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Indian Institute of Technology Hyderabad

Stochastic Processes

Communicating Classes, Class Properties, Irreducibility, Aperiodicity, Invariant Distribution

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Reachability

Definition (Reachability)

State $y \in \mathcal{X}$ is said to be reachable from state $x \in \mathcal{X}$ if there exists $n \in \mathbb{N} \cup \{0\}$ such that the probability of reaching y in n steps starting from x is strictly positive.

Notation: $x \longrightarrow y$.

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Remark: For a time-homogeneous Markov chain with state space $\mathcal{X}$ and TPM P,

$$x \longrightarrow y \quad \Longleftrightarrow \quad \exists n \in \mathbb{N} \cup \{0\} \text{ such that } P^n_{x,y} > 0.$$

Communication

Definition (Communication)

Two states x and y are said to communicate with each other if $x \longrightarrow y$ and $y \longrightarrow x$.

Notation: $x \longleftrightarrow y$.

Communication is an Equivalence Relation

Proposition (Communication is an Equivalence Relation)

$\longleftrightarrow$ defines an equivalence relation on $\mathcal{X} \times \mathcal{X}$. Formally:

1. (Reflexive): $x \longleftrightarrow x$ for all $x \in \mathcal{X}$.
2. (Symmetric): For all $x, y \in \mathcal{X}$,
$$x \longleftrightarrow y \quad \Longleftrightarrow \quad y \longleftrightarrow x.$$
3. (Transitive): For all $x, y, z \in \mathcal{X}$,
$$x \longleftrightarrow y, \quad y \longleftrightarrow z \implies x \longleftrightarrow z.$$

Proof of Proposition

- Reflexive:

$$P^0 = I \quad \Longleftrightarrow \quad P^0_{x,x} = 1 > 0 \quad \Longleftrightarrow \quad x \longleftrightarrow x.$$

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$$x \longleftrightarrow y, \;\; y \longleftrightarrow z \quad \Longrightarrow \quad \exists m, n \in \mathbb{N} \cup \{0\} \text{ such that } P^m_{x,y} > 0, \;\; P^n_{y,z} > 0$$

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$$P^{m+n}_{x,z} = \sum_{w \in \mathcal{X}} P^m_{x,w} P^n_{w,z}$$

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$$\begin{aligned} P^{m+n}_{x,z} \quad &= \quad \sum_{w \in \mathcal{X}} P^m_{x,w} P^n_{w,z} \\ &\geq P^m_{x,y} P^n_{y,z} > 0 \end{aligned}$$

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Communicating Class

Definition (Communicating Class)

The communication relation $\longleftrightarrow$ creates a partition of the state space $\mathcal{X}$.
Each element of the partition is referred to as a communicating class.

Open and Closed Communicating Classes

Definition (Open and Closed Communicating Classes)

A communicating class $\mathcal{C}$ is said to be open if there exists an edge that leaves the class, i.e., there exists $x \in \mathcal{C}$ and $y \in \mathcal{C}^c$ such that $P_{x,y} > 0$.

A communicating class $\mathcal{C}$ is said to be closed if there is no edge leaving the class, i.e.,

$$P_{x,y} = 0 \qquad \forall x \in \mathcal{C}, \;\; y \in \mathcal{C}^c.$$

Irreducibility and Periodicity

Irreducible Markov Chain

Definition (Irreducible Markov Chain)

A time-homogeneous DTMC is said to be irreducible if its entire state space constitutes a single communicating class.
That is, for all $x, y \in \mathcal{X}$, there exists $n \in \mathbb{N} \cup \{0\}$ such that $P^n_{x,y} > 0$.

Remark:
Some textbooks (particularly on RL) refer to irreducibility as unichain property.

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- Consider a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ and TPM P
- For any $x \in \mathcal{X}$, let

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The period of a state $x \in \mathcal{X}$, denoted $d(x)$, is defined as the greatest common divisor of the set of return times, i.e.,

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A state x is called aperiodic if $d(x) = 1$.
If $d(x) > 1$, the state x is said to be periodic.

Aperiodic Markov Chain

Definition (Aperiodic Markov Chain)

A time-homogeneous DTMC is said to be aperiodic if

1. The Markov chain is irreducible, and
2. The period of every state is 1.

Class Properties

Period is a Class Property

Proposition (Period is a Class Property)

If $x \longleftrightarrow y$, then $d(x) = d(y)$.
Thus, all states within a communicating class possess the same period.

Proof of Proposition – 1

- Observe that

$$x \longleftrightarrow y \quad \implies \quad \exists m, n \in \mathbb{N} \cup \{0\} \text{ such that } P^m_{x,y} > 0, \;\; P^n_{y,x} > 0.$$

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$$P^{n+m}_{y,y} \geq P^n_{y,x} \cdot P^m_{x,y} > 0, \qquad P^{n+r+m}_{y,y} \geq P^n_{y,x} \cdot P^r_{x,x} \cdot P^m_{x,y} > 0,$$

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$$n + m \in \mathcal{R}(y), \qquad n + m + r \in \mathcal{R}(y)$$

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$$\Longrightarrow \quad d(y) \mid n + m, \qquad d(y) \mid n + m + r$$

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$$\begin{aligned} & n+m \in \mathcal{R}(y), \qquad n+m+r \in \mathcal{R}(y) \\ \Longrightarrow \quad & d(y) \mid n+m, \qquad d(y) \mid n+m+r \\ \Longrightarrow \quad & d(y) \mid r \end{aligned}$$

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- Switching the roles of x and y, we can establish that $d(x) \leq d(y)$

Transience and Recurrence are Class Properties

Proposition (Transience and Recurrence are Class Properties)

Transience and recurrence are class properties, i.e., the states within a communicating class are either all transient or all recurrent.

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&= \frac{1}{P^m_{x,y} P^n_{y,x}} \sum_{s\in\mathbb{N}} P^m_{x,y} P^s_{y,y} P^n_{y,x} \\
&\leq \frac{1}{P^m_{x,y} P^n_{y,x}} \sum_{s\in\mathbb{N}} P^{m+n+s}_{x,x} \leq \frac{1}{P^m_{x,y} P^n_{y,x}} \sum_{k\in\mathbb{N}} P^k_{x,x} < +\infty
\end{aligned}$$

Positive/Null Recurrence are Class Properties

Proposition (Positive/Null Recurrence are Class Properties)

Positive recurrence and null recurrence are class properties, i.e., the states within a communicating class are either all positive recurrent or all null recurrent.

A guided proof of this will be demonstrated in the homework.

An Important Result About Open and Closed Communicating Classes

Proposition (Result about Open/Closed Communicating Classes)

1. If $\mathcal{C}$ is an open communicating class, then every state within $\mathcal{C}$ is transient.
2. If $\mathcal{C}$ is a closed communicating class, and $|\mathcal{C}| < +\infty$, then every state within $\mathcal{C}$ is positive recurrent.

   As a corollary, an irreducible DTMC with a finite state space is positive recurrent.