भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 15: Singular Random Variables, Multiple Random Variables, Joint CDF, Joint PMF, Marginal CDFs from Joint CDF, Marginal PMFs from Joint PMF, Conditional CDF

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

23 September 2025

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Discrete Random Variable

Definition (Discrete Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **discrete** if there exists a **countable** set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that

$$\mathbb{P}_X(E) = 1.$$

PMF $\longrightarrow$ CDF for a Discrete RV

The following implications are noteworthy:

$$p_X \quad \underset{\text{any } X}{\overset{X \text{ discrete}}{\rightleftarrows}} \quad \mathbb{P}_X \quad \underset{\text{any } X}{\overset{\text{any } X}{\rightleftarrows}} \quad F_X.$$

PMF $=$ complete probabilistic description for discrete RV.

Continuous Random Variable

Definition (Continuous Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **continuous** if $\mathbb{P}_X \ll \lambda$, i.e.,

$$\lambda(B) = 0 \quad \Longrightarrow \quad \mathbb{P}_X(B) = 0.$$

PDF $\longrightarrow$ CDF for a Continuous RV

The following implications are noteworthy:

$$f_X \quad \underset{X \text{ continuous}}{\overset{X \text{ continuous}}{\rightleftarrows}} \quad F_X \quad \underset{\text{any } X}{\overset{\text{any } X}{\rightleftarrows}} \quad \mathbb{P}_X.$$

PDF $=$ complete probabilistic description for continuous RV.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Singular Random Variables

Singular Random Variable

Definition (Singular Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **singular** if:

- $\mathbb{P}_X(\{x\}) = 0$ for every $x \in \mathbb{R}$.
- There exists an **uncountable** set $U \subseteq \mathbb{R}$ such that

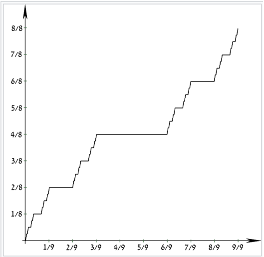
$$\lambda(U) = 0, \qquad \text{whereas} \qquad \mathbb{P}_X(U) = 1.$$

As usual, λ denotes the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

- $\mathbb{P}_X$ and U act in opposing ways on U!
- If X is singular, then $\mathbb{P}_X(B) = 0$ for every countable $B \in \mathscr{B}(\mathbb{R})$

The Cantor Function

An Example of a Singular Random Variable's CDF



- If X is a random variable having the above CDF, then

$$\mathbb{P}_X(K^{\complement}) = 0 \implies \mathbb{P}_X(K) = 1, \qquad \lambda(K) = 0, \quad \mathbb{P}_X(K) = 1$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Multiple Random Variables

Understanding $\mathscr{B}(\mathbb{R}^2)$

- Consider the special class of semi-infinite rectangles in $\mathbb{R}^2$, given by
$$\mathscr{P} = \Big\{(-\infty, x] \times (-\infty, y] : \; x, y \in \mathbb{R}\Big\}.$$
- $\mathscr{B}(\mathbb{R}^2) = \sigma\big(\mathscr{P}\big)$
- Other example sets in $\mathscr{B}(\mathbb{R}^2)$:
  - $(-\infty, x] \times \mathbb{R}, \quad (-\infty, x) \times \mathbb{R}, \quad [x, \infty) \times \mathbb{R}, \quad (x, \infty) \times \mathbb{R}, \quad x \in \mathbb{R}$
  - $\mathbb{R} \times (-\infty, y], \quad \mathbb{R} \times (-\infty, y), \quad \mathbb{R} \times [y, \infty), \quad \mathbb{R} \times (y, \infty), \quad y \in \mathbb{R}$
  - $\mathbb{R} \times (a, b), \quad (a, b) \times \mathbb{R}, \quad a, b \in \mathbb{R}$
  - $(a, b) \times (c, d), \quad a, b, c, d \in \mathbb{R}$
  - Circle of radius r centered at the origin, $\quad r > 0$

Important

$$\mathscr{B}(\mathbb{R}^2) \quad \neq \quad \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) = \Big\{B_1 \times B_2 : \; B_1, B_2 \in \mathscr{B}(\mathbb{R})\Big\}.$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Two Random Variables (Bivariate Random Vector)

Definition (Bivariate Random Vector)

Fix a measurable space $(\Omega, \mathscr{F})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables (with respect to $\mathscr{F}$).
We say $(X, Y) : \Omega \to \mathbb{R}^2$ is a **bivariate random vector** with respect to $\mathscr{F}$ if

$$\forall\, B \in \mathscr{B}(\mathbb{R}^2), \qquad (X,Y)^{-1}(B) = \underbrace{\{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\}}_{\text{pre-image of } B} = \{(X,Y) \in B\} \in \mathscr{F}.$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2)) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Bivariate Random Vector

Theorem (Equivalent Characterization of Bivariate Random Vector)

Fix a measurable space $(\Omega, \mathscr{F})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables (with respect to $\mathscr{F}$).
Then,

$$(X, Y) \text{ random vector} \qquad \Longleftrightarrow \qquad (X, Y)^{-1}(B) \in \mathscr{F} \quad \forall\, B \in \mathscr{P},$$

where $\mathscr{P}$ is the collection $\mathscr{P} = \Big\{(-\infty, x] \times (-\infty, y] : \; x, y \in \mathbb{R}\Big\}$.

Bivariate Random Vector Simplified

Fix a measurable space $(\Omega, \mathscr{F})$, and let X, Y be random variables.
$(X, Y) : \Omega \to \mathbb{R}^2$ is a bivariate random vector **if and only if** for all $x, y \in \mathbb{R}$,

$$(X, Y)^{-1}\big((\infty,\ x] \times (-\infty,\ y]\big) = \underbrace{\{\omega \in \Omega : X(\omega) \leq x\} \cap \{\omega \in \Omega : Y(\omega) \leq y\}}_{\text{pre-image of } (-\infty,\ x] \times (-\infty,\ y]} \in \mathscr{F}.$$

Joint Probability Law

Definition (Joint Probability Law)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
The **joint probability law of X and Y** is a function $\mathbb{P}_{X,Y} : \mathscr{B}(\mathbb{R}^2) \to [0, 1]$, defined as

$$\forall\, B \in \mathscr{B}(\mathbb{R}^2), \qquad \mathbb{P}_{X,Y}(B) = \mathbb{P}((X,Y)^{-1}(B)) = \mathbb{P}(\{(X,Y) \in B\}).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "{(X,Y)}"] & & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

On $\mathbb{P}_{X,Y}$

$\mathbb{P}_{X,Y}$ is a **probability measure** on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$.

Joint CDF

$$\begin{tikzcd}
(\Omega,\mathscr{F},\mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2,\mathscr{B}(\mathbb{R}^2),{\color{red}\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$ {\color{red}\mathbb{P}_{X,Y}}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

- $\mathbb{P}_{X,Y}(B) \in [0,1]$ for every $B \in \mathscr{B}(\mathbb{R}^2)$
- In particular, $\mathbb{P}_X\big((-\infty,\ x] \times (-\infty,\ y]\big) \in [0,1]$ for all $x, y \in \mathbb{R}$
- We thus have a mapping

$$(x,y) \ \mapsto \ \mathbb{P}_X\big((-\infty,\ x] \times (-\infty,\ y]\big)$$

- The above mapping (or function) is called the **joint CDF** of X and Y, denoted by $F_{X,Y}$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Joint CDF

Definition (Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector.
The **joint CDF of X and Y (or CDF of the vector (X, Y))** is a function $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ defined as

$$\forall\, x, y \in \mathbb{R}, \qquad F_{X,Y}(x, y) = \mathbb{P}_{X,Y}\Big((-\infty, x] \times (-\infty, y]\Big) = \mathbb{P}\Big(\{X \le x\} \cap \{Y \le y\}\Big).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
\{X \le x\} \cap \{Y \le y\} \arrow[u] & (-\infty, x] \times (-\infty, y] \arrow[l, "\text{pre-image}"]
\end{tikzcd}$$

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}((-\infty, x] \times (-\infty, y]) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\}), \quad x, y \in \mathbb{R}$$

Properties of Joint CDF

Lemma (Properties of Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector with CDF $F_{X,Y}$.
Then, $F_{X,Y}$ satisfies the following properties.

1. **(Monotonicity)** If $x_1 \leq x_2$ and $y_1 \leq y_2$, then $F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$.
2. If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are any two sequences such that $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} F_{X,Y}(x_n, y_n) = 0$.
3. If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are any two sequences such that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = +\infty$, then $\lim_{n\to\infty} F_{X,Y}(x_n, y_n) = 1$.
4. **(Continuity from Top-Right Quadrant)**

   $F_{X,Y}$ is continuous from the top-right quadrant at each point in its domain.
   More formally, for each $(x, y) \in \mathbb{R}^2$,

   $$x_n > x \ \forall n \in \mathbb{N}, \quad y_n > y \ \forall n \in \mathbb{N}, \quad \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y \implies \lim_{n\to\infty} F_{X,Y}(x_n, y_n) = F_{X,Y}(x, y).$$

Another Important Function

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), {\color{red}\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$${\color{red}\mathbb{P}_{X,Y}}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

- Taking $B = (-\infty,\ x] \times (-\infty,\ y]$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}\big((-\infty,\ x] \times (-\infty,\ y]\big)$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Another Important Function

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, bend left, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

- Taking $B = (-\infty,\ x] \times (-\infty,\ y]$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}\big((-\infty,\ x] \times (-\infty,\ y]\big)$$

- Taking $B = \{x\} \times \{y\}$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}(\{x\} \times \{y\})$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Another Important Function

$$
\begin{tikzcd}
(\Omega,\mathscr{F},\mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2,\mathscr{B}(\mathbb{R}^2),{\color{red}\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$
{\color{red}\mathbb{P}_{X,Y}}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)
$$

- Taking $B = (-\infty,\ x] \times (-\infty,\ y]$, and varying x, y, we get a mapping

$$
(x,y) \quad \mapsto \quad \mathbb{P}_{X,Y}\big((-\infty,\ x]\times(-\infty,\ y]\big)
$$

- Taking $B = \{x\} \times \{y\}$, and varying x, y, we get a mapping

$$
(x,y) \quad \mapsto \quad \mathbb{P}_{X,Y}(\{x\} \times \{y\})
$$

- The above map is called the **joint CDF**, denoted $F_{X,Y}$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Another Important Function

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \color{red}{\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\color{red}{\mathbb{P}_{X,Y}}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

- Taking $B = (-\infty,\ x] \times (-\infty,\ y]$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}\big((-\infty,\ x] \times (-\infty,\ y]\big)$$

- The above map is called the **joint CDF**, denoted $F_{X,Y}$

- Taking $B = \{x\} \times \{y\}$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}(\{x\} \times \{y\})$$

- The above map is called the **joint PMF**, denoted $p_{X,Y}$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Another Important Function

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "{(X,Y)}"] & & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

- Taking $B = (-\infty, \ x] \times (-\infty, \ y]$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}\big((-\infty, \ x] \times (-\infty, \ y]\big)$$

- The above map is called the **joint CDF**, denoted $F_{X,Y}$
- $F_{X,Y}(x, y) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\})$

- Taking $B = \{x\} \times \{y\}$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}(\{x\} \times \{y\})$$

- The above map is called the **joint PMF**, denoted $p_{X,Y}$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Another Important Function

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "{(X,Y)}"] & & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \color{red}{\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$\color{red}{\mathbb{P}_{X,Y}}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

- Taking $B = (-\infty,\ x] \times (-\infty,\ y]$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}\big((-\infty,\ x]\times(-\infty,\ y]\big)$$

- The above map is called the **joint CDF**, denoted $F_{X,Y}$
- $F_{X,Y}(x, y) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\})$

- Taking $B = \{x\} \times \{y\}$, and varying x, y, we get a mapping

$$(x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}(\{x\} \times \{y\})$$

- The above map is called the **joint PMF**, denoted $p_{X,Y}$
- $p_{X,Y}(x, y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})$

Joint PMF

Definition (Joint PMF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector.
Let $\mathbb{P}_{X,Y}$ denote the joint probability law of X and Y.
The **joint PMF of X and Y (or PMF of the vector (X, Y))** is a function $p_{X,Y} : \mathbb{R}^2 \to [0, 1]$ defined as

$$\forall\, x, y \in \mathbb{R}, \qquad p_{X,Y}(x, y) = \mathbb{P}_{X,Y}(\{x\} \times \{y\}) = \mathbb{P}(\{X = x\} \cap \{Y = y\}).$$

- **Joint CDF ($F_{X,Y}$) and joint PMF ($p_{X,Y}$) are always defined for any two RVs X and Y**

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Marginal CDFs from Joint CDF

Theorem (Marginal CDFs from Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector. Let $F_{X,Y}$ denote the joint CDF of X and Y.
Then, the following properties hold.

1. **(Marginalization of Y)**

   If $y_1, y_2, \ldots$ is any sequence of real numbers such that $\lim_{n\to\infty} y_n = +\infty$, then

   $$\forall\, x \in \mathbb{R}, \qquad \lim_{n\to\infty} F_{X,Y}(x,\ y_n) = F_X(x).$$

2. **(Marginalization of X)**

   If $x_1, x_2, \ldots$ is any sequence of real numbers such that $\lim_{n\to\infty} x_n = +\infty$, then

   $$\forall\, y \in \mathbb{R}, \qquad \lim_{n\to\infty} F_{X,Y}(x_n,\ y) = F_Y(y).$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Conditional CDF

Definition (Conditional CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector.

1. Fix $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$.
   The **conditional CDF of X, conditioned on A**, is defined as

$$F_{X|A} : \mathbb{R} \to [0,1], \qquad F_{X|A}(x) := \frac{\mathbb{P}(\{X \leq x\} \cap A)}{\mathbb{P}(A)}, \qquad x \in \mathbb{R}.$$

2. The **conditional CDF of X, conditioned on Y**, is defined as

$$\forall\, x \in \mathbb{R}, \qquad F_{X|Y}(x|y) := \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \leq y\})}{\mathbb{P}(\{Y \leq y\})},$$

   whenever denominator is non-zero.