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Indian Institute of Technology Hyderabad

Stochastic Processes

Random Vectors, Sequences of Random Variables, $\liminf$, $\limsup$, $\lim$ of Sequences of Random Variables

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

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Dedication

Figure: Kalyanapuram Rangachari Parthasarathy (1936-2023).

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Random Vectors and Sequences of Random Variables

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Two Random Variables - Bivariate Random Vector

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Bivariate Random Vector)

Given two functions $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$, we say $(X, Y) : \Omega \to \mathbb{R}^2$ is a bivariate random vector with respect to $\mathscr{F}$ if

$$(X, Y)^{-1}(B) = \{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\} \in \mathscr{F} \qquad \forall B \in \mathscr{B}(\mathbb{R}^2).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2)) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\forall B \in \mathscr{B}(\mathbb{R}^2), \quad (X,Y)^{-1}(B) \in \mathscr{F}$$

Understanding $\mathscr{B}(\mathbb{R}^2)$

- Consider the special class of semi-infinite rectangles in $\mathbb{R}^2$, given by

$$\mathcal{E} = \Big\{(-\infty, x] \times (-\infty, y] \,:\, x, y \in \mathbb{R}\Big\}.$$

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- Other example sets in $\mathscr{B}(\mathbb{R}^2)$:
  - $(-\infty, x] \times \mathbb{R}, \quad (-\infty, x) \times \mathbb{R}, \quad [x, \infty) \times \mathbb{R}, \quad (x, \infty) \times \mathbb{R}, \quad x \in \mathbb{R}$

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  - $\mathbb{R} \times (-\infty, y], \quad \mathbb{R} \times (-\infty, y), \quad \mathbb{R} \times [y, \infty), \quad \mathbb{R} \times (y, \infty), \quad y \in \mathbb{R}$

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  - $\mathbb{R} \times (a, b), \quad (a, b) \times \mathbb{R}, \quad a, b \in \mathbb{R}$
  - $(a, b) \times (c, d), \quad a, b, c, d \in \mathbb{R}$

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  - $\mathbb{R} \times (a, b), \quad (a, b) \times \mathbb{R}, \quad a, b \in \mathbb{R}$
  - $(a, b) \times (c, d), \quad a, b, c, d \in \mathbb{R}$
  - Circle of radius r centered at the origin, $\quad r > 0$

Bivariate Random Vector – Implications

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Bivariate Random Vector)

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Thus, X is a random variable w.r.t. $\mathscr{F}$

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Thus, Y is a random variable w.r.t. $\mathscr{F}$

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Implication

$$(X, Y) \text{ random vector} \quad \Longrightarrow \quad X \text{ RV}, \quad Y \text{ RV}$$

Is the other side implication also true?

Bivariate Random Vector – Implications

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Is the other side implication also true? YES!

$$X \text{ RV}, \quad Y \text{ RV} \quad \Longrightarrow \quad (X, Y)^{-1}(B) \in \mathscr{F} \ \ \forall B \in \mathcal{E} \quad \Longrightarrow \quad (X, Y)^{-1}(B) \in \mathscr{F} \ \ \forall B \in \sigma(\mathcal{E})$$

Bivariate Random Vector Simplified

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

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Equivalent Definition of Bivariate Random Vector

(X, Y) is a random vector if

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Bivariate Random Vector Simplified

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

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(X, Y) random vector $\iff$ X is RV, Y is RV

Probability Law of a Bivariate Random Vector

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint Probability Law of a Bivariate Random Vector

Given two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$, their joint probability law $\mathbb{P}_{X,Y} : \mathscr{B}(\mathbb{R}^2) \to [0, 1]$, is the probability measure defined as

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}(\{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\}), \qquad B \in \mathscr{B}(\mathbb{R}^2).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "(X,Y)"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
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\end{tikzcd}$$

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

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Joint CDF of Two Random Variables

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
\{X \le x\} \cap \{Y \le y\} \arrow[u] & (-\infty, x] \times (-\infty, y] \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\color{red}{F_{X,Y}}\color{black}(x, y) = \mathbb{P}_{X,Y}((-\infty, x] \times (-\infty, y]) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\}), \quad x, y \in \mathbb{R}$$

Definition (Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Given random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ with respect to $\mathscr{F}$, their joint CDF $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ is defined as

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\}), \qquad x, y \in \mathbb{R}.$$

Joint CDF $\longleftrightarrow$ Joint Probability Law

- If we know $\mathbb{P}_{X,Y} = \{\mathbb{P}_{X,Y}(B) : B \in \mathscr{B}(\mathbb{R}^2)\}$, then we can extract the CDF $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ by using the formula

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big), \qquad x, y \in \mathbb{R}.$$

- Given the joint CDF $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$, let

$$\mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big) = F_{X,Y}(x, y), \qquad x, y \in \mathbb{R}.$$

  Then, by Caratheodory's extension theorem, there exists a unique extension of $\mathbb{P}_{X,Y}$ to all Borel subsets of $\mathbb{R}^2$

Higher Dimensional Random Vectors

Fix a measurable space $(\Omega, \mathscr{F})$. Fix $n \in \mathbb{N}$.

Definition (Random Vector)

Given random variables $X_1, \dots, X_n$ defined with respect to $\mathscr{F}$, we say $(X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$ is a random vector with respect to $\mathscr{F}$ if

$$(X_1, \dots, X_n)^{-1}(B) = \{\omega \in \Omega : \big(X_1(\omega), \dots, X_n(\omega)\big) \in B\} \in \mathscr{F} \qquad \forall B \in \mathscr{B}(\mathbb{R}^n).$$

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$$\forall B \in \mathscr{B}(\mathbb{R}^n), \quad (X_1, \dots, X_n)^{-1}(B) \in \mathscr{F}$$

Understanding $\mathscr{B}(\mathbb{R}^n)$ for $n > 2$

- Consider the special class of semi-infinite rectangles in $\mathbb{R}^n$, given by

$$\mathcal{E} = \Big\{(-\infty, x_1] \times \cdots \times (-\infty, x_n] : \; x_1, \ldots, x_n \in \mathbb{R}\Big\}.$$

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Equivalently, $(X_1, \ldots, X_n)$ is a random vector if

$$(X_1, \ldots, X_n)^{-1}\big((-\infty, x_1] \times \cdots \times (-\infty, x_n]\big) =$$

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$$(X_1, \ldots, X_n)^{-1}\big((-\infty, x_1] \times \cdots \times (-\infty, x_n]\big) = \bigcap_{i=1}^{n} \{X_i \leq x_i\} \in \mathscr{F} \qquad \forall x_1, \ldots, x_n \in \mathbb{R}.$$

$(X_1, \ldots, X_n)$ random vector $\iff$ X_1 RV $\;\ldots\;$ X_n RV

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Probability Law of Random Vectors

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Probability Law of Random Vectors)

Given random variables $X_1, \ldots, X_n$ defined with respect to $\mathscr{F}$, their joint probability law is the probability measure $\mathbb{P}_{X_1,\ldots,X_n} : \mathscr{B}(\mathbb{R}^n) \to [0, 1]$ defined as

$$\mathbb{P}_{X_1,\ldots,X_n}(B) = \mathbb{P}(\{\omega \in \Omega : \big(X_1(\omega), \ldots, X_n(\omega)\big) \in B\}), \qquad B \in \mathscr{B}(\mathbb{R}^n).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, bend left, "{(X_1, \ldots, X_n)}"] & & (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mathbb{P}_{X_1,\ldots,X_n}) \arrow[d] \\
(X_1, \ldots, X_n)^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$\mathbb{P}_{X_1,\ldots,X_n}(B) = \mathbb{P}((X_1, \ldots, X_n)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^n)$$

Joint CDF of Multiple Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint CDF)

Given random variables $X_1, \ldots, X_n$ defined w.r.t. $\mathscr{F}$, their joint CDF $F_{X_1,\ldots,X_n} : \mathbb{R}^n \to [0,1]$ is defined as

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \mathbb{P}_{X_1,\ldots,X_n}\big((-\infty, x_1] \times \cdots \times (-\infty, x_n]\big)$$

$$= \mathbb{P}$$

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$$\begin{aligned} F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= \mathbb{P}_{X_1,\ldots,X_n}\big((-\infty, x_1] \times \cdots \times (-\infty, x_n]\big) \\ &= \mathbb{P}\left(\bigcap_{i=1}^{n} \{X_i \leq x_i\}\right), \qquad x_1, \ldots, x_n \in \mathbb{R}. \end{aligned}$$

Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Sequence of Random Variables)

A sequence of random variables is a collection $\{X_n\}_{n=1}^{\infty}$ such that

$$\forall n \in \mathbb{N},\ \forall k_1, \dots, k_n \in \mathbb{N}, \qquad (X_{k_1}, \dots, X_{k_n}) \text{ is a random vector.}$$

Note

When an outcome $\omega \in \Omega$ results from the random experiment, the entire sequence $X_1(\omega), X_2(\omega), \dots$ realizes at once.

Finite Dimensional Distributions

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Sequence of Random Variables)

Consider a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ w.r.t. $\mathscr{F}$. The collection

$$\left\{ F_{X_{k_1},\dots,X_{k_n}}(x_1,\dots,x_n) : n \in \mathbb{N},\ k_1,\dots,k_n \in \mathbb{N},\ x_1,\dots,x_n \in \mathbb{R} \right\}$$

is called the set of all finite-dimensional distributions of the sequence $\{X_n\}_{n=1}^{\infty}$.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$\liminf$, $\limsup$, $\lim$ of Sequence of Random Variables

$\liminf$ of Sequence of Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition ($\liminf$ of Sequence of Random Variables)

The limit infimum of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X_\star : \Omega \to [-\infty, +\infty]$ such that

$$X_\star(\omega) = \sup_{n\geq 1} \inf_{k\geq n} X_k(\omega) \qquad \forall \omega \in \Omega.$$

Notation: $\liminf_{n\to\infty} X_n$.