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Stochastic Processes

Stopping Times, Wald’s Lemma, Strong Independence Property,
Properties of Stopping Times, Markov Chains (Intro)

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Wald's Lemma

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma (Wald's Lemma [Wal45])

Let $\{X_n\}_{n=1}^{\infty}$ be an IID process w.r.t. $\mathscr{F}$, with $\mathbb{E}|X_1| < +\infty$.
For each $n \in \mathbb{N}$, let

$$S_n = \sum_{i=1}^{n} X_i.$$

If τ is a stopping time w.r.t. the process $\{X_n\}_{n=1}^{\infty}$, with $\mathbb{E}|\tau| < +\infty$, then

$$\mathbb{E}[S_\tau] = \mathbb{E}\left[\sum_{i=1}^{\tau} X_i\right] = \mathbb{E}[\tau] \cdot \mathbb{E}[X_1].$$

Proof of Wald's Lemma

$$\mathbb{E}[S_\tau] = \mathbb{E}\left[\sum_{i=1}^{\tau} X_i\right]$$

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Proof of Wald's Lemma

$$\begin{aligned}\mathbb{E}[S_\tau] &= \mathbb{E}\left[\sum_{i=1}^{\tau} X_i\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{\infty} X_i\,\mathbf{1}_{\{\tau \geq i\}}\right]\end{aligned}$$

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\begin{aligned}
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&= \mathbb{E}\left[\sum_{i=1}^{\infty} X_i \, \mathbf{1}_{\{\tau \geq i\}}\right] \\
&\overset{(a)}{=} \sum_{i=1}^{\infty} \mathbb{E}\left[X_i \, \mathbf{1}_{\{\tau \geq i\}}\right]
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&= \sum_{i=1}^{\infty} \mathbb{E}[X_i] \cdot \mathbb{E}\left[\mathbf{1}_{\{\tau \geq i\}}\right]
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&= \mathbb{E}[X_1] \cdot \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{\tau \geq i\}}\right]
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&= \mathbb{E}[X_1] \cdot \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{\tau \geq i\}}\right] \overset{(b)}{=} \mathbb{E}[X_1] \cdot \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\tau \geq i\}}\right]
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&= \mathbb{E}[X_1] \cdot \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{\tau \geq i\}}\right] \overset{(b)}{=} \mathbb{E}[X_1] \cdot \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\tau \geq i\}}\right] \overset{(c)}{=} \mathbb{E}[X_1] \cdot \mathbb{E}[\tau]
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\end{aligned}
$$

(a), (b) follow from MCT & $\mathbb{E}|X_1| < +\infty$, (c) follows from $\mathbb{E}|\tau| < +\infty$

Monotone Convergence Theorem

Steps (a) and (b) in the proof of Wald's lemma may be justified using the monotone convergence theorem (MCT).

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Steps (a) and (b) in the proof of Wald's lemma may be justified using the monotone convergence theorem (MCT).

Theorem (Monotone Convergence)

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of RVs such that

$$0 \leq Y_1(\omega) \leq Y_2(\omega) \leq Y_3(\omega) \leq \cdots \qquad \forall \omega \in \Omega.$$

Suppose that $Y_n \xrightarrow{\text{pointwise}} Y$. Then,

$$\mathbb{E}[Y] = \mathbb{E}\left[\lim_{n\to\infty} Y_n\right] = \lim_{n\to\infty} \mathbb{E}[Y_n].$$

Justification for (a) in Wald's Lemma

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- Justification for (a):
  - $Y_n = \sum_{i=1}^{n} (X_i)_+ \mathbf{1}_{\{\tau \geq i\}}, \qquad Z_n = \sum_{i=1}^{n} (X_i)_- \mathbf{1}_{\{\tau \geq i\}}$

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  - For all $\omega \in \Omega$, we have

$$0 \leq Y_1(\omega) \leq Y_2(\omega) \leq Y_3(\omega) \leq \cdots, \qquad 0 \leq Z_1(\omega) \leq Z_2(\omega) \leq Z_3(\omega) \leq \cdots$$

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  - Also, we have

$$Y_n \xrightarrow{\text{pointwise}}$$

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  - Also, we have

$$Y_n \xrightarrow{\text{pointwise}} \sum_{i=1}^{\infty} (X_i)_+ \mathbf{1}_{\{\tau \geq i\}}, \qquad Z_n \xrightarrow{\text{pointwise}} \sum_{i=1}^{\infty} (X_i)_- \mathbf{1}_{\{\tau \geq i\}}$$

  - Using MCT,

$$\mathbb{E}\left[\sum_{i=1}^{\infty} (X_i)_+ \mathbf{1}_{\{\tau \geq i\}}\right] =$$

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  - Also, we have

$$Y_n \overset{\text{pointwise}}{\longrightarrow} \sum_{i=1}^{\infty} (X_i)_+ \mathbf{1}_{\{\tau \geq i\}}, \qquad Z_n \overset{\text{pointwise}}{\longrightarrow} \sum_{i=1}^{\infty} (X_i)_- \mathbf{1}_{\{\tau \geq i\}}$$

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  - Also, we have
$$Y_n \xrightarrow{\text{pointwise}} \sum_{i=1}^{\infty} (X_i)_+ \mathbf{1}_{\{\tau \geq i\}}, \qquad Z_n \xrightarrow{\text{pointwise}} \sum_{i=1}^{\infty} (X_i)_- \mathbf{1}_{\{\tau \geq i\}}$$
  - Using MCT,
$$\mathbb{E}\left[\sum_{i=1}^{\infty} (X_i)_+ \mathbf{1}_{\{\tau \geq i\}}\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[(X_i)_+ \mathbf{1}_{\{\tau \geq i\}}\right], \quad \mathbb{E}\left[\sum_{i=1}^{\infty} (X_i)_- \mathbf{1}_{\{\tau \geq i\}}\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[(X_i)_- \mathbf{1}_{\{\tau \geq i\}}\right]$$
  - $\mathbb{E}|X_1| < +\infty \implies$ RHS quantities are finite $\implies (a)$

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Example

- Suppose $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Geometric}(0.5)$.
  For each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} X_i$.
  Let τ be defined as

$$\tau := \inf\left\{n \geq 1 : S_n = 33\right\}.$$

  Determine $\mathbb{E}[\tau]$.

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Example

- Suppose $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(1,1)$.
  For each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} X_i$.
  Let τ be defined as
  $$\tau := \inf\left\{n \geq 1 : S_n = \frac{\pi}{2}\right\}.$$
  Determine $\mathbb{E}[\tau]$.

Strong Independence Property

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma (Strong Independence Property)

Let $\{X_n\}_{n=1}^{\infty}$ be an independent process w.r.t. $\mathscr{F}$.
Let τ be a stopping time w.r.t. $\{X_n\}_{n=1}^{\infty}$. Then,

$$(X_1, \ldots, X_\tau) \perp\!\!\!\perp (X_{\tau+1}, X_{\tau+2}, \ldots).$$

Properties of Stopping Times

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\{\mathcal{G}_t : t \in \mathcal{T}\}$ be a filtration w.r.t. $\mathscr{F}$.

Lemma (Properties of Stopping Times)

Let τ_1, τ_2 be two stopping times w.r.t. the filtration $\{\mathcal{G}_t : t \in \mathcal{T}\}$.

1. $\min\{\tau_1, \tau_2\}$ is a stopping time.
2. If $\mathcal{T} = \mathbb{R}_+$, then
$$\tau_1 + \tau_2 \quad \text{is a stopping time.}$$

Proof of Lemma

For any $t \in \mathcal{T}$,

$$\{\min\{\tau_1, \tau_2\} > t\} =$$

Proof of Lemma

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$$\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{G}_t,$$

$$\{\tau_1 + \tau_2 \leq t\} = \bigcup_{\substack{q \in \mathcal{Q}: \\ 0 \leq q \leq t}} \{\tau_1 \leq q \leq t - \tau_2\}$$

Proof of Lemma

For any $t \in \mathcal{T}$,

$$\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{G}_t,$$

$$\begin{aligned}\{\tau_1 + \tau_2 \leq t\} &= \bigcup_{\substack{q \in \mathbb{Q}: \\ 0 \leq q \leq t}} \{\tau_1 \leq q \leq t - \tau_2\} \\ &= \bigcup_{\substack{q \in \mathbb{Q}: \\ 0 \leq q \leq t}} \{\tau_1 \leq q\} \cap \{\tau_2 \leq t - q\}\end{aligned}$$

Markov Chain

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Markov Chain)

A process $\{X_t : t \in \mathcal{T}\}$ is called a Markov chain if for any $t \in \mathcal{T}$,

$$(X_s : s < t) \perp\!\!\!\perp (X_s : s > t) \mid X_t,$$

References

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Some generalizations of the theory of cumulative sums of random variables.
The Annals of Mathematical Statistics, 16(3):287–293, 1945.