AI 5090: Stochastic Processes
Homework 5

Topics: Recurrence, Transience, Communicating Classes, Invariant Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing below are defined with respect to this probability space.

1. Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous DTMC on the state space $\mathcal{X} = \{0, 1\}$, with the TPM

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

   for some fixed $p, q \in (0, 1)$.
   Argue that the above TPM admits a unique stationary distribution, and compute the stationary distribution.

2. Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous DTMC on the state space $\mathcal{X} = \{0, 1, 2, 3, 4, 5\}$, with the TPM as shown below.

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix} \end{array}$$

   (a) Classify the states as transient and recurrent.
   (b) Let $T_{\{0,1\}}$ be defined as the random variable

$$T_{\{0,1\}} := \inf\{n \in \mathbb{N} : X_n \in \{0, 1\}\}.$$

   For each $x \in \{0, 1, 2, 3, 4, 5\}$, determine the value of $\mathbb{P}(T_{\{0,1\}} < +\infty \mid X_0 = x)$.

3. There are N empty boxes and infinite collection of balls.
   At each time step $n \in \mathbb{N}$, a box is chosen uniformly at random and a ball is placed in the chosen box.
   Let X_n denote the number of empty boxes at the end of time step n.
   (a) Taking $X_0 = N$, show that $\{X_n\}_{n=0}^{\infty}$ is a DTMC.
   (b) Identify the state space and specify the TPM of the above DTMC.
   (c) Classify the states as transient, null recurrent, or positive recurrent.

4. Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible, positive recurrent, and time-homogeneous DTMC on $\mathcal{X} = \{0, 1, 2, 3, \ldots\}$.
   Fix a state $y \in \mathcal{X}$, and assume that $X_0 = y$. Fix another state $x \in \mathcal{X}$, $x \neq y$.
   For each $k \in \mathbb{N}$, let $V_x^{(k)}$ denote the number of visits to state x between the $(k-1)$th and kth return times to state y.
   That is, $V_x^{(1)}$ is the number of visits to x until the time of first return to y, $V_x^{(2)}$ is the number of visits to x after the time of first return to y and until the time of second return to y, and so on.
   Show that $\{V_x^{(k)}\}_{k \in \mathbb{N}}$ is an IID sequence of random variables.

5. Fix $N \in \mathbb{N}$, and let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous DTMC on the state space $\mathcal{X} = \{0, 1, \ldots, N\}$, with transition probabilities given by

$$P_{i,j} = \begin{cases} 1 - \dfrac{i}{N}, & j = i+1, \\ \dfrac{i}{N}, & j = i-1. \end{cases}$$

Show that the above TPM is irreducible and positive recurrent, and compute μ_{ii} for all $i \in \mathcal{X}$.

6. In Lecture 25, we saw the recursive relations to compute the expected number of times a fair coin must be tossed before observing the pattern "HTH" for the first time.
   Derive these recursive relations from first principles.