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Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 01: Functions, Cardinality, Countability

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Functions

Definition (Function)

Given two sets A, B, a function $f : A \rightarrow B$ is a rule that maps each element of A to a unique element of B.

- For every $x \in A$,

$$f : x \mapsto f(x) \in B$$

- A is called the domain of f
- B is called the co-domain of f

Note

While every element of A is mapped to some element of B, the converse may not always be true.

Range of a Function

Definiton (Range)

The range of a function $f : A \to B$, denoted by $R(f)$, is the subset of B defined as

$$R(f) = \left\{ y \in B : y = f(x) \text{ for some } x \in A \right\}.$$

- Given $x \in A$, if $f(x) = y$, then y is called the image of x (under f)
- Given $y \in B$, the set $f^{-1}(y) := \{x \in A : f(x) = y\}$ is called the pre-image of y

Image and Pre-Image

- A function $f : A \to B$ is said to be injective if f is *one-one*, i.e., each element of $R(f)$ has a unique pre-image
- A function $f : A \to B$ is said to be surjective if it is *onto*, i.e., range $=$ codomain
- A function $f : A \to B$ is said to be bijective if it is both injective and surjective

Note

- If $f : A \to B$ is bijective, then for each $y \in B$, there exists a unique element $x \in A$ such that $f^{-1}(y) = \{x\}$. In this case, we simply write $f^{-1}(y) = x$.
- Alternatively, if $f : A \to B$ is bijective, we have $f^{-1} : B \to A$.

Cardinality

Definition (Cardinality)

Notation: $|A|$ = cardinality of set A

- Two sets A and B are said to be equicardinal ($|A| = |B|$) if there exists $f : A \to B$ bijective.
- $|B| \geq |A|$ if there exists $f : A \to B$ injective
- $|B| > |A|$ if there exists $f : A \to B$ injective, and A and B are not equicardinal (i.e., no bijective function mapping A to B exists)

Note

$|A|$ is representative of the number of elements in A.

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Countability

- A set A is said to be finite if A is empty or $|A| = |\{1, \ldots, n\}| = n$ for some $n \in \mathbb{N}$
- A set A is said to be countably infinite if $|A| = |\mathbb{N}|$, where $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers
- A set A is countable if either $|A| < +\infty$ or $|A| = |\mathbb{N}|$

Remark

If A is countably infinite, then its elements may be listed as $A = \{a_1, a_2, \ldots\}$.

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Examples of Countable Sets

- Set of odd natural numbers, set of even natural numbers
- Set of integers, $\mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$
- Set of prime numbers
- Set of rational numbers, $\mathbb{Q}$

$\mathbb{Q}$ is Countable – Proof

Step 1: $\mathbb{Q} \cap [0, 1]$ is countable. Indeed, note that

$$\mathbb{Q} \cap [0, 1] = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots\right\}.$$

Step 2: "Countable union of countable sets is countable."

Lemma

Let $\mathcal{I}$ be a countable index set, and let $\{A_i : i \in \mathcal{I}\}$ be a countable collection of countable sets. Then, $\bigcup_{i\in\mathcal{I}} A_i$ is countable.

Step 3: Complete the proof using the above lemma.

Uncountable Sets

Definition (uncountable sets)

A set A is said to be uncountable if it is not countable, i.e., if $|A| > |\mathbb{N}|$.

Some examples of uncountable sets:

- Unit interval, $[0, 1]$
- Set of all real numbers, $\mathbb{R}$
- Set of all irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$
- Set of all infinite length binary strings, denoted commonly as $\{0, 1\}^{\mathbb{N}}$ or $\{0, 1\}^{\infty}$
- Power set of $\mathbb{N}$ (collection of all subsets of $\mathbb{N}$), denoted $2^{\mathbb{N}}$

$\{0,1\}^{\mathbb{N}}$ is Uncountable – Proof

It suffices to demonstrate that there exists an injective map but no bijective map from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$.

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$$f(n) = \text{infinite binary string with '1' in the } n\text{th index.}$$

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No bijective map: Suppose there exists a bijective map $g : \mathbb{N} \to \{0, 1\}^{\mathbb{N}}$. Let

$$g : n \mapsto a_{n1}\, a_{n2}\, a_{n3}\, \cdots ,$$

where $a_{nj} \in \{0, 1\}$ for all n, j.

$\{0, 1\}^{\mathbb{N}}$ is Uncountable – Proof

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Cantor's diagonalisation argument: Consider the binary string

$$b = \bar{a}_{11}\, \bar{a}_{22}\, \bar{a}_{33} \cdots\ ,$$

where $\bar{a}_{jj} = 1 - a_{jj}$ for all $j \in \mathbb{N}$. Then, $\nexists\, n \in \mathbb{N}$ such that $g(n) = b$. Thus, g is not a bijection.

$[0, 1]$ is Uncountable – Proof

Let

$$\mathcal{D} = \left\{ d_1 = \frac{1}{2}, d_2 = \frac{1}{4}, d_3 = \frac{3}{4}, d_4 = \frac{1}{8}, \dots \right\}$$ – set of dyadic rational numbers

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Define $g : \{0, 1\}^{\mathbb{N}} \to [0, 1]$ defined as

$$g : b = (b_1\, b_2\, \cdots) \mapsto \begin{cases} \sum_{k=1}^{\infty} \frac{b_k}{2^k}, & b \notin \mathcal{D}, \\ d_1, & b = (100\cdots) \\ d_2, & b = (011\cdots) \\ d_3, & b = (0100\cdots) \\ d_4, & b = (0011\cdots) \\ \vdots \end{cases}$$

Prove that g is a bijection!

- $2^{\mathbb{N}}$ is uncountable – exercise!
- $\mathbb{R}$ is uncountable
  Consider the function $f : [0, 1] \to \mathbb{R}$ defined via

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right), \quad x \in [0, 1].$$

- $\mathbb{R} \setminus \mathbb{Q}$ is uncountable
  Write $\mathbb{R}$ as

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}.$$

Reading Exercise

To be acquainted with the formal proof of the lemma introduced on slide 7, see [Royden and Fitzpatrick, 2010, Section 1.3].

Royden, H. and Fitzpatrick, P. M. (2010).
Real Analysis.
China Machine Press.