AI5090 / EE5910: Stochastic Processes
Homework 02

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All random variables appearing below are assumed to be defined with respect to a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. Let $(\Omega, \mathscr{F}, \mathbb{P}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_{\mathsf{Exp}})$, where $\mathbb{P}_{\mathsf{Exp}}$ denotes the Exponential probability measure specified by

$$\mathbb{P}_{\mathsf{Exp}}((-\infty, x]) = \begin{cases} 1 - e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued random variables defined with respect to $\mathscr{F}$ via

$$X_n(\omega) = \begin{cases} 0, & \omega < n, \\ e^{n/2}, & \omega \geq n. \end{cases}$$

(a) Show that the above sequence converges pointwise, and identify the pointwise limit.

(b) Show that the above sequence does not converge in the mean-squared sense to the pointwise limit random variable identified in part (a).

Note: This example shows that a sequence may converge pointwise but not in the mean-squared sense.

2. Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathsf{Ber}(0.5)$.

(a) Prove that $\{X_n\}_{n=1}^{\infty}$ converges in distribution. Identify a limit random variable.

(b) Prove that $\{X_n\}_{n=1}^{\infty}$ cannot converge in probability to any random variable.

Hint for part (b):
Suppose there exists a limit random variable X such that $X_n \xrightarrow{\text{p.}} X$. Then, from triangle inequality, we have

$$|X_n - X_{n+1}| \leq |X_n - X| + |X_{n+1} - X|.$$

Use the above inequality to prove that for any $\varepsilon > 0$,

$$\mathbb{P}(\{|X_n - X_{n+1}| > \varepsilon\}) \leq \mathbb{P}\left(\left\{|X_n - X| > \frac{\varepsilon}{2}\right\}\right) + \mathbb{P}\left(\left\{|X_{n+1} - X| > \frac{\varepsilon}{2}\right\}\right).$$

In particular, compute the left-hand side of the above inequality for $\varepsilon = 0.5$, and prove that convergence in probability does not hold.

3. Let $\{X_n\}_{n \in \mathbb{N}}$ be any given sequence of random variables. Show that

$$X_n \xrightarrow{\text{p.}} 0 \qquad \text{if and only if} \qquad \lim_{n \to \infty} \mathbb{E}\left[\frac{|X_n|}{1 + |X_n|}\right] = 0.$$

Hint:
Let $Z_n := \frac{|X_n|}{1+|X_n|}$. For the if part, fix $\varepsilon > 0$, and write

$$\mathbb{P}(|X_n| > \varepsilon) = \mathbb{E}[\mathbf{1}_{\{|X_n| > \varepsilon\}}] = \mathbb{E}\left[\frac{1}{1 + |X_n|} \mathbf{1}_{\{|X_n| > \varepsilon\}}\right] + \mathbb{E}\left[\frac{|X_n|}{1 + |X_n|} \mathbf{1}_{\{|X_n| > \varepsilon\}}\right].$$

Upper bound the right-hand side of the above relation carefully and show that it goes to 0 as $n \to \infty$.

For the only if part, fix an arbitrary $\varepsilon > 0$ and write

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_n \mathbf{1}_{\{|X_n| > \varepsilon\}}] + \mathbb{E}[Z_n \mathbf{1}_{\{|X_n| \leq \varepsilon\}}].$$

Upper bound the right-hand side of the above relation carefully to show that it can be made negligible as $n \to \infty$.

4. Suppose $V_1, V_2, \ldots \overset{\text{i.i.d.}}{\sim}$ Uniform$[0, 1]$. Let $Y_0 := 1$, and for each $n \in \mathbb{N}$, let

$$Y_n = \min\left\{Y_{n-1},\ \frac{Y_{n-1} + V_n}{2}\right\}.$$

   (a) Argue that Y_n converges pointwise.
   Hint: Establish monotonicity, and conclude that every bounded, monotone sequence must converge.

   (b) Denote by Y the pointwise limit random variable of part (a).
   In this part, we will work our way through a series of logical steps to formally prove that $\mathbb{P}(Y = 0) = 1$, thereby proving that $Y_n \xrightarrow{\text{a.s.}} 0$.
   Fix $\delta > 0$, and define the event B_δ as

$$B_\delta := \{Y \geq \delta\}.$$

   Furthermore, for each $n \in \mathbb{N}$, define

$$A_{n,\delta} := \left\{V_n \leq \frac{\delta}{2}\right\}.$$

   i. Keeping δ fixed, compute the value of

$$\mathbb{P}\left(\underbrace{\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_{n,\delta}}_{A_\delta}\right).$$

   ii. We will now show that $A_\delta \cap B_\delta = \emptyset$. Suppose, on the contrary, that this is non-empty.
   For any $\omega \in A_\delta \cap B_\delta$, show that

$$Y_n(\omega) \leq \frac{3}{4}\, Y_{n-1}(\omega) \qquad \text{for infinitely many values of } n, \ \text{ say } n_1 < n_2 < \cdots.$$

   Considering the subsequence $Y_{n_1}(\omega), Y_{n_2}(\omega), \ldots$, we must have

$$\lim_{k\to\infty} Y_{n_k}(\omega) = 0.$$

   Argue that the above contradicts the assumption that $\omega \in B_\delta$.

   iii. Using the results of parts (a) and (b), show that

$$\mathbb{P}(\{Y \geq \delta\}) = 0 \qquad \forall \delta > 0.$$

   Show that this in turn implies that $\mathbb{P}(\{Y = 0\}) = 1$.

5. **(Portfolio allocation.)**
   Suppose that you are given one unit of money (for e.g., 1 million). At the start of each day, you bet a fraction α of your total earnings on a fair coin toss. If you win, you get back double the amount you bet. If you lose, you get back half of the amount you bet. Denote W_n as the total wealth you have accumulated at the end of day $n \in \mathbb{N}$. By convention, $W_0 = 1$.

   (a) Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with

$$M_n = \begin{cases} 1 + \alpha, & \text{w.p. } 1/2, \\ 1 - \frac{\alpha}{2}, & \text{w.p. } 1/2. \end{cases}$$

   For each $n \in \mathbb{N}$, show that $W_n = W_{n-1} M_n$.

   (b) Using the strong law of large numbers, determine (in terms of α) the limit to which the sequence $\left\{\frac{1}{n}\ \log W_n\right\}_{n\in\mathbb{N}}$ converges almost-surely.

   (c) Let $\mu(\alpha)$ denote the limit in part (ii) above. Determine the value of α that maximizes $\mu(\alpha)$.

6. **(Symmetric random walk on the integer line.)**

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. *Radamacher* random variables, i.e.,

$$\mathbb{P}(\xi_n = +1) = \frac{1}{2} = \mathbb{P}(\xi_n = -1) \qquad \forall\, n \in \mathbb{N}.$$

Let $S_0 = 0$, and for each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} \xi_i$ denote the partial sum of the first n Radamacher variables. The sequence $\{S_n\}_{n\in\mathbb{N}}$ is called a random walk sequence.
Fix integers $m \geq 1$ and $N \geq 1$, and let

$$h_{m,N}(i) \coloneqq \mathbb{P}\Big(\underbrace{\Big\{\text{starting from } i,\ \text{ the random walk hits } m \text{ before hitting } -N\Big\}}_{\mathcal{E}_{m,N}(i)}\Big).$$

Clearly, $h_{m,N}(m) = 1$ and $h_{m,N}(-N) = 0$.

(a) For any $-N < i < m$, write a difference equation to express $h_{m,N}(i)$ in terms of $h_{m,N}(i-1)$ and $h_{m,N}(i+1)$.

(b) Solve the above difference equation using the boundary conditions $h_{m,N}(m) = 1$ and $h_{m,N}(-N) = 0$. Obtain a closed-form solution for the probability $h_{m,N}(i)$ for each $-N < i < m$.

(c) Fixing m, compute $\mathbb{P}\left(\bigcup\limits_{N\in\mathbb{N}} \mathcal{E}_{m,N}(0)\right)$ (hint: continuity of probability).
Interpret in plain English the event $\mathcal{E}_m \coloneqq \bigcup\limits_{N\in\mathbb{N}} \mathcal{E}_{m,N}(0)$.

(d) Compute $\mathbb{P}\left(\bigcap_{m\in\mathbb{N}} \mathcal{E}_m\right)$.
For any $\omega \in \bigcap_{m\in\mathbb{N}} \mathcal{E}_m$, what can you say about the value of $\limsup_{n\to\infty} S_n(\omega)$?

(e) For each $-N < i < m$, let

$$g_{m,N}(i) \coloneqq \mathbb{P}\Big(\underbrace{\Big\{\text{starting from } i,\ \text{ the random walk hits } -N \text{ before hitting } m\Big\}}_{\mathcal{G}_{m,N}(i)}\Big).$$

Compute the probabilities

$$\mathbb{P}\left(\underbrace{\bigcup_{m\in\mathbb{N}} \mathcal{G}_{m,N}(0)}_{\mathcal{G}_N}\right), \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \mathcal{G}_N\right)$$

For any $\omega \in \bigcap_{N\in\mathbb{N}} \mathcal{G}_N$, what can you say about $\liminf_{n\to\infty} S_n(\omega)$?

7. **(Bonus question.)** Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Exp}(1)$.
For each $n \in \mathbb{N}$, let $Y_n = \max\{X_1, \ldots, X_n\}$.
In this exercise, we shall prove formally that almost surely, Y_n grows as $\log n$ for large n.
If the base of the logarithm is not mentioned explicitly, it should be considered to be e.

(a) Show formally that

$$\frac{Y_n}{\log n} \xrightarrow{\text{d.}} 1,$$

and use the reverse implication from class to conclude that $\frac{Y_n}{\log n} \xrightarrow{\text{p.}} 1$.

(b) Based on the conclusion in part (a), show that

$$\frac{Y_n}{\log_2 n} \xrightarrow{\text{p.}} \log 2.$$

(c) Consider the subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ given by $n_k = 2^k$ for all $k \in \mathbb{N}$.
Fix $\varepsilon > 0$. For each k, let

$$x_k \coloneqq e^{-(\varepsilon + \log 2)k}.$$

Prove that

$$(1 - x_k)^{n_k} \geq \exp\left(-\frac{n_k\, x_k}{1 - x_k}\right) \quad \forall k.$$

Further, deduce that

$$(1 - x_k)^{n_k} \geq \exp(-2 n_k x_k)$$

for all sufficiently large values of k.

Hint for part (c):
To prove the first part, use the relation $\log x \geq 1 - \frac{1}{x}$ for any $x > 0$ (this is another way of seeing the well-known inequality $\log x \leq x - 1$ for all $x > 0$).
To deduce the second part, use the fact that x_k converges to 0 as $k \to \infty$, and therefore $x_k < \frac{1}{2}$ for all sufficiently large values of k.

(d) Using the result in the second half of part (c), prove that for every $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{Y_{n_k}}{\log_2 n_k} > \log 2 + \varepsilon\right) < +\infty.$$

Then, using the Borel–Cantelli lemma, conclude that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \limsup_{k \to \infty} \frac{Y_{n_k}(\omega)}{\log_2 n_k} \leq \log 2\right\}\right) = 1. \tag{1}$$

(e) Fix $0 < \varepsilon < \log 2$. For each $k \in \mathbb{N}$, let

$$y_k \coloneqq e^{-(\log 2 - \varepsilon)k}.$$

Using the facts that $1 - x \leq e^{-x}$ and $e^x > x$ for all $x \geq 0$ (again, alternative ways to see the inequality $\log x \leq x - 1 < x$), prove that

$$(1 - y_k)^{n_k} \leq e^{-\varepsilon k} \quad \forall k.$$

Use this relation to prove that for every $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{Y_{n_k}}{\log_2 n_k} \leq \log 2 - \varepsilon\right) < +\infty,$$

and hence conclude from the Borel–Cantelli lemma that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \liminf_{k \to \infty} \frac{Y_{n_k}(\omega)}{\log_2 n_k} \geq \log 2\right\}\right) = 1. \tag{2}$$

Epilogue for question 7:

Combining the results in (1) and (2), we see that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{k\to\infty} \frac{Y_{n_k}(\omega)}{\log_2 n_k} = \log 2\right\}\right) = 1.$$

That is, $\frac{Y_{n_k}}{\log_2 n_k} \xrightarrow{\text{a.s.}} \log 2$.
This proves that the subsequence $\{Y_{n_k}/(\log_2 n_k)\}_{k=1}^{\infty}$ converges almost surely to the constant random variable taking the value $\log 2$.
We can use this to prove that the entire sequence $\{Y_n/(\log_2 n)\}_{n=1}^{\infty}$ must also converge to the same constant random variable $\log 2$, as follows.

Given any $n \in \mathbb{N}$, find k such that $n_k \le n < n_k + 1$ (you can always find at least one such k).
Because $Y_1 \le Y_2 \le Y_3 \le \cdots$, it follows that

$$\begin{aligned}
& Y_{n_k} \le Y_n < Y_{n_k+1} \\
\Longrightarrow \quad & \frac{Y_{n_k}}{\log_2 n} \le \frac{Y_n}{\log_2 n} < \frac{Y_{n_k+1}}{\log_2 n} \\
\Longrightarrow \quad & \frac{Y_{n_k}}{\log_2 (n_k+1)} \le \frac{Y_n}{\log_2 n} < \frac{Y_{n_k+1}}{\log_2 n_k} \\
\Longrightarrow \quad & \frac{\log_2(n_k+1)}{\log_2 n_k} \cdot \frac{Y_{n_k}}{\log_2 n_k} \le \frac{Y_n}{\log_2 n} < \frac{Y_{n_k+1}}{\log_2(n_k+1)} \cdot \frac{\log_2 n_k}{\log_2(n_k+1)}.
\end{aligned}$$

Using (1) and (2), along with the fact that $\lim_{k\to\infty} \frac{\log_2(n_k+1)}{\log_2 n_k} = 1$ gives us the desired result.