AI 5090: STOCHASTIC PROCESSES

LECTURE 03

SCRIBE: ARUNKUMAR KUDARIHAL, MANNAVA VENKATASAI

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भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Contents

In the previous lecture, we learned about sample spaces, σ-algebras, the Borel σ-algebra of subsets of $\mathbb{R}$ (denoted $\mathscr{B}(\mathbb{R})$), probability measure, and random variables. In this lecture, we will explore further details about $\mathscr{B}(\mathbb{R})$, how random variables are derived from them, and examples of random variables. In addition, we will discuss the concept of probability law of a random variable, and how the cumulative distribution function (CDF) is linked to the probability law.

Topics Covered: Random variables, some fundamental information about $\mathscr{B}(\mathbb{R})$, the probability law of a random variable, the cumulative distribution function (CDF), properties of CDF, random vectors, and sequences of random variables.

1 Borel σ-Algebra

Before introducing the Borel σ-algebra, let us first discuss probability measures on uncountable sample spaces. This discussion will highlight the importance of the Borel σ-algebra.

1.1 Uncountable probability spaces

Consider the experiment of choosing a real number uniformly at random from the interval $\Omega = [0, 1]$, where every number is equally likely to be chosen. To construct a probability measure on Ω, if we take $\mathscr{F}$ = 2^{Ω}, the collection of all subsets of Ω, it becomes impossible to assign probabilities to all subsets of Ω due to the following reasons:

- Suppose that each elementary outcome $\omega \in \Omega$ is assigned a positive probability $p > 0$, i.e,

  $$\mathbb{P}(\{\omega\}) = p > 0 \text{ for all } \omega \in \Omega.$$

  Noting that the interval $[0, 1]$ contains countably infinite rational numbers, and that the probability of the set of all rational numbers in $[0, 1]$ can be no greater than the probability of the set of all real numbers in $[0, 1]$, we get

  $$\begin{aligned} \mathbb{P}([0,1]) &\geq \mathbb{P}(\mathbb{Q} \cap [0,1]) \\ &= \sum_{\omega \in \mathbb{Q}\cap[0,1]} \mathbb{P}(\{\omega\}) \\ &= \sum_{\omega \in \mathbb{Q}\cap[0,1]} p \\ &= +\infty, \end{aligned}$$

  leading to a contradiction. This demonstrates that if we try to assign a positive probability to each real number in $[0, 1]$, then the probability of an event with infinitely many elements would become unbounded (undefined).

- In order to overcome the above problem, if we assign zero probability to each elementary outcome, we cannot determine the probability of an uncountable subset of Ω, say $\left[\frac{1}{4}, 1\right]$. In this case, we would need to sum the probabilities of all the elements in the set $\left[\frac{1}{4}, 1\right]$. However, the interval $\left[\frac{1}{4}, 1\right]$ contains uncountably many elements, so we cannot sum the probabilities of the elements in the interval because probability measures are not additive over uncountable disjoint unions.

The solution to this problem is to avoid focusing on assigning probabilities to each element of Ω (i.e., to singleton sets). When dealing with uncountable sample spaces, it is not possible to assign probabilities if we use $\mathscr{F} = 2^{\Omega}$, the power set of Ω. To solve this, we assign probabilities directly to certain subsets of Ω. This requires constructing a new σ-algebra called the **Borel σ-algebra**.

1.2 Definition of Borel σ-Algebra

Let $\Omega = \mathbb{R}$, and consider the collection of sets $\mathscr{D}_1$ = $\{(-\infty, x] : x \in \mathbb{R}\}$. The Borel σ-algebra is defined as $\mathscr{B}(\mathbb{R})$ = $\sigma(\mathscr{D}_1)$. i.e., start from the collection of sets in $\mathscr{D}_1$ and add all the sets to $\mathscr{D}_1$ to form a σ-algebra. Stop in the very first instance when you obtain a σ algebra. Do not go until $2^{\mathbb{R}}$. The smallest σ algebra in which you end up is called the **Borel σ-Algebra**. A standard result from measure theory states that the cardinality of the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ is equal to the cardinality of $\mathbb{R}$. The sets in $\mathscr{B}(\mathbb{R})$ are referred to as **Borel sets**.

1.3 Different ways to generate Borel σ-algebra

In Section 1.2, we saw that $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_1)$, where $\mathscr{D}_1 = \{(-\infty, x] : x \in \mathbb{R}\}$. Interestingly, $\mathscr{B}(\mathbb{R})$ can also be constructed by starting with other collections of sets, such as $\mathscr{D}_2$, $\mathscr{D}_3$, $\mathscr{D}_4$, $\mathscr{D}_5$, or $\mathscr{D}_6$ which are outlined below.
Below are the ways in which we can generate the Borel σ-algebra:

- Let $\mathscr{D}_2 = \{(-\infty, x) : x \in \mathbb{R}\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_2)$.
- Let $\mathscr{D}_3 = \{[x, \infty) : x \in \mathbb{R}\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_3)$.
- Let $\mathscr{D}_4 = \{(x, \infty) : x \in \mathbb{R}\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_4)$.
- Let $\mathscr{D}_5 = \{[a, b] : a < b, \forall a, b \in \mathbb{R}\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_5)$.
- Let $\mathscr{D}_6 = \{(a, b) : a < b, \forall a, b \in \mathbb{R}\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_6)$.

Here, $\sigma(\mathscr{D}_i)$ denotes the σ-algebra generated by the collection $\mathscr{D}_i$ for $i \in \{1, \ldots, 6\}$.

Remark 1. *Every set in $\mathscr{B}(\mathbb{R})$ can be expressed exclusively in terms of countable unions, complements, and countable intersections of sets from any of $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$, $\mathscr{D}_4$, $\mathscr{D}_5$ or $\mathscr{D}_6$.*

To generate the Borel σ-algebra $\mathscr{B}(\mathbb{R})$, we must carefully select a starting collection $\mathscr{D}$ of sets. Importantly, the collection $\mathscr{D}$ should *generate* $\mathscr{B}(\mathbb{R})$, i.e., any set in $\mathscr{B}(\mathbb{R})$ should be expressible in terms of countable unions, countable intersections, and complements of sets from the collection. It is customary to choose the collection $\mathscr{D}$ that is simplest in terms of its structural properties. For instance, it is common to consider $\mathscr{D}$ that is a *π-system*–a collection of sets closed under finite intersections. Another popular structural choice for $\mathscr{D}$ is that of an *algebra*–a collection of sets that contains the sample space, and is closed under complements and finite unions. The careful reader will observe that each of the collections $\mathscr{D}_1, \ldots, \mathscr{D}_6$ described above is a π-system that generates $\mathscr{B}(\mathbb{R})$.
We now present some examples to demonstrate that $\mathscr{D}_i$ generates $\mathscr{B}(\mathbb{R})$ for each $i \in \{1, \ldots, 6\}$.

1. **Express $(2,3)$ via countable unions, countable intersections, and complements of sets from $\mathscr{D}_1$.**
   We have
$$\begin{aligned}
(2,3) &= (2,\infty) \cap (-\infty, 3) \\
&= (-\infty, 2]^c \cap [3, \infty)^c \\
&= (-\infty, 2]^c \cap \left(\bigcap_{n=1}^{\infty} \left(3 - \frac{1}{n}, \infty\right)\right)^c \\
&= (-\infty, 2]^c \cap \left(\bigcup_{n=1}^{\infty} \left(-\infty, 3 - \frac{1}{n}\right]\right) \\
(2,3) &= (-\infty, 2]^c \cap \left(\bigcup_{n=1}^{\infty} \left(-\infty, 3 - \frac{1}{n}\right]\right).
\end{aligned}$$

2. **Express $\{5.5\}$ via countable unions, countable intersections, and complements of sets from $\mathscr{D}_2$.**
   We have
$$\begin{aligned}
\{5.5\} &= (-\infty,\ 5.5] \cap [5.5,\ \infty) \\
&= (-\infty,\ 5.5] \cap (-\infty,\ 5.5)^c \\
&= \bigcap_{n=1}^{\infty} \left(-\infty,\ 5.5 + \frac{1}{n}\right) \cap (-\infty,\ 5.5)^c.
\end{aligned}$$

3. **Express $(-3,-2]$ via countable unions, countable intersections, and complements of sets from $\mathscr{D}_3$.**
   We have
$$\begin{aligned}
(-3,-2] &= (-3, \infty) \cap (-\infty, -2] \\
&= (-3, \infty) \cap (-2, \infty)^c.
\end{aligned}$$

We now note that

$$\begin{aligned}(-3,\infty) &= (\infty,-3]^c \\ &= \left(\bigcap_{n=1}^{\infty}\left(-\infty,-3+\frac{1}{n}\right)\right)^c \\ &= \bigcup_{n=1}^{\infty}\left[-3+\frac{1}{n},\infty\right).\end{aligned}$$

Along similar lines, we have

$$\begin{aligned}(-\infty,-2] &= (-2,\infty)^c \\ &= \left(\bigcup_{n=1}^{\infty}\left[-2+\frac{1}{n},\infty\right)\right)^c \\ &= \bigcap_{n=1}^{\infty}\left[-2+\frac{1}{n},\infty\right)^c.\end{aligned}$$

Combining the above expressions, we get

$$(-3,-2] = \bigcup_{n=1}^{\infty}\left[-3+\frac{1}{n},\infty\right) \cap \left(\bigcap_{n=1}^{\infty}\left[-2+\frac{1}{n},\infty\right)^c\right)$$

4. **Express $[-6,5]$ via countable unions, countable intersections, and complements of sets from $\mathscr{D}_4$.**
   We have

$$\begin{aligned}[-6,5] &= [-6,\infty) \cap (-\infty,5] \\ &= [-6,\infty) \cap (5,\infty)^c.\end{aligned}$$

We now observe that

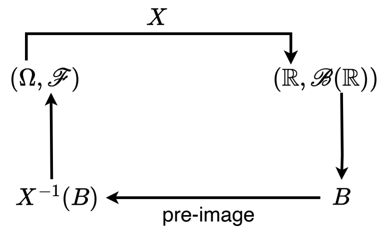
$$\begin{aligned}[-6,\infty) &= (-\infty,-6)^c \\ &= \left(\bigcup_{n=1}^{\infty}\left(-\infty,-6-\frac{1}{n}\right]\right)^c \\ &= \bigcap_{n=1}^{\infty}\left(-\infty,-6-\frac{1}{n}\right]^c \\ &= \bigcap_{n=1}^{\infty}\left(-6-\frac{1}{n},\infty\right).\end{aligned}$$

We thus have

$$[-6,5] = \bigcap_{n=1}^{\infty}\left(-6-\frac{1}{n},\infty\right) \cap (5,\infty)^c.$$

2 Random Variable

When performing an experiment, we might not always be interested in a particular elementary outcome. Instead, we would like to observe some numerical function of the random outcome that occurs. For example, on tossing a coin n times, the number of times heads showed up might be of interest to us rather than the exact sequence of the outcomes. That is, the quantity of interest may be a *function* of the outcome or a collection of outcomes. This motivates the definition of a random variable.



$$\forall B \in \mathscr{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathscr{F}$$

Figure 1: Pictorial representation of a random variable.

2.1 Definition of Random Variable

Definition 1. *A function $X : \Omega \to \mathbb{R}$ is called a random variable with respect to $\mathscr{F}$ if*

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}, \quad \text{for all } B \in \mathscr{B}(\mathbb{R}).$$

Remark 2. *A random variable is a deterministic function that maps elements from a sample space Ω to real numbers $\mathbb{R}$, and is measurable with respect to a σ-algebra $\mathscr{F}$ on Ω, meaning that the pre-images of Borel sets in $\mathbb{R}$ are events in $\mathscr{F}$. The term random arises from the fact that the output of the function depends on the realization of an elementary outcome ω from the sample space Ω. (Randomness is associated with the realization of this elementary outcome ω).*

2.2 Examples of Random Variable

Example 2.1. *Consider a six-sided die. Consider a function X that maps each outcome to the number that comes face up. Specifically, consider $X : \Omega \to \mathbb{R}$, where $\Omega = \{1, 2, 3, 4, 5, 6\}$ is the sample space, defined via*

$$X(\omega) = \omega, \qquad \omega \in \Omega.$$

As we shall soon see, to verify that the above function is indeed a random variable, we first need to fix an appropriate σ-algebra, and then verify the conditions of Definition 1 with respect to the σ-algebra under consideration.

Example 2.2. *Let X represent the number of heads obtained by flipping a coin once, with $\Omega = \{H, T\}$. The function X can therefore take two possible values, as given below:*

$$X(\omega) = \begin{cases} 1 & \omega = H, \\ 0 & \omega = T. \end{cases}$$

It is easy to verify that the above function is a random variable with respect to $\mathscr{F} = 2^{\Omega}$.

2.3 Random Variable Simplified

Let us recall the definition of a random variable.
A function $X : \Omega \to \mathbb{R}$ is called a random variable with respect to $\mathscr{F}$ if

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}, \quad \text{for all } B \in \mathscr{B}(\mathbb{R}).$$

From the above definition, we can say that if we take any Borel set B from the Borel σ-algebra and compute its pre-image $X^{-1}(B)$, the pre-image $X^{-1}(B)$ should belong to the σ-algebra $\mathscr{F}$. However, it is not necessary to explicitly verify this for every set in the Borel σ-algebra. It suffices to show this property for sets of the form $(-\infty, x]$, because the collection $\mathscr{D}_1 = \{(-\infty, x] : x \in \mathbb{R}\}$ generates the Borel σ-algebra, and every set in the Borel σ-algebra can be expressed in terms of this collection.
Here, $\mathscr{D}_1$ is only a representative choice for the collection. Indeed, we can also consider sets from one of $\mathscr{D}_2, \mathscr{D}_3, \mathscr{D}_4, \mathscr{D}_5$, or $\mathscr{D}_6$, and check that pre-images from these collections belong to $\mathscr{F}$. This leads to the following result outlining alternative, yet equivalent, definitions for a random variable.

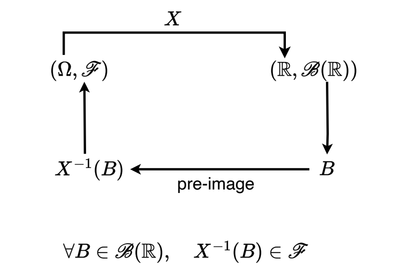


Figure 2: Pictorial representation for definition of random variable

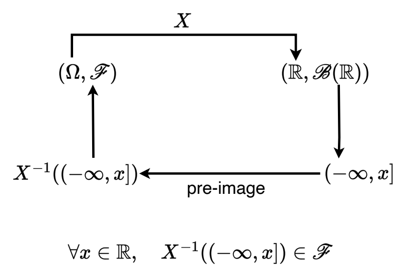


Figure 3: pictorial representation of definition of simplified random variable

Lemma 1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be a function. The following statements are equivalent.*

1. $X^{-1}(B) \in \mathscr{F}$ *for all* $B \in \mathscr{B}(\mathbb{R})$.
2. $X^{-1}(B) \in \mathscr{F}$ *for all* $B \in \mathscr{D}_1$.
3. $X^{-1}(B) \in \mathscr{F}$ *for all* $B \in \mathscr{D}_2$.
4. $X^{-1}(B) \in \mathscr{F}$ *for all* $B \in \mathscr{D}_3$.
5. $X^{-1}(B) \in \mathscr{F}$ *for all* $B \in \mathscr{D}_4$.
6. $X^{-1}(B) \in \mathscr{F}$ *for all* $B \in \mathscr{D}_5$.
7. $X^{-1}(B) \in \mathscr{F}$ *for all* $B \in \mathscr{D}_6$.

In other words, for X to be a random variable, the pre-image of any set B under X must belong to the σ-algebra $\mathscr{F}$, for each set B from one of the specified collections $\mathscr{B}(\mathbb{R})$, $\mathscr{D}_1, \ldots, \mathscr{D}_6$. The proof of Lemma 1 follows by noting that

$$X^{-1}(B^c) = (X^{-1}(B))^c, \qquad X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} X^{-1}(B_n).$$

2.4 Examples

1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathscr{F} = \{\emptyset, \Omega\}$, and $X(\omega) = \omega$. Is X a random variable? What other functions X are random variables with respect to $\mathscr{F}$?

   Solution: To determine whether a function X is a random variable, it must satisfy the following condition:

   Consider the collection of sets $\mathscr{D}_1 = \{(-\infty, x] : x \in \mathbb{R}\}$.
   Check for $X^{-1}((-\infty, x]) \in \mathscr{F} \quad \forall\, x \in \mathbb{R}$.
   Given that $X(\omega) = \omega$ and $\mathscr{F} = \{\emptyset, \Omega\}$

   $$\text{for } x = 0, \quad X^{-1}((-\infty, 0]) = \{\omega \in \Omega : X(\omega) \leq 0\} = \emptyset \in \mathscr{F}.$$
   $$\text{for } x = 1, \quad X^{-1}((-\infty, 1]) = \{\omega \in \Omega : X(\omega) \leq 1\} = \{1\} \notin \mathscr{F}.$$

So, for $x = 1$, the condition fails. Therefore, the given function $X(\omega) = \omega$ is not a random variable.
For the given σ-algebra $\mathscr{F} = \{\emptyset, \Omega\}$, only constant functions are random variables. Indeed, we note that if $X(\omega) = c$ for all $\omega \in \Omega$, for some fixed constant $c \in \mathbb{R}$, then

$$X^{-1}((-\infty, x]) = \begin{cases} \emptyset, & x < c, \\ \Omega, & x \geq c. \end{cases}$$

2. Let $\Omega = [0, 1]$ and $\mathscr{F} = \{\emptyset, \Omega, A, A^c\}$ for some fixed $A \subseteq \Omega$. What functions X are random variables?

   Solution:

   (a) Constant functions are definitely random variables for the given $(\Omega, \mathscr{F})$.

   (b) Let X be defined via

   $$X(\omega) = \begin{cases} c_1 & \text{if } \omega \in A, \\ c_2 & \text{if } \omega \in A^c. \end{cases}$$

   for some fixed constants $c_1, c_2 \in \mathbb{R}$. Such a function is sometimes referred to as being "piecewise constant" (taking a constant value c_1 on A and a possibly different constant value c_2 on A^c). If $c_1 = c_2$, we get a constant function.
   Without loss of generality, let $c_1 < c_2$. Then, we have

   $$X^{-1}((-\infty, x]) = \begin{cases} \emptyset, & x < c_1, \\ A, & c_1 \leq x < c_2, \\ \Omega, & x \geq c_2, \end{cases}$$

   thus proving that X, as defined above, is a random variable with respect to $\mathscr{F}$. It is easy to verify that for any function X to be a random variable with respect to $\mathscr{F}$, X must necessarily possess a "piecewise constant" structure as above.

3. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, and $\mathscr{F} = \sigma(\{\{1\}, \{2, 3\}\})$. What functions X are random variables?

   Solution:

   (a) First, we note that
   $\mathscr{F} = \{\emptyset, \Omega, \{1\}, \{2, 3\}, \{1, 2, 3\}, \{4, 5, 6\}, \{1, 4, 5, 6\}, \{2, 3, 4, 5, 6\}\}$.

   (b) Consider a function $X : \Omega \to \mathbb{R}$ defined via

   $$X(\omega) = \begin{cases} c_1 & \text{if } \omega = 1, \\ c_2 & \text{if } \omega \in \{2, 3\}, \\ c_3 & \text{if } \omega \in \{4, 5, 6\}. \end{cases}$$

   for some constants $c_1, c_2, c_3 \in \mathbb{R}$. Without loss of generality, let $c_1 \leq c_2 \leq c_3$. Then,

   $$X^{-1}((-\infty, x]) = \begin{cases} \emptyset, & x < c_1, \\ \{1\}, & c_1 \leq x < c_2, \\ \{1, 2, 3\}, & c_2 \leq x < c_3, \\ \Omega, & x \geq c_3, \end{cases}$$

   hence demonstrating that X is a random variable. In fact, the only functions that are random variables with respect to the given $\mathscr{F}$ are those which have the above structure.

4. $\Omega = \mathbb{N}, \mathscr{F} = 2^\Omega$. What functions X are random variables?

   Solution: Because $\mathscr{F} = 2^\Omega$, *every* function X is a random variable.

3 Probability Law of a Random Variable

The probability law of a random variable describes how probabilities are distributed over the possible outcomes in Borel σ-algebra i.e. ($B \in \mathscr{B}(\mathbb{R})$).

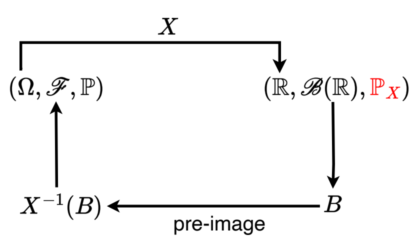
What is the need to define $\mathbb{P}_X$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$?

In real life, we cannot directly observe the underlying randomness or the exact mechanism that governs the outcomes of a random experiment. Instead, we only observe the results or outcomes of random variables, which are influenced by the probability law. For example, when flipping a coin, we cannot observe the precise factors (such as air resistance, initial velocity, or exact position of the coin) that contribute to its randomness. Instead, we observe the result heads or tails and we can infer the probability law governing the coin flip (such as a 50% chance for each side landing up). Similarly, in fields such as weather forecasting or stock market analysis, we can only observe historical data or outcomes, but we cannot directly access the underlying randomness that influences these outcomes.

3.1 Definition for Probability Law of a Random Variable

Definition 2. *For a random variable $X : \Omega \to \mathbb{R}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, its probability law $\mathbb{P}_X$ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, where $\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. The probability law $\mathbb{P}_X$ is defined by:*

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}\left(\{\omega \in \Omega : X(\omega) \in B\}\right), \quad B \in \mathscr{B}(\mathbb{R}).$$



$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

Figure 4: Probability law of a random variable

The probability law $\mathbb{P}_X$ captures how the values of the random variable are distributed along the real line. It tells us the probability of X falling within any given set of real numbers. The distribution can be described in various forms, depending on whether X is discrete or continuous.

- **For Discrete Random Variables:** If X is a discrete random variable, the probability law assigns probabilities to specific values that X can take. In this case, the probability law is typically represented by a probability mass function (PMF), say p_X, which gives the probability of each individual outcome. That is,

$$\mathbb{P}_X(\{x\}) = p_X(x), \qquad x \in \mathbb{R}.$$

- **For Continuous Random Variables:** If X is continuous, the probability law is often represented by a probability density function (PDF), which describes how the probability is distributed over the real line. The probability law is given by the integral of the PDF over a set B, and the probability of X falling in an interval $[a, b]$ is given by:

$$\mathbb{P}_X([a, b]) = \int_a^b f_X(x)\, dx,$$

where $f_X(x)$ is the PDF of X.

4 Cumulative Distribution Function (CDF)

In this section, we discuss the Cumulative Distribution Function (CDF) of a random variable.

The Cumulative Distribution Function (CDF) is another way to describe the probability law of a random variable. Recall from Section 1.3 that if $\mathscr{D}_1 = \{(-\infty, x] : x \in \mathbb{R}\}$, then $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_1)$.

Note that $(-\infty, x] \in \mathscr{B}(\mathbb{R})$ for each $x \in \mathbb{R}$. Thus, we can talk about the probability law of these half-open sets which are present in the Borel σ-algebra ($\mathscr{B}(\mathbb{R})$).

4.1 Definition of Cumulative Distribution Function (CDF)

Definition 3. *Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Given a random variable $X : \Omega \to \mathbb{R}$ with respect to $\mathscr{F}$, its cumulative distribution function (CDF) $F_X : \mathbb{R} \to [0,1]$ is defined as*

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}\left(X^{-1}((-\infty, x])\right), \quad x \in \mathbb{R}.$$

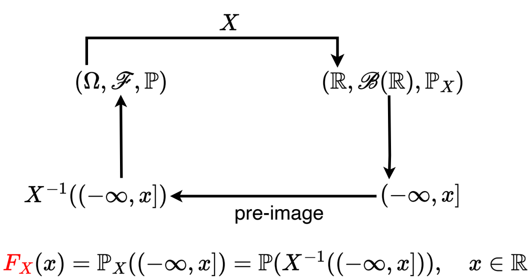


Figure 5: Cumulative Distribution Function

- $F_X(x)$ denotes the CDF of the random variable X evaluated at $x \in \mathbb{R}$, and is defined as the probability that X takes a value less than or equal to x.
- The symbol $\mathbb{P}_X$ denotes the probability law of the random variable X, and $X^{-1}((-\infty, x])$ represents the pre-image of the interval $(-\infty, x]$, which consists of all points $\omega \in \Omega$ where $X(\omega) \in (-\infty, x]$.

Remark 3. *The Cumulative Distribution Function (CDF) is another way to describe the probability law of a random variable.*

4.2 Properties Of Cumulative Distribution Function (CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$ and CDF F_X.

- $\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1$
- (Monotonicity) If $x \leq y$, then $F_X(x) \leq F_X(y)$, i.e., the CDF is monotonic and non-decreasing.
- (Right-Continuity) F_X is right-continuous, i.e., for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} F_X(x + \varepsilon) = F_X(x).$$

4.3 Relation between CDF and Probability Law

The Cumulative Distribution Function (CDF) and the probability law are closely related concepts in probability theory, because the CDF is derived directly from the probability law (PMF for discrete variables or PDF for continuous variables).

- If we know $\mathbb{P}_X = \{\mathbb{P}_X(B) : B \in \mathscr{B}(\mathbb{R})\}$, then we can extract the CDF $F_X : \mathbb{R} \to [0,1]$ by using the formula

$$F_X(x) = \mathbb{P}_X((-\infty, x]), \quad x \in \mathbb{R}.$$

- Given the CDF $F_X : \mathbb{R} \to [0,1]$, let

$$\mathbb{P}_X((-\infty, x]) = F_X(x), \quad x \in \mathbb{R}.$$

  Then, there exists a unique extension of $\mathbb{P}_X$ to all Borel subsets of $\mathbb{R}$. For a proof of this statement, see [Fol99, Theorem 1.16].

4.4 Examples of CDF

Example 4.1. *Consider a discrete random variable X with the following probability mass function (PMF):*

$$p_X(x) = \begin{cases} \frac{1}{4}, & x = 1 \\ \frac{1}{4}, & x = 2 \\ \frac{1}{2}, & x = 3 \\ 0, & \textit{otherwise} \end{cases}$$

The Cumulative Distribution Function (CDF) is defined as $F_X(x) = \mathbb{P}(X \leq x)$. For this example, we compute the CDF:

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{4}, & 1 \leq x < 2 \\ \frac{1}{2}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

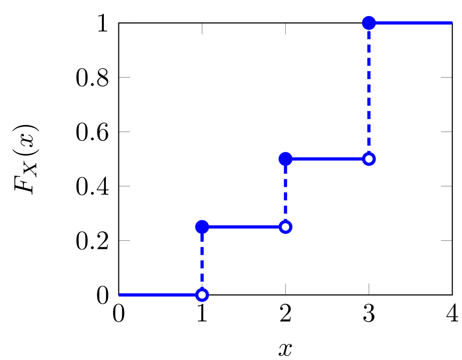


Figure 6: CDF of Discrete Random Variable.

Example 4.2. *For a continuous random variable, the CDF is a smooth curve. Consider the following continuous probability density function (PDF):*

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

The CDF of this continuous random variable is given by:

$$F_X(x) = \int_0^x 2t\,dt = x^2 \quad \textit{for } 0 \leq x \leq 1.$$

The CDF of the continuous random variable is plotted below:

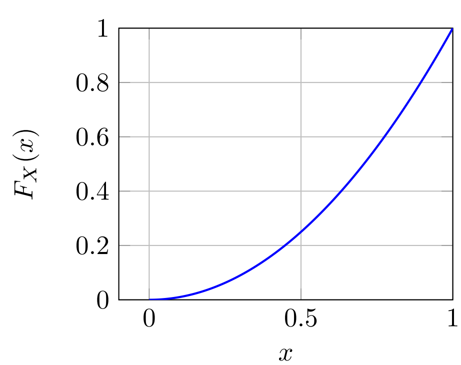


Figure 7: CDF of Continuous Random Variable.

5 Structured Assignment of Probabilities to sets in $\mathscr{B}(\mathbb{R})$

In the previous section we discussed about Cumulative Distribution Function (CDF) of Random Variable. Using the properties of a CDF there is a structured way to come up with a probability measure $\mathbb{Q}$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ space.

By operating only on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ space we can generate the probability measure without knowing the information about $\Omega, \mathscr{F}$.

1. Consider $\mathscr{D}_1 = \{(-\infty, x] : x \in \mathbb{R}\}$.
2. Consider a function $G : \mathbb{R} \to [0,1]$ such that:
   (a) $\lim_{x\to-\infty} G(x) = 0, \quad \lim_{x\to\infty} G(x) = 1.$
   (b) G is non-decreasing.
   (c) G is right-continuous.
3. Set $\mathbb{Q}((-\infty, x]) = G(x) \ \ \forall x \in \mathbb{R}$.
4. Using **Carathéodory's extension theorem**, get $\mathbb{Q}(B)$ for all $B \in \mathscr{B}(\mathbb{R})$.

6 Random Vectors

6.1 Definition of Random Vector

Fix a measurable space $(\Omega, \mathscr{F})$. Fix $n \in \mathbb{N}$.

Given random variables $X_1, \ldots, X_n$ defined with respect to $\mathscr{F}$, we say $(X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$ is a random vector with respect to $\mathscr{F}$ if

$$(X_1, \ldots, X_n)^{-1}(B) = \{\omega \in \Omega : (X_1(\omega), \ldots, X_n(\omega)) \in B\} \in \mathscr{F} \quad \text{for every Borel set } B \subseteq \mathbb{R}^n.$$

7 Sequence of Random Variables

7.1 Definition of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

A sequence of random variables is a collection $\{X_n\}_{n=1}^{\infty}$ such that

$$\forall n \in \mathbb{N}, \forall k_1, \ldots, k_n \in \mathbb{N}, \quad (X_{k_1}, \ldots, X_{k_n}) \text{ is a random vector.}$$

References

[Fol99] Gerald B Folland. *Real analysis: modern techniques and their applications*. John Wiley & Sons, 1999.