CS 6660: Mathematical Foundations of Data Science (Probability)

Instructor: Dr. Karthik P. N.

Practice Problems 04

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are defined with respect to $\mathscr{F}$.

1. Let X and Y be jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx^2 + \frac{xy}{3}, & 0 \leq x \leq 1,\ 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the constant c.

   (b) Are X and Y independent?

   (c) Calculate $\mathrm{Cov}(X, Y)$.

2. Let X and Y be independent random variables distributed uniformly on $[0, 1]$. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Calculate $\mathrm{Cov}(U, V)$.

3. Suppose U and V are jointly uniformly distributed on the square with corners at $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Let $X = UV$. Determine the PDF of X, and compute $\mathbb{E}[X]$.

4. Let $X_1, X_2, \ldots$ be independent random variables distributed uniformly on $(0, 1)$. Let N be defined as

$$N = \min\{n \geq 1 : X_{n+1} > X_n\}.$$

   Determine the PMF of N and compute $\mathbb{E}[N]$.

5. Let $X = (X_1, \ldots, X_n)^\top$ be a random vector with mean $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and covariance matrix K. Given an $n \times n$ matrix A, determine the mean vector and covariance matrix of the random vector $Y = AX$.

6. Let $X \sim \mathcal{N}(0, 1)$. Let W be a discrete random variable independent of X and having the PMF

$$\mathbb{P}(\{W = w\}) = \begin{cases} \frac{1}{2}, & w = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Define a new random variable Y as $Y = WX$.

   (a) Show that $Y \sim \mathcal{N}(0, 1)$.

   (b) Show that X and Y are uncorrelated.

   (c) Demonstrate that

$$X + Y = \begin{cases} 0, & \text{with probability } \frac{1}{2}, \\ 2X, & \text{with probability } \frac{1}{2}, \end{cases}$$

   hence proving that $X + Y$ is not Gaussian, and therefore X and Y are not jointly Gaussian.

   (d) Finally, show that X and Y are not independent.

   (e) Let X and Y be jointly Gaussian with mean vector $\boldsymbol{\mu} = \mathbf{0}$ and covariance matrix K, where

$$K = \begin{pmatrix} \sigma_X^2 & \rho\,\sigma_X\,\sigma_Y \\ \rho\,\sigma_X\,\sigma_Y & \sigma_Y^2 \end{pmatrix}.$$

   Suppose that $|\rho| < 1$ (thereby implying that K is invertible). Show that the conditional PDF $f_{Y|X=x}(y)$ is the PDF of a one-dimensional Gaussian distribution with mean $x\rho\,\frac{\sigma_Y}{\sigma_X}$ and variance $\sigma_Y^2(1 - \rho^2)$.

7. Compute $\mathbb{E}[Y|X]$ and $\mathbb{E}[X|Y]$ for question 3 in practice problems set 2.