Name:
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Program: BTech / MTech TA / MTech RA / PhD (Tick one)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

AI5090/EE5817: Probability and Stochastic Processes

Quiz 03

Date: 12 September 2025

Question	1	2	Total
Marks Scored			

1. Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be non-negative numbers with $\sum_{n\in\mathbb{N}} \mathbb{P}_n = 1$.
Let $\mathbb{P} : \mathscr{B}(\mathbb{R}) \to [0,1]$ be defined as

$$\mathbb{P}(A) := \sum_{n\in\mathbb{N}} \mathbb{P}_n\, \delta_n(A), \qquad \text{where for any } n \in \mathbb{N}, \quad \delta_n(A) = \begin{cases} 1, & n \in A, \\ 0, & n \notin A. \end{cases}$$

(a) **(2 Marks)**
Verify that $\mathbb{P}$ is a valid probability measure on $\mathscr{B}(\mathbb{R})$.

(b) **(1 Mark)**
Let $\mathbb{P}_n = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{n-1}$ for all $n \in \mathbb{N}$. If E is the set of even natural numbers, compute $\mathbb{P}(E)$.

Solution.

(a) Observe that $\delta_n(\emptyset) = 0$ for all $n \in \mathbb{N}$. Similarly, we have $\delta_n(\Omega) = \delta_n(\mathbb{R})$, as $\mathbb{N} \subseteq \mathbb{R}$. We now proceed to check each of the conditions for $\mathbb{P}$ to be a probability measure. First, we note that

$$\mathbb{P}(\emptyset) \;=\; \sum_{n\in\mathbb{N}} \mathbb{P}_n\, \delta_n(\emptyset) \;=\; \sum_{n\in\mathbb{N}} \mathbb{P}_n \cdot 0 \;=\; 0.$$

Next, we note that

$$\mathbb{P}(\mathbb{R}) \;=\; \sum_{n\in\mathbb{N}} \mathbb{P}_n\, \delta_n(\mathbb{R}) \;=\; \sum_{n\in\mathbb{N}} \mathbb{P}_n \cdot 1 \;=\; \sum_{n\in\mathbb{N}} \mathbb{P}_n \;=\; 1.$$

Finally, consider any pairwise disjoint collection $\{A_i\}_{i\in\mathbb{N}}$. For each $i \in \mathbb{N}$, there exists $n \in \mathbb{N}$ (potentially depending on i) such that $n \in A_i$. Using this observation, we have

$$\delta_n\Big(\bigsqcup_{i\in\mathbb{N}} A_i\Big) = \mathbf{1}_{A_i}(n) = \sum_{i\in\mathbb{N}} \mathbf{1}_{A_i}(n) = \sum_{i\in\mathbb{N}} \delta_n(A_i), \tag{1}$$

because disjointness implies at most one term in the sum equals 1. Hence,

$$\begin{aligned}
\mathbb{P}\Big(\bigsqcup_{i\in\mathbb{N}} A_i\Big) &= \sum_{n\in\mathbb{N}} \mathbb{P}_n\, \delta_n\Big(\bigsqcup_{i\in\mathbb{N}} A_i\Big) \overset{\text{from (1)}}{=} \sum_{n\in\mathbb{N}} \mathbb{P}_n \sum_{i\in\mathbb{N}} \delta_n(A_i) \\
&= \sum_{i\in\mathbb{N}} \sum_{n\in\mathbb{N}} \mathbb{P}_n\, \delta_n(A_i) \qquad \text{(as } \mathbb{P}_n \geq 0\ \forall n \in \mathbb{N}\text{, we can exchange order of summation)} \\
&= \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).
\end{aligned}$$

Thus $\mathbb{P}$ is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

(b) We are given

$$\mathbb{P}_n = \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}, \quad n \in \mathbb{N}.$$

Let $E = \{n \in \mathbb{N} : n \text{ is even}\}$ be the set of even natural numbers. We want to compute $\mathbb{P}(E)$.

By definition,

$$\mathbb{P}(E) = \sum_{n\in\mathbb{N}} \mathbb{P}_n \delta_n(E).$$

Since $\delta_n(E) = 1$ iff n is even (and 0 otherwise), this simplifies to

$$\mathbb{P}(E) = \sum_{\substack{n\in\mathbb{N} \\ n \text{ even}}} \mathbb{P}_n.$$

Writing $n = 2k$ for $k \in \mathbb{N}$, we have

$$\mathbb{P}(E) = \sum_{k\in\mathbb{N}} p_{2k} = \sum_{k\in\mathbb{N}} \frac{1}{4}\left(\frac{3}{4}\right)^{2k-1} = \frac{3}{16} \cdot \frac{1}{1-\frac{9}{16}} = \frac{3}{16} \cdot \frac{16}{7} = \frac{3}{7}.$$

2. Let $(\Omega, \mathscr{F}) = (\mathbb{N}, 2^{\mathbb{N}})$. For each $n \in \mathbb{N}$, let $\mathbb{P}_n : \mathscr{F} \to [0,1]$ be defined as

$$\mathbb{P}_n(A) := \frac{|A \cap \{1,2,\ldots,n\}|}{n}, \qquad A \in \mathscr{F}.$$

Given a set $A \in \mathscr{F}$, its density $D(A)$ is defined as

$$D(A) := \lim_{n\to\infty} \mathbb{P}_n(A), \qquad \text{provided the limit exists.}$$

Let $\mathscr{D}$ be the collection of all sets whose density is well-defined.

(a) **(1 Mark)**
Show that $\mathscr{D}$ is closed under complements, i.e., if $A \in \mathscr{D}$, then $A^{\complement} \in \mathscr{D}$.

(b) **(1 Mark)**
Let $M = \{3k : k = 1, 2, \ldots\}$. Find $D(M)$.

Solution.

(a) Fix $A \subseteq \mathbb{N}$ with $D(A)$ well defined.
For every $n \in \mathbb{N}$, the sets $A \cap \{1,2,\ldots,n\}$ and $A^{\complement} \cap \{1,2,\ldots,n\}$ constitute a partition of $\{1,2,\ldots,n\}$. Hence,

$$|A \cap \{1,2,\ldots,n\}| + \left|A^{\complement} \cap \{1,2,\ldots,n\}\right| = |\{1,2,\ldots,n\}| = n.$$

Dividing by n yields

$$\mathbb{P}_n(A) + \mathbb{P}_n(A^{\complement}) = 1 \qquad \text{for all } n \in \mathbb{N},$$

so that

$$\mathbb{P}_n(A^{\complement}) = 1 - \mathbb{P}_n(A).$$

Taking limits as $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty} \mathbb{P}_n(A^{\complement}) = \lim_{n\to\infty} 1 - \mathbb{P}_n(A) = 1 - \lim_{n\to\infty} \mathbb{P}_n(A) = 1 - D(A).$$

Thus, we observe that $\lim_{n\to\infty} \mathbb{P}_n(A^{\complement})$ is well-defined. Therefore, it follows that $D(A^{\complement}) = \lim_{n\to\infty} \mathbb{P}_n(A^{\complement})$ exists whenever $D(A)$ exists, i.e., $\mathscr{D}$ is closed under complements.

(b) Let $M = \{3k : k \in \mathbb{N}\}$. Then $|M \cap \{1, 2, \dots, n\}| = \lfloor n/3 \rfloor$, so

$$\mathbb{P}_n(M) = \frac{|M \cap \{1, 2, \dots, n\}|}{n} = \frac{\lfloor n/3 \rfloor}{n}.$$

Using $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ (or equivalently $x - 1 < \lfloor x \rfloor \leq x$) with $x = n/3$, we obtain

$$\frac{1}{3} - \frac{1}{n} < \frac{\lfloor n/3 \rfloor}{n} \leq \frac{1}{3}.$$

Taking limits as $n \to \infty$ and using the sandwich theorem for limits, we get

$$\lim_{n \to \infty} \mathbb{P}_n(M) = D(M) = \frac{1}{3}.$$