AI5030 / EE5817: Probability and Stochastic Processes

Homework 01

Functions, Countable Sets, Uncountable Sets

1. **Cantor's pairing function and countability of finite cartesian products**

   Let $\mathbb{W}$ denote the set of whole numbers, i.e., $\mathbb{W} = \mathbb{N} \cup \{0\}$. Consider the following depiction of the elements of the set $\mathbb{W} \times \mathbb{W}$ in which the rows are indexed by $m \in \mathbb{W}$, columns are indexed by $n \in \mathbb{W}$, and all pairs of whole numbers (m, n) with a constant value of $m + n$ have been colored identical (these pairs constitute the "diagonals" in the picture extending from bottom left to top right). For any $k \in \mathbb{W}$, let

   $$D_k := \{(m, n) \in \mathbb{W} \times \mathbb{W} \, : \, m + n = k\}$$

   denote the kth diagonal.

   $$\begin{array}{ccccccccc}
   (0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & \cdots \\
   (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (1,7) & \cdots \\
   (2,0) & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & (2,7) & \cdots \\
   (3,0) & (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) & (3,7) & \cdots \\
   (4,0) & (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) & (4,7) & \cdots \\
   (5,0) & (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) & (5,7) & \cdots \\
   (6,0) & (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) & (6,7) & \cdots \\
   (7,0) & (7,1) & (7,2) & (7,3) & (7,4) & (7,5) & (7,6) & (7,7) & \cdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
   \end{array}$$

   (a) If T_k denotes the number of pairs present on or to the left of the $(k-1)$th diagonal, show that $T_k = \frac{k(k+1)}{2}$.

   (b) Let $(0, 0)$ be assigned index 0, $(1, 0)$ be assigned index 1, $(0, 1)$ be assigned index 2, $(2, 0)$ be assigned index 3, $(1, 1)$ be assigned index 4, and so on. Show that the index of (m, n) is given by $\frac{(m+n)(m+n+1)}{2} + n$.
   Hint: Use the expression for T_k derived in part (a).

   (c) Let $f : \mathbb{W} \times \mathbb{W} \to \mathbb{W}$ denote the index assignment function of part (b), i.e.,

   $$f(m, n) = \frac{(m+n)(m+n+1)}{2} + n, \qquad (m, n) \in \mathbb{W} \times \mathbb{W}.$$

   The function f as defined above is called *Cantor's pairing function*. Show that f is bijective, and conclude that $\mathbb{W} \times \mathbb{W}$ is countably infinite.

   (d) Using the principle of mathematical induction, show that $\underbrace{\mathbb{W} \times \cdots \times \mathbb{W}}_{d \text{ times}}$ is countably infinite for every $d \in \mathbb{N}$.

2. **Countably infinite cartesian products of countable sets is uncountable**

   In this exercise, we will show that the countably infinite cartesian product of natural numbers, $\mathbb{N}^{\mathbb{N}} := \mathbb{N} \times \mathbb{N} \times \cdots$, is uncountable and has cardinality $\aleph_1$ (aleph$_1$).
   For this exercise, we use the fact from class that $|\{0, 1\}^{\mathbb{N}}| = \aleph_1$.

   (a) Construct an injective map from $\{0, 1\}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$. Prove formally that the constructed map is an injection, and hence conclude that $|\mathbb{N}^{\mathbb{N}}| \geq \aleph_1$.

   (b) Given a sequence of natural numbers $(a_1 a_2 a_3 \cdots) \in \mathbb{N}^{\mathbb{N}}$, consider the map

   $$g : (a_1 a_2 a_3 \cdots) \mapsto \underbrace{1 \cdots 1}_{a_1} 0 \underbrace{1 \cdots 1}_{a_2} 0 \underbrace{1 \cdots 1}_{a_3} 0 \cdots$$

   Show that the above map $g : \mathbb{N}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ is injective, and hence conclude that $|\mathbb{N}^{\mathbb{N}}| \leq \aleph_1$.

From parts (a) and (b) above, conclude that $|\mathbb{N}^{\mathbb{N}}| = \aleph_1$.

3. Suppose $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are bijective.
   Is $g \circ f : \mathcal{X} \to \mathcal{Z}$ bijective? Prove formally or give a counterexample.

4. Let $\mathscr{C} \subset \{0, 1\}^{\mathbb{N}}$ denote the subset of all infinite binary strings with finitely many 1s in them.
   For instance, $(\bar{0})$, $(01\bar{0})$, $(1\bar{0})$, $(11\bar{0}\cdots)$, $(101\bar{0})$, and so on are elements of $\mathscr{C}$; here, $\bar{0}$ is a shorthand for a countably infinite string of consecutive zeros.
   Show that $\mathscr{C}$ is countably infinite.
   Hint: Use the fact that countable union of countable sets is countable.

5. Fix a countable set A.

   (a) For any $n \in \mathbb{N}$, let B_n denote the collection of all possible n-tuples of the form $(a_1, a_2, \ldots, a_n)$, where $a_k \in A$ for each $k \in \{1, 2, \ldots, n\}$. Show that B_n is countable.
   Hence argue that $\bigcup_{n \in \mathbb{N}} B_n$ is countable.

   (b) A real number $x_0 \in \mathbb{R}$ is called *algebraic* if it is a root of a polynomial with rational coefficients. For example, $x_0 = \sqrt{2}$ is an algebraic number, as it is a root of the polynomial $x^2 - 2 = 0$ (whose coefficients are $1, -2$).
   Using the result in part (a) above, show that the set of all algebraic numbers is countable.
   Hint: Show that the there are only countably many polynomials with rational coefficients.

6. Let $\mathscr{D}$ denote the collection of all finite-length binary strings.

   (a) What is the cardinality of $\mathscr{D}$?

   (b) How does $\mathscr{D}$ differ from $\{0, 1\}^{\mathbb{N}}$? What are their cardinalities?

   (c) Produce an example of an element of $\mathscr{D}$ that is not present in $\{0, 1\}^{\mathbb{N}}$.