Introduction to Relations and Functions

Linguist 130A/230A Section

January 20, 2015

1 Ordered pairs

Last week, we said that order doesn't matter in sets. So, the set $\{a, b\}$ is the same as the set $\{b, a\}$. As we'll see, though, sometimes order matters, so we are also going to define **ordered pairs**, which we'll write with angle brackets:

Definition 1.1. The **ordered pair** $\langle a, b\rangle$ is picked out by the set $\{\{a\}, \{a, b\}\}$.

For the most part, we won't need the set definition of an ordered pair, but note that this makes $\langle a, b\rangle$ not equal to $\langle b, a\rangle$, since the first is $\{\{a\}, \{a, b\}\}$ and the second is $\{\{b\}, \{a, b\}\}$.

Note: Repetitions are now meaningful: $\langle a, a\rangle$ is an ordered pair with two elements the same, and $\langle a\rangle$ is an "ordered" 1-tuple.

Question: How would you write the ordered pair $\langle a, a\rangle$ using the set definition?

More generally, we can have **ordered n-tuples**. An ordered triple has three elements $\langle a, b, c\rangle$, an ordered n-tuple has n: $\langle a_1, a_2, \ldots, a_n\rangle$.

2 Relations

2.1 Basic definitions and examples

Informally, we can think of a **relation** as something that holds or doesn't hold between two objects. So a verb (predicate) like *loves* can be a relation: x and y are in the *loves* relation if x *loves* y. This obviously isn't the same as if y *loves* x, so we need ordered pairs!

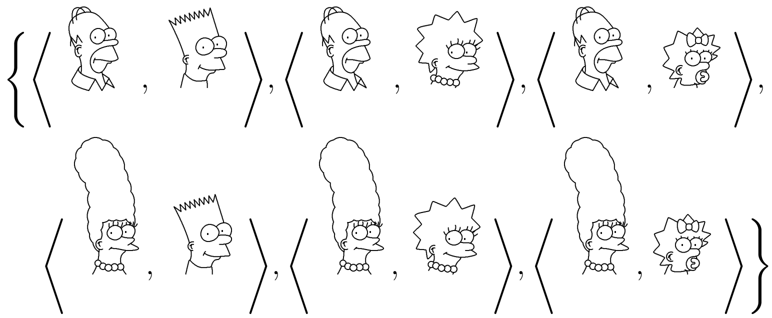
Definition 2.1. A **relation** is a set of ordered pairs.

Some relations that might be familiar to you are *equality*, $<$, and $>$.

Example 2.2. For the set of prime numbers less than 10 (recall: $\{2, 3, 5, 7\}$), the following set defines the *less than* ($<$) relation:

$$\{\langle 2, 3\rangle, \langle 2, 5\rangle, \langle 2, 7\rangle, \langle 3, 5\rangle, \langle 3, 7\rangle, \langle 5, 7\rangle\}$$

Example 2.3. If we limit ourselves to members of the Simpsons family, the following set defines the *is a parent of* relation:



We could also write this as:

$$\{\langle x, y\rangle : x \textit{ is a parent of } y \text{ and } x, y \text{ are in the Simpsons family}\}$$

2.2 Some additional definitions

Given two sets, A and B, we can take the set of all of the ordered pairs with their first element from A and their second element from B.

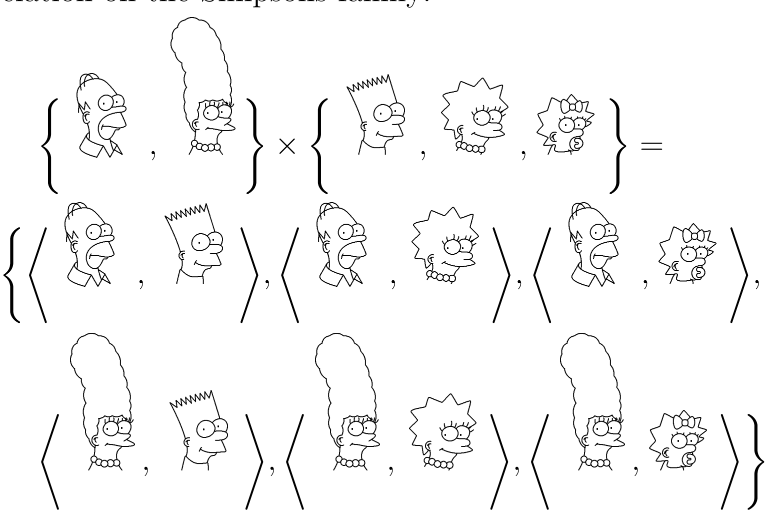
Definition 2.4. The **Cartesian product** $A \times B$ of two sets A and B is given by:

$$A \times B = \{\langle a, b\rangle : a \in A \text{ and } b \in B\}$$

Observe: The Cartesian product of A and B is a relation between A and B.

Question: What is the Cartesian product of A and B if either A or B is the empty set?

Example 2.5. The Cartesian product of parent Simpsons with child Simpsons gives us the *is a parent of* relation on the Simpsons family:



Sometimes we want to pick out just the things that occur as the first element of pairs in a relation, or just the things that occur as a second element.

Definition 2.6. Given a relation $R = A \times B$, the set of first coordinates A is the **domain** of R.

Definition 2.7. Given a relation $R = A \times B$, the set of second coordinates B is the **range** of R.

2.3 Properties of relations

We noticed that a for a relation like *loves*, $\langle x, y \rangle \in$ *loves* does not mean that $\langle y, x \rangle \in$ *loves* (if it did, there would be much less angsty poetry). But this isn't always true: a relation like *is a sibling of* works both ways.

Definition 2.8. A relation R is **symmetric** if $\langle x, y \rangle \in R$ means that $\langle y, x \rangle \in R$.

So, the *sibling* relationship is symmetric, but the *loves* relationship is not.

Some other possible properties a relation can have:

Definition 2.9. A relation R is **reflexive** if, for every x that is a first coordinate of a pair in R, $\langle x, x \rangle \in R$.

Example 2.10. Equality is a reflexive relation; for any x, $x = x$.

Definition 2.11. A relation R is **transitive** if, for any x, y, z with $\langle x, y \rangle$ and $\langle y, z \rangle$ in R, we have $\langle x, z \rangle$ in R.

Example 2.12. *Less than* is a transitive relation: if $x < y$ and $y < z$, we have $x < z$.

3 Functions

Functions are relations with a special property:

Definition 3.1. A relation f is a **function** if and only if every x in the domain of F there is at most one y such that $\langle x, y \rangle \in F$.

What this means is that no object appears more than once as the first element in an ordered pair in a function. You can think of a function from A to B as an input-output machine: you put in something from A and get back a unique thing from B (with a relation, the machine might not know what to give you, because the element of A can be paired with more than one element of B).

We write $f : A \to B$, which means that f is a function that takes elements of the set A to elements of the set B. A is the domain, and B is the range (sometimes called the co-domain).

3.1 Some properties of functions

Here are some additional properties that a function may or may not have:

Definition 3.2. A function $f : A \to B$ is called **total** if, for every $a \in A$, there is a $b \in B$ such that $f(a) = b$. If there is some a with no b value, f is called a **partial function**.

Example 3.3. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^{0,+}$ be the set of positive real numbers and 0. Let $f : \mathbb{R}^{0,+} \to \mathbb{R}$ be the square root function. Then f is total.

Question: Suppose f is still the square root function, but we define it as a function from $\mathbb{R}$ to $\mathbb{R}^{0,+}$. Is it a total or a partial function?

Definition 3.4. A function $f : A \to B$ is called **one-to-one** if for every $b \in B$, there is at most one $a \in A$ such that $f(a) = b$.

Example 3.5. Let A and B both be $\mathbb{Z}$, the integers, and let $f : A \to B$ be defined by $f(a) = a + 2$. Then f is *one-to-one*. Any $b \in B$ is uniquely mapped to by $b - 2 \in A$: $f(b - 2) = (b - 2) + 2 = b$.

Question: Let $A = \mathbb{R}$, the real numbers. Consider the function $f : A \to A$ which is defined by $f(a) = a^2$. Is this *one-to-one*? Why/why not?

Definition 3.6. A function $f : A \to B$ is called **onto** if, for every $b \in B$, there is a n $a \in A$ with $f(a) = b$.

Example 3.7. Let $A = \mathbb{Z}$ (the integers) and $B = 2\mathbb{Z}$ (the even integers). Then $f : A \to B$ defined by $f(a) = 2a$ is *onto* since every even integer is a multiple by 2 of some integer.

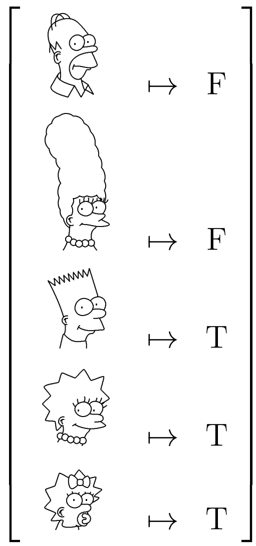
Question: $f : A \to A$ defined $f(a) = 2a$ is not onto. Why not?

Definition 3.8. If a function f is total, one-to-one, and onto, we call it a **bijection**.

3.2 Truth values and characteristic functions

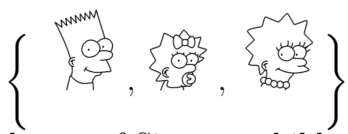
Many of the functions we will care about in this class are into the domain of truth values $\{\mathrm{T}, \mathrm{F}\}$.

Example 3.9. Let f be the function defined by mapping the set of Simpsons into the set of truth values according to the rule: *x is a Simpson child*. Then $f(x) =$T if x is a Simpson and a child and $f(x) =$F if x is either not a Simpson or not a child. Here is a way of depicting this:



If we know what our domain is (in this case, the Simpsons), we can just write down the set of elements that map to true as a kind of shorthand. This is called the **characteristic set** of a function.

Example 3.10. The characteristic set of the function depicted in the previous example is given by:



Unsurprisingly, given f, this is the set of Simpson children.

We'll also sometimes talk about **characteristic functions**:

Definition 3.11. If A is a set and D is the universe of objects we are concerned with, then the **characteristic function of** A is the function f which sends the elements of A to T and the elements of $D - A$ to F.

Example 3.12. Let D be the set of characters on the Simpsons show. Then the characteristic function of the set of members of the Simpsons family is the function f which returns T for Homer, Marge, Bart, Lisa, and Maggie, and F for everyone else.