

ALGEBRAS, σ -ALGEBRAS

1. (a) Let $\Omega = \{1, \dots, 6\}$. For each $i \in \{1, 2, 3, 4\}$, construct a σ -algebra \mathcal{F}_i of subsets of Ω such that $|\mathcal{F}_i| = 2^i$.
 (b) Let Ω be a finite sample space with $|\Omega| = n$ for some $n \in \mathbb{N}$. Let \mathcal{F} be a σ -algebra of subsets of Ω . Show that $|\mathcal{F}| = 2^k$ for some $1 \leq k \leq n$.

2. Let Ω be an arbitrary set (finite, countably infinite, or uncountable).

(a) Let \mathcal{A} be a collection of subsets of Ω satisfying the property that if $A, B \in \mathcal{A}$, then $A \cap B^c \in \mathcal{A}$. Show that \mathcal{A} must be an algebra (of subsets of Ω).

(b) Suppose \mathcal{F} is a collection of subsets of Ω satisfying the following properties:

- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (closure under complements).
- If A, B are two **disjoint** subsets of Ω , then $A \cup B \in \mathcal{F}$ (closure under finite **disjoint** unions).

Construct an explicit example to demonstrate that \mathcal{F} need not be an algebra.

3. Let Ω be an arbitrary set (finite, countably infinite, or uncountable).

(a) Let \mathcal{F}_1 denote the collection of all finite subsets of Ω , i.e.,

$$\mathcal{F}_1 := \left\{ A \subseteq \Omega : |A| \in \mathbb{N} \right\}.$$

Is \mathcal{F}_1 an algebra?

(b) Let \mathcal{F}_2 denote the collection of all finite subsets of Ω , plus all subsets of Ω whose complement is finite, i.e.,

$$\mathcal{F}_2 := \left\{ A \subseteq \Omega : A \text{ is finite or } (\Omega \setminus A) \text{ is finite or both} \right\}.$$

Show that \mathcal{F}_2 is an algebra.

Construct an example to demonstrate that \mathcal{F}_2 need not necessarily be a σ -algebra.

(c) Let \mathcal{F}_3 denote the collection of all countable subsets of Ω , plus all subsets of Ω whose complement is countable, i.e.,

$$\mathcal{F}_3 := \left\{ A \subseteq \Omega : A \text{ is countable or } (\Omega \setminus A) \text{ is countable or both} \right\}.$$

Show that \mathcal{F}_3 is a σ -algebra.

Note: Countable means finite or countably infinite.

4. Let $\Omega = \mathbb{R}$. Let \mathcal{P} denote the collection

$$\mathcal{P} := \left\{ [a, b) : a, b \in \mathbb{R}, a < b \right\}.$$

Clearly, \mathcal{P} consists of uncountably infinitely many subsets of Ω .

In [Lecture 6](#), we saw that $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$, i.e., \mathcal{P} is a generating class for $\mathcal{B}(\mathbb{R})$.

In this exercise, we will see an alternative construction of $\mathcal{B}(\mathbb{R})$ starting from a **countably infinite** collection of subsets of Ω .

Consider the collection \mathcal{C} given by

$$\mathcal{C} := \left\{ [a, b) : a \leq b, a, b \text{ are dyadic rational numbers} \right\}.$$

Note: A dyadic rational number is of the form $m/2^n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$.

- (a) Given $x \in \mathbb{R}$, express $\{x\}$ in terms of sets in \mathcal{C} using countable set operations.

Hint: Note that $\lfloor 2^n x \rfloor \leq 2^n x \leq \lceil 2^n x \rceil$ for all $n \in \mathbb{N}$. Therefore,

$$\frac{\lfloor 2^n x \rfloor}{2^n} \leq x \leq \frac{\lceil 2^n x \rceil}{2^n} \quad \forall n \in \mathbb{N}.$$

- (b) Given $a, b \in \mathbb{R}$ with $a < b$, express $[a, b]$ in terms of sets in \mathcal{C} using countable set operations.
 (c) Using the result in part (b), what can you say about the relationship between \mathcal{P} and $\sigma(\mathcal{C})$?
 (d) What can you say about the relationship between \mathcal{C} and $\sigma(\mathcal{P})$?
 (e) Using the results of parts (c), (d), what can you say about the relationship between $\sigma(\mathcal{C})$ and $\sigma(\mathcal{P})$?

5. Let Ω be an arbitrary set (finite, countably infinite, or uncountable).

- (a) Let \mathcal{C} denote the collection of all singleton subsets of Ω . What is $\sigma(\mathcal{C})$?

Hint: See Question 3c.

- (b) Consider the collection

$$\mathcal{D} := \left\{ (a, b] \cup [-b, -a) : a, b \in \mathbb{R}, a \leq b \right\}.$$

Show that $\sigma(\mathcal{D}) \subsetneq \mathcal{B}(\mathbb{R})$ by constructing a non-empty set $B \in \mathcal{B}(\mathbb{R}) \setminus \sigma(\mathcal{D})$.

- (c) Fix two elementary outcomes $a, b \in \Omega$.

Let $\mathcal{E}_{a,b}$ denote the collection of all those subsets of Ω which either contain both a and b or do not contain both.

Let $\mathcal{F} = \sigma(\mathcal{E}_{a,b})$. Show that every set in \mathcal{F} has the same property as the sets in $\mathcal{E}_{a,b}$.