AI 5030: PROBABILITY AND STOCHASTIC PROCESSES
INSTRUCTOR: DR. KARTHIK P. N.

HOMEWORK 9
TOPICS: CONDITIONAL EXPECTATIONS, LAW OF ITERATED EXPECTATIONS

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2 respectively.
   Determine $\mathbb{E}[X|X+Y]$ (this should be a function of $X+Y$).
   Hence compute $\mathbb{E}[X]$ using the law of iterated expectations.

   Solution: We know that $X+Y \sim \text{Poisson}(\lambda_1+\lambda_2)$. Thus, for any $n \in \mathbb{N} \cup \{0\}$, we have

$$
\begin{aligned}
\mathbb{P}(\{X=k\}|\{X+Y=n\}) &= \frac{\mathbb{P}(\{X=k\}\cap\{X+Y=n\})}{\mathbb{P}(\{X+Y=n\})} \\
&= \frac{\mathbb{P}(\{X=k\}\cap\{Y=n-k\})}{\mathbb{P}(\{X+Y=n\})} \\
&\overset{(*)}{=} \frac{\mathbb{P}(\{X=k\})\cdot\mathbb{P}(\{Y=n-k\})}{\mathbb{P}(\{X+Y=n\})} \\
&= \frac{\frac{e^{-\lambda_1}\,\lambda_1^k}{k!}\cdot\frac{e^{-\lambda_2}\,\lambda_2^{n-k}}{(n-k)!}}{\frac{e^{-\lambda_1+\lambda_2}\,(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!}\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\cdot\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}, \qquad k \in \{0,\ldots,n\}.
\end{aligned}
$$

   In the above set of equalities, $(*)$ follows because $X \perp\!\!\!\perp Y$. Thus, conditioned on the event $\{X+Y=n\}$, the random variable X is distributed as Binomial $\left(n,\ \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$. Noting that the mean of a binomial random variable with parameters (n,p) is equal to np, it follows that for all $n \in \mathbb{N}\cup\{0\}$,

$$\mathbb{E}[X|\{X+Y=n\}] = n\cdot\frac{\lambda_1}{\lambda_1+\lambda_2},$$

   from which it follows that

$$\mathbb{E}[X|X+Y] = (X+Y)\cdot\frac{\lambda_1}{\lambda_1+\lambda_2}.$$

   Applying $\mathbb{E}[\cdot]$ on both sides of the above equation, and using the law of iterated expectations, we get

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|X+Y]] = \mathbb{E}[X+Y]\cdot\frac{\lambda_1}{\lambda_1+\lambda_2} = (\lambda_1+\lambda_2)\cdot\frac{\lambda_1}{\lambda_1+\lambda_2} = \lambda_1.$$

2. Let X and Y be jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}\,y\,e^{-xy}, & x>0,\ 0<y<2, \\ 0, & \text{otherwise.}\end{cases}$$

   Compute $\mathbb{E}[e^{X/2}|Y]$.

   Solution: For any given $y \in (0,2)$, we first compute the conditional PDF of X, conditioned on the event $\{Y=y\}$. Towards this, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_0^{\infty}\frac{1}{2}\,y\,e^{-xy}\,\mathrm{d}x = \frac{1}{2}, \qquad y \in (0,2).$$

   Then, for any $y \in (0,2)$, we have

$$f_{X|\{Y=y\}}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = y\,e^{-xy}, \qquad x>0,$$

from which it follows that

$$\mathbb{E}[e^{X/2}|\{Y=y\}] = \int_0^\infty e^{x/2}\, f_{X|\{Y=y\}}(x)\,\mathsf{d}x = \int_0^\infty e^{x/2}\, y e^{-xy}\,\mathsf{d}x = \int_0^\infty y e^{-x(y-1/2)}\,\mathsf{d}x = \begin{cases} \frac{2y}{2y-1}, & \frac{1}{2} < y < 2, \\ +\infty, & 0 < y \le \frac{1}{2}. \end{cases},$$

which in turn implies that

$$\mathbb{E}[e^{X/2}|Y] = \frac{2Y}{2Y-1}\,\mathbf{1}_{\{\frac{1}{2}<Y<2\}} + (+\infty)\,\mathbf{1}_{\{0<Y\le\frac{1}{2}\}}.$$

3. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx(y-x)\,e^{-y}, & 0 \le x \le y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the constant c.
   (b) Determine $\mathbb{E}[X|Y]$.
   (c) Determine $\mathbb{E}[Y|X]$.

Solution: We present the solution to each part below.

(a) Setting

$$1 = \int_0^\infty \int_x^\infty cx(y-x)e^{-y}\,\mathsf{d}y\,\mathsf{d}x,$$

we get $c = 1$.

(b) From question 3 of homework 6, we note that for any $y \ge 0$,

$$f_{X|\{Y=y\}}(x) = \begin{cases} 6x(y-x)y^{-3}, & 0 \le x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

We thus have

$$\mathbb{E}[X|\{Y=y\}] = \int_0^y x\, f_{X|\{Y=y\}}(x)\,\mathsf{d}x = \int_0^y 6x^2(y-x)y^{-3}\,\mathsf{d}x = \frac{y}{2},$$

from which it follows that $\mathbb{E}[X|Y] = \frac{Y}{2}$.
Along similar lines, from question 3 of homework 6, we note that for any $x \in (0,\infty)$,

$$f_{Y|\{X=x\}}(y) = \begin{cases} (y-x)\,e^{-(y-x)}, & y \ge x, \\ 0, & \text{otherwise.} \end{cases}$$

We thus have

$$\mathbb{E}[Y|\{X=x\}] = \int_x^\infty y\, f_{Y|\{X=x\}}(y)\,\mathsf{d}y = \int_x^\infty y\,(y-x)\,e^{-(y-x)}\,\mathsf{d}y = x+2,$$

from which it follows that $\mathbb{E}[Y|X] = X + 2$.

4. Suppose that a fair coin is tossed repeatedly until the pattern "HTHH" is observed for the first time in succession. Determine the expected number of coin tosses required.

Hint: Let N denote the number of tosses required. Let $X_n \in \{H,T\}$ denote the outcome of the nth toss for $n \in \mathbb{N}$.
Write $\mathbb{E}[N] = \mathbb{E}[N|\{X_1=H\}]\cdot\mathbb{P}(\{X_1=H\}) + \mathbb{E}[N|\{X_1=T\}]\cdot\mathbb{P}(\{X_1=T\})$. Justify this step.
Express $\mathbb{E}[N|\{X_1=T\}]$ in terms of $\mathbb{E}[N]$. Justify the steps.
Write $\mathbb{E}[N|\{X_1=H\}] = \mathbb{E}[N|\{X_1=H\}\cap\{X_2=H\}]\cdot\mathbb{P}(\{X_2=H\}) + \mathbb{E}[N|\{X_1=H\}\cap\{X_2=T\}]\cdot\mathbb{P}(\{X_2=T\})$. Again, justify this step.
Express $\mathbb{E}[N|\{X_1=H\}\cap\{X_2=H\}]$ in terms of $\mathbb{E}[N]$. Justify the steps.
Proceed recursively as above.

Solution: Let $\mathbb{E}[N] = \alpha$. By the law of iterated expectations, we have

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N|X_1]].$$

Noting that the outer expectation is with respect to the distribution of X_1, we have

$$\mathbb{E}[N] = \mathbb{E}[N|\{X_1 = H\}] \cdot \mathbb{P}(\{X_1 = H\}) + \mathbb{E}[N|\{X_1 = T\}] \cdot \mathbb{P}(\{X_1 = T\}).$$

We then have

$$\mathbb{E}[N|\{X_1 = T\}] = 1 + \mathbb{E}[N] = 1 + \alpha.$$

On the other hand, using the law of total probability, we have

$$\mathbb{E}[N|\{X_1 = H\}] = \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = H\}] \cdot \mathbb{P}(\{X_2 = H\}) + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\}] \cdot \mathbb{P}(\{X_2 = T\}),$$

and along similar lines as before,

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = H\}] = 2 + \mathbb{E}[N] = 2 + \alpha.$$

Continuing, we get

$$\begin{aligned}\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\}] = \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\}] \cdot \mathbb{P}(\{X_3 = H\}) \\ + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = T\}] \cdot \mathbb{P}(\{X_3 = T\}).\end{aligned}$$

We then note that

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = T\}] = 3 + \mathbb{E}[N] = 3 + \alpha,$$

while

$$\begin{aligned}\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\}] = \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = H\}] \cdot \mathbb{P}(\{X_4 = H\}) \\ + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = T\}] \cdot \mathbb{P}(\{X_4 = T\}).\end{aligned}$$

Now, we have

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = T\}] \cdot \mathbb{P}(\{X_4 = T\}) = 4 + \mathbb{E}[N] = 4 + \alpha,$$

whereas

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = H\}] = 4 + 0 = 4;$$

here, the term '0' on the right-hand side represents the end of the coin tossing experiment, as the desired pattern is obtained at this stage.

Combining each of the expressions obtained above, we have

$$\alpha = \frac{1}{2}(1+\alpha) + \frac{1}{2}\left(\frac{1}{2}(2+\alpha) + \frac{1}{2}\left(\frac{1}{2}(3+\alpha) + \frac{1}{2}\left(\frac{1}{2}(4+\alpha) + \frac{1}{2} \cdot 4\right)\right)\right) = \frac{30}{16} + \frac{15\alpha}{16},$$

from which we get $\alpha = 30$. Thus, we have $\mathbb{E}[N] = 30$.

5. Let X and Y be jointly uniformly distributed over the right-angled triangle with vertices at $(0,0)$, $(1,0)$, and $(0,2)$. Compute $\mathbb{E}[X|\{Y > 1\}]$.

Solution: Note that the joint PDF of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x \le 1,\ 0 \le y \le 2 - 2x, \\ 0, & \text{otherwise.} \end{cases}$$

To compute $\mathbb{E}[X|\{Y > 1\}]$, we first compute the conditional CDF of X, conditioned on $\{Y > 1\}$. Towards this, we note that

$$F_{X|\{Y>1\}}(x) = \mathbb{P}(\{X \le x\}|\{Y > 1\})$$

$$= \frac{\mathbb{P}(\{X \le x\} \cap \{Y > 1\})}{\mathbb{P}(\{Y > 1\})}.$$

On the one hand, we then note that for any $y \in [0, 2]$,

$$f_Y(y) = \int_0^{\frac{2-y}{2}} \mathsf{d}x = \frac{2-y}{2},$$

from which we have

$$\mathbb{P}(\{Y > 1\}) = \int_1^2 f_Y(y)\,\mathsf{d}y = \int_1^2 \frac{2-y}{2}\,\mathsf{d}y = \frac{1}{4}.$$

On the other hand, we have

$$\mathbb{P}(\{X \le x\} \cap \{Y > 1\}) = \begin{cases} 0, & x < 0, \\ \int_0^x \int_1^{2-2u} \mathsf{d}v\,\mathsf{d}u, & 0 \le x < \frac{1}{2}, \\ \int_0^1 \int_1^{2-2u} \mathsf{d}v\,\mathsf{d}u, & x \ge \frac{1}{2} \end{cases} = \begin{cases} 0, & x < 0, \\ x(1-x), & 0 \le x < \frac{1}{2}, \\ \frac{1}{4}, & x \ge \frac{1}{2}, \end{cases}$$

from which we get

$$F_{X|\{Y>1\}}(x) = \begin{cases} 0, & x < 0, \\ 4x(1-x), & 0 \le x < \frac{1}{2}, \\ 1, & x \ge \frac{1}{2}. \end{cases}$$

Differentiating the above conditional CDF with respect to x, we get the conditional PDF expression as below:

$$f_{X|\{Y>1\}}(x) = \begin{cases} 4(1-2x), & 0 < x < \frac{1}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

Finally, we have

$$\mathbb{E}[X|\{Y > 1\}] = \int_0^{\frac{1}{2}} x\, f_{X|\{Y>1\}}(x)\,\mathsf{d}x = \int_0^{\frac{1}{2}} (4x - 8x^2)\,\mathsf{d}x = \frac{1}{6}.$$

6. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 3y, & -1 \le x \le 1,\ 0 \le y \le |x|, \\ 0, & \text{otherwise}. \end{cases}$$

   (a) Determine $\mathbb{E}[Y|\{X \ge Y + 0.5\}]$.
   (b) Evaluate $\mathbb{E}[Y|X]$, and verify the relation $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$.

Solution: We first note that the range of possible values that Y can take is $[0, 1]$, and for any $y \in [0, 1]$,

$$f_Y(y) = \int_{\{x:|x| \ge y\}} f_{X,Y}(x,y)\,\mathsf{d}x = \int_{-1}^{-y} 3y\,\mathsf{d}x + \int_y^1 3y\,\mathsf{d}x = 6y(1-y),$$

from which it follows that

$$\mathbb{E}[Y] = \int_0^1 6y^2(1-y)\,\mathsf{d}y = \frac{1}{2}.$$

We now provide the solution to each of the parts below.

(a) To obtain the value of $\mathbb{E}[Y|\{X \ge Y + 0.5\}]$, we first compute the conditional PDF of Y, conditioned on the event $A = \{X \ge Y + 0.5\}$. Towards this, we note that

$$\begin{aligned} F_{Y|\{X \ge Y+0.5\}}(y) &= \mathbb{P}(\{Y \le y\}|A) \\ &= \frac{\mathbb{P}(\{Y \le y\} \cap A)}{\mathbb{P}(A)}. \end{aligned}$$

On the one hand, we have

$$\mathbb{P}(A) = \mathbb{P}(\{Y \le X - 0.5\}) = \int_{\frac{1}{2}}^{1} \int_{0}^{x-0.5} 3y \, \mathsf{d}y \, \mathsf{d}x = \frac{1}{16}.$$

On the other hand, we have

$$\begin{aligned}
\mathbb{P}(\{Y \le y\} \cap A) &= \mathbb{P}(\{Y \le y\} \cap \{Y \le X - 0.5\}) \\
&= \mathbb{P}(\{Y \le \mathsf{min}\{y, X - 0.5\}) \\
&\overset{(*)}{=} \mathbb{P}(\{Y \le \mathsf{min}\{y, X - 0.5\} \cap \{X - 0.5 < y\}) + \mathbb{P}(\{Y \le \mathsf{min}\{y, X - 0.5\} \cap \{X - 0.5 \ge y\}) \\
&= \mathbb{P}(\{Y \le X - 0.5\} \cap \{X - 0.5 < y\}) + \mathbb{P}(\{Y \le y\} \cap \{X - 0.5 \ge y\}) \\
&= \int_{\frac{1}{2}}^{\frac{1}{2}+y} \int_{0}^{u-0.5} 3v \, \mathsf{d}v \, \mathsf{d}u + \int_{\frac{1}{2}+y}^{1} \int_{0}^{y} 3v \, \mathsf{d}v \, \mathsf{d}u = \frac{y^3}{2} + \frac{3y^2}{4} - \frac{3y^3}{2} = \frac{3y^2}{4} - y^3,
\end{aligned}$$

where $(*)$ follows from the law of total probability. We thus have

$$F_{Y|A}(y) = \begin{cases} 0, & y < 0, \\ 12y^2 - 16y^3, & 0 \le y < \frac{1}{2}, \\ 1, & y \ge \frac{1}{2}, \end{cases}$$

from which we get

$$f_{Y|A}(y) = \begin{cases} 24y - 48y^2, & 0 < y < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we have

$$\mathbb{E}[Y|A] = \int_{0}^{\frac{1}{2}} y \, f_{Y|A}(y) \, \mathsf{d}y = \int_{0}^{\frac{1}{2}} (24y^2 - 48y^3) \, \mathsf{d}y = \frac{1}{4}.$$

(b) For any $x \in [-1, 1]$, we have

$$f_X(x) = \int_{0}^{|x|} 3y \, \mathsf{d}y = \frac{3x^2}{2},$$

Noting that $f_X(x) = 0$ for $x = 0$, we have for all $x \in [-1, 1] \setminus \{0\}$ that

$$f_{Y|\{X=x\}}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{2y}{x^2}, \qquad 0 \le y \le |x|.$$

Thus, for any $x \in [-1, 1] \setminus \{0\}$, we have

$$\mathbb{E}[Y|\{X = x\}] = \int_{0}^{|x|} y \, \frac{2y}{x^2} \, \mathsf{d}y = \frac{2|x|}{3},$$

from which it follows that $\mathbb{E}[Y|X] = \frac{2|X|}{3}$.
Finally, using the law of iterated expectations, we have

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \frac{2}{3} \, \mathbb{E}[|X|].$$

We now note that

$$\mathbb{E}[|X|] = \int_{-1}^{1} |x| \cdot \frac{3x^2}{2} \, \mathsf{d}x = \int_{0}^{1} 3x^3 \, \mathsf{d}x = \frac{3}{4},$$

from which we get $\mathbb{E}[Y] = \frac{2}{3} \, \mathbb{E}[|X|] = \frac{1}{2}$.

7. Define $\mathsf{Var}(X|Y)$ as

$$\mathsf{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y] = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2.$$

Verify the relation

$$\mathsf{Var}(X) = \mathbb{E}[\mathsf{Var}(X|Y)] + \mathsf{Var}(\mathbb{E}[X|Y]).$$

Solution: From the given formula for $\mathsf{Var}(X|Y)$, we have

$$\begin{aligned}\mathbb{E}[\mathsf{Var}(X|Y)] &= \mathbb{E}[\mathbb{E}[X^2|Y]] - \mathbb{E}[(\mathbb{E}[X|Y])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2],\end{aligned}$$

where the last line above follows from the law of iterated expectations. Also, noting from the law of total expectations that $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$, we have

$$\begin{aligned}\mathsf{Var}(\mathbb{E}[X|Y]) &= \mathbb{E}[(\mathbb{E}[X|Y] - \mathbb{E}[X])^2] \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] + (\mathbb{E}[X])^2 - 2\,\mathbb{E}[\mathbb{E}[X]\,\mathbb{E}[X|Y]] \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2.\end{aligned}$$

Combining the above results, we get

$$\mathbb{E}[\mathsf{Var}(X|Y)] + \mathsf{Var}(\mathbb{E}[X|Y]) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathsf{Var}(X).$$