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Indian Institute of Technology Hyderabad

Stochastic Processes

Sampling Techniques: Inverse Transform Technique, Rejection Sampling

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Sampling from a Given Distribution

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The basic ingredient: Uniform $(0, 1)$ sample(s) via PRNGs

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- **Claim:** The CDF of X is exactly equal to F, i.e., $F_X = F$

Example

- Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 0.1, & x = 10, \\ 0.2, & x = 20, \\ 0.3, & x = 30, \\ 0.4, & x = 40, \\ 0, & \text{otherwise.} \end{cases}$$

Use the inverse transform method to generate a sample from the above distribution.

Example

- **[Generating a Sample from Rayleigh Distribution]**

  The PDF of the Rayleigh distribution is given by

$$f(r) = r\, e^{-r^2/2}, \quad r > 0.$$

  Use the inverse transform method to generate a sample from the above distribution.

Gaussian Samples on Python via ITT

- Python's built-in module

$$\texttt{numpy.random.normal(loc, scale, size)}$$

  generates n independent samples from $\mathcal{N}(\mu, \sigma^2)$, where

$$n = \texttt{size}, \qquad \mu = \texttt{loc}, \qquad \sigma = \texttt{scale}.$$

- In principle, the above module uses the inverse transform technique

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Gaussian Samples on Python via ITT

1. Let $U_1, U_2 \overset{\text{i.i.d.}}{\sim} \text{Unif}(0, 1)$
2. Let R and Θ be two random variables defined via
$$R = F_1^{-1}(U_1), \qquad \Theta = 2\pi U_2,$$
where F_1 is the CDF of the Rayleigh distribution
3. Let Y_1 and Y_2 be defined as
$$Y_1 = R\,\cos(\Theta), \qquad Y_2 = R\,\sin(\Theta).$$
4. Then, $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$
5. To get $X \sim \mathcal{N}(\mu, \sigma^2)$, simply discard Y_2, and
$$X = \sigma\, Y_1 + \mu.$$
6. Repeat steps 1-5 a total of n times to get $X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$

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- The CDF corresponding to the above PDF is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ 3x^2 - 2x^3, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

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- **Alternative solution:** Rejection sampling!

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$$f(x) \leq a f_Z(x) \qquad \forall x \in \mathbb{R},$$

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Theorem (Rejection Sampling)

Let $E = \{a\, U f_Z(Z) \leq f(Z)\}$. Then,

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(Z \leq x \mid E).$$

Example

- Design an algorithm to generate a sample $X \sim f$, where

$$f(x) = 6x(1-x), \qquad x \in [0,1].$$

Example

- For fixed constants $\lambda, t > 0$, the $\mathrm{Gamma}(\lambda, t)$ PDF is given by

$$f(x) = e^{-\lambda x} \frac{\lambda^t x^{t-1}}{\Gamma(t)}, \qquad x > 0.$$

1. When $t \in \mathbb{N}$, suggest a technique to generate a sample $X \sim f$ via ITT.
2. When $t \notin \mathbb{N}$, design an algorithm to sample $X \sim f$ via rejection sampling.
   Hint: Take $Z \sim \mathrm{Exponential}(1/t)$.