AI 5090: STOCHASTIC PROCESSES

LECTURE 05

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Contents

The lecture introduces sequences of random variables and their limiting behavior, focusing on lim inf, lim sup, and point-wise limits. Understanding these concepts is crucial for analyzing convergence in probability theory.

Topics Covered: Lim inf, Lim sup, and Limit of Sequence of Random Variables, Pointwise Limit of Sequence of Random Variables, Pointwise Convergence

Note: This scribe considers $(\Omega, \mathscr{F}, \mathbb{P})$ as the measure space & $(\Omega, \mathscr{F})$ as the measurable space for all definitions.

1 Lim inf, Lim sup, and Limit of Sequence of Random Variables

Recall the definition of a sequence of random variables:

Definition 1. *A sequence of random variables is a collection $\{X_n\}_{n=1}^{\infty}$ such that*

$$(X_{k_1}, \dots, X_{k_n}) \text{ is a random vector} \qquad \forall n \in \mathbb{N}, \forall k_1, \dots, k_n \in \mathbb{N},$$

1.1 Motivation

Sequences of random variables arise naturally in probability and statistics, especially in convergence analysis. The concepts of Liminf & Limsup help describe the long-term behaviour of such sequences.
Intuitively:

- Lim inf represents the eventual lower bound of the sequence.
- Lim sup represents the eventual upper bound of the sequence.

Understanding these limits helps prove theorems about almost sure convergence and the stability of stochastic processes.

1.2 Lim inf of Sequence of Random Variables

Definition 2. *The* **limit infimum** *of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X_\star : \Omega \to [-\infty, +\infty]$ such that*

$$X_\star(\omega) = \sup_{n \geq 1} \inf_{k \geq n} X_k(\omega) \quad \forall \omega \in \Omega.$$

Notation:

$$\liminf_{n \to \infty} X_n.$$

Note: The range of $X_\star$ is typically taken to be $[-\infty, +\infty]$ and not $\mathbb{R}$. This is to account for the fact that the limit infimum can be $\pm\infty$.

Lemma 1. $X_\star = \liminf_{n \to \infty} X_n : \Omega \to [-\infty, +\infty]$ *is a random variable w.r.t. $\mathscr{F}$.*

Proof of Lemma 1. To establish that $X_\star$ is a random variable, we need to show that it is measurable with respect to $\mathscr{F}$. We accomplish this by considering inverse images of generating sets.

Noting that $\{(-\infty, x) : x \in \mathbb{R}\}$ generates $\mathscr{B}(\mathbb{R})$, we have that for each $x \in \mathbb{R}$,

$$\begin{aligned} X_\star^{-1}((-\infty, x)) &= \{\omega \in \Omega : \sup_{n\geq 1} \inf_{k\geq n} X_k(\omega) < x\} \\ &= \bigcap_{n\geq 1} \{\omega \in \Omega : Y_n(\omega) < x\} \quad \text{where} \quad Y_n(\omega) \coloneqq \inf_{k\geq n} X_k(\omega) \\ &= \bigcap_{n\geq 1} \bigcup_{k\geq n} \{\omega \in \Omega : X_k(\omega) < x\} \\ &= \bigcap_{n\geq 1} \underbrace{\bigcup_{k\geq n} \underbrace{X_k^{-1}((-\infty, x))}_{\in\mathscr{F} \text{ for every } n\geq 1, \text{ for every } k\geq n}}_{\in\mathscr{F} \text{ for every } n\geq 1} . \end{aligned}$$

Is the aforementioned $Y_n(\omega)$ a Random Variable? Yes! Since each Y_n is a pointwise infimum of measurable functions, it is measurable.

Inverse image of $\{-\infty\}$

$$\begin{aligned} X_\star^{-1}(\{-\infty\}) &= \{\omega \in \Omega : \sup_{n\geq 1} \inf_{k\geq n} X_k(\omega) = \{-\infty\}\} \\ &= \bigcap_{N\geq 1} \{\omega \in \Omega : \{Y_N(\omega) < -N\}\} \quad where \quad Y_N(\omega) = \inf_{k\geq N} X_k(\omega) \\ &= \bigcap_{N\geq 1} \{\omega \in \Omega : \{\inf_{k\geq N} X_k(\omega) < -N\}\} \\ &= \bigcap_{N\geq 1} \bigcup_{k\geq N} \{\omega \in \Omega : \{X_k(\omega) < -N\}\} \\ &= \bigcap_{N\geq 1} \bigcup_{k\geq N} X_k^{-1}((-\infty, -N)) \\ &= \bigcap_{N\geq 1} \underbrace{\bigcup_{k\geq N} \underbrace{X_k^{-1}((-\infty, -N))}_{\in\mathscr{F} \text{ for every N, for every k } \geq N}}_{\in\mathscr{F} \text{for every k } \geq N} \end{aligned}$$

Similar steps as above can be followed for finding inverse image of $\{+\infty\}$

We now know that inverse images of $\{+\infty\}$,$\{-\infty\}$, sets of the form $(-\infty, x)$. This implies that we know inverse images of all borel sets in $[-\infty, +\infty]$, as they can be formed by taking countable unions & countable intersections of $\{+\infty\}$, $\{-\infty\}$ & $(-\infty, x)$.

Intuitive Understanding of lim inf X_n: Lim inf captures the smallest value that the sequence persistently approaches (not just visits briefly).
Example: Consider the sequence $X_n = \frac{(-1)^n}{n}$. The sequence oscillates but lim inf $X_n = 0$ as it is the lowest value approached infinitely often.

□

1.3 Lim sup of Sequence of Random Variables

Definition 3. *The* **limit supremum** *of a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined w.r.t. $\mathscr{F}$ is a function $X_\star : \Omega \to [-\infty, +\infty]$ such that*

$$X^\star(\omega) = \inf_{n\geq 1} \sup_{k\geq n} X_k(\omega) \quad \forall \omega \in \Omega.$$

Notation:

$$\limsup_{n\to\infty} X_n.$$

Lemma 2. $X^\star = \limsup_{n\to\infty} X_n : \Omega \to [-\infty, +\infty]$ *is a random variable w.r.t.* $\mathscr{F}$.

Proof of Lemma 2. To establish that lim sup is a random variable, we need to show that it is measurable with respect to $\mathscr{F}$. We achieve this by considering inverse images of the generating set $(x, +\infty)$.

Let's find the inverse image of the generating set $(x, +\infty)$ for $x \in \mathscr{R}$

$$\begin{aligned}
X^{\star -1}((x,+\infty)) &= \{\omega \in \Omega : \inf_{n\geq 1} \sup_{k\geq n} X_k(\omega) > x\} \\
&= \bigcap_{n\geq 1} \{\omega \in \Omega : \{Y_n(\omega) > x\}\} \quad where \quad Y_n(\omega) = \sup_{k\geq n} X_k(\omega) \\
&= \bigcap_{n\geq 1} \{\omega \in \Omega : \{\sup_{k\geq n} X_k(\omega) > x\}\} \\
&= \bigcap_{n\geq 1} \bigcup_{k\geq n} \{\omega \in \Omega : \{X_k(\omega) > x\}\} \\
&= \bigcap_{n\geq 1} \bigcup_{k\geq n} X_k^{-1}((x,+\infty)) \\
&= \bigcap_{n\geq 1} \underbrace{\bigcup_{k\geq n} \underbrace{X_k^{-1}((x,+\infty))}_{\in\mathscr{F} \text{ for every } n\geq 1, \text{ for every } k\geq n}}_{\in\mathscr{F} \text{ for every } n\geq 1} .
\end{aligned}$$

Can you generate sets of the form $(x, +\infty]$ using the generating set? No! But, can you argue why? Refer to the argument we made during lim inf.

Is the $Y_n(\omega)$ mentioned above, a Random Variable? Yes! Can you argue why?

Since we found pre-image of $\{-\infty\}$ incase of liminf, let's find inverse image of $\{+\infty\}$ this time

$$\begin{aligned}
X^{\star -1}(\{+\infty\}) &= \{\omega \in \Omega : \inf_{n\geq 1} \sup_{k\geq n} X_k(\omega) = +\infty\} \\
&= \bigcap_{N\geq 1} \{\omega \in \Omega : Y_N(\omega) > N\} \quad \text{where} \quad Y_N(\omega) = \sup_{k\geq N} X_k(\omega) \\
&= \bigcap_{N\geq 1} \{\omega \in \Omega : \sup_{k\geq 1} X_k(\omega) > N\} \\
&= \bigcap_{N\geq 1} \bigcup_{k\geq N} X_k^{-1}((N,+\infty)) \\
&= \bigcap_{N\geq 1} \underbrace{\bigcup_{k\geq N} \underbrace{X_k^{-1}((N,+\infty))}_{\in\mathscr{F} \text{ for every } N\geq 1, \text{ for every } k\geq N}}_{\in\mathscr{F} \text{ for every } N\geq 1} .
\end{aligned}$$

Once we show that the inverse images of sets $\{-\infty\}$,$\{+\infty\}$, sets of the form $(x, +\infty)$ are present in $\mathscr{F}$. We can show that pre-image of all borel sets in $[-\infty, +\infty]$ belong $\mathscr{F}$, as they can be formed by taking countable unions & countable intersections of $\{+\infty\}$, $\{-\infty\}$ & $(x, +\infty)$. □

1.4 Conclusion (Sequence of Random Variables)

We have established that Liminf($X_\star$) and Limsup($X^\star$) are random variables. These limits help us study convergence of sequences of random variables, particularly in almost sure convergence, laws of large numbers.

2 Pointwise Convergence and Almost Sure Convergence of Sequence of Random Variables

This section discusses two different notions of convergence of a sequence of random variables- **pointwise** convergence and **almost sure** convergence.

2.1 Pointwise Convergence of Sequence of Random Variables

Definition 4 (Pointwise Limit of Sequence of Random Variables)**.** *The* **pointwise** *limit of a sequence of random variables* $\{X_n\}_{n=1}^{\infty}$ *defined w.r.t.* $\mathscr{F}$ *is a function* $X : \Omega \to [-\infty, +\infty]$ *such that*

$$\liminf_{n\to\infty} X_n = X(\omega) = \limsup_{n\to\infty} X_n \quad \forall \omega \in \Omega.$$

Notation: $\lim_{n\to\infty} X_n$.

In words, for each fixed outcome $\omega \in \Omega$, the sequence of real numbers $\{X_n(\omega)\}_{n=1}^{\infty}$ converges to a single real value if the lim inf and lim sup of the sequence are equal at that outcome. When this condition holds the common value of lim inf and lim sup is defined as the limit $X(\omega)$.

Lemma 3. $X = \lim_{n\to\infty} X_n : \Omega \to [-\infty, +\infty]$ *if it exists, is a random variable w.r.t.* $\mathscr{F}$

Proof of Lemma 3. We have, for each Borel measurable set $B \in \mathscr{B}([-\infty, +\infty])$,

$$\begin{aligned} X^{-1}(B) &= \{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \in B\} \\ &= \{\omega \in \Omega : \limsup_{n\to\infty} X_n(\omega) \in B\} \cap \{\omega \in \Omega : \liminf_{n\to\infty} X_n(\omega) \in B\}. \end{aligned}$$

But we know that lim inf and lim sup of a sequence of random variables are themselves random variables. Therefore, sets of the form $\{\omega \in \Omega : \liminf_{n\to\infty} X_n(\omega) \in B\}$ and $\{\omega \in \Omega : \limsup_{n\to\infty} X_n(\omega) \in B\}$ belong to $\mathscr{F}$. Their intersection also belongs to $\mathscr{F}$. So, $X^{-1}(B)$ belongs to $\mathscr{F}$ for all $B \in \mathscr{B}([-\infty, +\infty])$. This completes the proof.

□

Example 2.1. *Suppose that* $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$.

- *For each* $n \in \mathbb{N}$*, let*

  $$X_n(\omega) = \begin{cases} 1, & \omega \in [0, \frac{1}{n}), \\ 0, & \textit{otherwise}, \end{cases} \qquad \omega \in [0,1].$$

  Identify the pointwise limit.

  Clearly, the sample path traced by $\omega = 0$ *is 1,1,1,... For any other* $\omega \in \Omega$, $X_n \to 0$*. That is,* $\forall \omega \in (0,1], \forall \epsilon > 0$*, there exists* $N_\epsilon = \lceil \frac{1}{\epsilon} \rceil \in \mathbb{N}$ *such that*

  $$|X_n(\omega) - 0| < \epsilon \quad \forall n > N_\epsilon.$$

  Thus, the pointwise limit random variable is,

  $$X_n(\omega) = \begin{cases} 1, & \textit{if } \omega = 0, \\ 0, & \textit{otherwise}, \end{cases} \qquad \omega \in [0,1].$$

  The limit random variable can be concisely written as $\mathbf{1}_{\{0\}}(\omega)$.

- *For each* $n \in \mathbb{N}$*, let*

  $$X_n(\omega) = \omega^n, \qquad \omega \in \Omega$$

  Identify the limit random variable X.

  For $\omega = 1, X(\omega)$ *is the constant sequence* $1, 1, 1 \ldots$*, which converges to* 1 *trivially. For* $\omega \in [0,1)$*,* ω^n *decays geometrically, and therefore converges to* 0*. Thus, the limit random variable is given by* $X = \mathbf{1}_{\{1\}}$.

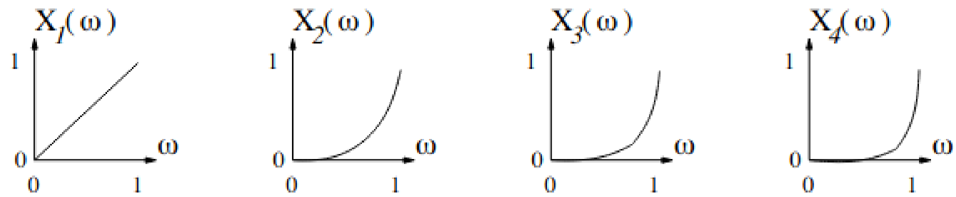

Figure 1: $X_n(\omega) = \omega^n$ on the standard unit-interval probability space. Figure credits: [Haj14].

- **Moving Rectangles** *Consider a sequence of random variables defined as follows:*

$$X_1 = \mathbf{1}_{\{[0,1]\}}(\omega)$$
$$X_2 = \mathbf{1}_{\{[0,\frac{1}{2}]\}}(\omega), \quad X_3 = \mathbf{1}_{\{[\frac{1}{2},1]\}}(\omega)$$
$$X_4 = \mathbf{1}_{\{[0,\frac{1}{4}]\}}(\omega), \quad X_5 = \mathbf{1}_{\{[\frac{1}{4},\frac{1}{2}]\}}(\omega), \quad X_6 = \mathbf{1}_{\{[\frac{1}{2},\frac{3}{4}]\}}(\omega), \quad X_7 = \mathbf{1}_{\{[\frac{3}{4},1]\}}(\omega), \text{ and so on.}$$

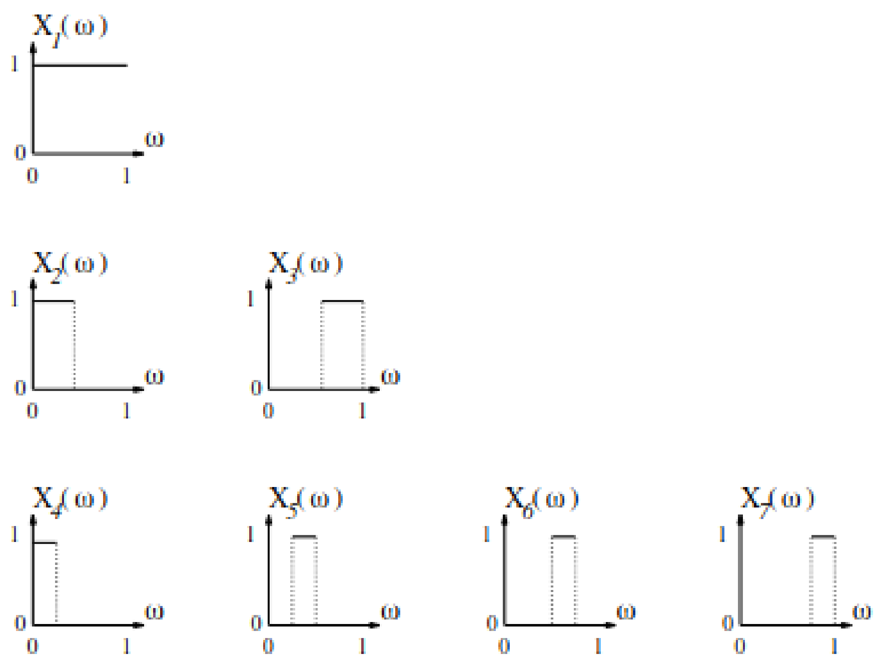

Figure 2: Moving rectangles X_1 to X_7. Figure credits: [Haj14].

Fix an arbitrary $\omega \in \Omega$. Now, consider the sequence $\{X_n(\omega)\}_{n=1}^{\infty}$. For each $k \geq 1$, there is one value of n with $2^k \leq n < 2^{k+1}$ such that $X_n(\omega) = 1$ and $X_n(\omega) = 0$ for all other n. That is, a 1 keeps on appearing in every interval of length 2^k. As k gets larger even though the gap between subsequent 1's increases, they never cease to appear. Therefore, $\lim_{n\to\infty} X_n(\omega)$ *does not exist. Which means the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ does not converge pointwise. In fact, there does not exist even a single $\omega \in \Omega$ such that $\{X_n(\omega)\}_{n=1}^{\infty}$ exists. However, observe that, for large n, $\mathbb{P}(X_n = 0)$ is close to one. This suggests that X_n converges to the zero random variable in some weaker sense.*

Example 2.2. *For each $n \in \mathbb{N}$, let*

$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$

Notice that X_n follows a Bernoulli distribution with probability of "success" $\frac{1}{n^2}$. However, we do not have any information about the exact mapping $\omega \mapsto X_n(\omega)$, and therefore we cannot fix an $\omega \in \Omega$ and analyse the convergence of the sequence $\{X_n(\omega)\}_{n=1}^{\infty}$. Therefore, while checking for pointwise convergence is infeasible, observe that the probability of "success" goes to zero as n gets larger. In some sense, the zero random variable can be considered as a limit of the given sequence of random variables. However, we cannot verify if this is the pointwise limit we defined earlier, as pointwise convergence requires evaluating the sequence for each ω, which is not possible in this case. To address such cases where $(\Omega, \mathscr{F}, \mathbb{P})$ and the sequence $\{X_n\}_{n=1}^{\infty}$ are not explicitly specified we need other notions of convergence.

Fix a measurable space $(\Omega, \mathscr{F}, \mathbb{P})$

Definition 5 (Pointwise Convergence)**.** *Given a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ and a random variable X, all defined w.r.t. $\mathscr{F}$, we say that the sequence converges* **pointwise** *to X if*

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \forall \omega \in \Omega$$

Notation:

$$X_n \xrightarrow{\text{pointwise}} X$$

Pointwise convergence is a strong notion of convergence. In practice, proving pointwise convergence is often infeasible, unless detailed information about $(\Omega, \mathscr{F}, \mathbb{P})$ is provided. This limitation naturally raises the question: are there other notions of convergence that can be applied in such cases? To explore this, we consider the following example.

Example 2.3. *Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$. For each $n \in \mathbb{N}$, let*

$$X_n(\omega) = \begin{cases} (-1)^n, & \text{if } \omega = 0, \\ \omega^n, & \text{if } \omega \in (0,1), \\ 0, & \text{if } \omega = 0. \end{cases} \quad \omega \in [0,1].$$

The sequence $\{X_n(0)\}_{n=1}^{\infty}$ is $(-1)^n$ which oscillates between -1 and $+1$ and does not converge. For $\omega \in (0,1)$, $\{X_n(\omega)\}_{n=1}^{\infty}$ is a geometric sequence with common ratio less than one, thereby converging to 0. The sequence $\{X_n(1)\}_{n=1}^{\infty}$ is the zero sequence, which trivially converges to zero.

In the above example,

$$\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\} = (0,1] \neq \Omega.$$

However, intuitively, the zero random variable serves as a natural candidate for the limit, even though the sequence fails to converge at a single point. Disregarding the limiting behaviour entirely due to absence of convergence at a single point seems overly restrictive. We need a way to capture this limiting behaviour more effectively. This motivates us to define the following set and the lemma states an important result about its measurability.

Fix a measurable space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\{X_n\}_{n=1}^{\infty}$ and X be random variables defined w.r.t. $\mathscr{F}$.

Lemma 4. *Let*

$$A_{\text{lim}} := \{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}.$$

Then, $A_{\text{lim}} \in \mathscr{F}$. Thus, we may assign a probability to A_{lim}.

Proof of Lemma 4. We prove the lemma by considering several cases.

1. Case 1: Consider $\omega \in A_{\text{lim}}$ such that $\{X_n(\omega)\}_{n=1}^{\infty}$ converges to some real number $X(\omega)$. In this case, we note that

$$\begin{aligned} \omega \in A_{\text{lim}} &\iff \lim_{n\to\infty} X_n(\omega) = X(\omega) \\ &\iff \forall \epsilon > 0, \exists N_\epsilon(\omega) \text{ such that } |X_n(\omega) - X(\omega)| < \epsilon \, \forall n \geq N_\epsilon(\omega) \\ &\iff \forall q \in \mathbb{Q}^+, \exists N_q(\omega) \text{ such that } |X_n(\omega) - X(\omega)| < q \, \forall n \geq N_q(\omega) \\ &\iff \omega \in \bigcap_{q\in\mathbb{Q}^+} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \{|X_n - X| < q\}. \end{aligned}$$

Thus we have,

$$A_{\text{lim}} = \bigcap_{q \in \mathbb{Q}^+} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{|X_n - X| < q\}.$$

Observe that the sequence $\{X_n\}_{n=1}^{\infty}$ and X are random variables defined w.r.t. $\mathscr{F}$. The absolute value of the difference between two random variables is also a random variable. Which means the set $\{|X_n - X| < q\}$ is an $\mathscr{F}$-measurable set. Countable union/intersection of measurable sets is also measurable. Therefore, A_{lim} expressed as a countable union and intersection of measurable sets, is also measurable.

Observe that in the above proof we have replaced $\epsilon > 0$ with $q \in \mathbb{Q}_+$, where $\mathbb{Q}_+$ is the set of all positive rationals. This is valid because for any $\epsilon > 0$, we can find $0 < q < \epsilon$, and conversely, for any $q \in \mathbb{Q}_+$, we can find an $\epsilon > 0$ such that $0 < \epsilon < q$; this follows as a consequence of rationals being *dense* in the set of real numbers. This enables us to convert the outer uncountable intersection into a countable intersection.

2. Case 2: Consider $\omega \in A_{\text{lim}}$ such that $\{X_n(\omega)\}_{n=1}^{\infty}$ converges to $+\infty$. In this case, we note that

$$\begin{aligned} \omega \in A_{\text{lim}} &\iff \lim_{n \to \infty} X_n(\omega) = +\infty \\ &\iff \forall M \in \mathbb{N}, \exists N_M(\omega) \text{ such that } X_n(\omega) > M \,\forall n \geq N_M(\omega) \\ &\iff \omega \in \bigcap_{M \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{X_n > M\}. \end{aligned}$$

Here also, sets of the form $\{X_n > M\} \in \mathscr{F}$. A_{lim} expressed as a countable union and intersection of such sets, also belongs to $\mathscr{F}$.

3. Case 3: Consider $\omega \in A_{\text{lim}}$ such that $\{X_n(\omega)\}_{n=1}^{\infty}$ converges to $-\infty$. In this case, we have

$$\begin{aligned} \omega \in A_{\text{lim}} &\iff \lim_{n \to \infty} X_n(\omega) = -\infty \\ &\iff \forall M \in \mathbb{N}, \exists N_M(\omega) \text{ such that } X_n(\omega) < -M \,\forall n \geq N_M(\omega) \\ &\iff \omega \in \bigcap_{M \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{X_n < -M\}. \end{aligned}$$

A similar argument as above shows that the above set is also $\mathscr{F}$-measurable, thereby completing the proof.

□

The result above demonstrates how the set of points where the sequence of random variables converges is a measurable set. More importantly, the set of points where the sequence of random variables does not converge (i.e., the complement of A_{lim}) is also measurable. This allows us to extend the notion of convergence from pointwise to *almost sure* convergence, where convergence may fail on a negligible subset of zero probability. Let us formalize this concept.

Fix a measurable space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n=1}^{\infty}$ and X be random variables defined w.r.t. $\mathscr{F}$.

Definition 6 (Almost-Sure Convergence)**.** *We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X almost surely (a.s.) if,*

$$\mathbb{P}(A_{\textit{lim}}) = 1$$

Notation:

$$X_n \xrightarrow{\text{a.s.}} X$$

References

[Haj14] Bruce Hajek. *Random processes for engineers*. Cambridge university press, 2014.