భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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AI5030: Probability and Stochastic Processes

Mid-Term Exam 01

Date: 24 September 2025

Instructions:

- **This question paper consists of 6 questions, and is worth for a total of 40 MARKS.**
- **Questions 1-5 are mandatory and constitute 30 MARKS. Question 6 is a bonus question and is optional.**
- **Combined marks scored in questions 1-5 will be scaled to 20 (contributing 20% towards overall assessment).**
- **Any marks scored in the bonus question will be used as is at the time of grade assignment.**
- **You may use any result covered in class/homeworks directly without proof.**
- **Hints are provided for some questions.**
  However, it is NOT mandatory to solve the question using the approach in the hints.
- **Show all your working clearly.**
  We want to see your thought process, and possibly provide partial credit for the intermediate logical steps.
- **Plagiarism will NOT be entertained at any length.**
  If you are caught cheating during the exam, you will be awarded 0 MARKS.

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing in the questions below are defined with respect to $\mathscr{F}$.

1. (a) **(1 Mark)**
   If $\mathbb{P}(A|B) = \mathbb{P}(A|B^{\complement})$, then show that $A \perp\!\!\!\perp B$.

   (b) **(2 Marks)**
   On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, for events $A, B \in \mathscr{F}$, we say that A *suggests* B if

   $$\mathbb{P}(A \cap B) \;\geq\; \mathbb{P}(A)\,\mathbb{P}(B).$$

   Let $A, B, C \in \mathscr{F}$. Suppose that A suggests B and B suggests C.
   Must it follow that A suggests C? Either prove the implication or provide a concrete counterexample.

   (c) **(2 Marks)**
   Consider events $A, B, C \in \mathscr{F}$ such that $A \perp\!\!\!\perp B$ and $A \perp\!\!\!\perp C$. Show that

   $$A \perp\!\!\!\perp B \cup C \quad\Longleftrightarrow\quad A \perp\!\!\!\perp B \cap C.$$

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2. **(4x1.5 = 6 Marks)**
   State whether the following statements are True or False. Give a brief justification.

   (a) The limit of a sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} A_n$, exists only if $\{A_n\}_{n\in\mathbb{N}}$ is either increasing or decreasing.

   (b) An event A can be self independent only if $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

   (c) Let S be the collection of all singleton subsets of $\mathbb{R}$. Then, $\sigma(S) = 2^{\mathbb{R}}$.

   (d) A finite union of σ-algebras is a σ-algebra.

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3. Let Ω be the unit square $[0,1] \times [0,1]$.
   Let $\mathscr{B}([0,1])$ denote the Borel σ-algebra of subsets of $[0,1]$. Let $\mathscr{F}$ denote the collection

   $$\mathscr{F} := \Big\{ B \times [0,1] : B \in \mathscr{B}([0,1]) \Big\}.$$

   Let $\mathbb{P} : \mathscr{F} \to [0,1]$ be given by

   $$\mathbb{P}(B \times [0,1]) = \lambda(B),$$

   where λ is the Lebesgue measure on $\mathscr{B}([0,1])$.

   (a) **(3 Marks)**
   Show that $\mathscr{F}$ is a σ-algebra of subsets of Ω.

   (b) **(2 Marks)**
   Show that $\mathbb{P} : \mathscr{F} \to [0,1]$ is a probability measure.

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4. (a) **(2 Marks)**
   Buses arrive at ten-minute intervals starting at noon. A man arrives at the bus stop at a random time X minutes post noon, where X has the CDF

   $$F_X(x) = \begin{cases} 0, & x < 0, \\ \dfrac{x}{60}, & 0 \leq x \leq 60, \\ 1, & x > 60. \end{cases}$$

   What is the probability that the man waits more than six minutes for a bus?

   (b) Given $p \in \mathbb{R}$ and $\theta \in \mathbb{R}$, let X be a random variable with CDF F given by

   $$F(x) = \begin{cases} 0, & x < 0, \\ p, & 0 \leq x < 1, \\ p + (1-p)\,\dfrac{x-1}{\theta}, & 1 \leq x < 1+\theta, \\ 1, & x \geq 1+\theta. \end{cases}$$

i. **(1 Mark)**
Find all values of p and θ for which F is a *valid* CDF on $\mathbb{R}$.

ii. **(3 Marks)**
Specify the range of values for p and θ such that X is:
A. Continuous.
B. Discrete.
C. Mixed.

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5. Let $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, where λ is Lebesgue measure.
Define $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ as

$$\forall\, \omega \in \Omega, \qquad X(\omega) = \left\lfloor \frac{1}{\omega} \right\rfloor, \qquad Y(\omega) = \text{sgn}(\sin(2\pi\omega)),$$

where sgn is defined as

$$\text{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0 \end{cases}$$

(a) **(4 Marks)**
Determine $\mathbb{P}_X, \mathbb{P}_Y$.

(b) **(2 Marks)**
Let $\sigma(Y)$ be defined as the collection

$$\sigma(Y) := \left\{ Y^{-1}(B) : B \in \mathscr{B}(\mathbb{R}) \right\}.$$

Determine $\sigma(Y)$ explicitly for Y as defined above.

(c) **(2 Marks)**
Determine $\mathbb{P}(E)$, where E is defined as

$$E := \{\omega \in \Omega : X(\omega) \geq Y(\omega)\}.$$

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6. **(Bonus Question)**

(a) **(3 Marks)**
Consider the following two-player game between A and B.
- Both players are constructing the decimal expansion of a number in $[0,1]$.
- They take turns writing digits: A writes the first digit D_1, then B writes D_2, then A writes D_3, and so on indefinitely, where $D_i \in \{0, 1, \ldots, 9\}$ for each $i \in \mathbb{N}$.

$$x \;=\; 0.D_1 D_2 D_3 \cdots \in [0,1].$$

The winning rule is as follows:
- If x is **rational**, then A wins.
- If x is **irrational**, then B wins.

Show that B can always win.

(b) **(3 Marks)**
Consider one of our standard probability spaces $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, where λ is the Lebesgue measure. To every element $\omega \in \Omega$ we assign its infinite decimal representation. We disallow decimal representations that end with an infinite string of nines (thus, for e.g., $1/10 = 0.1$ and not $0.099999\cdots$).
Define an event E as

$$E \;:=\; \{\omega \in [0,1] : \text{the decimal expansion of } \omega \text{ contains all digits in } \{0, 1, \ldots, 9\}\}.$$

Find $\lambda(E)$.