# AI5030 / EE5817: Probability and Stochastic Processes

## Homework 05

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### PDF, PMF, Joint Distributions

1. (a) Show that if $X \sim \text{Poisson}(\lambda)$ for some fixed $\lambda > 0$, then its PMF $p_X$ satisfies the relation

$$p_X(k-1) \cdot p_X(k+1) \leq (p_X(k))^2 \qquad \forall\, k \in \mathbb{N}.$$

   (b) Give an example for a PMF $p_X$ satisfying the relation

$$p_X(k-1) \cdot p_X(k+1) = (p_X(k))^2 \qquad \forall\, k \in \mathbb{N}.$$

2. Suppose that $X$ and $Y$ are jointly discrete, integer-valued random variables, with the joint PMF

$$p_{X,Y}(m,n) = \frac{\lambda^n\, e^{-2\lambda}}{m!\,(n-m)!}, \qquad 0 \leq m \leq n < +\infty, \quad m,n \in \{0,1,2,\ldots\}.$$

   (a) Determine a countable set $E \subset \mathbb{R}^2$ such that $\mathbb{P}_{X,Y}(E) = 1$.
   (b) Using the result of part (a) above, determine $E_1, E_2 \subseteq \mathbb{R}$, defined as

$$E_1 \coloneqq \{x \in \mathbb{R} :\ \exists y \in \mathbb{R} \text{ such that } (x,y) \in E\}, \qquad E_2 \coloneqq \{y \in \mathbb{R} :\ \exists x \in \mathbb{R} \text{ such that } (x,y) \in E\}.$$

   (c) Determine the marginal PMFs of $X$ and $Y$.
   (d) Determine the conditional PMF of $X$ conditioned on the event $\{Y = n\}$ for some fixed $n \in \{0,1,2,\ldots\}$.
   (e) Are $X$ and $Y$ independent? Justify.

3. A total of $n$ coins, each with probability of heads $p$, are tossed independently of one another. Each coin that lands up heads is tossed again. Determine the PMF of the number of heads that shows up after the second round of tossing.

4. Suppose that $X$ and $Y$ are jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c\,x\,(y-x)e^{-y}, & 0 \leq x \leq y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

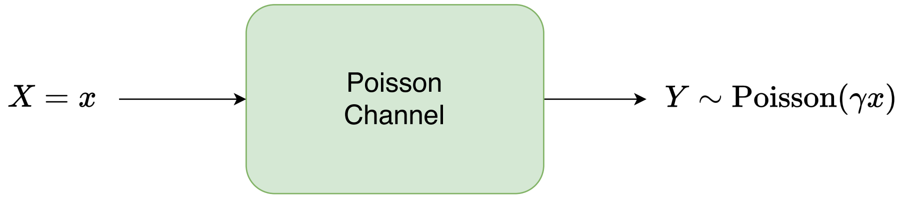
   (a) Determine the constant $c$.
   (b) Show that

$$f_{X|Y=y}(x) = \begin{cases} 6x(y-x)y^{-3}, & 0 \leq x \leq y, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{Y|X=x}(y) = \begin{cases} (y-x)e^{x-y}, & y \geq x, \\ 0, & \text{otherwise.} \end{cases}$$

5. Suppose that $X$ and $Y$ are jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cy, & -1 \leq x \leq 1,\ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the constant $c$.
   (b) Are $X$ and $Y$ independent?
   (c) Evaluate $\mathbb{P}(\{X \geq Y + 0.5\})$.
   (d) Compute the conditional PDF of $X$, conditioned on the event $\{Y > 0.5\}$.
   Using the above conditional PDF, evaluate $\mathbb{P}(\{X > 0.75\} | \{Y > 0.5\})$.




6. **(The Poisson Channel)**

   The Poisson channel was developed around 45 years ago as a model for an optical communication link. Suppose that $X \sim$ Exponential$(1)$. Let $Y$ be related to $X$ as in the figure above.
   Here, $\gamma > 0$ is a fixed constant known as channel signal-to-noise ratio (SNR). You make assume $\gamma = 1$.

   (a) For a fixed $x > 0$, what is the conditional PMF of $Y$, conditioned on $\{X = x\}$?

   (b) Using the law of total probability, determine the (unconditional) PMF of $Y$.

   (c) Specify a countable set $E \subset \mathbb{R}$ such that $\mathbb{P}_Y(E) = 1$.

   (d) For a fixed $y \in E$, determine the conditional PDF of $X$, conditioned on $\{Y = y\}$.

   (e) Determine $\mathbb{P}(\{X \in B_1\} \cap \{Y \in B_2\})$, where $B_1 = (1, \infty)$ and $B_2 = \{1\}$.