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ON THE LOGIC OF ABILITY

One popular view of free will has it that, for example, when I raise my arm I am acting freely if I have the ability to raise my arm and also have the ability to do otherwise. For any theory incorporating such a view as this, it is important to have a coherent account of the logical aspects of ability.

It appears that we should represent the **can** of ability as an operator which, acting on a sentence **A**, would produce a sentence saying something like that I can bring about circumstances in which **A** will be true (if $\mathscr{P}_A$ is the proposition expressed by **A**: that I can bring it about that $\mathscr{P}_A$). It seems plausible to look to modal logic to help in characterizing such an operator. But in [6] Anthony Kenny concludes that no modal logic with a formal semantics based on possible worlds can be adequate as a logic of ability.[1]

The weakest system of modal logic normally thought suitable for alethic interpretations of the operators is the system KT.[2] One axiomatization of KT contains (among other principles) the axiom schemata

(T$\Diamond$) $\quad \vdash \mathbf{A} \rightarrow \Diamond \mathbf{A}$ and

(C$\Diamond$) $\quad \vdash \Diamond(\mathbf{A} \vee \mathbf{B}) \rightarrow (\Diamond \mathbf{A} \vee \Diamond \mathbf{B})$.

Kenny points out that neither of these schemata is correct if the possibilitation operator (the diamond) is interpreted as the **can** of ability.

To say that I am able to hit the bull's-eye at darts is not merely to say that I can try and may succeed, but to say that my success is (at least) reliable and (perhaps) reproducible. Kenny holds that "abilities are inherently general; there are no genuine abilities which are abilities to do things only on one particular occasion".[3] Thus Kenny holds out for reproducibility. This may seem too strong a constraint. Even if opportunity only knocks once, I may be able to act on it, and may be culpable for doing so, or for failing to do so.

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But, whether or not we disagree with Kenny about this, we must at least hold that nothing will count as a (morally relevant) ability unless it is reliable. Chance accomplishments do not establish morally relevant abilities. There are such things as accidents, happy and otherwise, for which no-one is held responsible. My hitting the bull's-eye once by chance at darts doesn't establish any ability (even an ability at that very moment) for the successful exercise of which I can be held responsible. Here we are confining ourselves to a theory of those claims to ability that are morally relevant.[4] Consequently the **can** of ability doesn't satisfy schema $T\diamond$.[5]

Neither does it satisfy the schema $C\diamond$. If asked to draw a card from a deck of cards, I certainly can do so. If the deck is held face up, so I can see the colors of the cards, I can draw a red card. If the deck is now turned face down, so the colors of the cards are concealed, I am able to draw a card which will have one of the colors red or black, but I cannot draw a red card, and I cannot draw a black card. When I draw, the card will always be of one or the other of the two colors, but I have no control over which color it will be. I cannot *reliably* draw a red card, nor can I *reliably* draw a black one, though I can reliably draw a card which will satisfy this condition: it is either red or black.

Again, I may be good enough at darts that I am able to hit the board, and in hitting the board I will certainly hit one of the areas of the board, possibly the bull's-eye. But whether I hit the bull's-eye or not may be a matter of luck, not of ability. In such a case, if something of moral consequence hangs on my throw of the dart I can reasonably be held responsible for failing to exercise my ability to hit the board, but not for not being lucky enough to hit the bull's-eye.

The fact that the **can** of ability violates both of the schemata characteristic of the weakest system thought suitable for alethic modal logic may not seem much to the point, since we're not looking for an alethic interpretation of the system anyway. And there certainly are weaker systems of modal logic. The system K, for example, differs from KT in lacking the schema $T\diamond$ (and its dual schema T). But even the system K has the schema $C\diamond$, which makes K (and any extension of K) unsuitable as a vehicle for the logic of ability. Putting the point semantically, K is the weakest system we can create that uses possible worlds, has a relevance relation[6] that relates some worlds to others, and has the usual truth conditions for $\Diamond\mathbf{A}$ and $\Box\mathbf{A}$ at every world. Kenny, having had his attention drawn to these facts by Stalnaker, concludes that no system of modal logic based on possible worlds can be adequate as an account of ability.[7] Others have been led by the same evidence to doubt positions to which they were otherwise attracted.[8]

Such pessimism is a bit hasty. The general idea of a possible-worlds semantics is a very flexible one: there are other ways in which we can give a semantics based on possible worlds.[9] In particular, there are systems based on minimal models, as in Segerberg [12] and Chellas [2], in which in effect we have a relevance relation between a world and a *set* of possible worlds or, as I shall call such a set, a *cluster*.[10]

To establish a point of reference, it may be helpful to look at one standard use of minimal models. If we wish to retain an analog of an alethic necessitation operator $\Box$ in a system based on minimal models, the truth conditions will have $\Box\mathbf{A}$ true at a given world α in a minimal model $\mathscr{M}$ iff the cluster $\|\mathbf{A}\|^{\mathscr{M}}$ is relevant to α, where $\|\mathbf{A}\|^{\mathscr{M}}$ is the set of all worlds in $\mathscr{M}$ at which $\mathbf{A}$ is true. Often $\|\mathbf{A}\|^{\mathscr{M}}$ is thought of as corresponding to the proposition expressed by $\mathbf{A}$ according to the model $\mathscr{M}$, and under this interpretation the relevance relation is construed as singling out, as relevant to a given world, all and only the propositions that are (in the relevant respect) necessarily true at that world in that model. If we restrict attention to minimal models in which the necessity considered relevant fits one of the usual alethic interpretations, then we will get the corresponding alethic modal logic. The truth conditions for $\Diamond\mathbf{A}$ are as usual dual to those for $\Box\mathbf{A}$. The system of all the modal formulas valid in all minimal models under these truth conditions – the weakest classical[11] system – is the system E, in which neither $\mathrm{T}\Diamond$ nor $\mathrm{C}\Diamond$ is a theorem.

All the distinctive modal content of the system E can be summarized in the rule

(RE$\Diamond$) If $\vdash \mathbf{A} \leftrightarrow \mathbf{B}$, then $\vdash \Diamond\mathbf{A} \leftrightarrow \Diamond\mathbf{B}$.

If the diamond were interpreted as expressing the **can** of ability, this would surely be an acceptable principle. Hence it is open to us, as far as syntax is concerned, to use some extension of the system E as the logic of ability.

If we take the diamond operator characterized by truth conditions given above and try to interpret it as expressing the **can** of ability, however, we will have to view the semantics in some such fashion as this: the propositions which are relevant, at a given world, are the propositions whose falsity it is (at that world) not within my ability to arrange. This seems a bit unnatural. The truth conditions which work so naturally for the alethic interpretation seem awkward as conditions for the truth of statements involving the **can** of ability.

On the other hand, the system E can also be axiomatized by substituting for the rule RE$\Diamond$ the rule

(RE) If $\vdash \mathbf{A} \leftrightarrow \mathbf{B}$, then $\vdash \Box\mathbf{A} \leftrightarrow \Box\mathbf{B}$.

Consequently we could get a syntactically acceptable logic of the **can** of ability by interpreting the square as expressing the **can** of ability. We then get a more natural interpretation of the semantics. It seems odd to represent the *ability* operator with a symbol usually used to express various kinds of *necessity* operators, but that is merely a notational point. We could just interchange the symbols '$\Diamond$' and '$\Box$' throughout our representation of the system E and leave all else as usual.

But we can do better than this. The notion of a minimal model affords us opportunities to define a wide variety of distinct modal operators, using different truth conditions. In fact the method is quite general: any propositional operator in possible worlds semantics can be represented in some way using minimal models.[12] In particular we can introduce an operator with slightly different, and more appropriate, semantics designed specially for the **can** of ability.

The idea is the following. When I say that I can bring it about that **A** is true, I can be understood to mean that there is an action open to me, the execution of which would assure that **A** would be true.[13] But performing such an action need not (and should not) be understood to determine absolutely every detail of the ensuing state of affairs. To say that I can hit the dart board would be to say that there is an action open to me (throwing the dart in a certain way), the exercise of which will assure that the dart hits the board, and not, say, the surrounding wall. There are perhaps many possible ways in which my throw would lead to the dart's hitting the board – perhaps when I throw the dart that way the dart will land inside the bull's-eye, for example, but perhaps it will land outside. In saying that I can hit the board, I need not be saying anything about which of these particular possibilities will be realized.

We can take different clusters of worlds to correspond to different logically possible actions (including, perhaps, the null action), and take *relevant* clusters to correspond to choices of actions of which I am actually *capable*. Then a given cluster will be the set of all worlds in which I take a certain possible action. Or (equivalently, in view of some) I can take each cluster to be, or to correspond to, the proposition that I take a certain possible action. Either way, a relevant cluster can normally be expected to contain more than one world, because my choice of an action can normally be expected to leave some details of the outcome undetermined. Proximately, I may be able to bring it about that the dart will hit the dartboard, yet not have fine control over which part of the board it hits. The dart hits one part of the board in some worlds in the cluster corresponding to my act of throwing the dart, and different parts in some others. More remotely, my acting freely at this moment will not normally preclude the possibility that I (and other agents) will have further opportunities to act freely at various future times.

If we introduce the special modal operator $\diamondplus$ to correspond to the **can** of ability, we can express the semantics as follows: $\diamondplus$**A** will be true at a given world iff *there exists* a relevant cluster of worlds, at *every* world of which **A** is true.[14] Cast in this form, the truth condition involves *two* metalinguistic quantifiers (one existential and one universal). The presence of the universal quantifier in the semantics for $\diamondplus$ corresponds to the fact, noted by Kenny, that ability has a *general* character. (Kenny had reproducibility – generality within a world – in mind, but I have emphasized reliability – generality across worlds in a cluster.) It also helps explain why Kenny briefly considers the possibility that the **can** of ability is more like a necessitation operator than like a possibilitation operator.[15] In fact, with truth conditions like these, it should be a little like each. We have already noted that both the necessitation and the possibilitation operators in the weak classical system E resemble operators expressing the **can** of ability. The symbol which I have used above hints at this double character by having an outer shape reminiscent of the diamond of possibilitation (corresponding to a metalingustic existential quantifier) and having the inner shape of a necessitation operator (corresponding to a metalinguistic universal quantifier). Since there are four possible combinations of two quantifiers, we should expect to be able to characterize three additional modal operators, analogous to $\Diamond\mkern-13mu\Box$, and we should expect some interesting interplay among the four operators.

We interpret a formula $\Diamond\mkern-13mu\Box\mathbf{A}$ to mean that I can (reliably) bring about circumstances in which **A** is true. The claims of ability formed using $\Diamond\mkern-13mu\Box$ are to be construed neither timelessly nor impersonally, but both the temporal index and the personal index for the agent will be left tacit[16] and will be assumed to remain constant.[17] (For the sake of definiteness of illustrations, I take the agent to be myself, and the time to be now.) The working presumption is that if we can give a reasonable account of ability as it applies to a single agent at a single time under conditions in which appropriate opportunities are available,[18] we will be well on our way towards a more general theory.

Alternatively, if the formula **A** expresses the proposition $\mathscr{P}_A$, and $\mathscr{A}_A$ is an action (by me, now) whose defining outcome is asserted by the proposition $\mathscr{P}_A$ to occur, then we can interpret $\Diamond\mkern-13mu\Box\mathbf{A}$ to mean that I can do $\mathscr{A}_A$.[19]

The dual notion will be expressed by the formula $\Box\mkern-13mu\Diamond\mathbf{A}$, equivalent to $\neg\Diamond\mkern-13mu\Box\neg\mathbf{A}$, and interpreted to mean that I can't do $\mathscr{A}_{\neg A}$, i.e. that I can't (reliably) bring about circumstances in which **A** is false, can't (reliably) prevent **A** from being true. Whereas $\Diamond\mkern-13mu\Box\mathbf{A}$ will be true at a world iff *there exists* a relevant cluster, in *every* world of which **A** is true, $\Diamond\mkern-13mu\Box\mathbf{A}$ will be true at a world iff in *every* relevant cluster *there exists* at least one world at which **A** is true. It is common, I believe, to express this notion by saying that I *might* do $\mathscr{A}_A$. So, for example, I could be warned off from throwing my dart just now by someone who warns "If you throw now, you might hit Joe", meaning that (under the circumstances at the moment) if I throw the dart at the dartboard, I can't guarantee not to hit Joe. But we need to exercise extreme caution in using 'might' as the dual of 'can', since in other contexts of use, 'might' may function quite differently.[20] In some contexts, for example, to say that I *might* do $\mathscr{A}_A$ (meaning, roughly: $\mathscr{A}_A$ is one of my options still under consideration) would imply that I *can* do $\mathscr{A}_A$, and no such implication is appropriate for the dual of the ability operator. There is nothing I can do to guarantee that you will win the lottery. It doesn't follow that there is anything I can do that will guarantee that you won't win, either.

The formula ⧈**A** will be true at a world iff **A** is true at every world in every relevant cluster. We may interpret ⧈**A** as meaning that, in a certain weak sense, no matter what I choose to do, **A** will be true. The sense involved is a weak one, because we do not (at least initially) rule out the possibility that either or both of the metalinguistic universal quantifiers can be vacuously satisfied, either because there is nothing I am able to do (there are no relevant clusters) or because there is only one thing I can do, and it is a degenerate choice (the only relevant cluster is the empty set). Thus ⧈**A** will be vacuously true if there is nothing at all that I can do, and also if the one thing I can do is to eliminate all possibilities whatsoever. In Section 5 we will examine systems in which these bizarre eventualities are ruled out.

I believe the sense of ⧈**A** is sometimes expressed by saying that I *will* do $\mathscr{A}_A$, as when someone says "if I throw the dart now, I will hit Joe", meaning that if I now launch the dart (in anything like the way which, in the context, might be expected), the dart will unavoidably strike Joe. Again, however, we must use extreme caution in so expressing ourselves, since in other contexts, as in expressions of intention, 'will' has different properties.

The dual of the formula ⧈**A** is ◈**A**, which will be true iff there is a relevant cluster in which there is at least one world at which **A** is true. The sense of the formula ◈**A** is sometimes expressed by saying that I *might* do $\mathscr{A}_A$. Thus, it could be said that I might win the lottery, meaning merely that there is at least one choice open to me (buying a lottery ticket) which leaves open the possibility of my winning the lottery. The **might** in ◈**A** is different from the dual of the **can** of ability in ⊠**A**, which is only applicable if *every* course of action open to me leaves open the possibility that $\mathscr{P}_A$. This again points up the importance of caution in using the word 'might' at all in discussions of ability and free will. I shall refer to ⟐ as the **can** operator, to ⊠ as the **might**$_c$ operator (the subscript indicating that it is the dual of the

can operator), to ⧈ as the **will** operator, and to ◈ as the $\mathbf{might_w}$ operator (the dual of the **will** operator).

In this paper I formulate and investigate some systems of modal logic which employ the semantics just described. We will thus have systems in which we simultaneously explore aspects of the behavior of some formal analogs of the modal verbs 'can', 'will' and (two uses of) 'might'. To the best of my knowledge this is the first modal logic that systematically connects these notions. The most fundamental system based on this semantics will be called the system $\mathscr{V}$.[21]

SECTION 1. THE SYSTEM $\mathscr{V}$: PROOF THEORY

Let $\mathscr{L}$ be a language for modal logic whose primitive symbols are the atomic sentences $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, . . . , $\mathbf{P}_n$, . . . , the sentential connectives $\bot$ and $\rightarrow$, and the modal operators ⟐ and ◈. Define ¬**A** to be **A** → ⊥, and adopt the usual definitions for ∧, ∨, ↔, and the following definitions for modal operators ⊠ and ⧈:

(Df ⊠) ⊠**A** $=_{\text{df}}$ ¬⟐¬**A**;

(Df ⧈) ⧈**A** $=_{\text{df}}$ ¬◈¬**A**.

Assume the obvious definition for well-formed formulas.

Let the system $\mathscr{V}$ of sentential modal logic be the system whose axiom schemata and rules are:

(**PC1**) ⊢**A** → (**B** → **A**)

(**PC2**) ⊢(**A** → (**B** → **C**)) → ((**A** → **B**) → (**A** → **C**))

(**PC3**) ⊢¬¬**A** → **A**

(**C**◈) ⊢◈(**A** ∨ **B**) → (◈**A** ∨ ◈**B**)

(**V**) ⊢⟐(**A** ∨ **B**) → (◈**A** ∨ ⟐**B**)

(**W**) ⊢◈**A** → (⧈**B** → ⟐**B**)

(**MP**) If ⊢**A** and ⊢**A** → **B**, then ⊢**B**

(**RM**◈) If ⊢**A** → **B**, then ⊢◈**A** → ◈**B**

(**RM**⟐) If ⊢**A** → **B**, then ⊢⟐**A** → ⟐**B**

(**RN**⧈) If ⊢**A**, then ⊢⧈**A**.

We identify the system $\mathscr{V}$ with the set of theorems generated by these axioms and rules, and write that $\vdash\mathbf{A}$ to indicate that $\mathbf{A}$ is a theorem of $\mathscr{V}$. By an *extension* of $\mathscr{V}$ (or a $\mathscr{V}$*-extension*) we mean any set $\mathscr{S}$ of wffs of $\mathscr{L}$ such that $\mathscr{S}$ is closed under MP and $\mathscr{V} \subseteq \mathscr{S}$. By a *reliable* $\mathscr{V}$*-extension*, we mean any $\mathscr{V}$-extension closed under RM◈, RM⟐, and RN⧈. If $\mathscr{S}$ is any $\mathscr{V}$-extension, $\mathbf{A}$ any wff, and $\mathbf{T}$ any set of wffs, we write that $\vdash_{\mathscr{S}} \mathbf{A}$ iff $\mathbf{A} \in \mathscr{S}$, and write that

$$\mathbf{T} \vdash_{\mathscr{S}} \mathbf{A} \text{ iff for some } \mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n \in \mathbf{T} \quad (\mathbf{n} \in \mathbb{N})$$
$$\vdash_{\mathscr{S}} (\mathbf{B}_1 \wedge (\mathbf{B}_2 \ldots \wedge (\mathbf{B}_{n-1} \wedge \mathbf{B}_n) \ldots)) \rightarrow \mathbf{A},$$

where $\mathbb{N}$ is the set of the natural numbers. Since PC1-3 and MP give a complete system of non-modal sentential logic, every instance of any non-modal theorem schema will be a theorem of any $\mathscr{V}$-extension $\mathscr{S}$. Consequently, we can easily establish the deduction theorem for any $\mathscr{V}$-extension $\mathscr{S}$, namely that if $\mathbf{T} \cup \{\mathbf{A}\} \vdash_{\mathscr{S}} \mathbf{B}$ then $\mathbf{T} \vdash_{\mathscr{S}} \mathbf{A} \rightarrow \mathbf{B}$. A derived rule of substitutivity of logical equivalents can be shown to hold in any reliable $\mathscr{V}$-extension, as can the following derived rules:

(**RM◇⃞**) If ⊢**A** → **B** then ⊢◇⃞**A** → ◇⃞**B**;

(**RM⧈**) If ⊢**A** → **B** then ⊢⧈**A** → ⧈**B**;

(**RV⧈◇⃞**) If ⊢(**B**$_1$ ∧ . . . ∧ **B**$_n$ ∧ **C**) → **A** then
⊢(⧈**B**$_1$ ∧ . . . ∧ ⧈**B**$_n$ ∧ ◇⃞**C**) → ◇⃞**A**.

In addition, the following classes of theorems can be proved:

⊢◈(**A**$_1$ ∨ . . . ∨ **A**$_n$) → (◈**A**$_1$ ∨ . . . ∨ ◈**A**$_n$) $(\mathbf{n} \in \mathbb{N})$.

⊢(⧈**A**$_1$ ∧ . . . ∧ ⧈**A**$_n$) → ⧈(**A**$_1$ ∧ . . . ∧ **A**$_n$) $(\mathbf{n} \in \mathbb{N})$.

We list a few theorems of the system $\mathscr{V}$ in the table on the next page. Theorems in the right column are duals of the corresponding theorems or axioms in the left column.

SOME THEOREMS OF THE SYSTEM $\mathscr{V}$

	$\boxbox(\mathbf{A} \vee \neg\mathbf{A})$	$\neg\diamonddiamond(\mathbf{A} \wedge \neg\mathbf{A})$
	$\boxdiamond\mathbf{A} \to (\boxbox\mathbf{A} \vee \diamonddiamond\mathbf{A})$	$(\diamonddiamond\mathbf{A} \wedge \boxbox\mathbf{A}) \to \diamondplus\mathbf{A}$
	$\boxbox\mathbf{A} \to (\diamondplus\mathbf{A} \vee \boxdiamond\mathbf{A})$	$(\boxdiamond\mathbf{A} \wedge \diamondplus\mathbf{A}) \to \diamonddiamond\mathbf{A}$
	$\diamondplus\mathbf{A} \to \diamondplus(\mathbf{B} \vee \neg\mathbf{B})$	$\boxdiamond(\mathbf{A} \wedge \neg\mathbf{A}) \to \boxdiamond\mathbf{B}$
	$\boxdiamond\mathbf{A} \to \boxdiamond(\mathbf{B} \vee \neg\mathbf{B})$	$\diamondplus(\mathbf{A} \wedge \neg\mathbf{A}) \to \diamondplus\mathbf{B}$
	$\diamonddiamond\mathbf{A} \to \diamonddiamond(\mathbf{B} \vee \neg\mathbf{B})$	$\boxbox(\mathbf{A} \wedge \neg\mathbf{A}) \to \boxbox\mathbf{B}$
	$\diamonddiamond\mathbf{A} \to \diamondplus(\mathbf{B} \vee \neg\mathbf{B})$	$\boxdiamond(\mathbf{A} \wedge \neg\mathbf{A}) \to \boxbox\mathbf{B}$
	$\boxbox\mathbf{A} \to \boxbox(\mathbf{B} \vee \neg\mathbf{B})$	$\diamonddiamond(\mathbf{A} \wedge \neg\mathbf{A}) \to \diamonddiamond\mathbf{B}$
(V)	$\diamondplus(\mathbf{A} \vee \mathbf{B}) \to (\diamonddiamond\mathbf{A} \vee \diamondplus\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \boxdiamond\mathbf{B}) \to \boxdiamond(\mathbf{A} \wedge \mathbf{B})$
	$\boxdiamond(\mathbf{A} \vee \mathbf{B}) \to (\diamonddiamond\mathbf{A} \vee \boxdiamond\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \diamondplus\mathbf{B}) \to \diamondplus(\mathbf{A} \wedge \mathbf{B})$
(C$\diamonddiamond$)	$\diamonddiamond(\mathbf{A} \vee \mathbf{B}) \to (\diamonddiamond\mathbf{A} \vee \diamonddiamond\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \boxbox\mathbf{B}) \to \boxbox(\mathbf{A} \wedge \mathbf{B})$
	$\boxbox(\mathbf{A} \vee \mathbf{B}) \to (\diamonddiamond\mathbf{A} \vee \boxbox\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \diamonddiamond\mathbf{B}) \to \diamonddiamond(\mathbf{A} \wedge \mathbf{B})$
	$\boxbox(\mathbf{A} \vee \mathbf{B}) \to (\diamondplus\mathbf{A} \vee \boxdiamond\mathbf{B})$	$(\boxdiamond\mathbf{A} \wedge \diamondplus\mathbf{B}) \to \diamonddiamond(\mathbf{A} \wedge \mathbf{B})$
	$\diamondplus(\mathbf{A} \wedge \mathbf{B}) \to (\diamondplus\mathbf{A} \wedge \diamondplus\mathbf{B})$	$(\boxdiamond\mathbf{A} \vee \boxdiamond\mathbf{B}) \to \boxdiamond(\mathbf{A} \vee \mathbf{B})$
	$\boxdiamond(\mathbf{A} \wedge \mathbf{B}) \to (\boxdiamond\mathbf{A} \wedge \boxdiamond\mathbf{B})$	$(\diamondplus\mathbf{A} \vee \diamondplus\mathbf{B}) \to \diamondplus(\mathbf{A} \vee \mathbf{B})$
	$\diamonddiamond(\mathbf{A} \wedge \mathbf{B}) \to (\diamonddiamond\mathbf{A} \wedge \diamonddiamond\mathbf{B})$	$(\boxbox\mathbf{A} \vee \boxbox\mathbf{B}) \to \boxbox(\mathbf{A} \vee \mathbf{B})$
	$\boxbox(\mathbf{A} \wedge \mathbf{B}) \to (\boxbox\mathbf{A} \wedge \boxbox\mathbf{B})$	$(\diamonddiamond\mathbf{A} \vee \diamonddiamond\mathbf{B}) \to \diamonddiamond(\mathbf{A} \vee \mathbf{B})$
	$\diamondplus\mathbf{A} \to (\boxdiamond\mathbf{B} \to \diamonddiamond\mathbf{B})$	$(\boxdiamond\mathbf{A} \wedge \boxbox\neg\mathbf{A}) \to \boxdiamond\mathbf{B}$
	$\diamondplus\mathbf{A} \to (\boxbox\mathbf{B} \to \diamondplus\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \boxdiamond\neg\mathbf{A}) \to \boxdiamond\mathbf{B}$
	$\boxdiamond\mathbf{A} \to (\diamondplus\mathbf{B} \to \diamonddiamond\mathbf{B})$	$(\diamondplus\mathbf{A} \wedge \boxbox\neg\mathbf{A}) \to \diamondplus\mathbf{B}$
	$\boxdiamond\mathbf{A} \to (\boxbox\mathbf{B} \to \boxdiamond\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \diamondplus\neg\mathbf{A}) \to \diamondplus\mathbf{B}$
	$\diamonddiamond\mathbf{A} \to (\boxdiamond\mathbf{B} \to \diamonddiamond\mathbf{B})$	$(\boxdiamond\mathbf{A} \wedge \boxbox\neg\mathbf{A}) \to \boxbox\mathbf{B}$
(W)	$\diamonddiamond\mathbf{A} \to (\boxbox\mathbf{B} \to \diamondplus\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \boxdiamond\neg\mathbf{A}) \to \boxbox\mathbf{B}$
	$\diamonddiamond\mathbf{A} \to (\boxbox\mathbf{B} \to \diamonddiamond\mathbf{B})$	$(\boxbox\mathbf{A} \wedge \boxbox\neg\mathbf{A}) \to \boxbox\mathbf{B}$

SECTION 2. THE SYSTEM $\mathscr{V}$: SEMANTICS

By a *minimal model* we mean any ordered triple $\mathscr{M} = \langle \mathbf{W}, \mathbf{R}, \mathbf{P} \rangle$ such that $\mathbf{R} \subseteq \mathbf{W} \times \mathscr{P}\mathbf{W}$ and $\mathbf{P}: \mathbb{N} \rightarrow \mathscr{P}\mathbf{W}$. $\mathbf{W}$ is called the set of possible worlds in $\mathscr{M}$. $\mathbf{R}$ is called the relevance relation, and holds between a world and a set (from now on: a *cluster*) of worlds, in contrast to the usual arrangement in which relevance is a relation between one world and another. Finally, for any $\mathbf{n} \in \mathbb{N}$, $\mathbf{P}(\mathbf{n})$ gives the worlds at which the atomic sentence $\mathbf{P}_n$ is true. Thus $\mathbf{P}$ assigns truth values to the atomic sentences $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n, \ldots$. The notation

$$\alpha, \mathscr{M}, \mathscr{S} \vDash \mathbf{A}$$

abbreviates the claim that the formula $\mathbf{A}$ is true at the world α in the model $\mathscr{M}$ for the system $\mathscr{S}$, and the notation

$$\alpha, \mathscr{M}, \mathscr{S} \nvDash \mathbf{A}$$

abbreviates the claim that $\mathbf{A}$ is not true at α in the model $\mathscr{M}$ for $\mathscr{S}$. Then the notations

$$\mathscr{M}, \mathscr{S} \vDash \mathbf{A}, \mathscr{M}, \mathscr{S} \nvDash \mathbf{A}$$

and

$$\mathscr{S} \vDash \mathbf{A}, \qquad \mathscr{S} \nvDash \mathbf{A}$$

abbreviate the claims that $\mathbf{A}$ is true at all the worlds in the model $\mathscr{M}$ for $\mathscr{S}$, that $\mathbf{A}$ is not true at all the worlds in the model $\mathscr{M}$ for $\mathscr{S}$, that $\mathbf{A}$ is true at all the worlds in every model for $\mathscr{S}$ (i.e. that $\mathbf{A}$ is *valid* in $\mathscr{S}$), and that $\mathbf{A}$ is not true at all the worlds in every model for $\mathscr{S}$, respectively. For the special case of the system $\mathscr{V}$, we write simply that $\vDash\mathbf{A}$ to record that $\mathbf{A}$ is valid in $\mathscr{V}$. Using this notation, we can give the following truth conditions for the formulas of $\mathscr{L}$: for any world α in any minimal model $\mathscr{M} = \langle \mathbf{W}, \mathbf{R}, \mathbf{P} \rangle$ for $\mathscr{S}$, and for any formula $\mathbf{A}$ of $\mathscr{L}$:

$$\alpha, \mathscr{M}, \mathscr{S} \nvDash \perp;$$

$$\alpha, \mathscr{M}, \mathscr{S} \vDash \mathbf{A} \rightarrow \mathbf{B} \text{ iff } \alpha, \mathscr{M}, \mathscr{S} \nvDash \mathbf{A} \text{ or } \alpha, \mathscr{M}, \mathscr{S} \vDash \mathbf{B};$$

$$\alpha, \mathscr{M}, \mathscr{S} \vDash \oplus\mathbf{A} \text{ iff } (\exists \mathbf{K} \subseteq \mathbf{W})[\alpha\mathbf{R}\mathbf{K} \ \& \ (\forall \gamma \in \mathbf{K})[\gamma, \mathscr{M}, \mathscr{S} \vDash \mathbf{A}]];$$

$$\alpha, \mathscr{M}, \mathscr{S} \vDash \diamondsuit\mathbf{A} \text{ iff } (\exists \mathbf{K} \subseteq \mathbf{W})[\alpha\mathbf{R}\mathbf{K} \ \& \ (\exists \gamma \in \mathbf{K})[\gamma, \mathscr{M}, \mathscr{S} \vDash \mathbf{A}]].$$

It is straightforward to establish that $\mathscr{V}$ is sound, i.e. that if $\vdash \mathbf{A}$ then $\models \mathbf{A}$.

SECTION 3. THE SYSTEM $\mathscr{V}$: COMPLETENESS

Fairly routine Henkin-style proofs can be given of the completeness of $\mathscr{V}$ and various of its reliable extensions, once we give the appropriate definition of a canonical model. If $\mathscr{S}$ is any consistent reliable $\mathscr{V}$-extension, then by the *canonical model for* $\mathscr{S}$ we shall mean the minimal model $\mathscr{M}[\mathscr{S}] = \langle \mathbf{W}[\mathscr{S}], \mathbf{R}[\mathscr{S}], \mathbf{P}[\mathscr{S}] \rangle$ defined by the following conditions:

$\mathbf{W}[\mathscr{S}]$ is the set of all maximal consistent $\mathscr{S}$-extensions;

$\alpha\mathbf{R}[\mathscr{S}]\mathbf{K}$ iff $\alpha \in \mathbf{W}[\mathscr{S}]$ and $\mathbf{K} \subseteq \mathbf{W}[\mathscr{S}]$, with

$$\{\oplus\mathbf{B}: \mathbf{B} \in \cap\mathbf{K}\} \subseteq \alpha \text{ and}$$

$$\{\diamondsuit\mathbf{B}: \mathbf{B} \in \cup\mathbf{K}\} \subseteq \alpha; \text{ and}$$

$\mathbf{P}[\mathscr{S}](\mathbf{n})$ is $\{\alpha \in \mathbf{W}[\mathscr{S}]: \mathbf{P}_n \in \alpha\}$.

Clearly $\mathscr{M}[\mathscr{S}]$, as thus defined, is a minimal model. (Throughout, we take $\cup\mathbf{K}$ and $\cap\mathbf{K}$ to be restricted to formulas, so if $\mathbf{K} = \phi$, $\cup\mathbf{K} = \phi$ and $\cap\mathbf{K}$ is the set of all formulas.)

It is routine to show that in the canonical model for $\mathscr{S}$, if α, $\beta \in \mathbf{W}[\mathscr{S}]$ and $\mathbf{K} \subseteq \mathbf{W}[\mathscr{S}]$, then:

(i) $\{\diamondsuit\mathbf{B}: \mathbf{B} \in \cup\mathbf{K}\} \subseteq \alpha$ iff $\{\mathbf{B}: \boxdot\mathbf{B} \in \alpha\} \subseteq \cap\mathbf{K}$;

(ii) $\{\oplus\mathbf{C}: \mathbf{C} \in \cap\mathbf{K}\} \subseteq \alpha$ iff $\{\mathbf{C}: \boxtimes\mathbf{C} \in \alpha\} \subseteq \cup\mathbf{K}$.

It is not quite so trivial to prove the following lemmas.

LEMMA 1. *In the canonical model for* $\mathscr{S}$,

if α, $\beta \in \mathbf{W}[\mathscr{S}]$ *and* $\{\mathbf{B}: \boxdot\mathbf{B} \in \alpha\} \subseteq \beta$,
then $(\exists\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})[\beta \in \mathbf{K}]$.

Proof: Let $\mathscr{B} = \{\mathbf{B}: \boxdot\mathbf{B} \in \alpha\}$ and let $\mathscr{C} = \{\mathbf{C}: \boxtimes\mathbf{C} \in \alpha\}$. Suppose $\mathscr{B} \subseteq \beta$. For each $\mathbf{C} \in \mathscr{C}$, let $\mathbf{K}_\mathbf{C} = \{\gamma \in \mathbf{W}[\mathscr{S}]: \mathscr{B} \subseteq \gamma \ \&\ \mathbf{C} \in \gamma\}$. Let $\mathbf{K}$ be $\{\beta\} \cup \bigcup_{\mathbf{C} \in \mathscr{C}} \mathbf{K}_\mathbf{C}$.

By construction, $\mathscr{B} \subseteq \cap\mathbf{K}$ and $\beta \in \mathbf{K}$. By the results stated above, it remains only to show that $\mathscr{C} \subseteq \cup\mathbf{K}$. Suppose not. Then for some $\mathbf{C} \in \mathscr{C}$, $\mathbf{C} \notin \cup\mathbf{K}$, so $\mathbf{K}_{\mathrm{C}} = \phi$. It follows that $\mathscr{B} \cup \{\mathbf{C}\}$ is inconsistent, i.e. $\mathscr{B} \vdash \neg\mathbf{C}$. So for some $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathscr{B}$ we have $\vdash(\mathbf{B}_1 \wedge \ldots \wedge \mathbf{B}_n) \rightarrow \neg\mathbf{C}$. By results from the proof theory, we get $\vdash(\boxdot\mathbf{B}_1 \wedge \ldots \wedge \boxdot\mathbf{B}_n) \rightarrow \boxdot\neg\mathbf{C}$. But since $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathscr{B}$, we have $\boxdot\mathbf{B}_1, \ldots, \boxdot\mathbf{B}_n \in \alpha$. So $\boxdot\neg\mathbf{C} \in \alpha$, since α is closed under modus ponens. But $\mathbf{C} \in \mathscr{C}$, so $\boxed{\diamond}\mathbf{C} \in \alpha$. By definition it follows that $\neg\Diamond\!\!\!\!+\neg\mathbf{C} \in \alpha$. So by W it follows that *any* formula $\mathbf{A}$ we must have $\neg\Diamond\!\!\!\!+\mathbf{A} \in \alpha$. In particular $\neg\diamondsuit\neg\mathbf{C} \in \alpha$, i.e. $\boxdot\mathbf{C} \in \alpha$. It follows that $\mathbf{C} \in \mathscr{B}$, so $\mathbf{C} \in \beta$, so $\mathbf{C} \in \cup\mathbf{K}$, which is a contradiction.

LEMMA 2. *In the canonical model for* $\mathscr{S}$, *if* $\alpha \in \mathbf{W}[\mathscr{S}]$ *then*:

(i) $\boxdot\mathbf{A} \in \alpha$ *iff* $(\forall\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\forall\beta \in \mathbf{K})[\mathbf{A} \in \beta)$;

(ii) $\diamondsuit\mathbf{A} \in \alpha$ *iff* $(\exists\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\exists\beta \in \mathbf{K})[\mathbf{A} \in \beta]$.

Proof: For part (i), suppose $\boxdot\mathbf{A} \in \alpha$, and let $\alpha\mathbf{R}[\mathscr{S}]\mathbf{K}$ and $\beta \in \mathbf{K}$. Since $\alpha\mathbf{R}[\mathscr{S}]\mathbf{K}$, $\{\mathbf{B}: \boxdot\mathbf{B} \in \alpha\} \subseteq \cap\mathbf{K}$. So $\mathbf{A} \in \cap\mathbf{K}$. So $\mathbf{A} \in \beta$. Conversely, suppose

$$(\forall\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\forall\beta \in \mathbf{K})[\mathbf{A} \in \beta].$$

We need to show that $\boxdot\mathbf{A} \in \alpha$. Let $\mathscr{B} = \{\mathbf{B}: \boxdot\mathbf{B} \in \alpha\}$. Let

$$\mathbf{K}_\alpha = \{\gamma \in \mathbf{W}[\mathscr{S}]: (\exists\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})[\gamma \in \mathbf{K}]\}.$$

Then by Lemma 1, every maximal consistent $\mathscr{S}$-extension of $\mathscr{B}$ is an element of $\mathbf{K}_\alpha$. So $\mathbf{A}$ is an element of every maximal consistent $\mathscr{S}$-extension of $\mathscr{B}$. By the Lindenbaum Extension Theorem it follows that $\mathscr{B} \vdash \mathbf{A}$ in $\mathscr{S}$. So for some $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathscr{B}$ we have $\vdash(\mathbf{B}_1 \wedge \ldots \wedge \mathbf{B}_n) \rightarrow \mathbf{A}$. By the proof theory, we get $\vdash(\boxdot\mathbf{B}_1 \wedge \ldots \wedge \boxdot\mathbf{B}_n) \rightarrow \boxdot\mathbf{A}$. But since $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathscr{B}$, we have $\boxdot\mathbf{B}_1, \ldots, \boxdot\mathbf{B}_n \in \alpha$. So $\boxdot\mathbf{A} \in \alpha$. Part (ii) follows from part (i) trivially.

LEMMA 3. *In the canonical model for* $\mathscr{S}$, *if* $\alpha \in \mathbf{W}[\mathscr{S}]$ *then*:

(i) $\boxed{\diamond}\mathbf{A} \in \alpha$ *iff* $(\forall\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\exists\beta \in \mathbf{K})[\mathbf{A} \in \beta]$;

(ii) $\Diamond\!\!\!\!+\mathbf{A} \in \alpha$ *iff* $(\exists\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\forall\beta \in \mathbf{K})[\mathbf{A} \in \beta]$.

Proof: For part (i), suppose $\boxtimes\mathbf{A} \in \alpha$, and let $\alpha\mathbf{R}[\mathscr{S}]\mathbf{K}$. Then since $\alpha\mathbf{R}[\mathscr{S}]\mathbf{K}$, $\{\mathbf{C}: \boxtimes\mathbf{C} \in \alpha\} \subseteq \cup\mathbf{K}$. Thus $\mathbf{A} \in \cup\mathbf{K}$, and hence $(\exists\beta \in \mathbf{K})[\mathbf{A} \in \beta]$. For the contrapositive of the converse, suppose $\boxtimes\mathbf{A} \notin \alpha$. Then $\neg\boxtimes\mathbf{A} \in \alpha$. We need to show that $\neg(\forall\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\exists\beta \in \mathbf{K})[\mathbf{A} \in \beta]$, i.e. that $(\exists\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\forall\beta \in \mathbf{K})[\mathbf{A} \notin \beta]$, i.e. that $(\exists\mathbf{K}: \alpha\mathbf{R}[\mathscr{S}]\mathbf{K})(\forall\beta \in \mathbf{K})[\neg\mathbf{A} \in \beta]$. Let $\mathscr{B} = \{\mathbf{B}: \boxdot\mathbf{B} \in \alpha\}$ and let $\mathscr{C} = \{\mathbf{C}: \boxtimes\mathbf{C} \in \alpha\}$. For each $\mathbf{C} \in \mathscr{C}$, let $\mathbf{K}^{\mathbf{C}} = \{\gamma \in \mathbf{W}[\mathscr{S}]: \mathscr{B} \subseteq \gamma \ \&\ \neg\mathbf{A} \in \gamma \ \&\ \mathbf{C} \in \gamma\}$. Let $\mathbf{K}$ be the union of the sets $\mathbf{K}^{\mathbf{C}}$, for all $\mathbf{C} \in \mathscr{C}$. Certainly $(\forall\beta \in \mathbf{K})[\neg\mathbf{A} \in \beta]$, by construction. Certainly, also, $\mathscr{B} \subseteq \cap\mathbf{K}$. It remains only to show that $\mathscr{C} \subseteq \cup\mathbf{K}$. Suppose not. Then for some $\mathbf{C} \in \mathscr{C}$, $\mathbf{C} \notin \cup\mathbf{K}$, so $\mathbf{C} \notin \cup\mathbf{K}^{\mathbf{C}}$, so $\mathbf{K}^{\mathbf{C}} = \phi$. It follows that $\mathscr{B} \cup \{\neg\mathbf{A}, \mathbf{C}\}$ is inconsistent, and thus that $\mathscr{B} \cup \{\mathbf{C}\} \vdash \mathbf{A}$. So for some $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathscr{B}$ we have $\vdash(\mathbf{B}_1 \wedge \ldots \wedge \mathbf{B}_n \wedge \mathbf{C}) \rightarrow \mathbf{A}$. By the derived rule RV$\boxdot\boxtimes$, we get $\vdash((\boxdot\mathbf{B}_1 \wedge \ldots \wedge \boxdot\mathbf{B}_n) \wedge \boxtimes\mathbf{C}) \rightarrow \boxtimes\mathbf{A}$. But since $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathscr{B}$, we have $\boxdot\mathbf{B}_1, \ldots, \boxdot\mathbf{B}_n \in \alpha$, and since $\mathbf{C} \in \mathscr{C}$, we have $\boxtimes\mathbf{C} \in \alpha$. So $\boxtimes\mathbf{A} \in \alpha$, contradicting our assumption. Part (ii) follows trivially from part (i).

We can then prove that $\alpha, \mathscr{M}[\mathscr{S}], \mathscr{S} \models \mathbf{A}$ iff $\mathbf{A} \in \alpha$, by induction on the complexity of $\mathbf{A}$, and the completeness theorem follows straightforwardly.

We can now see that the system $\mathscr{V}$ cannot have any theorems of any of the forms $\diamondplus\mathbf{A}$, $\boxtimes\mathbf{A}$, or $\diamonddot\mathbf{A}$, since no formula of any of these forms can be valid. In Section 5 below we will consider a natural extension of $\mathscr{V}$ which contains theorems of each of these forms.

We could give a complete axiomatization for a subsystem of $\mathscr{V}$ using only the operator $\diamonddot$ and its dual $\boxdot$. The obvious adaptation of the system K would do. We could also give a complete axiomatization of a subsystem using only the operator $\diamondplus$ and its dual $\boxtimes$. The obvious adaptation of the classical system E would serve. But it does not appear possible to view the system $\mathscr{V}$ as the simple sum of two such separately axiomatized systems. The distinctive axiom schemata V and W are not derivable from any combination of single-operator schemata and rules valid in $\mathscr{V}$.

SECTION 4. THE SYSTEM $\mathscr{V}$: RELATIONSHIP TO THE SYSTEM K

We can get some understanding of the nature and behavior of the system $\mathscr{V}$ by noting that there is a systematic connection between formulas which are valid in $\mathscr{V}$ and a subset of the formulas valid in the basic normal modal system K, whose models are all the models involving the usual relevance relation between worlds.

To prove that there is such a connection between $\mathscr{V}$ and K, and to see more precisely what that connection is, we need to define a certain transformation from formulas of $\mathscr{L}$ to the language $\mathscr{L}_K$ for the system K. For formulas $\mathbf{A}$ of $\mathscr{L}$, let $\tau(\mathbf{A})$ be defined recursively as follows:

if $\mathbf{A}$ is $\mathbf{P}_i$ for some number $\mathbf{i}$,	then $\tau(\mathbf{A}) = \mathbf{P}_i$;
if $\mathbf{A}$ is $\bot$,	then $\tau(\mathbf{A}) = \bot$;
if $\mathbf{A}$ is $\mathbf{B} \to \mathbf{C}$, for $\mathbf{B}$ and $\mathbf{C}$ in $\mathscr{L}$,	then $\tau(\mathbf{A}) = \tau(\mathbf{B}) \to \tau(\mathbf{C})$;
if $\mathbf{A}$ is $\oplus\mathbf{B}$, for $\mathbf{B}$ in $\mathscr{L}$,	then $\tau(\mathbf{A}) = \Diamond\Box\tau(\mathbf{B})$;
if $\mathbf{A}$ is $\circledast\mathbf{B}$, for $\mathbf{B}$ in $\mathscr{L}$,	then $\tau(\mathbf{A}) = \Diamond\Diamond\tau(\mathbf{B})$.

We can then show that, for any formula $\mathbf{A}$ in $\mathscr{L}$, $\mathbf{A}$ is valid in $\mathscr{V}$ iff $\tau(\mathbf{A})$ is valid in K. First we show that if $\mathbf{A}$ is valid in $\mathscr{V}$ then $\tau(\mathbf{A})$ is valid in K. Then we show the converse.

To show that if $\mathscr{V} \vDash \mathbf{A}$ then $\mathrm{K} \vDash \tau(\mathbf{A})$, we need to make use of a way of constructing a model for $\mathscr{V}$ out of any arbitrarily chosen model for K. If $\mathscr{M} = \langle \mathbf{W}, \mathbf{R}, \mathbf{P} \rangle$ is any model for K, with $\mathbf{R} \subseteq \mathbf{W}^2$ and $\mathbf{P}\colon \mathbb{N} \to \mathscr{P}\mathbf{W}$, then let $\mathbf{R}_{\mathscr{V}} = \{\langle \alpha, \mathbf{K} \rangle\colon \alpha \in \mathbf{W}\ \&\ (\exists \beta \in \mathbf{W})[\alpha\mathbf{R}\beta\ \&$ $\mathbf{K} = \{\gamma \in \mathbf{W}\colon \beta\mathbf{R}\gamma\}]\}$. Then let $\mathscr{V}(\mathscr{M}) = \langle \mathbf{W}, \mathbf{R}_{\mathscr{V}}, \mathbf{P} \rangle$. It is then trivial to show that $\mathscr{V}(\mathscr{M})$ is a model for $\mathscr{V}$, and that $\alpha, \mathscr{M}, \mathrm{K} \vDash \tau(\mathbf{A})$ iff $\alpha, \mathscr{V}(\mathscr{M}), \mathscr{V} \vDash \mathbf{A}$, from which it follows that if $\mathscr{V} \vDash \mathbf{A}$ then $\mathrm{K} \vDash \tau(\mathbf{A})$.

To show that if $\mathrm{K} \vDash \tau(\mathbf{A})$ then $\mathscr{V} \vDash \mathbf{A}$, we need to make use of a way of constructing a model for K out of any arbitrarily chosen model for $\mathscr{V}$. If $\mathscr{M} = \langle \mathbf{W}, \mathbf{R}, \mathbf{P} \rangle$ is any model for $\mathscr{V}$, with $\mathbf{R} \subseteq \mathbf{W} \times \mathscr{P}\mathbf{W}$, and $\mathbf{P}\colon \mathbb{N} \to \mathscr{P}\mathbf{W}$, then

let $\mathbf{W}_K = \mathbf{W} \cup \{\langle \alpha, \mathbf{K} \rangle\colon \alpha\mathbf{R}\mathbf{K}\}$, and

let $\mathbf{R}_K = \{\langle \alpha, \beta \rangle\colon \alpha, \beta \in \mathbf{W}_K\ \&\ ((\exists \mathbf{K} \subseteq \mathbf{W})[[\beta = \langle \alpha, \mathbf{K} \rangle] \vee$ $(\exists \gamma \in \mathbf{W})[\alpha = \langle \gamma, \mathbf{K} \rangle\ \&\ \beta \in \mathbf{K}]])\}$.

Then let $\mathrm{K}(\mathscr{M}) = \langle \mathbf{W}_K, \mathbf{R}_K, \mathbf{P} \rangle$.

It is then trivial to show that if $\mathscr{M}$ is any model for $\mathscr{V}$, then $\mathrm{K}(\mathscr{M})$ is a model for K. Note that we need to treat the artificial objects

$\langle \alpha, \mathbf{K} \rangle$ as worlds in $\mathrm{K}(\mathscr{M})$. If we just think of possible worlds as "pegs" on which to hang collections of true formulas or true propositions, this is no problem: the intrinsic nature of the "peg" will matter only to the degree that it affects the question which "pegs" are distinct from which. Otherwise, we can think of the elements of $\mathrm{W_K}$ as being *names* of distinct possible worlds, rather than as being the worlds themselves.

Note also that, since nothing has been done to augment the truth-assignment function $\mathbf{P}$, all atomic formulas are false at each of the artificially constructed worlds in $\mathbf{W_K}$, i.e. in each of the worlds in $\mathbf{W_K}$ that weren't worlds in $\mathbf{W}$. This doesn't matter, as it happens, since it turns out to be irrelevant to our purposes just what truth values atomic sentences may have at these artificial worlds. Assigning all atomic sentences the value **false** at these worlds just happens to be the most convenient thing to do.

It is easy to show that if we let $\mathscr{M} = \langle \mathbf{W}, \mathbf{R}, \mathbf{P} \rangle$ be any model for $\mathscr{V}$, let α be any world in $\mathbf{W}$, and let $\mathbf{A}$ be any formula in $\mathscr{L}$, then

$$\alpha, \mathscr{M}, \mathscr{V} \models \mathbf{A} \text{ iff } \alpha, \mathrm{K}(\mathscr{M}), \mathrm{K} \models \tau(\mathbf{A}),$$

from which it follows easily that if $\mathrm{K} \models \tau(\mathbf{A})$ then $\mathscr{V} \models (\mathbf{A})$.

Together, these two results establish that the valid formulas of $\mathscr{V}$ are just the formulas whose transforms in the language $\mathscr{L}_\mathrm{K}$ are valid in K. In a certain way, then, the valid formulas of $\mathscr{V}$ correspond to a subset of those of K. On the other hand, there is another way in which the valid formulas of K correspond to a subset of those of $\mathscr{V}$. To see this, we need to define another transformation, this time from formulas of K to formulas of $\mathscr{V}$. For formulas $\mathbf{A}$ of $\mathscr{L}_\mathrm{K}$, let $\sigma(\mathbf{A})$ be defined recursively as follows:

if $\mathbf{A}$ is $\mathbf{P}_i$ for some number $\mathbf{i}$,	then $\sigma(\mathbf{A}) = \mathbf{P}_i$;
if $\mathbf{A}$ is $\perp$,	then $\sigma(\mathbf{A}) = \perp$;
if $\mathbf{A}$ is $\mathbf{B} \to \mathbf{C}$, for $\mathbf{B}$, $\mathbf{C}$ in $\mathscr{L}_\mathrm{K}$,	then $\sigma(\mathbf{A}) = \sigma(\mathbf{B}) \to \sigma(\mathbf{C})$;
if $\mathbf{A}$ is $\Diamond\mathbf{B}$, for $\mathbf{B}$ in $\mathscr{L}_\mathrm{K}$,	then $\sigma(\mathbf{A}) = \diamondsuit\sigma(\mathbf{B})$.

It is then trivial to show that for any formula $\mathbf{A}$ in $\mathscr{L}_\mathrm{K}$, if $\mathbf{A}$ is a theorem in K then $\sigma(\mathbf{A})$ is a theorem in $\mathscr{V}$.

To show that if $\sigma(\mathbf{A})$ is valid in $\mathscr{V}$ then $\mathbf{A}$ is valid in K, we need to find an appropriate way of constructing models of K out of arbitrary models for $\mathscr{V}$. The method used before isn't suitable for our present purpose, but the following method is: if $\mathscr{M} = \langle \mathbf{W}, \mathbf{R}, \mathbf{P} \rangle$ is any minimal model for $\mathscr{V}$, let $\mathbf{R}^{\mathscr{V}} = \{\langle \alpha, \beta \rangle \in \mathbf{W}^2: (\exists \mathbf{K}: \alpha \mathbf{RK})[\beta \in \mathbf{K}]\}$. Then let $\mathscr{M}^{\mathscr{V}} = \langle \mathbf{W}, \mathbf{R}^{\mathscr{V}}, \mathbf{P} \rangle$.

Then it is easy to show that $\mathscr{M}^{\mathscr{V}}$ is a model for K. Then we can show that if we let α be any world in $\mathbf{W}$, and let $\mathbf{A}$ be any formula in $\mathscr{L}_{\mathrm{K}}$, then

$$\alpha, \mathscr{M}, \mathscr{V} \models \boldsymbol{\sigma}(\mathbf{A}) \text{ iff } \alpha, \mathscr{M}^{\mathscr{V}}, \mathrm{K} \models \mathbf{A}.$$

From these results it follows that for any formula $\mathbf{A}$ in $\mathscr{L}_{\mathrm{K}}$, $\mathrm{K} \models \mathbf{A}$ iff $\mathscr{V} \models \boldsymbol{\sigma}(\mathbf{A})$.

Together, the results in this section establish a relation between systems $\mathscr{V}$ and K analogous to the relation between classical logic and intuitionist logic in this respect: the set of theorems of either can be seen as corresponding in a fairly natural way to a subset of the theorems of the other, though of course the correspondences are different in the two directions.

SECTION 5. RELIABLE EXTENSIONS OF $\mathscr{V}$

We can get complete axiomatizations for various reliable extensions of $\mathscr{V}$ by adding axiom schemata, provided we impose corresponding restrictions on the models. Below I give a sampler of axiom schemata which could be added to $\mathscr{V}$, together with the corresponding conditions on models required for completeness. Names are given to the schemata by analogy with terminology in Chellas [2].

(N⊕)	$\vdash \neg \oplus \bot$	$(\forall \alpha \in \mathbf{W})(\forall \mathbf{K}: \alpha \mathbf{RK})[\mathbf{K} \neq \phi]$
(D⊕)	$\vdash \oplus \mathbf{A} \to \diamondsuit \mathbf{A}$	$(\forall \alpha \in \mathbf{W})(\forall \mathbf{K}: \alpha \mathbf{RK})[\mathbf{K} \neq \phi]$
(D⊠)	$\vdash \boxtimes \mathbf{A} \to \diamondsuit \mathbf{A}$	$(\forall \alpha \in \mathbf{W})(\exists \mathbf{K} \subseteq \mathbf{W})[\alpha \mathbf{RK}]$
(D⧈)	$\vdash ⧈ \mathbf{A} \to \diamondsuit \mathbf{A}$	$(\forall \alpha \in \mathbf{W})(\exists \mathbf{K}: \alpha \mathbf{RK})[\mathbf{K} \neq \phi]$
(T◈)	$\vdash \mathbf{A} \to \diamondsuit \mathbf{A}$	$(\forall \alpha \in \mathbf{W})(\exists \mathbf{K}: \alpha \mathbf{RK})[\alpha \in \mathbf{K}]$
(T⊕)	$\vdash \mathbf{A} \to \oplus \mathbf{A}$	$(\forall \alpha \in \mathbf{W})(\exists \mathbf{K}: \alpha \mathbf{RK})[\mathbf{K} \subseteq \{\alpha\}]$
($\mathrm{T_c}$⧈)	$\vdash \mathbf{A} \to ⧈ \mathbf{A}$	$(\forall \alpha \in \mathbf{W})(\forall \mathbf{K}: \alpha \mathbf{RK})[\mathbf{K} \subseteq \{\alpha\}]$
(B⧈)	$\vdash \mathbf{A} \to ⧈ \diamondsuit \mathbf{A}$	$(\forall \alpha, \beta \in \mathbf{W})[(\exists \mathbf{K}: \alpha \mathbf{RK})[\beta \in \mathbf{K}] \Rightarrow (\exists \mathbf{K}': \beta \mathbf{RK}')[\alpha \in \mathbf{K}']]$
($\mathrm{D_c}$⊕)	$\vdash \diamondsuit \mathbf{A} \to \oplus \mathbf{A}$	$(\forall \alpha \in \mathbf{W})(\forall \mathbf{K}: \alpha \mathbf{RK})[\lvert \mathbf{K} \rvert \leqslant 1]$

The proof that the schemata listed above are sound under the associated conditions on models is straightforward. Proofs that the conditions are necessary are more complex, involving techniques like those in the proof of the lemmas in Section 3 above.

Perhaps the two most natural assumptions to add to the semantics for the system $\mathscr{V}$ would be the assumptions corresponding to the schemata $\mathrm{N}\oplus$ (or, equivalently, $\mathrm{D}\oplus$) and $\mathrm{D}\boxtimes$, namely the assumptions that no relevant cluster is empty, and that every world has some cluster relevant to it. As we can see, the schemata involved seem entirely reasonable under the intended interpretation.

The schema $\mathrm{N}\oplus$ tells us that we can't do the impossible.[22] The equivalent schema $\mathrm{D}\oplus$ tells us that anything I *can* do, I *might* do. This could have been false in the fundamental system $\mathscr{V}$, but only in a bizarre-sounding situation. The only way it could be true that I was able to hit the bull's-eye and yet be false that I might hit the bull's-eye would be if I were able to hit the bull's-eye only in the trivial sense of eliminating all possibility that I not hit the bull's-eye, by eliminating *all* possibilities whatsoever. In short I would have to be able to bring possibilities to a screeching halt, so to speak. In such a situation I would be able to do the impossible. In any less bizarre situation, it will surely be correct to say that what I *can* do I *might* do.

The contrapositive of $\mathrm{D}\boxtimes$ (a strengthening of axiom schema W) tells us that anything I *will* do, I *can* do, and this seems so reasonable that $\mathrm{D}\boxtimes$ seems clearly a desirable thesis. Taken directly, $\mathrm{D}\boxtimes$ tells us that if I might$_c$ do something, in the sense that I can't prevent it from happening, then I might$_w$ do it, in the sense that it might be one upshot of my actions. If I can't (reliably) make the dart avoid the bull's-eye, then the dart might hit the bull's-eye. This could have been false in the fundamental system $\mathscr{V}$, because in that system the reason I couldn't make the dart avoid the bull's-eye might have been that there just wasn't anything at all I could do, and that would be perfectly compatible with situations in which nothing could make the dart hit the bull's-eye either.

$\mathrm{D}\boxdot$ is a logical consequence of $\mathrm{D}\oplus$ and $\mathrm{D}\boxtimes$. It assures us that anything I *will* do, I *might* do, and this corresponds to assuring us that the force of the 'will' involved is not as weak as in the fundamental system $\mathscr{V}$, where it could be true that I *will* spend money and yet, no matter what I do, never be possible that I *do* spend it. This combination of circumstances could come about because the import of saying that I *will* spend money is only that everything I *can* do will involve my spending money, and in $\mathscr{V}$ this condition would be vacuously satisfied if, as it happened, there wasn't a single thing that I could do.

The principle expressed by schema T◈ is perhaps also plausible, given the acceptance of D⊠ and D⟠. For if, as is implied by accepting D⊠, there is always at least one thing I can do and, as is implied by accepting D⟠, the things I can do are non-trivial, it seems reasonable to suppose that I can do something which might turn out as in fact things do turn out. But that is what T◈ says.

The principles expressed by schemata T⟠, T_c⧈, and B⧈ are more dubious. The implausibility of T⟠ was what first led us to seek a modal system weaker than KT. Similar, and perhaps even more dramatic, difficulties attend the acceptance of the other schemata just mentioned. Examination of the corresponding constraints on models shows them to be implausible for our intended interpretation.

D_c⟠ is the converse of the plausible schema D⟠. But if adopted in addition to D⟠, it would collapse the modalities ◈ and ⟠ into one another, thereby collapsing their duals as well. As a result, the system $\mathscr{V}$D⟠D_c⟠ would collapse into the system K. Semantically, this is because every cluster would be constrained to contain exactly one world, so that relevance of a cluster to a world would be isomorphic to relevance of a world to a world, as in K.

It is tempting to consider adding a schema 4⟠: ⊢ ⟠⟠**A** → ⟠**A**, which would say that if I can bring about circumstances in which I can bring about circumstances in which $\mathscr{P}_A$, then I can bring about circumstances in which $\mathscr{P}_A$. It is at present an open question whether there is any corresponding condition on models which would yield completeness for the system $\mathscr{V}$4⟠. The analog of a transitivity condition, namely $(\forall\alpha)(\forall \mathbf{K}\colon \alpha\mathbf{RK})(\forall\beta\colon \beta \in \mathbf{K})(\forall \mathbf{K}^*\colon \beta\mathbf{RK}^*)[\alpha\mathbf{RK}^*]$, is sufficient but too strong.

It's not clear that the schema 4⟠ represents a desirable addition, anyway. It amounts to an acceptance of composition of actions. If a pair of successive actions will bring about a given result, it is initially plausible to construe them as constituting a single action. But on the view implied by our semantics, it's not clear that this is correct. Suppose that at world α there is a relevant cluster $\mathbf{K} = \{\beta, \gamma\}$, and that there are clusters $\mathbf{K}_\beta$ and $\mathbf{K}_\gamma$ such that $\beta\mathbf{R}\mathbf{K}_\beta$ and $\gamma\mathbf{R}\mathbf{K}_\gamma$ and at all the worlds in $\mathbf{K}_\beta$ and $\mathbf{K}_\gamma$, $\mathbf{A}$ is true. Then I can choose to act in the way corresponding to the cluster $\mathbf{K}$. This will actualize one of the two worlds β and γ, but I cannot in advance determine which (and there is no reason to suppose that $\mathbf{A}$ is yet true at either β or γ). If β is the one which is actualized, then there is a way in which I can bring about circumstances in which $\mathbf{A}$ will be true, namely by choosing the action associated with cluster $\mathbf{K}_\beta$, but if it is γ that is actualized, then I must choose a different action, the one associated with cluster $\mathbf{K}_\gamma$. Since I cannot determine in advance how my first action will turn out, I cannot determine in advance what subsequent action may be required to bring about circumstances in which $\mathbf{A}$ is true. Suppose I am a skillful enough golfer that on the short par 3 hole I can hit the green in one stroke, and that, no matter where on the green the ball lands, I can then putt out in one additional stroke. Nonetheless, until I know where the ball lands on the green I don't know which further action to take to get the ball into the hole. It may not be true that I am able to get a hole in one, nor even that there is some pair of strokes I can choose in advance that will assure the ball's going into the hole. I have used epistemic language for expository convenience, but the point is really not an epistemic one, but a metaphysical one: when the choice of a second action must be dependent on uncontrolled details of the outcome of the first action, the combination of the two actions cannot assure the desired result.

This illustrates nicely how attention to the semantics given here can bring to light subtleties and issues which might otherwise elude attention. Even if, in the end, the semantics given here is found inadequate for the logic of ability, it can serve to bring out some of the questions which must be answered if a better account is to be found.

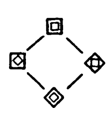
Because the first four schemata listed above seem so appropriate under the intended interpretation of our formalism, it appears that for a full-bodied logic of ability we want at least the augmented system $\mathscr{V}\mathrm{D}\diamondplus\mathrm{D}\boxdiamond$ (equivalently: $\mathscr{V}\mathrm{N}\diamondplus\mathrm{D}\boxdiamond$) in which these schemata are appended to $\mathscr{V}$. The table below gives a sample of theorems available in this augmented system in addition to those already available in $\mathscr{V}$.

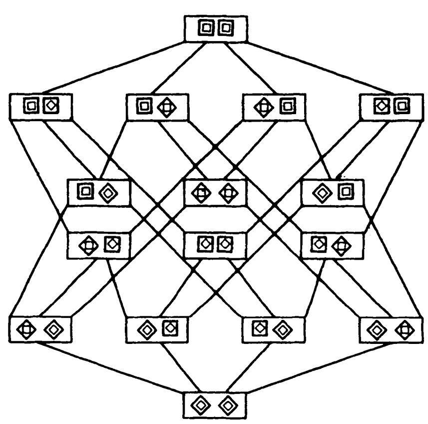
A variety of other results can be gotten from theorems of $\mathscr{V}$ by straightforward application of the results in the first three lines of the table. Pairs of theorems marked with an asterisk between them are ones which would also be theorems of the system $\mathscr{V}$T◈. Adding the schema T◈ to the system $\mathscr{V}$D⟐D⊠ doesn't seem to add any significant theorems other than T◈ itself.

SOME THEOREMS OF THE SYSTEM $\mathscr{V}$D⟐D⊠

(D⟐)	⟐A → ◈A		⧈A → ⊠A	
(D⊠)	⊠A → ◈A	*	⧈A → ⟐A	
(D⧈)	⧈A → ◈A	*	(self-dual)	
(N⊠)	⊠(A ∨ ¬A)		¬⟐(A ∧ ¬A)	(N⟐)
	⟐(A ∨ ¬A)	*	¬⊠(A ∧ ¬A)	
	◈(A ∨ ¬A)	*	¬⧈(A ∧ ¬A)	
	⟐(A ∨ B) → (◈A ∨ ◈B)		(⧈A ∧ ⧈B) → ⊠(A ∧ B)	
	⊠(A ∨ B) → (◈A ∨ ◈B)	*	(⧈A ∧ ⧈B) → ⟐(A ∧ B)	
	⧈(A ∨ B) → (◈A ∨ ⊠B)		(⧈A ∧ ⟐B) → ◈(A ∧ B)	
	⧈(A ∨ B) → (⟐A ∨ ◈B)	*	(⊠A ∧ ⧈B) → ◈(A ∧ B)	
	⧈(A ∨ B) → (◈A ∨ ◈B)		(⧈A ∧ ⧈B) → ◈(A ∧ B)	

The relations among the modalities in the augmented systems $\mathscr{V}$D⟐D⊠ and $\mathscr{V}$D⟐D⊠T◈ are indicated in the diagrams below. As can be seen from the diagrams, they form complete Boolean lattices.





Nothing prevents us from entertaining alternative interpretations of the system $\mathscr{V}$ and its reliable extensions. I have already noted the possibility of a deontic interpretation (see note 14). In addition, there is nothing to prevent our construing worlds as corresponding to states of affairs at moments in time, and interpreting clusters as corresponding to intervals of time. Under this interpretation, if the relevant clusters are intervals of the future, for example, then the formula $\oplus$A could be interpreted to be true iff there will be an interval of time throughout which A will be true. Similar readings can be provided for formulas involving the other three operators, and the resulting interpretation of the system $\mathscr{V}$ turns out to be entirely plausible. If, instead of having only future intervals relevant, we allow all broken intervals to be relevant, then we might find it plausible to accept the schema T$\diamondsuit$ after all. We leave it to another occasion to explore the question whether such a temporal logic would compare favourably or unfavourably with extant systems.[23]

NOTES

[1] Pp. 122–144.

[2] I use the conventions and terminology of [2] for naming axiom schemata, rules, and systems of modal logic. What Chellas calls the system KT is called the system T by Lemmon [10], and the system M by Kenny [6].

[3] [6], p. 135.

[4] Often, when 'can' is used in negative constructions, mere possibility is all that is involved. If I claim I *can't* win the lottery, I will normally be construed as saying that it is not possible (the drawing is rigged) or that it is *as if* it were not possible (the odds are so low). Of course either of these construals will entail that I can't be relied on to win the lottery, and that I cannot be held morally responsible for failing to do so. But since my abilities in this regard were never really in question, my remark about not being able to win the lottery is construed as cynicism or pessimism, rather than as a disclaimer of ability. The fact that such contexts for the use of 'can't' are common may help account for the fact that it sounds paradoxical to say, for example, that I can't do it but I did.

[5] Cross [3] disagrees. This is in keeping with (though not formally a consequence of) his view that to say I have the ability to bring it about that $\mathscr{P}_A$ is to say that under appropriate test conditions I would *sometimes* bring it about that $\mathscr{P}_A$. In the notation I will introduce below, Cross takes ability to behave more like my operator $\diamondsuit$, for which a schema T$\diamondsuit$ is plausible, than like my operator $\oplus$, for which the corresponding schema T$\oplus$ is not. See Section 5 for discussion of these schemata.

[6] This is usually called an alternativeness relation or an accessibility relation. But 'alternativeness' suggests a symmetry we do not wish to assume, and 'accessibility' carries unwelcome connotations of spatial connections. Moreover, there is an awkwardness about the fact that if worlds α and β have the relation $\alpha R \beta$, this can be described both by saying that β is accessible *to* α and that β is accessible *from* α, a fact which tends to cause confusion about the direction of the relation. For these reasons I follow the suggestion in Chellas [2], p. 68, and substitute the term 'relevant'. Although the respect in which one possible world is relevant to another can be expected to vary from system to system, nonetheless in general the term 'relevant' will at least be literally correct (though not terribly specific), whereas the terms 'accessible' and 'alternative' are more likely to be metaphorical.

[7] [6], p. 140.

[8] For example Elster, in [5], explicitly notes Kenny's arguments and tentatively reconsiders the position he had taken earlier, in [4].

[9] Systems resembling S2, for example, are even weaker than K, and in many of these systems the schemata T$\Diamond$ and C$\Diamond$ will both be invalid. (See [1].) However the systems of this sort that are prominent in the literature share a feature which makes them in fact unacceptable for Kenny's purposes, and may even lead one to question whether they should be called modal logics at all: they incorporate as "possible worlds" some abnormal worlds in which the usual interpretations of $\Diamond$ and $\Box$ are, explicitly or in effect, interchanged. So at abnormal worlds $\Diamond$ behaves the way $\Box$ would at normal worlds, and vice versa. Presumably such systems are unsuitable for a logic of ability, even if some such system can be tailored to produce the right formulas as theorems.

[10] Such sets are sometimes called intervals, in models in which the worlds can be taken to be ordered. They are also sometimes called neighborhoods, as in Segerberg [12]. Again, sets of possible worlds are often taken to be, or to correspond to, propositions. Since it is not in general, and for all applications of the general semantical scheme, appropriate to assume any ordering of worlds, or to assume the closure of the family of relevant sets of worlds under set-theoretic operations, or to adopt a position on the nature of propositions, I choose the more non-committal term 'cluster'. Segerberg [12] also uses the term 'cluster', but for a more specialized kind of set of worlds.

[11] Here we adopt the terminology of [2], according to which a classical modal system is any modal system containing propositional logic and DF$\Diamond$ and closed under modus ponens and RE.

[12] The most general notion of a propositional operator is given in Montague [11]. The propositional operators that can be defined using minimal models are all specializations of this most general notion to the limitations of possible worlds semantics. To give only one kind of well-known example (which, however, is not usually thought of in this way) the theories of counterfactuals given in Lewis [8] and Stalnaker [13] are in effect based on minimal models. What Lewis refers to as the system of spheres about a given world, for example, can be redescribed as the set of all clusters relevant to that world. The conditions which Lewis imposes on systems of spheres can then be redescribed as conditions on relevant clusters. The truth conditions for counterfactual conditionals are of course not such as to make Lewis' or Stalnaker's counterfactual operators definable in terms of the usual necessitation operator of the system E and its classical extensions.

[13] It is technically convenient to construe the ability operators as meaningfully applicable even to necessarily true sentences. This is not without consequences, however. Thus, to the extent that it would be odd to say that I can bring about the truth of '$2 + 2 = 4$', we should avoid shortening "can bring about circumstances in which **A** is true" to "can bring about the truth of **A**". More importantly, it means that if my *causing* my arm to rise is considered a necessary feature of my exercising of an ability to raise my arm, and if the ability operator under study is taken to include this causal condition, then we must construe causal claims in such a way that it makes sense to say I can cause $2 + 2$ to equal 4. If causal connection is an important consideration, however, it makes better sense to hold that the ability operator under study here captures the non-causal component of ability, and that a fuller account of ability will be given by conjoining the present weak ability operator with some suitable causal connective.

[14] I am indebted to Richmond Thomason for calling my attention to the fact that van Fraassen [14] uses essentially the same truth conditions to characterize a deontic operator for unconditional obligation. On this interpretation, a cluster **K** is to be considered relevant to world α iff there is some imperative **I**, in force at α, such that **K** is the set of worlds at which the imperative **I** is fulfilled. Axiomatically, the van Fraassen system extends the classical system E (see Chellas [2]) by adding the axiom schema N⟐ discussed in Section 5 below, and is a proper subsystem of the system $\mathscr{V}$N⟐ which results from adding the same schema to the system $\mathscr{V}$ developed in this paper. Since van Fraassen's system does not incorporate deontic analogs of the operators ◈ and ⧈ which I define below, his axiom system perforce lacks many of the principles characteristic of the system $\mathscr{V}$.

[15] [6], p. 139n.

[16] Often it is appropriate to construe epistemic and deontic operators similarly, as involving a fixed but tacit agent, but in those subjects the picture is obscured by the fact that it is plausible to give the modal operators a subjectless reading as well. So, for example, if we adopt an epistemic operator **[k]**, then the formula **[k]A** might read as ⌜It is known that **A**⌝ or as ⌜(The knowing agent under discussion) knows that **A**⌝. Similarly, with a deontic operator **[o]**, the formula **[o]A** might read as ⌜It is obligatory that **A**⌝ or as ⌜(The moral agent under discussion) ought to bring it about that **A**⌝. Our views about what theorems epistemic or deontic logic ought to provide may be influenced by our choice between these interpretations of the operator, although many formulas will be plausible under both readings. In the logic of ability, however, only the latter sort of reading seems plausible. ⌜It is able that **A**⌝ and even ⌜There is an ability to bring it about that **A**⌝ won't seem very appealing. It appears that there are no disembodied abilities, so to speak. In any case I shall not assume that there are any.

[17] The implicit subject must be held constant throughout any given context of argument, on pain of fallacy. Consequently we cannot assume the agent to be any of the people (if any) indicated in the embedded clause **A**. The need for care on this point is well argued in Cross [3], pp. 54ff. Similarly, the implicit time must be held constant throughout any argument, and will not necessarily coincide with any time indicated in the embedded clause. Since the embedded clause cannot be assumed to have the agent in question as its subject, we cannot assume that constructions involving the ability operator always correspond to the result of some such linguistic transformation as Equi-NP Deletion.

[18] I presuppose opportunity, setting aside cases (presumably to be handled in some later, more comprehensive theory capable of expressing the concept of opportunity) in which, for example, I have the ability to play the piano, but have no piano available. On the other hand, opportunity is only *presupposed*, not *asserted* as a (tacit) conjunct. Otherwise negations of claims of ability would become (tacit) disjunctions, giving occasion for confusion.

[19] It is tempting to take some liberties with syntax and with the use/mention distinction, and say that the formula $\diamondplus \mathbf{A}$ expresses the claim that I can do **A** (rather than that I can do $\mathscr{A}_{\mathrm{A}}$), but of course this will not be an accurate way of putting the point.

[20] For discussion of the context-dependence of such modalities, see Kratzer [7] and Lewis [9].

[21] The system is called $\mathscr{V}$ in honor of Peter van Inwagen, who made the original suggestion for truth conditions for the **can** of ability, from which this paper then grew. I am indebted to him more generally for pointing out how flexible the idea of a possible-worlds semantics is.

[22] The analog of this schema is the only modal axiom schema van Fraassen [14] uses in his axiomatization of unconditional deontic logic.

[23] A short early version of this paper was delivered at the April 1986 meetings of the Association for Symbolic Logic in Indianapolis. I am grateful to the anonymous referees and to Brian Chellas, John Hawthorne, Anthony Kenny, Thomas McKay, Margery Naylor, Robert Stalnaker, Robert van Gulick, and especially Richmond Thomason and Peter van Inwagen for helpful suggestions and encouragement.

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