1. (a) **(1 Mark)**
   If $\mathbb{P}(A|B) = \mathbb{P}(A|B^{\complement})$, then show that $A \perp\!\!\!\perp B$.
   (b) **(2 Marks)**
   On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, for events $A, B \in \mathscr{F}$, we say that A *suggests* B if

$$\mathbb{P}(A \cap B) \;\geq\; \mathbb{P}(A)\,\mathbb{P}(B).$$

   Let $A, B, C \in \mathscr{F}$. Suppose that A suggests B and B suggests C.
   Must it follow that A suggests C? Either prove the implication or provide a concrete counterexample.
   (c) **(2 Marks)**
   Consider events $A, B, C \in \mathscr{F}$ such that $A \perp\!\!\!\perp B$ and $A \perp\!\!\!\perp C$. Show that

$$A \perp\!\!\!\perp B \cup C \quad \iff \quad A \perp\!\!\!\perp B \cap C.$$

Solution.

(a) We define conditional probability if $\mathbb{P}(B) \neq 0$.
By the Law of Total Probability,

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^{\complement})\mathbb{P}(B^{\complement}) \\ &= \mathbb{P}(A \mid B)\,\mathbb{P}(B) + \mathbb{P}(A \mid B)\,(1 - \mathbb{P}(B)) \quad \text{since, } \mathbb{P}(A \mid B) = \mathbb{P}(A \mid B^{\complement}) \\ &= \mathbb{P}(A \mid B)\end{aligned}$$

From above,

$$\begin{aligned}\mathbb{P}(A \mid B) = \mathbb{P}(A) \Rightarrow \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} &= \mathbb{P}(A) \\ \Rightarrow \mathbb{P}(A \cap B) &= \mathbb{P}(A) \cdot \mathbb{P}(B)\end{aligned}$$

Hence, it follows that $A \perp\!\!\!\perp B$.

(b) Let $A, B, C \in \mathscr{F}$ be events, such that

$$\mathbb{P}(B) = 0, \qquad \mathbb{P}(A) > 0, \qquad \mathbb{P}(C) > 0, \qquad A \cap C = \emptyset.$$

Then

$$\begin{aligned}\mathbb{P}(A \cap B) \geq 0 = \mathbb{P}(A)\mathbb{P}(B) \quad &\Rightarrow \quad A \text{ suggests } B, \\ \mathbb{P}(B \cap C) \geq 0 = \mathbb{P}(B)\mathbb{P}(C) \quad &\Rightarrow \quad B \text{ suggests } C,\end{aligned}$$

but

$$\mathbb{P}(A \cap C) = 0 \;<\; \mathbb{P}(A)\mathbb{P}(C) \quad \Rightarrow \quad A \text{ does not suggest } C.$$

Hence “suggests” need not be transitive.

(c) Remember

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \qquad \mathbb{P}(X \cup Y) = \mathbb{P}(X) + \mathbb{P}(Y) - \mathbb{P}(X \cap Y).$$

(⇒) Assume $A \perp\!\!\!\perp (B \cup C)$, i.e. $\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A)\mathbb{P}(B \cup C)$. Then

$$\begin{aligned}\mathbb{P}(A)\mathbb{P}(B \cup C) &= \mathbb{P}(A \cap (B \cup C)) \\ &= \mathbb{P}\big((A \cap B) \cup (A \cap C)\big) \\ &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C) \\ &= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A \cap B \cap C) \quad \text{as } A \perp\!\!\!\perp B, A \perp\!\!\!\perp C\end{aligned}$$

Rearranging and using $\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)$, we get

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)[\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cup C)]$$

$$= \mathbb{P}(A)\mathbb{P}(B \cap C).$$

Hence $A \perp\!\!\!\perp (B \cap C)$.
($\Leftarrow$) Assume $A \perp\!\!\!\perp (B \cap C)$, i.e. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B \cap C)$. Then

$$\begin{aligned}
\mathbb{P}\big(A \cap (B \cup C)\big) &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C) \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B \cap C) \quad \text{as } A \perp\!\!\!\perp B, A \perp\!\!\!\perp C \,\&\, A \perp\!\!\!\perp (B \cap C) \\
&= \mathbb{P}(A)\big[\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)\big] \\
&= \mathbb{P}(A)\mathbb{P}(B \cup C)
\end{aligned}$$

so $A \perp\!\!\!\perp (B \cup C)$.

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2. **(4x1.5 = 6 Marks)**

State whether the following statements are True or False. Give a brief justification.

(a) The limit of a sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} A_n$, exists only if $\{A_n\}_{n\in\mathbb{N}}$ is either increasing or decreasing.
(b) An event A can be self independent only if $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.
(c) Let S be the collection of all singleton subsets of $\mathbb{R}$. Then, $\sigma(S) = 2^{\mathbb{R}}$.
(d) A finite union of σ-algebras is a σ-algebra.

Solution.

(a) *False.* A limit can exist without monotonicity. Work on $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$ and fix any $B \in \mathscr{F}$. Define

$$A_n := B \cup \{n\}, \qquad n \in \mathbb{N}.$$

The sequence is neither increasing nor decreasing. For any $\omega \in \Omega$, $\omega \in \{n\}$ holds for at most one n, hence ω belongs to only finitely many of the singletons. Therefore

$$\limsup_{n\to\infty} A_n = \bigcap_{m\geq 1} \bigcup_{n\geq m} A_n = B, \qquad \liminf_{n\to\infty} A_n = \bigcup_{m\geq 1} \bigcap_{n\geq m} A_n = B.$$

Thus $\lim_{n\to\infty} A_n$ exists and equals B, even though (A_n) is not monotone.

(b) *True.* Self-independence means $\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$. Since $A \cap A = A$,

$$\mathbb{P}(A) = \mathbb{P}(A)^2 \quad \Rightarrow \quad \mathbb{P}(A)\left(1 - \mathbb{P}(A)\right) = 0,$$

so $\mathbb{P}(A) \in \{0, 1\}$.

(c) *False.* Let $S = \big\{\{x\} : x \in \mathbb{R}\big\}$. The σ-algebra generated by S is

$$\sigma(S) = \big\{A \subseteq \mathbb{R} : \ A \text{ is countable or } A^{\complement} \text{ is countable}\big\}.$$

Take the Cantor set $K \subset [0,1]$, which is uncountable with uncountable complement; hence $K \notin \sigma(S)$ while $K \in 2^{\mathbb{R}}$. Therefore $\sigma(S) \neq 2^{\mathbb{R}}$.

(d) *False.* Take $\Omega = \{1, 2, 3\}$ and

$$\mathscr{F}_1 = \{\emptyset, \Omega, \{1\}, \{2,3\}\}, \qquad \mathscr{F}_2 = \{\emptyset, \Omega, \{2\}, \{1,3\}\}.$$

Both $\mathscr{F}_1, \mathscr{F}_2$ are σ-algebras. However, in $\mathscr{F}_1 \cup \mathscr{F}_2$ we have $\{1\} \in \mathscr{F}_1$ and $\{2\} \in \mathscr{F}_2$, but

$$\{1\} \cup \{2\} = \{1, 2\} \notin \mathscr{F}_1 \cup \mathscr{F}_2,$$

so $\mathscr{F}_1 \cup \mathscr{F}_2$ fails to be a σ-algebra (not closed under finite unions).

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3. Let Ω be the unit square $[0,1]\times[0,1]$.
Let $\mathscr{B}([0,1])$ denote the Borel σ-algebra of subsets of $[0,1]$. Let $\mathscr{F}$ denote the collection

$$\mathscr{F} := \Big\{ B\times[0,1] : B\in\mathscr{B}([0,1]) \Big\}.$$

Let $\mathbb{P}:\mathscr{F}\to[0,1]$ be given by

$$\mathbb{P}(B\times[0,1]) = \lambda(B),$$

where λ is the Lebesgue measure on $\mathscr{B}([0,1])$.

(a) **(3 Marks)**
Show that $\mathscr{F}$ is a σ-algebra of subsets of Ω.

(b) **(2 Marks)**
Show that $\mathbb{P}:\mathscr{F}\to[0,1]$ is a probability measure.

Solution.

(a) i. We notice, $[0,1]\in\mathscr{B}([0,1])$,

$$\Omega = [0,1]\times[0,1] = [0,1]\times[0,1]\in\mathscr{F}.$$

ii. *Closure under complements (relative to Ω).*
Let $A\in\mathscr{F}$. Then $A = B\times[0,1]$ for some $B\in\mathscr{B}([0,1])$. For complements taken in Ω,

$$A^{\complement} = \Omega\setminus(B\times[0,1]) = \{(x,y)\in[0,1]\times[0,1]:\ x\notin B\} = (B^{\complement})\times[0,1],$$

where $B^{\complement} = [0,1]\setminus B\in\mathscr{B}([0,1])$. Hence $A^{\complement}\in\mathscr{F}$.

iii. *Closure under countable unions.* Let $(A_n)_{n\in\mathbb{N}}\in\mathscr{F}$, $A_n = B_n\times[0,1]$ with $B_n\in\mathscr{B}([0,1])$. Then

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}}(B_n\times[0,1]) = \Big(\bigcup_{n\in\mathbb{N}} B_n\Big)\times[0,1].$$

Since $\mathscr{B}([0,1])$ is a σ-algebra, $\bigcup_n B_n\in\mathscr{B}([0,1])$; hence the union lies in $\mathscr{F}$.

Therefore $\mathscr{F}$ is a σ-algebra on Ω.

(b) i.

$$\mathbb{P}(\emptyset) = \mathbb{P}(\emptyset\times[0,1]) = \lambda(\emptyset) = 0, \qquad \mathbb{P}(\Omega) = \mathbb{P}([0,1]\times[0,1]) = \lambda([0,1]) = 1.$$

ii. *Countable additivity on disjoint sets.* Let $(A_n)_{n\in\mathbb{N}}\in\mathscr{F}$ be pairwise disjoint, with $A_n = B_n\times[0,1]$ and $B_n\in\mathscr{B}([0,1])$.
Now

$$\mathbb{P}\Big(\bigsqcup_{n\in\mathbb{N}} A_n\Big) = \mathbb{P}\bigg(\Big(\bigcup_{n\in\mathbb{N}} B_n\Big)\times[0,1]\bigg) = \lambda\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \sum_{n\in\mathbb{N}}\lambda(B_n) = \sum_{n\in\mathbb{N}}\mathbb{P}(A_n),$$

using countable additivity of λ on the disjoint (B_n).

Thus $\mathbb{P}$ satisfies the axioms of a probability measure on $(\Omega,\mathscr{F})$.

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4. (a) **(2 Marks)**
Buses arrive at ten-minute intervals starting at noon. A man arrives at the bus stop at a random time X minutes post noon, where X has the CDF

$$F_X(x) = \begin{cases} 0, & x<0, \\ \dfrac{x}{60}, & 0\le x\le 60, \\ 1, & x>60. \end{cases}$$

What is the probability that the man waits more than six minutes for a bus?

(b) Given $p \in \mathbb{R}$ and $\theta \in \mathbb{R}$, let X be a random variable with CDF F given by

$$F(x) = \begin{cases} 0, & x < 0, \\ p, & 0 \le x < 1, \\ p + (1-p)\,\dfrac{x-1}{\theta}, & 1 \le x < 1+\theta, \\ 1, & x \ge 1+\theta. \end{cases}$$

i. **(1 Mark)**
Find all values of p and θ for which F is a *valid* CDF on $\mathbb{R}$.

ii. **(3 Marks)**
Specify the range of values for p and θ such that X is:

A. Continuous.
B. Discrete.
C. Mixed.

Solution. .

(a) X is the arrival time (in minutes) post noon with CDF F_X.
Our goal is to compute $\mathbb{P}(\text{wait} > 6 \text{ minutes})$.
Buses arrive at times $0, 10, 20, 30, 40, 50, 60$. The man waits more than six minutes iff he arrives in the first 4 minutes after a bus within each 10-minute block. Thus

$$\{\text{wait} > 6\} = \bigsqcup_{i=0}^{5} (10i,\ 10i+4).$$

These six intervals are disjoint and endpoints have probability 0 (by absolute continuity), hence

$$\mathbb{P}(\text{wait} > 6) = \mathbb{P}\big(X \in \bigsqcup_{i=0}^{5}(10i,\ 10i+4)\big) = \sum_{i=0}^{5} \mathbb{P}\big(10i < X < 10i+4\big) = \sum_{i=0}^{5} \big[F_X(10i+4) - F_X(10i)\big].$$

Using $F_X(x) = x/60$ on $[0, 60]$,

$$F_X(10i+4) - F_X(10i) = \frac{10i+4}{60} - \frac{10i}{60} = \frac{4}{60}.$$

Therefore,

$$\mathbb{P}(\text{wait} > 6) = 6 \cdot \frac{4}{60} = \frac{24}{60} = \frac{2}{5}.$$

(b) i. We can clearly see that $0 \le p \le 1$ & $\theta \ne 0$. If $\theta < 0$, then the regions $[0, 1)$ and $[1+\theta, \infty)$ non-decreasing property is violated. Hence we must have $\theta > 0$.

$$F_X \text{ is a valid CDF iff } \theta \ge 0 \text{ and } 0 \le p \le 1.$$

ii.

$$\begin{aligned} &\text{Continuous} && \iff && p = 0 && \big(f_X(x) = \tfrac{1}{\theta}\mathbf{1}_{[1,1+\theta]}(x)\big), \\ &\text{Discrete} && \iff && p = 1 && \big(\mathbb{P}(X = 0) = 1\big), \\ &\text{Mixed} && \iff && 0 < p < 1 && \big(\mathbb{P}(X=0) = p,\ X_{\text{cont}} \sim \text{Unif}[1, 1+\theta] \text{ of weight } 1-p\big). \end{aligned}$$

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5. Let $(\Omega, \mathscr{F}, \mathbb{P}) = ((0,1], \mathscr{B}((0,1]), \lambda)$, where λ is Lebesgue measure.
Define $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ as

$$\forall\, \omega \in \Omega, \qquad X(\omega) = \left\lfloor \frac{1}{\omega} \right\rfloor, \qquad Y(\omega) = \text{sgn}(\sin(2\pi\omega)),$$

where sgn is defined as

$$\text{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0 \end{cases}$$

(a) **(4 Marks)**
Determine $\mathbb{P}_X, \mathbb{P}_Y$.

(b) **(2 Marks)**
Let $\sigma(Y)$ be defined as the collection

$$\sigma(Y) := \Big\{Y^{-1}(B) : B \in \mathscr{B}(\mathbb{R})\Big\}.$$

Determine $\sigma(Y)$ explicitly for Y as defined above.

(c) **(2 Marks)**
Determine $\mathbb{P}(E)$, where E is defined as

$$E := \{\omega \in \Omega : X(\omega) \geq Y(\omega)\}.$$

Solution.

(a) **Probability Law of X.** For $k \in \mathbb{N}$,

$$\{\omega \in [0,1] : \ X(\omega) = k\} = \{\omega \in (0,1] : \ \lfloor 1/\omega \rfloor = k\} = \{\omega \in (0,1] : \ k \leq 1/\omega < k+1\}.$$

Solving the inequalities gives

$$k \leq \frac{1}{\omega} < k+1 \iff \frac{1}{k+1} < \omega \leq \frac{1}{k}.$$

Hence, for $k \in \mathbb{N}$,

$$\mathbb{P}(X = k) = \lambda\left(\left(\frac{1}{k+1}, \frac{1}{k}\right]\right) = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}.$$

Therefore,

$$P_X(\{k\}) = \frac{1}{k(k+1)} \quad \forall k \in \mathbb{N}$$

P_X is a probability law.

Probability Law of Y.
Observe that

$$\sin(2\pi\omega) \begin{cases} > 0, & \omega \in (0, \frac{1}{2}), \\ = 0, & \omega \in \{\frac{1}{2}, 1\}, \\ < 0, & \omega \in (\frac{1}{2}, 1), \end{cases} \quad \Rightarrow \quad Y(\omega) = \begin{cases} 1, & \omega \in (0, \frac{1}{2}), \\ 0, & \omega \in \{\frac{1}{2}, 1\}, \\ -1, & \omega \in (\frac{1}{2}, 1). \end{cases}$$

Thus

$$\{\omega \in \Omega : \ Y(\omega) = 1\} = (0, \tfrac{1}{2}), \quad \{\omega \in \Omega : \ Y(\omega) = -1\} = (\tfrac{1}{2}, 1), \quad \{\omega \in \Omega : \ Y(\omega) = 0\} = \{\tfrac{1}{2}, 1\}.$$

Therefore,

$$\mathbb{P}(Y = 1) = \lambda(0, \tfrac{1}{2}) = \tfrac{1}{2}, \qquad \mathbb{P}(Y = -1) = \lambda(\tfrac{1}{2}, 1) = \tfrac{1}{2}, \qquad \mathbb{P}(Y = 0) = \lambda\{\tfrac{1}{2}, 1\} = 0.$$

Hence

$$P_Y(\{1\}) = \tfrac{1}{2}, \quad P_Y(\{-1\}) = \tfrac{1}{2}, \quad P_Y(\{0\}) = 0\,.$$

(b) Define the three atoms (inverse images of singletons):

$$A_{+1} := Y^{-1}(\{1\}) = (0, \tfrac{1}{2}), \qquad A_0 := Y^{-1}(\{0\}) = \{\tfrac{1}{2}, 1\}, \qquad A_{-1} := Y^{-1}(\{-1\}) = (\tfrac{1}{2}, 1).$$

Since the range of Y is $\{-1, 0, 1\}$, for any $B \in \mathscr{B}(\mathbb{R})$,

$$Y^{-1}(B) = \bigcup_{y \in \{-1,0,1\} \cap B} Y^{-1}(\{y\})\,,$$

so the measurable sets generated by Y are precisely all unions of the atoms A_{+1}, A_0, A_{-1}. Hence

$$\sigma(Y) = \sigma\big(\{A_{+1}, A_0, A_{-1}\}\big)$$

Explicitly, this is the collection of 8 sets

$$\Big\{\emptyset,\ A_{+1},\ A_{-1},\ A_0,\ A_{+1} \cup A_{-1},\ A_{+1} \cup A_0,\ A_{-1} \cup A_0,\ (0,1]\Big\}.$$

(c) Notice $E = (0, 1]$ as $\forall \omega \in (0, 1]$, $X(\omega) = \lfloor 1/\omega \rfloor \geq 1$ and $Y(\omega) \in \{-1, 0, 1\}$, hence

$$\mathbb{P}(E) = \lambda((0, 1]) = 1.$$

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6. **(Bonus Question)**

(a) **(3 Marks)**

Consider the following two-player game between A and B.

- Both players are constructing the decimal expansion of a number in $[0, 1]$.
- They take turns writing digits: A writes the first digit D_1, then B writes D_2, then A writes D_3, and so on indefinitely, where $D_i \in \{0, 1, \dots, 9\}$ for each $i \in \mathbb{N}$.

$$x \; = \; 0.D_1 D_2 D_3 \cdots \in [0, 1].$$

The winning rule is as follows:

- If x is **rational**, then A wins.
- If x is **irrational**, then B wins.

Show that B can always win.

(b) **(3 Marks)**

Consider one of our standard probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$ = $([0, 1], \mathscr{B}([0, 1]), \lambda)$, where λ is the Lebesgue measure. To every element $\omega \in \Omega$ we assign its infinite decimal representation. We disallow decimal representations that end with an infinite string of nines (thus, for e.g., $1/10 = 0.1$ and not $0.099999 \cdots$).

Define an event E as

$$E \; \coloneqq \; \{\omega \in [0, 1] : \text{the decimal expansion of } \omega \text{ contains all digits in } \{0, 1, \dots, 9\}\}.$$

Find $\lambda(E)$.

Solution.

(a) **Diagonalization argument**

Let list the rationals in $[0, 1]$ as $\mathbb{Q} \cap [0, 1] = \{q_1, q_2, q_3, \dots\}$, and for each $q \in \mathbb{Q} \cap [0, 1]$ fix its *canonical* decimal expansion (the one *not* ending in an infinite tail of 9s), writing $q = 0.d_1(q)d_2(q)d_3(q) \cdots$ with digits $d_k(q) \in \{0, 1, \dots, 9\}$.

On B's i-th move (i.e., when choosing the digit D_{2i}), use the rule

$$D_{2i} \; = \; \begin{cases} 1, & \text{if } d_{2i}(q_i) = 0, \\ 0, & \text{if } d_{2i}(q_i) \neq 0. \end{cases}$$

Then for every $i \in \mathbb{N}$, $D_{2i} \neq d_{2i}(q_i)$. Consequently, the final number $x = 0.D_1 D_2 D_3 \cdots$ differs from q_i at the $2i$-th digit, hence $x \neq q_i$ for all i. Therefore $x \notin \mathbb{Q} \cap [0, 1]$, i.e., x is irrational, and B wins regardless of A's moves.

(b) **Variation of Cantor's Set**

For $k \in \mathbb{N}$, let $D_k(\omega) \in \{0, 1, \dots, 9\}$ be the k-th decimal digit of ω.

Define Cylinder sets and their measure

For any finite word $a_1 \dots a_n \in \{0, 1, \dots, 9\}^n$, define the cylinder

$$C(a_1 \dots a_n) := \{\omega \in [0, 1] : D_1(\omega) = a_1, \dots, D_n(\omega) = a_n\}.$$

By the choice of canonical expansions, $C(a_1 \dots a_n) = \left[\frac{m}{10^n}, \frac{m+1}{10^n}\right)$ where $m = \sum_{j=1}^{n} a_j \, 10^{\,n-j}$; hence

$$\lambda(C(a_1 \dots a_n)) = \frac{1}{10^n}.$$

Therefore, for each fixed n, the 10^n cylinders of length n have equal measure $\frac{1}{10^n}$ and are disjoint.

Probability that a fixed digit never appears.
Fix $i \in \{0, 1, \dots, 9\}$ and let

$$A_i^{(n)} := \{\omega : D_1(\omega) \neq i, \dots, D_n(\omega) \neq i\}$$

be the event that digit i does not occur among the first n digits. There are exactly 9^n length-n cylinders avoiding i, so

$$\lambda(A_i^{(n)}) = 9^n \cdot \frac{1}{10^n} = \left(\frac{9}{10}\right)^n.$$

Let $A_i := \bigcap_{n=1}^{\infty} A_i^{(n)}$ be the event "digit i never appears." Since $A_i^{(n)} \downarrow A_i$ as $n \to \infty$ (decreasing sequence), by continuity of measure for decreasing events,

$$\lambda(A_i) = \lim_{n\to\infty} \lambda(A_i^{(n)}) = \lim_{n\to\infty} \left(\frac{9}{10}\right)^n = 0.$$

All digits appear at least once.

Let $E_i := A_i^{\complement}$ be the event "digit i appears at least once," and let

$$E := \bigcap_{i=0}^{9} E_i = \{\omega \in [0,1] : \text{every digit } 0, 1, \dots, 9 \text{ occurs at least once}\}.$$

Then $E^{\complement} = \bigcup_{i=0}^{9} A_i$, so by subadditivity and the computation above,

$$\lambda(E^{\complement}) \;\le\; \sum_{i=0}^{9} \lambda(A_i) \;=\; 0, \qquad \Rightarrow \qquad \lambda(E) = 1 - \lambda(E^{\complement}) = 1 - 0 = 1.$$

Grading Rubric for Mid-01

Qn / Part	Key Steps / Criteria	Marks
1(a)	Apply Law of Total Probability or equivalent + deduce independence final conclusion $A \perp\!\!\!\perp B$.	0.5 M 0.5 M = 1 M
1(b)	Construct counterexample (e.g. with $P(B) = 0$) verify $A \to B, B \to C$ show $A \not\to C$.	- 1 M 1 M = 2 M
1(c)	Forward proof using inclusion–exclusion Backward proof	1.0 M 1 M = 2 M
2(a–d)	For each: correct T/F Justification or example/counterexample if required.	- 0.5 M 1 M = 1.5 M each
3(a)	Show F is a σ-algebra: contains Ω, closure under complements, closure under unions.	- 1 M 1 M 1 M = 3 M
3(b)	Show P is a probability measure: $P(\emptyset) = 0, P(\Omega) = 1$ countable additivity.	- 1 M 1 M = 2 M
4(a)	Identify intervals Sum probabilities (CDF) correctly + final result $2/5$.	1 M 1 M = 2 M
4(b)	For each: continuous / discrete / mixed Identify admissible ranges for p, θ justify.	- 0.5 M 0.5 M = 1 M(each)
5(a)	Correct PMF of X correct PMF of Y.	2 M 2 M = 4 M
5(b)	Identify atoms list 8 sets of $\sigma(Y)$.	1 M 1 M = 2 M
5(c)	Argue $E = (0, 1]$ compute probability = 1.	1 M 1 M = 2 M
6(a)	Diagonalization strategy: show differs from all rationals conclude B wins.	2.5 M 0.5 M = 3 M
6(b)	Lebesgue measure of Cylinder set = 0 (argument) conclude $\lambda(E) = 1$.	2.5 M 0.5 M = 3 M