



Probability and Stochastic Processes

Lecture 12: Probability Law, Cumulative Distribution Function (CDF),
Properties of CDF

Karthik P. N.

Assistant Professor, Department of AI

Email: pnkarthik@ai.iith.ac.in

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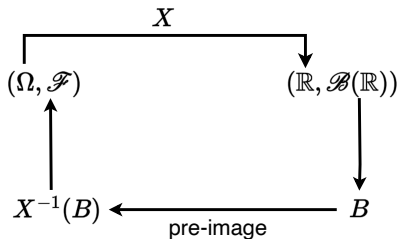
Random Variable

Definition (Random Variable)

Fix a measurable space (Ω, \mathcal{F}) .

A function $X : \Omega \rightarrow \mathbb{R}$ is called a **random variable** if it is measurable, i.e.,

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad \underbrace{X^{-1}(B)}_{\text{pre-image of } B} = \{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\} \in \mathcal{F}.$$



Properties of a Random Variable

Proposition (Random Variable Properties)

Let (Ω, \mathcal{F}) be a measurable space, and let $X : \Omega \rightarrow \mathbb{R}$ be a random variable.

1. For any $B \subseteq \mathbb{R}$, $X^{-1}(B^c) = (X^{-1}(B))^c$.
2. For any $B_1 \subseteq \mathbb{R}, B_2 \subseteq \mathbb{R}, \dots$,

$$X^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} X^{-1}(B_n).$$

3. Let \mathcal{B}_1 denote the collection

$$\mathcal{B}_1 := \left\{ B \subseteq \mathbb{R} : X^{-1}(B) \in \mathcal{F} \right\}.$$

Then, \mathcal{B}_1 is a σ -algebra of subsets of \mathbb{R} . Furthermore, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_1$.

Generating Classes for $\mathcal{B}(\mathbb{R})$

$\mathcal{B}(\mathbb{R})$

$$\mathcal{P}_1 = \left\{ (a, b) : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_2 = \left\{ [a, b] : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_3 = \left\{ [a, b) : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_4 = \left\{ (a, b] : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_5 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_6 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_7 = \left\{ (x, +\infty) : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_8 = \left\{ [x, +\infty) : x \in \mathbb{R} \right\}$$

Equivalent Definitions of Random Variable

Fix a measurable space (Ω, \mathcal{F}) .

Theorem (Equivalent Definitions)

$X : \Omega \rightarrow \mathbb{R}$ is a random variable **if and only if** any one of the following holds:

1. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_1$.
2. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_2$.
3. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_3$.
4. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_4$.
5. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_5$.
6. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_6$.
7. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_7$.
8. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_8$.

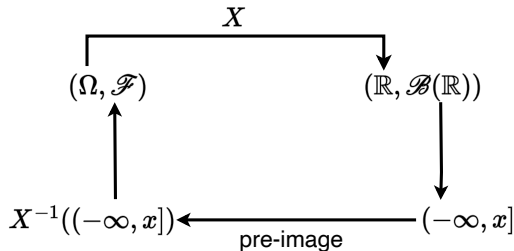
Random Variable Simplified

Definition (Random Variable)

Fix a measurable space (Ω, \mathcal{F}) .

A function $X : \Omega \rightarrow \mathbb{R}$ is called a random variable with respect to \mathcal{F} if and only if

$$\forall x \in \mathbb{R}, \quad \underbrace{X^{-1}((-\infty, x])}_{\text{pre-image of } (-\infty, x]} = \{\omega \in \Omega : X(\omega) \leq x\} = \{X \leq x\} \in \mathcal{F}.$$



Indicator Functions

Fix a sample space Ω .

Fix a subset $A \subseteq \Omega$.

Definition (Indicator Function)

The **indicator function** of set A is the function $\mathbf{1}_A : \Omega \rightarrow \mathbb{R}$ defined as

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

Exercise

Fix a measurable space (Ω, \mathcal{F}) . Show that

$$\mathbf{1}_A \text{ is a random variable} \iff A \in \mathcal{F}.$$

Probability Law of a Random Variable

Probability Law of a Random Variable

- Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- If $X : \Omega \rightarrow \mathbb{R}$ is a random variable, then

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathcal{F}.$$

Therefore, it makes sense to talk about $\mathbb{P}(X^{-1}(B))$ for each $B \in \mathcal{B}(\mathbb{R})$

- We then have a mapping from $\mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$:

$$B \mapsto \mathbb{P}(X^{-1}(B))$$

- The above mapping is called the **probability law** of the random variable X

Probability Law of a Random Variable

Definition (Probability Law)

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable (with respect to \mathcal{F}).

The **probability law** of X is a function $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ defined as

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)), \quad B \in \mathcal{B}(\mathbb{R}).$$

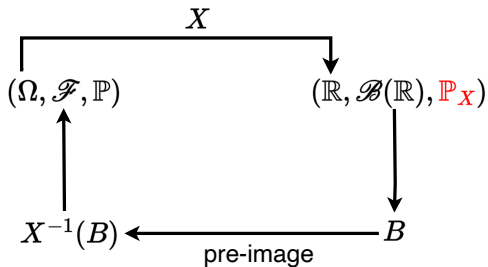
Remarks:

- \mathbb{P}_X is sometimes referred to the **pushforward** of \mathbb{P} under the random variable X
- \mathbb{P}_X is sometimes denoted as $\mathbb{P} \circ X^{-1}$

Proposition (Probability Law)

\mathbb{P}_X is a **probability measure** on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

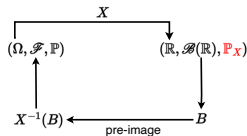
Completing the Picture



$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

Figure: Pictorial representation of probability law

Proof that \mathbb{P}_X is a Probability Measure



$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

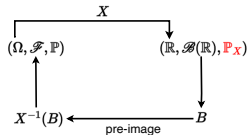
- First, we note that

$$\mathbb{P}_X(\emptyset) = \mathbb{P}(X^{-1}(\emptyset)) = \mathbb{P}(\emptyset) = 0.$$

- Next, we note that

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1.$$

Proof that \mathbb{P}_X is a Probability Measure



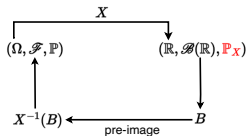
$$\mathbf{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

- Finally, if $B_1, B_2, \dots \in \mathcal{B}(\mathbb{R})$ are mutually disjoint, then

$$\mathbb{P}_X \left(\bigsqcup_{n \in \mathbb{N}} B_n \right) = \mathbb{P} \left(X^{-1} \left(\bigsqcup_{n \in \mathbb{N}} B_n \right) \right) = \mathbb{P} \left(\bigsqcup_{n \in \mathbb{N}} X^{-1}(B_n) \right) = \sum_{n \in \mathbb{N}} \mathbb{P} \left(X^{-1}(B_n) \right) = \sum_{n \in \mathbb{N}} \mathbb{P}_X(B_n).$$

Cumulative Distribution Function

Cumulative Distribution Function (CDF)



$$\mathbf{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

- $\mathbb{P}_X(B) \in [0, 1]$ for every $B \in \mathcal{B}(\mathbb{R})$
- In particular, $\mathbb{P}_X((-\infty, x]) \in [0, 1]$ for every $x \in \mathbb{R}$
- We thus have a mapping

$$x \mapsto \mathbb{P}_X((-\infty, x])$$

- The above mapping (or function) is called the **cumulative distribution function** of the random variable X , denoted by F_X

Cumulative Distribution Function (CDF)

Definition (Cumulative Distribution Function (CDF))

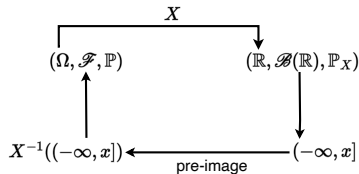
Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable.

The function $F_X : \mathbb{R} \rightarrow [0, 1]$ defined by

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) = \mathbb{P}(\{X \leq x\}), \quad x \in \mathbb{R},$$

is called the **cumulative distribution function (CDF)** of X .



$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}$$

Properties of CDF

Lemma (Properties of CDF)

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable with CDF F_X . Then, F_X satisfies the following properties.

1. **(Monotonicity)** If $x \leq y$, then $F_X(x) \leq F_X(y)$.
2. If x_1, x_2, \dots is any sequence such that $\lim_{n \rightarrow \infty} x_n = -\infty$, then $\lim_{n \rightarrow \infty} F_X(x_n) = 0$.
3. If x_1, x_2, \dots is any sequence such that $\lim_{n \rightarrow \infty} x_n = +\infty$, then $\lim_{n \rightarrow \infty} F_X(x_n) = 1$.

4. **(Right-Continuity)**

F_X is right-continuous at every point in the domain.

More formally, for each $x \in \mathbb{R}$, if x_1, x_2, \dots is a sequence such that $x_1 \geq x_2 \geq \dots$ and $\lim_{n \rightarrow \infty} x_n = x$, then

$$\lim_{n \rightarrow \infty} F_X(x_n) = F_X(x).$$

Proof of Lemma - 1

- Note that

$$\mathbb{F}_X(x) = \mathbb{P}_X((-\infty, x]), \quad \mathbb{F}_X(y) = \mathbb{P}_X((-\infty, y])$$

- If $x \leq y$, then

$$(-\infty, x] \subseteq (-\infty, y]$$

- Using monotonicity property of \mathbb{P}_X , it follows that

$$\mathbb{P}_X((-\infty, x]) \leq \mathbb{P}_X((-\infty, y])$$

Proof of Lemma - 2

- Suppose x_1, x_2, \dots is **monotone decreasing** sequence such that

$$x_1 \geq x_2 \geq \dots, \quad \lim_{n \rightarrow \infty} x_n = -\infty.$$

- Then, we have

$$(-\infty, x_1] \supseteq (-\infty, x_2] \supseteq \dots$$

- Therefore,

$$\lim_{n \rightarrow \infty} F_X(x_n) = \lim_{n \rightarrow \infty} \mathbb{P}_X((-\infty, x_n]) = \mathbb{P}_X\left(\bigcap_{n \in \mathbb{N}} (-\infty, x_n]\right) = \mathbb{P}_X(\emptyset) = 0.$$

Proof of Lemma - 2

- Suppose x_1, x_2, \dots is **any** sequence such that

$$\lim_{n \rightarrow \infty} x_n = -\infty$$

- Let $\gamma_1, \gamma_2, \dots$ be a new sequence defined as

$$\gamma_n = \sup_{k \geq n} x_k, \quad n \in \mathbb{N}$$

- Then, it follows that

$$\gamma_1 \geq \gamma_2 \geq \dots, \quad \lim_{n \rightarrow \infty} \gamma_n = -\infty, \quad \gamma_n \geq x_n \quad \forall n \in \mathbb{N}.$$

- From the previous result for non-increasing sequences,

$$\lim_{n \rightarrow \infty} F_X(\gamma_n) = 0.$$

- $F_X(x_n) \leq F_X(\gamma_n) \quad \forall n \in \mathbb{N} \quad \implies \quad \lim_{n \rightarrow \infty} F_X(x_n) \leq \lim_{n \rightarrow \infty} F_X(\gamma_n) = 0.$

Proof of Lemma – 3

- Left as exercise.

Proof of Lemma - 4

- Fix $x \in \mathbb{R}$
- Suppose that x_1, x_2, \dots is any monotone decreasing sequence such that

$$x_1 \geq x_2 \geq \dots, \quad \lim_{n \rightarrow \infty} x_n = x.$$

- Then, note that

$$(-\infty, x_1] \supseteq (-\infty, x_2] \supseteq \dots, \quad \bigcap_{n \in \mathbb{N}} (-\infty, x_n] = (-\infty, x].$$

- We then have

$$\lim_{n \rightarrow \infty} F_X(x_n) = \lim_{n \rightarrow \infty} \mathbb{P}_X((-\infty, x_n]) = \mathbb{P}_X\left(\bigcap_{n \in \mathbb{N}} (-\infty, x_n]\right) = \mathbb{P}_X((-\infty, x]) = F_X(x).$$

Example

- Fix a measurable space (Ω, \mathcal{F}) . Fix $A \in \mathcal{F}$.
Plot the CDF of the random variable $\mathbf{1}_A$.