Random Processes 2018

TA Session 1

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(Who is solely responsible for the mistakes that may appear in this document)

Agenda:

- A brief history of Kolmogorov's theory of probability
- Algebras and σ-algebras – examples and exercises
- "Infinitely often" and "all but finitely many" events.

A brief history of Kolmogorov's theory of probability

- Bernoulli (1713) and De-Moivre (1718) gave the first definition for probability of occurrence of an event:

$$\text{probability of an event} = \frac{\#\text{ cases in favour of the event}}{\text{Total }\#\text{ cases possible under the given circumstances}}$$

De-Moivre also gave the following two theorems:

1. Addition theorem (or the theorem of total probability)

2. Multiplication theorem (or the theorem of compound probability)

- Then, the notion of relative prob was introduced (with areas & volumes of sets replacing counting of occurrences of events)

Cournot (1843) then gave a principle:
" An event with very small probability is morally impossible
An event with very high probability is morally certain"

The French mathematicians of those days were in support of the classical theory of frequentist approach to probability but not the German and English mathematicians

The first evidences of dissatisfaction in the classical theory came when certain Paradoxes of Bernstein could not be explained satisfactorily (see the reference article for more details)

- The invention of measure theory by Borel and Lebesgue in 1894 was a huge contribution to the field of mathematics.
- The genius of Kolmogorov was in uniting the classical theory of probability with the richness of measure theory introduced by Borel and Lebesgue. Hence came into being the theory of probability as we know it today.

Algebras and σ-algebras

With reference to sets, the three important (and perhaps exhaustive) operations are

a) complementation

b) unions

c) intersections.

Back in those days, at the time when the theory of probability was maturing, there was a need felt for formalism and rigor: collect all "events" of interest, and then talk about probabilities of those events (we do not yet know what events mean. It suffices to think of them as sets for now).

Such an attempt at collecting sets (i.e., a set of sets, or a basket of sets) was made by Borel, and the first such construction we shall study is that of an algebra, which is defined as follows:

Def^n (Algebra): Let Ω be any non-empty set. A collection $\mathcal{A}$ of subsets of Ω is called an algebra (of subsets of Ω) if:

a) $\Omega \in \mathcal{A}$

b) For any $A \in \mathcal{A}$, we have $A^c \in \mathcal{A}$ (closed under complements)

c) If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ (A, B need not be disjoint)
(closed under finite unions)

<u>Remarks</u>: 1. From the above def^n for algebra, it follows that nullset $\phi \in \mathcal{A}$. Also, it follows that for any two sets $A, B \in \mathcal{A}$, we have $A \cap B \in \mathcal{A}$.

Thus,
an algebra is a collection that is closed under all the important set operations.
(seems like a nice collection to start with, that gives a lot of information about events).

<u>Examples</u>:

1. $\Omega = \{H, T\}$

$$\mathcal{A}_1 = \{\phi, \Omega\}$$

$$\mathcal{A}_2 = \{\phi, \Omega, \{H\}, \{T\}\} = 2^{\Omega}$$

2. $\Omega = \{1, 2, 3, 4\}$

$$\mathcal{A}_1 = \{\phi, \Omega\}$$

$$\mathcal{A}_2 = \{\phi, \Omega, \{1\}, \{2,3,4\}\}$$

3. Let Ω be any non-empty set, and let $A \subseteq \Omega$ be any subset of Ω. Then,

$$\mathcal{A} = \{\phi, \Omega, A, A^c\} \text{ is an algebra.}$$

4. Let $\Omega = \{r \in \mathbb{Q} : 0 \leq r \leq 1\}$, the set of all rational numbers in the interval $[0, 1]$.

Let

$$\mathcal{A} = \left\{ \biguplus_{i=1}^{n} A_i : \begin{array}{l} A_1, \ldots, A_n \text{ are subsets of } \Omega; \\ \text{for each } i = 1, \ldots, n, \ A_i \text{ is a set of} \\ \text{one of the following forms:} \\ \{r \in \mathbb{Q} : a_i < r < b_i\}, \\ \{r \in \mathbb{Q} : a_i \leq r < b_i\}, \\ \{r \in \mathbb{Q} : a_i < r \leq b_i\}, \\ \{r \in \mathbb{Q} : a_i \leq r \leq b_i\} \text{ for any choice} \\ \text{of } a_i \text{ and } b_i \text{ satisfying} \\ 0 \leq a_i \leq b_i \leq 1; \\ n \in \mathbb{N} \text{ is any natural number} \end{array} \right\}$$

Interpretation: Choose any favourite $n \in \mathbb{N}$ of your choice. For the chosen n, choose your favorite $a_1, b_1, \ldots, a_n, b_n$ such that

$0 \leq a_i \leq b_i \leq 1$ for each $i = 1, \ldots, n$.

Then, for each $i = 1, \dots, n$, pick A_i as one of the following sets:

$$\{r \in \mathbb{Q} : a_i < r < b_i\},$$
$$\{r \in \mathbb{Q} : a_i \leq r < b_i\},$$
$$\{r \in \mathbb{Q} : a_i < r \leq b_i\},$$
$$\{r \in \mathbb{Q} : a_i \leq r \leq b_i\}$$

Make sure all the A_i's are disjoint

Repeat this for each $n = 1, 2, 3, \dots$. Put all the sets you have generated into $\mathcal{A}$ (Seems like $\mathcal{A}$ is a very ugly looking collection which is perhaps not amenable to clear visualisation. But still we can play with it ! :-))

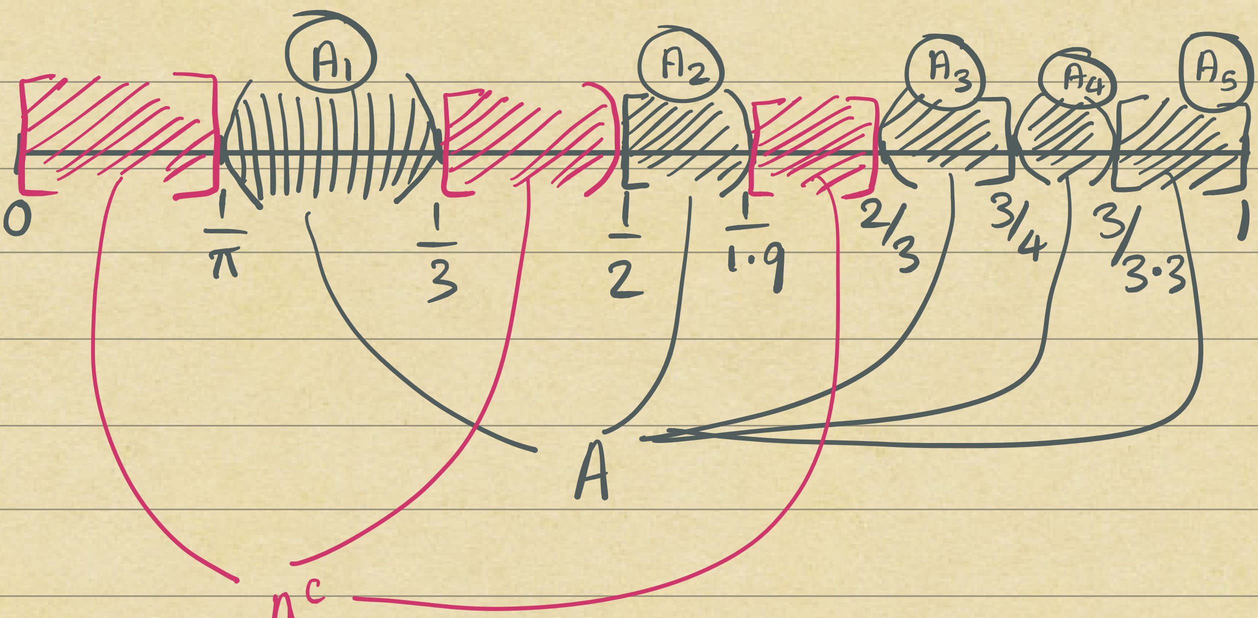
<u>To show $\mathcal{A}$ is an algebra</u>:

a) $\Omega \in \mathcal{A}$ because $\Omega = \{r \in \mathbb{Q} : 0 \leq r \leq 1\}$. In the process of constructing $\mathcal{A}$, you will be taking $n = 1$, $a_1 = 0$, $b_1 = 1$ as one of the choices for n, a_n and b_n. This defines Ω.

b) Suppose $A \in \mathcal{A}$. We need to show $A^c \in \mathcal{A}$. To do so, we will first work out a small example to understand what this proof entails. Let's say

$$A = \overbrace{\left\{ r : \frac{1}{\pi} < r < \frac{1}{3} \right\}}^{A_1} \cup \overbrace{\left\{ \frac{1}{2} \leq r < \frac{1}{1.9} \right\}}^{A_2} \cup \overbrace{\left\{ r : \frac{2}{3} < r \leq \frac{3}{4} \right\}}^{A_3} \cup \overbrace{\left\{ r : 3/4 < r < \frac{3}{3.3} \right\}}^{A_4}$$

$$\cup \underbrace{\left\{ r : \frac{3}{3.3} \leq r \leq 1 \right\}}_{A_5}$$



We see that A_1 and A_4 are of similar type ( ).

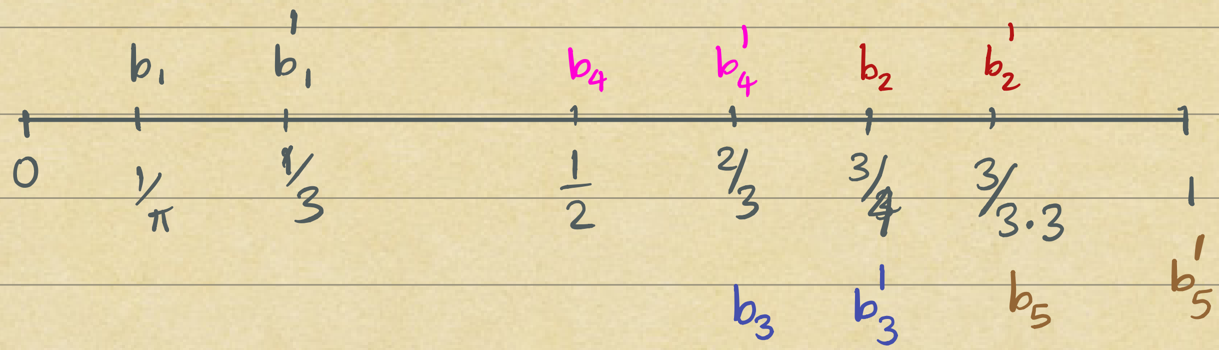
So, we can relabel $A_1, \cdots, A_5$ as $B_1, \cdots, B_5$ where

$B_1 = A_1$, $B_2 = A_4$, $B_3 = A_3$, $B_4 = A_2$, $B_5 = A_5$

Now, observe that

$$A = \biguplus_{i=1}^{5} A_i = \biguplus_{i=1}^{5} B_i .$$

The reason we do this is so that it becomes easy to write the complement set A^c. Let us define the boundary points of each B_i as b_i, b_i'. We now have the following picture.



Notice that some b_i's are overlapping with b_i's, but that's OK. It's just our labeling for convenience. In defining B_i's, we just made sure that the first few sets are of the form ( ), next come ( ], next come [ ) and finally come [ ] (this is just my way of relabeling. You can use yours).

Now can you write A^c, and take this idea for any $A = \biguplus_{i=1}^{n} A_i$?

c) Let $A, B \in \mathcal{A}$. We have to show $A \cup B \in \mathcal{A}$.

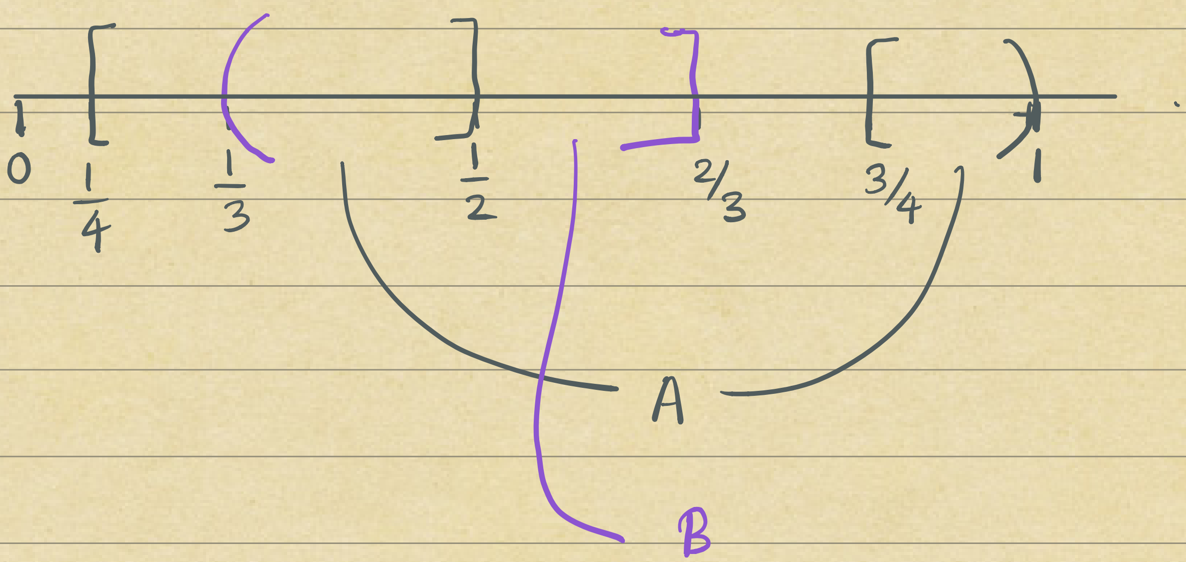
Let

$$A = \biguplus_{i=1}^{m} A_i \,, \qquad B = \biguplus_{j=1}^{n} B_j \,, \text{ where}$$

without loss of generality, we can assume $m \leq n$.

Now, the problem is that some of A_i's could be overlapping with some of the B_j's (for eg, A_1 could be same as B_2 or A_1 and B_2 might have a non-empty overlap).

Let us use an example to see the problem of overlap discussed above.



Say

$$A = \{r : 1/4 \leq r \leq 1/2\} \cup \{r : 3/4 \leq r < 1\}$$

$$B = \{r : 1/3 < r \leq 2/3\}$$

(I have taken $\substack{m=2, \\ n=1}$ to illustrate, but this problem exists even with $m \leq n$.)

What we need to do is break the regions of overlap into disjoint sets, and write the union of A & B in terms of these disjoint sets.

So, we have

$$A \cup B = \left\{x : \tfrac{1}{4} \leq x \leq \tfrac{1}{3}\right\} \cup \left\{x : \tfrac{1}{3} < x \leq \tfrac{1}{2}\right\} \cup \left\{x : \tfrac{1}{2} < x \leq \tfrac{2}{3}\right\}$$
$$\cup \left\{x : 3/4 \leq x < 1\right\}$$

We could have clubbed the first three sets above and just written $\left\{x : \tfrac{1}{4} \leq x \leq \tfrac{2}{3}\right\}$, but the point is that here we know the nos $\tfrac{1}{4}$ & $\tfrac{2}{3}$. What do we do in a general setting?

Can you take this idea to general A and B?

We have discussed about algebras till now. Mathematicians could have stopped at this and not really worried about going further in constructing informative collections of sets. So, why did they do it at all, then?

Let us observe the previous example carefully. Since $\Omega = \{r \in \mathbb{Q} : 0 \leq r \leq 1\}$ is a set which has countably many elements, we can draw a one-one and onto map with the set of natural numbers. Upon doing so, let the elements of Ω be denoted $r_1, r_2, \ldots$.
That is,

$$\Omega = \{r_1, r_2, r_3, \ldots\}$$
$$= \biguplus_{i=1}^{\infty} \{r_i\}.$$

Since this representation of Ω has a union of countably infinite number of sets, and the collection $\mathcal{A}$ in the previous example has only finite unions of disjoint sets, we might (wrongly) conclude that $\Omega \notin \mathcal{A}$, but we know this contradicts our earlier knowledge. There seems to be a discrepancy like this one can possibly run into!

In order to circumvent such discrepancies, mathematicians constructed another collection of sets, namely a σ-algebra.

<u>Def</u>

Def^n: Let Ω be a non-empty set. A collection $\mathcal{F}$ of subsets of Ω is called a σ-algebra (of subsets of

Ω) if :

a) $\Omega \in \mathcal{F}$

b) $A \in \mathcal{F} \Leftrightarrow A^c \in \mathcal{F}$ (closed under complements)

c) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

($A_1, A_2, \ldots$ need not be disjoint. In fact, they can all be the same as well).

(closed under countable unions)

Corollaries :

1. $\mathcal{F}$ as defined above is closed under finite unions, i.e., if $A_1, \ldots, A_n \in \mathcal{F}$ then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$ for all $n = 1, 2, 3, \ldots$

2. $\mathcal{F}$ as defined above is closed under countable intersections, i.e., if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

(Try to prove these).

Remarks: Apart from algebras and σ-algebras, there are many other collections of sets mathematicians constructed, such as π-system, d-system, λ-system; which we do not touch upon here. What is important to note is that each of these collections had a flaw (like the one we pointed out for algebra), which eventually led the mathematicians to conclude that a σ-algebra is the most useful and informative construction that could be used further.

A σ-algebra is also referred to as an event space and its elements are referred to as events.

Two important events

Suppose Ω is a non-empty set, and $\mathcal{F}$ is a σ-algebra on Ω (i.e, σ-algebra of subsets of Ω). Further, let $A_1, A_2, \ldots \in \mathcal{F}$ be a countably infinite number of sets in $\mathcal{F}$. Let us denote the elements of Ω as ω (little omega)

1) $\omega \in \bigcap_{n=1}^{\infty} A_n \Rightarrow \omega$ occurs in all the A_n's

2) $\omega \in \bigcup_{n=1}^{\infty} A_n \Rightarrow \omega$ occurs in atleast one of the A_n's

$$\Rightarrow \exists\, n_0 \in \mathbb{N} \text{ s.t. } \omega \in A_{n_0}.$$

Notice that both of the above sets belong to $\mathcal{F}$.

3) How do we interpret the set

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \ ?$$

Does it belong to the σ- algebra $\mathcal{F}$?

Answer: Let us define

$$B_n = \bigcup_{k=n}^{\infty} A_k \quad , \quad n=1,2,3,\dots$$

Then, we have

$$A = \bigcap_{n=1}^{\infty} B_n .$$

$\omega \in A \Rightarrow \omega$ occurs in all the B_n's

$\Rightarrow \omega \in B_1$ and $\omega \in B_2$ and $\omega \in B_3$ and so on

$\Rightarrow \omega \in \bigcup_{k=1}^{\infty} A_k$ and $\omega \in \bigcup_{k=2}^{\infty} A_k$ and $\omega \in \bigcup_{k=3}^{\infty} A_k$ ....

$\Rightarrow$ ω occurs in atleast one of $A_1, A_2, \dots$

and

ω occurs in atleast one of $A_2, A_3, \dots$

and

ω occurs in atleast one of $A_3, A_4, \dots$

and so on

$\Rightarrow$ $\exists\ n_1 \geq 1$ s.t. $\omega \in A_{n_1}$

and

$\exists\ n_2 \geq 2$ s.t. $\omega \in A_{n_2}$

and

$\exists\ n_3 \geq 3$ s.t. $\omega \in A_{n_3}$, and so on. .

$\Rightarrow$ ω occurs in infinitely many of the A_n's.

$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ is called "ω infinitely often" or "A_n infinitely often" set, often abbreviated as "ω i.o." or "A_n i.o." respectively.

Clearly, $B_n \in \mathcal{F}$ for each $n \geq 1$, and therefore

$$A = \bigcap_{n=1}^{\infty} B_n \in \mathcal{F}.$$

<u>Remark</u>: Notice that in the above def^n of B_n, we have

$$B_1 \supseteq B_2 \supseteq B_3 \ldots\ldots$$

4) How do we interpret the set

$$B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \ ?$$

Does $B \in \mathcal{F}$?

<u>Answer</u>: Define

$$C_n = \bigcap_{k=n}^{\infty} A_k \quad , \quad n = 1, 2, 3, \dots$$

Then, $$B = \bigcup_{n=1}^{\infty} C_n$$

$\omega \in B \Rightarrow \omega$ occurs in atleast one of the C_n's

$\Rightarrow \exists\, n_0 \in \mathbb{N}$ s.t. $\omega \in C_{n_0}$

$$\Rightarrow \omega \in \bigcap_{k=n_0}^{\infty} A_k$$

$\Rightarrow \omega$ occurs in all A_k's for $k \geqslant n_0$.

(here $n_0 \geqslant 1$. It could be that $n_0 = 1$, but it could also be that $n_0 \geqslant 2$).

$\Rightarrow \omega$ occurs in all $\underbrace{\text{but}}_{\text{except}}$ finitely many of the A_n's

$B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ is called "ω all but finitely many"

or "all but finitely many A_n's" set.

Clearly, $C_n \in \mathcal{F}$ for each $n \geq 1$, and hence $B \in \mathcal{F}$.
Further, notice that $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$

Exercises

1. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ (written in short as $(\mathcal{F}_n)_{n \geq 1}$) be a countably infinite number of σ-algebras on a set Ω. Prove or disprove

   a) $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n$ is a σ-algebra on Ω

   b) $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a σ-algebra on Ω.

2. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \ldots$ be an increasing sequence of σ-algebras on a set Ω. Prove or disprove that

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n \text{ is a } \sigma\text{-algebra on } \Omega.$$

3. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be an increasing sequence of subsets of a set Ω. What are "A_n i.o" and "all but finitely many of the A_n's" sets in this case?

4. Let $A_1, A_2, \dots$ be a sequence of sets. Convince yourself that

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k .$$

5. Let Ω be a finite, nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. Argue that $\mathcal{F}$ should necessarily contain an even number of sets in it.

6. Let $\Omega = \{1, 2, 3, 4, 5\}$

   a) Construct a σ-algebra of size 2

   b) Construct a σ-algebra of size 4

   c) Construct σ-algebras having size 8, 16 and 32.

Summary:

σ-algebra is a way of collecting informative events. It is a complete collection in terms of set operations: closed under complements countable intersections & unions

Attached below are the first pages of two articles that explain the history and development of the theory of probability.

Read them at your leisure.

The origins and legacy of Kolmogorov's *Grundbegriffe*

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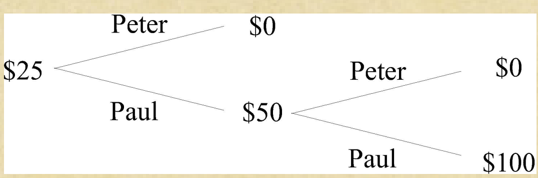

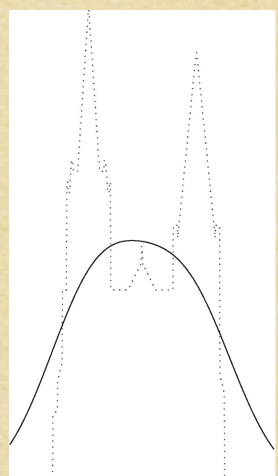

The Game-Theoretic Probability and Finance Project

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FROM THE HERITAGE OF A. N. KOLMOGOROV: THE THEORY OF PROBABILITY*

A. N. KOLMOGOROV

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In 1956 the AN USSR Publishing House published three volumes of the monograph *Mathematics*: *Its Content*, *Methods*, *and Meaning* which was elaborated by the Steklov Mathematical Institute RAN. A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev were the members of the editorial board. In order for the mathematical community to have an opportunity to discuss the monograph, 350 copies of it were printed in 1953 as a manuscript.

Kolmogorov's idea was that it would be good to have two books: a first which informally was planned as "Anti-Courant" (see the introduction to the 3rd Russian edition of R. Courant and H. Robbins, *What is Mathematics*? *An Elementary Approach to Ideas and Methods*, Oxford University Press, London, New York, 1996), i.e., a book for everybody who wants in vivid and simple form to get to know the elements of higher mathematics, to test the level of his abilities in mathematics, and, for a young reader, to consider choosing mathematics as his profession, and a second book "intended for more advanced readers including ourselves, mathematicians, who very often are helpless in estimating future trends of their science as a whole." Finally, three volumes containing 20 chapters showed the best correlation with the first of the variants indicated above. This follows additionally from the introduction which says that "the purpose of the author was to acquaint a wide Soviet circle with the content and methods of separate mathematical disciplines, their material resources, and paths of development."

In Chapter XI of the second volume of this monograph, the Kolmogorov paper was published, which is reprinted in the present jubilee issue together with the Khinchin referee report and selected correspondence of A. D. Aleksandrov (Editor-in-Chief of the monograph) with A. N. Kolmogorov, which are interesting both for their view on the content of the variant of the paper presented by Kolmogorov and for the philosophical and methodological aspects of probability theory.

A. N. Shiryaev

THE THEORY OF PROBABILITY

1. The laws of probability. The simplest laws of natural science are those that state the conditions under which some event of interest to us will either certainly occur or certainly not occur; i.e., these conditions may be expressed in one of the following two forms:

1. If a complex (i.e., a set or collection) of conditions S is realized, then event A certainly occurs;
2. If a complex of conditions S is realized, then event A cannot occur.

In the first case the event A, with respect to the complex of conditions S, is called a "certain" or "necessary" event, and in the second an "impossible" event. For example, under atmospheric pressure and at temperature t between 0° and 100° (the complex of conditions S) water necessarily occurs in the liquid state (the event A_1 is certain) and cannot occur in a gaseous or solid state (events A_2 and A_3 are impossible).

*Reprinted from A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev, *Mathematics*: *Its Content*, *Methods*, *and Meaning, 2nd ed., Vol.* 2, MIT Press, Cambridge, MA, 1969, Part 4, Chap. XI, pp. 229–264, with permission of the MIT Press.

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