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Indian Institute of Technology Hyderabad

Stochastic Processes

Random Process: Definition, Finite Dimensional Distributions (FDDs), Mean, Autocorrelation, and Autocovariance, Stationary and Wide-Sense Stationary Processes, IID Processes

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Random Process: Definition

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Random Process)

Fix an index set $\mathcal{T}$.
A collection of random variables $\{X_t : t \in \mathcal{T}\}$ indexed by the elements of $\mathcal{T}$ and defined w.r.t. $\mathscr{F}$ is called random process.

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- Sometimes, $\mathcal{T}$ has the interpretation of time
- Random processes are also referred to as stochastic processes (after the Greek word $\sigma\tau o\chi\alpha\sigma\tau\iota\kappa\acute{o}\zeta$ which means 'to proceed by guesswork')

Ways to Think of a Random Process

- $X_t : \Omega \to \mathbb{R}$ is a random variable w.r.t. $\mathscr{F}$ for each $t \in \mathcal{T}$
- $X_\cdot(\omega) : \mathcal{T} \to \mathbb{R}$ is a sample path of the process for each $\omega \in \Omega$
- $X : \mathcal{T} \times \Omega \to \mathbb{R}$
  - $X_t(\omega)$ is a real number for each $\omega \in \Omega$ and $t \in \mathcal{T}$

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Note

In this course, we will typically consider $\mathcal{T}$ to be one of $\mathbb{R}_+$, $\mathbb{R}$, $\mathbb{Z}$, or $\mathbb{N}$.

Finite Dimensional Distributions

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Finite Dimensional Distributions)

Let $\{X_t : t \in \mathcal{T}\}$ be a random process defined w.r.t. $\mathscr{F}$.

- Given $n \in \mathbb{N}$ and $\mathbf{t} = (t_1, \ldots, t_n) \in \mathcal{T}^n$, the joint CDF of $X_{t_1}, \ldots, X_{t_n}$ is given by

$$F_{\mathbf{t}}(\mathbf{x}) = \mathbb{P}(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n), \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

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- The collection

$$\left\{F_{\mathbf{t}} : \;\; n \in \mathbb{N},\, \mathbf{t} \in \mathcal{T}^n\right\}$$

is referred to as the collection of finite dimensional distributions (FDDs) of the process $\{X_t : t \in \mathcal{T}\}$.

A Note on FDDs

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$$\text{FDDs} = \left\{F_1, F_2, F_3, F_{1,2}, F_{2,3}, F_{1,3}, F_{1,2,3}\right\}$$

  - We observe that

$$F_2(x) = F_{1,2}(x, \infty), \quad F_{2,3}(x, y) = F_{1,2,3}(\infty, x, y), \quad \ldots$$

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- In some sense, the FDDs have to be consistent

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Consistency of FDDs

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_t : t \in \mathcal{T}\}$ be a random process defined w.r.t. $\mathscr{F}$.

Definition (Consistency of FDDs)

The FDDs of the process $\{X_t : t \in \mathcal{T}\}$ are said to be consistent if for any $m, n \in \mathbb{N}$ with $m < n$, and subsets $\mathcal{T}_m \subset \mathcal{T}_n \subset \mathcal{T}$ with $|\mathcal{T}_m| = m$ and $|\mathcal{T}_n| = n$, we have

$$F_{\mathbf{t}}(x_1, \ldots, x_m) = F_{\mathbf{s}}(\underbrace{\infty, \cdots, \infty, x_1, \infty, \cdots, \infty, x_2, \infty, \cdots, \infty, x_m, \infty, \cdots, \infty}_{n}),$$

where

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where

- $\mathbf{t} \in \mathcal{T}_m, \quad \mathbf{s} \in \mathcal{T}_n$ and contains the coordinates in $\mathbf{t}$.
- The coordinates in $\mathbf{s}$ corresponding to those not in $\mathbf{t}$ are shown as ∞ on the RHS.

Random Processes with Consistent FDDs

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Our interest is in the study of random processes whose FDDs are consistent.

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Examples of processes with consistent FDDs include:

- IID processes.
- Bernoulli processes.
- Gaussian processes.
- Markov processes (or Markov chains).
- Poisson process.
- Lévy process.
- Brownian motion and diffusions.

Properties of Random Processes

Mean, Autocorrelation, and Autocovariance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_t : t \in \mathcal{T}\}$ be a random process defined w.r.t. $\mathscr{F}$.

Definition (Mean, Autocorrelation, Autocovariance)

- The mean of the process $\{X_t : t \in \mathcal{T}\}$ is a function $M_X : \mathcal{T} \to [-\infty, +\infty]$ defined as

$$M_X(t) = \mathbb{E}[X_t], \qquad t \in \mathcal{T}.$$

- The autocorrelation and autocovariance of the process $\{X_t : t \in \mathcal{T}\}$ are functions $R_X, C_X : \mathcal{T} \times \mathcal{T} \to [-\infty, +\infty]$, defined as

$$R_X(t,s) = \mathbb{E}[X_t X_s], \qquad C_X(t,s) = \mathrm{Cov}(X_t, X_s), \qquad t, s \in \mathcal{T}.$$

Stationary Process

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_t : t \in \mathbb{R}_+\}$ be a random process defined w.r.t. $\mathscr{F}$.

Definition (Stationary Process)

$\{X_t : t \geq 0\}$ is said to be (strictly) stationary if all FDDs are translation invariant, i.e., for any $n \in \mathbb{N}$, $\mathbf{t} \in \mathbb{R}_+^n$, and $h \in \mathbb{R}_+$,

$$F_{\mathbf{t}} = F_{\mathbf{t}+h}.$$

Here, $\mathbf{t} + h$ is a vector with each coordinate incremented by h with respect to the corresponding coordinate in $\mathbf{t}$.

Weakly Stationary Process

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $\{X_t : t \in \mathbb{R}_+\}$ be a random process defined w.r.t. $\mathscr{F}$.

Definition (Stationary Process)

$\{X_t : t \in \mathbb{R}_+\}$ is said to be weakly stationary (or wide-sense stationary) if for all $t_1, t_2 \in \mathbb{R}_+$ and $h \in \mathbb{R}_+$:

1. $M_X(t_1) = M_X(t_2)$.
2. $C_X(t_1, t_2) = C_X(t_1 + h, t_2 + h)$.

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Remarks:

- A process is weakly stationary iff it has constant mean, and $C_X(t, t+h) = C_X(0, h)$ for all $t, h \in \mathbb{R}_+$ (proof: exercise!)

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Remarks:

- A process is weakly stationary iff it has constant mean, and $C_X(t, t+h) = C_X(0, h)$ for all $t, h \in \mathbb{R}_+$ (proof: exercise!)
- Every stationary process with finite variance is wide-sense stationary (proof: exercise!)

IID Processes

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IID Process

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Definition (IID Process)

$\{X_t : t \in \mathbb{R}_+\}$ is said to be an IID process with the common CDF F if for any $n \in \mathbb{N}$ and $\mathbf{t} \in \mathbb{R}_+^n$,

$$F_{\mathbf{t}}(\mathbf{x}) = \prod_{i=1}^{n} F(x_i), \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Some Results on IID Processes

Lemma

Suppose that $\{X_t : t \in \mathbb{R}_+\}$ is an IID process.

1. The FDDs of $\{X_t : t \in \mathbb{R}_+\}$ are consistent.
2. $\{X_t : t \in \mathbb{R}_+\}$ is strictly stationary.
   That is, every IID process is stationary.

Example

- Let $X_1, X_2, \ldots$ be an $\mathbb{N}$-valued IID process.
  Let $S_0 := 0$, and for each $n \in \mathbb{N}$, let

$$S_n = \sum_{i=1}^{n} X_i.$$

  - Determine M_S and C_S for the process $\{S_n\}_{n=0}^{\infty}$.

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  - Determine M_S and C_S for the process $\{S_n\}_{n=0}^{\infty}$.
  - Is $\{S_n\}_{n=0}^{\infty}$ wide-sense stationary?