AE 331 HEAT TRANSFER

Sinan Eyi
Department of Aerospace Engineering
Middle East Technical University
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Transient Conduction

One-Dimensional, Steady-State Conduction

- Unsteady, or transient, problems typically arise when the boundary conditions of a system are changed.
- Our objective in this chapter is to develop procedures for determining the time dependence of the temperature distribution within a solid during a transient process, as well as for determining heat transfer between the solid and its surroundings.

One-Dimensional, Steady-State Conduction

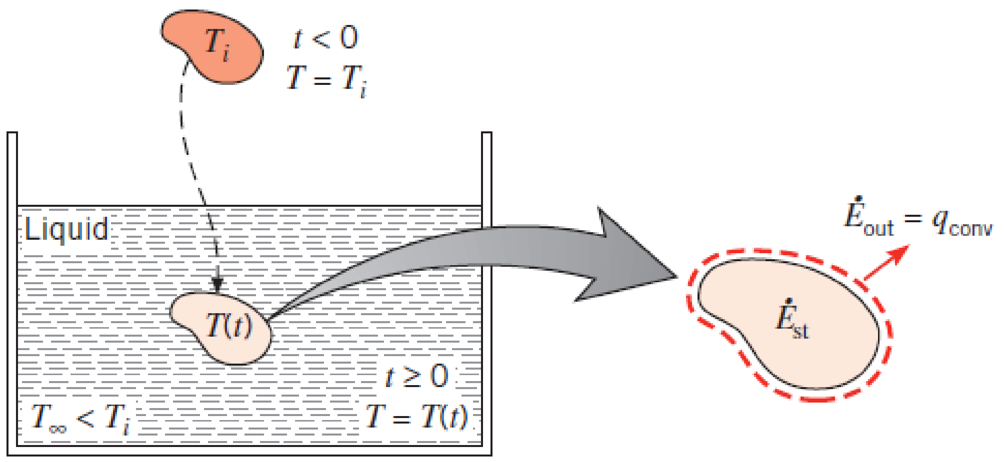
- The nature of the procedure depends on assumptions that may be made for the process.
- If, for example, temperature gradients within the solid may be neglected, a comparatively simple approach, termed the lumped capacitance method, may be used to determine the variation of temperature with time.
- Under conditions for which temperature gradients are not negligible, but heat transfer within the solid is one-dimensional, exact solutions to the heat equation may be used to compute the dependence of temperature on both location and time.

The Lumped Capacitance Method

- A simple, yet common, transient conduction problem is one for which a solid experiences a sudden change in its thermal environment.
- Consider a hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T<T_i$.
- If the quenching is said to begin at time $t=0$, the temperature of the solid will decrease for time $t>0$, until it eventually reaches T_∞.
- This reduction is due to convection heat transfer at the solid–liquid interface.

The Lumped Capacitance Method

- The essence of the lumped capacitance method is the assumption that the temperature of the solid is *spatially uniform* at any instant during the transient process.
- This assumption implies that temperature gradients within the solid are negligible.



The Lumped Capacitance Method

- From Fourier's law, heat conduction in the absence of a temperature gradient implies the existence of infinite thermal conductivity.
- Such a condition is clearly impossible.
- However, the condition is closely approximated if the resistance to conduction within the solid is small compared with the resistance to heat transfer between the solid and its surroundings.

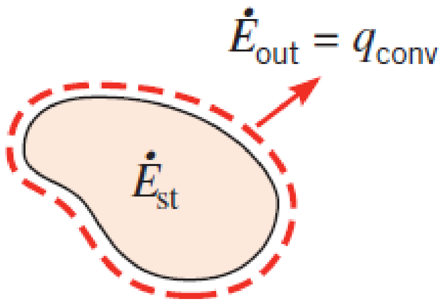
The Lumped Capacitance Method

- The transient temperature response is determined by formulating an overall energy balance on the entire solid.
- This balance must relate the rate of heat loss at the surface to the rate of change of the internal energy.
- Applying the conservation of energy for the control volume.

$$-\dot{E}_{out} = \dot{E}_{st}$$

or

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt}$$



The Lumped Capacitance Method

- Introducing the temperature difference as

$$\theta = T - T_\infty \qquad \Rightarrow \qquad \frac{d\theta}{dt} = \frac{dT}{dt}$$

- The previous equation can be written as

$$-hA_s\theta = \rho V c \frac{d\theta}{dt} \qquad \Rightarrow \qquad \frac{\rho V c}{hA_s}\frac{d\theta}{dt} = -\theta$$

The Lumped Capacitance Method

- Separating variables and integrating from the initial condition, for which t=0 and T(0)=T_i, we then obtain

$$\frac{\rho V c}{h A_s} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t dt$$

where $\theta_i = T_i - T_\infty$

Evaluating integral gives

$$\frac{\rho V c}{h A_s} \ln \frac{\theta_i}{\theta} = t \qquad (1)$$

The Lumped Capacitance Method

- Above equation can also be written as

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c} \right) t \right] \qquad (2)$$

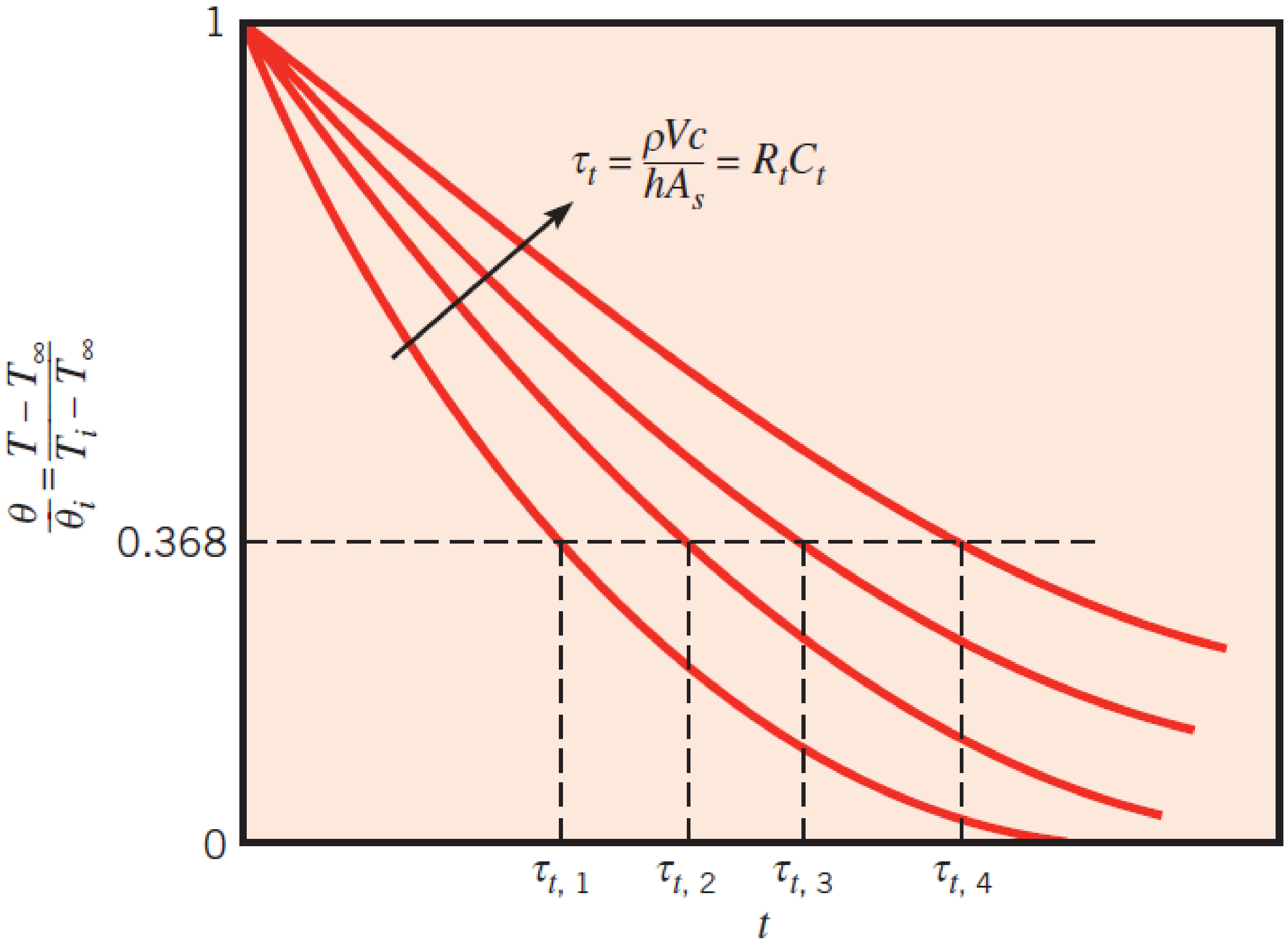
- Eq. (1) may be used to determine the time required for the solid to reach some temperature.
- Eq. (2) may be used to compute the temperature reached by the solid at some time *t.*

The Lumped Capacitance Method

- Eq. (2) indicates that the difference between the solid and fluid temperatures must decay exponentially to zero as *t* approaches infinity.
- From Eq. (2), it is evident that the quantity (Vc/hA_s) may be interpreted as a *thermal time constant* expressed as

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho Vc) = R_t C_t$$

The Lumped Capacitance Method



The Lumped Capacitance Method

- To determine the total energy transfer Q occurring up to some time t, we simply write.

$$Q = \int_0^t q\, dt = hA_s \int_0^t \theta\, dt$$

- Substituting for θ into above equation and then integrating the equations gives.

$$Q = (\rho V c)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t} \right) \right]$$

The Lumped Capacitance Method

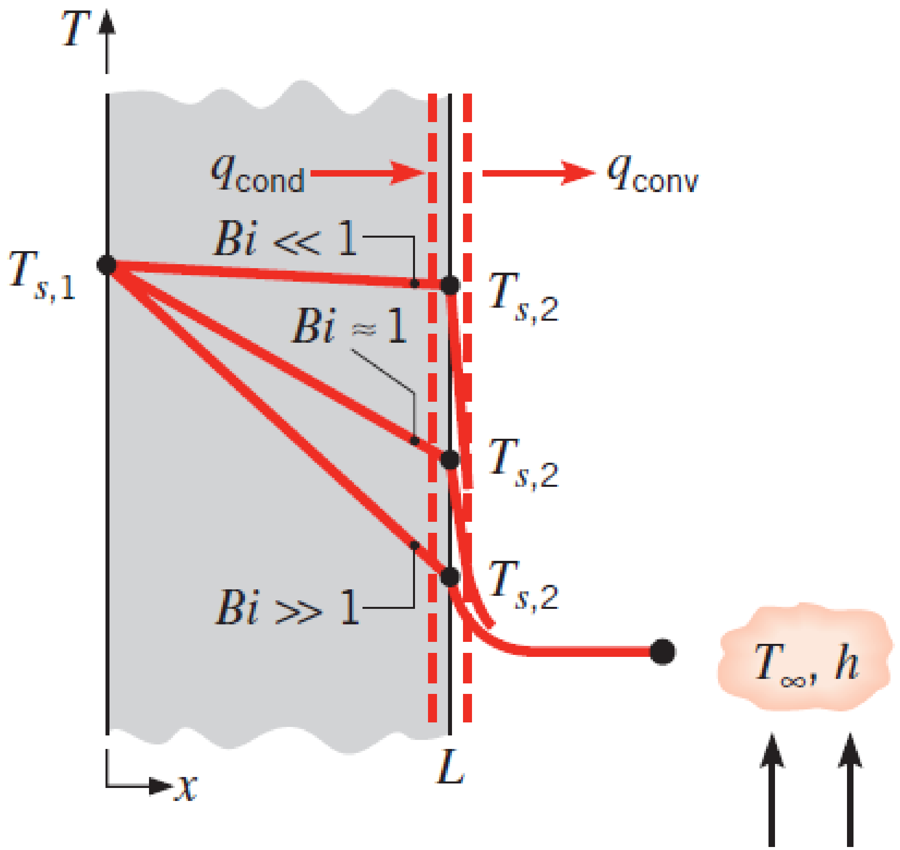
- The quantity Q is related to the change in the internal energy of the solid.

$$-Q = \Delta E_{st}$$

Validity of the Lumped Capacitance Method

- The Lumped Capacitance Method is certainly the simplest and most convenient method that can be used to solve transient heating and cooling problems.
- Hence it is important to determine under what conditions it may be used with reasonable accuracy.
- To develop a suitable criterion consider steady-state conduction through the plane wall of area A.
- Although we are assuming steady-state conditions, the following criterion is readily extended to transient processes.

Validity of the Lumped Capacitance Method



Validity of the Lumped Capacitance Method

- One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$.
- The temperature of this surface will be some intermediate value $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$.
- Hence, under steady-state conditions, the surface energy balance equation can be written as

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

where k is the thermal conductivity of the solid

Validity of the Lumped Capacitance Method

- Rearranging the terms, we then obtain

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_{\infty}} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,\mathrm{cond}}}{R_{t,\mathrm{conv}}} = \frac{hL}{k} \equiv Bi$$

- The quantity (hL/k) appearing in above equation is a dimensionless parameter and it is called the Biot number.
- Biot number may be interpreted as a ratio of thermal resistances.
- If Bi << 1, the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer.
- Hence, the assumption of a uniform temperature distribution within the solid is reasonable if the Biot number is small.

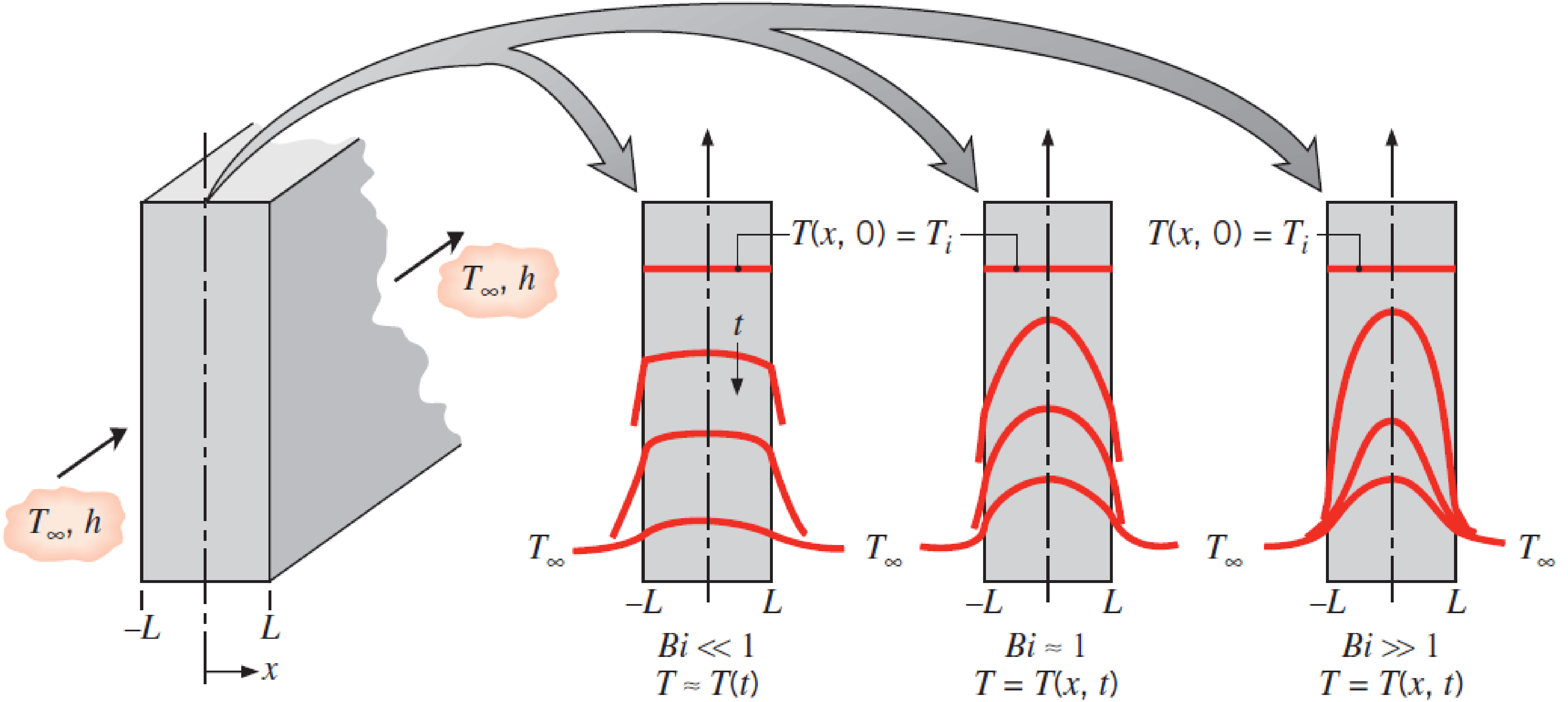
Validity of the Lumped Capacitance Method

- Although we have discussed the Biot number in the context of steady-state conditions, we are reconsidering this parameter because of its significance to transient conduction problems.
- Consider the plane wall which is initially at a uniform temperature T_i and experiences convection cooling when it is immersed in a fluid of $T_\infty < T_i$.
- The problem may be treated as one-dimensional in x, and we are interested in the temperature variation with position and time, $T(x, t)$.

Validity of the Lumped Capacitance Method

- This variation is a strong function of the Biot number, and three conditions are shown in the following figure.
- For Bi << 1, the temperature gradients in the solid are small and the assumption of a uniform temperature distribution, $T(x, t) \approx T(t)$ is reasonable.
- For moderate to large values of the Biot number, however, the temperature gradients within the solid are significant.
- for Bi >> 1, the temperature difference across the solid is much larger than that between the surface and the fluid.

Validity of the Lumped Capacitance Method



Validity of the Lumped Capacitance Method

- Before we us the limped capacitance method, the first thing we should do is to calculate the Biot number.
- If the following condition is satisfied, the error associated with using the lumped capacitance method is small.

$$\boxed{Bi = \frac{hL_c}{k} < 0.1}$$

- For convenience, it is customary to define the *characteristic length* L_c as

$$L_c \equiv V/A_s$$

Validity of the Lumped Capacitance Method

- Such a definition facilitates calculation of Lc for solids ofb complicated shape and reduces to the half-thickness L for a plane wall of thickness 2L to $r_o/2$ for a long cylinder, and to $r_o/3$ for a sphere.
- Remember the equation we developed using the Lumped Capacitance Method.

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c}\right) t \right]$$

Validity of the Lumped Capacitance Method

- The exponent term in this equation can be written as

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

or

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo$$

where $\boxed{Fo \equiv \frac{\alpha t}{L_c^2}}$

Fo is a dimensionless number and it is called Fourier number

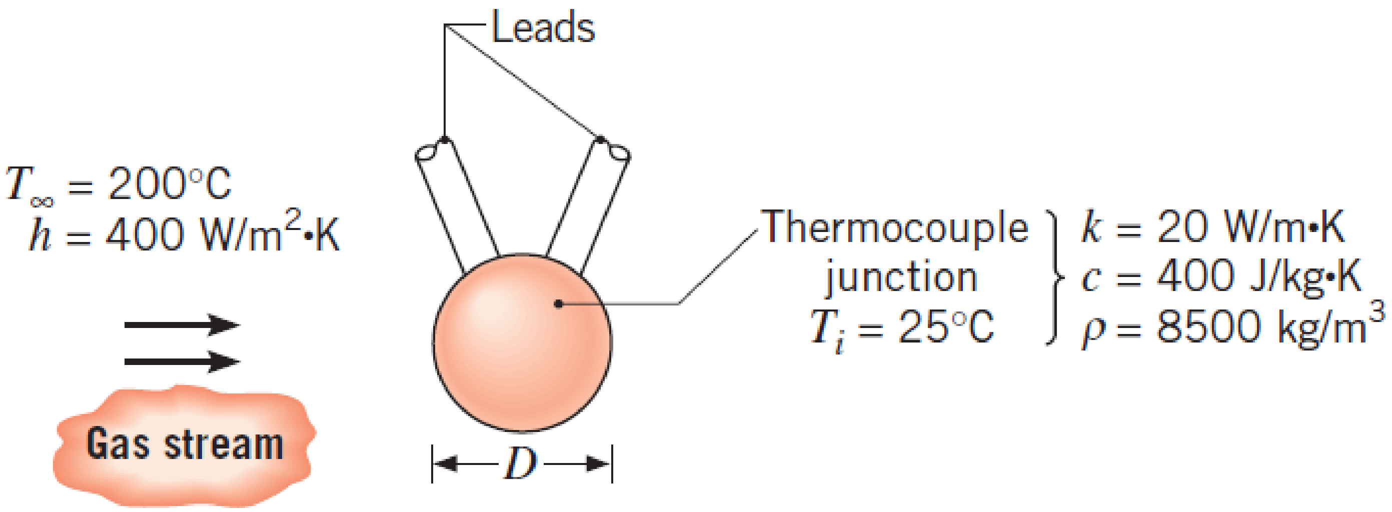
Validity of the Lumped Capacitance Method

- Using above relations, we can obtain

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$$

EXAMPLE

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is $h = 400\ \mathrm{W/m^2 \cdot K}$, and the junction thermophysical properties are $k = 20\ \mathrm{W/m \cdot K}$, $c = 400\ \mathrm{J/kg \cdot K}$, and $\rho = 8500\ \mathrm{kg/m^3}$. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?



EXAMPLE

- Because the junction diameter is unknown, it is not possible to begin the solution by determining whether the criterion for using the lumped capacitance method is satisfied.
- However, a reasonable approach is to use the method to find the diameter and to then determine whether the criterion is satisfied.
- For sphere, we can calculate

$$A_s = \pi D^2 \qquad V = \pi D^3/6$$

EXAMPLE

- The thermal time constant has already been defined as.

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho V c)$$

- The thermal time constant for sphere can be calculated as.

$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho \pi D^3}{6} c$$

- Above equation can be rearranged as

$$D = \frac{6h\tau_t}{\rho c} = \frac{6 \times 400 \text{ W/m}^2 \cdot \text{K} \times 1 \text{ s}}{8500 \text{ kg/m}^3 \times 400 \text{ J/kg} \cdot \text{K}} = 7.06 \times 10^{-4} \text{ m}$$

EXAMPLE

- With $L_c = r_o/3$.

$$Bi = \frac{h(r_o/3)}{k} = \frac{400 \text{ W/m}^2 \cdot \text{K} \times 3.53 \times 10^{-4} \text{ m}}{3 \times 20 \text{ W/m} \cdot \text{K}} = 2.35 \times 10^{-3}$$

- Hence, the criteria for Lumped Capacitance Method is satisfied.

EXAMPLE

2. The time required for the junction to reach T=199^{o}C is

$$t = \frac{\rho(\pi D^3/6)c}{h(\pi D^2)} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho D c}{6h} \ln \frac{T_i - T_\infty}{T - T_\infty}$$

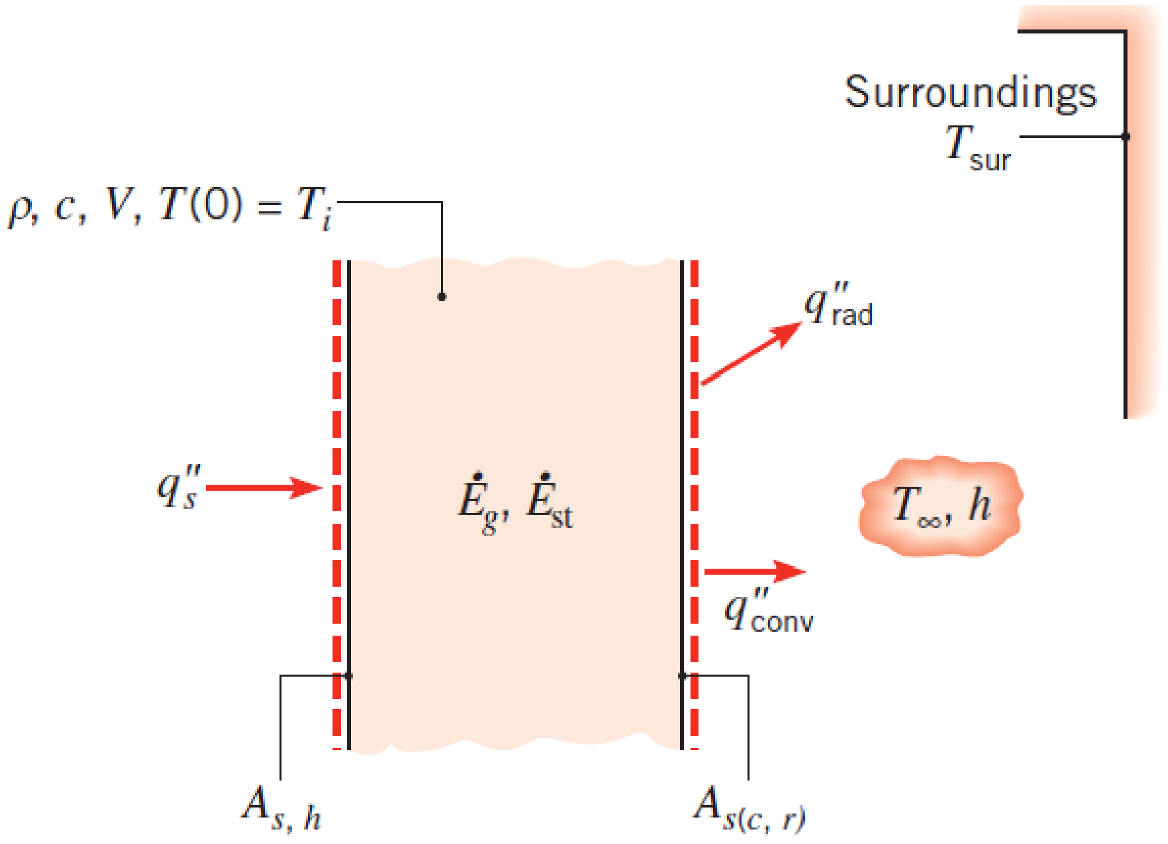
$$t = \frac{8500\,\text{kg/m}^3 \times 7.06 \times 10^{-4}\,\text{m} \times 400\,\text{J/kg}\cdot\text{K}}{6 \times 400\,\text{W/m}^2\cdot\text{K}} \ln \frac{25 - 200}{199 - 200}$$

$$t = 5.2\,\text{s} \approx 5\tau_t$$

General Lumped Capacitance Analysis

- Although transient conduction in a solid is commonly initiated by convection heat transfer to or from an adjoining fluid, other processes may induce transient thermal conditions within the solid.
- For example, a solid may be separated from large surroundings by a gas or vacuum.
- If the temperatures of the solid and surroundings differ, radiation exchange could cause the internal thermal energy, and hence the temperature, of the solid to change.
- Temperature changes could also be induced by applying a heat flux at a portion, or all, of the surface or by initiating thermal energy generation within the solid.
- Surface heating could be applied by attaching a film or sheet electrical heater to the surface, while thermal energy could be generated by passing an electrical current through the solid.

General Lumped Capacitance Analysis



General Lumped Capacitance Analysis

- In a general situation, the thermal condition within a solid may be influenced simultaneously by convection, radiation, an applied surface heat flux, and internal energy generation.
- It is presumed that, initially (t=0), the temperature of the solid T_i differs from that of the fluid T_∞, and the surroundings T_{sur}, and that both surface and volumetric heating (q_s'' and $\dot{q}$) are initiated.
- The imposed heat flux q_s'' and the convection–radiation heat transfer occur at mutually exclusive portions of the surface, $A_{s(h)}$ and $A_{s(c,r)}$, respectively, and convection–radiation transfer is presumed to be from the surface.

General Lumped Capacitance Analysis

- Moreover, although convection and radiation have been prescribed for the same surface, the surfaces may, in fact, differ ($A_{s,c} \neq A_{s,r}$).
- Applying conservation of energy at any instant t,

$$q_s'' A_{s,h} + \dot{E}_g - (q''_{conv} + q''_{rad}) A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

- Substituting convection and radiation relation into above equation gives

$$q_s'' A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)] A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

General Lumped Capacitance Analysis

- Above equation is nonlinear, first-order, nonhomogeneous, ordinary differential equation that cannot be integrated to obtain an exact solution.
- However, exact solutions may be obtained for simplified versions of the equation.

Radiation Only

- If there is no imposed heat flux or generation and convection is either nonexistent (a vacuum) or negligible relative to radiation, the previous equation reduces to.

$$\rho V c \frac{dT}{dt} = -\varepsilon A_{s,r} \sigma (T^4 - T_{\text{sur}}^4)$$

- Separating variables and integrating from the initial condition to any time t gives

$$\frac{\varepsilon A_{s,r} \sigma}{\rho V c} \int_0^t dt = \int_{T_i}^{T} \frac{dT}{T_{\text{sur}}^4 - T^4}$$

Radiation Only

- Evaluating both integrals and rearranging, the time required to reach the temperature T becomes.

$$t = \frac{\rho V c}{4\varepsilon A_{s,r}\sigma T_{sur}^3}\left\{\ln\left|\frac{T_{sur}+T}{T_{sur}-T}\right| - \ln\left|\frac{T_{sur}+T_i}{T_{sur}-T_i}\right| + 2\left[\tan^{-1}\left(\frac{T}{T_{sur}}\right) - \tan^{-1}\left(\frac{T_i}{T_{sur}}\right)\right]\right\}$$

Negligible Radiation

- An exact solution for General Lumped Capacitance equation can be obtained if radiation is neglected.
- Introducing a temperature difference $\theta = T - T_\infty$, where $d\theta/dt = dT/dt$, the General Lumped Capacitance equation reduces to

$$\frac{d\theta}{dt} + a\theta - b = 0$$

- This is a linear, first-order, nonhomogeneous differential equation, where $a \equiv (hA_{s,c}/\rho Vc)$ and $b \equiv [(q_s''A_{s,h} + \dot{E}_g)/\rho Vc]$.

Negligible Radiation

- The nonhomogeneity can be eliminated by introducing the transformation.

$$\theta' \equiv \theta - \frac{b}{a}$$

- Since $d\theta'/dt = d\theta/dt$, the differential equation can be written as

$$\frac{d\theta'}{dt} + a\theta' = 0$$

- Separating variables and integrating from 0 to *t* (θ'_i to θ') gives

$$\frac{\theta'}{\theta'_i} = \exp(-at)$$

Negligible Radiation

- Writing above equation in terms of *T*.

$$\frac{T - T_\infty - (b/a)}{T_i - T_\infty - (b/a)} = \exp(-at)$$

Hence,

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T_i - T_\infty}[1 - \exp(-at)]$$

Negligible Radiation

- Without volumetric and surface heat flux b becomes zero.

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at)$$

This is the same relation obtained in Eq. (2)

Convection Only with Variable Convection Coefficient

- In some cases, such as those involving free convection or boiling, the convection coefficient h varies with the temperature difference between the object and the fluid.
- In these situations, the convection coefficient can often be approximated with an expression of the form

$$h = C(T - T_\infty)^n$$

where *n* is a constant and the parameter *C* has units of $W/m^2K^{(1+n)}$.

Convection Only with Variable Convection Coefficient

- If radiation, surface heating, and volumetric generation are negligible, the conservation of energy equation may be written as

$$-C(T - T_\infty)^n A_{s,c}(T - T_\infty) = -CA_{s,c}(T - T_\infty)^{1+n} = \rho Vc\,\frac{dT}{dt}$$

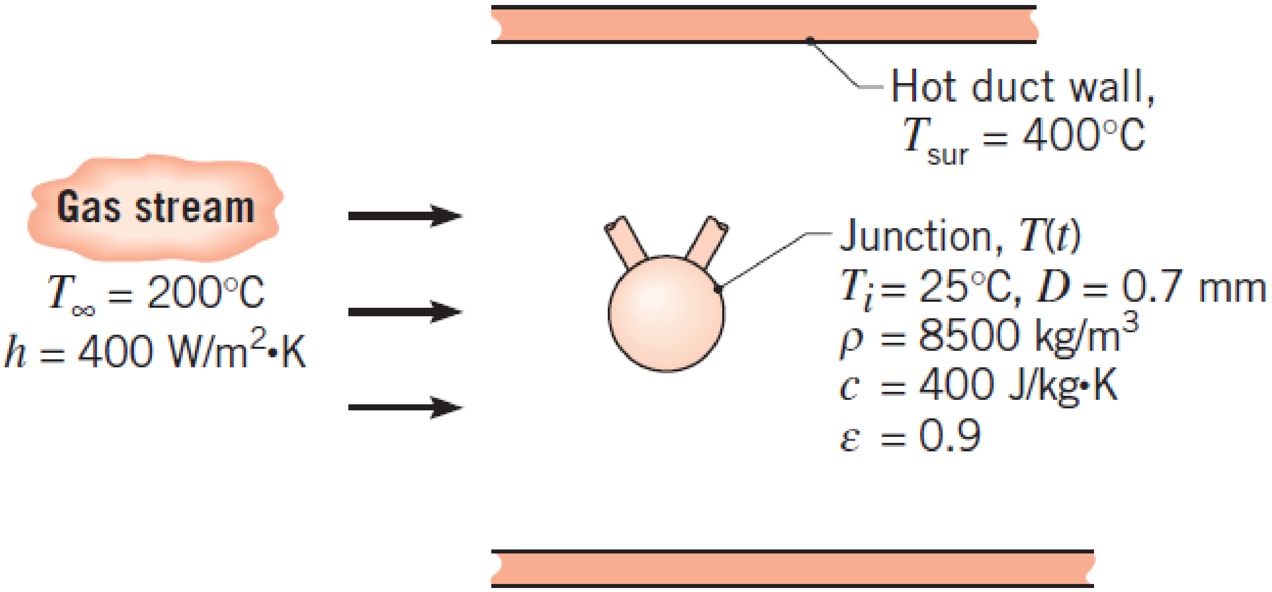
- Substituting θ and $d\theta/dt=dT/dt$ into the preceding expression, separating variables and integrating yields

$$\frac{\theta}{\theta_i} = \left[\frac{nCA_{s,c}\theta_i^n}{\rho Vc}\,t + 1\right]^{-1/n}$$

EXAMPLE

Consider the thermocouple and convection conditions of previous example, but now allow for radiation exchange with the walls of a duct that encloses the gas stream. If the duct walls are at 400^oC and the emissivity of the thermocouple bead is 0.9, calculate the steady-state temperature of the junction. Also, determine the time for the junction temperature to increase from an initial condition of 25^oC to a temperature that is within 1^oC of its steady-state value.

EXAMPLE



Hot duct wall,
$T_{sur} = 400°C$
Gas stream
$T_\infty = 200°C$
$h = 400\ W/m^2 \cdot K$
Junction, $T(t)$
$T_i = 25°C$, $D = 0.7$ mm
$\rho = 8500\ kg/m^3$
$c = 400\ J/kg \cdot K$
$\varepsilon = 0.9$

EXAMPLE

- For steady-state conditions, the energy balance on the thermocouple junction has the form

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0$$

- Recognizing that net radiation to the junction must be balanced by convection from the junction to the gas, the energy balance may be expressed as

$$[\varepsilon\sigma(T_{\text{sur}}^4 - T^4) - h(T - T_\infty)]A_s = 0$$

- Substituting numerical values, we obtain

$$T = 218.7°\text{C}$$

EXAMPLE

- The temperature-time history, $T(t)$, for the junction, initially at $T(0) = T_i = 25^oC$, follows from the energy balance for transient conditions,

$$\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{st}$$

- The energy balance may be expressed as

$$-[h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)]A_s = \rho V c \frac{dT}{dt}$$

- The solution to this first-order differential equation can be obtained by numerical integration, giving the result, $T(4.9\text{ s})$ 217.7^oC.
- Hence, the time required to reach a temperature that is within 1^oC of the steady-state value is $t = 4.9$ s.

Spatial Effects

- If the Biot number is not small, the temperature gradients within the medium are no longer negligible.
- For this condition, the use of the lumped capacitance method would yield incorrect results.
- In this case, the partial differential equations need to be solved.
- One dimensional and unsteady heat conduction equation in cartesian coordinate system can be written as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Spatial Effects

- To solve above equation, it is necessary to specify an initial condition and two boundary conditions.
- The initial condition is $T(x, 0)=Ti$
- The boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L, t) - T_\infty]$$

- Uniform temperature distribution is assumed at time t=0.
- Symmetry condition is assumed at x=0.
- Convection boundary condition is used at x=L.
- We are looking for a solution in the form of

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h)$$

Spatial Effects

- This problem may be solved analytically or numerically.
- Define a non-dimensional temperature as

$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

Accordingly, θ^* must lie in the range $0 \leq \theta^* \leq 1$.

- A dimensionless spatial coordinate may be defined as

$$x^* \equiv \frac{x}{L}$$

where L is the half-thickness of the plane wall,

- A dimensionless time may be defined as

$$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$$

where t^* is equivalent to the dimensionless *Fourier number*,

Spatial Effects

- Substituting the non-dimensional variables into the governing equation.

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo}$$

- Substituting the non-dimensional variables into the initial and boundary conditions

$$\theta^*(x^*, 0) = 1 \qquad \left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0 \quad \text{and} \quad \left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1, t^*)$$

where the Biot number is Bi = hL/k.

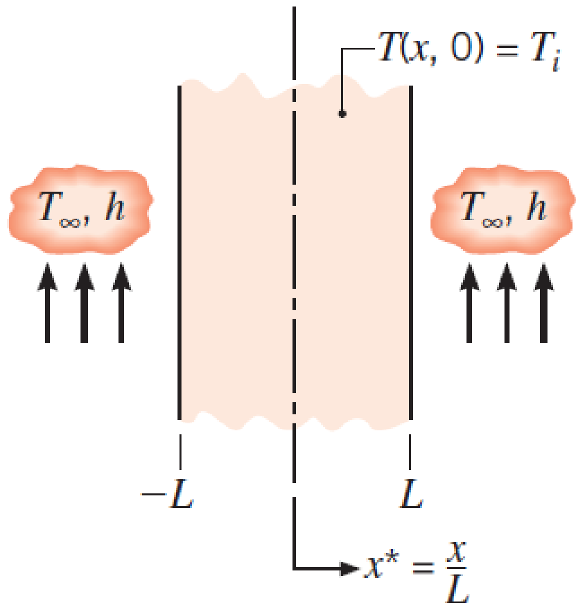
- We are looking for a solution in the form of $\theta^* = f(x^*, Fo, Bi)$

The Plane Wall with Convection

- Exact, analytical solutions to transient conduction problems have been obtained for many simplified geometries and boundary conditions.
- Several mathematical techniques, including the method of separation of variables may be used for this purpose.
- In general, the solution for the dimensionless temperature distribution is in the form of an infinite series.
- However, except for very small values of the Fourier number, this series may be approximated by a single term, considerably simplifying its evaluation.

Exact Solution

- Consider the plane wall of thickness 2L.



Exact Solution

- If the thickness is small relative to the width and height of the wall, it is reasonable to assume that conduction occurs exclusively in the x-direction.
- Assume that the wall is initially at a uniform temperature of $T(x,0) = T_i$, and is suddenly immersed in a fluid of $T_\infty \neq T_i$.
- Since the convection conditions for the surfaces at $x^* = \mp 1$ are the same, the temperature distribution at any instant must be symmetrical about the midplane ($x^* = 0$).

Exact Solution

- An exact solution to this problem is of the form

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right) \cos\left(\zeta_n x^*\right)$$

- where Fo = $\alpha t/L^2$, the coefficient C_n is

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin\left(2\zeta_n\right)}$$

- The discrete values of *n* (*eigenvalues*) are positive roots of the transcendental equation

$$\zeta_n \tan \zeta_n = Bi$$

Exact Solution

- The first four roots of this equation are given in Appendix B.3.
- The exact solution is valid for any time, $0 \leq Fo \leq \infty$.

Approximate Solution

- For values of *Fo* > 0.2, the infinite series solution can be approximated by the first term of the series, *n*=1.
- Using this approximation the dimensionless form of the temperature distribution becomes

$$\theta^* = C_1 \exp\left(-\zeta_1^2 Fo\right) \cos\left(\zeta_1 x^*\right)$$

or

$$\theta^* = \theta_o^* \cos\left(\zeta_1 x^*\right)$$

where $\theta_o^* \equiv (T_o - T_\infty)/(T_i - T_\infty)$ represents the midplane $(x^* = 0)$ temperature

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

Approximate Solution

TABLE 5.1 Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction

	Plane Wall		Infinite Cylinder		Sphere	
Bi^a	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1723	1.0049	0.2440	1.0075	0.2991	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2218	1.0082	0.3143	1.0124	0.3854	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3709	1.0173	0.4551	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6609	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8447	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5994	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	1.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2733	2.4050	1.6018	3.1415	2.0000

[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.6.

Total Energy Transfer

- In many situations it is useful to know the total energy that has left (or entered) the wall up to any time t in the transient process.
- Using the conservation of energy for the time interval bounded by the initial condition (t 0) and any time t > 0

$$E_{in} - E_{out} = \Delta E_{st}$$

- Equating the energy transferred from the wall Q to E_{out} and setting E_{in}=0 and ΔE_{st}=$E(t)$-$E(0)$, it follows that

$$Q = -[E(t) - E(0)]$$

Total Energy Transfer

or

$$Q = -\int \rho c[T(x, t) - T_i]dV$$

where the integration is performed over the volume of the wall.

- It is convenient to nondimensionalize this result by introducing the quantity

$$Q_o = \rho c V(T_i - T_\infty)$$

which may be interpreted as the initial internal energy of the wall relative to the fluid temperature.

- It is also the *maximum* amount of energy transfer that could occur if the process were continued to time *t=∞*.

Total Energy Transfer

- Hence, assuming constant properties, the ratio of the total energy transferred from the wall over the time interval t to the maximum possible transfer is

$$\frac{Q}{Q_o} = \int \frac{-[T(x,t) - T_i]}{T_i - T_\infty} \frac{dV}{V} = \frac{1}{V} \int (1 - \theta^*) dV$$

- Employing the approximate form of the temperature distribution for the plane wall, above integration can be performed to obtain

$$\frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1} \theta_o^*$$

where θ_o^* is the temperature at the midplane.

Radial Systems with Convection

- For an infinite cylinder or sphere of radius r_o (uniform initial temperature and experiences convective heat transfer), similar solutions can be obtained
- An infinite series solution may be obtained for the time dependence of the radial temperature distribution.
- A one-term approximation may be used for most conditions.
- The infinite cylinder is an idealization that permits the assumption of one-dimensional conduction in the radial direction.
- It is a reasonable approximation for cylinders having $L/r_o>10$.

Exact Solutions

- For a uniform initial temperature and convective boundary conditions, the exact solutions, applicable at any time (*Fo*>0), are as follows.
- ***Infinite Cylinder:*** In dimensionless form, the temperature is

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n r^*)$$

where $Fo = \alpha t / r_o^2$,

$$C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)}$$

Exact Solutions

and the discrete values of ζ_n are positive roots of the transcendental equation

$$\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi$$

where $Bi = hr_o/k$. The quantities J_1 and J_0 are Bessel functions of the first kind, and their values are tabulated in Appendix B.4.

- ***Sphere:*** Similarly, for the sphere

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \frac{1}{\zeta_n r^*} \sin(\zeta_n r^*) \qquad \text{where } Fo = \alpha t/r_o^2,$$

Exact Solutions

$$C_n = \frac{4[\sin(\zeta_n) - \zeta_n \cos(\zeta_n)]}{2\zeta_n - \sin(2\zeta_n)}$$

and the discrete values of ζ_n are positive roots of the transcendental equation

$$1 - \zeta_n \cot \zeta_n = Bi$$

where $Bi = hr_o/k$.

Approximate Solutions

- For the infinite cylinder and sphere, the foregoing series solutions can again be approximated by a single term, *n*=1, for *Fo*>0.2.
- Hence, as for the case of the plane wall, the time dependence of the temperature at any location within the radial system is the same as that of the centerline or center point.
- ***Infinite Cylinder:*** The one-term approximation

$$\theta^* = C_1 \exp\left(-\zeta_1^2 Fo\right) J_0(\zeta_1 r^*)$$

or

$$\theta^* = \theta_o^* J_0(\zeta_1 r^*)$$

Approximate Solutions

where θ_o^* represents the centerline temperature and is of the form

$$\theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

- ***Sphere:*** The one-term approximation

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*)$$

or

$$\theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*)$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

Total Energy Transfer

- An energy balance may be performed to determine the total energy transfer from the infinite cylinder or sphere over the time interval $\Delta t = t$.
- Substituting from the approximate solutions and introducing Q_o

- ***Infinite Cylinder***

$$\frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1} J_1(\zeta_1)$$

- ***Sphere***

$$\frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3} [\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)]$$

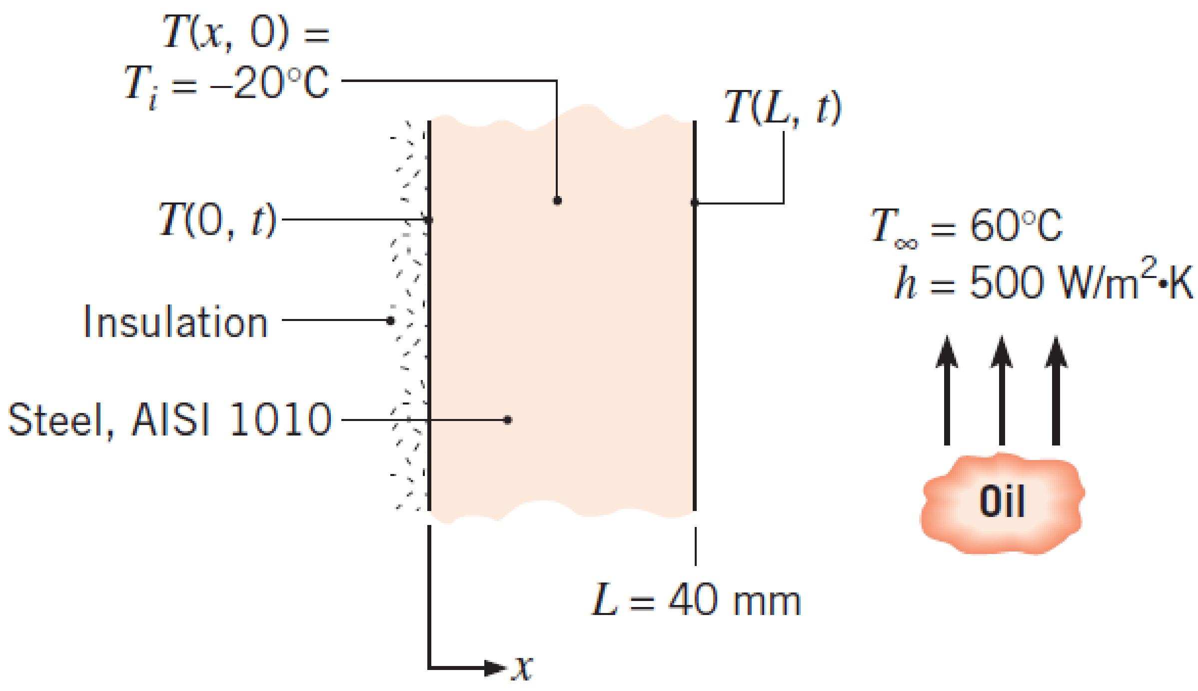
EXAMPLE

Consider a steel pipeline (AISI 1010) that is 1 m in diameter and has a wall thickness of 40 mm. The pipe is heavily insulated on the outside, and, before the initiation of flow, the walls of the pipe are at a uniform temperature of -20^{o}C. With the initiation of flow, hot oil at 60^{o}C is pumped through the pipe, creating a convective condition corresponding to h=500 W/m^{2}K at the inner surface of the pipe.

Assumptions:

1. Pipe wall can be approximated as plane wall, since thickness is much less than diameter.
2. Constant properties.
3. Outer surface of pipe is adiabatic.

EXAMPLE



Properties: Table A.1, steel type AISI 1010 $[\bar{T} = (-20 + 60)°C/2 \approx 300\ K]$: $\rho = 7832\ kg/m^3$, $c = 434\ J/kg \cdot K$, $k = 63.9\ W/m \cdot K$, $\alpha = 18.8 \times 10^{-6}\ m^2/s$.

EXAMPLE

1. What are the appropriate Biot and Fourier numbers 8 min after the initiation of flow?

2. At t 8 min, what is the temperature of the exterior pipe surface covered by the insulation?

3. What is the heat flux q''(W/m^2) to the pipe from the oil at t= 8 min ?

4. How much energy per meter of pipe length has been transferred from the oil to the pipe at t=8 min?

EXAMPLE

1. At t=8 min, the Biot and Fourier numbers can be calculated as

$$Bi = \frac{hL}{k} = \frac{500\ \mathrm{W/m^2 \cdot K} \times 0.04\ \mathrm{m}}{63.9\ \mathrm{W/m \cdot K}} = 0.313$$

$$Fo = \frac{\alpha t}{L^2} = \frac{18.8 \times 10^{-6}\ \mathrm{m^2/s}\ \times 8\ \mathrm{min} \times 60\ \mathrm{s/min}}{(0.04\ \mathrm{m})^2} = 5.64$$

2. With Bi=0.313, use of the lumped capacitance method is inappropriate.

However, since Fo>0.2 and transient conditions in the insulated pipe wall of thickness L correspond to those in a plane wall of thickness 2L experiencing the same surface condition, the desired results may be obtained from the one-term approximation for a plane wall.

EXAMPLE

The midplane temperature can be determined as

$$\theta_o^* = \frac{T_o - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo)$$

where, with $Bi = 0.313$, $C_1 = 1.047$ and $\zeta_1 = 0.531$ rad from Table 5.1. With $Fo = 5.64$,

$$\theta_o^* = 1.047 \exp[-(0.531 \text{ rad})^2 \times 5.64] = 0.214$$

- Hence after 8 min, the temperature of the exterior pipe surface, which corresponds to the midplane temperature of a plane wall, is

$$T(0, 8 \text{ min}) = T_\infty + \theta_o^*(T_i - T_\infty) = 60°\text{C} + 0.214(-20 - 60)°\text{C} = 42.9°\text{C}$$

EXAMPLE

3. Heat transfer to the inner surface at x=L is by convection, and at any time t the heat flux may be obtained from Newton's law of cooling. Hence at t=480 s,

$$q''_x(L, 480\ \text{s}) \equiv q''_L = h[T(L, 480\ \text{s}) - T_\infty]$$

Using the one-term approximation for the surface temperature,

$$\theta^* = \theta_o^* \cos(\zeta_1)$$

$$T(L, t) = T_\infty + (T_i - T_\infty)\theta_o^* \cos(\zeta_1)$$

$$T(L, 8\ \text{min}) = 45.2°\text{C}$$

EXAMPLE

The heat flux at t 8 min is then

$$q_x''(L, 480\ \mathrm{s}) \equiv q_L'' = h[T(L, 480\ \mathrm{s}) - T_\infty]$$

4. The energy transfer to the pipe wall over the 8-min interval may be obtained

$$\frac{Q}{Q_o} = 1 - \frac{\sin(\zeta_1)}{\zeta_1}\theta_o^*$$

$$\frac{Q}{Q_o} = 1 - \frac{\sin(0.531\ \mathrm{rad})}{0.531\ \mathrm{rad}} \times 0.214 = 0.80$$

EXAMPLE

it follows that

$$Q = 0.80\, \rho c V(T_i - T_\infty)$$

or with a volume per unit pipe length of *V' =DL*,

$$Q' = 0.80\, \rho c \pi D L(T_i - T_\infty)$$

$$Q' = 0.80 \times 7832\ \text{kg/m}^3 \times 434\ \text{J/kg} \cdot \text{K}$$

$$\times\, \pi \times 1\ \text{m} \times 0.04\ \text{m}\ (-20 - 60)°\text{C}$$

$$Q' = -2.73 \times 10^7\ \text{J/m}$$

The Semi-Infinite Solid

- An important simple geometry for which analytical solutions may be obtained is the *semi-infinite solid.*
- If a sudden change of conditions is imposed at this surface, transient, one-dimensional conduction will occur within the solid.
- The semi-infinite solid provides a useful idealization for many practical problems.
- It may be used to determine transient heat transfer near the surface of the earth or to approximate the transient response of a finite solid, such as a thick slab.

The Semi-Infinite Solid

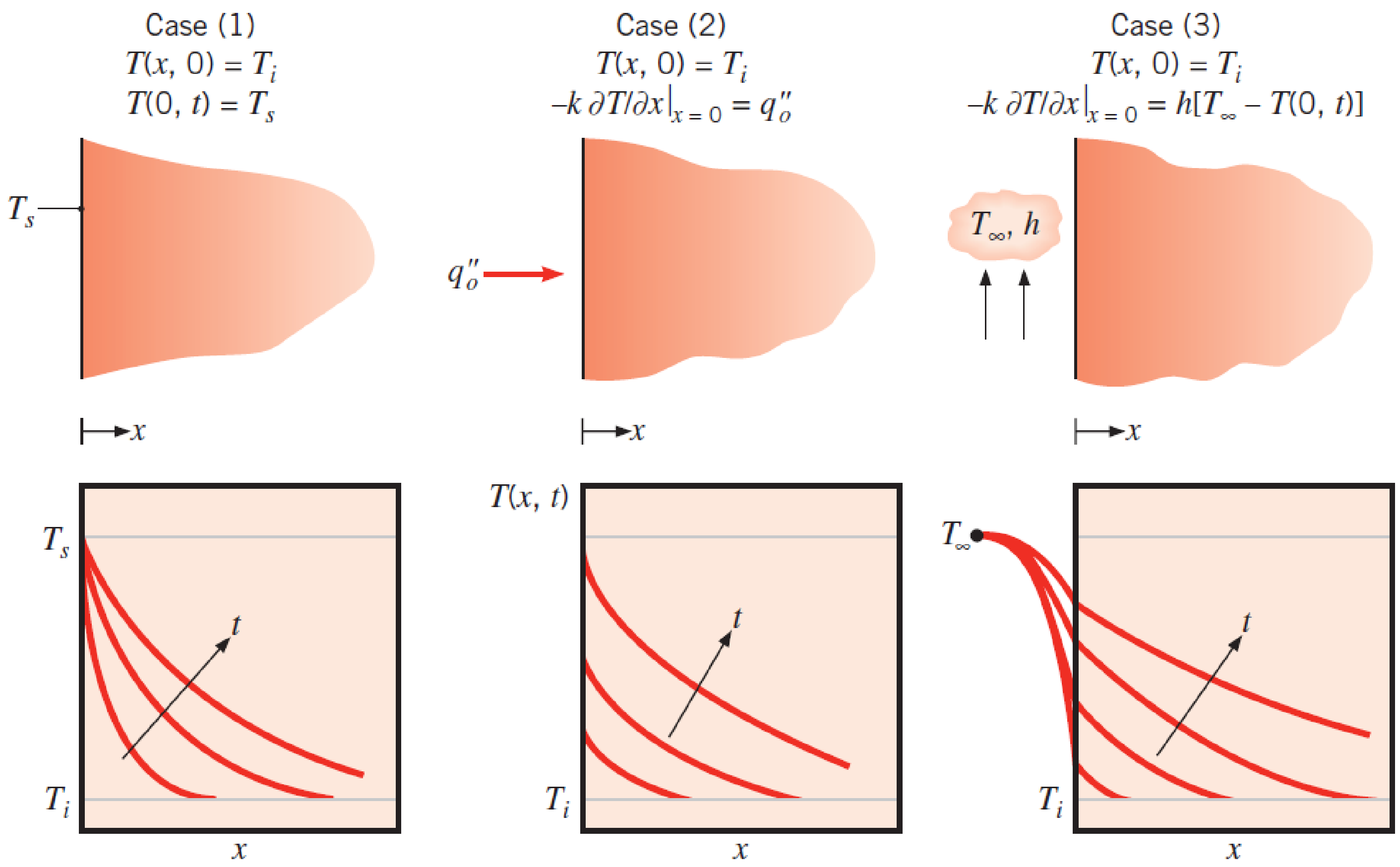


FIGURE 5.7 Transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.

The Semi-Infinite Solid

- The solution may be obtained by defining the similarity parameter η.
- This is a general technique to convert a partial differential equations into an ordinary differential equation.
- Define $\eta \equiv x/(4\alpha t)^{1/2}$
- By using the similarity parameter, we will try to evaluate the partial derivatives in the following governing equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

The Semi-Infinite Solid

- Evaluation of $\dfrac{\partial^2 T}{\partial x^2}$

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{(4\alpha t)^{1/2}}\frac{dT}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left[\frac{\partial T}{\partial x}\right]\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$$

- Evaluation of $\dfrac{\partial T}{\partial t}$

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{x}{2t(4\alpha t)^{1/2}}\frac{dT}{d\eta}$$

The Semi-Infinite Solid

- Substituting above relations into the governing equation

$$\frac{d^2T}{d\eta^2} = -2\eta \frac{dT}{d\eta}$$

- Initial and boundary conditions can be expressed as
- With *x*=0 corresponding to η=0, the surface condition may be expressed as

$$T(\eta = 0) = T_s$$

The Semi-Infinite Solid

- With $x \to \infty$, *as well as* $t = 0$, *corresponding to* $\eta \to \infty$, *both the initial condition and the interior boundary condition correspond to the single requirement that*

$$T(\eta \to \infty) = T_i$$

- Above transformation shows that the governing equation and the initial/boundary conditions can be written as function of the similarity variable.

The Semi-Infinite Solid

- By rearranging terms, the governing equation can be written as

$$\frac{d(dT/d\eta)}{(dT/d\eta)} = -2\eta \, d\eta$$

- Integrating above equation gives

$$\ln(dT/d\eta) = -\eta^2 + C_1'$$

or

$$\frac{dT}{d\eta} = C_1 \exp(-\eta^2)$$

The Semi-Infinite Solid

- Integrating above equation once more

$$T = C_1 \int_0^{\eta} \exp(-u^2)\, du + C_2$$

- where u is a dummy variable
- Applying the boundary condition at η=0 gives C_2=T_s
- Temperature distribution becomes

$$T_i = C_1 \int_0^{\infty} \exp(-u^2)\, du + T_s$$

The Semi-Infinite Solid

- Evaluating the definite integral

$$C_1 = \frac{2(T_i - T_s)}{\pi^{1/2}}$$

- Hence the temperature distribution may be expressed as

$$\frac{T - T_s}{T_i - T_s} = (2/\pi^{1/2}) \int_0^{\eta} \exp(-u^2)\, du \equiv \operatorname{erf} \eta$$

- *where the Gaussian error function, erfη, is a standard mathematical function that is tabulated in Appendix B.*

The Semi-Infinite Solid

- The surface heat flux may be obtained by applying Fourier's law at x=0

$$q_s'' = -k \frac{\partial T}{\partial x}\bigg|_{x=0} = -k(T_i - T_s)\frac{d(\mathrm{erf}\,\eta)}{d\eta}\frac{\partial \eta}{\partial x}\bigg|_{\eta=0}$$

$$q_s'' = k(T_s - T_i)(2/\pi^{1/2})\exp(-\eta^2)(4\alpha t)^{-1/2}\big|_{\eta=0}$$

$$q_s'' = \frac{k(T_s - T_i)}{(\pi\alpha t)^{1/2}}$$

The Semi-Infinite Solid

- Analytical solutions may also be obtained for the case 2 and case 3 surface conditions, and results for all three cases are summarized as follows

Case 1 Constant Surface Temperature: $T(0, t) = T_s$

$$\frac{T(x, t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

The Semi-Infinite Solid

Case 2 Constant Surface Heat Flux: $q''_s = q''_o$

$$T(x,t) - T_i = \frac{2q''_o(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q''_o x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Case 3 Surface Convection: $-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

where $\operatorname{erfc} w \equiv 1 - \operatorname{erf} w$

Discretization of the Heat Equation: Explicit Method

- Analytical solutions to transient problems are restricted to simple geometries and boundary conditions, such as the one dimensional cases
- In two or three dimensional geometries with complex boundary conditions, numerical methods can be used to solve transient problems.
- In this section we consider explicit and implicit methods of finite-difference solutions to transient conduction problems.
- Such methods can easily be simplified for steady-state problems.

Discretization of the Heat Equation: Explicit Method

- Consider the two-dimensional system under transient conditions with constant properties and no internal generation.

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

- To obtain the finite-difference form of this equation, we may use the central-difference approximations to the spatial derivatives and forward-difference approximation to time derivatives.

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} \qquad \text{where} \quad t = p\Delta t$$

Discretization of the Heat Equation: Explicit Method

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$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} \qquad \text{where} \quad t = p\Delta t$$

- The superscript *p* is used to denote the time dependence of *T*, and the time derivative is expressed in terms of the difference in temperatures associated with the *new* (*p*+1) and *previous* (*p*) times.

Discretization of the Heat Equation: Explicit Method

- Similarly,

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T^P_{m+1,n} - 2T^P_{m,n} + T^P_{m-1,n}}{\left(\Delta x\right)^2}$$

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{T^P_{m,n+1} - 2T^P_{m,n} + T^P_{m,n-1}}{\left(\Delta y\right)^2}$$

- The finite-difference form of the governing equation becomes

$$\frac{1}{\alpha}\frac{T^{p+1}_{m,n} - T^p_{m,n}}{\Delta t} = \frac{T^p_{m+1,n} + T^p_{m-1,n} - 2T^p_{m,n}}{(\Delta x)^2} + \frac{T^p_{m,n+1} + T^p_{m,n-1} - 2T^p_{m,n}}{(\Delta y)^2}$$

Discretization of the Heat Equation: Explicit Method

- Solving for the nodal temperature at the new (p+1) time and assuming that $\Delta x = \Delta y$,

$$T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}) + (1 - 4Fo)T_{m,n}^{p}$$

- where Fo is a finite-difference form of the Fourier number

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2}$$

- This approach can easily be extended to one- or three dimensional systems. In one-dimensional system:

$$T_{m}^{p+1} = Fo(T_{m+1}^{p} + T_{m-1}^{p}) + (1 - 2Fo)T_{m}^{p}$$

Discretization of the Heat Equation: Explicit Method

- Above equations are explicit because unknown nodal temperatures for the new time are determined exclusively by known nodal temperatures at the previous time.
- Since the temperature of each interior node is known at $t=0$ ($p=0$) from prescribed initial conditions, the calculations begin at $t=\Delta t$ ($p=1$).
- After the temperatures are known for $t=\Delta t$, the finite-difference equation is applied at each node to determine its temperature at $t=2\Delta t$ ($p=2$).
- In this way, the transient temperature distribution is obtained by marching out in time, using intervals of Δt.

Discretization of the Heat Equation: Explicit Method

- The accuracy of the finite-difference solution may be improved by decreasing the values of Δx and Δt.
- However, the computation time increases with decreasing Δx and Δt.
- After the temperatures are known for $t=\Delta t$, the finite-difference equation is applied at each node to determine its temperature at $t=2\Delta t$ ($p=2$).
- In this way, the transient temperature distribution is obtained by marching out in time, using intervals of Δt.

Discretization of the Heat Equation: Implicit Method

- In the explicit finite-difference scheme, the temperature of any node at $t + \Delta t$ may be calculated from knowledge of temperatures at the same and neighboring nodes for the preceding time t.
- Hence determination of a nodal temperature at some time is independent of temperatures at other nodes for the same time.
- Although the method offers computational convenience, it suffers from limitations on the selection of Δt.
- This dictates the use of extremely small values of Δt, and a very large number of time intervals may be necessary to obtain a solution.

Discretization of the Heat Equation: Implicit Method

- A reduction in the amount of computation time may often be realized by employing an implicit, rather than explicit, finite-difference scheme.
- The implicit form of a finite-difference equation may be derived by approximating the spatial derivatives using the temperatures at the new (p+1) time, instead of the previous (p) time.

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\Delta t}=\frac{T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}-2T_{m,n}^{p+1}}{(\Delta x)^2}$$

$$+\frac{T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}-2T_{m,n}^{p+1}}{(\Delta y)^2}$$

Discretization of the Heat Equation: Implicit Method

- Rearranging the previous equation and assuming $\Delta x = \Delta y$

$$(1 + 4Fo)T_{m,n}^{p+1} - Fo(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p}$$

- From above equation, it is evident that the new temperature of the (m, n) node depends on the new temperatures of its adjoining nodes, which are, in general, unknown.
- Hence, to determine the unknown nodal temperatures at $t+\Delta t$, the corresponding nodal equations must be solved simultaneously.
- Such a solution may be effected by using Gauss–Seidel iteration or matrix inversion.