



Stochastic Processes

Lecture 09

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Monotone Convergence Theorem (MCT)

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Theorem (Monotone Convergence Theorem)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X, Y be two other extended real-valued RVs.

1. Let $X_n \geq Y$ for all $n \in \mathbb{N}$ with $\mathbb{E}[Y] > -\infty$.

Furthermore, suppose that

$$Y \leq X_1 \leq X_2 \leq X_3 \leq \cdots \leq X, \quad X_n \xrightarrow{\text{pointwise}} X \quad \left(X = \lim_{n \rightarrow \infty} X_n \right).$$

Then, we have

$$\mathbb{E}[Y] \leq \mathbb{E}[X_1] \leq \mathbb{E}[X_2] \leq \mathbb{E}[X_3] \leq \cdots \leq \mathbb{E}[X], \quad \lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X] = \mathbb{E}\left[\lim_{n \rightarrow \infty} X_n\right].$$

2. An exact analogue of the above statement holds when X_1, X_2, \dots are non-increasing and $X_n \leq Y$ for all $n \in \mathbb{N}$, with $\mathbb{E}[Y] < +\infty$

Applications of MCT

- If X_1, X_2, \dots are **non-negative**, real-valued random variables, then

$$\mathbb{E}\left[\sum_{n \in \mathbb{N}} X_n\right] = \mathbb{E}\left[\lim_{N \rightarrow \infty} \sum_{n=1}^N X_n\right] \stackrel{\text{MCT}}{=} \lim_{N \rightarrow \infty} \mathbb{E}\left[\sum_{n=1}^N X_n\right] = \lim_{N \rightarrow \infty} \sum_{n=1}^N \mathbb{E}[X_n] = \sum_{n \in \mathbb{N}} \mathbb{E}[X_n].$$

- If τ is a **non-negative**, integer-valued random variable, then

$$\mathbb{E}[\tau] = \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbf{1}_{\{\tau \geq n\}}\right] \stackrel{\text{MCT}}{=} \sum_{n \in \mathbb{N}} \mathbb{P}(\{\tau \geq n\}).$$

Fatou's Lemma

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Fatou's Lemma)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let Y be another real-valued RV.

1. If $X_n \geq Y$ for all $n \in \mathbb{N}$ and $\mathbb{E}[Y] > -\infty$, then

$$\mathbb{E}[Y] \leq \mathbb{E}\left[\liminf_{n \rightarrow \infty} X_n\right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n].$$

2. If $X_n \leq Y$ for all $n \in \mathbb{N}$ and $\mathbb{E}[Y] < +\infty$, then

$$\mathbb{E}[Y] \geq \mathbb{E}\left[\limsup_{n \rightarrow \infty} X_n\right] \geq \limsup_{n \rightarrow \infty} \mathbb{E}[X_n].$$

3. If $|X_n| \leq Y$ for all $n \in \mathbb{N}$, with $|\mathbb{E}[Y]| < +\infty$, then

$$-\mathbb{E}[Y] \leq \mathbb{E}\left[\liminf_{n \rightarrow \infty} X_n\right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n] \leq \limsup_{n \rightarrow \infty} \mathbb{E}[X_n] \leq \mathbb{E}\left[\limsup_{n \rightarrow \infty} X_n\right] \leq \mathbb{E}[Y].$$

Proof of Lemma 2, Part 1

- For each $n \in \mathbb{N}$, let

$$Y_n = \inf_{k \geq n} X_k.$$

- Because $X_k \geq Y$ for all k , it follows that

$$Y_n \geq Y \quad \forall n \in \mathbb{N}$$

- Furthermore, note that

$$Y \leq Y_1 \leq Y_2 \leq Y_3 \leq \cdots, \quad \lim_{n \rightarrow \infty} Y_n = \sup_{n \geq 1} Y_n = \liminf_{n \rightarrow \infty} X_n.$$

- Applying MCT, we get

$$\mathbb{E}\left[\liminf_{n \rightarrow \infty} X_n\right] = \mathbb{E}\left[\lim_{n \rightarrow \infty} Y_n\right] \stackrel{\text{MCT}}{=} \lim_{n \rightarrow \infty} \mathbb{E}[Y_n] = \liminf_{n \rightarrow \infty} \mathbb{E}[Y_n] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n].$$

Dominated Convergence Theorem (DCT)

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Theorem (Dominated Convergence Theorem)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X, Y be extended real-valued RVs.

Suppose that $|X_n| \leq Y$ for all $n \in \mathbb{N}$, with $\mathbb{E}[Y] < +\infty$. ($\mathbb{E}[|X_n|] \leq \mathbb{E}[Y] < +\infty$ for all $n \in \mathbb{N}$.)

If $X_n \xrightarrow{\text{a.s.}} X$, then:

1. $\mathbb{E}[|X|] < +\infty$, i.e., X is integrable.

2. Furthermore,

$$\mathbb{E}[X] = \mathbb{E}\left[\lim_{n \rightarrow \infty} X_n\right] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n].$$

3. Additionally,

$$X_n \xrightarrow{\mathcal{L}^1} X.$$

Proof of Theorem 3

- Let $A_{\lim} = \left\{ \lim_{n \rightarrow \infty} X_n = X \right\}$; we are given that $\mathbb{P}(A_{\lim}) = 1$
- For every $\omega \in A_{\lim}$, we have

$$|X_n(\omega)| \leq Y(\omega) \quad \implies \quad |X(\omega)| = \left| \lim_{n \rightarrow \infty} X_n(\omega) \right| = \lim_{n \rightarrow \infty} |X_n(\omega)| \leq Y(\omega).$$

- **Proof of Part 1:** We have

$$\mathbb{E}[|X|] = \mathbb{E}[|X| \mathbf{1}_{A_{\lim}}] \leq \mathbb{E}[Y \mathbf{1}_{A_{\lim}}] = \mathbb{E}[Y] < +\infty.$$

- **Proof of Part 2:** By Fatou's lemma, we have

$$\mathbb{E}\left[\liminf_{n \rightarrow \infty} X_n\right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n] \leq \limsup_{n \rightarrow \infty} \mathbb{E}[X_n] \leq \mathbb{E}\left[\limsup_{n \rightarrow \infty} X_n\right]$$

Proof of Theorem 3

- We note that

$$\begin{aligned}\mathbb{E}\left[\liminf_{n \rightarrow \infty} X_n\right] &= \mathbb{E}\left[\left(\liminf_{n \rightarrow \infty} X_n\right) \mathbf{1}_{A_{\lim}}\right] = \mathbb{E}\left[X \mathbf{1}_{A_{\lim}}\right] = \mathbb{E}[X], \\ \mathbb{E}\left[\limsup_{n \rightarrow \infty} X_n\right] &= \mathbb{E}\left[\left(\limsup_{n \rightarrow \infty} X_n\right) \mathbf{1}_{A_{\lim}}\right] = \mathbb{E}\left[X \mathbf{1}_{A_{\lim}}\right] = \mathbb{E}[X].\end{aligned}$$

- **Proof of Part 3:** For every $\omega \in A_{\lim}$, we have

$$|X_n(\omega)| \leq Y(\omega), \quad |X(\omega)| \leq Y \quad \implies \quad |X_n(\omega) - X(\omega)| \leq 2Y(\omega).$$

Exercise: Complete the remainder of the proof along the exact same lines as that of parts [1](#), [2](#)

Some Useful Consequences of MCT and DCT

Some Useful Consequences of MCT and DCT

- **Bounded convergence theorem (BCT):** Special case of DCT when Y is a constant RV
- If $\mathbb{E}[|X|] < +\infty$, then

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[|X| \mathbf{1}_{\{|X| > n\}} \right] = 0.$$

Proof:

- For each $n \in \mathbb{N}$, let

$$X_n := |X| \mathbf{1}_{\{|X| > n\}}.$$

- Observe that

$$X_1 \geq X_2 \geq X_3 \geq \cdots, \quad \lim_{n \rightarrow \infty} X_n(\omega) = 0 \quad \forall \omega \in \Omega.$$

- Furthermore, $X_n \leq |X|$, $\mathbb{E}[|X|] < +\infty$
- An application of MCT yields the desired result

Reverse Implication p. \implies m.s.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\{X_n\}_{n \in \mathbb{N}}$ be real-valued random variables and let X be an extended real-valued random variable.

Proposition (Reverse Implication p. \implies m.s.)

Suppose that the following conditions hold:

1. $\mathbb{E}[X_n^2] < +\infty$ for all $n \in \mathbb{N}$.
2. $|X_n| \leq Y$ for all n , with $\mathbb{E}[Y^2] < +\infty$.

Then,

$$X_n \xrightarrow{\text{p.}} X \implies X_n \xrightarrow{\text{m.s.}} X.$$

Step 1: Prove that $\mathbb{P}(\{|X| \leq Y\}) = 1$

- To prove that $\mathbb{P}(\{|X| \leq Y\}) = 1$, we will prove that

$$\mathbb{P}(\{|X| > Y + \varepsilon\}) = 0 \quad \forall \varepsilon > 0.$$

- Fix an arbitrary $\varepsilon > 0$, and note that

$$\begin{aligned} \mathbb{P}(\{|X| > Y + \varepsilon\}) &= \mathbb{P}(\{|X - X_n + X_n| > Y + \varepsilon\}) \\ &\leq \mathbb{P}(\{|X_n - X| + |X_n| > Y + \varepsilon\}) \\ &\leq \mathbb{P}(\{|X_n - X| > \varepsilon\}) + \mathbb{P}(\{|X_n| > Y\}) \\ &= \mathbb{P}(\{|X_n - X| > \varepsilon\}). \end{aligned}$$

- Taking limits as $n \rightarrow \infty$ and using the fact that $X_n \xrightarrow{p.} X$, we arrive at the desired result

Theory and Applications of Stochastic (Random) Processes

Random (Stochastic¹) Process: Definition

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Random Process)

Fix an **index set** \mathcal{T} .

A collection of random variables $X = \{X_t : t \in \mathcal{T}\}$ indexed by the elements of \mathcal{T} is called **random process**.

- When \mathcal{T} is finite, say $\mathcal{T} = \{1, \dots, d\}$, we have a **random vector** $\mathbf{X} = [X_1, \dots, X_d]^\top$
- If \mathcal{T} is a countably infinite set, we say $\{X_t : t \in \mathcal{T}\}$ is a **discrete parameter** process
- If \mathcal{T} is uncountably infinite, we say $\{X_t : t \in \mathcal{T}\}$ is a **continuous parameter** process
- Sometimes, \mathcal{T} has the interpretation of time

¹after the Greek word $\sigma\tau\omicron\chi\alpha\sigma\tau\iota\kappa\acute{o}\varsigma$ which means 'to proceed by guesswork'

Useful Ways to Think of a Random Process

- **Fixing $t \in \mathcal{T}$:** $X_t : \Omega \rightarrow \mathbb{R}$ is a random variable defined with respect to \mathcal{F}
- **Fixing $\omega \in \Omega$:** $X_{(\cdot)}(\omega) : \mathcal{T} \rightarrow \mathbb{R}$ denotes a **sample path** of the process
- $X : \mathcal{T} \times \Omega \rightarrow \mathbb{R}$
 - $X_t(\omega)$ is a real number for each $\omega \in \Omega$ and $t \in \mathcal{T}$

Note

In this course, we will typically consider \mathcal{T} to be one of \mathbb{R}_+ , \mathbb{R} , \mathbb{Z} , or \mathbb{N} .

Finite Dimensional Distributions

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Finite Dimensional Distributions)

Let $\{X_t : t \in \mathcal{T}\}$ be a random process.

- Given $n \in \mathbb{N}$ and $\mathbf{t} = (t_1, \dots, t_n) \in \mathcal{T}^n$, the joint CDF of X_{t_1}, \dots, X_{t_n} is given by

$$F_{\mathbf{t}}(\mathbf{x}) = \mathbb{P}(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

- The collection of CDFs

$$\left\{ F_{\mathbf{t}} : n \in \mathbb{N}, \mathbf{t} \in \mathcal{T}^n \right\}$$

is referred to as **finite dimensional distributions (FDDs)** of the process $\{X_t : t \in \mathcal{T}\}$.



Stochastic Processes

Lecture 10

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Some Useful Consequences of MCT and DCT

- **Bounded convergence theorem (BCT):** Special case of DCT when Y is a constant RV
- If $\mathbb{E}[|X|] < +\infty$, then

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[|X| \mathbf{1}_{\{|X| > n\}} \right] = 0.$$

Proof:

- For each $n \in \mathbb{N}$, let

$$X_n := |X| \mathbf{1}_{\{|X| > n\}}.$$

- Observe that

$$X_1 \geq X_2 \geq X_3 \geq \cdots, \quad \lim_{n \rightarrow \infty} X_n(\omega) = 0 \quad \forall \omega \in \Omega.$$

- Furthermore, $X_n \leq |X|$, $\mathbb{E}[|X|] < +\infty$
- An application of MCT yields the desired result

Reverse Implication p. \implies m.s.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\{X_n\}_{n \in \mathbb{N}}$ be real-valued random variables and let X be an extended real-valued random variable.

Proposition (Reverse Implication p. \implies m.s.)

Suppose that the following conditions hold:

1. $\mathbb{E}[X_n^2] < +\infty$ for all $n \in \mathbb{N}$.
2. $|X_n| \leq Y$ for all n , with $\mathbb{E}[Y^2] < +\infty$.

Then,

$$X_n \xrightarrow{\text{p.}} X \implies X_n \xrightarrow{\text{m.s.}} X.$$

Step 1: Prove that $\mathbb{P}(\{|X| \leq Y\}) = 1$

- To prove that $\mathbb{P}(\{|X| \leq Y\}) = 1$, we will prove that

$$\mathbb{P}(\{|X| > Y + \varepsilon\}) = 0 \quad \forall \varepsilon > 0.$$

- Fix an arbitrary $\varepsilon > 0$, and note that

$$\begin{aligned} \mathbb{P}(\{|X| > Y + \varepsilon\}) &= \mathbb{P}(\{|X - X_n + X_n| > Y + \varepsilon\}) \\ &\leq \mathbb{P}(\{|X_n - X| + |X_n| > Y + \varepsilon\}) \\ &\leq \mathbb{P}(\{|X_n - X| > \varepsilon\}) + \mathbb{P}(\{|X_n| > Y\}) \\ &= \mathbb{P}(\{|X_n - X| > \varepsilon\}). \end{aligned}$$

- Taking limits as $n \rightarrow \infty$ and using the fact that $X_n \xrightarrow{p.} X$, we arrive at the desired result

Step 2: Completing the Proof

- Notice that

$$|X_n| \leq Y, \quad \mathbb{P}(\{|X| \leq Y\}) = 1 \quad \implies \quad \mathbb{P}\left(\underbrace{\{|X_n - X| \leq 2Y\}}_A\right) = 1.$$

- For every choice of $\varepsilon > 0$:

— We have

$$\begin{aligned} \mathbb{E}[(X_n - X)^2] &= \mathbb{E}[(X_n - X)^2 \mathbf{1}_A] \\ &= \mathbb{E}\left[(X_n - X)^2 \mathbf{1}_{A \cap \{|X_n - X| > \varepsilon\}}\right] + \mathbb{E}\left[(X_n - X)^2 \mathbf{1}_{A \cap \{|X_n - X| \leq \varepsilon\}}\right] \\ &\leq \mathbb{E}\left[4Y^2 \mathbf{1}_{A \cap \{|X_n - X| > \varepsilon\}}\right] + \mathbb{E}\left[\varepsilon^2 \mathbf{1}_{A \cap \{|X_n - X| \leq \varepsilon\}}\right] \\ &\leq \mathbb{E}\left[4Y^2 \mathbf{1}_{\{|X_n - X| > \varepsilon\}}\right] + \varepsilon^2 \end{aligned}$$

Completing the Proof

- For any $M > 0$, we have

$$\begin{aligned}\mathbb{E} \left[4Y^2 \mathbf{1}_{\{|X_n - X| > \varepsilon\}} \right] &= \mathbb{E} \left[4Y^2 \mathbf{1}_{\{|X_n - X| > \varepsilon\} \cap \{Y > M\}} \right] + \mathbb{E} \left[4Y^2 \mathbf{1}_{\{|X_n - X| > \varepsilon\} \cap \{Y \leq M\}} \right] \\ &\leq \mathbb{E} \left[4Y^2 \mathbf{1}_{\{Y > M\}} \right] + 4M^2 \mathbb{P}(\{|X_n - X| > \varepsilon\})\end{aligned}$$

- Thanks to MCT, we have

$$\lim_{M \rightarrow \infty} \mathbb{E} \left[4Y^2 \mathbf{1}_{\{Y > M\}} \right] = 0.$$

Therefore, we may choose M large enough so that

$$\mathbb{E} \left[4Y^2 \mathbf{1}_{\{Y > M\}} \right] \leq \varepsilon.$$

- We then have

$$\mathbb{E}[(X_n - X)^2] \leq \varepsilon + 4M^2 \mathbb{P}(\{|X_n - X| > \varepsilon\}) + \varepsilon^2.$$

- Taking limits as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \mathbb{E}[(X_n - X)^2] \leq \varepsilon + \varepsilon^2$$

- Because the above is true for every choice of $\varepsilon > 0$, we get the desired result

Theory and Applications of Stochastic (Random) Processes

Random (Stochastic¹) Process: Definition

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Definition (Random Process)

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Useful Ways to Think of a Random Process

- **Fixing $t \in \mathcal{T}$:** $X_t : \Omega \rightarrow \mathbb{R}$ is a random variable defined with respect to \mathcal{F}
- **Fixing $\omega \in \Omega$:** $X_{(\cdot)}(\omega) : \mathcal{T} \rightarrow \mathbb{R}$ denotes a **sample path** of the process
- **$\mathbf{X} : \mathcal{T} \times \Omega \rightarrow \mathbb{R}$**
 - $X_t(\omega)$ is a real number for each $\omega \in \Omega$ and $t \in \mathcal{T}$
 - Let \mathcal{T} be the σ -algebra on \mathcal{T}
Let \mathcal{F} be the σ -algebra on Ω
Consider the product σ -algebra on $\mathcal{T} \times \Omega$ given by

$$\Sigma = \sigma(\mathcal{T} \times \mathcal{F}).$$

Then, $\mathbf{X} : \mathcal{T} \times \Omega \rightarrow \mathbb{R}$ is a random variable, i.e.,

$$\mathbf{X}^{-1}(B) \in \Sigma \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

Note

In this course, we will typically consider \mathcal{T} to be one of \mathbb{R}_+ , \mathbb{R} , \mathbb{Z} , or \mathbb{N} .

Finite Dimensional Distributions

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Finite Dimensional Distributions)

Let $\{X_t : t \in \mathcal{T}\}$ be a random process.

- Given $n \in \mathbb{N}$ and $\mathbf{t} = (t_1, \dots, t_n) \in \mathcal{T}^n$, the joint CDF of X_{t_1}, \dots, X_{t_n} is given by

$$F_{\mathbf{t}}(\mathbf{x}) = \mathbb{P}(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

- The collection of CDFs

$$\left\{ F_{\mathbf{t}} : n \in \mathbb{N}, \mathbf{t} \in \mathcal{T}^n \right\}$$

is referred to as **finite dimensional distributions (FDDs)** of the process $\{X_t : t \in \mathcal{T}\}$.

A Note on FDDs

- Is it necessary to specify **all finite-length** joint CDFs?
- Suppose that we wish to define a process $\{X_n\}_{n \in \mathbb{N}}$ as follows:
 - Any two random variables in the process are jointly Gaussian with mean vector and covariance matrix given by

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

- For the above example, the joint CDFs of any two pairs of random variables is well-defined
- Taking X_1, X_2, X_3 , we find that their covariance matrix is given by

$$K = K_{X_1, X_2, X_3} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

It is easy to check that $\det(K) = -1$ (not p.s.d.!) and hence not a valid covariance matrix!

- We need to specify the joint CDFs of **all finite collections of RVs** from the process

A Note on FDDs

- FDDs are **NOT** simply an arbitrary collection of joint CDFs
- Suppose that $\mathcal{T} = \{1, 2, 3\}$

— In this case,

$$\text{FDDs} = \left\{ F_1, F_2, F_3, F_{1,2}, F_{2,3}, F_{1,3}, F_{1,2,3} \right\}$$

— We observe that

$$F_2(x) = F_{1,2}(x, \infty), \quad F_{2,3}(x, y) = F_{1,2,3}(\infty, x, y), \quad \dots$$

- In some sense, the FDDs have to be **consistent**

Consistency of FDDs

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\{X_t : t \in \mathcal{T}\}$ be a random process defined w.r.t. \mathcal{F} .

Definition (Consistency of FDDs)

The FDDs of the process $\{X_t : t \in \mathcal{T}\}$ are said to be **consistent** if for any $m, n \in \mathbb{N}$ with $m < n$, and subsets $\mathcal{T}_m \subset \mathcal{T}_n \subset \mathcal{T}$ with $|\mathcal{T}_m| = m$ and $|\mathcal{T}_n| = n$, we have

$$F_{\mathbf{t}}(x_1, \dots, x_m) = F_{\mathbf{s}}(\underbrace{\infty, \dots, \infty, x_1, \infty, \dots, \infty, x_2, \infty, \dots, \infty, x_m, \infty, \dots, \infty}_n),$$

where

- $\mathbf{t} \in \mathcal{T}_m$, $\mathbf{s} \in \mathcal{T}_n$, \mathbf{s} contains the coordinates in \mathbf{t} .
- The coordinates in \mathbf{s} corresponding to those not in \mathbf{t} are marked with ∞ on the RHS.

Random Processes with Consistent FDDs

Note

Our interest is in the study of random processes whose FDDs are consistent.

Examples of processes with consistent FDDs include:

- IID processes.
- Bernoulli processes.
- Gaussian processes.
- Markov processes (or Markov chains).
- Poisson process.
- Lévy process.
- Brownian motion and diffusions.

Properties of Random Processes

Mean, Autocorrelation, and Autocovariance

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathbf{X} = \{X_t : t \in \mathcal{T}\}$ be a random process defined w.r.t. \mathcal{F} .

Definition (Mean, Autocorrelation, Autocovariance)

- The **mean** of the process $\mathbf{X} = \{X_t : t \in \mathcal{T}\}$ is a function $M_X : \mathcal{T} \rightarrow [-\infty, +\infty]$ defined as

$$M_X(t) = \mathbb{E}[X_t], \quad t \in \mathcal{T}.$$

- The **autocorrelation** and **autocovariance** of the process $\mathbf{X} = \{X_t : t \in \mathcal{T}\}$ are functions $R_X, C_X : \mathcal{T} \times \mathcal{T} \rightarrow [-\infty, +\infty]$, defined as

$$R_X(t, s) = \mathbb{E}[X_t X_s], \quad C_X(t, s) = \text{Cov}(X_t, X_s), \quad t, s \in \mathcal{T}.$$

Stationary Process

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathbf{X} = \{X_t : t \in \mathbb{R}_+\}$ be a random process defined w.r.t. \mathcal{F} .

Definition (Stationary Process)

The process $\mathbf{X} = \{X_t : t \geq 0\}$ is said to be **stationary** if **all FDDs are translation invariant**, i.e., for any $n \in \mathbb{N}$, $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, and $h \in \mathbb{R}_+$,

$$F_{\mathbf{t}} = F_{X_{t_1}, \dots, X_{t_n}} = F_{X_{t_1+h}, \dots, X_{t_n+h}} = F_{\mathbf{t}+h}.$$

Here, $\mathbf{t} + h = (t_1 + h, \dots, t_n + h)$.

Remark: A stationary process is also referred to as **strictly/strongly stationary** process

Weakly Stationary Process

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathbf{X} = \{X_t : t \in \mathbb{R}_+\}$ be a random process defined w.r.t. \mathcal{F} .

Definition (Weakly Stationary Process)

The process $\mathbf{X} = \{X_t : t \in \mathbb{R}_+\}$ is said to be **weakly stationary** (or **wide-sense stationary**) if for all $t_1, t_2 \in \mathbb{R}_+$ and $h \in \mathbb{R}_+$:

1. $M_X(t_1) = M_X(t_2)$.
2. $C_X(t_1, t_2) = C_X(t_1 + h, t_2 + h)$.

Exercises:

- A process is weakly stationary iff it has constant mean and $C_X(t, t + h) = C_X(0, h)$ for all $t, h \in \mathbb{R}_+$
- Every stationary process with finite variance is wide-sense stationary

IID Processes

IID Process

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathbf{X} = \{X_t : t \in \mathbb{R}_+\}$ be a random process defined w.r.t. \mathcal{F} .

Definition (IID Process)

The process $\mathbf{X} = \{X_t : t \in \mathbb{R}_+\}$ is said to be an **IID process** with the **common CDF** F if for any $n \in \mathbb{N}$, $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, and $\mathbf{x} = (x_1, \dots, x_n)$,

$$F_{\mathbf{t}}(\mathbf{x}) = F_{X_{t_1}, \dots, X_{t_n}}(x_1, \dots, x_n) = \prod_{i=1}^n F_{X_{t_i}}(x_i) = \prod_{i=1}^n F(x_i).$$

Exercises:

- The FDDs of $\mathbf{X} = \{X_t : t \in \mathbb{R}_+\}$ are consistent.
- $\mathbf{X} = \{X_t : t \in \mathbb{R}_+\}$ is strictly stationary. That is, **every IID process is stationary**.

Example

- Let X_1, X_2, \dots be an \mathbb{N} -valued IID process.
Let $S_0 := 0$, and for each $n \in \mathbb{N}$, let

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- Determine M_S and C_S for the process $\{S_n\}_{n \geq 0}$.

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- Determine the joint PMF of S_1, \dots, S_n .
- If X_i 's are IID and \mathbb{R} -valued with a common PDF f_X , determine the joint PDF of S_1, \dots, S_n .