AI 5030: Probability and Stochastic Processes

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Homework 5

Topics: CDFs, Joint CDFs, Discrete and Continuous Random Variables, Jointly Discrete Random Variables, Independence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Let X be a random variable. Determine, in each case below, if the function therein can be a valid CDF of X. If not, provide at least one valid justification. For each valid CDF, compute $\mathbb{P}(\{X > 5\})$.

   (a) $F_X(x) = \begin{cases} \frac{e^{-x^2}}{4}, & x < 0, \\ 1 - \frac{e^{-x^2}}{4}, & x \geq 0. \end{cases}$

   (b) $F_X(x) = \begin{cases} 0, & x < 0, \\ 0.5 + e^{-x}, & 0 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$

   (c) $F_X(x) = \begin{cases} 0, & x < 0, \\ 0.5 + \frac{x}{20}, & 0 \leq x < 10, \\ 1, & x \geq 10. \end{cases}$

2. Let X be a Geometric random variable. Show that for all $n, k \geq 1$,

$$\mathbb{P}(\{X > n + k\} | \{X > n\}) = \mathbb{P}(\{X > k\}).$$

   This is called the memoryless property of a Geometric distribution.

   Conversely, show that if the random variable X satisfies the above property, then it must be Geometric.

3. Let X and Y be continuous random variables with PDFs f_X and f_Y respectively. For any $\alpha \in [0, 1]$, argue that $\alpha\, f_X + (1 - \alpha)\, f_Y$ is a valid PDF. Can you think of a random variable Z whose PDF is $f_Z = \alpha\, f_X + (1 - \alpha)\, f_Y$?

4. We learnt in class that random variables X and Y are independent if

$$\{X \leq x\} \perp\!\!\!\perp \{Y \leq y\} \qquad \forall x, y \in \mathbb{R}. \tag{1}$$

   In this exercise, we will show through a series of logical steps that the above definition of independence implies that

$$\{X = x\} \perp\!\!\!\perp \{Y = y\} \qquad \forall x, y \in \mathbb{R}. \tag{2}$$

   Fix arbitrary $x, y \in \mathbb{R}$.

   (a) Show that if A, B are events such that $A \perp\!\!\!\perp B$, then $A \perp\!\!\!\perp B^c$.

   (b) Show that if A, B, C are events such that $A \perp\!\!\!\perp B$ and $A \perp\!\!\!\perp C$, then $A \perp\!\!\!\perp (B \cup C)$ if and only if $A \perp\!\!\!\perp (B \cap C)$.

   (c) Let $B_1, B_2, \ldots \in \mathscr{F}$ be such that $B_n \supseteq B_{n+1}$ for all $n \in \mathbb{N}$. If $A \perp\!\!\!\perp B_n$ for all $n \in \mathbb{N}$, show $A \perp\!\!\!\perp \bigcap_{n=1}^{\infty} B_n$.

   (d) Apply the result in part (c) to the sets $A = \{X \leq x\}$ and $B_n = \{Y > y - \frac{1}{n}\}$. Argue using the result in part (a) that $A \perp\!\!\!\perp B_n$ for all n. Hence conclude from the result in part (c) that $\{X \leq x\} \perp\!\!\!\perp \{Y \geq y\}$.

   (e) Apply the result in part (b) to the sets $A = \{X \leq x\}$, $B = \{Y \leq y\}$, and $C = \{Y \geq y\}$, and conclude using the results in parts (d) and (a) that $\{X \leq x\} \perp\!\!\!\perp \{Y = y\}$. What are $B \cap C$ and $B \cup C$ here?

   (f) Repeat the series of logical steps in parts (d) and (e) with $A = \{Y = y\}$ to arrive at $\{X = x\} \perp\!\!\!\perp \{Y = y\}$.

   Remark: This exercise shows that if X and Y are independent, then (1) implies (2). However, in general, (2) DOES NOT imply (1). If X and Y are jointly discrete, then (2) implies (1).

5. Suppose that two batteries are chosen simultaneously and uniformly at random from the following group of 12 batteries : 3 new, 4 used (yet working), 5 defective. You may assume that all batteries within a particular group are identical. Let X be the number of new batteries chosen, and let Y be the number of used batteries chosen. Determine the joint PMF of X and Y, and compute $\mathbb{P}(\{|X - Y| \leq 1\})$.

6. Let $X \sim$ Uniform$([0, 1])$ and $Y = 1 - X$. Derive the joint CDF of X and Y.

7. Let X be a random variable with CDF F_X as shown in the figure below. Compute $\mathbb{P}(\{X \in [3, 15]\}|\{X > 4\})$.