# CS 6660: Mathematical Foundations of Data Science (Probability)

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## Practice Problems 02

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are defined with respect to $\mathscr{F}$.

1. Virat and Anushka have a date at $7$ pm. Each will arrive at the meeting place with a delay that is distributed uniformly randomly between $0$ minutes and $60$ minutes, independent of the delay of the other. The first to arrive will wait for $15$ minutes and leave if the other does not arrive within $15$ minutes. Find the probability that both meet.

2. Let $X$ and $Y$ be continuous random variables with PDFs $f_X$ and $f_Y$ respectively. For any $\alpha \in [0,1]$, argue that $\alpha\, f_X + (1-\alpha)\, f_Y$ is a valid PDF. Can you think of a random variable $Z$ whose PDF is $f_Z = \alpha\, f_X + (1-\alpha)\, f_Y$?

3. Let $X$ and $Y$ be jointly continuous random variables with the joint PDF
$$f_{X,Y}(x,y) = \begin{cases} cx(y-x)e^{-y}, & 0 \leq x \leq y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Determine the constant $c$.
   (b) Show that
$$f_{X|Y=y}(x) = \begin{cases} 6x(y-x)y^{-3}, & 0 \leq x \leq y, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{Y|X=x}(y) = \begin{cases} (y-x)e^{x-y}, & 0 \leq x \leq y < +\infty, \\ 0, & \text{otherwise,} \end{cases}$$

4. Let $X$ and $Y$ be independent Poisson variables with parameters $\lambda$ and $\mu$ respectively. Fix $n \in \mathbb{N}$. Determine the conditional PMF of $X$, conditioned on the event $\{X+Y=n\}$.

5. Suppose that two batteries are chosen simultaneously and uniformly at random from the following group of 12 batteries : 3 new, 4 used (yet working), 5 defective. You may assume that all batteries within a particular group are identical. Let $X$ denote the number of new batteries chosen, and let $Y$ denote the number of used batteries chosen. Determine the joint PMF of $X$ and $Y$, and compute $\mathbb{P}(\{|X-Y| \leq 1\})$.

6. Suppose that $X$ and $Y$ have the joint PDF
$$f_{X,Y}(x,y) = \begin{cases} cy, & -1 \leq x \leq 1,\ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Determine the constant $c$.
   (b) Are $X$ and $Y$ independent?
   (c) Evaluate $\mathbb{P}(\{X \geq Y + 0.5\})$.
   (d) Evaluate $\mathbb{P}(\{X > 0.75\} | \{Y > 0.5\})$.