Stochastic Processes

DTMCs: Recap of Important Results, Ergodicity, Convergence to Stationary Distribution

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  In addition, if x is aperiodic, then

$$\lim_{n\to\infty} P^n_{x,x} = \frac{1}{\mu_{xx}}.$$

- For an irreducible Markov chain,

$$\text{Unique stationary distribution exists} \iff \text{Markov chain is positive recurrent.}$$

  Furthermore, in this case, $\pi_x = \frac{1}{\mu_{xx}}$ for all x.

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Ergodicity and Ergodic Theorem

Ergodicity

Definition (Ergodic Markov Chain)

A time-homogeneous DTMC (on a finite or countably infinite state space) with TPM P is said to be ergodic if P is irreducible, aperiodic, and positive recurrent.

Ergodicity and Convergence to Stationary Distribution

Theorem (Ergodicity and Convergence to Stationary Distribution)

Consider a time-homogeneous DTMC $\{X_n\}_{n=0}^{\infty}$ on a discrete state space $\mathcal{X}$ with TPM P.
Assume that $X_0 = x$, and for each $n \in \mathbb{N}$, let π_n denote the PMF of X_n.
If P is ergodic with associated stationary distribution π, then

$$\lim_{n\to\infty} d_{\mathrm{TV}}(\pi_n, \pi) = \lim_{n\to\infty} \frac{1}{2} \|\pi_n - \pi\|_1 = 0,$$

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Remark: We shall present the proof only for $\mathcal{X}$ finite.
We note that the result holds even when $\mathcal{X}$ is countably infinite.

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- By Chapman–Kolmogorov,

$$Q^n_{(x,w),(y,z)} = P^n_{x,y} \cdot P^n_{w,z} \qquad \forall n \in \mathbb{N}.$$

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- **Important observation:**
$\lambda = 1$ is a simple eigenvalue of Q^n for all $n \geq N$.
Therefore, there exists a unique probability vector $\boldsymbol{\theta}$ such that $\boldsymbol{\theta} = \boldsymbol{\theta} Q$.
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- A simple observation shows that $\boldsymbol{\theta}$ defined via
$$\theta(x, w) = \pi(x) \cdot \pi(w)$$
is a stationary distribution for Q, and the only such one

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- From the coupling time onwards, the two processes $\{X_n\}_{n=0}^{\infty}$ and $\{Y_n\}_{n=0}^{\infty}$ will have identical statistics:

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&= \Big|\mathbb{P}(X_n = y,\ \tau > n) - \mathbb{P}(Y_n = y, \tau > n)\Big| \\
&\leq 2\,\mathbb{P}(\tau > n)
\end{aligned}$$

- Taking limits as $n \to \infty$, and noting that $\mathbb{P}(\tau < +\infty) = 1$, we arrive at the result