AI 5030: PROBABILITY AND STOCHASTIC PROCESSES
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HOMEWORK 6
TOPICS: JOINT PDFS, CONDITIONAL PDFS, TRANSFORMATIONS

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Virat and Anushka have a date at 7 pm. Each will arrive at the meeting place with a delay that is distributed uniformly randomly between 0 minutes and 60 minutes, independent of the delay of the other. The first to arrive will wait for 15 minutes and leave if the other does not arrive within 15 minutes. Find the probability that both meet.

Solution: Let X denote the delay in Virat's arrival, and let Y denote the delay in Anushka's arrival. If both are to meet, then the event of interest is $\{|X-Y| \leq 15\}$. To compute this probability, we simply note that

$$\mathbb{P}(\{|X-Y| \leq 15\}) = \int_{(x,y):|x-y|\leq 15} f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_{(x,y):|x-y|\leq 15} \frac{1}{60^2}\, \mathrm{d}x\, \mathrm{d}y.$$

We now note that

$$\begin{aligned}&\mathbb{P}(\{|X-Y| \leq 15\})\\ &= \mathbb{P}(\{|X-Y| \leq 15\} \cap \{X \leq 15\}) + \mathbb{P}(\{|X-Y| \leq 15\} \cap \{15 < X < 45\}) + \mathbb{P}(\{|X-Y| \leq 15\} \cap \{X \geq 45\}).\end{aligned} \tag{1}$$

The first term on the right-hand side of (1) is given by

$$\begin{aligned}\mathbb{P}(\{|X-Y| \leq 15\} \cap \{X \leq 15\}) &= \mathbb{P}(\{X \leq 15\} \cap \{0 \leq Y \leq X+15\})\\ &= \int_0^{15} \int_0^{x+15} \frac{1}{60^2}\, \mathrm{d}x\, \mathrm{d}y = \frac{3}{32}.\end{aligned}$$

By symmetry, it follows that the last term on the right-hand side of (1) is also equal to $3/32$. The second term on the right-hand side of (1) is given by

$$\mathbb{P}(\{|X-Y| \leq 15\} \cap \{15 < X < 45\}) = \int_{15}^{45} \int_{x-15}^{x+15} \frac{1}{60^2}\, \mathrm{d}x\, \mathrm{d}y = \frac{1}{4}. \tag{2}$$

Combining the above results, we get

$$\mathbb{P}(\{|X-Y| \leq 15\}) = \frac{3}{32} + \frac{1}{4} + \frac{3}{32} = \frac{7}{16}.$$

2. Let $X_1, X_2, X_3 \overset{\text{i.i.d.}}{\sim} \text{Unif}((0,1))$.

   (a) Compute $\mathbb{P}(\{X_1 + X_2 > X_3\})$.
   Solution without Convolution

$$\begin{aligned}\mathbb{P}(\{X_1 + X_2 > X_3\}) &= \int_0^1 \int_0^1 \int_0^{\min(1, x_1+x_2)} f_{X_1,X_2,X_3}(x_1,x_2,x_3)\, \mathrm{d}x_1 \mathrm{d}x_2\, \mathrm{d}x_3\\ &= \int_0^1 \int_0^1 \int_0^{\min(1, x_1+x_2)} f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) \mathrm{d}x_1 \mathrm{d}x_2\, \mathrm{d}x_3 \text{ (Using } X_1 \perp\!\!\!\perp X_2)\\ &= \int_0^1 \int_0^1 \min(1, x_1+x_2) \mathrm{d}x_1\, \mathrm{d}x_2\\ &= \int_0^1 \int_0^{1-x_1} (x_1+x_2) \mathrm{d}x_1 \mathrm{d}x_2 + \int_0^1 \int_{1-x_1}^1 \mathrm{d}x_1 \mathrm{d}x_2\end{aligned}$$

$$
\begin{aligned}
&= 1/3 + 1/2 \\
&= 5/6.
\end{aligned}
$$

Solution via Convolution Let $Y := X_1 + X_2$. Y is a random variable (being a continuous function of $X_1,\ X_2$) that takes values in $(0, 2)$. We first derive the PDF of Y, which will be used later. We have that $f_{X_1}(a) = f_{X_2}(a) = \begin{cases} 1, & a \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$

Now, we derive $f_Y(a)$ when $a \in (0, 2)$. As X_1, X_2 are independent, we can use the convolution formula

$$
\begin{aligned}
f_Y(a) &= \int_{-\infty}^{\infty} f_{X_1}(a-y) f_{X_2}(y) \mathsf{d}y \\
&= \int_0^1 f_{X_1}(a-y) f_{X_2}(y) \mathsf{d}y \\
&= \int_0^1 f_{X_1}(a-y) \mathsf{d}y
\end{aligned}
$$

We can now see that depending on the value a, we will get a finer range of y which ensures $(a - y) \in (0, 1)$.
Case $a \in (1, 2)$: With this, we need $1 > y > a - 1$ to ensure $a - y \in (0, 1)$.
$f_Y(a) = \int_{a-1}^{1} \mathsf{d}y = 2 - a$.
Case $a \in (0, 1)$: With this, we need $0 < y < a$ to ensure $a - y \in (0, 1)$.
$f_Y(a) = \int_0^a \mathsf{d}y = a$.

$$
\begin{aligned}
\mathbb{P}(\{Y > X_3\}) =& 1 - \mathbb{P}(\{Y \le X_3\}) \\
=& 1 - \int_0^1 \mathbb{P}(\{Y \le x_3\}) f_{X_3}(x_3)\ \mathsf{d}x_3 \\
=& 1 - \int_0^1 \int_0^{x_3} f_Y(y)\ \mathsf{d}y\ f_{X_3}(x_3)\ \mathsf{d}x_3 \\
=& 1 - \int_0^1 \int_0^{x_3} y\ \mathsf{d}y\ \mathsf{d}x_3 \\
=& 1 - \int_0^1 y^2/2\ \mathsf{d}x_3 \\
=& 1 - |y^3/6|_0^1 \\
=& 5/6.
\end{aligned}
$$

(b) Derive the CDF of the random variable $X = X_1 X_2$.
Solution: It is easy to see that $\mathbb{P}(\{X_1 X_2 \le y\}) = 0$ for $y \le 0$ as $X_1, X_2 \sim U((0, 1))$. We derive the CDF for $y \in (0, 1]$.
We note that the random variable y/X_2 can also take values greater than 1. Hence, we need to accordingly adjust the integration limits.

$$
\begin{aligned}
\mathbb{P}(\{X_1 X_2 \le y\}) &= \int_0^1 \int_0^{\mathsf{min}(1, y/x_2)} f_{X_1, X_2}(x_1, x_2)\ \mathsf{d}x_1\ \mathsf{d}x_2 \\
&= \int_0^1 \int_0^{\mathsf{min}(1, y/x_2)} f_{X_1}(x_1) f_{X_2}(x_2)\ \mathsf{d}x_1\ \mathsf{d}x_2 \ \text{(Using } X_1 \perp\!\!\!\perp X_2\text{)} \\
&= \int_0^1 \mathsf{min}(1, y/x_2) \mathsf{d}x_2 \\
&= \int_0^y \mathsf{d}x_2 + \int_y^1 y/x_2 \mathsf{d}x_2 \\
&= y + y\, \mathsf{ln}(x_2)|_y^1 \\
&= y - y\, \mathsf{ln}(y).
\end{aligned}
$$

(c) **Solution**

$$
\begin{aligned}
\mathbb{P}(X_1X_2 \le X_3^2) &= \int_0^1 \mathbb{P}(X_1X_2 \le x_3^2) f_{X_3}(x)\,\mathsf{d}x \\
&= \int_0^1 (x^2 - x^2 \ln x^2)\,\mathsf{d}x \text{ (Using part (b))} \\
&= 1/3 - \int_0^1 2x^2 \ln x\,\mathsf{d}x \\
&= 5/9 \text{ (Using Integration by parts rule).}
\end{aligned}
$$

Solution1 without using part (b):

$$
\begin{aligned}
\mathbb{P}(\{X_1X_2 \le X_3^2\}) &= 1 - \mathbb{P}(\{X_1X_2 > X_3^2\}) \\
&= 1 - \int_0^1 \int_0^1 \int_0^{\sqrt{x_1x_2}} f_{X_1,X_2,X_3}(x_1,x_2,x_3)\,\mathsf{d}x_3\mathsf{d}x_1\mathsf{d}x_2 \\
&= 1 - \int_0^1 \int_0^1 \int_0^{\sqrt{x_1x_2}} f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3)\,\mathsf{d}x_3\mathsf{d}x_1\mathsf{d}x_2 \text{ (Using independence)} \\
&= 1 - \int_0^1 \int_0^1 \sqrt{x_1x_2}\,\mathsf{d}x_1\mathsf{d}x_2 \\
&= 5/9.
\end{aligned}
$$

Solution2 without using part (b):

$$
\begin{aligned}
\mathbb{P}(\{X_1X_2 \le X_3^2\}) &= \int_0^1 \int_0^1 \int_0^{\min(1,x_3^2/x_2)} f_{X_1,X_2,X_3}(x_1,x_2,x_3)\,\mathsf{d}x_1\mathsf{d}x_2\mathsf{d}x_3 \\
&= \int_0^1 \int_0^1 \int_0^{\min(1,x_3^2/x_2)} f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3)\,\mathsf{d}x_1\mathsf{d}x_2\mathsf{d}x_3 \text{ (Using independence)} \\
&= \int_0^1 \int_0^1 \min(1, x_3^2/x_2)\,\mathsf{d}x_2\mathsf{d}x_3 \\
&= \int_0^1 \int_0^{x_3^2} \mathsf{d}x_2\mathsf{d}x_3 + \int_0^1 \int_{x_3^2}^1 \frac{x_3^2}{x_2}\,\mathsf{d}x_2\mathsf{d}x_3 \\
&= \int_0^1 x_3^2\,\mathsf{d}x_3 - \int_0^1 x_3^2 \ln x_3^2\,\mathsf{d}x_3 \\
&= 1/3 + 2/9 \;\; \text{(the second integration uses integration by parts rule)} \\
&= 5/9
\end{aligned}
$$

3. Let X and Y be jointly continuous random variables with the joint PDF

$$
f_{X,Y}(x,y) = \begin{cases} cx(y-x)e^{-y}, & 0 \le x \le y < +\infty, \\ 0, & \text{otherwise.} \end{cases}
$$

(a) Determine the value of the constant c.

(b) Show that

$$
f_{X|Y=y}(x) = \begin{cases} 6x(y-x)y^{-3}, & 0 \le x \le y, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{Y|X=x}(y) = \begin{cases} (y-x)e^{x-y}, & y \ge x, \\ 0, & \text{otherwise,} \end{cases}
$$

Solution: We present the solution to each of the parts below.

(a) To determine the constant c, we set the integral of the joint PDF to 1. Doing so, we obtain $c = 1$.

(b) From the joint PDF expression, we first obtain the marginal PDFs of X and Y. For any $0 \leq y < +\infty$, we note that

$$f_Y(y) = \int_0^y x(y-x)\, e^{-y}\, \mathsf{d}x = \frac{y^3\, e^{-y}}{6},$$

from which it follows that

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 6x(y-x)y^{-}3, & 0 \leq x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

Along similar lines, for any $0 \leq x < +\infty$, we have

$$f_X(x) = \int_x^{\infty} x(y-x)e^{-y}\, \mathsf{d}y = xe^{-x},$$

from which it follows that

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} (y-x)\, e^{-(y-x)}, & x \leq y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

4. Suppose that X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cy, & -1 \leq x \leq 1,\ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the constant c.

(b) Are X and Y independent?

(c) Evaluate $\mathbb{P}(\{X \geq Y + 0.5\})$.

(d) Compute the conditional PDF of X, conditioned on the event $\{Y > 0.5\}$.
Using the above conditional PDF, evaluate $\mathbb{P}(\{X > 0.75\}|\{Y > 0.5\})$.

Solution: We present the solution to each part below.

(a) To determine the constant c, we integrate the joint PDF and set the integral to 1. Doing so, we get

$$1 = \int_{-1}^{1} \int_0^{|x|} cy\, \mathsf{d}y\, \mathsf{d}x = \int_{-1}^{1} c\, \frac{x^2}{2}\, \mathsf{d}x = \frac{c}{3},$$

from which it follows that $c = 3$.

(b) To determine if X is independent of Y, or otherwise, we first compute the marginal PDFs of X and Y. For any $x \in [-1, 1]$, we have

$$f_X(x) = \int_0^{|x|} 3y\, \mathsf{d}y = \frac{3x^2}{2}.$$

Similarly, for any $y \in [0, 1]$, we have

$$f_Y(y) = \int_{-1}^{-y} 3y\, \mathsf{d}x + \int_y^1 3y\, \mathsf{d}x = 6y(1-y).$$

Clearly, $f_{X,Y}(1,1) = 3 \neq 0 = f_X(1) f_Y(1)$, thereby proving that $X \not\perp\!\!\!\perp Y$.

(c) The desired probability is given by

$$\mathbb{P}(\{X \geq Y + 0.5\}) = \int_{0.5}^{1} \int_0^{x-0.5} 3y\, \mathsf{d}y\, \mathsf{d}x = \int_{0.5}^{1} \frac{3(x-0.5)^2}{2}\, \mathsf{d}x = \frac{1}{16}.$$

(d) We first compute the conditional CDF of X, conditioned on the event $A = \{Y > 0.5\}$. First, we note that

$$\mathbb{P}(A) = \int_{0.5}^{1} 6y(1-y)\,\mathrm{d}y = \frac{1}{2}.$$

Next, we note that

$$F_{X|A}(x) = \frac{\mathbb{P}(\{X \le x\} \cap A)}{\mathbb{P}(A)} = \begin{cases} 0, & x < -1, \\ \dfrac{\int_{-1}^{-x}\int_{0.5}^{|u|} 3v\,\mathrm{d}v\,\mathrm{d}u}{1/2}, & -1 \le x < -\frac{1}{2}, \\ \dfrac{\int_{-1}^{-0.5}\int_{0.5}^{|u|} 3v\,\mathrm{d}v\,\mathrm{d}u}{1/2}, & -\frac{1}{2} \le x < \frac{1}{2}, \\ \dfrac{\int_{-1}^{-0.5}\int_{0.5}^{|u|} 3v\,\mathrm{d}v\,\mathrm{d}u + \int_{0.5}^{x}\int_{0.5}^{|u|} 3v\,\mathrm{d}v\,\mathrm{d}u}{1/2}, & \frac{1}{2} \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Simplifying the integrals in the above expression, we get

$$F_X(x) = \begin{cases} 0, & x < -1, \\ 1 + x^3 - \frac{3}{4}(1+x), & -1 \le x < -\frac{1}{2}, \\ \frac{1}{2}, & -\frac{1}{2} \le x < \frac{1}{2}, \\ \frac{1}{2} + x^3 - \frac{3x}{4} + \frac{1}{4}, & \frac{1}{2} \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Differentiating the above CDF expression with respect to x, we get

$$f_{X|A}(x) = \begin{cases} 3x^2 - \frac{3}{4}, & x \in [-1, -0.5] \cup [0.5, 1], \\ 0, & \text{otherwise.} \end{cases}$$

We then have

$$\mathbb{P}(\{X > 0.75\} | \{Y > 0.5\}) = \int_{0.75}^{1} f_{X|A}(x)\,\mathrm{d}x = \int_{0.75}^{1} \left(3x^2 - \frac{3}{4}\right)\,\mathrm{d}x = \frac{25}{64}.$$

5. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim}$ Exponential(λ). Using the bivariate Jacobian transformation formula, compute the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$, and show that $Y_1 \perp\!\!\!\perp Y_2$.

Solution: The transformation function $g(x_1, x_2) := \begin{bmatrix} x_1 + x_2 \\ x_1/x_2 \end{bmatrix}$, where $g_1(x_1, x_2) = x_1 + x_2,\ g_2 = x_1/x_2$ are differentiable with continuous first order partial derivatives.
We now test if g is one-one. Let $g(x_1, x_2) = g(x_1', x_2')$. On solving the equations $x_1 + x_2 = x_1' + x_2'$ and $x_1/x_2 = x_1'/x_2'$, we get $x_1 = x_1'$ and $x_2 = x_2'$. This implies that g is a one-one function.
The Jacobian matrix for g is $J_g(x_1, x_2) = \begin{bmatrix} 1 & 1 \\ 1/x_2 & -x_1/x_2^2 \end{bmatrix}$. The corresponding Jacobian = Det($J_g(x_1, x_2)|$) $= -\frac{x_1+x_2}{x_2^2}$. As $X_1, X_2 \sim$ Exp(λ), the Jacobian is never 0.
Now, we can apply the transformation formula. To find $g^{-1}(y_1, y_2)$, we solve $x_1 + x_2 = y_1$ and $x_1/x_2 = y_2$ for x_1

and x_2. Thus, $g^{-1}(y_1, y_2) = \begin{bmatrix} \frac{y_1 y_2}{1+y_2} \\ \frac{y_1}{1+y_2} \end{bmatrix}$ and Det $\left(J_g\left(\frac{y_1 y_2}{1+y_2}, \frac{y_1}{1+y_2}\right)\right) = -\frac{(1+y_2)^2}{y_1}$. Finally, we put these in the transformation formula $f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{f_{X_1,X_2}(g^{-1}(y_1,y_2))}{|\text{Det}(J_g(g^{-1}(y_1,y_2)))|}, & y_1 > 0, y_2 \geq 0 \\ 0 & \text{Otherwise} \end{cases}$ and use that X_1 is independent of X_2, we get $f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{y_1 e^{-\lambda y_1}}{(y_2+1)^2}\lambda^2, & y_1 > 0, y_2 \geq 0 \\ 0 & \text{Otherwise} \end{cases}$.

By marginalizing out y_2, we get $f_{Y_1}(y_1) = \begin{cases} \int_0^\infty \frac{y_1 e^{-\lambda y_1}}{(y_2+1)^2}\lambda^2 \, \mathrm{d}y_2, & y_1 > 0 \\ 0, & \text{Otherwise} \end{cases} = \begin{cases} \frac{y_1 e^{-\lambda y_1}\lambda^2}{1}, & y_1 > 0 \\ 0, & \text{Otherwise.} \end{cases}$

Similarly, by marginalizing out y_1, we get $f_{Y_2}(y_2) = \begin{cases} \int_0^\infty \frac{y_1 e^{-\lambda y_1}}{(y_2+1)^2}\lambda^2 \, \mathrm{d}y_1, & y_2 > 0 \\ 0, & \text{Otherwise} \end{cases} \overset{(1)}{=} \begin{cases} \frac{1}{(1+y_2)^2}, & y_2 > 0 \\ 0, & \text{Otherwise,} \end{cases}$ where step (1) uses integration by parts rule with the first function as $f(y) = y$ and the second function as $f(y) = e^{-\lambda y}$. We can now see that $f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$ which is a necessary and sufficient condition for $Y_1 \perp\!\!\!\perp Y_2$. Hence, proved.

6. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim}$ Exponential(λ). Let $X = X_1$ and $Y = X_1 + X_2$. Determine the joint PDF of X and Y using the bivariate Jacobian transformation formula. Further, for any $y > 0$, show that the conditional PDF of X, conditioned on the event $\{Y = y\}$, is the uniform PDF on the interval $[0, y]$.

   Solution: The transformation function $g(x_1, x_2) := \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$, where $g_1(x_1, x_2) = x_1,\ g_2 = x_1 + x_2$ are differentiable with continuous first order partial derivatives. It is also clear that g is one-one. The Jacobian matrix for g is $J_g(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. The corresponding Jacobian = Det($J_g(x_1, x_2)|) = 1 > 0$.
   $g^{-1}(y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 - y_1 \end{bmatrix}$. Due to $X_1 \perp\!\!\!\perp X_2$, we have $f_{X_1,X_2}(g^{-1}(y_1, y_2)) = \lambda^2 e^{-\lambda y_2}$.

   Now, applying the formula, we get $f_{X,Y}(x, y) = \begin{cases} e^{-\lambda y}\lambda^2 & , y > 0 \\ 0, \text{ Otherwise} \end{cases}$.

   Using the formula for conditional PDF, $f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{e^{-\lambda y}\lambda^2}{f_Y(y)} \overset{(1)}{=} \frac{1}{y}, & y > 0 \\ 0, & \text{Otherwise} \end{cases}$ where in step (1), we used that Y is the sum of two I.I.D. Exp(λ) random variables so $f_Y(y) = \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda a} e^{-\lambda(y-a)} \, \mathrm{d}a = \int_0^y \lambda^2 e^{-\lambda a} e^{-\lambda(y-a)} \, \mathrm{d}a = \lambda^2 y e^{-\lambda y}$.
   The conditional PDF of X conditioned on $\{Y = y\}$ follows the PDF of a uniform distribution on $[0,\ y]$. Hence proved.

7. Let $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Let $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan \frac{Y}{X} = \tan^{-1}\left(\frac{Y}{X}\right)$. Determine the joint PDF of R and Θ, and show that $R \perp\!\!\!\perp \Theta$. What are the marginal PDFs of R and Θ?

   Solution: Let $g_1(x, y) = \sqrt{x^2 + y^2}$, and $g_2(x, y) = \tan^{-1}(y/x)$. We then have $R = g_1(X, Y)$ and $\Theta = g_2(X, Y)$. Let $g(x, y) = [g_1(x, y) g_2(x, y)]^\top$. Then, the Jacobian of g at any point $(x, y) \in \mathbb{R}^2$ is given by

$$\det(J_g(x, y)) = \det\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \det\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}}.$$

   Setting $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$, and using the Jacobian transformation formula, we get

$$f_{R,\Theta}(r, \theta) = \frac{f_{X,Y}(g^{-1}(r, \theta)}{\left|\det(J_g(g^{-1}(r, \theta)))\right|}.$$

   We now note that $g^{-1}(r, \theta) = (r \cos \theta, r \sin \theta)$, and hence

$$f_{R,\Theta}(r, \theta) = \frac{\frac{1}{2\pi} e^{-r^2/2}}{1/r} = \frac{1}{2\pi} \cdot r e^{-r^2/2}, \qquad r > 0,\ \theta \in [0, 2\pi].$$

From the above expression, we see that the marginal PDFs of R and Θ are given by

$$f_R(r) = re^{-r^2/2}, \quad r > 0, \qquad\qquad f_\Theta(\theta) = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi].$$

Here, R follows the *Rayleigh* distribution, while Θ is uniformly distributed in $[0, 2\pi]$, and $R \perp\!\!\!\perp \Theta$.

8. Let R and Θ be two random variables with the joint PDF

$$f_{R,\Theta}(r, \theta) = re^{-r^2/2} \cdot \frac{1}{2\pi}, \qquad r \geq 0, \quad \theta \in [0, 2\pi].$$

   (a) Compute the marginal PDFs of R and Θ, and show that $R \perp\!\!\!\perp \Theta$.

   (b) Let $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. Show that $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$.

Solution: We present the solution to each of the parts below.

   (a) From the solution to Question 7, it is clear that $R \perp\!\!\!\perp \Theta$.

   (b) Let $g_1(r, \theta) = r\cos(\theta)$ and $g_2(r, \theta) = r\sin\theta$. Then, we have $X = g_1(R, \Theta)$ and $Y = g_2(R, \Theta)$. Furthermore, setting $x = r\cos\theta$ and $y = r\sin\theta$, the inverse of the mapping $g(r, \theta) = (g_1(r, \theta), g_2(r, \theta))$ may be easily shown to be $g^{-1}(x, y) = (\sqrt{x^2 + y^2}, \tan^{-1}(y/x))$. Noting that

$$\det(J_g(r, \theta)) = \det\begin{pmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \end{pmatrix} = det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r(\cos^2\theta + \sin^2\theta) = r,$$

   and using the Jacobian transformation formula, we get

$$f_{X,Y}(x, y) = \frac{f_{R,\Theta}(g^{-1}(x, y))}{\left|\det(J_g(g^{-1}(x, y)))\right|} = \frac{e^{-(x^2+y^2)/2}}{2\pi},$$

   from which it follows that $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$.

9. Let X be a random variable with CDF $F : \mathbb{R} \to [0, 1]$.
Define a second function $g_F : (0, 1) \to \mathbb{R}$ as

$$g_F(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \qquad u \in (0, 1).$$

   (a) Argue that $F(g_F(u)) \geq u$ for all $u \in (0, 1)$.

   (b) Let $U \sim \text{Unif}((0, 1))$. Show that the random variable $Z = g_F(U)$ has the same CDF as X, i.e., $F_Z = F$. Here, the random variable Z is defined as

$$Z(\omega) = g_F(U(\omega)), \qquad \omega \in \Omega.$$

   Remark: This result is the premise for sampling from a custom distribution function F in computer simulations. To generate a sample $X \sim F$, we first generate a random sample U uniformly from $(0, 1)$ and set $X = g_F(U)$.
   Hint: Fix an arbitrary $z \in \mathbb{R}$.
   Using the result in part (a) and the fact that F is non-decreasing, argue that $\{g_F(U) \leq z\} \subseteq \{U \leq F(z)\}$.
   On the other hand, using the definition of g_F, show that $\{U \leq F(z)\} \subseteq \{g_F(U) \leq z\}$.
   (To show $A \subseteq B$, you need to argue that $\omega \in A$ implies $\omega \in B$.)

   (c) Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 0.1, & x = 10, \\ 0.2, & x = 20, \\ 0.3, & x = 30, \\ 0.4, & x = 40, \\ 0, & \text{otherwise.} \end{cases}$$

Explicitly compute the CDF F and the function g_F corresponding to the above PMF.
Write a Python program to generate $N = 100,000$ independent samples uniformly from $(0, 1)$. Call these samples $\{u_1, u_2, \ldots, u_N\}$. For each $i \in \{1, \ldots, N\}$, set $x_i = g_F(u_i)$. Plot the histogram of the samples $\{x_1, \ldots, x_N\}$, and verify that the histogram matches closely with the PMF p_X.
You may use the NumPy module `random.random()` to generate samples uniformly from $(0, 1)$.

(d) How will you generate a random sample from Exponential(1) distribution on a computer?

Solution: We present the solution to each of the parts below.

(a) Given any $u \in (0, 1)$, we have the following cases:

- There exists $x_0 \in \mathbb{R}$ such that $F(x_0) = u$. This is the case, for instance, if F is continuous at the point x_0. In this case, we have

$$g_F(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\} = \inf\{x \in \mathbb{R} : F(x) = u\} \leq x_0.$$

  The last inequality follows because F may be flat with a y-value equal to u, in which case there may be other points $x \neq x_0$ for which $F(x) = u$. In any case, we will have $F(g_F(u)) = u = F(x_0)$.

- There exists $x_0 \in \mathbb{R}$ such that

$$\lim_{\varepsilon \downarrow 0} F(x_0 - \varepsilon) \leq u < F(x_0).$$

  This is the case, for instance, if F has a jump at the point x_0. In this case, we have

$$g_F(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\} = x_0,$$

  and furthermore, $F(g_F(u)) = F(x_0) \geq u$.

Combining the results of the above cases, we see that $F(g_F(u)) \geq u$ for all $u \in (0, 1)$.

(b) For any $z \in \mathbb{R}$, noting that $F(z) \in [0, 1]$ and $\mathbb{P}(\{U \leq u\}) = u$ for all $u \in [0, 1]$, we have

$$\begin{aligned} \mathbb{P}(\{Z \leq z\}) &= \mathbb{P}(\{g_F(U) \leq z\}) \\ &\overset{(*)}{=} \mathbb{P}(\{U \leq F(z)\}) \\ &= F(z), \end{aligned}$$

where $(*)$ above follows by noting that $\{g_F(U) \leq z\} = \{U \leq F(z)\}$, a formal proof of which is as follows.

- <u>Proving that $\{g_F(U) \leq z\} \subseteq \{U \leq F(z)\}$:</u>

  Suppose that $\omega \in \{g_F(U) \leq z\}$. This implies that $g_F(U(\omega)) \leq z$. From the monotonicity property of F, we then have $F(g_F(U(\omega)) \leq F(z)$. Combining this with the result in part (a) above, we have

$$U(\omega) \leq F(g_F(U(\omega))) \leq F(z),$$

  thus proving that $\omega \in \{U \leq F(z)\}$.

- <u>Proving that $\{U \leq F(z)\} \subseteq \{g_F(U) \leq z\}$:</u>

  Suppose that $\omega \in \{U \leq F(z)\}$. This implies that $U(\omega) \leq F(z)$ (or equivalently, $F(z) \geq U(\omega)$, which in turn implies that

$$z \in \{x \in \mathbb{R} : F(x) \geq U(\omega)\}.$$

  Hence, it follows that

$$g_F(U(\omega)) = \inf\{x \in \mathbb{R} : F(x) \geq U(\omega)\} \leq z,$$

  thereby proving that $\omega \in \{g_F(U) \leq z\}$.

(c) We first note that the CDF of X may be expressed as

$$F_X(x) = \begin{cases} 0, & x < 10, \\ 0.1, & 10 \leq x < 20, \\ 0.3, & 20 \leq x < 30, \\ 0.6, & 30 \leq x < 40, \\ 1, & x \geq 40. \end{cases}$$

Using the above expression, for any $u \in (0,1)$, we have

$$g_F(u) = \mathsf{inf}\{x \in \mathbb{R} : F(x) \geq u\} = \begin{cases} 10, & 0 < u \leq 0.1, \\ 20, & 0.1 < u \leq 0.3 \\ 30, & 0.3 < u \leq 0.6 \\ 40, & 0.6 < u < 1. \end{cases}$$

The programming part is left as exercise.

(d) We first note that if $X \sim$ Exponential(1), then

$$F_X(x) = \begin{cases} 1 - e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Hence, it follows that for any $u \in (0,1)$,

$$\begin{aligned} g_F(u) &= \mathsf{inf}\{x \in \mathbb{R} : F(x) \geq u\} \\ &= \mathsf{inf}\{x \in \mathbb{R} : 1 - e^{-x} \geq u\} \\ &= \mathsf{inf}\{x \in \mathbb{R} : e^{-x} \leq 1 - u\} \\ &= \mathsf{inf}\{x \in \mathbb{R} : -x \leq \mathsf{log}(1-u)\} \\ &= \mathsf{inf}\{x \in \mathbb{R} : x \geq -\mathsf{log}(1-u)\} \\ &= -\mathsf{log}(1-u). \end{aligned}$$

Thus, to generate a random sample $X \sim$ Exponential(1) distribution, we first generate a random sample $U \sim$ Unif$(0,1)$, and simply set $X = -\mathsf{log}(1-U)$.