AI5030 / EE5817: Probability and Stochastic Processes

Homework 04

Borel-Cantelli Lemma, Conditional Probability, Indpendence, Random Variables, CDF

1. At the end of each day Professor Coen puts his glasses in his drawer with probability 0.9, leaves them on the table with probability 0.06, leaves them in his briefcase with probability 0.03, and he actually leaves them at the office with probability 0.01. The next morning he has no recollection of where he left the glasses. He looks for them, but each time he looks in a place the glasses are actually located, he misses finding them with probability 0.1, whether or not he already looked in the same place. (After all, he doesn't have his glasses on and he is in a hurry.)

   (a) Specify $(\Omega, \mathscr{F}, \mathbb{P})$ for the above search exercise of the professor.

   (b) Given that Professor Coen didn't find the glasses in his drawer after looking one time, what is the conditional probability the glasses are on the table?

   (c) Given that he didn't find the glasses after looking for them in the drawer and on the table once each, what is the conditional probability they are in the briefcase?

   (d) Given that he failed to find the glasses after looking in the drawer twice, on the table twice, and in the briefcase once, what is the conditional probability he left the glasses at the office?

2. (a) Prove the following.

   i. $\mathbb{P}\big(A \mid A \cup B\big) = \dfrac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(B)}$, where A, B are disjoint, and $\mathbb{P}(A \cup B) > 0$.

   ii. If $\mathbb{P}(A \mid C) > \mathbb{P}(B \mid C)$ and $\mathbb{P}(A \mid C^{\complement}) > \mathbb{P}(B \mid C^{\complement})$, then $\mathbb{P}(A) > \mathbb{P}(B)$.

   iii. $\mathbb{P}(A \mid B) = \mathbb{P}\big(A \mid B \cap C\big)\, \mathbb{P}(C \mid B) + \mathbb{P}\big(A \mid B \cap C^{\complement}\big)\, \mathbb{P}\big(C^{\complement} \mid B\big)$.

   (b) Let A, B, C be pairwise independent, equiprobable events such that $A \cap B \cap C = \emptyset$. Find the largest possible value of the probability $\mathbb{P}(A)$. Construct an example and verify that this holds.

3. Fix an probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

   (a) Let $X : \Omega \to \mathbb{R}$ be a random variable. Define $\sigma(X)$ as the collection

   $$\sigma(X) \coloneqq \Big\{ X^{-1}(B) : B \in \mathscr{B}(\mathbb{R}) \Big\}.$$

   That is, every set in $\sigma(X)$ is a pre-image of some Borel set under the random variable X.
   Show that $\sigma(X)$ is a σ-algebra of subsets of Ω.

   (b) Set $\Omega = \{1, 2, 3, 4, 5, 6\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(\{\omega\}) = \frac{1}{6}$ for all $\omega \in \Omega$.
   Consider the events $A =$ "outcome is even", $B =$ "outcome ≤ 4".
   Let $X = \mathbf{1}_A$ and $Y = \mathbf{1}_B$. Determine $\sigma(X)$ and $\sigma(Y)$.

   Remark: $\sigma(X)$ is known as the σ-algebra generated by the random variable X.

4. (a) Let $A, B, A_1, A_2, \ldots$ be events. Suppose that for each k, we have $A_k \subseteq A_{k+1}$, and that B is independent of $A_k, \forall k \geq 1$. Let $A = \bigcup_{k \geq 1} A_k$. Show that B is independent of A.

   (b) Let A_n be a sequence of independent events with $\mathbb{P}(A_n) < 1$ for all n, and $\mathbb{P}\left(\bigcup_{n \in \mathbb{N}} A_n\right) = 1$. Show that

   $$\mathbb{P}\left(\limsup_{n \to \infty} A_n\right) = 1.$$

(c) Let $\Omega = \{0,1\}^{\mathbb{N}}$, $\quad \mathscr{F} = \mathscr{B}(\{0,1\}^{\mathbb{N}})$.
Let $\mathscr{C}$ be the collection of all cylinder sets.
Define $\mathbb{P}_0 \colon \mathscr{C} \to [0,1]$ as

$$\mathbb{P}_0([b]) = \frac{1}{2^{|b|}}, \qquad [b] \in \mathscr{C}.$$

Let $\mathbb{P} \colon \mathscr{B}(\{0,1\}^{\mathbb{N}}) \to [0,1]$ be the probability measure obtained by extending $\mathbb{P}_0$ to the whole of $\mathscr{B}(\{0,1\}^{\mathbb{N}})$.
The measure $\mathbb{P}$ is the Lebesgue measure on $\mathscr{B}(\{0,1\}^{\mathbb{N}})$.

For an outcome $\omega \in \Omega$, we say that "a run length of k occurs" if ω contains a sequence of k consecutive 1s.

i. For a fixed $k \in \mathbb{N}$, let $E^{(k)}$ denote the event that infinitely many runs of length k occur.
Show that $\mathbb{P}(E^{(k)}) = 1$.
Hint: Construct a sequence of cylinder sets $E_1^{(k)}, E_2^{(k)}, \ldots$ which are independent, with $\mathbb{P}(E_n^{(k)}) < 1$ for all $n \in \mathbb{N}$ and $\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} E_n^{(k)}\right) = 1$. Use the result of part (b) to arrive at the answer.
Alternatively, construct a sequence of independent cylinder sets $E_1^{(k)}, E_2^{(k)}, \ldots$ with $\sum_{n\in\mathbb{N}} \mathbb{P}(E_n^{(k)}) = +\infty$. Apply the Borel–Cantelli Lemma to arrive at the answer.

ii. What is the probability that every finite run length occurs infinitely many times?
Hint: Express the event "**every** finite run length occurs infinitely many times" in terms of $E^{(1)}, E^{(2)}, \ldots$, and use the result of part (i) above.

5. (a) Which of the following are valid CDFs? For each that is not valid, state at least one reason why. For each valid CDF, determine $\mathbb{P}\big(\{\omega \in \Omega : (X(\omega))^2 > 5\}\big)$.

$$F_1(x) = \begin{cases} \dfrac{e^{-x^2}}{4}, & x < 0, \\[2ex] 1 - \dfrac{e^{-x^2}}{4}, & x \geq 0, \end{cases}$$

$$F_2(x) = \begin{cases} 0, & x < 0, \\ 0.5 + e^{-x}, & 0 \leq x < 3, \\ 1, & x \geq 3, \end{cases}$$

$$F_3(x) = \begin{cases} 0, & x < 0, \\ 0.5 + \dfrac{x}{20}, & 0 \leq x \leq 10, \\ 1, & x \geq 10. \end{cases}$$

(b) Let F_1, F_2 be valid CDFs. Under what conditions would the following function be a CDF?

$$F_3(x) = \begin{cases} F_1(x), & x < 1, \\ (2-x)\,F_1(1) + (x-1)\,F_2(2), & 1 \leq x < 2, \\ F_2(x), & x \geq 2. \end{cases}$$

6. Fix a measurable space $(\Omega, \mathscr{F})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable (with respect to $\mathscr{F}$).
Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function (for the definition of a measurable function $g : E \to G$, where E and G are endowed with σ-algebras $\mathscr{E}$ and $\mathscr{G}$ respectively, see the lecture slides. We set $E = G = \mathbb{R}$ and $\mathscr{E} = \mathscr{G} = \mathscr{B}(\mathbb{R})$).
Show that $Y : \Omega \to \mathbb{R}$ defined as $Y = g(X)$ is a random variable with respect to $\mathscr{F}$.