AI5090 / EE5910: Stochastic Processes
Homework 01

All random variables appearing below are assumed to be defined with respect to a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences of real numbers.

   (a) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, show that
   $$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n.$$

   (b) Show that
   $$\lim_{n\to\infty} a_n \in \mathbb{R} \text{ exists} \qquad \text{if and only if} \qquad \underbrace{\forall\, \varepsilon > 0,\ \exists\, N_\varepsilon \in \mathbb{N} \text{ such that } |a_n - a_m| \leq \varepsilon \quad \forall\, n, m \geq N_\varepsilon}_{\text{Cauchy criterion for convergence to a real number}}.$$
   Here, "$\lim_{n\to\infty} a_n \in \mathbb{R}$ exists" means, by definition, that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n \in \mathbb{R}$.

   (c) For $a, a' \in \mathbb{R}$, show that
   $$\inf_{n\in\mathbb{N}} a_n \geq a \qquad \text{if and only if} \qquad a_n \geq a \quad \forall\, n \in \mathbb{N},$$
   $$\sup_{n\in\mathbb{N}} a_n \leq a \qquad \text{if and only if} \qquad a_n \leq a \quad \forall\, n \in \mathbb{N}.$$

   (d) Show that
   $$\lim_{n\to\infty} a_n = a \in \mathbb{R} \setminus \{0\} \qquad \text{implies} \qquad \lim_{n\to\infty} \frac{a_n}{a} = 1.$$

   (e) Show that
   $$\lim_{n\to\infty} a_n = a \in \mathbb{R} \qquad \text{implies} \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i = a.$$

2. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables.

   (a) For any $x \in \mathbb{R}$, show that the event
   $$\left\{\limsup_{n\to\infty} X_n < x\right\} \in \mathscr{F}.$$
   Hence conclude that $\limsup_{n\to\infty} X_n$ is a random variable.
   Hint:
   Express the above event as countable unions and/or intersections of events associated with X_n. Use 1(c).

   (b) For any $x \in \mathbb{R}$, show that the event
   $$\left\{\liminf_{n\to\infty} X_n > x\right\} \in \mathscr{F}.$$
   Hence conclude that $\liminf_{n\to\infty} X_n$ is a random variable.
   Hint:
   Express the above event as countable unions and/or intersections of events associated with X_n. Use 1(c).

3. Let $(\Omega, \mathscr{F}, \mathbb{P}) = ((0,1], \mathscr{B}(0,1], \lambda)$, where λ denotes the Lebesgue measure.

   (a) For each $n \in \mathbb{N}$, let
   $$X_n(\omega) = n\omega - \lfloor n\omega \rfloor, \quad \omega \in \Omega.$$
   Here, $\lfloor x \rfloor$ denotes the smallest integer lesser than or equal to x (i.e., "floor" of x).
   Does $\{X_n\}_{n\in\mathbb{N}}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.

   (b) For each $n \in \mathbb{N}$, let
   $$Y_n(\omega) = n^2 \omega\, \mathbf{1}_{(0,\frac{1}{n})}(\omega), \quad \omega \in \Omega.$$
   Does $\{Y_n\}_{n\in\mathbb{N}}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.

(c) For each $n \in \mathbb{N}$, let

$$Z_n(\omega) = \sin(2\pi n\omega), \quad \omega \in \Omega.$$

Does $\{Z_n\}_{n=1}^{\infty}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.

4. For each $n \in \mathbb{N}$, let

$$X_n \sim \mathcal{N}\left(0, \frac{1}{n}\right).$$

(a) For any choice of $\varepsilon > 0$, show that

$$\mathbb{P}\left(\{|X_n| > \varepsilon\}\right) = 2\,\mathbb{P}(\{X_n > \varepsilon\}).$$

Furthermore, using the Chernoff bound, show that

$$\mathbb{P}(\{X_n > \varepsilon\}) \leq \exp\left(-\frac{n\varepsilon^2}{2}\right).$$

(b) Using the result of part (a) above, conclude that $X_n \xrightarrow{\text{a.s.}} 0$.