AI5030 / EE5817: Probability and Stochastic Processes

Homework 05

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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PDF, PMF, Joint Distributions

1. (a) Show that if $X \sim$ Poisson(λ) for some fixed $\lambda > 0$, then its PMF p_X satisfies the relation

$$p_X(k-1) \cdot p_X(k+1) \leq (p_X(k))^2 \qquad \forall\, k \in \mathbb{N}.$$

   (b) Give an example for a PMF p_X satisfying the relation

$$p_X(k-1) \cdot p_X(k+1) = (p_X(k))^2 \qquad \forall\, k \in \mathbb{N}.$$

Solution:

(a) Recall that if $X \sim$ Poisson(λ), then

$$p_X(k) = e^{-\lambda}\, \frac{\lambda^k}{k!}, \qquad k \in \{0, 1, 2, \ldots\}.$$

Notice that

$$\frac{p_X(k)}{p_X(k-1)} = \frac{\lambda}{k}, \qquad k \in \mathbb{N}.$$

Noting that $\frac{p_X(k)}{p_X(k-1)}$ is a strictly decreasing sequence in k, it follows that

$$\frac{p_X(k+1)}{p_X(k)} < \frac{p_X(k)}{p_X(k-1)} \qquad \forall k \in \mathbb{N}.$$

Rearranging terms, we arrive at the desired relation.

(b) Notice that the desired PMF must satisfy the relation

$$\frac{p_X(k+1)}{p_X(k)} = \frac{p_X(k)}{p_X(k-1)} \qquad \forall k \in \mathbb{N},$$

i.e., the ratio between consecutive terms of the PMF must be constant, say $\lambda > 0$. Then, we have

$$p_X(k) = \lambda^k\, p_X(0) \qquad \forall k \in \mathbb{N} \cup \{0\}.$$

Using the fact that the PMF values must sum to 1, we get

$$1 = \sum_{k \in \mathbb{N} \cup \{0\}} p_X(k) = p_X(0) \sum_{k \in \mathbb{N} \cup \{0\}} \lambda^k = \frac{p_X(k)}{1-\lambda},$$

from which we deduce that $p_X(0) = 1 - \lambda$, and therefore

$$p_X(k) = (1-\lambda)\, \lambda^k, \qquad k \in \mathbb{N}.$$

2. Suppose that X and Y are jointly discrete, integer-valued random variables, with the joint PMF

$$p_{X,Y}(m,n) = \frac{\lambda^n\, e^{-2\lambda}}{m!\,(n-m)!}, \qquad 0 \leq m \leq n < +\infty, \quad m, n \in \{0, 1, 2, \ldots\}.$$

   (a) Determine a countable set $E \subset \mathbb{R}^2$ such that $\mathbb{P}_{X,Y}(E) = 1$.

   (b) Using the result of part (a) above, determine $E_1, E_2 \subseteq \mathbb{R}$, defined as

$$E_1 := \{x \in \mathbb{R} :\ \exists y \in \mathbb{R} \text{ such that } (x,y) \in E\}, \qquad E_2 := \{y \in \mathbb{R} :\ \exists x \in \mathbb{R} \text{ such that } (x,y) \in E\}.$$

(c) Determine the marginal PMFs of X and Y.

(d) Determine the conditional PMF of X conditioned on the event $\{Y = n\}$ for some fixed $n \in \{0, 1, 2, \ldots\}$.

(e) Are X and Y independent? Justify.

Solution:

(a) Let $\mathbb{W} := \mathbb{N} \cup \{0\}$. Consider the set

$$E = \{(m, n) \in \mathbb{W} : m \leq n\}.$$

We then have

$$\begin{aligned}
\mathbb{P}_{X,Y}(E) = \sum_{(m,n)\in E} p_{X,Y}(m, n) &= \sum_{n\in\mathbb{W}} \sum_{\substack{m\in\mathbb{W}:\\ m\leq n}} \frac{\lambda^n\, e^{-2\lambda}}{m!\,(n-m)!} \\
&= \sum_{n\in\mathbb{W}} \frac{\lambda^n\, e^{-2\lambda}}{n!} \underbrace{\sum_{m=0}^{n} \frac{n!}{m!(n-m)!}}_{=\,2^n} \\
&= \sum_{n\in\mathbb{W}} \frac{(2\lambda)^n\, e^{-2\lambda}}{n!} = 1.
\end{aligned}$$

(b) For E as defined in part (a) above, we have $E_1 = E_2 = \mathbb{W}$.

(c) The marginal PMF of X may be obtained as

$$\begin{aligned}
p_X(m) = \sum_{n:\,(m,n)\in E} p_{X,Y}(m, n) &= \sum_{n=m}^{\infty} \frac{\lambda^n\, e^{-2\lambda}}{m!\,(n-m)!} \\
&= \sum_{\ell=0}^{\infty} \frac{\lambda^{\ell+m}\, e^{-2\lambda}}{m!\ \ell!} \qquad \text{(setting } \ell = n - m\text{)} \\
&= \frac{\lambda^m\, e^{-2\lambda}}{m!} \underbrace{\sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}}_{e^\lambda} \\
&= \frac{\lambda^m\, e^{-\lambda}}{m!}, \qquad m \in \mathbb{W}.
\end{aligned}$$

That is, $X \sim$ Poisson(λ). Along similar lines, the marginal PMF of Y may be obtained as

$$p_Y(n) = \sum_{m:\,(m,n)\in E} p_{X,Y}(m, n) = \sum_{m=0}^{n} \frac{\lambda^n\, e^{-2\lambda}}{m!\,(n-m)!} = \frac{(2\lambda)^n\, e^{-2\lambda}}{n!}, \qquad n \in \mathbb{W}.$$

That is, $Y \sim$ Poisson(2λ).

(d) For any fixed $n \in \mathbb{W}$, the conditional PMF of X, conditioned on $\{Y = n\}$, is given by

$$p_{X|Y=n}(m) = \frac{p_{X,Y}(m, n)}{p_Y(n)} = \frac{\dfrac{\lambda^n\, e^{-2\lambda}}{m!\,(n-m)!}}{\dfrac{(2\lambda)^n\, e^{-2\lambda}}{n!}} = \frac{n!}{m!\,(n-m)!} \left(\frac{1}{2}\right)^n, \qquad m \in \{0, \ldots, n\}.$$

Thus, it follows that conditioned on $\{Y = n\}$, the random variable X follows Binomial$(n, \frac{1}{2})$ distribution.

(e) Observe that

$$p_{X,Y}(0, 0) = e^{-2\lambda}, \qquad p_X(0) \cdot p_Y(0) = e^{-3\lambda},$$

and therefore $p_{X,Y}(1, 1) \neq p_X(1) \cdot p_Y(1)$. This shows that X, Y are NOT independent.

3. A total of n coins, each with probability of heads p, are tossed independently of one another. Each coin that lands up heads is tossed again. Determine the PMF of the number of heads that shows up after the second round of tossing.

Solution: Let N_1 denote the number of heads that appear after the first round of tossing, and let N_2 denote the number of heads that appear after the second round of tossing. Notice that conditioned on $\{N_1 = k\}$, the maximum value that N_2 can take is equal to k. For any fixed $k \in \{0, \ldots, n\}$, the conditional PMF of N_2, conditioned on $\{N_1 = k\}$, is given by

$$p_{N_2|N_1=k}(\ell) = \binom{k}{\ell} p^{\ell}(1-p)^{k-\ell}, \qquad \ell \in \{0, \ldots, k\}.$$

That is, conditioned on $\{N_1 = k\}$, the random variable N_2 follows Binomial(k, p) distribution.

Using the law of total probability, we may then express the unconditional PMF of N_2 as

$$p_{N_2}(\ell) = \sum_{k=\ell}^{n} p_{N_2|N_1=k}(\ell) \cdot p_{N_1}(k).$$

Noting that $N_1 \sim$ Binomial(n, p), we have

$$\begin{aligned}
p_{N_2}(\ell) &= \sum_{k=\ell}^{n} \binom{k}{\ell} p^{\ell}(1-p)^{k-\ell} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \binom{n}{\ell} p^{\ell}\,(1-p)^{n-\ell} \sum_{k=\ell}^{n} \frac{(n-\ell)!}{(n-k)!\,(k-\ell)!}\, p^k \\
&= \binom{n}{\ell} p^{\ell}\,(1-p)^{n-\ell} \sum_{m=0}^{n} \frac{(n-\ell)!}{m!\,(n-m-\ell)!}\, p^{\ell+m} \qquad \text{(setting } k-\ell = m\text{)} \\
&= \binom{n}{\ell} p^{2\ell}\,(1-p)^{n-\ell} \underbrace{\sum_{m=0}^{n} \binom{n-\ell}{m} p^m\, 1^{n-\ell-m}}_{=\,(1+p)^{n-\ell}} \\
&= \binom{n}{\ell} p^{2\ell}\,(1-p^2)^{n-\ell}, \qquad \ell \in \{0, \ldots, n\}.
\end{aligned}$$

That is, $N_2 \sim$ Binomial(n, p^2).

4. Suppose that X and Y are jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c\,x\,(y-x)e^{-y}, & 0 \le x \le y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the constant c.

   (b) Show that

$$f_{X|Y=y}(x) = \begin{cases} 6x(y-x)y^{-3}, & 0 \le x \le y, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{Y|X=x}(y) = \begin{cases} (y-x)e^{x-y}, & y \ge x, \\ 0, & \text{otherwise.} \end{cases}$$

Solution: We present the solution to each of the parts below.

   (a) To determine the constant c, we set the integral of the joint PDF to 1. Doing so, we obtain $c = 1$.

   (b) From the joint PDF expression, we first obtain the marginal PDFs of X and Y. For any $0 \le y < +\infty$, we note that

$$f_Y(y) = \int_0^y x(y-x)\,e^{-y}\,\mathrm{d}x = \frac{y^3\,e^{-y}}{6},$$

   from which it follows that

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 6x(y-x)y^{-3}, & x \in [0, y], \\ 0, & \text{otherwise.} \end{cases}$$

Along similar lines, for any $0 \leq x < +\infty$, we have

$$f_X(x) = \int_x^\infty x(y-x)e^{-y}\,\mathsf{d}y = xe^{-x},$$

from which it follows that

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} (y-x)\,e^{-(y-x)}, & y \in [x,\,+\infty), \\ 0, & \text{otherwise.} \end{cases}$$

5. Suppose that X and Y are jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cy, & -1 \leq x \leq 1,\ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the constant c.

(b) Are X and Y independent?

(c) Evaluate $\mathbb{P}(\{X \geq Y + 0.5\})$.

(d) Compute the conditional PDF of X, conditioned on the event $\{Y > 0.5\}$.
Using the above conditional PDF, evaluate $\mathbb{P}(\{X > 0.75\}|\{Y > 0.5\})$.

Solution: We present the solution to each part below.

(a) To determine the constant c, we integrate the joint PDF and set the integral to 1. Doing so, we get

$$1 = \int_{-1}^{1}\int_0^{|x|} cy\,\mathsf{d}y\,\mathsf{d}x = \int_{-1}^{1} c\,\frac{x^2}{2}\,\mathsf{d}x = \frac{c}{3},$$

from which it follows that $c = 3$.

(b) To determine if X is independent of Y or otherwise, we first compute the marginal PDFs of X and Y. For any $x \in [-1, 1]$, we have

$$f_X(x) = \int_0^{|x|} 3y\,\mathsf{d}y = \frac{3x^2}{2}.$$

Similarly, for any $y \in [0, 1]$, we have

$$f_Y(y) = \int_{-1}^{-y} 3y\,\mathsf{d}x + \int_y^1 3y\,\mathsf{d}x = 6y(1-y).$$

Clearly, $f_{X,Y}(1,1) = 3 \neq 0 = f_X(1)f_Y(1)$, thereby proving that X, Y are not independent.

(c) The desired probability is given by

$$\mathbb{P}(\{X \geq Y + 0.5\}) = \int_{0.5}^{1}\int_0^{x-0.5} 3y\,\mathsf{d}y\,\mathsf{d}x = \int_{0.5}^{1} \frac{3(x-0.5)^2}{2}\,\mathsf{d}x = \frac{1}{16}.$$

(d) We first compute the conditional CDF of X, conditioned on the event $A = \{Y > 0.5\}$. First, we note that

$$\mathbb{P}(A) = \int_{0.5}^{1} 6y(1-y)\,\mathsf{d}y = \frac{1}{2}.$$

Next, we note that

$$F_{X|A}(x) = \frac{\mathbb{P}(\{X \le x\} \cap A)}{\mathbb{P}(A)} = \begin{cases} 0, & x < -1, \\ \dfrac{\int\limits_{-1}^{-x} \int\limits_{0.5}^{|u|} 3v \, \mathrm{d}v \, \mathrm{d}u}{1/2}, & -1 \le x < -\frac{1}{2}, \\ \dfrac{\int\limits_{-1}^{-0.5} \int\limits_{0.5}^{|u|} 3v \, \mathrm{d}v \, \mathrm{d}u}{1/2}, & -\frac{1}{2} \le x < \frac{1}{2}, \\ \dfrac{\int\limits_{-1}^{-0.5} \int\limits_{0.5}^{|u|} 3v \, \mathrm{d}v \, \mathrm{d}u + \int\limits_{0.5}^{x} \int\limits_{0.5}^{|u|} 3v \, \mathrm{d}v \, \mathrm{d}u}{1/2}, & \frac{1}{2} \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Simplifying the integrals in the above expression, we get

$$F_X(x) = \begin{cases} 0, & x < -1, \\ 1 + x^3 - \frac{3}{4}(1+x), & -1 \le x < -\frac{1}{2}, \\ \frac{1}{2}, & -\frac{1}{2} \le x < \frac{1}{2}, \\ \frac{1}{2} + x^3 - \frac{3x}{4} + \frac{1}{4}, & \frac{1}{2} \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Differentiating the above CDF expression with respect to x, we get

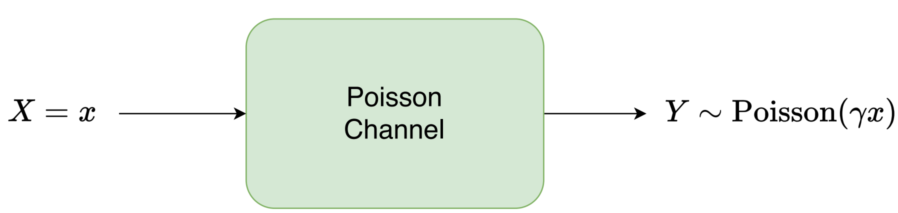
$$f_{X|A}(x) = \begin{cases} 3x^2 - \frac{3}{4}, & x \in [-1, -0.5] \cup [0.5, 1], \\ 0, & \text{otherwise.} \end{cases}$$

We then have

$$\mathbb{P}(\{X > 0.75\} | \{Y > 0.5\}) = \int_{0.75}^{1} f_{X|A}(x) \, \mathrm{d}x = \int_{0.75}^{1} \left(3x^2 - \frac{3}{4}\right) \mathrm{d}x = \frac{25}{64}.$$

6. **(The Poisson Channel)**

The Poisson channel was developed around 45 years ago as a model for an optical communication link. Suppose that $X \sim$ Exponential(1). Let Y be related to X as in the figure above.
Here, $\gamma > 0$ is a fixed constant known as channel signal-to-noise ratio (SNR). You may assume $\gamma = 1$.



(a) For a fixed $x > 0$, what is the conditional PMF of Y, conditioned on $\{X = x\}$?

(b) Using the law of total probability, determine the (unconditional) PMF of Y.

(c) Specify a countable set $E \subset \mathbb{R}$ such that $\mathbb{P}_Y(E) = 1$.

(d) For a fixed $k \in E$, determine the conditional PDF of X, conditioned on $\{Y = k\}$.

(e) Determine $\mathbb{P}(\{X \in B_1\} \cap \{Y \in B_2\})$, where $B_1 = (1, \infty)$ and $B_2 = \{1\}$.

Solution:

(a) For any fixed $x > 0$, the conditional PMF of Y, conditioned on $\{X = x\}$, is Poisson(x), i.e.,

$$p_{Y|X=x}(k) = e^{-x}\,\frac{x^k}{k!}, \qquad k \in \{0, 1, 2, \ldots\}.$$

(b) The unconditional PMF of Y is given by

$$p_Y(k) = \int_{-\infty}^{\infty} p_{Y|X=x}(k)\, f_X(x)\, \mathrm{d}x.$$

Noting that $f_X(x) = e^{-x}\, \mathbf{1}\{x \geq 0\}$, we get

$$p_Y(k) = \int_0^{\infty} e^{-2x}\,\frac{x^k}{k!}\,\mathrm{d}x = \frac{1}{k!}\int_0^{\infty} x^k\, e^{-2x}\,\mathrm{d}x.$$

Using the standard result

$$\int_0^{\infty} x^n\, e^{-\mu\, x}\,\mathrm{d}x = \frac{n!}{\mu^{n+1}}, \qquad n \in \mathbb{N},\ \mu > 0,$$

and plugging $n = k$ and $\mu = 2$, we get

$$p_Y(k) = \frac{1}{k!}\,\frac{k!}{2^{k+1}} = \frac{1}{2^{k+1}}, \qquad k \in \{0, 1, 2, \ldots\}.$$

(c) From the PMF expression derived in part (b) above, we may set $E = \{0, 1, 2, \ldots\}$, noting that

$$\mathbb{P}_Y(E) = \sum_{k \in E} p_Y(k) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

(d) For any fixed $k \in \{0, 1, 2, \ldots\}$, the conditional PDF of X, conditioned on $\{Y = k\}$, may be obtained as

$$\begin{aligned} f_{X|Y=k}(x) &= \frac{p_{Y|X=x}(k)\, f_X(x)}{p_Y(k)} \qquad &\text{(Bayes' Theorem)} \\ &= \frac{e^{-2x}\, x^k\, 2^{k+1}}{k!}, \qquad &x > 0. \end{aligned}$$

(e) The desired joint probability is given by

$$\begin{aligned} \mathbb{P}(\{X > 1\} \cap \{Y = 1\}) &= \int_1^{\infty} f_{X|Y=1}(x)\, p_Y(1)\,\mathrm{d}x \\ &= \int_1^{\infty} x\, e^{-2x}\,\mathrm{d}x \\ &= \frac{1}{4}. \end{aligned}$$