1.

(a) For any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\left|L_n - 0\right| > \varepsilon\right) = \mathbb{P}\left(\left|\frac{V}{n} + \frac{Y_n}{\sqrt{n}}\right| > \varepsilon\right)$$

$$\leq \mathbb{P}\left(\frac{|V|}{n} + \frac{|Y_n|}{\sqrt{n}} > \varepsilon\right)$$ (because $|a+b| \leq |a| + |b|$ by Δ^{le} ineq.)

$$\leq \mathbb{P}\left(\frac{|V|}{n} > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\frac{|Y_n|}{\sqrt{n}} > \varepsilon/2\right) \longrightarrow (1)$$

Now,

$$\mathbb{P}\left(\frac{|V|}{n} > \frac{\varepsilon}{2}\right) = \mathbb{P}\left(|V| > n\varepsilon/2\right)$$

$$= \mathbb{P}\left(V^2 > n^2\varepsilon^2/4\right)$$

$$\leq \frac{\mathbb{E}[V^2]}{n^2\varepsilon^2/4}.$$

($\mathbb{E}[V^2] < +\infty$ because $\mathrm{Var}(V) < +\infty$)

On the other hand,

$$\mathbb{P}\left(|Y_n| > \sqrt{n}\ \varepsilon/2\right) = \mathbb{P}\left(Y_n > \sqrt{n}\cdot\varepsilon/2\right) + \mathbb{P}\left(Y_n < -\sqrt{n}\cdot\varepsilon/2\right)$$

$$= 2\,\mathbb{P}\left(Y_n > \sqrt{n}\cdot\varepsilon/2\right)$$ (because PDF of Y_n is symmetric about the origin)

$$\leq 2 \inf_{t>0} \frac{\mathbb{E}\left[e^{tY_n}\right]}{e^{t\cdot\sqrt{n}\cdot\varepsilon/2}}$$

Chernoff bound

$$= 2\cdot \inf_{t>0} \exp\left(-\sqrt{n}\cdot t\cdot \varepsilon/2 + \frac{t^2}{2}\right)$$

$$= 2\cdot \exp\left(-\frac{n\varepsilon^2}{8}\right).$$

Therefore, we get

$$\mathbb{P}\left(|L_n - 0| > \varepsilon\right) \leq \frac{\mathbb{E}[V^2]}{n^2\frac{\varepsilon^2}{4}} + 2e^{-n\varepsilon^2/8} \quad \forall\, n \in \mathbb{N},\ \forall\, \varepsilon > 0$$

$$\Rightarrow \sum_{n\in\mathbb{N}} \mathbb{P}\left(|L_n - 0| > \varepsilon\right) < +\infty \quad \forall\, \varepsilon > 0$$

$$\Rightarrow \mathbb{P}\left(\{|L_n - 0| > \varepsilon\}\ \text{i.o.}\right) = 0 \quad \forall\, \varepsilon > 0$$

Borel – Cantelli Lemma

$$\Rightarrow L_n \xrightarrow{a.s.} 0.$$

b) We have

$$C_{M_n}(s) = \mathbb{E}\left[e^{jsM_n}\right]$$
$$= \mathbb{E}\left[e^{js\left(\frac{V}{\sqrt{n}} + Y_n\right)}\right]$$
$$= \mathbb{E}\left[e^{j\frac{s}{\sqrt{n}}V}\right] \cdot \mathbb{E}\left[e^{jsY_n}\right] \quad \text{(because } V \perp\!\!\!\perp Y_n \ \forall\, n \in \mathbb{N}\text{)}$$
$$= \left(1 + j\frac{s}{\sqrt{n}} \cdot \mathbb{E}[V] - \frac{s^2}{n}\mathbb{E}[V^2] + o\left(\frac{s^2}{n}\right)\right) \cdot e^{-s^2/2}$$

Taylor's series expansion for $C_V(s/\sqrt{n})$, noting that $E[V^2] < +\infty$

$$\Rightarrow \lim_{n\to\infty} C_{M_n}(s) = e^{-s^2/2} \qquad \forall\, s \in \mathbb{R}$$

$$\Rightarrow \quad M_n \xrightarrow[d]{} M, \qquad M \sim N(0,1).$$

2.

<u>If part</u>: Suppose $\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] \xrightarrow[n\to\infty]{} 0$. Then, for any $\varepsilon > 0$,

$$\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{E}\left[\mathbb{1}_{\{|X_n| > \varepsilon\}}\right]$$

$$= \mathbb{E}\Big[\underbrace{\frac{1}{1+|X_n|}}_{\le \frac{1}{1+\varepsilon} \text{ whenever } |X_n|>\varepsilon} \mathbb{1}_{\{|X_n|>\varepsilon\}}\Big] + \mathbb{E}\Big[\frac{|X_n|}{1+|X_n|} \underbrace{\mathbb{1}_{\{|X_n|>\varepsilon\}}}_{\le 1}\Big]$$

$$\le \frac{1}{1+\varepsilon} \cdot \underbrace{\mathbb{E}\left[\mathbb{1}_{\{|X_n|>\varepsilon\}}\right]}_{=\mathbb{P}(|X_n|>\varepsilon)} + \mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right]$$

$$\Rightarrow \left(\frac{\varepsilon}{1+\varepsilon}\right) \mathbb{P}(|X_n| > \varepsilon) \le \mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right].$$

$$\Rightarrow \left(\frac{\varepsilon}{1+\varepsilon}\right) \lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) \le \lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) = 0.$$

Because the above holds for any $\varepsilon > 0$, it follows that $X_n \xrightarrow{P} 0$.

Only if part: Suppose $X_n \xrightarrow{p.} 0$, i.e., $\mathbb{P}(|X_n| > \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0.$

Then, for any $\varepsilon > 0$,

$$\mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] = \mathbb{E}\left[\underbrace{\frac{|X_n|}{1+|X_n|}}_{\leq 1} \cdot \mathbb{1}_{\{|X_n| > \varepsilon\}}\right] + \mathbb{E}\left[\underbrace{\frac{|X_n|}{1+|X_n|}}_{\leq \varepsilon \text{ whenever } |X_n| \leq \varepsilon} \mathbb{1}_{\{|X_n| \leq \varepsilon\}}\right]$$

$$\leq \mathbb{E}\left[\mathbb{1}_{\{|X_n| > \varepsilon\}}\right] + \varepsilon \underbrace{\mathbb{E}\left[\mathbb{1}_{\{|X_n| \geq \varepsilon\}}\right]}_{\leq 1}$$

$$= \mathbb{P}(|X_n| > \varepsilon) + \varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] \leq \lim_{n\to\infty} \mathbb{P}(|X_n| > \varepsilon) + \varepsilon$$

$$= \varepsilon$$

Because the above holds for any $\varepsilon > 0$, it follows that

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n|}{1+|X_n|}\right] = 0.$$

3.

a) Note that $\{X_n\}_{n\in\mathbb{N}}$ is an i.i.d. sequence of RVs with PMF

$$\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = -1).$$

Suppose $p \neq 1/2$. Then, by SLLN,

$$\frac{S_n}{n} \xrightarrow[a.s.]{} \mathbb{E}[X_1] = 2p - 1 \neq 0.$$

Without loss of generality, suppose that $p > 1/2$. Then, and

$$\frac{S_n}{n} \xrightarrow[a.s.]{} 2p - 1 > 0.$$

$$\Rightarrow S_n \xrightarrow[a.s.]{} +\infty$$

$$\Rightarrow \mathbb{P}(\{S_n = 0\} \text{ i.o.}) = 0.$$

b) Note that $\mathbb{P}(A_n^c) = \frac{1}{2^n}$, and $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n^c) < +\infty.$

By the Borel-Cantelli lemma,

$$\mathbb{P}(A_n^c \text{ i.o.}) = 0$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c\right) = 1$$

$$\Rightarrow \mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 1 - \mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c\right)$$

$$= 1 - 0$$

$$= 1.$$

4.

(a) Given: Y_n is a random variable w.r.t. $\mathcal{F}_n$, for each $n \in \mathbb{N}$.

$$\Rightarrow Y_n^{-1}((-\infty, x]) \in \mathcal{F}_n \quad \forall x \in \mathbb{R}, \text{ for each } n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$Y_N^{-1}((-\infty, x]) \cap \{N = n\} = \{\omega \in \Omega : Y_N(\omega) \le x\} \cap \{N = n\}$$

$$= \underbrace{\{\omega \in \Omega : Y_n(\omega) \le x\}}_{\in \mathcal{F}_n} \cap \underbrace{\{N = n\}}_{\in \mathcal{F}_n \text{ because } N \text{ is a stopping time}}$$

$$\in \mathcal{F}_n$$

$$\Rightarrow Y_N^{-1}((-\infty, x]) \in \mathcal{F}_N \quad \forall x \in \mathbb{R}$$

$$\Rightarrow Y_N \text{ is a RV w.r.t. } \mathcal{F}_N.$$

b) Notice that

$$\{N > n\} = \bigcap_{i=1}^{n} \{Y_i \notin \{1,3,5,7\}\} \in \sigma(Y_1, \dots, Y_n) \quad \forall n \in \mathbb{N}.$$

Furthermore,

$$\mathbb{P}(N > n) = \left[\mathbb{P}(Y_1 \notin \{1,3,5,7\})\right]^n$$

Y_i's are iid

$$= \left(\frac{6}{10}\right)^n$$

$$\Rightarrow \lim_{n \to \infty} \mathbb{P}(N > n) = 0 \Rightarrow \mathbb{P}(N = +\infty) = \lim_{n \to \infty} \mathbb{P}(N > n) = 0$$

$$\Rightarrow \mathbb{P}(N < +\infty) = 1.$$

Thus,

$$\{N > n\}^c = \{N \le n\} \in \sigma(Y_1, \dots, Y_n) \quad \forall\, n \in \mathbb{N}, \text{ and}$$

$$\mathbb{P}(N < +\infty) = 1.$$

$$\Rightarrow N \text{ is a Stopping time w.r.t. } \{Y_n\}_{n \in \mathbb{N}}.$$

Lastly,

$$E[N] = \sum_{n \in \mathbb{N}} n \cdot \mathbb{P}(N = n) + (+\infty)\, \mathbb{P}(N = +\infty)$$

$$= \sum_{n \in \mathbb{N}} n \cdot \mathbb{P}(N = n)$$

$$= \sum_{n \in \mathbb{N} \cup \{0\}} \mathbb{P}(N > n)$$

$$= \sum_{n \ge 0} (6/10)^n = \frac{10}{4} = 2.5.$$

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5. The solution to this problem follows along the lines of the solution to question 3 of homework 3, with $K = 100$.

The desired quantity is $E[T_{100}]$, and we get

$$E[T_{100}] = \sum_{k=0}^{K-1} \frac{K}{K-k} = \sum_{k=0}^{99} \frac{100}{100-k}.$$

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6. Given: $X_t = 2A + Bt$, $t \in \mathbb{R}$, with $A, B \overset{iid}{\sim} \text{Unif}\{-1, +1\}$.

Notice that

$$X_t \sim \text{Unif}\{-2-t,\ -2+t,\ 2-t,\ 2+t\}.$$

OR EQUIVALENTLY

$$X_t \sim \text{Unif}\{-(t+2),\ t-2,\ -(t-2),\ t+2\}.$$

a) For any $t \in \mathbb{R} \setminus \{\pm 2\}$, two of the values of X_t will be > 0, and two of the values of X_t will be < 0.

Thus, $\mathbb{P}(X_t \ge 0) = 1/2 \quad \forall\, t \in \mathbb{R} \setminus \{2\}.$

$P(X_2=x) \to 1/4 \quad 1/2 \quad 1/4$

For $t=\pm 2$, we have $X_2 \in \{-4, 0, 4\}$, and therefore

$$\mathbb{P}(X_2 \geq 0) = 1/2 + 1/4 = 3/4.$$

Thus,

$$\mathbb{P}(X_t \geq 0) = \begin{cases} 1/2, & t \neq \pm 2 \\ 3/4, & t \in \{-2, 2\}. \end{cases}$$

b) $$\mathbb{P}(X_t \geq 0 \quad \forall t \in \mathbb{R}) = \mathbb{P}(2A + Bt \geq 0 \quad \forall t \in \mathbb{R})$$

$$= \mathbb{P}(A=-1,\ B=-1,\ -2-t \geq 0 \quad \forall t \in \mathbb{R})$$

$$+ \mathbb{P}(A=-1,\ B=1,\ -2+t \geq 0 \quad \forall t \in \mathbb{R})$$

$$+ \mathbb{P}(A=1,\ B=-1,\ 2-t \geq 0 \quad \forall t \in \mathbb{R})$$

$$+ \mathbb{P}(A=1,\ B=1,\ 2+t \geq 0 \quad \forall t \in \mathbb{R})$$

$$= 0.$$

7.

a) $$\mathbb{P}(N(t)=n) = \mathbb{P}(T_n \leq t,\ T_{n+1} > t)$$

$$= \int_0^t \int_t^\infty f_{T_n, T_{n+1}}(s_1, s_2)\, ds_2\, ds_1$$

$$= \int_0^t \int_t^\infty f_{T_1, X_{n+1}}(s_1, s_2 - s_1)\, ds_2\, ds_1$$

$X_{n+1} \perp\!\!\!\perp T_n \leftarrow$

$$= \int_0^t \int_t^\infty f_{T_1}(s_1) \cdot f_{X_{n+1}}(s_2 - s_1)\, ds_2\, ds_1$$

$$= \int_0^t f_{T_1}(s_1) \cdot \left(\int_t^\infty \lambda e^{-\lambda(s_2 - s_1)}\, ds_2 \right) ds_1$$

$$= \int_0^t f_{T_1}(s_1) \cdot e^{-\lambda t} \cdot e^{\lambda s_1}\, ds_1$$

$$= \int_0^t e^{-\lambda t} \cdot \lambda^n \frac{s_1^{n-1}}{(n-1)!}\, ds_1$$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \quad \forall\ n \in \mathbb{N} \cup \{0\}.$$

$$\Rightarrow \quad N(t) \sim \text{Poisson}(\lambda t).$$

b) It is easy to show that

$$M_{N(t)}(s) = e^{\lambda t(e^s - 1)}, \quad s \in \mathbb{R}.$$

c) Notice that $Y_m \sim \text{Poisson}(m\lambda)$, $m \in \mathbb{N}$.

Thus, $\mathbb{E}\left[\frac{Y_m}{m}\right] = \lambda$, and

$$\mathbb{E}\left[\left(\frac{Y_m}{m} - \lambda\right)^2\right] = \text{Var}\left(\frac{Y_m}{m}\right) = \frac{1}{m^2}\,\text{Var}(Y_m) = \frac{m\lambda}{m^2} = \frac{\lambda}{m}$$

$$\Rightarrow \quad \mathbb{E}\left[\left(\frac{Y_m}{m} - \lambda\right)^2\right] \xrightarrow[m \to \infty]{} 0$$

$$\Rightarrow \quad \frac{Y_m}{m} \xrightarrow[m.s.]{} \lambda$$

d) Using Chernoff bound, for any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\frac{Y_m}{m} > \lambda + \varepsilon\right) = \mathbb{P}\left(Y_m > m(\lambda + \varepsilon)\right)$$

$$\leq \inf_{t>0} \frac{\mathbb{E}\left[e^{tY_m}\right]}{e^{tm(\lambda+\varepsilon)}}$$

$$= \inf_{t>0} \frac{e^{m\lambda(e^t - 1)}}{e^{mt(\lambda+\varepsilon)}}$$

$$= \exp\left(-m \cdot \sup_{t>0} \left\{\lambda(e^t - 1) - t(\lambda+\varepsilon)\right\}\right)$$

$$= \exp\left(-m\left\{(\lambda+\varepsilon)\log\frac{\lambda+\varepsilon}{\lambda} - (\lambda+\varepsilon) + \lambda\right\}\right) \rightarrow 1$$

e) Because $\frac{Y_m}{m}$ is a non-negative random variable,

$$\mathbb{P}\left(\frac{Y_m}{m} < \lambda - \varepsilon\right) = 0 \quad \forall \ \varepsilon \geq \lambda. \rightarrow 2$$

For any $0 < \varepsilon < \lambda$,

$$\mathbb{P}\left(\frac{Y_m}{m} < \lambda - \varepsilon\right) = \mathbb{P}\left(Y_m < m(\lambda-\varepsilon)\right)$$

$$= \mathbb{P}\left(tY_m > tm(\lambda-\varepsilon)\right) \qquad \forall\ t<0$$

$$= \mathbb{P}\left(e^{tY_m} > e^{tm(\lambda-\varepsilon)}\right) \qquad \forall\ t<0$$

Chernoff bound

$$\leq \inf_{t<0} \frac{\mathbb{E}\left[e^{tY_m}\right]}{e^{tm(\lambda-\varepsilon)}}$$

$$= \exp\left(-m \cdot \sup_{t<0}\left\{\lambda(e^t-1) - t(\lambda-\varepsilon)\right\}\right)$$

$$= \exp\left(-m \cdot \left\{(\lambda-\varepsilon)\log\left(\frac{\lambda-\varepsilon}{\lambda}\right) - (\lambda-\varepsilon) + \lambda\right\}\right) \to 3$$

f) From 1, 2, and 3,

$$\sum_{m\in\mathbb{N}} \mathbb{P}\left(\left|\frac{Y_m}{m} - \lambda\right| > \varepsilon\right) < +\infty \qquad \forall\ \varepsilon>0$$

Borel–Cantelli lemma

$$\Rightarrow \mathbb{P}\left(\left\{\left|\frac{Y_m}{m} - \lambda\right| > \varepsilon\right\} \text{ i.o.}\right) \qquad \forall\ \varepsilon>0$$

$$\Rightarrow \frac{Y_m}{m} \xrightarrow[a.s.]{} \lambda.$$