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Stochastic Processes

Convergence Notions: Pointwise Convergence, Almost-Sure Convergence, Borel–Cantelli Lemma, Mean-Squared Convergence, Convergence in Probability, Convergence in Distribution, Examples

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24 January 2025

Dedication

Figure: Prof. Vivek Shripad Borkar, IIT Bombay (1954-).

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Recap

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Pointwise Convergence)

Given a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ and a random variable X, all defined w.r.t. $\mathscr{F}$, we say that the sequence converges pointwise to X if

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \qquad \forall \omega \in \Omega.$$

Notation:

$$X_n \xrightarrow{\text{pointwise}} X$$

Uniqueness of Pointwise Limit

The pointwise limit RV, whenever it exists, is always unique.

Recap

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Lemma

$$A_{\text{lim}} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} \in \mathscr{F}.$$

Thus, we may assign probability to A_{lim}.

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Proof of Lemma

Lemma

$$A_{\lim} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} \in \mathscr{F}.$$

$$\omega \in A_{\lim} \implies \lim_{n\to\infty} X_n(\omega) = X(\omega)$$

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$$\begin{aligned} \omega \in A_{\lim} &\implies \lim_{n\to\infty} X_n(\omega) = X(\omega) \\ &\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \ \text{ such that } \ |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \end{aligned}$$

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&\implies \forall \varepsilon > 0,\ \exists N_\varepsilon(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < \varepsilon \ \ \forall n \geq N_\varepsilon(\omega) \\
&\implies \forall q \in \mathbb{Q}_+,\ \exists N_q(\omega) \quad \text{such that} \quad |X_n(\omega) - X(\omega)| < q \ \ \forall n \geq N_q(\omega)
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&\implies \omega \in \bigcap_{q\in\mathbb{Q}_+}
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Proof of Lemma

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\omega \in A_{\lim} &\iff \lim_{n\to\infty} X_n(\omega) = X(\omega) \\
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$$A_{\lim} = \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{|X_n - X| < q\}.$$

Almost-Sure Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Almost-Sure Convergence)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X almost surely (a.s.) if

$$\mathbb{P}(A_{\text{lim}}) = 1.$$

Notation:

$$X_n \xrightarrow{\text{a.s.}} X.$$

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Revisiting Examples

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

- For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} 1, & \omega \in \left[0, \frac{1}{n}\right), \\ 0, & \text{otherwise}, \end{cases} \qquad \omega \in [0,1].$$

Identify the pointwise limit and an almost-sure limit.

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Is the almost-sure limit unique?

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Identify the pointwise limit and an almost-sure limit.

Is the almost-sure limit unique?

Note

The almost-sure limit is only specified up to sets of zero probability. That is,

$$X_n \xrightarrow{\text{a.s.}} X, \;\; X_n \xrightarrow{\text{a.s.}} Y \quad \implies \quad \mathbb{P}(X = Y) = 1.$$

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Borel–Cantelli Lemma and Almost-Sure Convergence

The $\liminf$ and $\limsup$ Events – 1

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $A_1, A_2, \ldots$ be events in $\mathscr{F}$.

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Let $A_1, A_2, \ldots$ be events in $\mathscr{F}$.

Definition (The $\liminf$ Event)

The limit infimum of the sequence $\{A_n\}_{n=1}^{\infty}$ is defined as the set

$$A_\star := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Clearly, $A_\star \in \mathscr{F}$.

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Interpretation

$$\omega \in A_\star \implies$$

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Clearly, $A_\star \in \mathscr{F}$.

Interpretation

$$\begin{aligned} \omega \in A_\star \quad &\Longrightarrow \quad \exists n \in \mathbb{N} \text{ such that } \omega \in A_k \text{ for all } k \geq n \\ &\Longrightarrow \quad \omega \text{ belongs to all but finitely many of the } A_n's \end{aligned}$$

The $\liminf$ and $\limsup$ Events – 2

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $A_1, A_2, \ldots$ be events in $\mathscr{F}$.

The $\liminf$ and $\limsup$ Events – 2

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $A_1, A_2, \ldots$ be events in $\mathscr{F}$.

Definition (The $\limsup$ Event)

The limit supremum of the sequence $\{A_n\}_{n=1}^{\infty}$ is defined as the set

$$A^\star := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Clearly, $A^\star \in \mathscr{F}$.

The $\liminf$ and $\limsup$ Events – 2

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $A_1, A_2, \ldots$ be events in $\mathscr{F}$.

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Clearly, $A^\star \in \mathscr{F}$.

Interpretation

$$\omega \in A^\star \quad \Longrightarrow$$

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Clearly, $A^\star \in \mathscr{F}$.

Interpretation

$$\omega \in A^\star \implies \forall n \in \mathbb{N},\ \exists k \geq n \text{ such that } \omega \in A_k$$

$$\implies \omega \text{ belongs to infinitely many of the } A_n's$$

Remarks on $\liminf$ and $\limsup$ Events

- We have

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n \qquad (A_\star \subseteq A^\star).$$

- Some texts use the phrase "A_n infinitely often" or "A_n i.o." to refer to $\limsup_{n\to\infty} A_n$

Borel–Cantelli Lemma

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma (Borel–Cantelli Lemma)

1. Suppose $A_1, A_2, \ldots \in \mathscr{F}$ are such that $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < +\infty$. Then,
$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0.$$
2. Suppose $A_1, A_2, \ldots \in \mathscr{F}$ are independent and satisfy $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = +\infty$. Then,
$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 1.$$

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$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 1.$$

The above lemma can be used to verify almost-sure convergence property in some scenarios

Borel–Cantelli Lemma and Almost-Sure Convergence

- For each $n \in \mathbb{N}$, let
$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$
Identify an almost-sure limit.

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$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$
Identify an almost-sure limit.
- For each $n \in \mathbb{N}$, let
$$\mathbb{P}(X_n = 1) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0).$$
Furthermore, suppose that $X_1, X_2, \ldots$ are mutually independent.
What can we say about the convergence of the above sequence?