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Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 09: Probability Assignment for Uncountable Sample Spaces

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

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Recap: Probability Assignment for Countable Sample Spaces

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- Assign $\mathbb{P}(\{\omega_i\}) = p_i, \quad i \in \{1, \ldots, n\}$
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  - $\sum_{i=1}^{n} p_i = 1$

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Probability Assignment for Countable Sample Spaces

When Ω is countable, it suffices to assign probabilities to **singleton subsets** of Ω.

Important Points to Keep in Mind

On Sets with $0/1$ Probability

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The following points must be borne in mind.

- $\mathbb{P}(A) = 0 \quad \not\Rightarrow \quad A = \emptyset.$
- $\mathbb{P}(A) = 1 \quad \not\Rightarrow \quad A = \Omega.$

Equality in Union Bound

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Suppose that for a given collection $A_1, A_2, \ldots \in \mathscr{F}$, we find that

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_n).$$

What can we say about $A_1, A_2, \ldots$?

Answer:

$$\mathbb{P}\left(A_i \cap A_j\right) = 0 \qquad \forall\, i \neq j.$$

Note that

$$\mathbb{P}(A_i \cap A_j) = 0 \quad \not\Rightarrow \quad A_i \cap A_j = \emptyset.$$

Example

- Let $(\Omega, \mathscr{F})$ be given by

$$\Omega = \{1, \ldots, 6\}, \qquad \mathscr{F} = \Big\{\emptyset, \Omega, \{1,2\}, \{3,4\}, \{5,6\}, \{1,2,3,4\}, \{3,4,5,6\}, \{1,2,5,6\}\Big\}.$$

- Suppose

$$\mathbb{P}(\{1,2\}) = 0, \qquad \mathbb{P}(\{3,4\}) = 0, \qquad \mathbb{P}(\{5,6\}) = 1.$$

- Then, we have

$$\mathbb{P}(\underbrace{\{1,2\}}_{A_1} \cup \underbrace{\{1,2,5,6\}}_{A_2}) = \mathbb{P}(A_1) + \mathbb{P}(A_2), \qquad A_1 \cap A_2 \neq \emptyset.$$

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Probability Assignment for Uncountable Sample Spaces

Probability Assignment for Uncountable Sample Spaces

- $\Omega = (0, 1)$
- As before, suppose we start by assigning probabilities to all **singleton subsets**
- More specifically, let

$$\mathbb{P}(\emptyset) = 0, \qquad \mathbb{P}(\Omega) = 1, \qquad \mathbb{P}(\{\omega\}) = p_\omega, \quad \omega \in \Omega.$$

- **What is $\mathbb{P}((0, \frac{1}{2}))$?**

  This cannot be derived from the probabilities of singleton subsets! (Why?)

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An Important Result from Measure Theory

Theorem

Suppose Ω is an uncountable set, and $\mathscr{F} = 2^{\Omega}$.
If $\mathbb{P}$ is a valid probability measure on $\mathscr{F}$ (satisfying the three axioms of probability), then there exists a **countable subset** $S \subseteq \Omega$ such that $\mathbb{P}(S) = 1$.
Furthermore, for any $A \in \mathscr{F}$, we have

$$\mathbb{P}(A) = \sum_{\omega \in A \cap S} \mathbb{P}(\{\omega\}).$$

Takeaway

When Ω is uncountable, the only interesting probability measures on 2^{Ω} are discrete measures!

Interesting Measures on $\mathscr{B}(\Omega)$

Example 1: Lebesgue Measure on $\Omega = (0, 1)$

- Let $(\Omega, \mathscr{F}) = ((0,1), \mathscr{B}(0,1))$
- Consider the collection

$$\mathscr{S} = \Big\{(a, b] : \; 0 \le a \le b \le 1\Big\}.$$

  Observe that:
  - $\emptyset \in \mathscr{S}$
  - $\mathscr{S}$ is **closed under finite intersections**
  - For any $A, B \in \mathscr{S}$, the set $A \setminus B$ may be expressed as

$$A \setminus B = \bigsqcup_{i=1}^{n} C_i,$$

    for some disjoint sets $C_1, \ldots, C_n \in \mathscr{S}$
- The collection $\mathscr{S}$ is called a **semiring**

Example 1: Lebesgue Measure on $\Omega = (0, 1)$

- Consider the collection
$$\mathscr{S} = \Big\{ (a, b] : \; 0 \leq a \leq b \leq 1 \Big\}.$$
- Let $m : \mathscr{S} \to [0, 1]$ be an assignment satisfying the following properties:
  - $m(\emptyset) = 0$
  - $m(\Omega) = 1$
  - $m((a, b]) = b - a$
  - **Finite additivity**

Caratheodory's Extension Theorem

There exists a unique extension of m to the whole of $\mathscr{B}(0, 1)$.
The extended measure is called the **Lebesgue measure** on $\mathscr{B}(0, 1)$, denoted by λ.
In particular,
$$\lambda(A) = m(A) \qquad \forall\, A \in \mathscr{S}.$$

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Example 2: Lebesgue Measure on $\Omega = \mathbb{R}$

- Consider the collection

$$\mathscr{S} = \Big\{(a, b] : \; -\infty \le a \le b < +\infty\Big\}.$$

- Let $m : \mathscr{S} \to [0, +\infty]$ be an assignment satisfying the following properties:
  - $m(\emptyset) = 0$
  - $m(\Omega) = +\infty$
  - $m((a, b]) = b - a$
  - **Finite additivity**

Caratheodory's Extension Theorem

There exists a unique extension of m to the whole of $\mathscr{B}(\mathbb{R})$.
The extended measure is called the **Lebesgue measure** on $\mathscr{B}(\mathbb{R})$, denoted by λ.
In particular,

$$\lambda(A) = m(A) \qquad \forall\, A \in \mathscr{S}.$$

Properties of Lebesgue Measure on $\mathscr{B}(\mathbb{R})$

Consider the measure space $(\Omega, \mathscr{F}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

- $\lambda(\{x\}) = 0$ for all $x \in \mathbb{R}$
- $\lambda(a, b) = \lambda((a, b]) = \lambda([a, b)) = \lambda([a, b]) = b - a$
- $\lambda(\mathbb{Q}) = 0$
- **Exercise:** $\lambda(K) = 0$, where K denotes the Cantor set