



Stochastic Processes

Random Vectors, Sequences of Random Variables, \liminf , \limsup ,
 \lim of Sequences of Random Variables

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Dedication



Figure: Kalyanapuram Rangachari Parthasarathy (1936-2023).

Random Vectors and Sequences of Random Variables

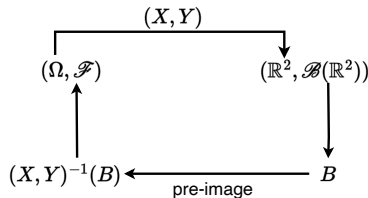
Two Random Variables - Bivariate Random Vector

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Bivariate Random Vector)

Given two functions $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega \rightarrow \mathbb{R}$, we say $(X, Y) : \Omega \rightarrow \mathbb{R}^2$ is a **bivariate random vector** with respect to \mathcal{F} if

$$(X, Y)^{-1}(B) = \{\omega \in \Omega : (X(\omega), Y(\omega)) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}(\mathbb{R}^2).$$



$$\forall B \in \mathcal{B}(\mathbb{R}^2), \quad (X, Y)^{-1}(B) \in \mathcal{F}$$

Understanding $\mathcal{B}(\mathbb{R}^2)$

- Consider the special class of semi-infinite rectangles in \mathbb{R}^2 , given by

$$\mathcal{E} = \left\{ (-\infty, x] \times (-\infty, y] : x, y \in \mathbb{R} \right\}.$$

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 - Circle of radius r centered at the origin, $r > 0$

Bivariate Random Vector – Implications

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Bivariate Random Vector)

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$$(X, Y)^{-1}((-\infty, x] \times \mathbb{R}) =$$

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Thus, **X is a random variable w.r.t. \mathcal{F}**

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Thus, **Y is a random variable w.r.t. \mathcal{F}**

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Implication

$$(X, Y) \text{ random vector} \implies X \text{ RV}, Y \text{ RV}$$

Is the other side implication also true?

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$$X \text{ RV}, Y \text{ RV} \implies (X, Y)^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{E} \implies (X, Y)^{-1}(B) \in \mathcal{F} \quad \forall B \in \sigma(\mathcal{E})$$

Bivariate Random Vector Simplified

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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Equivalent Definition of Bivariate Random Vector

(X, Y) is a random vector if

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$$(X, Y) \text{ random vector} \iff X \text{ is RV, } Y \text{ is RV}$$

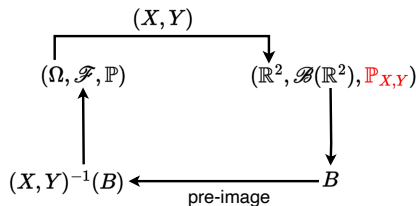
Probability Law of a Bivariate Random Vector

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Joint Probability Law of a Bivariate Random Vector)

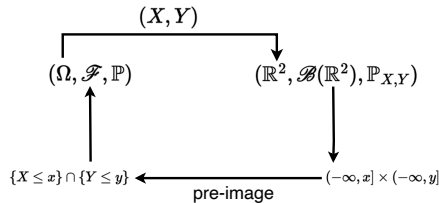
Given two random variables $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega \rightarrow \mathbb{R}$ defined with respect to \mathcal{F} , their **joint probability law** $\mathbb{P}_{X,Y} : \mathcal{B}(\mathbb{R}^2) \rightarrow [0, 1]$, is the probability measure defined as

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}(\{\omega \in \Omega : (X(\omega), Y(\omega)) \in B\}), \quad B \in \mathcal{B}(\mathbb{R}^2).$$



$$\mathbb{P}_{X,Y}(B) = \mathbb{P}((X, Y)^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R}^2)$$

Joint CDF of Two Random Variables



$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}((-\infty, x] \times (-\infty, y]) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}), \quad x, y \in \mathbb{R}$$

Definition (Joint CDF)

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Given random variables $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega \rightarrow \mathbb{R}$ with respect to \mathcal{F} , their **joint CDF** $F_{X,Y} : \mathbb{R}^2 \rightarrow [0, 1]$ is defined as

$$F_{X,Y}(\mathbf{x}, \mathbf{y}) = \mathbb{P}_{X,Y}((-\infty, \mathbf{x}] \times (-\infty, \mathbf{y}]) = \mathbb{P}(\{X \leq \mathbf{x}\} \cap \{Y \leq \mathbf{y}\}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}.$$

Joint CDF \longleftrightarrow Joint Probability Law

- If we know $\mathbb{P}_{X,Y} = \{\mathbb{P}_{X,Y}(B) : B \in \mathcal{B}(\mathbb{R}^2)\}$, then we can extract the CDF $F_{X,Y} : \mathbb{R}^2 \rightarrow [0, 1]$ by using the formula

$$F_{X,Y}(\mathbf{x}, \mathbf{y}) = \mathbb{P}_{X,Y}((-\infty, \mathbf{x}] \times (-\infty, \mathbf{y}]), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}.$$

- Given the joint CDF $F_{X,Y} : \mathbb{R}^2 \rightarrow [0, 1]$, let

$$\mathbb{P}_{X,Y}((-\infty, \mathbf{x}] \times (-\infty, \mathbf{y}]) = F_{X,Y}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}.$$

Then, by **Caratheodory's extension theorem**, there exists a unique extension of $\mathbb{P}_{X,Y}$ to all Borel subsets of \mathbb{R}^2

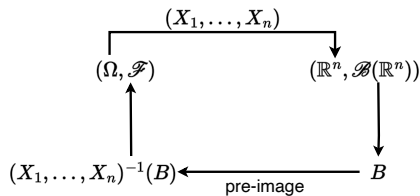
Higher Dimensional Random Vectors

Fix a measurable space (Ω, \mathcal{F}) . Fix $n \in \mathbb{N}$.

Definition (Random Vector)

Given random variables X_1, \dots, X_n defined with respect to \mathcal{F} , we say $(X_1, \dots, X_n) : \Omega \rightarrow \mathbb{R}^n$ is a **random vector** with respect to \mathcal{F} if

$$(X_1, \dots, X_n)^{-1}(B) = \{\omega \in \Omega : (X_1(\omega), \dots, X_n(\omega)) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}(\mathbb{R}^n).$$



$$\forall B \in \mathcal{B}(\mathbb{R}^n), \quad (X_1, \dots, X_n)^{-1}(B) \in \mathcal{F}$$

Understanding $\mathcal{B}(\mathbb{R}^n)$ for $n > 2$

- Consider the special class of semi-infinite rectangles in \mathbb{R}^n , given by

$$\mathcal{E} = \left\{ (-\infty, x_1] \times \cdots \times (-\infty, x_n] : x_1, \dots, x_n \in \mathbb{R} \right\}.$$

Understanding $\mathcal{B}(\mathbb{R}^n)$ for $n > 2$

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Equivalently, (X_1, \dots, X_n) is a random vector if

$$(X_1, \dots, X_n)^{-1}((-\infty, x_1] \times \dots \times (-\infty, x_n]) =$$

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$$(X_1, \dots, X_n)^{-1}((-\infty, x_1] \times \dots \times (-\infty, x_n]) = \bigcap_{i=1}^n \{X_i \leq x_i\} \in \mathcal{F} \quad \forall x_1, \dots, x_n \in \mathbb{R}.$$

$$(X_1, \dots, X_n) \text{ random vector} \iff X_1 \text{ RV} \quad \dots \quad X_n \text{ RV}$$

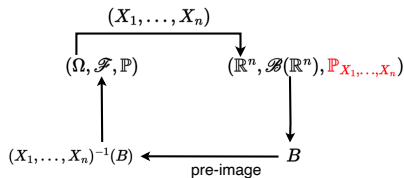
Probability Law of Random Vectors

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Probability Law of Random Vectors)

Given random variables X_1, \dots, X_n defined with respect to \mathcal{F} , their **joint probability law** is the probability measure $\mathbb{P}_{X_1, \dots, X_n} : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, 1]$ defined as

$$\mathbb{P}_{X_1, \dots, X_n}(B) = \mathbb{P}(\{\omega \in \Omega : (X_1(\omega), \dots, X_n(\omega)) \in B\}), \quad B \in \mathcal{B}(\mathbb{R}^n).$$



$$\mathbb{P}_{X_1, \dots, X_n}(B) = \mathbb{P}((X_1, \dots, X_n)^{-1}(B)) \quad \forall B \in \mathcal{B}(\mathbb{R}^n)$$

Joint CDF of Multiple Random Variables

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition (Joint CDF)

Given random variables X_1, \dots, X_n defined w.r.t. \mathcal{F} , their **joint CDF** $F_{X_1, \dots, X_n} : \mathbb{R}^n \rightarrow [0, 1]$ is defined as

$$\begin{aligned} F_{X_1, \dots, X_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \mathbb{P}_{X_1, \dots, X_n} \left((-\infty, \mathbf{x}_1] \times \dots \times (-\infty, \mathbf{x}_n] \right) \\ &= \mathbb{P} \end{aligned}$$

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Definition (Joint CDF)

Given random variables X_1, \dots, X_n defined w.r.t. \mathcal{F} , their **joint CDF** $F_{X_1, \dots, X_n} : \mathbb{R}^n \rightarrow [0, 1]$ is defined as

$$\begin{aligned} F_{X_1, \dots, X_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \mathbb{P}_{X_1, \dots, X_n} \left((-\infty, \mathbf{x}_1] \times \dots \times (-\infty, \mathbf{x}_n] \right) \\ &= \mathbb{P} \left(\bigcap_{i=1}^n \{X_i \leq \mathbf{x}_i\} \right), \quad \mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}. \end{aligned}$$

Sequence of Random Variables

Fix a measurable space (Ω, \mathcal{F}) .

Definition (Sequence of Random Variables)

A **sequence** of random variables is a collection $\{X_n\}_{n=1}^{\infty}$ such that

$$\forall n \in \mathbb{N}, \forall k_1, \dots, k_n \in \mathbb{N}, \quad (X_{k_1}, \dots, X_{k_n}) \text{ is a random vector.}$$

Note

When an outcome $\omega \in \Omega$ results from the random experiment, the **entire** sequence $X_1(\omega), X_2(\omega), \dots$ realizes at once.

Finite Dimensional Distributions

Fix a measurable space (Ω, \mathcal{F}) .

Definition (Sequence of Random Variables)

Consider a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ w.r.t. \mathcal{F} . The collection

$$\left\{ F_{X_{k_1}, \dots, X_{k_n}}(x_1, \dots, x_n) : n \in \mathbb{N}, k_1, \dots, k_n \in \mathbb{N}, x_1, \dots, x_n \in \mathbb{R} \right\}$$

is called the set of all **finite-dimensional distributions** of the sequence $\{X_n\}_{n=1}^{\infty}$.

\liminf , \limsup , \lim of Sequence of Random Variables

\liminf of Sequence of Random Variables

Fix a measurable space (Ω, \mathcal{F}) .

Definition (\liminf of Sequence of Random Variables)

The **limit infimum** of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. \mathcal{F} is a function $X_{\star} : \Omega \rightarrow [-\infty, +\infty]$ such that

$$X_{\star}(\omega) = \sup_{n \geq 1} \inf_{k \geq n} X_k(\omega) \quad \forall \omega \in \Omega.$$

Notation: $\liminf_{n \rightarrow \infty} X_n$.