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Probability and Stochastic Processes

Lecture 17: Jointly Discrete/Continuous Random Variables, Joint PMF, Joint PDF, Conditional PMF, Conditional PDF, Multiple Random Variables

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Jointly Discrete Random Variables

Definition (Jointly Discrete Random Variables)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
X and Y are said to be **jointly discrete** if the vector (X, Y) is a discrete random variable, i.e., there exists a **countable** set $E \subseteq \mathbb{R}^2$ such that

$$\mathbb{P}_{X,Y}(E) = \mathbb{P}(\{(X, Y) \in E\}) = 1.$$

- Define E_1 and E_2 as

$$E_1 := \{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } (x, y) \in E\}, \qquad E_2 = \{y \in \mathbb{R} : \exists x \in \mathbb{R} \text{ such that } (x, y) \in E\}.$$

- If (X, Y) is discrete, then X and Y are individually discrete, as

$$\mathbb{P}_X(E_1) = \mathbb{P}_{X,Y}(E_1 \times \mathbb{R}) = 1, \qquad \mathbb{P}_Y(E_2) = \mathbb{P}_{X,Y}(\mathbb{R} \times E_2) = 1.$$

- (X, Y) discrete $\implies$ X discrete, Y discrete.

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X and Y are said to be **jointly discrete** if the vector (X, Y) is a discrete random variable, i.e., there exists a **countable** set $E \subseteq \mathbb{R}^2$ such that

$$\mathbb{P}_{X,Y}(E) = \mathbb{P}(\{(X, Y) \in E\}) = 1.$$

- Suppose X and Y are individually discrete with

$$\mathbb{P}_X(E_1) = 1, \qquad \mathbb{P}_Y(E_2) = 1,$$

  for some countable sets $E_1, E_2 \subset \mathbb{R}$
- Then, (X, Y) is discrete: $E_1 \times E_2$ countable,

$$\mathbb{P}_{X,Y}(E_1 \times E_2) = \mathbb{P}_{X,Y}\big((E_1 \times \mathbb{R}) \cap (\mathbb{R} \times E_2)\big) = 1.$$

- X discrete, Y discrete $\implies$ (X, Y) discrete .

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Marginal PMFs and Conditional PMFs

Marginal PMFs from Joint PMF

For Jointly Discrete Random Variables

Theorem (Marginal PMFs from Joint PMF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be jointly discrete, with $\mathbb{P}_{X,Y}(E) = 1$ for some countable $E \subset \mathbb{R}^2$.
Then, the following properties hold.

1. The joint PMF on the range must sum to 1, i.e.,

$$\sum_{(x,y)\in E} p_{X,Y}(x,y) = 1.$$

2. **(Marginalization Property)**

$$\forall\, x \in \mathbb{R}, \qquad \sum_{y:\ (x,y)\in E} p_{X,Y}(x,y) = p_X(x),$$

$$\forall\, y \in \mathbb{R}, \qquad \sum_{x:\ (x,y)\in E} p_{X,Y}(x,y) = p_Y(y).$$

Independence for Jointly Discrete Random Variables

Proposition (Independence for Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be **jointly discrete** random variables with $\mathbb{P}_{X,Y}(E) = 1$ for some countable $E \subset \mathbb{R}^2$. Then,

$$X \perp\!\!\!\perp Y \qquad \Longleftrightarrow \qquad p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \quad \forall x, y \in \mathbb{R}.$$

- If $X \perp\!\!\!\perp Y$, then by definition,

$$\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_X(B_1) \cdot \mathbb{P}_Y(B_2) \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

- Taking $B_1 = \{x\}$ and $B_2 = \{y\}$, we get

$$\underbrace{\mathbb{P}_{X,Y}(\{x\} \times \{y\})}_{p_{X,Y}(x,y)} = \underbrace{\mathbb{P}_X(\{x\})}_{p_X(x)} \cdot \underbrace{\mathbb{P}_Y(\{y\})}_{p_Y(y)}$$

- This proves the $\implies$ direction

Independence for Jointly Discrete Random Variables

Proposition (Independence for Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be **jointly discrete** random variables with $\mathbb{P}_{X,Y}(E) = 1$ for some countable $E \subset \mathbb{R}^2$. Then,

$$X \perp\!\!\!\perp Y \qquad \Longleftrightarrow \qquad p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \quad \forall x, y \in \mathbb{R}.$$

- Suppose that $\quad p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \quad$ for all $x, y \in \mathbb{R}$
- $E_1 := \{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } (x,y) \in E\}, \quad E_2 = \{y \in \mathbb{R} : \exists x \in \mathbb{R} \text{ such that } (x,y) \in E\}$
- Notice that: $\quad E_1 \times E_2 \supseteq E, \quad \mathbb{P}_{X,Y}(E_1 \times E_2) = 1, \quad \mathbb{P}_Y(E_2) = 1.$
- $(\Longleftarrow) \qquad$ For any $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_{X,Y}\Big((B_1 \times B_2) \cap (E_1 \times E_2)\Big) = \sum_{x \in B_1 \cap E_1} \sum_{y \in B_2 \cap E_2} p_{X,Y}(x,y)$$

$$= \sum_{x \in B_1 \cap E_1} p_X(x) \cdot \sum_{y \in B_2 \cap E_2} p_Y(y) = \mathbb{P}_X(B_1 \cap E_1) \cdot \mathbb{P}_Y(B_2 \cap E_2) = \mathbb{P}_X(B_1) \cdot \mathbb{P}_Y(B_2).$$

Conditional PMF

Definition (Conditional PMF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be jointly discrete random variables.

1. Fix $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$.
   The **conditional PMF of X, conditioned on event A**, is defined as

$$p_{X|A}(x) := \frac{\mathbb{P}(\{X = x\} \cap A)}{\mathbb{P}(A)}, \qquad x \in \mathbb{R}.$$

2. The **conditional PMF of X, conditioned on Y**, is defined as

$$p_{X|Y}(x|y) = \frac{\mathbb{P}(\{X = x\} \cap \{Y = y\})}{\mathbb{P}(\{Y = y\})} = \frac{p_{X,Y}(x, y)}{p_Y(y)}, \qquad x \in \mathbb{R},$$

   valid for all $y \in \mathbb{R}$ such that $p_Y(y) = \mathbb{P}(\{Y = y\}) > 0$.

Conditional PMF and Independence

Proposition (Conditional PMF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be jointly discrete random variables. Then,

$$X \perp\!\!\!\perp Y \qquad \Longleftrightarrow \qquad p_{X|Y}(x|y) = p_X(x) \quad \forall x, \text{ feasible } y.$$

A proof of this is left as **exercise**.

Jointly Continuous Random Variables

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Lebesgue Measure $\lambda^{(2)}$ on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$

- Consider the collection

$$\mathscr{S} = \Big\{ (a,b] \times (c,d] : \ a \le b, \ c \le d, \quad a,b,c,d \in \mathbb{R} \Big\}.$$

- The collection $\mathscr{S}$ forms a **semiring**; $\quad \sigma(\mathscr{S}) = \mathscr{B}(\mathbb{R}^2)$
- Let $m : \mathscr{S} \to [0, +\infty]$ be an assignment satisfying the following properties:
  - $m(\emptyset) = 0, \quad m(\mathbb{R}^2) = +\infty, \quad m\big((a,b] \times (c,d]\big) = (b-a) \cdot (d-c)$
  - **Countable additivity**: if $A_1, A_2, \ldots \in \mathscr{S}$ are disjoint and satisfy $\bigsqcup_{n \in \mathbb{N}} A_n \in \mathscr{S}$, then

$$m\left(\bigsqcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} m(A_n).$$

  - By Caratheodory's extension theorem, there exists a **unique extension** of m to $\mathscr{B}(\mathbb{R}^2)$
    The extended measure is called the Lebesgue measure, denoted $\lambda^{(2)}$
  - $\lambda^{(2)}\big((a,b] \times (c,d]\big) = \lambda\big((a,b]\big) \cdot \lambda\big((c,d]\big), \qquad \lambda$: 1-D Lebesgue measure

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Jointly Continuous Random Variables

Definition (Jointly Continuous Random Variables)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
X and Y are said to be said to be **jointly continuous** if $\quad \mathbb{P}_{X,Y} \ll \lambda^{(2)}, \quad$ i.e.,

$$\lambda^{(2)}(E) = 0 \qquad \Longrightarrow \qquad \mathbb{P}_{X,Y}(E) = 0.$$

- X and Y jointly continuous $\quad \Longrightarrow \quad$ X continuous, Y continuous
- **Proof:** Suppose $E \subseteq \mathbb{R}$ with $\lambda(E) = 0$

$$\lambda^{(2)}(E \times \mathbb{R}) = \lambda^{(2)}\left(\bigcup_{n \in \mathbb{N}} E \times [-n, n]\right) = \lim_{n \to \infty} \lambda^{(2)}(E \times [-n, n]) = \lim_{n \to \infty} (\lambda(E) \cdot 2n) = 0$$

- Therefore,
$\mathbb{P}_X(E) = \mathbb{P}_{X,Y}(E \times \mathbb{R}) = 0 \quad \Longrightarrow \quad \mathbb{P}_X \ll \lambda \quad \Longrightarrow \quad X$ continuous

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Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
X and Y are said to be **jointly continuous** if $\mathbb{P}_{X,Y} \ll \lambda^{(2)}$, i.e.,

$$\lambda^{(2)}(E) = 0 \implies \mathbb{P}_{X,Y}(E) = 0.$$

- X and Y jointly continuous $\implies$ X continuous, Y continuous
- **Proof:** Suppose $E \subseteq \mathbb{R}$ with $\lambda(E) = 0$

$$\lambda^{(2)}(\mathbb{R} \times E) = \lambda^{(2)}\left(\bigcup_{n\in\mathbb{N}} [-n,\, n] \times E\right) = \lim_{n\to\infty} \lambda^{(2)}([-n,\, n] \times E) = \lim_{n\to\infty} \left(2n \cdot \lambda(E)\right) = 0$$

- Therefore,
  $\mathbb{P}_Y(E) = \mathbb{P}_{X,Y}(\mathbb{R} \times E) = 0 \implies \mathbb{P}_Y \ll \lambda \implies Y$ continuous

Jointly Continuous Random Variables

Definition (Jointly Continuous Random Variables)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
X and Y are said to be **jointly continuous** if $\quad \mathbb{P}_{X,Y} \ll \lambda^{(2)}, \quad$ i.e.,

$$\lambda^{(2)}(E) = 0 \qquad \Longrightarrow \qquad \mathbb{P}_{X,Y}(E) = 0.$$

- **Caution!** In general, $\quad X$ continuous, Y continuous $\quad \not\Longrightarrow \quad X$ and Y jointly continuous
- Suppose X is a continuous random variable. $\quad$ Let $Y = X$. $\quad$ (X continuous, Y continuous)
- Let $\quad B := \{(x, y) \in \mathbb{R}^2 : x = y\}$
- Notice that

$$\lambda^{(2)}(B) = 0, \qquad \mathbb{P}_{X,Y}(B) = 1 \qquad \Longrightarrow \qquad \mathbb{P}_{X,Y} \not\ll \lambda^{(2)}.$$

Remark: $\quad$ In the above example, the bivariate vector (X, Y) is **singular**!

Joint PDF

- If X and Y are jointly continuous, then

$$\mathbb{P}_{X,Y} \ll \lambda^{(2)}$$

- By the Radon-Nikodym theorem, there exists a **non-negative, measurable function** $f_{X,Y} : \mathbb{R}^2 \to [0, +\infty)$ such that

$$\forall\, x, y \in \mathbb{R}, \qquad F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\, \mathrm{d}v\, \mathrm{d}u.$$

- $f_{X,Y}$ is called the **joint PDF** of X and Y

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Marginal PDFs and Conditional PDF

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Marginal PDFs from Joint PDF

For Jointly Continuous Random Variables

Theorem (Marginal PDFs from Joint PDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be jointly continuous. Then, the following properties hold.

1. The joint PDF must integrate to 1, i.e.,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u, v)\, \mathrm{d}v\, \mathrm{d}u = 1.$$

2. **(Marginalization Property)**

$$\forall\, x \in \mathbb{R}, \qquad \int_{-\infty}^{\infty} f_{X,Y}(x, v)\, \mathrm{d}v = f_X(x),$$

$$\forall\, y \in \mathbb{R}, \qquad \int_{-\infty}^{\infty} f_{X,Y}(u, y)\, \mathrm{d}u = f_Y(y).$$

Independence for Jointly Continuous Random Variables

Proposition (Independence for Jointly Continuous Random Variables)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X, Y be **jointly continuous** random variables. Then,

$$X \perp\!\!\!\perp Y \qquad \Longleftrightarrow \qquad f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \quad \forall x, y \text{ except possibly on a set } B \text{ with } \lambda^{(2)}(B) = 0.$$

Conditional CDF and Conditional PDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be **jointly continuous** random variables.

Objective

We would like to define the conditional CDF of X, conditioned on observing the value of Y.
Mathematically, our interest is in $\mathbb{P}(\{X \leq x\} | \{Y = y\})$.
However, this conditional probability is not defined because $\mathbb{P}(\{Y = y\}) = 0$ for all $y \in \mathbb{R}$.
So, then, how do we actually compute $\mathbb{P}(\{X \leq x\} | \{Y = y\})$?

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So, then, how do we actually compute $\mathbb{P}(\{X \leq x\}|\{Y = y\})$?

Remedy:
Fix $y \in \mathbb{R}$ and $\varepsilon > 0$ such that $\mathbb{P}\big(\{Y \in (y - \varepsilon, y + \varepsilon)\}\big) > 0$.
Define conditional probability with respect to the event $\{Y \in (y - \varepsilon, y + \varepsilon)\}$, and let $\varepsilon \downarrow 0$.

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Conditional CDF and Conditional PDF

$$
\begin{aligned}
\mathbb{P}(\{X \le x\} | \{Y \in (y-\varepsilon, y+\varepsilon)\}) &= \frac{\mathbb{P}(\{X \le x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})} \\
&= \frac{\mathbb{P}_{X,Y}\big((-\infty,\ x] \times (y-\varepsilon,\ y+\varepsilon)\big)}{\mathbb{P}_Y\big((y-\varepsilon,\ y+\varepsilon)\big)} \\
&= \frac{\int\limits_{-\infty}^{x} \int\limits_{y-\varepsilon}^{y+\varepsilon} f_{X,Y}(u,v)\, dv\, \mathrm{d}u}{\int\limits_{y-\varepsilon}^{y+\varepsilon} f_Y(v)\, dv} \approx \frac{\int\limits_{-\infty}^{x} f_{X,Y}(u,y)\, \mathrm{d}u \cdot 2\varepsilon}{f_Y(y) \cdot 2\varepsilon} \\
&= \int_{-\infty}^{x} \underbrace{\frac{f_{X,Y}(u,y)}{f_Y(y)}}_{\text{conditional PDF}}\ \mathrm{d}u
\end{aligned}
$$

Conditional CDF for Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be **jointly continuous** random variables.

Definition (Conditional CDF for Jointly Continuous Random Variables)

Fix y such that $f_Y(y) > 0$.
The **conditional CDF** of X, conditioned on the event $\{Y = y\}$, is the function $F_{X|Y=y} : \mathbb{R} \to [0, 1]$ defined as

$$F_{X|Y=y}(x) = \int_{-\infty}^{x} \underbrace{\frac{f_{X,Y}(u, y)}{f_Y(y)}}_{\text{conditional PDF}} \, du, \qquad x \in \mathbb{R}.$$

Conditional PDF for Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be **jointly continuous** random variables.

Definition (Conditional PDF for Jointly Continuous Random Variables)

Fix y such that $f_Y(y) > 0$.
The **conditional PDF** of X, conditioned on the event $\{Y = y\}$, is the function $f_{X|Y=y} : \mathbb{R} \to [0, +\infty)$ defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \qquad x \in \mathbb{R}.$$

Remark: Some textbooks use the notation $f_{X|Y}(x|y)$ to denote the conditional PDF.