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Tutorial 14: Random Processes

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Contents

We begin this lecture with a problem on stopping time.

Problem 1: Consider an iid sequence of random variables $(X_n)_{n\geq 1}$ such that

$$P(X_n = 1) = \alpha, \quad P(X_n = -1) = \beta,$$

where $0 < \alpha + \beta \leq 1$.

Let $i \in \mathbb{Z}$ be any fixed integer (could be positive, negative or zero), and let $S_0 = i$. For each $n \geq 1$, define S_n as

$$S_n := S_0 + \sum_{k=1}^{n} X_k.$$

Suppose $\alpha < \beta$. Define a new random variable N as follows: for some $j < i$,

$$N := \inf\{n \in \mathbb{N} : S_n = j\}.$$

Show that N is a stopping time with respect to the process $(X_n)_{n\geq 1}$. Further, show that $P(N < \infty) = 1$, and subsequently compute $E[N]$.
(Note: do not compute $E[N]$ first and then argue that $E[N] < \infty$ implies that $P(N < \infty) = 1$.)

Solution:
First, we note that $(X_n)_{n\geq 1}$ is an iid sequence, with $E[X_1] = \alpha - \beta$. Thus, by the strong law of large numbers, we have

$$\begin{aligned}\frac{S_n - S_0}{n} &= \frac{1}{n}\sum_{k=1}^{n} X_k \\ &\xrightarrow{\text{a.s.}} \alpha - \beta < 0.\end{aligned}$$

This implies that there exists a set A such that $P(A) = 1$, and for $\omega \in A$, we have

$$\frac{S_n(\omega) - S_0}{n} \longrightarrow \alpha - \beta < 0 \quad \text{as } n \to \infty.$$

This implies that

$$\lim_{n\to\infty} S_n(\omega) - S_0 = -\infty \quad \text{for all } \omega \in A.$$

Thus, with probability 1, $S_n - S_0$ will cross any negative integer value in finite time. In particular, this implies that for every $\omega \in A$, there exists a natural number $n^* = n^*(\omega)$ (n^* could depend on ω) such that $S_{n^*(\omega)}(\omega) - S_0 = j - i$ since $j - i$ is a negative integer, which in turn implies that $N(\omega) \leq n^*(\omega)$.

Thus, we have proved that for all $\omega \in A$, there exists $n^*(\omega) \in \mathbb{N}$ such that

$$N(\omega) \leq n^*(\omega) < \infty,$$

thereby resulting in the fact that occurrence of event A implies occurrence of the event $N < \infty$. That is, $A \subseteq \{N < \infty\}$, and since $1 = P(A) \leq P(N < \infty)$, we get $P(N < \infty) = 1$.

We now use Wald's lemma to compute $E[N]$. Towards this, we note that $S_N = j$ almost surely, and therefore, $E[S_N] = j$. By Wald's lemma, we have

$$E[S_N - S_0] = E\left[\sum_{k=1}^{N} X_k\right] = E[N] \cdot E[X_1].$$

Thus, we get

$$E[N] = \frac{E[S_N] - S_0}{E[X_1]} = \frac{j - i}{\alpha - \beta}.$$

14.1 A Recap of Results on Hitting Times and Recurrence

We now briefly recall some important results on hitting times and recurrence in the context of DTMCs. Let $(X_n)_{n\geq 0}$ be a DTMC on a countable state space $\mathcal{X}$, with transition probabilities $(p_{xy})_{x,y\in\mathcal{X}}$. For any state $x \in \mathcal{X}$, we denote by H_x the first hitting time of state x after time zero, i.e.,

$$H_x := \inf\{n \geq 1 : X_n = x\}.$$

Remark 1. Note that H_x takes values in the set $\{1, 2, \ldots\}$. In other words, if $X_0 = x$, this does not contribute to counting H_x as equal to 1.

Note that for each $n \geq 1$,

$$\begin{aligned}\{H_x = n\} &= \{X_1 \neq x, \ldots, X_{n-1} \neq x, X_n = x\} \\ &\in \sigma(X_1, \ldots, X_n),\end{aligned}$$

therefore implying that H_x is a stopping time with respect to the process $(X_n)_{n\geq 0}$.

Suppose $X_0 = x$. Then, we denote by $f_{xx}^{(n)}$ the probability of hitting x for the first time after n steps starting from x, i.e.,

$$f_{xx}^{(n)} = P(H_x = n | X_0 = x).$$

Analogously, we may define $f_{xy}^{(n)}$ for any two states $x, y \in \mathcal{X}$ as the probability of hitting state y for the first time after n time steps starting from x.

Remark 2. Note that $f_{xy}^{(n)}$ denotes the probability of hitting state y after n time steps for the first time starting from x, whereas $p_{xy}^{(n)}$ denotes the probability of being in state y at time n starting from x at time zero (not necessarily for the first time). Thus, for all $x, y \in \mathcal{X}$, we have $p_{xy}^{(n)} \geq f_{xy}^{(n)}$.

We also denote by f_{xx} the probability of ever hitting state x starting from x, i.e.,

$$f_{xx} = P(H_x < \infty | X_0 = x) = \sum_{n=1}^{\infty} P(H_x = n | X_0 = x) = \sum_{n=1}^{\infty} f_{xx}^{(n)}.$$

14.1.1 Recurrence and Transience

A state $x \in \mathcal{X}$ is called transient if $f_{xx} < 1$. A state x is called recurrent if $f_{xx} = 1$.

Remark 3. Note that for each $n \geq 1$, $0 \leq f_{xx}^{(n)} \leq 1$. Thus, if $f_{xx} = 1$, then $(f_{xx}^{(n)})_{n\geq 1}$ forms a probability distribution on the set $\{1, 2, \ldots\}$.

Following up on the above remark, when x is recurrent, we denote by μ_{xx} the mean recurrence time of state x, which is defined as the mean of the distribution $(f_{xx}^{(n)})_{n\geq 1}$, i.e,

$$\mu_{xx} = \sum_{n=1}^{\infty} n f_{xx}^{(n)}.$$

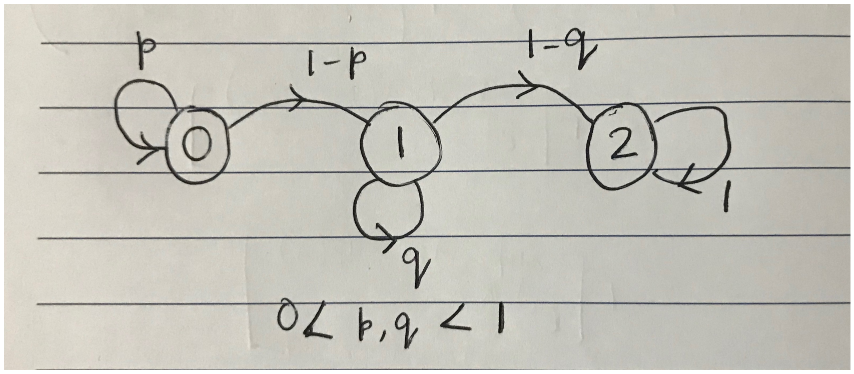
If $\mu_{xx} < \infty$, then x is called a positively recurrent state. If $\mu_{xx} = +\infty$, then x is called a null recurrent state.

Remark 4. Notice that for $x \in \mathcal{X}$ transient, since $f_{xx} < 1$, we have

$$0 < 1 - f_{xx} = P(H_x = +\infty | X_0 = x).$$

In other words, with a strictly positive probability, the Markov chain never visits state a transient state x. For a recurrent state x, $P(H_x = +\infty | X_0 = x) = 0$. Thus, the Markov chain always visits the state x in finite time with probability 1 whenever it starts from x.

Problem 2:
For $0 < p, q < 1$, consider a Markov chain $(X_n)_{n\geq 0}$ whose transition diagram is as shown below. Evaluate f_{00}, f_{11}, f_{22}, f_{01} and f_{12}.



Solution:
Note that $f_{00}^{(1)} = p$, and $f_{00}^{(n)} = 0$ for all $n \geq 2$. Hence, $f_{00} = p < 1$, implying that 0 is a transient state. Similarly, for each $n \geq 1$, $f_{01}^{(n)} = p^{n-1}(1-p)$, and therefore $f_{01} = 1$. The other quantities may be evaluated on similar lines.

We note here that remark 4 above suggests that on an average, the number of visits of a Markov chain to a transient state is finite, and that to a recurrent state is infinite. This is indeed the case as shown in the following problem.

Problem 3:
For any state $x \in \mathcal{X}$, let

$$N_x := \sum_{n=1}^{\infty} 1_{\{X_n = x\}}$$

denote the total number of times the Markov chain $(X_n)_{n\geq 0}$ visits state x. Show that the following are true.

1. $E[N_x | X_0 = x] = \sum_{n=1}^{\infty} p_{xx}^{(n)}$.
2. $E[N_x | X_0 = x] = \begin{cases} \frac{f_{xx}}{1-f_{xx}}, & f_{xx} < 1, \\ +\infty, & f_{xx} = 1. \end{cases}$
3. A state x is transient if and only if $\sum_{n=1}^{\infty} p_{xx}^{(n)} < \infty$.
4. A state x is recurrent if and only if $\sum_{n=1}^{\infty} p_{xx}^{(n)} = +\infty$.

Solution:

1. We have

$$\begin{aligned}
E[N_x|X_0 = x] &= E\left[\sum_{n=1}^{\infty} 1_{\{X_n=x\}} \middle| X_0 = x\right] \\
&\stackrel{(a)}{=} \sum_{n=1}^{\infty} E[1_{\{X_n=x\}}|X_0 = x] \\
&= \sum_{n=1}^{\infty} P(X_n = x|X_0 = x) \\
&= \sum_{n=1}^{\infty} p_{xx}^{(n)},
\end{aligned}$$

   where (a) above follows from linearity of expectations and monotone convergence theorem.

2. We first compute the conditional pmf of N_x, given $\{X_0 = x\}$, and then evaluate the desired conditional expectation. Towards this, suppose first that $x \in \mathcal{X}$ is transient. Then, $f_{xx} < 1$, and for any $m \geq 0$, we have

$$P(N_x = m|X_0 = x) = (f_{xx})^m(1 - f_{xx}),$$

   from which it follows that

$$\begin{aligned}
E[N_x|X_0 = x] &= \sum_{m=0}^{\infty} m\, P(N_x = m|X_0 = x) \\
&= \sum_{m=0}^{\infty} m(f_{xx})^m(1 - f_{xx}) \\
&= \frac{f_{xx}}{1 - f_{xx}}.
\end{aligned}$$

   Next, suppose $x \in \mathcal{X}$ is recurrent. Then, $f_{xx} = 1$, and it follows that for all $m \geq 0$,

$$P(N_x = m|X_0 = x) = (f_{xx})^m(1 - f_{xx}) = 0.$$

   Since N_x is a non-negative random variable, this implies that $P(N_x = +\infty|X_0 = x) = 1$, from which it follows that $E[N_x|X_0 = x] = +\infty$.

3. If x is transient, by the argument made above in part 2, it follows that $\sum_{n=1}^{\infty} p_{xx}^{(n)} < \infty$. Conversely, suppose that $\sum_{n=1}^{\infty} p_{xx}^{(n)} < \infty$. That is, $\sum_{n=1}^{\infty} P(X_n = x|X_0 = x) < \infty$. Defining

$$A_n := \{X_n = x\}, \quad n \geq 1,$$

   we have that $\sum_{n=1} P(A_n|X_0 = x) < \infty$. Thus, by the first part of Borel-Cantelli lemma, we get $P(A_n \text{ i.o.}|X_0 = x) = 0$, i.e.,

$$P(X_n = x \text{ i.o.}|X_0 = x) = 0.$$

   This implies that with probability 1, X_n does not visit the state x infinitely many times, which means that with probability 1, X_n visits state x only finitely many times. Therefore, we get

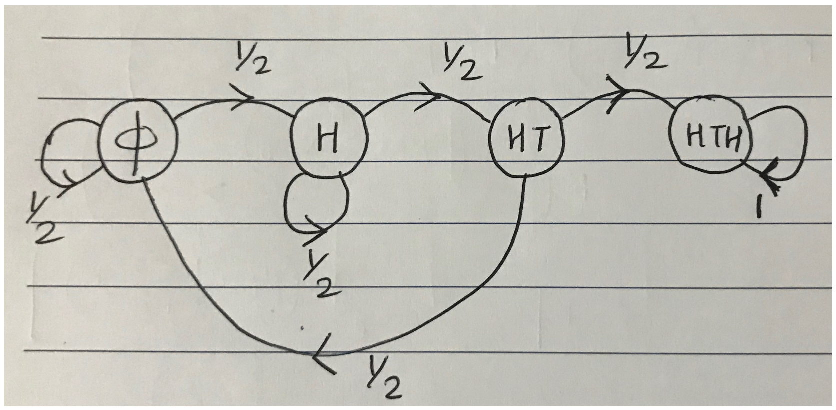
$$P(N_x < \infty|X_0 = x) = 1,$$

   from which it follows that $E[N_x|X_0 = x] < \infty$.

4. This is an immediate corollary of the result in part 3 above.

Problem 4:
A fair coin is tossed repeatedly and independently until the pattern "HTH" is observed for the first time. Find the expected number of tosses required.

Solution:
The above problem can be cast in the form of a problem on Markov chains (since every iid sequence trivially forms a Markov chain) with the transition diagram as shown below. In the figure, ϕ represents a null state to



which the Markov chain resets itself whenever an outcome of coin toss disrupts the desired pattern required. We wish to compute the expected number of steps required to go from state ϕ to state "HTH" in the transition diagram.

Towards this, we set up some notation. Let

$\psi(0)$ denote the expected number of steps required to go from state ϕ to state HTH,
$\psi(1)$ denote the expected number of steps required to go from state H to state HTH,
$\psi(2)$ denote the expected number of steps required to go from state HT to state HTH,
$\psi(3)$ denote the expected number of steps required to go from state HTH to state HTH.

We then have the following set of equations:

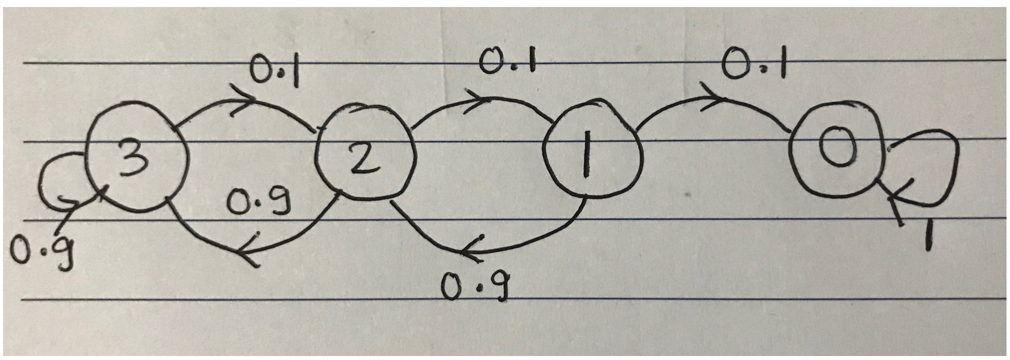
$$\begin{aligned}
\psi(3) &= 0 \\
\psi(2) &= \frac{1}{2}(1+\psi(0)) + \frac{1}{2}(1+\psi(3)) \\
\psi(1) &= \frac{1}{2}(1+\psi(1)) + \frac{1}{2}(1+\psi(2)) \\
\psi(0) &= \frac{1}{2}(1+\psi(0)) + \frac{1}{2}(1+\psi(1))
\end{aligned}$$

Solving the above set of equations, we get $\psi(0) = 10$.

Exercise: Does it require more tosses (in an expected sense) to observe the pattern "HHT" rather than "HTH" in the context of the above problem? Justify your answer. What is your answer if the bias of the coin is $p \in (0, 0.5)$?

Problem 5:
Imagine that you are playing a game that involves tossing a coin of bias 0.9 repeatedly and independently, with 3 lives at the start of the game. At each round, if the coin toss results in a tail, you lose one life. However, if the coin toss results in a head, you gain back your lost lives one at a time. If you have all 3 lives and the coin toss results in a head, nothing changes, and you continue tossing. You are allowed to play until you lose all your lives. What is the expected number of times you will play this game?

Solution:
The above problem can be cast as a problem on Markov chains with the transition diagram as shown below. Denoting by $\psi(i)$ the expected number of steps required to go from state i to state 0, where $i \in \{0, 1, 2, 3\}$,



we have the following set of equations:

$$\begin{aligned}
\psi(0) &= 0 \\
\psi(1) &= 0.9(1 + \psi(2)) + 0.1(1 + \psi(0)) \\
\psi(2) &= 0.9(1 + \psi(3)) + 0.1(1 + \psi(1)) \\
\psi(3) &= 0.9(1 + \psi(3)) + 0.1(1 + \psi(2)).
\end{aligned}$$

Noting that we wish to compute $\psi(3)$, by solving the above set of equations, we get $\psi(3) = 1020$.

14.2 Invariant Distributions / Stationary Distributions

Let $(X_n)_{n\geq 0}$ be a time homogeneous DTMC on a countable state space $\mathcal{X}$ with transition matrix $P = (p_{xy})_{x,y\in\mathcal{X}}$. Then, a probability distribution $\pi = (\pi_x)_{x\in\mathcal{X}}$ on $\mathcal{X}$ is called an invariant distribution or stationary distribution of the Markov chain $(X_n)_{n\geq 0}$ if it satisfies the relation

$$\pi = \pi P.$$

The above equation when applied recursively n times results in $\pi = \pi P^n$ for all $n \geq 1$.

The term "stationary" in stationary distribution is due to the following fact: if π is an invariant distribution, and the initial state $X_0 \sim \pi$, then the time homogeneous Markov chain $(X_n)_{n\geq 0}$ is a stationary process.

This can be easily checked as follows: first, we note that if $X_0 \sim \pi$, then we have

$$\begin{aligned} P(X_1 = y) &= \sum_{x \in \mathcal{X}} P(X_1 = y | X_0 = x) P(X_0 = x) \\ &= \sum_{x \in \mathcal{X}} \pi_x p_{xy} \\ &= \pi_y \end{aligned}$$

for all $y \in \mathcal{X}$. Thus, $X_1 \sim \pi$. The above argument may be extended by induction to conclude that $X_n \sim \pi$ for all $n \geq 1$. Next, it is an easy exercise to check using time homogeneity and Markov property of $(X_n)_{n \geq 0}$ that if $X_0 \sim \pi$, then

$$P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x) = P(X_k = x_0, \ldots, X_{n+k-1} = x_{n-1}, X_{n+k} = x)$$

for all $n \geq 1$, for all $k \geq 0$, and for all $x_0, \ldots, x_{n-1}, x \in \mathcal{X}$, thus implying that the process $(X_n)_{n \geq 0}$ is stationary.

We now introduce two more definitions, of irreducibility of the state space of a Markov process and period of a state.

Definition 14.2.1. (Irreducibility) The on state space $\mathcal{X}$ of a DTMC $(X_n)_{n \geq 0}$ is said to be irreducible if every state can be reached from every other state in finite time, i.e., for all $x, y \in \mathcal{X}$, there exists $n \geq 0$ such that $p_{xy}^{(n)} > 0$.

Remark 5. If the state space is not irreducible, it can always be partitioned into equivalence classes called communicating classes, such that in every communicating class, each state is reachable from every other state of the communicating class. Looked at this way, irreducibility is a way of saying that the entire state space constitutes a single communicating class.

Definition 14.2.2. (Period) Given a state $x \in \mathcal{X}$, its period is denoted by $d(x)$ and is defined as

$$d(x) := \gcd\{n \geq 1 : p_{xx}^{(n)} > 0\}.$$

Remark 6. It can be shown that $d(x)$ is also equal to the following:

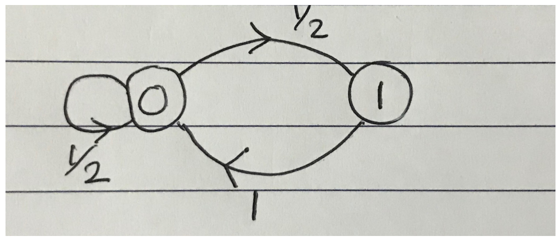
$$d(x) = \gcd\{n \geq 1 : f_{xx}^{(n)} > 0\}.$$

14.2.1 Some Important Results

We now list some important properties associated with invariant distributions, irreducibility, recurrence and periodicity.

1. Recurrence, transience and period are class properties, i.e., the states belonging to a communicating class are either all transient, all positive recurrent or all null recurrent. Also, all of them will have the same period.
2. Open communicating classes are transient. Finite closed communicating classes are positive recurrent.
3. A state x with $p_{xx}^{(1)} > 0$ will always have period 1.
4. An irreducible Markov chain is positive recurrent if and only if there exists a unique stationary distribution π such that $\pi_x > 0$ for all $x \in \mathcal{X}$. Furthermore, in this case, $\pi_x = \frac{1}{\mu_{xx}}$ for all $x \in \mathcal{X}$, where μ_{xx} denotes the mean recurrence time of state x.

Problem 6:
Consider a DTMC whose state transition diagram is as depicted below.



1. Is the Markov chain irreducible?
2. Is the Markov chain aperiodic?
3. Classify the states as transient, positive recurrent or null recurrent.
4. Compute a stationary distribution for this Markov chain.

Solution:

1. Yes.
2. Yes, since $p_{00}^{(1)} = 0.5 > 0$.
3. The state space is finite, and the Markov chain is irreducible. Hence, all the states are positive recurrent.
4. Let $\pi = (\pi_0, \pi_1)$ denote a stationary distribution. Then, the following relations hold:

$$\pi_0 = \frac{\pi_0}{2} + \pi_1$$
$$\pi_1 = \frac{\pi_0}{2}.$$

Solving, we get $\pi_0 = \frac{2}{3}$ and $\pi_1 = \frac{1}{3}$. Thus, we see that $\pi = \left(\frac{2}{3}, \frac{1}{3}\right)$ is the unique stationary distribution of the Markov chain. Also, it can be shown that $\mu_{00} = \frac{3}{2}$ and $\mu_{11} = 3$, thus verifying that $\pi_x = \frac{1}{\mu_{xx}}$ for $x = 0, 1$.

Problem 7:
Consider a DTMC on $\mathcal{X} = \{0, 1, 2, \ldots\}$ whose transition probabilities are given by

$$p_{0i} = \left(\frac{1}{2}\right)^i, \quad p_{i,i+1} = \frac{1}{2} = p_{i0}$$

for all $i \in \{1, 2, \ldots\}$.

1. Is the Markov chain irreducible?
2. What is $f_{00}^{(n)}$?

3. Evaluate μ_{ii} for all $i \in \{1, 2, \ldots\}$.

Solution

1. Yes.
2. We note that $f_{00}^{(1)} = 0$, and for any $n \geq 2$,

$$f_{00}^{(n)} = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{n-2} \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{n-1},$$

where the first term inside the summation corresponds to moving from state 0 to any state $i \in \{1, 2, \ldots\}$, the second term corresponds to making $n-2$ forward steps starting from state i, and the last term corresponds to making a backward step to state 0 from state $i+n-2$. Therefore, we get $f_{00} = 1$, implying that 0 and hence all states are recurrent.

Furthermore, we have

$$\mu_{00} = \sum_{n=2}^{\infty} n f_{00}^{(n)} = 3,$$

implying that 0 (and hence all states) are positively recurrent.

3. We note here that evaluating μ_{ii} by computing $f_{ii}^{(n)}$ is too cumbersome. Thus, we resort to computing μ_{ii} by means of stationary distributions. Recall that if a Markov chain is irreducible and positive recurrent, then it has a unique stationary distribution $\pi = (\pi_x)_{x \in \mathcal{X}}$ with the property that $\pi_x > 0$ for all $x \in \mathcal{X}$, and $\pi_x = \frac{1}{\mu_{xx}}$. We thus find the stationary distribution, which then yields μ_{ii} for all $i \in \{1, 2, \ldots\}$.

Towards computing the stationary distribution, we have the following set of equations:

$$\begin{aligned} \pi_0 &= \frac{1}{\mu_{00}} = \frac{1}{3}, \\ \pi_1 &= \frac{\pi_0}{2} = \frac{1}{6}, \\ \pi_i &= \left(\frac{1}{2}\right)^i \pi_0 + \frac{1}{2}\pi_{i-1}, \quad i \geq 2, \end{aligned}$$

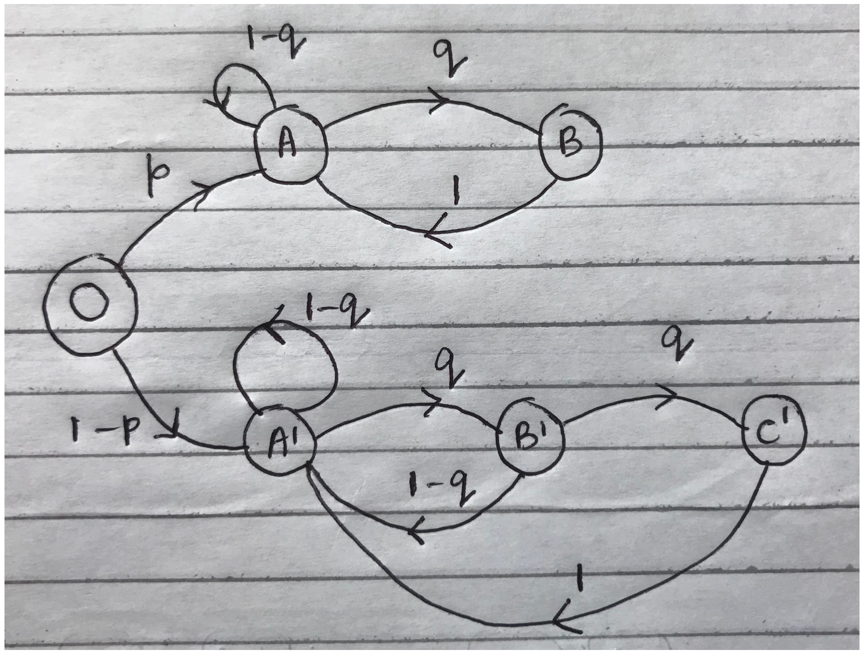
whence upon solving the last equation above, we get $\pi_i = i\left(\frac{1}{2}\right)^i \pi_0$ for all $i \in \{2, 3, \ldots\}$. Thus, we have

$$\mu_{ii} = \frac{1}{\pi_i} = \frac{2^i}{i\pi_0}, \quad i \geq 2.$$

Problem 8: Consider a Markov chain whose transition diagram is as shown in the figure below. Here, $0 < p, q < 1$.

1. Is the Markov chain irreducible?
2. Does there exist a unique stationary distribution? If not, give examples of at least two such distributions.

Solution:



1. No. The Markov chain is not irreducible because starting from state 'A', for all $n \in \mathbb{N}$, we have $p_{A0}^{(n)} = 0$.
2. We note that
$$\mathcal{C}_1 = \{0\}, \quad \mathcal{C}_2 = \{A, B\}, \quad \mathcal{C}_3 = \{A', B', C'\}$$
constitute three finite communicating classes. Further, $\mathcal{C}_1$ is open while $\mathcal{C}_2$ and $\mathcal{C}_3$ are closed, hence implying that state 0 is transient and the remaining states are positive recurrent.

   Since each of $\mathcal{C}_2$ and $\mathcal{C}_3$ is irreducible and positive recurrent, it individually admits a unique stationary distribution. Let $\pi_{\mathcal{C}_2}$ and $\pi_{\mathcal{C}_3}$ denote respectively the stationary distributions of $\mathcal{C}_2$ and $\mathcal{C}_3$. Then, we have
$$\pi_{\mathcal{C}_2}(A) = (1-q)\pi_{\mathcal{C}_2}(A) + \pi_{\mathcal{C}_2}(B), \quad \pi_{\mathcal{C}_2}(A) + \pi_{\mathcal{C}_2}(B) = 1,$$
from which it follows that
$$\pi_{\mathcal{C}_2}(A) = \frac{1}{1+q}, \quad \pi_{\mathcal{C}_2}(B) = \frac{q}{1+q}.$$
Similarly, we have
$$\begin{aligned}
\pi_{\mathcal{C}_3}(A') &= (1-q)\pi_{\mathcal{C}_3}(A') + (1-q)\pi_{\mathcal{C}_3}(B') + \pi_{\mathcal{C}_3}(C') \\
\pi_{\mathcal{C}_3}(B') &= q\pi_{\mathcal{C}_3}(A') \\
\pi_{\mathcal{C}_3}(C') &= q\pi_{\mathcal{C}_3}(B') \\
\pi_{\mathcal{C}_3}(A') + \pi_{\mathcal{C}_3}(B') + \pi_{\mathcal{C}_3}(C') &= 1,
\end{aligned}$$
from which we get
$$\pi_{\mathcal{C}_3}(A') = \frac{1}{1+q+q^2}, \quad \pi_{\mathcal{C}_3}(B') = \frac{q}{1+q+q^2}, \quad \pi_{\mathcal{C}_3}(C') = \frac{q^2}{1+q+q^2}.$$
We now observe for all $\alpha_1, \alpha_2 \in [0,1]$, with $\alpha_1+\alpha_2 = 1$, the probability vector $\pi = (\pi_0, \pi_A, \pi_B, \pi_{A'}, \pi_{B'}, \pi_{C'})$ given by
$$\pi = \left(0, \frac{\alpha_1}{1+q}, \frac{\alpha_1 q}{1+q}, \frac{\alpha_2}{1+q+q^2}, \frac{\alpha_2 q}{1+q+q^2}, \frac{\alpha_2 q^2}{1+q+q^2}\right)$$

constitutes a stationary distribution for the Markov chain. Thus, the stationary distribution is not unique.

Problem 9:
Consider a discrete-time single-server queue with arbitrarily large buffer space. Let the queue state at time n be denoted by Q_n, where $n \in \{0, 1, 2, \ldots\}$. At each $n \geq 1$, there is an independent arrival of into the queue denoted by the random variable $A_n \in \{0, 2\}$, where the arrival probability is $p \in (0, 0.5)$. That is $(A_n : n \in \mathbb{N})$ is an iid process, with each A_n being independent of the queue state at time $n-1$. There is a unit service in each time slot, and hence the evolution of the single-server queue can be written as

$$Q_n = \max\{0, Q_{n-1} + A_n - 1\}, \ n \in \mathbb{N}.$$

Show that the queue state process $(Q_n : n \in \mathbb{N})$ is a time homogeneous discrete time Markov chain that is irreducible, aperiodic, and positive recurrent.

Solution:
We first show that $(Q_n)_{n \geq 0}$ is a time homogeneous DTMC by using the random representation mapping theorem. Towards this, let $f(x, y) := \max\{0, x + y - 1\}$. Thus, we have $Q_n = f(Q_{n-1}, A_n)$ for all $n \in \mathbb{N}$. Now, for all valid $q, q' \in \{0, 1, 2, \ldots\}$, we have

$$\begin{aligned} P(Q_1 = q' | Q_0 = q) &= P(f(q, A_1) = q' | Q_0 = q) \\ &= P(f(q, A_1) = q'), \end{aligned}$$

where the last line follows since A_1 is independent of Q_0. Similarly, we note that for all $n \geq 1$ and for all valid $q_0, \ldots, q_{n-1}, q, q' \in \{0, 1, 2, \ldots\}$, we have

$$\begin{aligned} &P(Q_{n+1} = q' | Q_n = q, Q_{n-1} = q_{n-1}, \ldots, Q_1 = q_1, Q_0 = q_0) \\ &= P(f(Q_n, A_{n+1}) = q' | Q_n = q, Q_{n-1} = q_{n-1}, \ldots, Q_1 = q_1, Q_0 = q_0) \\ &= f(f(q, A_{n+1}) = q' | Q_n = q, Q_{n-1} = q_{n-1}, \ldots, Q_1 = q_1, Q_0 = q_0) \\ &\overset{(a)}{=} P(f(q, A_{n+1}) = q') \\ &\overset{(b)}{=} P(f(q, A_1) = q') \\ &= P(Q_1 = q' | Q_0 = q), \end{aligned}$$

where (a) above follows from independence of A_{n+1} and $Q_0, \ldots, Q_{n-1}$, and (b) follows from the fact that $(A_n)_{n \geq 1}$ is an iid sequence of random variables. Thus, we have proved that $(Q_n)_{n \geq 0}$ is a time homogeneous DTMC on the state space $\mathcal{X} = \{0, 1, 2, \ldots\}$.

Next, we note that for all $i \in \{1, 2, \ldots\}$,

$$\begin{aligned} p_{i,i+1} &= p, \quad p_{i,i-1} = 1 - p, \\ p_{0,1} &= p, \quad p_{0,0} = 1 - p. \end{aligned}$$

Thus, we can transition from state 0 to any state $i \in \{1, 2, \ldots\}$ in n steps with probability

$$p_{0i}^{(n)} = p^n > 0.$$

Similarly, we can transition from any state $i \in \{1, 2, \ldots\}$ to state 0 in n steps with probability

$$p_{i0}^{(n)} = (1 - p)^n > 0.$$

Hence, it follows that there is a single communicating class in the state space, and therefore the Markov chain is irreducible. Furthermore, since $p_{00}^{(1)} = 1 - p > 0$, the Markov chain is aperiodic.

To show positive recurrence, we find the unique positive stationary distribution for this Markov chain. We see that the following distribution $\pi = (\pi_i : i \in \{0, 1, 2, \ldots\})$ that satisfies

$$\pi_i = \left(\frac{1-2p}{1-p}\right)\left(\frac{p}{1-p}\right)^i, \quad i \in \{0, 1, 2, \ldots\},$$

is a stationary distribution distribution for the Markov chain. Since this is a positive distribution, the Markov chain is positive recurrent.