Transformations of Random Variables

1. Let $X, Y \overset{\text{IID}}{\sim} \mathcal{N}(0, 1)$, where X and Y are both defined on a common underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

   (a) Argue formally that $\{X = Y\} = \{\omega \in \Omega : X(\omega) = Y(\omega)\} \in \mathscr{F}$.

   (b) Compute $\mathbb{P}(\{X = Y\})$.

2. Fix $q \in (0, 1)$. Let $U \sim \text{Unif}((0, 1))$, and let
$$X = \lfloor \log_q U \rfloor + 1,$$
   where $\lfloor x \rfloor$ denotes the largest integer lesser than or equal to x (for e.g., $\lfloor 0.3 \rfloor = 0$, $\lfloor 4.99 \rfloor = 4$, $\lfloor 2 \rfloor = 2$, and so on).
   Here, $\log_q U$ denotes the logarithm of U to the base q.

   Determine the PMF of X.

   Hint: List down the possible values of $\lfloor \log_q U \rfloor$.

3. Let X, Y be jointly continuous with the joint PDF
$$f_{X,Y}(x, y) = \begin{cases} \dfrac{1}{x}, & 0 \leq y \leq x \leq 1, \\[2mm] 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the PDF of $Z = X + Y$.

   (b) Compute $\mathbb{P}(\{Z \leq 1\})$.

4. Numbers from $[0, 1]$ are picked uniformly, independently, and sequentially over time.
   Let X_n denote the number picked at time n, where $n \in \{0, 1, 2, \ldots\}$.
   Let N be the random variable defined as
$$N = \min\{n \geq 1 : X_n < X_0\}.$$

   That is, N denotes the first time index $n \geq 1$ at which the value of X_n goes below the value of X_0.

   (a) For any fixed $n \in \mathbb{N}$, determine $\mathbb{P}(\{N = n\})$.
      Hint: The event that $N = n$ is identical to the event that
$$X_1 \geq X_0 \quad \text{and} \quad X_2 \geq X_0 \quad \text{and} \quad \cdots \quad \text{and} \quad X_{n-1} \geq X_0 \quad \text{and} \quad X_n < X_0.$$

   (b) Compute $\mathbb{P}(\{N > 2\})$.

5. Let $X_1, X_2 \overset{\text{IID}}{\sim} \text{Exponential}(\lambda)$ for a fixed $\lambda > 0$.

   (a) Determine the PDF of $Y_1 = \dfrac{X_1}{X_2}$.
      Hint: Compute the CDF $\mathbb{P}(\{\frac{X_1}{X_2} \leq x\}) = \mathbb{P}(\{X_1 \leq x\, X_2\})$ and differentiate it to get the PDF.

   (b) Determine the PDF of $Y_2 = X_1 - X_2$.
      Hint: Compute the CDF $\mathbb{P}(\{X_1 - X_2 \leq x\})$ and differentiate it to get the PDF.

6. **(Memoryless Property)**

   (a) Show that a discrete random variable $X \sim \text{Geometric}(p)$ for some $p < 1$ if and only if satisfies the following memoryless property:
$$\mathbb{P}(\{X > k + n\} \mid \{X > k\}) = \mathbb{P}(\{X > n\}) \qquad \forall k, n \in \mathbb{N}.$$

(b) Show that a continuous random variable $X \sim \text{Exponential}(\mu)$ for some $\mu > 0$ if and only if satisfies the following memoryless property:

$$\mathbb{P}(\{X > t + s\} \mid \{X > t\}) = \mathbb{P}(\{X > s\}) \qquad \forall s, t > 0.$$

Hint: In either case, to show the "only if" part, define the functions

$$f(n) = \mathbb{P}(\{X > n\}), \ n \in \mathbb{N}, \qquad g(t) = \mathbb{P}(\{X > t\}), \ t > 0.$$

Show that the unique solutions to the functional equations

$$f(n + k) = f(n) \cdot f(k) \ \forall k, n \in \mathbb{N}, \qquad g(t + s) = g(t) \cdot g(s) \ \forall s, t > 0,$$

are the Geometric PMF and the complementary Exponential CDF respectively.