CS 6660: Mathematical Foundations of Data Science (Probability)
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Practice Problems 03

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are defined with respect to $\mathscr{F}$.

1. Let $X_1, X_2, X_3 \overset{\text{i.i.d.}}{\sim} \text{Unif}((0,1))$.

    (a) Compute $\mathbb{P}(\{X_1 + X_2 > X_3\})$.

    (b) Derive the CDF of the random variable $X = X_1 X_2$.

    (c) Using the result from part (b), show that $\mathbb{P}(\{X_1 X_2 \leq X_3^2\}) = \frac{5}{9}$.

2. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\lambda)$. Compute the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$, and show that $Y_1 \perp\!\!\!\perp Y_2$.

3. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\lambda)$. Let $X = X_1$ and $Y = X_1 + X_2$. Determine the joint PDF of X and Y. Also determine the conditional PDF of X, conditioned on the event $\{Y = y\}$.

4. Let $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. Let $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan \frac{Y}{X} = \tan^{-1}\left(\frac{Y}{X}\right)$. Determine the joint PDF of R and Θ, and show that $R \perp\!\!\!\perp \Theta$. What are the marginal PDFs of R and Θ?

5. Let R and Θ be two random variables with the joint PDF

$$f_{R,\Theta}(r, \theta) = r e^{-r^2/2} \cdot \frac{1}{2\pi}, \qquad r \geq 0, \quad \theta \in [0, 2\pi].$$

    (a) Show that $R \perp\!\!\!\perp \Theta$.

    (b) Let $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. Show that $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.