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Probability and Stochastic Processes

Lecture 07: Probability Measure and its Properties

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Measure

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Measure)

A function $\mu : \mathscr{F} \to [0, +\infty]$ is called a **measure** on $(\Omega, \mathscr{F})$ if it satisfies the following properties:

1. $\mu(\emptyset) = 0$.
2. If $A_1, A_2, \ldots$ is a countable collection of **disjoint** sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mu\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

Property 2 above is called the property of **countable additivity**.

The triplet $(\Omega, \mathscr{F}, \mu)$ is called a **measure space**.

Measure

- When $\mu(\Omega) < +\infty$, the measure μ is called a **finite measure**
- When $\mu(\Omega) = +\infty$, the measure μ is called an **infinite measure**
- When $\mu(\Omega) = 1$, the measure μ is called a **probability measure**, and denoted by $\mathbb{P}$.

Probability Measure

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Probability Measure)

A function $\mathbb{P} : \mathscr{F} \to [0, 1]$ is called a **probability measure** if the following properties are satisfied:

1. $\mathbb{P}(\emptyset) = 0.$
2. $\mathbb{P}(\Omega) = 1.$
3. If $A_1, A_2, \ldots$ is a countable collection of **disjoint** sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mathbb{P}(A_i).$$

Properties of Probability Measure

- **(Finite Additivity)**

  Fix $n \in \mathbb{N}$.
  If $A_1, \ldots, A_n$ is a finite collection of **disjoint** sets, with $A_i \in \mathscr{F}$ for each $i \in \{1, \ldots, n\}$, then

$$\mathbb{P}\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

Properties of Probability Measure

- **(Complements)**

  For any $A \in \mathscr{F}$,

$$\mathbb{P}\left(A^{\complement}\right) = 1 - \mathbb{P}(A).$$

Properties of Probability Measure

- **(Monotonicity)**
  If $A, B \in \mathscr{F}$ with $A \subseteq B$, then
  $$\mathbb{P}(A) \leq \mathbb{P}(B).$$

Properties of Probability Measure

(Inclusion-Exclusion)

- For any two events $A_1, A_2 \in \mathscr{F}$,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).$$

- More generally, for any $n \in \mathbb{N}$ and events $A_1, \ldots, A_n \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\, \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

Properties of Probability Measure

(Inclusion-Exclusion)

- For any two events $A_1, A_2 \in \mathscr{F}$,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).$$

- More generally, for any $n \in \mathbb{N}$ and events $A_1, \ldots, A_n \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\, \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

Continuity of Probability Measure

Union and Intersection Events

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **union** is defined by

$$A_{\text{union}} = \bigcup_{k \in \mathbb{N}} A_k.$$

Interpretation: $\omega \in A_{\text{union}} \implies \exists\, k \in \mathbb{N} : \ \omega \in A_k$

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **intersection** is defined by

$$A_{\text{intersection}} = \bigcap_{k \in \mathbb{N}} A_k.$$

Interpretation: $\omega \in A_{\text{intersection}} \implies \omega \in A_k \quad \forall\, k \in \mathbb{N}$

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The Limit Infimum (liminf) Event

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **limit infimum (liminf)** is defined by

$$A_{\text{liminf}} = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

Interpretation: For each $n \in \mathbb{N}$, let $B_n := \bigcap_{k\geq n} A_k$.

$$\begin{aligned}
\omega \in A_{\text{liminf}} \quad &\implies \quad \omega \in \bigcup_{n\in\mathbb{N}} B_n \\
&\implies \quad \exists\, n \in \mathbb{N} : \quad \omega \in B_n \\
&\implies \quad \exists\, n \in \mathbb{N} : \quad \omega \in \bigcap_{k\geq n} A_k \\
&\implies \quad \exists\, n \in \mathbb{N} : \quad \omega \in A_k \ \forall\, k \geq n.
\end{aligned}$$

The Limit Supremum (limsup) Event

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **limit supremum (limsup)** is defined by

$$A_{\text{limsup}} = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k.$$

Interpretation: For each $n \in \mathbb{N}$, let $B_n := \bigcup_{k \geq n} A_k$.

$$\begin{aligned}
\omega \in A_{\text{limsup}} \quad & \implies \quad \omega \in \bigcap_{n \in \mathbb{N}} B_n \\
& \implies \quad \forall\, n \in \mathbb{N}, \;\; \omega \in B_n \\
& \implies \quad \forall\, n \in \mathbb{N}, \;\; \omega \in \bigcup_{k \geq n} A_k \\
& \implies \quad \forall\, n \in \mathbb{N}, \;\; \exists\, k \geq n : \;\; \omega \in A_k.
\end{aligned}$$

The Limit Event

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, if

$$A_{\text{liminf}} = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k = A_{\text{limsup}},$$

  then we say that the **limit of $A_1, A_2, \ldots$ exists**, and is defined by

$$A_{\text{limit}} = A_{\text{liminf}} = A_{\text{limsup}}.$$

Examples

- **(Moving Singletons)**

  Let $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$. Let
$$A_n = \{n\}, \qquad n \in \mathbb{N}.$$

  1. What is A_{liminf}?
  2. What is A_{limsup}?
  3. Identify A_{limit} if it exists.

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Examples

- **(Odd-Even Stabilization)**

  Let $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$. Let

$$A_n = \{i \leq n : \; i \text{ odd}\} \cup \{i > n : \; i \text{ even}\}, \qquad n \in \mathbb{N}.$$

1. What is A_{liminf}?
2. What is A_{limsup}?
3. Identify A_{limit} if it exists.

Examples

- **(Sliding Window)**

  Let $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$.
  Fix $m \in \mathbb{N}$. Let

  $$A_n = \{n, n+1, \ldots, n+m\}, \qquad n \in \mathbb{N}.$$

  1. What is A_{liminf}?
  2. What is A_{limsup}?
  3. Identify A_{limit} if it exists.

Examples

- **(Non-Decreasing Sets)**

  Let $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$.
  Suppose that $A_1, A_2, \ldots \in \mathscr{F}$ satisfy

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

1. What is A_{liminf}?
2. What is A_{limsup}?
3. Identify A_{limit} if it exists.

Some Tidbits About liminf, limsup, and limit Events

Fix a measurable space $(\Omega, \mathscr{F})$. Let $A_1, A_2, \ldots \in \mathscr{F}$.

- A_{liminf} and A_{limsup} belong to $\mathscr{F}$
- A_{limit}, if it exists, belongs to $\mathscr{F}$
- If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, then
$$A_{\text{limit}} = \bigcup_{n \in \mathbb{N}} A_n.$$
- If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, then
$$A_{\text{limit}} = \bigcap_{n \in \mathbb{N}} A_n.$$
- A_{liminf} is sometimes denoted more explicitly as $\liminf_{n\to\infty} A_n$
  A_{limsup} is sometimes denoted more explicitly as $\limsup_{n\to\infty} A_n$
  A_{limit} is sometimes denoted more explicitly as $\lim_{n\to\infty} A_n$

Properties of Probability Measure

(Continuity of Probability)
Fix a measurable space $(\Omega, \mathscr{F})$.

- Let $A_1, A_2, \ldots \in \mathscr{F}$ be a collection of events for which $A_{\text{limit}} = \lim_{n\to\infty} A_n$ exists. Then,

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

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Continuity of Probability: Proof – 1

- Case 1: $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$
  - Let $B_1, B_2, \ldots$ be defined as

$$B_1 = A_1, \quad B_2 = A_2 \setminus A_1, \quad B_3 = A_3 \setminus A_2, \quad \ldots$$

  - **Claim 1:** $B_i \cap B_j = \emptyset$ for all $i \neq j$
  - **Claim 2:** We have

$$\bigsqcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k \quad \forall\, n \in \mathbb{N}, \qquad \bigsqcup_{k\in\mathbb{N}} B_k = \bigcup_{k\in\mathbb{N}} A_k.$$

  - Therefore, it follows that

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \mathbb{P}\left(\bigcup_{k\in\mathbb{N}} A_k\right) \overset{\text{Claim 2}}{=} \mathbb{P}\left(\bigsqcup_{k\in\mathbb{N}} B_k\right) = \sum_{k\in\mathbb{N}} \mathbb{P}(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(B_k) = \lim_{n\to\infty} \mathbb{P}\Big(\bigsqcup_{k\ldots}$$

Continuity of Probability: Proof – 2

- Case 2: $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$
  - Clearly, $A_1^{\complement} \subseteq A_2^{\complement} \subseteq A_3^{\complement} \subseteq \cdots$
  - We then have

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = 1 - \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n^{\complement}\right) = 1 - \mathbb{P}\left(\lim_{n\to\infty} A_n^{\complement}\right) = 1 - \lim_{n\to\infty} \mathbb{P}(A_n^{\complement})$$

$$= \lim_{n\to\infty} 1 - \mathbb{P}(A_n^{\complement}) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Continuity of Probability: Proof – 3

- General case: $\lim_{n\to\infty} A_n$ exists, i.e., $A_{\text{liminf}} = A_{\text{limsup}} = \lim_{n\to\infty} A_n$
  - For any $n \in \mathbb{N}$,

$$\underbrace{\bigcap_{k\geq n} A_k}_{B_n} \subseteq A_n \subseteq \underbrace{\bigcup_{k\geq n} A_k}_{C_n}$$

  - Clearly,

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots, \qquad C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots, \qquad \mathbb{P}(B_n) \leq \mathbb{P}(A_n) \leq \mathbb{P}(C_n).$$

  - Observe that

$$A_{\text{liminf}} = \bigcup_{n\in\mathbb{N}} B_n, \qquad A_{\text{limsup}} = \bigcap_{n\in\mathbb{N}} C_n.$$

$$\mathbb{P}(A_{\text{liminf}}) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n) \leq \lim_{n\to\infty} \mathbb{P}(A_n) \leq \lim_{n\to\infty} \mathbb{P}(C_n) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} C_n\right)$$

$$= \mathbb{P}(A_{\text{limsup}})$$

Continuity of Probability: Proof – 3

- General case: $\lim_{n\to\infty} A_n$ exists, i.e., $A_{\text{liminf}} = A_{\text{limsup}} = \lim_{n\to\infty} A_n$
  - We then have

$$\mathbb{P}(A_{\text{liminf}}) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n) \leq \lim_{n\to\infty} \mathbb{P}(A_n) \leq \lim_{n\to\infty} \mathbb{P}(C_n) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} C_n\right)$$
$$= \mathbb{P}(A_{\text{limsup}})$$

  - If $A_{\text{liminf}} = A_{\text{limsup}}$, it follows that

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right)$$