1. On the one hand, we have (using the Markov property)

$$\mathbb{P}(X_0 = x_0, \dots, X_n = x_n) = \mathbb{P}(X_0 = x_0, X_1 = x_1) \cdot \prod_{i=2}^{n} \mathbb{P}(X_i = x_i \mid X_{i-1} = x_{i-1}),$$

whereas on the other hand, we have

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdot \prod_{i=2}^{n} \mathbb{P}(X_i = x_i \mid X_{i-1} = x_{i-1}).$$

Dividing the two equations, we arrive at the desired answer.

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2. We note that for any $n \in \mathbb{N}$ and $x_0, \dots, x_n \in \mathcal{X}$,

$$\mathbb{P}(Y_n = x_n \mid Y_{n-1} = x_{n-1}, \dots, Y_0 = x_0) = \mathbb{P}(X_{kn} = x_n \mid X_{kn-k} = x_{n-1}, \dots, X_0 = x_0)$$

$$= \mathbb{P}(X_{kn} = x_n \mid X_{kn-k} = x_{n-1})$$

$$= \mathbb{P}(Y_n = x_n \mid Y_{n-1} = x_{n-1}).$$

Thus, $\{Y_n\}_{n=0}^{\infty}$ is a DTMC. Furthermore,

$$\mathbb{P}(Y_n = y \mid Y_{n-1} = x) = \mathbb{P}(X_{kn} = y \mid X_{kn-k} = x)$$

$$= \mathbb{P}(X_k = y \mid X_0 = x)$$

$$= P^k_{x,y} \qquad \forall\, n \in \mathbb{N},\ x, y \in \mathcal{X},$$

thus proving that $\{Y_n\}_{n=0}^{\infty}$ is time-homogeneous with TPM P^k.

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3. We note that

a)
$$P^2 = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

$$= \begin{pmatrix} (1-p)^2 + pq & p(2-p-q) \\ q(2-p-q) & (1-q)^2 + pq \end{pmatrix}.$$

Now,

$$\mathbb{P}(X_1 = 0 \mid X_0 = 0, X_2 = 0) = \frac{\mathbb{P}(X_0 = 0, X_1 = 0, X_2 = 0)}{\mathbb{P}(X_0 = 0, X_2 = 0)}$$

$$= \frac{\mathbb{P}(X_0=0)\cdot\mathbb{P}(X_1=0\mid X_0=0)\cdot\overbrace{\mathbb{P}(X_2=0\mid X_1=0)}^{\text{same as } \mathbb{P}(X_1=0\mid X_0=0)}}{\mathbb{P}(X_0=0)\cdot\mathbb{P}(X_2=0\mid X_0=0)}$$

$$= \frac{\frac{1}{2}\cdot(1-p)^2}{\frac{1}{2}\cdot\left((1-p)^2+pq\right)} = \frac{(1-p)^2}{(1-p)^2+pq}.$$

b) By the law of total probability,

$$\mathbb{P}(X_1\neq X_2) = \mathbb{P}(X_0=0,\ X_1\neq X_2) + \mathbb{P}(X_0=1,\ X_1\neq X_2)$$

$$= \mathbb{P}(X_0=0, X_1=0, X_2=1) + \mathbb{P}(X_0=0, X_1=1, X_2=0)$$

$$+ \mathbb{P}(X_0=1, X_1=0, X_2=1) + \mathbb{P}(X_0=1, X_1=1, X_2=0)$$

$$= \tfrac{1}{2}\cdot(1-p)\cdot p + \tfrac{1}{2}\cdot p\cdot q + \tfrac{1}{2}\cdot q\cdot p + \tfrac{1}{2}\cdot(1-q)\,q$$

$$= \tfrac{1}{2}\left(p(1-p)+q(1-q)+2pq\right).$$

c) In this case,

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix},$$

$$P^4 = P^2\cdot P^2 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}.$$

Then,

$$\mathbb{P}(X_{n+4}=0\mid X_n=0) = \mathbb{P}(X_4=0\mid X_0=0) = P^4_{0,0} = 0.5749.$$

---

4. Note that we may express X_n as

a) $$X_n = A_n + B_{n-1}, \quad \text{where}$$

$A_n \rightarrow$ no. of families who arrive / check-in to the hotel on day n,

$B_{n-1} \rightarrow$ no. of families who continue to remain from previous day

Note that $A_n \perp\!\!\!\perp B_{n-1}$. Therefore, $\{X_n\}_{n=0}^{\infty}$ is a DTMC.

Furthermore, we note that for all $n\in\mathbb{N}$,

$$B_{n-1} \mid X_{n-1} \sim \text{Bin}(X_{n-1}, 1-p),$$

$$A_n \sim \text{Poisson}(\lambda), \quad A_n \perp\!\!\!\perp B_{n-1}, \quad \{A_n\}_{n=0}^{\infty} \text{ iid.}$$

Thus, $\mathbb{P}(X_n = y \mid X_{n-1} = x) = \mathbb{P}(X_{n-1} = y \mid X_{n-2} = x)$,

thereby showing that $\{X_n\}_{n=0}^{\infty}$ is a time-homogeneous DTMC.

b) We have

Conditioned on $X_{n-1} = x$, its range is $\{0, \dots, x\}$

$$\mathbb{P}(X_n = y \mid X_{n-1} = x) = \mathbb{P}(A_n + B_{n-1} = y \mid X_{n-1} = x)$$

$$= \sum_{k=0}^{\min\{x,y\}} \mathbb{P}(B_{n-1} = k,\ A_n = y-k \mid X_{n-1} = x)$$

$$= \sum_{k=0}^{\min\{x,y\}} \mathbb{P}(B_{n-1} = k \mid X_{n-1} = x) \cdot \mathbb{P}(A_n = y-k)$$

$$= \sum_{k=0}^{\min\{x,y\}} {}^{x}C_k \cdot (1-p)^k \cdot p^{x-k} \cdot e^{-\lambda} \cdot \frac{\lambda^{y-k}}{(y-k)!}.$$

---

5. We have

$$\mathbb{P}\left(\tau_y^{(1)} < +\infty,\ X_{\tau_y^{(1)}+n} = y \mid X_0 = y\right) = \sum_{k \in \mathbb{N}} \mathbb{P}\left(\tau_y^{(1)} = k,\ X_{k+n} = y \mid X_0 = y\right)$$

$$= \sum_{k \in \mathbb{N}} \mathbb{P}\left(\tau_y^{(1)} = k \mid X_0 = y\right) \cdot \mathbb{P}\left(X_{k+n} = y \mid \overbrace{\tau_y^{(1)} = k}^{\text{function of } X_0, \dots, X_k \text{ only}},\ X_0 = y\right)$$

$$= \sum_{k \in \mathbb{N}} \mathbb{P}\left(\tau_y^{(1)} = k\right) \cdot \mathbb{P}\left(X_{k+n} = y \mid \underbrace{X_k = y,\ \tau_y^{(1)} = k,\ X_0 = y}\right)$$

$\{\tau_y^{(1)} = k\} \subseteq \{X_k = y\}$, so

$\{X_k = y\} \cap \{\tau_y^{(1)} = k\} \cap \{X_0 = k\}$

$= \{\tau_y^{(1)} = k\} \cap \{X_0 = k\}$

$$= \sum_{k \in \mathbb{N}} \mathbb{P}\left(\tau_y^{(1)} = k\right) \cdot \mathbb{P}\left(X_{k+n} = y \mid X_k = y\right)$$

$$= \sum_{k \in \mathbb{N}} \mathbb{P}\left(\tau_y^{(1)} = k\right) \cdot P^n_{y,y}$$

$$= \mathbb{P}\left(\tau_y^{(1)} < +\infty\right) \cdot P^n_{y,y},$$

from which it follows that

$$\mathbb{P}\left(X_{\tau_y^{(1)}+n} = y \,\middle|\, X_0 = y,\ \tau_y^{(1)} < +\infty\right) = P^n_{y,y}.$$

6.

a) We have

$$\lambda x_{i^*} = \sum_i x_i P_{i^*,i}$$

$$= x_{i^*} \cdot P_{i^*,i^*} + \sum_{i \neq i^*} x_i P_{i^*,i}$$

from which we have

$$|\lambda - P_{i^*,i^*}| = \frac{\left|\sum_{i\neq i^*} x_i P_{i^*,i}\right|}{|x_{i^*}|}$$

triangle inequality

$$\leq \sum_{i\neq i^*} \frac{|x_i|}{|x_{i^*}|} \cdot P_{i^*,i}$$

$$\leq \sum_{i\neq i^*} P_{i^*,i} = R_{i^*}.$$

$\frac{|x_i|}{|x_{i^*}|} \leq 1$

b) For any i, we have by triangle inequality

$$|\lambda| = |\lambda - P_{i^*,i^*} + P_{i^*,i^*}|$$

$$\leq |\lambda - P_{i^*,i^*}| + P_{i^*,i^*}$$

$$\leq R_{i^*} + P_{i^*,i^*}$$

$$= \sum_{i\neq i^*} P_{i^*,i} + P_{i^*,i^*}$$

$$= \sum_i P_{i^*,i} = 1.$$

c) Let x be eigenvector corresponding to $\lambda = 1$.
Then, by triangle inequality,

$$|\lambda x_i| = \left|\sum_j P_{i,j}\, x_j\right|$$

$$\overset{①}{\leq} \sum_j P_{i,j} \, |x_j|$$

$$\overset{②}{\leq} \sum_j P_{i,j} \cdot \max_k |x_k| \cdot = \max_k |x_k| \cdot$$

If P has strictly positive entries, the only case when ① and ② hold with equality are when $\underline{x} = [x_1 \dots x_d]$ has identical entries.

That is to say that any eigenvector $\underline{x}$ s.t. $\underline{x} = P\underline{x}$ must have all identical entries, and any $\underline{x}$ having identical entries must satisfy $\underline{x} = P\underline{x}$.

d) Conversely, suppose $\lambda \neq 1$ has an associated eigenvector $\underline{y}$. Then,

$$\forall i, \quad |\lambda y_i| < \max_k |y_k| .$$

$$\Rightarrow |\lambda| \cdot \max_i |y_i| < \max_k |y_k| \Rightarrow |\lambda| < 1 .$$

A general remark on when triangle inequality holds with equality

Triangle inequality, in its simplest form, may be expressed as

$$|x+y| \leq |x| + |y| \quad \text{for all } x, y \in \mathbb{R} .$$

Now, suppose equality holds in the above inequality, i.e.,

$$|x+y| = |x| + |y| .$$

$$\Leftrightarrow |x+y|^2 = (|x| + |y|)^2$$

$$\Leftrightarrow x^2 + y^2 + 2xy = x^2 + y^2 + 2|xy|$$

$$\Leftrightarrow xy = |xy|$$

$$\Leftrightarrow x \text{ and } y \text{ have same sign } (x \geq 0, y \geq 0 \text{ OR } x \leq 0, y \leq 0)$$

Applying this to question 6(c) above,

$$\left| \sum_j P_{i,j} x_j \right| = \sum_j P_{i,j} |x_j| \quad \text{if and only if}$$

$$\forall j, \; P_{i,j} x_j \text{ has same sign as } \sum_{k \neq j} P_{i,k} \cdot x_k$$

$$\Leftrightarrow x_1, \dots, x_d \; \underline{\text{have the same sign}} \; (\because P_{i,j} \geq 0 \; \forall i, j)$$