AI5030 / EE5817: PROBABILITY AND STOCHASTIC PROCESSES

HOMEWORK 03 - SOLUTIONS

MEASURES, PROBABILITY MEASURES

1. Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

   (a) Given $c \in \mathbb{R}$, define $\delta_c : \mathscr{F} \to [0,1]$ as

   $$\delta_c(A) = \begin{cases} 1, & c \in A, \\ 0, & c \notin A, \end{cases} \qquad A \in \mathscr{F}.$$

   Show that δ_c is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

   Remark: δ_c is called the Dirac measure concentrated at c.

   (b) Let $\mu : \mathscr{F} \to [0, +\infty]$ be defined as

   $$\mu(A) = \sum_{n \in \mathbb{N}} \delta_n(A), \qquad A \in \mathscr{F}.$$

   Show that μ is a measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. What does $\mu(A)$ for any $A \in \mathscr{F}$ represent?
   You may want to use the fact that if $\{a_{n,k}\}_{n,k \in \mathbb{N}}$ is a sequence of non-negative real numbers, then

   $$\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} a_{n,k} = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} a_{n,k}.$$

   Remark: μ is called the counting measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Solution.

(a) Observe that $\delta_c(\emptyset) = 0$ and $\delta_c(\Omega) = \delta_c(\mathbb{R}) = 1$, as $c \in \mathbb{R}$.
Let $A_1, A_2, \ldots \in \mathscr{B}(\mathbb{R})$ be a countable disjoint collection of sets.
Case 1. $c \notin \bigsqcup_{i \in \mathbb{N}} A_i$.
In this case, $c \notin A_i$ for each $i \in \mathbb{N}$, thus implying that $\delta_c(A_i) = 0$ for every $i \in \mathbb{N}$. Hence,

$$\delta_c\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = 0 = \sum_{i \in \mathbb{N}} \delta_c(A_i).$$

Case 2. $c \in \bigsqcup_{i \in \mathbb{N}} A_i$.
In this case, using the fact that $A_1, A_2, \ldots$ are disjoint, there exists a unique $i \in \mathbb{N}$ such that $c \in A_i$ and $c \notin A_j$ for $j \neq i$. Thus,

$$\delta_c\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = 1 = \sum_{i \in \mathbb{N}} \delta_c(A_i).$$

In either case, we have demonstrated that

$$\delta_c\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \delta_c(A_i),$$

thereby proving that δ_c satisfies countably additivity. This shows that δ_c is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

(b) Observe that

$$\mu(\emptyset) = \sum_{n \in \mathbb{N}} \delta_n(\emptyset) = 0,$$

as $\delta_n(\emptyset) = 0$ for all $n \in \mathbb{N}$. Also,

$$\mu(\mathbb{R}) = \sum_{n\in\mathbb{N}} \delta_n(\mathbb{R}) = \sum_{n\in\mathbb{N}} 1 = +\infty,$$

noting that $\delta_n(\mathbb{R}) = 1$ for all $n \in \mathbb{N}$.
Suppose that $A_1, A_2, \ldots$ is a countable disjoint collection of sets. Then,

$$\begin{aligned}
\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) &= \sum_{n\in\mathbb{N}} \delta_n\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) \\
&= \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \delta_n(A_k) \\
&= \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} \delta_n(A_k) \qquad \text{(using the hint provided, with } a_{n,k} = \delta_n(A_k)\text{)} \\
&= \sum_{k\in\mathbb{N}} \mu(A_k).
\end{aligned}$$

We have thus shown that μ is an infinite measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.
Interpretation: For any $A \in \mathscr{F}$,

$$\mu(A) = \sum_{n\in\mathbb{N}} \mathbf{1}_{\{n\in A\}} = |A \cap \mathbb{N}|.$$

That is, $\mu(A)$ counts how many natural numbers lie in A.

2. Let $\Omega = \mathbb{N}$. Let $\mathscr{A}$ be defined as the collection

$$\mathscr{A} := \left\{ A \subseteq \Omega : \ |A| < +\infty \quad \text{or} \quad |\Omega \setminus A| < +\infty \right\}.$$

We know from Question 3(b) of Homework 2 that $\mathscr{A}$ is an algebra, but not a σ-algebra.
Define $\mathbb{P}_0 : \mathscr{A} \to [0, 1]$ as

$$\mathbb{P}_0(A) = \begin{cases} 0, & |A| < +\infty, \\ 1, & |\Omega \setminus A| < +\infty. \end{cases}$$

(a) Show that for any two disjoint sets $A, B \in \mathscr{A}$,

$$\mathbb{P}_0(A \cup B) = \mathbb{P}_0(A) + \mathbb{P}_0(B).$$

(By induction, it follows that $\mathbb{P}_0$ satisfies the property of finite additivity on $\mathscr{A}$.)

(b) Show that $\mathbb{P}_0$ does not necessarily satisfy countable additivity property.
That is, construct an explicit sequence of disjoint events $A_1, A_2, \ldots \in \mathscr{A}$ such that

$$\bigsqcup_{n\in\mathbb{N}} A_n \in \mathscr{A}, \qquad \mathbb{P}_0\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) \neq \sum_{n\in\mathbb{N}} \mathbb{P}_0(A_n).$$

(c) Construct a non-increasing sequence of sets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ such that

$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset, \qquad \lim_{n\to\infty} \mathbb{P}_0(A_n) \neq 0.$$

Solution.

(a) Let $A, B \in \mathscr{A}$ with $A \cap B = \emptyset$. Let, $C = A \sqcup B$.
We consider the possible cases:

i. **A finite, B finite.**
In this case, note that C is finite. Hence,

$$\mathbb{P}_0(C) = 0 = \mathbb{P}_0(A) + \mathbb{P}_0(B).$$

ii. A **finite,** B **co-finite (i.e.,** $B^{\complement}$ **finite).**
In this case, note that C is co-finite (i.e., $C^{\complement}$ is finite). Hence,

$$\mathbb{P}_0(C) = 1 = 0 + 1 = \mathbb{P}_0(A) + \mathbb{P}_0(B).$$

iii. A **co-finite,** B **finite.**
This case is symmetric to case (ii) above.

iv. A **co-finite,** B **co-finite (i.e.,** $A^{\complement}$ **finite,** $B^{\complement}$ **finite.)**
We note that this is an impossible scenario. To see this, suppose $|A^{\complement}| = |\Omega \setminus A| < +\infty$ and $|B^{\complement}| = |\Omega \setminus B| < +\infty$. In this case, noting that A and B are disjoint, we must have $|A^{\complement} \cup B^{\complement}| < +\infty$. This then implies that $A \cap B$ is co-finite (or equivalently, $A \cap B$ is countably infinite), which is clearly a contradiction as $A \cap B = \emptyset$.

In all valid cases depicted above, we have demonstrated that

$$\mathbb{P}_0(A \cup B) = \mathbb{P}_0(A) + \mathbb{P}_0(B).$$

(b) Define $A_i = \{i\}, \quad \forall i \in \mathbb{N}$.
Then $A_i \in \mathscr{A}$ are disjoint. Moreover,

$$\bigsqcup_{i \in \mathbb{N}} A_i = \mathbb{N} \in \mathscr{A}.$$

Therefore, we have

$$\mathbb{P}_0\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \mathbb{P}_0(\mathbb{N}) = 1.$$

However, noting that $|A_i| < +\infty$ for all $i \in \mathbb{N}$, we have $\mathbb{P}_0(A_i) = 0$ for all $i \in \mathbb{N}$. Clearly, then,

$$1 = \mathbb{P}_0\left(\bigcup_{i \in \mathbb{N}} A_i\right) \neq \sum_{i \in \mathbb{N}} \mathbb{P}_0(A_i) = 0.$$

This shows that $\mathbb{P}_0$ is not countably additive.

(c) Define

$$A_n := \{n+1, n+2, \dots\} = \mathbb{N} \setminus \{1, 2, \dots, n\}.$$

Clearly, $A_1 \supseteq A_2 \supseteq \cdots$, and

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset.$$

Because $|\Omega \setminus A_n| = n$, we have $\mathbb{P}_0(A_n) = 1$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n \to \infty} \mathbb{P}_0(A_n) = 1 \neq 0 = \mathbb{P}_0\left(\bigcap_{n \in \mathbb{N}} A_n\right).$$

3. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathscr{G}$ be defined as the collection

$$\mathscr{G} := \left\{A \in \mathscr{F} : \ \mathbb{P}(A) = 0 \quad \text{or} \quad \mathbb{P}(A) = 1\right\}.$$

Show that $\mathscr{G}$ is a σ-algebra of subsets of Ω.

Solution.

By definition, $\mathscr{G} \subseteq \mathscr{F}$. We verify the three defining properties of a σ-algebra. First, observe that $\Omega \in \mathscr{F}$ and $\mathbb{P}(\Omega) = 1$. Therefore, it follows that $\Omega \in \mathscr{G}$. Next, suppose that $A \in \mathscr{G}$. Because $A^{\complement} \in \mathscr{F}$ and $\mathbb{P}(A^{\complement}) = 1 - \mathbb{P}(A)$, it follows that if $\mathbb{P}(A) \in \{0, 1\}$, then $\mathbb{P}(A^{\complement}) \in \{0, 1\}$, thus implying that $A^{\complement} \in \mathscr{G}$. Lastly, suppose that $A_1, A_2, \ldots \in \mathscr{G}$. Let $A := \bigcup_{n \in \mathbb{N}} A_n$. We show $\mathbb{P}(A) \in \{0, 1\}$.

Case 1: $\exists j \in \mathbb{N}$ such that $\mathbb{P}(A_j) = 1$. In this case,

$$\begin{aligned} A = \bigcup_{n \in \mathbb{N}} A_n &\supseteq A_j \\ &\Rightarrow \mathbb{P}\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) \geq \mathbb{P}(A_j) \\ &\Rightarrow \mathbb{P}\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) \geq 1 \\ &\Rightarrow \mathbb{P}\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = 1. \end{aligned}$$

That is, $\mathbb{P}(A) = \mathbb{P}\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = 1, \quad \bigcup_{n \in \mathbb{N}} A_n \in \mathscr{F}$, thus implying that $A \in \mathscr{G}$.

Case 2: $\forall n \in \mathbb{N}, \mathbb{P}(A_n) = 0$. In this case, let

$$B_1 \coloneqq A_1, \qquad B_n \coloneqq A_n \setminus \bigcup_{k=1}^{n-1} A_k \quad \forall\, n \geq 2.$$

Then, we note that

$$\bigsqcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n = A,$$

and $B_n \subseteq A_n$ for every $n \in \mathbb{N}$. By countable additivity on disjoint unions and monotonicity,

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}\Big(\bigsqcup_{n \in \mathbb{N}} B_n\Big) \\ &\leq \mathbb{P}\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) \\ &\leq 0, \end{aligned}$$

thus implying that $\mathbb{P}(A) = 0$, and hence $A \in \mathscr{G}$.

From the above exposition, it follows that $\mathscr{G}$ is a σ-algebra of subsets of Ω.

4. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $A_1, A_2, \ldots \in \mathscr{F}$.

   (a) Show formally that

   $$\liminf_{n \to \infty} A_n \in \mathscr{F}, \qquad \limsup_{n \to \infty} A_n \in \mathscr{F}.$$

   (b) Prove that

   $$\liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n.$$

   Provide an example construction in which $\liminf_{n \to \infty} A_n \subsetneq \limsup_{n \to \infty} A_n$ (strict inclusion).

   (c) Let $B_1, B_2, \ldots \in \mathscr{F}$ be sequence of disjoint sets, i.e., $B_i \cap B_j = \emptyset \quad \forall i \neq j$. Prove that

   $$\lim_{n \to \infty} \mathbb{P}(B_n) = 0.$$

Solution.

(a) By definition,

$$\liminf_{n \to \infty} A_n = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k.$$

For each $n \in \mathbb{N}$, let

$$B_n \coloneqq \bigcap_{k \geq n} A_k.$$

Since each $A_k \in \mathscr{F}$ for each k, and $\mathscr{F}$ is closed under countable intersections, we have $B_n \in \mathscr{F}$ for each $n \in \mathbb{N}$. Because $\mathscr{F}$ is also closed under countable unions,

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} B_n \in \mathscr{F}.$$

Similarly, note that by definition,

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k.$$

For each $n \in \mathbb{N}$, let

$$C_n \coloneqq \bigcup_{k\geq n} A_k.$$

Since each $A_k \in \mathscr{F}$ for each k, and $\mathscr{F}$ is closed under countable unions, $C_n \in \mathscr{F}$ for every $n \in \mathbb{N}$. Because $\mathscr{F}$ is also closed under countable intersections,

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} C_n \in \mathscr{F}.$$

Thus, both $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ belong to $\mathscr{F}$.

(b) Show $\liminf\limits_{n\to\infty} A_n \subseteq \limsup\limits_{n\to\infty} A_n$.

Recall the liminf & limsup definitions

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k, \qquad \limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k.$$

For liminf,

$$\begin{aligned} x \in \liminf_{n\to\infty} A_n = x \in \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k \\ &\Rightarrow \exists N \in \mathbb{N} \quad \text{such that} \quad x \in \bigcap_{k\geq N} A_k \\ &\Rightarrow \exists N \in \mathbb{N}, \forall k \geq N, x \in A_k \end{aligned}$$

For limsup,

$$\begin{aligned} x \in \limsup_{n\to\infty} A_n &\Rightarrow x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \\ &\Rightarrow \forall n \in \mathbb{N},\ x \in \bigcup_{k\geq n} A_k \\ &\Rightarrow \forall n \in \mathbb{N},\ \exists k \geq n \ \text{ such that } x \in A_k. \end{aligned}$$

Now fix an arbitrary $n \in \mathbb{N}$ and let $k^* := \max\{n, N\}$. From the definition of liminf set, $x \in A_{k^*}$. As $k^* \geq n$, we have $x \in \bigcup_{k\geq n} A_k$. Because this holds for every $n \in \mathbb{N}$, it follows that

$$x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k = \limsup_{n\to\infty} A_n.$$

Thus, every element belonging to the liminf set also belongs to the limsup set, thus proving that $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

Strict inclusion example. Let $\Omega = \mathbb{R}$, $\mathscr{F} = \mathscr{B}(\mathbb{R})$, and let

$$A_n = \begin{cases} \left(-\infty, -\frac{1}{n}\right], & n \text{ odd}, \\ (-\infty, 0], & n \text{ even}. \end{cases}$$

We leave it as exercise to verify that $\liminf\limits_{n\to\infty} A_n = (-\infty, 0)$, while $\limsup\limits_{n\to\infty} A_n = (-\infty, 0]$.

5. Let $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$. For each $n \in \mathbb{N}$, let $\mathbb{P}_n : \mathscr{F} \to [0,1]$ be defined as

$$\mathbb{P}_n(A) = \frac{|A \cap \{1, \ldots, n\}|}{n}, \qquad A \in \mathscr{F}.$$

(a) Show that P_n is a probability measure on $\mathscr{F}$ for each $n \in \mathbb{N}$.

(b) Given a set $A \in \mathscr{F}$, its **density** $D(A)$ is defined as

$$D(A) \coloneqq \lim_{n \to \infty} \mathbb{P}_n(A),$$

provided the above limit exists. Let $\mathscr{D}$ denote the collection of all sets whose density is well-defined, i.e.,

$$\mathscr{D} \coloneqq \left\{ A \in \mathscr{F} : \lim_{n \to \infty} \mathbb{P}_n(A) \text{ is well-defined} \right\}.$$

Show that D is finitely additive on $\mathscr{D}$.
Construct an example to show that D is not necessarily countably additive on $\mathscr{D}$.

Solution.

(a) We verify that $\mathbb{P}_n$ is a probability measure on $(\Omega, \mathscr{F})$ for each $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. Note that

$$\begin{aligned}
\mathbb{P}_n(\Omega) &= \frac{|\Omega \cap \{1, 2, \ldots, n\}|}{n} \\
&= \frac{n}{n} \qquad \text{as } |\mathbb{N} \cap \{1, 2, \ldots, n\}| = n \\
&= 1.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{P}_n(\emptyset) &= \frac{|\emptyset \cap \{1, 2, \ldots, n\}|}{n} \\
&= \frac{0}{n} \qquad \text{as } |\emptyset \cap \{1, 2, \ldots, n\}| = 0 \\
&= 0.
\end{aligned}$$

Let $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathscr{F}$ be pairwise disjoint. Observe that

$$\begin{aligned}
\mathbb{P}_n\Big(\bigsqcup_{i \in \mathbb{N}} A_i\Big) &= \frac{\big|\big(\bigsqcup_{i \in \mathbb{N}} A_i\big) \cap \{1, 2, \ldots, n\}\big|}{n} \\
&= \frac{\big|\bigsqcup_{i \in \mathbb{N}} \big(A_i \cap \{1, 2, \ldots, n\}\big)\big|}{n}.
\end{aligned}$$

Since $\{A_i\}$ are disjoint, their intersections with $\{1, 2, \ldots, n\}$ remain disjoint. Hence,

$$\left| \bigsqcup_{i \in \mathbb{N}} (A_i \cap \{1, 2, \ldots, n\}) \right| = \sum_{i \in \mathbb{N}} |A_i \cap \{1, 2, \ldots, n\}|,$$

and therefore

$$\mathbb{P}_n\Big(\bigsqcup_{i \in \mathbb{N}} A_i\Big) = \sum_{i \in \mathbb{N}} \frac{|A_i \cap \{1, 2, \ldots, n\}|}{n} = \sum_{i \in \mathbb{N}} \mathbb{P}_n(A_i).$$

Thus, it follows that $\mathbb{P}_n$ is a probability measure.

(b) Consider $A, B \in \mathscr{D}$ disjoint. Then, for every $n \in \mathbb{N}$,

$$\mathbb{P}_n(A \cup B) = \frac{|(A \cup B) \cap \{1, \ldots, n\}|}{n} = \frac{|A \cap \{1, \ldots, n\}| + |B \cap \{1, \ldots, n\}|}{n} = \mathbb{P}_n(A) + \mathbb{P}_n(B).$$

Because $D(A)$ and $D(B)$ are well defined, we obtain

$$D(A \cup B) = \lim_{n \to \infty} \mathbb{P}_n(A \cup B) = \lim_{n \to \infty} \big(\mathbb{P}_n(A) + \mathbb{P}_n(B)\big) = D(A) + D(B).$$

Hence D is finitely additive on $\mathscr{D}$.

Failure of Countable Additivity. Consider the sequence of disjoint singleton sets

$$A_k := \{k\}, \qquad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, note that

$$\mathbb{P}_n(A_k) = \frac{1}{n} \qquad \forall n \geq k.$$

Thus, it follows that

$$D(A_k) = \lim_{n\to\infty} \mathbb{P}_n(A_k) = 0.$$

However, observe that

$$\bigsqcup_{k=1}^{\infty} A_k = \mathbb{N}, \qquad D\left(\bigsqcup_{k=1}^{\infty} A_k\right) = D(\mathbb{N}) = \lim_{n\to\infty} \mathbb{P}_n(\mathbb{N}) = 1.$$

Therefore, we have

$$D\left(\bigsqcup_{k=1}^{\infty} A_k\right) = 1 \neq 0 = \sum_{k\in\mathbb{N}} D(A_k).$$

This shows D is not countably additive on $\mathscr{D}$.