



# Stochastic Processes

Lecture 02

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## Limit of a Sequence of Random Variables

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$

Let  $\{X_n\}_{n \in \mathbb{N}} = \{X_1, X_2, \dots\}$  be a collection of random variables w.r.t.  $\mathcal{F}$

- Fix  $\omega \in \Omega$ , and consider the sequence of real numbers

$$X_1(\omega), X_2(\omega), \dots$$

- A limit may or may not exist for the above sequence

### Lemma (An Important Set and its Measurability)

*The set of all  $\omega \in \Omega$  for which  $\lim_{n \rightarrow \infty} X_n(\omega)$  exists is a valid event, i.e.,*

$$A_{\lim} := \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists} \right\} \in \mathcal{F}.$$

## Proof of Lemma 1

$$\begin{aligned}
 \omega \in A_{\lim} &\iff \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists } (= x_\omega, \text{ say}) \\
 &\iff \forall \varepsilon > 0, \exists N_\varepsilon(\omega) : |X_n(\omega) - x_\omega| < \varepsilon \quad \forall n \geq N_\varepsilon(\omega) \\
 &\iff \forall q \in \mathbb{Q}, q > 0, \exists N_q(\omega) : |X_n(\omega) - x_\omega| < q \quad \forall n \geq N_q(\omega) \\
 &\iff \omega \in \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ \omega' \in \Omega : |X_n(\omega') - x_{\omega'}| < q \right\}
 \end{aligned}$$

### The Set $A_{\lim}$ and its Measurability

$$A_{\lim} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ \omega' \in \Omega : |X_n(\omega') - \lim_{k \rightarrow \infty} X_k(\omega')| < q \right\} \in \mathcal{F}.$$

## Properties of $A_{\lim}$

### Theorem (Properties of $A_{\lim}$ )

Let  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of random variables on a common underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .  
Let  $A_{\lim}$  denote the set

$$A_{\lim} = \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists} \right\}.$$

1.  $A_{\lim}$  belongs to the **tail  $\sigma$ -algebra**, i.e.,

$$A_{\lim} \in \underbrace{\bigcap_{k \in \mathbb{N}} \sigma(X_k, X_{k+1}, \dots)}_{\text{tail } \sigma\text{-algebra}} \subseteq \mathcal{F}.$$

2. **Kolmogorov's 0-1 Law:** If  $X_1, X_2, \dots$  are **independent**, then every event in the tail  $\sigma$ -algebra has either probability 0 or probability 1, in which case  $\mathbb{P}(A_{\lim}) \in \{0, 1\}$ .

## Proof of Theorem 2, Part 1

- The set  $A_{\lim}$  can be expressed as

$$A_{\lim} = \underbrace{\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \in \mathbb{R} \right\}}_{A_{\lim}^{(1)}} \cup \underbrace{\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = +\infty \right\}}_{A_{\lim}^{(2)}} \cup \underbrace{\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = -\infty \right\}}_{A_{\lim}^{(3)}}.$$

- To show  $A_{\lim}^{(2)}$  belongs to the tail  $\sigma$ -algebra:

—  $A_{\lim}^{(2)}$  can be expressed as

$$A_{\lim}^{(2)} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{X_n > q\} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \geq k} \bigcap_{n \geq N} \{X_n > q\} \quad \forall k \in \mathbb{N}.$$

— We then note that

$$\forall k \in \mathbb{N}, q \in \mathbb{Q}, q > 0, \quad \bigcup_{N \geq k} \bigcap_{n \geq N} \{X_n > q\} \in \sigma(X_k, X_{k+1}, \dots)$$

- Exercise:** Along similar lines, it can be shown that  $A_{\lim}^{(3)}$  belongs to the tail  $\sigma$ -algebra

## Proof of Theorem 2, Part 1

- To show that  $A_{\lim}^{(1)}$  belongs to the tail  $\sigma$ -algebra, we use the **Cauchy criterion for convergence**:

### Cauchy Criterion for Convergence

Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence of real numbers. Let  $x \in \mathbb{R}$ . Then,

**Exercise:**  $\lim_{n \rightarrow \infty} x_n = x \iff \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : |x_n - x_m| < \varepsilon \quad \forall n, m \geq N_\varepsilon.$

As usual,

$$\text{for every choice of } \varepsilon > 0 \iff \text{for every choice of } \varepsilon \in \mathbb{Q}, \varepsilon > 0.$$

- Using the above criterion, we may write

$$A_{\lim}^{(1)} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{|X_n - X_m| < q\} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \bigcup_{N \geq k} \bigcap_{n \geq N} \{|X_n - X_m| < q\} \quad \forall k \in \mathbb{N}.$$

- Same arguments as before can be used to show that  $A_{\lim}^{(1)} \in \sigma(X_k, X_{k+1}, \dots)$  for every  $k \in \mathbb{N}$

## Proof of Theorem 2, Part 2

- Will be given as homework exercise

## Forms of Convergence of Sequences of Random Variables

# Pointwise Convergence

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . All random variables are assumed to be defined on this space.

## Definition (Pointwise Convergence)

Let  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of real-valued RVs, and let  $X$  an extended real-valued RV.

We say that the sequence  $\{X_n\}_{n \in \mathbb{N}}$  converges **pointwise** to  $X$  if

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \quad \forall \omega \in \Omega;$$

**Notation:**  $X_n \xrightarrow{\text{pointwise}} X.$

Equivalently, we have  $A_{\lim} = \Omega$ .

# Almost-Sure Convergence

Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . All random variables are assumed to be defined on this space.

## Definition (Almost-Sure Convergence)

Let  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of real-valued RVs, and let  $X$  an extended real-valued RV.

We say that the sequence  $\{X_n\}_{n \in \mathbb{N}}$  converges to  $X$  **almost surely (a.s.)** if

$$\mathbb{P} \left( \left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right\} \right) = 1;$$

**Notation:**  $X_n \xrightarrow{\text{a.s.}} X$ .

### Note:

- The above definition DOES NOT require  $X_1, X_2, \dots$  to be independent
- If  $X_1, X_2, \dots$  are independent, then  $\mathbb{P}(A_{\lim}) \in \{0, 1\}$  by Kolmogorov's 0-1 law  
This has no bearing on the above definition in any way
- **Pointwise convergence implies almost-sure convergence**, i.e.,

$$X_n \xrightarrow{\text{pointwise}} X \implies X_n \xrightarrow{\text{a.s.}} X.$$

In general, the converse may not be true

## Example: Shrinking Pulse

Suppose that  $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$ ,

$\lambda$ : Lebesgue measure

- For each  $n \in \mathbb{N}$ , let

$$X_n(\omega) = \begin{cases} 1, & \omega \in [0, \frac{1}{n}) , \\ 0, & \text{otherwise.} \end{cases}$$

Does this sequence converge pointwise? If so, what is the pointwise limit RV?

- For each  $n \in \mathbb{N}$ , let

$$X_n(\omega) = \omega^n, \quad \omega \in \Omega.$$

Does this sequence converge pointwise? If so, what is the pointwise limit RV?

## Example: Moving Rectangles

Suppose that  $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$ ,  $\lambda$ : Lebesgue measure

- Consider the sequence of random variables given by:

$$X_1 = \mathbf{1}_{[0,1]}$$

$$X_2 = \mathbf{1}_{[0, \frac{1}{2}]}, \quad X_3 = \mathbf{1}_{[\frac{1}{2}, 1]}$$

$$X_4 = \mathbf{1}_{[0, \frac{1}{4}]}, \quad X_5 = \mathbf{1}_{[\frac{1}{4}, \frac{1}{2}]}, \quad X_6 = \mathbf{1}_{[\frac{1}{2}, \frac{3}{4}]}, \quad X_7 = \mathbf{1}_{[\frac{3}{4}, 1]}, \quad \text{and so on.}$$

## Example: Moving Rectangles

Suppose that  $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$ ,  $\lambda$ : Lebesgue measure

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Does the above sequence admit a pointwise limit? If so, what is the pointwise limit RV?

## Example: Going Beyond Pointwise Convergence

- For each  $n \in \mathbb{N}$ , let

$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$

### Remarks on this Example

- In cases where  $(\Omega, \mathcal{F}, \mathbb{P})$  and the sequence  $\{X_n\}_{n \in \mathbb{N}}$  are not explicitly specified, it is not possible to identify the pointwise limit.
- In such cases, we start with a guess for the limit RV and prove convergence in other forms (starting with almost-sure convergence)
- In many cases (including the current example), we need a way to infer almost-sure convergence merely based on probabilities

**In enters Borel–Cantelli Lemma!**