AI5030 / EE5817: Probability and Stochastic Processes

Homework 02 Solutions

Algebras, σ-Algebras

Solutions Prepared By: K.A. Midhunkumar & Deekshith Rathod K.

1. (a) Let $\Omega = \{1, \dots, 6\}$. For each $i \in \{1, 2, 3, 4\}$, construct a σ-algebra $\mathscr{F}_i$ of subsets of Ω such that $|\mathscr{F}_i| = 2^i$.
   (b) Let Ω be a finite sample space with $|\Omega| = n$ for some $n \in \mathbb{N}$. Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Show that $|\mathscr{F}| = 2^k$ for some $1 \le k \le n$.

Solution:

(a) Following are some example σ-algebras with desired sizes:

$$\begin{aligned}
\mathscr{F}_1 &= \{\emptyset, \Omega\}, \\
\mathscr{F}_2 &= \{\emptyset, \Omega, \{1,2,3\}, \{4,5,6\}\}, \\
\mathscr{F}_3 &= \{\emptyset, \Omega, \{1,2\}, \{3,4\}, \{5,6\}, \{1,2,3,4\}, \{1,2,5,6\}, \{3,4,5,6\}\}, \\
\mathscr{F}_4 &= \{\emptyset, \Omega, \{1,2\}, \{3,4\}, \{5\}, \{6\}, \{1,2,3,4\}, \{1,2,5\}, \{1,2,6\}, \\
&\qquad \{3,4,5\}, \{3,4,6\}, \{5,6\}, \{1,2,3,4,5\}, \{1,2,3,4,6\}, \{3,4,5,6\}, \{1,2,5,6\}\}.
\end{aligned}$$

(b) For each $x \in \Omega$, define

$$A_x = \bigcap_{A \in \mathscr{F} : x \in A} A \quad \text{where, } A \subseteq \Omega$$

By construction $\forall x \in A_x$, and $A_x \in \mathscr{F}$ since $\mathscr{F}$ is closed under countable intersections.

<u>Claim:</u> A_x is an atom, i.e., if $B \subseteq A_x$ and $B \in \mathscr{F}$, then either $B = A_x$ or $B = \emptyset$.

Proof. Given,

$$\begin{aligned}
B \subseteq A_x &\Rightarrow x \in B \quad \text{by definition} \\
&\Rightarrow A_x \subseteq B
\end{aligned}$$

This gives $B = A_x$.

If instead $B \nsubseteq A_x$ and $x \notin B$, then B cannot be a non-empty proper subset of A_x in $\mathscr{F}$.

$$\begin{aligned}
B \nsubseteq A_x,\ x \notin B \quad &\Rightarrow A_x \setminus B \in \mathscr{F} && \text{(closure under complements)} \\
&\Rightarrow A_x \setminus B \subset A_x, \quad x \in A_x \setminus B && \text{(since } x \notin B \text{ but } x \in A_x) \\
&\text{contradicts minimality of } A_x \\
&\Rightarrow B = \emptyset.
\end{aligned}$$

Thus $B = \emptyset$.

□

Hence A_x is an atom.
If A_x and A_y are two atoms

$$A_x \cap A_y \neq \emptyset \Rightarrow A_x \cap A_y = A_y$$

So $A_x = A_y$. Thus, distinct atoms are disjoint.

Therefore, the collection of atoms forms a partition of Ω. Let the atoms be $B_1, B_2, \dots, B_k$.

Now take any $A \in \mathscr{F}$. Then

$$A = A \cap \Omega = A \cap \left(\bigsqcup_{i=1}^{k} B_i \right) = \bigsqcup_{i=1}^{k} (A \cap B_i) = \bigsqcup_{i : A \cap B_i \neq \emptyset} B_i.$$

Thus A equals the union of those atoms B_i that intersect A.
Since each $B_i \in \mathscr{F}$ and finite unions of atoms belong to $\mathscr{F}$, every $A \in \mathscr{F}$ can be expressed as a union of atoms. Hence the number of distinct sets in $\mathscr{F}$ is bounded by

$$|\mathscr{F}| \leq 2^k. \tag{1}$$

But also, for every subset $I \subseteq [k]$, the union $\bigcup_{i \in I} B_i \in \mathscr{F}$, and there are exactly 2^k such unions. Hence

$$|\mathscr{F}| \geq 2^k. \tag{2}$$

Combining (1) and (2), we conclude

$$|\mathscr{F}| = 2^k.$$

2. Let Ω be an arbitrary set (finite, countably infinite, or uncountable).
   (a) Let $\mathscr{A}$ be a collection of subsets of Ω satisfying the following properties:
      - $\Omega \in \mathscr{A}$.
      - If $A, B \in \mathscr{A}$, then $A \cap B^{\complement} \in \mathscr{A}$.

   Show that $\mathscr{A}$ must be an algebra (of subsets of Ω).
   (b) Suppose $\mathscr{F}$ is a collection of subsets of Ω satisfying the following properties:
      - $\Omega \in \mathscr{F}$.
      - If $A \in \mathscr{F}$, then $A^{\complement} \in \mathscr{F}$ (closure under complements).
      - If A, B are two **disjoint** subsets of Ω, then $A \cup B \in \mathscr{F}$ (closure under finite **disjoint** unions).

   Construct an explicit example to demonstrate that $\mathscr{F}$ need not be an algebra.

Solution.

(a) By definition, we have $\Omega \in \mathscr{A}$. Setting $A = B = \Omega$ and using the property that $A \cap B^{\complement} \in \mathscr{A}$, we get that $\Omega \cap \Omega^{\complement} = \emptyset \in \mathscr{A}$. Next, we show that $\mathscr{A}$ is closed under complements. Suppose that $B \in \mathscr{A}$. Then, choosing $A = \Omega$, and noting that $A \cap B^{\complement} \in \mathscr{A}$, we get that $\Omega \cap B^{\complement} = B^{\complement} \in \mathscr{A}$. This proves that $\mathscr{A}$ is closed under complements. Finally, to show that $\mathscr{A}$ is closed under finite unions, suppose that $B_1, B_2 \in \mathscr{A}$. Taking $A = B_1^{\complement}$ and $B = B_2$, and using the fact that $A \cap B^{\complement} \in \mathscr{A}$, we get that $B_1^{\complement} \cap B_2^{\complement} \in \mathscr{A}$. Because $\mathscr{A}$ is closed under complements, it follows that $(B_1^{\complement} \cap B_2^{\complement})^{\complement} = B_1 \cup B_2 \in \mathscr{A}$. This establishes closure under finite unions, and thereby the fact that $\mathscr{A}$ is an algebra of subsets of Ω.

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Consider the collection

$$\mathscr{F} = \{\emptyset, \Omega, \{1, 2\}, \{2, 3\}, \{3, 4, 5, 6\}, \{1, 4, 5, 6\}\}.$$

Clearly $\mathscr{F}$ is a collection of subsets of Ω that satisfy the given properties. However,

$$\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\} \notin \mathscr{F}.$$

Thus, it follows that $\mathscr{F}$ is not closed under finite unions, and hence not an algebra.

3. Let Ω be an arbitrary set (finite, countably infinite, or uncountable).
   (a) Let $\mathscr{F}_1$ denote the collection of all finite subsets of Ω, i.e.,

   $$\mathscr{F}_1 := \Big\{A \subseteq \Omega : \ |A| \in \mathbb{N}\Big\}.$$

   Is $\mathscr{F}_1$ an algebra?
   (b) Let $\mathscr{F}_2$ denote the collection of all finite subsets of Ω, plus all subsets of Ω whose complement is finite, i.e.,

   $$\mathscr{F}_2 := \Big\{A \subseteq \Omega : \ A \text{ is finite or } (\Omega \setminus A) \text{ is finite or both}\Big\}.$$

   Show that $\mathscr{F}_2$ is an algebra.
   Construct an example to demonstrate that $\mathscr{F}_2$ need not necessarily be a σ-algebra.

(c) Let $\mathscr{F}_3$ denote the collection of all countable subsets of Ω, plus all subsets of Ω whose complement is countable, i.e.,

$$\mathscr{F}_3 := \Big\{ A \subseteq \Omega : \ A \text{ is countable or } (\Omega \setminus A) \text{ is countable or both} \Big\}.$$

Show that $\mathscr{F}_3$ is a σ-algebra.

Note: Countable means finite or countably infinite.

Solution.

(a) If Ω is either countably infinite or uncountable, then $\Omega \notin \mathscr{F}_1$, in which case $\mathscr{F}_1$ is not an algebra. However, if Ω is finite, then $\mathscr{F}_1 = 2^{\Omega}$ and hence trivially an algebra.

(b) (i) $\Omega \in \mathscr{F}_2$, since $|\Omega^{\complement}| = |\emptyset| = 0$.

(ii) Closure under complements. Let $A \in \mathscr{F}_2$. Then either A is finite or $A^{\complement}$ is finite. Hence $A^{\complement} \in \mathscr{F}_2$.

(iii) Closure under finite unions. Let $A, B \in \mathscr{F}_2$. We check all possibilities.

1. A, B **finite.** Then $A \cup B$ is finite, hence in $\mathscr{F}_2$.
2. A **finite,** B **co-finite (i.e.,** $B^{\complement}$ **finite).** In this case,

$$(A \cup B)^{\complement} \ = \ A^{\complement} \cap B^{\complement}.$$

Here $A^{\complement}$ is co-finite and $B^{\complement}$ is finite, so $A^{\complement} \cap B^{\complement}$ is finite. Hence $(A \cup B)^{\complement}$ is finite, i.e. $A \cup B$ is co-finite, thus $A \cup B \in \mathscr{F}_2$.

3. A **co-finite,** B **finite.** Symmetric to case 2.
4. A, B **co-finite.** Then $A^{\complement}, B^{\complement}$ are finite, and

$$(A \cup B)^{\complement} \ = \ A^{\complement} \cap B^{\complement}$$

is an intersection of two finite sets, hence finite. Thus $A \cup B$ is co-finite, so $A \cup B \in \mathscr{F}_2$.

Therefore, $\mathscr{F}_2$ is closed under finite unions. Combining (i), (ii) & (iii), it follows that $\mathscr{F}_2$ is an *algebra* on Ω.

Counterexample to show $\mathscr{F}_2$ is not a σ-algebra:
Let $\Omega = \mathbb{N}$. All singletons belong to $\mathscr{F}_2$, since they are finite. Consider the set

$$A = \bigcup_{k \in \mathbb{N}} \{2k - 1\}.$$

Clearly, both A (the set of odd numbers) and $A^{\complement}$ (the set of even numbers) are countably infinite, thus implying that $A \notin \mathscr{F}_2$. This demonstrates that $\mathscr{F}_2$ is not closed under countable unions, and hence not a σ-algebra.

(c) We show that $\mathscr{F}_3$ is a σ-algebra on Ω by verifying the three defining properties:

i. $\Omega \in \mathscr{F}_3$:
$\emptyset$ is countable, hence $\emptyset \in \mathscr{F}_3$. Also $\Omega^{\complement} = \emptyset$ is countable, so Ω is co-countable; thus $\Omega \in \mathscr{F}_3$.

ii. **Closed under complements:** If $A \in \mathscr{F}_3$, then $A^{\complement} \in \mathscr{F}_3$.
There are two cases by definition of $\mathscr{F}_3$.

- If A is countable, then $A^{\complement}$ is co-countable; hence $A^{\complement} \in \mathscr{F}_3$.
- If $A^{\complement}$ is countable, then $(A^{\complement})^{\complement} = A$ is co-countable; in particular $A^{\complement} \in \mathscr{F}_3$ directly by assumption.

Thus $\mathscr{F}_3$ is closed under complements.

iii. **Closed under countable unions:** If $A_1, A_2, \dots \in \mathscr{F}_3$, then $\bigcup\limits_{n \in \mathbb{N}} A_n \in \mathscr{F}_3$.

Partition the indices. Let

$$I := \{\, n \in \mathbb{N} : A_n \text{ is countable} \,\} \qquad \text{and} \qquad J := \Big\{\, n \in \mathbb{N} : A_n^{\complement} \text{ is countable} \,\Big\}.$$

Since each $A_n \in \mathscr{F}_3$, $\mathbb{N} = I \cup J$ (disjoint union).

Step 1 (the I-part is countable). We know, the union of countably many countable sets is countable:

$$U_I := \bigcup_{n\in I} A_n \quad \text{is countable.}$$

Step 2 (the J-part is co-countable). Consider

$$U_J := \bigcup_{n\in J} A_n.$$

Its complement is

$$U_J^{\complement} \;=\; \bigcap_{n\in J} A_n^{\complement}.$$

Each $A_n^{\complement}$ is countable for $n \in J$. Since an intersection is a subset of each operand,

$$\bigcap_{n\in J} A_n^{\complement} \;\subseteq\; A_{n_0}^{\complement} \quad \text{for any fixed } n_0 \in J,$$

hence $U_J^{\complement}$ is a subset of a countable set and therefore countable by. Thus U_J is co-countable, i.e., $U_J \in \mathscr{F}_3$. (If $J = \emptyset$, then $U_J = \emptyset$ is countable, so again $U_J \in \mathscr{F}_3$.)
Step 3 (combine the two parts). We have

$$\bigcup_{n\in\mathbb{N}} A_n \;=\; U_I \cup U_J.$$

If $J = \emptyset$, then the union is U_I, which is countable and hence in $\mathscr{F}_3$. If $J \neq \emptyset$, then U_J is co-countable and U_I is countable. In that case

$$(U_I \cup U_J)^{\complement} \;=\; U_I^{\complement} \cap U_J^{\complement} \;\subseteq\; U_J^{\complement},$$

and $U_J^{\complement}$ is countable, so $(U_I \cup U_J)^{\complement}$ is countable; hence $U_I \cup U_J$ is co-countable. Therefore $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{F}_3$.

Properties (i), (ii) & (iii) prove that $\mathscr{F}_3$ is a σ-algebra on Ω.

4. Let $\Omega = \mathbb{R}$. Let $\mathscr{P}$ denote the collection

$$\mathscr{P} := \Big\{ [a, b) : \;\; a, b \in \mathbb{R}, \;\; a < b \Big\}.$$

Clearly, $\mathscr{P}$ consists of uncountably infinitely many subsets of Ω.
In Lecture 6, we saw that $\sigma(\mathscr{P}) = \mathscr{B}(\mathbb{R})$, i.e., $\mathscr{P}$ is a generating class for $\mathscr{B}(\mathbb{R})$.
In this exercise, we will see an alternative construction of $\mathscr{B}(\mathbb{R})$ starting from a **countably infinite** collection of subsets of Ω.

Consider the collection $\mathscr{C}$ given by

$$\mathscr{C} := \Big\{ [a, b) : a \le b, \;\; a, b \text{ are dyadic rational numbers} \Big\}.$$

Note: A dyadic rational number is of the form $m/2^n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$.

(a) Given $x \in \mathbb{R}$, express $\{x\}$ in terms of sets in $\mathscr{C}$ using countable set operations.
Hint: Note that $\lfloor 2^n x \rfloor \le 2^n x \le \lceil 2^n x \rceil$ for all $n \in \mathbb{N}$. Therefore,

$$\frac{\lfloor 2^n x \rfloor}{2^n} \le x \le \frac{\lceil 2^n x \rceil}{2^n} \qquad \forall n \in \mathbb{N}.$$

(b) Given $a, b \in \mathbb{R}$ with $a < b$, express $[a, b)$ in terms of sets in $\mathscr{C}$ using countable set operations.

(c) Using the result in part (b), what can you say about the relationship between $\mathscr{P}$ and $\sigma(\mathscr{C})$?

(d) What can you say about the relationship between $\mathscr{C}$ and $\sigma(\mathscr{P})$?

(e) Using the results of parts (c), (d), what can you say about the relationship between $\sigma(\mathscr{C})$ and $\sigma(\mathscr{P})$?

Solution.

(a) Expressing $\{x\}$ in terms of sets in $\mathscr{C}$:
Case 1: $x \in \mathbb{R}$ is dyadic. Define

$$A_n = \left[x,\, x + \tfrac{1}{2^n}\right), \quad n \in \mathbb{N}.$$

Clearly, $A_n \in \mathscr{C}$ for all $n \in \mathbb{N}$.
We claim:

$$\bigcap_{n \in \mathbb{N}} A_n = \{x\}.$$

Proof: By definition, $x \in A_n$ for every $n \in \mathbb{N}$, so $x \in \cap_n A_n$. Suppose $y \neq x$ with $y \in \cap_n A_n$. Let $|y - x| = \varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\varepsilon < \frac{1}{2^N}$. But then $A_N = [x,\, x + \frac{1}{2^N})$ does not contain y, a contradiction. Hence, $\cap_n A_n = \{x\}$.
Case 2: $x \in \mathbb{R}$ is non-dyadic. We know

$$\frac{\lfloor 2^n x \rfloor}{2^n} < x < \frac{\lceil 2^n x \rceil}{2^n}, \quad n \in \mathbb{N}.$$

Define

$$A_n = \left[\frac{\lfloor 2^n x \rfloor}{2^n}, \frac{\lceil 2^n x \rceil}{2^n}\right), \quad n \in \mathbb{N}.$$

Then $x \in A_n$ for every n, and since $\lceil 2^n x \rceil - \lfloor 2^n x \rfloor = 1$, the interval A_n shrinks around x as $n \to \infty$. Hence,

$$\bigcap_{n \in \mathbb{N}} A_n = \{x\}.$$

Thus, in both cases, $\{x\}$ can be expressed as a countable intersection of sets in $\mathscr{C}$.

(b) Expressing $[a, b)$ in terms of sets in $\mathscr{C}$:
If a, b are dyadic, then $[a, b) \in \mathscr{C}$.
If not, let

$$A_n = \left[\tfrac{\lceil 2^n a \rceil}{2^n},\, \tfrac{\lfloor 2^n b \rfloor}{2^n}\right)$$

Then

$$[a, b) = \bigcup_{n \in \mathbb{N}} A_n \,\cup\, \{a\}.$$

Hence, $[a, b)$ is expressed as a countable union of sets in $\mathscr{C}$.

(c) Relation between $\mathscr{P}$ and $\sigma(\mathscr{C})$:
The class $\mathscr{P}$ contains all sets of the form $[a, b)$ with $a \leq b$, $a, b \in \mathbb{R}$. From part (b), any such set can be expressed using countable unions of sets in $\mathscr{C}$. Hence,

$$\mathscr{P} \subseteq \sigma(\mathscr{C}).$$

(d) Relation between $\mathscr{C}$ and $\sigma(\mathscr{P})$:
Clearly, $\mathscr{C} \subseteq \mathscr{P}$. Hence,

$$\sigma(\mathscr{C}) \subseteq \sigma(\mathscr{P}).$$

(e) Final conclusion:
Combining (c) and (d), we obtain

$$\sigma(\mathscr{C}) = \sigma(\mathscr{P}).$$

But we know $\sigma(\mathscr{P}) = \mathscr{B}(\mathbb{R})$. Therefore,

$$\sigma(\mathscr{C}) = \mathscr{B}(\mathbb{R}),$$

i.e.$\mathscr{C}$ is a countable generating class for $\mathscr{B}(\mathbb{R})$.

5. Let Ω be an arbitrary set (finite, countably infinite, or uncountable).

   (a) Let $\mathscr{C}$ denote the collection of all singleton subsets of Ω. What is $\sigma(\mathscr{C})$?
   Hint: See Question 3c.

(b) Fix two elementary outcomes $a, b \in \Omega$.
Let $\mathscr{E}_{a,b}$ denote the collection of all those subsets of Ω which either contain both a and b or do not contain both. Let $\mathscr{F} = \sigma(\mathscr{E}_{a,b})$. Show that every set in $\mathscr{F}$ has the same property as the sets in $\mathscr{E}_{a,b}$.

Solution

(a) **Case 1: Ω is countable.** For any $A \subseteq \Omega$ we can write $A = \bigcup_{x\in A}\{x\}$ as a *countable* union of singletons. Hence $A \in \sigma(\mathscr{C})$. Therefore

$$\sigma(\mathscr{C}) = \mathcal{P}(\Omega).$$

Case 2: Ω is uncountable. Define the countable–co–countable family

$$\mathscr{A} := \Big\{ A \subseteq \Omega : \ A \text{ is countable or } A^{\complement} \text{ is countable} \Big\}.$$

Claim 1: $\mathscr{A}$ is a σ–algebra.

Indeed, if $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then either some A_n is co-countable, in which case $\bigcup_n A_n$ is co-countable, or else every A_n is countable, and $\bigcup_n A_n$ is a countable union of countable sets, hence countable. Thus $\bigcup_n A_n \in \mathscr{A}$. Therefore $\mathscr{A}$ is a σ–algebra.
Claim 2: $\mathscr{C} \subseteq \mathscr{A}$. Every singleton is countable; hence $\mathscr{C} \subseteq \mathscr{A}$ and by minimality of $\sigma(\mathscr{C})$ we obtain

$$\sigma(\mathscr{C}) \subseteq \mathscr{A}.$$

Claim 3: $\mathscr{A} \subseteq \sigma(\mathscr{C})$. If $A \in \mathscr{A}$ is countable, then $A = \bigcup_{x\in A}\{x\}$ is a countable union of elements of $\mathscr{C}$, hence $A \in \sigma(\mathscr{C})$. If $A \in \mathscr{A}$ is co-countable, then $A^{\complement}$ is countable, and can be expressed as a countable union of singletons, hence implying that $A^{\complement} \in \sigma(\mathscr{C})$. But because $\sigma(\mathscr{C})$ is a σ-algebra and hence closed under complements, we must have $(A^{\complement})^{\complement} = A \in \sigma(\mathscr{C})$.
From Claims 2–3 we conclude $\sigma(\mathscr{C}) = \mathscr{A}$.

(b) We first show that $\mathscr{E}_{a,b}$ is a σ-algebra on Ω. Then $\mathscr{F} = \sigma(\mathscr{E}_{a,b}) = \mathscr{E}_{a,b}$, which immediately implies that every set in $\mathscr{F}$ has the same property as sets in $\mathscr{E}_{a,b}$.

$$\mathscr{E}_{a,b} = \{A \subseteq \Omega : A \cap \{a,b\} \in \{\phi, \{a,b\}\}$$

We show $\mathscr{E}_{a,b}$ is a σ-algebra

(1) $\Omega \in \mathscr{E}_{a,b}$*:* To see this, observe that

$$\Omega \subseteq \Omega, \quad \Omega \cap \{a,b\} = \{a,b\} \in \{\phi, \{a,b\}\}.$$

(2) *Closure under complements.* Let $A \in \mathscr{E}_{a,b}$. There are two cases:

(i) $a, b \in A \Rightarrow a, b \notin A^{\complement}$ or (ii) $a, b \notin A \Rightarrow a, b \in A^{\complement}$.

In case (i), $a, b \notin A^{\complement}$; in case (ii), $a, b \in A^{\complement}$. Thus in either case $A^{\complement} \in \mathscr{E}_{a,b}$. Since,

$$A^{\complement} \subseteq \Omega, \quad A^{\complement} \cap \{a,b\} \in \{\phi, \{a,b\}\} \implies A^{\complement} \in \mathscr{E}_{a,b}$$

(3) *Closure under countable unions.* Let $A_1, A_2, \ldots \in \mathscr{E}_{a,b}$. Then,

$$a \in \bigcup_{i\in\mathbb{N}} A_i \iff \exists j \in \mathbb{N} : \ a \in A_j \iff b \in A_j \iff b \in \bigcup_{i\in\mathbb{N}} A_i.$$

Hence

$$\bigcup_{i\in\mathbb{N}} A_i \in \mathscr{E}_{a,b}.$$

From (1)–(3), $\mathscr{E}_{a,b}$ is a σ-algebra, so $\sigma(\mathscr{E}_{a,b}) = \mathscr{E}_{a,b}$. Consequently, every set in $\mathscr{F}$ either contains both points or contains neither.

6. Consider the collection
$$\mathscr{D} := \Big\{(a,b] \cup [-b,-a) : \ a,b \in \mathbb{R}, \ a \le b\Big\}.$$
Show that $\sigma(\mathscr{D}) \subsetneq \mathscr{B}(\mathbb{R})$ by constructing a non-empty set $B \in \mathscr{B}(\mathbb{R}) \setminus \sigma(\mathscr{D})$.

Solution

We want to study the collection
$$\mathscr{D} := \{(a,b] \cup [-b,-a) : a,b \in \mathbb{R}, \ a \le b\},$$
and show that the σ-algebra it generates satisfies
$$\sigma(\mathscr{D}) \subsetneq \mathscr{B}(\mathbb{R}),$$
where $\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. The idea is to show that every set in $\sigma(\mathscr{D})$ is symmetric about 0, whereas $\mathscr{B}(\mathbb{R})$ also contains asymmetric sets.

Reflection operator and symmetric sets. For $A \subseteq \mathbb{R}$, define the reflection
$$-A := \{-x : x \in A\}.$$
We say that A is *symmetric about* 0 if $A = -A$. Let
$$\mathcal{S} := \{A \subseteq \mathbb{R} : -A = A\}$$
denote the collection of all symmetric sets.

Claim 1: $\mathscr{D} \subseteq \mathcal{S}$. Take $E = (a,b] \cup [-b,-a) \in \mathscr{D}$. Reflection swaps the two halves:
$$-(a,b] \to [-b,-a) \quad \text{and} \quad -[-b,-a) \to (a,b],$$
so indeed $-E = E$. Hence E is symmetric, and therefore $\mathscr{D} \subseteq \mathcal{S}$.

Claim 2: $\mathcal{S}$ is a σ-algebra. We verify the defining properties:

(i) $\Omega = \mathbb{R}$. Clearly $-\mathbb{R} = \mathbb{R}$, so $\mathbb{R}$ is symmetric, $\therefore \mathbb{R} \in \mathcal{S}$.

(ii) Closure under complements. Let $A \in \mathcal{S}$, so $A = -A$. For any $x \in \mathbb{R}$,
$$x \in -(A^{\mathsf{C}}) \iff -x \in A^{\mathsf{C}} \iff -x \notin A \iff x \notin -A \iff x \notin A \iff x \in A^{\mathsf{C}}.$$
Thus $-(A^{\mathsf{C}}) = A^{\mathsf{C}}$, so A^{C} is symmetric. Therefore $A^{\mathsf{C}} \in \mathcal{S}$.

(iii) Closure under countable unions. Let $A_1, A_2, \dots \in \mathcal{S}$. For any $x \in \mathbb{R}$,
$$x \in -\Big(\bigcup_{n=1}^{\infty} A_n\Big) \iff -x \in \bigcup_{n=1}^{\infty} A_n \iff \exists n \text{ such that } -x \in A_n.$$
Since each A_n is symmetric, this is equivalent to $\exists n$ with $x \in A_n$. Hence
$$-\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} A_n.$$
So the union is symmetric.

Together, (i),(ii) &(iii) show that $\mathcal{S}$ is a σ-algebra.

Since $\mathcal{S}$ is a σ-algebra containing $\mathscr{D}$, by the minimality of $\sigma(\mathscr{D})$ we have
$$\sigma(\mathscr{D}) \subseteq \mathcal{S}.$$
Therefore every set in $\sigma(\mathscr{D})$ is symmetric about 0.

Consider the Borel set $B := (0,\infty)$. It is not symmetric, since
$$-B = (-\infty, 0) \neq (0,\infty).$$
Thus $B \notin \mathcal{S}$, and so $B \notin \sigma(\mathscr{D})$. On the other hand $B \in \mathscr{B}(\mathbb{R})$.