Stochastic Processes

Lecture 05

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

27 January 2026

Equivalent Statements for Almost-Sure Convergence

When X is a Extended Real-Valued Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Equivalent Statements for Almost-Sure Convergence)

*Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X be an **extended real-valued** RV. The following statements are equivalent.*

1. $X_n \xrightarrow{\text{a.s.}} X$.

2. *All of the following hold simultaneously.*

   2.1 *For every choice of $\varepsilon > 0$,* $\quad \mathbb{P}\left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \left\{ |X_n - X| > \varepsilon \right\} \cap \{X \in \mathbb{R}\} \right) = 0.$

   2.2 *For every choice of $\varepsilon > 0$,* $\quad \mathbb{P}\left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \left\{ X_n \leq \varepsilon \right\} \cap \{X = +\infty\} \right) = 0.$

   2.3 *For every choice of $\varepsilon > 0$,* $\quad \mathbb{P}\left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \left\{ X_n \geq -\varepsilon \right\} \cap \{X = -\infty\} \right) = 0.$

Convergence in Probability

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in Probability)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n \in \mathbb{N}}$ converges to X **in probability (p.)** if:

1. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{|X_n - X| > \varepsilon\} \cap \{X \in \mathbb{R}\}\right) \overset{n \to \infty}{\longrightarrow} 0$.

2. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{X_n \leq \varepsilon\} \cap \{X = +\infty\}\right) \overset{n \to \infty}{\longrightarrow} 0$.

3. For every choice of $\varepsilon > 0$, $\quad \mathbb{P}\left(\{X_n \geq -\varepsilon\} \cap \{X = -\infty\}\right) \overset{n \to \infty}{\longrightarrow} 0$.

Notation: $X_n \overset{\text{p.}}{\longrightarrow} X$.

Remark:

- If X is a real-valued RV, then the conditions in 2, 3 hold trivially

Convergence in p-Norm $(p \geq 1)$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in p-Norm)

Fix $p \geq 1$. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n \in \mathbb{N}}$ converges to X **in p-norm** if

1. We have
$$\mathbb{E}\left[\left|X_n\right|^p\right] < +\infty \quad \text{for all } n \in \mathbb{N}, \qquad \mathbb{E}\left[\left|X\right|^p\right] < +\infty.$$

2. We have
$$\mathbb{E}\left[\left|X_n - X\right|^p\right] \overset{n \to \infty}{\longrightarrow} 0; \qquad \textbf{Notation:} \quad X_n \xrightarrow{\mathcal{L}^p} X.$$

Remarks:

- If $p = 2$, then the convergence in 2-norm is called **mean-squared convergence (m.s.)** and denoted
$$X_n \xrightarrow{\text{m.s.}} X$$

- $\mathbb{E}\left[\left|X\right|^p\right] < +\infty \qquad \implies \qquad \mathbb{P}(|X|^p < +\infty) = 1 \qquad \implies \qquad \mathbb{P}(X \in \mathbb{R}) = 1.$

Convergence in Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Convergence in Distribution)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV.
We say that the sequence $\{X_n\}_{n \in \mathbb{N}}$ converges to X **in distribution (d.)** if:

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \qquad \forall x \in C_{F_X},$$

where C_{F_X} denotes the **points of continuity** of F_X.

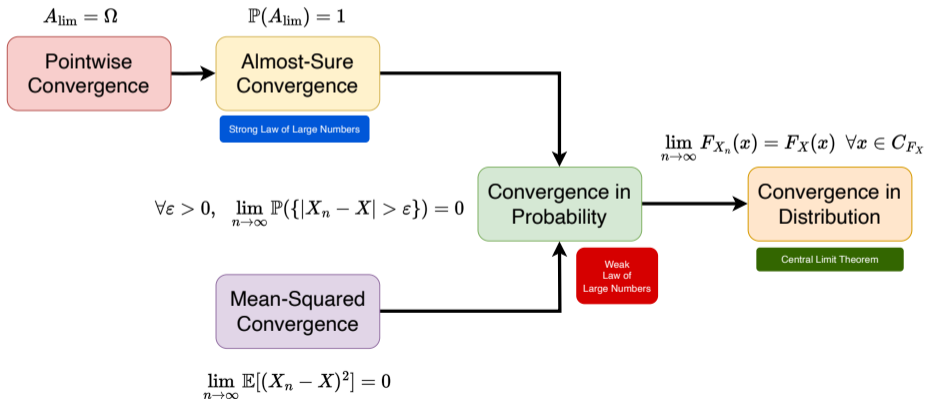**Notation:** $\quad X_n \xrightarrow{\text{d.}} X$.

Remarks:

- We assume that the CDFs of all random variables functions from $\mathbb{R} \cup \{\pm\infty\}$ to $[0, 1]$
- If $F : \mathbb{R} \cup \{\pm\infty\} \to [0, 1]$ is a valid CDF, then we define

$$F(+\infty) := \lim_{x \to +\infty} F(x), \qquad F(-\infty) := \lim_{x \to -\infty} F(x).$$

Convergence – The Full Picture

$$A_{\lim} = \left\{ \omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega) \right\}$$

$A_{\lim} = \Omega$

$\mathbb{P}(A_{\lim}) = 1$

Pointwise Convergence → Almost-Sure Convergence

Strong Law of Large Numbers

$\forall \varepsilon > 0, \quad \lim_{n \to \infty} \mathbb{P}(\{|X_n - X| > \varepsilon\}) = 0$

Convergence in Probability

$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \ \forall x \in C_{F_X}$

Convergence in Distribution

Weak Law of Large Numbers

Central Limit Theorem

Mean-Squared Convergence

$$\lim_{n \to \infty} \mathbb{E}[(X_n - X)^2] = 0$$

Proofs of Implications

Proofs of Implications $(\text{a.s.} \implies \text{p.})$

We will show that $\quad$ 2.1, 2.2, 2.3 $\quad \implies \quad$ 1, 2, 3 $\quad$ respectively

- **Proof of** $\quad$ 2.1 $\implies$ 1:

$$2.1 \quad \iff \quad \forall \varepsilon > 0, \quad \mathbb{P}\left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \{|X_n - X| > \varepsilon\}\right) = 0$$

$$\implies \quad \forall \varepsilon > 0, \quad \lim_{N \to \infty} \mathbb{P}\left(\bigcup_{n \geq N} \{|X_n - X| > \varepsilon\}\right) = 0$$

$$\implies \quad \forall \varepsilon > 0, \quad \lim_{N \to \infty} \mathbb{P}\left(\{|X_N - X| > \varepsilon\}\right) = 0$$

- Proof of $\quad$ 2.2 $\implies$ 2, $\quad$ 2.3 $\implies$ 3: $\quad$ **exercise**

Proofs of Implications ($\mathcal{L}^p \implies$ p.)

- For a fixed $p \geq 1$, we need to prove $\quad X_n \xrightarrow{\mathcal{L}^p} X \quad \implies \quad X_n \xrightarrow{\text{p.}} X$

- Recall that

$$X_n \xrightarrow{\mathcal{L}^p} X \quad \Longleftrightarrow \quad \lim_{n \to \infty} \mathbb{E}[|X_n - X|^p] = 0.$$

- Because $X_n \xrightarrow{\mathcal{L}^p} X$ implicitly means that $\mathbb{P}(\{X \in \mathbb{R}\}) = 1$, we simply have to prove that

$$X_n \xrightarrow{\mathcal{L}^p} X \quad \implies \quad \forall \, \varepsilon > 0, \quad \lim_{n \to \infty} \mathbb{P}(\{|X_n - X| > \varepsilon\}) = 0.$$

- **Proof of implication:** For every $\varepsilon > 0$, we have

$$
\begin{aligned}
\mathbb{P}(\{|X_n - X| > \varepsilon\}) &= \mathbb{P}(\{|X_n - X|^p > \varepsilon^p\}) \\
&\leq \frac{\mathbb{E}[|X_n - X|^p]}{\varepsilon^p} \qquad \text{(Markov's inequality)}.
\end{aligned}
$$

- Taking limits as $n \to \infty$ on both sides, we get the desired implication

Proofs of Implications (p. $\implies$ d.**)**

Given: $X_n \xrightarrow{\text{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n \to \infty} F_{X_n}(x) = F_X(x)$

- Because we have

$$\lim_{n \to \infty} F_{X_n}(+\infty) = F(+\infty) = 1, \qquad \lim_{n \to \infty} F_{X_n}(-\infty) = F(-\infty) = 0,$$

we simply need to demonstrate convergence of CDFs for $x \in C_{F_X} \cap \mathbb{R}$

- Fix an arbitrary $x \in C_{F_X} \cap \mathbb{R}$.
  For every choice of $\varepsilon > 0$, we have

$$\begin{aligned}
F_{X_n}(x) = \mathbb{P}(\{X_n \le x\}) &= \mathbb{P}(\{X_n \le x\} \cap \{X \le x + \varepsilon\}) + \mathbb{P}(\{X_n \le x\} \cap \{X > x + \varepsilon\}) \\
&\le \mathbb{P}(\{X \le x + \varepsilon\}) + \mathbb{P}(\{X_n - X < -\varepsilon\}) \\
&\le \mathbb{P}(\{X \le x + \varepsilon\}) + \mathbb{P}(\{|X_n - X| > \varepsilon\}) \\
&= F_X(x + \varepsilon) + \mathbb{P}(\{|X_n - X| > \varepsilon\}).
\end{aligned}$$

- Taking lim as $n \to \infty$ on both sides, we get that

$$\forall \varepsilon > 0, \qquad \lim_{n \to \infty} F_{X_n}(x) \le F_X(x + \varepsilon) + \lim_{n \to \infty} \mathbb{P}(|X_n - X| > \varepsilon) = F_X(x + \varepsilon). \tag{1}$$

Proofs of Implications (p. $\implies$ d.**)**

- Fix an arbitrary $x \in C_{F_X} \cap \mathbb{R}$.
  For every choice of $\varepsilon > 0$, we have

$$
\begin{aligned}
F_X(x - \varepsilon) &= \mathbb{P}(\{X \leq x - \varepsilon\}) \\
&= \mathbb{P}(\{X \leq x - \varepsilon\} \cap \{X_n \leq x\}) \ + \ \mathbb{P}(\{X \leq x - \varepsilon\} \cap \{X_n > x\}) \\
&\leq \mathbb{P}(\{X_n \leq x\}) \ + \ \mathbb{P}(\{X_n - X > \varepsilon\}) \\
&\leq \mathbb{P}(\{X_n \leq x\}) \ + \ \mathbb{P}(\{|X_n - X| > \varepsilon\}) \\
&= F_{X_n}(x) \ + \ \mathbb{P}(\{|X_n - X| > \varepsilon\}).
\end{aligned}
$$

- Taking limits as $n \to \infty$ on both sides, we get

$$
\forall \, \varepsilon > 0, \qquad F_X(x - \varepsilon) \ \leq \ \lim_{n \to \infty} F_{X_n}(x) \ + \lim_{n \to \infty} \mathbb{P}(|X_n - X| > \varepsilon) \ = \ \lim_{n \to \infty} F_{X_n}(x). \tag{2}
$$

- Combining (1) and (2), we get

$$\forall \, \varepsilon > 0, \ x \in C_{F_X}, \qquad F_X(x - \varepsilon) \ \leq \ \lim_{n \to \infty} F_{X_n}(x) \ \leq \ F_X(x + \varepsilon).$$

- Because the above equation holds for every $\varepsilon > 0$, we can take limits as $\varepsilon \downarrow 0$ to get

$$\forall \, x \in C_{F_X}, \qquad \lim_{\varepsilon \downarrow 0} F_X(x - \varepsilon) \ \leq \ \lim_{n \to \infty} F_{X_n}(x) \ \leq \ \lim_{\varepsilon \downarrow 0} F_X(x + \varepsilon)$$

- The blue term is equal to $F_X(x)$ because F_X is right continuous at the point x
  The red term is equal to $F_X(x)$ because $x \in C_{F_X}$, and therefore F_X is also left continuous at the point x

Problems

Problems

- Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of **independent** random variables with

$$\mathbb{P}\left(X_n = \frac{1}{2}\left(1 - \frac{1}{n}\right)\right) = \mathbb{P}\left(X_n = \frac{1}{2}\left(1 + \frac{1}{n}\right)\right) = \frac{1}{2}.$$

  — Determine whether the above sequence converges in the mean-squared sense.
  — Determine whether the above sequence converges in the almost-sure sense.

Problems

- Let $U_1, U_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Unif}[0, 1]$.
  For each $n \in \mathbb{N}$, let

  $$X_n = \min\{U_1, \ldots, U_n\}.$$

  Identify a limit, and determine the forms of convergence.

- Suppose that for each $n \in \mathbb{N}$,

  $$Y_n = n \cdot \min\{U_1, \ldots, U_n\}.$$

  Show that Y_n converges in distribution. What is the limit?

Stochastic Processes

Lecture 06

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

30 January 2026

- Let $W_1, W_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ for some fixed $\sigma > 0$.
  Let $X_0 = 0$, and for each $n \in \mathbb{N}$, let

  $$X_{n+1} = \frac{X_n + W_{n+1}}{2}.$$

  Prove that $\{X_n\}_{n \in \mathbb{N}}$ converges in distribution.

- Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Exp}(1)$.
  For each $n \in \mathbb{N}$, let

  $$Y_n = \max\{X_1, \ldots, X_n\}.$$

  — Compute the CDF of Y_n.
  — For each $a \in \mathbb{R}$, evaluate

  $$\lim_{n \to \infty} \mathbb{P}(Y_n \leq a \log n).$$

  What can you conclude from this result?

Applications to AI/ML Domain

Empirical Risk Minimisation

Suppose that $\quad X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, 1), \qquad \mu$: **unknown**

- Because μ is unknown, you guess it to be some $\theta \in \mathbb{R}$. An incorrect guess incurs a penalty (loss). Define the **loss (empirical risk)** under θ as

$$\ell(X_1, \ldots, X_n; \theta) := \frac{1}{n} \sum_{i=1}^{n} (X_i - \theta)^2.$$

  Give a closed-form expression for $\widehat{\theta}_n$, the guess that minimizes the empirical risk.
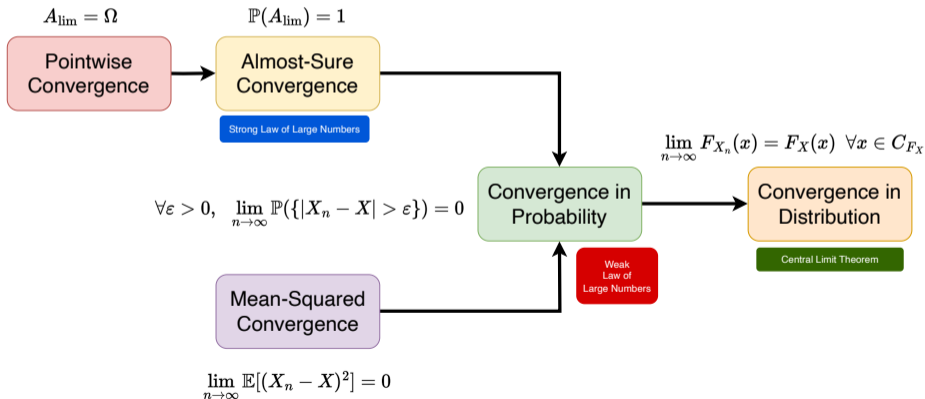
- Show that $\quad \widehat{\theta}_n \overset{\text{p.}}{\longrightarrow} \mu$.

- Show that the risk-minimizing $\widehat{\theta}_n$ is indeed a good guess, i.e.,

$$\widehat{\theta}_n \overset{\text{a.s.}}{\longrightarrow} \mu.$$

Limit Theorems

Convergence – The Full Picture

$$A_{\lim} = \left\{ \omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega) \right\}$$

$A_{\lim} = \Omega$

Pointwise Convergence

$\mathbb{P}(A_{\lim}) = 1$

Almost-Sure Convergence

Strong Law of Large Numbers

$\forall \varepsilon > 0, \quad \lim_{n \to \infty} \mathbb{P}(\{|X_n - X| > \varepsilon\}) = 0$

Convergence in Probability

Weak Law of Large Numbers

$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \ \forall x \in C_{F_X}$

Convergence in Distribution

Central Limit Theorem

Mean-Squared Convergence

$$\lim_{n \to \infty} \mathbb{E}[(X_n - X)^2] = 0$$

Weak Law of Large Numbers (WLLN)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Theorem (Weak Law of Large Numbers)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of **independent and identically distributed (i.i.d.)** real-valued RVs with $\mathbb{E}[|X_1|] < +\infty$. Further, let $\mathbb{E}[X_1] = \mu \in \mathbb{R}$. For each $n \in \mathbb{N}$, let

$$S_n := \sum_{i=1}^{n} X_i$$

denote the partial sum of random variables up to time n. Then,

$$\frac{S_n}{n} \xrightarrow{\text{p.}} \mu. \qquad \lim_{n\to\infty} \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) = 0 \qquad \forall\, \varepsilon > 0.$$

Proof of WLLN – Using Finite Variance Assumption

- We shall first see a simple proof of WLLN under a finite variance assumption
- We then provide a proof relaxing the finite variance assumption
- Suppose that $\mathrm{Var}(X_1) = \sigma^2 < +\infty$
- Then, for any choice of $\varepsilon > 0$,

$$
\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) &\leq \frac{1}{\varepsilon^2}\,\mathbb{E}\left[\left(\frac{S_n}{n} - \mu\right)^2\right] \qquad \text{(Chebyshev's inequality)} \\
&= \frac{\sigma^2}{n}
\end{aligned}
$$

- It follows that

$$
\forall\, \varepsilon > 0, \qquad \lim_{n\to\infty} \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n} = 0,
$$

thus proving that $X_n \xrightarrow{\text{p.}} \mu$

Two Important Results

+

Proof of WLLN Without Finite Variance Assumption

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Reverse Implication d. $\implies$ p.**)**

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs
For any $c \in \mathbb{R}$,

$$X_n \xrightarrow{\text{d.}} c \quad \implies \quad X_n \xrightarrow{\text{p.}} c$$

Characteristic Function

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let X be a random variable w.r.t. $\mathscr{F}$.

Definition (Characteristic Function)

The **characteristic function** of X is a function $C_X : \mathbb{R} \to \mathbb{C}$, defined as

$$C_X(s) = \mathbb{E}[e^{jsX}] = \mathbb{E}[\cos sX] + j\,\mathbb{E}[\sin sX], \qquad s \in \mathbb{R}.$$

- Observe that

$$\left| C_X(s) \right| \leq 1 \qquad \forall\, s \in \mathbb{R}.$$

- If X is extended real-valued, then its characteristic function is not defined

Taylor Expansion for Characteristic Functions

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Lemma (Taylor Expansion for Characteristic Functions)

Suppose that X is a random variable such that $\mathbb{E}[|X|^k] < +\infty$ for some $k \in \mathbb{N}$. Then,

$$C_X(s) = \sum_{\ell=0}^{k} \frac{\mathbb{E}[X^\ell]}{\ell!} (js)^\ell + o(s^k), \qquad s \in \mathbb{R}.$$

For a proof, see [KT08].

- Given two sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$, we say that $a_n = o(b_n)$ if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$

- For instance,

$$\sqrt{n} = o(n), \qquad \log n = o(n), \qquad \frac{1}{n^2} = o\left(\frac{1}{n}\right), \qquad \frac{1}{n} = o(1).$$

Characteristic Functions and Convergence in Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Characteristic Functions and Convergence in Distribution)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV. Then,

$$X_n \xrightarrow{\text{d.}} X \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} C_{X_n}(s) = C_X(s) \quad \forall\, s \in \mathbb{R}.$$

That is, **pointwise convergence of characteristic functions** is equivalent to convergence in distribution.

Proof is based on Skorokhod's representation theorem [GS20, Section 7.2].

References

Geoffrey Grimmett and David Stirzaker.
Probability and random processes.
Oxford university press, 2020.

John Frank Charles Kingman and Samuel James Taylor.
Introdction to Measure and Probability.
Cambridge University Press, 2008.