AI 5090: Stochastic Processes
Homework 6

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Topics: Markov Chains, Sampling Techniques

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing below are defined with respect to this probability space.

1. Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible, time-homogeneous DTMC on a finite state space $\mathcal{X}$, with TPM P and unique stationary distribution π. Given two disjoint sets $A, B \subseteq \mathcal{X}$, $A \cap B = \emptyset$, define the **probability flux** from A to B as

$$\Phi(A, B) \coloneqq \sum_{x \in A} \sum_{y \in B} \pi_x \, P_{x,y}.$$

   (a) Let $\mathcal{X}_1 \subset \mathcal{X}$ and $\mathcal{X}_2 \subset \mathcal{X}$ form a partition of $\mathcal{X}$. Show that probability flux balances, i.e.,

$$\Phi(\mathcal{X}_1, \mathcal{X}_2) = \Phi(\mathcal{X}_2, \mathcal{X}_1).$$

   (b) Rewrite the relation $\pi = \pi P$ as a flux balance equation.

2. Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous, **ergodic** Markov chain on a discrete state space $\mathcal{X}$, with TPM P and unique stationary distribution π.
Let $\{Y_n\}_{n=0}^{\infty}$ be an independent copy of $\{X_n\}_{n=0}^{\infty}$ with the same TPM P and $Y_0 \sim \pi$.
Let $\tau = \inf\{n \geq 0 : X_n = Y_n\}$ be the coupling time.
In class, we showed that if $X_0 = x$, then

$$|P^n_{x,y} - \pi(y)| \leq 2\,\mathbb{P}(\tau > n \mid X_0 = x) \qquad \forall y \in \mathcal{X}.$$

   (a) Show that there exists $\lambda \in (0, 1)$ such that

$$\mathbb{P}(\tau > n \mid X_0 = x) \leq \lambda^n \qquad \forall n \geq 0.$$

   Hint: Note that

$$\mathbb{P}(\tau > n \mid X_0 = x) = \mathbb{P}(X_0 \neq Y_0, \ldots, X_n \neq Y_n \mid X_0 = x).$$

   Express the probability on the right-hand side above using the entries of P and π.
   What do you know about the entries of π?

   (b) Let $N_y(n) = \sum_{k=1}^{n} \mathbf{1}_{\{X_k = y\}}$ denote the number of visits to state y up to time n.
   Using the result of part (a), prove that starting from state x,

$$\frac{N_y(n)}{n} \xrightarrow{\text{m.s.}} \pi_y.$$

   (c) Using the result of part (a) and the Borel–Cantelli lemma, prove that starting from state x,

$$\frac{N_y(n)}{n} \xrightarrow{\text{a.s.}} \pi_y.$$

3. Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ and TPM P.
Assume that $P_{x,x} < 1$ for all $x \in \mathcal{X}$. Let

$$T_1 \coloneqq \inf\{n \in \mathbb{N} : X_n \neq X_0\},$$

and for $m \geq 2$, let

$$T_m \coloneqq \inf\{n \in \mathbb{N} : X_n \neq X_{T_{m-1}}\}.$$

Thus, T_m's denote the random times at which the DTMC changes state.

   (a) For each $m \in \mathbb{N}$, show that T_m is a stopping time w.r.t. the natural filtration of the process $\{X_n\}_{n=0}^{\infty}$.

(b) Let $Z_0 = X_0$, and for each $m \in \mathbb{N}$, let $Z_m = X_{T_m}$. Prove that $\{Z_m\}_{m=0}^{\infty}$ is a time-homogeneous DTMC with TPM $\widetilde{P}$ given by

$$\widetilde{P}_{x,y} = \begin{cases} 0, & x = y, \\ \dfrac{P_{x,y}}{1 - P_{x,x}}, & x \neq y. \end{cases}$$

(c) If P admits a unique stationary distribution π, determine the stationary distribution of $\widetilde{P}$.

4. Fix a sufficiently large number $N \in \mathbb{N}$, and let $\{X_n\}_{n=0}^{N}$ be an irreducible and positive recurrent DTMC with TPM P and stationary distribution π. Suppose that $X_0 \sim \pi$.
For each $n \in \{0, \ldots, N\}$, define $Y_n = X_{N-n}$. The process $\{Y_n\}_{n=0}^{N}$ is called the time-reversed chain.
   (a) From an earlier homework, we know that $\{Y_n\}_{n=0}^{N}$ is a DTMC. Identify its TPM.
   (b) What conditions should the entries of P and π satisfy for $\{X_n\}_{n=0}^{N}$ and $\{Y_n\}_{n=0}^{N}$ to have identical TPMs?
   Remark: A DTMC $\{X_n\}_{n=0}^{N}$ which starts off in its stationary distribution is said to be **reversible** chain if its TPM is identical to the TPM of its time-reversed version.

5. We saw the rejection sampling technique in class. Prove formally that

$$\mathbb{P}(Z \leq x \mid E) = F(x),$$

where $E = \{a\, U\, f_Z(Z) \leq f(Z)\}$, the function f is the PDF corresponding to the CDF F, and (U, Z) satisfy the conditions outlined in the lecture with

$$f(x) \leq a\, f_Z(x) \qquad \forall x.$$

6. **Another variant of rejection sampling.**

Fix a target PDF f with range $[0, \infty)$. Let $U_1, U_2 \overset{\text{i.i.d.}}{\sim} \text{Unif}(0,1)$. Define $R = U_2/U_1$, and let E denote the event

$$E = \left\{U_1 \leq \sqrt{f(U_2/U_1)}\right\}.$$

Show that

$$\mathbb{P}(R \leq x \mid E) = \int_0^x f(t)\, \mathrm{d}t.$$

Hint: Use the transformation $S = U_2/U_1, T = U_1$.
The idea is that we sample two independent uniform $(0,1)$ random variables, and check for the condition in event E. If this condition is satisfied, we accept the uniform samples and simply output R as the desired sample. If not, we reject the uniform samples and repeat the procedure till the condition in E holds. This variant of rejection sampling technique holds only when the target sample desired has non-negative support.