CS 6660: Mathematical Foundations of Data Science (Probability)

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Practice Problems 01

1. A box contains one coupon labelled 1, two coupons labelled 2, and so on up to 10 coupons labelled 10. Two coupons are drawn simultaneously and uniformly at random from the box.
   (a) Specify Ω and $\mathbb{P}$ for the experiment, assuming that $\mathscr{F} = 2^{\Omega}$.
   (b) Find the probability of the event that the two coupons carry the same number.

2. Let $\Omega = \{H, T\}^3$ and $\mathscr{F} = 2^{\Omega}$.
   Construct a probability measure $\mathbb{P}$ and events $A, B, C \in \mathscr{F}$ such that <u>both</u> of the below conditions are met:
   (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B), \quad \mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C), \quad \mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C).$
   (b) $\mathbb{P}(A \cap B \cap C) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$

3. Let $\Omega = \{H, T\}^3$ and $\mathscr{F} = 2^{\Omega}$.
   Construct a probability measure $\mathbb{P}$ and events $A, B, C \in \mathscr{F}$ such that <u>both</u> of the below conditions are met:
   (a) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$
   (b) $\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B), \quad \mathbb{P}(B \cap C) \neq \mathbb{P}(B) \cdot \mathbb{P}(C), \quad \mathbb{P}(A \cap C) \neq \mathbb{P}(A) \cdot \mathbb{P}(C).$

4. Out of all the students in a class, 60% wear glasses, 70% watch Sherlock, and 40% belong to both the categories. Determine the probability that a student selected uniformly at random neither wears glasses nor watches Sherlock.

5. Let X and Y be two independent random variables defined on the same measurable space $(\Omega, \mathscr{F})$, with CDFs F_X and F_Y respectively. Define
$$Z = \max\{X, Y\}, \qquad W = \min\{X, Y\}.$$
   That is, $Z(\omega) = \max\{X(\omega), Y(\omega)\}$ and $W(\omega) = \min\{X(\omega), Y(\omega)\}$ for all $\omega \in \Omega$.
   (a) Prove that Z and W are random variables on the measurable space $(\Omega, \mathscr{F})$ from first principles.
   (b) Derive the CDFs of Z and W in terms of the CDFs F_X and F_Y.

6. Let $\Omega = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$, and $\mathscr{F} = 2^{\Omega}$.
   Let $\mathbb{P}(\{\omega\}) = 1/9$ for all $\omega \in \Omega$.
   Let $X : \Omega \to \mathbb{R}$ be defined as
$$X((a,b)) = a + b, \qquad a, b \in \{1, 2, 3\}.$$
   (a) Verify from first principles that X, as defined above, is a random variable with respect to $\mathscr{F}$.
   (b) Specify the CDF and PMF of X.
   (c) Compute $\mathbb{P}(\{X \in (1,5)\})$ and $\mathbb{P}(\{X = 4\})$.

7. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.
   Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$ with CDF
$$F_X(x) = \begin{cases} 0, & x < -1, \\ 1 - p, & -1 \leq x < 0, \\ 1 - p + xp, & 0 \leq x \leq 1, \\ 1, & x > 1, \end{cases}$$
   where $p \in (0,1)$ is a fixed constant. Sketch the CDF F_X, and compute the values of $\mathbb{P}(\{X = -1\})$, $\mathbb{P}(\{X = 0\})$, and $\mathbb{P}(\{X \geq 1\})$.