue.

What is wrong with this ?

We could have smaller and smaller intervals. For example length of first interval can be $\frac{1}{3}$, length of second interval can be $\frac{1}{9}$, length of third interval can be $\frac{1}{27}$ and so on. So its a geometric progression and it will sum less than 1. So i just may not get there in finite number of steps.

Question 3

Suppose X is a topological space that is compact and 5.2 (Take x ro be a compact metric space if you like). Prove that given disjoint compact subsets K and L, there are disjoint open sets U and V with $K \subset U$ and $L \subset V$(First do it for $K =$ single point)

In the above exercise we could have replaced the word compact with another word which is closed because X is given to be compact so any closed set will be compact and in a Housdorff space compact subset is also closed.

Note:-

Cauchy Sequence in Metric space need not converge. For example $(0, 1)$ and take the sequence $\frac{1}{n}$. It wants to converge to 0 but 0 is not there.

Theorem 5.2.4 Haine Borel Theorem - If Part: Step 1

$[0, 1]$ is compact in $\mathbb{R}$

Proof: Let $\{V_\alpha\}_{\alpha \in I}$ be a family of open sets in $\mathbb{R}$ covering $[0, 1]$.

Let $S = \{a \in [0, 1] \mid [0, a]$ can be covered by a finite number of V_α's$\}$. Our goal is to prove $1 \in S$.

Let $0 \leq x < y \leq 1$. So $[0, x] \subset [0, y]$. This $y \in S \implies x \in S$ i.e $x \notin S \implies y \notin S$. Now S is nonempty because $0 \in S$ and S is bounded. Let $u =$ lub of S. Clearly $0 \leq u \leq 1$. Hence it is enough to show $u = 1$ and $u \in S$.

$0 \in$ some open set V_α. Hence $\exists\ \epsilon > 0\ B_\epsilon(0) \subset V_\alpha$. Hence $\forall$ point $x \in [0, \epsilon)\ x \in S$

For $a \in [0, u)$, $a \in S$ (otherwise a itself would be an upper bound for S). As $\{V_\alpha\}_{\alpha \in I}$ cover $[0, 1]$, $u \in V_\beta$. So $\exists\ \epsilon > 0$ such that $(u - \epsilon, u + \epsilon) \subset V_\beta$ As $u - \epsilon \in S$ we have $V_{\alpha_1} \sup V_{\alpha_2} \sup \cdots V_{\alpha_k} \supset [0, u - \epsilon]$ Then $V_{\alpha_\beta} \cup V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots V_{\alpha_k} \supset \left[0, u + \frac{\epsilon}{2}\right]$. So $u = 1$ because otherwise some $u + \delta \in S$ contradicting that u is an upper bound. $\square$

Question 4

Can the strategy from the last time be made to work ti actually extract a finite subcover of a given cover.

Theorem 5.2.5

Suppose $X \xrightarrow{f} Y$ continuous and $K \subset X$ is compact. Then $f(K)$ is compact

Proof: Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of $f(k)$ by open sets V_α of Y. So

$$\bigcup_\alpha V_\alpha \supset f(K) \implies f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}\left(V_\alpha\right) \supset f^{-1}(f(K)) \supset K$$

Thus $\left\{f^{-1}(V_\alpha)\right\}_{\alpha \in I}$ is an open (because of continuity Theorem **??**) cover of K.

Extract a finite subcover

$$\begin{aligned} & f^{-1}\left(V_{\alpha_1}\right) \cup f^{-1}\left(V_{\alpha_2}\right) \cup \cdots f^{-1}\left(V_{\alpha_m}\right) \supset K \\ \implies & f\left(f^{-1}\left(V_{\alpha_2}\right) \cup \cdots f^{-1}\left(V_{\alpha_m}\right)\right) \supset f(K) \\ \implies & \bigcup_{i=1}^{m} f\left(f^{-1}\left(V_{\alpha_i}\right)\right) \supset f(K) \end{aligned}$$

As $V_{\alpha_i} \supset f\left(f^{-1}\left(V_{\alpha_i}\right)\right)$ we have $\bigcup_{i=1}^{m} V_{\alpha_i} \supset f(K)$ ☐

Question 5

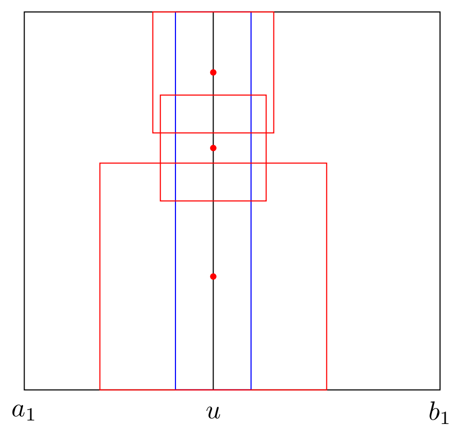
f(Sequentially compact K) is sequentially compact

Theorem 5.2.6 Haine Borel Theorem - If Part: Step ②

$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact in $\mathbb{R}^n$

Proof: Induction on n. $n = 1$ we already proved in Theorem 5.2.4.Let $\mathcal{F} = \{V_\alpha\}_{\alpha \in I}$ be a cover of K by open sets in $\mathbb{R}^n$. Fix $u \in [a_1, b_1]$ and consider $\{u\} \times \underbrace{[a_2, b_2] \times \cdots \times [a_n, b_n]}_{\substack{=C \text{ is compact} \\ \text{by induction on } n}}$ Hence $\{u\} \times C$ is compact because $\mathbb{R}^{n-1} \to \mathbb{R}^n$ which maps $(y_2, y \cdots, y_n) \mapsto (u, y_2, \cdots, y_n)$ or $f(C) = \{u\} \times C$ is continuous.

For each $p = (u, y_2, \cdots, y_n)$ in $\{u\} \times C$ pick an open neighborhood $V_p \in \mathcal{F}$. Hence $V_p \supset (x - \epsilon, x + \epsilon) \times \underbrace{(y_2 - \epsilon, y_2 + \epsilon) \times \cdots \times (y_n - \epsilon, y_n + \epsilon)}_{W_p}$ for some $\epsilon = \epsilon_p$ depending on p



By compactness of $\{u\} \times C$, extract a finite subcover of the cover $\{W_p\}$. Hence $W_{p_1} \cup W_{p_2} \cup \times \cup W_{p_k} \supset \{u\} \times C$. Since its a union of open sets we have in fact $W_{p_1} \cup W_{p_2} \cup \times \cup W_{p_k} \supset (u - \epsilon, u + \epsilon) \times C$ where $\epsilon = \min\{\epsilon_{p_1}, \epsilon_{p_2}, \cdots, \epsilon_{p_k}\}$. Let $\mathcal{F}_u = \{V_{p_1}, V_{p_2} < \cdots, V_{p_k}\}$. So

$$V_{p_1} \cup V_{p_2} \cup \cdots \cup V_{p_k} \supset W_{p_1} \cup W_{p_2} \cup \cdots \cup W_{p_k} \supset (u - \epsilon, u + \epsilon) \times C$$

i.e. this finite subcover $\mathcal{F}_u$ cover not just the slice but a tube around it.

Now as u varies in $[a_1, b_1]$, $(u - \epsilon_u, u + \epsilon_u)$ gives an open cover. Extract a finite subcover $(u_1 - \epsilon_{u_1}, u_1 + \epsilon_{u_1}), (u_2 - \epsilon_{u_2}, u_2 + \epsilon_{u_2}), \cdots, (u_l + \epsilon_{u_l}, u_l + \epsilon_{u_l})$. Then $\mathcal{F}_{u_1} \cup \mathcal{F}_{u_2} \cup \cdots \cup \mathcal{F}_{u_l}$ is a finite subcover of $[a_1, b_1] \times C = K$ ☐

Question 6

Why the map $\mathbb{R}^{n-1} \to \mathbb{R}^n$ which maps $(y_2, y \cdots, y_n) \mapsto (u, y_2, \cdots, y_n)$ or $f(C) = \{u\} \times C$ is continuous ?

Question 7

X, Y are topological spaces. $K \subset X$ and $Y \subset Y$ are compact subsets. Then $K \times L$ is compact subset of $X \times Y$ where Open sets of $X \times Y$ are $\bigcup$(Open set of X)$\times$(Open set in Y)

Theorem 5.2.7

All norms on $\mathbb{R}^n$ are equivalent

Proof: Enough to show any norm $f \sim \| \cdot \|$

$$\text{i.e } \alpha \|u\| \leq f(u) \leq \beta \|u\| \forall\ u$$

$$\text{i.e } \alpha \leq \frac{f(u)}{\|u\|} \leq \beta\ \forall\ u \forall\ u \neq 0$$

Note that $\frac{f(x)}{\|x\|} = f\left(\frac{x}{\|x\|}\right) = f(u)$ where $u = \frac{x}{\|x\|}$, so $\|u\| = 1$. Hence it is enough to show that

$$\alpha \leq f(u) \leq \beta$$

for any u with $\|u\| = 1$

Let $S = \{u \mid \|u\| = 1\}$ is the unit sphere in $\mathbb{R}^n$, which is closed and bounded

$$\begin{aligned} S \text{ is closed and bounded} \quad &\implies \quad S \text{ is compact} \\ &\implies \quad f(S) \text{ is compact in } \mathbb{R} \\ &\implies \quad f(S) \text{ is closed and bounded in } \mathbb{R} \\ &\implies \quad f(S) \text{ has largest element in } \beta \text{ and smallest} \\ &\qquad\quad \text{element } \alpha \text{ such that } \alpha \leq f(S) \leq \beta \end{aligned}$$

□