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Probability and Stochastic Processes

Lecture 12: Probability Law, Cumulative Distribution Function (CDF), Properties of CDF

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

12 September 2025

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Random Variable

Definition (Random Variable)

Fix a measurable space $(\Omega, \mathscr{F})$.
A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if it is measurable, i.e.,

$$\forall\, B \in \mathscr{B}(\mathbb{R}), \qquad \underbrace{X^{-1}(B)}_{\text{pre-image of } B} = \{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\} \in \mathscr{F}.$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[rr, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"]
\end{tikzcd}$$

Properties of a Random Variable

Proposition (Random Variable Properties)

Let $(\Omega, \mathscr{F})$ be a measurable space, and let $X : \Omega \to \mathbb{R}$ be a random variable.

1. For any $B \subseteq \mathbb{R}$, $\quad X^{-1}(B^{\complement}) = \left(X^{-1}(B)\right)^{\complement}$.
2. For any $B_1 \subseteq \mathbb{R}, B_2 \subseteq \mathbb{R}, \ldots,$

$$X^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \bigcup_{n\in\mathbb{N}} X^{-1}(B_n).$$

3. Let $\mathscr{B}_1$ denote the collection

$$\mathscr{B}_1 := \left\{B \subseteq \mathbb{R} : X^{-1}(B) \in \mathscr{F}\right\}.$$

   Then, $\mathscr{B}_1$ is a σ-algebra of subsets of $\mathbb{R}$. Furthermore, $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{B}_1$.

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Generating Classes for $\mathscr{B}(\mathbb{R})$

$\mathscr{B}(\mathbb{R})$

$$\mathscr{P}_1 = \Big\{(a,b) : \ a,b \in \mathbb{R}, \ a \leq b\Big\}$$

$$\mathscr{P}_2 = \Big\{[a,b] : \ a,b \in \mathbb{R}, \ a \leq b\Big\}$$

$$\mathscr{P}_3 = \Big\{[a,b) : \ a,b \in \mathbb{R}, \ a \leq b\Big\}$$

$$\mathscr{P}_4 = \Big\{(a,b] : \ a,b \in \mathbb{R}, \ a \leq b\Big\}$$

$$\mathscr{P}_5 = \Big\{(-\infty, \ x) : \ x \in \mathbb{R}\Big\}$$

$$\mathscr{P}_6 = \Big\{(-\infty, \ x] : \ x \in \mathbb{R}\Big\}$$

$$\mathscr{P}_7 = \Big\{(x, \ +\infty) : \ x \in \mathbb{R}\Big\}$$

$$\mathscr{P}_8 = \Big\{[x, \ +\infty) : \ x \in \mathbb{R}\Big\}$$

Equivalent Definitions of Random Variable

Fix a measurable space $(\Omega, \mathscr{F})$.

Theorem (Equivalent Definitions)

$X : \Omega \to \mathbb{R}$ is a random variable **if and only if** any one of the following holds:

1. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_1$.
2. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_2$.
3. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_3$.
4. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_4$.
5. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_5$.
6. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_6$.
7. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_7$.
8. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_8$.

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Random Variable Simplified

Definition (Random Variable)

Fix a measurable space $(\Omega, \mathscr{F})$.
A function $X : \Omega \to \mathbb{R}$ is called a random variable with respect to $\mathscr{F}$ if and only if

$$\forall x \in \mathbb{R}, \qquad \underbrace{X^{-1}((-\infty, x])}_{\text{pre-image of } (-\infty,\, x]} = \{\omega \in \Omega : X(\omega) \leq x\} = \{X \leq x\} \in \mathscr{F}.$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[rr, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & & (-\infty, x] \arrow[ll, "\text{pre-image}"]
\end{tikzcd}$$

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Indicator Functions

Fix a sample space Ω.
Fix a subset $A \subseteq \Omega$.

Definition (Indicator Function)

The **indicator function** of set A is the function $\mathbf{1}_A : \Omega \to \mathbb{R}$ defined as

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

Exercise

Fix a measurable space $(\Omega, \mathscr{F})$. Show that

$$\mathbf{1}_A \text{ is a random variable} \quad \Longleftrightarrow \quad A \in \mathscr{F}.$$

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Probability Law of a Random Variable

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Probability Law of a Random Variable

- Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
- If $X : \Omega \to \mathbb{R}$ is a random variable, then

$$\forall\, B \in \mathscr{B}(\mathbb{R}), \qquad X^{-1}(B) \in \mathscr{F}.$$

  Therefore, it makes sense to talk about $\mathbb{P}(X^{-1}(B))$ for each $B \in \mathscr{B}(\mathbb{R})$
- We then have a mapping from $\mathscr{B}(\mathbb{R}) \to [0, 1]$:

$$B \quad \mapsto \quad \mathbb{P}(X^{-1}(B))$$

- The above mapping is called the **probability law** of the random variable X

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Probability Law of a Random Variable

Definition (Probability Law)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable (with respect to $\mathscr{F}$).
The **probability law** of X is a function $\mathbb{P}_X : \mathscr{B}(\mathbb{R}) \to [0, 1]$ defined as

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)), \qquad B \in \mathscr{B}(\mathbb{R}).$$

Remarks:

- $\mathbb{P}_X$ is sometimes referred to the **pushforward** of $\mathbb{P}$ under the random variable X
- $\mathbb{P}_X$ is sometimes denoted as $\mathbb{P} \circ X^{-1}$

Proposition (Probability Law)

$\mathbb{P}_X$ is a **probability measure** on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

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Completing the Picture

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

Figure: Pictorial representation of probability law

Proof that $\mathbb{P}_X$ is a Probability Measure

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

- First, we note that

$$\mathbb{P}_X(\emptyset) = \mathbb{P}(X^{-1}(\emptyset)) = \mathbb{P}(\emptyset) = 0.$$

- Next, we note that

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1.$$

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Proof that $\mathbb{P}_X$ is a Probability Measure

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, bend left, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}
$$

$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

- Finally, if $B_1, B_2, \ldots \in \mathscr{B}(\mathbb{R})$ are mutually disjoint, then

$$\mathbb{P}_X\left(\bigsqcup_{n\in\mathbb{N}} B_n\right) = \mathbb{P}\left(X^{-1}\left(\bigsqcup_{n\in\mathbb{N}} B_n\right)\right) = \mathbb{P}\left(\bigsqcup_{n\in\mathbb{N}} X^{-1}(B_n)\right) = \sum_{n\in\mathbb{N}} \mathbb{P}\left(X^{-1}(B_n)\right) = \sum_{n\in\mathbb{N}} \mathbb{P}_X(B_n).$$

Cumulative Distribution Function

Cumulative Distribution Function (CDF)

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

- $\mathbb{P}_X(B) \in [0, 1]$ for every $B \in \mathscr{B}(\mathbb{R})$
- In particular, $\mathbb{P}_X\big((-\infty, x]\big) \in [0, 1]$ for every $x \in \mathbb{R}$
- We thus have a mapping

$$x \mapsto \mathbb{P}_X\big((-\infty, x]\big)$$

- The above mapping (or function) is called the **cumulative distribution function** of the random variable X, denoted by F_X

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Cumulative Distribution Function (CDF)

Definition (Cumulative Distribution Function (CDF))

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.
The function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(x) = \mathbb{P}_X\big((-\infty, x]\big) = \mathbb{P}\big(\{\omega \in \Omega : X(\omega) \leq x\}\big) = \mathbb{P}\big(\{X \leq x\}\big), \qquad x \in \mathbb{R},$$

is called the **cumulative distribution function (CDF)** of X.

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & (-\infty, x] \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}$$

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Properties of CDF

Lemma (Properties of CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable with CDF F_X. Then, F_X satisfies the following properties.

1. **(Monotonicity)** If $x \leq y$, then $F_X(x) \leq F_X(y)$.
2. If $x_1, x_2, \ldots$ is any sequence such that $\lim_{n\to\infty} x_n = -\infty$, then $\lim_{n\to\infty} F_X(x_n) = 0$.
3. If $x_1, x_2, \ldots$ is any sequence such that $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} F_X(x_n) = 1$.
4. **(Right-Continuity)**

   F_X is right-continuous at every point in the domain.
   More formally, for each $x \in \mathbb{R}$,

$$x_n > x \ \forall\, n \in \mathbb{N}, \quad \lim_{n\to\infty} x_n = x \quad \Longrightarrow \quad \lim_{n\to\infty} F_X(x_n) = F_X(x).$$

Proof of Lemma – 1

- Note that

$$\mathbb{F}_X(x) = \mathbb{P}_X\big((-\infty,\, x]\big), \qquad \mathbb{F}_X(y) = \mathbb{P}_X\big((-\infty,\, y]\big)$$

- If $x \leq y$, then

$$(-\infty,\, x] \;\subseteq\; (-\infty,\, y]$$

- Using monotonicity property of $\mathbb{P}_X$, it follows that

$$\mathbb{P}_X\big((-\infty,\, x]\big) \;\leq\; \mathbb{P}_X\big((-\infty,\, y]\big)$$

Proof of Lemma – 2

- Suppose $x_1, x_2, \ldots$ is **monotone decreasing** sequence such that

$$x_1 \geq x_2 \geq \cdots, \qquad \lim_{n\to\infty} x_n = -\infty.$$

- Then, we have

$$(-\infty,\ x_1] \supseteq (-\infty,\ x_2] \supseteq \cdots$$

- Therefore,

$$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbb{P}_X\big((-\infty,\ x_n]\big) = \mathbb{P}_X\left(\bigcap_{n\in\mathbb{N}} (-\infty, x_n]\right) = \mathbb{P}_X(\emptyset) = 0.$$

Proof of Lemma – 2

- Suppose $x_1, x_2, \ldots$ is **any** sequence such that

$$\lim_{n \to \infty} x_n = -\infty$$

- Let $y_1, y_2, \ldots$ be a new sequence defined as

$$y_n = \sup_{k \geq n} x_k, \qquad n \in \mathbb{N}$$

- Then, it follows that

$$y_1 \geq y_2 \geq \cdots, \qquad \lim_{n \to \infty} y_n = -\infty, \qquad y_n \geq x_n \ \forall\, n \in \mathbb{N}.$$

- From the previous result for non-increasing sequences,

$$\lim_{n \to \infty} F_X(y_n) = 0.$$

- $F_X(x_n) \leq F_X(y_n) \ \forall\, n \in \mathbb{N} \quad \Longrightarrow \quad \lim_{n \to \infty} F_X(x_n) \leq \lim_{n \to \infty} F_X(y_n) = 0.$

Proof of Lemma – 3

- Left as exercise.

Proof of Lemma – 4

- Fix $x \in \mathbb{R}$
- Suppose that $x_1, x_2, \ldots$ is any monotone decreasing sequence such that

$$x_1 \geq x_2 \geq \cdots, \qquad \lim_{n\to\infty} x_n = x.$$

- Then, note that

$$(-\infty,\ x_1] \supseteq (-\infty,\ x_2] \supseteq \cdots, \qquad \bigcap_{n\in\mathbb{N}} (-\infty, x_n] = (-\infty, x].$$

- We then have

$$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbb{P}_X\big((-\infty,\ x_n]\big) = \mathbb{P}_X\left(\bigcap_{n\in\mathbb{N}} (-\infty,\ x_n]\right) = \mathbb{P}_X\big((-\infty,\ x]\big) = F_X(x).$$

Example

- Fix a measurable space $(\Omega, \mathscr{F})$. Fix $A \in \mathscr{F}$.
  Plot the CDF of the random variable $\mathbf{1}_A$.