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Probability and Stochastic Processes

Lecture 08: Probability Measure and its Properties, Examples of Probability Assignment

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Probability Measure

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Probability Measure)

A function $\mathbb{P} : \mathscr{F} \to [0, 1]$ is called a **probability measure** if the following properties are satisfied:

1. $\mathbb{P}(\emptyset) = 0$.
2. $\mathbb{P}(\Omega) = 1$.
3. If $A_1, A_2, \ldots$ is a countable collection of **disjoint** sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

The triplet $(\Omega, \mathscr{F}, \mathbb{P})$ is called **probability space.**

Properties of Probability Measure

- **(Finite Additivity)**

  Fix $n \in \mathbb{N}$.
  If $A_1, \ldots, A_n$ is a finite collection of **disjoint** sets, with $A_i \in \mathscr{F}$ for each $i \in \{1, \ldots, n\}$, then
  $$\mathbb{P}\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

- **(Complements)**

  For any $A \in \mathscr{F}$,
  $$\mathbb{P}\left(A^{\complement}\right) = 1 - \mathbb{P}(A).$$

- **(Monotonicity)**

  If $A, B \in \mathscr{F}$ with $A \subseteq B$, then
  $$\mathbb{P}(A) \leq \mathbb{P}(B).$$

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Properties of Probability Measure

(Inclusion-Exclusion)

- For any two events $A_1, A_2 \in \mathscr{F}$,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).$$

- More generally, for any $n \in \mathbb{N}$ and events $A_1, \ldots, A_n \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\, \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

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Continuity of Probability Measure

Union and Intersection Events

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **union** is defined by

$$A_{\text{union}} = \bigcup_{k \in \mathbb{N}} A_k .$$

Interpretation: $\omega \in A_{\text{union}} \implies \exists\, k \in \mathbb{N} : \ \omega \in A_k$

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **intersection** is defined by

$$A_{\text{intersection}} = \bigcap_{k \in \mathbb{N}} A_k .$$

Interpretation: $\omega \in A_{\text{intersection}} \implies \omega \in A_k \quad \forall\, k \in \mathbb{N}$

The Limit Infimum (liminf) Event

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **limit infimum (liminf)** is defined by

$$A_{\mathrm{liminf}} = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

Interpretation: For each $n \in \mathbb{N}$, let $B_n := \bigcap_{k\geq n} A_k$.

$$\begin{aligned}
\omega \in A_{\mathrm{liminf}} \quad &\implies \quad \omega \in \bigcup_{n\in\mathbb{N}} B_n \\
&\implies \quad \exists\, n \in \mathbb{N} : \quad \omega \in B_n \\
&\implies \quad \exists\, n \in \mathbb{N} : \quad \omega \in \bigcap_{k\geq n} A_k \\
&\implies \quad \exists\, n \in \mathbb{N} : \quad \omega \in A_k \ \forall\, k \geq n.
\end{aligned}$$

The Limit Supremum (limsup) Event

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, their **limit supremum (limsup)** is defined by

$$A_{\text{limsup}} = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k.$$

Interpretation: For each $n \in \mathbb{N}$, let $B_n := \bigcup_{k \geq n} A_k$.

$$\begin{aligned}
\omega \in A_{\text{limsup}} \quad & \implies \quad \omega \in \bigcap_{n \in \mathbb{N}} B_n \\
& \implies \quad \forall\, n \in \mathbb{N}, \ \ \omega \in B_n \\
& \implies \quad \forall\, n \in \mathbb{N}, \ \ \omega \in \bigcup_{k \geq n} A_k \\
& \implies \quad \forall\, n \in \mathbb{N}, \ \ \exists\, k \geq n : \ \ \omega \in A_k.
\end{aligned}$$

The Limit Event

Fix a measurable space $(\Omega, \mathscr{F})$.

- Given sets $A_1, A_2, \ldots \in \mathscr{F}$, if

$$A_{\text{liminf}} = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k = A_{\text{limsup}},$$

  then we say that the **limit of $A_1, A_2, \ldots$ exists**, and is defined by

$$A_{\text{limit}} = A_{\text{liminf}} = A_{\text{limsup}}.$$

Some Tidbits About liminf, limsup, and limit Events

Fix a measurable space $(\Omega, \mathscr{F})$. Let $A_1, A_2, \ldots \in \mathscr{F}$.

- A_{liminf} and A_{limsup} are valid events, i.e.,

$$A_{\text{liminf}} \in \mathscr{F}, \qquad A_{\text{limsup}} \in \mathscr{F}.$$

- A_{limit}, if it exists, is a valid event
- In general, $A_{\text{liminf}} \subseteq A_{\text{limsup}}$. If equality holds, then the limit set is well defined
- If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, then

$$A_{\text{limit}} = \bigcup_{n \in \mathbb{N}} A_n.$$

- If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, then

$$A_{\text{limit}} = \bigcap_{n \in \mathbb{N}} A_n.$$

- A_{liminf} is sometimes denoted more explicitly as $\liminf_{n\to\infty} A_n$
  A_{limsup} is sometimes denoted more explicitly as $\limsup_{n\to\infty} A_n$
  A_{limit} is sometimes denoted more explicitly as $\lim_{n\to\infty} A_n$

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Properties of Probability Measure

(Continuity of Probability)
Fix a measurable space $(\Omega, \mathscr{F})$.

Proposition (Continuity of Probability)

Let $A_1, A_2, \ldots \in \mathscr{F}$ be a collection of events for which $A_{\text{limit}} = \lim_{n\to\infty} A_n$ exists. Then,

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

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Continuity of Probability: Proof – 1

- Case 1: $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$
  - Let $B_1, B_2, \ldots$ be defined as
$$B_1 = A_1, \quad B_2 = A_2 \setminus A_1, \quad B_3 = A_3 \setminus A_2, \quad \ldots$$
  - **Claim 1:** $B_i \cap B_j = \emptyset$ for all $i \neq j$
  - **Claim 2:** We have
$$\bigsqcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k \quad \forall\, n \in \mathbb{N}, \qquad \bigsqcup_{k \in \mathbb{N}} B_k = \bigcup_{k \in \mathbb{N}} A_k.$$
  - Therefore, it follows that
$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \mathbb{P}\left(\bigcup_{k \in \mathbb{N}} A_k\right) \overset{\text{Claim 2}}{=} \mathbb{P}\left(\bigsqcup_{k \in \mathbb{N}} B_k\right) = \sum_{k \in \mathbb{N}} \mathbb{P}(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(B_k) \quad = \lim_{n\to\infty} \mathbb{P}\Bigg(\bigsqcup_{k=}$$

Continuity of Probability: Proof – 2

- Case 2: $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$
  - Clearly, $A_1^{\complement} \subseteq A_2^{\complement} \subseteq A_3^{\complement} \subseteq \cdots$
  - We then have

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = 1 - \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n^{\complement}\right) = 1 - \mathbb{P}\left(\lim_{n\to\infty} A_n^{\complement}\right) = 1 - \lim_{n\to\infty} \mathbb{P}(A_n^{\complement})$$

$$= \lim_{n\to\infty} 1 - \mathbb{P}(A_n^{\complement}) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Continuity of Probability: Proof – 3

- General case: $\lim_{n\to\infty} A_n$ exists, i.e., $A_{\text{liminf}} = A_{\text{limsup}} = \lim_{n\to\infty} A_n$
  - For any $n \in \mathbb{N}$,

$$\underbrace{\bigcap_{k\geq n} A_k}_{B_n} \subseteq A_n \subseteq \underbrace{\bigcup_{k\geq n} A_k}_{C_n}$$

  - Clearly,

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots, \qquad C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots, \qquad \mathbb{P}(B_n) \leq \mathbb{P}(A_n) \leq \mathbb{P}(C_n) \quad \forall n \in \mathbb{N}.$$

  - Observe that

$$A_{\text{liminf}} = \bigcup_{n\in\mathbb{N}} B_n, \qquad A_{\text{limsup}} = \bigcap_{n\in\mathbb{N}} C_n.$$

$$\mathbb{P}(A_{\text{liminf}}) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n) \leq \lim_{n\to\infty} \mathbb{P}(A_n) \leq \lim_{n\to\infty} \mathbb{P}(C_n) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} C_n\right)$$
$$= \mathbb{P}(A_{\text{limsup}})$$

Continuity of Probability: Proof – 3

- General case: $\lim_{n\to\infty} A_n$ exists, i.e., $A_{\text{liminf}} = A_{\text{limsup}} = \lim_{n\to\infty} A_n$
  - We then have

$$\mathbb{P}(A_{\text{liminf}}) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n) \le \lim_{n\to\infty} \mathbb{P}(A_n) \le \lim_{n\to\infty} \mathbb{P}(C_n) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} C_n\right)$$
$$= \mathbb{P}(A_{\text{limsup}})$$

  - If $A_{\text{liminf}} = A_{\text{limsup}}$, it follows that

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right)$$

Properties of Probability Measure

(Union Bound)

For any $A_1, A_2, \ldots \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mathbb{P}(A_n).$$

Equality in Union Bound

Equality in Union Bound

Suppose that for a given collection $A_1, A_2, \ldots \in \mathscr{F}$, we find that

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_n).$$

What can we say about $A_1, A_2, \ldots$?

Stay tuned for the answer!

Examples of Probability Assignment

For **Countable** (Finite or Countably Infinite) Sample Spaces

Examples of Probability Assignment

- Suggest a probability assignment $\mathbb{P}$ for

$$\Omega = \{H, T\}, \qquad \mathscr{F} = 2^{\Omega} = \Big\{\emptyset, \Omega, \{H\}, \{T\}\Big\}.$$

Examples of Probability Assignment

- Suggest a probability assignment $\mathbb{P}$ for

$$\Omega = \{1, \ldots, 6\}, \qquad \mathscr{F} = \Big\{\emptyset, \Omega, \{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 2, 3, 4\}, \{3, 4, 5, 6\}, \{1, 2, 5, 6\}\Big\}.$$

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Important Points to Keep in Mind

On Sets with $0/1$ Probability

On Sets with $0/1$ Probability

The following points must be borne in mind.

- $\mathbb{P}(A) = 0 \quad \not\Rightarrow \quad A = \emptyset.$
- $\mathbb{P}(A) = 1 \quad \not\Rightarrow \quad A = \Omega.$

Examples of Probability Assignment

- Fix $n \in \mathbb{N}$. Suggest a probability assignment $\mathbb{P}$ for

$$\Omega = \{1, \ldots, n\}, \qquad \mathscr{F} = \mathscr{B}(\Omega) = 2^{\Omega}.$$

Examples of Probability Assignment

- Suggest a probability assignment $\mathbb{P}$ for

$$\Omega = \mathbb{N}, \qquad \mathscr{F} = \mathscr{B}(\mathbb{N}) = 2^{\mathbb{N}}.$$

Some Popular Probability Assignments

- $\Omega = \{0, 1\}, \qquad \mathscr{F} = 2^{\Omega} = \Big\{\emptyset, \Omega, \{0\}, \{1\}\Big\}$

  Fix $p \in [0, 1]$. The measure $\mathbb{P}$ defined by

  $$\mathbb{P}(\{1\}) = p, \qquad \mathbb{P}(\{0\}) = 1 - p,$$

  is popularly called **Bernoulli(p) measure**

Some Popular Probability Assignments

- Fix $n \in \mathbb{N}$.
  $\Omega = \{1, \ldots, n\}, \qquad \mathscr{F} = 2^{\Omega}$

  The measure $\mathbb{P}$ defined by

  $$\mathbb{P}(\{k\}) \quad = \quad \frac{1}{n}, \qquad k \in \Omega,$$

  is popularly called **discrete uniform measure**

Some Popular Probability Assignments

- Fix $n \in \mathbb{N}$.
  $\Omega = \{0, \ldots, n\}, \qquad \mathscr{F} = 2^{\Omega}.$

  Fix $p \in [0, 1]$. The measure $\mathbb{P}$ defined by

$$\mathbb{P}(\{k\}) \quad = \quad \binom{n}{k} p^k (1-p)^{n-k}, \qquad k \in \Omega,$$

  is popularly called **Binomial**(n, p) **measure**

Some Popular Probability Assignments

- $\Omega = \mathbb{N}, \qquad \mathscr{F} = 2^{\Omega}.$

  Fix $p \in [0, 1]$. The measure $\mathbb{P}$ defined by

  $$\mathbb{P}(\{k\}) \quad = \quad (1-p)^{k-1}\, p, \qquad k \in \Omega,$$

  is popularly called **Geometric(p) measure**

Some Popular Probability Assignments

- $\Omega = \mathbb{N} \cup \{0\}, \qquad \mathscr{F} = 2^{\Omega}.$

  Fix $\lambda \in (0,\ +\infty)$. The measure $\mathbb{P}$ defined by

  $$\mathbb{P}(\{k\}) \quad = \quad \exp(-\lambda)\, \frac{\lambda^k}{k!}, \qquad k \in \Omega,$$

  is popularly called **Poisson(λ) measure**