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Stochastic Processes

Lecture 01: Supremum and infimum of a set of real numbers and sequences of random variables, limit supremum, limit infimum, and limit of sequences of real numbers and random variables

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Supremum

Let $A \subseteq \mathbb{R}$ be a subset of real numbers

- The **supremum** of the set A is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is an **upper bound** for the set A, i.e.,

$$\forall\, y \in A, \qquad y \leq x.$$

  - For any arbitrary choice of $\varepsilon > 0$, the number $x - \varepsilon$ is not an upper bound for A (x is the **least** among all upper bounds for the set A)
    Mathematically,

$$\forall \varepsilon > 0, \quad \exists\, y_\varepsilon \in A \quad \text{such that} \quad y_\varepsilon > x - \varepsilon.$$

  - Notation: $x = \sup A$

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers.

- The **supremum** of the sequence $\{x_n\}_{n \in \mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is an **upper bound** for the sequence, i.e., $x_n \leq x$ for all $n \in \mathbb{N}$
  - x is the **least** among all upper bounds for the sequence, i.e.,

$$\forall \varepsilon > 0, \quad \exists\, N_\varepsilon \in \mathbb{N} \quad \text{such that} \quad x_{N_\varepsilon} > x - \varepsilon.$$

  - Notation: $x = \sup_{n \geq 1}\ x_n$

Tidbits About Supremum

- Example: suppose $A = (-2, 3)$, then $\sup A = 3$
- Supremum of a set **need not be** an element of the set
  If supremum of a set belongs to the set, it is called the **maximum**
- By convention, if $A = \emptyset$, then $\sup A = -\infty$
- In the definition of supremum,

$$\text{for every choice of } \varepsilon > 0 \qquad \Longleftrightarrow \qquad \text{for every choice of } \varepsilon \in \mathbb{Q},\ \varepsilon > 0$$

  This holds true because **rational numbers are dense in the set of real numbers**

Supremum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a sequence of random variables w.r.t. $\mathscr{F}$

Definition (Supremum of Sequence of Random Variables)

The **supremum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \sup_{n\geq 1} X_n(\omega), \qquad \omega \in \Omega.$$

- The supremum of a sequence of random variables is a random variable
- Indeed, for any $x \in \mathbb{R} \cup \{\pm\infty\}$,

$$\{X \leq x\} = \bigcap_{n\in\mathbb{N}} \{X_n \leq x\}.$$

Infimum

Let $A \subseteq \mathbb{R}$ be a subset of real numbers

- The **infimum** of the set A is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is a **lower bound** for the set A, i.e.,
    $$\forall\, y \in A, \qquad y \geq x.$$
  - For any arbitrary choice of $\varepsilon > 0$, the number $x + \varepsilon$ is not a lower bound for A (x is the **greatest** among all lower bounds for the set A)
    Mathematically,
    $$\forall \varepsilon > 0, \quad \exists\, y_\varepsilon \in A \quad \text{such that} \quad y_\varepsilon < x + \varepsilon.$$
  - Notation: $x = \inf A$

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers.

- The **infimum** of the sequence $\{x_n\}_{n \in \mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that
  - x is a **lower bound** for the sequence, i.e., $x_n \geq x$ for all $n \in \mathbb{N}$, and
  - x is the **greatest** among all lower bounds for the sequence, i.e.,
    $$\forall \varepsilon > 0, \quad \exists\, N_\varepsilon \in \mathbb{N} \quad \text{such that} \quad x_{N_\varepsilon} < x + \varepsilon.$$
  - Notation: $x = \inf_{n \geq 1}\ x_n$

Tidbits About Infimum

- Example: suppose $A = (-2, 3)$, then $\inf A = -2$
- Infimum of a set **need not be** an element of the set
  If infimum of a set belongs to the set, then it is called the **minimum**
- By convention, if $A = \emptyset$, then $\inf A = +\infty$
- For any non-empty set A,
$$\inf A \leq \sup A.$$
- In the definition of infimum,
$$\text{for every choice of } \varepsilon > 0 \quad \Longleftrightarrow \quad \text{for every choice of } \varepsilon \in \mathbb{Q},\ \varepsilon > 0$$
  This holds true because **rational numbers are dense in the set of real numbers**

Infimum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a sequence of random variables w.r.t. $\mathscr{F}$

Definition (Infimum of a Sequence of Random Variables)

The **infimum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \inf_{n\geq 1} X_n(\omega), \qquad \omega \in \Omega.$$

- The infimum of a sequence of random variables is a random variable
- Indeed, for any $x \in \mathbb{R} \cup \{\pm\infty\}$,

$$\{X \geq x\} = \bigcap_{n\in\mathbb{N}} \{X_n \geq x\}.$$

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Limit Supremum, Limit Infimum, Limit

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Limit Supremum of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit Supremum of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit Supremum of a Sequence of Real Numbers

The **limit supremum** of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$x = \inf_{n\geq 1} \sup_{k\geq n} x_k; \qquad \textbf{Notation:} \quad x = \limsup_{n\to\infty} x_n.$$

- For each $n \in \mathbb{N}$, let $\quad y_n = \sup_{k\geq n} x_k \qquad x = \inf_{n\geq 1} y_n$
- If $x = \limsup_{n\to\infty} x_n$, then:
  - For every choice of primary index $n \in \mathbb{N}$, there exists a secondary index $k \geq n$ such that $x_k \geq x$ **(Infinitely many elements of the sequence are $\geq x$)**
  - For every choice of $\varepsilon > 0$, there exists an index $N_\varepsilon \in \mathbb{N}$ such that

$$x_n \leq x + \varepsilon \qquad \forall\, n \geq N_\varepsilon.$$

(All but finitely many elements of the sequence are $\leq x + \varepsilon$)

Limit Supremum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a collection of random variables w.r.t. $\mathscr{F}$

Definition (Limit Supremum of a Sequence of Random Variables)

The **limit supremum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \inf_{n\geq 1} \sup_{k\geq n} X_k(\omega), \qquad \omega \in \Omega.$$

Notation: $X = \limsup_{n\to\infty} X_n$

- **Exercise:** The limit supremum of a sequence of random variables is a random variable

Limit Infimum of a Sequence of Real Numbers

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers

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Limit Infimum of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit Infimum of a Sequence of Real Numbers

The **limit infimum** of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an element $x \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$x = \sup_{n\geq 1} \inf_{k\geq n} x_k; \qquad \textbf{Notation:} \quad x = \liminf_{n\to\infty} x_n.$$

- For each $n \in \mathbb{N}$, let $\quad y_n = \inf_{k\geq n} x_k \qquad x = \sup_{n\geq 1} y_n$
- If $x = \liminf_{n\to\infty} x_n$, then:
  - For every choice of primary index $n \in \mathbb{N}$, there exists a secondary index $k \geq n$ such that $x_k \leq x$ **(Infinitely many elements of the sequence are $\leq x$)**
  - For every choice of $\varepsilon > 0$, there exists an index $N_\varepsilon \in \mathbb{N}$ such that

$$x_n \geq x - \varepsilon \qquad \forall\, n \geq N_\varepsilon.$$

(All but finitely many elements of the sequence are $\geq x - \varepsilon$)

Limit Infimum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$

Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a collection of random variables w.r.t. $\mathscr{F}$

Definition (Limit Infimum of a Sequence of Random Variables)

The **limit infimum** of a sequence of random variables $\{X_1, X_2, \ldots\}$ is a function $X : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$X(\omega) = \sup_{n\geq 1} \inf_{k\geq n} X_k(\omega), \qquad \omega \in \Omega.$$

Notation: $X = \liminf_{n\to\infty} X_n$

- **Exercise:** The limit infimum of a sequence of random variables is a random variable

Limit of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit of a Sequence of Real Numbers

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers

Limit of a Sequence of Real Numbers

We say that $x \in \mathbb{R} \cup \{\pm\infty\}$ is the **limit** of the sequence $\{x_n\}_{n\in\mathbb{N}}$ if

$$\liminf_{n\to\infty} x_n = x = \limsup_{n\to\infty} x_n .$$

Equivalently, for every choice of $\varepsilon > 0$, there exists an index $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \qquad \forall\, n \geq N_\varepsilon .$$

Notation: $x = \lim_{n\to\infty} x_n .$

- The limit of a sequence, if it exists, is unique
- Not every sequence admits a limit
  However, every sequence admits a unique limit supremum and a unique limit infimum
- As before,

$$\text{for every choice of } \varepsilon > 0 \quad \iff \quad \text{for every choice of } \varepsilon \in \mathbb{Q},\ \varepsilon > 0$$

Limit of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $\{X_n\}_{n\in\mathbb{N}} = \{X_1, X_2, \ldots\}$ be a collection of random variables w.r.t. $\mathscr{F}$

- Fix $\omega \in \Omega$, and consider the sequence of real numbers

$$X_1(\omega), X_2(\omega), \ldots$$

- A limit may or may not exist for the above sequence

Lemma (An Important Set and its Measurability)

The set of all $\omega \in \Omega$ for which $\lim\limits_{n\to\infty} X_n(\omega)$ exists is a valid event, i.e.,

$$A_{\lim} := \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\right\} \in \mathscr{F}.$$

Proof of Lemma 1