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Stochastic Processes

Recurrence, Transience, Communicating Classes, Class Properties, Irreducibility, Periodicity

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Some Tidbits

- For $x, y \in \mathcal{X}$, let

$$f_{xy}^{(n)} := \mathbb{P}(\tau_y^{(1)} = n \mid X_0 = x).$$

$f_{xy}^{(n)}$: **probability of first visit to state y at time n, starting from state x.**

- Let f_{xy} be defined as

$$f_{xy} = \mathbb{P}(\tau_y^{(1)} < +\infty \mid X_0 = x) = \sum_{n \in \mathbb{N}} f_{xy}^{(n)}.$$

f_{xy}: **probability of eventually visiting state y, starting from state x.**

Some Tidbits

- By the law of total probability,

$$\mathbb{P}(\tau_y^{(1)} < +\infty) = \sum_{x \in \mathcal{X}} \mathbb{P}(\tau_y^{(1)} < +\infty \mid X_0 = x) \cdot \mathbb{P}(X_0 = x) = \sum_{x \in \mathcal{X}} f_{xy} \cdot \mathbb{P}(X_0 = x).$$

- If $f_{xy} = 1$ for all $x \in \mathcal{X}$, then $\mathbb{P}(\tau_y^{(1)} < +\infty) = 1$, and hence $\tau_y^{(1)}$ is a stopping time.
- $1 - f_{xy}$: probability that starting from x, the state y is **never** visited

Recurrent and Transient States

Definition (Recurrent and Transient States)

A state $x \in \mathcal{X}$ is called recurrent if $f_{xx} = 1$.
If $f_{xx} < 1$, then x is called a transient state.

Remarks:

- The collection

$$\{f_{xx}^{(1)}, f_{xx}^{(2)}, \ldots, \; 1 - f_{xx}\}$$

  defines a valid PMF on $\mathbb{N} \cup \{+\infty\}$.
- The above PMF is called first recurrence time distribution.

Mean Recurrence Time

Definition (Mean Recurrence Time)

The mean recurrence time of a state $x \in \mathcal{X}$ is denoted by μ_{xx} and is defined by

$$\mu_{xx} := \mathbb{E}[\tau_x^{(1)} \mid X_0 = x].$$

Remarks:

- If x is transient, then $\mu_{xx} = +\infty$.
- If x is recurrent, then

$$\mu_{xx} = \sum_{n \in \mathbb{N}} n f_{xx}^{(n)}.$$

Positive Recurrent and Null Recurrent States

Definition (Positive / Null Recurrent States)

A recurrent state $x \in \mathcal{X}$ is called positive recurrent if $\mu_{xx} < +\infty$.
Else, if $\mu_{xx} = +\infty$, then x is called null recurrent.

Proposition

Proposition

Fix $x, y \in \mathcal{X}$. Let $N_y = \#$ visits to state y. Then,

$$\mathbb{P}(N_y = k \mid X_0 = x) = \begin{cases} 1 - f_{xy}, & k = 0, \\ f_{xy}\left(f_{yy}\right)^{k-1}\left(1 - f_{yy}\right), & k \in \mathbb{N}. \end{cases}$$

Corollary to Proposition

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Corollary

For any $x, y \in \mathcal{X}$.

$$\mathbb{P}(N_y < +\infty \mid X_0 = x) =$$

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Corollary

For any $x, y \in \mathcal{X}$.

$$\mathbb{P}(N_y < +\infty \mid X_0 = x) = \begin{cases} 1, & f_{yy} < 1, \\ 1 - f_{xy}, & f_{yy} = 1. \end{cases}$$

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Corollary

For any $x, y \in \mathcal{X}$,

$$\mathbb{P}(N_y = +\infty \mid X_0 = x) = \begin{cases} 0, & f_{yy} < 1, \\ f_{xy}, & f_{yy} = 1. \end{cases}$$

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Corollary to Proposition

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Corollary

For any $x, y \in \mathcal{X}$,

$$\mathbb{E}[N_y \mid X_0 = x] =$$

Corollary to Proposition

Proposition

Fix $x, y \in \mathcal{X}$. Then,

$$\mathbb{P}(N_y = k \mid X_0 = x) = \begin{cases} 1 - f_{xy}, & k = 0, \\ f_{xy}\left(f_{yy}\right)^{k-1}\left(1 - f_{yy}\right), & k \in \mathbb{N}. \end{cases}$$

Corollary

For any $x, y \in \mathcal{X}$,

$$\mathbb{E}[N_y \mid X_0 = x] = \begin{cases} \frac{f_{xy}}{1-f_{yy}}, & f_{yy} < 1, \\ +\infty, & f_{yy} = 1,\ f_{xy} > 0, \\ 0, & f_{yy} = 1, f_{xy} = 0. \end{cases}$$

Transience and Recurrence in a Nutshell

Proposition

Consider a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ with TPM P.

1. A state $y \in \mathcal{X}$ is transient if and only if $\mathbb{E}[N_y \mid X_0 = y] < +\infty$.
2. A state $y \in \mathcal{X}$ is recurrent if and only if $\mathbb{E}[N_y \mid X_0 = y] = +\infty$.

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Remark: Note that $N_y = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{X_n = y\}}$. Therefore,

$$\mathbb{E}[N_y \mid X_0 = y] = \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbf{1}_{\{X_n = y\}} \mid X_0 = y\right] = \sum_{n \in \mathbb{N}} \mathbb{P}(X_n = y | X_0 = y) = \sum_{n \in \mathbb{N}} P^n_{y,y}.$$

Transience and Recurrence in a Nutshell

- With probability 1, a transient state y is visited only finitely many times, no matter what initial state x the Markov chain starts in

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- If y is recurrent and the Markov chain starts in y, then y will be visited infinitely many times with probability 1

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- If y is recurrent, Markov chain starts in some state x, and $f_{xy} = 1$, then y will be visited infinitely many times with probability 1

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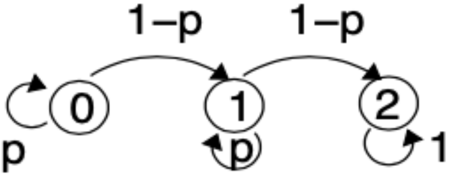
- If y is recurrent, Markov chain starts in some state x, and $f_{xy} = 1$, then y will be visited infinitely many times with probability 1
- If $|\mathcal{X}| < +\infty$, then all states in $\mathcal{X}$ cannot be transient

Example

- Consider a time-homogeneous DTMC with following transition graph.



Compute f_{00}, f_{11}, and f_{22}.
When $p \in (0, 1)$, classify the states into transient and positive/null recurrent.

Communicating Classes

Reachability

Definition (Reachability)

State $y \in \mathcal{X}$ is said to be reachable from state $x \in \mathcal{X}$ if there exists $n \in \mathbb{N} \cup \{0\}$ such that the probability of reaching y in n steps starting from x is strictly positive.

Notation: $x \longrightarrow y.$

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Remark: For a time-homogeneous Markov chain with state space $\mathcal{X}$ and TPM P,

$$x \longrightarrow y \quad \iff \quad \exists n \in \mathbb{N} \cup \{0\} \text{ such that } P^n_{x,y} > 0.$$

Communication

Definition (Communication)

Two states x and y are said to communicate with each other if $x \longrightarrow y$ and $y \longrightarrow x$.

Notation: $x \longleftrightarrow y$.

Communication is an Equivalence Relation

Proposition (Communication is an Equivalence Relation)

$\longleftrightarrow$ defines an equivalence relation on $\mathcal{X} \times \mathcal{X}$. Formally:

1. (Reflexive): $x \longleftrightarrow x$ for all $x \in \mathcal{X}$.
2. (Symmetric): For all $x, y \in \mathcal{X}$,
$$x \longleftrightarrow y \quad \Longleftrightarrow \quad y \longleftrightarrow x.$$
3. (Transitive): For all $x, y, z \in \mathcal{X}$,
$$x \longleftrightarrow y, \quad y \longleftrightarrow z \implies x \longleftrightarrow z.$$