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Stochastic Processes

Some Reverse Implications, Limit Theorems: Weak Law of Large Numbers, Strong Law of Large Numbers, Central Limit Theorem

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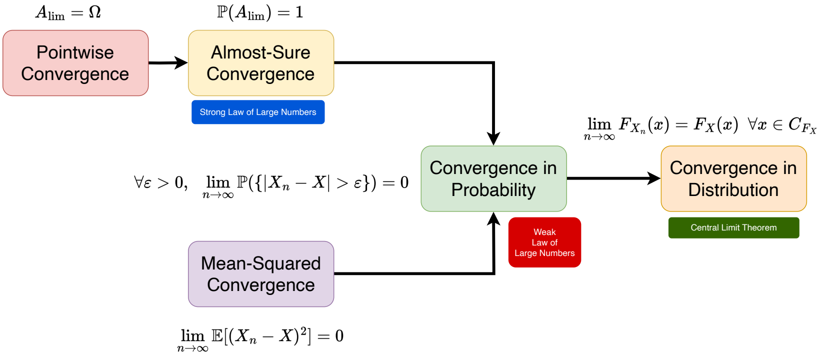
Recall

$$A_{\text{lim}} = \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\}$$



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Some Reverse Implications

Reverse Implication $\mathrm{p.} \implies \mathrm{m.s.}$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Proposition (Reverse Implication $\mathrm{p.} \implies \mathrm{m.s.}$)

Suppose that the following conditions hold:

1. $\mathbb{E}[X_n^2] < +\infty$ for all $n \in \mathbb{N}$.
2. $\mathbb{P}(|X_n| \leq Y) = 1$ for all n, with $\mathbb{E}[Y^2] < +\infty$.

Then,

$$X_n \xrightarrow{\mathrm{p.}} X \quad \implies \quad X_n \xrightarrow{\mathrm{m.s.}} X.$$

Reverse Implication $\mathrm{d.} \implies \mathrm{p.}$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ be defined w.r.t. $\mathscr{F}$.

Proposition (Reverse Implication $\mathrm{d.} \implies \mathrm{p.}$)

For any $c \in \mathbb{R}$,

$$X_n \xrightarrow{\mathrm{d.}} c \quad \implies X_n \xrightarrow{\mathrm{p.}} c$$

Proof of Reverse Implication $\mathrm{d.} \implies \mathrm{p.}$

Fix $\varepsilon > 0$ arbitrarily.

$$\mathbb{P}(|X_n - c| > \varepsilon) \quad = \quad \mathbb{P}(X_n > c + \varepsilon) + \mathbb{P}(X_n < c - \varepsilon) \quad \forall n$$

Proof of Reverse Implication d. $\implies$ p.

Fix $\varepsilon > 0$ arbitrarily.

$$\mathbb{P}(|X_n - c| > \varepsilon) \quad = \quad \mathbb{P}(X_n > c + \varepsilon) + \mathbb{P}(X_n < c - \varepsilon) \quad \forall n$$

$$\implies \quad \lim_{n\to\infty} \mathbb{P}(|X_n - c| > \varepsilon) = \lim_{n\to\infty} \mathbb{P}(X_n > c + \varepsilon) + \lim_{n\to\infty} \mathbb{P}(X_n < c - \varepsilon)$$

Proof of Reverse Implication d. $\implies$ p.

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$$\mathbb{P}(|X_n - c| > \varepsilon) \quad = \quad \mathbb{P}(X_n > c + \varepsilon) + \mathbb{P}(X_n < c - \varepsilon) \quad \forall n$$

$$\implies \quad \lim_{n\to\infty} \mathbb{P}(|X_n - c| > \varepsilon) = \lim_{n\to\infty} \mathbb{P}(X_n > c + \varepsilon) + \lim_{n\to\infty} \mathbb{P}(X_n < c - \varepsilon) = 0$$

Limit Theorems

Characteristic Function

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be a random variable w.r.t. $\mathscr{F}$.

Definition (Characteristic Function

The characteristic function of the random variable X is a function $C_X : \mathbb{R} \to \mathbb{C}$, defined as

$$C_X(s) = \mathbb{E}[e^{jsX}] = \mathbb{E}[\cos sX] + j\,\mathbb{E}[\sin sX], \qquad s \in \mathbb{R}.$$

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Remark:

$$\left| C_X(s) \right| \leq 1 \qquad \forall s \in \mathbb{R}.$$

Taylor Expansion for Characteristic Functions

Lemma (Taylor Expansion for Characteristic Functions)

Suppose that $\mathbb{E}[|X|^k] < +\infty$ for some $k \in \mathbb{N}$. Then,

$$C_X(s) = \sum_{j=0}^{k} \frac{\mathbb{E}[X^j]}{j!} (is)^j + o(s^k), \qquad s \in \mathbb{R}.$$

For a proof, see [Kingman and Taylor, 2008].

Characteristic Functions and Convergence in Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be random variables defined w.r.t. $\mathscr{F}$.

Proposition (Characteristic Functions and Convergence in Distribution)

We have

$$X_n \xrightarrow{\text{d.}} X \iff C_{X_n}(s) \xrightarrow{n\to\infty} C_X(s) \quad \forall s \in \mathbb{R}.$$

Proof is based on Skorokhod's representation theorem [Grimmett and Stirzaker, 2020, Section 7.2].

Weak Law of Large Numbers (WLLN)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\{X_n\}_{n=1}^{\infty}$ be defined w.r.t. $\mathscr{F}$.

Theorem (Weak Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. with $\mathbb{E}[|X_1|] < +\infty$. Further, let $\mathbb{E}[X_1] = \mu$. Let

$$S_n = \sum_{i=1}^{n} X_i.$$

Then,

$$\frac{S_n}{n} \xrightarrow{\text{p.}} \mu.$$

More formally, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = 0.$$

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Strong Law of Large Numbers (SLLN)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\{X_n\}_{n=1}^{\infty}$ be defined w.r.t. $\mathscr{F}$.

Theorem (Strong Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. with $\mathbb{E}[|X_1|] < +\infty$. Further, let $\mathbb{E}[X_1] = \mu$. Let

$$S_n = \sum_{i=1}^{n} X_i.$$

Then,

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mu.$$

More formally,

$$\mathbb{P}\left(\lim_{n \to \infty} \frac{S_n}{n} = \mu \right) = 1.$$

Proof of WLLN – Using Finite Variance Assumption

We shall first see a simple proof of WLLN under a finite variance assumption.
Suppose that $\mathrm{Var}(X_1) = \sigma^2 < +\infty$.

$$\forall \varepsilon > 0, \qquad \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \quad \leq \quad \frac{1}{\varepsilon^2}\,\mathbb{E}\left[\left(\frac{S_n}{n} - \mu\right)^2\right]$$

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$$= \frac{\sigma^2}{n}$$

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$$
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\forall \varepsilon > 0, \qquad \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \quad &\leq \quad \frac{1}{\varepsilon^2}\,\mathbb{E}\left[\left(\frac{S_n}{n} - \mu\right)^2\right] \\
&= \quad \frac{\sigma^2}{n} \\
&\longrightarrow \quad 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Proof of WLLN – Without Using Finite Variance Assumption

$X_1, X_2, \cdots$ i.i.d. with $\mathbb{E}[|X_1|] < +\infty, \quad \mathbb{E}[X_1] = \mu.$

$$\forall s \in \mathbb{R}, \qquad C_{\frac{S_n}{n}}(s) \quad = \quad \left(C_{X_1}\left(\frac{s}{n}\right)\right)^n$$

Proof of WLLN – Without Using Finite Variance Assumption

$X_1, X_2, \cdots$ i.i.d. with $\mathbb{E}[|X_1|] < +\infty, \quad \mathbb{E}[X_1] = \mu.$

$$\forall s \in \mathbb{R}, \qquad C_{\frac{S_n}{n}}(s) = \left(C_{X_1}\left(\frac{s}{n}\right)\right)^n$$

$$= \left(1 + \frac{\mathbb{E}[X]}{1!} \cdot \frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n$$

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$$= \left(1 + \frac{\mathbb{E}[X]}{1!} \cdot \frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n$$

$$\forall s \in \mathbb{R}, \qquad \lim_{n\to\infty} C_{\frac{S_n}{n}}(s) = \lim_{n\to\infty} \left(1 + \frac{\mu}{1!} \cdot \frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n$$

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$X_1, X_2, \cdots$ i.i.d. with $\mathbb{E}[|X_1|] < +\infty$, $\quad \mathbb{E}[X_1] = \mu$.

$$\forall s \in \mathbb{R}, \qquad C_{\frac{S_n}{n}}(s) = \left(C_{X_1}\left(\frac{s}{n}\right)\right)^n = \left(1 + \frac{\mathbb{E}[X]}{1!} \cdot \frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n$$

$$\forall s \in \mathbb{R}, \qquad \lim_{n\to\infty} C_{\frac{S_n}{n}}(s) = \lim_{n\to\infty} \left(1 + \frac{\mu}{1!} \cdot \frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n = e^{js\mu}.$$

Proof of WLLN – Without Using Finite Variance Assumption

$X_1, X_2, \cdots$ i.i.d. with $\mathbb{E}[|X_1|] < +\infty, \quad \mathbb{E}[X_1] = \mu.$

$$\begin{aligned}
\forall s \in \mathbb{R}, \qquad C_{\frac{S_n}{n}}(s) &= \left(C_{X_1}\left(\frac{s}{n}\right)\right)^n \\
&= \left(1 + \frac{\mathbb{E}[X]}{1!} \cdot \frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n
\end{aligned}$$

$$\begin{aligned}
\forall s \in \mathbb{R}, \qquad \lim_{n\to\infty} C_{\frac{S_n}{n}}(s) &= \lim_{n\to\infty} \left(1 + \frac{\mu}{1!} \cdot \frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n \\
&= e^{js\mu}.
\end{aligned}$$

Thus,

$$\frac{S_n}{n} \xrightarrow{\text{d.}} \mu \quad \Longrightarrow \quad \frac{S_n}{n} \xrightarrow{\text{p.}} X.$$

Proof of SLLN Under Additional Assumptions

Suppose that $\mathbb{E}[X_1^4] < +\infty$. Assume, without loss of generality that $\mathbb{E}[X_1] = 0$.
We want to show

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0 \, .$$

Fix an arbitrary $\varepsilon > 0$.

$$\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) \quad \leq \quad \frac{\mathbb{E}[(S_n)^4]}{n^4\,\varepsilon^4}$$

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We want to show

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0 \, .$$

Fix an arbitrary $\varepsilon > 0$.

$$\begin{aligned}
\mathbb{P}\left(\left| \frac{S_n}{n} \right| > \varepsilon \right) \quad & \leq \quad \frac{\mathbb{E}[(S_n)^4]}{n^4 \, \varepsilon^4} \\
& = \quad \frac{\mathbb{E}[(X_1 + \cdots + X_n)^4]}{n^4 \, \varepsilon^4}
\end{aligned}$$

Proof of SLLN Under Additional Assumptions

Suppose that $\mathbb{E}[X_1^4] < +\infty$. Assume, without loss of generality that $\mathbb{E}[X_1] = 0$.
We want to show

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0\ .$$

Fix an arbitrary $\varepsilon > 0$.

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) \quad &\leq \quad \frac{\mathbb{E}[(S_n)^4]}{n^4\,\varepsilon^4} \\
&= \quad \frac{\mathbb{E}[(X_1 + \cdots + X_n)^4]}{n^4\,\varepsilon^4} \\
&= \quad \frac{n\,\mathbb{E}[X_1^4] + 6\binom{n}{2}\,(\mathrm{Var}(X_1))^2}{n^4}
\end{aligned}$$

Proof of SLLN Under Additional Assumptions

Suppose that $\mathbb{E}[X_1^4] < +\infty$. Assume, without loss of generality that $\mathbb{E}[X_1] = 0$. We want to show

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Fix an arbitrary $\varepsilon > 0$.

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) &\leq \frac{\mathbb{E}[(S_n)^4]}{n^4\,\varepsilon^4} \\
&= \frac{\mathbb{E}[(X_1 + \cdots + X_n)^4]}{n^4\,\varepsilon^4} \\
&= \frac{n\,\mathbb{E}[X_1^4] + 6\binom{n}{2}\left(\mathrm{Var}(X_1)\right)^2}{n^4} \\
&\leq \frac{\mathbb{E}[X_1^4]}{n^3} + 3\left(\mathrm{Var}(X_1)\right)^2 \frac{1}{n^2}
\end{aligned}$$

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Central Limit Theorem (CLT)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\{X_n\}_{n=1}^{\infty}$ be defined w.r.t. $\mathscr{F}$.

Theorem (Central Limit Theorem)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. with $\mathrm{Var}(X_1) < +\infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Then,

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{\mathrm{d}} X, \qquad X \sim \mathcal{N}(0,1).$$

More formally,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} \leq x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,\mathrm{d}t \qquad \forall x \in \mathbb{R}.$$

References

- Grimmett, G. and Stirzaker, D. (2020). *Probability and random processes*. Oxford university press.
- Kingman, J. F. C. and Taylor, S. J. (2008). *Introdction to Measure and Probability*. Cambridge University Press.