Name:
Roll Number:

CS6660: Mathematical Foundations of Data Science (Probability)

Quiz 3

Date: 14 September 2024

Question	1	2(a)	2(b)	Total
Marks Scored				

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

1. **(1 Mark)**
Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let X and Y be jointly continuous random variables with the joint PDF

$$f_{X,Y}(x,y) = \frac{1}{x}, \qquad 0 \le y \le x \le 1.$$

Let $Z = X + Y$.
If the value of $\mathbb{P}(\{0 \le Z \le 1\})$ is expressed as $\log \alpha$, where the logarithm is the natural logarithm, then what is the value of α?

Solution:
Setting $W = X$, we let

$$g_1(x,y) := x + y, \qquad g_2(x,y) = x, \qquad x, y \in \mathbb{R}.$$

We then note from the joint PDF of X and Y that $Z = g_1(X,Y)$ takes values between 0 and 2, and $W = g_2(X,Y)$ takes values between 0 and 1. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$g(x,y) = (x+y, x).$$

Writing $z = x + y$ and $w = x$, we see that the mapping $(z,w) \mapsto (w, z-w)$ defines g^{-1}.
We now use the Jacobian formula to compute the joint PDF of Z and W. To do so, we first note that for any (x,y),

$$J_g(x,y) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and hence $\left|\det(J_g(x,y))\right| = 1$ for all x, y. Using the Jacobian formula, we have

$$f_{Z,W}(z,w) = \frac{f_{X,Y}(g^{-1}(z,w))}{\left|\det(J_g(g^{-1}(z,w)))\right|} = f_{X,Y}(w, z-w) = \frac{1}{w}, \qquad 0 \le z - w \le w \le 1.$$

Observe that

$$0 \le z - w \qquad \text{implies} \qquad w \le z,$$

$$z - w \leq w \qquad \text{implies} \qquad w \geq \frac{z}{2}.$$

Combining the above together with $w \leq 1$, we get that for any fixed z, the range of W is $[\frac{z}{2}, \min\{z, 1\}]$. We thus have

$$f_Z(z) = \int_{\frac{z}{2}}^{\min\{z,1\}} f_{Z,W}(z,w)\, dw = \int_{\frac{z}{2}}^{\min\{z,1\}} \frac{1}{w}\, dw = \log \min\{z, 1\} - \log \frac{z}{2}, \qquad 0 \leq z \leq 2.$$

Finally, we have

$$\mathbb{P}(\{0 \leq Z \leq 1\}) = \int_0^1 f_Z(z)\, dz \overset{(*)}{=} \int_0^1 \left(\log z - \log \frac{z}{2}\right)\, dz = \log 2,$$

where $(*)$ follows from the observation that $\min\{z, 1\} = z$ for $0 \leq z \leq 1$. Hence, we have $\alpha = 2$.

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2. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that all random variables appearing below are defined with respect to $\mathscr{F}$.

   Let $X, Y \overset{\text{i.i.d.}}{\sim}$ Exponential(λ).

   (a) **(3 Marks)**
   Determine the joint PDF of $Z = X + Y$ and $W = \frac{X}{X+Y}$.
   Clearly specify the range of Z and the range of W in the joint PDF expression.

   (b) **(1 Mark)**
   Compute $\mathbb{P}(\{W \leq \frac{1}{3}\})$.

   Solution: We solve each of the parts below.

   (a) Defining g_1 and g_2 as the mappings $g_1 : (x, y) \mapsto x + y$ and $g_2 : (x, y) \mapsto \frac{x}{x+y}$, we note that

$$J_g(x,y) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{pmatrix},$$

   from which it follows that

$$\left|\text{det}(J_g(x,y))\right| = \frac{1}{x+y}.$$

   Setting $z = x+y$ and $w = \frac{x}{x+y}$, we have $x = wz$ and $y = z - wz$, and hence it follows that $g^{-1} : (z, w) \mapsto (wz, z - wz)$. Noting that the range of Z is the set of non-negative real numbers, and the range of W is $[0, 1]$, using the Jacobian formula, we get

$$f_{Z,W}(z,w) = \frac{f_{X,Y}(g^{-1}(z,w))}{\left|\text{det}(J_g(g^{-1}(z,w)))\right|} = \lambda^2 \, z \, e^{-\lambda z}, \qquad z \geq 0, w \in [0,1].$$

   We then get

$$f_W(w) = \int_0^\infty f_{Z,W}(z) \, dz = 1, \qquad w \in [0,1].$$

   Thus, $W \sim$ Unif$([0, 1])$.

   (b) The desired probability is

$$\mathbb{P}(\{W \leq 1/3\}) = F_W(1/3) = 1/3.$$