

Agenda:

Primer on real analysis:

- Supremum, infimum of sets on \mathbb{R}
- Limsup, liminf of \mathbb{R} -valued sequences
- Limits of \mathbb{R} -valued sequences



Andrey Nikolaevich
Kolmogorov (1903-1987)



Kolmogorov during his visit to
ISI Kolkata (Feb 1962)



Kolmogorov preparing for a talk

\mathbb{N} - set of natural numbers = $\{1, 2, 3, \dots\}$

\mathbb{Z} - set of integers = $\{0, \pm 1, \pm 2, \pm 3, \dots\}$

\mathbb{Q} - set of rational numbers

$$= \left\{ p/q : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}$$

\mathbb{R} - set of real numbers

Sequences of real numbers

A sequence of real numbers is a mapping $f: \mathbb{N} \rightarrow \mathbb{R}$.

$$\left. \begin{array}{l} 1 \mapsto f(1) = a_1 \\ 2 \mapsto f(2) = a_2 \\ 3 \mapsto f(3) = a_3 \\ \vdots \end{array} \right\} \{a_n\}_{n=1}^{\infty}$$

Supremum of a set of real numbers

Let $A \subseteq \mathbb{R}$. The Supremum of A is an element x^* such that:

i) x^* is an upper bound for A , i.e.,

$$\forall x \in A, \quad x \leq x^*$$

for all

ii) x^* is the least upper bound, i.e.,

$\forall \varepsilon > 0$, $x^* - \varepsilon$ is NOT an upper bound

$$\Rightarrow \forall \varepsilon > 0, \exists x \in A \text{ s.t. } x > x^* - \varepsilon.$$

Examples:

i) $A = (0, 1)$



$$\boxed{\sup A = 1}$$

$$(ii) A = \{1, 2, 3\}$$

$$\boxed{\sup A = 3}$$

$$(iv) A = \emptyset$$

$$\boxed{\sup A = -\infty}$$

$$(iii) A = \mathbb{N}$$

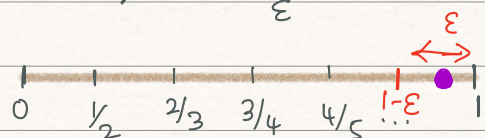
$$\boxed{\sup A = +\infty}$$

$$(v) A = \left\{1 - \frac{1}{n} : n \geq 1\right\}$$

* 1 is an upper bound

$$1 - \frac{1}{n} \leq 1 \quad \forall n \geq 1$$

$$* \forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ s.t. } 1 - \frac{1}{n} > 1 - \varepsilon$$



$$1 - \frac{1}{n} > 1 - \varepsilon$$

$$\Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$$

$$\boxed{\sup A = 1}$$

Infimum of a set of real numbers

Let $A \subseteq \mathbb{R}$. The infimum of A is a number x_* s.t.

$$i) x_* \text{ is a } \underline{\text{lower bound}} \text{ for } A, \text{ i.e.,}$$

$$\forall x \in A, \quad x \geq x_*$$

$$ii) x_* \text{ is the greatest lower bound for } A$$

$$\Rightarrow \forall \varepsilon > 0, x_* + \varepsilon \text{ is NOT a lower bound for } A$$

$$\Rightarrow \forall \varepsilon > 0, \exists x \in A \text{ s.t. } x < x_* + \varepsilon$$

Examples:

i) $A = (0, 1)$

$$\inf A = 0$$

iv) $\inf \emptyset = +\infty$

v) $A = \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}$

$$\inf A = 0$$

ii) $A = \{1, 2, 3\}$

$$\inf A = 1$$

ii) $A = \mathbb{N}$

$$\inf \mathbb{N} = 1$$

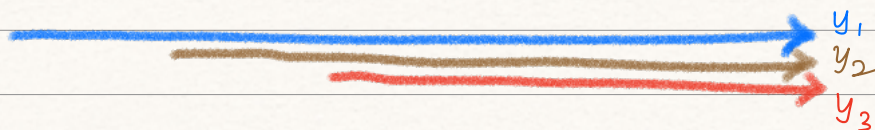
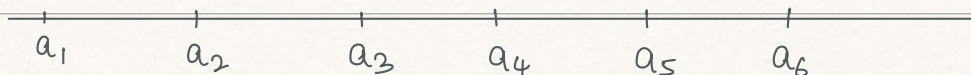
Limit supremum & Limit Infimum of sequences

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

Defⁿ (Limit supremum)

Limit supremum of $\{a_n\}_{n=1}^{\infty}$ is a number L s.t.

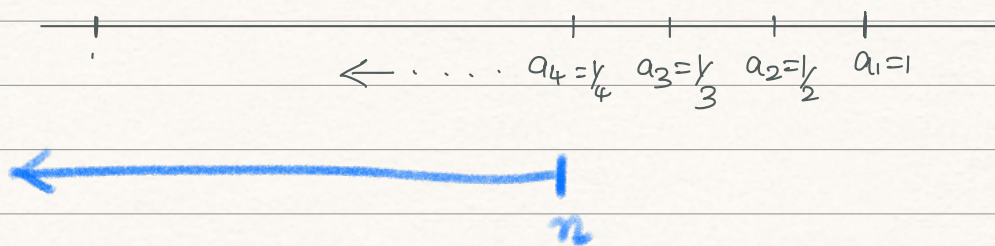
$$\bar{L} = \inf_{n \geq 1} \sup_{k \geq n} a_k$$



Notation: $\limsup_{n \rightarrow \infty} a_n$

$$y_n = \sup_{k \geq n} a_k$$

Eg: $a_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$



$$y_n = \sup_{k \geq n} a_k = \frac{1}{n}$$

$$\limsup_{n \rightarrow \infty} a_n = \inf_{n \geq 1} y_n = \inf \left\{ \frac{1}{n} : n \geq 1 \right\} = 0.$$

Formally, we say that

$$\limsup_{n \rightarrow \infty} a_n = \bar{L}$$

if:

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \text{ s.t.}$$

$$a_n < \bar{L} + \varepsilon \quad \forall n \geq N_\varepsilon.$$



$$y_n = \sup_{k \geq n} a_k$$

$$\limsup_{n \rightarrow \infty} a_n = \inf_{n \geq 1} y_n$$

$$\inf_{n \geq 1} y_n = \bar{L}$$

$$\text{i) } y_n \geq \bar{L} \quad \forall n \geq 1$$

$$\text{ii) } \forall \varepsilon > 0, \exists N_\varepsilon \text{ s.t.}$$

$$\sup_{k \geq N_\varepsilon} a_k \leftarrow y_{N_\varepsilon} < \bar{L} + \varepsilon$$

$$\Rightarrow a_k < \bar{L} + \varepsilon \quad \forall k \geq N_\varepsilon$$

Defⁿ (Limit infimum)

The limit infimum of a sequence $\{a_n\}_{n=1}^{\infty}$ is a number L s.t.

$$\underline{L} = \sup_{n \geq 1} \inf_{k \geq n} a_k$$

Formally, $\liminf_{n \rightarrow \infty} a_n = \underline{L}$ if

$\forall \varepsilon > 0, \exists N_{\varepsilon} \in \mathbb{N}$ s.t.

$$a_n > \underline{L} - \varepsilon \quad \forall n \geq N_{\varepsilon}$$

Result:

$$\inf A \leq \sup A \quad \forall A \neq \emptyset$$
$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

Limit:

We say that $\{a_n\}_{n=1}^{\infty}$ admits a limit, say L , if

$$\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = L$$

Mathematically, we say

$$\lim_{n \rightarrow \infty} a_n = L$$

if:

$\forall \varepsilon > 0, \exists N_{\varepsilon} \in \mathbb{N}$ s.t.

$$L - \varepsilon < a_n < L + \varepsilon \quad \forall n \geq N_{\varepsilon}$$

$$|a_n - L| < \varepsilon$$