भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 14: Continuous Random Variables, Probability Density Function (PDF), Example PDFs

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

19 September 2025

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Discrete Random Variable

Definition (Discrete Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **discrete** if there exists a **countable** set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that

$$\mathbb{P}_X(E) = 1.$$

- By countable additivity,

$$1 = \mathbb{P}_X(E) = \mathbb{P}_X\left(\bigsqcup_{i \in \mathbb{N}} \{e_i\}\right) = \sum_{i \in \mathbb{N}} \mathbb{P}_X(\{e_i\}) = \sum_{i \in \mathbb{N}} p_X(e_i).$$

- For any Borel set $B \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}_X(B) = \mathbb{P}_X(B \cap E) = \sum_{i:\ e_i \in B} \mathbb{P}_X(\{e_i\}) = \sum_{i:\ e_i \in B} p_X(e_i).$$

Discrete Random Variable

Definition (Discrete Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **discrete** if there exists a **countable** set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that

$$\mathbb{P}_X(E) = 1.$$

PMF $\longrightarrow$ CDF for a Discrete RV

The following implications are noteworthy:

$$p_X \quad \underset{\text{any } X}{\overset{X \text{ discrete}}{\rightleftarrows}} \quad \mathbb{P}_X \quad \underset{\text{any } X}{\overset{\text{any } X}{\rightleftarrows}} \quad F_X.$$

PMF $=$ complete probabilistic description for discrete RV.

Continuous Random Variables

Continuous Random Variable

Definition (Continuous Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **continuous** if $\mathbb{P}_X$ satisfies the following property:

$$\text{If} \quad \lambda(B) = 0, \qquad \text{then} \quad \mathbb{P}_X(B) = 0.$$

Here, λ denotes the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

- The above property
$$\text{“If} \quad \lambda(B) = 0, \qquad \text{then} \quad \mathbb{P}_X(B) = 0\text{”}$$
is called **absolute continuity property**
- In words: “P_X is absolutely continuous with respect to λ”
- Mathematical notation: $\mathbb{P}_X \ll \lambda$

An Important Result from Measure Theory

Theorem (Radon–Nikodym Theorem — A Special Case)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable.
Let $\mathbb{P}_X$ and λ denote respectively the probability law and the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

If $\mathbb{P}_X \ll \lambda$, then there exists a **non-negative, measurable function** $f_X : \mathbb{R} \to [0, +\infty)$ such that

$$\forall\, x \in \mathbb{R}, \qquad \mathbb{P}_X\big((-\infty, x]\big) = \int_{-\infty}^{x} f_X(t)\, \mathrm{d}t.$$

- The function f_X is called the **probability density function (PDF)** of X

Important

PDF in probability theory has its roots in the Radon–Nikodym Theorem of measure theory.

CDF of Continuous RV

Proposition (CDF of Continuous RV)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable.
If X is continuous, then the following properties hold.

1. The CDF F_X is **continuous** at every point $x \in \mathbb{R}$. **That is, the CDF does not have any jumps.** Therefore, for all $x, a, b \in \mathbb{R}$ and $a < b$,

$$\mathbb{P}_X\big((-\infty, x)\big) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t \quad \forall x \in \mathbb{R},$$

$$\mathbb{P}_X((a,b)) = \mathbb{P}_X([a,b)) = \mathbb{P}_X((a,b]) = \mathbb{P}_X([a,b]) = \int_a^b f_X(t)\,\mathrm{d}t \quad \forall -\infty < a < b < +\infty.$$

2. The CDF F_X is **absolutely continuous**.
3. The CDF is differentiable $\underbrace{\text{everywhere except possibly on a Borel set } B \text{ with } \lambda(B) = 0}_{\text{almost everywhere (a.e.).}}$.

Tidbits About PDFs

About PDFs

PDF $\neq$ **probability**, **Integral of PDF** $=$ **probability**,

$$f_X \underset{X \text{ continuous}}{\overset{X \text{ continuous}}{\rightleftarrows}} F_X \underset{\text{any } X}{\overset{\text{any } X}{\rightleftarrows}} \mathbb{P}_X.$$

- A PDF can be **any arbitrary non-negative function** that satisfies: $\int_{-\infty}^{\infty} f_X(t)\, \mathrm{d}t = 1.$
- **A proof of why PDF must integrate to 1:**

$$\int_{-\infty}^{\infty} f_X(t)\, \mathrm{d}t = \lim_{n\to\infty} \int_{-\infty}^{n} f_X(t)\, \mathrm{d}t = \lim_{n\to\infty} \mathbb{P}_X\big((-\infty, n]\big) = \mathbb{P}_X\left(\bigcup_{n\in\mathbb{N}} (-\infty, n]\right) = \mathbb{P}_X(\mathbb{R}) = 1.$$

- If X is continuous, then its PMF p_X satisfies $p_X(x) = 0$ for all $x \in \mathbb{R}$

Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $a, b \in \mathbb{R}$ with $a < b$.
  X is said to be **Uniformly distributed** on $[a, b]$ (denoted $X \sim \text{Unif}[a, b]$) if:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b], \\ 0, & \text{otherwise}. \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x < b, \\ 1, & x \geq b. \end{cases}$$

  Notice that:

  - F_X is differentiable everywhere except at $x \in \underbrace{\{a, b\}}_{B}$, and $\lambda(B) = 0$
  - $\mathbb{P}(\{X \in [a, b]\}) = 1$, i.e., X takes values in $[a, b]$ with probability 1

Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $\mu > 0$.
  X is said to be **Exponentially distributed** with parameter μ (denoted $X \sim \mathrm{Exp}(\mu)$) if:

$$f_X(x) = \begin{cases} \mu\, e^{-\mu\, x}, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\mu\, x}, & x \geq 0. \end{cases}$$

  Notice that:

  - F_X is differentiable everywhere except $x = 0$, and $\lambda(\{0\}) = 0$
  - $\mathbb{P}(\{X \geq 0\}) = 1$, i.e., X takes non-negative values with probability 1
  - F_X **does not touch** the value 1 anywhere!
  - F_X is strictly increasing on $[0, +\infty)$

Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $\mu \in \mathbb{R}$, $\sigma > 0$.
  X is said to be **Gaussian distributed** with parameters (μ, σ^2) (denoted $X \sim \mathcal{N}(\mu, \sigma^2)$) if:

$$\forall\, x \in \mathbb{R}, \quad f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t = \mathrm{Erf}(x)$$

  Notice that:

  – F_X is differentiable everywhere
  – $\mathbb{P}(\{X \in \mathbb{R}\}) = 1$, i.e., X takes all real values with probability 1
  – F_X does not touch either 0 or 1 anywhere
  – F_X is strictly increasing
  – If $\mu = 0$ and $\sigma = 1$, then X is said to be **Normal distributed (denoted $X \sim \mathcal{N}(0, 1)$)**