AI 5030: Probability and Stochastic Processes

Instructor: Dr. Karthik P. N.

Homework 7

Topics: Abstract Integrals, Expectations of Discrete Random Variables

1. Fix $n \in \mathbb{N}$. Consider the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, where λ denotes the Lebesgue measure. Compute $\int_{\mathbb{R}} f \, \mathrm{d}\lambda$ for each of the following cases.

   (a) $f : \mathbb{R} \to \mathbb{R}$ is defined as
   $$f(\omega) = \begin{cases} \omega, & \omega \in \{0, 1, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

   (b) $f : \mathbb{R} \to \mathbb{R}$ is defined as
   $$f(\omega) = \begin{cases} 1, & \omega \in \mathbb{Q}^c \cap [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

   (c) $f : \mathbb{R} \to \mathbb{R}$ is defined as
   $$f(\omega) = \begin{cases} n, & \omega \in \mathbb{Q}^c \cap [0, n], \\ 0, & \text{otherwise.} \end{cases}$$

2. Fix $n \in \mathbb{N}$. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$, $\mathscr{F} = 2^{\Omega}$, and $\mathbb{P}(\{\omega_i\}) = \frac{1}{n}$ for all $i \in \{1, \ldots, n\}$.
   Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathscr{F}$.
   Compute $\mathbb{E}[X] = \int_{\Omega} X \, \mathrm{d}\mathbb{P}$ for the following cases.

   (a) $X = \mathbf{1}_A$, where $A = \{\omega_1, \ldots, \omega_m\}$, with $1 \leq m \leq n$.

   (b) X is defined as
   $$X(\omega) = \begin{cases} i, & \omega = \omega_i, \ \omega_i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

3. Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$. For a fixed $c \in \mathbb{R}$, define $\delta_c : \mathscr{F} \to [0, 1]$ as
   $$\delta_c(A) = \begin{cases} 1, & c \in A, \\ 0, & c \notin A. \end{cases}$$

   (a) Show that δ_c is a probability measure on $(\Omega, \mathscr{F})$.
   Remark: δ_c is called the Dirac measure at c.
   It is referred to as "unit impulse" in the engineering literature, and sometimes (incorrectly) called a Dirac delta "function".

   (b) For any simple function $g : \Omega \to \mathbb{R}$, show that $\int_{\Omega} g \, \mathrm{d}\delta_c = g(c)$.

   (c) Extend the result in part (b) above to the case when g is non-negative.

   (d) Let $\mu : \mathscr{F} \to [0, +\infty]$ be defined as
   $$\mu(A) = \sum_{n=1}^{\infty} \delta_n(A), \qquad A \in \mathscr{F}.$$
   Show that for any simple function $g : \Omega \to \mathbb{R}$,
   $$\int_{\Omega} g \, \mathrm{d}\mu = \sum_{n=1}^{\infty} g(n).$$
   Extend the above result to the case when g is non-negative.
   Remark: Here, μ is a measure on $(\Omega, \mathscr{F})$, and is called the "counting" measure.
   For any given $A \in \mathscr{F}$, $\mu(A)$ is equal to the count of the number of positive integers present in the set A.

The above exercise shows that every summation is simply an integral with respect to the counting measure.

4. Suppose that N is a discrete random variable taking values in $\mathbb{N}$. Prove that

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}(\{N > n\}).$$

Hint: Notice that $N = \sum_{n=0}^{N-1} 1 = \sum_{n=0}^{\infty} \mathbf{1}_{\{N>n\}}$.
Apply expectations on both sides and use MCT to justify passing the expectation inside the infinite summation.

5. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a non-negative random variable with respect to $\mathscr{F}$.

   (a) Suppose that $\mathbb{E}[X] < +\infty$. Then, show that

$$\lim_{n\to\infty} n\,\mathbb{P}(\{X > n\}) = 0. \tag{1}$$

   Hint: Write $X = X\mathbf{1}_{\{X\le n\}} + X\mathbf{1}_{\{X>n\}}$.
   For each $n \in \mathbb{N}$, let $X_n = X\mathbf{1}_{\{X\le n\}}$.
   Show that $0 \le X_n \le X_{n+1}$ for all n, and $X_n \xrightarrow{\text{pointwise}} X$. Using MCT, compute $\lim_{n\to\infty} \mathbb{E}[X_n]$.
   Show that $\lim_{n\to\infty} \mathbb{E}[X\mathbf{1}_{\{X>n\}}] = 0$.
   Finally, argue that $0 \le \lim_{n\to\infty} n\,\mathbb{P}(\{X > n\}) \le \lim_{n\to\infty} \mathbb{E}[X\mathbf{1}_{\{X>n\}}] = 0$.

   (b) Produce an example of a random variable X for which $\mathbb{E}[X] = +\infty$, and

$$\lim_{n\to\infty} n\,\mathbb{P}(\{X > n\}) > 0.$$

   This exercise shows that (1) holds only when $\mathbb{E}[X] < +\infty$.

6. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to [0, +\infty]$ be a non-negative, extended real-valued random variable with respect to $\mathscr{F}$.
(Here, X is allowed take the value $+\infty$.)

   (a) Show that $\{X = +\infty\} = \{\omega \in \Omega : X(\omega) = +\infty\} \in \mathscr{F}$.
   Hint: If $X(\omega) = +\infty$, then $X(\omega) > N$ for all $N \in \mathbb{N}$.

   (b) Show that $\mathbb{E}[X] < +\infty$ implies that

$$\mathbb{P}(\{X < +\infty\}) = 1.$$

   Hint: We have to show that $\mathbb{P}(\{X = +\infty\}) = 0$. We will do this by contradiction.
   Let $L = \mathbb{E}[X]$. Suppose that $\mathbb{P}(\{X = +\infty\}) = p > 0$.
   Let $C = \{X > 2L/p\}$. Using the reasoning of part (a), argue that $\mathbb{P}(C) \ge p$.
   From class, we know that there exists a sequence of simple random variables $\{X_n\}_{n=1}^{\infty}$ such that $X_n \xrightarrow{\text{pointwise}} X$.
   Using the pointwise convergence property and MCT, argue that

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n] \ge \lim_{n\to\infty} \mathbb{E}[X_n \mathbf{1}_C] \ge \frac{2L}{p}\,\mathbb{P}(C) \ge 2L,$$

   thereby leading to a contradiction.

   (c) Construct an example of a non-negative random variable for which $\mathbb{P}(\{X < +\infty\}) = 1$, yet $\mathbb{E}[X] = +\infty$.
   This exercise shows that $\mathbb{P}(\{X < +\infty\}) = 1$ does not imply $\mathbb{E}[X] < +\infty$.

7. A biased coin with heads probability $p \in (0, 1)$ is tossed repeatedly.
Let $X_n \in \{0, 1\}$ denote the outcome of the nth toss, $n \in \mathbb{N}$.
Let N be defined as the random variable

$$N := \min\{n \ge 2 : X_n = 1 - X_1\}.$$

That is, N is the first time index $n \ge 2$ for which the outcome X_n is the complement of the first outcome.

   (a) Compute the PMF of N.

   (b) Show that

$$\mathbb{E}[N] = \frac{p}{q} + \frac{q}{p}.$$