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Mathematical Foundations for Data Science (Probability)

Expectations of Random Variables, Variance, Covariance, Correlation Coefficient, Cauchy–Schwartz Inequality, Conditional Expectations

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Expectations of Random Variables

Recap of Expectations: Simple Random Variables

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.

For a simple random variable X in its canonical form

$$X(\omega) = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}(\omega), \qquad \omega \in \Omega,$$

we define $\int_\Omega X \, d\mathbb{P}$ as

$$\int_\Omega X \, d\mathbb{P} := \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i).$$

Recap of Expectations: Non-Negative Random Variables

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ be any random variable with respect to $\mathscr{F}$ such that

$$X(\omega) \geq 0 \qquad \forall \omega \in \Omega.$$

Let

$$\mathcal{S}(X) := \left\{ q : \Omega \to \mathbb{R} : q \text{ simple}\,,\ q(\omega) \leq X(\omega)\ \ \forall \omega \in \Omega \right\}.$$

Then, the expectation of the non-negative random variable X under $\mathbb{P}$ is defined as

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P} := \sup_{q \in \mathcal{S}(X)} \int_\Omega q \, d\mathbb{P}.$$

Remark: It is possible that $\mathbb{E}[X] = +\infty$.

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Recap of Expectations: Arbitrary Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be any random variable with respect to $\mathscr{F}$.

Define

$$X_+(\omega) := \max\{X(\omega), 0\}, \quad \omega \in \Omega, \qquad X_-(\omega) := -\min\{X(\omega), 0\}, \quad \omega \in \Omega.$$

Clearly, both X_+ and X_- are non-negative random variables with respect to $\mathscr{F}$.
We define the expectation of X under $\mathbb{P}$ as

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P} := \mathbb{E}[X_+] - \mathbb{E}[X_-],$$

provided $\min\{\mathbb{E}[X_+], \ \mathbb{E}[X_-]\} < +\infty.$

Recap of Expectations: The Abstract Integral

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be any random variable with respect to $\mathscr{F}$.

For any event $A \in \mathscr{F}$, we define the abstract integral $\int_A X \, d\mathbb{P}$ as

$$\int_A X \, d\mathbb{P} = \int_\Omega (X \cdot \mathbf{1}_A) \, d\mathbb{P},$$

provided the right-hand side is well-defined (i.e., not of the form $\infty - \infty$).

Properties of Expectations - 1

1. For any $A \in \mathscr{F}$, we have
$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$

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3. If $X(\omega) \geq 0$ for all $\omega \in \Omega$, and $\mathbb{P}(\{X = 0\}) = 1$, then
$$\mathbb{E}[X] = 0.$$

Properties of Expectations - 1

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4. If $X(\omega) \geq 0$ for all $\omega \in \Omega$, and $\mathbb{E}[X] = 0$, then
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Properties of Expectations - 2

1. If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, then for any random variable X, we have

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Properties of Expectations - 2

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Properties of Expectations - 2

1. If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, then for any random variable X, we have
$$\mathbb{E}[X \cdot \mathbf{1}_A] = 0.$$
2. If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 1$, then for any random variable X, we have
$$\mathbb{E}[X \cdot \mathbf{1}_A] = \mathbb{E}[X].$$
3. For any $a, b \in \mathbb{R}$ and random variables X and Y,
$$\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y],$$
provided the right-hand side is well defined (i.e., not of the form $\infty - \infty$)

Expectations for Discrete and Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.

- If X is a discrete random variable with PMF p_X, then

$$\mathbb{E}[X] = \sum_{x} x \cdot p_X(x).$$

- If X is a continuous random variable with PDF f_X, then

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx.$$

Expectations of Functions of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.

- If X is a discrete random variable with PMF p_X, then

$$\mathbb{E}[g(X)] = \sum_x g(x) \cdot p_X(x).$$

- If X is a continuous random variable with PDF f_X, then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx.$$

Examples

- Let X be a discrete random variable with the PMF

$$p_X(x) = \begin{cases} 0.1, & x = 1, \\ 0.2, & x = -2, \\ 0.2, & x = 3, \\ 0.5, & x = -4, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\mathbb{E}[X]$.

Examples

- Let X be a discrete random variable with the PMF

$$p_X(x) = \begin{cases} \dfrac{1}{x(x+1)}, & x \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\mathbb{E}[X]$.

Examples

- Let X be a discrete random variable with the PMF

$$p_X(x) = \begin{cases} \dfrac{3}{\pi^2} \cdot \dfrac{1}{x^2}, & x \in \mathbb{Z}, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\mathbb{E}[X]$.

Examples

- Let X be a continuous random variable with the PDF

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \qquad x \in \mathbb{R}.$$

What is $\mathbb{E}[X]$?

Examples

- Compute the mean of $X \sim \mathrm{Ber}(p)$.
- What is the mean of $X \sim \mathrm{Poisson}(\lambda)$?
- What is the mean of $X \sim \mathrm{Unif}([a, b])$?
- What is the mean of $X \sim \mathrm{Exponential}(\mu)$?
- What is the mean of $X \sim \mathcal{N}(\mu, \sigma^2)$?

Variance, Covariance, and Correlation

Variance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be random variable with respect to $\mathscr{F}$.
Let $\mathbb{E}[X]$ be well defined (i.e., not of the form $\infty - \infty$).

Variance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be random variable with respect to $\mathscr{F}$.
Let $\mathbb{E}[X]$ be well defined (i.e., not of the form $\infty - \infty$).

Definition (Variance)

The variance of X is defined as

$$\operatorname{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Remarks:

Variance

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Remarks:

- $\operatorname{Var}(X) \geq 0$.

Variance

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Definition (Variance)

The variance of X is defined as

$$\mathrm{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Remarks:

- $\mathrm{Var}(X) \geq 0$.
- The quantity $\sqrt{\mathrm{Var}(X)}$ is called the standard deviation of X.

A Result on Zero Variance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be random variable with respect to $\mathscr{F}$.
Let $\mathbb{E}[X]$ be well defined (i.e., not of the form $\infty - \infty$).

Lemma (Zero Variance)

The variance of X is zero if and only

$$\mathbb{P}(\{X = c\}) = 1 \qquad \text{for some constant } c.$$

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An Alternative Expression for Variance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be random variable with respect to $\mathscr{F}$.

Alternative Expression for Variance

Let $\mathbb{E}[X]$ be well defined (i.e., not of the form $\infty - \infty$).

1. If $\left|\mathbb{E}[X]\right| = +\infty$, then $\mathrm{Var}(X) = +\infty$.
2. If $\left|\mathbb{E}[X]\right| < +\infty$, then

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2.$$

Examples

- Compute the variance of $X \sim \mathrm{Ber}(p)$.
- What is the variance of $X \sim \mathrm{Poisson}(\lambda)$?
- What is the variance of $X \sim \mathrm{Unif}([a, b])$?
- What is the variance of $X \sim \mathrm{Exponential}(\mu)$?
- What is the variance of $X \sim \mathcal{N}(\mu, \sigma^2)$?
- Give an example of a random variable X for which $\left|\mathbb{E}[X]\right| < +\infty$, but $\mathrm{Var}(X) = +\infty$.

Covariance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be random variables with respect to $\mathscr{F}$.
Let $\mathbb{E}[X], \mathbb{E}[Y]$ be well defined (i.e., not of the form $\infty - \infty$).

Definition (Covariance)

The covariance of X and Y is defined as

$$\mathrm{Cov}(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[\mathbb{Y}])],$$

provided the expectation on the right-hand side is well defined (i.e., not $\infty - \infty$).
Furthermore,

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y],$$

provided the right-hand side is not of the form $\infty - \infty$.

Uncorrelated Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be random variables with respect to $\mathscr{F}$.
Let $\mathbb{E}[X], \mathbb{E}[Y]$ be well defined (i.e., not of the form $\infty - \infty$).

Definition (Uncorrelated Random Variables)

X and Y are said to be uncorrelated if

$$\mathrm{Cov}(X, Y) = 0.$$

Uncorrelatedness and Independence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be random variables with respect to $\mathscr{F}$.
Let $\mathbb{E}[X], \mathbb{E}[Y]$ be well defined (i.e., not of the form $\infty - \infty$).

Theorem (Uncorrelatedness and Independence)

If $X \perp\!\!\!\perp Y$, then
$$\mathrm{Cov}(X, Y) = 0.$$

The converse is not true in general. For example, consider

$$X \sim \mathcal{N}(0, 1).$$

Let $Y = X^2$. Then, it is easy to verify that $\mathbb{E}[XY] = \mathbb{E}[X^3] = 0$, and $\mathbb{E}[X]\mathbb{E}[Y] = 0$.
Therefore, $\mathrm{Cov}(X, Y) = 0$, but $X \not\perp\!\!\!\perp Y$.

Variance of Sum of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be random variables with respect to $\mathscr{F}$.

Lemma (Variance of Sum of Two Random Variables)

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y).$$

Variance of Sum of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be random variables with respect to $\mathscr{F}$.

Lemma (Variance of Sum of Two Random Variables)

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y).$$

Remark:
If X, Y are uncorrelated, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

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Correlation Coefficient and Cauchy–Schwartz Inequality

Correlation Coefficient

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be random variables with respect to $\mathscr{F}$.

Definition (Correlation Coefficient)

The correlation coefficient of X and Y is defined as

$$\rho_{X,Y} := \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}.$$

Correlation Coefficient

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
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The correlation coefficient of X and Y is defined as

$$\rho_{X,Y} := \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}.$$

Remark:

$\rho_{X,Y}$ can be positive, negative, or zero

The Cauchy–Schwartz Inequality

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Theorem (Cauchy–Schwartz Inequality)

For any two random variables X and Y,

$$-1 \leq \rho_{X,Y} \leq 1.$$

Furthermore, the following hold.

1. If $\rho_{X,Y} = 1$, then there exists $a > 0$ such that

$$\mathbb{P}(\{Y - \mathbb{E}[Y] = a\,(X - \mathbb{E}[X])\}) = 1.$$

2. If $\rho_{X,Y} = -1$, then there exists $a < 0$ such that

$$\mathbb{P}(\{Y - \mathbb{E}[Y] = a\,(X - \mathbb{E}[X])\}) = 1.$$

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Conditional Expectations

Conditional Expectation

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be random variables with respect to $\mathscr{F}$.

Objective

To define the following quantities:

- $\mathbb{E}[X|\{Y = y\}]$, for any $y \in \mathbb{R}$.
- $\mathbb{E}[X|Y]$.

Programme:
We shall define the above quantities by considering X discrete/continuous, and Y discrete/continuous.

Case 1: X Discrete, Y Discrete

Let X, Y have the joint PMF $p_{X,Y}$.

- Step 1: Conditional PMF of X, conditioned on the event $\{Y = y\}$:

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x, y)}{p_Y(y)}, \qquad x \in \mathbb{R}.$$

Case 1: X Discrete, Y Discrete

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- Step 1: Conditional PMF of X, conditioned on the event $\{Y = y\}$:

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \qquad x \in \mathbb{R}.$$

- Step 2: The quantity $\mathbb{E}[X|\{Y = y\}]$ is defined as the expectation with respect to the conditional PMF $p_{X|Y=y}$, i.e.,

$$\mathbb{E}[X|\{Y = y\}] := \sum_{x \in \mathbb{R}} x \cdot p_{X|Y=y}(x).$$

Case 1: X Discrete, Y Discrete

Let X, Y have the joint PMF $p_{X,Y}$.

- Step 3: Define the function $\psi_1 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_1(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & p_Y(y) > 0, \\ 0, & p_Y(y) = 0. \end{cases}$$

Case 1: X Discrete, Y Discrete

Let X, Y have the joint PMF $p_{X,Y}$.

- Step 3: Define the function $\psi_1 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_1(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & p_Y(y) > 0, \\ 0, & p_Y(y) = 0. \end{cases}$$

- Step 4: The quantity $\mathbb{E}[X|Y]$ is simply defined as

$$\mathbb{E}[X|Y] = \psi_1(Y).$$

Case 2: X Continuous, Y Continuous

Let X, Y have the joint PDF $f_{X,Y}$.

- Step 1: Conditional PDF of X, conditioned on the event $\{Y = y\}$:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \qquad x \in \mathbb{R}.$$

Case 2: X Continuous, Y Continuous

Let X, Y have the joint PDF $f_{X,Y}$.

- Step 1: Conditional PDF of X, conditioned on the event $\{Y = y\}$:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad x \in \mathbb{R}.$$

- Step 2: The quantity $\mathbb{E}[X|\{Y = y\}]$ is defined as the expectation with respect to the conditional PDF $f_{X|Y=y}$, i.e.,

$$\mathbb{E}[X|\{Y = y\}] := \int_{-\infty}^{+\infty} x \cdot f_{X|Y=y}(x).$$

Case 2: X Continuous, Y Continuous

Let X, Y have the joint PDF $p_{X,Y}$.

- Step 3: Define the function $\psi_2 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_2(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & f_Y(y) > 0, \\ 0, & f_Y(y) = 0. \end{cases}$$

Case 2: X Continuous, Y Continuous

Let X, Y have the joint PDF $p_{X,Y}$.

- Step 3: Define the function $\psi_2 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_2(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & f_Y(y) > 0, \\ 0, & f_Y(y) = 0. \end{cases}$$

- Step 4: The quantity $\mathbb{E}[X|Y]$ is simply defined as

$$\mathbb{E}[X|Y] = \psi_2(Y).$$

Case 3: X Continuous, Y Discrete

- Step 1: Conditional CDF of X, conditioned on the event $\{Y = y\}$:

$$F_{X|Y=y}(x) = \mathbb{P}(\{X \leq x\}|\{Y = y\}), \qquad x \in \mathbb{R}.$$

Case 3: X Continuous, Y Discrete

- Step 1: Conditional CDF of X, conditioned on the event $\{Y = y\}$:

$$F_{X|Y=y}(x) = \mathbb{P}(\{X \leq x\} | \{Y = y\}), \qquad x \in \mathbb{R}.$$

- Step 2: Get the conditional PDF of X, conditioned on the event $\{Y = y\}$:

$$h_y(x) = \frac{d}{dx} F_{X|Y=y}(x).$$

Case 3: X Continuous, Y Discrete

- Step 1: Conditional CDF of X, conditioned on the event $\{Y = y\}$:

$$F_{X|Y=y}(x) = \mathbb{P}(\{X \leq x\}|\{Y = y\}), \qquad x \in \mathbb{R}.$$

- Step 2: Get the conditional PDF of X, conditioned on the event $\{Y = y\}$:

$$h_y(x) = \frac{d}{dx} F_{X|Y=y}(x).$$

- Step 3: The quantity $\mathbb{E}[X|\{Y = y\}]$ is defined as the expectation with respect to the conditional PDF $h_y(x)$, i.e.,

$$\mathbb{E}[X|\{Y = y\}] := \int_{-\infty}^{+\infty} x \cdot h_y(x).$$

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Case 3: X Continuous, Y Discrete

- Step 4: Define the function $\psi_3 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_3(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & p_Y(y) > 0, \\ 0, & p_Y(y) = 0. \end{cases}$$

Case 3: X Continuous, Y Discrete

- Step 4: Define the function $\psi_3 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_3(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & p_Y(y) > 0, \\ 0, & p_Y(y) = 0. \end{cases}$$

- Step 5: The quantity $\mathbb{E}[X|Y]$ is simply defined as

$$\mathbb{E}[X|Y] = \psi_3(Y).$$

Case 4: X Discrete, Y Continuous

- Step 1: The joint probability

$$i(x, y) = \mathbb{P}(\{X = x\} \cap \{Y \leq y\}).$$

Case 4: X Discrete, Y Continuous

- Step 1: The joint probability

$$i(x, y) = \mathbb{P}(\{X = x\} \cap \{Y \leq y\}).$$

- Step 2: Construction of the conditional PMF of X, conditioned on the event $\{Y = y\}$:

$$i_y(x) = \frac{1}{f_Y(y)} \cdot \frac{d}{dy} i(x, y), \qquad x \in \mathbb{R}.$$

Case 4: X Discrete, Y Continuous

- Step 1: The joint probability

$$i(x, y) = \mathbb{P}(\{X = x\} \cap \{Y \leq y\}).$$

- Step 2: Construction of the conditional PMF of X, conditioned on the event $\{Y = y\}$:

$$i_y(x) = \frac{1}{f_Y(y)} \cdot \frac{d}{dy} i(x, y), \qquad x \in \mathbb{R}.$$

- Step 3: The quantity $\mathbb{E}[X|\{Y = y\}]$ is defined as the expectation with respect to the conditional PMF $i_y(x)$, i.e.,

$$\mathbb{E}[X|\{Y = y\}] := \sum_{x \in \mathbb{R}} x \cdot i_y(x).$$

Case 4: X Discrete, Y Continuous

- Step 4: Define the function $\psi_4 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_4(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & p_Y(y) > 0, \\ 0, & p_Y(y) = 0. \end{cases}$$

Case 4: X Discrete, Y Continuous

- Step 4: Define the function $\psi_4 : \mathbb{R} \to \mathbb{R}$ as

$$\psi_4(y) := \begin{cases} \mathbb{E}[X|\{Y = y\}], & p_Y(y) > 0, \\ 0, & p_Y(y) = 0. \end{cases}$$

- Step 5: The quantity $\mathbb{E}[X|Y]$ is simply defined as

$$\mathbb{E}[X|Y] = \psi_4(Y).$$

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Examples