Name:
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Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

AI5090: Stochastic Processes

Quiz 1

Date: 02 February 2026

Question	1	2	Total
Marks Scored			

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that all random variables are defined with respect to this common space.

1. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables.

   (a) **(2 Marks)**
   Show from first principles that for every choice of $n \in \mathbb{N}$ and $x \in \mathbb{R}$, the set

   $$\left\{\sup_{k\geq n} X_k < x\right\} \in \mathscr{F}.$$

   (b) **(1 Mark)**
   Using the result of part (a), justify why

   $$\left\{\limsup_{n\to\infty} X_n \geq 2\right\} \in \mathscr{F}.$$

   Show every step of reasoning clearly.

Solutions.

(a) Fixing $n \in \mathbb{N}$ and $x \in \mathbb{R}$ arbitrarily, we note that

$$\begin{aligned}\forall\,\omega\in\Omega, \qquad \sup_{k\geq n} X_k(\omega) < x \quad &\iff \quad \exists\, q\in\mathbb{Q},\ q>0: \quad \sup_{k\geq n} X_k(\omega) \leq x - q \\ &\iff \quad \exists\, q\in\mathbb{Q},\ q>0: \quad X_k(\omega) \leq x - q \quad \forall\, k\geq n\end{aligned}$$

Thus, it follows that

$$\left\{\sup_{k\geq n} X_k < x\right\} = \bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \left\{\sup_{k\geq n} X_k \leq x - q\right\} = \bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \bigcap_{k\geq n} \left\{X_k \leq x - q\right\}.$$

Because X_k is a random variable for each k, we have $\left\{X_k \leq x - q\right\} \in \mathscr{F}$. The result then follows by noting that countable unions and/or countable intersections of sets in $\mathscr{F}$ also belong to $\mathscr{F}$.

(b) We note that

$$\left\{\limsup_{n\to\infty} X_n \geq 2\right\} = \left\{\inf_{n\in\mathbb{N}} \sup_{k\geq n} X_k \geq 2\right\} = \bigcap_{n\in\mathbb{N}} \left\{\sup_{k\geq n} X_k \geq 5\right\} = \bigcap_{n\in\mathbb{N}} \left\{\sup_{k\geq n} X_k < 5\right\}^{\complement}.$$

From part (a), we have

$$\left\{\sup_{k\geq n} X_k < 5\right\}^{\complement} \in \mathscr{F} \qquad \forall\, n\in\mathbb{N}.$$

The desired result then follows by noting that countable intersections of sets in $\mathscr{F}$ belong to $\mathscr{F}$.

2. **(2 Marks)**

Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of **independent**, real-valued random variables, with

$$\mathbb{P}\left(\left\{X_n = \frac{1}{2}\Big(1-\frac{1}{n}\Big)\right\}\right) = \mathbb{P}\left(\left\{X_n = \frac{1}{2}\Big(1+\frac{1}{n}\Big)\right\}\right) = \frac{1}{2}\left(1-\frac{1}{n}\right), \qquad \mathbb{P}(\{X_n=1\}) = \frac{1}{n}.$$

Does the above sequence convergence almost-surely?
If yes, identify a limit random variable and prove the almost-sure convergence.
If not, justify why the sequence does not converge almost-surely.

Solution.

For each $n\in\mathbb{N}$, let

$$A_n := \left\{X_n = \frac{1}{2}\Big(1-\frac{1}{n}\Big)\right\}, \qquad B_n := \left\{X_n = \frac{1}{2}\Big(1+\frac{1}{n}\Big)\right\}, \qquad C_n := \{X_n = 1\}.$$

Then, we have

$$\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \sum_{n\in\mathbb{N}} \frac{1}{2}\Big(1-\frac{1}{n}\Big) = +\infty, \qquad \sum_{n\in\mathbb{N}} \mathbb{P}(B_n) = \sum_{n\in\mathbb{N}} \frac{1}{2}\Big(1-\frac{1}{n}\Big) = +\infty, \qquad \sum_{n\in\mathbb{N}} \mathbb{P}(C_n) = \sum_{n\in\mathbb{N}} \frac{1}{n} = +\infty.$$

Furthermore, noting that $X_1, X_2, \ldots$ are independent, it follows that $\{A_n\}_{n\in\mathbb{N}}$, $\{B_n\}_{n\in\mathbb{N}}$, and $\{C_n\}_{n\in\mathbb{N}}$ are sequences of independent events. Applying the Borel–Cantelli lemma, we get that

$$\mathbb{P}\left(\underbrace{\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_n}_{A}\right) = 1, \qquad \mathbb{P}\left(\underbrace{\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} B_n}_{B}\right) = 1, \qquad \mathbb{P}\left(\underbrace{\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} C_n}_{C}\right) = 1.$$

We then have $\mathbb{P}(A\cap B\cap C) = 1$. For any $\omega \in A\cap B\cap C$, we note that

$$X_n(\omega) = \frac{1}{2}\left(1-\frac{1}{n}\right) \quad \text{for infinitely many indices } n,$$
$$X_n(\omega) = \frac{1}{2}\left(1+\frac{1}{n}\right) \quad \text{for infinitely many indices } n,$$
$$X_n(\omega) = 1 \quad \text{for infinitely many indices } n,$$

which in turn implies that $\lim_{n\to\infty} X_n(\omega)$ does not exist. Thus, we have

$$A\cap B\cap C \;\subseteq\; \left\{\lim_{n\to\infty} X_n \text{ does not exist}\right\},$$

thereby demonstrating that

$$\mathbb{P}\left(\left\{\lim_{n\to\infty} X_n \text{ exists}\right\}\right) = 0.$$

This shows that $\{X_n\}_{n\in\mathbb{N}}$ does not admit any almost-sure limit.