

CS 6660: MATHEMATICAL FOUNDATIONS OF DATA SCIENCE
(PROBABILITY)

INSTRUCTOR: DR. KARTHIK P. N.



PRACTICE PROBLEMS 02

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All random variables appearing below are defined with respect to \mathcal{F} .

1. Virat and Anushka have a date at 7 pm. Each will arrive at the meeting place with a delay that is distributed uniformly randomly between 0 minutes and 60 minutes, independent of the delay of the other. The first to arrive will wait for 15 minutes and leave if the other does not arrive within 15 minutes. Find the probability that both meet.
2. Let X and Y be continuous random variables with PDFs f_X and f_Y respectively. For any $\alpha \in [0, 1]$, argue that $\alpha f_X + (1 - \alpha) f_Y$ is a valid PDF. Can you think of a random variable Z whose PDF is $f_Z = \alpha f_X + (1 - \alpha) f_Y$?
3. Let X and Y be jointly continuous random variables with the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} cx(y - x)e^{-y}, & 0 \leq x \leq y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Determine the constant c .
- (b) Show that

$$f_{X|Y=y}(x) = \begin{cases} 6x(y - x)y^{-3}, & 0 \leq x \leq y, \\ 0, & \text{otherwise,} \end{cases} \quad f_{Y|X=x}(y) = \begin{cases} (y - x)e^{x-y}, & 0 \leq x \leq y < +\infty, \\ 0, & \text{otherwise,} \end{cases}$$

4. Let X and Y be independent Poisson variables with parameters λ and μ respectively. Fix $n \in \mathbb{N}$. Determine the conditional PMF of X , conditioned on the event $\{X + Y = n\}$.
5. Suppose that two batteries are chosen simultaneously and uniformly at random from the following identical group of 12 batteries : 3 new, 4 used (yet working), 5 defective. You may assume that all batteries within a particular group are identical. Let X denote the number of new batteries chosen, and let Y denote the number of used batteries chosen. Determine the joint PMF of X and Y , and compute $\mathbb{P}(\{|X - Y| \leq 1\})$.
6. Suppose that X and Y have the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} cy, & -1 \leq x \leq 1, 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Determine the constant c .
- (b) Are X and Y independent?
- (c) Evaluate $\mathbb{P}(\{X \geq Y + 0.5\})$.
- (d) Evaluate $\mathbb{P}(\{X > 0.75\}|\{Y > 0.5\})$.