भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 18: Recipes for Conditional Distributions and Conditional Probability Computations

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X and Y Jointly Discrete

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Suppose X and Y are **jointly discrete**, with $\mathbb{P}_{X,Y}(E) = 1$ for some countable $E \subset \mathbb{R}^2$

- $E_1 := \{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } (x, y) \in E\}$, $\quad E_2 = \{y \in \mathbb{R} : \exists x \in \mathbb{R} \text{ such that } (x, y) \in E\}$
- Conditional PMF:

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x, y)}{p_Y(y)}, \qquad x \in \mathbb{R}, \;\; y : p_Y(y) > 0.$$

- For any $B \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B\}|\{Y = y\}) = \sum_{x \in B \cap E_1} p_{X|Y=y}(x).$$

- For any $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B_1\}|\{Y \in B_2\}) = \frac{\mathbb{P}(\{X \in B_1\} \cap \{Y \in B_2\})}{\mathbb{P}(\{Y \in B_2\})} = \frac{\sum_{x \in B_1 \cap E_1} \sum_{y \in B_2 \cap E_2} p_{X,Y}(x, y)}{\sum_{y \in B_2 \cap E_2} p_Y(y)}$$

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X and Y Jointly Continuous

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Suppose X and Y are **jointly continuous** with joint PDF $f_{X,Y}$.

- Conditional PDF:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad x \in \mathbb{R}, \;\; y : f_Y(y) > 0.$$

- For any $B \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B\} | \{Y = y\}) = \int_{-\infty}^{\infty} \mathbf{1}_B(u)\, f_{X|Y=y}(u)\, \mathrm{d}u.$$

- For any $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B_1\} | \{Y \in B_2\}) = \frac{\mathbb{P}(\{X \in B_1\} \cap \{Y \in B_2\})}{\mathbb{P}(\{Y \in B_2\})} = \frac{\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \mathbf{1}_{B_1}(u)\, \mathbf{1}_{B_2}(v)\, f_{X,Y}(u,v)\, \mathrm{d}v\, \mathrm{d}u}{\int\limits_{-\infty}^{\infty} \mathbf{1}_{B_2}(v)\, f_Y(v)\, \mathrm{d}v}.$$

X Continuous, Y Discrete

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Suppose X is a **continuous** RV with PDF f_X and Y is a **discrete** RV with PMF p_Y.

- Conditional PDF:

$$f_{X|Y=y}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X|Y=y}(x), \qquad F_{X|Y=y}(x) = \frac{\mathbb{P}(\{X \leq x\} \cap \{Y = y\})}{\mathbb{P}(\{Y = y\})}, \qquad x \in \mathbb{R}, \;\; y : p_Y(y) > 0.$$

- For any $B \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B\} | \{Y = y\}) = \int_{-\infty}^{\infty} \mathbf{1}_B(u) \, f_{X|Y=y}(u) \, \mathrm{d}u.$$

- There exists **countable** $E \subset \mathbb{R}$ such that $\mathbb{P}_Y(E) = 1$
- For any $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B_1\} | \{Y \in B_2\}) = \frac{\mathbb{P}(\{X \in B_1\} \cap \{Y \in B_2\})}{\mathbb{P}(\{Y \in B_2\})} = \frac{\int\limits_{-\infty}^{\infty} \sum\limits_{y \in B_2 \cap E} \mathbf{1}_{B_1}(u) \, f_{X|Y=y}(u) \, p_Y(y) \, \mathrm{d}u}{\sum\limits_{y \in B_2 \cap E} p_Y(y)}.$$

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X Discrete, Y Continuous

Suppose X is a **discrete** RV with PMF p_X and Y is a **continuous** RV with PDF f_Y.

- There exists **countable** $E \subset \mathbb{R}$ such that $\mathbb{P}_X(E) = 1$
- Conditional PMF:

$$p_{X|Y=y}(x) = \frac{f_{Y|X=x}(y)\, p_X(x)}{\underbrace{\sum_{x \in E} f_{Y|X=x}(y)\, p_X(x)}_{f_Y(y)}}, \qquad x \in \mathbb{R}, \quad y : f_Y(y) > 0.$$

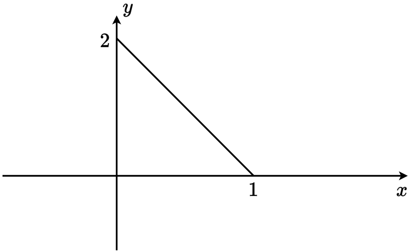
- For any $B \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B\} | \{Y = y\}) = \sum_{x \in B \cap E} p_{X|Y=y}(x).$$

- For any $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}(\{X \in B_1\} | \{Y \in B_2\}) = \frac{\mathbb{P}(\{X \in B_1\} \cap \{Y \in B_2\})}{\mathbb{P}(\{Y \in B_2\})} = \frac{\sum\limits_{x \in B_1 \cap E} \int\limits_{-\infty}^{\infty} p_{X|Y=v}(x)\, \mathbf{1}_{B_2}(v) f_Y(v)\, \mathrm{d}v}{\int\limits_{-\infty}^{\infty} f_Y(v)\, \mathrm{d}v}.$$

Example



Let $f_{X,Y}(x, y) = 1$ inside the triangle, and 0 elsewhere.

- Compute f_X and f_Y.
- Compute $f_{X|Y=y}$ for all feasible y.
- Compute $f_{Y|X=x}$ for all feasible x.
- Is $X \perp\!\!\!\perp Y$? Justify.