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Stochastic Processes

DTMCs: Some Important Results and Their Proofs, Examples Problems

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

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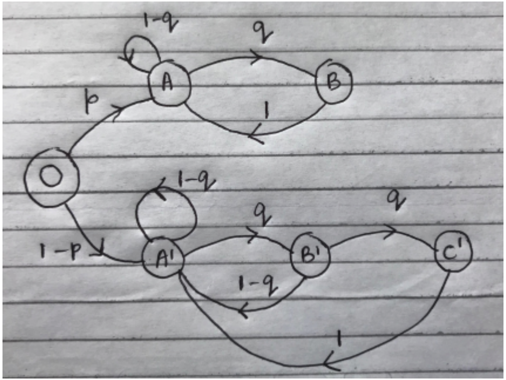
Example

- Consider a DTMC with the following transition graph.



1. Is this Markov chain irreducible?
2. Does there exist a unique stationary distribution?

Some Important Results – 1

Lemma (Regarding a Transient State)

Consider a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ with TPM P.
Let $x \in \mathcal{X}$ be transient. Then,

$$\lim_{n \to \infty} P^n_{x,x} = 0.$$

Furthermore,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k_{x,x} = 0.$$

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Some Important Results – 2

Lemma (Regarding a Recurrent State)

Consider a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ and TPM P.
Let $x \in \mathcal{X}$ be recurrent. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{x,x}^{k} = \frac{1}{\mu_{xx}}.$$

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  denote the time instant of kth visit to x
- Because x is recurrent, $\mathbb{P}(\tau_x^{(k)} < +\infty) = 1$ for all $k \in \mathbb{N}$

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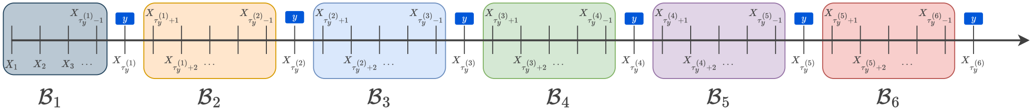

- Consider $H_k = \tau_x^{(k)} - \tau_x^{(k-1)}$, $k \in \mathbb{N}$
- From IID block structure, we know that $\{H_k\}_{k\in\mathbb{N}}$ is an IID process
- **Claim:**
  For each $n \in \mathbb{N}$, the random variable $N_x(n) + 1$ is a stopping time w.r.t. $\{H_k\}_{k\in\mathbb{N}}$

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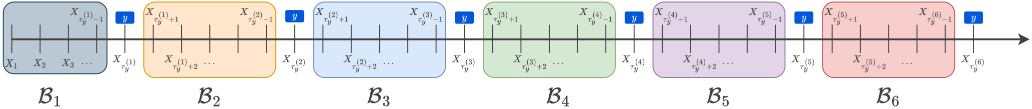

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$$\{N_x(n) + 1 = m\} = \{N_x(n) = m - 1\} = \left\{ \sum_{k=1}^{m-1} H_k \le n < \sum_{k=1}^{m} H_k \right\}$$

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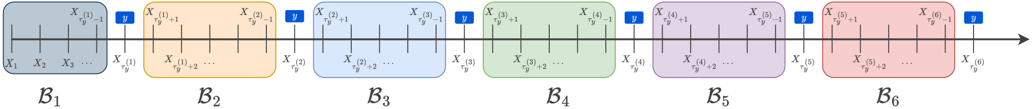

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$$N_x(n) \le n \implies \mathbb{P}(N_x(n) < +\infty) = 1, \quad \mathbb{E}[N_x(n)] \le n < +\infty$$

Proof of Lemma – Lower Bound

- Observe that

$$\sum_{k=1}^{N_x(n)+1} H_k > n$$

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- Taking limits as $n \to \infty$ on either sies, we get the desired lower bound

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- Let
$$\bar{\mu}_{xx} := \mathbb{E}[\bar{H}_1].$$
- Then, $\bar{\mu}_{xx} \leq \mu_{xx}$

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- Let $\bar{\tau}_x^{(0)} = 0$, and let

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Clearly, $\bar{\tau}_x^{(k)} \leq \tau_x^{(k)}$

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- Observe that $\bar{N}_x(n) \geq N_x(n)$

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- Observe that $\bar{N}_x(n) \geq N_x(n)$
- **Claim:**
  $\bar{N}_x(n) + 1$ is a stopping time w.r.t. $\{\bar{H}_k\}_{k \in \mathbb{N}}$

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$$\leq n + M$$

- Applying $\mathbb{E}[\cdot]$ and using Wald's lemma,

$$\mathbb{E}[\bar{N}_x(n) + 1] \cdot \bar{\mu}_{xx} \leq n + M.$$

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- Applying $\mathbb{E}[\cdot]$ and using Wald's lemma,

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- Noting that $N_x(n) \leq \bar{N}_x(n)$, we have

$$\mathbb{E}[N_x(n) + 1] \cdot \bar{\mu}_{xx} \leq \mathbb{E}[\bar{N}_x(n) + 1] \cdot \bar{\mu}_{xx} \leq n + M.$$

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Proof of Lemma – Upper Bound

- Dividing both sides by n and taking limits as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_x(n)]}{n} \leq \frac{1}{\bar{\mu}_{xx}} = \frac{1}{\mathbb{E}[\bar{H}_1]} = \frac{1}{\mathbb{E}[\min\{H_1, M\}]}$$

- Left-hand side does not depend on M, and above equation holds for all $M \in \mathbb{N}$
- Taking $M \to \infty$ on the right hand side, we get the desired upper bound

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Recap – Invariant Distributions

Proposition (On Existence and Uniqueness of Invariant Distribution)

Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible, time-homogeneous DTMC on a discrete state space $\mathcal{X}$ with TPM P.

Then, a unique stationary distribution π exists if and only if P is positive recurrent.

In this case, $\pi_x = \frac{1}{\mu_{xx}} > 0$ for all $x \in \mathcal{X}$.

Example

- Consider a time-homogeneous DTMC with $\mathcal{X} = \{0, 1, 2, \ldots\}$ whose transition probabilities are given by

$$P_{0,i} = \left(\frac{1}{2}\right)^i, \qquad P_{i,i+1} = \frac{1}{2} = P_{i,0}$$

for all $i \in \mathbb{N}$.

1. Draw the transition graph.
2. Is the Markov chain irreducible?
3. Classify the states as transient, positive recurrent, or null recurrent.
4. Compute μ_{ii} for all $i \in \{0, 1, 2, \ldots\}$.

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Example

- A fair coin is tossed repeatedly and independently until the pattern "HTH" is observed for the first time.
  How many tosses will be required on the average?

Example

- **[Goldmann Sachs Interview Question @IISc] [2017-18 Placement Season]**

  Imagine that you are playing a game that involves tossing a coin of bias 0.9 repeatedly and independently, with 3 lives at the start of the game. At each round, if the coin toss results in a tail, you lose one life. However, if the coin toss results in a head, you gain back your lost lives one at a time. If you have all 3 lives and the coin toss results in a head, nothing changes, and you continue tossing. You are allowed to play until you lose all your lives. What is the expected number of times you will play this game?