Name:
Roll Number:

CS6660: Mathematical Foundations of Data Science (Probability)

Quiz 2

Date: 14 September 2024

Question	1	2(a)	2(b)	Total
Marks Scored				

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

1. **(1 Mark)**
   Suppose that two batteries are chosen simultaneously and uniformly at random from the following group of 12 batteries : 3 new, 4 used (yet working), 5 defective. You may assume that all batteries within a particular group are identical.
   Let X be the number of **new** batteries chosen, and let Y be the number of **defective** batteries chosen.
   If the value of $\mathbb{P}(\{X \geq Y\})$ is expressed as $\frac{\alpha}{\beta}$, determine the value of $\frac{\alpha+\beta}{\beta-\alpha}$. Give your answer up to 1 decimal place only.
   Solution:
   Observe that $X \in \{0,1,2\}$, $Y \in \{0,1,2\}$, and $X+Y \leq 2$. The joint PMF of X and Y may be expressed as follows:

$$p_{X,Y}(x,y) = \begin{cases} \frac{\binom{4}{2}}{\binom{12}{2}}, & x=0, y=0, \\ \frac{\binom{4}{1}\cdot\binom{5}{1}}{\binom{12}{2}}, & x=0, y=1, \\ \frac{\binom{5}{2}}{\binom{12}{2}}, & x=0, y=2, \\ \frac{\binom{3}{1}\cdot\binom{4}{1}}{\binom{12}{2}}, & x=1, y=0, \\ \frac{\binom{3}{1}\cdot\binom{5}{1}}{\binom{12}{2}}, & x=1, y=1, \\ \frac{\binom{3}{2}}{\binom{12}{2}}, & x=2, y=0, \\ 0, & \text{otherwise.} \end{cases}$$

   We then note that

$$\mathbb{P}(\{X \geq Y\}) = p_{X,Y}(0,0) + p_{X,Y}(1,0) + p_{X,Y}(2,0) + p_{X,Y}(1,1) = \frac{36}{66} = \frac{6}{11}.$$

   We thus have $\alpha = 6, \beta = 11$, and therefore $\frac{\alpha+\beta}{\beta-\alpha} = \frac{17}{5} = 3.4.$

2. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
   Assume that all random variables appearing below are defined with respect to $\mathscr{F}$.

   Numbers from $[0,1]$ are picked uniformly, independently, and sequentially over time.
   Let X_n denote the number picked at time n, where $n \in \{0,1,2,\ldots\}$. Let N be the random variable defined as

$$N = \min\{n \geq 1 : X_n > X_0\}.$$

   That is, N denotes the first time index n at which the value of X_n exceeds the value of X_0.

   (a) **(3 Marks)**
   For any fixed $n \in \mathbb{N}$, determine $\mathbb{P}(\{N = n\})$.
   Hint: The event that $N = n$ is identical to the event that $X_1 \leq X_0, \ldots, X_{n-1} \leq X_0,\ X_n > X_0$.
   Solution: For any fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}(\{N = n\}) &= \mathbb{P}(\{X_1 \leq X_0\} \cap \cdots \cap \{X_{n-1} \leq X_0\} \cap \{X_n > X_0\}) \\
&= \int_0^1 \underbrace{\int_0^{x_0} \cdots \int_0^{x_0}}_{n-1 \text{ times}} \int_{x_0}^1 f_{X_0,\ldots,X_n}(x_0,\ldots,x_n)\, dx_n dx_{n-1} \ldots dx_0 \\
&\overset{(a)}{=} \int_0^1 \underbrace{\int_0^{x_0} \cdots \int_0^{x_0}}_{n-1 \text{ times}} \int_{x_0}^1 dx_n dx_{n-1} \ldots dx_0 \\
&= \int_0^1 x_0^{n-1}(1 - x_0)\, dx_0 \\
&= \frac{1}{n} - \frac{1}{n+1},
\end{aligned}$$

   where (a) above follows from the fact that $X_0, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Unif}([0,1])$.

   (b) **(1 Mark)**
   Compute $\mathbb{P}(\{N > 2\})$.
   Solution: We have

$$\mathbb{P}(\{N > 2\}) = \sum_{n=3}^{\infty} \mathbb{P}(\{N = n\}) = \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots = \frac{1}{3}.$$