Behind the Scenes of $Ax = b$: Axioms and an Open Question

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Based on joint work with Rajesh Sundaresan

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Linear Inverse Problem

Linear Inverse Problem

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$$\checkmark A \qquad\qquad x \qquad\qquad b \checkmark$$

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Is there a principled approach to solve the problem?

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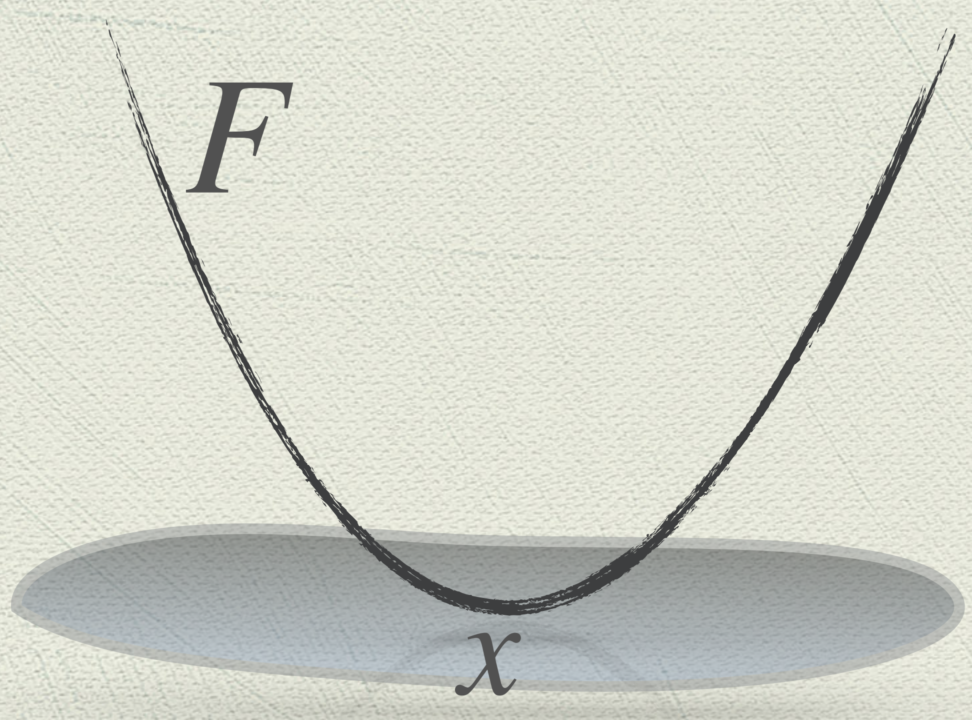
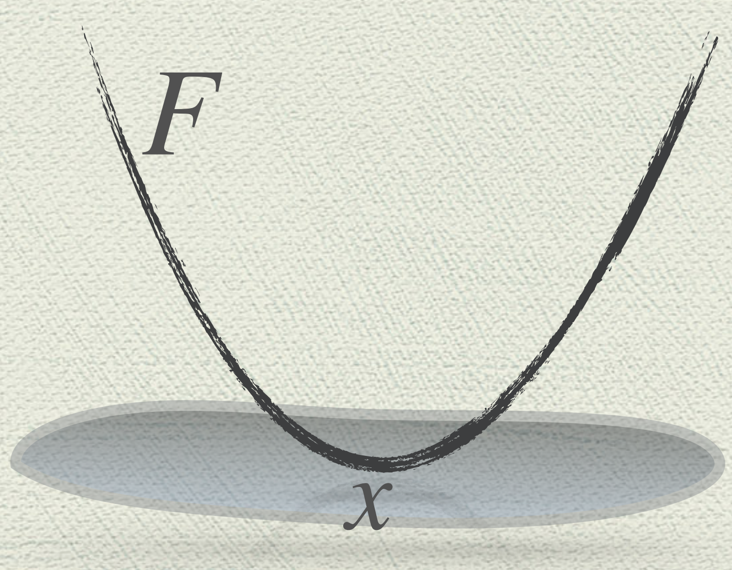


$$L = \{w \in \mathbb{R}^n : Aw = b\}$$

$$x = \arg\min_{w \in L} F(w)$$

Solution using Optimisation Techniques



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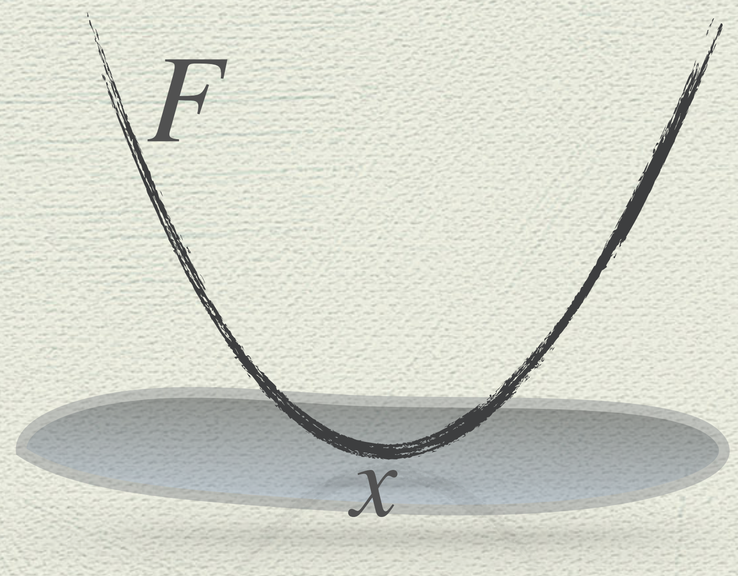

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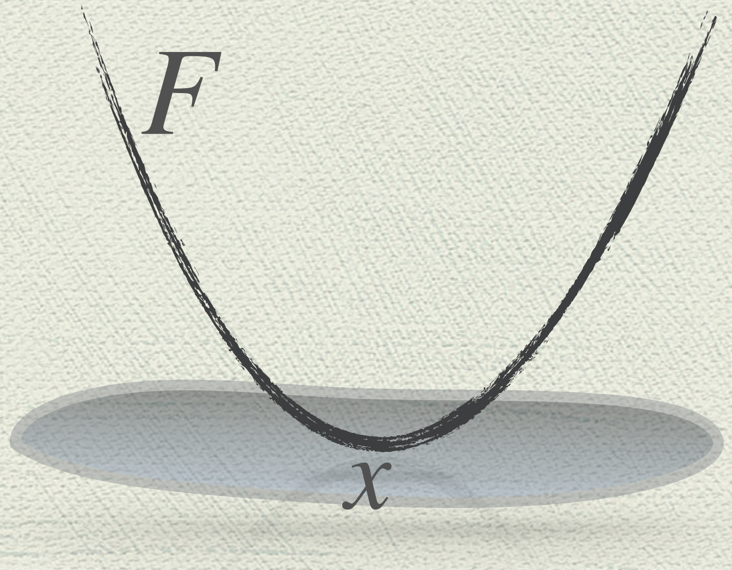

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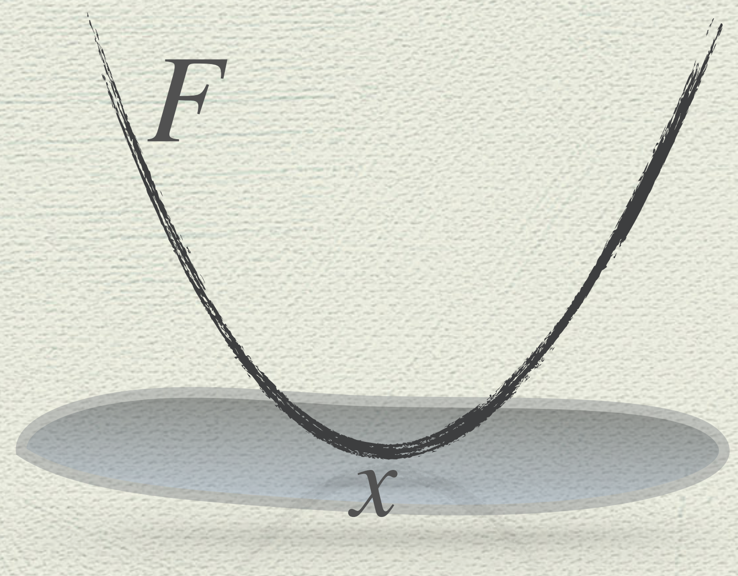

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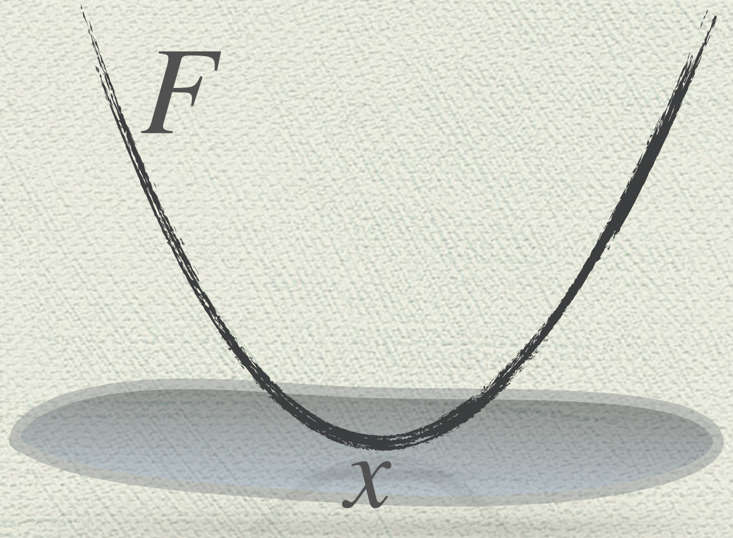

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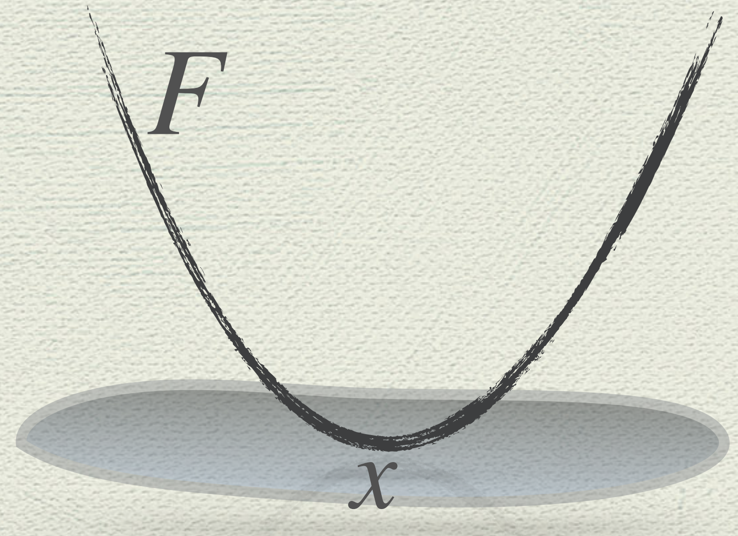

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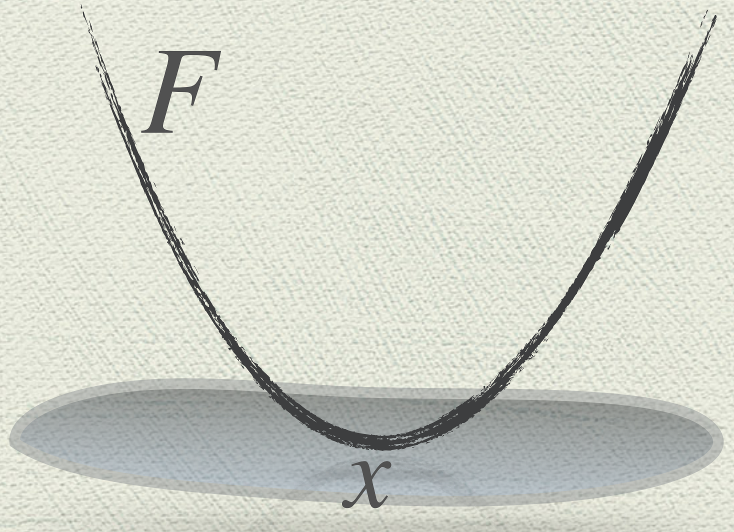

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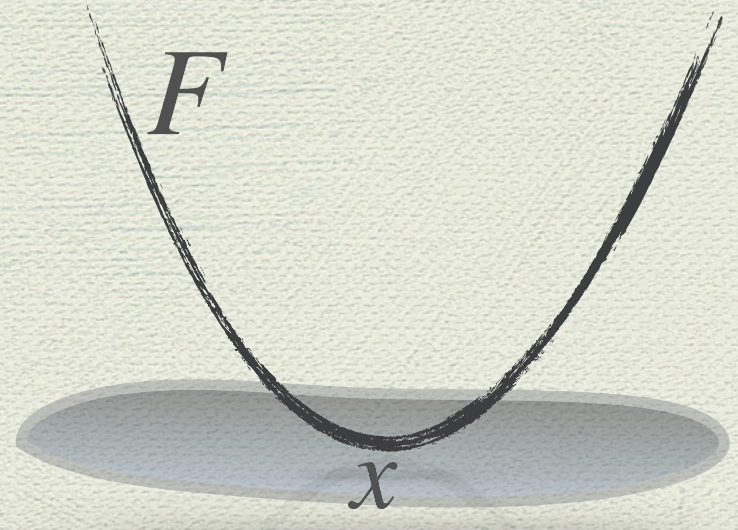

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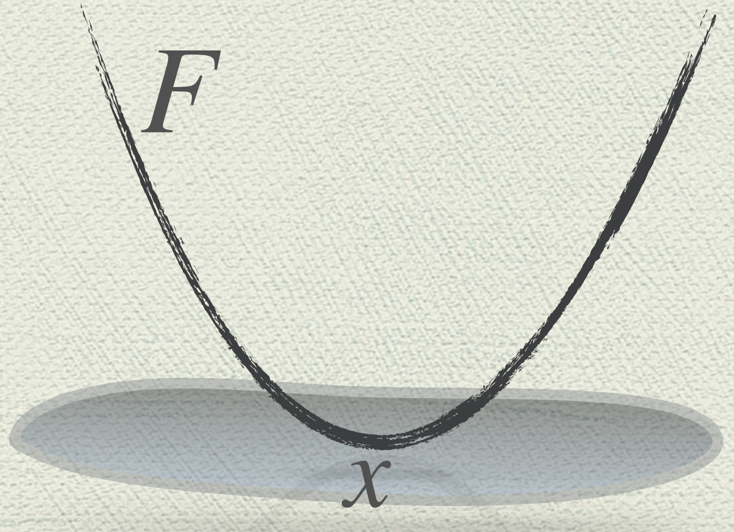

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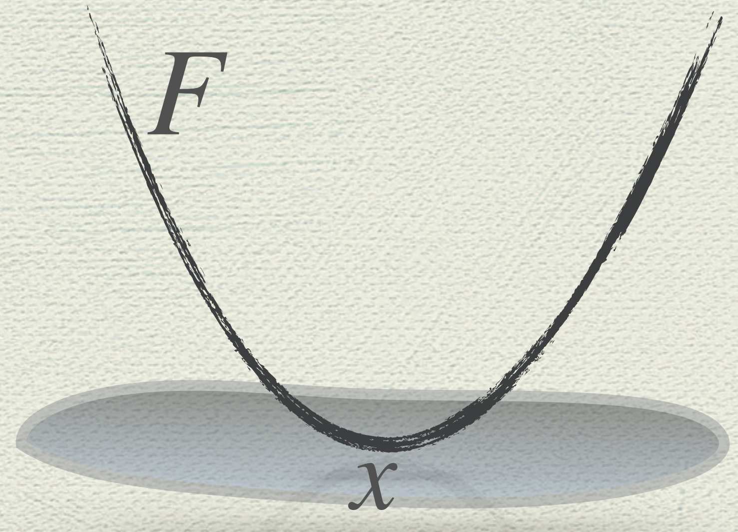

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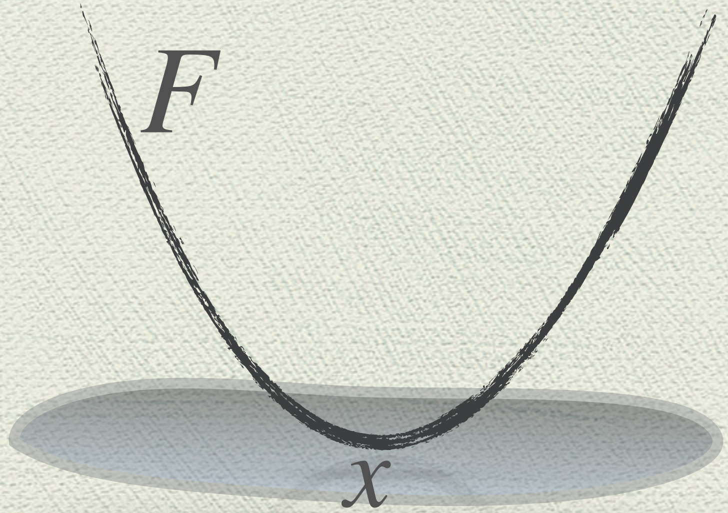

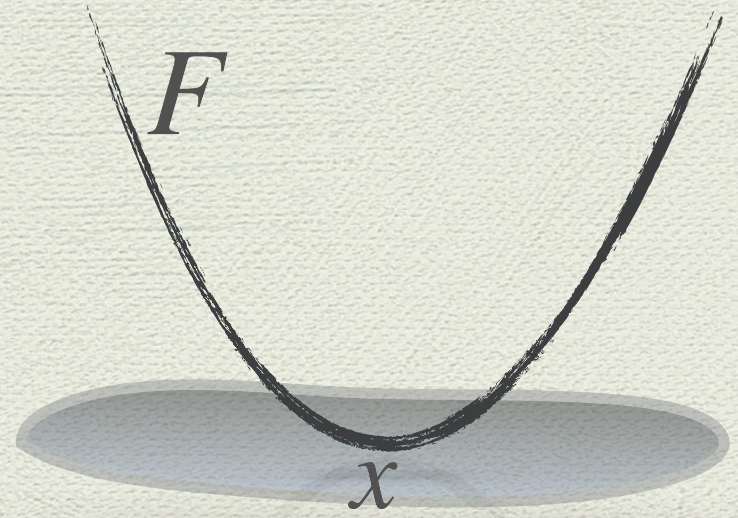

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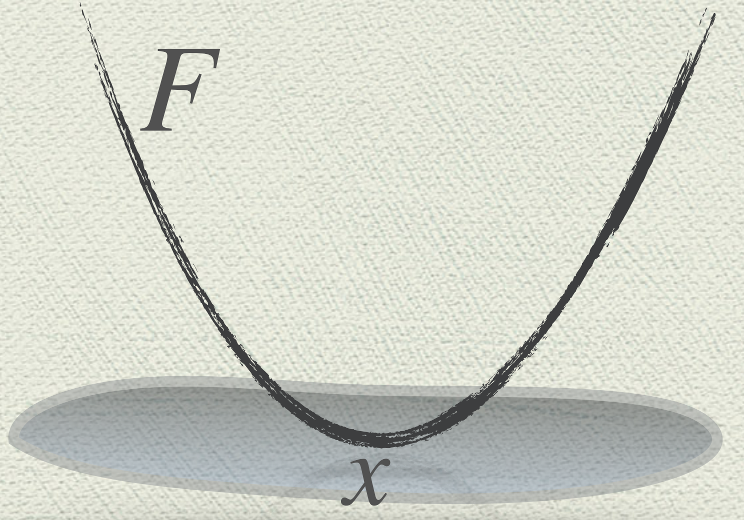

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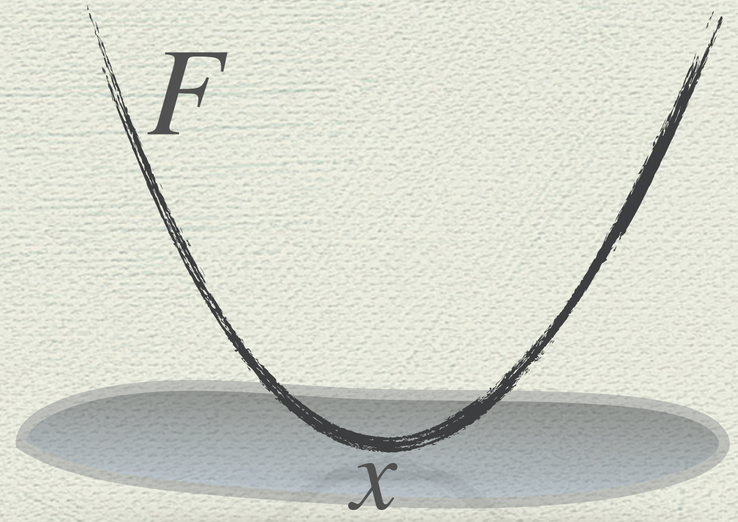

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Projection Rules

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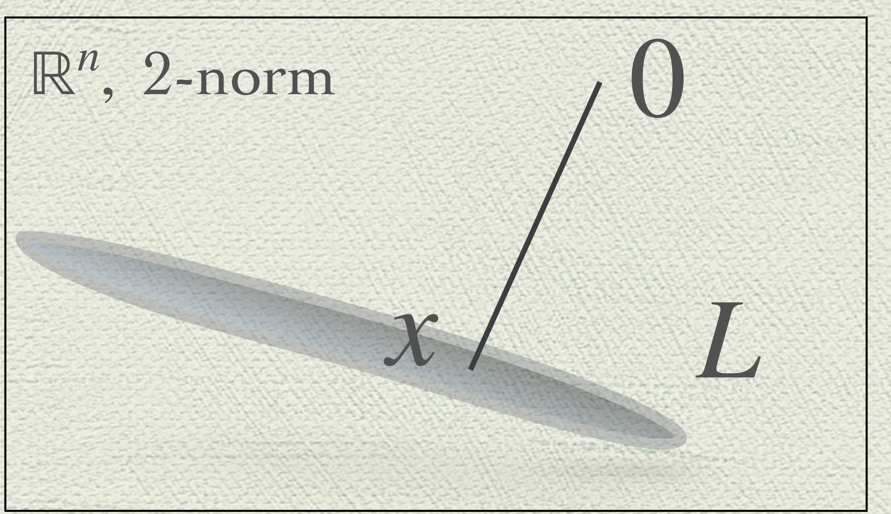

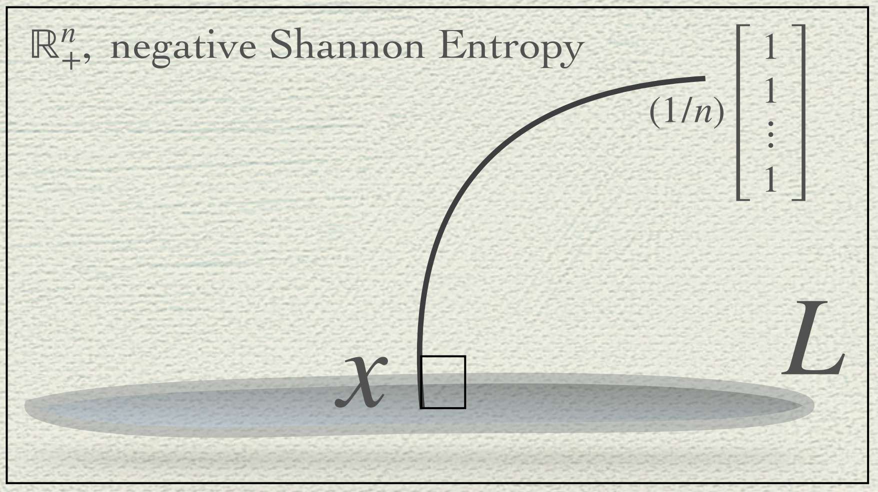

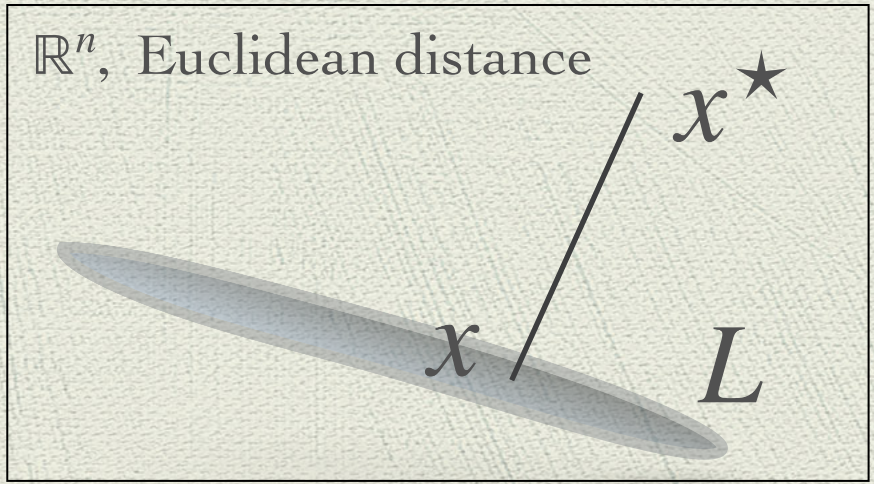

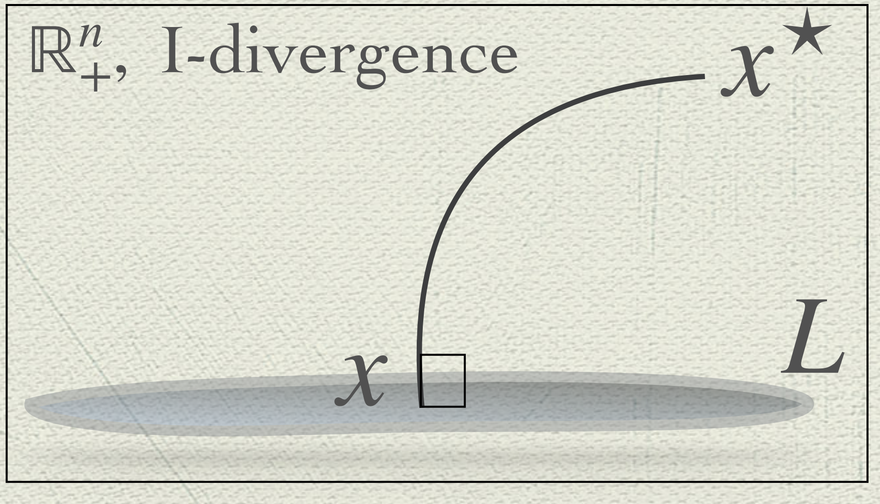

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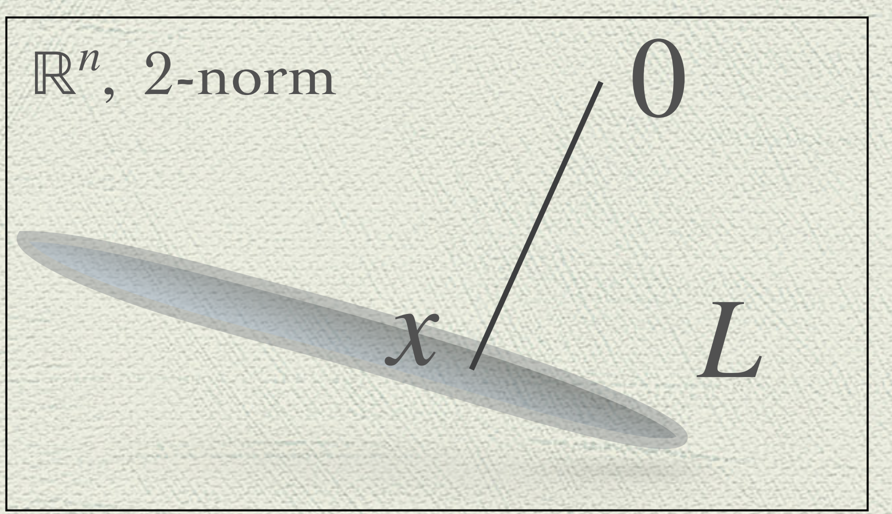
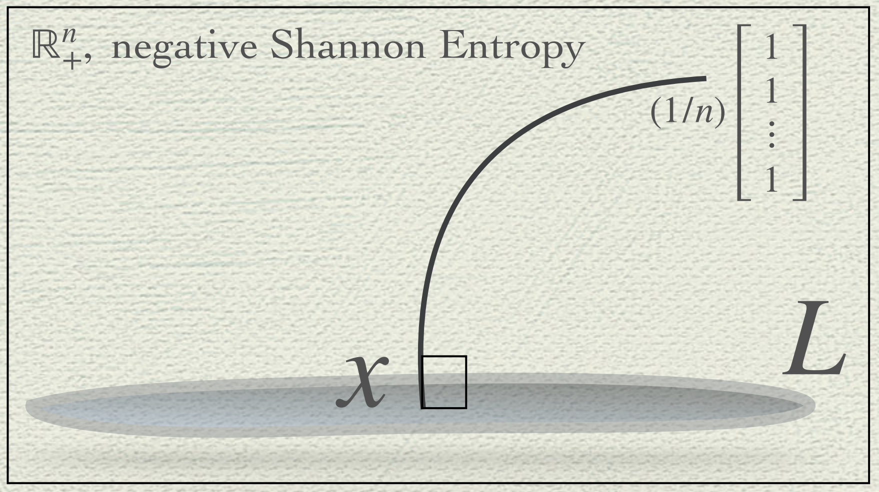
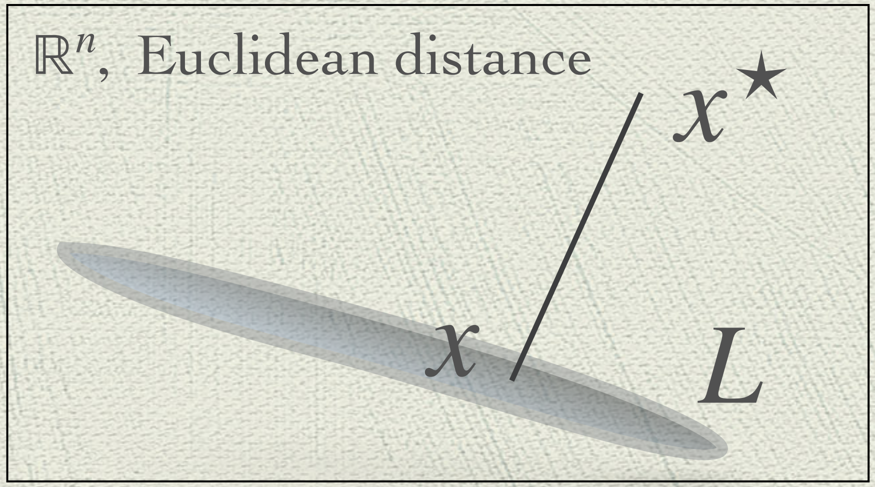
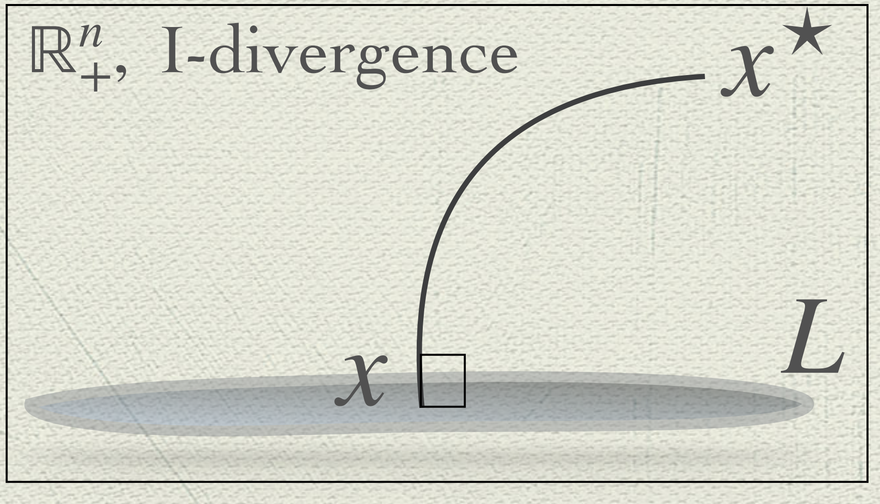




Function-minimisation
=
projection rule





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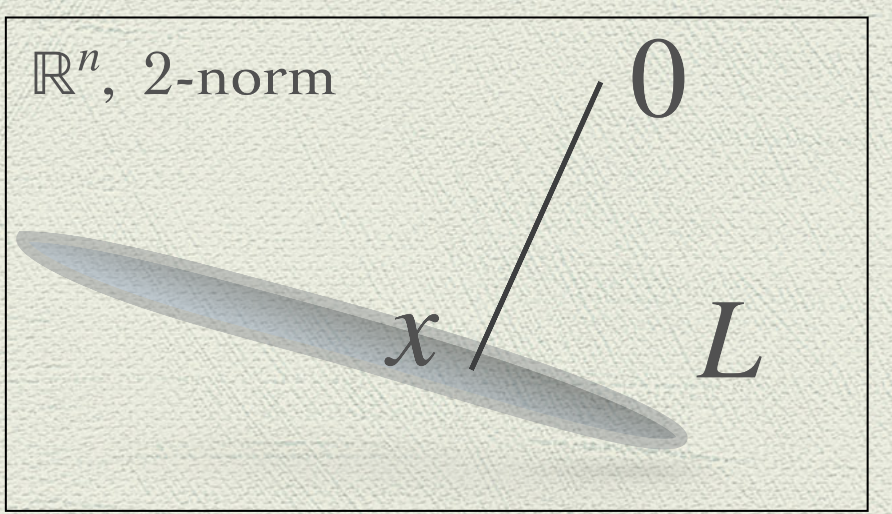

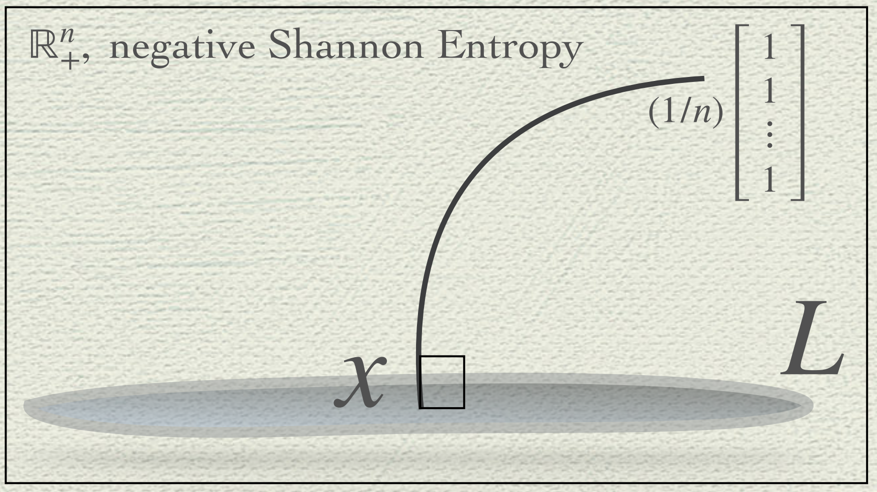

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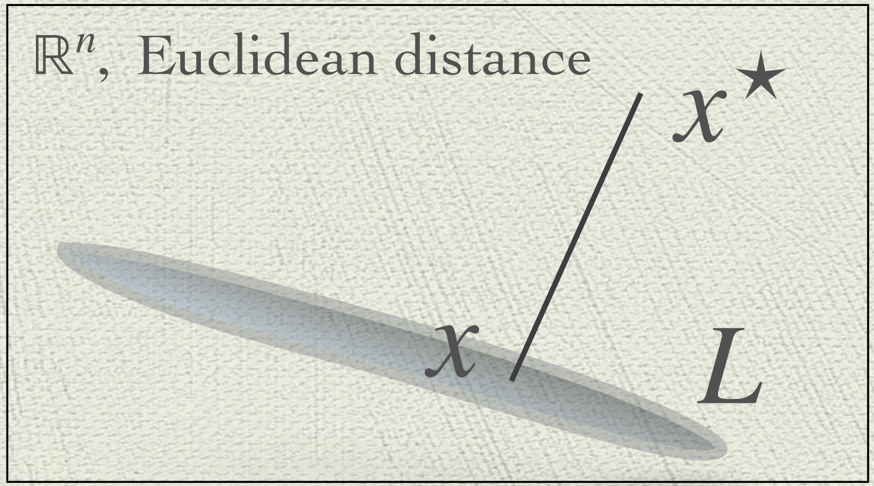
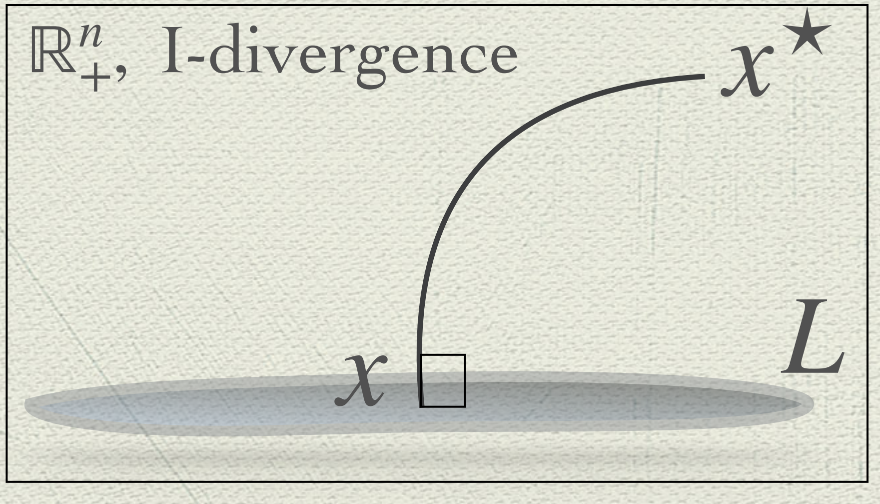
=

projection rule



$\Pi : (L, x^\star) \mapsto \Pi(L \,|\, x^\star) \in L$



Projection Rules and Function Minimisation

Projection Rules and Function Minimisation

$$\begin{aligned} F : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (x^\star, w) &\mapsto F(w \,|\, x^\star) \end{aligned}$$

Projection Rules and Function Minimisation

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$(x^\star, w) \mapsto F(w \,|\, x^\star)$$

$\Pi : \mathscr{L} \times \mathbb{R}^n \to \mathbb{R}^n$ projection rule

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<u>Definition:</u> <u>Π is generated by F,</u> if

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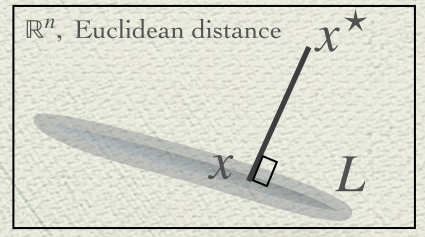
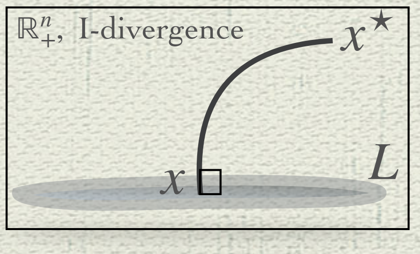


$F(w \,|\, x^\star) = \|w - x^\star\|^2$



$F(w \,|\, x^\star) = \sum_i w_i \log \frac{w_i}{x_i^\star}$

Is every projection rule generated by some function?

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- Answers to this question were provided by Csiszár in 1991

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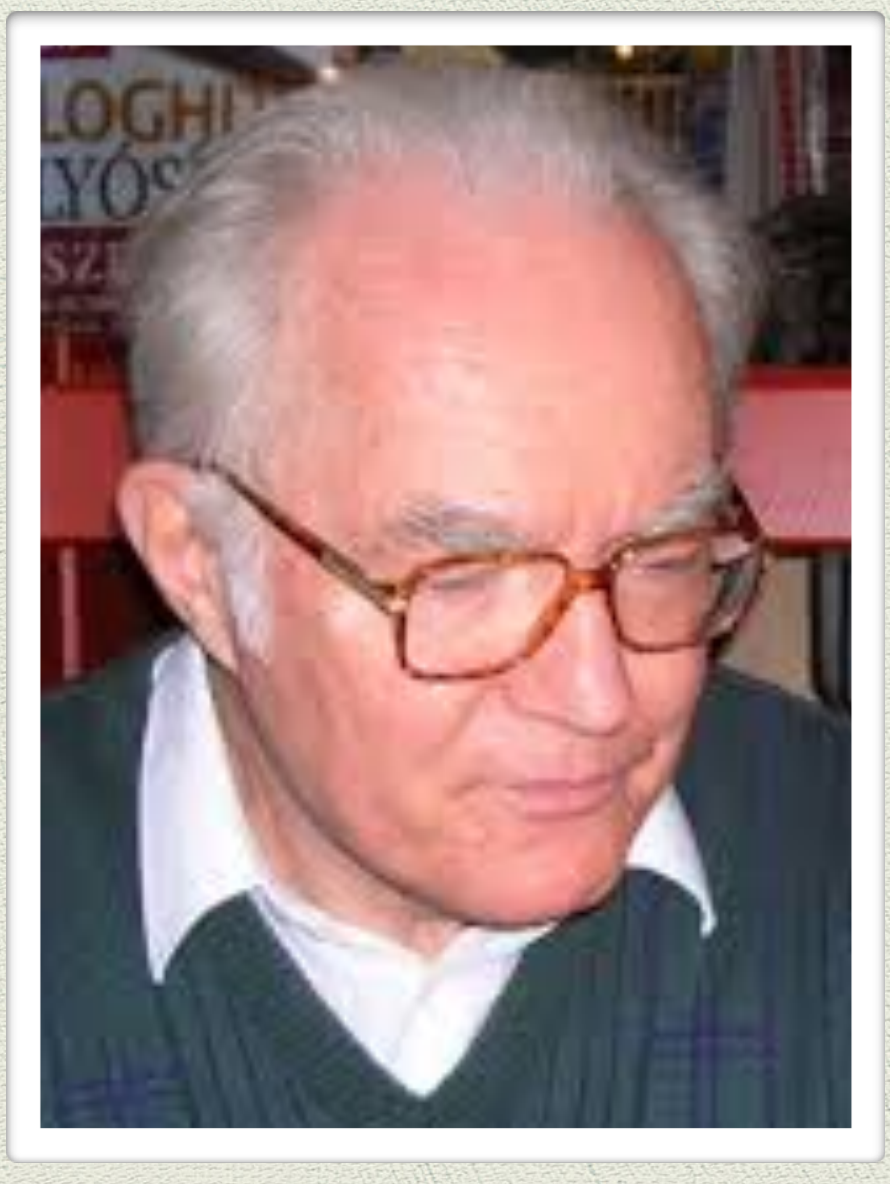

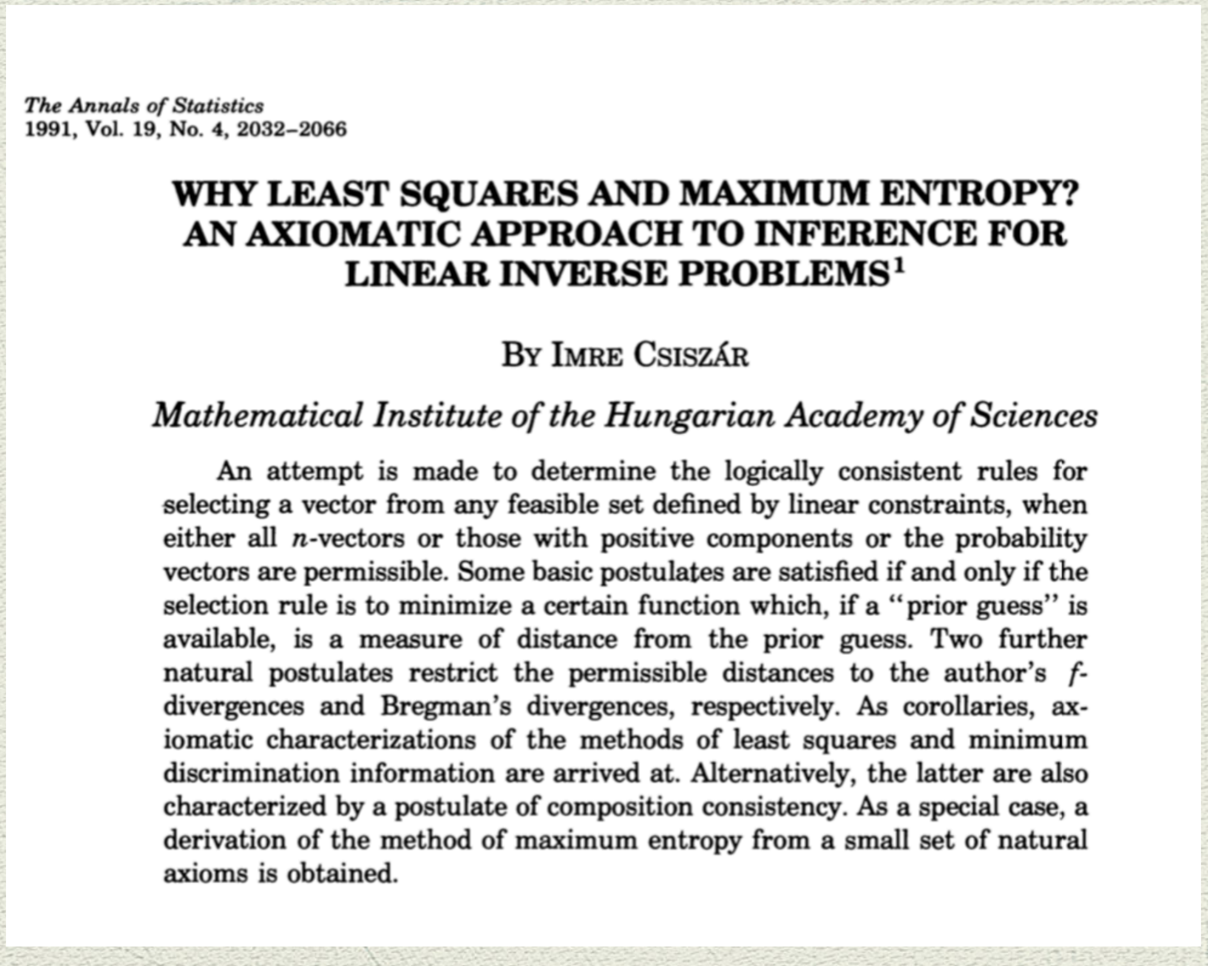

The Annals of Statistics
1991, Vol. 19, No. 4, 2032–2066

WHY LEAST SQUARES AND MAXIMUM ENTROPY? AN AXIOMATIC APPROACH TO INFERENCE FOR LINEAR INVERSE PROBLEMS[1]

BY IMRE CSISZÁR

Mathematical Institute of the Hungarian Academy of Sciences

An attempt is made to determine the logically consistent rules for selecting a vector from any feasible set defined by linear constraints, when either all n-vectors or those with positive components or the probability vectors are permissible. Some basic postulates are satisfied if and only if the selection rule is to minimize a certain function which, if a "prior guess" is available, is a measure of distance from the prior guess. Two further natural postulates restrict the permissible distances to the author's f-divergences and Bregman's divergences, respectively. As corollaries, axiomatic characterizations of the methods of least squares and minimum discrimination information are arrived at. Alternatively, the latter are also characterized by a postulate of composition consistency. As a special case, a derivation of the method of maximum entropy from a small set of natural axioms is obtained.

The seminal work of Csiszár

Approach 2: Axioms

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- Csiszár demonstrated that if projection rules satisfy some naturally appealing axioms, they must be generated by "nice" functions

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Projection rules satisfies the axioms of	Projection rule is generated by
Regularity + Locality	$F(w\|x^\star) = \sum_{i=1}^{n} f_i(w_i\|x_i^\star)$, f_i continuously differentiable and strictly convex
Regularity+ Locality + Subspace Transitivity	$F(w\|x^\star) =$ Bregman's divergence
Regularity+ Locality + Subspace Transitivity + Statistical	$F(w\|x^\star) = \sum_{i=1}^{n} w_i \log \frac{w_i}{x_i^\star}$
Regularity + Locality + Subspace Transitivity + Location Invariance + Scale Invariance	$F(w\|x^\star) = \sum_{i=1}^{n} (w_i - x_i^\star)^2$

Why are these axioms of interest to us?

- Kumar and Sundaresan ('15) discovered a family of projection rules satisfying

Regularity ✓

Subspace transitivity ✓

Locality ✗

- These projection rules are generated by a parametric divergence known as relative α-entropy
(related to the well-known Rényi divergence)

M. A. Kumar and R. Sundaresan. (2015). Minimization problems based on relative α-entropy I: Forward projection. IEEE Transactions on Information Theory, 61(9), 5063-5080.

M. A. Kumar and R. Sundaresan. (2015). Minimization problems based on relative α-entropy II: Reverse projection. IEEE Transactions on Information Theory, 61(9), 5081-5095.

"Are the projection rules generated by relative α-entropy the only rules which are regular, subspace transitive and nonlocal?"

Regularity ✔

Subspace transitivity ✔

Locality ✘

Our Quest

- Csiszár's results provide a necessary and sufficient axiomatic characterisation of many projection rules

Regularity + Locality $\Longleftrightarrow$ Projection rule generated by $F(w\,|\,x^\star) = \sum_{i=1}^{n} f_i(w_i\,|\,x_i^\star)$

Regularity + Locality + Subspace Transitivity $\Longleftrightarrow$ Projection rule generated by Bregman's divergence

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$\boxed{\text{Regularity + Subspace Transitivity} \Longleftrightarrow \text{Projection rule generated by ??}}$

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The axiom of regularity

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$$\mathscr{M} = \left\{ \{w \in \mathbb{R}^n : a^T w = b\},\ a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R} \right\}$$

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Π satisfies regularity if, for all $x^\star \in \mathbb{R}^n$,

Regularity

(Consistency)

$$L' \subset L,\ \Pi(L \,|\, x^\star) \in L' \implies \Pi(L' \,|\, x^\star) = \Pi(L \,|\, x^\star)$$

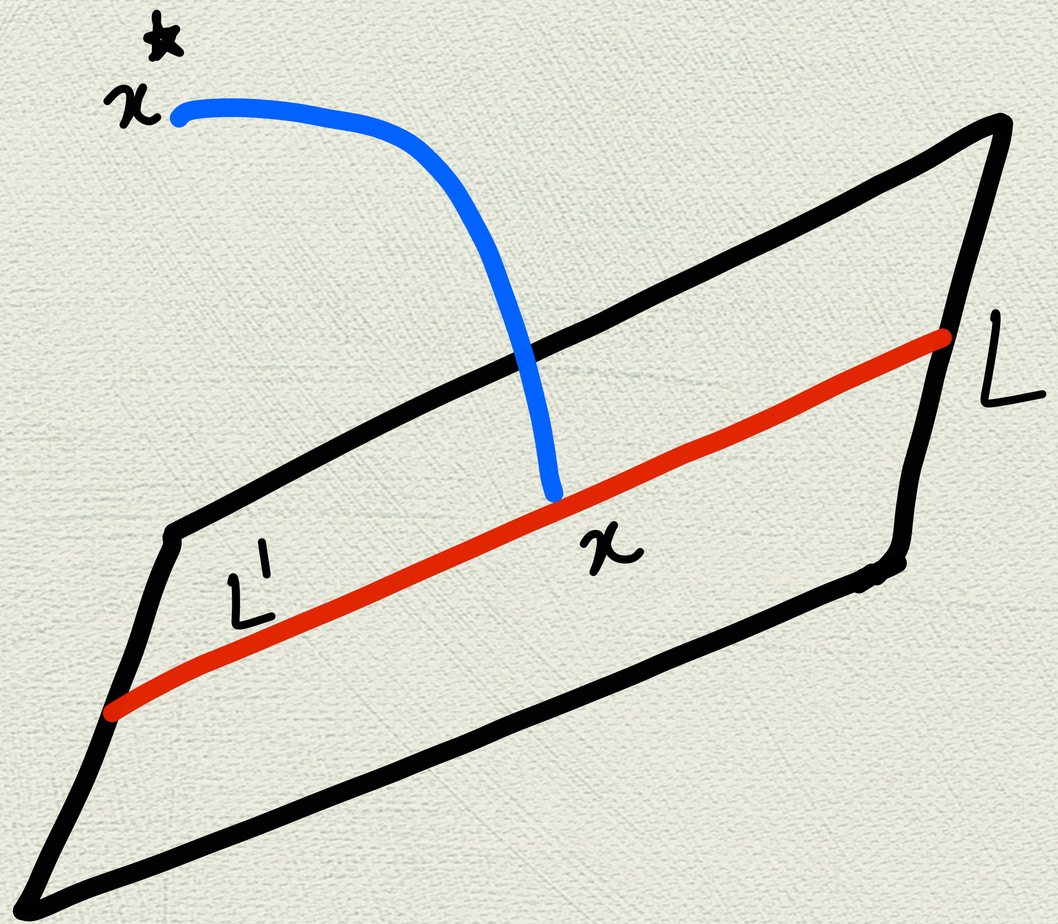
(Distinctness)

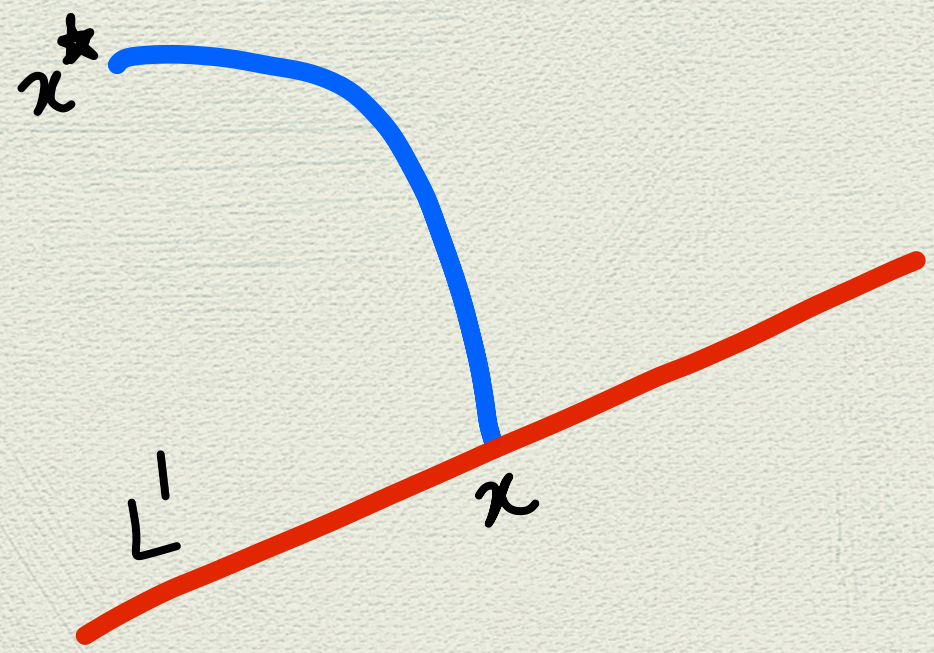
$$L, \tilde{L} \in \mathscr{M},\ L \neq \tilde{L},\ x^\star \notin L \cap \tilde{L} \implies \Pi(L \,|\, x^\star) \neq \Pi(\tilde{L} \,|\, x^\star)$$

(Continuity)

$\Pi(\,\cdot\,|\,x^\star)$ restricted to any fixed dimension is continuous

The axiom of consistency





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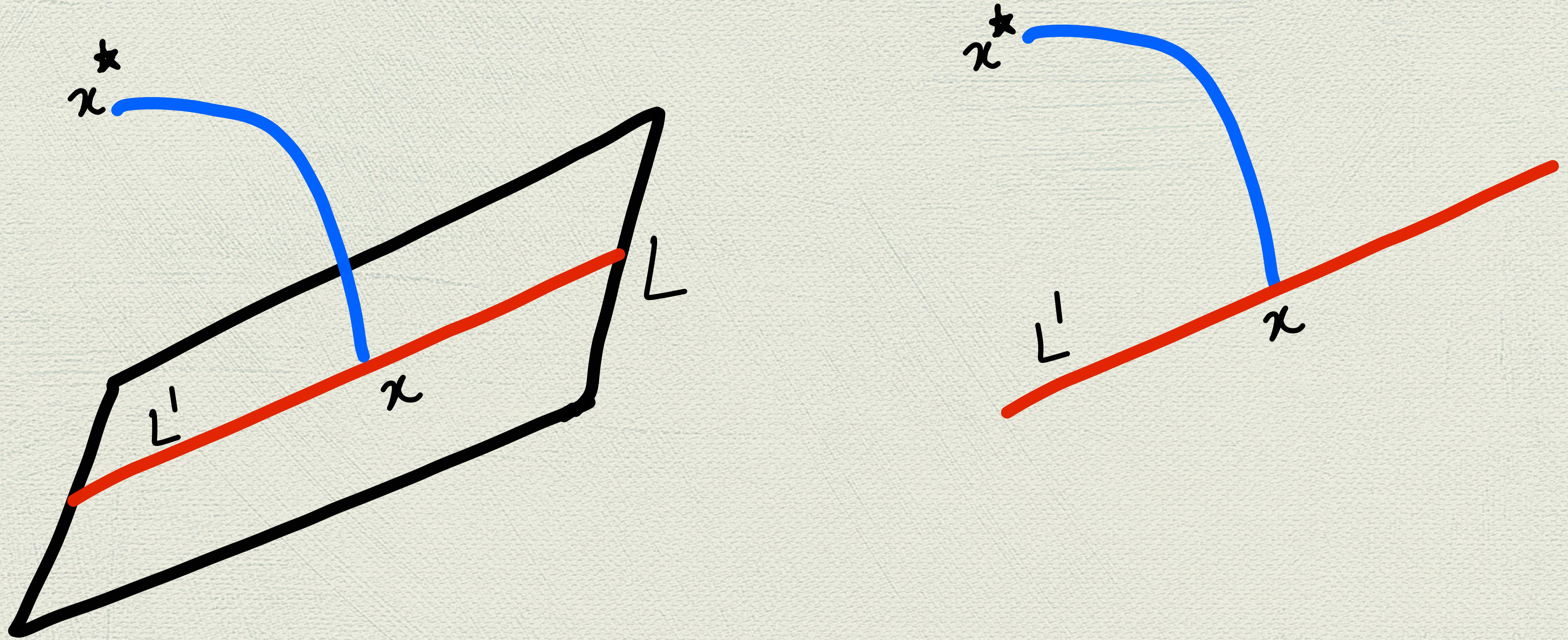

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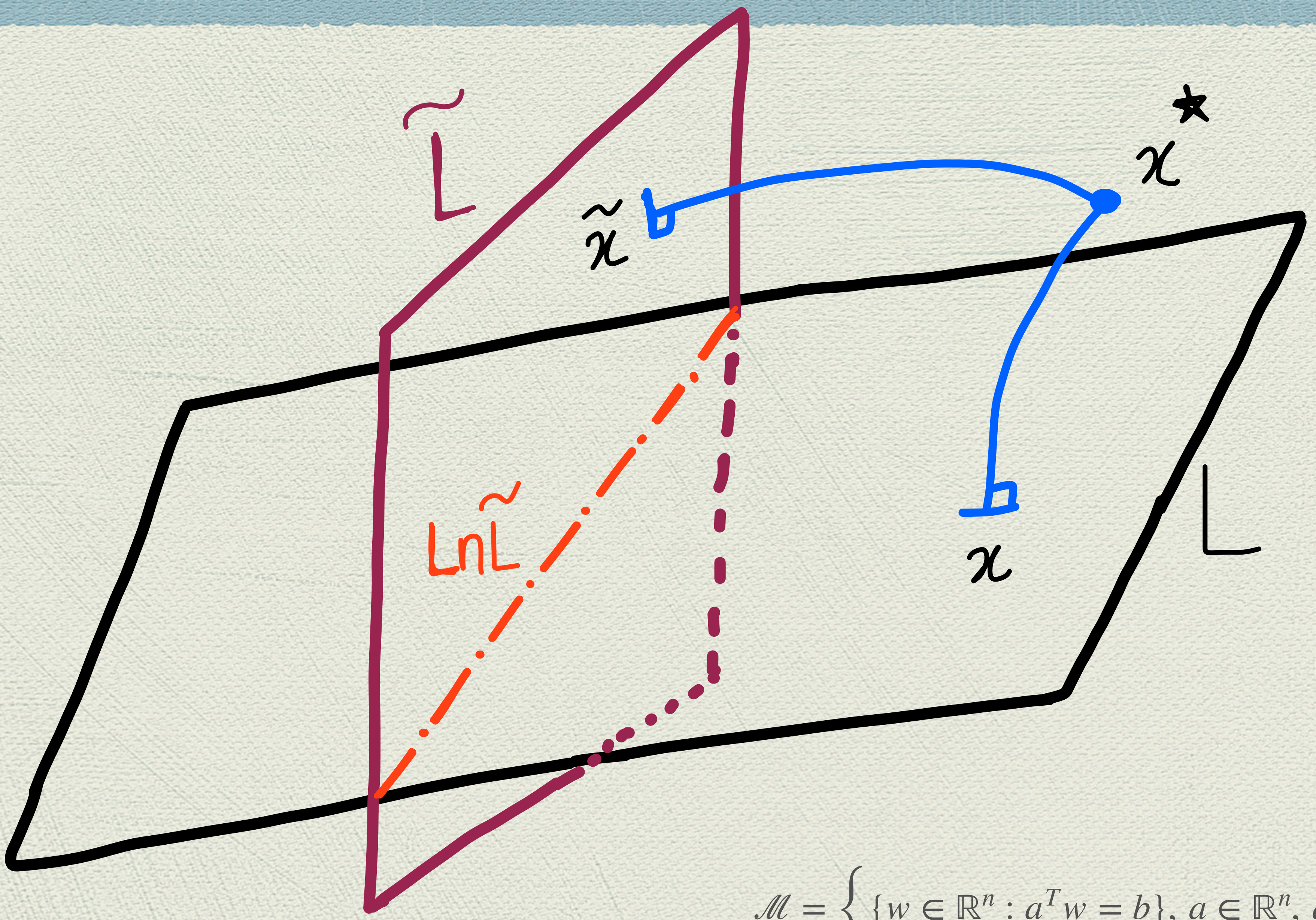
Additional observations do not give a reason to change the original selection

The axiom of distinctness



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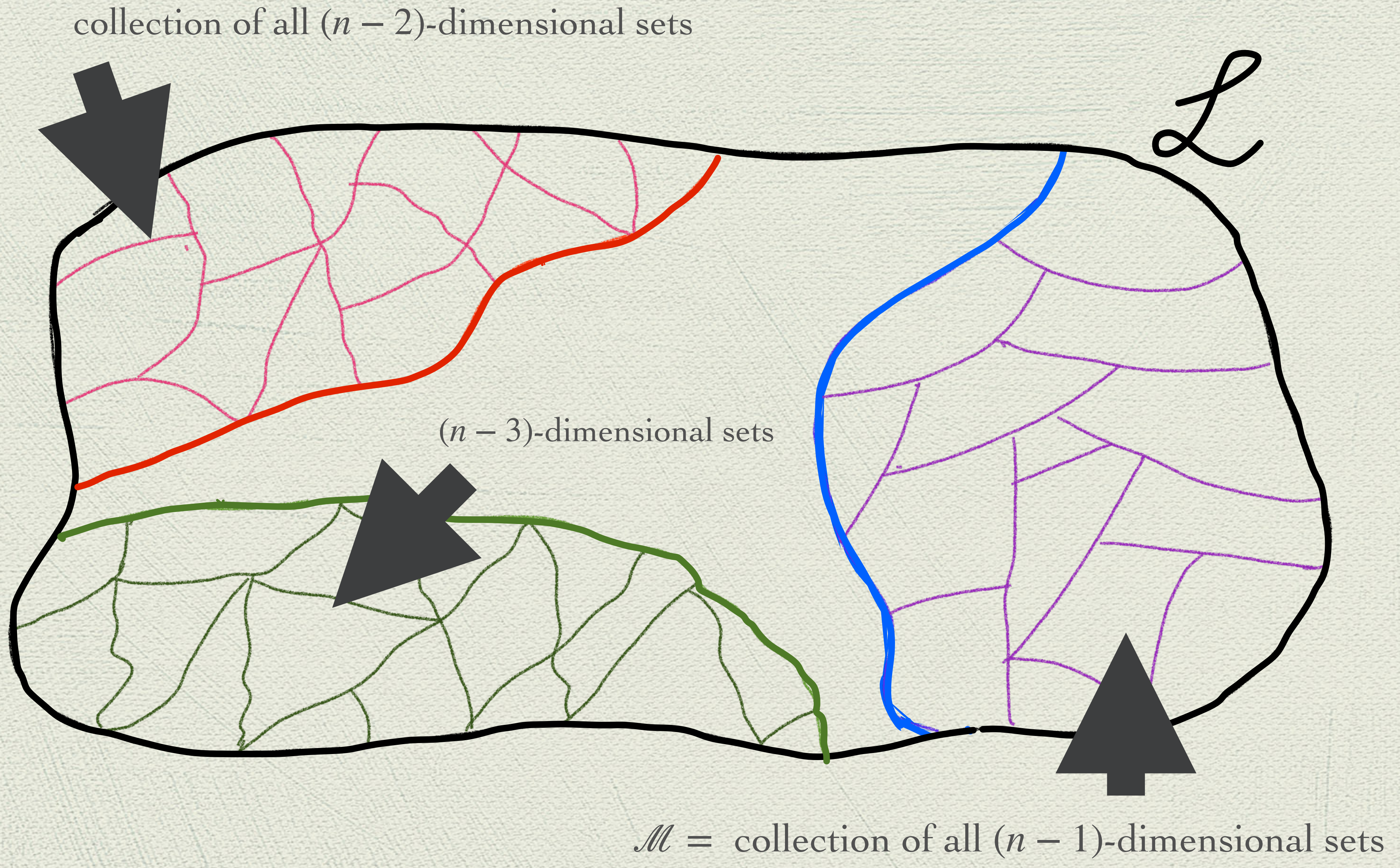
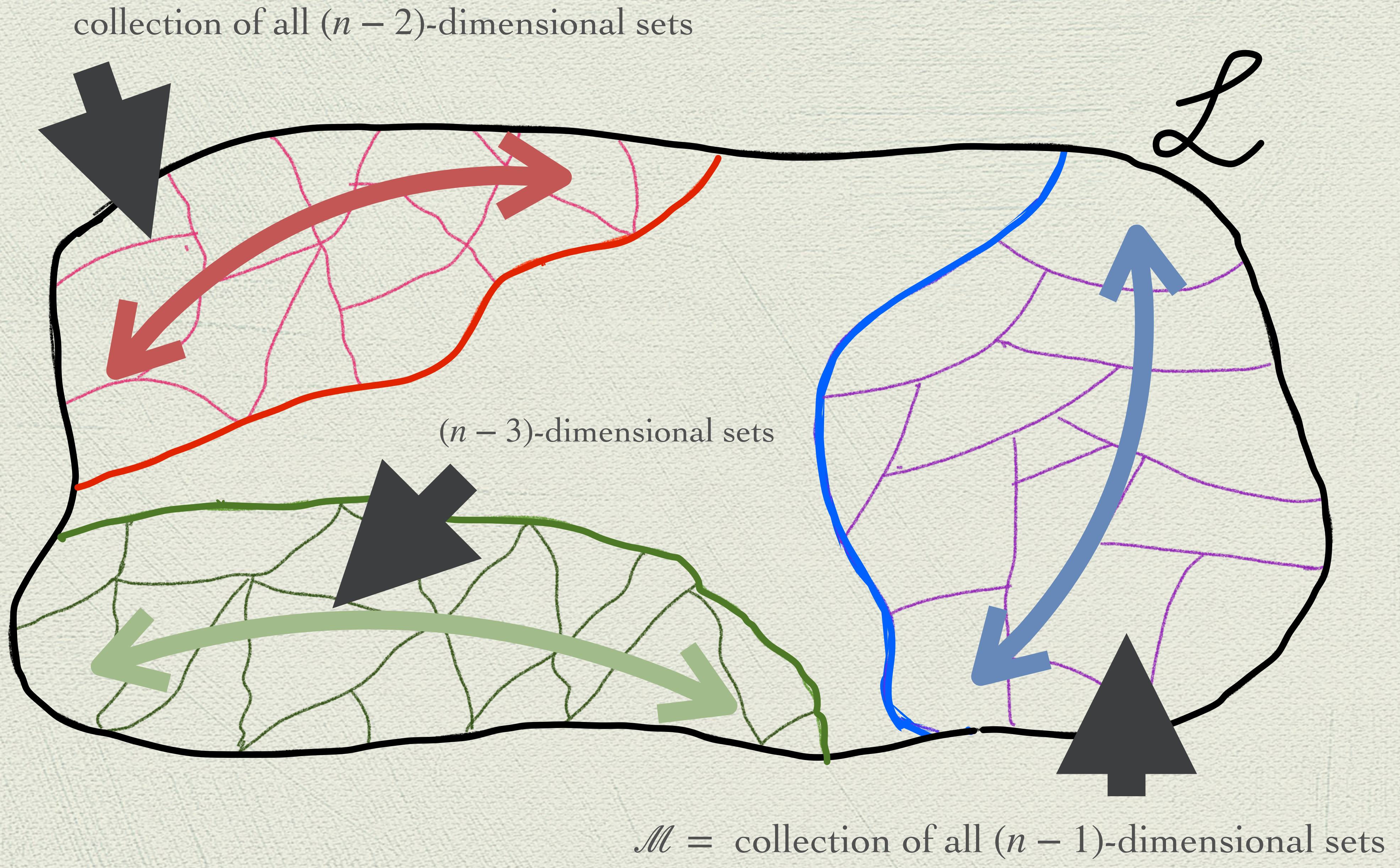
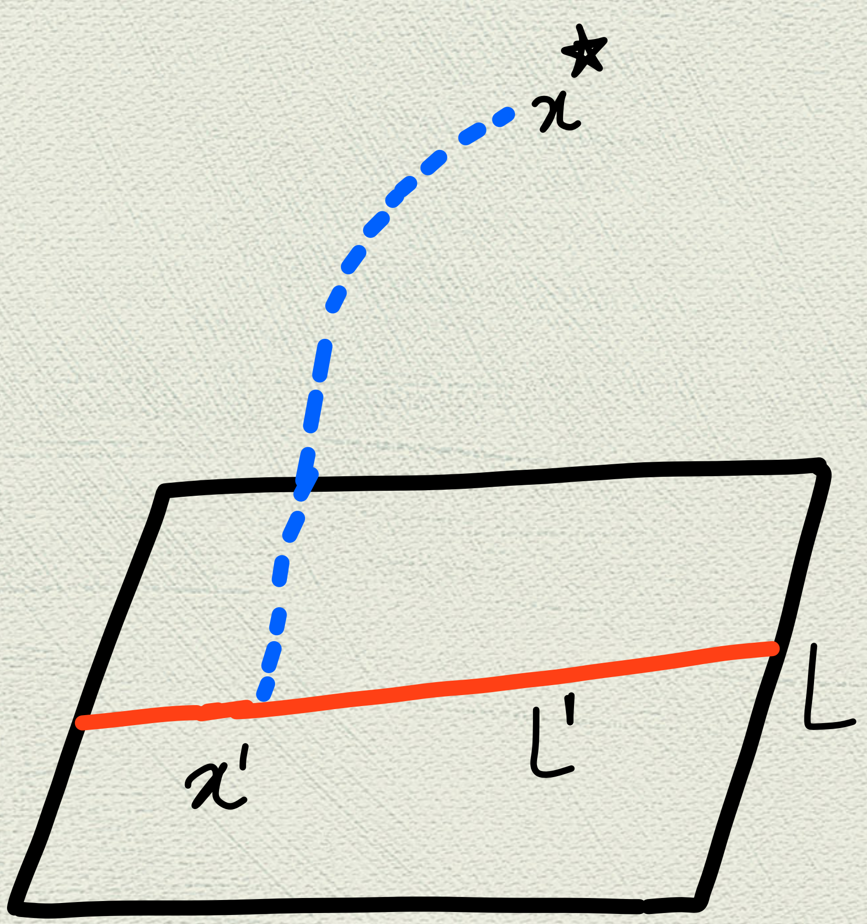


The axiom of continuity



$\Pi(\,\cdot\,|\,x^{\star})$ restricted to any fixed dimension is continuous

Axiom of subspace transitivity



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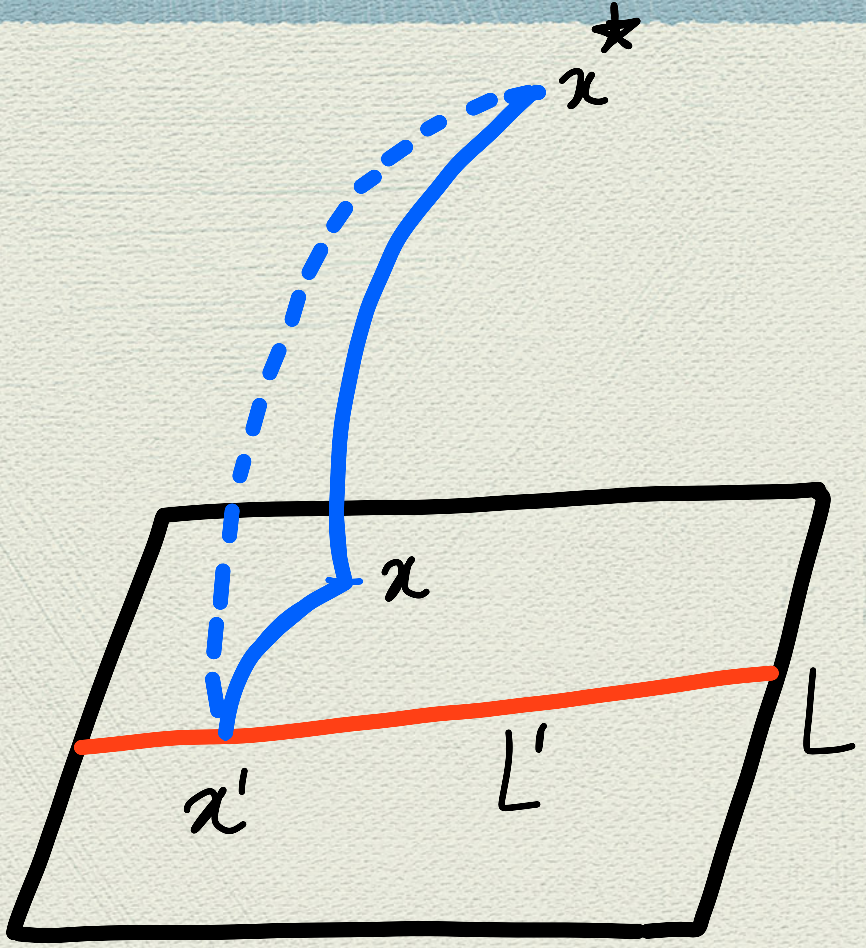

$$L' \subset L \implies \Pi(L' \,|\, x^\star) = \Pi(\Pi(L \,|\, x^\star) \,|\, x^\star)$$

A Key Implication of Regularity

$$\mathscr{L}^0(x^\star) = \left\{ L \in \mathscr{L} : \Pi(L \,|\, x^\star) = x^\star \right\}, \quad x^\star \in \mathbb{R}^n$$

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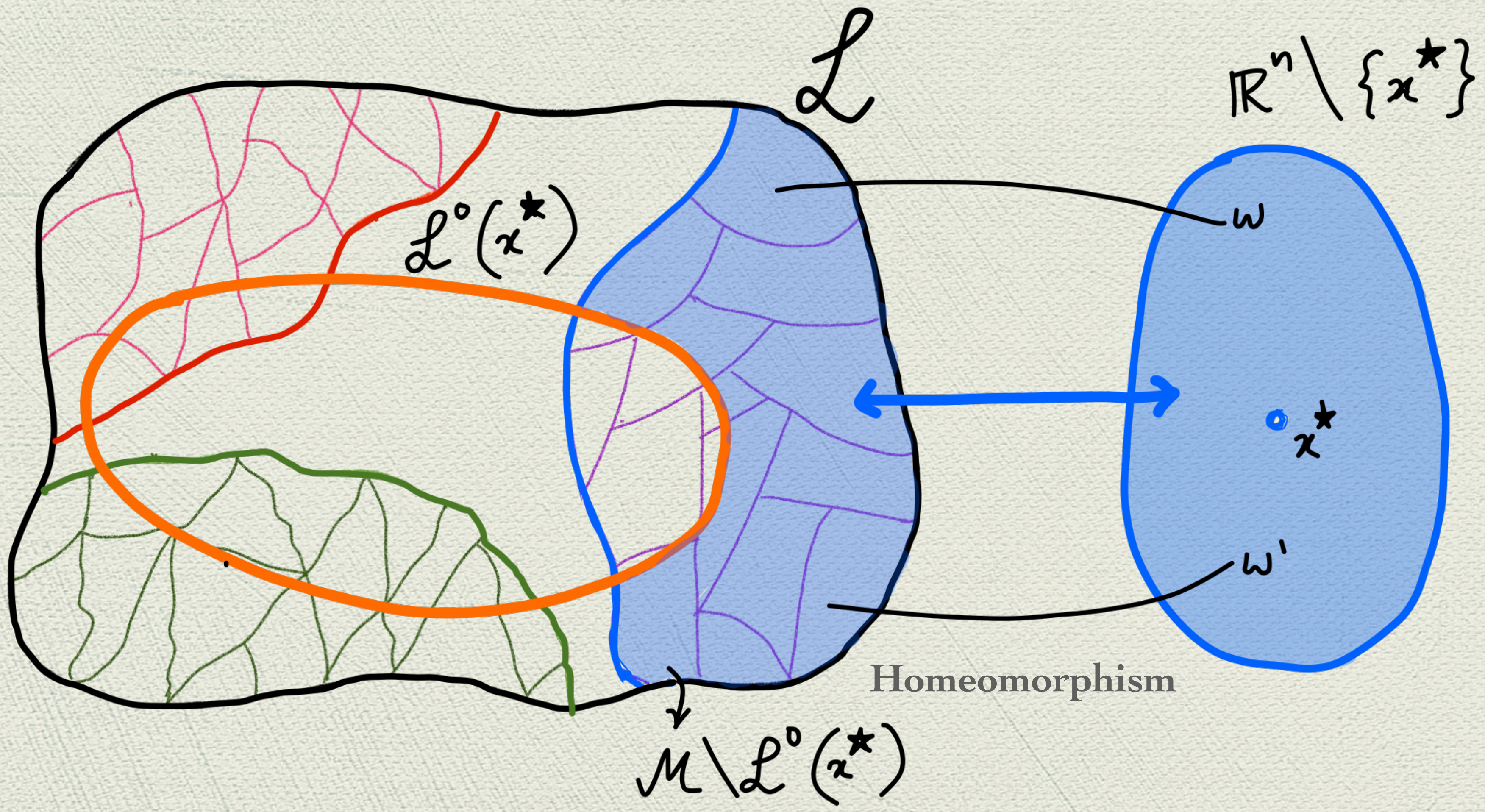
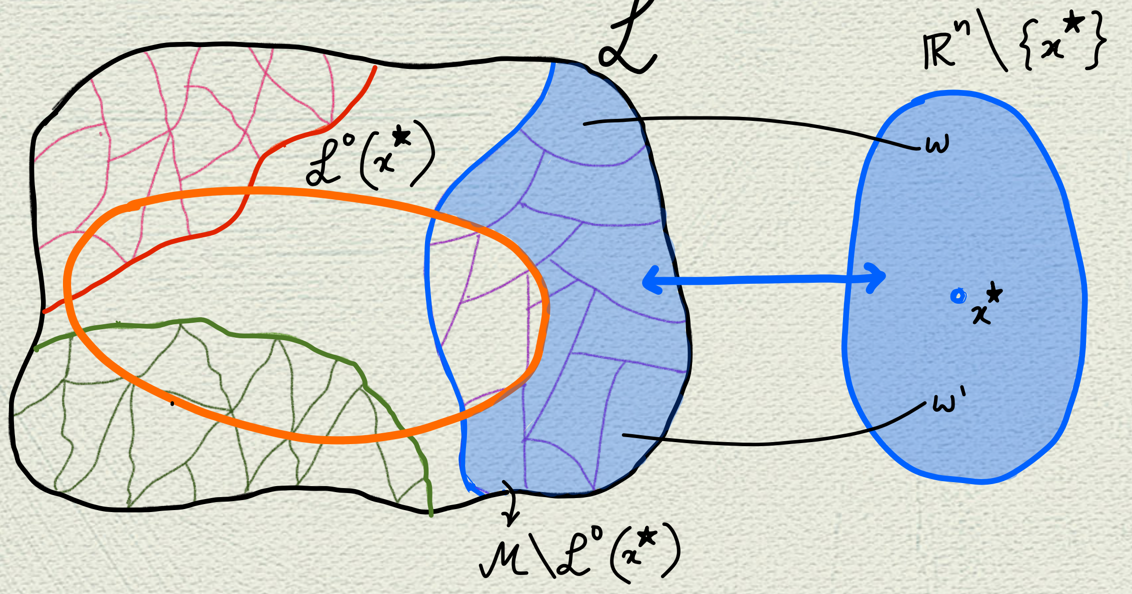


A Key Implication of Regularity

$\Pi(\cdot \mid x^\star)\colon \mathscr{M} \backslash \mathscr{L}^0(x^\star) \to \mathbb{R}^n \backslash \{x^\star\}$ is a homeomorphism

This means that for every $w \neq x^\star$,
there exists a unique set L of dimension $(n-1)$
such that $\Pi(L \mid x^\star) = w$. Denote this L as $L(w \mid x^\star)$



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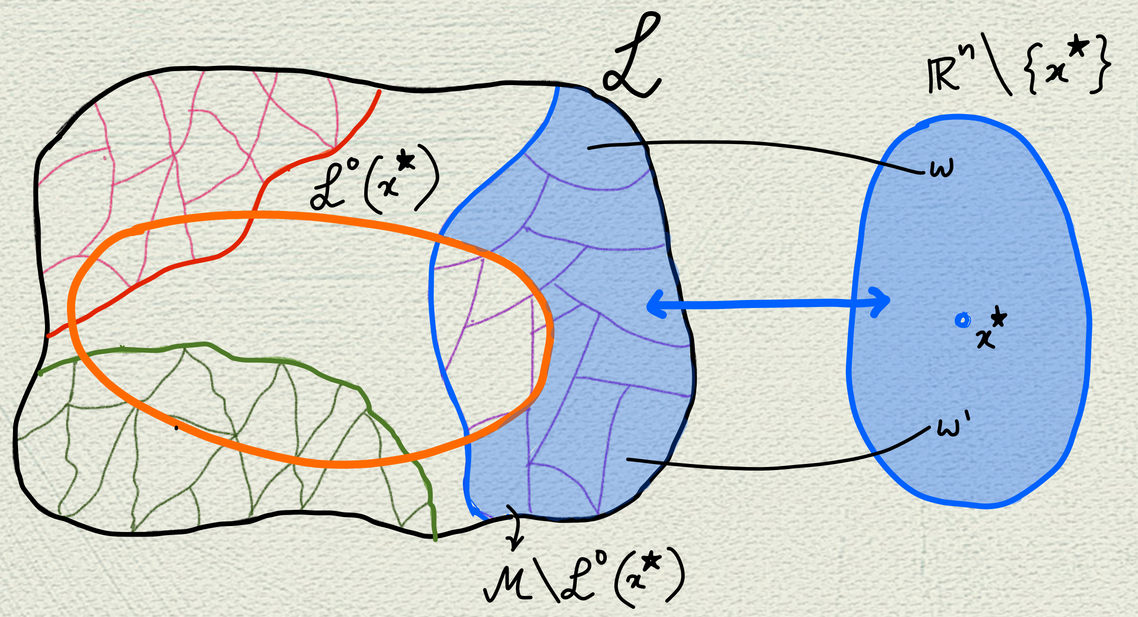

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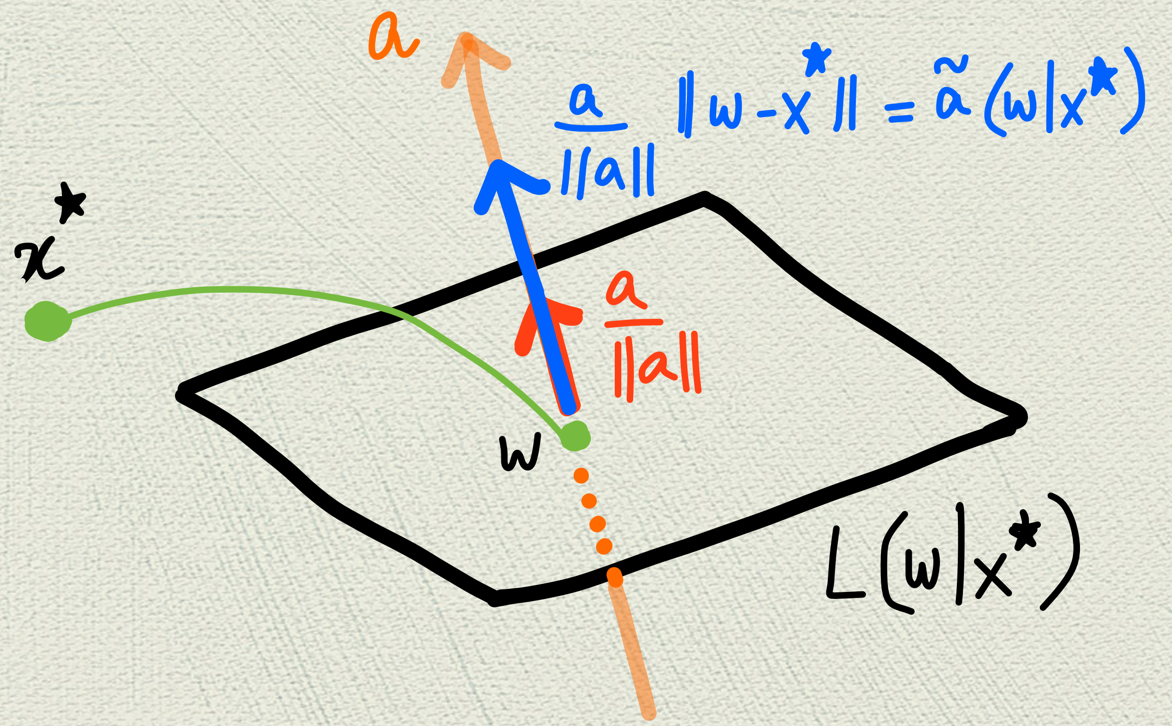

$$
\begin{aligned}
L(w|x^\star) &= \{y \in \mathbb{R}^n : a^T y = b\} \\
&= \{y \in \mathbb{R}^n : a^T y = a^T w\} \\
&= \{y \in \mathbb{R}^n : \tfrac{a^T}{\|a\|}(y-w) = 0\} \\
&= \{y \in \mathbb{R}^n : \tilde{a}(w|x^\star)^T (y-w) = 0\}
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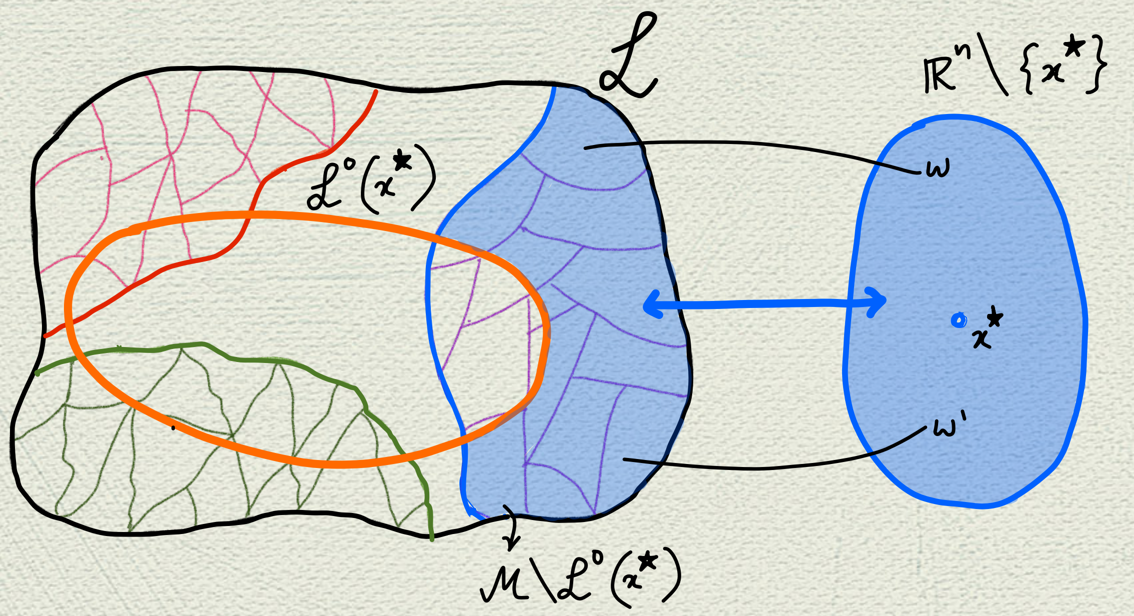

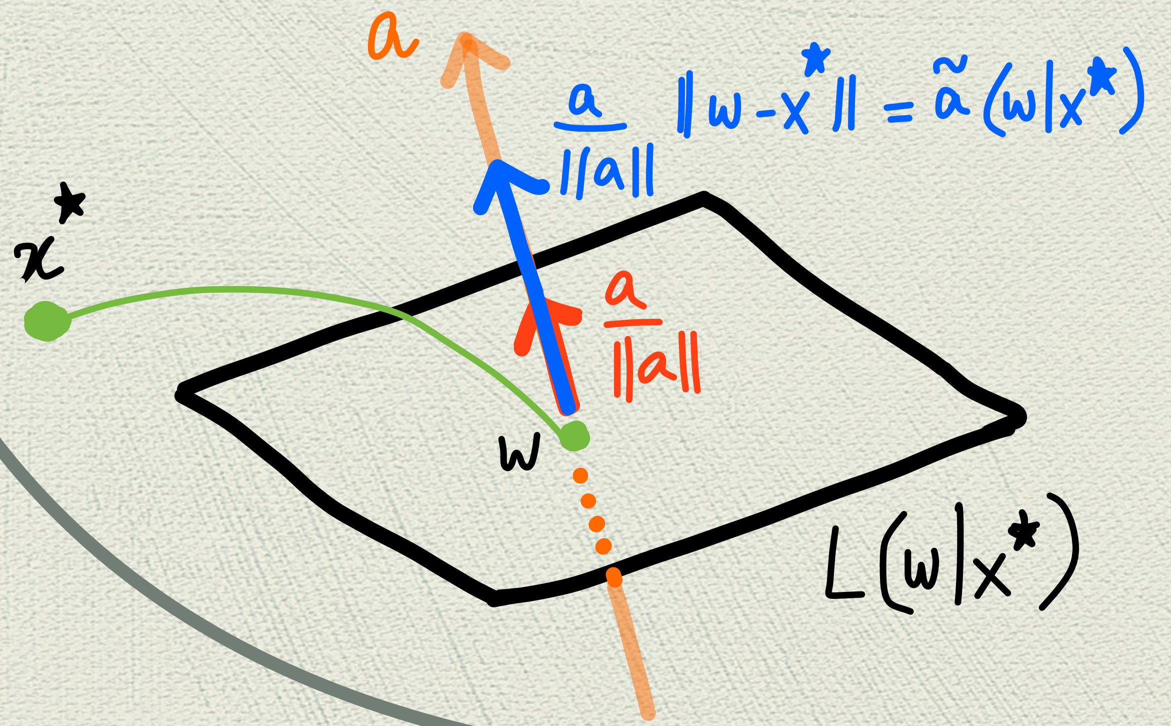

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$w \mapsto \tilde{a}(w\,|\,x^\star)$ is continuous

Regularity and vector fields

- Mere regularity helps us extract a **vector field**

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$$\tilde{a}(\,\cdot\,|x^\star) : \mathbb{R}^n \to \mathbb{R}^n$$

$$\tilde{a}(\,\cdot\,|x^\star) \text{ continuous}$$

$$\tilde{a}(x^\star|x^\star) = 0,$$

$$\tilde{a}(w|x^\star) \neq 0 \text{ for all } w \neq x^\star$$

Existence of a Non-Zero Scaling Function

$\tilde{a}(\cdot\,|\,x^\star) : \mathbb{R}^n \to \mathbb{R}^n$

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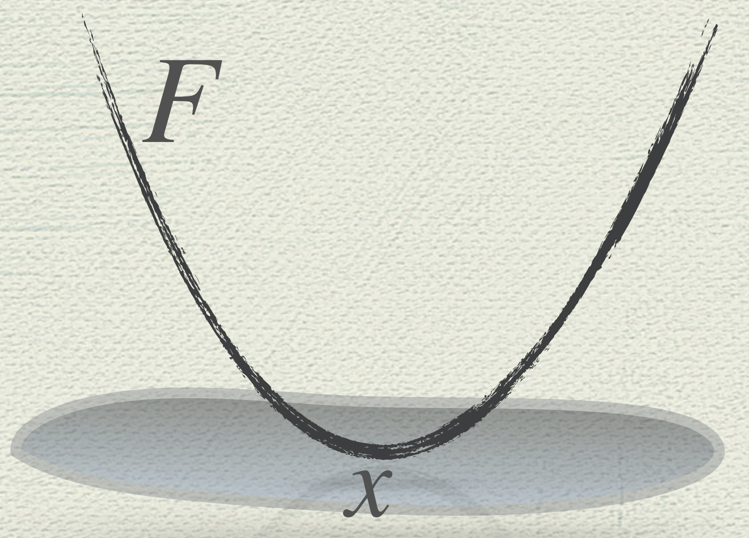

$$\min_{w \in L} F(w)$$

$$L = \{w \in \mathbb{R}^n : a^T w = b\}$$

$$G(w, \lambda) = F(w) - \lambda(a^T w - b)$$

$$\nabla_w G(w, \lambda) = 0 \iff \boxed{\nabla F(w) = \lambda a}$$

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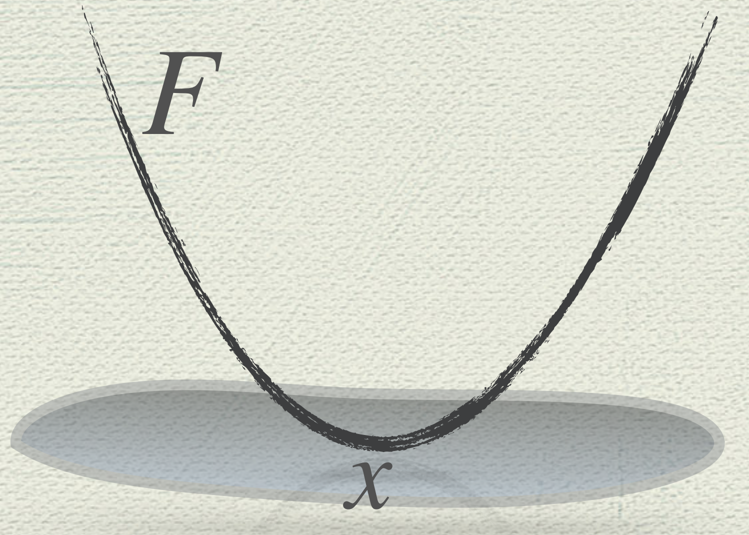

$$\min_{w \in L} F(w)$$

$$L = \{w \in \mathbb{R}^n : a^T w = b\}$$

Is there $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that $\lambda(w) \cdot \tilde{a}(w\,|\,x^\star) = \nabla F(w\,|\,x^\star)$?

$$G(w, \lambda) = F(w) - \lambda(a^T w - b)$$

$$\nabla_w G(w, \lambda) = 0 \iff \boxed{\nabla F(w) = \lambda a}$$

Conservative Vector Fields and Regularity

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$\boxed{\Pi(\,\cdot\,|\,x^{\star})\text{ regular}}$

Conservative Vector Fields and Regularity

$\boxed{\Pi(\,\cdot\,|\,x^\star)\text{ regular}} \longrightarrow$

Conservative Vector Fields and Regularity

$\boxed{\Pi(\,\cdot\,|\,x^\star) \text{ regular}} \longrightarrow \boxed{\tilde{a}(\,\cdot\,|\,x^\star) \text{ continuous vector field}}$

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$\downarrow$

Conservative Vector Fields and Regularity

$\boxed{\Pi(\,\cdot\,|\,x^\star)\text{ regular}} \longrightarrow \boxed{\tilde{a}(\,\cdot\,|\,x^\star)\text{ continuous vector field}}$

$\downarrow$

Does there exist a nonzero scaling $\lambda(\,\cdot\,|\,x^\star)$

$\lambda(w\,|\,x^\star) \neq 0$ for all $w \neq x^\star$,

$\lambda(\,\cdot\,|\,x^\star)$ continuous except at $x^\star$ such that

$\lambda(w\,|\,x^\star)\tilde{a}(w\,|\,x^\star) = \nabla F(w\,|\,x^\star)$ for some smooth F

Conservative Vector Fields and Regularity

$\boxed{\Pi(\,\cdot\,|\,x^\star) \text{ regular}}$ $\longrightarrow$ $\boxed{\tilde{a}(\,\cdot\,|\,x^\star) \text{ continuous vector field}}$

$\downarrow$

$\longleftarrow$

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$\longleftarrow$

Projection rule generated by F

Conservative Vector Fields and Regularity

$\Pi(\,\cdot\,|\,x^\star)$ regular → $\tilde{a}(\,\cdot\,|\,x^\star)$ continuous vector field

↓

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←

Projection rule generated by F

😀

Conservative Vector Fields and Regularity

$\Pi(\cdot\,|\,x^\star)$ regular → $\tilde{a}(\cdot\,|\,x^\star)$ continuous vector field

Open question

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→ Projection rule generated by F

An Open Question

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Given a continuous vector field, does there exist a scaling function such that the **scaled vector field is conservative**?

An Open Question

Given a continuous vector field, does there exist a scaling function such that the scaled vector field is conservative?

$a : \mathbb{R}^n \to \mathbb{R}^n$ continuous

$\lambda : \mathbb{R}^n \to \mathbb{R}$

$\lambda \cdot a = \nabla F$?

Instances Where the Answers are Known / Not Unknown

Given a continuous vector field, does there exist a scaling function such that the scaled vector field is conservative?

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$\lambda \cdot a = \nabla F$?

Instances Where the Answers are Known / Not Unknown

$$a(w) = \frac{w}{\|w\|} \qquad \lambda(w) = \|w\|$$

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$$\lambda \cdot a = \nabla F?$$

$$a(w) = \frac{w}{\|w\|} \qquad \lambda(w) = \|w\|$$

$$\lambda(w)\ a(w) = \nabla F(w) \qquad F(w) = \frac{1}{2}\|w\|^2$$

Instances Where the Answers are Known / Not Unknown

Given a continuous vector field, does there exist a scaling function such that the scaled vector field is conservative?

$$a : \mathbb{R}^n \to \mathbb{R}^n \text{ continuous}$$

$$\lambda : \mathbb{R}^n \to \mathbb{R}$$

$$\lambda \cdot a = \nabla F ?$$

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$$a(x, y, z) = \begin{pmatrix} y \\ 0 \\ 1 \end{pmatrix} \implies \text{there is no } \lambda \text{ such that } \lambda \cdot a = \nabla F$$

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$\lambda \cdot a = \nabla F$?

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Can this arise from a regular projection rule?

Summary

- Two approaches for solving systems of linear equations:
  - Function-minimisation approach
  - Axiomatic approach

Projection rules satisfies the axioms of	Projection rule is generated by
Regularity + Locality	$F(w \mid x^\star) = \sum_{i=1}^{n} f_i(w_i \mid x_i^\star)$, f_i continuously differentiable and strictly convex
Regularity+ Locality + Subspace Transitivity	$F(w \mid x^\star) =$ Bregman's divergence
Regularity+ Locality + Subspace Transitivity + Statistical	$F(w \mid x^\star) = \sum_{i=1}^{n} w_i \log \frac{w_i}{x_i^\star}$
Regularity + Locality + Subspace Transitivity + Location Invariance + Scale Invariance	$F(w \mid x^\star) = \sum_{i=1}^{n} (w_i - x_i^\star)^2$

- Regularity: a fundamental axiom
- Regularity has connections with conservative vector fields
- Given a continuous vector field that is not necessarily conservative, is there a continuous scaling that can make the product vector field conservative? Open!

Thank you!

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