Stochastic Processes

Primer on Probability: Random Variables, Some Basic Facts about Borel σ-Algebra, Random Variables, Random Vectors, Sequences of Random Variables

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Dedication

Figure: Henri Lebesgue (1875-1941).

Random Variables

Random Variable – Definition

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Random Variables)

A function $X : \Omega \to \mathbb{R}$ is called a random variable with respect to $\mathscr{F}$ if

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F} \qquad \forall B \in \mathscr{B}(\mathbb{R}).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\forall B \in \mathscr{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathscr{F}$$

Some Facts About $\mathscr{B}(\mathbb{R})$

- Let $\mathscr{D}_1 = \left\{(-\infty, x] : x \in \mathbb{R}\right\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_1)$.

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- Let $\mathscr{D}_2 = \left\{(-\infty, x) : x \in \mathbb{R}\right\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_2)$.

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- Let $\mathscr{D}_3 = \left\{[x, \infty) : x \in \mathbb{R}\right\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_3)$.

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- Let $\mathscr{D}_3 = \left\{[x, \infty) : x \in \mathbb{R}\right\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_3)$.
- Let $\mathscr{D}_4 = \left\{(x, \infty) : x \in \mathbb{R}\right\}$. Then, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D}_4)$.

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IMPORTANT

Every set in $\mathscr{B}(\mathbb{R})$ can be expressed exclusively in terms of countable unions, complements, and countable intersections of sets from any one of $\mathscr{D}_1$ or $\mathscr{D}_2$ or $\mathscr{D}_3$ or $\mathscr{D}_4$.

Examples

- Express $(2, 3)$ in terms of sets from $\mathscr{D}_1$

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- Express $(-3, -2]$ in terms of sets from $\mathscr{D}_3$

Examples

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- Express $\{5.5\}$ in terms of sets from $\mathscr{D}_2$
- Express $(-3,-2]$ in terms of sets from $\mathscr{D}_3$
- Express $[-6,5]$ in terms of sets from $\mathscr{D}_4$

Equivalent Definitions of Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and a function $X : \Omega \to \mathbb{R}$.

Theorem (Equivalent Definitions of Random Variable)

The following statements are equivalent.

1. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{B}(\mathbb{R})$.
2. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{D}_1$.
3. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{D}_2$.
4. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{D}_3$.
5. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{D}_4$.

Proof follows by noting that $X^{-1}(B^c) = \left(X^{-1}(B)\right)^c$ and $X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} X^{-1}(B_n)$

Random Variable Simplified

$$
\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[rr, bend left, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"]
\end{tikzcd}
$$

$$\forall B \in \mathscr{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathscr{F}$$

$$
\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[rr, bend left, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & & (-\infty, x] \arrow[ll, "\text{pre-image}"]
\end{tikzcd}
$$

$$\forall x \in \mathbb{R}, \quad X^{-1}((-\infty, x]) \in \mathscr{F}$$

Examples

- $\Omega = \{1, 2, \ldots, 6\}, \qquad \mathscr{F} = \Big\{\emptyset, \Omega\Big\}, \qquad X(\omega) = \omega$

  Is X a RV?
  What functions X are RVs?

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- $\Omega = [0, 1], \qquad \mathscr{F} = \Big\{\emptyset, \Omega, A, A^c\Big\}$ for a fixed $A \subseteq \Omega$

  What functions X are RVs?

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- $\Omega = \{1, 2, 3, 4, 5\}, \qquad \mathscr{F} = \sigma\Big(\Big\{\{1\}, \{2, 3\}\Big\}\Big)$

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Examples

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  What functions X are RVs?

- $\Omega = \mathbb{N}, \qquad \mathscr{F} = 2^{\Omega}$

  What functions X are RVs?

Probability Law and CDF

Probability Law of a Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$

Definition (Probability Law)

Given a random variable $X : \Omega \to \mathbb{R}$ with respect to $\mathscr{F}$, its probability law $\mathbb{P}_X$ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ defined as

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\}) \quad \forall B \in \mathscr{B}(\mathbb{R}).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

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Cumulative Distribution Function (CDF)

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & (-\infty, x] \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}$$

Definition (Cumulative Distribution Function)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Given a random variable $X : \Omega \to \mathbb{R}$ with respect to $\mathscr{F}$, its cumulative distribution function (CDF) $F_X : \mathbb{R} \to [0, 1]$ is defined as

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}(\infty, x]), \qquad x \in \mathbb{R}.$$

Properties of CDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$

Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$ with CDF F_X

- $\lim_{x \to -\infty} F_X(x) = 0, \qquad \lim_{x \to +\infty} F_X(x) = 1$

- (Monotonicity) If $x \le y$, then $F_X(x) \le F_X(y)$

- (Right-Continuity) F_X is right-continuous, i.e., for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} F_X(x + \varepsilon) = F_X(x).$$

CDF $\longleftrightarrow$ Probability Law

- If we know $\mathbb{P}_X = \{\mathbb{P}_X(B) : B \in \mathscr{B}(\mathbb{R})\}$, then we can extract the CDF $F_X : \mathbb{R} \to [0,1]$ by using the formula

$$F_X(x) = \mathbb{P}_X((-\infty, x]), \qquad x \in \mathbb{R}.$$

- Given the CDF $F_X : \mathbb{R} \to [0,1]$, let

$$\mathbb{P}_X((-\infty, x]) = F_X(x), \qquad x \in \mathbb{R}.$$

  Then, there exists a unique extension of $\mathbb{P}_X$ to all Borel subsets of $\mathbb{R}$
  For a proof of this, see [Folland, 1999, Theorem 1.16]

Notation

- $\{\omega \in \Omega : X(\omega) \leq x\} = \{X \leq x\}$
- $\mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) = \mathbb{P}(X \leq x)$

Structured Assignment of Probabilities to Sets in $\mathscr{B}(\mathbb{R})$

Question

Is there a structured way to come up with a probability measure $\mathbb{Q}$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$?

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- Set $\mathbb{Q}((-\infty, x]) = G(x)$ for all $x \in \mathbb{R}$.

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  - G is non-decreasing.
  - G is right-continuous.
- Set $\mathbb{Q}((-\infty, x]) = G(x)$ for all $x \in \mathbb{R}$.
- Using Caratheodory's extension theorem, get $\mathbb{Q}(B)$ for all $B \in \mathscr{B}(\mathbb{R})$.

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Random Vectors and Sequences of Random Variables

Random Vectors

Fix a measurable space $(\Omega, \mathscr{F})$. Fix $n \in \mathbb{N}$.

Definition (Random Vector)

Given random variables $X_1, \ldots, X_n$ defined with respect to $\mathscr{F}$, we say $(X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$ is a random vector with respect to $\mathscr{F}$ if

$$(X_1, \ldots, X_n)^{-1}(B) = \{\omega \in \Omega : \big(X_1(\omega), \ldots, X_n(\omega)\big) \in B\} \in \mathscr{F} \qquad \forall B \in \mathscr{B}(\mathbb{R}^n).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "{(X_1, \ldots, X_n)}"] & (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)) \arrow[d] \\
(X_1, \ldots, X_n)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\forall B \in \mathscr{B}(\mathbb{R}^n), \quad (X_1, \ldots, X_n)^{-1}(B) \in \mathscr{F}$$

Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Sequence of Random Variables)

A sequence of random variables is a collection $\{X_n\}_{n=1}^{\infty}$ such that

$$\forall n \in \mathbb{N},\ \forall k_1, \ldots, k_n \in \mathbb{N}, \qquad (X_{k_1}, \ldots, X_{k_n}) \text{ is a random vector.}$$

References

Folland, G. B. (1999).
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