Stamatics,Discrete Mathematics A-1

Panuganti Neha

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1. Proof that there is no positive integer n such that $n^2 + n^3 = 100$

Assume for the sake of contradiction that there exists a positive integer n such that $n^2 + n^3 = 100$.

$$n^3 + n^2 = 100$$

Rewriting, we get:

$$n^3 = 100 - n^2$$

Since $n > 0$, it follows that $n^3 > 0$. Therefore:

$$100 - n^2 > 0 \implies 100 > n^2 \implies n^2 < 100 \implies n < 10$$

This implies that n must be less than 10. Now, rewriting the equation as:

$$n^2 = 100 - n^3$$

Since $n > 0$, it follows that $n^2 > 0$. Therefore:

$$100 - n^3 > 0 \implies 100 > n^3 \implies n^3 < 100 \implies n < \sqrt[3]{100} \approx 4.64$$

This implies that n must be less than approximately 4.64. Since n is a positive integer, the possible values for n are 1, 2, 3, or 4.

Let's test these values:

- For $n = 1$:

$$1^2 + 1^3 = 1 + 1 = 2 \neq 100$$

- For $n = 2$:

$$2^2 + 2^3 = 4 + 8 = 12 \neq 100$$

- For $n = 3$:

$$3^2 + 3^3 = 9 + 27 = 36 \neq 100$$

- For $n = 4$:

$$4^2 + 4^3 = 16 + 64 = 80 \neq 100$$

Thus, there is no positive integer n such that $n^2 + n^3 = 100$.

Additionally, if we assume $n > 10$, then:

$$n^3 > 1000 \implies n^2 + n^3 > 1000 \implies 100 > 1000 \quad \text{(contradiction)}$$

Hence, our assumption is false, and there is no positive integer n such that $n^2 + n^3 = 100$.

2. Prove that $n^2 + 1 \geq 2n$ when n is a positive integer with $1 \leq n \leq 4$.

For $n = 1$:

$$1^2 + 1 = 2 \quad \text{and} \quad 2 \times 1 = 2 \quad \text{(true)}$$

For $n = 2$:

$$2^2 + 1 = 5 \quad \text{and} \quad 2 \times 2 = 4 \quad \text{(true)}$$

For $n = 3$:

$$3^2 + 1 = 10 \quad \text{and} \quad 2 \times 3 = 6 \quad \text{(true)}$$

For $n = 4$:

$$4^2 + 1 = 17 \quad \text{and} \quad 2 \times 4 = 8 \quad \text{(true)}$$

Thus, $n^2 + 1 \geq 2n$ holds for $1 \leq n \leq 4$.

3. Find a compound proposition involving the propositional variables p, q, r, and s that is true when exactly three of these propositional variables are true and is false otherwise.

The compound proposition can be expressed as:

$$(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$$

4. Show that $\exists x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \exists x Q(x)$ always have the same truth value

To prove that $\exists x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \exists x Q(x)$ always have the same truth value, we will construct truth tables for each expression and compare them.

Truth Table for $\exists x(P(x) \rightarrow Q(x))$

$P(x)$	$Q(x)$	$P(x) \rightarrow Q(x)$	$\exists x(P(x) \rightarrow Q(x))$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth Table for $\forall x P(x) \to \exists x Q(x)$

$P(x)$	$Q(x)$	$\forall x P(x)$	$\exists x Q(x)$	$\forall x P(x) \to \exists x Q(x)$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	F	T

Analysis

From the truth tables:

- For $\exists x(P(x) \to Q(x))$: - The expression is true when at least one $P(x) \to Q(x)$ is true.

- For $\forall x P(x) \to \exists x Q(x)$: - The expression is true when $\forall x P(x)$ is false or $\exists x Q(x)$ is true.

Examining the truth tables, we observe the following: - When $P(x)$ is true and $Q(x)$ is true, both expressions are true. - When $P(x)$ is true and $Q(x)$ is false, both expressions are false. - When $P(x)$ is false and $Q(x)$ is true, both expressions are true. - When $P(x)$ is false and $Q(x)$ is false, both expressions are true.

Therefore, the truth values of $\exists x(P(x) \to Q(x))$ and $\forall x P(x) \to \exists x Q(x)$ match in all cases. Hence, they always have the same truth value.

5. Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that $A \subseteq B$?

Yes, it follows that $A \subseteq B$.

The **power set** of a set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A. Formally,

$$\mathcal{P}(A) = \{S \mid S \subseteq A\}$$

Assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We need to show that $A \subseteq B$. To prove this, consider an element $x \in A$. The set $\{x\}$ is a subset of A, so $\{x\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $\{x\} \in \mathcal{P}(B)$. Thus, $x \in B$. Therefore, $A \subseteq B$.

6. Let A and B be sets. Show that $A \subseteq B$ if and only if $A \cap B = A$.

First, assume $A \subseteq B$. Then, every element of A is also an element of B. Thus, $A \cap B = A$. Conversely, assume $A \cap B = A$. Then, every element $x \in A$ must also be in B because $x \in A$ implies $x \in A \cap B$. Therefore, $A \subseteq B$.Thus, $A \subseteq B$ if and only if $A \cap B = A$.