AI 5090: Stochastic Processes
Homework 2

Convergence of Sequences of Random Variables

Fix a probability space $(\Omega, \mathscr{F}), \mathbb{P}$.
Aassume that all random variables appearing below are defined with respect to this probability space.

1. Let $\Theta \sim$ Unif$[0, 2\pi]$. For each $n \in \mathbb{N}$, let

$$Y_n = \left|1 - \frac{\Theta}{\pi}\right|^n.$$

   Argue in which of the four senses (a.s., m.s., p., d.) does the sequence $\{Y_n\}_{n=1}^{\infty}$ converge.
   Identify the limit random variable in each case, and justify your answers.

2. Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim}$ Ber(0.5).

   (a) Prove that $\{X_n\}_{n=1}^{\infty}$ converges in distribution. Identify a limit random variable.

   (b) Prove that $\{X_n\}_{n=1}^{\infty}$ cannot converge in probability to any random variable.

   Hint for part (b):
   Suppose there exists a limit random variable X such that $X_n \xrightarrow{\text{p.}} X$. Then, from triangle inequality, we have

$$|X_n - X_{n+1}| \leq |X_n - X| + |X_{n+1} - X|.$$

   Use the above inequality to prove that for any $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X_{n+1}| > \varepsilon) \leq 2\,\mathbb{P}\left(|X_n - X| > \frac{\varepsilon}{2}\right).$$

   In particular, compute the left-hand side of the above inequality for $\varepsilon = 0.5$, and prove that convergence in probability does not hold.

3. Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim}$ Poisson(1).
   For each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} X_i$.

   (a) Show that $S_n \sim$ Poisson(n) for each $n \in \mathbb{N}$.

   (b) Without using the central limit theorem, prove from first principles that

$$\frac{S_n - n}{\sqrt{n}} \xrightarrow{\text{d.}} X, \qquad X \sim \mathcal{N}(0, 1).$$

   (c) Prove that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

   Hint for part (c): Use the result in part (b).

4. Suppose that $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim}$ Ber(p) for some fixed constant $p \in (0, 1)$.
   For each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} X_i$.

   (a) Derive the moment generating function of S_n.

   (b) Show that

$$\mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \varepsilon\right) = 0$$

   for all $\varepsilon > 1$.

(c) Prove that for any $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\frac{S_n}{n} - p > \varepsilon\right) \begin{cases} = 0, & p+\varepsilon \geq 1, \\ \leq \exp\left(-n\, D((p+\varepsilon) \,\|\, p)\right), & 0 < p+\varepsilon < 1, \end{cases}$$

where $D((p+\varepsilon) \,\|\, p) := (p+\varepsilon) \log \frac{p+\varepsilon}{p} + (1-p-\varepsilon)\, \log \frac{1-p-\varepsilon}{1-p}$ is the Kullback–Leibler divergence between the probability distributions Ber$(p+\varepsilon)$ and Ber(p).

Hint for part (b):
Use Chernoff's bound to obtain an upper bound for $\mathbb{P}(S_n > n(p+\varepsilon))$, and optimise the upper bound.

(d) Prove that for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{S_n}{n} - p < -\varepsilon\right) \begin{cases} = 0, & p-\varepsilon \leq 0, \\ \leq \exp\left(-n\, D((1-p+\varepsilon) \,\|\, (1-p))\right), & 0 < p-\varepsilon < 1. \end{cases}$$

Hint for part (d):
Set $Y_n = 1 - X_n$ for each $n \in \mathbb{N}$, and use the result in part (c) on the sum of Y_n's.

(e) Combining the results in parts (b), (c), and (d) above, prove that

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} p.$$

5. Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables satisfying the property

$$\sum_{n=1}^{\infty} \mathbb{E}\left[|X_n - 2|^p\right] < +\infty$$

for some fixed $p > 1$.
Prove that $\{X_n\}_{n=1}^{\infty}$ converges almost surely. What is the limit random variable?

6. Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Exp}(1)$.
For each $n \in \mathbb{N}$, let $Y_n = \max\{X_1, \ldots, X_n\}$.
In this exercise, we shall prove formally that almost surely, Y_n grows as $\log n$ for large n.
If the base of the logarithm is not mentioned explicitly, it should be considered to be e.

(a) Show formally that

$$\frac{Y_n}{\log n} \xrightarrow{\text{d.}} 1,$$

and use the reverse implication from class to conclude that $\frac{Y_n}{\log n} \xrightarrow{\text{p.}} 1$.

(b) Based on the conclusion in part (a), show that

$$\frac{Y_n}{\log_2 n} \xrightarrow{\text{p.}} \log 2.$$

(c) Consider the subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ given by $n_k = 2^k$ for all $k \in \mathbb{N}$.
Fix $\varepsilon > 0$. For each k, let

$$x_k := e^{-(\varepsilon + \log 2)k}.$$

Prove that

$$(1 - x_k)^{n_k} \geq \exp\left(-\frac{n_k\, x_k}{1 - x_k}\right) \quad \forall k.$$

Further, deduce that

$$(1 - x_k)^{n_k} \geq \exp(-2 n_k x_k)$$

for all sufficiently large values of k.

Hint for part (c):
To prove the first part, use the relation $\log x \geq 1 - \frac{1}{x}$ for any $x > 0$ (this is another way of seeing the well-known inequality $\log x \leq x - 1$ for all $x > 0$).
To deduce the second part, use the fact that x_k converges to 0 as $k \to \infty$, and therefore $x_k < \frac{1}{2}$ for all sufficiently large values of k.

(d) Using the result in the second half of part (c), prove that for every $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{Y_{n_k}}{\log_2 n_k} > \log 2 + \varepsilon\right) < +\infty.$$

Then, using the Borel–Cantelli lemma, conclude that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \limsup_{k\to\infty} \frac{Y_{n_k}(\omega)}{\log_2 n_k} \le \log 2\right\}\right) = 1. \tag{1}$$

(e) Fix $0 < \varepsilon < \log 2$. For each $k \in \mathbb{N}$, let

$$y_k := e^{-(\log 2 - \varepsilon)k}.$$

Using the facts that $1 - x \le e^{-x}$ and $e^x > x$ for all $x \ge 0$ (again, alternative ways to see the inequality $\log x \le x - 1 < x$), prove that

$$(1 - y_k)^{n_k} \le e^{-\varepsilon k} \quad \forall k.$$

Use this relation to prove that for every $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \left(\frac{Y_{n_k}}{\log_2 n_k} \le \log 2 - \varepsilon\right) < +\infty,$$

and hence conclude from the Borel–Cantelli lemma that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \liminf_{k\to\infty} \frac{Y_{n_k}(\omega)}{\log_2 n_k} \ge \log 2\right\}\right) = 1. \tag{2}$$

Epilogue for question 6:
Combining the results in (1) and (2), we see that

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{k\to\infty} \frac{Y_{n_k}(\omega)}{\log_2 n_k} = \log 2\right\}\right) = 1.$$

That is, $\frac{Y_{n_k}}{\log_2 n_k} \xrightarrow{\text{a.s.}} \log 2$.
This proves that the subsequence $\{Y_{n_k}/(\log_2 n_k)\}_{k=1}^{\infty}$ converges almost surely to the constant random variable taking the value $\log 2$.
We can use this to prove that the entire sequence $\{Y_n/(\log_2 n)\}_{n=1}^{\infty}$ must also converge to the same constant random variable $\log 2$, as follows.
Given any $n \in \mathbb{N}$, find k such that $n_k \le n < n_k + 1$ (you can always find at least one such k).
Because $Y_1 \le Y_2 \le Y_3 \le \cdots$, it follows that

$$\begin{aligned}
& Y_{n_k} \le Y_n < Y_{n_k+1} \\
\implies \quad & \frac{Y_{n_k}}{\log_2 n} \le \frac{Y_n}{\log_2 n} < \frac{Y_{n_k+1}}{\log_2 n} \\
\implies \quad & \frac{Y_{n_k}}{\log_2 (n_k+1)} \le \frac{Y_n}{\log_2 n} < \frac{Y_{n_k+1}}{\log_2 n_k} \\
\implies \quad & \frac{\log_2(n_k+1)}{\log_2 n_k} \cdot \frac{Y_{n_k}}{\log_2 n_k} \le \frac{Y_n}{\log_2 n} < \frac{Y_{n_k+1}}{\log_2(n_k+1)} \cdot \frac{\log_2 n_k}{\log_2(n_k+1)}.
\end{aligned}$$

Using (1) and (2), along with the fact that $\lim_{k\to\infty} \frac{\log_2(n_k+1)}{\log_2 n_k} = 1$ gives us the desired result.