Almost-Optimal Best Restless Markov Arm Identification with Fixed Confidence

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National University of Singapore
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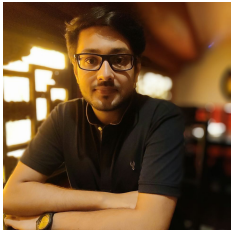
Joint Work With

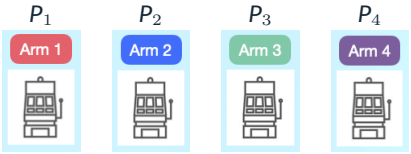


Arpan Mukherjee
RPI, New York

Vincent Tan
NUS, Singapore

Ali Tajer
RPI, New York

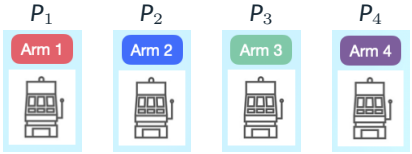
Problem Setup



Problem Setup

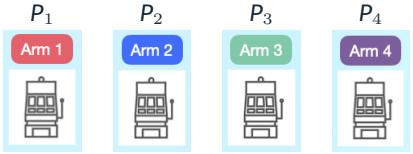
P_1	P_2	P_3	P_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$

Problem Setup



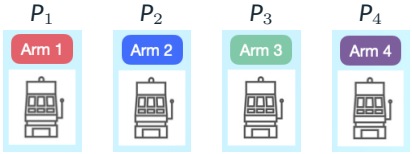
Time (n)	Arm (A_n)	Observation ($\bar{X}_n$)
0	1	$X_{0,1}$

Problem Setup



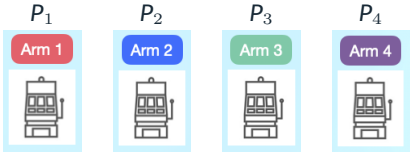
Time (n)	Arm (A_n)	Observation ($\bar{X}_n$)
0	1	$X_{0,1}$
1	2	$X_{1,2}$

Problem Setup



Time (n)	Arm (A_n)	Observation ($\bar{X}_n$)
0	1	$X_{0,1}$
1	2	$X_{1,2}$
2	3	$X_{2,3}$

Problem Setup



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
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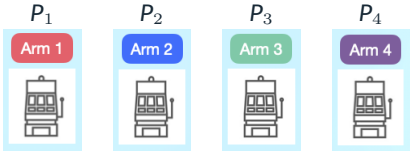
Problem Setup



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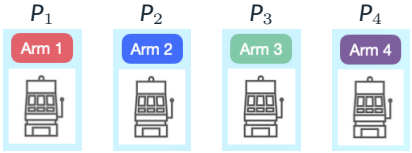
5	3	$X_{5,4}$
6	2	$X_{6,2}$
7	1	$X_{7,1}$

Problem Setup



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5	3	$X_{5,4}$
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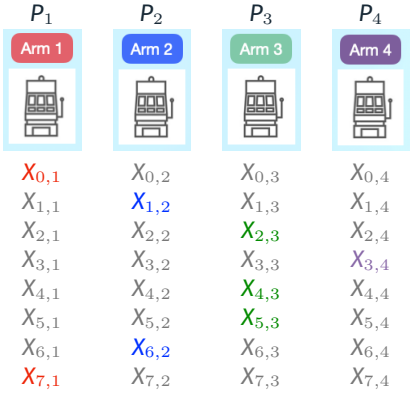
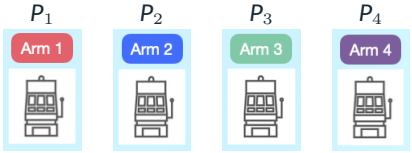
Goal: fixed-confidence best arm identification

Problem Setup



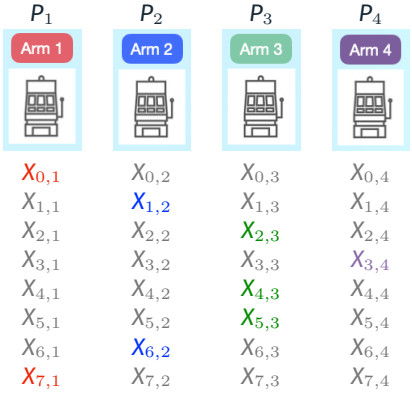
- K-armed bandit, $K \geq 2$

Problem Setup



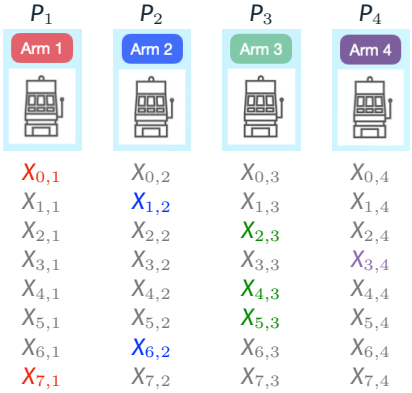
- K-armed bandit, $K \geq 2$
- Arm a: homogeneous DTMC $\{X_{t,a}\}_{t \geq 0}$

Problem Setup



- K-armed bandit, $K \geq 2$
- Arm a: homogeneous DTMC $\{X_{t,a}\}_{t \geq 0}$
- State space of each arm: $\mathcal{S}$, finite

Problem Setup



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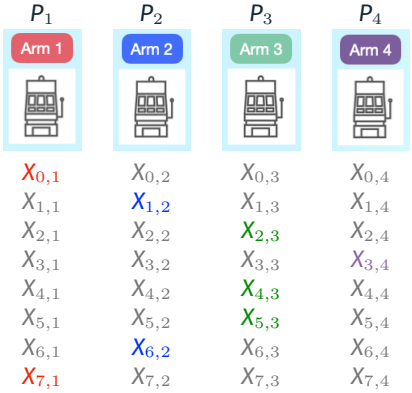
- State space of each arm: $\mathcal{S}$, finite
- Restless arms

Problem Setup



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- P_a : TPM of arm a, ergodic

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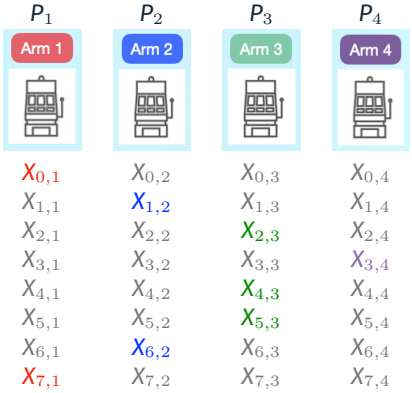

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- μ_a : stationary distribution of arm a

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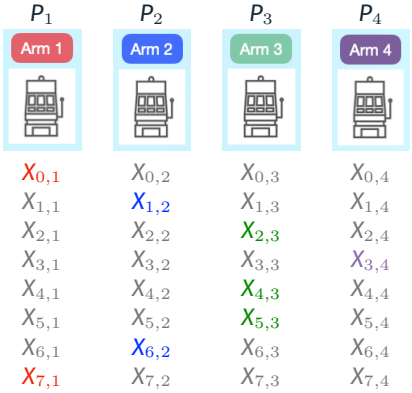

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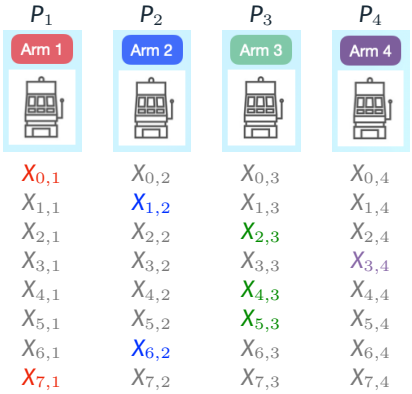

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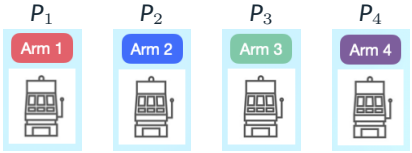
- Known reward function $f : \mathcal{S} \to \mathbb{R}$

Problem Setup



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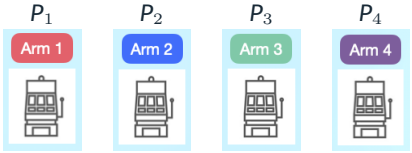
- $\eta_a = \langle f, \mu_a \rangle = \sum_{i \in \mathcal{S}} f(i)\, \mu_a(i)$

Problem Setup



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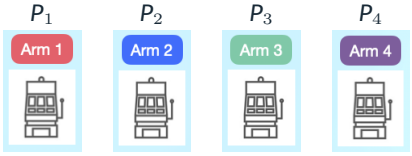
- $\eta_a = \langle f, \mu_a \rangle = \sum_{i \in \mathcal{S}} f(i)\, \mu_a(i)$
- Best arm $a^\star = \arg\max_a \eta_a$

Objective



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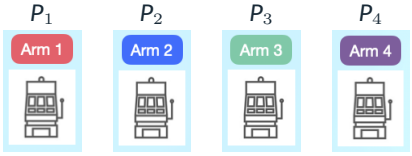
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$

π - policy for best arm identification
τ_π - stopping time of policy π

Objective



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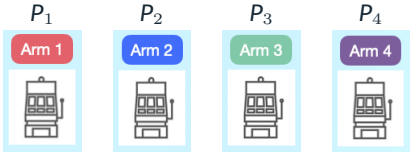

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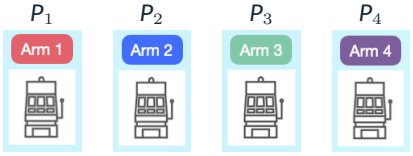
1. $\Pr(\tau_\pi < +\infty) = 1$

Objective



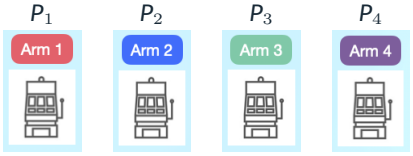
π - policy for best arm identification
τ_π - stopping time of policy π

1. $\Pr(\tau_\pi < +\infty) = 1$
2. Fixed-confidence setting: given any $\delta \in (0,1)$,

Instance	Arm 1	Arm 4	Other arms
P_1 P_2 P_3 P_4	$\geq 1-\delta$	$\leq \delta$	$\leq \delta$
Q_1 Q_2 Q_3 Q_4	$\leq \delta$	$\geq 1-\delta$	$\leq \delta$

$$\Pr(a_\pi \neq a^\star) \leq \delta \quad \forall \text{ instances}$$

Objective



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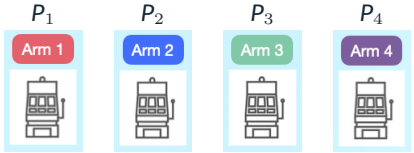
$Q_1\ Q_2\ Q_3\ \boxed{Q_4}$	$\leq \delta$	$\geq 1-\delta$	$\leq \delta$

$\Pr(a_\pi \neq a^\star) \leq \delta \quad \forall$ instances

3. $\Pi(\delta) = \{\pi$ satisfying 1. and 2.$\}$

Objective



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Fixed-confidence BAI: $\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi]$

Model: Exponential Family of TPMs

$\Theta \subset \mathbb{R}$ - known parameter space
P – irreducible TPM on $\mathcal{S}$ (generator)
Known reward function $f : \mathcal{S} \to \mathbb{R}$

$$\theta \mapsto \tilde{P}_\theta(i,j) = P(i,j)\, e^{\theta f(j)}, \quad i,j \in \mathcal{S}$$

Property of $\tilde{P}_\theta$: there exist vectors u_θ, v_θ such that

$$u_\theta(i) > 0 \quad \forall i, \qquad v_\theta(i) > 0 \quad \forall i, \qquad \tilde{P}_\theta\, u_\theta = \rho(\theta)\, u_\theta, \qquad v_\theta^\top\, \tilde{P}_\theta = \rho(\theta)\, v_\theta^\top$$

$$\theta \mapsto P_\theta(i,j) = \frac{v_\theta(j)}{\rho(\theta)\, v_\theta(i)} \times \tilde{P}_\theta(i,j), \quad i,j \in \mathcal{S}$$

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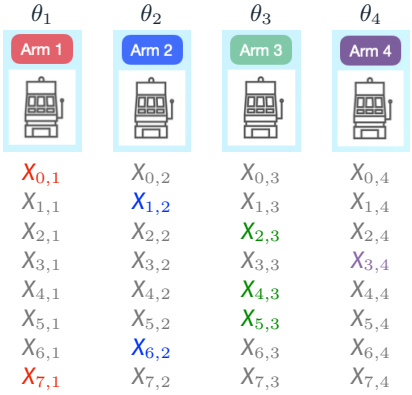
$$\theta \mapsto P_\theta(i,j) = \frac{v_\theta(j)}{\rho(\theta)\, v_\theta(i)} \times \tilde{P}_\theta(i,j), \quad i,j \in \mathcal{S}$$

Ergodic ($P_\theta \iff \mu_\theta = u_\theta \odot v_\theta$) ✔

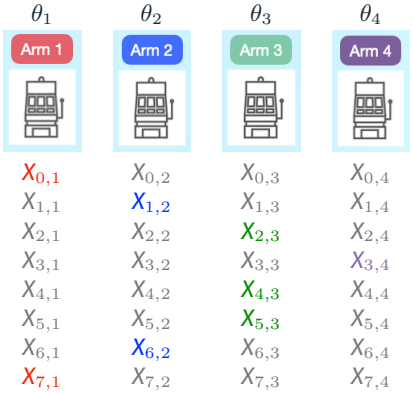
$\theta \mapsto \eta_\theta = \langle f, \mu_\theta \rangle$ strictly increasing bijection ✔

Objective



- $\boldsymbol{\theta} = [\theta_a : a \in [K]]^\top$ unknown
- $a^\star = \arg\max_a \eta_{\theta_a} = \arg\max_a \langle f, \mu_{\theta_a} \rangle$

Objective



- $\boldsymbol{\theta} = [\theta_a : a \in [K]]^\top$ unknown
- $a^\star = \arg\max_a \eta_{\theta_a} = \arg\max_a \langle f, \mu_{\theta_a} \rangle$

Fixed-confidence BAI: $\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi]$

Markov Decision Process

Markov Decision Process

θ_1	θ_2	θ_3	θ_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$

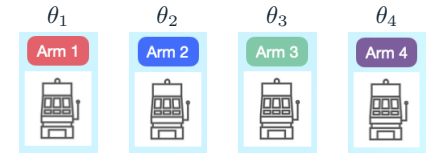
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

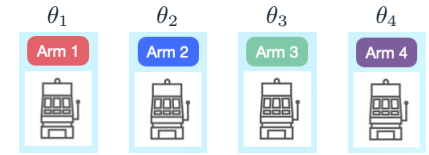
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9				

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9				

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

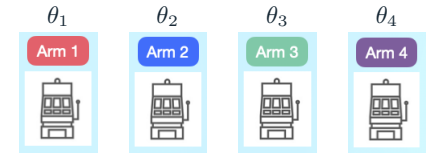
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2			

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$			

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

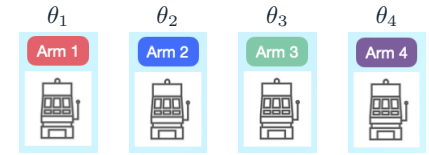
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1		

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$		

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

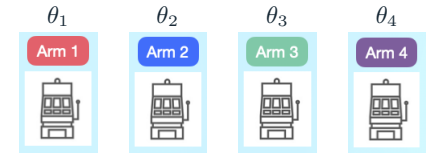
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$

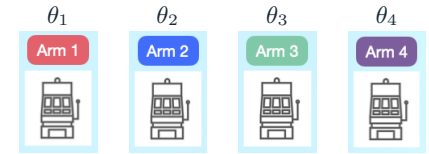
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

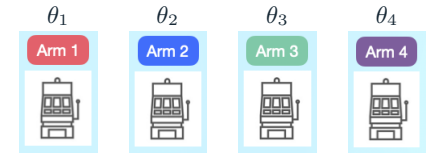
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10				

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10				

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3			

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$			

Markov Decision Process

θ_1	θ_2	θ_3	θ_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2		

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$		

Markov Decision Process

θ_1	θ_2	θ_3	θ_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

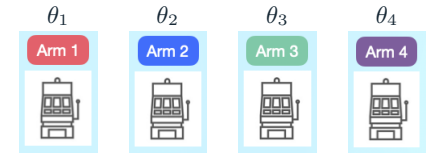
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

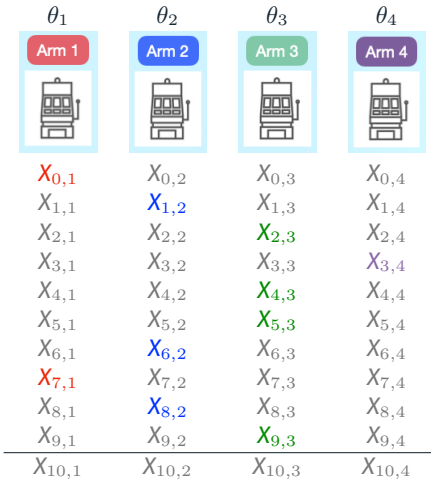
Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

Markov Decision Process



Arm delays

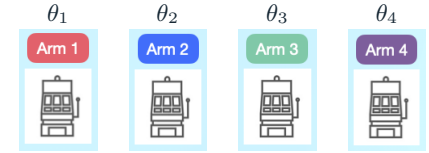
n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\underline{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\underline{i}(n) = [i_a(n) : a \in [K]]^\top$$

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\underline{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\underline{i}(n) = [i_a(n) : a \in [K]]^\top$$

$$(\underline{d}(n), \underline{i}(n)) \longrightarrow A_n \longrightarrow (\underline{d}(n+1), \underline{i}(n+1)) \longrightarrow A_{n+1} \longrightarrow (\underline{d}(n+2), \underline{i}(n+2)) \longrightarrow \cdots$$

Markov Decision Process

θ_1	θ_2	θ_3	θ_4
Arm 1	Arm 2	Arm 3	Arm 4
			
$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

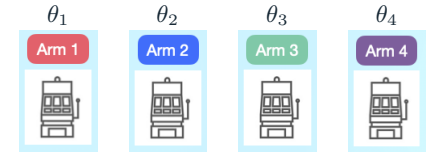
Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\underline{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\underline{i}(n) = [i_a(n) : a \in [K]]^\top$$

State space: $\mathbb{S} = \{(\underline{d}, \underline{i})\} \subset \mathbb{N}^K \times \mathcal{S}^K$ countably infinite

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Action space: $[K]$ finite

Arm delays

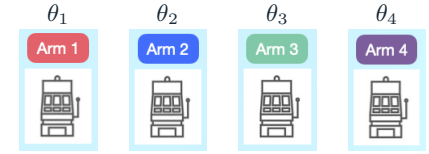
n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\underline{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\underline{i}(n) = [i_a(n) : a \in [K]]^\top$$

Markov Decision Process



Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

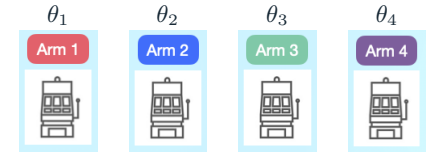
Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\underline{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\underline{i}(n) = [i_a(n) : a \in [K]]^\top$$

Transition probabilities: $\Pr(\underline{d}(9), \underline{i}(9) \mid \underline{d}(8), \underline{i}(8), A_8 = 2) = \Pr(X_{8,2} | X_{6,2}) = P^2_{\theta_2}(X_{6,2}, X_{8,2})$

Markov Decision Process



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

Arm delays

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

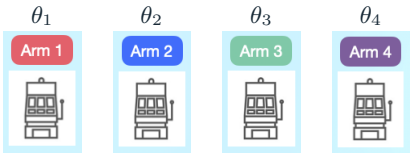
Last observed states

n	$i_1(n)$	$i_2(n)$	$i_3(n)$	$i_4(n)$
8	$X_{7,1}$	$X_{6,2}$	$X_{5,3}$	$X_{3,4}$
9	$X_{7,1}$	$X_{8,2}$	$X_{5,3}$	$X_{3,4}$
10	$X_{7,1}$	$X_{8,2}$	$X_{9,3}$	$X_{3,4}$

$$\underline{d}(n) = [d_a(n) : a \in [K]]^\top$$
$$\underline{i}(n) = [i_a(n) : a \in [K]]^\top$$

Transition probabilities: $\Pr(\underline{d}(10), \underline{i}(10) \mid \underline{d}(9), \underline{i}(9), A_9 = 3) = P^4_{\theta_3}(X_{5,3}, X_{9,3})$ TPM powers

Flow Constraint



$X_{0,1}$	$X_{0,2}$	$X_{0,3}$	$X_{0,4}$
$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$
$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$
$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$
$X_{6,1}$	$X_{6,2}$	$X_{6,3}$	$X_{6,4}$
$X_{7,1}$	$X_{7,2}$	$X_{7,3}$	$X_{7,4}$
$X_{8,1}$	$X_{8,2}$	$X_{8,3}$	$X_{8,4}$
$X_{9,1}$	$X_{9,2}$	$X_{9,3}$	$X_{9,4}$
$X_{10,1}$	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$

$$(\underline{d}(n), \underline{i}(n)) \longrightarrow A_n \xrightarrow{\text{entry}} (\underline{d}(n+1), \underline{i}(n+1)) \xrightarrow{\text{exit}} \cdots$$
$$A_{n+1} \xrightarrow{\text{entry}} (\underline{d}(n+2), \underline{i}(n+2)) \xrightarrow{\text{exit}} A_{n+2} \xrightarrow{\text{entry}} \cdots$$

$$\xrightarrow{\text{entry}} \boxed{(\underline{d}, \underline{i})} \xrightarrow{\text{exit}}$$

$$N(n, \underline{d}, \underline{i}, a) = \sum_{t=K}^{n} \mathbf{1}_{\{\underline{d}(t)=\underline{d},\ \underline{i}(t)=\underline{i},\ A_t=a\}}$$

$$N(n, \underline{d}, \underline{i}) = \sum_{a=1}^{K} N(n, \underline{d}, \underline{i}, a)$$

Flow Constraint

Under every policy π,

$$\left| \mathbb{E}[N(n, \underline{d}', \underline{i}')] - \sum_{(\underline{d}, \underline{i}) \in \mathbb{S}} \sum_{a=1}^{K} \mathbb{E}[N(n, \underline{d}, \underline{i}, a)] \, Q_{\theta}(\underline{d}', \underline{i}' | \underline{d}, \underline{i}, a) \right| \leq 1 \quad \forall (\underline{d}', \underline{i}') \in \mathbb{S}, \quad \forall n \geq K.$$

Maximum Delay Constraint

- Fix $R \in \mathbb{N}$ such that $R \gg K$
- Forcefully pull an arm if its delay equals R at any given time
- $\mathbb{S}_R = \{(\underline{d}, \underline{i}) \in \mathbb{S} : \max_a d_a \leq R\}, \qquad \mathbb{S}_{R,a} = \{(\underline{d}, \underline{i}) \in \mathbb{S}_R : d_a = R\}$ finite state space
- $\xrightarrow{\text{entry}}$ $(\underline{d}(n), \underline{i}(n)) \in \mathbb{S}_{R,a}$ $\xrightarrow{\text{exit}}$ $A_n = a$
- $Q_\theta \longrightarrow Q_{\theta,R}$

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

Maximum Delay Constraint

- Fix $R \in \mathbb{N}$ such that $R \gg K$
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- $\mathbb{S}_R = \{(\underline{d}, \underline{i}) \in \mathbb{S} : \max_a d_a \leq R\}$, $\mathbb{S}_{R,a} = \{(\underline{d}, \underline{i}) \in \mathbb{S}_R : d_a = R\}$ finite state space
- $\xrightarrow{\text{entry}}$ $(\underline{d}(n), \underline{i}(n)) \in \mathbb{S}_{R,a}$ $\xrightarrow{\text{exit}} A_n = a$
- $Q_\theta \longrightarrow Q_{\theta,R}$

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
8	1	2	3	5
9	2	1	4	6
10	3	2	1	7

$R = 6$

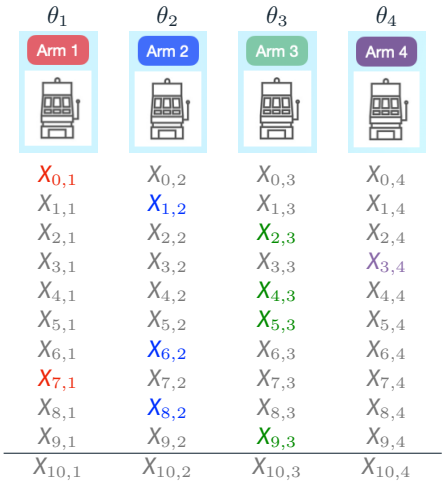
$(\underline{d}(9), \underline{i}(9)) \longrightarrow A_{10} = 4$ ✔

Flow Constraint + R-Max-Delay Constraint

$$\left| \mathbb{E}[N(n, \underline{d}', \underline{i}')] - \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \mathbb{E}[N(n, \underline{d}, \underline{i}, a)]\, Q_{\theta,R}(\underline{d}', \underline{i}' | \underline{d}, \underline{i}, a) \right| \leq 1 \quad \forall (\underline{d}', \underline{i}') \in \mathbb{S}_R, \quad \forall n \geq K,$$

$$N(n, \underline{d}, \underline{i}, a) = N(n, \underline{d}, \underline{i}) \qquad \forall (\underline{d}, \underline{i}) \in \mathbb{S}_{R,a},\ a \in [K]$$

Objective Restated



- $\boldsymbol{\theta} = [\theta_a : a \in [K]]^\top$ unknown
- $a^\star(\boldsymbol{\theta}) = \arg\max_a \eta_{\theta_a} = \arg\max_a \langle f, \mu_{\theta_a} \rangle$

Objective: $\inf_\pi \mathbb{E}[\tau_\pi]$

- ✔ $\pi \in \Pi(\delta)$
- ✔ π satisfies *R*-max-delay constraint

Converse

Converse

- Fix $\boldsymbol{\theta}$. Assume $a^\star(\boldsymbol{\theta}) = 1$
- Set of alternative instances:

$$\begin{aligned}\textsc{Alt}(\boldsymbol{\theta}) &= \{\boldsymbol{\lambda} \in \Theta^K : \exists a \neq a^\star(\boldsymbol{\theta}) \text{ such that } \eta_{\lambda_a} > \eta_{\lambda_1}\} \\ &= \{\boldsymbol{\lambda} \in \Theta^K : \exists a \neq a^\star(\boldsymbol{\theta}) \text{ such that } \lambda_a > \lambda_1\} = \bigcup_{a=1}^{K} \{\boldsymbol{\lambda} \in \Theta^K : \lambda_a > \lambda_1\}\end{aligned}$$

Converse

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- Set of alternative instances:

$$\begin{aligned}\textsc{Alt}(\boldsymbol{\theta}) &= \{\boldsymbol{\lambda} \in \Theta^K : \exists a \neq a^\star(\boldsymbol{\theta}) \text{ such that } \eta_{\lambda_a} > \eta_{\lambda_1}\} \\ &= \{\boldsymbol{\lambda} \in \Theta^K : \exists a \neq a^\star(\boldsymbol{\theta}) \text{ such that } \lambda_a > \lambda_1\} = \bigcup_{a=1}^{K} \{\boldsymbol{\lambda} \in \Theta^K : \lambda_a > \lambda_1\}\end{aligned}$$

Proposition

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T^\star_R(\boldsymbol{\theta})},$$

$$T^\star_R(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \textsc{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d}, \underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d}, \underline{i}, a)\, D_{KL}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d}, \underline{i}, a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d}, \underline{i}, a)),$$

$$\Sigma_R(\boldsymbol{\theta}) = \Big\{\nu : \checkmark \textit{prob. dist.}, \ \checkmark \boxed{\textit{flow constraint}}, \ \checkmark \boxed{\textit{R-max-delay constraint}}\Big\}$$

Converse – 1

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T_R^\star(\boldsymbol{\theta})}$$

Converse – 1

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T_R^\star(\boldsymbol{\theta})} \sim \frac{\log \frac{1}{\delta}}{T_R^\star(\boldsymbol{\theta})}$$

Converse – 1

$$\inf_{\pi \in \Pi(\delta)} \mathbb{E}[\tau_\pi] \geq \frac{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}{T_R^\star(\boldsymbol{\theta})} \sim \frac{\log \frac{1}{\delta}}{T_R^\star(\boldsymbol{\theta})}$$

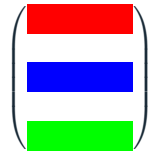
$$\liminf_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}[\tau_\pi]}{\log(1/\delta)} \geq \frac{1}{T_R^\star(\boldsymbol{\theta})}$$

Converse – 2

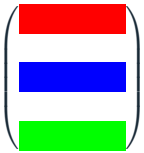
$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d}, \underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d}, \underline{i}, a)\, D_{\mathrm{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d}, \underline{i}, a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d}, \underline{i}, a))$$

Converse – 2

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathsf{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d},\underline{i},a)\, D_{\mathsf{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d},\underline{i},a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d},\underline{i},a))$$



$Q_{\boldsymbol{\theta},R}, \quad \boldsymbol{\theta} = [\theta_a : a \in [K]]^\top$



$Q_{\boldsymbol{\lambda},R}, \quad \boldsymbol{\lambda} = [\lambda_a : a \in [K]]^\top$

$$Q_{\boldsymbol{\theta},R}(\underline{d}', \underline{i}' \mid \underline{d}, \underline{i}, a) = \begin{cases} P_{\theta_a}^{d_a}(i_a, i_a'), & (\underline{d},\underline{i}) \xrightarrow{a} (\underline{d}', \underline{i}'), \\ 0, & \text{otherwise} \end{cases}$$

Converse – 3

$$\Sigma_R(\boldsymbol{\theta}) = \Big\{\nu : \checkmark\text{prob. dist.}, \quad \checkmark\boxed{\text{flow constraint}}, \quad \checkmark\boxed{\text{R-max-delay constraint}}\Big\}$$

$$\Sigma_R(\boldsymbol{\theta}) = \Bigg\{\nu : \quad \nu(\underline{d}, \underline{i}, a) \geq 0 \quad \forall (\underline{d}, \underline{i}, a),$$

$$\sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d}, \underline{i}, a) = 1,$$

$$\sum_{a=1}^{K} \nu(\underline{d}', \underline{i}', a) = \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d}, \underline{i}, a)\, Q_{\boldsymbol{\theta},R}(\underline{d}', \underline{i}' | \underline{d}, \underline{i}, a) \quad \forall (\underline{d}', \underline{i}') \in \mathbb{S}_R,$$

$$\nu(\underline{d}, \underline{i}, a) = \sum_{a'=1}^{K} \nu(\underline{d}, \underline{i}, a') \quad \forall (\underline{d}, \underline{i}) \in \mathbb{S}_{R,a},\ a \in [K] \Bigg\}$$

Converse – 4

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d},\underline{i},a)\, D_{\text{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d},\underline{i},a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d},\underline{i},a))$$

$$\psi(\nu,\boldsymbol{\theta}) = \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d},\underline{i},a)\, D_{\text{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d},\underline{i},a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d},\underline{i},a)), \quad \nu \in \Sigma_R(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \Theta^K$$

Lemma

Converse – 4

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d},\underline{i},a)\, D_{\mathrm{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d},\underline{i},a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d},\underline{i},a))$$

$$\psi(\nu,\boldsymbol{\theta}) = \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d},\underline{i},a)\, D_{\mathrm{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d},\underline{i},a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d},\underline{i},a)), \quad \nu \in \Sigma_R(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \Theta^K$$

Lemma

- ψ *is continuous*

Continuity of ψ – Berge's maximum theorem for non-compact sets (Feinberg et al., 2014)

Converse – 4

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d},\underline{i},a)\, D_{\mathrm{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d},\underline{i},a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d},\underline{i},a))$$

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Lemma

- ψ *is continuous*
- $\boldsymbol{\theta} \mapsto \mathcal{W}^\star(\boldsymbol{\theta}) = \{\nu \in \Sigma_R(\boldsymbol{\theta}) : \psi(\nu,\boldsymbol{\theta}) = T_R^\star(\boldsymbol{\theta})\}$ *is upper-semicontinuous*

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Key idea for achievability: $\left[\frac{N(n,\underline{d},\underline{i},a)}{n} : (\underline{d},\underline{i},a) \in \mathbb{S}_R \times [K]\right]^\top \longrightarrow \mathcal{W}^\star(\boldsymbol{\theta})$

Lower Bounds from Prior Works

Restless arms

$$T_R^\star(\boldsymbol{\theta}) = \sup_{\nu \in \Sigma_R(\boldsymbol{\theta})} \inf_{\boldsymbol{\lambda} \in \mathsf{Alt}(\boldsymbol{\theta})} \sum_{(\underline{d}, \underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d}, \underline{i}, a)\, D_{\mathsf{KL}}(Q_{\boldsymbol{\theta},R}(\cdot|\underline{d}, \underline{i}, a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d}, \underline{i}, a))$$

Rested arms Moulos (2019)

$$T^\star(\boldsymbol{\theta}) = \sup_{\nu} \inf_{\boldsymbol{\lambda} \in \mathsf{Alt}(\boldsymbol{\theta})} \sum_{i \in \mathcal{S}} \sum_{a=1}^{K} \nu(a)\, \mu_{\theta_a}(i)\, D_{\mathsf{KL}}(P_{\theta_a}(i, \cdot) \| P_{\lambda_a}(i, \cdot))$$

Independent observations from arms Garivier and Kaufmann (2016)

$$T^\star(\boldsymbol{\theta}) = \sup_{\nu} \inf_{\boldsymbol{\lambda} \in \mathsf{Alt}(\boldsymbol{\theta})} \sum_{a=1}^{K} \nu(a)\, D_{\mathsf{KL}}(\theta_a, \lambda_a)$$

Achievability

Uniform Policy and Ergodicity

$$\pi^{\mathsf{unif}}(a|\underline{d},\underline{i}) = \begin{cases} \frac{1}{K}, & (\underline{d},\underline{i}) \in \bigcup_{a'=1}^{K} \mathbb{S}_{R,a'}, \\ 1, & (\underline{d},\underline{i}) \in \mathbb{S}_{R,a}, \\ 0, & (\underline{d},\underline{i}) \in \bigcup_{a' \neq a} \mathbb{S}_{R,a'}. \end{cases}$$

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Transition kernel under $\boldsymbol{\theta} \in \Theta^K$ and π^{unif}:

$$Q_{\boldsymbol{\theta},\pi^{\text{unif}}}(\underline{d}',\underline{i}'|\underline{d},\underline{i}) = \sum_{a=1}^{K} Q_{\boldsymbol{\theta},R}(\underline{d}',\underline{i}'|\underline{d},\underline{i},a) \cdot \pi^{\text{unif}}(a|\underline{d},\underline{i}) \quad \forall (\underline{d},\underline{i}), (\underline{d}',\underline{i}') \in \mathbb{S}_R$$

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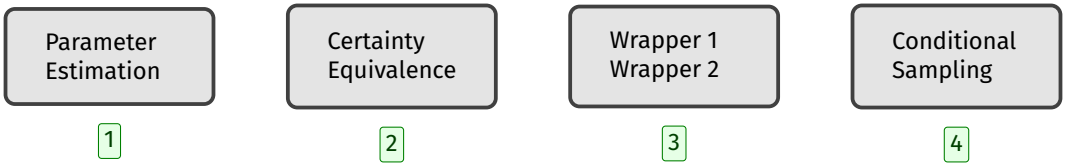
Lemma

$Q_{\boldsymbol{\theta},\pi^{unif}}$ is ergodic for all $\boldsymbol{\theta} \in \Theta^K$

Stationary distribution of $Q_{\boldsymbol{\theta},\pi^{\text{unif}}} = \mu_{\boldsymbol{\theta}}^{\text{unif}} > 0$,

$$\nu_{\boldsymbol{\theta}}^{\text{unif}}(\underline{d},\underline{i},a) = \mu_{\boldsymbol{\theta}}^{\text{unif}}(\underline{d},\underline{i}) \cdot \pi^{\text{unif}}(a|\underline{d},\underline{i}) \qquad \forall (\underline{d},\underline{i},a) \in \mathbb{S}_R \times [K]$$

Achievability – Arm Selection Rule



Arm Selection Rule – 1

Parameter Estimation

Certainty Equivalence

Wrapper 1
Wrapper 2

Conditional Sampling

1

2

3

4

$\hat{\boldsymbol{\theta}}(n)$

$$N_a(n) = \sum_{t=0}^{n} \mathbf{1}_{\{A_t = a\}}$$

$$\hat{\eta}^a(n) = \begin{cases} 0, & N_a(n) = 0, \\ \frac{1}{N_a(n)} \sum_{t=0}^{n} \mathbf{1}_{\{A_t = a\}} f(\bar{X}_t), & N_a(n) > 0 \end{cases}$$

$$\hat{\eta}_a(n) \Longleftrightarrow \hat{\theta}_a(n)$$

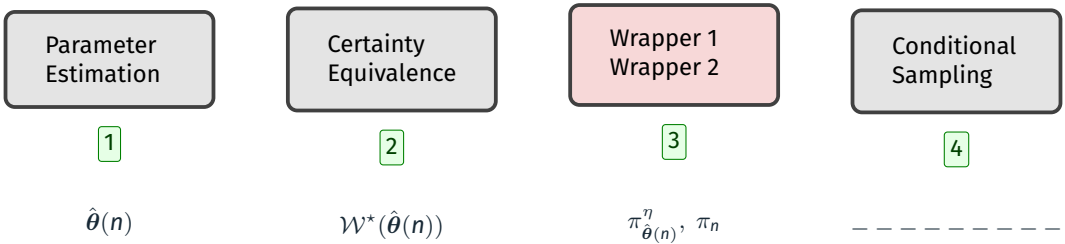
$$\hat{\boldsymbol{\theta}}(n) = [\hat{\theta}_a(n) : a \in [K]]$$

Arm Selection Rule – 2

Parameter Estimation	Certainty Equivalence	Wrapper 1 Wrapper 2	Conditional Sampling
1	2	3	4
$\hat{\boldsymbol{\theta}}(n)$	$\mathcal{W}^\star(\hat{\boldsymbol{\theta}}(n))$	– – – – – – – – –	– – – – – – – – –

$$\mathcal{W}^\star(\hat{\boldsymbol{\theta}}(n)) = \underset{\nu \in \Sigma_R(\hat{\boldsymbol{\theta}}(n))}{\arg\sup} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\hat{\boldsymbol{\theta}}(n))} \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \nu(\underline{d},\underline{i},a)\, D_{\text{KL}}(Q_{\hat{\boldsymbol{\theta}}(n),R}(\cdot|\underline{d},\underline{i},a) \| Q_{\boldsymbol{\lambda},R}(\cdot|\underline{d},\underline{i},a))$$

Arm Selection Rule – 3



Fix $\eta \in (0,1)$ and $\varepsilon_n \to 0$.
Pick $\nu_n^\star \in \mathcal{W}^\star(\hat{\theta}(n))$

Wrapper 1

$$\pi^\eta_{\hat{\theta}(n)}(a|\underline{d},\underline{i}) = \frac{\eta\, \nu^{\text{unif}}_{\hat{\theta}(n)}(\underline{d},\underline{i},a) + (1-\eta)\, \nu_n^\star(\underline{d},\underline{i},a)}{\eta\, \mu^{\text{unif}}_{\hat{\theta}(n)}(\underline{d},\underline{i}) + (1-\eta)\, \sum_{a'=1}^{K} \nu_n^\star(\underline{d},\underline{i},a')}, \quad (\underline{d},\underline{i},a) \in \mathbb{S}_R \times [K]$$

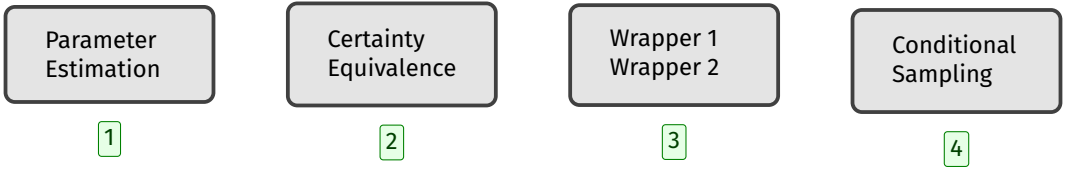
Wrapper 2

$$\pi_n = \varepsilon_n \pi^{\text{unif}} + (1-\varepsilon_n)\, \pi^\eta_{\hat{\theta}(n-1)} \quad \forall n \geq K$$

Arm Selection Rule – 4

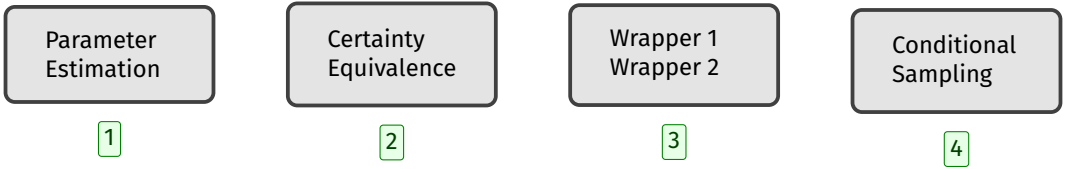
Parameter Estimation	Certainty Equivalence	Wrapper 1 Wrapper 2	Conditional Sampling
1	2	3	4
$\hat{\theta}(n)$	$\mathcal{W}^\star(\hat{\theta}(n))$	$\pi^{\eta}_{\hat{\theta}(n)},\ \pi_n$	$A_n \sim \pi_n(\cdot \vert \underline{d}(n), \underline{i}(n))$

Achievability – Arm Selection Rule



Lemma

Achievability – Arm Selection Rule



Lemma

For $S_R = |\mathbb{S}_R|$ *and* $\varepsilon_n = n^{-\frac{1}{2(1+S_R)}}$,

$$\lim_{n\to\infty} d_\infty \left(\left[\frac{N(n, \underline{d}, \underline{i}, a)}{n} : (\underline{d}, \underline{i}, a) \in \mathbb{S}_R \times [K] \right]^\top, \mathcal{W}_\eta^\star(\boldsymbol{\theta}) \right) = 0 \quad a.s.,$$

where $\mathcal{W}_\eta^\star(\boldsymbol{\theta}) = \{\eta\, \nu_{\boldsymbol{\theta}}^{unif} + (1-\eta)\, \nu : \nu \in \mathcal{W}^\star(\boldsymbol{\theta})\}$

Achievability – Stopping Rule

Empirical transition matrix:

$$\widehat{Q}_n(\underline{d}', \underline{i}' | \underline{d}, \underline{i}, a) = \begin{cases} \dfrac{1}{N(n, \underline{d}, \underline{i}, a)} \sum_{t=K}^{n} \mathbf{1}_{\{(\underline{d}(t), \underline{i}(t)) = (\underline{d}, \underline{i}),\, A_t = a,\, (\underline{d}(t+1), \underline{i}(t+1)) = (\underline{d}', \underline{i}')\}}, & N(n, \underline{d}, \underline{i}, a) > 0, \\ \dfrac{1}{S_R}, & N(n, \underline{d}, \underline{i}, a) = 0 \end{cases}$$

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Test statistic

$$Z(n) = \inf_{\lambda \in \mathrm{Alt}(\hat{\theta}(n))} \sum_{(\underline{d}, \underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} N(n, \underline{d}, \underline{i}, a)\, D_{\mathrm{KL}}(\hat{Q}_n(\cdot | \underline{d}, \underline{i}, a) \| Q_{\lambda, R}(\cdot | \underline{d}, \underline{i}, a))$$

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Threshold:

$$\zeta(n, \delta) = \log\left(\frac{1}{\delta}\right) + (S_R - 1) \sum_{(\underline{d},\underline{i}) \in \mathbb{S}_R} \sum_{a=1}^{K} \log\left(e\left[1 + \frac{N(n, \underline{d}, \underline{i}, a)}{S_R - 1}\right]\right)$$

Stopping rule:

$$\tau_\delta = \inf\{n \geq K : Z(n) \geq \zeta(n, \delta)\}$$

Recommendation rule:

$$\hat{a} = \arg\max_a \hat{\eta}_a(n)$$

Performance

- Error prob. $\leq \delta$
- Almost sure upper bound

$$\limsup_{\delta \downarrow 0} \frac{\tau_\delta}{\log(1/\delta)} \leq \frac{1}{(1-\eta)\, T_R^\star(\boldsymbol{\theta})} \quad \text{a.s.}$$

- Upper bound in expectation

$$\limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{(1-\eta)\, T_R^\star(\boldsymbol{\theta})}$$

$$\limsup_{\eta \downarrow 0} \limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{T_R^\star(\boldsymbol{\theta})}$$

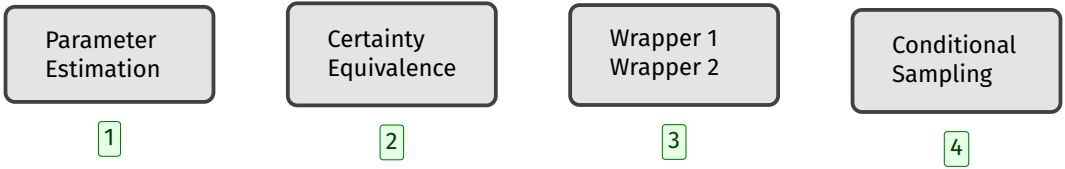
Algorithm 1 Restless Tracking

Input: $K, R, \eta, \delta > 0$
Output: $\hat{a} \in [K]$ (best arm).

Initialization: $n = 0$, $N_a(n) = 0$, $\hat{\eta}_a(n) = 0$ for all $a \in [K]$, $N(n, \underline{d}, \underline{i}, a) = 0$ for all $(\underline{d}, \underline{i}, a) \in \mathbb{S}_R \times [K]$, $\texttt{stop} = 0$

for $n < K$ **do**
  Select arm $A_n = n + 1$.
end for
while $\texttt{stop} == 0$ **do**
  Update $(\underline{d}(n), \underline{i}(n))$; update $\hat{\eta}_a(n)$ for each $a \in [K]$
  $\hat{\boldsymbol{\eta}}(n) \iff \hat{\boldsymbol{\theta}}(n)$
  Evaluate $Z(n)$
  if $Z(n) \geq \zeta(n, \delta)$ **then**
    $\texttt{stop} = 1$
    $\tau_\delta = n$
    $\hat{a} = \arg\max_a \hat{\eta}_a(n)$
  else
    Select $A_n \sim \pi_n(\cdot | \underline{d}(n), \underline{i}(n))$
    $n \leftarrow n + 1$.
  end if
end while
return $\hat{a}$.

Main Result



Theorem

$$\frac{1}{T_R^\star(\boldsymbol{\theta})} \leq \liminf_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}[\tau_\pi]}{\log(1/\delta)} \leq \limsup_{\eta \downarrow 0} \limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{T_R^\star(\boldsymbol{\theta})}$$

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Independent observations from arms ✔

Restless arms – OPEN

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Restless arms – OPEN

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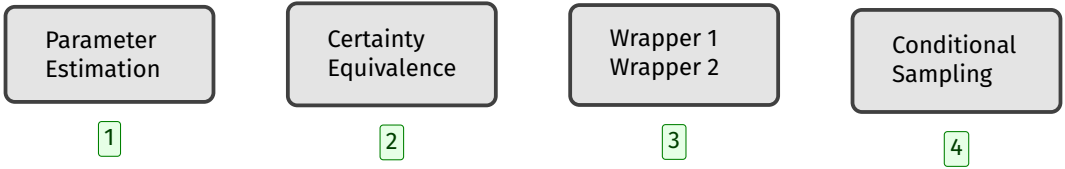
References

Feinberg, E. A., Kasyanov, P. O., and Voorneveld, M. (2014). Berge's maximum theorem for noncompact image sets. *Journal of Mathematical Analysis and Applications*, 413(2):1040–1046.

Garivier, A. and Kaufmann, E. (2016). Optimal best arm identification with fixed confidence. In *Conference on Learning Theory*, pages 998–1027. PMLR.

Moulos, V. (2019). Optimal best Markovian arm identification with fixed confidence. *Advances in Neural Information Processing Systems*, 32.

Main Result



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