# Probability and Stochastic Processes

## Lecture 02: Countable Sets, Uncountable Sets

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01 August 2025

# Countability

- A set $A$ is said to be finite if $A$ is empty or $|A| = |\{1, \ldots, n\}| = n$ for some $n \in \mathbb{N}$
- A set $A$ is said to be countably infinite if $|A| = |\mathbb{N}|$, where $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers
- A set $A$ is countable if either $|A| < +\infty$ or $|A| = |\mathbb{N}|$

**Remark**

If $A$ is countably infinite, then its elements may be listed as $A = \{a_1, a_2, \ldots\}$.

# Examples of Countable Sets

- Set of odd natural numbers, set of even natural numbers
- Set of integers, $\mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$
- Set of prime numbers
- Set of rational numbers, $\mathbb{Q}$

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# $\mathbb{Q}$ is Countable – Proof

**Step 1:** $\mathbb{Q} \cap [0, 1]$ is countable. Indeed, we have

$$\mathbb{Q} \cap [0, 1] = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots \right\}.$$

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Step 2: "Countable union of countable sets is countable."

**Lemma**

If $A_1, A_2, \dots$ is a collection of countable sets, then their union is countable.

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**Lemma**

If $A_1, A_2, \ldots$ is a collection of countable sets, then their union is countable.

Step 3: Use above lemma in conjunction with

$$\mathbb{Q} = \mathbb{Q} \cap \left(\bigcup_{i \in \mathbb{Z}} [i, i+1]\right) = \bigcup_{i \in \mathbb{Z}} \mathbb{Q} \cap [i, i+1].$$

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# Proof of Lemma – 1

**Lemma (Countable Union of Countable Sets is Countable)**

If $A_1, A_2, \ldots$ is a countably infinite collection of countable sets, then their union is countable.

- Let $A = A_1 \cup A_2 \cup \cdots$; we want to prove that $A$ is countable (i.e., finite or countably infinite)
- Fix any $n \in \mathbb{N}$. Because $A_n$ is countable, there exists an injection

$$f_n : A_n \to \mathbb{N}.$$

- For any $a \in A$, let $f : A \to \mathbb{N} \times \mathbb{N}$ be defined as

$$f(a) = (n_a, f_{n_a}(a)),$$

where $n_a = \min\{n \in \mathbb{N} : a \in A_n\}$

# Proof of Lemma – 2

- **Claim:** $f$ is an injection. Indeed, for any $a, b \in A$,

$$\begin{aligned} f(a) = f(b) &\implies (n_a, f_{n_a}(a)) = (n_b, f_{n_b}(b)) \\ &\implies n_a = n_b = n \text{ (say)}, \qquad a, b \in A_n, \qquad f_n(a) = f_n(b) \\ &\implies a = b \qquad \text{(because } f_n \text{ is injective)} \end{aligned}$$

- **Homework:** $\mathbb{N} \times \mathbb{N}$ is countably infinite
- Putting the pieces together,

$$f : A \to \mathbb{N} \times \mathbb{N} \text{ injective}, \quad \mathbb{N} \times \mathbb{N} \text{ countably infinite} \quad \implies \quad A \text{ is countable.}$$

# Uncountable Sets

**Definition (uncountable sets)**

A set $A$ is said to be uncountable if it is not countable, i.e., if $|A| > |\mathbb{N}|$.

Some examples of uncountable sets:

- Set of all countably infinite length binary strings, denoted commonly as $\{0, 1\}^{\mathbb{N}}$
- Unit interval, $[0, 1]$
- Set of all real numbers, $\mathbb{R}$
- Set of all irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$
- Power set of $\mathbb{N}$ (collection of all subsets of $\mathbb{N}$), denoted $2^{\mathbb{N}}$

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Suffices to show that there exists an injective map but no bijective map from $\mathbb{N}$ to $\{0, 1\}^{\mathbb{N}}$.

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No bijective map: Suppose there exists a bijective map $g : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$. Let

$$g : n \mapsto a_{n1}\, a_{n2}\, a_{n3}\, \cdots ,$$

where $a_{nj} \in \{0,1\}$ for all $n, j$.

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Cantor's diagonalization argument: Consider the binary string

$$b = \bar{a}_{11}\, \bar{a}_{22}\, \bar{a}_{33} \cdots ,$$

where $\bar{a}_{jj} = 1 - a_{jj}$ for all $j \in \mathbb{N}$. Then, $\nexists\, n \in \mathbb{N}$ such that $g(n) = b$. Thus, $g$ is not a bijection.

# $[0, 1]$ is Uncountable – Proof

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Let

$$\mathcal{D} = \left\{ d_1 = \frac{1}{2}, d_2 = \frac{1}{4}, d_3 = \frac{3}{4}, d_4 = \frac{1}{8}, \ldots \right\} \quad - \quad \text{set of dyadic rational numbers}$$

Define $g : \{0,1\}^{\mathbb{N}} \to [0,1]$ as

$$g : b = (b_1\, b_2\, \cdots) \mapsto \begin{cases} \sum_{k=1}^{\infty} \frac{b_k}{2^k}, & b \notin \mathcal{D}, \\ d_1, & b = (100\cdots) \\ d_2, & b = (011\cdots) \\ d_3, & b = (0100\cdots) \\ d_4, & b = (0011\cdots) \\ \vdots \end{cases}$$

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**Claim:** $g$ is a bijection!

# Examples of Uncountable Sets

- $[0, 1]$
- $2^{\mathbb{N}} =$ power set of $\mathbb{N}$ (**exercise**)
- $\mathbb{R}$: the set of real numbers.
  Hint: consider the function $f : [0, 1] \to \mathbb{R}$ defined via

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right), \quad x \in [0, 1].$$

- $\mathbb{R} \setminus \mathbb{Q}$: the set of irrational numbers.
  Hint: write $\mathbb{R}$ as

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}.$$

- Cantor set