1 Sample Space, Algebra, σ-Algebra

1. Let Ω be a sample space, and let $\mathscr{F}$ be a σ-algebra of subsets of Ω.
Argue that $\mathscr{F}$ is closed under countable intersections.
Hint: Apply De Morgan's laws.

Solution: Let $A_1, A_2, \ldots \in \mathscr{F}$. Because $\mathscr{F}$ is closed under set complements, it follows that $A_1^c, A_2^c, \ldots \in \mathscr{F}$. Noting that $\mathscr{F}$ is closed under countable unions, it then follows that $\bigcup_{i=1}^{\infty} A_i^c \in \mathscr{F}$. Using De Morgan's law, we have

$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathscr{F}.$$

This proves the desired result.

2. Let Ω be a sample space. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two σ-algebras of subsets of Ω.
Show, via an example, that $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ is not necessarily a σ-algebra.
Note: This exercise shows that union of σ-algebras is not necessarily a σ-algebra.

Solution: Consider the following example:

$$\begin{aligned}\Omega &= \{1,2,3,4,5,6\},\\ \mathscr{F}_1 &= \{\phi, \Omega, \{1\}, \{2,3,4,5,6\}\},\\ \mathscr{F}_2 &= \{\phi, \Omega, \{2\}, \{1,3,4,5,6\}\}.\end{aligned}$$

Notice that $\{1\} \in \mathscr{F}_1 \cup \mathscr{F}_2$, $\{2\} \in \mathscr{F}_1 \cup \mathscr{F}_2$, but $\{1,2\} \notin \mathscr{F}_1 \cup \mathscr{F}_2$. Therefore, $\mathscr{F}_1 \cup \mathscr{F}_2$ is not a σ-algebra.

3. Let Ω be a sample space.

(a) Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two σ-algebras of subsets of Ω. Show that $\mathscr{F} = \mathscr{F}_1 \cap \mathscr{F}_2$ is also a σ-algebra.

(b) More generally, let $\mathcal{I}$ be an arbitrary index set (finite, countably infinite, or uncountable), and for each $i \in \mathcal{I}$, let $\mathscr{F}_i$ be a σ-algebra of subsets of Ω. Show that

$$\mathscr{F} = \bigcap_{i \in \mathcal{I}} \mathscr{F}_i$$

is also a σ-algebra.

This exercise shows that intersection of σ-algebras is necessarily a σ-algebra.

Solution: We prove the result in part (b) above, and note that the result in part (a) simply follows by setting $\mathcal{I} = \{1,2\}$. First, we note that $\Omega \in \mathscr{F}_i$ for every $i \in \mathcal{I}$, and therefore $\Omega \in \mathscr{F}$. Next, suppose that $A \in \mathscr{F}$. This implies that $A \in \mathscr{F}_i$ for every $i \in \mathcal{I}$, which in turn implies that $A^c \in \mathscr{F}_i$ for each $i \in \mathcal{I}$, and therefore $A^c \in \bigcap_{i \in \mathcal{I}} \mathscr{F}_i$. Lastly, suppose that $A_1, A_2, \ldots \in \mathscr{F}$ (or equivalently, $\{A_1, A_2, \ldots\} \subseteq \mathscr{F}$). This implies that $\{A_1, A_2, \ldots\} \subseteq \mathscr{F}_i$ for every $i \in \mathcal{I}$, from which it follows that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}_i$ for each $i \in \mathcal{I}$, thereby implying that $\bigcup_{j=1}^{\infty} A_i \in \mathscr{F}$. This demonstrates that $\mathscr{F}$ is a σ-algebra.

4. Let Ω be a sample space, and let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Fix $B \in \mathscr{F}$, and consider the collection

$$\mathscr{G} = \{A \cap B : A \in \mathscr{F}\}.$$

That is, $\mathscr{G}$ is a collection of subsets of B formed by taking the intersection of each set in $\mathscr{F}$ with B.
Show that $\mathscr{G}$ is a σ-algebra of subsets of B.

Solution:

(a) To see that $B \in \mathscr{G}$, we simply note that $B = \Omega \cap B$, and $\Omega \in \mathscr{F}$.

(b) Suppose that $C \in \mathscr{G}$. We now show that the complement of C with respect to B, i.e., $B \setminus C$, is an element of $\mathscr{G}$. Because $C \in \mathscr{G}$, it follows that $C = A \cap B$ for some $A \in \mathscr{F}$. Clearly, $A^c = \Omega \setminus A \in \mathscr{F}$. Furthermore, $B \setminus C = B \cap C^c = B \cap (B^c \cup A^c) = B \cap A^c$, where the complements A^c, B^c, C^c are with respect to Ω. Thus, we have $B \setminus C = A^c \cap B$, and noting that $A^c \in \mathscr{F}$, it follows that $B \setminus C \in \mathscr{G}$.

(c) Suppose that $C_1, C_2, \ldots \in \mathscr{G}$. Then, by definition, there exist sets $A_1, A_2, \ldots \in \mathscr{F}$ such that $C_1 = A_1 \cap B$, $C_2 = A_2 \cap B$, etc. We then note that $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$, and therefore $B \cap (\bigcup_{i=1}^{\infty} A_i) \in \mathscr{G}$. Using the distributive law of sets, we note that $B \cap (\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} B \cap A_i = \bigcup_{i=1}^{\infty} C_i$, thus proving that $\bigcup_{i=1}^{\infty} C_i \in \mathscr{G}$.

The above properties collectively demonstrate that $\mathscr{G}$ is σ-algebra of subsets of B.

5. Let Ω be a sample space. Consider the collection

$$\mathscr{A}_1 = \{A \subseteq \Omega : A \text{ is finite or } \Omega \setminus A \text{ is finite}\}. \tag{1}$$

(a) Prove that $\mathscr{A}_1$ is an algebra.

(b) Construct an example to show that $\mathscr{A}_1$ is not necessarily a σ-algebra.
Hint: Consider $\Omega = \mathbb{R}$ and $A = \mathbb{Q}$, the set of rational numbers. What do you know about $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$?

Solution:

(a) First, we note that $\Omega \in \mathscr{A}_1$, as $\Omega \setminus \Omega = \emptyset$ is finite. Next, suppose that $A \in \mathscr{A}_1$. Then, by definition, either A is finite or $\Omega \setminus A$ is finite. Equivalently, $\Omega \setminus A$ is finite or $\Omega \setminus (\Omega \setminus A) = A$ is finite, thereby proving that $\Omega \setminus A \in \mathscr{A}_1$. Lastly, fix $n \in \mathbb{N}$, and suppose that $A_1, A_2, \ldots, A_n \in \mathscr{A}_1$. Let $\mathcal{I} \subseteq \{1, \ldots, n\}$ be such that A_i is finite for each $i \in \mathcal{I}$. Notice that

$$\bigcup_{i=1}^{n} A_i = \left(\bigcup_{i \in \mathcal{I}} A_i\right) \cup \left(\bigcup_{i \notin \mathcal{I}} A_i\right).$$

If $\mathcal{I} = \{1, \ldots, n\}$, then it follows that $\bigcup_{i \in \mathcal{I}} A_i = \bigcup_{i=1}^{n} A_i$ is finite, and therefore belongs to $\mathscr{A}_1$. On the other hand, if $\mathcal{I} \subset \{1, \ldots, n\}$, then $\Omega \setminus A_i$ is finite for every $i \notin \mathcal{I}$. This implies that $\Omega \setminus (\bigcup_{i=1}^{n} A_i) \subset \bigcap_{i \notin \mathcal{I}} (\Omega \setminus A_i)$ is finite, and therefore $\Omega \setminus (\bigcup_{i=1}^{n} A_i) \in \mathscr{A}_1$. This proves that $\bigcup_{i=1}^{n} A_i \in \mathscr{A}_1$, thereby demonstrating that $\mathscr{A}_1$ is an algebra.

(b) Consider $\Omega = \mathbb{R}$, $A = \mathbb{N}$. Let $A_i = \{i\}$ for all $i \in \mathbb{N}$. Clearly, A_i is finite for each $i \in \mathbb{N}$. We now claim that $A = \bigcup_{i=1}^{\infty} A_i \notin \mathscr{A}_1$. Indeed, we have $A = \mathbb{N}$, and therefore neither A nor $\Omega \setminus A$ is finite. This shows that $\mathscr{A}_1$ is not closed under countable intersections, thereby failing to meet the requirements of a σ-algebra.

6. Let Ω be a sample space. Consider the collection

$$\mathscr{A}_2 = \{A \subseteq \Omega : A \text{ is countable or } \Omega \setminus A \text{ is countable}\}. \tag{2}$$

Prove that $\mathscr{A}_2$ is a σ-algebra.
Hint: Recall that countable means finite or countably infinite.
Use the lemma "countable union of countable sets is countable" covered in class.

Solution: First, we note that $\Omega \in \mathscr{A}_2$, as $\Omega \setminus \Omega = \emptyset$ is finite (hence countable). Next, suppose that $A \in \mathscr{A}_2$. Then, by definition, either A is countable or $\Omega \setminus A$ is countable. Equivalently, $\Omega \setminus A$ is countable or $\Omega \setminus (\Omega \setminus A) = A$ is countable, thereby proving that $\Omega \setminus A \in \mathscr{A}_2$. Lastly, suppose that $A_1, A_2, \ldots \in \mathscr{A}_2$. Let $\mathcal{I} \subseteq \{1, 2, \ldots\}$ be such that A_i is countable for each $i \in \mathcal{I}$. Notice that

$$\bigcup_{i=1}^{\infty} A_i = \left(\bigcup_{i \in \mathcal{I}} A_i\right) \cup \left(\bigcup_{i \notin \mathcal{I}} A_i\right).$$

If $\mathcal{I} = \{1, 2, \ldots\}$, then it follows that $\bigcup_{i \in \mathcal{I}} A_i = \bigcup_{i=1}^{\infty} A_i$ is countable (this follows from the fact that countable union of countable sets is countable), and therefore belongs to $\mathscr{A}_2$. On the other hand, if $\mathcal{I} \subset \{1, 2, \ldots\}$, then $\Omega \setminus A_i$ is countable for every $i \notin \mathcal{I}$. This implies that $\Omega \setminus (\bigcup_{i=1}^{\infty} A_i) \subset \bigcap_{i \notin \mathcal{I}} (\Omega \setminus A_i)$ is at most countable, and therefore $\Omega \setminus (\bigcup_{i=1}^{\infty} A_i) \in \mathscr{A}_1$. This proves that $\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}_2$, thereby demonstrating that $\mathscr{A}_2$ is a σ-algebra.