KAN Networks for Conformal Bootstrap

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Top Down vs Bottom Up Approach in High Energy Physics

- Top Down: start with a particular UV model (for example **string theory**) and then calculate its predictions on IR observable quantities.
  - **Strengths:**
    - Conceptual elegance.
  - **Weaknesses:**
    - Relies on the validity of the initial UV model, which might not match reality.
- Bottom Up: start with the physical principles (**unitarity, locality, causality**) and IR data and work towards building UV theory.
  - **Strengths:**
    - Practical realism.
  - **Weaknesses:**
    - Computationally intensive for large systems.
    - May miss overarching connections or deeper principles.

Bootstrap approach

- **Self-Consistency:** Use principles like **symmetry**, **unitarity**, and **causality** to constrain the theory.
- **Minimal Assumptions:** Avoid assuming specific microscopic details; rely on general properties and consistency conditions.
- **Emergence:** Physical properties arise as solutions of consistency equations, rather than being derived from fundamental building blocks.

Bootstrap as a Search Problem

- **Bootstrap Framework:**
  - The bootstrap is fundamentally a search problem for solutions that satisfy a set of **self-consistency constraints**.
- **Challenge:**
  - The solution space is typically high-dimensional and non-linear, requiring advanced computational methods for exploration.
- **Role of Computational Techniques:**
  - **Optimization:** Efficiently solve nonlinear equations or maximize consistency.
  - **Sampling:** Explore vast parameter spaces to identify allowed regions.
  - **Pattern Discovery:** Detect structures or emergent behaviors in solution spaces.
- **Examples of Solutions:**
  - Constraining the CFT spectrum.
  - Constraining the spectrum and interaction of quantum gravity theories.

Four-Point Function in CFT

The four-point function of primary scalar operators depends on the coordinates through conformally invariant cross-ratios. For four operators $\mathcal{O}(x_i)$ with scaling dimensions Δ_i, the four-point function can be written as:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}} G(u,v)$$

where

$$x_{ij} = x_i - x_j, \quad \Delta_{ij} = \Delta_i - \Delta_j, \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

Crossing symmetry

We parametrize u, v as:

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z})$$

Swapping states 2 and 4 should leave the correlator invariant therefore,

$$(z\bar{z})^{-\Delta_\phi} G(z,\bar{z}) = \left[(1-z)(1-\bar{z})\right]^{-\Delta_\phi} G(1-z, 1-\bar{z}). \tag{1}$$

Operator Product Expansion (OPE)

The OPE expresses the product of two local operators as a sum over other operators:

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k C_{ij}^k(x-y)\mathcal{O}_k(y),$$

where $C_{ij}^k(x-y)$ are the OPE coefficients. The 4-point function becomes a sum over contributions from different intermediate operators:

$$G(z,\bar{z}) = \sum_k \lambda_k^2 \mathcal{F}_k(z,\bar{z}),$$

where $\mathcal{F}_k(z,\bar{z})$ are **conformal blocks**.

Unitarity

$$G(z,\bar{z}) = \sum_k \lambda_k^2 \mathcal{F}_k(z,\bar{z}),$$

Unitarity implies λ_k is real so $C_k \equiv \lambda_k^2$ must be positive:

$$C_k > 0,$$

in **unitary** CFTs.

Bootstrapping the four-point function

Therefore, we look for $G(z,\bar{z})$ that is **symmetric**:

$$(z\bar{z})^{-\Delta_\phi} G(z,\bar{z}) = \left[(1-z)(1-\bar{z})\right]^{-\Delta_\phi} G(1-z, 1-\bar{z}). \tag{2}$$

and admits a **positive** OPE expansion:

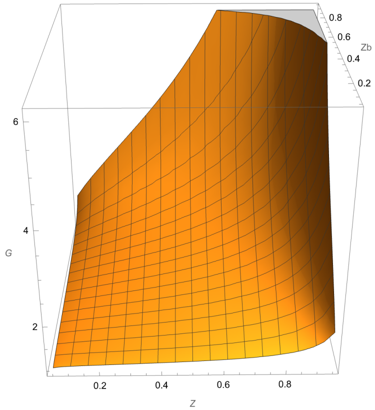
$$G(z,\bar{z}) = \sum_k C_k \mathcal{F}_k(z,\bar{z}),$$

where

$$C_k > 0.$$

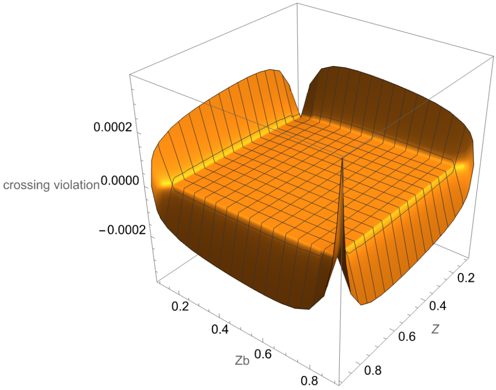
3d Ising CFT

We consider a four point function of identitial scalar operators with scalling dimension $\Delta_\phi = 0.5181489$ of 3d Ising model CFT. By expanding $G(z, \bar{z})$ in z around the symmetric point $z = 1/2$ and imposing the constraints order by order in z we can find a candidate solution in the vicinity of $z = 1/2$:



3d Ising CFT

It obeys crossing around $z = 1/2$ but not in the entire z plane:



Neural network ansatz

We try to extend this to a solution everywhere by parameterizing $G(z, \bar{z})$ as a **neural network**, $f(z, \bar{z})$. First we require:

- $(z\bar{z})^{-\Delta_\phi} f(z, \bar{z}) = \left[(1-z)(1-\bar{z})\right]^{-\Delta_\phi} f(1-z, 1-\bar{z})$
- $f(z, \bar{z}) = G_{candidate}(z, \bar{z})$ in a region around $z = 1/2$.

Neural network ansatz

In order to impose symmetry we write:

$$f(z,\bar{z}) = \left[g_W(z,\bar{z}) + g_W(1-z,1-\bar{z})\right](z\bar{z})^{\Delta_\phi}$$

where $g_W(z,\bar{z})$ is a **neural network**.

Neural network ansatz

We then train it using the following loss function:

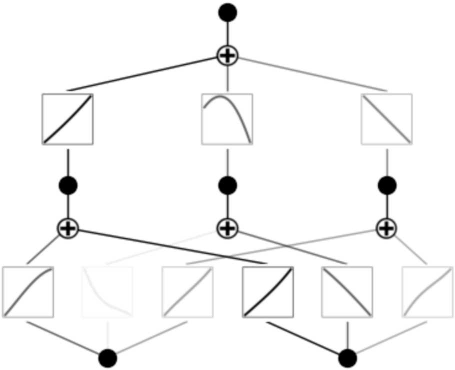
$$\text{Loss} = \frac{1}{N}\sum_{i=1}^{N} \left| G_{\text{candidate}}(z_i, \bar{z}_i) - f(z, \bar{z}) \right|^2 ,$$

where:

- $z_i, \bar{z}_i$ are N points sampled from a truncated Gaussian distribution centered around $z = \bar{z} = 1/2$, with a specified standard deviation σ and maximum radius.
- $G_{\text{candidate}}(z, \bar{z})$ is the value of the candidate 4-point function in the sampled region.
- $f(z, \bar{z})$ is the neural network prediction.

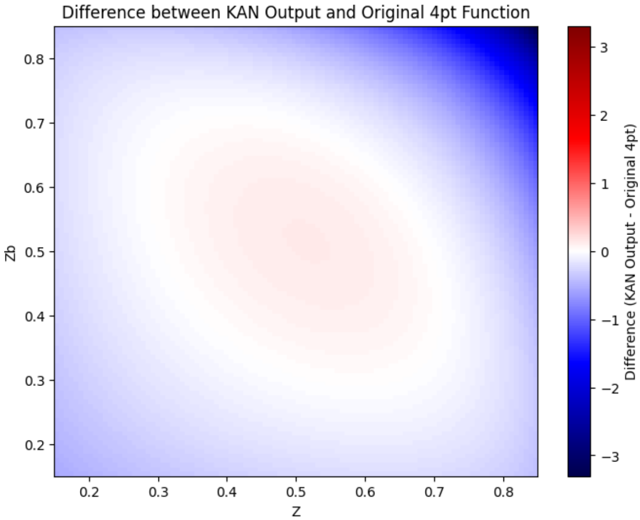
Interpretability

In order to be able to extract analytic formula of the solution we use interpretable **Kolmogorov-Arnold Networks (KAN)**. A small number of neurons is sufficient to fit the four-point function.



Result

The obtained function is crossing symmetric everywhere and it matches the solution around $z = 1/2$



Analytic expression

By choosing basis for the activation functions we can extract an **analytic formula**:

$$
f(z,\bar{z}) = \begin{aligned}[t] \Big[& -0.108\, z - 0.102\, \bar{z} - 1.086 \log \Big(1.075 \sqrt{1 - 0.915\, \bar{z}} \\
& - 0.412 \sin(2.831\, z + 4.998) - 0.215 \Big) \\
& + 0.783 \sin \Big(1.592 \sin(0.637\, \bar{z} - 1.208) \\
& + 1.159 + 1.056 \exp(-3.975\, z) \Big) + 1.193 \Big]. \end{aligned}
$$

Unitarity

We did not explicitly impose the positivity of OPE coefficients, therefore this function is not guaranteed to be compatible with **unitarity**.

Lorentz inversion formula

In order to extract OPE coefficients we need to do a complicated integral transform:

$$
\begin{aligned}
c^t(h,\bar{h}) &= \frac{\kappa_{2\bar{h}}}{4}\int_0^1 dzd\bar{z}\mu(z,\bar{z})g^{r,s}_{d-1-h,\bar{h}}(z,\bar{z})\mathrm{dDisc}_t[\mathcal{G}(z,\bar{z})]\,,\\
\kappa_{2\bar{h}} &\equiv \frac{\Gamma(\bar{h}+r)\Gamma(\bar{h}-r)\Gamma(\bar{h}+s)\Gamma(\bar{h}-s)}{2\pi^2\Gamma(2\bar{h}-1)\Gamma(2\bar{h})}\,,\\
\mu(z,\bar{z}) &= \left|\frac{z-\bar{z}}{z\bar{z}}\right|^{d-2}\frac{\left((1-z)(1-\bar{z})\right)^{s-r}}{\left(z\bar{z}\right)^2}\,.
\end{aligned}
\tag{3}
$$

Here, d is the spacetime dimension and we have introduced the labels

$$
\begin{aligned}
h = \frac{\Delta-\ell}{2} = \frac{\tau}{2}\,, &\qquad \bar{h} = \frac{\Delta+\ell}{2} = \frac{\tau}{2}+\ell\,,\\
r = h_{12}\,, &\qquad s = h_{34}\,,
\end{aligned}
\tag{4}
$$

Integrals of neural networks

In order to impose unitarity constraints on four-point function directly we need an efficient way of **integrating** functions represented by neural networks.

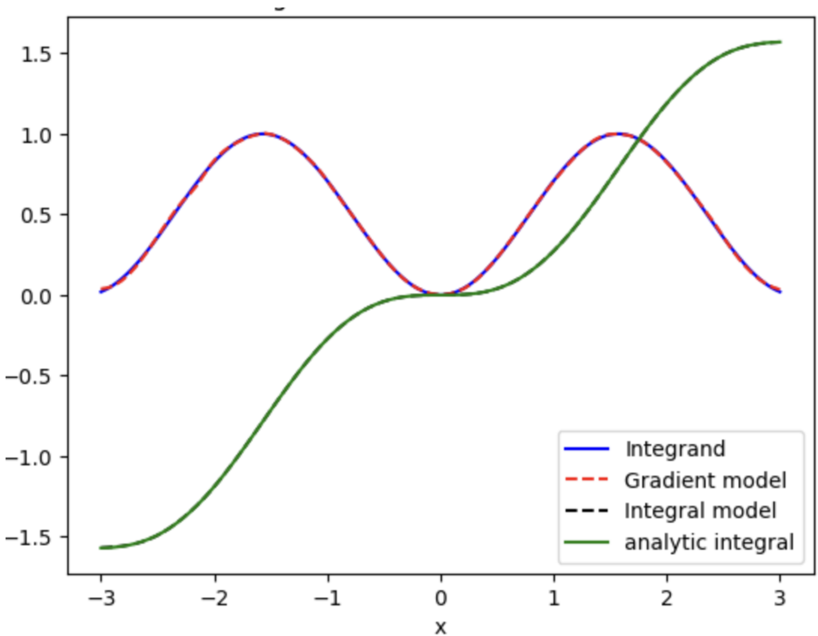
Integrals of neural networks

For that we use the following trick [Lindell, Martel, Wetzstein, 2021]:

1. Define a network $f_{int}(x)$.
2. Use automatic differentiation to define its derivative network $f_{grad}(x) = \partial_x f_{int}(x)$.
3. Fit the function we want to integrate, $f(x)$ to $f_{grad}(x)$, with the **same weights**.
4. Obtain the indefinite integral of $f(x)$, by forward passing it to $f_{int}(x)$.

$$f_{int}(x) = \int f_{grad}(x)dx.$$

Simple example $f(x) = sin(x)^2$



Future directions

We are working on applying the integral trick to do integral transforms of KAN neural networks in order to extract OPE coefficients and impose their positivity in KAN training.

Also we plan to extend this to S-Matrix bootstrap for the search of consistent scattering amplitudes.