1. $X(t)$ is wide-sense stationary if and only if

$\mathbb{E}[X(t)]$ is constant wrt t, $\quad Cov(X(t), X(t+h)) = Cov(X(0), X(h))$
$\forall\ t, h \in \mathbb{R}_+$.

Observe that

$$\mathbb{E}[X(t)] = \mathbb{E}[X_1] \cos(2\pi f_c t) + \mathbb{E}[X_2] \sin(2\pi f_c t).$$

In particular,

$$\mathbb{E}[X(0)] = \mathbb{E}[X_1], \qquad \mathbb{E}[X(1/f_c)] = \mathbb{E}[X_2].$$

It then follows that

$$\mathbb{E}[X(t)] \text{ is constant wrt } t \iff \mathbb{E}[X_1] = 0 = \mathbb{E}[X_2]. \rightarrow (1)$$

Furthermore, for any $h \in \mathbb{R}_+$,

$$Cov(X(0), X(h)) = Cov(\underbrace{X_1}_{0 \text{ mean}}, \underbrace{X_1 \cos(2\pi f_c h) + X_2 \sin(2\pi f_c h)}_{0 \text{ mean}})$$
$$= \mathbb{E}[X_1 \cdot (X_1 \cos(2\pi f_c h) + X_2 \sin(2\pi f_c h))]$$
$$= \mathbb{E}[X_1^2] \cos(2\pi f_c h) + \mathbb{E}[X_1 X_2] \sin(2\pi f_c h),$$

whereas for $t, h \in \mathbb{R}_+$,

$$Cov(X(t), X(t+h)) = \mathbb{E}[X(t)\ X(t+h)] \quad (\text{because } \mathbb{E}[X(t)] = 0 = \mathbb{E}[X(t+h)])$$
$$= \mathbb{E}[X_1^2] \cos(2\pi f_c t) \cos(2\pi f_c t + 2\pi f_c h)$$
$$+ \mathbb{E}[X_1 X_2] \big(\cos(2\pi f_c t) \sin(2\pi f_c t + 2\pi f_c h)$$
$$+ \sin(2\pi f_c t) \cos(2\pi f_c t + 2\pi f_c h) \big)$$
$$+ \mathbb{E}[X_2^2] \sin(2\pi f_c t) \sin(2\pi f_c t + \sin(2\pi f_c h))$$

add and subtract
$\mathbb{E}[X_1^2] \sin(2\pi f_c t) \sin(2\pi f_c (t+h))$;
use identities
$\cos(A-B) = \cos A \cos B + \sin A \sin B$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \underbrace{\mathbb{E}[X_1^2] \cos(2\pi f_c h)}_{\text{function of } h \text{ only}}$$
$$+ \mathbb{E}[X_1 X_2] \sin(4\pi f_c t + 2\pi f_c h)$$
$$+ (\mathbb{E}[X_2^2] - \mathbb{E}[X_1^2]) \sin(2\pi f_c t) \sin(2\pi f_c t + 2\pi f_c h)$$

We then see that $\mathbb{E}[X(t)\ X(t+h)]$ will be independent of t

if and only if

$$E[X_1 X_2] = 0, \qquad E[X_1^2] = E[X_2^2]. \longrightarrow (2)$$

Thus, $\{X(t) : t \geq 0\}$ is wide-sense stationary if and only if the conditions in (1) and (2) are satisfied, i.e.,

a) $E[X_1] = 0 = E[X_2]$.
b) $E[X_1 X_2] = 0$.
c) $E[X_1^2] = E[X_2^2]$.

---

2. We note that

$$E[X_n] = \begin{cases} E[U_n] = 0, & n \text{ odd} \\ \frac{1}{\sqrt{2}}\left(E[U_n^2] - 1\right) = 0, & n \text{ even}. \end{cases}$$

Thus, $E[X_n] = 0 \quad \forall\, n \in \mathbb{N}$. Furthermore, because $\{U_n\}_{n=1}^{\infty}$ is i.i.d., it follows that $E[X_m X_n] = 1_{\{m=n\}}$, thus proving that $\{X_n\}_{n=1}^{\infty}$ is wide-sense stationary.

To show that $\{X_n\}_{n=1}^{\infty}$ is not stationary, it suffices to show that the CDFs of X_1 and X_2 are different. Indeed,

$$\mathbb{P}(X_1 \geq 0) = \mathbb{P}(U_1 \geq 0) = 1/2, \quad \text{while}$$

$$\mathbb{P}(X_2 > 0) = \mathbb{P}(U_2^2 \geq \sqrt{2}) = \mathbb{P}(U_1^2 \geq \sqrt{2}) \neq \mathbb{P}(U_1^2 \geq 0).$$

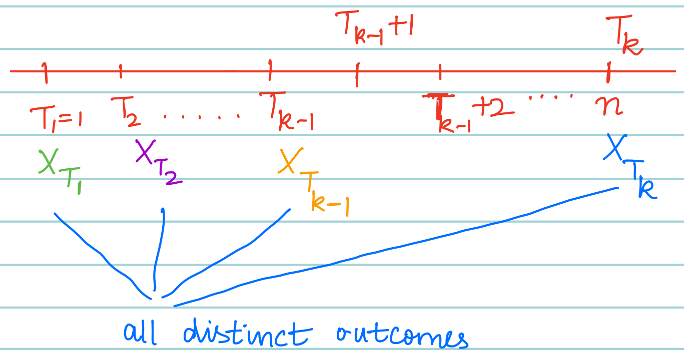
3.
a) T_k represents the first instant of observing the k^{th} distinct outcome.

b) Clearly, $T_1 = 1$ is a stopping time, as

$$\{T_1 = n\} = \begin{cases} \Omega, & n = 1, \\ \phi, & n \neq 1, \end{cases}$$

and therefore $\{T_1 = n\} \in \sigma(X_1, \ldots, X_n) \quad \forall\, n \in \mathbb{N}$.

Assume, by induction, that T_{k-1} is a stopping time for some $k>1$.



$$\{T_k=n\} = \{X_{T_{k-1}+1} \in \{X_{T_1},\dots,X_{T_{k-1}}\}, \dots, X_{n-1}\in\{X_{T_1},\dots,X_{T_{k-1}}\}, X_n \notin \{X_{T_1},\dots,X_{T_{k-1}}\}\}$$

$$= \bigcup_{m\le n} \{T_{k-1}=m, X_{m+1}\in\{X_1,\dots,X_m\}; \dots, X_{n-1}\in\{X_1,\dots,X_m\}, X_n \notin \{X_1,\dots,X_m\}\}$$

$$\in \sigma(X_1,\dots,X_n) \quad \forall\ n\in\mathbb{N}.$$

Furthermore, we have $\mathbb{P}(T_1<+\infty)=1$ by definition of T_1.
Assume, by induction, that $\mathbb{P}(T_{k-1}<+\infty)=1$ for some $k>1$.
Then,

all finite

$$T_k = T_k - T_{k-1} + T_{k-1} + T_{k-2} + \dots + T_2 - T_1 + T_1$$

Thus, $T_k<+\infty$ will occur if and only if $T_k - T_{k-1}<+\infty$ occurs.

$$\Rightarrow \{T_k<+\infty\} = \{T_k-T_{k-1}<+\infty\}.$$

$$\Rightarrow \mathbb{P}(T_k<+\infty) = \mathbb{P}(T_k-T_{k-1}<+\infty)$$

$$= \sum_{n\in\mathbb{N}} \mathbb{P}(T_k-T_{k-1}=n)$$

for $(n-1)$ time instants, one of the previously seen $(k-1)$ outcomes must occur, & at n^{th} instant, the k^{th} distinct outcome must occur

$$= \sum_{n\in\mathbb{N}} \left(\frac{k-1}{K}\right)^{n-1}\left(1-\frac{k-1}{K}\right)$$

$$= 1.$$

Thus, T_k is a stopping time wrt $\{X_n\}_{n=1}^{\infty}$ for all $k\in\{1,\dots,6\}$.

c) As already seen above,

$$S_k \sim \text{Geometric}\left(1-\frac{k-1}{K}\right), \quad k\in\{2,\dots,6\}.$$

d) We have

$$\mathbb{E}[T_k] = \mathbb{E}[T_k - T_{k-1}] + \mathbb{E}[T_{k-1} - T_{k-2}] + \dots + \mathbb{E}[T_2 - T_1] + \mathbb{E}[T_1]$$

$$= 1 + \sum_{i=2}^{k} \mathbb{E}[S_i]$$

$$= 1 + \sum_{i=2}^{k} \frac{1}{1 - \frac{i-1}{K}} \qquad \left(\mathbb{E}[X] = \frac{1}{p} \text{ if } X \sim \text{Geo}(p) \right)$$

$$= 1 + \sum_{i=1}^{k-1} \frac{1}{1 - \frac{i}{K}}$$

$$= 1 + \sum_{i=1}^{k-1} \frac{K}{K-i} \quad , \quad k \in \{2, \dots, 6\}.$$

4.

a) Observe that

$$\Omega = X^{-1}(\mathbb{R}) \quad , \quad \text{and} \quad \mathbb{R} \in \mathcal{B}(\mathbb{R}).$$

$$\Rightarrow \quad \Omega \in \sigma(X).$$

Next, suppose that $A \in \sigma(X)$.

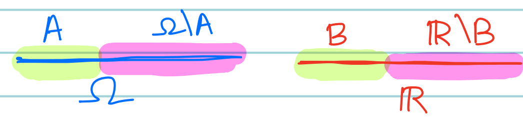
$$\Rightarrow \quad \exists B \in \mathcal{B}(\mathbb{R}) \text{ such that } A = X^{-1}(B).$$

$$\Rightarrow A = \{\omega \in \Omega : X(\omega) \in B\}$$

We then have

$$A^c = \Omega \setminus A$$

$$= \{\omega \in \Omega : X(\omega) \notin B\}$$



$$= \{\omega \in \Omega : X(\omega) \in \mathbb{R} \setminus B\}$$

$$= X^{-1}(\mathbb{R} \setminus B) \quad , \quad \text{and } \mathbb{R} \setminus B \in \mathcal{B}(\mathbb{R}) \text{ as } \mathbb{R}, B \in \mathcal{B}(\mathbb{R}).$$

$$\Rightarrow \quad A^c \in \sigma(X).$$

Finally, if $A_1, A_2, \dots \in \sigma(X)$, then $\exists B_1, B_2, \dots \in \mathcal{B}(\mathbb{R})$ such that

$$A_i = X^{-1}(B_i) \quad \forall \; i \in \mathbb{N}$$

$$\Rightarrow \quad \bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} X^{-1}(B_i)$$

$$= X^{-1}\left(\bigcup_{i \in \mathbb{N}} B_i \right) \quad , \quad \text{and } \bigcup_{i \in \mathbb{N}} B_i \in \mathcal{B}(\mathbb{R})$$

$$\Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \sigma(X).$$

Thus, $\sigma(X)$ is a σ-algebra of subsets of Ω.

---

b) First, we observe that $\Omega \in \mathcal{F}_\tau$, as

$$\Omega \cap \{\tau \le t\} = \{\tau \le t\} \in \mathcal{F}_t \quad \forall\ t \in \mathcal{T},$$ by the fact that τ is a stopping time.

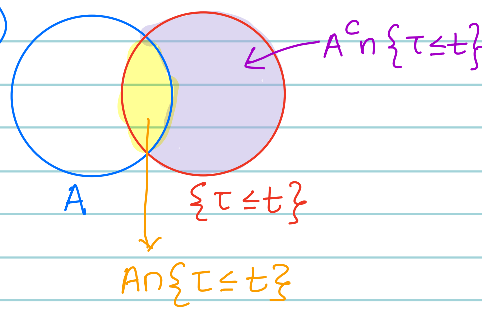
Next, suppose that $A \in \mathcal{F}_\tau$. That is,

$$A \cap \{\tau \le t\} \in \mathcal{F}_t \quad \forall\ t \in \mathcal{T}.$$

Then, for any $t \in \mathcal{T}$,

$$A^c \cap \{\tau \le t\} = \underbrace{\{\tau \le t\}}_{\in \mathcal{F}_t} \setminus \Big(\underbrace{A \cap \{\tau \le t\}}_{\in \mathcal{F}_t}\Big)$$

$$\in \mathcal{F}_t.$$



$$\Rightarrow A^c \cap \{\tau \le t\} \in \mathcal{F}_t \quad \forall\ t \in \mathcal{T}$$

$$\Rightarrow A^c \in \mathcal{F}_\tau.$$

Finally, suppose that $A_1, A_2, \ldots \in \mathcal{F}_\tau$

$$\Rightarrow A_1 \cap \{\tau \le t\} \in \mathcal{F}_t \quad \forall\ t \in \mathcal{T}$$

$$A_2 \cap \{\tau \le t\} \in \mathcal{F}_t \quad \forall\ t \in \mathcal{T}$$

$$\vdots$$

Then, for any $t \in \mathcal{T}$,

$$\Big(\bigcup_{i \in \mathbb{N}} A_i\Big) \cap \{\tau \le t\} \overset{\text{de-Morgan's law}}{=} \bigcup_{i \in \mathbb{N}} \Big(\underbrace{A_i \cap \{\tau \le t\}}_{\in \mathcal{F}_t}\Big)$$

$$\in \mathcal{F}_t.$$

$$\Rightarrow \Big(\bigcup_{i \in \mathbb{N}} A_i\Big) \cap \{\tau \le t\} \in \mathcal{F}_t \quad \forall\ t \in \mathcal{T}$$

$$\Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}_\tau .$$

Thus, $\mathcal{F}_\tau$ is a σ-algebra of subsets of Ω.

5.

a) Observe that

$$\{N > n\} = \{\text{no two consecutive ones are observed in } X_1, \dots, X_n\}$$

$$= \{X_1 X_2 = 0, \; X_2 X_3 = 0, \dots, X_{n-1} X_n = 0\} \quad (\text{each } X_i \in \{0,1\})$$

$$= \{X_1 X_2 + X_2 X_3 + \dots + X_{n-1} X_n = 0\}$$

$$\in \sigma(X_1, \dots, X_n). \qquad \forall \; n \in \mathbb{N}$$

$$\Rightarrow \{N \le n\} \in \sigma(X_1, \dots, X_n) \quad \forall \; n \in \mathbb{N}.$$

Furthermore, Let

$$a_n := \mathbb{P}(N > n) = \mathbb{P}(X_1 X_2 + \dots + X_{n-1} X_n = 0).$$

Now,

$$\{N > n\} \subseteq \{N > n-1\} \quad \forall \; n \in \mathbb{N}$$

$$\Rightarrow \bigcap_{n \in \mathbb{N}} \{N > n\} = \lim_{n \to \infty} \{N > n\} = \{N = +\infty\}.$$

$$\Rightarrow \mathbb{P}(N = +\infty) = \lim_{n \to \infty} \mathbb{P}(N > n) = \lim_{n \to \infty} a_n .$$

Continuity of Probability

We now show $\lim_{n \to \infty} a_n = 0$, thereby proving that $\mathbb{P}(N < +\infty) = 1$.

We have

$$a_n = \mathbb{P}(X_1 X_2 + \dots + X_{n-1} X_n = 0)$$

$$= \mathbb{P}(X_1 X_2 + \dots + X_{n-1} X_n = 0, \; X_1 = 0) + \mathbb{P}(X_1 X_2 + \dots + X_{n-1} X_n = 0, \; X_1 = 1)$$

Law of total prob.

$$= \mathbb{P}(X_2 X_3 + \dots + X_{n-1} X_n = 0, \; X_1 = 0)$$

$$+\ \mathbb{P}\left(X_2 + X_2X_3 + \cdots + X_{n-1}X_n = 0,\ X_1 = 1\right)$$

independence of X_i's

$$= \mathbb{P}(X_1=0)\cdot \mathbb{P}\left(X_2X_3+\cdots+X_{n-1}X_n=0\right)$$

$$+\ \mathbb{P}\left(X_1=1,\ X_2=0,\ X_3X_4+\cdots+X_{n-1}X_n=0\right)$$

independence of X_i's

$$= \frac{1}{2}\cdot \mathbb{P}\left(\text{no 2 consecutive ones in } X_2,\ldots,X_n\right)$$

$$+\ \mathbb{P}(X_1=1)\ \mathbb{P}(X_2=0)\cdot \mathbb{P}\left(\text{no 2 consecutive ones in } X_3,\ldots,X_n\right)$$

Noting that

identically distributed as

$$\left.\begin{aligned}(X_2,\ldots,X_n) &\overset{d}{=} (X_1,\ldots,X_{n-1}),\\ (X_3,\ldots,X_n) &\overset{d}{=} (X_1,\ldots,X_{n-2}).\end{aligned}\right\}$$

Stationarity of $\{X_i\}_{i\in\mathbb{N}}$, Courtesy of iid nature of X_i's.

it follows that

$$\mathbb{P}\left(X_2X_3+\cdots+X_{n-1}X_n=0\right) = \mathbb{P}\left(X_1X_2+\cdots+X_{n-2}X_{n-1}=0\right) = a_{n-1},$$

$$\mathbb{P}\left(X_3X_4+\cdots+X_{n-1}X_n=0\right) = \mathbb{P}\left(X_1X_2+\cdots+X_{n-3}X_{n-2}=0\right) = a_{n-2}.$$

Thus, we get

$$a_n = \tfrac{1}{2}\,a_{n-1} + \tfrac{1}{4}\,a_{n-2}.$$

Taking limits as $n\to\infty$ on both sides, we get

$$\lim_{n\to\infty} a_n = \tfrac{1}{2}\lim_{n\to\infty} a_{n-1} + \tfrac{1}{4}\lim_{n\to\infty} a_{n-2}$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0.$$

This proves that N is a stopping time w.r.t $\{X_n\}_{n\in\mathbb{N}}$.

b) We have

because $\mathbb{P}(N<+\infty)=1$

$$\mathbb{P}\left(X_{N+1}=0,\ X_{N+2}=0\right) = \mathbb{P}\left(X_{N+1}=0,\ X_{N+2}=0,\ N<+\infty\right)$$

Law of total prob.

$$= \sum_{n\in\mathbb{N}} \mathbb{P}\Big(X_{n+1}=0,\ X_{n+2}=0,\ \underbrace{N=n}_{\in\,\sigma(X_1,\ldots,X_n)}\Big)$$

N is a stopping time and X_i's are indep.

$$= \sum_{n\in\mathbb{N}} \mathbb{P}(X_{n+1}=0)\cdot \mathbb{P}(X_{n+2}=0)\cdot \mathbb{P}(N=n)$$

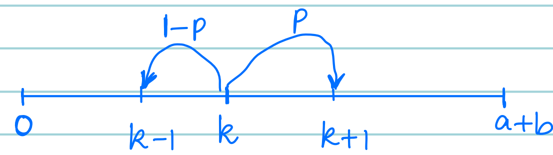
X_i's are i.d.

$$= \frac{1}{4} \sum_{n \in \mathbb{N}} \mathbb{P}(N = n)$$

$$= \frac{1}{4} \mathbb{P}(N < +\infty) = \frac{1}{4}.$$

6.

a) $a[0] = 0, \quad a[a+b] = 1.$



(b) Suppose A has \$k at some time n. Let X_n^A denote this value, i.e., $X_n^A = k$.

$$a[k] = \mathbb{P}(A \text{ wins} \mid X_n^A = k)$$

$$= \mathbb{P}(A \text{ wins}, X_{n+1}^A = k+1 \mid X_n^A = k) + \mathbb{P}(A \text{ wins}, X_{n+1}^A = k-1 \mid X_n^A = k)$$

$$= \mathbb{P}(X_{n+1}^A = k+1 \mid X_n^A = k) \cdot \mathbb{P}(A \text{ wins} \mid X_{n+1}^A = k+1, X_n^A = k)$$
$$+ \mathbb{P}(X_{n+1}^A = k-1 \mid X_n^A = k) \cdot \mathbb{P}(A \text{ wins} \mid X_{n+1}^A = k-1, X_n^A = k)$$

once we know X_{n+1}^A, we can discard X_n^A

$$= p \cdot \mathbb{P}(A \text{ wins} \mid X_{n+1}^A = k+1) + (1-p)\, \mathbb{P}(A \text{ wins} \mid X_{n+1}^A = k-1)$$

$$= p\, a[k+1] + (1-p)\, a[k-1]$$

(c) Rearranging terms, we get

$$p\left(a[k+1] - a[k]\right) = (1-p)\left(a[k] - a[k-1]\right)$$

$$\Rightarrow a[k+1] - a[k] = \left(\frac{1-p}{p}\right)\left(a[k] - a[k-1]\right)$$

$$= \left(\frac{1-p}{p}\right)^2 \left(a[k-1] - a[k-2]\right)$$

$$\vdots$$

$$= \left(\frac{1-p}{p}\right)^k \left(a[1] - a[0]\right)$$

Using the results of part (a), we get

$$a[2] - a[1] = \left(\frac{1-p}{p}\right)\left(a[1] - a[0]\right)$$

$$a[3] - a[2] = \left(\frac{1-p}{p}\right)^2 \left(a[1] - a[0]\right)$$

$\vdots$

$$a[k] = \left(a[k] - a[k-1]\right) + a[k-1] - a[k-2] + \cdots + a[2] - a[1] + a[1] - a[0] + a[0]$$

$$= \sum_{i=1}^{k} \left(a[i] - a[i-1]\right)$$

$$= \sum_{i=1}^{k} \left(\frac{1-p}{p}\right)^{i-1} \left(a[1] - a[0]\right)$$

$$= a[1] \cdot \sum_{i=0}^{k-1} \left(\frac{1-p}{p}\right)^{i}$$

i) If $p = 1/2$, then

$$a[k] = k\, a[1].$$

Using $a[a+b] = 1$, we get $a[1] = \frac{1}{a+b}$.

$$\Rightarrow \quad a[k] = \frac{k}{a+b}, \quad k \in \{0, \ldots, a+b\}.$$

ii) If $p > 1/2$, then $\frac{1-p}{p} < 1$, and

$$a[k] = a[1] \cdot \frac{1 - \left(\frac{1-p}{p}\right)^{k}}{1 - \left(\frac{1-p}{p}\right)}.$$

Using $a[a+b] = 1$, we get

$$a[1] = \frac{1 - \left(\frac{1-p}{p}\right)}{1 - \left(\frac{1-p}{p}\right)^{a+b}}, \quad \text{and hence}$$

$$a[k] = \frac{1 - \left(\frac{1-p}{p}\right)^{k}}{1 - \left(\frac{1-p}{p}\right)^{a+b}}, \quad k \in \{0, \ldots, a+b\}.$$

(iii) If $p < 1/2$, then $\frac{1-p}{p} > 1$, and it is easy to show that

$$a[k] = \frac{\left(\frac{1-p}{p}\right)^{k} - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1}, \quad k \in \{0, \ldots, a+b\}.$$

d) $\mathbb{P}(A \text{ ruins } B) = a[a] = \begin{cases} \dfrac{a}{a+b}, & p = 1/2, \\ \dfrac{1 - \left(\frac{1-p}{p}\right)^a}{1 - \left(\frac{1-p}{p}\right)^{a+b}}, & p > 1/2, \\ \dfrac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1}, & p < 1/2. \end{cases}$

7.

a) Observe that for any $n \in \mathbb{N}$,

$$\{N > n\} = \{X_1 \geq X_2 \geq \cdots \geq X_n\} \in \sigma(X_1, \ldots, X_n).$$

Furthermore,

$$\mathbb{P}(N > n) = \mathbb{P}(X_1 \geq X_2 \geq \cdots \geq X_n)$$

$$= \underbrace{\int_0^1}_{\text{limits for } x_1} \underbrace{\int_0^{x_1}}_{\text{limits for } x_2} \int_0^{x_2} \cdots \underbrace{\int_0^{x_{n-1}}}_{\text{limits for } x_n} dx_n \cdots dx_2 \, dx_1$$

$$= \frac{1}{n!}.$$

Thus, we have $\lim_{n \to \infty} \mathbb{P}(N > n) = 0$, from which it follows that

$$\mathbb{P}(N < +\infty) = 1 - \mathbb{P}(N = +\infty)$$

$$= 1 - \mathbb{P}\left(\bigcap_{n \in \mathbb{N}} \{N > n\}\right)$$

$$= 1 - \lim_{n \to \infty} \mathbb{P}(N > n)$$

$$= 1.$$

Continuity of probability, observing that $\{N > n\} \subseteq \{N > n-1\}$ $\forall n \in \mathbb{N}$

Thus, N is a stopping time wrt $\{X_n\}_{n \in \mathbb{N}}$.

b) We have

$$\mathbb{E}[N] = \sum_{n \in \mathbb{N}} n \cdot \mathbb{P}(N=n) + (+\infty)\,\mathbb{P}\underbrace{(N=+\infty)}_{=0}$$

$$= \sum_{n=2}^{\infty} n \cdot \mathbb{P}(N=n) \quad \left(\text{because } \mathbb{P}(N=1)=0 \text{ by def}^n\right)$$

$$= \sum_{n=2}^{\infty} n \left(\mathbb{P}(N>n-1) - \mathbb{P}(N>n) \right)$$

$$= \sum_{n=2}^{\infty} n \left(\frac{1}{(n-1)!} - \frac{1}{n!} \right)$$

$$= \sum_{n=2}^{\infty} \frac{1}{(n-2)!}$$

$$= \sum_{m=0}^{\infty} 1/m! = e. \quad \left(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)$$

Using Wald's Lemma,

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}[N] \cdot \mathbb{E}[X_1]$$

$$= \frac{e}{2}.$$