1. Because $p, q \in (0,1)$, $P_{i,j} > 0 \;\; \forall\, i,j \in \{0,1\}$, and hence P is irreducible.

Because an irreducible DTMC on a finite state space is positive recurrent, and

irreducibility + positive recurrence $\Rightarrow$ unique stationary dist^n exists,

we conclude that there exists a unique $\pi = [\pi_0 \;\; \pi_1]$ s.t.

$$\pi_0 \geq 0, \quad \pi_1 \geq 0, \quad \pi_0 + \pi_1 = 1.$$

Solving $\pi = \pi P$, we get

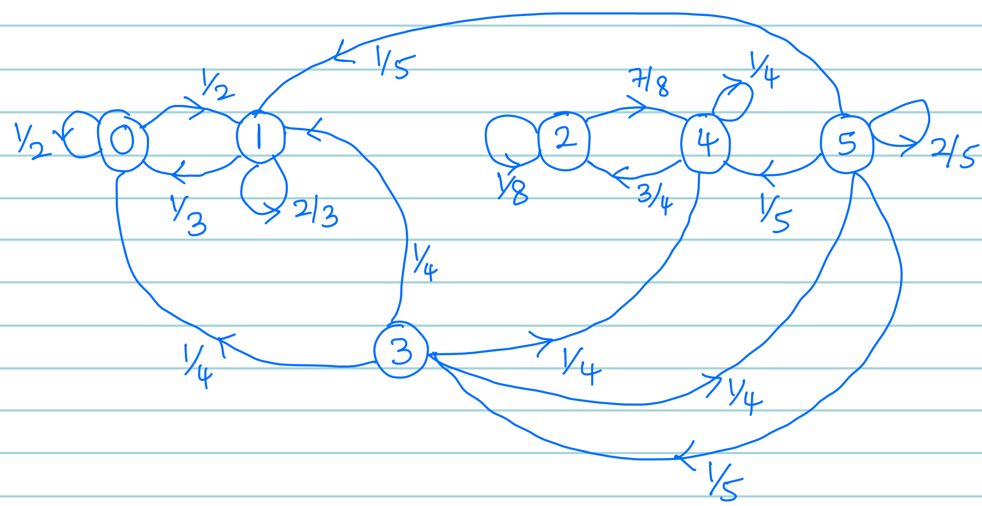
$$\pi_0 = (1-p)\pi_0 + q\pi_1 \quad \Rightarrow \quad \pi_1 = \frac{p}{q} \cdot \pi_0 .$$

Then, because $\pi_0 + \pi_1 = 1$, we get

$$\pi_0 = \frac{q}{p+q}, \quad \pi_1 = \frac{p}{p+q}.$$

---

2. The transition diagram is as follows.



a) It is evident from the transition graph that

$$\mathcal{C}_1 = \{0,1\}, \quad \mathcal{C}_2 = \{2,4\}, \quad \mathcal{C}_3 = \{3,5\}$$

are the communicating classes. Furthermore,

$\mathcal{C}_1 \to$ closed $\Rightarrow$ positive recurrent (because no of states in $\mathcal{C}_2$ is finite),

$\mathcal{C}_2 \to$ closed $\Rightarrow$ positive recurrent,

$\mathcal{C}_3 \to$ open $\Rightarrow$ transient.

b) If $x \in \mathcal{C}_1$, then $\mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = x) = 1$.

If $x \in \mathcal{C}_2$, then $\mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = x) = 0$.

If $x \in \mathcal{C}_3$, then :

$$\begin{aligned}
\mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = 3) &= \mathbb{P}(T_{\{0,1\}} < +\infty, X_1 = 0 \mid X_0 = 3) \\
&\quad + \mathbb{P}(T_{\{0,1\}} < +\infty, X_1 = 1 \mid X_0 = 3) \\
&\quad + \mathbb{P}(T_{\{0,1\}} < +\infty, X_1 = 4 \mid X_0 = 3) \\
&\quad + \mathbb{P}(T_{\{0,1\}} < +\infty, X_1 = 5 \mid X_0 = 3) \\
&= \left.\begin{aligned}
&\tfrac{1}{4} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 0, X_0 = 3) \\
&+ \tfrac{1}{4} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 1, X_0 = 3) \\
&+ \tfrac{1}{4} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 4, X_0 = 3) \\
&+ \tfrac{1}{4} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 5, X_0 = 3)
\end{aligned}\right\} \text{Can neglect } X_0 = 3 \text{ because of Markov property.} \\
&= \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 5) \\
&= \tfrac{1}{2} + \tfrac{1}{4}\, \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = 5) \quad \text{(again, Markov property)}
\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}
\mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = 5) &= \frac{2}{5} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 5) \\
&\quad + \tfrac{1}{5} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 3) \\
&\quad + \tfrac{1}{5} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_1 = 1) \\
&= \tfrac{1}{5} + \tfrac{1}{5} \cdot \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = 3) \\
&\quad + \frac{2}{5}\, \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = 5)
\end{aligned}$$

Setting $a = \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = 3)$

$b = \mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = 5)$, we have

$$a = \tfrac{1}{2} + b/4 \iff 4a = 2 + b$$

$$\tfrac{3}{5} b = \tfrac{1}{5} + a/5 \iff 3b = 1 + a$$

Then,

$$3(4a-2) = 1 + a \Rightarrow 11a = 7 \Rightarrow a = 7/11,$$

$$b = 6/11.$$

---

3. Let A_n denote the box chosen at time $n \in \mathbb{N}$.
We note that $A_1, A_2, \dots \overset{iid}{\sim} \text{Unif}\{1, \dots, N\}$, and $A_n \perp\!\!\!\perp X_{n-1}$ $\forall n$.

Furthermore, we have

$$X_n = X_{n-1} \mathbb{1}_{\{A_n \in \{A_1, \dots, A_{n-1}\}\}} + (X_{n-1} - 1) \mathbb{1}_{\{A_n \notin \{A_1, \dots, A_{n-1}\}\}}.$$

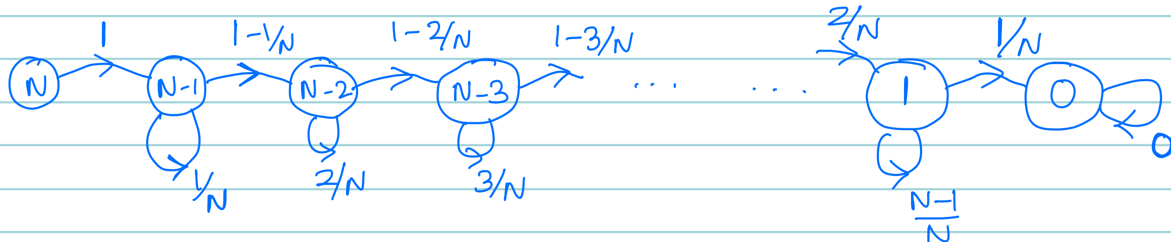
Because $A_n \perp\!\!\!\perp X_{n-1}$ and X_n is a function only of X_{n-1},

we conclude that $\{X_n\}_{n=0}^{\infty}$ is a DTMC.

b) The state space $\mathcal{X} = \{N, N-1, \dots, 0\}$.
We note that

$$\mathbb{P}(X_n = j \mid X_{n-1} = i) = \begin{cases} \dfrac{N-i}{N}, & \text{if } j = i, \\ i/N, & \text{if } j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

The transition graph is as follows.



c) Noting that 0 is an absorbing state,
we conclude that all states except 0 are transient,
and 0 is positive recurrent (because $\{0\}$ is a closed communicating class).

---

4. Let $T_0 = 0$, and for each $k \in \mathbb{N}$, let

$$T_k := \inf\{ n > T_{k-1} : X_n = y \}$$

denote the k^{th} return time to y. Note that

$$\mathbb{P}(T_1 < +\infty \mid X_0 = y) = f_{yy} = 1 \quad (\text{because } y \text{ is +ve recurrent}),$$

and for any $k \geq 2$, assuming $\mathbb{P}(T_{k-1} < +\infty \mid X_0 = y) = 1$, we have

$$\begin{aligned}
\mathbb{P}(T_k < +\infty \mid X_0 = y) &= \mathbb{P}(T_k < +\infty \mid X_{T_{k-1}} = y, \; T_{k-1} < +\infty, \; X_0 = y) \\
&= \mathbb{P}(T_k < +\infty \mid X_{T_{k-1}} = y, \; T_{k-1} < +\infty) \\
&\overset{\text{Strong Markov Property}}{=} f_{yy} = 1.
\end{aligned}$$

Hence, $\{T_k\}_{k=0}^{\infty}$ are a sequence of stopping times, and

$$\mathbb{P}(T_k - T_{k-1} < +\infty \mid X_0 = y) = 1 \quad \forall \, k \in \mathbb{N}.$$

We then note that

$$\begin{aligned}
\mathbb{P}(V_x^{(k)} = n \mid X_0 = y) &= \mathbb{P}(V_x^{(k)} = n, \; \underbrace{T_{k-1} < +\infty, \; T_k - T_{k-1} < +\infty}_{\text{probability 1 events}} \mid X_0 = y) \\
&\overset{\text{law of total probability}}{=} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}(V_x^{(k)} = n, \; T_{k-1} = m, \; T_k - T_{k-1} = l \mid X_0 = y) \\
&= \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}(V_x^{(k)} = n, \; T_{k-1} = m, \; X_m = y, \underbrace{X_{m+1} \neq y, \ldots, X_{m+l} \neq y}_{\text{says } T_k - T_{k-1} = l} \mid X_0 = y) \\
&= \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}(T_{k-1} = m, \; X_m = y \mid X_0 = y) \\
&\qquad \cdot \mathbb{P}(V_x^{(k)} = n, \; X_{m+1} \neq y, \ldots, X_{m+l} \neq y \mid X_0 = y, X_m = y, \underbrace{T_{k-1} = m}_{\substack{\text{function of } X_0, \ldots, X_m \\ \text{only because } T_{k-1} \text{ is a} \\ \text{stopping time}}}) \\
&= \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}(T_{k-1} = m, \; X_m = y \mid X_0 = y) \\
&\qquad \cdot \mathbb{P}(\underbrace{V_x^{(k)} = n, \; X_{m+1} \neq y, \ldots, X_{m+l} \neq y}_{\text{function of block } k} \mid X_m = y) \\
&\overset{\substack{\text{Markov prop.} \\ \text{and} \\ \text{IID block} \\ \text{Structure}}}{=} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}(T_{k-1} = m, \; X_m = y \mid X_0 = y) \cdot \mathbb{P}(V_x^{(1)} = n, \; X_1 \neq y, \ldots, X_l \neq y \mid X_0 = y) \\
&= \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{P}(T_{k-1} = m \mid X_0 = y) \cdot \mathbb{P}(V_x^{(1)} = n, \; T_1 = l \mid X_0 = y)
\end{aligned}$$

$$= \mathbb{P}\left(V_x^{(1)} = n \mid X_0 = y\right),$$

thus proving that $\{V_x^{(k)}\}_{k\in\mathbb{N}}$ are identically distributed.

We now prove independence of $V_x^{(k)}$ and $V_x^{(k+1)}$ for all $k \in \mathbb{N}$.
We have

$$\mathbb{P}\left(V_x^{(k)} = m,\ V_x^{(k+1)} = n \mid X_0 = y\right)$$

$$= \mathbb{P}\left(V_x^{(k)} = m,\ T_{k-1} < +\infty,\ T_k - T_{k-1} < +\infty,\ V_x^{(k+1)} = n,\ T_{k+1} - T_k < +\infty \mid X_0 = y\right)$$

$$= \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \mathbb{P}\left(V_x^{(k)} = m,\ T_{k-1} = l,\ T_k - T_{k-1} = r,\ V_x^{(k+1)} = n,\ T_{k+1} - T_k = s \mid X_0 = y\right)$$

Using similar arguments as before (identical distribution part) with IID block structure, it can be shown that

$$\mathbb{P}\left(V_x^{(k)} = m,\ V_x^{(k+1)} = n \mid X_0 = y\right) = \mathbb{P}\left(V_x^{(1)} = m \mid X_0 = y\right) \cdot \mathbb{P}\left(V_x^{(2)} = n \mid X_0 = y\right)$$

identical distribution of $\{V_x^{(k)}\}_{k\in\mathbb{N}}$.

$$= \mathbb{P}\left(V_x^{(k)} = m \mid X_0 = y\right) \cdot \mathbb{P}\left(V_x^{(k+1)} = n \mid X_0 = y\right)$$

---

5. We note that every state is accessible from state 0; in particular,

$$P_{0,i}^{i} = P_{0,1} \cdot P_{1,2} \cdots P_{i-1,i} = \left(1 - \frac{1}{N}\right) \cdots \left(1 - \frac{i-1}{N}\right)$$

$$= \frac{N!}{(N-i)!\ N^i} > 0.$$

Also, we note that

$$P_{i,0}^{i} = P_{i,i-1} \times P_{i-1,i-2} \times \cdots \times P_{1,0} = \frac{i!}{N^i} > 0.$$

Thus, the Markov chain is irreducible, and because state space is finite, the Markov chain is positive recurrent.

Let π denote the unique stationary distribution of P. Then,

$$\pi_0 = \frac{1}{N}\pi_1 \Rightarrow \pi_1 = N\pi_0 = {}^N C_1 \cdot \pi_0$$

$$\pi_1 = \left(1 - \frac{1}{N}\right)\pi_0 + \frac{2}{N}\pi_2 \Rightarrow \pi_2 = \frac{N(N-1)}{2}\pi_0 = {}^N C_2 \cdot \pi_0$$

$$\pi_2 = \left(1-\frac{2}{N}\right)\pi_0 + \frac{3}{N}\pi_3 \Rightarrow \pi_3 = \frac{N(N-1)(N-2)}{6}\pi_0 = {}^N C_3 \cdot \pi_0$$

$\vdots$

In general, we get $\pi_k = {}^N C_k \cdot \pi_0, \quad k \in \{0, \dots, N\}$.

Because $\sum_{k=0}^{N} \pi_k = 1$, we get

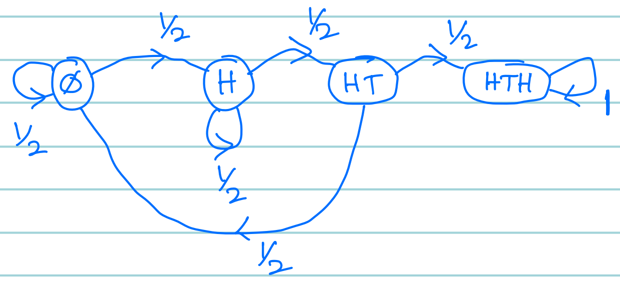
$$\pi_0 = \frac{1}{{}^N C_0 + \dots + {}^N C_N} = \frac{1}{2^N}.$$

Thus, $\pi_k = \frac{{}^N C_k}{2^N}, \quad k \in \{0, \dots, N\}$.

Then, it follows that $\mu_{kk} = \frac{1}{\pi_k} = \frac{2^N}{{}^N C_k}, \quad k \in \{0, \dots, N\}$.

---

6. Consider a time-homogeneous DTMC with the following transition graph.



Let $\psi_0 := E[\#\text{ tosses required to get "HTH"} \mid X_0 = \phi] = \sum_{n=1}^{\infty} n\, \mathbb{P}(\tau^{(1)}_{HTH} = n \mid X_0 = \phi)$

$\psi_1 := E[\#\text{ tosses required to get "HTH"} \mid X_0 = H] = \sum_{n=1}^{\infty} n \cdot \mathbb{P}(\tau^{(1)}_{HTH} = n \mid X_0 = H)$

$\psi_2 := E[\#\text{ tosses required to get "HTH"} \mid X_0 = HT] = \sum_{n=1}^{\infty} n \cdot \mathbb{P}(\tau^{(1)}_{HTH} = n \mid X_0 = HT)$

$\psi_3 := E[\#\text{ tosses required to get HTH} \mid X_0 = HTH] = 0.$

We then have

$$\mathbb{P}(\tau^{(1)}_{HTH} = n \mid X_0 = \phi) = \mathbb{P}(X_1 \neq HTH, \dots, X_{n-1} \neq HTH, X_n = HTH \mid X_0 = \phi)$$

$$= \mathbb{P}(X_1 = \phi, X_2 \neq HTH, \dots, X_{n-1} \neq HTH, X_n = HTH \mid X_0 = \phi)$$

$$+ \mathbb{P}(X_1 = H, X_2 \neq HTH, \dots, X_{n-1} \neq HTH, X_n = HTH \mid X_0 = \phi)$$

Markov Property; can neglect X_0 if given X_1

$$= \tfrac{1}{2} \cdot \mathbb{P}(X_2 \neq HTH, \dots, X_{n-1} \neq HTH, X_n = HTH \mid X_1 = \phi)$$

$$+ \tfrac{1}{2} \cdot \mathbb{P}(X_2 \neq HTH, \dots, X_{n-1} \neq HTH, X_n = HTH \mid X_1 = H).$$

We now see that by Markov property,

$$\mathbb{P}\left(X_2 \neq HTH, \ldots, X_{n-1} \neq HTH, X_n = HTH \,\middle|\, X_1 = \phi \right)$$

$$= \mathbb{P}\left(X_1 \neq HTH, \ldots, X_{n-2} \neq HTH, X_{n-1} = HTH \,\middle|\, X_0 = \phi \right)$$

$$= \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = \phi \right).$$

Similarly,

$$\mathbb{P}\left(X_2 \neq HTH, \ldots, X_{n-1} \neq HTH, X_n = HTH \,\middle|\, X_0 = H \right) = \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = H \right).$$

Thus, we have

$$\mathbb{P}\left(\tau^{(1)}_{HTH} = n \,\middle|\, X_0 = \phi \right) = \tfrac{1}{2} \cdot \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = \phi \right) + \tfrac{1}{2} \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = H \right),$$

from which we get

$$\psi_0 = \sum_{n=1}^{\infty} n \left(\tfrac{1}{2} \cdot \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = \phi \right) + \tfrac{1}{2} \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = H \right) \right)$$

$$= \sum_{n=2}^{\infty} n \left(\tfrac{1}{2} \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = \phi \right) + \tfrac{1}{2} \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = H \right) \right)$$

noting that $\mathbb{P}\left(\tau^{(1)}_{HTH} = 0 \,\middle|\, X_0 = \phi \right) = 0 = \mathbb{P}\left(\tau^{(1)}_{HTH} = 0 \,\middle|\, X_0 = H \right)$.

$$= \frac{1}{2} \left[\sum_{n=2}^{\infty} (n-1) \, \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = \phi \right) + \sum_{n=2}^{\infty} \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = \phi \right) \right]$$

$$+ \frac{1}{2} \left[\sum_{n=2}^{\infty} (n-1) \, \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = H \right) + \sum_{n=2}^{\infty} \mathbb{P}\left(\tau^{(1)}_{HTH} = n-1 \,\middle|\, X_0 = H \right) \right]$$

$$= \frac{1}{2} \left[\sum_{m=1}^{\infty} m \, \mathbb{P}\left(\tau^{(1)}_{HTH} = m \,\middle|\, X_0 = \phi \right) + \underbrace{\sum_{m=1}^{\infty} \mathbb{P}\left(\tau^{(1)}_{HTH} = m \,\middle|\, X_0 = \phi \right)}_{= \mathbb{P}\left(\tau^{(1)}_{HTH} < +\infty \,\middle|\, X_0 = \phi \right) = 1} \right]$$

$$+ \frac{1}{2} \left[\sum_{m=1}^{\infty} m \, \mathbb{P}\left(\tau^{(1)}_{HTH} = m \,\middle|\, X_0 = H \right) + \underbrace{\sum_{m=1}^{\infty} \mathbb{P}\left(\tau^{(1)}_{HTH} = m \,\middle|\, X_0 = H \right)}_{= \mathbb{P}\left(\tau^{(1)}_{HTH} < +\infty \,\middle|\, X_0 = H \right) = 1} \right]$$

because HTH is an absorbing state

$$= \frac{1}{2} \left[\psi_0 + 1 \right] + \frac{1}{2} \left[\psi_1 + 1 \right].$$

Along similar lines, we may derive/justify the other recursive relations.