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Probability and Stochastic Processes

Lecture 06: σ-Algebras (contd.), Construction of $\mathscr{B}[0, 1]$ and $\mathscr{B}(\mathbb{R})$, Generating Classes for $\mathscr{B}(\mathbb{R})$

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Construction of $\mathscr{B}(\{0,1\}^{\mathbb{N}})$

- Consider the "cylinder" base collection $\mathscr{C} = \Big\{[\mathbf{b}] : \mathbf{b} \in \mathscr{D}\Big\}$.
- Let $\mathscr{A} = \alpha(\mathscr{C})$ denote the smallest algebra constructed starting from $\mathscr{C}$
- Let A^* denote the set

$$A^* = \{\omega \in \{0,1\}^{\mathbb{N}} : \quad \omega_i = 1 \text{ for all } i \in \{2,4,6,8,\ldots\}\}$$

- Clearly, $A^* \notin \mathscr{A}$, as its occurrence/non-occurrence cannot be determined from only observing the first finitely many bits of any infinite binary string
- This shows that $\mathscr{A}$ is **not a σ-algebra**
- Let $\sigma(\mathscr{A})$ denote the smallest σ-algebra constructed starting from $\mathscr{A}$

The Borel σ-Algebra

The σ-algebra $\sigma(\mathscr{A})$ so constructed is called the **Borel σ-algebra** of subsets of $\{0,1\}^{\mathbb{N}}$. Henceforth, we shall denote the same by $\mathscr{B}(\{0,1\}^{\mathbb{N}})$.

Construction of $\mathscr{B}(0, 1)$

- Consider the collection

$$\mathscr{P} = \left\{(a, b) : \quad a, b \in \mathbb{R},\ \ 0 \leq a \leq b \leq 1\right\}.$$

- Is $\mathscr{P}$ a σ-algebra? **No!**
- Let $\sigma(\mathscr{P})$ denote the smallest σ-algebra constructed starting from $\mathscr{P}$

The Borel σ-Algebra

The σ-algebra $\sigma(\mathscr{P})$ so constructed is called the **Borel σ-algebra** of subsets of $(0, 1)$. Henceforth, we shall denote the same by $\mathscr{B}(0, 1)$.

Examples

- Let $\Omega = \{1, \ldots, 6\}$. Consider the collection

$$\mathscr{C} = \Big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\Big\}$$

What is $\mathscr{F} = \sigma(\mathscr{C})$?

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Examples

- Let $\Omega = \{1, \ldots, 6\}$. Consider the collection

$$\mathscr{C} = \Big\{ \{1,2\}, \{3,4\}, \{5,6\} \Big\}$$

  What is $\mathscr{F} = \sigma(\mathscr{C})$?

- What if $\mathscr{C} = \Big\{ \{1,2\}, \{3,4\} \Big\}$?

Examples

- Let $\Omega = \{1, \dots, 6\}$. Consider the collections

$$\mathscr{C}_1 = \Big\{\{1,2\},\{3,4\}\Big\}, \qquad \mathscr{C}_2 = \Big\{\{1,3\}\Big\}.$$

Let $\mathscr{F}_1 = \sigma(\mathscr{C}_1)$ and $\mathscr{F}_2 = \sigma(\mathscr{C}_2)$.

1. What is $\mathscr{F}_1 \cap \mathscr{F}_2$? Is it also a σ-algebra?
2. What is $\mathscr{F}_1 \cup \mathscr{F}_2$? Is it also a σ-algebra?

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Examples

- Let $\Omega = \{1, \ldots, 6\}$. Construct a σ-algebra $\mathscr{F}$ containing 16 events in it.

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Examples

- Let $\Omega = \mathbb{N}$. Consider the collection

$$\mathscr{C} = \Big\{ \{n\} : n \in \mathbb{N} \Big\}$$

What is $\mathscr{F} = \sigma(\mathscr{C})$?

Examples

- Let $\Omega = \mathbb{N}$. Construct a σ-algebra with 512 sets in it.

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Examples

- Let $\Omega = (0, 1)$. Construct a σ-algebra with 512 sets in it.

Examples

- Let $\Omega = (0, 1)$. For each $n \in \mathbb{N}$, let

$$A_n := \left(\frac{1}{5}, \ \frac{1}{3} + \frac{1}{n} \right), \qquad B_n := \left(\frac{1}{5} - \frac{1}{n}, \ \frac{1}{3} \right).$$

1. Evaluate $\displaystyle\bigcap_{n \in \mathbb{N}} A_n$.
2. Evaluate $\displaystyle\bigcup_{n \in \mathbb{N}} A_n$
3. Evaluate $\displaystyle\bigcap_{n \in \mathbb{N}} B_n$.
4. Evaluate $\displaystyle\bigcup_{n \in \mathbb{N}} B_n$.
5. Show that $\left[\frac{1}{5}, \ \frac{1}{3}\right] \in \mathscr{B}(0, 1)$.

Examples

- Let $\Omega = (0, 1)$.

  1. Show that $\{x\} \in \mathscr{B}(0, 1)$ for every $x \in (0, 1)$.
  2. Given $0 < a < b < 1$, show that $(a, b] \in \mathscr{B}(0, 1)$.
  3. Given $0 < a < b < 1$, show that $[a, b) \in \mathscr{B}(0, 1)$.
  4. Given $0 < a < b < 1$, show that $[a, b] \in \mathscr{B}(0, 1)$.

Construction of $\mathscr{B}[0,1]$

Question

Now that we know how to construct $\mathscr{B}(0,1)$, how do we construct $\mathscr{B}[0,1]$?

- Observe that

$$[0,1] = \{0\} \quad \cup \quad (0,1) \quad \cup \quad \{1\}$$

- We then have

$$\mathscr{B}[0,1] = \sigma\left(\Big\{\{0\},\{1\}\Big\} \quad \bigcup \quad \mathscr{B}(0,1)\right).$$

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Construction of $\mathscr{B}(\mathbb{R})$

- Experiment: measure the noise level at the receiver of a communication system
- Each outcome: $\omega \in \mathbb{R}$
- Sample space: $\Omega = \mathbb{R} = (-\infty, +\infty)$
- Consider the collection

$$\mathscr{P}_1 := \Big\{(a, b) : \ a, b \in \mathbb{R}, \ a \leq b\Big\}.$$

Borel σ-Algebra $\mathscr{B}(\mathbb{R})$

The smallest σ-algebra that can be constructed from $\mathscr{P}_1$ is called the Borel σ-algebra of subsets of $\mathbb{R}$. Henceforth, we shall denote the same by $\mathscr{B}(\mathbb{R})$.

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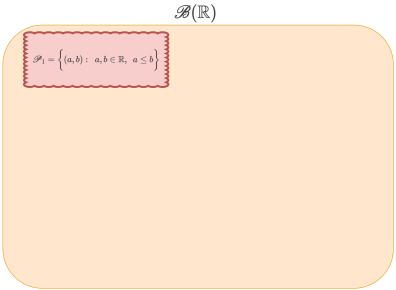
Demystifying $\mathscr{B}(\mathbb{R})$

Recall:

$$\mathscr{P}_1 = \Big\{(a,b) : \ a, b \in \mathbb{R}, \ a \le b\Big\}.$$

- Given $x \in \mathbb{R}$, show that $\{x\}$ can be expressed in terms of sets from $\mathscr{P}_1$.
- Given $x \in \mathbb{R}$, show that $(-\infty, x)$ can be expressed in terms of sets from $\mathscr{P}_1$.
- Given $x \in \mathbb{R}$, show that $(-\infty, x]$ can be expressed in terms of sets from $\mathscr{P}_1$.
- Given $x \in \mathbb{R}$, show that $(x, +\infty)$ can be expressed in terms of sets from $\mathscr{P}_1$.
- Given $x \in \mathbb{R}$, show that $[x, +\infty)$ can be expressed in terms of sets from $\mathscr{P}_1$.
- Given $x, y \in \mathbb{R}$ with $x < y$, show that $(x, y]$ can be expressed in terms of sets from $\mathscr{P}_1$.
- Given $x, y \in \mathbb{R}$ with $x < y$, show that $[x, y]$ can be expressed in terms of sets from $\mathscr{P}_1$.
- Given $x, y \in \mathbb{R}$ with $x < y$, show that $[x, y)$ can be expressed in terms of sets from $\mathscr{P}_1$.

$\mathscr{P}_1$ is a Generating Class for $\mathscr{B}(\mathbb{R})$!



$\mathscr{P}_1$ is a Generating Class

Any set in $\mathscr{B}(\mathbb{R})$ may be expressed via complements and/or countable unions and/or countable intersections of sets in $\mathscr{P}_1$, i.e., $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{P}_1)$.

Demystifying $\mathscr{B}(\mathbb{R})$

Consider the collection

$$\mathscr{P}_2 := \Big\{ [a, b] : \; a, b \in \mathbb{R}, \; a \leq b \Big\}.$$

- Given $x \in \mathbb{R}$, show that $\{x\}$ can be expressed in terms of sets from $\mathscr{P}_2$.
- Given $x \in \mathbb{R}$, show that $(-\infty, x)$ can be expressed in terms of sets from $\mathscr{P}_2$.
- Given $x \in \mathbb{R}$, show that $(-\infty, x]$ can be expressed in terms of sets from $\mathscr{P}_2$.
- Given $x \in \mathbb{R}$, show that $(x, +\infty)$ can be expressed in terms of sets from $\mathscr{P}_2$.
- Given $x \in \mathbb{R}$, show that $[x, +\infty)$ can be expressed in terms of sets from $\mathscr{P}_2$.
- Given $x, y \in \mathbb{R}$ with $x < y$, show that $(x, y]$ can be expressed in terms of sets from $\mathscr{P}_2$.
- Given $x, y \in \mathbb{R}$ with $x < y$, show that (x, y) can be expressed in terms of sets from $\mathscr{P}_2$.
- Given $x, y \in \mathbb{R}$ with $x < y$, show that $[x, y)$ can be expressed in terms of sets from $\mathscr{P}_2$.

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$\mathscr{P}_2$ is a Generating Class for $\mathscr{B}(\mathbb{R})$!

$\mathscr{B}(\mathbb{R})$

$\mathscr{P}_1 = \left\{(a, b) : \ a, b \in \mathbb{R}, \ a \leq b\right\}$

$\mathscr{P}_2 = \left\{[a, b] : \ a, b \in \mathbb{R}, \ a \leq b\right\}$

$\mathscr{P}_2$ is a Generating Class

Any set in $\mathscr{B}(\mathbb{R})$ may be expressed via complements and/or countable unions and/or countable intersections of sets in $\mathscr{P}_2$, i.e., $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{P}_2)$.

Generating Classes for $\mathscr{B}(\mathbb{R})$

- We already saw

$$\mathscr{P}_1 = \Big\{(a,b): \ a,b \in \mathbb{R}, \ a \leq b\Big\}, \qquad \mathscr{P}_2 = \Big\{[a,b]: \ a,b \in \mathbb{R}, \ a \leq b\Big\},$$

  are generating classes for $\mathscr{B}(\mathbb{R})$

- In simple words, $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{P}_1)$, $\quad \mathscr{B}(\mathbb{R}) = \sigma(\mathscr{P}_2)$
- Consider the collections

$$\mathscr{P}_3 = \Big\{[a,b): \ a,b \in \mathbb{R}, \ a \leq b\Big\}, \qquad \mathscr{P}_4 = \Big\{(a,b]: \ a,b \in \mathbb{R}, \ a \leq b\Big\},$$

$$\mathscr{P}_5 = \Big\{(-\infty, \ x): \ x \in \mathbb{R}\Big\}, \qquad \mathscr{P}_6 = \Big\{(-\infty, \ x]: \ x \in \mathbb{R}\Big\},$$

$$\mathscr{P}_7 = \Big\{(x, \ +\infty): \ x \in \mathbb{R}\Big\}, \qquad \mathscr{P}_8 = \Big\{[x, \ +\infty): \ x \in \mathbb{R}\Big\}.$$

- It is easy to show that $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{P}_i)$ for all $i \in \{3, \ldots, 8\}$

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Generating Classes for $\mathscr{B}(\mathbb{R})$

$\mathscr{B}(\mathbb{R})$

$$\mathscr{P}_1 = \left\{(a,b) : \ a,b \in \mathbb{R}, \ a < b\right\}$$

$$\mathscr{P}_2 = \left\{[a,b] : \ a,b \in \mathbb{R}, \ a < b\right\}$$

$$\mathscr{P}_3 = \left\{[a,b) : \ a,b \in \mathbb{R}, \ a < b\right\}$$

$$\mathscr{P}_4 = \left\{(a,b] : \ a,b \in \mathbb{R}, \ a < b\right\}$$

$$\mathscr{P}_5 = \left\{(-\infty, \ x) : \ x \in \mathbb{R}\right\}$$

$$\mathscr{P}_6 = \left\{(-\infty, \ x] : \ x \in \mathbb{R}\right\}$$

$$\mathscr{P}_7 = \left\{(x, \ +\infty) : \ x \in \mathbb{R}\right\}$$

$$\mathscr{P}_8 = \left\{[x, \ +\infty) : \ x \in \mathbb{R}\right\}$$

Figure: Various generating classes for $\mathscr{B}(\mathbb{R})$.