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Stochastic Processes

Hitting Times and Recurrence, Transience, Positive/Null Recurrence

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Hitting Times

Definition (Hitting Times)

Let $\{X_n\}_{n=1}^{\infty}$ be DTMC on a discrete state space $\mathcal{X}$ with TPM P.
Fix $y \in \mathcal{X}$.
Let $\tau_y^{(0)} := 0$, and

$$\tau_y^{(k)} = \inf\{n > \tau_y^{(k-1)} : X_n = y\}, \qquad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, the random variable $\tau_y^{(k)}$ is called the "kth hitting time of state y".

Exercise:
For each $k \in \mathbb{N}$, verify that $\{\tau_y^k = n\} \in \sigma(X_1, \ldots, X_n)$ for all n.

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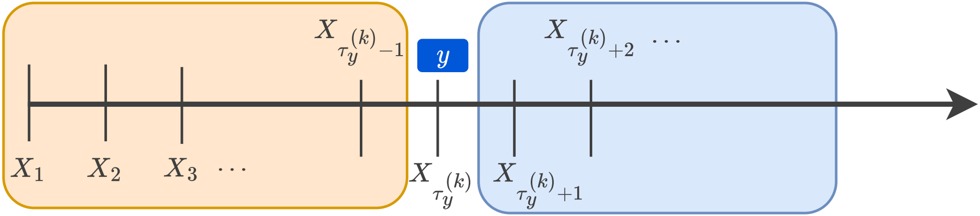
An Important Observation Regarding $\tau_y^{(k)}$

Lemma (An Important Observation Regarding $\tau_y^{(k)}$)

For each $k \in \mathbb{N}$, suppose that $\mathbb{P}(\tau_y^{(k)} < +\infty) = 1$.
Then, the history up to $\tau_y^{(k)}$ is independent of the future unconditionally, i.e.,

$$(X_1, \ldots, X_{\tau_y^{(k)}-1}) \perp\!\!\!\perp (X_{\tau_y^{(k)}+1}, X_{\tau_y^{(k)}+2}, \ldots).$$



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Proof of Lemma:

- According to strong Markov property,

$$(X_1, \ldots, X_{\tau_y^{(k)}-1}) \perp\!\!\!\perp (X_{\tau_y^{(k)}+1}, X_{\tau_y^{(k)}+2}, \ldots) \mid X_{\tau_y^{(k)}}.$$

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- However, $X_{\tau_y^{(k)}}$ is a constant random variable (taking the constant value y).
  Thus, conditioning on $X_{\tau_y^{(k)}}$ is as good as not conditioning at all.

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IID Block Structure

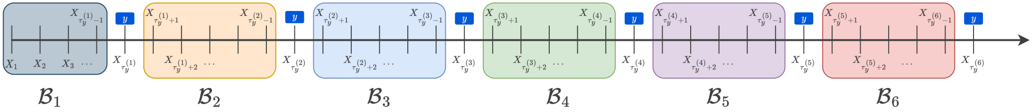
Lemma (IID Block Structure)

For each $k \in \mathbb{N} \cup \{0\}$, define the kth block $\mathcal{B}_k$ as

$$\mathcal{B}_k := \left(X_{\tau_y^{(k-1)}+1}, X_{\tau_y^{(k-1)}+2}, \ldots, X_{\tau_y^{(k)}-1}\right), \qquad k \in \mathbb{N} \cup \{0\}.$$

Suppose that $\mathbb{P}(\tau_y^{(k)} < +\infty) = 1$ for all k. Then, the following holds.

1. The collection $\{B_1, B_2, B_3, \ldots\}$ is independent.
2. The collection $\{B_2, B_3, \ldots\}$ is identically distributed.



Proof of Lemma

- $\left(X_1, \ldots, X_{\tau_y^{(k)}-1}\right) \perp\!\!\!\perp \left(X_{\tau_y^{(k)}+1}, X_{\tau_y^{(k)}+1}, \ldots\right) \implies B_k \perp\!\!\!\perp B_{k+1}$

Proof of Lemma

- $(X_1, \ldots, X_{\tau_y^{(k)}-1}) \perp\!\!\!\perp (X_{\tau_y^{(k)}+1}, X_{\tau_y^{(k)}+1}, \ldots) \implies B_k \perp\!\!\!\perp B_{k+1}$
- To show identical distribution of $\{B_2, B_3, \ldots\}$, it suffices to show that

$$X_{\tau_y^{(1)}+1} \stackrel{d.}{=} X_{\tau_y^{(2)}+1} \stackrel{d.}{=} X_{\tau_y^{(3)}+1} \stackrel{d.}{=} X_{\tau_y^{(4)}+1} \cdots$$

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- For any $k \in \mathbb{N}$, we have

$$\mathbb{P}(X_{\tau_y^{(k)}+1} = j) = \mathbb{P}(X_{\tau_y^{(k)}+1} = j \mid X_{\tau_y^{(k)}} = y)$$

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$$\begin{aligned} \mathbb{P}(X_{\tau_y^{(k)}+1} = j) &= \mathbb{P}(X_{\tau_y^{(k)}+1} = j \mid X_{\tau_y^{(k)}} = y) \\ &= \mathbb{P}(X_2 = j \mid X_1 = y) \end{aligned}$$

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Recurrence Times

Definition (Recurrence Time)

Let $\{X_n\}_{n=1}^{\infty}$ be DTMC on a discrete state space $\mathcal{X}$ with TPM P.
Fix $y \in \mathcal{X}$.
Let $\tau_y^{(0)} := 0$, and

$$\tau_y^{(k)} = \inf\{n > \tau_y^{(k-1)} : X_n = y\}, \qquad k \in \mathbb{N}.$$

The kth recurrence time associated with state y is defined as

$$H_y^{(k)} := \tau_y^{(k)} - \tau_y^{(k-1)}, \qquad k \in \mathbb{N}.$$

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$$H_y^{(k)} := \tau_y^{(k)} - \tau_y^{(k-1)}, \qquad k \in \mathbb{N}.$$

Note: $H_y^{(k)}$ is the (random) length of block $\mathcal{B}_k$.

I.I.D. Property of Recurrence Times

Lemma (I.I.D. Property of Recurrence Times)

Suppose that $\mathbb{P}(\tau_y^{(k)} < +\infty) = 1$ for all $k \in \mathbb{N}$.
Then, $\{H_y^{(k)}\}_{k=1}^{\infty}$ is a sequence of i.i.d. random variables.

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Proof of Lemma:

- We note that

$$\{H_y^{(k)} = n\} \in \sigma\left(X_{\tau_y^{(k-1)}+1}, \ldots, X_{\tau_y^{(k-1)}+n}\right)$$

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- We note that
$$\{H_y^{(k)} = n\} \in \sigma\left(X_{\tau_y^{(k-1)}+1}, \ldots, X_{\tau_y^{(k-1)}+n}\right)$$
- The result follows from the IID nature of the blocks $\{B_2, B_3, \ldots\}$.

Some Tidbits

- For $x, y \in \mathcal{X}$, let

$$f_{xy}^{(n)} := \mathbb{P}(\tau_y^{(1)} = n \mid X_1 = x).$$

$f_{xy}^{(n)}$: **probability of first visit to state y at time n, starting from state x.**

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- Let f_{xy} be defined as
$$f_{xy} = \mathbb{P}(\tau_y^{(1)} < +\infty \mid X_1 = x) = \sum_{n \in \mathbb{N}} f_{xy}^{(n)}.$$
f_{xy}: **probability of eventually visiting state y, starting from state x.**
- If $f_{xy} = 1$ for all $x \in \mathcal{X}$, then $\mathbb{P}(\tau_y^{(1)} < +\infty) = 1$, and hence $\tau_y^{(1)}$ is a stopping time.

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- If $f_{xy} = 1$ for all $x \in \mathcal{X}$, then $\mathbb{P}(\tau_y^{(1)} < +\infty) = 1$, and hence $\tau_y^{(1)}$ is a stopping time.
- $1 - f_{xy}$: probability that starting from x, the state y is **never** visited

Recurrent and Transient States

Definition (Recurrent and Transient States)

A state $x \in \mathcal{X}$ is called recurrent if $f_{xx} = 1$.
If $f_{xx} < 1$, then x is called a transient state.

Remarks:

- The collection
$$\{f_{xx}^{(1)}, f_{xx}^{(2)}, \ldots, \quad 1 - f_{xx}\}$$
defines a valid PMF on $\mathbb{N} \cup \{+\infty\}$.

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- The collection
$$\{f_{xx}^{(1)}, f_{xx}^{(2)}, \ldots, \quad 1 - f_{xx}\}$$
defines a valid PMF on $\mathbb{N} \cup \{+\infty\}$.
- The above PMF is called first recurrence time distribution.

Mean Recurrence Time

Definition (Mean Recurrence Time)

The mean recurrence time of a state $x \in \mathcal{X}$ is denoted by μ_{xx} and is defined by

$$\mu_{xx} := \mathbb{E}[\tau_x^{(1)} \mid X_1 = x].$$

Remarks:

- If x is transient, then $\mu_{xx} = +\infty$.

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Remarks:

- If x is transient, then $\mu_{xx} = +\infty$.
- If x is recurrent, then

$$\mu_{xx} = \sum_{n \in \mathbb{N}} n f_{xx}^{(n)}.$$

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Positive Recurrent and Null Recurrent States

Definition (Positive / Null Recurrent States)

A recurrent state $x \in \mathcal{X}$ is called positive recurrent if $\mu_{xx} < +\infty$.
Else, if $\mu_{xx} = +\infty$, then x is called null recurrent.

Proposition

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Fix $x, y \in \mathcal{X}$. Then,

$$\mathbb{P}(\text{state } y \text{ is visited exactly } k \text{ times} \mid X_1 = x) = \begin{cases} \quad , & k = 0, \\ \quad , & k \in \mathbb{N}. \end{cases}$$

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Proof of Proposition $(k \in \mathbb{N})$

$\mathbb{P}(\text{state } y \text{ is visited exactly } k \text{ times} \mid X_1 = x)$

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$$\begin{aligned} &\mathbb{P}(\text{state } y \text{ is visited exactly } k \text{ times} \mid X_1 = x) \\ = \quad &\mathbb{P}(\tau_y^{(1)} < +\infty,\ \tau_y^{(2)} < +\infty,\ \ldots,\ \tau_y^{(k)} < +\infty,\ \tau_y^{(k+1)} = +\infty \mid X_1 = x) \end{aligned}$$

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= \quad & \mathbb{P}(\tau_y^{(1)} < +\infty,\ \tau_y^{(2)} - \tau_y^{(1)} < +\infty,\ \ldots,\ \tau_y^{(k)} - \tau_1^{(k-1)} < +\infty,\ \tau_y^{(k+1)} - \tau_1^{(k)} = +\infty \mid X_1 = x)
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= \quad & \mathbb{P}(H_y^{(1)} < +\infty \mid X_1 = x) \times \left(\prod_{\ell=2}^{k} \mathbb{P}(H_y^{(\ell)} < +\infty) \right) \times \mathbb{P}(H_y^{(k+1)} = +\infty)
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= \quad & f_{xy} \times \left(\prod_{\ell=2}^{k} f_{yy} \right) \times (1 - f_{yy}).
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Corollary to Proposition

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Fix $x, y \in \mathcal{X}$. Then,

$$\mathbb{P}(\text{state } y \text{ is visited exactly } k \text{ times} \mid X_1 = x) = \begin{cases} 1 - f_{xy}, & k = 0, \\ f_{xy} f_{yy}^{k-1}(1 - f_{yy}), & k \in \mathbb{N}. \end{cases}$$

Corollary

For any $x, y \in \mathcal{X}$,

$$\mathbb{P}(\text{state } y \text{ is visited finitely many times} \mid X_1 = x) =$$

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Corollary

For any $x, y \in \mathcal{X}$,

$$\mathbb{P}(\text{state } y \text{ is visited finitely many times} \mid X_1 = x) = \begin{cases} 1, & f_{yy} < 1, \\ 1 - f_{xy}, & f_{yy} = 1. \end{cases}$$