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Probability and Stochastic Processes

Lecture 03: Uncountable Sets, Bertrand's Paradox, Probability Basics (Sample Space)

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Uncountable Sets

Definition (uncountable sets)

A set A is said to be uncountable if it is not countable, i.e., if $|A| > |\mathbb{N}|$.

Some examples of uncountable sets:

- Set of all countably infinite length binary strings, denoted commonly as $\{0, 1\}^{\mathbb{N}}$
- Unit interval, $[0, 1]$
- Set of all real numbers, $\mathbb{R}$
- Set of all irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$
- Power set of $\mathbb{N}$ (collection of all subsets of $\mathbb{N}$), denoted $2^{\mathbb{N}}$

$|\{0,1\}^{\mathbb{N}}| > |\mathbb{N}|$

To show: There exists an injective map but no bijective map from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$.
Injective map: Define $f : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ by

$$f(n) = \text{infinite binary string with '1' in the } n\text{th index.}$$

No bijective map: Suppose there exists a bijective map $g : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$. Let

$$g : n \mapsto a_{n1}\, a_{n2}\, a_{n3}\, \cdots\ ,$$

where $a_{nj} \in \{0,1\}$ for all n, j.

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Cantor's diagonalization argument: Consider the binary string

$$b = \bar{a}_{11}\, \bar{a}_{22}\, \bar{a}_{33} \cdots ,$$

where $\bar{a}_{jj} = 1 - a_{jj}$ for all $j \in \mathbb{N}$. Then, $\nexists\, n \in \mathbb{N}$ such that $g(n) = b$. Thus, g is not a bijection.

$[0, 1]$ is Uncountable – Proof

Let

$$\mathcal{D} = \left\{ d_1 = \frac{1}{2}, d_2 = \frac{1}{4}, d_3 = \frac{3}{4}, d_4 = \frac{1}{8}, \ldots \right\} \quad - \quad \text{set of dyadic rational numbers}$$

Define $g : \{0, 1\}^{\mathbb{N}} \to [0, 1]$ as

$$g : \mathbf{b} = (b_1\, b_2\, \cdots) \mapsto \begin{cases} \sum_{k=1}^{\infty} \frac{b_k}{2^k}, & \mathbf{b} \notin \mathcal{D}, \\ d_1, & \mathbf{b} = (100\cdots) \\ d_2, & \mathbf{b} = (011\cdots) \\ d_3, & \mathbf{b} = (0100\cdots) \\ d_4, & \mathbf{b} = (0011\cdots) \\ \vdots \end{cases}$$

Claim: g is a bijection!

Other Examples of Uncountable Sets

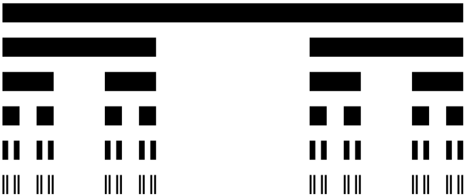
- $[0,1]$
- $\mathbb{R}$: the set of real numbers.
  Hint: the following function $f : [0,1] \to \mathbb{R}$ is a bijection:
  $$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right), \quad x \in [0,1].$$
- $\mathbb{R} \setminus \mathbb{Q}$: the set of irrational numbers.
  Hint: write $\mathbb{R}$ as
  $$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}.$$
- **Cantor set**
- $2^{\mathbb{N}} =$ power set of $\mathbb{N}$

The Cantor Set

Consider the interval $[0, 1]$

- $C_0 = [0, 1]$
- $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$
- $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

  $\vdots$



Pic credits: Jim Belk, Cornell

- Cantor set K is defined as

$$K := C_0 \cap C_1 \cap C_2 \cap \cdots$$

- **Exercise:** K is uncountable

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$|2^A| > |A|$ for any set A

Proposition

For any set A (finite, countably infinite, or uncountable), we have $|2^A| > |A|$, where 2^A denotes the power set of A (i.e., the collection of all subsets of A).

Proof of Proposition:

- Clearly, $x \mapsto \{x\}$ is an injection from A to 2^A. Therefore,
$$|2^A| \geq |A|$$
- Suppose $|2^A| = |A|$. Then, there exists a bijection $f : A \to 2^A$. Define
$$A^* = \{x \in A : x \notin f(x)\}$$
- Because f is surjective, there exists $x^* \in A$ such that $f(x^*) = A^*$
  - If $x^* \in A^*$, then $x^* \notin f(x^*) = A^*$ (contradiction)
  - If $x^* \notin A^*$, then $x^* \in f(x^*) = A^*$ (contradiction)

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Different Levels of Infinity

- $|\mathbb{N}| = \aleph_0$ – **countable infinity or aleph$_0$**
- $|2^{\mathbb{N}}| = 2^{\aleph_0} = \aleph_1$ – **uncountable infinity aleph$_1$**
- $\left|2^{2^{\mathbb{N}}}\right| = 2^{\aleph_1} = \aleph_2$ – **uncountable infinity aleph$_2$**
- and so on…
- Interestingly, $|2^{\mathbb{N}}| = |\mathbb{R}|$

$$\mathbb{R} \longleftrightarrow [0,1] \longleftrightarrow \{0,1\}^{\mathbb{N}} \xleftrightarrow{\text{exercise}} 2^{\mathbb{N}}$$

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A Formal Study of Probability

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Probability Theory – Humble Beginnings

- Bernoulli (1713) and de Moivre (1718) gave the first definition of probability:

$$\text{probability of an event} = \frac{\text{\# favourable outcomes}}{\text{total number of outcomes}}.$$

- Cournot (1843):
  "An event with very small probability is morally impossible; an event with very high probability is morally certain."
- French mathematicians of the day were satisfied with the "frequentist" approach to probability, but not the German and English mathematicians of the day
- Frequentist approach could not satisfactorily explain certain paradoxes

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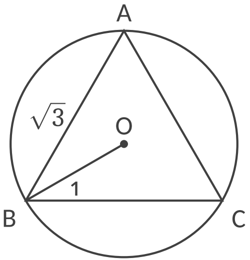
Why Axiomatic Theory? Bertrand's Paradox



Take a circle with unit radius and inscribe an equilateral triangle in it. Draw a "random chord". What is the probability that the length of this "random chord" is greater than $\sqrt{3}$?
Bertrand's perfectly valid arguments:

- Mid-point of chord should lie inside incircle of radius $1/2$
  Answer: $1/4$
- Angle between chord and tangent at A should be between $\pi/3$ and $2\pi/3$
  Answer: $1/3$
- Mid-point of chord should be between O and projection of O onto side BC
  Answer: $1/2$

Which answer is correct?

Which of the above answers is correct? Why multiple answers for one question?

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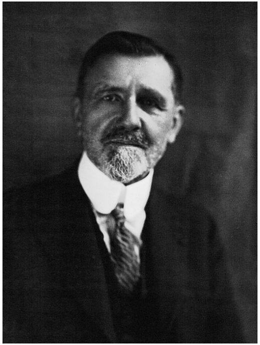
Borel to the Rescue

- Contributions to Measure Theory by **Emile Borel** (1894) provided a shift in perspective
- **Countable unions** played a key role in Borel's theory
- **Kolmogorov** applied Borel's theory to formalise the axioms of probability, laying the foundation stone for modern probability theory
- For more details on the history of probability, see [SV18] and [Kol04]



Sample Space

We begin with two universally accepted entities:

- Random experiment
- Outcome (denoted by ω) – source of randomness

Definition (Sample Space)

The sample space (denoted by Ω) of a random experiment is the set of all possible outcomes of the random experiment.

Example: Tossing a coin once

- If our interest is in the face that shows, then $\Omega = \{H, T\}$
- If our interest is in the velocity with which the coin lands on ground, then $\Omega = [0, \infty) = \mathbb{R}_+$
- If our interest is in the number of times coin flips in air, then $\Omega = \mathbb{N}$

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