# Probability and Stochastic Processes

Lecture 21: Transformations (Part 3)

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# General Transformations (Univariate Case)

## General Transformations (Univariate Case)

$X : \Omega \to \mathbb{R}$ **continuous RV**, $\quad g : \mathbb{R} \to \mathbb{R}$ **measurable**, $\quad Y = g(X)$

$g$ **monotone and differentiable throughout its domain**

$$f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{\left|g'(g^{-1}(y))\right|}, & y \in \text{Range}(g), \\ 0, & y \notin \text{Range}(g). \end{cases}$$

# Example

- Let $X \sim \mathcal{N}(0,1)$. Derive the PDF of $Y = e^X$ from first principles and using the transformation formula.
- Here, $g(x) = e^x, \qquad \mathrm{Domain}(g) = \mathbb{R}, \qquad \mathrm{Range}(g) = (0, +\infty)$
- $g$ monotone strictly and differentiable **throughout its domain**
- For any $y \in \mathrm{Range}(g)$ (i.e., $y > 0$)

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}, \qquad g^{-1}(y) = \ln y$$

$$= \frac{f_X(\ln y)}{g'(\ln y)}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}}}{e^{\ln y}} = \frac{1}{y\sqrt{2\pi}}\, e^{-\frac{(\ln y)^2}{2}} \qquad \text{(log-normal distribution)}$$

# General Transformations: $g$ Piecewise Monotone, Differentiable

## When $g : \mathbb{R} \to \mathbb{R}$ is piecewise monotone and differentiable

Suppose that $I_1, \ldots, I_n$ is a partition of $\mathbb{R}$.
Suppose that $g : \mathbb{R} \to \mathbb{R}$ is **piecewise monotone** and **differentiable** on $I_\ell$ for each $\ell \in \{1, \ldots, n\}$.
Let $g(I_\ell)$ denote the image of $I_\ell$ under $g$
Let $h_\ell$ denote the inverse of $g$ on $I_\ell$.
If $X$ is a continuous random variable with PDF $f_X$, then the PDF of $Y = g(X)$ is given by

$$f_Y(y) = \sum_{\ell=1}^{n} \frac{f_X\Big(h_\ell(y)\Big)}{\Big|g'\Big(h_\ell(y)\Big)\Big|} \mathbf{1}_{g(I_\ell)}(y).$$

# Example

Suppose $X \sim \mathcal{N}(0, 1)$. Derive the PDF of $Y = X^2$.

- Here, $g(x) = x^2$, $\quad \mathrm{Domain}(g) = \mathbb{R}$, $\quad \mathrm{Range}(g) = [0, +\infty)$
- $g$ is **differentiable** throughout its domain, and $g'(x) = 2x$ for all $x \in \mathbb{R}$
- $g$ is neither monotone increasing nor monotone decreasing throughout its domain
- Partition $\mathbb{R}$ into $I_1 = (-\infty, 0)$, $\quad I_2 = \{0\}$, $\quad I_3 = (0, +\infty)$
- $g(I_1) = (0, +\infty)$, $\quad g(I_2) = \{0\}$, $\quad g(I_3) = (0, +\infty)$
- $g$ is **strictly decreasing on $I_1$** and **strictly increasing on $I_2$**, $\quad h_1(y) = -\sqrt{y}$, $\quad h_2(y) = \sqrt{y}$
- For any $y \in \mathrm{Range}(g)$,

$$
\begin{aligned}
f_Y(y) &= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}}{\left| -2\sqrt{y} \right|} \mathbf{1}_{(0, +\infty)}(y) + 0 \cdot \mathbf{1}_{\{0\}}(y) + \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}}{\left| 2\sqrt{y} \right|} \mathbf{1}_{(0, +\infty)}(y) \\
&= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \mathbf{1}_{(0, +\infty)}(y)
\end{aligned}
$$

(Chi-Squared distribution with 1 degree of freedom).

# General Transformations: Multivariate Case

Let $X_1, \ldots X_n$ be **jointly continuous** random variables with joint PDF $f_{X_1,\ldots,X_n}$.
Fix **measurable** functions $g_1 : \mathbb{R}^n \to \mathbb{R}$, $g_2 : \mathbb{R}^n \to \mathbb{R}$, $\ldots$ $g_n : \mathbb{R}^n \to \mathbb{R}$.
Define $g : \mathbb{R}^n \to \mathbb{R}^n$ and random variables $Y_1, \ldots, Y_n$ as

$$g(x_1, \ldots, x_n) = \begin{pmatrix} g_1(x_1, \ldots, x_n) \\ \vdots \\ g_n(x_1, \ldots, x_n) \end{pmatrix}, \qquad \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} g_1(X_1, \ldots, X_n) \\ \vdots \\ g_n(X_1, \ldots, X_n) \end{pmatrix},$$

where $Y_\ell = g_\ell(X_1, \ldots, X_n)$, $\ell \in \{1, \ldots, n\}$, $g_1, \ldots, g_n$ are smooth[1] functions.

**General Transformations (Multivariate Case)**

**What is the joint PDF of $Y_1, \ldots, Y_n$ in terms of $f_{X_1,\ldots,X_n}$?**

[1]We will assume that $g_1, \ldots, g_n$ are differentiable with continuous first-order partial derivatives.

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# Jacobian Matrix

## Definition (Jacobian Matrix)

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$g(x_1, \dots, x_n) = \begin{pmatrix} g_1(x_1, \dots, x_n) \\ \vdots \\ g_n(x_1, \dots, x_n) \end{pmatrix}.$$

The **Jacobian matrix** of $g$ at the point $(x_1, \dots, x_n)$ is defined as

$$J_g(x_1, \dots, x_n) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial g_1}{\partial x_n}(x_1, \dots, x_n) \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1}(x_1, \dots, x_n) & \cdots & \frac{\partial g_n}{\partial x_n}(x_1, \dots, x_n) \end{pmatrix}$$

# Jacobi's Transformation Formula

Let $X_1, \ldots, X_n$ be jointly continuous with joint PDF $f_{X_1,\ldots,X_n}$.

## Jacobi's Transformation Formula

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be **one-one**, **differentiable** with **continuous first-order partial derivatives**, and **non-zero Jacobian throughout its domain**.

Let the individual components of $g$ be denoted by $g_1, \ldots, g_n$.

Let $Y_\ell = g_\ell(X_1, \ldots, X_n), \quad \ell \in \{1, \ldots, n\}$.

If $(X_1, \ldots, X_n) \sim f_{X_1,\ldots,X_n}$, then the joint PDF of $Y_1, \ldots, Y_n$ is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \begin{cases} \dfrac{f_{X_1,\ldots,X_n}\left(g^{-1}(y_1, \ldots, y_n)\right)}{\left|\det\ J_g\left(g^{-1}(y_1, \ldots, y_n)\right)\right|}, & (y_1, \ldots, y_n) \in \text{Range}(g), \\ 0, & (y_1, \ldots, y_n) \notin \text{Range}(g). \end{cases}$$

# Example

Let $X_1, X_2 \overset{\text{IID}}{\sim} \text{Exponential}(\lambda)$ for some fixed $\lambda > 0$.
Derive the joint PDF of $Y_1 = X_1$ and $Y_2 = X_1 + X_2$.
Also deduce the conditional PDF of $Y_1$, conditioned on the event $\{Y_2 = y_2\}$.

- $g_1(x_1, x_2) = x_1, \qquad g_2(x_1, x_2) = x_1 + x_2,$
  $\text{Domain}(g_1) = [0,\ +\infty) \times [0,\ +\infty) = \text{Domain}(g_2), \qquad \text{Range}(g_1) = [0, +\infty) = \text{Range}(g_2)$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$ and its Jacobian:

$$g(x_1, x_2) = \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix}, \qquad J_g(x_1, x_2) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1, x_2) & \frac{\partial g_1}{\partial x_2}(x_1, x_2) \\ \frac{\partial g_2}{\partial x_1}(x_1, x_2) & \frac{\partial g_2}{\partial x_2}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

- $\text{Domain}(g) = [0,\ +\infty) \times [0,\ +\infty), \qquad \text{Range}(g) = \{(x, y) : 0 \leq x \leq y < +\infty\}$
- $\left|\det\ J_g(x_1, x_2)\right| = 1 \qquad \forall (x_1, x_2) \in \text{Domain}(g)$
- $g^{-1}(y_1, y_2) = \begin{pmatrix} y_1 \\ y_2 - y_1 \end{pmatrix} \qquad \forall (y_1, y_2) \in \text{Range}(g)$

# Example

- For any $(y_1, y_2) \in \text{Range}(g)$,

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \frac{f_{X_1,X_2}\left(g^{-1}(y_1, y_2)\right)}{\left|\det\ J_g\left(g^{-1}(y_1, y_2)\right)\right|} \\ &= f_{X_1,X_2}\left(y_1,\ y_2 - y_1\right) \\ &= f_{X_1}(y_1) \cdot f_{X_2}(y_2 - y_1) \\ &= \lambda\, e^{-\lambda\, y_1} \cdot \lambda\, e^{-\lambda(y_2 - y_1)} \\ &= \lambda^2\, e^{-\lambda\, y_2}. \end{aligned}$$

- Conditional PDF:

$$f_{Y_1|Y_2=y_2}(y_1) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{\lambda^2\, e^{-\lambda\, y_2}}{\int\limits_{y_1:(y_1,y_2)\in\text{Range}(g)} f_{Y_1,Y_2}(y_1, y_2)\, \mathrm{d}y_1} = \frac{\lambda^2\, e^{-\lambda\, y_2}}{\int_0^{y_2} \lambda^2\, e^{-\lambda\, y_2}\, \mathrm{d}y_1} = \frac{1}{y_2}, \quad y_1 \in [0, y_2].$$

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# Jacobi's Transformation Formula: $g$ Piecewise Differentiable with Non-Zero Jacobian

## When $g : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise differentiable with non-zero Jacobian

Suppose that $I_1, \ldots, I_n$ is a partition of $\mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^n$ is **one-one**, **differentiable** with **continuous first-order partial derivatives**, and has non-zero Jacobian on $I_\ell$ for each $\ell \in \{1, \ldots, n\}$.
Let $h_\ell$ denote the inverse of $g$ on $I_\ell$.
Let the individual components of $g$ be $g_1, \ldots, g_n$. Let $Y_\ell = g_\ell(X_1, \ldots, X_n)$ for each $\ell \in \mathbb{N}$.
Let $X_1, \ldots, X_n$ be jointly continuous with joint PDF $f_{X_1,\ldots,X_n}$. Then, the joint PDF of $Y_1, \ldots, Y_n$ is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \sum_{\ell=1}^{n} \frac{f_{X_1,\ldots,X_n}\Big(h_\ell(y_1, \ldots, y_n)\Big)}{\Big|\det\, J_g\Big(h_\ell(y_1, \ldots, y_n)\Big)\Big|} \mathbf{1}_{g(I_\ell)}(y_1, \ldots, y_n).$$

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# Sampling from a Given Distribution:

## Inverse CDF Technique

# Sampling from a Given Distribution

**Objective**

**Given a CDF $F : \mathbb{R} \to [0, 1]$, generate a sample $X \sim F$.**

Remarks:

- The random variable $X$ may be discrete, continuous, singular, or any mixtures thereof
- If $X$ is discrete, we will be given its PMF $p$ instead of CDF $F$
- If $X$ is continuous, we will be given its PDF $f$ instead of CDF $F$

The basic ingredient: **IID Uniform $(0, 1)$** sample(s)

**Why $\mathrm{Unif}(0, 1)$?** Easy to generate on a computer!

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# The Inverse CDF Technique

## Objective

**Given a CDF $F : \mathbb{R} \to [0, 1]$, generate a sample $X \sim F$.**

$$U \sim \mathrm{Unif}(0, 1) \longrightarrow \boxed{F^{-1}} \longrightarrow X \sim F$$

- Define $F^{-1} : (0, 1) \to \mathbb{R}$ as

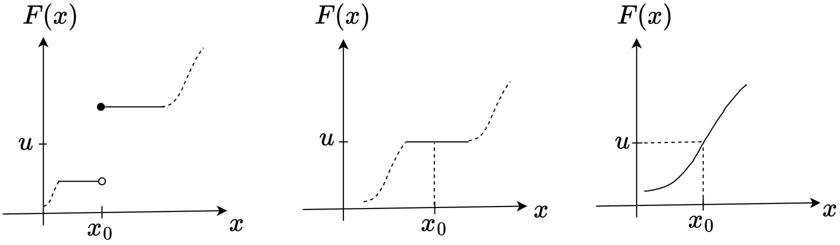
$$F^{-1}(u) := \inf \left\{ x \in \mathbb{R} : \; F(x) \geq u \right\}, \qquad u \in (0, 1)$$

- If $F$ is invertible (e.g., Gaussian CDF), then $F^{-1}$ as defined above coincides with the inverse of $F$

## Claim

If $U \sim \mathrm{Unif}(0, 1), \quad X = F^{-1}(U), \qquad$ then $\qquad F_X = F$

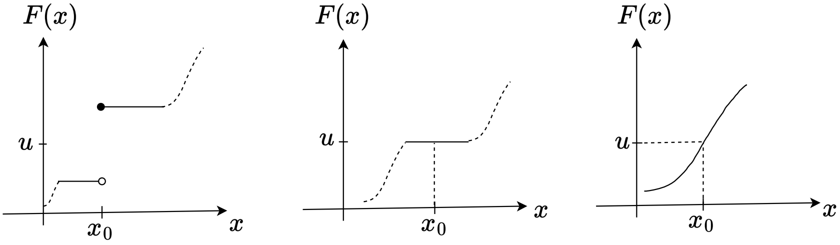
$$F\left(F^{-1}(u)\right) \geq u$$




- Case 1: $\exists\, x_0 \in \mathbb{R}$ such that $\lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon) < u \leq F(x_0)$
  In this case,

$$F^{-1}(u) = \inf\left\{x \in \mathbb{R} : \; F(x) \geq u\right\} = x_0, \qquad\qquad F\left(F^{-1}(u)\right) = F(x_0) \geq u$$

$F\left(F^{-1}(u)\right) \geq u$ **for all** $u \in (0,1)$

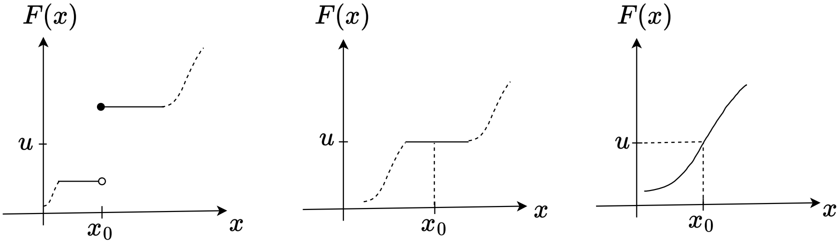



- Case 2: $\exists\, x_0 \in \mathbb{R}$ such that $F(x_0) = u$

  In this case,

$$F^{-1}(u) = \inf\left\{x \in \mathbb{R} : \; F(x) \geq u\right\} \leq x_0, \qquad F\left(F^{-1}(u)\right) = F(x_0) = u$$

$$F\left(F^{-1}(u)\right) \geq u \quad \textbf{for all} \quad u \in (0, 1)$$




- Case 2: $\exists$ **unique** $x_0 \in \mathbb{R}$ such that $F(x_0) = u$
  In this case,

$$F^{-1}(u) = \inf\left\{x \in \mathbb{R} : \; F(x) \geq u\right\} = x_0, \qquad F\left(F^{-1}(u)\right) = F(x_0) = u$$

# $\{F^{-1}(U) \leq x\} = \{U \leq F(x)\}$ **for all** $x \in \mathbb{R}$

- $\{F^{-1}(U) \leq x\} \subseteq \{U \leq F(x)\}$:

$$
\begin{aligned}
\omega \in \{F^{-1}(U) \leq x\} \quad &\Longrightarrow \quad F^{-1}\Big(U(\omega)\Big) \leq x \\
&\Longrightarrow \quad F\Big(F^{-1}(U(\omega))\Big) \leq F(x) \\
&\Longrightarrow \quad U(\omega) \leq F\Big(F^{-1}(U(\omega))\Big) \leq F(x) \\
&\Longrightarrow \quad \omega \in \{U \leq F(x)\}
\end{aligned}
$$

- $\{U \leq F(x)\} \subseteq \{F^{-1}(U) \leq x\}$:

$$
\begin{aligned}
\omega \in \{U \leq F(x)\} \quad &\Longrightarrow \quad U(\omega) \leq F(x) \\
&\Longrightarrow \quad x \in \Big\{x' \in \mathbb{R} : \;\; F(x') \geq U(\omega)\Big\} \\
&\Longrightarrow \quad F^{-1}\Big(U(\omega)\Big) = \inf \Big\{x' \in \mathbb{R} : \;\; F(x') \geq U(\omega)\Big\} \leq x \\
&\Longrightarrow \quad \omega \in \{F^{-1}(U) \leq x\}
\end{aligned}
$$

$$U \sim \mathrm{Unif}(0,1), \quad X = F^{-1}(U) \qquad \Longrightarrow \qquad F_X = F$$

For any $x \in \mathbb{R}$,

$$\begin{aligned} F_X(x) &= \mathbb{P}\Big(\{X \leq x\}\Big) \\ &= \mathbb{P}\Big(\{F^{-1}(U) \leq x\}\Big) \\ &= \mathbb{P}\Big(\{U \leq F(x)\}\Big) \\ &= F(x) \end{aligned}$$

# Examples

Generate a sample $X$ from the **standard Rayleigh distribution** whose PDF is given by

$$f(x) = x\,e^{-\frac{x^2}{2}}, \qquad x > 0.$$

- The CDF is given by

$$\forall x > 0, \qquad F(x) = \int_0^x u\,e^{-\frac{u^2}{2}}\,\mathrm{d}u = \int_0^{x^2/2} e^{-t}\,\mathrm{d}t = 1 - e^{-x^2/2}.$$

- For any $u \in (0, 1)$,

$$\begin{aligned} F^{-1}(u) &= \inf\left\{x \in \mathbb{R} : \; F(x) \geq u\right\} = \inf\left\{x > 0 : \; F(x) \geq u\right\} = \inf\left\{x > 0 : \; 1 - e^{-x^2/2} \geq u\right\} \\ &= \inf\left\{x > 0 : \; x^2 \geq -2\ln(1-u)\right\} = \inf\left\{x > 0 : \; x \geq \sqrt{-2\ln(1-u)}\right\} = \sqrt{-2\ln(1-u)} \end{aligned}$$

- Generate $U \sim \mathrm{Unif}(0, 1)$ on a computer, set $X = F^{-1}(U)$

# The Box–Muller Transform

Given $\mu \in \mathbb{R}$ and $\sigma > 0$, generate a sample $Z \sim \mathcal{N}(\mu, \sigma^2)$ using only $\text{Uniform}(0, 1)$ samples.

- Generate $U_1, U_2 \overset{\text{IID}}{\sim} \text{Uniform}(0, 1)$
- Set $R = \sqrt{-2\ln(1 - U_1)}, \quad \Theta = 2\pi U_2 \qquad U_1 \perp\!\!\!\perp U_2 \implies R \perp\!\!\!\perp \Theta$
- Set $X = R\cos\Theta, \quad Y = R\sin\Theta$
- **Exercise:** Using Jacobi's transformation formula (bivariate case),

$$\forall\, (x, y) \in \mathbb{R}^2, \qquad f_{X,Y}(x, y) = \frac{1}{2\pi}\, e^{-(x^2+y^2)/2} = \underbrace{\left(\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\right)}_{f_X(x)} \cdot \underbrace{\left(\frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\right)}_{f_Y(y)}$$

- Keep $X$, discard $Y$ Set $Z = \sigma X + \mu$
- **Exercise:** It can be shown that $Z \sim \mathcal{N}(\mu, \sigma^2)$