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Probability and Stochastic Processes

Lecture 22: Primer on Riemann Integration, Abstract Integrals

Karthik P. N.

Assistant Professor, Department of AI

Email: pnkarthik@ai.iith.ac.in

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Primer on Riemann Integration

The Key Question

Fix $a, b \in \mathbb{R}$ such that $a < b$.
Let $f : [a, b] \to \mathbb{R}$ be a **bounded** function.
How do we interpret the quantity

$$\int_a^b f(x)\,\mathrm{d}x, \qquad \text{or simply} \qquad \int_a^b f\,\mathrm{d}x \ ?$$

- Let us define a **partition** of $[a, b]$ as

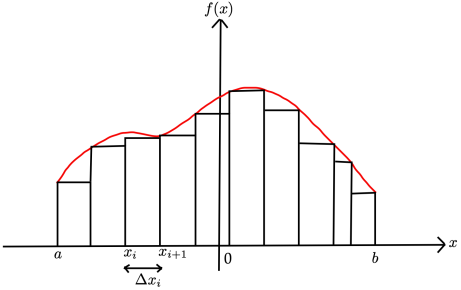
$$\mathcal{P} = \{x_0, x_1, \ldots, x_n\}, \qquad x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b.$$

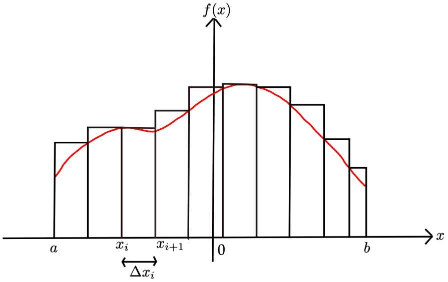
- We say that a partition $\mathcal{P}^*$ is a **refinement** of partition $\mathcal{P}$ if: $\mathcal{P}^* \supset \mathcal{P}$
- The **common refinement** of two partitions $\mathcal{P}, \mathcal{Q}$ is defined by: $\mathcal{P}^* = \mathcal{P} \cup \mathcal{Q}$

Primer on Riemann Integration

- Given a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$, with $x_0 = a$ and $x_n = b$, define

$$L(f, \mathcal{P}) := \sum_{\ell=1}^{n} \left(\inf_{x \in [x_{\ell-1},\, x_\ell]} f(x) \right) \cdot (x_\ell - x_{\ell-1}), \qquad U(f, \mathcal{P}) := \sum_{\ell=1}^{n} \left(\sup_{x \in [x_{\ell-1},\, x_\ell]} f(x) \right) \cdot (x_\ell - x_{\ell-1}).$$





Primer on Riemann Integration

- Given a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$, with $x_0 = a$ and $x_n = b$, define

$$L(f,\mathcal{P}) := \sum_{\ell=1}^{n} \left(\inf_{x\in[x_{\ell-1},\, x_\ell]} f(x) \right) \cdot (x_\ell - x_{\ell-1}), \qquad U(f,\mathcal{P}) := \sum_{\ell=1}^{n} \left(\sup_{x\in[x_{\ell-1},\, x_\ell]} f(x) \right) \cdot (x_\ell - x_{\ell-1}).$$

- $L(f,\mathcal{P})$ is called the **lower Riemann sum** of f under partition $\mathcal{P}$
  $U(f,\mathcal{P})$ is called the **upper Riemann sum** of f under partition $\mathcal{P}$
  Clearly, $\quad L(f,\mathcal{P}) \leq U(f,\mathcal{P})$
- If $\mathcal{P} \subset \mathcal{P}^*$, then

$$L(f,\mathcal{P}) \leq L(f,\mathcal{P}^*), \qquad U(f,\mathcal{P}) \geq U(f,\mathcal{P}^*).$$

- That is:
  Lower Riemann sum is **monotone increasing** in the partition
  Upper Riemann sum is **monotone decreasing** in the partition

$\mathcal{P} \subset \mathcal{P}^* \quad \Longrightarrow \quad L(f,\mathcal{P}) \leq L(f,\mathcal{P}^*)$

- Suppose that $\mathcal{P}^*$ has **one extra point** (say x^*) than $\mathcal{P}$. Let

$$\mathcal{P} = \{x_0, x_1, \ldots, x_n\}, \qquad x_0 = a < x_1 \cdots < x_{n-1} < x_n = b,$$
$$\mathcal{P}^* = \{x_0, x_1, \ldots, x_{m-1}, x^*, x_m, \cdots, x_n\}, \qquad x_0 = a < x_1 < \cdots < x_{m-1} < x^* < x_m < \cdots < x_n = b.$$

- Define w_1 and w_2 as

$$w_1 = \inf_{x \in [x_{m-1},\, x^*]} f(x), \qquad w_2 = \inf_{x \in [x^*,\, x_m]} f(x).$$

- Observe that

$$w_1, w_2 \geq \inf_{x \in [x_{m-1},\, x_m]} f(x).$$

- Therefore, it follows that

$$L(f,\mathcal{P}^*) - L(f,\mathcal{P}) = w_1(x^* - x_{m-1}) + w_2(x_m - x^*) - \left(\inf_{x \in [x_{m-1},\, x_m]} f(x)\right) \cdot (x_m - x_{m-1})$$
$$= \left(w_1 - \inf_{x \in [x_{m-1},\, x_m]} f(x)\right) \cdot (x^* - x_{m-1}) + \left(w_2 - \inf_{x \in [x_{m-1},\, x_m]} f(x)\right) \cdot (x_m - x^*) \geq 0.$$

- If $\mathcal{P}^*$ has k points more than $\mathcal{P}$, repeat the above exercise k times

Primer on Riemann Integration

- Define the **lower Riemann integral** and **upper Riemann integral as**

$$L_f := \sup_{\mathcal{P}} L(f,\mathcal{P}), \qquad U_f := \inf_{\mathcal{P}} U(f,\mathcal{P})$$

- **Claim:** $L_f \leq U_f$
- **Proof of Claim:**
  - Suppose $\mathcal{P}$ and $\mathcal{Q}$ are **any two arbitrary partitions** of $[a,b]$
    Let $\mathcal{P}^* = \mathcal{P} \cup \mathcal{Q}$
  - We then have

$$L(f,\mathcal{P}) \leq L(f,\mathcal{P}^*) \leq U(f,\mathcal{P}^*) \leq U(f,\mathcal{Q})$$

  - Fixing $\mathcal{Q}$ and taking supremum over $\mathcal{P}$, we get

$$L_f \leq U(f,\mathcal{Q})$$

  - Taking infimum over $\mathcal{Q}$, we get

$$L_f \leq U_f.$$

Primer on Riemann Integration

Riemann Integrability

In general,

$$L_f \leq U_f.$$

If $L_f = U_f$, then we say that f is **Riemann integrable** over the interval $[a, b]$.
In this case, the common value of L_f and U_f is denoted

$$\int_a^b f(x)\,\mathrm{d}x, \qquad \text{or simply} \qquad \int_a^b f\,\mathrm{d}x.$$

- Are there functions f for which $L_f < U_f$?
- Consider the **Dirichlet's function**

$$f(x) = \mathbf{1}_{\mathbb{Q}\cap[0,1]}(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

- For the above f, $\quad L_f = 0, \quad U_f = 1$

Where are we Heading Towards?

Abstract Integrals

Fix a measure space $(\Omega, \mathscr{F}, \mu)$, where μ is any measure (finite or infinite).
Let $f : \Omega \to \mathbb{R}$ be any **measurable** function.
We would like to define

$$\int_A f(x)\,\mathrm{d}\mu(x), \qquad \text{or simply} \qquad \int_A f\,\mathrm{d}\mu, \qquad A \in \mathscr{F}.$$

- Integration with respect to a variable $\longrightarrow$ Integration with respect to a **measure**
- Integrand: a **measurable** function
- Integration is over any **measurable set** $A \in \mathscr{F}$

Special Cases of Abstract Integrals

- If $(\Omega, \mathscr{F}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, λ – **Lebesgue measure**,

$$\int_B f \, \mathrm{d}\mu = \int_B f \, \mathrm{d}\lambda, \qquad B \in \mathscr{B}(\mathbb{R}),$$

  is called the **Lebesgue integral** of f over the Borel set B

  - Riemann integral is a special case of Lebesgue integral: $\int\limits_a^b f \, \mathrm{d}x = \int\limits_{[a,b]} f \, \mathrm{d}\lambda$
  - The Lebesgue integral of f may be defined even when the Riemann integral of f is not defined
    If Riemann integral is defined, then Lebesgue integral and Riemann integral are both identical

- If $(\Omega, \mathscr{F}, \mu) = (\Omega, \mathscr{F}, \mathbb{P})$, $\mathbb{P}$ – **probability measure**, $X : \Omega \to \mathbb{R}$ is a RV,

$$\int_\Omega X \, \mathrm{d}\mathbb{P}$$

  is called the **expectation** of X with respect to $\mathbb{P}$, denoted $\mathbb{E}[X]$

Theory of Expectations

Development of Theory of Expectations

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.
We will build the necessary machinery to be able to interpret an **abstract integral** of the form

$$\int_A X \, \mathrm{d}\mathbb{P} \quad = \quad \int_\Omega X \, \mathbf{1}_A \, \mathrm{d}\mathbb{P} \quad = \quad \mathbb{E}[X \, \mathbf{1}_A], \qquad A \in \mathscr{F}.$$

The theory will be developed in 3 stages:

- Definition of the abstract integral for **simple** random variables
- Definition of the abstract integral for **non-negative** random variables
- Definition of the abstract integral for **arbitrary** random variables

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Expectations of Simple Random Variables

Simple Random Variables

Definition (Simple Random Variable)

Fix a measurable space $(\Omega, \mathscr{F})$. Let $X : \Omega \to \mathbb{R}$ be a random variable.
X is said to be a **simple** random variable if it can be expressed as **weighted sum of indicators** as

$$X(\omega) = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}(\omega), \qquad \omega \in \Omega,$$

for some non-negative weights $a_1, \ldots, a_n \geq 0$ and sets $A_1, \ldots, A_n \in \mathscr{F}$.

Example of Simple Random Variable

- $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Consider $X : \mathbb{R} \to \mathbb{R}$ defined as

$$X(\omega) \quad = \quad \mathbf{1}_{[0,1]}(\omega) + \frac{3}{2}\,\mathbf{1}_{[1,3]}(\omega) \quad = \quad a_1\,\mathbf{1}_{A_1}(\omega) + a_2\,\mathbf{1}_{A_2}(\omega), \qquad \omega \in \Omega.$$

  where $a_1 = 1,\ \ a_2 = \frac{3}{2},\ \ A_1 = [0,1],\ \ A_2 = [1,3]$

- Mathematically, X can be expressed as

$$X(\omega) = \begin{cases} 1, & \omega \in [0,1), \\ \frac{5}{2}, & \omega = 1, \\ \frac{3}{2}, & \omega \in (1,3], \\ 0, & \text{otherwise.} \end{cases}$$

- Notice that X can also be represented as $X(\omega) = \mathbf{1}_{[0,3]}(\omega) + \frac{1}{2}\mathbf{1}_{[1,3]}(\omega) + \mathbf{1}_{\{1\}}(\omega)$.

The representation of simple random variables is not unique.

Canonical Form of a Simple Random Variable

Definition (Canonical Form of a Simple Random Variable)

A simple random variable X is said to be in **canonical form** if

$$X(\omega) = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}(\omega), \qquad \omega \in \Omega,$$

where the weights $a_1, \ldots, a_n \geq 0$ are **distinct**, and the sets $A_1, \ldots, A_n \in \mathscr{F}$ are **disjoint**.

The canonical representation of a simple random variable is unique.

Expectation of a Simple Random Variable

Definition (Expectation of a Simple RV

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable.
Suppose that X is **simple** with the canonical representation

$$X = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}, \qquad a_1, \ldots, a_n \geq 0 \ \textbf{distinct}, \quad A_1, \ldots, A_n \in \mathscr{F} \ \textbf{disjoint}.$$

Then, the **expectation of X**, denoted $\mathbb{E}[X]$, is defined as

$$\mathbb{E}[X] = \int_{\Omega} X \, \mathrm{d}\mathbb{P} := \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i).$$

- For any $A \in \mathscr{F}$,

$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$

- The expectation of a simple RV is a non-negative and finite real number

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Example: Dirichlet's Function

- $(\Omega, \mathscr{F}, \mathbb{P}) = ([0, 1], \mathscr{B}([0, 1]), \lambda),$ λ – Lebesgue measure
- If $X = \mathbf{1}_{\mathbb{Q}\cap[0,1]}$, then

$$\mathbb{E}[X] = \int_{\mathbb{Q}\cap[0,1]} X \, \mathrm{d}\lambda = \lambda\Big(\mathbb{Q} \cap [0, 1]\Big) = 0.$$

- Recall that X is not Riemann integrable

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Example: Bernoulli Random Variable

- Fix $(\Omega, \mathscr{F}, \mathbb{P})$
- Suppose that $X \sim \text{Bernoulli}(p)$ for some fixed $p \in [0, 1]$. **What is $\mathbb{E}[X]$?**
- Notice that X takes only two values: $a_1 = 0, \quad a_2 = 1$
- Define sets A_1, A_2 as: $A_1 = X^{-1}(\{0\}), \quad A_2 = X^{-1}(\{1\})$
- Then, X can be represented in canonical form as

$$X = 0 \cdot \mathbf{1}_{A_1} + 1 \cdot \mathbf{1}_{A_2}.$$

- Then, $\mathbb{E}[X]$ is given by

$$\mathbb{E}[X] = 0 \cdot \mathbb{P}(A_1) + 1 \cdot \mathbb{P}(A_2) = \mathbb{P}(A_2) = \mathbb{P}\Big(X^{-1}(\{1\})\Big) = \mathbb{P}_X(\{1\}) = p.$$

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Example: Binomial Random Variable

- Fix $(\Omega, \mathscr{F}, \mathbb{P})$
- Suppose that $X \sim \text{Binomial}(n, p)$ for some fixed $n \in \mathbb{N} \cup \{0\}$ and $p \in [0, 1]$. **What is $\mathbb{E}[X]$?**
- Notice that X takes $n+1$ distinct values: $a_1 = 0, \quad a_2 = 1, \quad \ldots \quad a_{n+1} = n$
- Define sets $A_1, \ldots, A_{n+1}$ as: $A_1 = X^{-1}(\{0\}), \quad A_2 = X^{-1}(\{1\}), \quad \ldots \quad A_{n+1} = X^{-1}(\{n\})$
- Then, X can be represented in canonical form as

$$X = \sum_{\ell=0}^{n} a_{\ell+1} \cdot \mathbf{1}_{A_{\ell+1}}.$$

- Then, $\mathbb{E}[X]$ is given by

$$\begin{aligned}\mathbb{E}[X] &= \sum_{\ell=0}^{n} a_{\ell+1} \cdot \mathbb{P}(A_{\ell+1}) = \sum_{\ell=0}^{n} \ell \cdot \binom{n}{\ell} p^{\ell} (1-p)^{n-\ell} = \sum_{\ell=1}^{n} \ell \cdot \binom{n}{\ell} p^{\ell} (1-p)^{n-\ell} \\ &= \sum_{\ell=1}^{n} \ell \cdot \frac{n!}{\ell!\,(n-\ell)!} p^{\ell} (1-p)^{n-\ell} = \sum_{\ell=1}^{n} \frac{n!}{(\ell-1)!\,(n-\ell)!} p^{\ell} (1-p)^{n-\ell} = np.\end{aligned}$$

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Expectations of Non-Negative Random Variables

Expectations of Non-Negative Random Variables

- Suppose X is **any non-negative random variable** (not necessarily simple)
- For each $n \in \mathbb{N}$, define X_n as

$$X_n = \sum_{\ell=0}^{n\,2^n-1} \frac{\ell}{2^n} \mathbf{1}_{\left\{\frac{\ell}{2^n} \le X < \frac{\ell+1}{2^n}\right\}} + n\,\mathbf{1}_{\{X \ge n\}}, \qquad X_n(\omega) = \begin{cases} 0, & 0 \le X(\omega) < \frac{1}{2^n}, \\ \frac{1}{2^n}, & \frac{1}{2^n} \le X(\omega) < \frac{2}{2^n}, \\ \vdots & \\ \frac{n\,2^n-1}{2^n}, & \frac{n\,2^n-1}{2^n} \le X(\omega) < n, \\ n, & X(\omega) \ge n. \end{cases}.$$

- X_n is a **simple** random variable for each $n \in \mathbb{N}$ $\quad$ $\mathbb{E}[X_n]$ is well-defined for each $n \in \mathbb{N}$
- X_n can be represented compactly as

$$X_n = \frac{\lfloor 2^n X \rfloor}{2^n} \mathbf{1}_{\{X<n\}} + n\,\mathbf{1}_{\{X \ge n\}}$$

$$\mathbb{E}[X] = \int_\Omega X \, \mathrm{d}\mathbb{P} := \lim_{n\to\infty} \mathbb{E}[X_n].$$

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Properties of $\{X_n\}$ - 1

- **Monotonicity:** For each $\omega \in \Omega$, we have $X_n(\omega) \le X_{n+1}(\omega) \quad \forall\, n \in \mathbb{N}$.
- **Proof:** For any $\omega \in \Omega$,

$$X_n(\omega) = \frac{\lfloor 2^n X(\omega) \rfloor}{2^n} \mathbf{1}_{\{X<n\}}(\omega) + n\,\mathbf{1}_{\{X\ge n\}}(\omega),$$

$$X_{n+1}(\omega) = \frac{\lfloor 2^{n+1} X(\omega) \rfloor}{2^{n+1}} \mathbf{1}_{\{X<n+1\}}(\omega) + (n+1)\,\mathbf{1}_{\{X\ge n+1\}}(\omega).$$

- If $X(\omega) \ge n+1$, then $X(\omega) \ge n$, and therefore

$$X_n(\omega) = n, \qquad X_{n+1}(\omega) = n+1 \qquad \Longrightarrow \qquad X_{n+1}(\omega) \ge X_n(\omega).$$

- If $n \le X(\omega) < n+1$, then

$$X_n(\omega) = n, \qquad X_{n+1}(\omega) = \frac{\lfloor 2\,2^n X(\omega) \rfloor}{2^{n+1}} \ge 2\cdot \frac{\lfloor 2^n X(\omega) \rfloor}{2^{n+1}} \ge 2\cdot \frac{\lfloor 2^n n \rfloor}{2^{n+1}} = n \qquad \Longrightarrow \qquad X_{n+1}(\omega) \ge X_n(\omega).$$

- If $X(\omega) < n$, then $X(\omega) < n+1$, and therefore

$$X_n(\omega) = \frac{\lfloor 2^n X(\omega) \rfloor}{2^n}, \quad X_{n+1}(\omega) = \frac{\lfloor 2\,2^n X(\omega) \rfloor}{2^{n+1}} \ge 2\cdot \frac{\lfloor 2^n X(\omega) \rfloor}{2^{n+1}} = \frac{\lfloor 2^n X(\omega) \rfloor}{2^n} \qquad \Longrightarrow \qquad X_{n+1}(\omega) \ge X_n(\omega).$$

Properties of $\{X_n\}$ - 2

- **Pointwise convergence:** For each $\omega \in \Omega$, we have $\lim_{n\to\infty} X_n(\omega) = X(\omega)$.
- **Proof:** For any $\omega \in \Omega$,

$$X_n(\omega) = \frac{\lfloor 2^n X(\omega) \rfloor}{2^n} \mathbf{1}_{\{X<n\}}(\omega) + n\,\mathbf{1}_{\{X\geq n\}}(\omega).$$

- For all sufficiently large values of n, we have $X(\omega) < n$, and therefore

$$\forall\, n \text{ sufficiently large}, \qquad X_n(\omega) = \frac{\lfloor 2^n X(\omega) \rfloor}{2^n}, \qquad \frac{2^n X(\omega) - 1}{2^n} < X_n(\omega) \leq X(\omega).$$

- Taking limits as $n \to \infty$,

$$\lim_{n\to\infty} X_n(\omega) = X(\omega).$$