Optimal Best Arm Identification with Fixed Confidence in Restless Bandits

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Joint Work With

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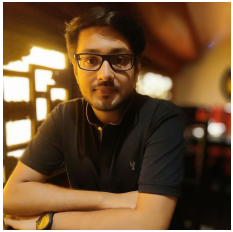
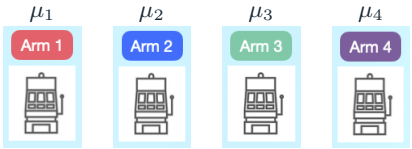
NUS, Singapore



Arpan Mukherjee
RPI, New York

Ali Tajer
RPI, New York

Preliminaries: Fixed-Confidence BAI



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Time (n)	Arm (A_n)	Observation ($\bar{X}_n$)
0	1	$X_{0,1}$

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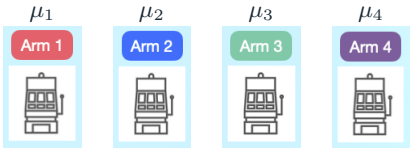

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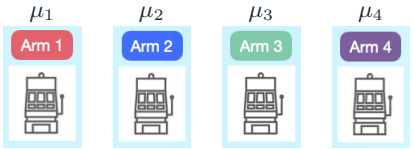

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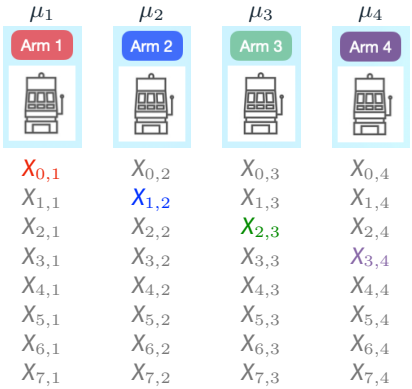

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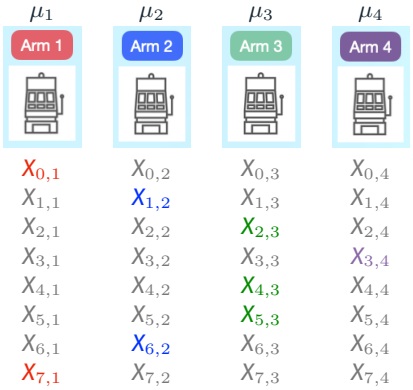

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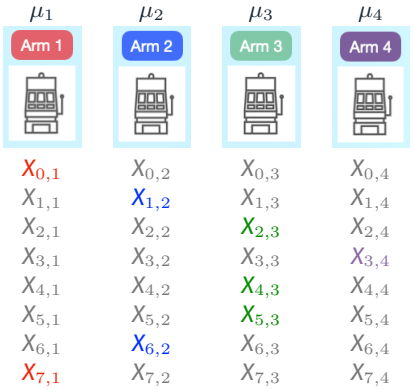

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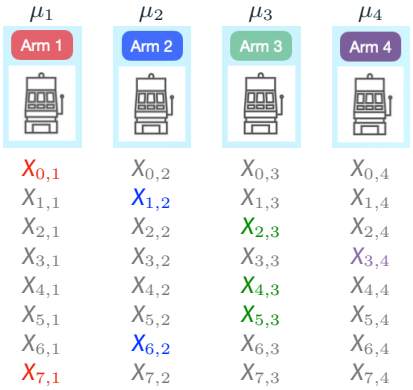

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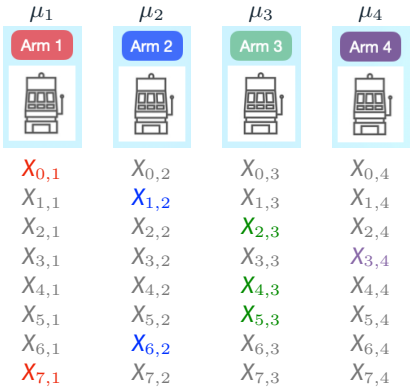

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a – guess of the best arm

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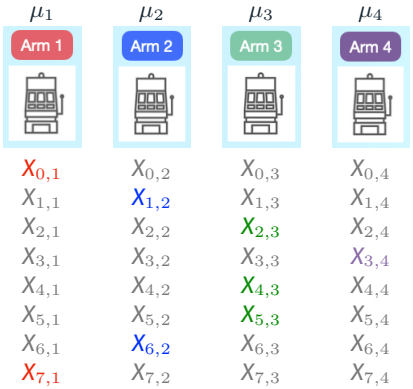

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Given $\delta \in (0,1)$:

$$\mathbb{P}_{\boldsymbol{\mu}}(\tau < +\infty) = 1, \quad \mathbb{P}_{\boldsymbol{\mu}}(a = a^\star(\boldsymbol{\mu})) \geq 1 - \delta$$

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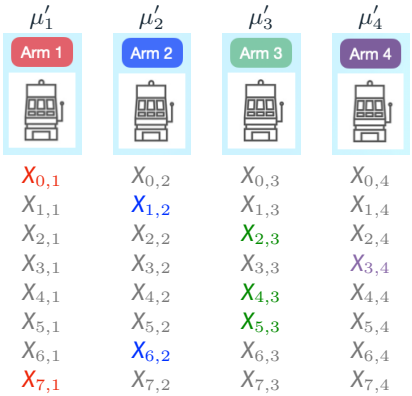

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- At time n: π_n – a rule for choosing $A_n \leftarrow (A_{0:n-1}, \bar{X}_{0:n-1})$ or stopping

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- Given $\delta \in (0, 1)$, our interest is in the class

$$\Pi(\delta) := \left\{ \pi : \mathbb{P}_{\boldsymbol{\mu}'}(\tau(\pi) < +\infty) = 1, \quad \mathbb{P}_{\boldsymbol{\mu}'}(a(\pi) = a^\star(\boldsymbol{\mu}')) \geq 1 - \delta \quad \forall \boldsymbol{\mu}' \right\}$$

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How does $\inf_{\pi \in \Pi(\delta)} \mathbb{E}_{\boldsymbol{\mu}}[\tau(\pi)]$ grow as $\delta \downarrow 0$ for any given $\boldsymbol{\mu}$?

Preliminaries: Fixed-Confidence BAI

Theorem

Garivier and Kaufmann (2016)
Assuming there are K arms and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$,

$$\lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_{\boldsymbol{\mu}}[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{T^\star(\boldsymbol{\mu})},$$

where the constant $T^\star(\boldsymbol{\mu})$ is defined as

$$T^\star(\boldsymbol{\mu}) := \sup_{\nu \in \Sigma_K} \inf_{\boldsymbol{\lambda} \in \mathrm{ALT}(\boldsymbol{\mu})} \sum_{a=1}^{K} \nu(a) \, \mathrm{KL}(\mu_a, \lambda_a)$$

$$\mathrm{ALT}(\boldsymbol{\mu}) := \left\{ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K) : a^\star(\boldsymbol{\lambda}) \neq a^\star(\boldsymbol{\mu}) \right\}$$

Preliminaries: Fixed-Confidence BAI

$$T^\star(\boldsymbol{\mu}) = \sup_{\nu \in \Sigma_K} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} \nu(a)\, \mathrm{KL}(\mu_a, \lambda_a)$$

$$\psi(\nu, \boldsymbol{\mu}) = \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})} \sum_{a=1}^{K} \nu(a)\, \mathrm{KL}(\mu_a, \lambda_a), \quad \nu \in \Sigma_K,\ \boldsymbol{\mu} \in \mathbb{R}^K, \qquad \nu^\star_{\boldsymbol{\mu}} = \arg\sup_{\nu} \psi(\nu, \boldsymbol{\mu})$$

Parameter Estimation

1

$$\widehat{\boldsymbol{\mu}}(n) = [\widehat{\mu}_1(n), \ldots, \widehat{\mu}_K(n)]^\top$$

Optimal Distribution

2

$$\nu^\star_{\widehat{\boldsymbol{\mu}}(n)} = \arg\sup_{\nu} \psi(\nu, \widehat{\boldsymbol{\mu}}(n))$$

Certainty Equivalence

3

$$A_{n+1} \sim \eta\, \nu^{\text{unif}} + (1-\eta)\, \nu^\star_{\widehat{\boldsymbol{\mu}}(n)}$$

Key ideas: $\widehat{\boldsymbol{\mu}}(n) \overset{n\to\infty}{\longrightarrow} \boldsymbol{\mu}$, $\quad \nu^\star_{\widehat{\boldsymbol{\mu}}(n)}(a) \overset{n\to\infty}{\longrightarrow} \nu^\star_{\boldsymbol{\mu}}(a)$, $\quad \dfrac{N_a(n)}{n} \overset{n\to\infty}{\longrightarrow} \nu^\star_{\boldsymbol{\mu}}(a)$ a.s. $\forall a$

Preliminaries: Fixed-Confidence BAI

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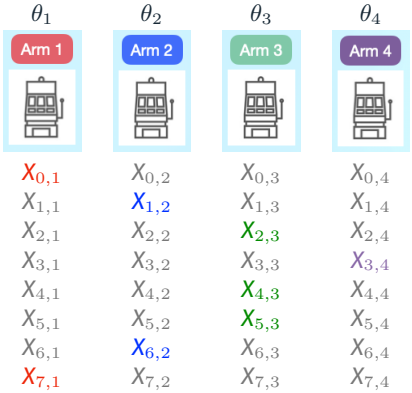
Theorem

For a threshold-based stopping rule using the GLLR along with the above arms selection rule,

$$\limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau]}{\log(1/\delta)} \leq \frac{1}{(1-\eta)\, T^\star(\boldsymbol{\mu})}$$

Preliminaries: Fixed-Confidence BAI



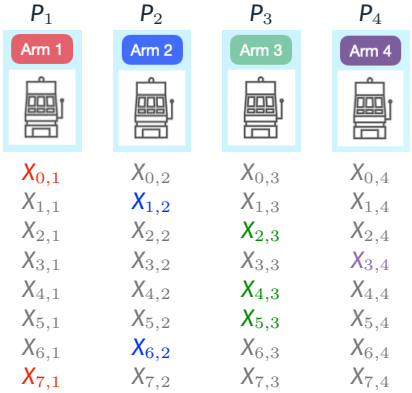
$$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_4)$$

$$a^\star(\boldsymbol{\theta}) = \arg\max_a \theta_a$$

- Single-parameter exponential family (SPEF)
- $T^\star(\boldsymbol{\theta}) = \sup_{\nu} \inf_{\lambda \in \mathsf{ALT}(\boldsymbol{\theta})} \sum_{a=1}^{K} \nu(a)\, \mathrm{KL}(\theta_a, \lambda_a)$
- $\theta \iff \mu_\theta = \mathbb{E}_\theta[X]$
  (one-to-one correspondence)

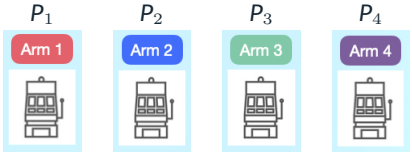
This talk: extensions to **restless Markovian** arms

BAI with Restless Markovian Arms



- Common, finite state space
- P_a is ergodic for each a
- μ_a – stationary distribution of P_a

BAI with Restless Markovian Arms



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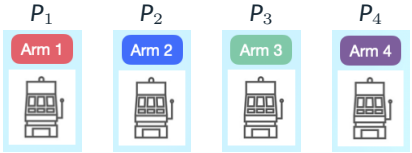

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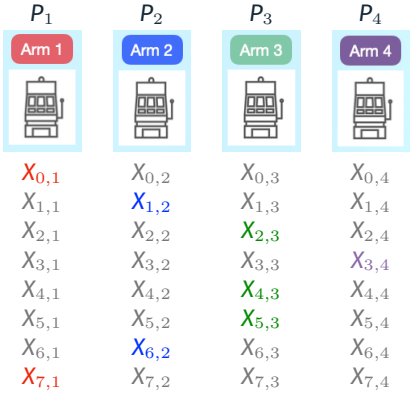

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Answer: SPEF for Markov chains!

Model: Exponential Family of TPMs

$\Theta \subset \mathbb{R}$ - known parameter space
P – irreducible TPM on $\mathcal{S}$ (generator)
Reward function $f : \mathcal{S} \to \mathbb{R}$

$$\theta \mapsto \tilde{P}_\theta(i,j) = P(i,j)\, e^{\theta f(j)}, \quad i,j \in \mathcal{S}$$

Property of $\tilde{P}_\theta$: there exist vectors u_θ, v_θ such that

$$u_\theta(i) > 0 \quad \forall i, \qquad v_\theta(i) > 0 \quad \forall i, \qquad \tilde{P}_\theta\, v_\theta = \rho(\theta)\, v_\theta, \qquad u_\theta^\top\, \tilde{P}_\theta = \rho(\theta)\, u_\theta^\top$$

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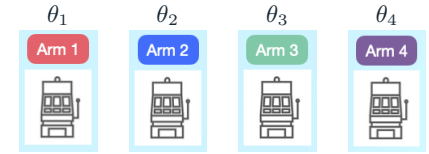
Ergodic ($P_\theta \iff \mu_\theta = u_\theta \odot v_\theta$) ✔

$\theta \mapsto \eta_\theta = \langle f, \mu_\theta \rangle$ strictly increasing bijection ✔

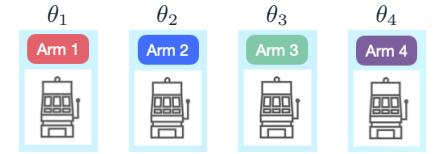
Analysis

Analysis



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$

Analysis



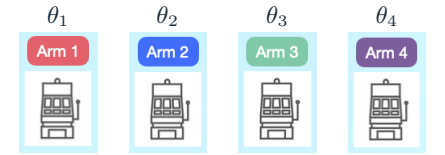
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$

Analysis



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

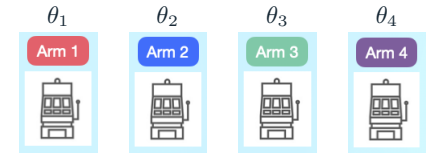
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9				

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9				

Analysis



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

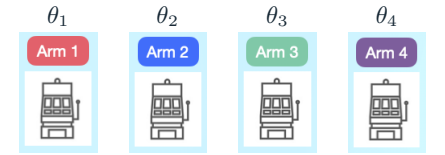
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2			

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$			

Analysis



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

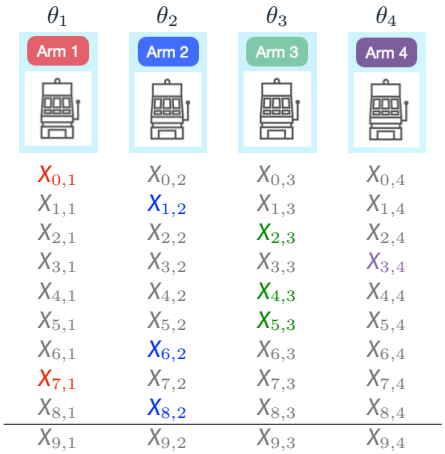
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1		

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$		

Analysis



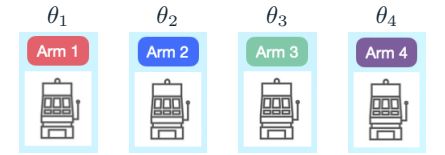
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	

Analysis



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

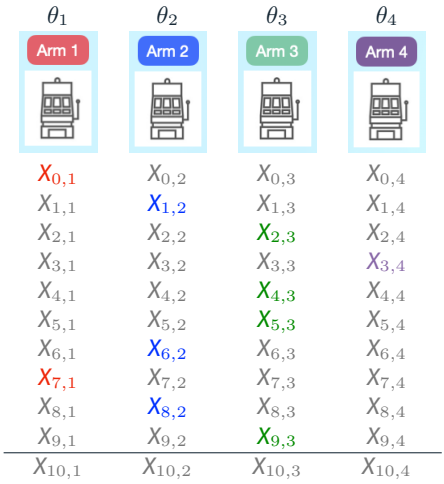
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$

Analysis



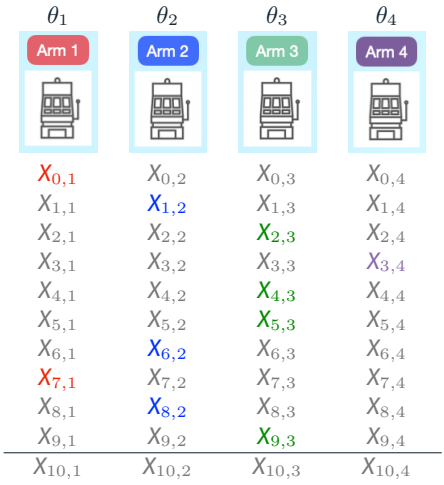
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10				

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10				

Analysis



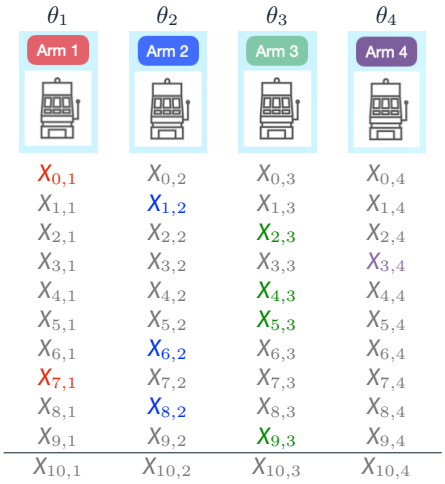
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3			

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$			

Analysis



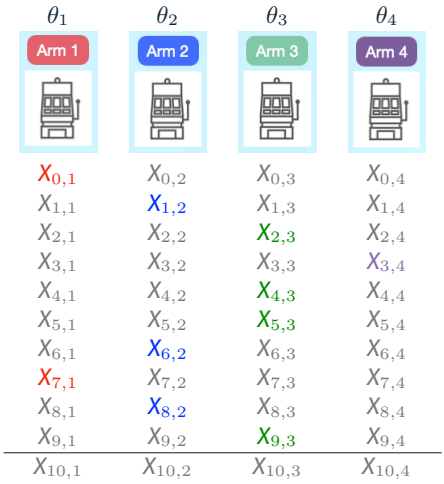
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2		

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$		

Analysis



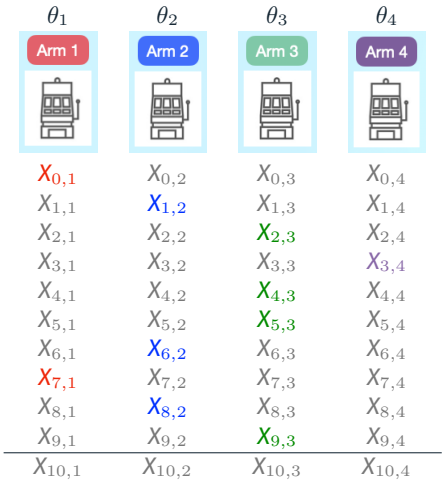
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	

Analysis



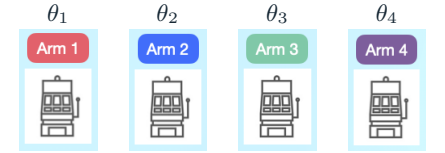
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

Analysis



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

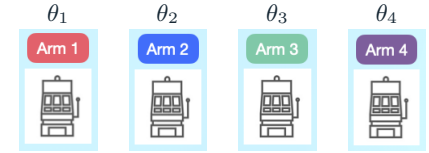
n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\mathbf{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\mathbf{i}(n) = [i_a(n) : a \in [K]]^\top$$

Analysis



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\mathbf{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\mathbf{i}(n) = [i_a(n) : a \in [K]]^\top$$

$$(\mathbf{d}(n), \mathbf{i}(n)) \longrightarrow A_n \longrightarrow (\mathbf{d}(n+1), \mathbf{i}(n+1)) \longrightarrow A_{n+1} \longrightarrow (\mathbf{d}(n+2), \mathbf{i}(n+2)) \longrightarrow \cdots$$

Markov Decision Process

θ_1	θ_2	θ_3	θ_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\mathbf{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\mathbf{i}(n) = [i_a(n) : a \in [K]]^\top$$

State space: $\mathbb{S} = \{(\mathbf{d}, \mathbf{i})\} \subset \mathbb{N}^K \times \mathcal{S}^K$ countably infinite

Markov Decision Process

θ_1	θ_2	θ_3	θ_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Action space: $[K]$ finite

Arm delays

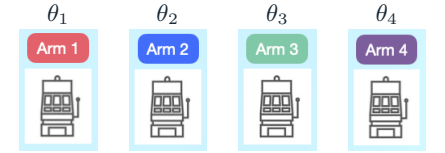
n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\mathbf{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\mathbf{i}(n) = [i_a(n) : a \in [K]]^\top$$

Markov Decision Process



Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\mathbf{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\mathbf{i}(n) = [i_a(n) : a \in [K]]^\top$$

Transition probabilities: $\Pr_{\boldsymbol{\theta}}(\mathbf{d}(9), \mathbf{i}(9) \mid \mathbf{d}(8), \mathbf{i}(8), A_8 = 2) = \Pr_{\boldsymbol{\theta}}(X_{8,2} | X_{6,2}) = P^2_{\theta_2}(X_{6,2}, X_{8,2})$

Markov Decision Process

θ_1	θ_2	θ_3	θ_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
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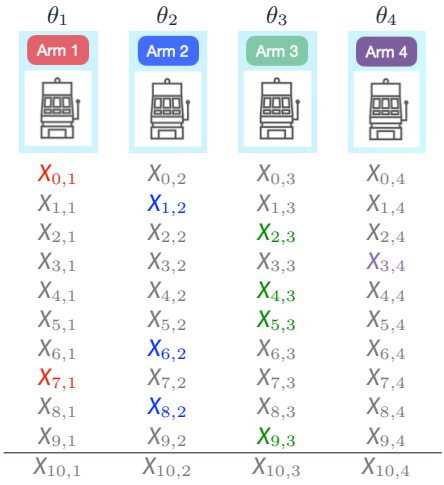
Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\mathbf{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\mathbf{i}(n) = [i_a(n) : a \in [K]]^\top$$

Transition probabilities: $\Pr_{\boldsymbol{\theta}}(\mathbf{d}(10), \mathbf{i}(10) \mid \mathbf{d}(9), \mathbf{i}(9), A_9 = 3) = P^4_{\theta_3}(X_{5,3}, X_{9,3})$ TPM powers

Flow Constraint



$$(\mathbf{d}(n), \mathbf{i}(n)) \longrightarrow A_n \xrightarrow{\text{entry}} (\mathbf{d}(n+1), \mathbf{i}(n+1)) \xrightarrow{\text{exit}} \cdots$$
$$A_{n+1} \xrightarrow{\text{entry}} (\mathbf{d}(n+2), \mathbf{i}(n+2)) \xrightarrow{\text{exit}} A_{n+2} \xrightarrow{\text{entry}} \cdots$$

$$\xrightarrow{\text{entry}} \boxed{(\mathbf{d}, \mathbf{i})} \xrightarrow{\text{exit}}$$

$$N(n, \mathbf{d}, \mathbf{i}, a) = \sum_{t=K}^{n} \mathbf{1}_{\{\mathbf{d}(t)=\mathbf{d},\ \mathbf{i}(t)=\mathbf{i},\ A_t=a\}}$$
$$N(n, \mathbf{d}, \mathbf{i}) = \sum_{a=1}^{K} N(n, \mathbf{d}, \mathbf{i}, a)$$

Flow Constraint

Under every policy π,

$$\left| \mathbb{E}[N(n, \mathbf{d}', \mathbf{i}')] - \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}} \sum_{a=1}^{K} \mathbb{E}[N(n, \mathbf{d}, \mathbf{i}, a)] \, \Pr_{\boldsymbol{\theta}}(\mathbf{d}', \mathbf{i}' | \mathbf{d}, \mathbf{i}, a) \right| \leq 1 \quad \forall (\mathbf{d}', \mathbf{i}') \in \mathbb{S}, \quad \forall n \geq K$$

Constraint on Maximum Delay

Maximum Delay Constraint

- Fix $R \in \mathbb{N}$ such that $R \gg K$
- Forcefully pull an arm if its delay equals R at any given time
- $\mathbb{S}_R = \{(\mathbf{d}, \mathbf{i}) \in \mathbb{S} : \max_a d_a \leq R\}, \qquad \mathbb{S}_{R,a} = \{(\mathbf{d}, \mathbf{i}) \in \mathbb{S}_R : d_a = R\}$ finite state space
- $\Pr_{\boldsymbol{\theta}} \longrightarrow \Pr_{\boldsymbol{\theta},R}$

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
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Maximum Delay Constraint

- Fix $R \in \mathbb{N}$ such that $R \gg K$
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- $\Pr_{\theta} \longrightarrow \Pr_{\theta,R}$

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

$R = 6$

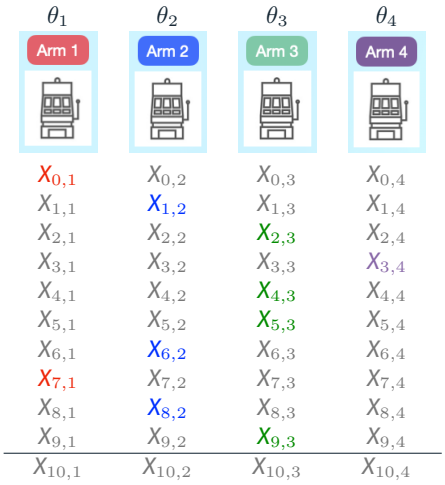
$(\mathbf{d}(9), \mathbf{i}(9)) \longrightarrow A_{10} = 4$ ✔

Flow Constraint + R-Max-Delay Constraint

$$\left| \mathbb{E}[N(n, \mathbf{d}', \mathbf{i}')] - \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \mathbb{E}[N(n, \mathbf{d}, \mathbf{i}, a)] \Pr_{\theta,R}(\mathbf{d}', \mathbf{i}' | \mathbf{d}, \mathbf{i}, a) \right| \leq 1 \quad \forall (\mathbf{d}', \mathbf{i}') \in \mathbb{S}_R, \quad \forall n \geq K,$$

$N(n, \mathbf{d}, \mathbf{i}, a) = N(n, \mathbf{d}, \mathbf{i}) \quad \text{if } d_a = R$

Objective Restated



- $\boldsymbol{\theta} = [\theta_a : a \in [K]]^\top$ unknown
- $a^\star(\boldsymbol{\theta}) = \arg\max_a \eta_{\theta_a} = \arg\max_a \langle f, \mu_{\theta_a} \rangle$

Objective: $\inf_\pi \mathbb{E}[\tau_\pi]$

- ✔ $\pi \in \Pi(\delta)$
- ✔ π satisfies *R*-max-delay constraint

Converse

Converse

- Fix $\boldsymbol{\theta}$. Assume $a^\star(\boldsymbol{\theta}) = 1$
- Set of alternative instances:

$$\begin{aligned}\text{Alt}(\boldsymbol{\theta}) &= \{\boldsymbol{\lambda} \in \Theta^K : \exists a \neq 1 \text{ such that } \eta_{\lambda_a} > \eta_{\lambda_1}\} \\ &= \{\boldsymbol{\lambda} \in \Theta^K : \exists a \neq 1 \text{ such that } \lambda_a > \lambda_1\} = \bigcup_{a=1}^{K} \{\boldsymbol{\lambda} \in \Theta^K : \lambda_a > \lambda_1\}\end{aligned}$$

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Proposition

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T_R^\star(\boldsymbol{\theta})},$$

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \textsc{Alt}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{KL}(\Pr_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a)),$$

$$\Sigma_R(\boldsymbol{\theta}) = \Big\{\nu : \checkmark \textit{prob. dist.}, \ \checkmark \boxed{\textit{flow constraint}}, \ \checkmark \boxed{\textit{R-max-delay constraint}}\Big\}$$

Converse – 1

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T^\star_R(\boldsymbol{\theta})}$$

Converse – 1

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T_R^\star(\boldsymbol{\theta})} \sim \frac{\log \frac{1}{\delta}}{T_R^\star(\boldsymbol{\theta})}$$

Converse – 1

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T_R^\star(\boldsymbol{\theta})} \sim \frac{\log \frac{1}{\delta}}{T_R^\star(\boldsymbol{\theta})}$$

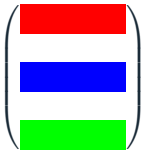
$$\liminf_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}[\tau_\pi]}{\log(1/\delta)} \geq \frac{1}{T_R^\star(\boldsymbol{\theta})}$$

Converse – 2

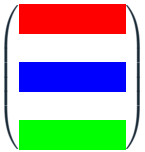
$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\text{KL}}(\Pr_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a))$$

Converse – 2

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\text{KL}}(\Pr_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a))$$



$\Pr_{\boldsymbol{\theta},R}, \quad \boldsymbol{\theta} = [\theta_a : a \in [K]]^\top$



$\Pr_{\boldsymbol{\lambda},R}, \quad \boldsymbol{\lambda} = [\lambda_a : a \in [K]]^\top$

$$\Pr_{\boldsymbol{\theta},R}(\mathbf{d}', \mathbf{i}' \mid \mathbf{d}, \mathbf{i}, a) = \begin{cases} P_{\theta_a}^{d_a}(i_a, i_a'), & (\mathbf{d},\mathbf{i}) \xrightarrow{a} (\mathbf{d}',\mathbf{i}'), \\ 0, & \text{otherwise} \end{cases}$$

Converse – 3

$$\Sigma_R(\boldsymbol{\theta}) = \Big\{\nu : \checkmark\text{prob. dist.}, \quad \checkmark\boxed{\text{flow constraint}}, \quad \checkmark\boxed{\text{R-max-delay constraint}}\Big\}$$

$$\Sigma_R(\boldsymbol{\theta}) = \Bigg\{\nu : \quad \nu(\mathbf{d},\mathbf{i},a) \geq 0 \quad \forall(\mathbf{d},\mathbf{i},a),$$

$$\sum_{(\mathbf{d},\mathbf{i})\in\mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a) = 1,$$

$$\sum_{a=1}^{K} \nu(\mathbf{d}',\mathbf{i}',a) = \sum_{(\mathbf{d},\mathbf{i})\in\mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, \Pr_{\boldsymbol{\theta},R}(\mathbf{d}',\mathbf{i}'|\mathbf{d},\mathbf{i},a) \quad \forall(\mathbf{d}',\mathbf{i}') \in \mathbb{S}_R,$$

$$\nu(\mathbf{d},\mathbf{i},a) = \sum_{a'=1}^{K} \nu(\mathbf{d},\mathbf{i},a') \quad \text{if } d_a = R,\ a \in [K] \Bigg\}$$

Converse – 4

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathsf{Alt}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\mathsf{KL}}(\Pr_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a))$$

$$\psi(\nu,\boldsymbol{\theta}) = \inf_{\boldsymbol{\lambda} \in \mathsf{Alt}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\mathsf{KL}}(\Pr_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a)), \quad \nu \in \Sigma_R(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \Theta^K$$

Lemma

Converse – 4

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathsf{ALT}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\mathsf{KL}}(\mathrm{Pr}_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \mathrm{Pr}_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a))$$

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Lemma

- ψ *is continuous*

Continuity of ψ – Berge's maximum theorem for non-compact sets (Feinberg et al., 2014)

Converse – 4

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Lemma

- ψ *is continuous*
- $\boldsymbol{\theta} \mapsto \mathcal{W}^\star(\boldsymbol{\theta}) = \{\nu \in \Sigma_R(\boldsymbol{\theta}) : \psi(\nu,\boldsymbol{\theta}) = T_R^\star(\boldsymbol{\theta})\}$ *is upper-semicontinuous*

Continuity of ψ – Berge's maximum theorem for non-compact sets (Feinberg et al., 2014)

Converse – 4

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathsf{ALT}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\mathsf{KL}}(\Pr_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a))$$

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Continuity of ψ – Berge's maximum theorem for non-compact sets (Feinberg et al., 2014)

Key idea for achievability: $\left[\frac{N(n,\mathbf{d},\mathbf{i},a)}{n} : (\mathbf{d},\mathbf{i},a) \in \mathbb{S}_R \times [K]\right]^\top \longrightarrow \mathcal{W}^\star(\boldsymbol{\theta})$

Comparison

Restless arms

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\text{KL}}(\Pr_{\boldsymbol{\theta},R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\boldsymbol{\lambda},R}(\cdot|\mathbf{d},\mathbf{i},a))$$

Independent observations from arms

Garivier and Kaufmann (2016)

$$T^\star(\boldsymbol{\theta}) = \sup_{\nu} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\theta})} \sum_{a=1}^{K} \nu(a)\, \text{KL}(\theta_a, \lambda_a)$$

Achievability

Uniform Policy and Ergodicity

$$\pi^{\mathsf{unif}}(a|\mathbf{d},\mathbf{i}) = \begin{cases} \frac{1}{K}, & (\mathbf{d},\mathbf{i}) \in \bigcup_{a'=1}^{K} \mathbb{S}_{R,a'}, \\ 1, & (\mathbf{d},\mathbf{i}) \in \mathbb{S}_{R,a}, \\ 0, & (\mathbf{d},\mathbf{i}) \in \bigcup_{a' \neq a} \mathbb{S}_{R,a'}. \end{cases}$$

Uniform Policy and Ergodicity

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Transition kernel under $\boldsymbol{\theta} \in \Theta^K$ and π^{unif}:

$$\Pr_{\boldsymbol{\theta},\pi^{\text{unif}}}(\mathbf{d}',\mathbf{i}'|\mathbf{d},\mathbf{i}) = \sum_{a=1}^{K} \Pr_{\boldsymbol{\theta},R}(\mathbf{d}',\mathbf{i}'|\mathbf{d},\mathbf{i},a) \cdot \pi^{\text{unif}}(a|\mathbf{d},\mathbf{i}) \quad \forall (\mathbf{d},\mathbf{i}),(\mathbf{d}',\mathbf{i}') \in \mathbb{S}_R$$

Uniform Policy and Ergodicity

$$\pi^{\text{unif}}(a|\mathbf{d},\mathbf{i}) = \begin{cases} \frac{1}{K}, & (\mathbf{d},\mathbf{i}) \in \bigcup_{a'=1}^{K} \mathbb{S}_{R,a'}, \\ 1, & (\mathbf{d},\mathbf{i}) \in \mathbb{S}_{R,a}, \\ 0, & (\mathbf{d},\mathbf{i}) \in \bigcup_{a' \neq a} \mathbb{S}_{R,a'}. \end{cases}$$

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$$\Pr_{\boldsymbol{\theta},\pi^{\text{unif}}}(\mathbf{d}',\mathbf{i}'|\mathbf{d},\mathbf{i}) = \sum_{a=1}^{K} \Pr_{\boldsymbol{\theta},R}(\mathbf{d}',\mathbf{i}'|\mathbf{d},\mathbf{i},a) \cdot \pi^{\text{unif}}(a|\mathbf{d},\mathbf{i}) \quad \forall (\mathbf{d},\mathbf{i}),(\mathbf{d}',\mathbf{i}') \in \mathbb{S}_R$$

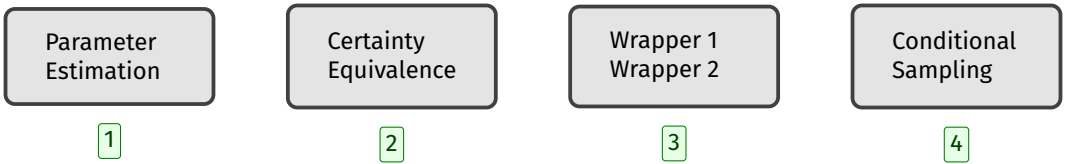
Lemma

$\Pr_{\boldsymbol{\theta},\pi^{unif}}$ *is ergodic for all* $\boldsymbol{\theta} \in \Theta^K$

Stationary distribution of $\Pr_{\boldsymbol{\theta},\pi^{\text{unif}}} = \mu_{\boldsymbol{\theta}}^{\text{unif}}$,

$$\nu_{\boldsymbol{\theta}}^{\text{unif}}(\mathbf{d},\mathbf{i},a) = \mu_{\boldsymbol{\theta}}^{\text{unif}}(\mathbf{d},\mathbf{i}) \cdot \pi^{\text{unif}}(a|\mathbf{d},\mathbf{i}) \qquad \forall (\mathbf{d},\mathbf{i},a) \in \mathbb{S}_R \times [K]$$

Achievability – Arm Selection Rule



Arm Selection Rule – 1

Parameter Estimation

Certainty Equivalence

Wrapper 1
Wrapper 2

Conditional Sampling

1

2

3

4

$\hat{\boldsymbol{\theta}}(n)$

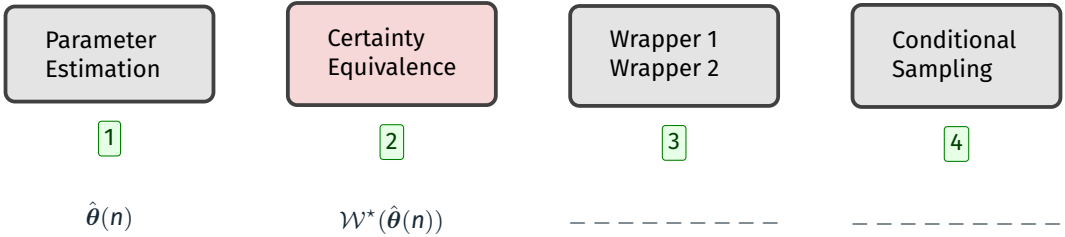
$$N_a(n) = \sum_{t=0}^{n} \mathbf{1}_{\{A_t = a\}}$$

$$\hat{\eta}^a(n) = \begin{cases} 0, & N_a(n) = 0, \\ \dfrac{1}{N_a(n)} \displaystyle\sum_{t=0}^{n} \mathbf{1}_{\{A_t = a\}} f(\bar{X}_t), & N_a(n) > 0 \end{cases}$$

$$\hat{\eta}_a(n) \Longleftrightarrow \hat{\theta}_a(n)$$

$$\hat{\boldsymbol{\theta}}(n) = [\hat{\theta}_a(n) : a \in [K]]$$

Arm Selection Rule – 2



$$\mathcal{W}^\star(\hat{\theta}(n)) = \underset{\nu \in \Sigma_R(\hat{\theta}(n))}{\arg\sup} \inf_{\lambda \in \text{Alt}(\hat{\theta}(n))} \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\text{KL}}(\Pr_{\hat{\theta}(n),R}(\cdot|\mathbf{d},\mathbf{i},a) \| \Pr_{\lambda,R}(\cdot|\mathbf{d},\mathbf{i},a))$$

Arm Selection Rule – 3

Parameter Estimation	Certainty Equivalence	Wrapper 1 Wrapper 2	Conditional Sampling
1	2	3	4
$\hat{\boldsymbol{\theta}}(n)$	$\mathcal{W}^\star(\hat{\boldsymbol{\theta}}(n))$	$\pi^{\eta}_{\hat{\boldsymbol{\theta}}(n)},\ \pi_n$	– – – – – – – – –

Fix $\eta \in (0,1)$ and $\varepsilon_n \to 0$.
Pick $\nu_n^\star \in \mathcal{W}^\star(\hat{\boldsymbol{\theta}}(n))$

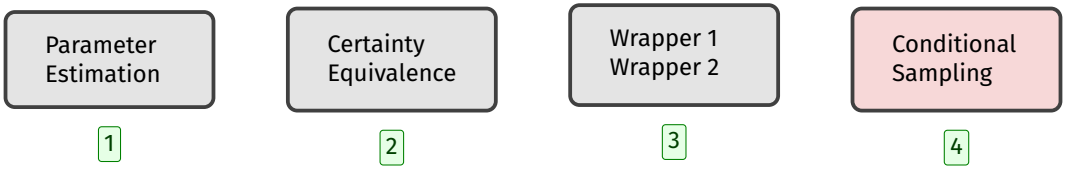
Wrapper 1

$$\pi^{\eta}_{\hat{\boldsymbol{\theta}}(n)}(a|\mathbf{d},\mathbf{i}) = \frac{\eta\, \nu^{\mathsf{unif}}_{\hat{\boldsymbol{\theta}}(n)}(\mathbf{d},\mathbf{i},a) + (1-\eta)\, \nu_n^\star(\mathbf{d},\mathbf{i},a)}{\eta\, \mu^{\mathsf{unif}}_{\hat{\boldsymbol{\theta}}(n)}(\mathbf{d},\mathbf{i}) + (1-\eta) \sum_{a'=1}^{K} \nu_n^\star(\mathbf{d},\mathbf{i},a')}$$

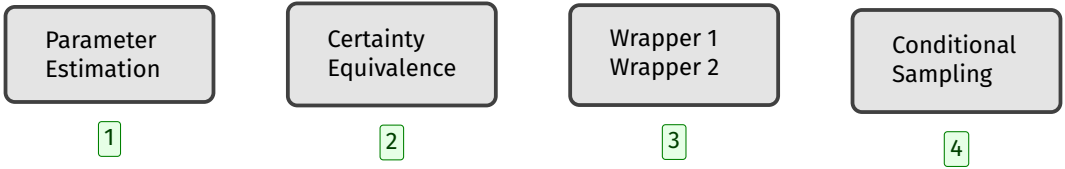
Wrapper 2

$$\pi_n = \varepsilon_n \pi^{\mathsf{unif}} + (1-\varepsilon_n)\, \pi^{\eta}_{\hat{\boldsymbol{\theta}}(n-1)} \quad \forall n \geq K$$

Arm Selection Rule – 4



Achievability – Arm Selection Rule



Lemma

Achievability – Arm Selection Rule

Parameter Estimation	Certainty Equivalence	Wrapper 1 Wrapper 2	Conditional Sampling
1	2	3	4
$\hat{\boldsymbol{\eta}}(n) \Longleftrightarrow \hat{\boldsymbol{\theta}}(n)$	$\mathcal{W}^\star(\hat{\boldsymbol{\theta}}(n))$	η, ε_n-mixtures	$A_n \sim \pi_n(\cdot \mid \mathbf{d}(n), \mathbf{i}(n))$

Lemma

For $S_R = |\mathbb{S}_R|$ *and* $\varepsilon_n = n^{-\frac{1}{2(1+S_R)}}$,

$$\lim_{n\to\infty} d_\infty \left(\left[\frac{N(n,\mathbf{d},\mathbf{i},a)}{n} : (\mathbf{d},\mathbf{i},a) \in \mathbb{S}_R \times [K] \right]^\top, \mathcal{W}_\eta^\star(\boldsymbol{\theta}) \right) = 0 \quad a.s.,$$

where $\mathcal{W}_\eta^\star(\boldsymbol{\theta}) = \{\eta\, \nu_{\boldsymbol{\theta}}^{unif} + (1-\eta)\, \nu : \nu \in \mathcal{W}^\star(\boldsymbol{\theta})\}$

Achievability – Stopping Rule

Empirical transition matrix:

$$\widehat{Q}_n(\mathbf{d}', \mathbf{i}' | \mathbf{d}, \mathbf{i}, a) = \begin{cases} \dfrac{1}{N(n, \mathbf{d}, \mathbf{i}, a)} \sum_{t=K}^{n} \mathbf{1}_{\{(\mathbf{d}(t), \mathbf{i}(t)) = (\mathbf{d}, \mathbf{i}),\, A_t = a,\, (\mathbf{d}(t+1), \mathbf{i}(t+1)) = (\mathbf{d}', \mathbf{i}')\}}, & N(n, \mathbf{d}, \mathbf{i}, a) > 0, \\ \dfrac{1}{S_R}, & N(n, \mathbf{d}, \mathbf{i}, a) = 0 \end{cases}$$

Achievability – Stopping Rule

Empirical transition matrix:

$$\widehat{Q}_n(\mathbf{d}', \mathbf{i}' | \mathbf{d}, \mathbf{i}, a) = \begin{cases} \frac{1}{N(n, \mathbf{d}, \mathbf{i}, a)} \sum_{t=K}^{n} \mathbf{1}_{\{(\mathbf{d}(t), \mathbf{i}(t)) = (\mathbf{d}, \mathbf{i}), A_t = a, (\mathbf{d}(t+1), \mathbf{i}(t+1)) = (\mathbf{d}', \mathbf{i}')\}}, & N(n, \mathbf{d}, \mathbf{i}, a) > 0, \\ \frac{1}{S_R}, & N(n, \mathbf{d}, \mathbf{i}, a) = 0 \end{cases}$$

Test statistic

$$Z(n) = \inf_{\boldsymbol{\lambda} \in \text{Alt}(\hat{\boldsymbol{\theta}}(n))} \sum_{(\mathbf{d}, \mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} N(n, \mathbf{d}, \mathbf{i}, a) \, D_{\text{KL}}(\hat{Q}_n(\cdot | \mathbf{d}, \mathbf{i}, a) \| \Pr_{\boldsymbol{\lambda}, R}(\cdot | \mathbf{d}, \mathbf{i}, a))$$

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Threshold:

$$\zeta(n, \delta) = \log\left(\frac{1}{\delta}\right) + (S_R - 1) \sum_{(\mathbf{d},\mathbf{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \log\left(e\left[1 + \frac{N(n, \mathbf{d}, \mathbf{i}, a)}{S_R - 1}\right]\right)$$

Stopping rule:

$$\tau_\delta = \inf\{n \geq K : Z(n) \geq \zeta(n, \delta)\}$$

Recommendation rule:

$$\hat{a} = \arg\max_a \hat{\eta}_a(n)$$

Performance

- Error prob. $\leq \delta$
- Almost sure upper bound

$$\limsup_{\delta \downarrow 0} \frac{\tau_\delta}{\log(1/\delta)} \leq \frac{1}{(1-\eta)\, T^\star_R(\boldsymbol{\theta})} \quad \text{a.s.}$$

- Upper bound in expectation

$$\limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{(1-\eta)\, T^\star_R(\boldsymbol{\theta})}$$

$$\limsup_{\eta \downarrow 0} \limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{T^\star_R(\boldsymbol{\theta})}$$

Algorithm 1 Restless Tracking

Input: $K, R, \eta, \delta > 0$
Output: $\hat{a} \in [K]$ (best arm).
Initialization: $n = 0$, $N_a(n) = 0$, $\hat{\eta}_a(n) = 0$ for all $a \in [K]$, $N(\mathbf{n}, \mathbf{d}, \mathbf{i}, a) = 0$ for all $(\mathbf{d}, \mathbf{i}, a) \in \mathbb{S}_R \times [K]$, $\texttt{stop} = 0$

for $n < K$ **do**
  Select arm $A_n = n + 1$.
end for
while $\texttt{stop} == 0$ **do**
  Update $(\mathbf{d}(n), \mathbf{i}(n))$; update $\hat{\eta}_a(n)$ for each $a \in [K]$
  $\hat{\boldsymbol{\eta}}(n) \iff \hat{\boldsymbol{\theta}}(n)$
  Evaluate $Z(n)$
  if $Z(n) \geq \zeta(n, \delta)$ **then**
    $\texttt{stop} = 1$
    $\tau_\delta = n$
    $\hat{a} = \arg\max_a \hat{\eta}_a(n)$
  else
    Select $A_n \sim \pi_n(\cdot | \mathbf{d}(n), \mathbf{i}(n))$
    $n \leftarrow n + 1$.
  end if
end while
return $\hat{a}$.

Main Result

Parameter Estimation	Certainty Equivalence	Wrapper 1 Wrapper 2	Conditional Sampling
1	2	3	4
$\hat{\boldsymbol{\eta}}(n) \Longleftrightarrow \hat{\boldsymbol{\theta}}(n)$	$\mathcal{W}^\star(\hat{\boldsymbol{\theta}}(n))$	η, ε_n-mixtures	$A_n \sim \pi_n(\cdot \mid \mathbf{d}(n), \mathbf{i}(n))$

Theorem

$$\frac{1}{T_R^\star(\boldsymbol{\theta})} \leq \liminf_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}[\tau_\pi]}{\log(1/\delta)} \leq \limsup_{\eta \downarrow 0} \limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{T_R^\star(\boldsymbol{\theta})}$$

Removing the Max-Delay Constraint

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Lemma (Montonicity)

$T_R^\star(\boldsymbol{\theta}) \leq T_{R+1}^\star(\boldsymbol{\theta})$ *for all* R

Removing the Max-Delay Constraint

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$T^\star_R(\boldsymbol{\theta}) \leq T^\star_{R+1}(\boldsymbol{\theta})$ for all R

$\lim_{R\to\infty} T^\star_R(\boldsymbol{\theta})$ exists, $\quad \lim_{R\to\infty} \Sigma_R(\boldsymbol{\theta}) = \Sigma(\boldsymbol{\theta}) = \left\{\nu : \checkmark\text{prob. dist.}, \quad \checkmark\boxed{\text{flow constraint}}\right\}$

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$$T^\star(\boldsymbol{\theta}) = \sup_{\nu\in\Sigma(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda}\in \text{Alt}(\boldsymbol{\theta})} \sum_{(\mathbf{d},\mathbf{i})\in\mathbb{S}} \sum_{a=1}^{K} \nu(\mathbf{d},\mathbf{i},a)\, D_{\text{KL}}(\Pr_{\boldsymbol{\theta}}(\cdot|\mathbf{d},\mathbf{i},a)\|\Pr_{\boldsymbol{\lambda}}(\cdot|\mathbf{d},\mathbf{i},a))$$

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$$\lim_{R\to\infty} T_R^\star(\boldsymbol{\theta}) = T^\star(\boldsymbol{\theta})??$$

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Independent observations from arms ✔

Restless arms – OPEN

Removing the Max-Delay Constraint

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Independent observations from arms ✔

Restless arms – OPEN

$$\frac{1}{T^\star(\boldsymbol{\theta})} \leq \frac{1}{T_R^\star(\boldsymbol{\theta})} \leq \liminf_{\delta\downarrow 0} \inf_{\pi\in\Pi(\delta)} \frac{\mathbb{E}[\tau_\pi]}{\log(1/\delta)} \leq \limsup_{\eta\downarrow 0} \limsup_{\delta\downarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{T_R^\star(\boldsymbol{\theta})} \overset{??}{\leq} \frac{1}{T^\star(\boldsymbol{\theta})}$$

Main Result

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References

Feinberg, E. A., Kasyanov, P. O., and Voorneveld, M. (2014). Berge's maximum theorem for noncompact image sets. *Journal of Mathematical Analysis and Applications*, 413(2):1040–1046.

Garivier, A. and Kaufmann, E. (2016). Optimal best arm identification with fixed confidence. In *Conference on Learning Theory*, pages 998–1027. PMLR.