AI5090 : STOCHASTIC PROCESSES (Lecture-2) 07 Jan 2025

Agenda : Primer on probability

- Sample space, σ-algebra, probability measure
- Random variables, CDF
- Random vectors and sequences of random variables

DEDICATION : This lecture is dedicated to ...

Émile Borel
(1871-1956)

Prob. theory assumes two important quantities:

- Random experiment → results of the experiment are not always predictable
- Outcome - result of the experiment.

$\boxed{\text{Def}^n \text{ (Sample space) : Set of all possible outcomes ; denoted } \Omega.}$

Eg : Coin toss - outcome depends on quantity of interest.

→ Face of coin that shows up → $\Omega = \{H, T\}$

→ Velocity with which coin lands → $\Omega = [0, \infty) = \mathbb{R}_+$

→ Number of flips in air → $\Omega = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$.

Def^n (σ-algebra) : Fix a sample space Ω. A σ-algebra of subsets of Ω is a collection of sets (denoted as $\mathcal{F}$) satisfying the following properties:

i) $\Omega \in \mathcal{F}$

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (closure under set complements)

$\rightarrow \Omega \setminus A$

iii) $A_1, A_2, A_3, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (closure under countable unions)

$\rightarrow A_1 \cup A_2 \cup A_3 \cup \dots$

Note: σ-algebra is also referred to as a σ-field.

Example: $\Omega = \{1, 2, \dots, 6\}$

- $\mathcal{F}_1 = \{\phi, \Omega\}$
  ϕ → \emptyset
- $\mathcal{F}_2 = 2^{\Omega} = \{A : A \subseteq \Omega\}$
- $\mathcal{F}_3 = \{\phi, \Omega, \{1\}, \{2, \dots, 6\}\}$
- $\mathcal{F}_4 = \{\phi, \Omega, \{1\}, \{2,3\}, \{4,5\}, \{2, \dots, 6\}, \{1,4,5,6\}, \{1,2,3,6\}, \{1,2,3\}, \{4,5,6\}, \{1,4,5\}, \{2,3,6\}, \{2,3,4,5\}, \{1,6\}, \{1,2,3,4,5\}, \{6\}\}$

Borel σ-algebra

Consider $\Omega = \mathbb{R}$

$$\mathcal{D} = \{(-\infty, x] : x \in \mathbb{R}\}$$

- $\mathcal{D}$ is closed under finite intersections, and is called a π-system. The smallest σ-algebra that can be constructed from $\mathcal{D}$ is called the Borel σ-algebra of subsets of $\mathbb{R}$, denoted using $\mathcal{B}(\mathbb{R})$.

\mathcal{B}

$$\boxed{\text{The pair } (\Omega, \mathcal{F}) \text{ is called a measurable space.}}$$

Defⁿ (probability measure): Fix $(\Omega, \mathcal{F})$. A probability measure $\mathbb{P} : \mathcal{F} \to [0,1]$ satisfying:

i) $\mathbb{P}(\Omega) = 1$

ii) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A). \quad \forall A \in \mathcal{F}.$

iii) For any collection of mutually disjoint sets $A_1, A_2, A_3, \ldots \in \mathcal{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

$$\boxed{(\Omega, \mathcal{F}, \mathbb{P}) \text{ is called a probability space.}}$$

<u>Example</u>: $\Omega = \{1,2,3,4,5,6\}$

$$\begin{array}{cccccc} \mathcal{F} & = \{ & \phi, & \Omega, & \{1\}, & \{2,\ldots,6\} \ \} \\ & & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{P} & & 0 & 1 & 1/6 & 5/6 \\ & & 0 & 1 & 0 & 1 \end{array}$$

<u>Aside</u>:

$$\mathbb{P}(A) = \lim_{N \to \infty} \frac{N(A)}{N} = \lim_{N \to \infty} \frac{\# A \text{ occurs}}{N}$$

<u>Remarks</u>:

- Ω is called the sure event.
- Any set $A \in \mathcal{F}$ s.t. $\mathbb{P}(A) = 1$ is called an almost-sure event.
- $\mathbb{P}(A) = 0 \not\Rightarrow A = \phi$
- $\mathbb{P}(A) = 1 \not\Rightarrow A = \Omega$.

<u>Random variables</u>:

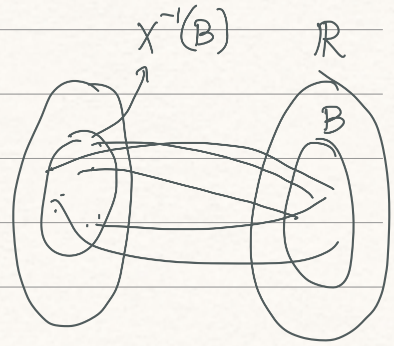
$$\begin{tikzcd} (\Omega, \mathcal{F}, \mathbb{P}) \arrow[rr, "X"] & & \mathbb{R}, \mathcal{B}(\mathbb{R}) \arrow[d] \\ X^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{Inverse image}"'] \end{tikzcd}$$

Fix $(\Omega, \mathcal{F})$.

A random variable $X : \Omega \to \mathbb{R}$ is a mapping satisfying :

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$$

measurability



Equivalently, $X : \Omega \to \mathbb{R}$ is a random variable if :

$$\forall x \in \mathbb{R}, \quad X^{-1}\big((-\infty, x]\big) = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$$