

## HOMEWORK 7

## TOPICS: ABSTRACT INTEGRALS, EXPECTATIONS OF DISCRETE RANDOM VARIABLES

1. Fix  $n \in \mathbb{N}$ . Consider the measure space  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ , where  $\lambda$  denotes the Lebesgue measure. Compute  $\int_{\mathbb{R}} f d\lambda$  for each of the following cases.

(a)  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined as

$$f(\omega) = \begin{cases} \omega, & \omega \in \{0, 1, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

(b)  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined as

$$f(\omega) = \begin{cases} 1, & \omega \in \mathbb{Q}^c \cap [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

(c)  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined as

$$f(\omega) = \begin{cases} n, & \omega \in \mathbb{Q}^c \cap [0, n], \\ 0, & \text{otherwise.} \end{cases}$$

2. Fix  $n \in \mathbb{N}$ . Let  $\Omega = \{\omega_1, \dots, \omega_n\}$ ,  $\mathcal{F} = 2^\Omega$ , and  $\mathbb{P}(\{\omega_i\}) = \frac{1}{n}$  for all  $i \in \{1, \dots, n\}$ . Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable defined with respect to  $\mathcal{F}$ . Compute  $\mathbb{E}[X] = \int_{\Omega} X d\mathbb{P}$  for the following cases.

(a)  $X = \mathbf{1}_A$ , where  $A = \{\omega_1, \dots, \omega_m\}$ , with  $1 \leq m \leq n$ .

(b)  $X$  is defined as

$$X(\omega) = \begin{cases} i, & \omega = \omega_i, \omega_i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

3. Let  $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . For a fixed  $c \in \mathbb{R}$ , define  $\delta_c : \mathcal{F} \rightarrow [0, 1]$  as

$$\delta_c(A) = \begin{cases} 1, & c \in A, \\ 0, & c \notin A. \end{cases}$$

(a) Show that  $\delta_c$  is a probability measure on  $(\Omega, \mathcal{F})$ .

**Remark:**  $\delta_c$  is called the Dirac measure at  $c$ .

It is referred to as “unit impulse” in the engineering literature, and sometimes (incorrectly) called a Dirac delta “function”.

(b) For any simple function  $g : \Omega \rightarrow \mathbb{R}$ , show that  $\int_{\Omega} g d\delta_c = g(c)$ .

(c) Extend the result in part (b) above to the case when  $g$  is non-negative.

(d) Let  $\mu : \mathcal{F} \rightarrow [0, +\infty]$  be defined as

$$\mu(A) = \sum_{n=1}^{\infty} \delta_n(A), \quad A \in \mathcal{F}.$$

Show that for any simple function  $g : \Omega \rightarrow \mathbb{R}$ ,

$$\int_{\Omega} g d\mu = \sum_{n=1}^{\infty} g(n).$$

Extend the above result to the case when  $g$  is non-negative.

**Remark:** Here,  $\mu$  is a measure on  $(\Omega, \mathcal{F})$ , and is called the “counting” measure.

For any given  $A \in \mathcal{F}$ ,  $\mu(A)$  is equal to the count of the number of positive integers present in the set  $A$ .

The above exercise shows that every summation is simply an integral with respect to the counting measure.

4. Suppose that  $N$  is a discrete random variable taking values in  $\mathbb{N}$ . Prove that

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}(\{N > n\}).$$

Hint: Notice that  $N = \sum_{n=0}^{N-1} 1 = \sum_{n=0}^{\infty} \mathbf{1}_{\{N > n\}}$ .

Apply expectations on both sides and use MCT to justify passing the expectation inside the infinite summation.

5. Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Let  $X : \Omega \rightarrow \mathbb{R}$  be a non-negative random variable with respect to  $\mathcal{F}$ .

- (a) Suppose that  $\mathbb{E}[X] < +\infty$ . Then, show that

$$\lim_{n \rightarrow \infty} n \mathbb{P}(\{X > n\}) = 0. \quad (1)$$

Hint: Write  $X = X \mathbf{1}_{\{X \leq n\}} + X \mathbf{1}_{\{X > n\}}$ .

For each  $n \in \mathbb{N}$ , let  $X_n = X \mathbf{1}_{\{X \leq n\}}$ .

Show that  $0 \leq X_n \leq X_{n+1}$  for all  $n$ , and  $X_n \xrightarrow{\text{pointwise}} X$ . Using MCT, compute  $\lim_{n \rightarrow \infty} \mathbb{E}[X_n]$ .

Show that  $\lim_{n \rightarrow \infty} \mathbb{E}[X \mathbf{1}_{\{X > n\}}] = 0$ .

Finally, argue that  $0 \leq \lim_{n \rightarrow \infty} n \mathbb{P}(\{X > n\}) \leq \lim_{n \rightarrow \infty} \mathbb{E}[X \mathbf{1}_{\{X > n\}}] = 0$ .

- (b) Produce an example of a random variable  $X$  for which  $\mathbb{E}[X] = +\infty$ , and

$$\lim_{n \rightarrow \infty} n \mathbb{P}(\{X > n\}) > 0.$$

This exercise shows that (1) holds only when  $\mathbb{E}[X] < +\infty$ .

6. Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Let  $X : \Omega \rightarrow [0, +\infty]$  be a non-negative, extended real-valued random variable with respect to  $\mathcal{F}$ .

(Here,  $X$  is allowed take the value  $+\infty$ .)

- (a) Show that  $\{X = +\infty\} = \{\omega \in \Omega : X(\omega) = +\infty\} \in \mathcal{F}$ .

Hint: If  $X(\omega) = +\infty$ , then  $X(\omega) > N$  for all  $N \in \mathbb{N}$ .

- (b) Show that  $\mathbb{E}[X] < +\infty$  implies that

$$\mathbb{P}(\{X < +\infty\}) = 1.$$

Hint: We have to show that  $\mathbb{P}(\{X = +\infty\}) = 0$ . We will do this by contradiction.

Let  $L = \mathbb{E}[X]$ . Suppose that  $\mathbb{P}(\{X = +\infty\}) = p > 0$ .

Let  $C = \{X > 2L/p\}$ . Using the reasoning of part (a), argue that  $\mathbb{P}(C) \geq p$ .

From class, we know that there exists a sequence of simple random variables  $\{X_n\}_{n=1}^{\infty}$  such that  $X_n \xrightarrow{\text{pointwise}} X$ . Using the pointwise convergence property and MCT, argue that

$$\mathbb{E}[X] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n] \geq \lim_{n \rightarrow \infty} \mathbb{E}[X_n \mathbf{1}_C] \geq \frac{2L}{p} \mathbb{P}(C) \geq 2L,$$

thereby leading to a contradiction.

- (c) Construct an example of a non-negative random variable for which  $\mathbb{P}(\{X < +\infty\}) = 1$ , yet  $\mathbb{E}[X] = +\infty$ .

This exercise shows that  $\mathbb{P}(\{X < +\infty\}) = 1$  does not imply  $\mathbb{E}[X] < +\infty$ .

7. A biased coin with heads probability  $p \in (0, 1)$  is tossed repeatedly.

Let  $X_n \in \{0, 1\}$  denote the outcome of the  $n$ th toss,  $n \in \mathbb{N}$ .

Let  $N$  be defined as the random variable

$$N := \min\{n \geq 2 : X_n = 1 - X_1\}.$$

That is,  $N$  is the first time index  $n \geq 2$  for which the outcome  $X_n$  is the complement of the first outcome.

- (a) Compute the PMF of  $N$ .

- (b) Show that

$$\mathbb{E}[N] = \frac{p}{q} + \frac{q}{p},$$

where  $q = 1 - p$ .