Decisions on the Fly: A Short, Surprising Journey Through Sequential Hypothesis Testing

Winter Workshop on Machine Learning and Artificial Intelligence, IIT Kanpur

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Department of Artificial Intelligence

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

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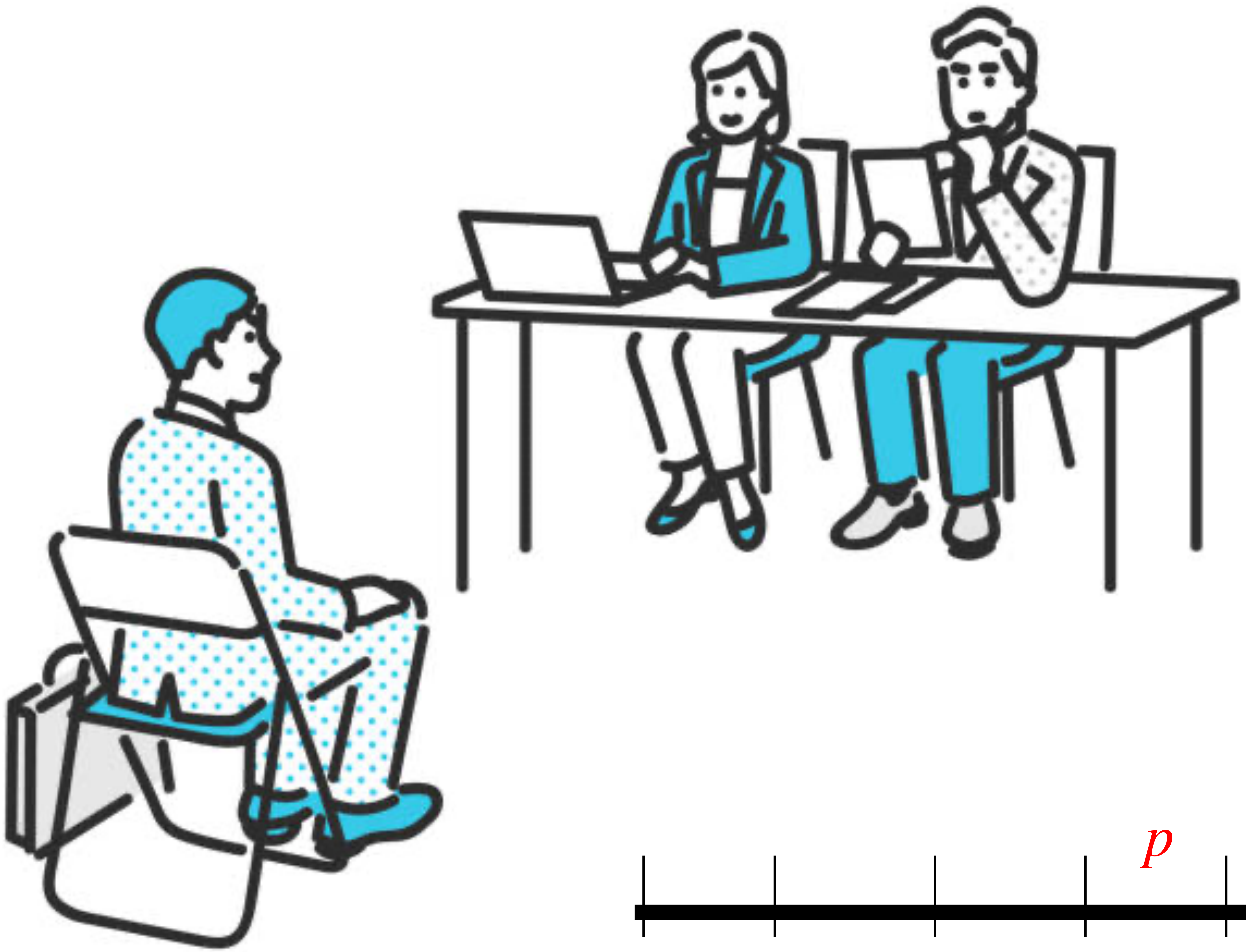

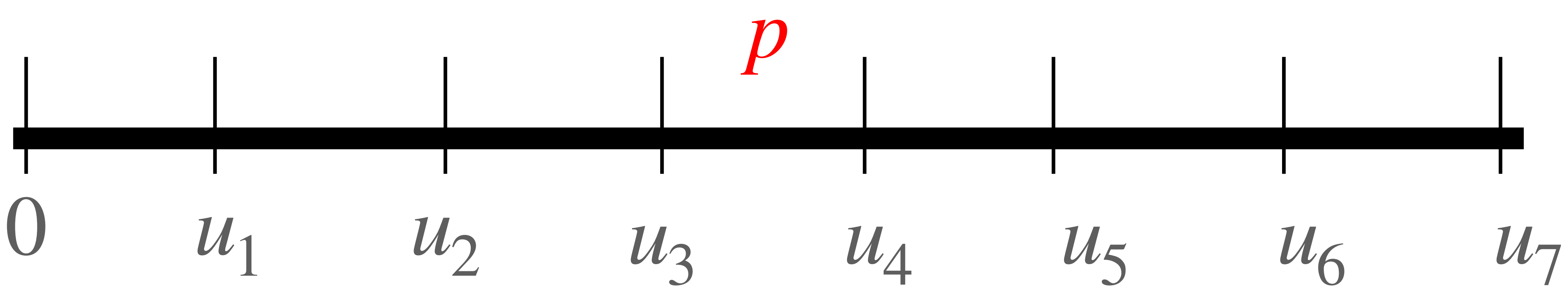
p
0
u_1
u_2
u_3
u_4
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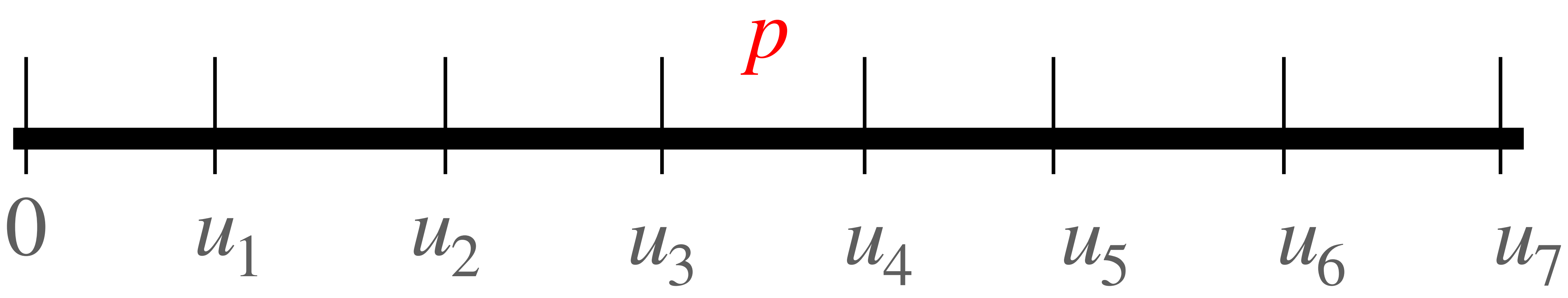
u_7

Difficulty Levels

1	2	3	4	5	6	7	8



Difficulty Levels

1	2	3	4	5	6	7	8

- Questions of different difficult levels can be asked sequentially

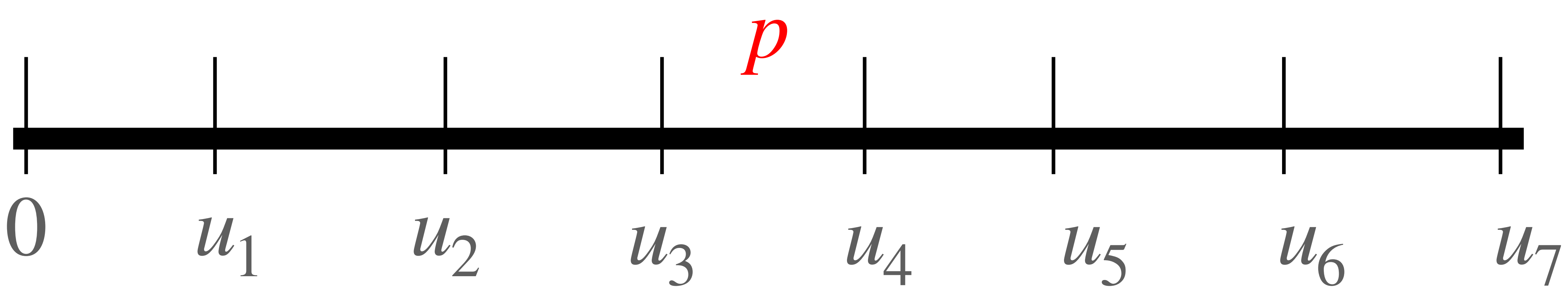
p

$0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7$

Difficulty Levels

1	2	3	4	5	6	7	8

- Questions of different difficult levels can be asked sequentially
- Binary reward: 1 if candidate answers correctly, 0 o.w.



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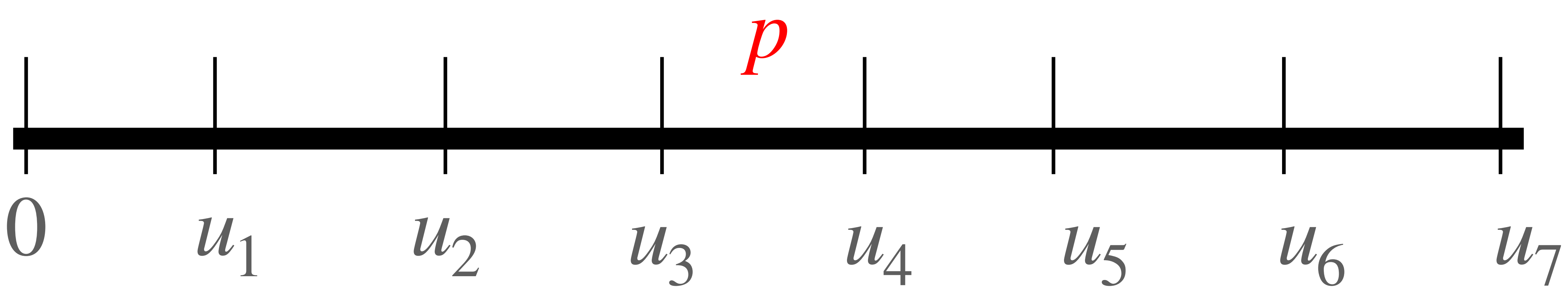
Difficulty Levels

1	2	3	4	5	6	7	8

- Questions of different difficult levels can be asked sequentially
- Binary reward: 1 if candidate answers correctly, 0 o.w.
- Goal: to determine the partition in which candidate's ability lies up to a desired accuracy



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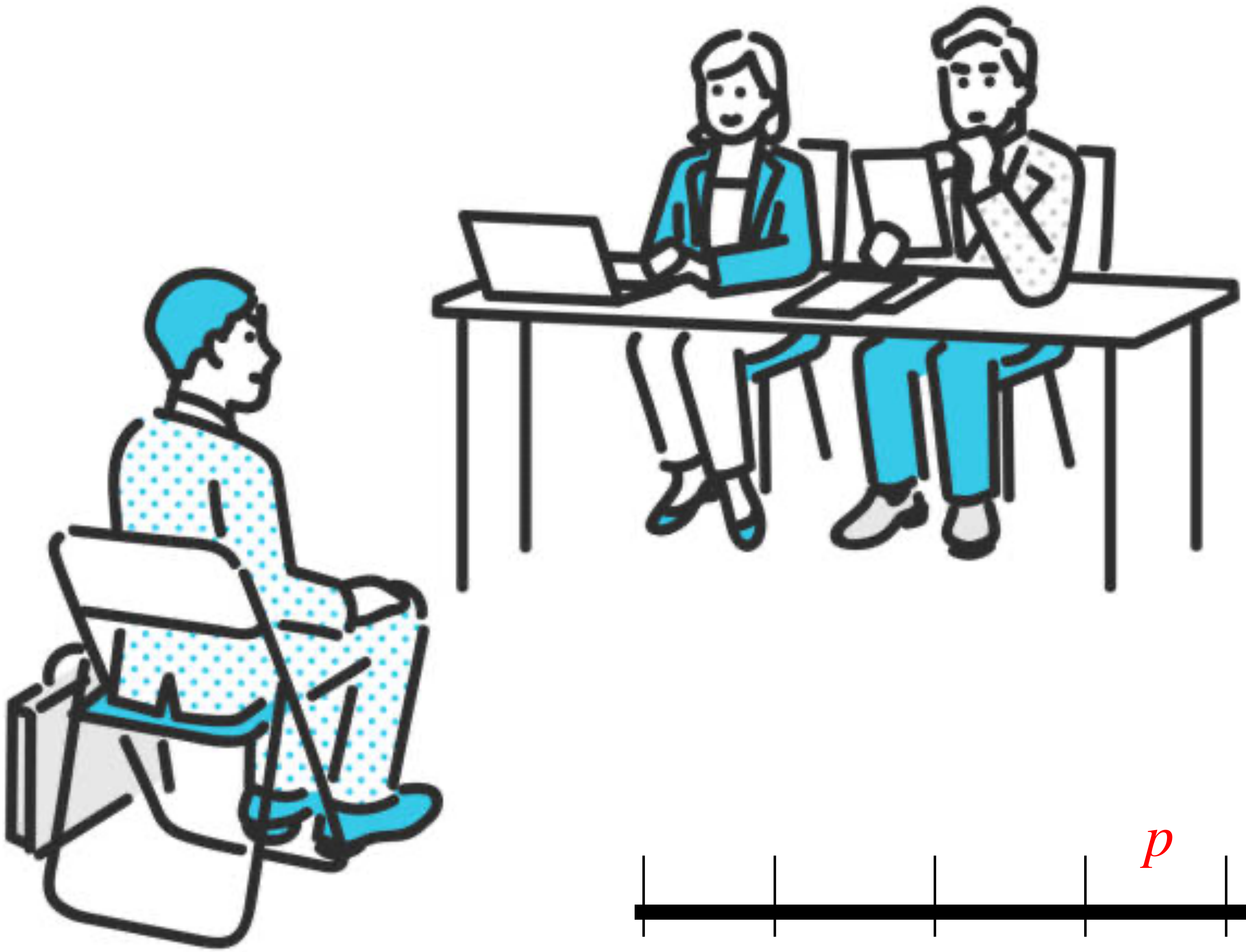
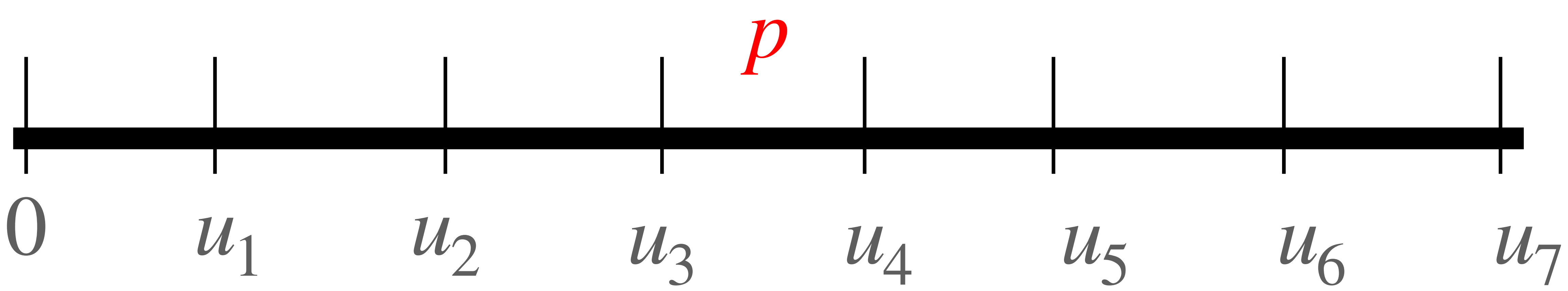
Difficulty Levels

1	2	3	4	5	6	7	8



- Questions of different difficult levels can be asked sequentially
- Binary reward: 1 if candidate answers correctly, 0 o.w.
- Goal: to determine the partition in which candidate's ability lies up to a desired accuracy
- How many questions need to be asked (on an average)?



Hypothesis Testing (HT)

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$$X_1, \ldots, X_\tau \sim P^\star$$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Hypothesis Testing (HT)

$$X_1, \ldots, X_\tau \sim P^\star$$

$$H_1 : \; P^\star \in \mathscr{P}_1$$

$$H_2 : \; P^\star \in \mathscr{P}_2$$

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

Hypothesis Testing (HT)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$X_1, \ldots, X_\tau \sim P^\star$$

data

unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \; P^\star \in \mathscr{P}_M$

known

Hypothesis Testing (HT)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

$$X_1, \ldots, X_\tau \sim P^\star$$

data — unknown

batch HT: τ deterministic, $X_1, \ldots, X_\tau$ available at start

Hypothesis Testing (HT)

$$X_1, \ldots, X_\tau \sim P^\star$$

data

unknown

$$H_1 : \ P^\star \in \mathscr{P}_1$$

known

$$H_2 : \ P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \ P^\star \in \mathscr{P}_M$$

known

batch HT: τ deterministic, $X_1, \ldots, X_\tau$ available at start

sequential HT: $X_1, X_2, \ldots$ available on demand, τ random

Hypothesis Testing (HT)

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$|\mathscr{P}_j| = 1$: **simple**

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

$$X_1, \ldots, X_\tau \sim P^\star$$

data   unknown

batch HT: τ deterministic, $X_1, \ldots, X_\tau$ available at start

sequential HT: $X_1, X_2, \ldots$ available on demand, τ random

Hypothesis Testing (HT)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$|\mathscr{P}_j| = 1$: **simple**

$|\mathscr{P}_j| > 1$: **composite**

$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

$$H_1 : \ P^\star \in \mathscr{P}_1$$

known

$$H_2 : \ P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \ P^\star \in \mathscr{P}_M$$

known

batch HT: τ deterministic, $X_1, \ldots, X_\tau$ available at start

sequential HT: $X_1, X_2, \ldots$ available on demand, τ random

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Hypothesis Testing (HT)

$|\mathscr{P}_j| = 1$: **simple**

$|\mathscr{P}_j| > 1$: **composite**

error probability

$\mathbb{P}(H_j$ **identified** $| \; H_i$ **true**$), \quad j \neq i$

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

$$X_1, \ldots, X_\tau \sim P^\star$$

data

unknown

batch HT: τ deterministic, $X_1, \ldots, X_\tau$ available at start

sequential HT: $X_1, X_2, \ldots$ available on demand, τ random

Batch vs Sequential HT

$$H_1 : \ \mu = 0$$

$$H_2 : \ \mu = 0.5$$

$$X_1, \ldots, X_\tau \sim \mathcal{N}(\mu, 1) \quad \text{IID}$$

Batch vs Sequential HT

$$X_1, \ldots, X_\tau \sim \mathcal{N}(\mu, 1) \quad \text{IID}$$

$$H_1 : \mu = 0$$

$$H_2 : \mu = 0.5$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq 0.05$

Batch vs Sequential HT

$$H_1 : \ \mu = 0$$

$$H_2 : \ \mu = 0.5$$

$$X_1, \ldots, X_\tau \sim \mathcal{N}(\mu, 1) \quad \text{IID}$$

	Batch	Sequential
τ	≥ 44	$\mathbb{E}[\tau] \approx 21.1$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq 0.05$

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data

unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data | unknown

confidence level: δ

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \quad \leq \quad \delta \quad \forall j \neq i$

$$H_1 : \ P^\star \in \mathscr{P}_1$$

known

$$H_2 : \ P^\star \in \mathscr{P}_2$$

known

$\vdots$

$$H_M : \ P^\star \in \mathscr{P}_M$$

known

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Fixed-Confidence Sequential HT

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$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

confidence level: δ

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \quad \leq \quad \delta \quad \forall j \neq i$

- Sequential test $\pi = \{\pi_1, \pi_2, \ldots\}$

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

confidence level: δ

$\mathbb{P}(H_j$ identified $| H_i$ true$) \leq \delta \quad \forall j \neq i$

- Sequential test $\pi = \{\pi_1, \pi_2, \ldots\}$
- $\pi_t : (x_1, \ldots, x_t) \mapsto \{$stop, continue$\}$

$H_1 : P^\star \in \mathscr{P}_1$

known

$H_2 : P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : P^\star \in \mathscr{P}_M$

known

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

$$H_1: \; P^\star \in \mathscr{P}_1$$

known

$$H_2: \; P^\star \in \mathscr{P}_2$$

known

$\vdots$

$$H_M: \; P^\star \in \mathscr{P}_M$$

known

confidence level: δ

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \quad \leq \quad \delta \quad \forall j \neq i$

- Sequential test $\pi = \{\pi_1, \pi_2, \ldots\}$
- $\pi_t : (x_1, \ldots, x_t) \mapsto \{$stop, continue$\}$
- $\tau = \tau(\pi)$: stopping time

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

⋮

$H_M : \; P^\star \in \mathscr{P}_M$

known

confidence level: δ

$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \quad \leq \quad \delta \quad \forall j \neq i$

- Sequential test $\pi = \{\pi_1, \pi_2, \ldots\}$
- $\pi_t : (x_1, \ldots, x_t) \mapsto \{\text{stop, continue}\}$
- $\tau = \tau(\pi)$: stopping time
- $\mathbb{P}^\pi(H_j \text{ identified} \mid H_i \text{ true}) \quad \leq \quad \delta \quad \forall j \neq i$

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \; P^\star \in \mathscr{P}_M$

known

$$\Pi(\delta) = \left\{ \pi : \quad \mathbb{P}^\pi(H_j \text{ identified} \mid H_i \text{ true}) \quad \leq \quad \delta \quad \forall j \neq i \right\}$$

$$\inf_{\pi \in \Pi(\delta)} \quad \mathbb{E}^\pi\big[\tau(\pi)\big] \quad \uparrow \quad \text{as} \quad \delta \downarrow 0$$

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

$$\Pi(\delta) = \left\{ \pi : \quad \mathbb{P}^\pi(H_j \text{ identified} \mid H_i \text{ true}) \quad \leq \quad \delta \quad \forall j \neq i \right\}$$

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \; P^\star \in \mathscr{P}_M$

known

Fixed-Confidence Sequential HT

$$X_1, \ldots, X_\tau \sim P^\star$$

data unknown

$$\Pi(\delta) = \left\{ \pi : \quad \mathbb{P}^\pi(H_j \text{ identified} \mid H_i \text{ true}) \quad \leq \quad \delta \quad \forall j \neq i \right\}$$

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \; P^\star \in \mathscr{P}_M$

known

Identified

True	i	j	Others
i	$\geq 1-\delta$	$\leq \delta$	$\leq \delta$
j	$\leq \delta$	$\geq 1-\delta$	$\leq \delta$

Fixed-Confidence Sequential HT

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$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$H_1 : \quad P^\star \in \mathscr{P}_1$$

known

$$H_2 : \quad P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \quad P^\star \in \mathscr{P}_M$$

known

Fixed-Confidence Sequential HT

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$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$H_1 : \; P^\star \in \mathscr{P}_1$ known

$H_2 : \; P^\star \in \mathscr{P}_2$ known

$\vdots$

$H_M : \; P^\star \in \mathscr{P}_M$ known

Data processing inequality:

$$\forall\, i \neq j, \quad \inf_{\pi \in \Pi(\delta)} \mathbb{E}^\pi\left[Z_{i,j}\big(\tau(\pi)\big)\right] \geq \; \delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}$$

Fixed-Confidence Sequential HT

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \; P^\star \in \mathscr{P}_M$

known

Data processing inequality:

$$\forall\, i \neq j, \quad \inf_{\pi \in \Pi(\delta)} \mathbb{E}^\pi\left[Z_{i,j}\big(\tau(\pi)\big)\right] \geq \underbrace{\delta \log \frac{\delta}{1-\delta} + (1-\delta) \log \frac{1-\delta}{\delta}}_{\Theta\left(\log \frac{1}{\delta}\right)}$$

Binary Sequential HT (Wald, 1945)

Wald, A. (1945). Sequential Tests of Statistical Hypotheses. Annals of Mathematical Statistics, 16.

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Binary Sequential HT (Wald, 1945)

Wald, A. (1945). Sequential Tests of Statistical Hypotheses. Annals of Mathematical Statistics, 16.

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

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$$H_2 : \; P^\star = P_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

error probability

$$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$$

Binary Sequential HT (Wald, 1945)

Wald, A. (1945). Sequential Tests of Statistical Hypotheses. Annals of Mathematical Statistics, 16.

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

error probability

$$\mathbb{P}(H_j \textbf{ identified} \mid H_i \textbf{ true}) \leq \delta$$

$$\mathbb{E}_i^\pi\left[Z_{i,j}\big(\tau(\pi)\big)\right] = \mathbb{E}_i^\pi\left[\sum_{s=1}^{\tau(\pi)} \log \frac{P_i(X_s)}{P_j(X_s)}\right] = \mathbb{E}_i^\pi[\tau(\pi)] \cdot D_{\mathrm{KL}}(P_i \parallel P_j)$$

Wald's Identity

Binary Sequential HT (Wald, 1945)

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathcal{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathcal{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathcal{L}(X_s \mid H_i \text{ true})}{\mathcal{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Binary Sequential HT (Wald, 1945)

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data　　unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

error probability

$$\mathbb{P}(H_j \textbf{ identified} \mid H_i \textbf{ true}) \leq \delta$$

$$\forall\, i \neq j, \quad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{D_{\mathrm{KL}}(P_i \,\|\, P_j)}$$

Binary Sequential HT (Wald, 1945)

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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

error probability

$$\mathbb{P}(H_j \textbf{ identified} \mid H_i \textbf{ true}) \leq \delta$$

$$\forall\, i \neq j, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \; \frac{\mathbb{E}_i^\pi[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{D_{\mathrm{KL}}(P_i \,\|\, P_j)}$$

amount of effort required to rule out H_j and identify H_i

Binary Sequential HT (Wald, 1945)

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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

Binary Sequential HT (Wald, 1945)

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathcal{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathcal{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathcal{L}(X_s \mid H_i \text{ true})}{\mathcal{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

under H_i : $\dfrac{Z_{i,j}(t)}{t} \longrightarrow D_{\mathrm{KL}}(P_i \parallel P_j) > 0$ a.s.

under H_j : $\dfrac{Z_{i,j}(t)}{t} \longrightarrow -D_{\mathrm{KL}}(P_j \parallel P_i) < 0$ a.s.

Binary Sequential HT (Wald, 1945)

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data   unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

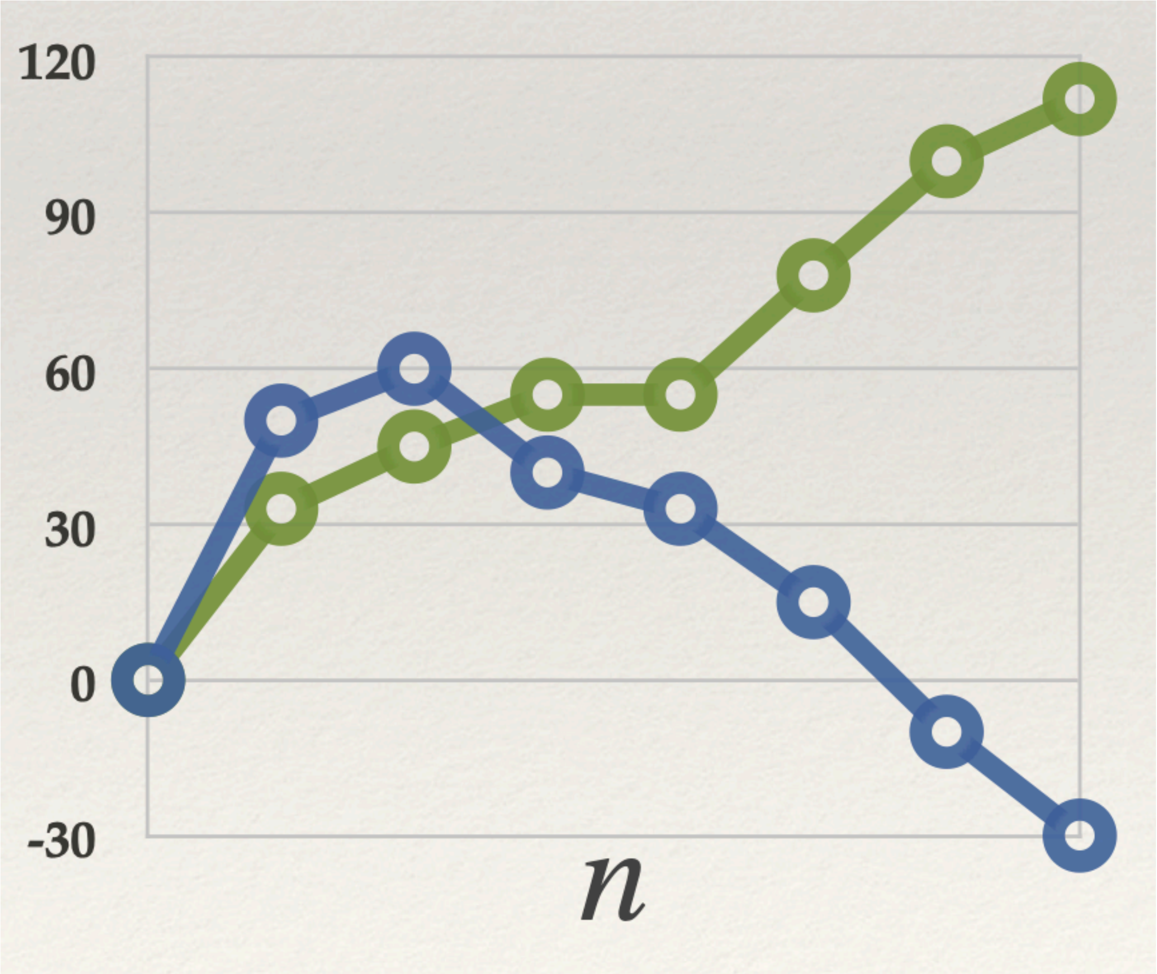
$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s)}{P_j(X_s)}$$

under H_i : $\dfrac{Z_{i,j}(t)}{t} \longrightarrow D_{\mathrm{KL}}(P_i \parallel P_j) > 0$ a.s.

under H_j : $\dfrac{Z_{i,j}(t)}{t} \longrightarrow -D_{\mathrm{KL}}(P_j \parallel P_i) < 0$ a.s.

$Z_{i,j}(n)$



under H_i

under H_j

Binary Sequential HT (Wald, 1945)

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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

sequential probability ratio test (SPRT)

$$\pi_t^{\mathrm{SPRT},\delta} = \begin{cases} \text{continue,} & -c_\delta \leq Z_{1,2}(t) \leq c_\delta \ , \\ \text{stop and announce } H_1, & Z_{1,2}(t) > c_\delta \ , \\ \text{stop and announce } H_2, & Z_{1,2}(t) < -c_\delta \ . \end{cases}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$c_\delta = O(\log(1/\delta))$

Binary Sequential HT (Wald, 1945)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

sequential probability ratio test (SPRT)

$$\pi_t^{\text{SPRT},\delta} = \begin{cases} \text{continue}, & -c_\delta \leq Z_{1,2}(t) \leq c_\delta \,, \\ \text{stop and announce } H_1, & Z_{1,2}(t) > c_\delta \,, \\ \text{stop and announce } H_2, & Z_{1,2}(t) < -c_\delta \,. \end{cases}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$c_\delta = O(\log(1/\delta))$

$$\pi^{\text{SPRT},\delta} \in \Pi(\delta), \qquad \lim_{\delta \downarrow 0} \frac{\mathbb{E}_i^{\pi^{\text{SPRT},\delta}}[\tau(\pi^{\text{SPRT},\delta})]}{\log(1/\delta)} \leq \frac{1}{D_{\text{KL}}(P_i \,\|\, P_j)} \quad \forall\, i \neq j$$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

Carl W Baum and Venugopal V Veeravalli. "A sequential procedure for multihypothesis testing". In: IEEE Transactions on Information Theory 40.6 (2002), pp. 1994–2007.

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

⋮

$$H_M : \; P^\star = P_M$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

Carl W Baum and Venugopal V Veeravalli. "A sequential procedure for multihypothesis testing". In: IEEE Transactions on Information Theory 40.6 (2002), pp. 1994–2007.

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$$\vdots$$

$$H_M : \; P^\star = P_M$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\forall\, i, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{\min_{j \neq i} D_{\mathrm{KL}}(P_i \,\|\, P_j)}$$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

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Carl W Baum and Venugopal V Veeravalli. “A sequential procedure for multihypothesis testing”. In: IEEE Transactions on Information Theory 40.6 (2002), pp. 1994–2007.

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

$\vdots$

$H_M : \; P^\star = P_M$

known

$$\forall\, i, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{\min_{j \neq i} D_{\mathrm{KL}}(P_i \,\|\, P_j)}$$

amount of effort required to rule out every hypothesis other than H_i

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

$$\underbrace{X_1, \ldots, X_\tau}_{\text{data}} \sim \underbrace{P^\star}_{\text{unknown}} \quad \text{IID}$$

$$H_1 : \; P^\star = \underbrace{P_1}_{\text{known}}$$

$$H_2 : \; P^\star = \underbrace{P_2}_{\text{known}}$$

$$\vdots$$

$$H_M : \; P^\star = \underbrace{P_M}_{\text{known}}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$$\vdots$$

$$H_M : \; P^\star = P_M$$

known

$$Z_i(t) = \min_{j \neq i} \; Z_{i,j}(t)$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

$\vdots$

$H_M : \; P^\star = P_M$

known

$$Z_i(t) = \min_{j \neq i} \; Z_{i,j}(t)$$

under H_i : $\quad Z_i(t) \longrightarrow +\infty \quad$ a.s.

under $H_j, \; j \neq i$: $\quad Z_i(t) \longrightarrow -\infty \quad$ a.s.

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

$\vdots$

$$H_M : \; P^\star = P_M$$

known

$$Z_i(t) = \min_{j \neq i} \; Z_{i,j}(t)$$

sequential M-SPRT

$$\pi_t^{\mathrm{M-SPRT},\delta} = \begin{cases} \text{continue,} & \max_i \; Z_i(t) \leq c_\delta \,, \\ \text{stop and announce } H_i, & Z_i(t) > c_\delta \,. \end{cases}$$

Sequential Multihypothesis Testing (Baum & Veeravalli, 2007)

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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data — unknown

$H_1 : \; P^\star = P_1$ known

$H_2 : \; P^\star = P_2$ known

$\vdots$

$H_M : \; P^\star = P_M$ known

$$Z_i(t) = \min_{j \neq i} \; Z_{i,j}(t)$$

sequential M-SPRT

$$\pi_t^{\mathrm{M-SPRT},\delta} = \begin{cases} \text{continue,} & \max_i \; Z_i(t) \leq c_\delta \; , \\ \text{stop and announce } H_i, & Z_i(t) > c_\delta \; . \end{cases}$$

$$\pi^{\mathrm{M-SPRT},\delta} \in \Pi(\delta), \qquad \lim_{\delta \downarrow 0} \frac{\mathbb{E}_i^{\pi^{\mathrm{M-SPRT},\delta}}[\tau(\pi^{\mathrm{M-SPRT},\delta})]}{\log(1/\delta)} \leq \frac{1}{\min_{j \neq i} \; D_{\mathrm{KL}}(P_i \parallel P_j)} \quad \forall \; i$$

Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1: \; P^\star = P_1$$

known

$$H_2: \; P^\star \in \mathscr{P}_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})}$$

error probability

$$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \le \delta$$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})}$$

error probability

$$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$$

$$\lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_1^\pi[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} D_{\mathrm{KL}}(P_1 \parallel Q)}$$

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Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid H_i \text{ true})}{\mathscr{L}(X_s \mid H_j \text{ true})}$$

error probability

$$\mathbb{P}(H_j \textbf{ identified} \mid H_i \textbf{ true}) \leq \delta$$

$$\lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_1^\pi[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} D_{\mathrm{KL}}(P_1 \parallel Q)}$$

$$\forall\, Q \in \mathscr{P}_2, \quad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid P^\star = Q]}{\log(1/\delta)} \geq \frac{1}{D_{\mathrm{KL}}(Q \parallel P_1)}$$

Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$Z(t) = \inf_{Q \in \mathscr{P}_2} \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P_1)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

$$H_1 : \ P^\star = P_1$$

known

$$H_2 : \ P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$Z(t) = \inf_{Q \in \mathscr{P}_2} \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P_1)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

under H_1 : $\quad Z(t) \longrightarrow +\infty \quad$ a.s.

under H_2 : $\quad Z(t) \longrightarrow -\infty \quad$ a.s.

Simple vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$Z(t) = \inf_{Q \in \mathscr{P}_2} \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P_1)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\pi_t^\delta = \begin{cases} \text{continue}, & -c_\delta \leq Z(t) \leq c_\delta \;, \\ \text{stop and announce } H_1, & Z(t) > c_\delta \;, \\ \text{stop and announce } H_2, & Z(t) < -c_\delta \;. \end{cases}$$

Composite vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\forall\, P \in \mathscr{P}_1, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid P^\star = P]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} D_{\mathrm{KL}}(P \,\|\, Q)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

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Composite vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\forall\, P \in \mathscr{P}_1, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid P^\star = P]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} D_{\mathrm{KL}}(P \,\|\, Q)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\forall\, Q \in \mathscr{P}_2, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid P^\star = Q]}{\log(1/\delta)} \geq \frac{1}{\inf_{P \in \mathscr{P}_1} D_{\mathrm{KL}}(Q \,\|\, P)}$$

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Composite vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$Z(t) = \sup_{P \in \mathscr{P}_1} \; \inf_{Q \in \mathscr{P}_2} \; \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite vs Composite SHT

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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$$Z(t) = \sup_{P \in \mathscr{P}_1} \; \inf_{Q \in \mathscr{P}_2} \; \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

under H_1 : $\quad Z(t) \longrightarrow +\infty \quad$ a.s.

under H_2 : $\quad Z(t) \longrightarrow -\infty \quad$ a.s.

Composite vs Composite SHT

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$Z(t) = \sup_{P \in \mathscr{P}_1} \; \inf_{Q \in \mathscr{P}_2} \; \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\pi_t = \begin{cases} \text{continue}, & -c_\delta \leq Z(t) \leq c_\delta \;, \\ \text{stop and announce } H_1, & Z(t) > c_\delta \;, \\ \text{stop and announce } H_2, & Z(t) < -c_\delta \;. \end{cases}$$

Composite Multihypothesis Testing

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \; P^\star \in \mathscr{P}_M$

known

$$\forall\, P \in \mathscr{P}_i\,, \qquad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid P^\star = P]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \bigcup_{j \neq i} \mathscr{P}_j} D_{\mathrm{KL}}(P \,\|\, Q)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$H_1: \; P^\star \in \mathscr{P}_1$

known

$H_2: \; P^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M: \; P^\star \in \mathscr{P}_M$

known

$$\forall\, P \in \mathscr{P}_i\,, \qquad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid P^\star = P]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \bigcup_{j \neq i} \mathscr{P}_j} D_{\mathrm{KL}}(P \,\|\, Q)}$$

$\mathrm{ALT}(P)$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star \in \mathcal{P}_1$$

known

$$H_2 : \; P^\star \in \mathcal{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathcal{P}_M$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$Z_{i,j}(t) = \sup_{P \in \mathscr{P}_i} \inf_{Q \in \mathscr{P}_j} \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$\vdots$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$Z_{i,j}(t) = \sup_{P \in \mathscr{P}_i} \inf_{Q \in \mathscr{P}_j} \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

$$Z_i(t) = \min_{j \neq i} Z_{i,j}(t)$$

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

⋮

$H_M : \; P^\star \in \mathscr{P}_M$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data

unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$\vdots$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

$$Z_{i,j}(t) = \sup_{P \in \mathscr{P}_i} \; \inf_{Q \in \mathscr{P}_j} \; \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

$$Z_i(t) = \min_{j \neq i} \; Z_{i,j}(t)$$

under H_i : $\quad Z_i(t) \longrightarrow +\infty \quad$ a.s.

under $H_j, \; j \neq i$: $\quad Z_i(t) \longrightarrow -\infty \quad$ a.s.

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Composite Multihypothesis Testing

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

⋮

$H_M : \; P^\star \in \mathscr{P}_M$

known

$$\pi_t^\delta = \begin{cases} \text{continue,} & \max_i \; Z_i(t) \leq c_\delta \; , \\ \text{stop and announce } H_i, & Z_i(t) > c_\delta \; . \end{cases}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

$$Z_{i,j}(t) = \sup_{P \in \mathscr{P}_i} \; \inf_{Q \in \mathscr{P}_j} \; \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

$$\pi_t^\delta = \begin{cases} \text{continue,} & \max_i \; Z_i(t) \leq c_\delta \;, \\ \text{stop and announce } H_i, & Z_i(t) > c_\delta \;. \end{cases}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Composite Multihypothesis Testing

$$X_1, \ldots, X_\tau \sim P^\star \quad \text{IID}$$

data unknown

$$H_1 : \; P^\star \in \mathscr{P}_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

$$\vdots$$

$$H_M : \; P^\star \in \mathscr{P}_M$$

known

$$Z_{i,j}(t) = \sup_{P \in \mathscr{P}_i} \; \inf_{Q \in \mathscr{P}_j} \; \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = P)}{\mathscr{L}(X_1, \ldots, X_t \mid P^\star = Q)}$$

$$Z_i(t) = \min_{j \neq i} \; Z_{i,j}(t)$$

$$\pi_t^\delta = \begin{cases} \text{continue,} & \max_i \; Z_i(t) \leq c_\delta \; , \\ \text{stop and announce } H_i, & Z_i(t) > c_\delta \; . \end{cases}$$

error probability

$$\mathbb{P}(H_j \textbf{ identified} \mid H_i \textbf{ true}) \leq \delta$$

Multiple Data Streams

Binary Active SHT (Simple vs Simple)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \mathbf{P}^\star = (Q_1, \ldots, Q_K)$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data　unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data　unknown

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data　unknown

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1: \ \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2: \ \mathbf{P}^\star = (Q_1, \ldots, Q_K)$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

Switch

Time	Stream (arm)	Observation
1	1	$X_1^{(1)}$
2	2	$X_2^{(2)}$
3	2	$X_3^{(2)}$
4	3	$X_4^{(3)}$

$H_1 : \ \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \ \mathbf{P}^\star = (Q_1, \ldots, Q_K)$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown

Switch

Time	Stream (arm)	Observation
1	1	$X_1^{(1)}$
2	2	$X_2^{(2)}$
3	2	$X_3^{(2)}$
4	3	$X_4^{(3)}$

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \ \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \ \mathbf{P}^\star = (Q_1, \ldots, Q_K)$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

At each round, we can pick any of the streams probabilitstically or deterministically

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \mathbf{P}^\star = (Q_1, \ldots, Q_K)$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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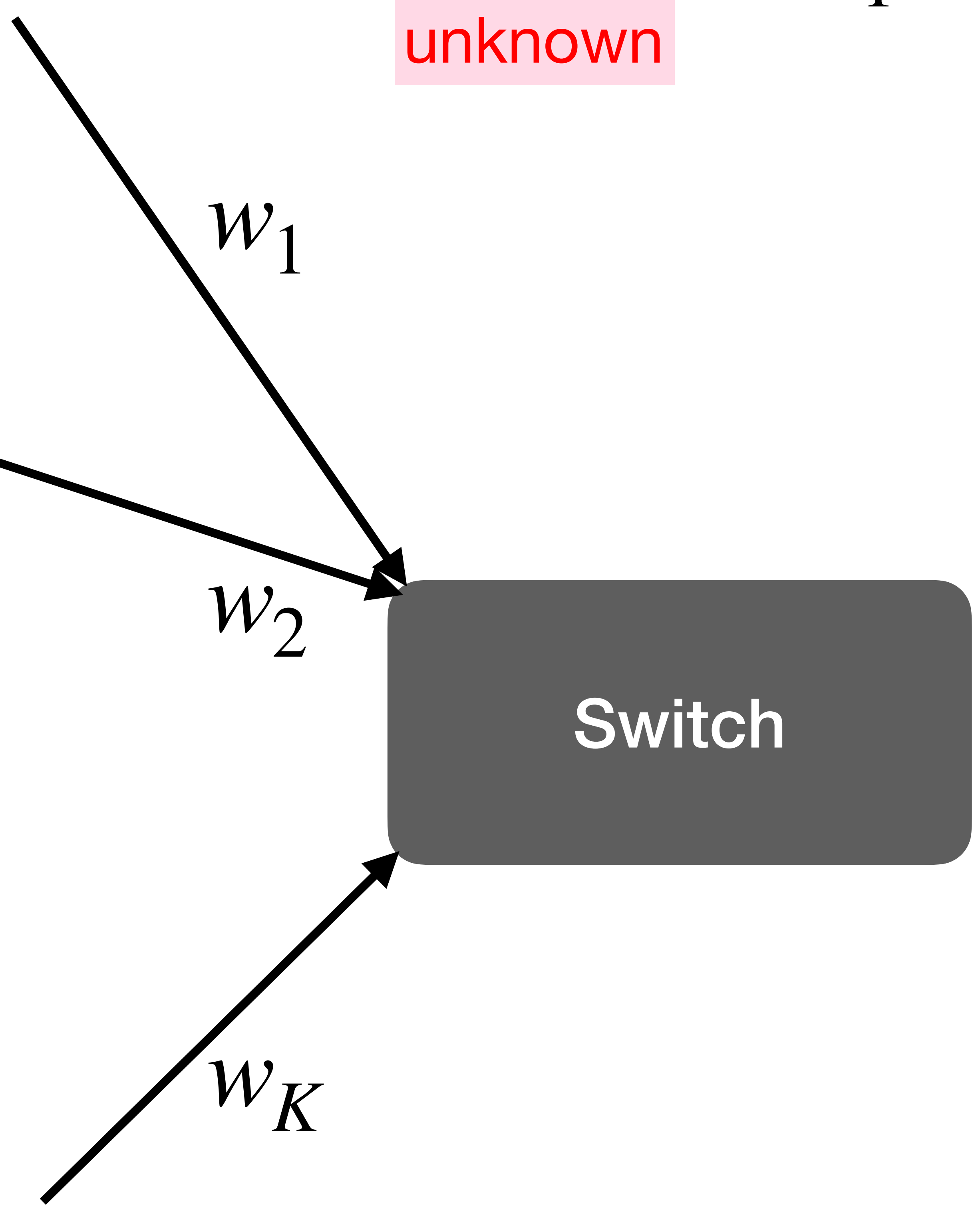
Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$
data unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$
data unknown

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$
data unknown



$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$
unknown

$H_1 : \mathbf{P}^\star = (P_1, \ldots, P_K)$
known

$H_2 : \mathbf{P}^\star = (Q_1, \ldots, Q_K)$
known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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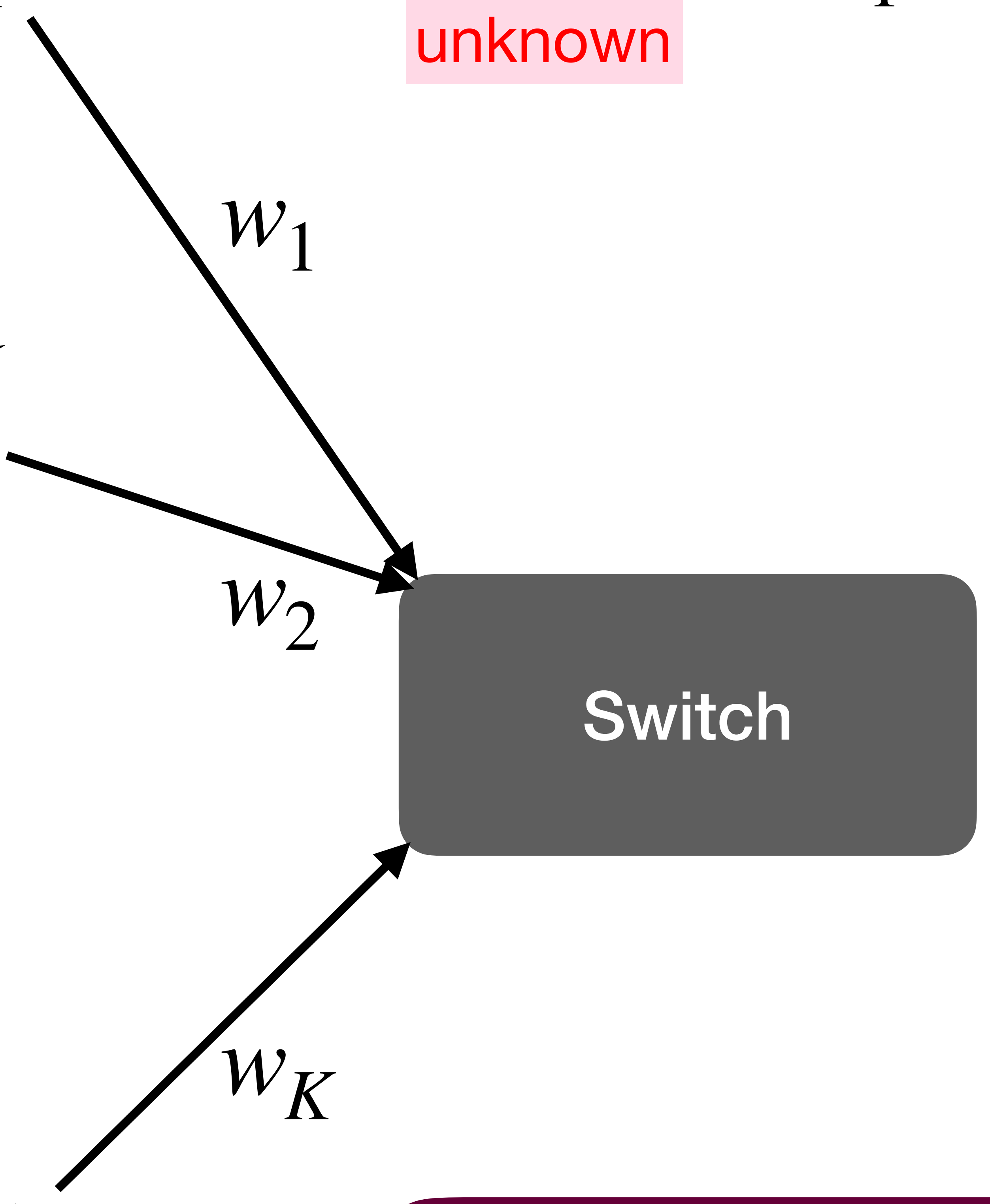
Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$
data unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$
data unknown

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$
data unknown



$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$
unknown

$H_1 : \mathbf{P}^\star = (P_1, \ldots, P_K)$
known

$H_2 : \mathbf{P}^\star = (Q_1, \ldots, Q_K)$
known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid H_1 \text{ true}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \sum_{a=1}^K w_a \, D_{\mathrm{KL}}(P_a \,\|\, Q_a)}$$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

w_1

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

w_2

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown

w_K

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \mathbf{P}^\star = (Q_1, \ldots, Q_K)$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Binary Active SHT (Simple vs Simple)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data — unknown

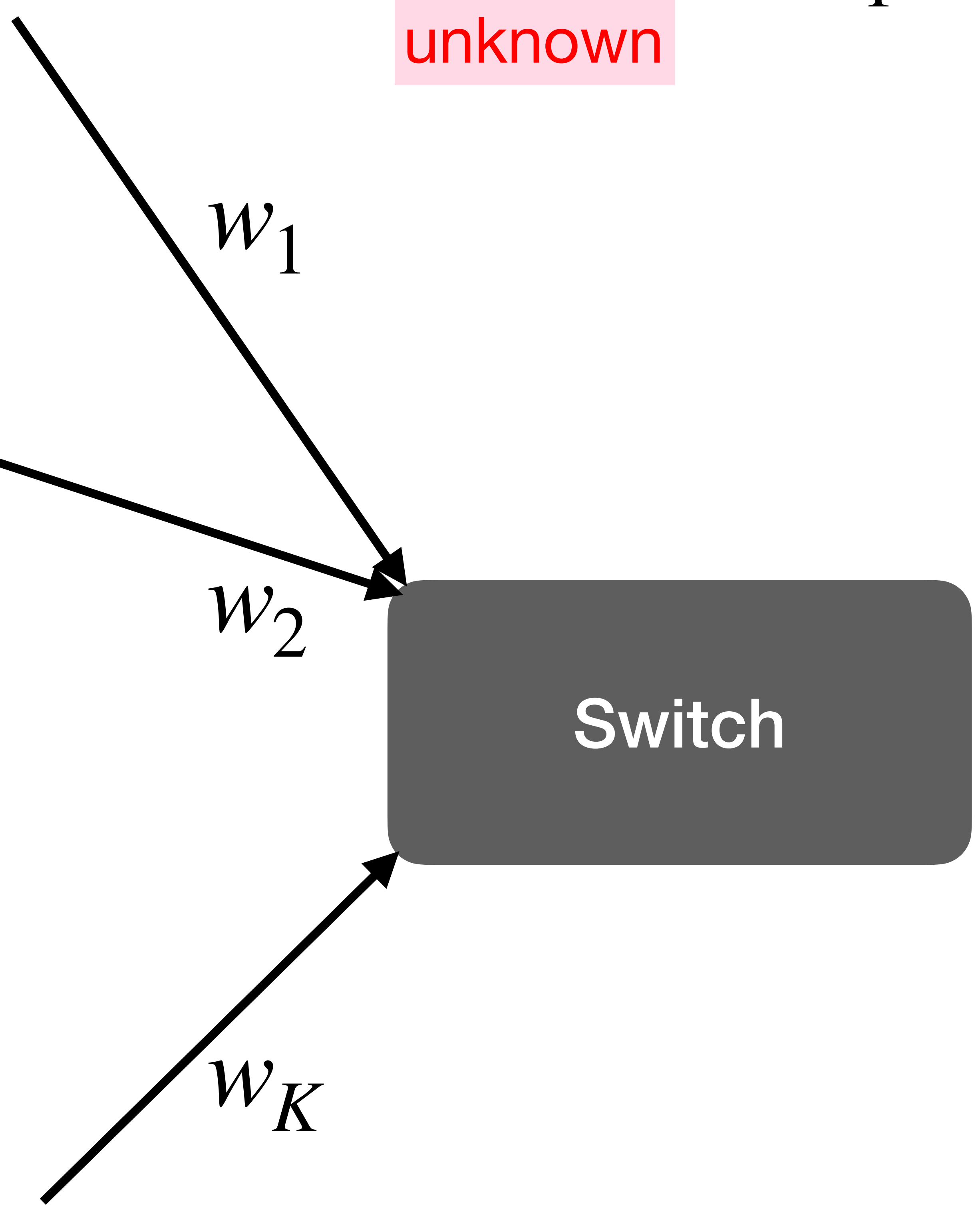
IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data — unknown

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data — unknown



$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \mathbf{P}^\star = (Q_1, \ldots, Q_K)$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

To achieve minimum sample complexity, always choose stream a for which $D_{\mathrm{KL}}(P_a \,\|\, Q_a)$ is the largest

Binary Active SHT (Simple vs Composite)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

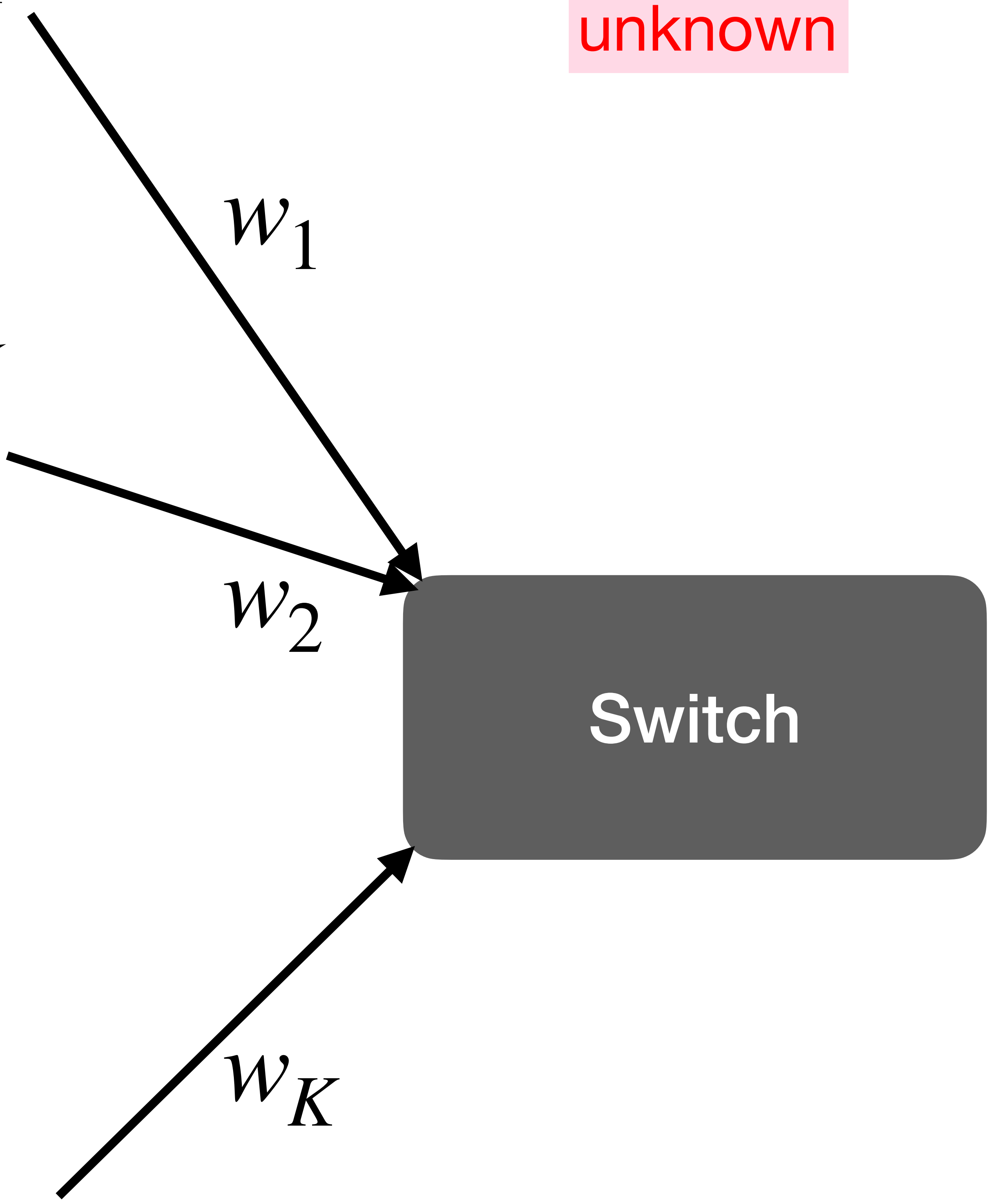
IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown



$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \mathbf{P}^\star \in \mathscr{P}_2$

known

Binary Active SHT (Simple vs Composite)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

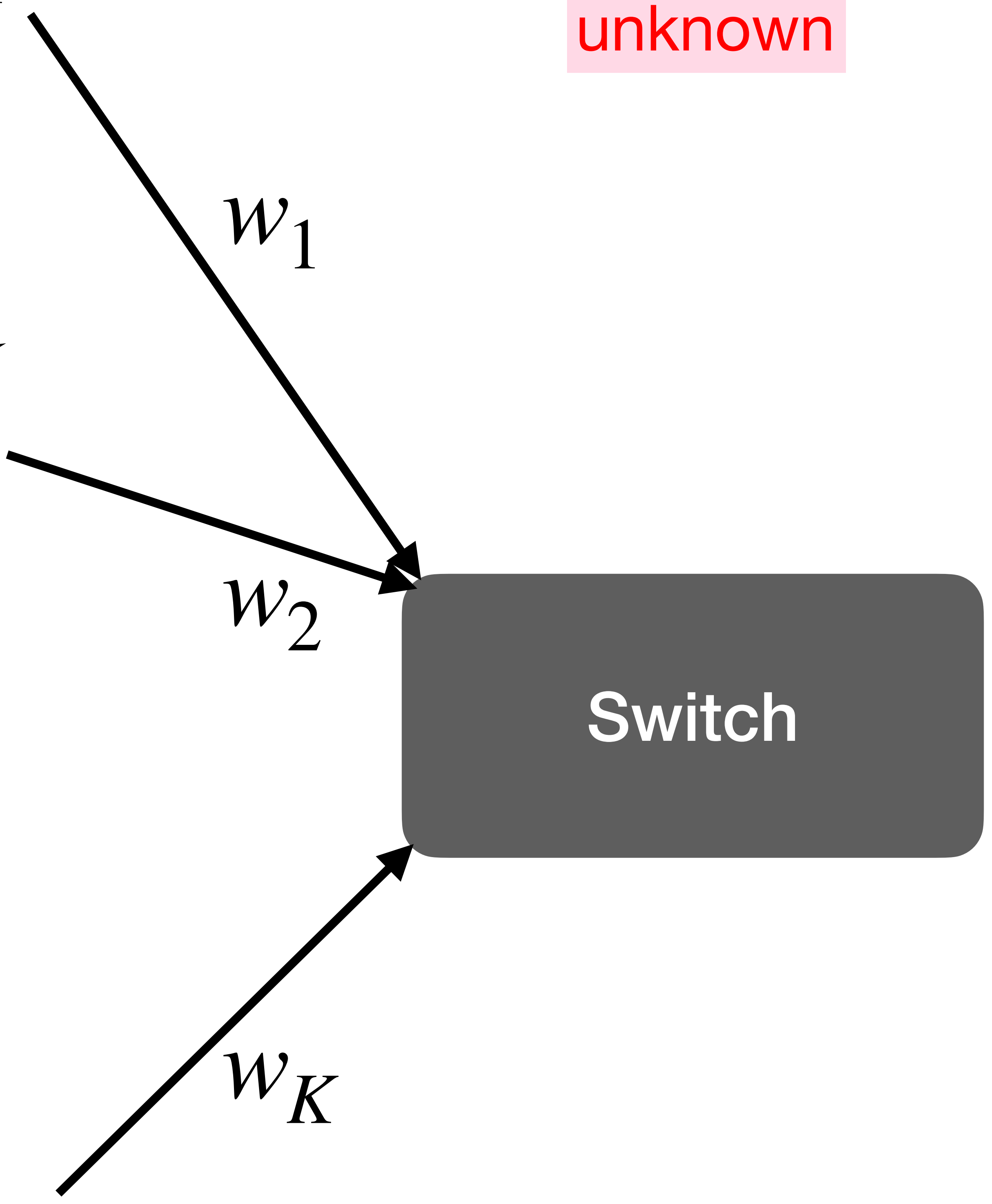
IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown



$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \ \mathbf{P}^\star = (P_1, \ldots, P_K)$

known

$H_2 : \ \mathbf{P}^\star \in \mathscr{P}_2$

known

$$\lim_{\delta \downarrow 0} \ \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid H_1 \text{ true}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \ \inf_{\mathbf{Q} \in \mathscr{P}_2} \ \sum_{a=1}^K w_a \ D_{\mathrm{KL}}(P_a \parallel Q_a)}$$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Binary Active SHT (Simple vs Composite)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$
data unknown

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$
data unknown

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$
data unknown

w_1 w_2 w_K

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$
unknown

$H_1: \ \mathbf{P}^\star = (P_1, \ldots, P_K)$
known

$H_2: \ \mathbf{P}^\star \in \mathscr{P}_2$
known

$$\lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid H_1 \text{ true}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \inf_{\mathbf{Q} \in \mathscr{P}_2} \sum_{a=1}^K w_a \ D_{\mathrm{KL}}(P_a \parallel Q_a)}$$

$$\forall\ \mathbf{Q} \in \mathscr{P}_2, \quad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid \mathbf{P}^\star = \mathbf{Q}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \sum_{a=1}^K w_a \ D_{\mathrm{KL}}(Q_a \parallel P_a)}$$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Binary Active SHT (Chernoff 1959)

Herman Chernoff. "Sequential design of experiments". In: The Annals of Mathematical Statistics 30.3 (1959), pp. 755–770.

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

w_1

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

w_2

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown

w_K

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \mathbf{P}^\star \in \mathscr{P}_2$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

w_2

⋮

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown

w_K

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \mathbf{P}^\star \in \mathscr{P}_2$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

At each round, we can pick any of the streams probabilitstically or deterministically

Binary Active SHT (Composite vs Composite)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$H_1 : \ \mathbf{P}^\star \in \mathscr{P}_1$$

known

$$H_2 : \ \mathbf{P}^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Binary Active SHT (Composite vs Composite)

$$\forall\, \mathbf{P} \in \mathscr{P}_1, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^{\pi}[\tau(\pi) \mid \mathbf{P}^{\star} = \mathbf{P}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \; \inf_{\mathbf{Q} \in \mathscr{P}_2} \sum_{a=1}^{K} w_a \; D_{\mathrm{KL}}(P_a \,\|\, Q_a)}$$

$H_1 : \; \mathbf{P}^{\star} \in \mathscr{P}_1$

known

$H_2 : \; \mathbf{P}^{\star} \in \mathscr{P}_2$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Binary Active SHT (Composite vs Composite)

$$\forall\, \mathbf{P} \in \mathscr{P}_1, \quad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^{\pi}[\tau(\pi) \mid \mathbf{P}^\star = \mathbf{P}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \inf_{\mathbf{Q} \in \mathscr{P}_2} \sum_{a=1}^{K} w_a \; D_{\mathrm{KL}}(P_a \,\|\, Q_a)}$$

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$$H_1 : \mathbf{P}^\star \in \mathscr{P}_1$$

known

$$H_2 : \mathbf{P}^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Active Multihypothesis Testing (Composite)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

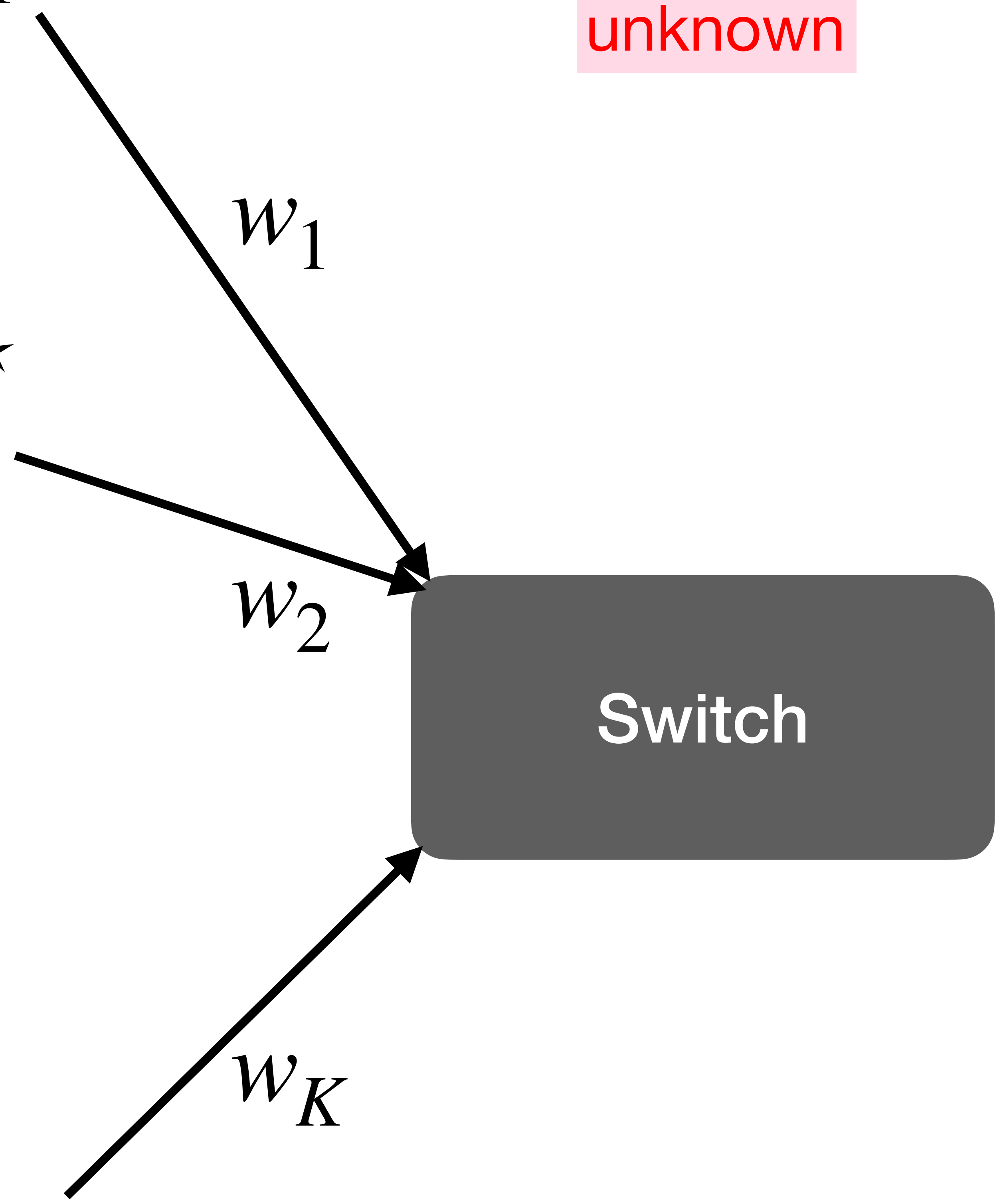
IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown



$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1: \ \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2: \ \mathbf{P}^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M: \ \mathbf{P}^\star \in \mathscr{P}_M$

known

error probability

$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$

Active Multihypothesis Testing (Composite)

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data unknown

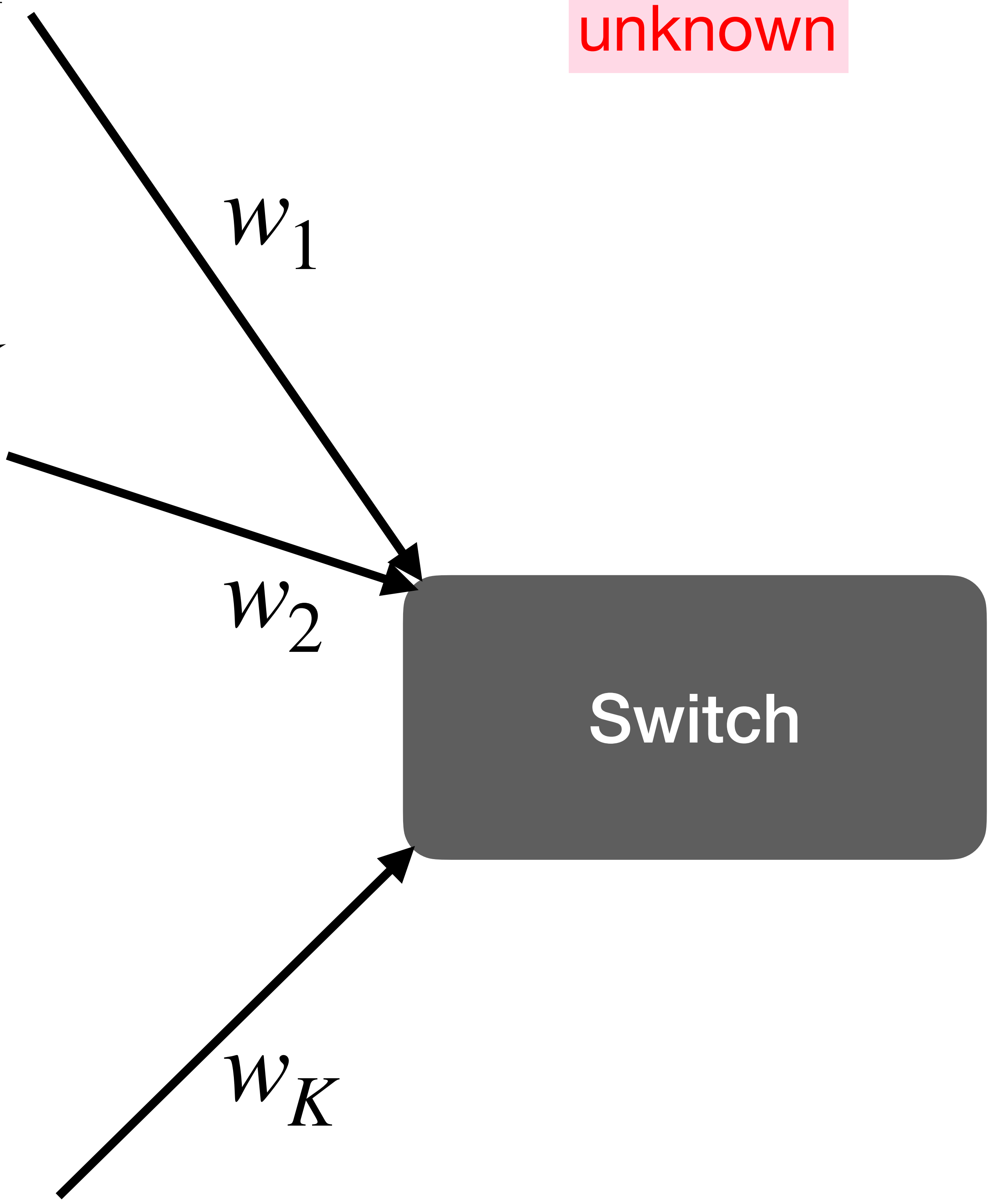
IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data unknown

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data unknown



$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \ \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \ \mathbf{P}^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \ \mathbf{P}^\star \in \mathscr{P}_M$

known

At each round, we can pick any of the streams probabilitstically or deterministically

error probability

$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$

Active Multihypothesis Testing (Composite)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$H_1 : \ \mathbf{P}^\star \in \mathscr{P}_1$$

known

$$H_2 : \ \mathbf{P}^\star \in \mathscr{P}_2$$

known

$\vdots$

$$H_M : \ \mathbf{P}^\star \in \mathscr{P}_M$$

known

error probability

$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$

Active Multihypothesis Testing (Composite)

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$$\forall\, \mathbf{P} \in \mathscr{P}_i\,, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^{\pi}[\tau(\pi) \mid \mathbf{P}^{\star} = \mathbf{P}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \; \inf_{\mathbf{Q} \in \bigcup_{j \neq i} \mathscr{P}_j} \sum_{a=1}^{K} w_a \; D_{\mathrm{KL}}(P_a \parallel Q_a)}$$

$H_1 : \; \mathbf{P}^{\star} \in \mathscr{P}_1$

known

$H_2 : \; \mathbf{P}^{\star} \in \mathscr{P}_2$

known

⋮

$H_M : \; \mathbf{P}^{\star} \in \mathscr{P}_M$

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error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Active Multihypothesis Testing (Composite)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$H_1 : \ \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \ \mathbf{P}^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \ \mathbf{P}^\star \in \mathscr{P}_M$

known

$$\forall\, \mathbf{P} \in \mathscr{P}_i\,, \quad \lim_{\delta \downarrow 0} \ \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid \mathbf{P}^\star = \mathbf{P}]}{\log(1/\delta)} \geq \frac{1}{\sup_{\mathbf{w}} \ \inf_{\mathbf{Q} \in \bigcup_{j \neq i} \mathscr{P}_j} \sum_{a=1}^{K} w_a \ D_{\mathrm{KL}}(P_a \parallel Q_a)}$$

$\mathrm{ALT}(\mathbf{P})$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Active Multihypothesis Testing (Composite)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$H_1 : \ \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \ \mathbf{P}^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \ \mathbf{P}^\star \in \mathscr{P}_M$

known

$$\forall\ \mathbf{P} \in \mathscr{P}_i\,, \quad \lim_{\delta \downarrow 0} \ \inf_{\pi \in \Pi(\delta)} \ \frac{\mathbb{E}^\pi[\tau(\pi) \mid \mathbf{P}^\star = \mathbf{P}]}{\log(1/\delta)} \ \geq \ \frac{1}{\sup_{\mathbf{w}} \ \inf_{\mathbf{Q} \in \bigcup_{j \neq i} \mathscr{P}_j} \ \sum_{a=1}^{K} w_a \ D_{\mathrm{KL}}(P_a \parallel Q_a)}$$

Amount of information discrimination to rule out every hypothesis other than H_i when choosing streams according to distribution $\mathbf{w}$

ALT($\mathbf{P}$)

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

Best Arm Identification (BAI)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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BAI with Exponential Families

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_{\theta_1^\star}$

data

IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_{\theta_2^\star}$

data

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_{\theta_K^\star}$

data

w_1, w_2, w_K

Switch

$\boldsymbol{\theta}^\star = (\theta_1^\star, \ldots, \theta_K^\star)$

unknown

$H_1 : \ \boldsymbol{\theta}^\star \in \mathscr{P}_1 = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$

known

$H_2 : \ \boldsymbol{\theta}^\star \in \mathscr{P}_2 = \{\boldsymbol{\theta} : \theta_2 > \max_{j \neq 2} \theta_j\}$

known

$\vdots$

$H_K : \ \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$

known

error probability

$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$

BAI with Exponential Families

IID $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_{\theta_1^\star}$

data

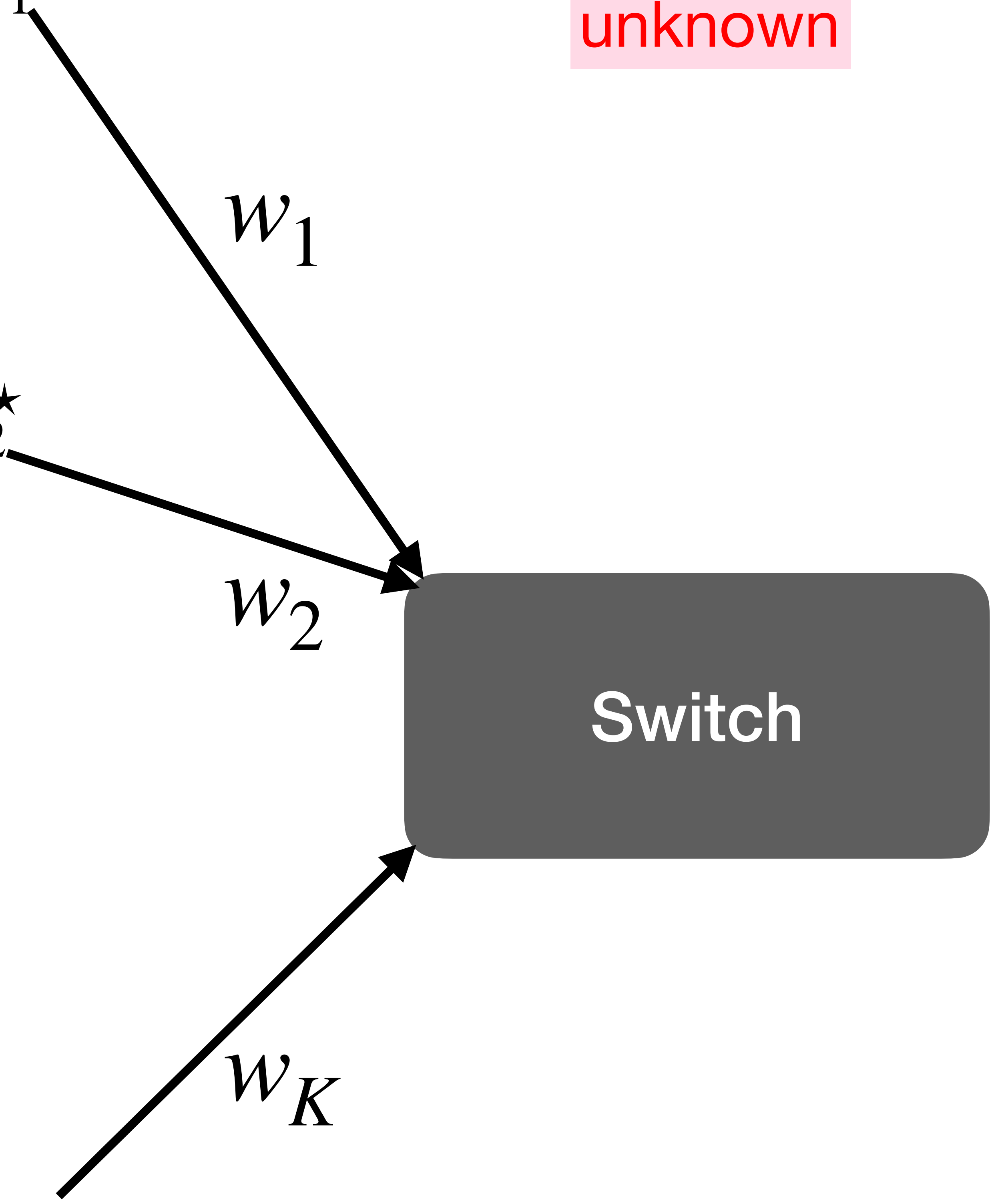
IID $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_{\theta_2^\star}$

data

$\vdots$

IID $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_{\theta_K^\star}$

data



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known

$\vdots$

$H_K : \ \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$

known

At each round, we can pick any of the streams probabilitstically or deterministically

error probability

$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$

BAI with Exponential Families

$$\underbrace{\boldsymbol{\theta}^\star}_{\text{unknown}} = (\theta_1^\star, \ldots, \theta_K^\star)$$

$$P_\theta(x) = h(x)\,\exp\big(\theta x - A(\theta)\big), \qquad \theta \in \mathbb{R}, \quad x \in \mathbb{R}$$

$$H_1 : \ \boldsymbol{\theta}^\star \in \underbrace{\mathscr{P}_1}_{\text{known}} = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$$

$$H_2 : \ \boldsymbol{\theta}^\star \in \underbrace{\mathscr{P}_2}_{\text{known}} = \{\boldsymbol{\theta} : \theta_2 > \max_{j \neq 2} \theta_j\}$$

$\vdots$

$$H_K : \ \boldsymbol{\theta}^\star \in \underbrace{\mathscr{P}_K}_{\text{known}} = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

BAI with Exponential Families

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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$$\underbrace{\boldsymbol{\theta}^\star}_{\text{unknown}} = (\theta_1^\star, \ldots, \theta_K^\star)$$

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$$\vdots$$

$$H_K : \ \boldsymbol{\theta}^\star \in \underbrace{\mathscr{P}_K}_{\text{known}} = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$$

$$P_\theta(x) = h(x) \exp\big(\theta x - A(\theta)\big), \qquad \theta \in \mathbb{R}, \ \ x \in \mathbb{R}$$

- Canonical exponential family parametrised by θ

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

BAI with Exponential Families

$$\underbrace{\boldsymbol{\theta}^\star}_{\text{unknown}} = (\theta_1^\star, \ldots, \theta_K^\star)$$

$$P_\theta(x) = h(x) \exp\big(\theta x - A(\theta)\big), \qquad \theta \in \mathbb{R}, \;\; x \in \mathbb{R}$$

- Canonical exponential family parametrised by θ
- $\mathbb{E}_\theta[X] = \theta;$ identifying true hypothesis = BAI

$$H_1 : \; \boldsymbol{\theta}^\star \in \underbrace{\mathscr{P}_1}_{\text{known}} = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$$

$$H_2 : \; \boldsymbol{\theta}^\star \in \underbrace{\mathscr{P}_2}_{\text{known}} = \{\boldsymbol{\theta} : \theta_2 > \max_{j \neq 2} \theta_j\}$$

$$\vdots$$

$$H_K : \; \boldsymbol{\theta}^\star \in \underbrace{\mathscr{P}_K}_{\text{known}} = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

BAI with Exponential Families

$$\boldsymbol{\theta}^\star = (\theta_1^\star, \ldots, \theta_K^\star)$$

unknown

$$H_1 : \ \boldsymbol{\theta}^\star \in \mathscr{P}_1 = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$$

known

$$H_2 : \ \boldsymbol{\theta}^\star \in \mathscr{P}_2 = \{\boldsymbol{\theta} : \theta_2 > \max_{j \neq 2} \theta_j\}$$

known

$\vdots$

$$H_K : \ \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$$

known

$$P_\theta(x) = h(x) \exp\big(\theta x - A(\theta)\big), \qquad \theta \in \mathbb{R}, \ \ x \in \mathbb{R}$$

- Canonical exponential family parametrised by θ
- $\mathbb{E}_\theta[X] = \theta;$ identifying true hypothesis = BAI
- Examples: Bernoulli, Gaussian with known variance

error probability

$\mathbb{P}(H_j$ **identified** $| \ H_i$ **true**$) \leq \delta$

BAI with Exponential Families

$$\boldsymbol{\theta}^\star = (\theta_1^\star, \ldots, \theta_K^\star)$$

unknown

$$P_\theta(x) = h(x)\exp\big(\theta x - A(\theta)\big), \qquad \theta \in \mathbb{R},\ \ x \in \mathbb{R}$$

- Canonical exponential family parametrised by θ
- $\mathbb{E}_\theta[X] = \theta$; identifying true hypothesis = BAI
- Examples: Bernoulli, Gaussian with known variance
- $D_{\text{KL}}(P_\theta \,\|P_\phi) = A(\phi) - A(\theta) - \dot{A}(\theta)\ (\phi - \theta)$

$$H_1 :\ \boldsymbol{\theta}^\star \in \mathscr{P}_1 = \{\boldsymbol{\theta} : \theta_1 > \max_{j\neq 1} \theta_j\}$$

known

$$H_2 :\ \boldsymbol{\theta}^\star \in \mathscr{P}_2 = \{\boldsymbol{\theta} : \theta_2 > \max_{j\neq 2} \theta_j\}$$

known

$$\vdots$$

$$H_K :\ \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j\neq K} \theta_j\}$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

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BAI with Exponential Families

$$\boldsymbol{\theta}^\star = (\theta_1^\star, \ldots, \theta_K^\star)$$

unknown

$H_1 : \boldsymbol{\theta}^\star \in \mathscr{P}_1 = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$

known

$H_2 : \boldsymbol{\theta}^\star \in \mathscr{P}_2 = \{\boldsymbol{\theta} : \theta_2 > \max_{j \neq 2} \theta_j\}$

known

⋮

$H_K : \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$

known

$$\forall\, \boldsymbol{\theta} \in \mathscr{P}_i\,, \quad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid \boldsymbol{\theta}^\star = \boldsymbol{\theta}]}{\log(1/\delta)} \geq \frac{1}{T^\star(\boldsymbol{\theta})}$$

$$T^\star(\boldsymbol{\theta}) = \sup_{\mathbf{w}} \inf_{\lambda \in \bigcup_{j \neq i} \mathscr{P}_j} \sum_{a=1}^{K} w_a \, D_{\mathrm{KL}}(P_{\theta_a} \,\|\, P_{\lambda_a})$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

BAI with Exponential Families

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$$\boldsymbol{\theta}^\star = (\theta_1^\star, \ldots, \theta_K^\star)$$

unknown

$$H_1 : \boldsymbol{\theta}^\star \in \mathscr{P}_1 = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$$

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$$H_2 : \boldsymbol{\theta}^\star \in \mathscr{P}_2 = \{\boldsymbol{\theta} : \theta_2 > \max_{j \neq 2} \theta_j\}$$

known

$$\vdots$$

$$H_K : \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$$

known

$$\forall\, \boldsymbol{\theta} \in \mathscr{P}_i\,, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}^\pi[\tau(\pi) \mid \boldsymbol{\theta}^\star = \boldsymbol{\theta}]}{\log(1/\delta)} \geq \frac{1}{T^\star(\boldsymbol{\theta})}$$

$$T^\star(\boldsymbol{\theta}) \quad = \quad \sup_{\mathbf{w}} \quad \inf_{\lambda \in \bigcup_{j \neq i} \mathscr{P}_j} \sum_{a=1}^{K} w_a \; D_{\mathrm{KL}}(P_{\theta_a} \,\|\, P_{\lambda_a})$$

$$w^\star(\boldsymbol{\theta}) \quad = \quad \arg\sup_{\mathbf{w}} \quad \inf_{\lambda \in \bigcup_{j \neq i} \mathscr{P}_j} \sum_{a=1}^{K} w_a \; D_{\mathrm{KL}}(P_{\theta_a} \,\|\, P_{\lambda_a})$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

BAI with Exponential Families

$$\boldsymbol{\theta}^\star = (\theta_1^\star, \ldots, \theta_K^\star)$$

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$$H_1 : \ \boldsymbol{\theta}^\star \in \mathscr{P}_1 = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$$

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known

$$\vdots$$

$$H_K : \ \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$$

known

D-tracking algorithm (Garivier and Kaufmann, 2016)

- Empirical means: $\widehat{\boldsymbol{\theta}}^\star(t) = \left(\widehat{\theta}_1(t), \ldots, \widehat{\theta}_K(t)\right)$
- Compute $w^\star\left(\widehat{\boldsymbol{\theta}}^\star(t)\right)$
- Sample at time $t+1$ according to distribution $w^\star\left(\widehat{\boldsymbol{\theta}}^\star(t)\right)$
- Keep sampling until stopping criterion is met

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

BAI with Exponential Families

$$\boldsymbol{\theta}^\star = (\theta_1^\star, \ldots, \theta_K^\star)$$

unknown

D-tracking algorithm (Garivier and Kaufmann, 2016)

$$\frac{Z(t)}{t} = \inf_{\lambda \in \mathrm{ALT}(\widehat{\boldsymbol{\theta}}^\star(t))} \sum_{a=1}^{K} w_a^\star\left(\widehat{\boldsymbol{\theta}}^\star(t)\right) \; D_{\mathrm{KL}}(P_{\widehat{\theta}_a(t)} \parallel \lambda_a)$$

$$\pi_t^\delta = \begin{cases} \text{continue}, & Z(t) \leq c_\delta \,, \\ \text{stop}, & Z(t) > c_\delta \,. \end{cases}$$

At stoppage, announce $H_{a^\star}$ such that $a^\star = \arg\max_a \widehat{\theta}_a(t)$

$$H_1 : \; \boldsymbol{\theta}^\star \in \mathscr{P}_1 = \{\boldsymbol{\theta} : \theta_1 > \max_{j \neq 1} \theta_j\}$$

known

$$H_2 : \; \boldsymbol{\theta}^\star \in \mathscr{P}_2 = \{\boldsymbol{\theta} : \theta_2 > \max_{j \neq 2} \theta_j\}$$

known

$$\vdots$$

$$H_K : \; \boldsymbol{\theta}^\star \in \mathscr{P}_K = \{\boldsymbol{\theta} : \theta_K > \max_{j \neq K} \theta_j\}$$

known

error probability

$$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$$

Markovian Data

SHT with Markovian Data

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$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

SHT with Markovian Data

$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid X_{s-1}, \;\; H_i \text{ true})}{\mathscr{L}(X_s \mid X_{s-1}, \;\; H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s \mid X_{s-1})}{P_j(X_s \mid X_{s-1})}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

SHT with Markovian Data

$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$Z_{i,j}(t) = \log \frac{\mathscr{L}(X_1, \ldots, X_t \mid H_i \text{ true})}{\mathscr{L}(X_1, \ldots, X_t \mid H_j \text{ true})}$$

$$= \sum_{s=1}^{t} \log \frac{\mathscr{L}(X_s \mid X_{s-1}, \;\; H_i \text{ true})}{\mathscr{L}(X_s \mid X_{s-1}, \;\; H_j \text{ true})} = \sum_{s=1}^{t} \log \frac{P_i(X_s \mid X_{s-1})}{P_j(X_s \mid X_{s-1})}$$

$$\mathbb{E}_i^\pi\left[Z_{i,j}\big(\tau(\pi)\big)\right] = \mathbb{E}_i^\pi\left[\sum_{s=1}^{\tau(\pi)} \log \frac{P_i(X_s \mid X_{s-1})}{P_j(X_s \mid X_{s-1})}\right] = \mathbb{E}_i^\pi[\tau(\pi)] \cdot D_{\mathrm{KL}}(P_i \parallel P_j \mid \mu_i)$$

Markovian Wald's Identity

SHT with Markovian Data

$$\mathbb{E}_i^{\pi}\left[Z_{i,j}\big(\tau(\pi)\big)\right] = \mathbb{E}_i^{\pi}\left[\sum_{s=1}^{\tau(\pi)} \log \frac{P_i(X_s \mid X_{s-1})}{P_j(X_s \mid X_{s-1})}\right] = \mathbb{E}_i^{\pi}[\tau(\pi)] \cdot D_{\mathrm{KL}}(P_i \parallel P_j \,|\, \mu_i)$$

Markovian Wald's Identity

$$H_1 : \; P^{\star} = P_1$$

known

$$H_2 : \; P^{\star} = P_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

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$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\mathbb{E}_i^\pi\left[Z_{i,j}\big(\tau(\pi)\big)\right] = \mathbb{E}_i^\pi\left[\sum_{s=1}^{\tau(\pi)} \log \frac{P_i(X_s \mid X_{s-1})}{P_j(X_s \mid X_{s-1})}\right] = \mathbb{E}_i^\pi[\tau(\pi)] \cdot D_{\mathrm{KL}}(P_i \,\|\, P_j \,|\, \mu_i)$$

Markovian Wald's Identity

$$D_{\mathrm{KL}}(P_i \,\|\, P_j \,|\, \mu_i) = \sum_x \sum_y \mu_i(x)\; P_i(y \,|\, x)\; \log \frac{P_i(y \,|\, x)}{P_j(y \,|\, x)}$$

SHT with Markovian Data

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star = P_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\mathbb{E}_i^\pi\left[Z_{i,j}\big(\tau(\pi)\big)\right] = \mathbb{E}_i^\pi\left[\sum_{s=1}^{\tau(\pi)} \log \frac{P_i(X_s \mid X_{s-1})}{P_j(X_s \mid X_{s-1})}\right] = \mathbb{E}_i^\pi[\tau(\pi)] \cdot D_{\mathrm{KL}}(P_i \,\|\, P_j \,|\, \mu_i)$$

Markovian Wald's Identity

$$D_{\mathrm{KL}}(P_i \,\|\, P_j \,|\, \mu_i) = \sum_x \sum_y \mu_i(x)\, P_i(y \,|\, x)\, \log \frac{P_i(y \,|\, x)}{P_j(y \,|\, x)}$$

Stationary distribution of P_i

SHT with Markovian Data

$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

SHT with Markovian Data

$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$H_1 : \; P^\star = P_1$

known

$H_2 : \; P^\star = P_2$

known

$$\forall\, i \neq j, \quad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \; \frac{\mathbb{E}_i^\pi[\tau(\pi)]}{\log(1/\delta)} \geq \frac{1}{D_{\mathrm{KL}}(P_i \,\|\, P_j | \mu_i)}$$

error probability

$\mathbb{P}(H_j$ **identified** $|$ H_i **true**$) \leq \delta$

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SHT with Markovian Data

$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

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$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data

unknown

$$H_1 : \; P^\star = P_1$$

known

$$H_2 : \; P^\star \in \mathscr{P}_2$$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi) \mid P^\star = P_1]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} D_{\mathrm{KL}}(P_1 \,\|\, Q \,|\, \mu_1)}$$

$$\forall Q \in \mathscr{P}_2, \qquad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi) \mid P^\star = Q]}{\log(1/\delta)} \geq \frac{1}{D_{\mathrm{KL}}(Q \,\|\, P_1 \,|\, \mu_Q)}$$

SHT with Markovian Data

$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

SHT with Markovian Data

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$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$$\forall P \in \mathscr{P}_1, \qquad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi) \mid P^\star = P]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} D_{\mathrm{KL}}(P \,\|\, Q \,|\, \mu_P)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

SHT with Markovian Data

$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data unknown

$H_1: \; P^\star \in \mathscr{P}_1$

known

$H_2: \; P^\star \in \mathscr{P}_2$

known

$$\forall P \in \mathscr{P}_1, \qquad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi) \mid P^\star = P]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} \; D_{\mathrm{KL}}(P \,\|\, Q \,|\, \mu_P)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\forall Q \in \mathscr{P}_2, \qquad \lim_{\delta \downarrow 0} \; \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi) \mid P^\star = Q]}{\log(1/\delta)} \geq \frac{1}{\inf_{P \in \mathscr{P}_1} \; D_{\mathrm{KL}}(Q \,\|\, P \,|\, \mu_Q)}$$

SHT with Markovian Data

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$X_1, X_2, \ldots \sim$ TPM $P^\star$ Markov chain (ergodic)

data

unknown

$H_1 : \; P^\star \in \mathscr{P}_1$

known

$H_2 : \; P^\star \in \mathscr{P}_2$

known

$$\forall P \in \mathscr{P}_1, \qquad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi) \mid P^\star = P]}{\log(1/\delta)} \geq \frac{1}{\inf_{Q \in \mathscr{P}_2} D_{\mathrm{KL}}(P \,\|\, Q \,|\, \mu_P)}$$

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

$$\forall Q \in \mathscr{P}_2, \qquad \lim_{\delta \downarrow 0} \inf_{\pi \in \Pi(\delta)} \frac{\mathbb{E}_i^\pi[\tau(\pi) \mid P^\star = Q]}{\log(1/\delta)} \geq \frac{1}{\inf_{P \in \mathscr{P}_1} D_{\mathrm{KL}}(Q \,\|\, P \,|\, \mu_Q)}$$

Simple LLR-based tests help achieve the lower bounds

Multiple Markovian Data Streams

Active SHT with Markovian Data

MC $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data

MC $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data

$\vdots$

MC $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \mathbf{P}^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \mathbf{P}^\star \in \mathscr{P}_M$

known

error probability

$\mathbb{P}(H_j \text{ identified} \mid H_i \text{ true}) \leq \delta$

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Active SHT with Markovian Data

MC $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data

MC $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data

$\vdots$

MC $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

$H_1 : \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \mathbf{P}^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \mathbf{P}^\star \in \mathscr{P}_M$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Active SHT with Markovian Data

MC $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$
data

MC $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$
data

$\vdots$

MC $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$
data

Switch

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$
unknown

$H_1: \mathbf{P}^\star \in \mathscr{P}_1$
known

$H_2: \mathbf{P}^\star \in \mathscr{P}_2$
known

$\vdots$

$H_M: \mathbf{P}^\star \in \mathscr{P}_M$
known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

At each round, we can pick any of the streams probabilitstically or deterministically

Active SHT with Markovian Data

MC $X_1^{(1)}, X_2^{(1)}, \ldots \sim P_1^\star$

data

MC $X_1^{(2)}, X_2^{(2)}, \ldots \sim P_2^\star$

data

$\vdots$

MC $X_1^{(K)}, X_2^{(K)}, \ldots \sim P_K^\star$

data

$\mathbf{P}^\star = (P_1^\star, \ldots, P_K^\star)$

unknown

Switch

Time	Stream (arm)	Rested	Restless
1	1	$X_1^{(1)}$	$X_1^{(1)}$
2	2	$X_1^{(2)}$	$X_2^{(2)}$
3	2	$X_2^{(2)}$	$X_3^{(2)}$
4	3	$X_1^{(3)}$	$X_4^{(3)}$

$H_1 : \mathbf{P}^\star \in \mathscr{P}_1$

known

$H_2 : \mathbf{P}^\star \in \mathscr{P}_2$

known

$\vdots$

$H_M : \mathbf{P}^\star \in \mathscr{P}_M$

known

error probability

$\mathbb{P}(H_j$ **identified** $\mid H_i$ **true**$) \leq \delta$

At each round, we can pick any of the streams probabilitstically or deterministically

Progress So Far

Indian Institute of Technology Hyderabad

Optimal Best Markovian Arm Identification with Fixed Confidence

Vrettos Moulos
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Abstract

We give a complete characterization of the sampling complexity of best Markovian arm identification in one-parameter Markovian bandit models. We derive instance specific nonasymptotic and asymptotic lower bounds which generalize those of the IID setting. We analyze the Track-and-Stop strategy, initially proposed for the IID setting, and we prove that asymptotically it is at most a factor of four apart from the lower bound. Our one-parameter Markovian bandit model is based on the notion of an exponential family of stochastic matrices for which we establish many useful properties. For the analysis of the Track-and-Stop strategy we derive a novel concentration inequality for Markov chains that may be of interest in its own right.

Optimal Best Arm Identification With Fixed Confidence in Restless Bandits

P. N. Karthik, *Member, IEEE*, Vincent Y. F. Tan, *Senior Member, IEEE*, Arpan Mukherjee, and Ali Tajer, *Senior Member, IEEE*

***Abstract*—We study best arm identification in a *restless* multi-armed bandit setting with finitely many arms. The discrete-time data generated by each arm forms a homogeneous Markov chain taking values in a common, finite-state space. The state transitions in each arm are captured by an *ergodic* transition probability matrix (TPM) that is a member of a single-parameter exponential family of TPMs. The real-valued parameters of the arm TPMs are *unknown* and belong to a given space. Given a function f defined on the common state space of the arms, the goal is to identify the best arm—the arm with the largest average value of f evaluated under the arm's stationary distribution—with the fewest number of samples, subject to an upper bound on the decision's error probability (i.e., the *fixed-confidence* regime). A lower bound on the growth rate of the expected stopping time is established in the asymptote of a vanishing error probability. Furthermore, a policy for best arm identification is proposed, and its expected stopping time is proved to have an asymptotic growth rate that matches the lower bound. It is demonstrated that tracking the long-term behavior of a certain Markov decision process and its state-action visitation proportions are the key ingredients in analyzing the converse and achievability bounds. It is shown that under every policy, the state-action visitation proportions satisfy a specific approximate flow conservation constraint and that these proportions match the optimal proportions dictated by the lower bound under any asymptotically optimal policy. The prior studies on best arm identification in restless bandits focus on *independent observations* from the arms, *rested* Markov arms, and restless Markov arms with *known* arm TPMs. In contrast, this work is the first to study best arm identification in restless bandits with unknown arm TPMs.**

***Index Terms*—Restless bandits, best arm identification (BAI), exponential family, transition probability matrix (TPM), Markov decision process (MDP).**

I. Introduction

MULTI-ARMED bandits constitute an effective probabilistic model for sequential decision-making under uncertainty. In the canonical multi-armed bandit models, each arm is assumed to yield random rewards generated by an unknown reward distribution. The arms are selected sequentially over time to optimize a pre-specified reward measure. The two common frameworks to formalize bandit algorithms are *regret minimization* and *pure exploration*. In regret minimization, the objective is to have an arm selection policy that minimizes the difference between the expected reward realized and the maximum reward achievable by an oracle that knows the true reward distributions. Minimizing such regret measures captures the inherent *exploration-exploitation* trade-off that specifies the balance between the desire to choose the arms with high expected rewards (exploitation) and the need to explore other arms to acquire better information discrimination (exploration). In this context, there exists a wide range of algorithms for different settings based on the notions of Upper Confidence Bound (UCB) [1], [2] and Thompson Sampling [3] An in-depth analysis of these algorithms and a detailed survey of other studies on regret minimization can be found in [4].

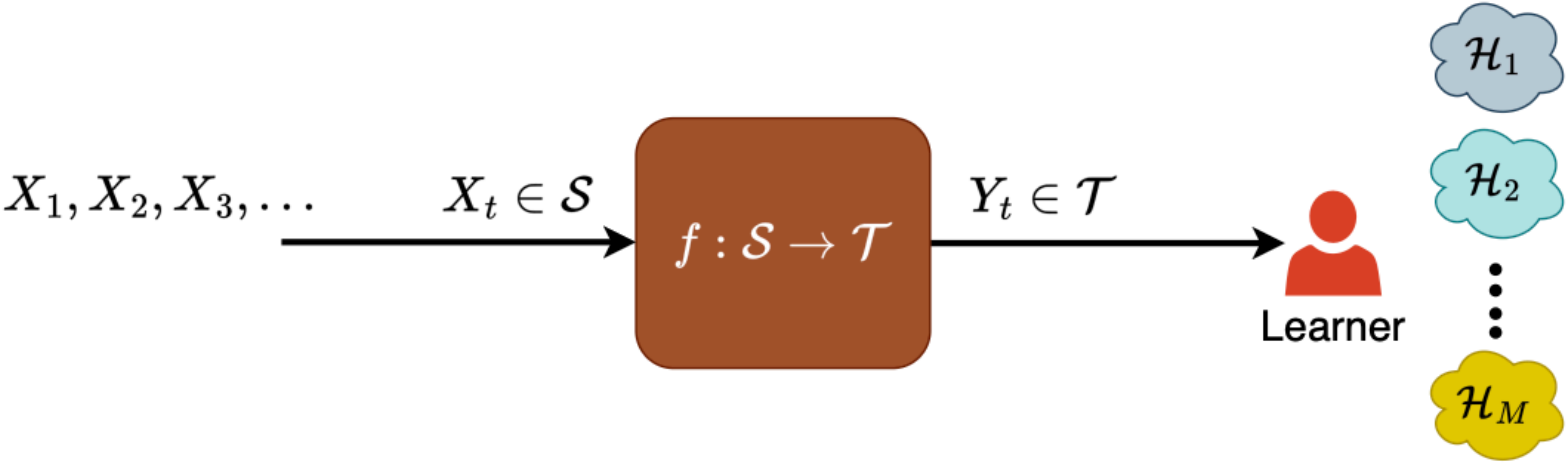
The pure exploration framework, on the other hand, focuses

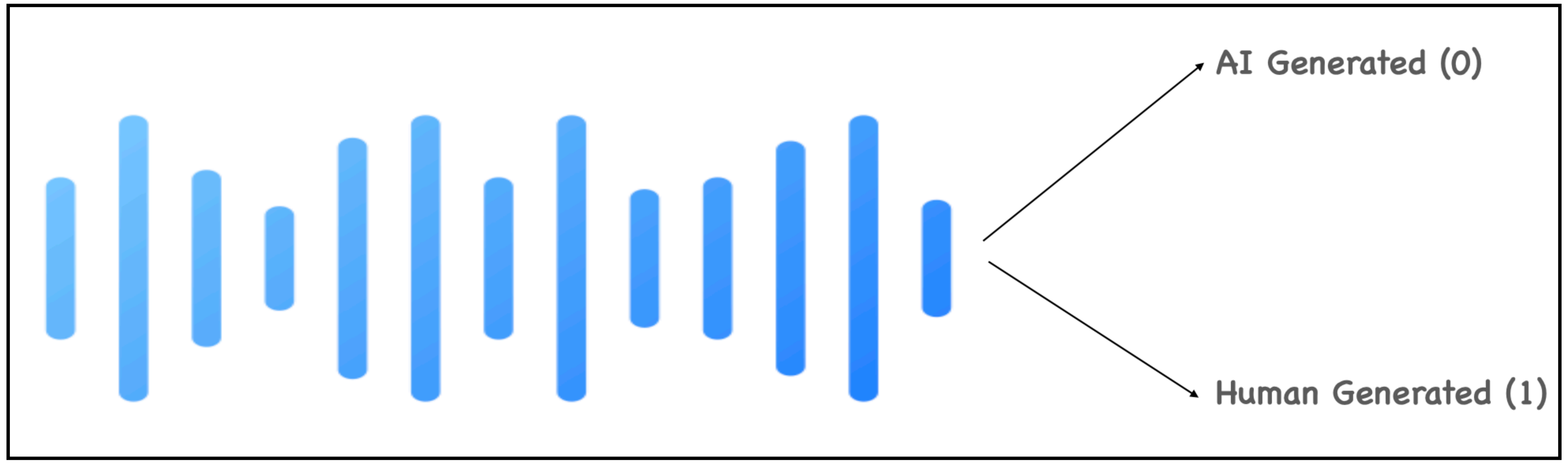
- Studies BAI in the rested setting
- Exponential family of Markov chains

- BAI in the restless setting
- Best policy identification in MDPs

Hidden Markov Data

Future Directions


$X_1, X_2, X_3, \ldots$
$X_t \in \mathcal{S}$
$f : \mathcal{S} \rightarrow \mathcal{T}$
$Y_t \in \mathcal{T}$
Learner
$\mathcal{H}_1$
$\mathcal{H}_2$
$\mathcal{H}_M$


AI Generated (0)
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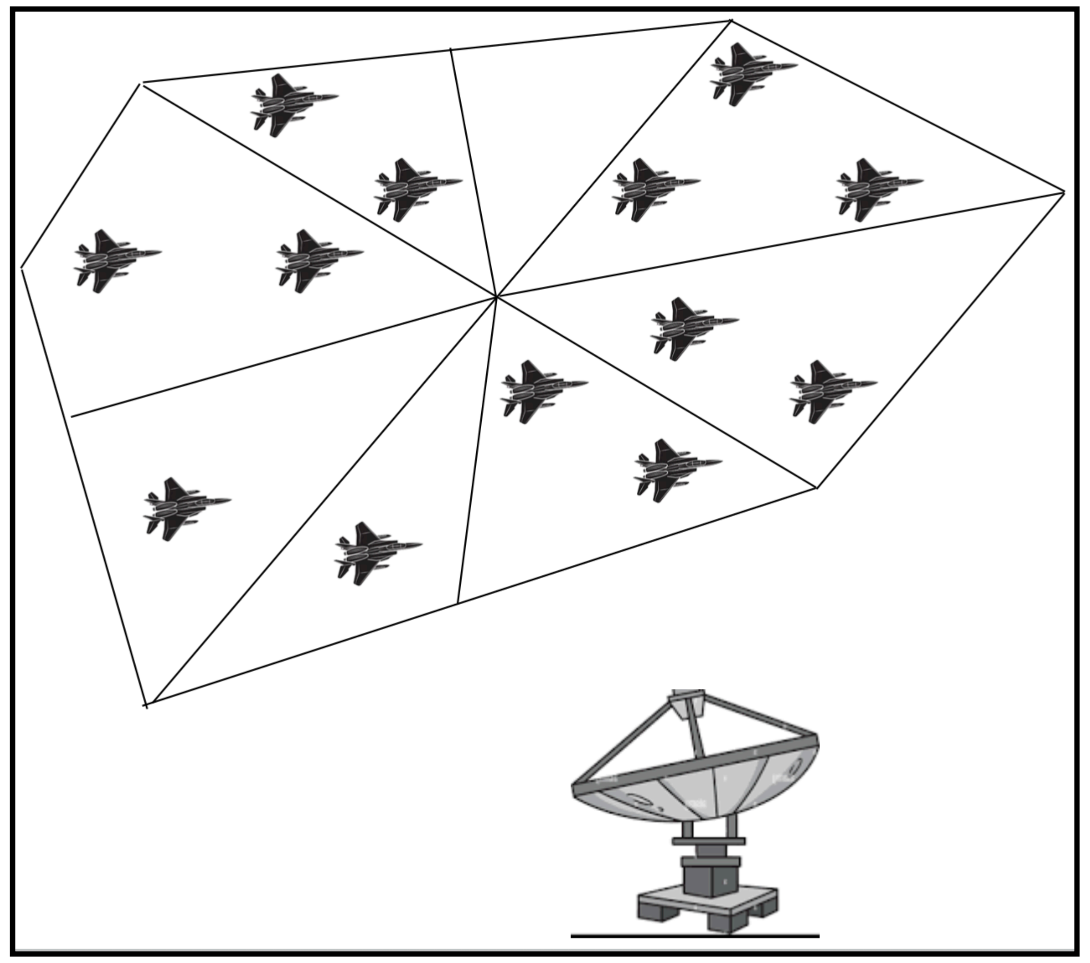
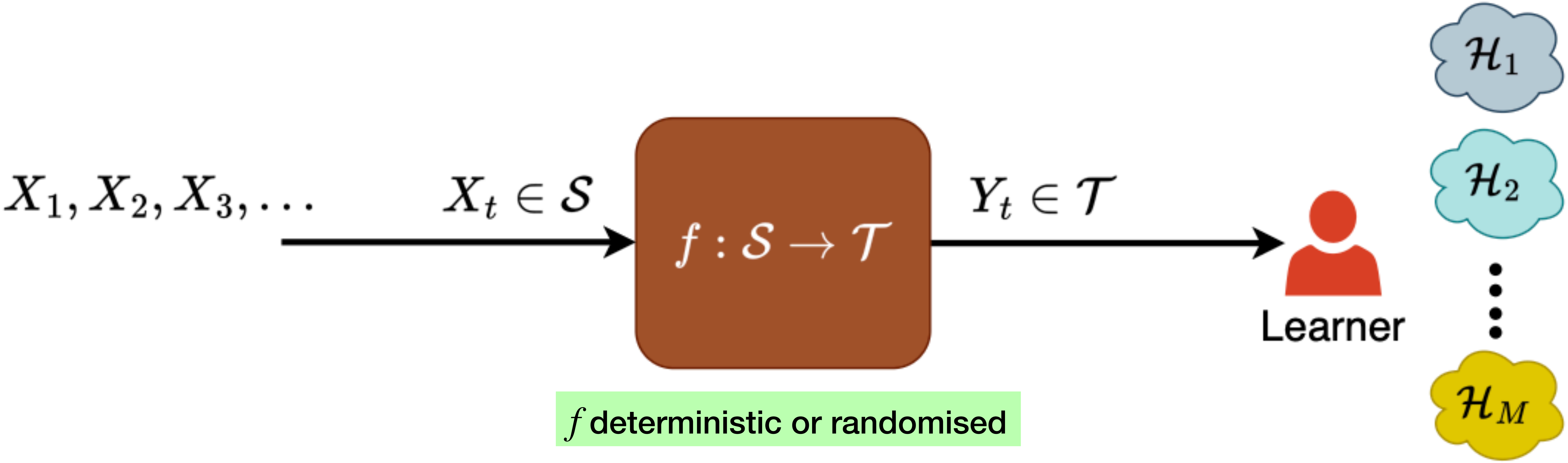
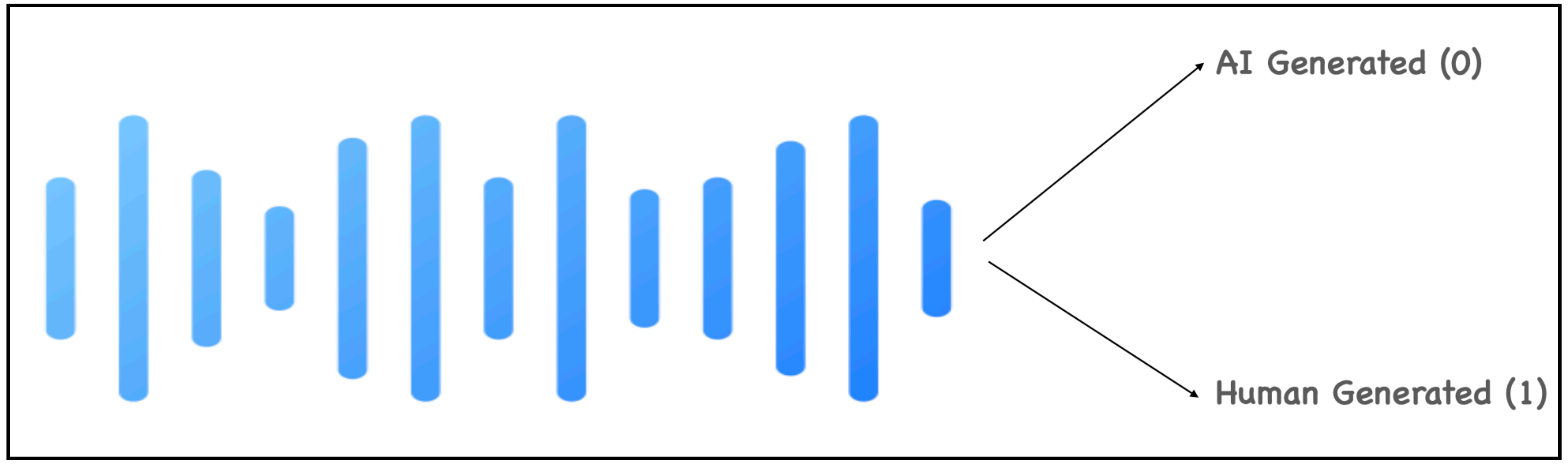
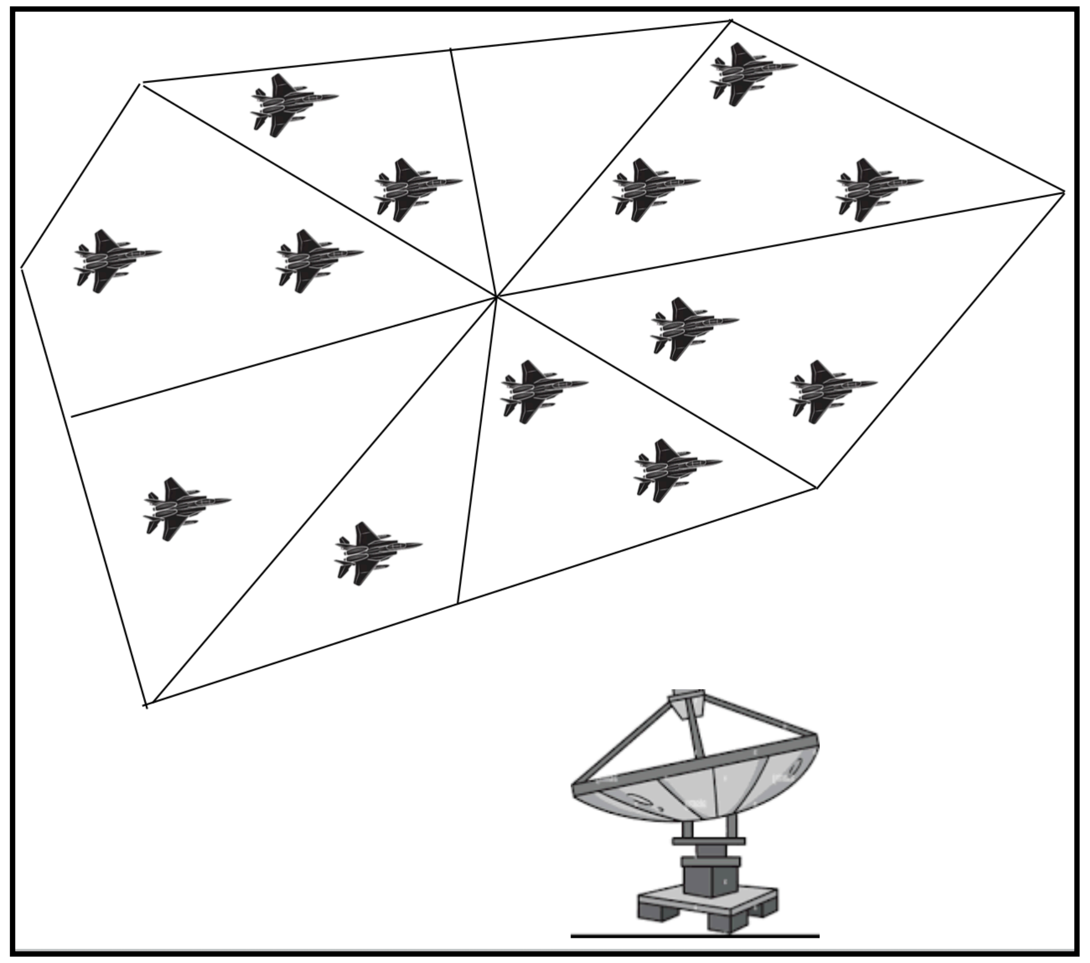



$X_1, X_2, X_3, \ldots$
$X_t \in \mathcal{S}$
$f : \mathcal{S} \to \mathcal{T}$
$Y_t \in \mathcal{T}$
Learner
$\mathcal{H}_1$
$\mathcal{H}_2$
$\mathcal{H}_M$
f deterministic or randomised


AI Generated (0)
Human Generated (1)



If you see a problem of regret minimization (RL, bandits, MDPs), there might be a compelling side to the story from the perspective of pure exploration, which may be worth investigating

Yours Truly

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