1. $$Y_n = \left|1 - \frac{\Theta}{\pi}\right|^n, \quad \Theta \sim \text{Unif}[0, 2\pi].$$

We notice that for all $\omega \in \Omega$ such that $\Theta(\omega) \in (0, 2\pi)$, we have

$$0 < \Theta(\omega) < 2\pi \iff -2\pi < -\Theta(\omega) < 0$$
$$\iff -2 < -\frac{\Theta(\omega)}{\pi} < 0$$
$$\iff -1 < 1 - \frac{\Theta(\omega)}{\pi} < 1$$
$$\iff \left|1 - \frac{\Theta(\omega)}{\pi}\right| < 1.$$

Thus, we have

$$\lim_{n\to\infty} Y_n(\omega) = \begin{cases} 0, & \omega \in \{\Theta \in (0, 2\pi)\}, \\ 1, & \omega \in \{\Theta = 0\} \cup \{\Theta = 2\pi\}. \end{cases}$$

It is then clear that

$$Y_n \xrightarrow{\text{Pointwise}} Y, \quad \text{where} \quad Y(\omega) = \begin{cases} 1, & \Theta(\omega) \in \{0, 2\pi\}, \\ 0, & \Theta(\omega) \in (0, 2\pi). \end{cases}$$

Also, noting that $\mathbb{P}(\Theta \in (0, 2\pi)) = 1$, we have

$$Y_n \xrightarrow{a.s.} 0 \Rightarrow Y_n \xrightarrow{P} 0, \quad Y_n \xrightarrow{d} 0.$$

Notice that $|Y_n| \le 1 \quad \forall\, n \in \mathbb{N}$.
Thus,

$$Y_n \xrightarrow{P} 0, \quad |Y_n| \le 1 \;\forall\, n \in \mathbb{N} \implies Y_n \xrightarrow{m.s.} 0.$$

using p. to m.s. reverse implication via dominance

2(a): $X_1, X_2, \ldots \overset{iid}{\sim} \text{Ber}(0.5)$.

Notice that $F_{X_1} = F_{X_2} = \ldots$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_{X_1}(x) = \begin{cases} 0, & x < 0, \\ 1/2, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Thus,

$$X_n \xrightarrow[d]{} X, \quad X \sim \text{Ber}(0.5).$$

Remark: The limit random variable is not unique. For instance, we could have written

$$X_n \xrightarrow{d} 1 - X, \quad X \sim \text{Ber}(0.5).$$

Only the CDF of the limit random variable matters, not the exact random variable itself, when speaking about convergence in distribution.

However, for other forms of convergence (a.s., m.s., p.), the limit random variable itself matters, not just its CDF.

2(b) Suppose there exists X s.t. $X_n \xrightarrow{P.} X$.
Then, by triangle inequality,

$$|X_{n+1} - X_n| \leq |X_n - X| + |X_{n+1} - X|.$$

Now, if $|X_{n+1} - X_n| > \varepsilon$, then we will have

$$|X_{n+1} - X_n| > \varepsilon \implies |X_n - X| + |X_{n+1} - X| > \varepsilon$$

$$\implies |X_n - X| > \varepsilon/2 \text{ OR } |X_{n+1} - X| > \frac{\varepsilon}{2} \text{ OR BOTH}$$

In set theoretic terms,

$$\{|X_{n+1} - X_n| > \varepsilon\} \subseteq \{|X_n - X| > \varepsilon/2\} \cup \{|X_{n+1} - X| > \varepsilon/2\}.$$

Using the union bound, we have for all $\varepsilon > 0$ that

$$\mathbb{P}(|X_{n+1} - X_n| > \varepsilon) \leq \mathbb{P}(|X_n - X| > \varepsilon/2) + \mathbb{P}(|X_{n+1} - X| > \varepsilon/2)$$

Applying $\lim_{n\to\infty}$ on both sides, and using the assumption that $X_n \xrightarrow{P.} X$, we get

$$\lim_{n\to\infty} \mathbb{P}(|X_{n+1} - X_n| > \varepsilon) \leq \underbrace{\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon/2)}_{\substack{=0 \\ \text{because} \\ X_n \xrightarrow{P} X}} + \underbrace{\lim_{n\to\infty} \mathbb{P}(|X_{n+1} - X| > \varepsilon/2)}_{=0 \text{ because } X_n \xrightarrow{P.} X}$$

$$= 0$$

$$\implies \boxed{\lim_{n\to\infty} \mathbb{P}(|X_{n+1} - X_n| > \varepsilon) = 0 \quad \forall\, \varepsilon > 0} \rightarrow (1)$$

On the other hand, for $\varepsilon = 1/2$, we note that

$$\forall n, \quad \mathbb{P}(|X_{n+1} - X_n| > 1/2) = \mathbb{P}(X_n = 0, X_{n+1} = 1) + \mathbb{P}(X_{n+1} = 0, X_n = 1)$$

because $X_n \perp\!\!\!\perp X_{n+1}$

$$= \mathbb{P}(X_n=0)\cdot\mathbb{P}(X_{n+1}=1) + \mathbb{P}(X_{n+1}=0)\cdot\mathbb{P}(X_n=1)$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{2}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \mathbb{P}\left(|X_{n+1}-X_n| > \tfrac{1}{2}\right) = \tfrac{1}{2} \neq 0.} \rightarrow (2)$$

(1) and (2) are contradictory.
Thus, the very assumption that $X_n \xrightarrow{P} X$ is incorrect!

3(a) If $X \sim \text{Poisson}(\lambda)$, then its MGF M_X is given by

$$M_X(t) = e^{\lambda(e^t-1)}, \quad t \in \mathbb{R}.$$ (this is a good exercise to do if you haven't done this before)

We then have

$$M_{S_n}(t) = E\left[e^{tS_n}\right] = E\left[e^{t\sum_{i=1}^n X_i}\right]$$

$$= E\left[\prod_{i=1}^n e^{tX_i}\right]$$

$$= \prod_{i=1}^n E\left[e^{tX_i}\right]$$ (because X_i's are independent)

$$= \left(E\left[e^{tX_1}\right]\right)^n$$ (because X_i's are identically distributed)

$$= e^{n\lambda(e^t-1)}, \quad t \in \mathbb{R}.$$

$$\Rightarrow S_n \sim \text{Poisson}(n\lambda).$$

Setting $\lambda = 1$ for the question, we get

$$S_n \sim \text{Poisson}(n) \quad \forall\, n \in \mathbb{N}.$$

3(b)

We have $E[S_n] = n$, $\text{Var}(S_n) = n$.

Let $D_n = S_n - n$, $n \in \mathbb{N}$.
Then,

$$E[D_n] = 0, \quad \text{Var}(D_n) = \text{Var}(S_n) = n.$$

Notice that

$$D_n = \sum_{i=1}^n X_i - E[X_i].$$

Noting that X_is are identically distributed, and Using the Taylor's approximation formula for the Characteristic function of $X_1 - E[X_1]$ upto upto order $k=2$, we get

$$C_{X_1}(s) = \sum_{m=0}^{2} E\left[\left(X_1 - E[X_1]\right)^m\right] \frac{(js)^m}{m!} + o(s^2)$$

$$= 1 - \frac{s^2}{2} + o(s^2), \quad s \in \mathbb{R}.$$

Consequently, we have

$$C_{\frac{D_n}{\sqrt{n}}}(s) = E\left[e^{j\left(\sum_{i=1}^{n} X_i - E[X_i]\right)\frac{s}{\sqrt{n}}}\right]$$

$$= \prod_{i=1}^{n} C_{X_i - E[X_i]}\left(\frac{s}{\sqrt{n}}\right)$$

$$= \left[C_{X_1 - E[X_1]}\left(\frac{s}{\sqrt{n}}\right)\right]^n$$

$$= \left[1 - \frac{s^2}{2n} + o\left(\frac{s^2}{n}\right)\right]^n.$$

Fix $s \in \mathbb{R}$. Let $a_n(s) = \left[1 - \frac{s^2}{2n} + o\left(\frac{s^2}{n}\right)\right]^n$

$$= \exp\left(n \log\left(1 - \frac{s^2}{2n} + o\left(\frac{s^2}{n}\right)\right)\right)$$

Notice that $1 - \frac{s^2}{2n} + o\left(\frac{s^2}{n}\right) > 0$ for all sufficiently large n.

Using the relation

$$\log x \leq x - 1 \quad \forall \; x > 0,$$

we get

$$a_n(s) \leq \exp\left(n\left(\frac{-s^2}{2n} + o\left(\frac{s^2}{n}\right)\right)\right)$$

$$= \exp\left(\frac{-s^2}{2} + n\, o\left(\frac{s^2}{n}\right)\right) \quad \text{for all sufficiently large values of } n.$$

Now, by definition of 'o' notation,

$$\lim_{n\to\infty} n\, o(s^2/n) = \lim_{n\to\infty} \frac{o(s^2/n)}{1/n} = 0.$$

Thus,

$$\lim_{n\to\infty} a_n(s) \leq \exp\left(\frac{-s^2}{2} + \lim_{n\to\infty} n\, o\left(\frac{s^2}{n}\right)\right)$$

$$= \exp\left(-s^2/2\right)$$

$$\Rightarrow \boxed{\lim_{n\to\infty} a_n(s) \le \exp\left(-s^2/2\right)} \longrightarrow (1)$$

On the other hand, using the relation

$$\log x \geq 1 - \frac{1}{x} \quad \forall \; x \geq 0,$$

$$\log x \geq 1 - \frac{1}{x}$$

$$\log 1-x \geq \frac{-x}{1-x}$$

we get that for all sufficiently large n,

$$a_n(s) = \exp\left(n \log\left[1 - \left(\frac{s^2}{2n} - o\left(\frac{s^2}{n}\right)\right)\right]\right)$$

$$\geq \exp\left(n \cdot - \left(\frac{s^2/2n - o(s^2/n)}{1 - s^2/2n + o(s^2/n)}\right)\right)$$

$$= \exp\left(\frac{-s^2/2 + n\,o(s^2/n)}{1 - s^2/2n + o(s^2/n)}\right)$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} a_n(s) \geq \exp\left(\frac{-s^2/2 + \lim\limits_{n\to\infty} n\,o(s^2/n)}{1 - \lim\limits_{n\to\infty} s^2/2n + \lim\limits_{n\to\infty} o(s^2/n)}\right)$$

$$= \exp\left(-s^2/2\right)$$

$$\Rightarrow \boxed{\lim_{n\to\infty} a_n(s) \geq \exp\left(-s^2/2\right)} \rightarrow (2)$$

From (1) and (2), we get

$$\lim_{n\to\infty} C_{\frac{D_n}{\sqrt{n}}}(s) = \lim_{n\to\infty} a_n(s) = e^{-s^2/2} \qquad \forall \; s \in \mathbb{R}$$

$$\Rightarrow \frac{D_n}{\sqrt{n}} \xrightarrow[d]{} N(0,1).$$

3(c)

Notice that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \mathbb{P}(S_n = k)$$

$$= \lim_{n\to\infty} \mathbb{P}(S_n \leq n)$$

$$= \lim_{n\to\infty} \mathbb{P}\left(S_n - n \le 0\right)$$

$$= \lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n}{\sqrt{n}} \le 0\right)$$

from 3(b) $\frac{S_n - n}{\sqrt{n}} \xrightarrow[d]{} N(0,1)$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$= 1/2.$$

4(a)

$$M_{S_n}(t) = \mathbb{E}\left[e^{t\sum_{i=1}^{n} X_i}\right]$$

$$= \left(\mathbb{E}\left[e^{tX_1}\right]\right)^n$$

$$= \left(1-p+pe^t\right)^n, \qquad t \in \mathbb{R}.$$

4(b) Because $X_i \in \{0,1\}$ $\forall\, i$, we have

$$S_n \in \{0, \dots, n\} \quad \forall\, n$$

$$\Rightarrow \frac{S_n}{n} \in \left\{0, \frac{1}{n}, \dots, 1\right\}$$

$$\Rightarrow \frac{S_n}{n} \in [0,1]$$

Therefore, we have

$$0 \le \frac{S_n}{n} \le 1$$

$$-1 < -p < 0 \qquad (\because 0 < p < 1)$$

$$\Rightarrow -1 < \frac{S_n}{n} - p < 1$$

$$\Rightarrow \left|\frac{S_n}{n} - p\right| < 1.$$

Thus,

$$\mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \varepsilon\right) = 0 \quad \forall\, \varepsilon > 1.$$

4(c). Noting that $\frac{S_n}{n} \in [0,1]$, we have $\mathbb{P}\left(\frac{S_n}{n} > p+\varepsilon\right) = 0$ if $p+\varepsilon \ge 1$.

Suppose $0 \le p+\varepsilon < 1$. Then, by Chernoff bound,

$$\mathbb{P}\left(\frac{S_n}{n} > p+\varepsilon\right) = \mathbb{P}\left(S_n > n(p+\varepsilon)\right)$$

$$\leq \inf_{t>0} \frac{\mathbb{E}\left[e^{tS_n}\right]}{e^{tn(p+\varepsilon)}}$$

$$= \inf_{t>0} \frac{\left(1-p+pe^t\right)^n}{e^{tn(p+\varepsilon)}}$$

$$= \exp\left\{-n\left(\sup_{t>0} \underbrace{(p+\varepsilon)t - \log\left(1-p+pe^t\right)}_{f(t)}\right)\right\}$$

The maximum of $f(t)$ occurs at t^* s.t.

$$p+\varepsilon = \frac{1}{1-p+pe^{t^*}} \cdot pe^{t^*}$$

$$\Rightarrow \frac{1-p+pe^{t^*}}{pe^{t^*}} = \frac{1}{p+\varepsilon}$$

$$\Rightarrow \frac{1-p}{pe^{t^*}} = \frac{1}{p+\varepsilon} - 1 = \frac{1-(p+\varepsilon)}{p+\varepsilon}$$

$$\Rightarrow e^{t^*} = \frac{(1-p)(p+\varepsilon)}{p\left(1-(p+\varepsilon)\right)}.$$

Furthermore,

$$\sup_{t>0} f(t) = f(t^*) = (p+\varepsilon)\log\frac{(1-p)(p+\varepsilon)}{p\left(1-(p+\varepsilon)\right)} - \log\left(1-p+\frac{(1-p)(p+\varepsilon)}{1-(p+\varepsilon)}\right)$$

$$= (p+\varepsilon)\log\frac{p+\varepsilon}{p} + \left(1-(p+\varepsilon)\right)\log\left(\frac{1-(p+\varepsilon)}{1-p}\right)$$

$$= D\left(p+\varepsilon \,\|\, p\right).$$

4(d) Because $\frac{S_n}{n} \in [0,1]$, we have $\mathbb{P}\left(\frac{S_n}{n} < p-\varepsilon\right) = 0$ if $p-\varepsilon \leq 0$.

For $0 < p-\varepsilon < 1$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{S_n}{n} - p < -\varepsilon\right) = \mathbb{P}\left(1-\frac{U_n}{n} - p < -\varepsilon\right)$$

$$= \mathbb{P}\left(\frac{U_n}{n} > 1-p+\varepsilon\right),$$

where $U_n = \sum_{i=1}^{n} 1 - X_i$

$$= n - S_n.$$

Noting that $U_n \sim \text{Bin}(n, 1-p)$, and applying the result of 4(c), we get

$$\mathbb{P}\left(\frac{U_n}{n} > 1-p+\varepsilon\right) \leq \exp\left(-n\, D\left(1-p+\varepsilon \,\|\, 1-p\right)\right).$$

4(e) Combining the results of 4(b), 4(c), and 4(d),

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \varepsilon\right) < +\infty \qquad \forall\ \varepsilon > 0$$

$$\Rightarrow \quad \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - p\right| > \varepsilon\right\} \text{ i.o.}\right) = 0 \qquad \forall\ \varepsilon > 0$$

Borel–Cantelli Lemma, part (i)

$$\Rightarrow \quad \frac{S_n}{n} \xrightarrow[a.s.]{} p.$$

5.

By Markov's inequality,

$$\mathbb{P}\left(|X_n - 2| > \varepsilon\right) = \mathbb{P}\left(|X_n - 2|^p > \varepsilon^p\right)$$

$$\leq \frac{\mathbb{E}\left[|X_n - 2|^p\right]}{\varepsilon^p} \qquad \forall\ \varepsilon > 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}\left(|X_n - 2| > \varepsilon\right) \leq \frac{1}{\varepsilon^p} \sum_{n=1}^{\infty} \mathbb{E}\left[|X_n - 2|^p\right]$$

$$< +\infty, \qquad \forall\ \varepsilon > 0$$

$$\Rightarrow \quad \mathbb{P}\left(\left\{|X_n - 2| > \varepsilon\right\} \text{ i.o.}\right) = 0 \qquad \forall\ \varepsilon > 0$$

Borel–Cantelli Lemma, part(i)

$$\Rightarrow \quad X_n \xrightarrow[a.s.]{} 2.$$

6(a) Note that

$$\mathbb{P}\left(\frac{Y_n}{\log n} \le a\right) = \mathbb{P}\left(Y_n \le a \log n\right)$$
$$= \mathbb{P}\left(\max\{X_1, \dots, X_n\} \le a \log n\right)$$
$$= \mathbb{P}\left(X_1 \le a \log n, \dots, X_n \le a \log n\right)$$
$$= \left[\mathbb{P}\left(X_1 \le a \log n\right)\right]^n \quad \text{(because of i.i.d. nature of } X_i\text{'s)}$$
$$= \left(1 - e^{-a \log n}\right)^n$$
$$= \exp\left(n \log\left(1 - e^{-a \log n}\right)\right)$$
$$= \exp\left(n \log\left(1 - \frac{1}{n^a}\right)\right).$$

Case 1 : $a < 1$:

Applying $\log 1-x \le -x$ with $x = \frac{1}{n^a}$, we get

$$\mathbb{P}\left(\frac{Y_n}{\log n} \le a\right) \le \exp\left(-\frac{n}{n^a}\right)$$
$$= \exp\left(-n^{1-a}\right).$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{\log n} \le a\right) \le \lim_{n\to\infty} e^{-n^{1-a}} = 0.$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{\log n} \le a\right) = 0 \quad \forall\ a < 0} \rightarrow (1)$$

Case 2 : $a > 1$:

Applying $\log x \ge 1 - \frac{1}{x}$ with $x = 1 - \frac{1}{n^a}$, we get

$$\mathbb{P}\left(\frac{Y_n}{\log n} \le a\right) \ge \exp\left(\frac{-n}{n^a - 1}\right)$$
$$\ge \exp\left(\frac{-n}{n^a/2}\right) \quad \forall \text{ large } n \quad \left(\text{because } n^a - 1 \ge \frac{n^a}{2}\ \forall \text{ large } n\right)$$
$$= \exp\left(\frac{-2}{n^{a-1}}\right)$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{\log n} \le a\right) \ge \lim_{n\to\infty} e^{-2/n^{a-1}}$$
$$= 1$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \mathbb{P}\left(\frac{Y_n}{\log n} \le a\right) = 1 \quad \forall\, a>1} \rightarrow (2)$$

From (1) and (2), we have

$$\frac{Y_n}{\log n} \xrightarrow{d} 1. \Rightarrow \frac{Y_n}{\log n} \xrightarrow{p.} 1.$$

6(b)

We know from 6(a) that

$$\frac{Y_n}{\log n} \xrightarrow[p.]{} 1 \iff \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{Y_n}{\log n} - 1\right| > \varepsilon\right) = 0 \quad \forall\, \varepsilon>0.$$

Now, for every $\varepsilon>0$, we have

$$\mathbb{P}\left(\left|\frac{Y_n}{\log_2 n} - \log 2\right| > \varepsilon\right) = \mathbb{P}\left(\left|\frac{Y_n}{\log n} \cdot \log 2 - \log 2\right| > \varepsilon\right)$$

$$= \mathbb{P}\left(\left|\frac{Y_n}{\log n} - 1\right| > \frac{\varepsilon}{\log 2}\right)$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{Y_n}{\log_2 n} - \log 2\right| > \varepsilon\right) = \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{Y_n}{\log n} - 1\right| > \frac{\varepsilon}{\log 2}\right)$$

$$= 0. \quad \left(\text{because } \frac{Y_n}{\log n} \xrightarrow{p.} 1\right)$$

Because the above relation holds for every $\varepsilon>0$, we get

$$\frac{Y_n}{\log_2 n} \xrightarrow{p.} \log 2.$$

6(c) We have

$$(1-x_k)^{n_k} = \exp\left(n_k \log(1-x_k)\right)$$

$$\ge \exp\left(-\frac{n_k x_k}{1-x_k}\right) \quad \left(\text{using } \log 1-x \ge \frac{-x}{1-x} \ \forall\, x\ge 0\right)$$

$$\ge \exp\left(-2 n_k x_k\right). \quad \forall \text{ sufficiently large } k$$

(because $x_k \to 0$, we have $x_k < \frac{1}{2}$ $\forall$ k large)

6(d) For any $\varepsilon>0$,

$$\mathbb{P}\left(\frac{Y_{n_k}}{\log_2 n_k} > (\log 2 + \varepsilon)\right) = \mathbb{P}\left(Y_{n_k} > k\left(\varepsilon + \log 2\right)\right) \quad \left(\text{because } \log_2 n_k = k\right)$$

$$= 1 - \mathbb{P}\left(Y_{n_k} \leq k(\varepsilon + \log 2)\right)$$

$$= 1 - \left[\mathbb{P}\left(X_1 \leq n_k(\varepsilon + \log 2)\right)\right]^k$$

$$= 1 - (1 - x_k)^{n_k}$$

$$\leq 1 - \exp\left(-2 n_k x_k\right) \quad \forall k \text{ sufficiently large}$$

$$\leq 2 n_k x_k \qquad \left(\text{using } 1 - e^{-x} \leq x \;\; \forall x \geq 0\right)$$

$$= 2 \cdot 2^k \cdot e^{-(\varepsilon + \log 2)k}$$

$$= 2 e^{-\varepsilon k}.$$

$$\Rightarrow \mathbb{P}\left(\frac{Y_{n_k}}{\log_2 n_k} > \varepsilon + \log 2\right) \leq 2 e^{-\varepsilon k} \quad \forall \; k \text{ sufficiently large}$$

$$\Rightarrow \sum_{k=1}^{\infty} \mathbb{P}\left(\frac{Y_{n_k}}{\log_2 n_k} > \varepsilon + \log 2\right) < +\infty \quad \forall \; \varepsilon > 0$$

$$\underset{\substack{\text{Borel-Cantelli} \\ \text{Lemma, part (i)}}}{\Rightarrow} \quad \mathbb{P}\left(\left\{\frac{Y_{n_k}}{\log_2 n_k} > \varepsilon + \log 2\right\} \text{ i.o.}\right) = 0 \quad \forall \; \varepsilon > 0$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{\frac{Y_{n_k}}{\log_2 n_k} \leq \varepsilon + \log 2\right\}\right) = 1 \quad \forall \; \varepsilon > 0$$

using "rationals are dense in reals"

$$\Rightarrow \mathbb{P}\Bigg(\underbrace{\bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{\frac{Y_{n_k}}{\log_2 n_k} \leq q + \log 2\right\}}_{=: A_q}\Bigg) = 1 \quad \forall \; q \in \mathbb{Q}_+$$

$$\Rightarrow \mathbb{P}\left(\bigcap_{q \in \mathbb{Q}_+} \left\{\limsup_{k \to \infty} \frac{Y_{n_k}}{\log_2 n_k} \leq \log 2 + q\right\}\right) = 1$$

(because if $\mathbb{P}(A_q) = 1$ $\forall q \in \mathbb{Q}_+$, then $\mathbb{P}\left(\bigcap_q A_q\right) = 1$)

$$\Rightarrow \mathbb{P}\left(\limsup_{k \to \infty} \frac{Y_{n_k}}{\log_2 n_k} \leq \log 2\right) = 1.$$

6(e) For all $k \in \mathbb{N}$, we have

$$(1-y_k)^{n_k} = \exp\left(n_k \log(1-y_k)\right)$$

$$\leq \exp\left(-n_k y_k\right) \quad \left(\text{using } \log x \leq x-1 \ \forall\, x \geq 0 \Rightarrow \log 1-x \leq (1-x)-1 = -x\right)$$

$$= \exp\left(-e^{\varepsilon k}\right)$$

$$\leq \exp(-\varepsilon k). \quad \left(\text{using } e^x > x \ \forall\, x \geq 0\right)$$

Then, we note that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{Y_{n_k}}{\log_2 n_k} \leq \log 2 - \varepsilon\right) = \sum_{n=1}^{\infty} \mathbb{P}\left(Y_{n_k} \leq k(\log 2 - \varepsilon)\right)$$

$$= \sum_{n=1}^{\infty} (1-y_k)^{n_k}$$

$$\leq \sum_{n=1}^{\infty} e^{-\varepsilon k} < +\infty.$$

$\Rightarrow$ Borel-Cantelli Lemma, part (i)

$$\mathbb{P}\left(\left\{\frac{Y_{n_k}}{\log_2 n_k} \leq \log 2 - \varepsilon\right\} \text{ i.o.}\right) = 0 \quad \forall\, \varepsilon > 0$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{\frac{Y_{n_k}}{\log_2 n_k} > \log 2 - \varepsilon\right\}\right) = 1 \quad \forall\, \varepsilon > 0$$

$\Rightarrow$ using "rationals are dense in reals"

$$\mathbb{P}\left(\bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{\frac{Y_{n_k}}{\log_2 n_k} > \log 2 - q\right\}\right) = 1 \quad \forall\, q \in \mathbb{Q}_+$$

$$\Rightarrow \mathbb{P}\left(\bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{\frac{Y_{n_k}}{\log_2 n_k} > \log 2 - q\right\}\right) = 1$$

$$\Rightarrow \mathbb{P}\left(\liminf_{k \to \infty} \frac{Y_{n_k}}{\log_2 n_k} \geq \log 2\right) = 1.$$
