Solution of Exercisesheet 2

Exercise 2-1

Test

Exercise 2-2

Exercise 2-3

f and g are negligible functions and q be a positive polynomial.

(a) Is e^{-x} negligible?
For any polynomial x^c, choose $N = c$, then for all $x > N$ holds:
$e^{-x} < \frac{1}{x^c}$, because $e^x > x^c$ for all $x > N = c$.
$\Rightarrow e^{-x}$ is negligible.

(b) Is $\frac{1}{x^{2021}+1}$ negligible?
For the polynomial x^{2022} there is no N, that for all $x > N$ holds:
$\frac{1}{x^{2021}+1} < \frac{1}{x^{2022}}$, because $x^{2021} + 1$ is smaller than x^{2022} for $x > 1$.
$\Rightarrow \frac{1}{x^{2021}+1}$ is not negligible.

(c) Is $h(x)$ negligible, when $h(x)$ is a positive function such that $h(x) < f(x)$ for all x?
For $f(x)$ holds: $f(x) < \frac{1}{p(x)}$ (Definition 0.1).
Because of $h(x) < f(x) < \frac{1}{p(x)}$ for all x, $h(x)$ is also negligible.

(d) Is $f(x) + g(x)$ negligible?
$f(x)$ negligible $\Rightarrow f(x) < \frac{1}{p(x)}$ (Definition 0.1).
$g(x)$ negligible $\Rightarrow g(x) < \frac{1}{p'(x)}$ (Definition 0.1).

$$\begin{aligned} \Rightarrow f(x) + g(x) &< \frac{1}{p(x)} + \frac{1}{p'(x)} \\ &= \frac{p'(x) + p(x)}{p(x) \cdot p'(x)} \\ &= \frac{1}{\frac{p(x) \cdot p'(x)}{p'(x)+p(x)}} \end{aligned}$$

Addition, multiplication and division of two polynomials results in another polynomial. $p(x)$ and $p'(x)$ can be any polynomials. Because of that the denominator ($\frac{p(x) \cdot p'(x)}{p'(x)+p(x)}$) can also be any polynomial.
$\Rightarrow f(x) + g(x)$ is negligible.

(e) Is $f(x) \cdot q(x)$ negligible?
$f(x)$ negligible $\Rightarrow f(x) < \frac{1}{p(x)}$ (Definition 0.1).

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$q(x)$ is a positive polynomial.

$$\Rightarrow f(x) < \frac{1}{p(x)} \qquad | \cdot q(x), q(x) positive$$
$$f(x) \cdot q(x) < \frac{q(x)}{p(x)}$$
$$f(x) \cdot q(x) < \frac{1}{\frac{p(x)}{q(x)}}$$

Division of two polynomials results in another polynomial. $p(x)$ can be any polynomial. So the denominator ($\frac{p(x)}{q(x)}$) can also be any polynomial.
$\Rightarrow f(x) \cdot q(x)$ is negligible.

(f) Is $\frac{f(x)}{g(x)}$ negligible?
$f(x)$ negligible $\Rightarrow f(x) < \frac{1}{p(x)}$ (Definition 0.1).
$g(x)$ negligible $\Rightarrow g(x) < \frac{1}{p'(x)}$ (Definition 0.1).

$$\Rightarrow \frac{f(x)}{g(x)} < \frac{\frac{1}{p(x)}}{\frac{1}{p'(x)}}$$
$$\frac{f(x)}{g(x)} < \frac{1}{\frac{p(x)}{p'(x)}}$$

Division of two polynomials results in another polynomial. $p(x)$ and $p'(x)$can be any polynomials. So the denominator ($\frac{p(x)}{p'(x)}$) can also be any polynomial.
$\Rightarrow \frac{f(x)}{g(x)}$ is negligible.

(g) Is $2^{-1024} = \frac{1}{2^{1024}}$ negligible?
For the polynomial x^{1025} there is no N, that for all $x > N$ holds:
$\frac{1}{2^{1024}} < \frac{1}{x^{1025}}$, because 2^{1024} is always smaller than x^{1025} for all $x > 1$.
$\Rightarrow 2^{-1024}$ is not negligible.

(h) Is $(f(x))^{\frac{1}{q(x)}}$ negligible?
$f(x) = e^{-x}$ is negligible (see (a))
$q(x) = x$ is a positive polynomial for all $x > 0$
$\Rightarrow (e^{-x})^{\frac{1}{x}} = e^{-1} = \frac{1}{e}$
For the polynomial x^2 there is no N, that for all $x > N$ holds:
$\frac{1}{e} < \frac{1}{x^2}$, because e is always smaller than x^2 for all $x \geq 2$.
$\Rightarrow e^{-1}$ is not negligible. $\Rightarrow (f(x))^{\frac{1}{q(x)}}$ is not negligible.

(i) Is $x^{-\log\log\log x}$ negligible?
For any polynomial x^c, choose $N = e^{e^{e^c}}$, then for all $x > N$ holds:
$x^{-\log\log\log x} < \frac{1}{x^c}$, because $x^{\log\log\log x} > x^c$ and $\log\log\log x > c$ for all $x > N = e^{e^{e^c}}$.
$\Rightarrow x^{-\log\log\log x}$ is negligible.

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Exercise 2-4

Exercise 2-5