



# Stochastic Processes

Primer on Probability: Random Variables, Some Basic Facts about Borel  $\sigma$ -Algebra, Random Variables, Random Vectors, Sequences of Random Variables

**Karthik P. N.**

**Assistant Professor, Department of AI**

**Email: [pnkarthik@ai.iith.ac.in](mailto:pnkarthik@ai.iith.ac.in)**

10 January 2025

## Dedication



Figure: Henri Lebesgue (1875-1941).

# Random Variables

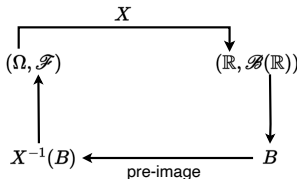
## Random Variable - Definition

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

### Definition (Random Variables)

A function  $X : \Omega \rightarrow \mathbb{R}$  is called a **random variable** with respect to  $\mathcal{F}$  if

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}(\mathbb{R}).$$



$$\forall B \in \mathcal{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathcal{F}$$

## Some Facts About $\mathcal{B}(\mathbb{R})$

- Let  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_1)$ .

## Some Facts About $\mathcal{B}(\mathbb{R})$

- Let  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_1)$ .
- Let  $\mathcal{D}_2 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_2)$ .

## Some Facts About $\mathcal{B}(\mathbb{R})$

- Let  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_1)$ .
- Let  $\mathcal{D}_2 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_2)$ .
- Let  $\mathcal{D}_3 = \left\{ [x, \infty) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_3)$ .

## Some Facts About $\mathcal{B}(\mathbb{R})$

- Let  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_1)$ .
- Let  $\mathcal{D}_2 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_2)$ .
- Let  $\mathcal{D}_3 = \left\{ [x, \infty) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_3)$ .
- Let  $\mathcal{D}_4 = \left\{ (x, \infty) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_4)$ .



## Some Facts About $\mathcal{B}(\mathbb{R})$

- Let  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_1)$ .
- Let  $\mathcal{D}_2 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_2)$ .
- Let  $\mathcal{D}_3 = \left\{ [x, \infty) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_3)$ .
- Let  $\mathcal{D}_4 = \left\{ (x, \infty) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_4)$ .

## Some Facts About $\mathcal{B}(\mathbb{R})$

- Let  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_1)$ .
- Let  $\mathcal{D}_2 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_2)$ .
- Let  $\mathcal{D}_3 = \left\{ [x, \infty) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_3)$ .
- Let  $\mathcal{D}_4 = \left\{ (x, \infty) : x \in \mathbb{R} \right\}$ . Then,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D}_4)$ .

### IMPORTANT

Every set in  $\mathcal{B}(\mathbb{R})$  can be expressed exclusively in terms of countable unions, complements, and countable intersections of sets from any one of  $\mathcal{D}_1$  or  $\mathcal{D}_2$  or  $\mathcal{D}_3$  or  $\mathcal{D}_4$ .

## Examples

- Express  $(2, 3)$  in terms of sets from  $\mathcal{D}_1$

## Examples

- Express  $(2, 3)$  in terms of sets from  $\mathcal{D}_1$
- Express  $\{5.5\}$  in terms of sets from  $\mathcal{D}_2$

## Examples

- Express  $(2, 3)$  in terms of sets from  $\mathcal{D}_1$
- Express  $\{5.5\}$  in terms of sets from  $\mathcal{D}_2$
- Express  $[-3, -2]$  in terms of sets from  $\mathcal{D}_3$

## Examples

- Express  $(2, 3)$  in terms of sets from  $\mathcal{D}_1$
- Express  $\{5.5\}$  in terms of sets from  $\mathcal{D}_2$
- Express  $(-3, -2]$  in terms of sets from  $\mathcal{D}_3$
- Express  $[-6, 5]$  in terms of sets from  $\mathcal{D}_4$

## Equivalent Definitions of Random Variable

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and a function  $X : \Omega \rightarrow \mathbb{R}$ .

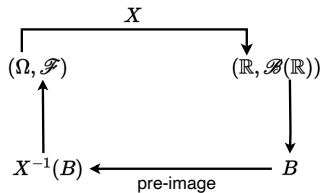
### Theorem (Equivalent Definitions of Random Variable)

The following statements are equivalent.

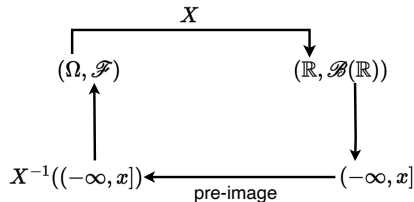
1.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{B}(\mathbb{R})$ .
2.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{D}_1$ .
3.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{D}_2$ .
4.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{D}_3$ .
5.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{D}_4$ .

Proof follows by noting that  $X^{-1}(B^c) = (X^{-1}(B))^c$  and  $X^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} X^{-1}(B_n)$

# Random Variable Simplified



$$\forall B \in \mathcal{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathcal{F}$$



$$\forall x \in \mathbb{R}, \quad X^{-1}((-\infty, x]) \in \mathcal{F}$$



## Examples

- $\Omega = \{1, 2, \dots, 6\}, \quad \mathcal{F} = \left\{ \emptyset, \Omega \right\}, \quad X(\omega) = \omega$

Is  $X$  a RV?

What functions  $X$  are RVs?

## Examples

- $\Omega = \{1, 2, \dots, 6\}, \quad \mathcal{F} = \left\{ \emptyset, \Omega \right\}, \quad X(\omega) = \omega$

Is  $X$  a RV?

What functions  $X$  are RVs?

- $\Omega = [0, 1], \quad \mathcal{F} = \left\{ \emptyset, \Omega, A, A^c \right\} \quad \text{for a fixed } A \subseteq \Omega$

What functions  $X$  are RVs?

## Examples

- $\Omega = \{1, 2, \dots, 6\}, \quad \mathcal{F} = \left\{ \emptyset, \Omega \right\}, \quad X(\omega) = \omega$

Is  $X$  a RV?

What functions  $X$  are RVs?

- $\Omega = [0, 1], \quad \mathcal{F} = \left\{ \emptyset, \Omega, A, A^c \right\} \quad \text{for a fixed } A \subseteq \Omega$

What functions  $X$  are RVs?

- $\Omega = \{1, 2, 3, 4, 5\}, \quad \mathcal{F} = \sigma \left( \left\{ \{1\}, \{2, 3\} \right\} \right)$

What functions  $X$  are RVs?

## Examples

- $\Omega = \{1, 2, \dots, 6\}, \quad \mathcal{F} = \left\{ \emptyset, \Omega \right\}, \quad X(\omega) = \omega$

Is  $X$  a RV?

What functions  $X$  are RVs?

- $\Omega = [0, 1], \quad \mathcal{F} = \left\{ \emptyset, \Omega, A, A^c \right\} \quad \text{for a fixed } A \subseteq \Omega$

What functions  $X$  are RVs?

- $\Omega = \{1, 2, 3, 4, 5\}, \quad \mathcal{F} = \sigma \left( \left\{ \{1\}, \{2, 3\} \right\} \right)$

What functions  $X$  are RVs?

- $\Omega = \mathbb{N}, \quad \mathcal{F} = 2^\Omega$

What functions  $X$  are RVs?

# Probability Law and CDF

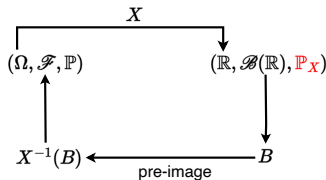
# Probability Law of a Random Variable

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$

## Definition (Probability Law)

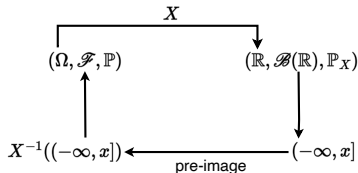
Given a random variable  $X : \Omega \rightarrow \mathbb{R}$  with respect to  $\mathcal{F}$ , its **probability law**  $\mathbb{P}_X$  is a probability measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  defined as

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\}) \quad \forall B \in \mathcal{B}(\mathbb{R}).$$



$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

## Cumulative Distribution Function (CDF)



$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}$$

### Definition (Cumulative Distribution Function)

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Given a random variable  $X : \Omega \rightarrow \mathbb{R}$  with respect to  $\mathcal{F}$ , its **cumulative distribution function (CDF)**  $F_X : \mathbb{R} \rightarrow [0, 1]$  is defined as

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}.$$

## Properties of CDF

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable with respect to  $\mathcal{F}$  with CDF  $F_X$

- $\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow +\infty} F_X(x) = 1$
- (**Monotonicity**) If  $x \leq y$ , then  $F_X(x) \leq F_X(y)$
- (**Right-Continuity**)  $F_X$  is right-continuous, i.e., for all  $x \in \mathbb{R}$ ,

$$\lim_{\varepsilon \downarrow 0} F_X(x + \varepsilon) = F_X(x).$$



## CDF $\longleftrightarrow$ Probability Law

- If we know  $\mathbb{P}_X = \{\mathbb{P}_X(B) : B \in \mathcal{B}(\mathbb{R})\}$ , then we can extract the CDF  $F_X : \mathbb{R} \rightarrow [0, 1]$  by using the formula

$$F_X(x) = \mathbb{P}_X((-\infty, x]), \quad x \in \mathbb{R}.$$

- Given the CDF  $F_X : \mathbb{R} \rightarrow [0, 1]$ , let

$$\mathbb{P}_X((-\infty, x]) = F_X(x), \quad x \in \mathbb{R}.$$

Then, there exists a **unique extension** of  $\mathbb{P}_X$  to all Borel subsets of  $\mathbb{R}$

For a proof of this, see [Folland, 1999, Theorem 1.16]

## Notation

- $\{\omega \in \Omega : X(\omega) \leq x\} = \{X \leq x\}$
- $\mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) = \mathbb{P}(X \leq x)$

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ?

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

- Consider  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ .

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

- Consider  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ .
- Consider a function  $G : \mathbb{R} \rightarrow [0, 1]$  such that:

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

- Consider  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ .
- Consider a function  $G : \mathbb{R} \rightarrow [0, 1]$  such that:
  - $\lim_{x \rightarrow -\infty} G(x) = 0, \quad \lim_{x \rightarrow +\infty} G(x) = 1.$

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

- Consider  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ .
- Consider a function  $G : \mathbb{R} \rightarrow [0, 1]$  such that:
  - $\lim_{x \rightarrow -\infty} G(x) = 0$ ,  $\lim_{x \rightarrow +\infty} G(x) = 1$ .
  - $G$  is non-decreasing.



## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

- Consider  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ .
- Consider a function  $G : \mathbb{R} \rightarrow [0, 1]$  such that:
  - $\lim_{x \rightarrow -\infty} G(x) = 0$ ,  $\lim_{x \rightarrow +\infty} G(x) = 1$ .
  - $G$  is non-decreasing.
  - $G$  is right-continuous.

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

- Consider  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ .
- Consider a function  $G : \mathbb{R} \rightarrow [0, 1]$  such that:
  - $\lim_{x \rightarrow -\infty} G(x) = 0$ ,  $\lim_{x \rightarrow +\infty} G(x) = 1$ .
  - $G$  is non-decreasing.
  - $G$  is right-continuous.
- Set  $\mathbb{Q}((-\infty, x]) = G(x)$  for all  $x \in \mathbb{R}$ .

## Structured Assignment of Probabilities to Sets in $\mathcal{B}(\mathbb{R})$

### Question

Is there a structured way to come up with a probability measure  $\mathbb{Q}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ? **YES!**

- Consider  $\mathcal{D}_1 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$ .
- Consider a function  $G : \mathbb{R} \rightarrow [0, 1]$  such that:
  - $\lim_{x \rightarrow -\infty} G(x) = 0$ ,  $\lim_{x \rightarrow +\infty} G(x) = 1$ .
  - $G$  is non-decreasing.
  - $G$  is right-continuous.
- Set  $\mathbb{Q}((-\infty, x]) = G(x)$  for all  $x \in \mathbb{R}$ .
- Using Caratheodory's extension theorem, get  $\mathbb{Q}(B)$  for all  $B \in \mathcal{B}(\mathbb{R})$ .

# Random Vectors and Sequences of Random Variables

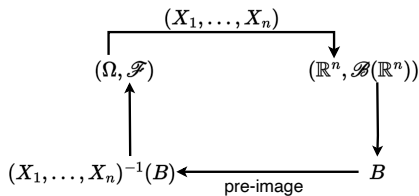
## Random Vectors

Fix a measurable space  $(\Omega, \mathcal{F})$ . Fix  $n \in \mathbb{N}$ .

### Definition (Random Vector)

Given random variables  $X_1, \dots, X_n$  defined with respect to  $\mathcal{F}$ , we say  $(X_1, \dots, X_n) : \Omega \rightarrow \mathbb{R}^n$  is a **random vector** with respect to  $\mathcal{F}$  if

$$(X_1, \dots, X_n)^{-1}(B) = \{\omega \in \Omega : (X_1(\omega), \dots, X_n(\omega)) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}(\mathbb{R}^n).$$



$$\forall B \in \mathcal{B}(\mathbb{R}^n), \quad (X_1, \dots, X_n)^{-1}(B) \in \mathcal{F}$$

## Sequence of Random Variables

Fix a measurable space  $(\Omega, \mathcal{F})$ .

### Definition (Sequence of Random Variables)

A **sequence** of random variables is a collection  $\{X_n\}_{n=1}^{\infty}$  such that

$$\forall n \in \mathbb{N}, \forall k_1, \dots, k_n \in \mathbb{N}, \quad (X_{k_1}, \dots, X_{k_n}) \text{ is a random vector.}$$

## References



Folland, G. B. (1999).

*Real analysis: modern techniques and their applications*, volume 40.

John Wiley & Sons.