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Indian Institute of Technology Hyderabad

Stochastic Processes

Class Properties, Irreducibility, Aperiodicity, Invariant Distribution

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Period is a Class Property

Proposition (Period is a Class Property)

If $x \longleftrightarrow y$, then $d(x) = d(y)$.
Thus, all states within a communicating class possess the same period.

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Transience and Recurrence are Class Properties

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Transience and recurrence are class properties, i.e., the states within a communicating class are either all transient or all recurrent.

Positive/Null Recurrence are Class Properties

Proposition (Positive/Null Recurrence are Class Properties)

Positive recurrence and null recurrence are class properties, i.e., the states within a communicating class are either all positive recurrent or all null recurrent.

An Important Result About Open and Closed Communicating Classes

Proposition (Result about Open/Closed Communicating Classes)

1. If $\mathcal{C}$ is an open communicating class, then every state within $\mathcal{C}$ is transient.
2. If $\mathcal{C}$ is a closed communicating class, and $|\mathcal{C}| < +\infty$, then every state within $\mathcal{C}$ is positive recurrent.

   As a corollary, an irreducible DTMC with a finite state space is positive recurrent.

An Important Property of Aperiodic State (Without Proof)

Proposition (Important Property about Aperiodic State)

If state x is aperiodic, then there exists $N_x \in \mathbb{N}$ (possibly large) such that

$$P^n_{x,x} > 0 \qquad \forall n \geq N_x.$$

An Important Property of Irreducible and Aperiodic Markov Chains (Without Proof)

Proposition

Consider a time-homogeneous DTMC with finite state space $\mathcal{X}$ and TPM P.
If P is irreducible and aperiodic, then there exists $r_0 \in \mathbb{N}$ such that

$$P^r_{x,y} > 0 \qquad \forall r \geq r_0, \;\; \forall x, y \in \mathcal{X}.$$

Invariant (Stationary) Distributions

Invariant (Stationary) Distribution

Definition (Invariant Distribution)

Consider a DTMC with a discrete state space $\mathcal{X}$ and TPM P.
A PMF π on $\mathcal{X}$ is called the invariant distribution for P if

$$\pi = \pi P \qquad \text{(global balance equation).}$$

That is, for all $y \in \mathcal{X}$,

$$\pi(y) = \sum_{x \in \mathcal{X}} \pi(x) P_{x,y}.$$

Remarks on Invariant Distribution

Invariant Distribution

$$\pi = \pi P, \qquad \pi(y) = \sum_{x \in \mathcal{X}} \pi(x) P_{x,y} \quad \forall y \in \mathcal{X}.$$

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$$\pi = \pi P^n \qquad \text{for any } n \in \mathbb{N}.$$

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- If a Markov chain is irreducible, and $\pi_x > 0$ for some x, then $\pi_y > 0$ for all $y \neq x$

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  - We can ask: does $\lim_{n\to\infty} \pi_n$ exist? If so, under what conditions?

On Existence and Uniqueness of Invariant Distribution

Proposition (On Existence and Uniqueness of Invariant Distribution)

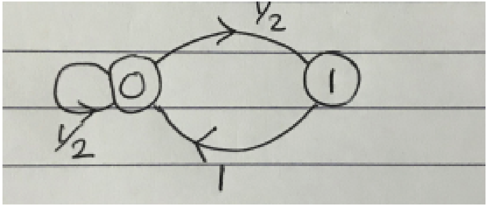
Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible, time-homogeneous DTMC on a discrete state space $\mathcal{X}$ with TPM P.

Then, a unique stationary distribution π exists if and only if P is positive recurrent.

In this case, $\pi_x = \frac{1}{\mu_{xx}} > 0$ for all $x \in \mathcal{X}$.

Example

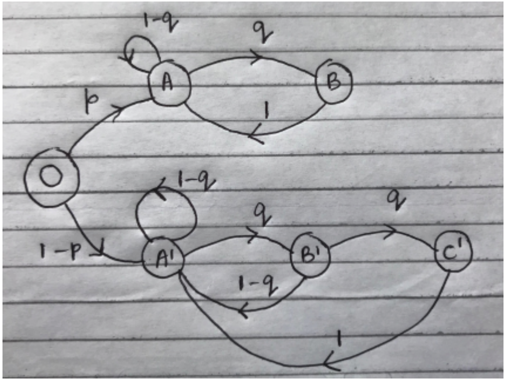
- Consider a DTMC with following transition graph.



1. Is the Markov chain irreducible?
2. Is the Markov chain aperiodic?
3. Classify the states as transient, positive recurrent, or null recurrent.
4. Does a stationary distribution exist for this Markov chain? If so, is it unique?

Example

- Consider a DTMC with the following transition graph.



1. Is this Markov chain irreducible?
2. Does there exist a unique stationary distribution?