Stochastic Processes

Convergence Notions, Proofs of Implications, Problems

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

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Almost-Sure Convergence and $\mathrm{i.o.}$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

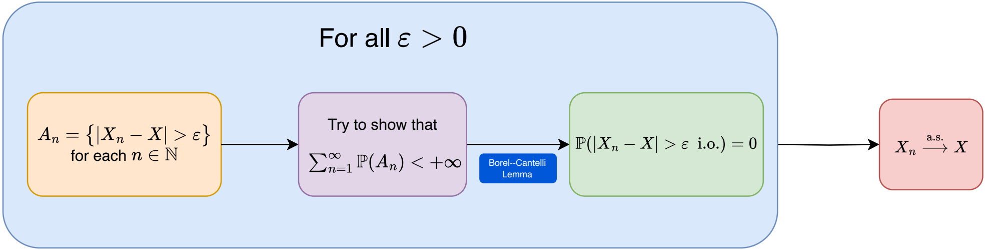
Proposition

The following statements are equivalent.

1. $X_n \xrightarrow{\text{a.s.}} X$.
2. For every $\varepsilon > 0$,
$$\mathbb{P}(\{|X_n - X| \geq \varepsilon\}\ \mathrm{i.o.}) = 0.$$

Borel–Cantelli Lemma and Almost-Sure Convergence

A Generic Template



Convergence in Probability

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Convergence in Probability)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X in probability (p.) if

$$\forall \varepsilon > 0, \qquad \lim_{n\to\infty} \mathbb{P}(\{|X_n - X| > \varepsilon\}) = 0.$$

Notation:

$$X_n \xrightarrow{\text{p.}} X.$$

Note

The in-probability limit is only specified up to sets of zero probability. That is,

$$X_n \xrightarrow{\text{p.}} X, \;\; X_n \xrightarrow{\text{p.}} Y \quad \Longrightarrow \quad \mathbb{P}(X = Y) = 1.$$

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Mean-Squared Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Mean-Squared Convergence)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X in mean-squared (m.s.) sense if

- $\mathbb{E}[X_n^2] < +\infty$ for all $n \in \mathbb{N}$.
- We have

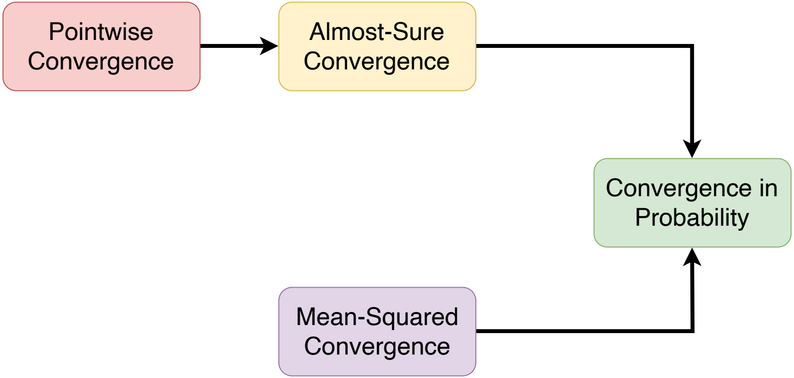
$$\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0.$$

Notation:

$$X_n \xrightarrow{\text{m.s.}} X.$$

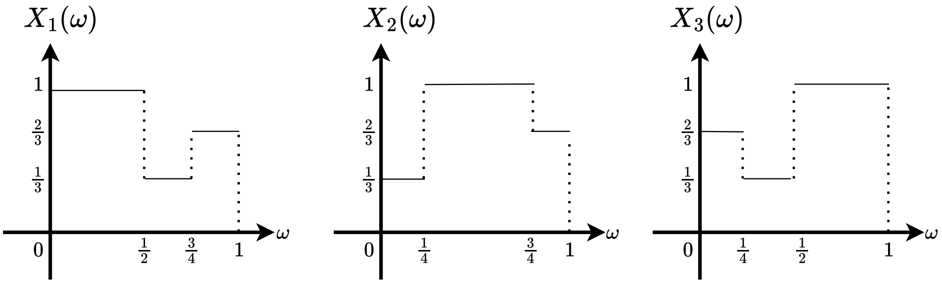
$$X_n \xrightarrow{\text{m.s.}} X, \;\; X_n \xrightarrow{\text{m.s.}} Y \quad \Longrightarrow \quad \mathbb{P}(X = Y) = 1.$$

A Picture to Have in Mind



Example

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.



Let $X_n = X_{n+3}$ for all $n \in \mathbb{N}$.
Identify forms of convergence and their corresponding limits.

Remarks on Previous Example

- The sequence of RVs do not converge pointwise, almost-surely, in mean-squared sense, or in probability
- However, the PMFs (hence CDFs) of X_1, X_2, X_3 are identical, hence there is convergence of CDFs

Convergence of CDFs – A Subtle Point

Let $U \sim \mathrm{Unif}[0,1]$.
For each $n \in \mathbb{N}$, let

$$X_n = \frac{(-1)^n\, U}{n}.$$

- Guess a limit RV.
- Identify forms of convergence to the above limit.
- Comment about convergence of the sequence of CDFs $\{F_{X_n}\}_{n=1}^{\infty}$.

Convergence in Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Convergence in Distribution)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X in distribution (d.) if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \qquad \forall x \in C_{F_X},$$

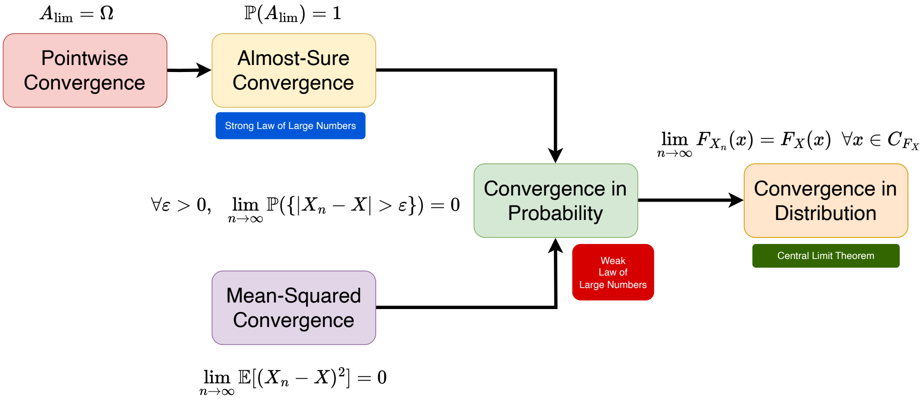
where C_{F_X} denotes the points of continuity of F_X.

Notation:

$$X_n \xrightarrow{\text{d.}} X.$$

Convergence – The Full Picture

$$A_{\lim} = \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\}$$



Proofs of Implications

Review: Continuity of Probability Measure

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, where $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then
$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$
- If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, where $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then
$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Proofs of Implications (a.s. $\implies$ p.)

Given: $X_n \xrightarrow{\text{a.s.}} X$

To prove: $\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

Proofs of Implications (a.s. $\implies$ p.)

Given: $X_n \xrightarrow{\text{a.s.}} X$

To prove: $\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

$$X_n \xrightarrow{\text{a.s.}} X \quad \implies \quad \forall \varepsilon > 0, \quad \mathbb{P}\left(\{|X_n - X| > \varepsilon\} \text{ i.o.}\right) = 0$$

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Given: $X_n \xrightarrow{\text{a.s.}} X$

To prove: $\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

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$$\implies \quad \forall \varepsilon > 0, \quad \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{|X_k - X| > \varepsilon\}\right) = 0$$

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\begin{aligned}
X_n \xrightarrow{\text{a.s.}} X \quad &\implies \quad \forall \varepsilon > 0, \quad \mathbb{P}\left(\{|X_n - X| > \varepsilon\} \text{ i.o.}\right) = 0 \\
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Proofs of Implications (m.s. $\implies$ p.)

Given: $X_n \xrightarrow{\text{m.s.}} X$

To prove: $\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

Proofs of Implications (m.s. $\implies$ p.)

Given: $X_n \xrightarrow{\text{m.s.}} X$

To prove: $\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

$$X_n \xrightarrow{\text{m.s.}} X \quad \implies \quad \lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$$

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Proofs of Implications (m.s. $\implies$ p.)

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To prove: $\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

$$X_n \xrightarrow{\text{m.s.}} X \quad \implies \quad \lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$$

For every $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X| > \varepsilon) \leq \frac{\mathbb{E}[(X_n - X)^2]}{\varepsilon^2}$$

$$\implies \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) \leq \frac{\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2]}{\varepsilon^2}$$

Proofs of Implications ($\text{m.s.} \implies \text{p.}$)

Given: $X_n \xrightarrow{\text{m.s.}} X$

To prove: $\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

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For every $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X| > \varepsilon) \leq \frac{\mathbb{E}[(X_n - X)^2]}{\varepsilon^2}$$

$$\implies \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) \leq \frac{\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2]}{\varepsilon^2} = 0$$

Proofs of Implications ($\mathrm{p.} \implies \mathrm{d.}$)

Given: $X_n \xrightarrow{\mathrm{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Pick arbitrary $x \in C_{F_X}$ and $\varepsilon > 0$

$$F_{X_n}(x) = \mathbb{P}(X_n \le x)$$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Pick arbitrary $x \in C_{F_X}$ and $\varepsilon > 0$

$$F_{X_n}(x) = \mathbb{P}(X_n \leq x) = \mathbb{P}(X_n \leq x,\ X \leq x + \varepsilon) + \mathbb{P}(X_n \leq x,\ X > x + \varepsilon)$$

Proofs of Implications ($\mathrm{p.} \implies \mathrm{d.}$)

Given: $X_n \xrightarrow{\mathrm{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Pick arbitrary $x \in C_{F_X}$ and $\varepsilon > 0$

$$F_{X_n}(x) = \mathbb{P}(X_n \le x) = \mathbb{P}(X_n \le x,\ X \le x+\varepsilon) + \mathbb{P}(X_n \le x,\ X > x+\varepsilon)$$
$$\le \mathbb{P}(X \le x+\varepsilon) + \mathbb{P}(X_n - X < -\varepsilon)$$

Proofs of Implications ($\text{p.} \implies \text{d.}$)

Given: $X_n \xrightarrow{\text{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Pick arbitrary $x \in C_{F_X}$ and $\varepsilon > 0$

$$\begin{aligned} F_{X_n}(x) &= \mathbb{P}(X_n \le x) = \mathbb{P}(X_n \le x,\ X \le x + \varepsilon) + \mathbb{P}(X_n \le x,\ X > x + \varepsilon) \\ &\le \mathbb{P}(X \le x + \varepsilon) + \mathbb{P}(X_n - X < -\varepsilon) \\ &\le F_X(x + \varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon) \end{aligned}$$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

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$$\implies \quad \lim_{n\to\infty} F_{X_n}(x) \le F_X(x+\varepsilon) + \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon)$$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

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\begin{aligned}
F_{X_n}(x) = \mathbb{P}(X_n \le x) &= \mathbb{P}(X_n \le x,\ X \le x+\varepsilon) + \mathbb{P}(X_n \le x,\ X > x+\varepsilon) \\
&\le \mathbb{P}(X \le x+\varepsilon) + \mathbb{P}(X_n - X < -\varepsilon) \\
&\le F_X(x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
\implies \quad & \lim_{n\to\infty} F_{X_n}(x) \le F_X(x+\varepsilon) + \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) \\
\implies \quad & \lim_{n\to\infty} F_{X_n}(x) \le F_X(x+\varepsilon) \qquad (1)
\end{aligned}
$$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Proofs of Implications ($\mathrm{p.} \implies \mathrm{d.}$)

Given: $X_n \xrightarrow{\mathrm{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Pick arbitrary $x \in C_{F_X}$ and $\varepsilon > 0$

$$F_X(x-\varepsilon) = \mathbb{P}(X \le x - \varepsilon)$$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Pick arbitrary $x \in C_{F_X}$ and $\varepsilon > 0$

$$\begin{aligned} F_X(x-\varepsilon) &= \mathbb{P}(X \le x-\varepsilon) \\ &= \mathbb{P}(X \le x-\varepsilon,\ X_n \le x) + \mathbb{P}(X \le x-\varepsilon,\ X_n > x) \\ &\le \mathbb{P}(X_n \le x) + \mathbb{P}(X_n - X > \varepsilon) \end{aligned}$$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

To prove: $\forall x \in C_{F_X}, \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Pick arbitrary $x \in C_{F_X}$ and $\varepsilon > 0$

$$\begin{aligned}
F_X(x-\varepsilon) &= \mathbb{P}(X \le x-\varepsilon) \\
&= \mathbb{P}(X \le x-\varepsilon,\ X_n \le x) + \mathbb{P}(X \le x-\varepsilon,\ X_n > x) \\
&\le \mathbb{P}(X_n \le x) + \mathbb{P}(X_n - X > \varepsilon) \\
&\le F_{X_n}(x) + \mathbb{P}(|X_n - X| > \varepsilon)
\end{aligned}$$

Proofs of Implications (p. $\implies$ d.)

Given: $X_n \xrightarrow{\text{p.}} X$

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&\le \mathbb{P}(X_n \le x) + \mathbb{P}(X_n - X > \varepsilon) \\
&\le F_{X_n}(x) + \mathbb{P}(|X_n - X| > \varepsilon)
\end{aligned}$$

$$\implies \quad F_X(x-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(x) + \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon)$$

Proofs of Implications (p. $\implies$ d.)

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\begin{aligned}
F_X(x-\varepsilon) &= \mathbb{P}(X \le x-\varepsilon) \\
&= \mathbb{P}(X \le x-\varepsilon,\ X_n \le x) + \mathbb{P}(X \le x-\varepsilon,\ X_n > x) \\
&\le \mathbb{P}(X_n \le x) + \mathbb{P}(X_n - X > \varepsilon) \\
&\le F_{X_n}(x) + \mathbb{P}(|X_n - X| > \varepsilon)
\end{aligned}
$$

$$
\implies \quad F_X(x-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(x) + \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon)
$$

$$
\implies \quad F_X(x-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(x) \tag{2}
$$

Proofs of Implications (p. $\implies$ d.)

Combining (1) and (2),

$$\forall x \in C_{F_X}, \;\; \varepsilon > 0, \qquad F_X(x-\varepsilon) \leq \lim_{n\to\infty} F_{X_n}(x) \leq F_X(x+\varepsilon)$$

Proofs of Implications (p. $\implies$ d.)

Combining (1) and (2),

$$\forall x \in C_{F_X},\ \ \varepsilon > 0, \qquad F_X(x-\varepsilon) \le \lim_{n\to\infty} F_{X_n}(x) \le F_X(x+\varepsilon)$$

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Combining (1) and (2),

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$$\implies \forall x \in C_{F_X}, \qquad F_X(x) \le \lim_{n\to\infty} F_{X_n}(x) \le F_X(x)$$