



Probability and Stochastic Processes

Lecture 11: Measurable Function, Random Variable, Probability Law,
Cumulative Distribution Function (CDF)

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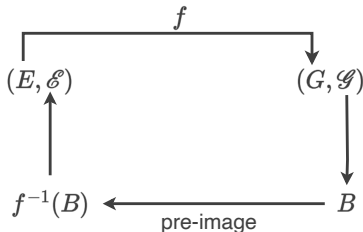
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Measurable Function

Definition (Measurable Function)

Let (E, \mathcal{E}) and (G, \mathcal{G}) be two measurable spaces. Consider a function $f : E \rightarrow G$. The function f is said to be **measurable** if

$$\forall B \in \mathcal{G}, \quad \underbrace{f^{-1}(B)}_{\text{pre-image of } B} = \{e \in E : f(e) \in B\} \in \mathcal{E}.$$



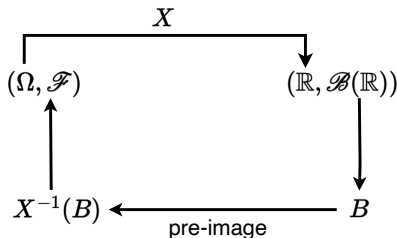
Random Variable

Definition (Random Variable)

Fix a measurable space (Ω, \mathcal{F}) .

A function $X : \Omega \rightarrow \mathbb{R}$ is called a **random variable** if it is measurable, i.e.,

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad \underbrace{X^{-1}(B)}_{\text{pre-image of } B} = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$



Random Variable

- A random variable is neither random nor a variable; it is a **deterministic function**
- A random variable assigns numerical values to outcomes
- The definition of a random variable is closely tied to the underlying σ -algebra \mathcal{F}
- If X is a random variable with respect to \mathcal{F} , it is said to be **\mathcal{F} -measurable**
- The definition of a random variable does not involve \mathbb{P}

Properties of a Random Variable

Proposition (Random Variable Properties)

Let (Ω, \mathcal{F}) be a measurable space, and let $X : \Omega \rightarrow \mathbb{R}$ be a random variable.

1. For any $B \subseteq \mathbb{R}$, $X^{-1}(B^c) = (X^{-1}(B))^c$.
2. For any $B_1 \subseteq \mathbb{R}, B_2 \subseteq \mathbb{R}, \dots$,

$$X^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} X^{-1}(B_n).$$

3. Let \mathcal{B}_1 denote the collection

$$\mathcal{B}_1 := \left\{ B \subseteq \mathbb{R} : X^{-1}(B) \in \mathcal{F} \right\}. \quad (1)$$

Then, \mathcal{B}_1 is a σ -algebra of subsets of \mathbb{R} . Furthermore, $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_1$.

Proof of Proposition – 1

$$\begin{aligned}\omega' \in X^{-1}(B^c) &\iff &\iff X(\omega') \notin B \\ &\iff \omega' \notin X^{-1}(B) \\ &\iff \omega' \in \Omega \setminus X^{-1}(B) \\ &\iff \omega' \in (X^{-1}(B))^c.\end{aligned}$$

Proof of Proposition – 2

$$\begin{aligned}\omega' \in X^{-1} \left(\bigcup_{n \in \mathbb{N}} B_n \right) &\iff \iff \exists n \in \mathbb{N} : X(\omega') \in B_n \\ &\iff \exists n \in \mathbb{N} : \omega' \in X^{-1}(B_n) \\ &\iff \omega' \in \bigcup_{n \in \mathbb{N}} X^{-1}(B_n).\end{aligned}$$

Proof of Proposition – 3

- To show that $\emptyset \in \mathcal{B}_1$, note that

$$X^{-1}(\emptyset) = \emptyset \in \mathcal{F}.$$

- Suppose $B \in \mathcal{B}_1$. That is, by definition, $X^{-1}(B) \in \mathcal{F}$.
Then, note that

$$X^{-1}(B^c) = (X^{-1}(B))^c \in \mathcal{F}.$$

This proves that $B^c \in \mathcal{B}_1$.

- For any $B_1, B_2, \dots \in \mathcal{B}_1$, we have

$$X^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} X^{-1}(B_n) \in \mathcal{F}.$$

This proves that $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}_1$.

Generating Classes for $\mathcal{B}(\mathbb{R})$

$\mathcal{B}(\mathbb{R})$

$$\mathcal{P}_1 = \left\{ (a, b) : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_2 = \left\{ [a, b] : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_3 = \left\{ [a, b) : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_4 = \left\{ (a, b] : a, b \in \mathbb{R}, a \leq b \right\}$$

$$\mathcal{P}_5 = \left\{ (-\infty, x) : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_6 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_7 = \left\{ (x, +\infty) : x \in \mathbb{R} \right\}$$

$$\mathcal{P}_8 = \left\{ [x, +\infty) : x \in \mathbb{R} \right\}$$

Equivalent Definitions of Random Variable

Fix a measurable space (Ω, \mathcal{F}) .

Theorem (Equivalent Definitions of Random Variable)

$X : \Omega \rightarrow \mathbb{R}$ is a random variable if and only if:

1. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_1$.
2. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_2$.
3. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_3$.
4. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_4$.
5. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_5$.
6. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_6$.
7. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_7$.
8. $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{P}_8$.

Proof of Theorem (Considering \mathcal{P}_6)

- Recall that

$$\mathcal{P}_6 = \left\{ (-\infty, x] : x \in \mathbb{R} \right\}.$$

- If X is a random variable, then by definition,

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathcal{F}.$$

Because $\mathcal{P}_6 \subseteq \mathcal{B}(\mathbb{R})$, it follows that

$$X \text{ random variable} \quad \implies \quad X^{-1}\left((-\infty, x]\right) \in \mathcal{F} \quad \forall x \in \mathbb{R}.$$

Proof of Theorem (Considering \mathcal{P}_6)

- Suppose now that $X^{-1}\left((-\infty, x]\right) \in \mathcal{F}$ for all $x \in \mathbb{R}$.

In other words,

$$X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{P}_6.$$

- This implies that

$$\mathcal{P}_6 \subseteq \mathcal{B}_1 \quad (\text{defined in (1)}).$$

- In turn, this implies that

$$\sigma(\mathcal{P}_6) \subseteq \sigma(\mathcal{B}_1), \quad \text{i.e.,} \quad \mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_1.$$

- This verifies that

$$X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{P}_6 \quad \implies \quad X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}(\mathbb{R}) \quad \implies \quad X \text{ random variable.}$$

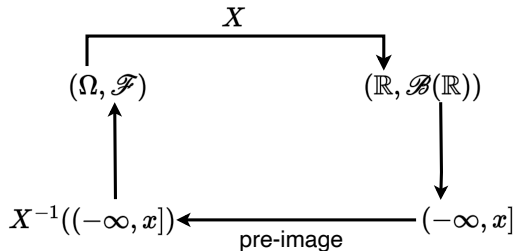
Random Variable Simplified

Definition (Random Variable)

Fix a measurable space (Ω, \mathcal{F}) .

A function $X : \Omega \rightarrow \mathbb{R}$ is called a random variable with respect to \mathcal{F} if and only if

$$\forall x \in \mathbb{R}, \quad \underbrace{X^{-1}((-\infty, x])}_{\text{pre-image of } (-\infty, x]} = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}.$$



Examples

- $\Omega = \{1, 2, \dots, 6\}, \quad \mathcal{F} = \left\{ \emptyset, \Omega \right\}, \quad X(\omega) = \omega$

Is X a random variable with respect to \mathcal{F} ?

- What functions X are random variables with respect to \mathcal{F} ?

Examples

- $\Omega = [0, 1]$, $\mathcal{F} = \left\{ \emptyset, \Omega, A, A^c \right\}$ for a fixed $A \subseteq \Omega$

What functions X are random variables with respect to \mathcal{F} ?

Examples

- $\Omega = \{1, 2, 3, 4, 5\}, \quad \mathcal{F} = \sigma \left(\left\{ \{1\}, \{2, 3\} \right\} \right)$
What functions X are random variables with respect to \mathcal{F} ?

Examples

- $\Omega = \mathbb{N}$, $\mathcal{F} = 2^\Omega$

What functions X are random variables with respect to \mathcal{F} ?

Examples

- Provide an example construction of (Ω, \mathcal{F}) and a function $X : \Omega \rightarrow \mathbb{R}$ that is NOT a random variable (with respect to \mathcal{F}).

Indicator Functions

Fix a sample space Ω .

Fix a subset $A \subseteq \Omega$.

Definition (Indicator Function)

The **indicator function** of set A is the function $\mathbf{1}_A : \Omega \rightarrow \mathbb{R}$ defined as

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

Exercise

Fix a measurable space (Ω, \mathcal{F}) . Show that

$$\mathbf{1}_A \text{ is a random variable} \iff A \in \mathcal{F}.$$