AI5030 / EE5817: PROBABILITY AND STOCHASTIC PROCESSES

HOMEWORK 05

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TRANSFORMATIONS OF RANDOM VARIABLES

1. Let $X, Y \overset{\text{IID}}{\sim} \mathcal{N}(0,1)$, where X and Y are both defined on a common underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
   (a) Argue formally that $\{X = Y\} = \{\omega \in \Omega : X(\omega) = Y(\omega)\} \in \mathscr{F}$.
   (b) Compute $\mathbb{P}(\{X = Y\})$.

2. Fix $q \in (0,1)$. Let $U \sim \text{Unif}((0,1))$, and let
   $$X = \lfloor \log_q U \rfloor + 1,$$
   where $\lfloor x \rfloor$ denotes the largest integer lesser than or equal to x (for e.g., $\lfloor 0.3 \rfloor = 0$, $\lfloor 4.99 \rfloor = 4$, $\lfloor 2 \rfloor = 2$, and so on). Here, $\log_q U$ denotes the logarithm of U to the base q.
   Determine the PMF of X.
   Hint: List down the possible values of $\lfloor \log_q U \rfloor$.

3. Let X, Y be jointly continuous with the joint PDF
   $$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{x}, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Determine the PDF of $Z = X + Y$.
   (b) Compute $\mathbb{P}(\{Z \le 1\})$.

4. Numbers from $[0,1]$ are picked uniformly, independently, and sequentially over time.
   Let X_n denote the number picked at time n, where $n \in \{0,1,2,\ldots\}$.
   Let N be the random variable defined as
   $$N = \min\{n \ge 1 : X_n < X_0\}.$$
   That is, N denotes the first time index $n \ge 1$ at which the value of X_n goes below the value of X_0.
   (a) For any fixed $n \in \mathbb{N}$, determine $\mathbb{P}(\{N = n\})$.
   Hint: The event that $N = n$ is identical to the event that
   $$X_1 \ge X_0 \quad \text{and} \quad X_2 \ge X_0 \quad \text{and} \quad \cdots \quad \text{and} \quad X_{n-1} \ge X_0 \quad \text{and} \quad X_n < X_0.$$
   (b) Compute $\mathbb{P}(\{N > 2\})$.

5. Let $X_1, X_2 \overset{\text{IID}}{\sim} \text{Exponential}(\lambda)$ for a fixed $\lambda > 0$.
   (a) Determine the PDF of $Y_1 = \frac{X_1}{X_2}$.
   Hint: Compute the CDF $\mathbb{P}(\{\frac{X_1}{X_2} \le x\}) = \mathbb{P}(\{X_1 \le x\, X_2\})$ and differentiate it to get the PDF.
   (b) Determine the PDF of $Y_2 = X_1 - X_2$.
   Hint: Compute the CDF $\mathbb{P}(\{X_1 - X_2 \le x\})$ and differentiate it to get the PDF.

6. **(Memoryless Property)**
   (a) Show that a discrete random variable $X \sim \text{Geometric}(p)$ for some $p < 1$ if and only if satisfies the following memoryless property:
   $$\mathbb{P}(\{X > k + n\} \mid \{X > k\}) = \mathbb{P}(\{X > n\}) \qquad \forall k, n \in \mathbb{N}.$$

(b) Show that a continuous random variable $X \sim$ Exponential(μ) for some $\mu > 0$ if and only if satisfies the following memoryless property:

$$\mathbb{P}(\{X > t + s\} \mid \{X > t\}) = \mathbb{P}(\{X > s\}) \qquad \forall s, t > 0.$$

Hint: In either case, to show the "only if" part, define the functions

$$f(n) = \mathbb{P}(\{X > n\}), \;\; n \in \mathbb{N}, \qquad\qquad g(t) = \mathbb{P}(\{X > t\}), \;\; t > 0.$$

Show that the unique solutions to the functional equations

$$f(n + k) = f(n) \cdot f(k) \;\; \forall k, n \in \mathbb{N}, \qquad\qquad g(t + s) = g(t) \cdot g(s) \;\; \forall s, t > 0,$$

are the Geometric PMF and the Exponential PDF respectively.