

REAL ANALYSIS BASICS, POINTWISE CONVERGENCE, ALMOST-SURE
CONVERGENCE

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $\lim_{n \rightarrow \infty} a_n = L$, for some $L \in \mathbb{R}$. Prove that the sequence must be bounded, i.e., there exists a finite number $M > 0$ such that

$$|a_n| \leq M \quad \forall n \in \mathbb{N}.$$

The above exercise proves that every convergent sequence is bounded.

Is the converse true? That is, is every bounded sequence convergent? Prove or give a counterexample.

2. Prove formally that if $\lim_{n \rightarrow \infty} a_n = L$ for some $L \in \mathbb{R}$, $L \neq 0$, then $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$.
3. Formally determine the lim sup and lim inf of the following sequences.

(a) $a_n = n \sin\left(\frac{n\pi}{2}\right)$ for all $n \in \mathbb{N}$.

(b) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots$

In the above sequence, first term is $\frac{1}{2}$, second and third terms are with denominator 3 and numerators 1, 2, next three terms are with denominator 4 and numerator 1, 2, 3, but we do not write $2/4$ as it is already covered before as $1/2$. Similarly, next few terms are with denominator 5, with numerator spanning 1, 2, 3, 4, only listing elements not covered before. The sequence goes on this way.

4. Fix $a \in \mathbb{R}$. For each $n \in \mathbb{N}$, let

$$A_n := \left(-\infty, a - \frac{1}{n}\right), \quad B_n := \left(-\infty, a + \frac{1}{n}\right), \quad C_n := \left(-\infty, a - \frac{1}{n}\right], \quad D_n := \left(-\infty, a + \frac{1}{n}\right].$$

Determine $\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} B_n$, $\liminf_{n \rightarrow \infty} C_n$, and $\limsup_{n \rightarrow \infty} D_n$.

5. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables, all defined on a common measurable space (Ω, \mathcal{F}) . Let

$$X^* := \limsup_{n \rightarrow \infty} X_n.$$

In this exercise, we will formally prove that $X^* : \Omega \rightarrow [-\infty, +\infty]$ is a random variable with respect to \mathcal{F} .

- (a) For each $x \in \mathbb{R}$, prove formally that $(X^*)^{-1}((x, \infty)) \in \mathcal{F}$.
- (b) Prove formally that $\{\omega \in \Omega : X^*(\omega) = -\infty\} \in \mathcal{F}$.
- (c) Prove formally that $\{\omega \in \Omega : X^*(\omega) = +\infty\} \in \mathcal{F}$.

6. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1], \mathcal{B}((0, 1]), \text{Unif})$.

- (a) For each $n \in \mathbb{N}$, let

$$X_n(\omega) = n\omega - \lfloor n\omega \rfloor, \quad \omega \in \Omega.$$

Here, $\lfloor x \rfloor$ denotes the smallest integer lesser than or equal to x (i.e., “floor” of x).

Does $\{X_n\}_{n=1}^{\infty}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.

- (b) For each $n \in \mathbb{N}$, let

$$Y_n(\omega) = n^2 \omega \mathbf{1}_{(0, \frac{1}{n})}(\omega), \quad \omega \in \Omega.$$

Does $\{Y_n\}_{n=1}^{\infty}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.

- (c) For each $n \in \mathbb{N}$, let

$$Z_n(\omega) = \sin(2\pi n\omega), \quad \omega \in \Omega.$$

Does $\{Z_n\}_{n=1}^{\infty}$ have pointwise and/or almost-sure limits? Justify and identify the limits, if any.