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# Probability and Stochastic Processes

Lecture 23: Expectations of Simple, Non-Negative, and Arbitrary Random Variables, Properties of Expectations

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# Expectation of a Simple Random Variable

## Definition (Expectation of a Simple RV)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable.
Suppose that $X$ is **simple** with the canonical representation

$$X = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}, \qquad a_1, \ldots, a_n \geq 0 \text{ **distinct**}, \quad A_1, \ldots, A_n \in \mathscr{F} \text{ **disjoint**}.$$

Then, the **expectation of $X$**, denoted $\mathbb{E}[X]$, is defined as

$$\mathbb{E}[X] = \int_{\Omega} X \, \mathrm{d}\mathbb{P} := \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i).$$

- For any $A \in \mathscr{F}$,

$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$

- The expectation of a simple RV is a non-negative and finite real number

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# Expectations of Non-Negative Random Variables

- Suppose $X$ is **any non-negative random variable** (not necessarily simple)
- For each $n \in \mathbb{N}$, define $X_n$ as

$$X_n = \sum_{\ell=0}^{n\,2^n-1} \frac{\ell}{2^n} \mathbf{1}_{\left\{\frac{\ell}{2^n} \le X < \frac{\ell+1}{2^n}\right\}} + n\,\mathbf{1}_{\{X \ge n\}}, \qquad X_n(\omega) = \begin{cases} 0, & 0 \le X(\omega) < \frac{1}{2^n}, \\ \frac{1}{2^n}, & \frac{1}{2^n} \le X(\omega) < \frac{2}{2^n}, \\ \vdots & \\ \frac{n\,2^n-1}{2^n}, & \frac{n\,2^n-1}{2^n} \le X(\omega) < n, \\ n, & X(\omega) \ge n. \end{cases}$$

- $X_n$ is a **simple** random variable for each $n \in \mathbb{N}$ $\mathbb{E}[X_n]$ is well-defined for each $n \in \mathbb{N}$
- $X_n$ can be represented compactly as

$$X_n = \frac{\lfloor 2^n X \rfloor}{2^n} \mathbf{1}_{\{X<n\}} + n\,\mathbf{1}_{\{X \ge n\}}$$

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P} := \lim_{n\to\infty} \mathbb{E}[X_n].$$

# Properties of $\{X_n\}$

- $X_n$ is a simple RV for each $n \in \mathbb{N}$
- **Monotonicity:** For each $\omega \in \Omega$, $X_n(\omega) \leq X_{n+1}(\omega) \quad \forall\, n \in \mathbb{N}$.
- **Pointwise convergence:** For each $\omega \in \Omega$, $\lim_{n\to\infty} X_n(\omega) = X(\omega)$.
- **Monotonicity of expectations:** $\mathbb{E}[X_n] \leq \mathbb{E}[X_{n+1}]$ for all $n \in \mathbb{N}$ **(exercise)**
  Therefore, $\lim_{n\to\infty} \mathbb{E}[X_n]$ exists
- $\lim_{n\to\infty} \mathbb{E}[X_n]$ may be equal to $+\infty$

**Remark**

If $X$ is a non-negative random variable, then $0 \leq \mathbb{E}[X] \leq +\infty$.

# Expectation of an Arbitrary Random Variable

- Suppose that $X$ is **any arbitrary random variable** (potentially taking both $+$ve and $-$ve values)
- Define $X_+$ and $X_-$ as

$$\forall \omega \in \Omega, \qquad X_+(\omega) = \max\{X(\omega), 0\}, \qquad X_-(\omega) = -\min\{X(\omega), 0\} = \max\{-X(\omega), 0\}.$$

- $X_+$ represents the **non-negative part** of $X$
  $X_-$ represents the **negative part** of $X$
- **Exercise:** $X_+$ and $X_-$ are random variables. Furthermore, they are **non-negative** RVs.
- $0 \leq \mathbb{E}[X_+] \leq +\infty, \qquad 0 \leq \mathbb{E}[X_-] \leq +\infty$

Suppose $X$ is an arbitrary random variable. Then, the **expectation** of $X$ is defined as

$$\mathbb{E}[X] = \int_\Omega X \, \mathrm{d}\mathbb{P} := \mathbb{E}[X_+] - \mathbb{E}[X_-],$$

**provided both $\mathbb{E}[X_+]$ and $\mathbb{E}[X_-]$ are not $+\infty$ simultaneously.**
If both $\mathbb{E}[X_+]$ and $\mathbb{E}[X_-]$ are equal to $+\infty$, then the expectation of $X$ is **not defined**.

# Properties of Expectations

# Property 1

For any $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, $\int_A X \, \mathrm{d}\mathbb{P} = \mathbb{E}[X \cdot \mathbf{1}_A] = 0.$

**Proof:**

- Suppose $X$ is **simple** with the canonical representation

$$X = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}, \qquad a_1, \ldots, a_n \text{ distinct}, \quad A_1, \ldots, A_n \text{ disjoint}.$$

- Then, $X \, \mathbf{1}_A$ is also a simple random variable, with the canonical representation

$$X \, \mathbf{1}_A = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i} \, \mathbf{1}_A = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i \cap A}.$$

- Taking expectations, we get

$$\mathbb{E}[X \, \mathbf{1}_A] = \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i \cap A) = 0 \qquad\qquad (\mathbb{P}(A_i) = 0)$$

# Property 1

For any $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, $\int_A X \, d\mathbb{P} = \mathbb{E}[X \cdot \mathbf{1}_A] = 0.$

**Proof:**

- Suppose $X$ is **non-negative**
- For each $n \in \mathbb{N}$, define $X_n$ as

$$X_n = \sum_{\ell=0}^{n\,2^n-1} \frac{\ell}{2^n} \mathbf{1}_{\left\{\frac{\ell}{2^n} \le X < \frac{\ell+1}{2^n}\right\}} + n\,\mathbf{1}_{\{X \ge n\}}, \qquad X_n \mathbf{1}_A = \left( \sum_{\ell=0}^{n\,2^n-1} \frac{\ell}{2^n} \mathbf{1}_{\left\{\frac{\ell}{2^n} \le X < \frac{\ell+1}{2^n}\right\}} + n\,\mathbf{1}_{\{X \ge n\}} \right) \mathbf{1}_A.$$

- We have

$$\mathbb{E}[X\,\mathbf{1}_A] = \lim_{n\to\infty} \mathbb{E}[X_n\,\mathbf{1}_A] = \lim_{n\to\infty} 0 = 0.$$

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# Property 1

For any $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, $\int_A X \, \mathrm{d}\mathbb{P} = \mathbb{E}[X \cdot \mathbf{1}_A] = 0.$

**Proof:**

- Suppose $X$ is **arbitrary**
- Define $X_+$ and $X_-$ as

$$X_+ = \max\{X, 0\}, \qquad X_- = -\min\{X, 0\}$$

- Observe that

$$\left(X\,\mathbf{1}_A\right)_+ = X_+\,\mathbf{1}_A, \qquad \left(X\,\mathbf{1}_A\right)_- = X_-\,\mathbf{1}_A$$

- We therefore have

$$\mathbb{E}[X\,\mathbf{1}_A] = \mathbb{E}[(X\,\mathbf{1}_A)_+] - \mathbb{E}[(X\,\mathbf{1}_A)_-] = \mathbb{E}[X_+\,\mathbf{1}_A] - \mathbb{E}[X_-\,\mathbf{1}_A] = 0 - 0 = 0.$$

# Property 2

For any $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 1$, $\int_A X \, \mathrm{d}\mathbb{P} = \mathbb{E}[X \cdot \mathbf{1}_A] = \mathbb{E}[X]$.

**Proof:**

- Suppose $X$ is **simple** with the canonical representation

$$X = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}, \qquad a_1, \ldots, a_n \text{ distinct}, \quad A_1, \ldots, A_n \text{ disjoint}.$$

- Then, $X \, \mathbf{1}_A$ is also a simple random variable, with the canonical representation

$$X \, \mathbf{1}_A = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i} \, \mathbf{1}_A = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i \cap A}.$$

- Taking expectations, we get

$$\mathbb{E}[X \, \mathbf{1}_A] = \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i \cap A) = \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i) = \mathbb{E}[X].$$

# Property 2

For any $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 1$, $\int_A X \,\mathrm{d}\mathbb{P} = \mathbb{E}[X \cdot \mathbf{1}_A] = \mathbb{E}[X]$.

**Proof:**

- Suppose $X$ is **non-negative**
- For each $n \in \mathbb{N}$, define $X_n$ as

$$X_n = \sum_{\ell=0}^{n\,2^n-1} \frac{\ell}{2^n} \mathbf{1}_{\left\{\frac{\ell}{2^n} \le X < \frac{\ell+1}{2^n}\right\}} + n\,\mathbf{1}_{\{X \ge n\}}, \qquad X_n\,\mathbf{1}_A = \left( \sum_{\ell=0}^{n\,2^n-1} \frac{\ell}{2^n} \mathbf{1}_{\left\{\frac{\ell}{2^n} \le X < \frac{\ell+1}{2^n}\right\}} + n\,\mathbf{1}_{\{X \ge n\}} \right) \mathbf{1}_A.$$

- We have

$$\mathbb{E}[X\,\mathbf{1}_A] = \lim_{n\to\infty} \mathbb{E}[X_n\,\mathbf{1}_A] = \lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

# Property 2

For any $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, $\int_A X \, d\mathbb{P} = \mathbb{E}[X \cdot \mathbf{1}_A] = 0.$

**Proof:**

- Suppose $X$ is **arbitrary**
- Define $X_+$ and $X_-$ as

$$X_+ = \max\{X, 0\}, \qquad X_- = -\min\{X, 0\}$$

- Observe that

$$\left(X\,\mathbf{1}_A\right)_+ = X_+\,\mathbf{1}_A, \qquad \left(X\,\mathbf{1}_A\right)_- = X_-\,\mathbf{1}_A$$

- We therefore have

$$\mathbb{E}[X\,\mathbf{1}_A] = \mathbb{E}[(X\,\mathbf{1}_A)_+] - \mathbb{E}[(X\,\mathbf{1}_A)_-] = \mathbb{E}[X_+\,\mathbf{1}_A] - \mathbb{E}[X_-\,\mathbf{1}_A] = \mathbb{E}[X_+] - \mathbb{E}[X_-] = \mathbb{E}[X].$$

# Property 3

If $\mathbb{P}(\{X \geq Y\}) = 1$, then
$$\mathbb{E}[X] \geq \mathbb{E}[Y].$$

**Proof:**

- Define $A = \{X \geq Y\}$. It is given that $\mathbb{P}(A) = 1$.
- We have
$$\mathbb{E}[X] = \mathbb{E}[X\,\mathbf{1}_A] = \int_A X(\omega)\,\mathrm{d}\mathbb{P}(\omega) \geq \int_A Y(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[Y\,\mathbf{1}_A] = \mathbb{E}[Y].$$

# Property 4

If $\mathbb{P}(\{X = Y\}) = 1$, then
$$\mathbb{E}[X] = \mathbb{E}[Y].$$

**Proof:**

- Define $A = \{X = Y\}$. It is given that $\mathbb{P}(A) = 1$.
- We have
$$\mathbb{E}[X] = \mathbb{E}[X\,\mathbf{1}_A] = \int_A X(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_A Y(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[Y\,\mathbf{1}_A] = \mathbb{E}[Y].$$

## Property 5

If $X$ is a **non-negative RV**, then

$$\mathbb{E}[X] = 0 \qquad \Longleftrightarrow \qquad \mathbb{P}(\{X = 0\}) = 1.$$

**Proof:** $(\Longleftarrow)$

- Let $A = \{X = 0\}$, and suppose that $\mathbb{P}(A) = 1$
- Then,

$$\mathbb{E}[X] = \mathbb{E}[X\,\mathbf{1}_A] = \int_{\{X=0\}} X(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_{\{X=0\}} 0\,\mathrm{d}\mathbb{P}(\omega) = 0.$$

## Property 5

If $X$ is a **non-negative RV**, then

$$\mathbb{E}[X] = 0 \qquad \iff \qquad \mathbb{P}(\{X = 0\}) = 1.$$

**Proof:** ($\implies$)

- Suppose $X$ is a **simple** RV with canonical representation

$$X = \sum_{i=1}^{m} a_i \, \mathbf{1}_{A_i}, \qquad a_1, \ldots, a_m \text{ distinct}, \quad A_1, \ldots, A_m \text{ disjoint and partition of } \Omega.$$

  Furthermore, suppose that $\mathbb{E}[X] = 0$
- We must have $\sum_{i=1}^{m} \mathbb{P}(A_i) = 1$
- Then, we have

$$0 = \mathbb{E}[X] = \sum_{i=1}^{m} a_i \, \mathbb{P}(A_i) \qquad \implies \qquad a_i \, \mathbb{P}(A_i) = 0 \;\; \forall i \in \{1, \ldots, m\}.$$

- There exists a unique $i^* \in \{1, \ldots, m\}$ such that $a_{i^*} = 0$ and $A_{i^*} = \{X = a_{i^*}\}$ has probability 1

# Property 5

If $X$ is a **non-negative RV**, then

$$\mathbb{E}[X] = 0 \qquad \Longleftrightarrow \qquad \mathbb{P}(\{X = 0\}) = 1.$$

**Proof:** ($\Longrightarrow$)

- Suppose $X$ is a **non-negative** RV with $\mathbb{E}[X] = 0$
- Let $\{X_n\}$ be the associated quantization sequence, with

$$0 \leq \mathbb{E}[X_1] \leq \mathbb{E}[X_2] \leq \cdots \qquad\qquad \lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X] = \mathbf{0}$$

- This implies that

$$\mathbb{E}[X_n] = 0 \qquad \forall\ n \in \mathbb{N}.$$

- For each $n \in \mathbb{N}$, the set $A_n = \{X_n = 0\}$ has probability 1, and hence $\mathbb{P}(\cap_{n\in\mathbb{N}} A_n) = 1$
- Observe that

$$\forall\, \omega \in \Omega, \quad \lim_{n\to\infty} X_n(\omega) = X(\omega), \qquad \{X = 0\} \supseteq \bigcap_{n\in\mathbb{N}} \{X_n = 0\}, \qquad \mathbb{P}(\{X = 0\}) \geq \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = 1.$$

# Property 6 (Linearity of Expectations)

For any two random variables $X, Y$,

$$\mathbb{E}[\alpha X] = \alpha\,\mathbb{E}[X] \quad \forall \alpha \in \mathbb{R}, \qquad\qquad \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y],$$

whenever the right-hand sides are well-defined

**Proof:**

- For any $\alpha \in \mathbb{R}$,

$$\mathbb{E}[\alpha X] = \int_\Omega \alpha X(\omega)\, \mathrm{d}\mathbb{P}(\omega) = \alpha \int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega) = \alpha\,\mathbb{E}[X].$$

# Property 6 (Linearity of Expectations)

For any two random variables $X, Y$,

$$\mathbb{E}[\alpha X] = \alpha\,\mathbb{E}[X] \quad \forall \alpha \in \mathbb{R}, \qquad\qquad \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y],$$

whenever the right-hand sides are well-defined

**Proof:**

- Suppose $X$ and $Y$ are **simple** RVs with canonical representations

$$X = \sum_{i=1}^{m} a_i\, \mathbf{1}_{A_i}, \qquad\qquad Y = \sum_{j=1}^{n} b_j\, \mathbf{1}_{B_j},$$

  where $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$ **each** partition $\Omega$
- Then, $X + Y$ has the representation

$$X + Y = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i + b_j)\, \mathbf{1}_{A_i \cap B_j}$$

- We may combine similar $(a_i + b_j)$ terms to bring $X + Y$ to canonical form

# Property 6 (Linearity of Expectations)

For any two random variables $X, Y$,

$$\mathbb{E}[\alpha X] = \alpha\,\mathbb{E}[X] \quad \forall \alpha \in \mathbb{R}, \qquad\qquad \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y],$$

whenever the right-hand sides are well-defined

**Proof:**

- We then have

$$\mathbb{E}[X+Y] = \sum_{i=1}^{m}\sum_{j=1}^{n}(a_i + b_j)\,\mathbb{P}(A_i \cap B_j) = \sum_{i=1}^{m} a_i \underbrace{\sum_{j=1}^{n}\mathbb{P}(A_i \cap B_j)}_{=\mathbb{P}(A_i)} + \sum_{j=1}^{n} b_j \underbrace{\sum_{i=1}^{m}\mathbb{P}(A_i \cap B_j)}_{=\mathbb{P}(B_j)}$$

$$= \sum_{i=1}^{m} a_i\,\mathbb{P}(A_i) + \sum_{j=1}^{n} b_j\,\mathbb{P}(B_j) = \mathbb{E}[X] + \mathbb{E}[Y].$$

# Property 6 (Linearity of Expectations)

For any two random variables $X, Y$,

$$\mathbb{E}[\alpha X] = \alpha\,\mathbb{E}[X] \quad \forall \alpha \in \mathbb{R}, \qquad\qquad \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y],$$

whenever the right-hand sides are well-defined

**Proof:**

- Suppose $X$ and $Y$ are **non-negative** random variables
- Let $\{X_n\}$ and $\{Y_n\}$ be the associated quantization sequences, with

$$\forall\, \omega \in \Omega, \quad \lim_{n\to\infty} X_n(\omega) + Y_n(\omega) = X(\omega) + Y(\omega), \qquad\qquad \lim_{n\to\infty} \mathbb{E}[X_n + Y_n] = \mathbb{E}[X+Y],$$

- Then,

$$\mathbb{E}[X+Y] = \lim_{n\to\infty} \mathbb{E}[X_n + Y_n] = \lim_{n\to\infty} \mathbb{E}[X_n] + \lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[X] + \mathbb{E}[Y].$$

**To be continued...**