AI 5090: Stochastic Processes

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Real Analysis Basics, Pointwise Convergence, Almost-Sure Convergence

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $\lim_{n\to\infty} a_n = L$, for some $L \in \mathbb{R}$. Prove that the sequence must be bounded, i.e., there exists a finite number $M > 0$ such that
$$|a_n| \leq M \quad \forall n \in \mathbb{N}.$$
The above exercise proves that every convergent sequence is bounded.
Is the converse true? That is, is every bounded sequence convergent? Prove or give a counterexample.

2. Prove formally that if $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

3. Formally determine the lim sup and lim inf of the following sequences.
   (a) $a_n = n \sin\left(\frac{n\pi}{2}\right)$ for all $n \in \mathbb{N}$.
   (b) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$.
   In the above sequence, first term is $\frac{1}{2}$, second and third terms are with denominator 3 and numerators $1, 2$, next three terms are with denominator 4 and numerator $1, 2, 3$, but we do not write $2/4$ as it is already covered before as $1/2$. Similarly, next few terms are with denominator 5, with numerator spanning $1, 2, 3, 4$, only listing elements not covered before. The sequence goes on this way.

4. For each $n \in \mathbb{N}$, let
$$A_n := \left(-\infty,\ a - \frac{1}{n}\right), \quad B_n := \left(-\infty,\ a + \frac{1}{n}\right), \quad C_n := \left(-\infty,\ a - \frac{1}{n}\right], \quad D_n := \left(-\infty,\ a + \frac{1}{n}\right].$$
Determine $\bigcap_{n=1}^{\infty} A_n$, $\quad \bigcup_{n=1}^{\infty} B_n$, $\quad \liminf_{n\to\infty} C_n$, $\quad$ and $\quad \limsup_{n\to\infty} D_n$.

5. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables, all defined on a common measurable space $(\Omega, \mathscr{F})$. Let
$$X^\star := \limsup_{n\to\infty} X_n.$$
In this exercise, we will formally prove that $X^\star : \Omega \to [-\infty, +\infty]$ is a random variable with respect to $\mathscr{F}$.
   (a) For each $x \in \mathbb{R}$, prove formally that $(X^\star)^{-1}((x, \infty)) \in \mathscr{F}$.
   (b) Prove formally that $\{\omega \in \Omega : X^\star(\omega) = -\infty\} \in \mathscr{F}$.
   (c) Prove formally that $\{\omega \in \Omega : X^\star(\omega) = +\infty\} \in \mathscr{F}$.

6. Let $(\Omega, \mathscr{F}, \mathbb{P}) = ((0, 1], \mathscr{B}((0, 1]), \text{Unif})$.
   (a) For each $n \in \mathbb{N}$, let
$$X_n(\omega) = n\omega - \lfloor n\omega \rfloor, \quad \omega \in \Omega.$$
   Here, $\lfloor x \rfloor$ denotes the smallest integer lesser than or equal to x (i.e., "floor" of x).
   Does $\{X_n\}_{n=1}^{\infty}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.
   (b) For each $n \in \mathbb{N}$, let
$$Y_n(\omega) = n^2 \omega\, \mathbf{1}_{(0, \frac{1}{n})}(\omega), \quad \omega \in \Omega.$$
   Does $\{Y_n\}_{n=1}^{\infty}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.
   (c) For each $n \in \mathbb{N}$, let
$$Z_n(\omega) = \sin(2\pi n\omega), \quad \omega \in \Omega.$$
   Does $\{Z_n\}_{n=1}^{\infty}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.