AI 5090: Stochastic Processes
Homework 3

Random Processes, Stopping Times, Wald's Lemma

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing below are defined with respect to this probability space.

1. A random process $\{X(t) : t \geq 0\}$ is defined in terms of two random variables X_1 and X_2 as

$$X(t) = X_1 \cos(2\pi f_c t) + X_2 \sin(2\pi f_c t), \qquad t \geq 0,$$

   for some fixed constant f_c.
   Determine the necessary and sufficient conditions on X_1 and X_2 for the process to be wide-sense stationary.

2. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined via

$$X_n = \begin{cases} U_n, & n \text{ odd}, \\ \frac{1}{\sqrt{2}}(U_n^2 - 1), & n \text{ even}, \end{cases}$$

   where $U_1, U_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.
   Show that $\{X_n\}_{n=1}^{\infty}$ is wide-sense stationary, but not stationary.

3. Fix $K \in \{2, 3, \ldots\}$.
   Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Unif}\{1, \ldots, K\}$.
   Let $T_1 := 1$, and for each $k \in \{2, \ldots, K\}$, define

$$T_k = \inf\left\{n > T_{k-1} : X_n \in \{1, \ldots, K\} \setminus \{X_{T_1}, \ldots, X_{T_{k-1}}\}\right\}.$$

   (a) Interpret T_k in words.
   (b) Prove formally that T_k is a stopping time w.r.t. the natural filtration of the process $\{X_n\}_{n=1}^{\infty}$ for each k.
   (c) For each $k \in \{2, \ldots, K\}$, let
   $$S_k := T_k - T_{k-1}.$$
   Compute the PMF of S_k, and use this to compute $\mathbb{E}[S_k]$.
   (d) Using the result in part (c) above, compute $\mathbb{E}[T_k]$ for each $k \in \{2, \ldots, K\}$.

4. (a) Given a random variable $X : \Omega \to \mathbb{R}$ defined w.r.t. $\mathscr{F}$, let

$$\sigma(X) := \left\{A \in \mathscr{F} : \exists B \in \mathscr{B}(\mathbb{R}) \text{ such that } A = X^{-1}(B)\right\}.$$

   Prove that $\sigma(X)$ is a σ-algebra of subsets of Ω.

   Remark: $\sigma(X)$ is called the σ-algebra generated by X.
   It is the smallest σ-algebra with respect to which X will be a random variable.

   (b) Fix a filtration $\{\mathscr{F}_t : t \in \mathcal{T}\}$ for some arbitrary index set $\mathcal{T}$.
   Let τ be a stopping time with respect to the above filtration. Let

$$\mathscr{F}_\tau := \left\{A \in \mathscr{F} : A \cap \{\tau \leq t\} \in \mathscr{F}_t \quad \forall t \in \mathcal{T}\right\}.$$

   Prove that $\mathscr{F}_\tau$ is a σ-algebra of subsets of Ω.

5. Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Ber}(0.5)$. Let

$$N := \inf\{n \geq 2 : X_{n-1} = X_n = 1\}$$

   denote the first time instant of observing two consecutive successes.

(a) Show that N is a stopping time with respect to the natural filtration associated with the process $\{X_n\}_{n=1}^{\infty}$.

(b) Determine $\mathbb{P}(X_{N+1} = X_{N+2} = 0)$.

Hint: Use the fact that N is a stopping time.

6. **(Gambler's Ruin)**

Two players A and B play a game with independent rounds where, in each round, one of the players wins $\$1$ from his opponent; A wins with probability p and B wins with probability $q = 1 - p$. A starts the game with $\$a$ and B with $\$b$. The game ends when one of the players is ruined (i.e., the player's earnings becomes 0).

(As a means of visualizing the above game, draw a straight line on a piece of paper, and mark $0, 1, 2, 3, \ldots$ on it. Imagine yourself as player A. Then, according to the game, you start from the integer a and at each step either move one integer forward with probability p or move one integer backward with probability $q = 1 - p$. Such a movement is known as a **one-dimensional random walk***. The game ends when you have reached either 0 (in which case your opponent has won) or $a + b$ (which is when you have won).)*

Assume that $p = q = 0.5$. For $k \in \mathbb{N}$, let

$$a[k] = \mathbb{P}(\text{A goes on to win the game starting with \$}k).$$

(a) Evaluate $a[0]$ and $a[a + b]$.

(b) Express $a[k]$ in terms of $a[k - 1]$ and $a[k + 1]$.

(c) Solve the difference equation obtained in part (b) above, using the initial conditions in part (a). Find a closed-form expression for $a[k]$.

(d) Compute the probability that A ruins B.

7. Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Unif}(0, 1)$. Let

$$N := \inf\{n \geq 2 : X_n > X_{n-1}\}.$$

(a) Show that N is a stopping time w.r.t. the natural filtration of the process $\{X_n\}_{n=1}^{\infty}$.

(b) Compute $\mathbb{E}\left[\sum_{i=1}^{N} X_i\right]$.