Stochastic Processes

$\liminf$, $\limsup$, $\lim$ of Sequences of Random Variables, Convergence Notions

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

21 January 2025

Dedication

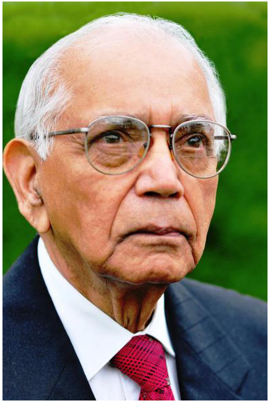

Figure: Prof. Calyampudi Radhakrishna Rao, FRS (1920-2023).

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$\liminf$, $\limsup$, $\lim$ of Sequence of Random Variables

$\liminf$ of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition ($\liminf$ of Sequence of Random Variables)

The limit infimum of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X_\star : \Omega \to [-\infty, +\infty]$ such that

$$X_\star(\omega) = \sup_{n \geq 1} \inf_{k \geq n} X_k(\omega) \qquad \forall \omega \in \Omega.$$

Notation: $\liminf_{n\to\infty} X_n$.

$\liminf$ of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition ($\liminf$ of Sequence of Random Variables)

The limit infimum of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X_\star : \Omega \to [-\infty, +\infty]$ such that

$$X_\star(\omega) = \sup_{n \geq 1} \inf_{k \geq n} X_k(\omega) \qquad \forall \omega \in \Omega.$$

Notation: $\liminf_{n\to\infty} X_n$.

Lemma

$X_\star = \liminf_{n\to\infty} X_n$ is a random variable w.r.t. $\mathscr{F}$.

Proof of Lemma

Lemma

$X_\star = \liminf_{n\to\infty} X_n : \Omega \to [-\infty, +\infty]$ is a random variable w.r.t. $\mathscr{F}$.

Proof of Lemma

Lemma

$X_\star = \liminf_{n\to\infty} X_n : \Omega \to [-\infty, +\infty]$ is a random variable w.r.t. $\mathscr{F}$.

- Inverse image of sets of the form $(-\infty, x)$ for $x \in \mathbb{R}$:

$$X_\star^{-1}((-\infty, x)) = \left\{ \sup_{n\geq 1} \inf_{k\geq n} X_k < x \right\}$$

Proof of Lemma

Lemma

$X_\star = \liminf_{n\to\infty} X_n : \Omega \to [-\infty, +\infty]$ is a random variable w.r.t. $\mathscr{F}$.

- Inverse image of sets of the form $(-\infty, x)$ for $x \in \mathbb{R}$:

$$X_\star^{-1}((-\infty, x)) = \left\{ \sup_{n\geq 1} \inf_{k\geq n} X_k < x \right\}$$

- Inverse image of $\{-\infty\}$:

$$X_\star^{-1}(\{-\infty\}) = \left\{ \sup_{n\geq 1} \inf_{k\geq n} X_k = -\infty \right\}$$

Proof of Lemma

Lemma

$X_\star = \liminf_{n\to\infty} X_n : \Omega \to [-\infty, +\infty]$ is a random variable w.r.t. $\mathscr{F}$.

- Inverse image of sets of the form $(-\infty, x)$ for $x \in \mathbb{R}$:

$$X_\star^{-1}((-\infty, x)) = \left\{ \sup_{n\geq 1} \inf_{k\geq n} X_k < x \right\}$$

- Inverse image of $\{-\infty\}$:

$$X_\star^{-1}(\{-\infty\}) = \left\{ \sup_{n\geq 1} \inf_{k\geq n} X_k = -\infty \right\}$$

- Inverse image of $\{+\infty\}$:

$$X_\star^{-1}(\{-\infty\}) = \left\{ \sup_{n\geq 1} \inf_{k\geq n} X_k = +\infty \right\}$$

$\limsup$ of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition ($\limsup$ of Sequence of Random Variables)

The limit supremum of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X^\star : \Omega \to [-\infty, +\infty]$ such that

$$X^\star(\omega) = \inf_{n \geq 1} \sup_{k \geq n} X_k(\omega) \qquad \forall \omega \in \Omega.$$

Notation: $\limsup_{n \to \infty} X_n$.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$\limsup$ of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition ($\limsup$ of Sequence of Random Variables)

The limit supremum of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X^\star : \Omega \to [-\infty, +\infty]$ such that

$$X^\star(\omega) = \inf_{n\geq 1} \sup_{k\geq n} X_k(\omega) \qquad \forall \omega \in \Omega.$$

Notation: $\limsup_{n\to\infty} X_n$.

Lemma

$X^\star = \limsup_{n\to\infty} X_n$ is a random variable w.r.t. $\mathscr{F}$.

Pointwise Limit of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Pointwise Limit of Sequence of Random Variables)

The pointwise limit of sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X : \Omega \to [-\infty, +\infty]$ such that

$$\liminf_{n\to\infty} X_n(\omega) = X(\omega) = \limsup_{n\to\infty} X_n(\omega) \qquad \forall \omega \in \Omega.$$

Notation: $\lim_{n\to\infty} X_n$.

Pointwise Limit of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Pointwise Limit of Sequence of Random Variables)

The pointwise limit of sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X : \Omega \to [-\infty, +\infty]$ such that

$$\liminf_{n\to\infty} X_n(\omega) = X(\omega) = \limsup_{n\to\infty} X_n(\omega) \qquad \forall \omega \in \Omega.$$

Notation: $\lim_{n\to\infty} X_n$.

Lemma

$X = \lim_{n\to\infty} X_n$, if it exists, is a random variable w.r.t. $\mathscr{F}$.

Examples

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} 1, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise}, \end{cases} \qquad \omega \in [0,1].$$

  Identify the pointwise limit.

Examples

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} 1, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise}, \end{cases} \qquad \omega \in [0,1].$$

  Identify the pointwise limit.

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \omega^n, \qquad \omega \in \Omega.$$

  Identify the limit RV X.

Examples

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

- Moving Rectangles

  Let $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

  $X_1 = \mathbf{1}_{[0,1]}$

  $X_2 = \mathbf{1}_{[0,\frac{1}{2}]}, \quad X_3 = \mathbf{1}_{[\frac{1}{2},1]}$

  $X_4 = \mathbf{1}_{[0,\frac{1}{4}]}, \quad X_5 = \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}, \quad X_6 = \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]}, \quad X_7 = \mathbf{1}_{[\frac{3}{4},1]},$ and so on.

Examples

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

- Moving Rectangles

  Let $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

  $X_1 = \mathbf{1}_{[0,1]}$

  $X_2 = \mathbf{1}_{[0,\frac{1}{2}]}, \quad X_3 = \mathbf{1}_{[\frac{1}{2},1]}$

  $X_4 = \mathbf{1}_{[0,\frac{1}{4}]}, \quad X_5 = \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}, \quad X_6 = \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]}, \quad X_7 = \mathbf{1}_{[\frac{3}{4},1]}, \quad$ and so on.

Examples

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

- Moving Rectangles
  Let $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.
  $X_1 = \mathbf{1}_{[0,1]}$
  $X_2 = \mathbf{1}_{[0,\frac{1}{2}]}, \quad X_3 = \mathbf{1}_{[\frac{1}{2},1]}$
  $X_4 = \mathbf{1}_{[0,\frac{1}{4}]}, \quad X_5 = \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}, \quad X_6 = \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]}, \quad X_7 = \mathbf{1}_{[\frac{3}{4},1]}, \quad$ and so on.

Note

The above sequence of random variables does not admit any pointwise limit.

Examples

- For each $n \in \mathbb{N}$, let
$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$

Examples

- For each $n \in \mathbb{N}$, let

$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$

Remarks on the Above Example

- In cases where $(\Omega, \mathscr{F}, \mathbb{P})$ and the sequence $\{X_n\}_{n=1}^{\infty}$ are not explicitly specified, it is not possible to identify the pointwise limit.
- In such cases, we start with a guess for the limit RV and prove convergence.

Pointwise Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Pointwise Convergence)

Given a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ and a random variable X, all defined w.r.t. $\mathscr{F}$, we say that the sequence converges pointwise to X if

$$\lim_{n \to \infty} X_n(\omega) = X(\omega) \qquad \forall \omega \in \Omega.$$

Notation:

$$X_n \xrightarrow{\text{pointwise}} X$$

Pointwise Convergence the Only Possibility?

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} (-1)^n, & \omega = 0, \\ \omega^n, & \omega \in (0,1), \\ 0, & \omega = 1. \end{cases}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Pointwise Convergence the Only Possibility?

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} (-1)^n, & \omega = 0, \\ \omega^n, & \omega \in (0,1), \\ 0, & \omega = 1. \end{cases}$$

Note

- In the above example,

$$\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\} = (0,1] \neq \Omega.$$

- Intuitively, the constant RV 0 is a limit, but in what sense? How to capture this limit?

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Going Beyond Pointwise Convergence

An Important Set and its Measurability

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

An Important Set and its Measurability

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Lemma

$$A_{\lim} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} \in \mathscr{F}.$$

Thus, we may assign probability to $A_{\lim}$.

Proof of Lemma

Lemma

$$A_{\lim} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} \in \mathscr{F}.$$

$$\omega \in A_{\lim} \implies \lim_{n\to\infty} X_n(\omega) = X(\omega)$$

Proof of Lemma

Lemma

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} \in \mathscr{F}.$$

$$\begin{aligned} \omega \in A_{\lim} &\implies \lim_{n\to\infty} X_n(\omega) = X(\omega) \\ &\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \ \text{ such that } \ |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \end{aligned}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Proof of Lemma

Lemma

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} \in \mathscr{F}.$$

$$\begin{aligned}
\omega \in A_{\lim} &\implies \lim_{n\to\infty} X_n(\omega) = X(\omega) \\
&\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \\
&\implies \forall q \in \mathbb{Q}_+,\ \exists N_q(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < q \ \ \forall n \geq N_q(\omega)
\end{aligned}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Proof of Lemma

Lemma

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} \in \mathscr{F}.$$

$$\begin{aligned}
\omega \in A_{\lim} &\implies \lim_{n\to\infty} X_n(\omega) = X(\omega) \\
&\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \\
&\implies \forall q \in \mathbb{Q}_+,\ \exists N_q(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < q \ \ \forall n \geq N_q(\omega) \\
&\implies \omega \in \bigcap_{q \in \mathbb{Q}_+}
\end{aligned}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Proof of Lemma

Lemma

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} \in \mathscr{F}.$$

$$\begin{aligned}
\omega \in A_{\lim} &\implies \lim_{n\to\infty} X_n(\omega) = X(\omega) \\
&\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \\
&\implies \forall q \in \mathbb{Q}_+,\ \exists N_q(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < q \ \ \forall n \geq N_q(\omega) \\
&\implies \omega \in \bigcap_{q\in\mathbb{Q}_+} \bigcup_{N\in\mathbb{N}}
\end{aligned}$$

Proof of Lemma

Lemma

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\} \in \mathscr{F}.$$

$$\begin{aligned}
\omega \in A_{\lim} &\implies \lim_{n\to\infty} X_n(\omega) = X(\omega) \\
&\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \\
&\implies \forall q \in \mathbb{Q}_+,\ \exists N_q(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < q \ \ \forall n \geq N_q(\omega) \\
&\implies \omega \in \bigcap_{q\in\mathbb{Q}_+} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N}
\end{aligned}$$

Proof of Lemma

Lemma

$$A_{\lim} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} \in \mathscr{F}.$$

$$\begin{aligned}
\omega \in A_{\lim} &\implies \lim_{n\to\infty} X_n(\omega) = X(\omega) \\
&\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \\
&\implies \forall q \in \mathbb{Q}_+,\ \exists N_q(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < q \ \ \forall n \geq N_q(\omega) \\
&\implies \omega \in \bigcap_{q\in\mathbb{Q}_+} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \{|X_n - X| < q\}
\end{aligned}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Proof of Lemma

Lemma

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega) \right\} \in \mathscr{F}.$$

$$\begin{aligned}
\omega \in A_{\lim} &\iff \lim_{n\to\infty} X_n(\omega) = X(\omega) \\
&\iff \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \\
&\iff \forall q \in \mathbb{Q}_+,\ \exists N_q(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < q \ \ \forall n \geq N_q(\omega) \\
&\iff \omega \in \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{|X_n - X| < q\}
\end{aligned}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$$A_{\lim} = \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{|X_n - X| < q\}.$$

Almost-Sure Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Almost-Sure Convergence)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X almost surely (a.s.) if

$$\mathbb{P}\left(A_{\text{lim}}\right) = 1.$$

Notation:

$$X_n \xrightarrow{\text{a.s.}} X.$$