AI5030 / EE5817: Probability and Stochastic Processes
Homework 03

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Measures, Probability Measures

1. Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

   (a) Given $c \in \mathbb{R}$, define $\delta_c : \mathscr{F} \to [0,1]$ as

   $$\delta_c(A) = \begin{cases} 1, & c \in A, \\ 0, & c \notin A, \end{cases} \qquad A \in \mathscr{F}.$$

   Show that δ_c is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

   Remark: δ_c is called the Dirac measure concentrated at c.

   (b) Let $\mu : \mathscr{F} \to [0, +\infty]$ be defined as

   $$\mu(A) = \sum_{n \in \mathbb{N}} \delta_n(A), \qquad A \in \mathscr{F}.$$

   Show that μ is a measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. What does $\mu(A)$ for any $A \in \mathscr{F}$ represent?
   You may want to use the fact that if $\{a_{n,k}\}_{n,k \in \mathbb{N}}$ is a sequence of non-negative real numbers, then

   $$\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} a_{n,k} = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} a_{n,k}.$$

   Remark: μ is called the counting measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

2. Let $\Omega = \mathbb{N}$. Let $\mathscr{A}$ be defined as the collection

   $$\mathscr{A} := \Big\{ A \subseteq \Omega : \ |A| < +\infty \quad \text{or} \quad |\Omega \setminus A| < +\infty \Big\}.$$

   We know from Question 3(b) of Homework 2 that $\mathscr{A}$ is an algebra, but not a σ-algebra.
   Define $\mathbb{P}_0 : \mathscr{A} \to [0,1]$ as

   $$\mathbb{P}_0(A) = \begin{cases} 0, & |A| < +\infty, \\ 1, & |\Omega \setminus A| < +\infty. \end{cases}$$

   (a) Show that for any two disjoint sets $A, B \in \mathscr{A}$,

   $$\mathbb{P}_0(A \cup B) = \mathbb{P}_0(A) + \mathbb{P}_0(B).$$

   (By induction, it follows that $\mathbb{P}_0$ satisfies the property of finite additivity on $\mathscr{A}$.)

   (b) Show that $\mathbb{P}_0$ does not necessarily satisfy countable additivity property.
   That is, construct an explicit sequence of disjoint events $A_1, A_2, \ldots \in \mathscr{A}$ such that

   $$\bigsqcup_{n \in \mathbb{N}} A_n \in \mathscr{A}, \qquad \mathbb{P}_0\left(\bigsqcup_{n \in \mathbb{N}} A_n \right) \neq \sum_{n \in \mathbb{N}} \mathbb{P}_0(A_n).$$

   (c) Construct a non-increasing sequence of sets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ such that

   $$\bigcap_{n \in \mathbb{N}} A_n = \emptyset, \qquad \lim_{n \to \infty} \mathbb{P}_0(A_n) \neq 0.$$

3. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathscr{G}$ be defined as the collection

$$\mathscr{G} := \Big\{ A \in \mathscr{F} : \ \mathbb{P}(A) = 0 \quad \text{or} \quad \mathbb{P}(A) = 1 \Big\}.$$

Show that $\mathscr{G}$ is a σ-algebra of subsets of Ω.

4. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $A_1, A_2, \ldots \in \mathscr{F}$.

   (a) Show formally that
   $$\liminf_{n\to\infty} A_n \in \mathscr{F}, \qquad \limsup_{n\to\infty} A_n \in \mathscr{F}.$$

   (b) Prove that
   $$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n.$$
   Provide an example construction in which $\liminf_{n\to\infty} A_n \subsetneq \limsup_{n\to\infty} A_n$ (strict inclusion).

   (c) Let $B_1, B_2, \ldots \in \mathscr{F}$ be sequence of disjoint sets, i.e., $B_i \cap B_j = \emptyset \quad \forall i \neq j$. Prove that
   $$\lim_{n\to\infty} \mathbb{P}(B_n) = 0.$$

5. Let $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$. For each $n \in \mathbb{N}$, let $\mathbb{P}_n : \mathscr{F} \to [0,1]$ be defined as

$$\mathbb{P}_n(A) = \frac{|A \cap \{1, \ldots, n\}|}{n}, \qquad A \in \mathscr{F}.$$

   (a) Show that P_n is a probability measure on $\mathscr{F}$ for each $n \in \mathbb{N}$.

   (b) Given a set $A \in \mathscr{F}$, its **density** $D(A)$ is defined as
   $$D(A) := \lim_{n\to\infty} \mathbb{P}_n(A),$$
   provided the above limit exists. Let $\mathscr{D}$ denote the collection of all sets whose density is well-defined, i.e.,
   $$\mathscr{D} := \Big\{ A \in \mathscr{F} : \ \lim_{n\to\infty} \mathbb{P}_n(A) \ \text{is well-defined} \Big\}.$$
   Show that D is finitely additive on $\mathscr{D}$.
   Construct an example to show that D is not necessarily countably additive on $\mathscr{D}$.