



Stochastic Processes

Lecture 02

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13 January 2026

Limit of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Let $\{X_n\}_{n \in \mathbb{N}} = \{X_1, X_2, \dots\}$ be a collection of random variables w.r.t. \mathcal{F}

- Fix $\omega \in \Omega$, and consider the sequence of real numbers

$$X_1(\omega), X_2(\omega), \dots$$

- A limit may or may not exist for the above sequence

Lemma (An Important Set and its Measurability)

The set of all $\omega \in \Omega$ for which $\lim_{n \rightarrow \infty} X_n(\omega)$ exists is a valid event, i.e.,

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists} \right\} \in \mathcal{F}.$$

Proof of Lemma 1

$$\begin{aligned}
 \omega \in A_{\lim} &\iff \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists } (= x_\omega, \text{ say}) \\
 &\iff \forall \varepsilon > 0, \exists N_\varepsilon(\omega) : |X_n(\omega) - x_\omega| < \varepsilon \quad \forall n \geq N_\varepsilon(\omega) \\
 &\iff \forall q \in \mathbb{Q}, q > 0, \exists N_q(\omega) : |X_n(\omega) - x_\omega| < q \quad \forall n \geq N_q(\omega) \\
 &\iff \omega \in \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ \omega' \in \Omega : |X_n(\omega') - x_{\omega'}| < q \right\}
 \end{aligned}$$

The Set A_{\lim} and its Measurability

$$A_{\lim} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ \omega' \in \Omega : |X_n(\omega') - \lim_{k \rightarrow \infty} X_k(\omega')| < q \right\} \in \mathcal{F}.$$

Properties of A_{\lim}

Theorem (Properties of A_{\lim})

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables on a common underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let A_{\lim} denote the set

$$A_{\lim} = \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists} \right\}.$$

1. A_{\lim} belongs to the **tail σ -algebra**, i.e.,

$$A_{\lim} \in \underbrace{\bigcap_{k \in \mathbb{N}} \sigma(X_k, X_{k+1}, \dots)}_{\text{tail } \sigma\text{-algebra}} \subseteq \mathcal{F}.$$

2. **Kolmogorov's 0-1 Law:** If X_1, X_2, \dots are **independent**, then every event in the tail σ -algebra has either probability 0 or probability 1, in which case $\mathbb{P}(A_{\lim}) \in \{0, 1\}$.

Proof of Theorem 2, Part 1

- The set A_{\lim} can be expressed as

$$A_{\lim} = \underbrace{\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \in \mathbb{R} \right\}}_{A_{\lim}^{(1)}} \cup \underbrace{\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = +\infty \right\}}_{A_{\lim}^{(2)}} \cup \underbrace{\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = -\infty \right\}}_{A_{\lim}^{(3)}}.$$

- To show $A_{\lim}^{(2)}$ belongs to the tail σ -algebra:

- $A_{\lim}^{(2)}$ can be expressed as

$$A_{\lim}^{(2)} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{X_n > q\} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{\substack{N \geq k \\ \text{red}}} \bigcap_{n \geq N} \{X_n > q\} \quad \forall k \in \mathbb{N}.$$

- We then note that

$$\forall k \in \mathbb{N}, q \in \mathbb{Q}, q > 0, \quad \bigcup_{\substack{N \geq k}} \bigcap_{n \geq N} \{X_n > q\} \in \sigma(X_k, X_{k+1}, \dots)$$

- Exercise:** Along similar lines, it can be shown that $A_{\lim}^{(3)}$ belongs to the tail σ -algebra

Proof of Theorem 2, Part 1

- To show that $A_{\lim}^{(1)}$ belongs to the tail σ -algebra, we use the **Cauchy criterion for convergence**:

Cauchy Criterion for Convergence

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers. Let $x \in \mathbb{R}$. Then,

Exercise: $\lim_{n \rightarrow \infty} x_n = x \iff \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : |x_n - x_m| < \varepsilon \quad \forall n, m \geq N_\varepsilon.$

As usual,

$$\text{for every choice of } \varepsilon > 0 \iff \text{for every choice of } \varepsilon \in \mathbb{Q}, \varepsilon > 0.$$

- Using the above criterion, we may write

$$A_{\lim}^{(1)} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ |X_n - X_m| < q \right\} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{\substack{N \geq k \\ \text{red}}} \bigcap_{n \geq N} \left\{ |X_n - X_m| < q \right\} \quad \forall k \in \mathbb{N}.$$

- Same arguments as before can be used to show that $A_{\lim}^{(1)} \in \sigma(X_k, X_{k+1}, \dots)$ for every $k \in \mathbb{N}$

Proof of Theorem 2, Part 2

- Will be given as homework exercise

Forms of Convergence of Sequences of Random Variables

Pointwise Convergence

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Pointwise Convergence)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV. We say that the sequence $\{X_n\}_{n \in \mathbb{N}}$ converges **pointwise** to X if

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \quad \forall \omega \in \Omega;$$

Notation: $X_n \xrightarrow{\text{pointwise}} X$.

Equivalently, we have $A_{\lim} = \Omega$.

Almost-Sure Convergence

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Almost-Sure Convergence)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV. We say that the sequence $\{X_n\}_{n \in \mathbb{N}}$ converges to X **almost surely (a.s.)** if

$$\mathbb{P} \left(\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right\} \right) = 1; \quad \text{Notation: } X_n \xrightarrow{\text{a.s.}} X.$$

Note:

- The above definition DOES NOT require X_1, X_2, \dots to be independent
- If X_1, X_2, \dots are independent, then $\mathbb{P}(A_{\lim}) \in \{0, 1\}$ by Kolmogorov's 0-1 law
This has no bearing on the above definition in any way
- **Pointwise convergence implies almost-sure convergence**, i.e.,

$$X_n \xrightarrow{\text{pointwise}} X \quad \implies \quad X_n \xrightarrow{\text{a.s.}} X.$$

In general, the converse may not be true

Example: Shrinking Pulse

Suppose that $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$, λ : Lebesgue measure

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} 1, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Does this sequence converge pointwise? If so, what is the pointwise limit RV?

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \omega^n, \quad \omega \in \Omega.$$

Does this sequence converge pointwise? If so, what is the pointwise limit RV?

Example: Moving Rectangles

Suppose that $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$, λ : Lebesgue measure

- Consider the sequence of random variables given by:

$$X_1 = \mathbf{1}_{[0,1]}$$

$$X_2 = \mathbf{1}_{[0, \frac{1}{2}]}, \quad X_3 = \mathbf{1}_{[\frac{1}{2}, 1]}$$

$$X_4 = \mathbf{1}_{[0, \frac{1}{4}]}, \quad X_5 = \mathbf{1}_{[\frac{1}{4}, \frac{1}{2}]}, \quad X_6 = \mathbf{1}_{[\frac{1}{2}, \frac{3}{4}]}, \quad X_7 = \mathbf{1}_{[\frac{3}{4}, 1]}, \quad \text{and so on.}$$

Example: Moving Rectangles

Suppose that $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$, λ : Lebesgue measure

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Does the above sequence admit a pointwise limit? If so, what is the pointwise limit RV?

Example: Going Beyond Pointwise Convergence

- For each $n \in \mathbb{N}$, let

$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$

Remarks on this Example

- In cases where $(\Omega, \mathcal{F}, \mathbb{P})$ and the sequence $\{X_n\}_{n \in \mathbb{N}}$ are not explicitly specified, it is not possible to identify the pointwise limit.
- In such cases, we start with a guess for the limit RV and prove convergence in other forms (starting with almost-sure convergence)
- In many cases (including the current example), we need a way to infer almost-sure convergence merely based on probabilities

In enters Borel–Cantelli Lemma!