Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

AI5090/EE5817: PROBABILITY AND STOCHASTIC PROCESSES

QUIZ 01

DATE: 12 AUGUST 2025

Question	1	2	Total
Marks Scored			

You may use any result covered in class without proof. Do not use any result from the homework unless suggested as a hint.

1. Suppose sets A and B are equicardinal.

   (a) **(1 Mark)**
   Prove or give a counterexample: every injective function from A to B must also be surjective.

   (b) **(1 Mark)**
   Prove or give a counterexample: every surjective function from A to B must also be injective.

Solutions:

(a) Let $A = \mathbb{N} = B$. Consider the function $g : A \to B$ defined as

$$g(n) = 2n, \qquad n \in \mathbb{N}.$$

Clearly, g is injective. Indeed,

$$g(m) = g(n) \implies 2m = 2n \implies m = n.$$

However, g is clearly not surjective, for the odd natural numbers do not have a pre-image under g.

(b) Let $A = \mathbb{R}$ and $B = \mathbb{R}_+ = [0, +\infty)$. Clearly, $|A| = |B|$. Consider the function $g : A \to B$ defined as

$$g(x) = x^2, \qquad x \in \mathbb{R}.$$

Clearly, g is surjective, as for any $y \in \mathbb{R}^+$,

$$g^{-1}(y) = \{x \in \mathbb{R} : x^2 = x\} = \{-\sqrt{y}, +\sqrt{y}\}$$

is non-empty. However, g is not injective, as $g(-\sqrt{y}) = y = g(+\sqrt{y})$ for any $y \in \mathbb{R}_+$.

2. For each $k \in \mathbb{N}$, let $\mathscr{C}_k \subset \{0,1\}^{\mathbb{N}}$ denote the set of all infinite binary strings containing **exactly** k ones in them. Furthermore, let $\mathscr{C} \subset \{0,1\}^{\mathbb{N}}$ denote the set of all infinite binary strings containing **finitely** many ones in them.

   (a) **(2 Marks)**
   Show that $\mathscr{C}_k$ is countably infinite for every $k \in \mathbb{N}$.
   You may use the fact that $\mathbb{N}^k := \mathbb{N} \underbrace{\times \cdots \times}_{k \text{ times}} \mathbb{N}$ is countably infinite for every $k \in \mathbb{N}$.

   (b) **(1 Mark)**
   Express $\mathscr{C}$ in terms of $\mathscr{C}_1, \mathscr{C}_2, \ldots$, and argue that $\mathscr{C}$ is countably infinite.

Solutions:

(a) Fix an arbitrary $k \in \mathbb{N}$. On the one hand, consider the mapping $f_k : \mathscr{C}_k \to \mathbb{N}^k$ given by

$$f_k(\mathbf{b}) = (n \in \mathbb{N} : b_n = 1), \qquad \mathbf{b} \in \mathscr{C}_k.$$

We claim that f_k is injective. Indeed,

$$f_k(\mathbf{b}) = f_k(\mathbf{b}') \quad \Longrightarrow \quad (n \in \mathbb{N} : b_n = 1) = (n \in \mathbb{N} : b'_n = 1) \quad \Longrightarrow \quad \mathbf{b} = \mathbf{b}'.$$

This proves that $|\mathscr{C}_k| \leq |\mathbb{N}^k| = \aleph_0$.
On the other hand, consider the mapping $g_k : \mathbb{N}^k \to \mathscr{C}_k$ given by

$$g_k : (n_1, \ldots, n_k) \mapsto \underbrace{1 \cdots 1}_{n_1}\, 0\, \underbrace{1 \cdots 1}_{n_2}\, 0 \cdots \underbrace{1 \cdots 1}_{n_k}\, \bar{0}, \qquad (n_1, \ldots, n_k) \in \mathbb{N}^k.$$

Then, we claim that g_k is injective. Indeed,

$$\begin{aligned} g(n_1, \ldots, n_k) = g(n'_1, \ldots, n'_k) &\implies \underbrace{1 \cdots 1}_{n_1}\, 0\, \underbrace{1 \cdots 1}_{n_2}\, 0 \cdots \underbrace{1 \cdots 1}_{n_k} \bar{0} = \underbrace{1 \cdots 1}_{n'_1}\, 0\, \underbrace{1 \cdots 1}_{n'_2}\, 0 \cdots \underbrace{1 \cdots 1}_{n'_k} \bar{0} \\ &\implies (n_1, \ldots, n_k) = (n'_1, \ldots, n'_k). \end{aligned}$$

This proves that $|\mathbb{N}^k| = \aleph_0 \leq |\mathscr{C}_k|$.
Combining the above arguments, we see that $|\mathscr{C}_k| = \aleph_0$, thus proving that $\mathscr{C}_k$ is countably infinite.

3. We note that

$$\mathscr{C} = \bigcup_{k \in \mathbb{N}} \mathscr{C}_k.$$

Using the result in part (a) and the fact that countable union of countable sets is countable, we deduce that $\mathscr{C}$ is countable.