Name:
Roll Number:

CS6660: Mathematical Foundations of Data Science (Probability)

Quiz 1

Date: 31 August 2024

Question	1	2(a)	2(b)	Total
Marks Scored				

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

1. **(1 Mark)**

   Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$ whose CDF is as depicted in Figure 1. Evaluate $\mathbb{P}(\{X \in [3, 6]\})$.

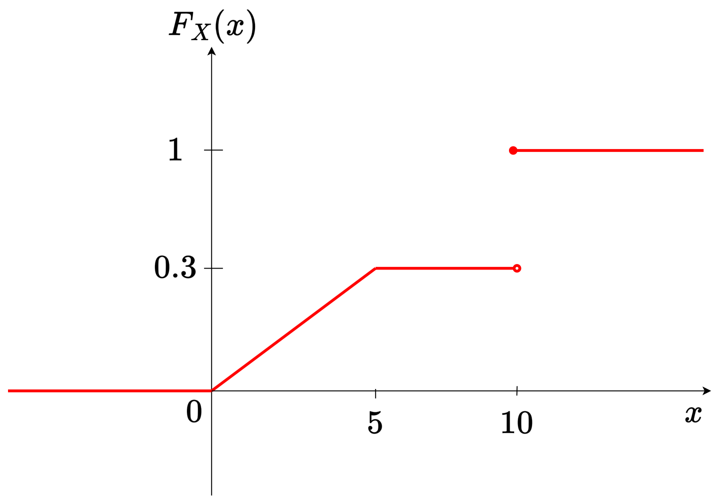

Figure 1: CDF of random variable X.

Solution:

The CDF of X may be expressed mathematically as

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{3x}{50}, & 0 \leq x < 5, \\ 0.3, & 5 \leq x < 10, \\ 1, & x \geq 10. \end{cases}$$

It then follows that

$$\mathbb{P}(\{X \in [3, 6]\}) = \mathbb{P}(\{X \leq 6\}) - \mathbb{P}(\{X < 3\}) \stackrel{(a)}{=} \mathbb{P}(\{X \leq 6\}) - \mathbb{P}(\{X \leq 3\}) = F_X(6) - F_X(3) = 0.3 - \frac{9}{50} = 0.12,$$

where (a) above follows from the fact that the F_X is continuous at the point $x = 3$.

2. A box contains one coupon labelled 1, two identical coupons labelled 2, and so on up to ten identical coupons labelled 10. Two coupons are drawn simultaneously and uniformly at random from the box.

   (a) **(2 Marks)**
   Specify Ω and $\mathbb{P}$ for the experiment, assuming that $\mathscr{F} = 2^{\Omega}$.

   Solution: Notice that there are a total of $1+2+\cdots+10 = 55$ coupons in the box. Writing (i,j) to denote the outcome in which one of the coupons drawn has the label i and the other coupon has label j, we have

$$\begin{aligned}\Omega = \Big\{&(1,2),(1,3),\ldots,(1,10),\\ &(2,1),(2,2),\ldots,(2,10),\\ &(3,1),(3,2),\ldots,(3,10),\\ &\vdots\\ &(10,1),(10,2),\ldots,(10,10)\Big\}.\end{aligned}$$

   There are 9 outcomes of the form $(1,\cdot)$, 10 outcomes of the form $(j,\cdot)$ for $j \in \{2,\ldots,10\}$. However, noting that the outcome (i,j) is same as (j,i), we get that there are a total of 54 outcomes in Ω, which may be enumerated as

$$\Omega = \Big\{(1,2),(1,3),\ldots,(1,10),(2,2),(2,3),\ldots,(2,10),\ldots\ldots\ldots,(9,10),(10,10)\Big\}$$

   Assuming that $\mathscr{F} = 2^{\Omega}$, we then have

$$\mathbb{P}(\{(i,j)\}) = \begin{cases} \dfrac{\binom{i}{1}\cdot\binom{j}{1}}{\binom{55}{2}}, & i \neq j,\\[2ex] \dfrac{\binom{i}{2}}{\binom{55}{2}}, & i = j.\end{cases}$$

   (b) **(2 Marks)**
   Determine the probability of the event that the two coupons carry the same label.

   Solution: Let E be the event in question. Then, we have $E = \bigcup_{i=2}^{10}\{(i,i)\} = \{(2,2),(3,3),\ldots,(10,10)\}$. Furthermore,

$$\mathbb{P}(E) = \sum_{i=2}^{10}\mathbb{P}(\{(i,i)\}) = \sum_{i=2}^{10}\frac{\binom{i}{2}}{\binom{55}{2}} = \frac{1}{9}.$$