



# Probability and Stochastic Processes

Lecture 12: Probability Law, Cumulative Distribution Function (CDF),  
Properties of CDF

**Karthik P. N.**

**Assistant Professor, Department of AI**

**Email: [pnkarthik@ai.iith.ac.in](mailto:pnkarthik@ai.iith.ac.in)**

12 September 2025

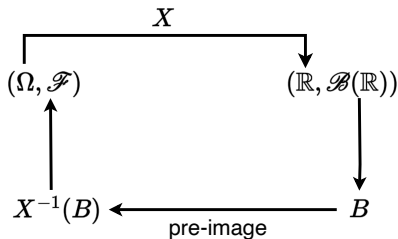
# Random Variable

## Definition (Random Variable)

Fix a measurable space  $(\Omega, \mathcal{F})$ .

A function  $X : \Omega \rightarrow \mathbb{R}$  is called a **random variable** if it is measurable, i.e.,

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad \underbrace{X^{-1}(B)}_{\text{pre-image of } B} = \{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\} \in \mathcal{F}.$$



## Properties of a Random Variable

### Proposition (Random Variable Properties)

Let  $(\Omega, \mathcal{F})$  be a measurable space, and let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable.

1. For any  $B \subseteq \mathbb{R}$ ,  $X^{-1}(B^c) = (X^{-1}(B))^c$ .
2. For any  $B_1 \subseteq \mathbb{R}, B_2 \subseteq \mathbb{R}, \dots$ ,

$$X^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} X^{-1}(B_n).$$

3. Let  $\mathcal{B}_1$  denote the collection

$$\mathcal{B}_1 := \left\{ B \subseteq \mathbb{R} : X^{-1}(B) \in \mathcal{F} \right\}.$$

Then,  $\mathcal{B}_1$  is a  $\sigma$ -algebra of subsets of  $\mathbb{R}$ . Furthermore,  $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_1$ .

## Generating Classes for $\mathcal{B}(\mathbb{R})$

$\mathcal{B}(\mathbb{R})$

$$\mathcal{P}_1 = \left\{ (a, b) : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_2 = \left\{ [a, b] : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_3 = \left\{ [a, b) : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_4 = \left\{ (a, b] : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_5 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_6 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_7 = \left\{ (x, +\infty) : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_8 = \left\{ [x, +\infty) : x \in \mathbb{R} \right\}$$

## Equivalent Definitions of Random Variable

Fix a measurable space  $(\Omega, \mathcal{F})$ .

### Theorem (Equivalent Definitions)

$X : \Omega \rightarrow \mathbb{R}$  is a random variable **if and only if** any one of the following holds:

1.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_1$ .
2.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_2$ .
3.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_3$ .
4.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_4$ .
5.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_5$ .
6.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_6$ .
7.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_7$ .
8.  $X^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{P}_8$ .

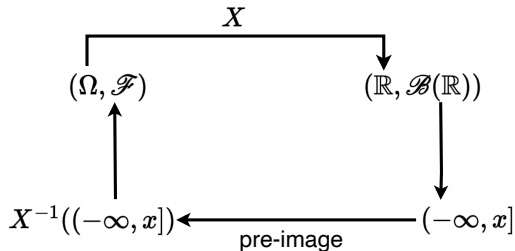
# Random Variable Simplified

## Definition (Random Variable)

Fix a measurable space  $(\Omega, \mathcal{F})$ .

A function  $X : \Omega \rightarrow \mathbb{R}$  is called a random variable with respect to  $\mathcal{F}$  if and only if

$$\forall x \in \mathbb{R}, \quad \underbrace{X^{-1}((-\infty, x])}_{\text{pre-image of } (-\infty, x]} = \{\omega \in \Omega : X(\omega) \leq x\} = \{X \leq x\} \in \mathcal{F}.$$



## Indicator Functions

Fix a sample space  $\Omega$ .

Fix a subset  $A \subseteq \Omega$ .

### Definition (Indicator Function)

The **indicator function** of set  $A$  is the function  $\mathbf{1}_A : \Omega \rightarrow \mathbb{R}$  defined as

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

### Exercise

Fix a measurable space  $(\Omega, \mathcal{F})$ . Show that

$$\mathbf{1}_A \text{ is a random variable} \iff A \in \mathcal{F}.$$

# Probability Law of a Random Variable



## Probability Law of a Random Variable

- Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$
- If  $X : \Omega \rightarrow \mathbb{R}$  is a random variable, then

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathcal{F}.$$

Therefore, it makes sense to talk about  $\mathbb{P}(X^{-1}(B))$  for each  $B \in \mathcal{B}(\mathbb{R})$

- We then have a mapping from  $\mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ :

$$B \mapsto \mathbb{P}(X^{-1}(B))$$

- The above mapping is called the **probability law** of the random variable  $X$

## Probability Law of a Random Variable

### Definition (Probability Law)

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable (with respect to  $\mathcal{F}$ ).

The **probability law** of  $X$  is a function  $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$  defined as

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)), \quad B \in \mathcal{B}(\mathbb{R}).$$

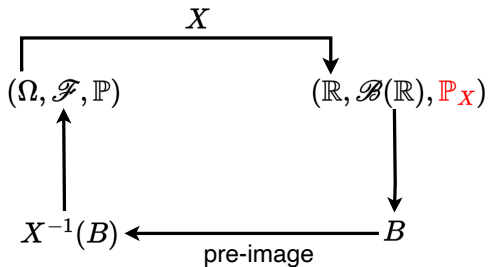
Remarks:

- $\mathbb{P}_X$  is sometimes referred to the **pushforward** of  $\mathbb{P}$  under the random variable  $X$
- $\mathbb{P}_X$  is sometimes denoted as  $\mathbb{P} \circ X^{-1}$

### Proposition (Probability Law)

$\mathbb{P}_X$  is a **probability measure** on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

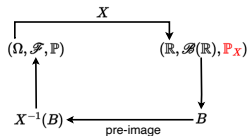
## Completing the Picture



$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

Figure: Pictorial representation of probability law

## Proof that $\mathbb{P}_X$ is a Probability Measure



$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

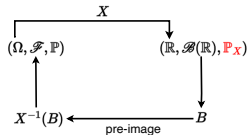
- First, we note that

$$\mathbb{P}_X(\emptyset) = \mathbb{P}(X^{-1}(\emptyset)) = \mathbb{P}(\emptyset) = 0.$$

- Next, we note that

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1.$$

## Proof that $\mathbb{P}_X$ is a Probability Measure



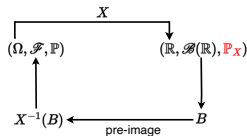
$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

- Finally, if  $B_1, B_2, \dots \in \mathcal{B}(\mathbb{R})$  are mutually disjoint, then

$$\mathbb{P}_X \left( \bigsqcup_{n \in \mathbb{N}} B_n \right) = \mathbb{P} \left( X^{-1} \left( \bigsqcup_{n \in \mathbb{N}} B_n \right) \right) = \mathbb{P} \left( \bigsqcup_{n \in \mathbb{N}} X^{-1}(B_n) \right) = \sum_{n \in \mathbb{N}} \mathbb{P} \left( X^{-1}(B_n) \right) = \sum_{n \in \mathbb{N}} \mathbb{P}_X(B_n).$$

# Cumulative Distribution Function

## Cumulative Distribution Function (CDF)



$$\mathbf{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

- $\mathbb{P}_X(B) \in [0, 1]$  for every  $B \in \mathcal{B}(\mathbb{R})$
- In particular,  $\mathbb{P}_X((-\infty, x]) \in [0, 1]$  for every  $x \in \mathbb{R}$
- We thus have a mapping

$$x \mapsto \mathbb{P}_X((-\infty, x])$$

- The above mapping (or function) is called the **cumulative distribution function** of the random variable  $X$ , denoted by  $F_X$

# Cumulative Distribution Function (CDF)

## Definition (Cumulative Distribution Function (CDF))

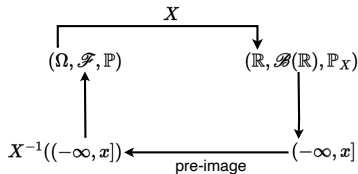
Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable.

The function  $F_X : \mathbb{R} \rightarrow [0, 1]$  defined by

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) = \mathbb{P}(\{X \leq x\}), \quad x \in \mathbb{R},$$

is called the **cumulative distribution function (CDF)** of  $X$ .



$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}$$



## Properties of CDF

### Lemma (Properties of CDF)

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable with CDF  $F_X$ . Then,  $F_X$  satisfies the following properties.

1. **(Monotonicity)** If  $x \leq y$ , then  $F_X(x) \leq F_X(y)$ .
2. If  $x_1, x_2, \dots$  is any sequence such that  $\lim_{n \rightarrow \infty} x_n = -\infty$ , then  $\lim_{n \rightarrow \infty} F_X(x_n) = 0$ .
3. If  $x_1, x_2, \dots$  is any sequence such that  $\lim_{n \rightarrow \infty} x_n = +\infty$ , then  $\lim_{n \rightarrow \infty} F_X(x_n) = 1$ .

4. **(Right-Continuity)**

$F_X$  is right-continuous at every point in the domain.

More formally, for each  $x \in \mathbb{R}$ ,

$$x_n > x \quad \forall n \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} x_n = x \quad \implies \quad \lim_{n \rightarrow \infty} F_X(x_n) = F_X(x).$$

## Proof of Lemma - 1

- Note that

$$\mathbb{F}_X(x) = \mathbb{P}_X((-\infty, x]), \quad \mathbb{F}_X(y) = \mathbb{P}_X((-\infty, y])$$

- If  $x \leq y$ , then

$$(-\infty, x] \subseteq (-\infty, y]$$

- Using monotonicity property of  $\mathbb{P}_X$ , it follows that

$$\mathbb{P}_X((-\infty, x]) \leq \mathbb{P}_X((-\infty, y])$$

## Proof of Lemma - 2

- Suppose  $x_1, x_2, \dots$  is **monotone decreasing** sequence such that

$$x_1 \geq x_2 \geq \dots, \quad \lim_{n \rightarrow \infty} x_n = -\infty.$$

- Then, we have

$$(-\infty, x_1] \supseteq (-\infty, x_2] \supseteq \dots$$

- Therefore,

$$\lim_{n \rightarrow \infty} F_X(x_n) = \lim_{n \rightarrow \infty} \mathbb{P}_X((-\infty, x_n]) = \mathbb{P}_X\left(\bigcap_{n \in \mathbb{N}} (-\infty, x_n]\right) = \mathbb{P}_X(\emptyset) = 0.$$

## Proof of Lemma - 2

- Suppose  $x_1, x_2, \dots$  is **any** sequence such that

$$\lim_{n \rightarrow \infty} x_n = -\infty$$

- Let  $\gamma_1, \gamma_2, \dots$  be a new sequence defined as

$$\gamma_n = \sup_{k \geq n} x_k, \quad n \in \mathbb{N}$$

- Then, it follows that

$$\gamma_1 \geq \gamma_2 \geq \dots, \quad \lim_{n \rightarrow \infty} \gamma_n = -\infty, \quad \gamma_n \geq x_n \quad \forall n \in \mathbb{N}.$$

- From the previous result for non-increasing sequences,

$$\lim_{n \rightarrow \infty} F_X(\gamma_n) = 0.$$

- $F_X(x_n) \leq F_X(\gamma_n) \quad \forall n \in \mathbb{N} \quad \implies \quad \lim_{n \rightarrow \infty} F_X(x_n) \leq \lim_{n \rightarrow \infty} F_X(\gamma_n) = 0.$

## Proof of Lemma – 3

- Left as exercise.

## Proof of Lemma - 4

- Fix  $x \in \mathbb{R}$
- Suppose that  $x_1, x_2, \dots$  is any monotone decreasing sequence such that

$$x_1 \geq x_2 \geq \dots, \quad \lim_{n \rightarrow \infty} x_n = x.$$

- Then, note that

$$(-\infty, x_1] \supseteq (-\infty, x_2] \supseteq \dots, \quad \bigcap_{n \in \mathbb{N}} (-\infty, x_n] = (-\infty, x].$$

- We then have

$$\lim_{n \rightarrow \infty} F_X(x_n) = \lim_{n \rightarrow \infty} \mathbb{P}_X((-\infty, x_n]) = \mathbb{P}_X\left(\bigcap_{n \in \mathbb{N}} (-\infty, x_n]\right) = \mathbb{P}_X((-\infty, x]) = F_X(x).$$

## Example

- Fix a measurable space  $(\Omega, \mathcal{F})$ . Fix  $A \in \mathcal{F}$ .  
Plot the CDF of the random variable  $\mathbf{1}_A$ .