CS 6660: Mathematical Foundations of Data Science (Probability)

Instructor: Dr. Karthik P. N.

Practice Problems 01

1. A box contains one coupon labelled 1, two coupons labelled 2, and so on up to 10 coupons labelled 10. Two coupons are drawn simultaneously and uniformly at random from the box.

   (a) Specify Ω and $\mathbb{P}$ for the experiment, assuming that $\mathscr{F} = 2^{\Omega}$.

   (b) Find the probability of the event that the two coupons carry the same number.

   Solution: We provide the solution to each of the parts below.

   (a) Notice that there are a total of $1+2+\cdots+10 = 55$ coupons in the box. Writing (i,j) to denote the outcome in which one of the coupons drawn has the label i and the other coupon has label j, we have

$$\Omega = \Big\{(1,2),(1,3),\ldots,(1,10),\\ (2,1),(2,2),\ldots,(2,10),\\ (3,1),(3,2),\ldots,(3,10),\\ \vdots\\ (10,1),(10,2),\ldots,(10,10)\Big\}.$$

   There are 9 outcomes of the form $(1,\cdot)$, 10 outcomes of the form $(j,\cdot)$ for $j \in \{2,\ldots,10\}$. However, noting that the outcome (i,j) is same as (j,i), we note that there are a total of 54 outcomes in Ω, which may be enumerated as

$$\Omega = \Big\{(1,2),(1,3),\ldots,(1,10),(2,2),(2,3),\ldots,(2,10),\ldots\ldots\ldots,(9,10),(10,10)\Big\}$$

   Assuming that $\mathscr{F} = 2^{\Omega}$, we then have

$$\mathbb{P}(\{(i,j)\}) = \begin{cases} \dfrac{\binom{i}{1}\cdot\binom{j}{1}}{\binom{55}{2}}, & i \neq j, \\ \dfrac{\binom{i}{2}}{\binom{55}{2}}, & i = j. \end{cases}$$

   (b) Let E be the event in question. Then, we have $E = \bigcup_{i=2}^{10}\{(i,i)\} = \{(2,2),(3,3),\ldots,(10,10)\}$. Furthermore,

$$\mathbb{P}(E) = \sum_{i=2}^{10} \mathbb{P}(\{(i,i)\}) = \sum_{i=2}^{10} \frac{\binom{i}{2}}{\binom{55}{2}} = \frac{1}{9}.$$

2. Let $\Omega = \{H,T\}^3$ and $\mathscr{F} = 2^{\Omega}$.
   Construct a probability measure $\mathbb{P}$ and events $A,B,C \in \mathscr{F}$ such that <u>both</u> of the below conditions are met:

   (a) $\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B), \quad \mathbb{P}(B\cap C) = \mathbb{P}(B)\cdot\mathbb{P}(C), \quad \mathbb{P}(A\cap C) = \mathbb{P}(A)\cdot\mathbb{P}(C).$

   (b) $\mathbb{P}(A\cap B\cap C) \neq \mathbb{P}(A)\cdot\mathbb{P}(B)\cdot\mathbb{P}(C).$

E	$\mathbb{P}(E)$
$\{HHH\}$	$1/4$
$\{HHT\}$	0
$\{HTH\}$	0
$\{HTT\}$	$1/4$
$\{THH\}$	0
$\{THT\}$	$1/4$
$\{TTH\}$	$1/4$
$\{TTT\}$	0

Table 1: Assignment of probabilities to demonstrate that for any 3 events, pairwise independence does not imply joint independence.

Solution: Consider the assignment of probabilities as depicted in Table 1.
Let A, B, C be events defined as follows.

$$A := \text{outcome of first coin is head},$$
$$B := \text{outcome of second coin is head},$$
$$C := \text{outcome of third coin is head}.$$

Then, it follows that

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{HHH\} \cup \{HHT\}) = \frac{1}{4},$$

while we have

$$\mathbb{P}(A) = \mathbb{P}(\{HHH\} \cup \{HHT\} \cup \{HTH\} \cup \{HTT\}) = \frac{1}{2},$$
$$\mathbb{P}(B) = \mathbb{P}(\{HHH\} \cup \{HHT\} \cup \{THH\} \cup \{THT\}) = \frac{1}{2}.$$

Thus, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$, i.e., $A \perp\!\!\!\perp B$. Along similar lines, it can be shown that $\mathbb{P}(C) = 1/2$, $B \perp\!\!\!\perp C$, and $A \perp\!\!\!\perp C$. However, we note that

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{HHH\}) = \frac{1}{4} \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$$

3. Let $\Omega = \{H, T\}^3$ and $\mathscr{F} = 2^{\Omega}$.
Construct a probability measure $\mathbb{P}$ and events $A, B, C \in \mathscr{F}$ such that both of the below conditions are met:
    (a) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$.
    (b) $\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B), \quad \mathbb{P}(B \cap C) \neq \mathbb{P}(B) \cdot \mathbb{P}(C), \quad \mathbb{P}(A \cap C) \neq \mathbb{P}(A) \cdot \mathbb{P}(C)$.

Solution: Consider the assignment of probabilities as depicted in Table 2.

E	$\mathbb{P}(E)$
$\{HHH\}$	$1/8$
$\{HHT\}$	$2/8$
$\{HTH\}$	0
$\{HTT\}$	$1/8$
$\{THH\}$	0
$\{THT\}$	$1/8$
$\{TTH\}$	$3/8$
$\{TTT\}$	0

Table 2: Assignment of probabilities to demonstrate that for any 3 events, joint independence does not imply pairwise independence.

Let A, B, C be events as defined in question 2 above. Then, it is easy to verify that A, B, C are pairwise independent, but not jointly independent.

4. Out of all the students in a class, 60% wear glasses, 70% watch Sherlock, and 40% belong to both the categories. Determine the probability that a student selected uniformly at random neither wears glasses nor watches Sherlock.

Solution: Let N be the total number of students in the class. Furthermore, let $\mathcal{G}$ and $\mathcal{S}$ denote the subsets of students wearing glasses and watching Sherlock respectively. Then, we have

$$\frac{|\mathcal{G}|}{N} = 0.6, \qquad \frac{|\mathcal{S}|}{N} = 0.7, \qquad \frac{|\mathcal{S} \cap \mathcal{G}|}{N} = 0.4.$$

Our interest is to compute $\frac{|\mathcal{S}^c \cap \mathcal{G}^c|}{N}$. We note that

$$\frac{|\mathcal{S}^c \cap \mathcal{G}^c|}{N} = 1 - \frac{|\mathcal{S} \cup \mathcal{G}|}{N} = 1 - \frac{|\mathcal{S}| + |\mathcal{G}| - |\mathcal{S} \cap \mathcal{G}|}{N} = 0.1$$

5. Let X and Y be two independent random variables defined on the same measurable space $(\Omega, \mathscr{F})$, with CDFs F_X and F_Y respectively. Define

$$Z = \max\{X, Y\}, \qquad W = \min\{X, Y\}.$$

That is, $Z(\omega) = \max\{X(\omega), Y(\omega)\}$ and $W(\omega) = \min\{X(\omega), Y(\omega)\}$ for all $\omega \in \Omega$.

(a) Prove that Z and W are random variables on the measurable space $(\Omega, \mathscr{F})$ from first principles.

(b) Derive the CDFs of Z and W in terms of the CDFs F_X and F_Y.

Solution: We provide the solution to each of the parts below.

(a) To show that Z is a random variable with respect to $\mathscr{F}$, we note that for any $z \in \mathbb{R}$,

$$\{Z \leq z\} = \{\max\{X, Y\} \leq z\} = \{X \leq z\} \cap \{Y \leq z\}.$$

Because X and Y are random variables with respect to $\mathscr{F}$, we note that $\{X \leq z\} \in \mathscr{F}$ and $\{Y \leq z\}$, and therefore it follows that $\{Z \leq z\}$. Because this is true for any $z \in \mathbb{R}$, we conclude that Z is a random variable with respect to $\mathscr{F}$. Along similar lines, using the fact that for any $w \in \mathbb{R}$,

$$\{W \leq w\} = \{\min\{X, Y\} \leq w\} = \{X \leq w\} \cup \{Y \leq w\} \in \mathscr{F},$$

we conclude that W is a random variable with respect to $\mathscr{F}$.

(b) For any $z \in \mathbb{R}$, we have

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\{X \leq z\} \cap \{Y \leq z\}) \\ &\overset{(a)}{=} \mathbb{P}(\{X \leq z\}) \cdot \mathbb{P}(\{Y \leq z\}) \\ &= F_X(z) \cdot F_Y(z), \end{aligned}$$

where (a) above follows from the fact that $X \perp\!\!\!\perp Y$. Along similar lines, for any $w \in \mathbb{R}$, we have

$$\begin{aligned} F_W(w) &= 1 - \mathbb{P}(\{W > w\}) \\ &= 1 - \mathbb{P}(\{X > w\} \cap \{Y > w\}) \\ &\overset{(*)}{=} 1 - \mathbb{P}(\{X > w\}) \cdot \mathbb{P}(\{Y > w\}) \\ &= 1 - (1 - F_X(w))(1 - F_Y(w)), \end{aligned}$$

where $(*)$ above follows from the fact that $X \perp\!\!\!\perp Y$.

6. Let $\Omega = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$, and $\mathscr{F} = 2^{\Omega}$.
Let $\mathbb{P}(\{\omega\}) = 1/9$ for all $\omega \in \Omega$.
Let $X : \Omega \to \mathbb{R}$ be defined as

$$X((a,b)) = a + b, \qquad a, b \in \{1, 2, 3\}.$$

(a) Verify from first principles that X, as defined above, is a random variable with respect to $\mathscr{F}$.

(b) Specify the CDF and PMF of X.

(c) Compute $\mathbb{P}(\{X \in (1,5)\})$ and $\mathbb{P}(\{X = 4\})$.

Solution: We provide the solution to each of the parts below.

(a) We first note that the range of X is $\{2,3,4,5,6\}$. We thus have

$$\{X \le x\} = \begin{cases} \emptyset, & x < 2, \\ \{(1,1)\}, & 2 \le x < 3, \\ \{(1,1),(1,2),(2,1)\}, & 3 \le x < 4, \\ \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}, & 4 \le x < 5, \\ \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)\}, & 5 \le x < 6, \\ \Omega, & x \ge 6. \end{cases}$$

It is then clear that $\{X \le x\} \in \mathscr{F}$ for all $x \in \mathbb{R}$, thereby establishing that X is a random variable w.r.t. $\mathscr{F}$.

(b) We note that

$$F_X(x) = \mathbb{P}(\{X \le x\}) = \begin{cases} 0, & x < 2, \\ 1/9, & 2 \le x < 3, \\ 3/9, & 3 \le x < 4, \\ 6/9, & 4 \le x < 5, \\ 8/9, & 5 \le x < 6, \\ 1, & x \ge 6. \end{cases}, \qquad p_X(x) = \mathbb{P}(\{X = x\}) = \begin{cases} 1/9, & x = 2, \\ 2/9, & x = 3, \\ 3/9, & x = 4, \\ 2/9, & x = 5, \\ 1/9, & x = 6, \\ 0, & x \notin \{2,3,4,5,6\}. \end{cases}$$

7. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.
Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$ with CDF

$$F_X(x) = \begin{cases} 0, & x < -1, \\ 1-p, & -1 \le x < 0, \\ 1-p+xp, & 0 \le x \le 1, \\ 1, & x > 1, \end{cases}$$

where $p \in (0,1)$ is a fixed constant. Sketch the CDF F_X, and compute the values of $\mathbb{P}(\{X = -1\})$, $\mathbb{P}(\{X = 0\})$, and $\mathbb{P}(\{X \ge 1\})$.

Solution: We have

$$\mathbb{P}(\{X = -1\}) = 1-p, \qquad \mathbb{P}(\{X = 0\}) = 1-p, \qquad \mathbb{P}(\{X \ge 1\}) = 1-\mathbb{P}(\{X < 1\}) \stackrel{(*)}{=} 1-\mathbb{P}(\{X \le 1\}) = 1-1 = 0,$$

where $(*)$ follows by noting that F_X is continuous at the point $x = 1$.