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Stochastic Processes

Lecture 03

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Pointwise Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Pointwise Convergence)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges **pointwise** to X if

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \qquad \forall\, \omega \in \Omega;$$

Notation: $X_n \xrightarrow{\text{pointwise}} X.$

Equivalently, we have $A_{\lim} = \Omega$.

Almost-Sure Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Almost-Sure Convergence)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to X **almost surely (a.s.)** if

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1;$$

Notation: $X_n \xrightarrow{\text{a.s.}} X.$

Note:

- The above definition DOES NOT require $X_1, X_2, \ldots$ to be independent
- If $X_1, X_2, \ldots$ are independent, then $\mathbb{P}(A_{\text{lim}}) \in \{0, 1\}$ by Kolmogorov's 0-1 law
  This has no bearing on the above definition in any way
- **Pointwise convergence implies almost-sure convergence**, i.e.,

$$X_n \xrightarrow{\text{pointwise}} X \qquad \Longrightarrow \qquad X_n \xrightarrow{\text{a.s.}} X.$$

  In general, the converse may not be true

Example: Shrinking Pulse

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, λ: Lebesgue measure

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} 1, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise.} \end{cases}$$

  Does this sequence converge pointwise? If so, what is the pointwise limit RV?

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \omega^n, \qquad \omega \in \Omega.$$

  Does this sequence converge pointwise? If so, what is the pointwise limit RV?

Example: Moving Rectangles

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, λ: Lebesgue measure

- Consider the sequence of random variables given by:

  $X_1 = \mathbf{1}_{[0,1]}$

  $X_2 = \mathbf{1}_{[0,\frac{1}{2}]}$, $X_3 = \mathbf{1}_{[\frac{1}{2},1]}$

  $X_4 = \mathbf{1}_{[0,\frac{1}{4}]}$, $X_5 = \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}$, $X_6 = \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]}$, $X_7 = \mathbf{1}_{[\frac{3}{4},1]}$, and so on.

  Does the above sequence converge pointwise?
  Does the above sequence converge almost-surely?

Example: Going Beyond Pointwise Convergence

- For each $n \in \mathbb{N}$, let

$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$

Remarks on this Example

- In cases where $(\Omega, \mathscr{F}, \mathbb{P})$ and the sequence $\{X_n\}_{n\in\mathbb{N}}$ are not explicitly specified, it is not possible to identify the pointwise limit.
- In such cases, we start with a guess for the limit RV and prove convergence in other forms (starting with almost-sure convergence)
- In many cases (including the current example), we need a way to infer almost-sure convergence merely based on probabilities

In enters Borel–Cantelli Lemma!

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Borel–Cantelli Lemma and Almost-Sure Convergence

Borel–Cantelli Lemma

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma (Borel–Cantelli Lemma)

1. *Suppose* $A_1, A_2, \ldots \in \mathscr{F}$ *are such that* $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) < +\infty$. *Then,*

$$\mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_n\right) = 0.$$

2. *Suppose* $A_1, A_2, \ldots \in \mathscr{F}$ *are* **independent** *and satisfy* $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = +\infty$. *Then,*

$$\mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_n\right) = 1.$$

The above lemma can be used to assert **almost-sure convergence** in some scenarios!

Borel–Cantelli Lemma and Almost-Sure Convergence

- For each $n \in \mathbb{N}$, let
$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$
Does the above sequence converge almost-surely? If so, identify the almost-sure limit.
- For each $n \in \mathbb{N}$, let
$$\mathbb{P}(X_n = 1) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0).$$
Furthermore, suppose that $X_1, X_2, \ldots$ are mutually independent.
What can you say about the almost-convergence of the above sequence?

Borel–Cantelli Lemma and Almost-Sure Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Borel–Cantelli Lemma and Almost-Sure Convergence)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
The following statements are equivalent.

1. $X_n \xrightarrow{\text{a.s.}} X$.
2. *One of the following holds.*
   2.1 *For every choice of* $\varepsilon > 0$, $\quad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{|X_n - X| > \varepsilon\right\} \cap \{X \in \mathbb{R}\}\right) = 0.$

   2.2 *For every choice of* $\varepsilon > 0$, $\quad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n < \varepsilon\right\} \cap \{X = +\infty\}\right) = 0.$

   2.3 *For every choice of* $\varepsilon > 0$, $\quad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n > -\varepsilon\right\} \cap \{X = -\infty\}\right) = 0.$

Note: If X is a real-valued RV, then the probabilities in 2.2 and 2.3 are zero

Proof of Lemma 2

- We have

$$A_{\lim} = \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} = A_{\lim}^{(1)} \cup A_{\lim}^{(2)} \cup A_{\lim}^{(3)},$$

  where

$$\begin{aligned}
A_{\lim}^{(1)} &:= \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \in \mathbb{R}\right\} = \bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| > q\},\\
A_{\lim}^{(2)} &:= \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) = +\infty\right\} = \bigcap_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \left\{X_n > q\right\},\\
A_{\lim}^{(3)} &:= \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) = -\infty\right\} = \bigcap_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \left\{X_n < -q\right\}.
\end{aligned}$$

- By definition of almost-sure convergence, we have

$$X_n \xrightarrow{\text{a.s.}} X \qquad \Longleftrightarrow \qquad \mathbb{P}(A_{\lim}) = \underbrace{\mathbb{P}(A_{\lim}^{(1)})}_{p_1} + \underbrace{\mathbb{P}(A_{\lim}^{(2)})}_{p_2} + \underbrace{\mathbb{P}(A_{\lim}^{(3)})}_{p_3} = 1.$$

Proof of Lemma 2

- Case 1: $p_1 = 1 \quad p_2 = 0 = p_3$
  In this case,

$$\mathbb{P}\left(\bigcap_{\substack{q\in\mathbb{Q}\\q>0}} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| \le q\}\right) = 1 \iff \mathbb{P}\left(\bigcup_{\substack{q\in\mathbb{Q}\\q>0}} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| > q\}\right) = 0$$

$$\iff \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| > q\}\right) = 0 \qquad \forall q \in \mathbb{Q},\ q > 0$$

$$\iff \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| > \varepsilon\}\right) = 0 \qquad \forall \varepsilon > 0.$$

Proof of Lemma 2

- Case 2: $p_2 = 1 \quad p_1 = 0 = p_3$
  In this case,

$$\mathbb{P}\left(\bigcap_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{X_n > q\}\right) = 1 \iff \mathbb{P}\left(\bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{X_n \le q\}\right) = 0$$

$$\iff \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{X_n \le q\}\right) = 0 \qquad \forall q \in \mathbb{Q},\ q > 0$$

$$\iff \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{X_n \le \varepsilon\}\right) = 0 \qquad \forall \varepsilon > 0.$$

- Case 3: $p_3 = 1 \quad p_1 = 0 = p_2$
  The proof is along similar lines as above and left as **exercise**

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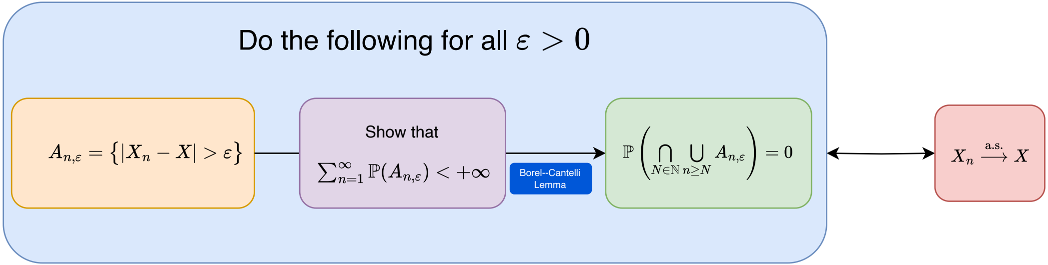
Borel–Cantelli Lemma and Almost-Sure Convergence

A Generic Template for the Case when X is Real-Valued



Example

- Suppose $U \sim \mathrm{Uniform}(0,1)$, and for each $n \in \mathbb{N}$, let

$$Z_n = \mathbf{1}_{\left\{\left|U - \frac{1}{2}\right| < \frac{1}{n}\right\}}.$$

  Does the sequence $\{Z_n\}_{n\in\mathbb{N}}$ converge pointwise?
  Does the sequence $\{Z_n\}_{n\in\mathbb{N}}$ converge almost-surely?

Example: Shrinking Pulse Revisited

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, λ: Lebesgue measure

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} 1, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Does this sequence converge pointwise?
Does this sequence converge almost-surely?
Is the almost-sure limit unique?

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Almost-Sure Uniqueness of Almost-Sure Limit RV

Lemma (Almost-Sure Uniqueness of Almost-Sure Limit RV)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs. Let X, Y be two extended real-valued random variables. If

$$X_n \xrightarrow{\text{a.s.}} X, \qquad X_n \xrightarrow{\text{a.s.}} Y,$$

then we have

$$\mathbb{P}(\{X = Y\}) = 1.$$

Proof of Lemma 3:

- We have

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1, \qquad \mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = Y(\omega)\right\}\right) = 1$$

- We then have

$$\{\omega \in \Omega : X(\omega) = Y(\omega)\} \supseteq \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} \cap \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = Y(\omega)\right\}$$

The result follows from the fact that intersection of two sets having probability 1 also has probability 1

Example: Moving Rectangles Revisited

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, λ: Lebesgue measure

- Consider the sequence of random variables given by:

  $X_1 = \mathbf{1}_{[0,1]}$

  $X_2 = \mathbf{1}_{[0,\frac{1}{2}]}$, $X_3 = \mathbf{1}_{[\frac{1}{2},1]}$

  $X_4 = \mathbf{1}_{[0,\frac{1}{4}]}$, $X_5 = \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}$, $X_6 = \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]}$, $X_7 = \mathbf{1}_{[\frac{3}{4},1]}$, and so on.

Note

- There is no pointwise limit or almost-sure limit for the above sequence
- However, we observe that $\mathbb{P}(X_n = 0) \approx 1$ for large n
  Intuitively, the constant RV 0 seems like a limit

In what sense is the constant RV 0 a limit here?

Convergence in Probability

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in Probability)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to X **in probability (p.)** if:

1. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{|X_n - X| > \varepsilon\} \cap \{X \in \mathbb{R}\}\right) \xrightarrow{n\to\infty} 0.$
2. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{X_n < \varepsilon\} \cap \{X = +\infty\}\right) \xrightarrow{n\to\infty} 0.$
3. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{X_n > -\varepsilon\} \cap \{X = -\infty\}\right) \xrightarrow{n\to\infty} 0.$

Notation: $\quad X_n \xrightarrow{\text{p.}} X.$

Remark:

- If X is a real-valued RV, then the conditions in 2, 3 hold trivially

Convergence in p-Norm $(p \geq 1)$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in p-Norm)

Fix $p \geq 1$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to X **in p-norm** if

1. We have
$$\mathbb{E}\left[\left|X_n\right|^p\right] < +\infty \quad \text{for all } n \in \mathbb{N}, \qquad \mathbb{E}\left[\left|X\right|^p\right] < +\infty.$$
2. We have
$$\mathbb{E}\left[\left|X_n - X\right|^p\right] \xrightarrow{n\to\infty} 0; \qquad \textbf{Notation:} \quad X_n \xrightarrow{\mathcal{L}^p} X.$$

Remark:

- If $p = 2$, then the convergence in 2-norm is called **mean-squared convergence (m.s.)** and denoted
$$X_n \xrightarrow{\text{m.s.}} X$$

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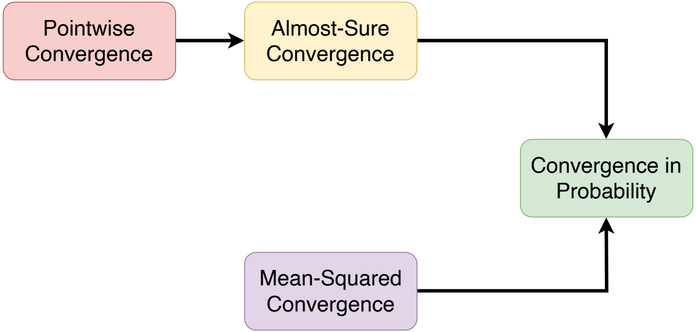
A Picture to Have in Mind



(proof of the implications to come later)

Example

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \text{Unif})$.
Fix a sequence of real numbers $\{a_n\}_{n \in \mathbb{N}}$.
For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} a_n, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise.} \end{cases}$$

- If $a_n = n$, does the sequence admit pointwise/almost-sure/mean-squared limits?
- Identify the conditions on $\{a_n\}_{n \in \mathbb{N}}$ for which the above sequence admits mean-squared limit

Some Remarks on the Previous Example

- If $a_n = n$, then
$$X_n \xrightarrow{\text{a.s.}} 0, \quad \text{but} \quad X_n \overset{\text{m.s.}}{\not\to} 0.$$
- Although $X_n \xrightarrow{\text{a.s.}} 0$, the value of $|X_n(\omega) - 0|$ can be arbitrarily large for some ω. This can lead to a large value for $\mathbb{E}[(X_n - X)^2]$ as $n \to \infty$.

Almost-Sure Convergence and Mean-Squared Convergence

In general,
$$X_n \xrightarrow{\text{a.s.}} 0 \quad \not\Rightarrow \quad X_n \xrightarrow{\text{m.s.}} 0, \qquad X_n \xrightarrow{\text{m.s.}} 0 \quad \not\Rightarrow \quad X_n \xrightarrow{\text{a.s.}} 0.$$

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Equivalent Statements for Almost-Sure Convergence

When X is a Real-Valued Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Borel–Cantelli Lemma and Almost-Sure Convergence)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X be a **real-valued** *RV.*
The following statements are equivalent.

1. $X_n \xrightarrow{\text{a.s.}} X$.
2. *For every choice of $\varepsilon > 0$,*

$$\mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \underbrace{\left\{|X_n - X| > \varepsilon\right\}}_{\text{bad event } A_{n,\varepsilon}}\right) = 0.$$

That is, the probability of the bad event occurring infinitely many times is zero (equivalently, after some stage, only good event will occur)

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Proof of Lemma 1

$$
\begin{aligned}
X_n \xrightarrow{\text{a.s.}} X \quad &\iff \quad \mathbb{P}\left(\bigcap_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| \le q\}\right) = 1 \\
&\iff \quad \mathbb{P}\left(\bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| > q\}\right) = 0 \\
&\iff \quad \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| > q\}\right) = 0 \qquad \forall q \in \mathbb{Q},\ q > 0 \\
&\iff \quad \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| > \varepsilon\}\right) = 0 \qquad \forall \varepsilon > 0.
\end{aligned}
$$

Equivalent Statements for Almost-Sure Convergence

When X is a Extended Real-Valued Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Equivalent Statements for Almost-Sure Convergence)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X be an **extended real-valued** *RV. The following statements are equivalent.*

1. $X_n \xrightarrow{\text{a.s.}} X$.
2. *All of the following hold simultaneously.*
   - 2.1 *For every choice of* $\varepsilon > 0$, $\mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{|X_n - X| > \varepsilon\right\} \cap \{X \in \mathbb{R}\}\right) = 0.$
   - 2.2 *For every choice of* $\varepsilon > 0$, $\mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \leq \varepsilon\right\} \cap \{X = +\infty\}\right) = 0.$
   - 2.3 *For every choice of* $\varepsilon > 0$, $\mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \geq -\varepsilon\right\} \cap \{X = -\infty\}\right) = 0.$

Proof of Lemma 2

- We have
$$A_{\lim} = \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} = A_{\lim}^{(1)} \cup A_{\lim}^{(2)} \cup A_{\lim}^{(3)},$$
where
$$A_{\lim}^{(1)} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \in \mathbb{R}\right\} = \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \{|X_n - X| \leq \varepsilon\} \cap \{X \in \mathbb{R}\},$$
$$A_{\lim}^{(2)} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) = +\infty\right\} = \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \left\{X_n > \varepsilon\right\} \cap \{X = +\infty\},$$
$$A_{\lim}^{(3)} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) = -\infty\right\} = \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \left\{X_n < -\varepsilon\right\} \cap \{X = -\infty\}.$$
- By definition of almost-sure convergence, we have
$$X_n \xrightarrow{\text{a.s.}} X \qquad \Longleftrightarrow \qquad \mathbb{P}(A_{\lim}) = 1.$$

Proof of Lemma 2 $(\implies)$

- Proof of $X_n \xrightarrow{\text{a.s.}} X \implies$ 2.1, 2.2, 2.3:
  For any choice of $\varepsilon > 0$, we note that

$$\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{|X_n - X| > \varepsilon\right\} \cap \{X \in \mathbb{R}\} \subseteq A_{\lim}^{\complement},$$

$$\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \leq \varepsilon\right\} \cap \{X = +\infty\} \subseteq A_{\lim}^{\complement},$$

$$\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \geq -\varepsilon\right\} \cap \{X = -\infty\} \subseteq A_{\lim}^{\complement}.$$

- The result follows from the fact that $\mathbb{P}(A_{\lim}^{\complement}) = 0$

Proof of Lemma 2 $\quad (\Longleftarrow)$

- Proof of $\quad X_n \xrightarrow{\text{a.s.}} X \quad \Longleftarrow \quad$ 2.1, 2.2, 2.3:
  Suppose that

$$\forall\, \varepsilon > 0, \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \Big\{|X_n - X| > \varepsilon\Big\} \cap \{X \in \mathbb{R}\}\right) = 0,$$

$$\forall\, \varepsilon > 0, \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \Big\{X_n \leq \varepsilon\Big\} \cap \{X = +\infty\}\right) = 0,$$

$$\forall\, \varepsilon > 0, \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \Big\{X_n \geq -\varepsilon\Big\} \cap \{X = -\infty\}\right) = 0.$$

Proof of Lemma 2 $\quad (\Longleftarrow)$

- Proof of $\quad X_n \xrightarrow{\text{a.s.}} X \quad \Longleftarrow \quad 2.1,\ 2.2,\ 2.3$:
  Then, it follows that

$$\forall\, q \in \mathbb{Q}, q > 0, \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{|X_n - X| > q\right\} \cap \{X \in \mathbb{R}\}\right) = 0,$$

$$\forall\, q \in \mathbb{Q}, q > 0, \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \leq q\right\} \cap \{X = +\infty\}\right) = 0,$$

$$\forall\, q \in \mathbb{Q}, q > 0, \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \geq -q\right\} \cap \{X = -\infty\}\right) = 0.$$

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Proof of Lemma 2 $\quad (\Longleftarrow)$

- Using the union bound, we have

$$\mathbb{P}\left(\underbrace{\bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{|X_n - X| > q\right\} \cap \{X \in \mathbb{R}\}}_{B_0}\right) = 0, \qquad \mathbb{P}\left(\underbrace{\bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \leq q\right\} \cap \{X = +\infty\}}_{B_1}\right) = 0,$$

$$\mathbb{P}\left(\underbrace{\bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{X_n \geq -q\right\} \cap \{X = -\infty\}}_{B_2}\right) = 0.$$

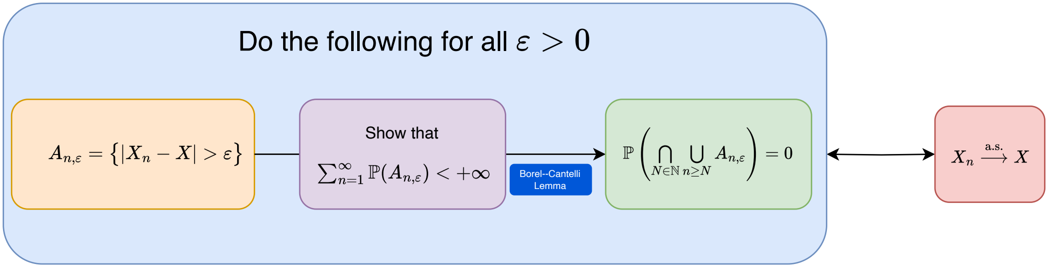
- If $B = B_0 \cup B_1 \cup B_2$, then $\mathbb{P}(B) = 0$

Proof of Lemma 2 $\quad (\Longleftarrow)$

- **Claim:** $B^{\complement} \subseteq A_{\lim}$
  - Case 1: $\omega \in B^{\complement}, \quad \omega \in \{X \in \mathbb{R}\}$
    In this case,
    $$\omega \in \bigcup_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \Big\{|X_n - X| > q\Big\},$$
    and hence it follows that
    $$\omega \in \bigcup_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \Big\{|X_n - X| > q\Big\} \cap \{X \in \mathbb{R}\} = A^{(1)}_{\lim} \subseteq A_{\lim}.$$
  - Case 2: $\omega \in B^{\complement}, \quad \omega \in \{X = +\infty\}$
    In this case, it can be shown that $\omega \in A^{(2)}_{\lim}$
  - Case 3: $\omega \in B^{\complement}, \quad \omega \in \{X = -\infty\}$
    In this case, it can be shown that $\omega \in A^{(3)}_{\lim}$

Borel–Cantelli Lemma and Almost-Sure Convergence

A Generic Template for the Case when X is Real-Valued



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Convergence in Probability

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in Probability)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to X **in probability (p.)** if:

1. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{|X_n - X| > \varepsilon\} \cap \{X \in \mathbb{R}\}\right) \xrightarrow{n\to\infty} 0.$
2. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{X_n \leq \varepsilon\} \cap \{X = +\infty\}\right) \xrightarrow{n\to\infty} 0.$
3. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{X_n \geq -\varepsilon\} \cap \{X = -\infty\}\right) \xrightarrow{n\to\infty} 0.$

Notation: $\quad X_n \xrightarrow{\text{p.}} X.$

Remark:

- If X is a real-valued RV, then the conditions in 2, 3 hold trivially

Convergence in p-Norm $(p \geq 1)$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in p-Norm)

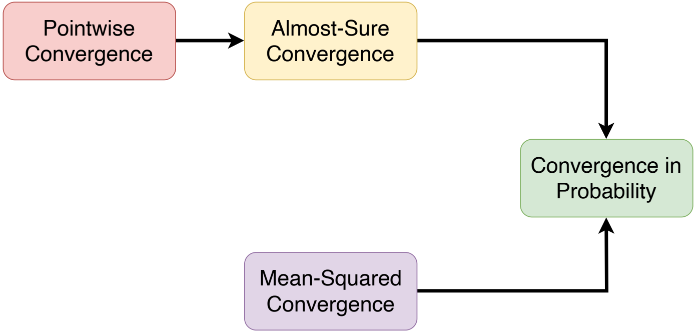
Fix $p \geq 1$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV. We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to X **in p-norm** if

1. We have
$$\mathbb{E}\left[\left|X_n\right|^p\right] < +\infty \quad \text{for all } n \in \mathbb{N}, \qquad \mathbb{E}\left[\left|X\right|^p\right] < +\infty.$$
2. We have
$$\mathbb{E}\left[\left|X_n - X\right|^p\right] \xrightarrow{n\to\infty} 0;$$
Notation: $X_n \xrightarrow{\mathcal{L}^p} X$.

Remarks:

- If $p = 2$, then the convergence in 2-norm is called **mean-squared convergence (m.s.)** and denoted
$$X_n \xrightarrow{\text{m.s.}} X$$
- $$\mathbb{E}\left[\left|X\right|^p\right] < +\infty \implies \mathbb{P}(|X|^p < +\infty) = 1 \implies \mathbb{P}(X \in \mathbb{R}) = 1.$$

A Picture to Have in Mind



(proof of the implications to come later)

Example

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \text{Unif})$.
Fix a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$.
For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} a_n, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise.} \end{cases}$$

- If $a_n = n$, does the sequence admit pointwise/almost-sure/mean-squared limits?
- Identify the conditions on $\{a_n\}_{n\in\mathbb{N}}$ for which the above sequence admits mean-squared limit

Some Remarks on the Previous Example

- If $a_n = n$, then

$$X_n \xrightarrow{\text{a.s.}} 0, \quad \text{but} \quad X_n \overset{\text{m.s.}}{\not\to} 0.$$

How can we understand this? What is going on?

- Although $X_n \xrightarrow{\text{a.s.}} 0$, the value of $|X_n(\omega) - 0|$ can be arbitrarily large for some ω. This can lead to a large value for $\mathbb{E}[(X_n - X)^2]$.

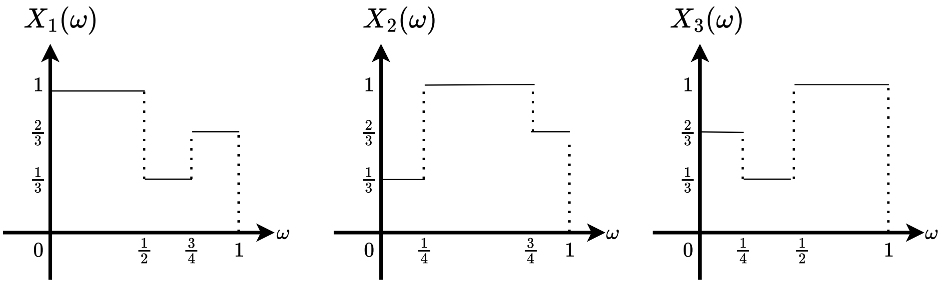
Almost-Sure Convergence and Mean-Squared Convergence

In general,

$$X_n \xrightarrow{\text{a.s.}} 0 \quad \not\Longrightarrow \quad X_n \xrightarrow{\text{m.s.}} 0, \qquad X_n \xrightarrow{\text{m.s.}} 0 \quad \not\Longrightarrow \quad X_n \xrightarrow{\text{a.s.}} 0.$$

Example

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.



Let $X_n = X_{n+3}$ for all $n \in \mathbb{N}$.
Identify forms of convergence and their corresponding limits.

No Convergence in Probability for Previous Example

$$X_n \xrightarrow{\text{p.}} X \text{ (real-valued)} \implies \forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(\{|X_n - X| > \varepsilon\}) = 0$$

$$\overset{\text{periodicity}}{\implies} \forall \varepsilon > 0, \quad \mathbb{P}(\{|X_n - X| > \varepsilon\}) = 0 \quad \text{for} \quad n \in \{1, 2, 3\}$$

Taking X_1:

$$\begin{aligned}
\forall \varepsilon > 0, \quad \mathbb{P}(\{|X_1 - X| > \varepsilon\}) = 0 \quad &\implies \quad \forall q \in \mathbb{Q},\ q > 0, \quad \mathbb{P}(\{|X_1 - X| > q\}) = 0 \\
&\implies \quad \mathbb{P}\left(\bigcap_{\substack{q\in\mathbb{Q}\\ q>0}} \{|X_1 - X| > q\}\right) = 0 \\
&\implies \quad \mathbb{P}(|X_1 - X| > 0) = 0 \\
&\implies \quad \mathbb{P}(X_1 = X) = 1
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X_1 = X) = 1, \quad \mathbb{P}(X_2 = X) = 1, \quad \mathbb{P}(X_3 = X) = 1 \quad &\implies \quad \mathbb{P}(X_1 = X_2 = X_3 = X) = 1 \\
&\implies \quad \mathbb{P}(X_1 = X_2 = X_3) = 1 \quad \text{(contradiction!)}
\end{aligned}$$

Remarks on Previous Example

- The sequence of RVs do not converge pointwise, almost-surely, in mean-squared sense, or in probability
- However, the PMFs (hence CDFs) of X_1, X_2, X_3 are identical, hence there is **convergence of CDFs**
- **A subtle point on convergence of CDFs:**
  Let $U \sim \text{Unif}[0,1]$. For each $n \in \mathbb{N}$, let
  $$X_n = \frac{(-1)^n U}{n}.$$
  - Guess a limit RV.
  - Identify forms of convergence to the above limit.
  - Comment about the convergence of the sequence of CDFs $\{F_{X_n}\}_{n \in \mathbb{N}}$.

Convergence in Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in Distribution)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n\in\mathbb{N}}$ converges to X **in distribution (d.)** if:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \qquad \forall x \in C_{F_X},$$

where C_{F_X} denotes the **points of continuity** of F_X.

Notation: $X_n \xrightarrow{\text{d.}} X$.

Remarks:

- We assume that the CDFs of all random variables functions from $\mathbb{R} \cup \{\pm\infty\}$ to $[0, 1]$
- If $F : \mathbb{R} \cup \{\pm\infty\} \to [0, 1]$ is a valid CDF, then we define

$$F(+\infty) := \lim_{x\to+\infty} F(x), \qquad F(-\infty) := \lim_{x\to-\infty} F(x).$$

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Convergence – The Full Picture

$$A_{\text{lim}} = \left\{ \omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega) \right\}$$