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Probability and Stochastic Processes

Lecture 20: Transformations (Part 2)

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

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Sums of Random Variables

Let X and Y be random variables.

- Show that $X + Y$ is a random variable.
- In the cases when X and Y are jointly discrete/continuous, derive the PMF/PDF of $X + Y$.
- Simplify the PMF/PDF when X and Y are independent.

Showing that $X+Y$ is a Random Variable

For any $x \in \mathbb{R}$,

$$\begin{aligned}\left\{X+Y<x\right\} &= \left\{X<x-Y\right\}\\ &= \left\{\exists\, q \in \mathbb{Q}:\ X<q<x-Y\right\}\\ &= \bigcup_{q\in\mathbb{Q}} \left\{X<q,\ q<x-Y\right\}\\ &= \bigcup_{q\in\mathbb{Q}} \left\{X<q\right\} \cap \left\{Y<x-q\right\} \quad \in \quad \mathscr{F}\end{aligned}$$

PMF of $X + Y$ for Jointly Discrete X, Y

- X, Y jointly discrete $\implies$ X discrete, Y discrete $\implies$ $X + Y$ discrete
- There exists countable $E \subset \mathbb{R}^2$ such that $\mathbb{P}_{X,Y}(E) = 1$
- Extract sets $E_1, E_2 \subset \mathbb{R}$ such that

$$E_1 = \{x \in \mathbb{R} : \ \exists y \in \mathbb{R} \text{ such that } (x, y) \in E\}, \qquad \mathbb{P}_X(E_1) = 1,$$
$$E_2 = \{y \in \mathbb{R} : \ \exists x \in \mathbb{R} \text{ such that } (x, y) \in E\}, \qquad \mathbb{P}_Y(E_2) = 1$$

- PMF of $Z = X + Y$: for any $z \in \mathbb{R}$,

$$p_Z(z) = \mathbb{P}(\{Z = z\}) = \mathbb{P}(\{X + Y = z\}) = \mathbb{P}_{X+Y}(\{z\}) = \mathbb{P}_{X,Y}\Big(\underbrace{\{(x, y) : x + y = z\}}_{B}\Big)$$

$$= \sum_{(x,y)\in B\cap E} p_{X,Y}(x, y) = \sum_{\substack{(x,y)\in E:\\ x+y=z}} p_{X,Y}(x, y) = \sum_{\substack{(x,y)\in E:\\ x+y=z}} p_{X,Y}(x,\ z - x) = \sum_{\substack{x:\ x\in E_1,\\ z-x\ \in\ E_2}} p_{X,Y}(x,\ z - x)$$

$$\overset{X\perp\!\!\!\perp Y}{=} \sum_{\substack{x:\ x\in E_1,\\ z-x\ \in\ E_2}} p_X(x)\, p_Y(z - x) \quad = \quad p_X * p_Y(z)$$

Sum of Two Independent Poissons

Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$. Assume $X \perp\!\!\!\perp Y$. Determine the distribution of $Z = X + Y$.

- Here, the sets $E \subset \mathbb{R}^2$ and $E_1, E_2 \subset \mathbb{R}$ are as follows:

$$E = \mathbb{W} \times \mathbb{W}, \qquad E_1 = \mathbb{W}, \qquad E_2 = \mathbb{W},$$

  where $\mathbb{W} = \mathbb{N} \cup \{0\}$

- For any $z \in \mathbb{W}$,

$$p_Z(z) = \sum_{\substack{x:\ x \in \mathbb{W}, \\ z-x\ \in\ \mathbb{W}}} p_X(x)\, p_Y(z-x) = \sum_{x=0}^{z} e^{-\lambda_1} \frac{\lambda_1^x}{x!}\, e^{-\lambda_2} \frac{\lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \sum_{x=0}^{z} \frac{z!}{x!\,(z-x)!}\, \lambda_1^x\, \lambda_2^{z-x} = e^{-(\lambda_1+\lambda_2)}\, \frac{(\lambda_1+\lambda_2)^z}{z!}$$

PDF of $X + Y$ for Jointly Continuous X, Y

- X, Y jointly continuous $\implies$ X continuous, Y continuous $\implies$ $X + Y$ continuous
- CDF of $Z = X + Y$: for any $z \in \mathbb{R}$,

$$F_Z(z) = \mathbb{P}(\{Z \leq z\}) = \mathbb{P}(\{X + Y \leq z\}) = \mathbb{P}_{X+Y}\Big((-\infty, z]\Big) = \mathbb{P}_{X,Y}\Big(\underbrace{\{(x, y) : x + y \leq z\}}_{B}\Big)$$

$$= \iint_B f_{X,Y}(u, v)\, \mathrm{d}v\, \mathrm{d}u = \int_{-\infty}^{\infty} \int_{-\infty}^{z-u} f_{X,Y}(u, v)\, \mathrm{d}v\, \mathrm{d}u \overset{t=u+v}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{z} f_{X,Y}(u, t-u)\, \mathrm{d}t\, \mathrm{d}u$$

$$= \int_{-\infty}^{z} \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(u, t-u)\, \mathrm{d}u}_{g(t)}\, \mathrm{d}t$$

- Thus, it follows that the PDF of Z is given by

$$\forall\, z \in \mathbb{R}, \qquad f_Z(z) = g(z) = \int_{-\infty}^{\infty} f_{X,Y}(u, z-u)\, \mathrm{d}u \overset{X \perp\!\!\!\perp Y}{=} \int_{-\infty}^{\infty} f_X(u)\, f_Y(z-u)\, \mathrm{d}u = f_X * f_Y(z)$$

Sum of Two Independent Exponentials

Let $X \sim \text{Exponential}(\mu_1)$ and $Y \sim \text{Exponential}(\mu_2)$. Assume $X \perp\!\!\!\perp Y$. Determine the PDF of $Z = X + Y$.

- For any $z \in (0, +\infty)$,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(u)\, f_Y(z-u)\,\mathrm{du} = \int_0^{\infty} \mu_1\, e^{-\mu_1 u}\, f_Y(z-u)\,\mathrm{du} = \int_0^{z} \mu_1\, e^{-\mu_1 u}\, \mu_2\, e^{-\mu_2 (z-u)}\,\mathrm{du}$$

$$= \mu_1\,\mu_2\, e^{-\mu_2 z} \int_0^{z} e^{(\mu_2-\mu_1)\,u}\,\mathrm{du} = \begin{cases} \underbrace{\mu^2\, z\, e^{-\mu z}}_{\text{Erlang}(2)}, & \mu_1 = \mu_2 = \mu, \\[2ex] \dfrac{\mu_1\,\mu_2}{\mu_2 - \mu_1}\left(e^{-\mu_1 z} - e^{-\mu_2 z}\right), & \mu_1 \neq \mu_2. \end{cases}$$

Functions of Independent Random Variables are Independent Random Variables

Let X and Y be two random variables. Let $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be **measurable**. Show that

$$X \perp\!\!\!\perp Y \quad\Longrightarrow\quad g(X) \perp\!\!\!\perp h(Y).$$

For any $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$\begin{aligned}
P_{g(X),h(Y)}(B_1 \times B_2) &= \mathbb{P}\Big(\{g(X)\in B_1\}\cap\{h(Y)\in B_2\}\Big) \\
&= \mathbb{P}\Big(\{X\in g^{-1}(B_1)\}\cap\{Y\in h^{-1}(B_2)\}\Big) = \mathbb{P}_{X,Y}(g^{-1}(B_1)\times h^{-1}(B_2)) \\
&\overset{X\perp\!\!\!\perp Y}{=} \quad \mathbb{P}_X(g^{-1}(B_1))\cdot\mathbb{P}_Y(h^{-1}(B_2)) \quad = \quad \mathbb{P}_{g(X)}(B_1)\cdot\mathbb{P}_{h(Y)}(B_2).
\end{aligned}$$

Sum of Random Number of Random Variables

Let $\{X_1, X_2, \ldots\}$ be a countably infinite collection of **IID** random variables having a common CDF F.
Let N be a discrete random variable taking values in natural numbers and having PMF p_N.
Let N be **independent** of $\{X_i : i \in \mathbb{N}\}$.
Consider the sum

$$S_N := \sum_{i=1}^{N} X_i; \qquad S_N(\omega) = \sum_{i=1}^{N(\omega)} X_i(\omega), \quad \omega \in \Omega.$$

- Show that $S_N : \Omega \to \mathbb{R}$ is a random variable with respect to $\mathscr{F}$.
- Determine the CDF of S_N.

Showing that S_N is a Random Variable

- Let us define sets
$$E_n := \{N = n\}, \quad n \in \mathbb{N}.$$
- Then, $\{E_n\}$ forms a partition of Ω, i.e.,
$$\Omega = \bigsqcup_{n\in\mathbb{N}} E_n.$$
- For any $x \in \mathbb{R}$,
$$\{S_N \leq x\} = \bigsqcup_{n\in\mathbb{N}} \Big(\{S_N \leq x\} \cap E_n\Big) = \bigsqcup_{n\in\mathbb{N}} \Big(\{S_N \leq x\} \cap \{N = n\}\Big) = \bigsqcup_{n\in\mathbb{N}} \Big(\{S_n \leq x\} \cap \{N = n\}\Big) \in \mathscr{F}.$$

CDF of S_N

- For any $x \in \mathbb{R}$,

$$
\begin{aligned}
F_{S_N}(x) = \mathbb{P}(\{S_N \le x\}) &= \sum_{n \in \mathbb{N}} \mathbb{P}\Big(\{S_N \le x\} \cap E_n\Big) = \sum_{n \in \mathbb{N}} \mathbb{P}\Big(\{S_N \le x\} \cap \{N = n\}\Big) \\
&= \sum_{n \in \mathbb{N}} \mathbb{P}\Big(\{S_n \le x\} \cap \{N = n\}\Big) = \sum_{n \in \mathbb{N}} \mathbb{P}(\{S_n \le x\}) \cdot \mathbb{P}(\{N = n\}) \\
&= \sum_{n \in \mathbb{N}} \mathbb{P}(\{S_n \le x\}) \cdot p_N(n).
\end{aligned}
$$

Sum of Geometrically Many Exponential RVs

Evaluate the CDF of S_N for the case when $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathrm{Exp}(\lambda)$ and $N \sim \mathrm{Geom}(p)$, with $N \perp\!\!\!\perp \{X_1, X_2, \ldots\}$.

- The sum S_n of n IID exponentials follows $\mathrm{Erlang}(n)$ distribution with PDF

$$f_{S_n}(x) = \frac{\lambda^n\, x^{n-1}\, e^{-\lambda x}}{(n-1)!}, \qquad x \geq 0.$$

- CDF of S_N:

$$\begin{aligned} F_{S_N}(x) &= \sum_{n \in \mathbb{N}} \mathbb{P}(\{S_n \leq x\}) \cdot p_N(n) \\ &= \sum_{n \in \mathbb{N}} \left(\int_0^x \frac{\lambda^n\, u^{n-1}\, e^{-\lambda u}}{(n-1)!} \,\mathrm{d}u \right) \cdot p(1-p)^{n-1} \\ &= \begin{cases} 1 - e^{-p\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases} \end{aligned}$$

Thus, $S_N \sim \mathrm{Exponential}(p\lambda)$

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General Transformations and the Jacobian Formula

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General Transformations

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with PDF f_X.

Given $g : \mathbb{R} \to \mathbb{R}$ that is monontone and differentiable with non-zero derivative throughout its domain, what is the PDF of $Y = g(X)$?

General Transformations

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with PDF f_X.

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Note that g admits an inverse, say g^{-1}.

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General Transformations: g Monotone Increasing

$$
\begin{aligned}
F_Y(y) &= \mathbb{P}(\{Y \le y\}) \\
&= \mathbb{P}(\{g(X) \le y\}) \\
&= \mathbb{P}(\{X \le g^{-1}(y)\}) \\
&= \int_{-\infty}^{g^{-1}(y)} f_X(u)\, \mathrm{d}u \\
&\text{make substitution } g(u) = v \\
&= \int_{-\infty}^{y} \underbrace{\frac{f_X(g^{-1}(v))}{g'(g^{-1}(v))}}_{f_Y(v)}\, \mathrm{d}v
\end{aligned}
$$

Thus, we have

$$
f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}, & y \in \mathrm{Range}(g), \\ 0, & y \notin \mathrm{Range}(g). \end{cases}
$$

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General Transformations: g Monotone Decreasing

$$
\begin{aligned}
F_Y(y) &= \mathbb{P}(\{Y \leq y\}) \\
&= \mathbb{P}(\{g(X) \leq y\}) \\
&= \mathbb{P}(\{X \geq g^{-1}(y)\}) \\
&= \int_{g^{-1}(y)}^{+\infty} f_X(u)\,\mathrm{d}u \\
&= \int_{y}^{\infty} \frac{f_X(g^{-1}(v))}{g'(g^{-1}(v))}\,\mathrm{d}v
\end{aligned}
$$

make substitution $g(u) = v$

General Transformations: g Monotone Decreasing

$$
\begin{aligned}
F_Y(y) &= \mathbb{P}(\{Y \leq y\}) \\
&= \mathbb{P}(\{g(X) \leq y\}) \\
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&= \int_{y}^{\infty} \frac{f_X(g^{-1}(v))}{g'(g^{-1}(v))}\,\mathrm{d}v
\end{aligned}
$$

make substitution $g(u) = v$

It thus follows that

$$
f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{-g'(g^{-1}(y))}, & y \in \mathrm{Range}(g), \\ 0, & y \notin \mathrm{Range}(g). \end{cases}
$$

General Transformations: g Monotone, Differentiable

When g is monotone and differentiable throughout its domain:

$$Y = g(X), \qquad f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{\left|g'(g^{-1}(y))\right|}, & y \in \mathrm{Range}(g), \\[2ex] 0, & y \notin \mathrm{Range}(g). \end{cases}$$