AI5030 / EE5817: Probability and Stochastic Processes
Homework 07

Multivariate Transformations, Expectations of Discrete Random Variables

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

1. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim}$ Exponential(λ). Using the bivariate Jacobian transformation formula, compute the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$, and show that $Y_1 \perp\!\!\!\perp Y_2$.

2. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim}$ Exponential(λ). Let $X = X_1$ and $Y = X_1 + X_2$. Determine the joint PDF of X and Y using the bivariate Jacobian transformation formula. Further, for any $y > 0$, show that the conditional PDF of X, conditioned on the event $\{Y = y\}$, is the uniform PDF on the interval $[0, y]$.

3. **(On the Box–Muller Transform)**

   (a) Let $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. Let $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan \frac{Y}{X} = \tan^{-1}\left(\frac{Y}{X}\right)$. Determine the joint PDF of R and Θ, and show that $R \perp\!\!\!\perp \Theta$. What are the marginal PDFs of R and Θ?

   (b) Let R and Θ be two random variables with the joint PDF

   $$f_{R,\Theta}(r, \theta) = r e^{-r^2/2} \cdot \frac{1}{2\pi}, \qquad r \geq 0, \quad \theta \in [0, 2\pi].$$

       i. Compute the marginal PDFs of R and Θ, and show that $R \perp\!\!\!\perp \Theta$.

       ii. Let $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. Show that $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

4. **(Caution to be Exercised When Conditioning on Zero-Probability Events)**

   Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim}$ Exponential(λ).

   Two Ph.D. students of IIT Hyderabad (let us call them S_1 and S_2) are on a mission to come up with their own definitions for what it means to "condition on" the event $\{X_1 = X_2\}$.

   - **First Definition:** Student S_1 reasons that $X_1 = X_2$ if and only if $\frac{X_1}{X_2} = 1$, and therefore finds it apt to define conditioning on the event $\{X_1 = X_2\}$ as conditioning on the event $\left\{\frac{X_1}{X_2} = 1\right\}$.

   - **Second Definition:** Student S_2 reasons that $X_1 = X_2$ if and only if $X_1 - X_2 = 0$, and therefore finds it apt to define conditioning on the event $\{X_1 = X_2\}$ as conditioning on the event $\left\{X_1 - X_2 = 0\right\}$.

   (a) Show formally that $\mathbb{P}(\{X_1 = X_2\}) = 0$.

   (b) Determine the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_2}$.

   (c) Determine the conditional PDF of Y_1, conditioned on the event $\{Y_2 = 1\}$.

   (d) Determine the joint PDF of Y_1 (as defined above) and $Y_3 = X_1 - X_2$.

   (e) Determine the conditional PDF of Y_1, conditioned on the event $\{Y_3 = 0\}$.

   It is clear that $Y_3(\omega) = 0$ if and only if $Y_2(\omega) = 1$. Despite this, conditioning on the event $\{Y_3 = 0\}$ yields a different conditional PDF than conditioning on $\{Y_2 = 1\}$. This problem shows that when conditioning on a zero probability event, one must exercise care to specify the exact definition of conditioning.

5. Fix $n \in \mathbb{N}$. Consider the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, where λ denotes the Lebesgue measure. Compute $\int_{\mathbb{R}} f \, d\lambda$ for each of the following cases.

   (a) $f : \mathbb{R} \to \mathbb{R}$ is defined as

   $$f(\omega) = \begin{cases} \omega, & \omega \in \{0, 1, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $f : \mathbb{R} \to \mathbb{R}$ is defined as
$$f(\omega) = \begin{cases} 1, & \omega \in \mathbb{Q}^c \cap [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

(c) $f : \mathbb{R} \to \mathbb{R}$ is defined as
$$f(\omega) = \begin{cases} n, & \omega \in \mathbb{Q}^c \cap [0,n], \\ 0, & \text{otherwise.} \end{cases}$$

6. Fix $n \in \mathbb{N}$. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$, $\mathscr{F} = 2^\Omega$, and $\mathbb{P}(\{\omega_i\}) = \frac{1}{n}$ for all $i \in \{1, \ldots, n\}$.
   Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathscr{F}$.
   Compute $\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$ for the following cases.

   (a) $X = \mathbf{1}_A$, where $A = \{\omega_1, \ldots, \omega_m\}$, with $1 \le m \le n$.

   (b) X is defined as
   $$X(\omega) = \begin{cases} i, & \omega = \omega_i, \ \omega_i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

7. **(Summation as an Abstract Integral with Respect to the Counting Measure)**
   Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$. For a fixed $c \in \mathbb{R}$, define $\delta_c : \mathscr{F} \to [0,1]$ as

   $$\delta_c(A) = \begin{cases} 1, & c \in A, \\ 0, & c \notin A. \end{cases}$$

   (a) Show that δ_c is a probability measure on $(\Omega, \mathscr{F})$.
      Remark: δ_c is called the Dirac measure at c.

   (b) For any simple function $g : \Omega \to \mathbb{R}$, show that $\int_\Omega g \, d\delta_c = g(c)$.

   (c) Extend the result in part (b) above to the case when g is non-negative.

   (d) Let $\mu : \mathscr{F} \to [0, +\infty]$ be defined as

   $$\mu(A) = \sum_{n=1}^{\infty} \delta_n(A), \qquad A \in \mathscr{F}.$$

   Show that for any simple function $g : \Omega \to \mathbb{R}$,

   $$\int_\Omega g \, d\mu = \sum_{n=1}^{\infty} g(n).$$

   Extend the above result to the case when g is non-negative.
   Remark: Here, μ is a measure on $(\Omega, \mathscr{F})$, and is called the "counting" measure.
   For any given $A \in \mathscr{F}$, $\mu(A)$ is equal to the count of the number of positive integers present in the set A.

8. Suppose that N is a discrete random variable taking values in $\mathbb{N}$. Prove that

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}(\{N > n\}).$$

Hint: Express N as an infinite sum of indicators and use MCT.