भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Stochastic Processes

Lecture 07

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Weak Law of Large Numbers (WLLN)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Theorem (Weak Law of Large Numbers)

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of **independent and identically distributed (i.i.d.)** real-valued RVs with $\mathbb{E}[|X_1|] < +\infty$. Further, let $\mathbb{E}[X_1] = \mu \in \mathbb{R}$. For each $n \in \mathbb{N}$, let

$$S_n := \sum_{i=1}^{n} X_i$$

denote the partial sum of random variables up to time n. Then,

$$\frac{S_n}{n} \xrightarrow{\text{p.}} \mu. \qquad \lim_{n \to \infty} \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) = 0 \qquad \forall\, \varepsilon > 0.$$

Proof of WLLN – Using Finite Variance Assumption

- Suppose that $\mathrm{Var}(X_1) = \sigma^2 < +\infty$
- Then, for any choice of $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) &\leq \frac{1}{\varepsilon^2}\,\mathbb{E}\left[\left(\frac{S_n}{n} - \mu\right)^2\right] \qquad \text{(Chebyshev's inequality)} \\ &= \frac{\sigma^2}{n\varepsilon^2} \end{aligned}$$

- It follows that

$$\forall\, \varepsilon > 0, \qquad \lim_{n\to\infty} \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0,$$

thus proving that $X_n \xrightarrow{\text{p.}} \mu$

Two Important Results

+

Proof of WLLN Without Finite Variance Assumption

Reverse Implication $\mathrm{d.} \implies \mathrm{p.}$

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Reverse Implication $\mathrm{d.} \implies \mathrm{p.}$)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs
For any $c \in \mathbb{R}$,

$$X_n \xrightarrow{\mathrm{d.}} c \quad \implies \quad X_n \xrightarrow{\mathrm{p.}} c$$

Proof of Reverse Implication $\mathrm{d.} \implies \mathrm{p.}$

- Notice that

$$X_n \xrightarrow{\mathrm{d.}} c \qquad \iff \qquad \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0, & x < c, \\ 1, & x > c. \end{cases}$$

  Convergence of CDFs for $x = c$ may or may not hold (we will not make any assumption)

- **To prove:** For every choice of $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(\{|X_n - c| > \varepsilon\}) \; = \; 0.$$

- Fixing an arbitrary $\varepsilon > 0$, we note that for each $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(\{|X_n - c| > \varepsilon\}) \quad &= \quad \mathbb{P}(\{X_n > c + \varepsilon\}) + \mathbb{P}(\{X_n < c - \varepsilon\}) \\ &= \quad 1 - \mathbb{P}(\{X_n \le c + \varepsilon\}) + \mathbb{P}(\{X_n < c - \varepsilon\}) \\ &\le \quad 1 - \mathbb{P}(\{X_n \le c + \varepsilon\}) + \mathbb{P}(\{X_n \le c - \varepsilon\}) \\ &= \quad 1 - F_{X_n}(c + \varepsilon) + F_{X_n}(c - \varepsilon). \end{aligned}$$

- Taking limits as $n \to \infty$, we get

$$\forall\, \varepsilon > 0, \qquad \lim_{n\to\infty} \mathbb{P}(\{|X_n - c| > \varepsilon\}) \; \le \; 1 \; - \; \lim_{n\to\infty} F_{X_n}(c+\varepsilon) \; + \; \lim_{n\to\infty} F_{X_n}(c-\varepsilon) \; = \; 0.$$

Characteristic Function

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let X be a random variable w.r.t. $\mathscr{F}$.

Definition (Characteristic Function)

The **characteristic function** of X is a function $C_X : \mathbb{R} \to \mathbb{C}$, defined as

$$C_X(s) = \mathbb{E}[e^{jsX}] = \mathbb{E}[\cos sX] + j\,\mathbb{E}[\sin sX], \qquad s \in \mathbb{R}.$$

- Observe that
$$\left|C_X(s)\right| \leq 1 \qquad \forall\, s \in \mathbb{R}.$$
- If X is extended real-valued, then its characteristic function is not defined

Taylor Expansion for Characteristic Functions

Lemma (Taylor Expansion for Characteristic Functions)

Suppose that X is a random variable such that $\mathbb{E}[|X|^k] < +\infty$ for some $k \in \mathbb{N}$. Then,

$$C_X(s) = \sum_{\ell=0}^{k} \frac{\mathbb{E}[X^\ell]}{\ell!} (js)^\ell + o(s^k), \qquad s \in \mathbb{R}.$$

For a proof, see [KT08].

- Given two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, we say that $a_n = o(b_n)$ if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

- For instance,

$$\sqrt{n} = o(n), \qquad \log n = o(n), \qquad \frac{1}{n^2} = o\left(\frac{1}{n}\right), \qquad \frac{1}{n} = o(1).$$

Characteristic Functions and Convergence in Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Definition (Characteristic Functions and Convergence in Distribution)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X an extended real-valued RV. Then,

$$X_n \xrightarrow{\text{d.}} X \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} C_{X_n}(s) = C_X(s) \quad \forall\, s \in \mathbb{R}.$$

That is, **pointwise convergence of characteristic functions** is equivalent to convergence in distribution.

Proof is based on Skorokhod's representation theorem [GS20, Section 7.2].

Proof of WLLN – Without Finite Variance Assumption

- We need to prove that if $\{X_n\}_{n\in\mathbb{N}}$ is an i.i.d. sequence with finite mean μ, then $\frac{S_n}{n} \xrightarrow{\text{p.}} \mu$
- For this, we will show that

$$\lim_{n\to\infty} C_{\frac{S_n}{n}}(s) = e^{js\mu} \qquad \forall\, s \in \mathbb{R}.$$

  This will then imply that $\frac{S_n}{n} \xrightarrow{\text{d.}} \mu$, which in turn implies that $\frac{S_n}{n} \xrightarrow{\text{p.}} \mu$
- Fixing an arbitrary $s \in \mathbb{R}\setminus\{0\}$, we have

$$\begin{aligned} C_{\frac{S_n}{n}}(s) &= \left(C_{X_1}\left(\frac{s}{n}\right)\right)^n \\ &= \left(1 + \frac{\mathbb{E}[X]}{1!}\cdot\frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n \qquad \text{(using Taylor expansion formula).} \end{aligned}$$

- Taking limits as $n \to \infty$, we get

$$\forall\, s \in \mathbb{R}\setminus\{0\}, \qquad \lim_{n\to\infty} C_{\frac{S_n}{n}}(s) = \lim_{n\to\infty}\left(1 + \frac{\mu}{1!}\cdot\frac{js}{n} + o\left(\frac{s}{n}\right)\right)^n \overset{(a)}{=} \lim_{n\to\infty}\left(1 + \frac{\mu}{1!}\cdot\frac{js}{n}\right)^n = e^{js\mu}.$$

Proof of WLLN – Without Finite Variance Assumption

- Justification for (a):
  - For a fixed $s \in \mathbb{R} \setminus \{0\}$, let $\{g_n(s)\}_{n\in\mathbb{N}}$ be any sequence such that

$$g_n(s) = o\left(\frac{s}{n}\right), \qquad n \in \mathbb{N}.$$

  - Then, we have

$$\lim_{n\to\infty} \frac{g_n(s)}{s/n} = 0 \quad \iff \quad \lim_{n\to\infty} \frac{g_n(s)}{1/n} = 0 \quad \iff \quad g_n(s) = o(1/n).$$

Linear Regression in Large Language Models (LLMs)

Consider the token embeddings generated at some intermediate layer (say L) of an LLM. Each token embedding is a vector in $\mathbb{R}^d$ for some large d. Let $\{X_n\}_{n\in\mathbb{N}}$ denote the first coordinate readings of the vector embeddings across various contexts.

Model: $X_1, X_2, \cdots \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$.

Let Y denote a downstream performance metric (e.g., ROUGE/BLEU scores), and suppose that

$$Y_i = \beta X_i + \varepsilon_i,$$

where $\beta \in \mathbb{R}$ **is unknown**, and $\varepsilon_1, \varepsilon_2 \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$, independent of $X_1, X_2, \ldots$, and σ^2 **is known**

- Considering n data points $\{(X_i, Y_i)\}_{i=1}^n$, write down the expression for the ℓ_2 loss (a.k.a. squared loss) between input and output assuming a guess θ for the unknown β.
- Give a closed-form expression for $\widehat{\beta}_n$, the value of θ that minimizes the ℓ_2 loss.
- Prove that

$$\widehat{\beta}_n \xrightarrow{\text{a.s.}} \beta.$$

Strong Law of Large Numbers (SLLN)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Theorem (Strong Law of Large Numbers)

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denote the partial sum of random variables up to time n. Then,

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mu. \qquad \mathbb{P}\left(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) = 0 \qquad \forall\, \varepsilon > 0.$$

Consequences of SLLN

- **Estimating the unknown bias of a coin:**

  Suppose that $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathrm{Ber}(p)$, where p is **unknown**
  According to SLLN, we have
  $$\frac{1}{n}\sum_{\ell=1}^{n} X_\ell \xrightarrow{\text{a.s.}} p.$$

- **Empirical risk minimization:**

  Suppose $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, 1)$, where μ is **unknown**
  According to SLLN, we have
  $$\frac{1}{n}\sum_{\ell=1}^{n} X_\ell \xrightarrow{\text{a.s.}} \mu.$$

- **Linear regression in LLMs:**

  Suppose $Y_i = \beta X_i + \varepsilon_i$, where $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, $\varepsilon_1, \varepsilon_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ are independent of $\{X_n\}_{n\in\mathbb{N}}$. Here, $\beta \in \mathbb{R}$ is **unknown**, while $\sigma > 0$ is **known**. According to SLLN, we have
  $$\frac{\sum_{\ell=1}^{n} X_\ell Y_\ell}{\sum_{\ell=1}^{n} X_\ell^2} \xrightarrow{\text{a.s.}} \frac{\mathbb{E}[X_1 Y_1]}{\mathbb{E}[X_1^2]} = \beta.$$

SLLN Example: Estimating the Unknown Bias of a Coin

```
Experiment 1: First 10 samples: [0 0 1 1 1 1 1 1 1 0]
Experiment 1: Sample mean = 0.5934
Experiment 2: First 10 samples: [1 1 0 0 1 1 1 1 0 1]
Experiment 2: Sample mean = 0.6004
Experiment 3: First 10 samples: [1 1 0 0 0 0 0 0 0 0]
Experiment 3: Sample mean = 0.5951
Experiment 4: First 10 samples: [0 1 0 1 0 0 0 1 1 0]
Experiment 4: Sample mean = 0.5957
Experiment 5: First 10 samples: [0 0 1 0 0 0 1 1 1 0]
Experiment 5: Sample mean = 0.6009
Experiment 6: First 10 samples: [1 1 1 0 1 1 1 0 0 1]
Experiment 6: Sample mean = 0.6002
Experiment 7: First 10 samples: [1 0 1 1 1 0 1 0 1 1]
Experiment 7: Sample mean = 0.5993
Experiment 8: First 10 samples: [1 0 1 1 1 0 1 0 1 0]
Experiment 8: Sample mean = 0.6005
Experiment 9: First 10 samples: [1 0 1 1 1 1 0 1 1 0]
Experiment 9: Sample mean = 0.6037
Experiment 10: First 10 samples: [0 0 1 1 0 0 1 1 1 1]
Experiment 10: Sample mean = 0.6031
Experiment 11: First 10 samples: [0 0 1 0 0 0 0 0 0 0]
Experiment 11: Sample mean = 0.6015
Experiment 12: First 10 samples: [0 0 1 0 1 1 0 0 0 1]
Experiment 12: Sample mean = 0.6095
Experiment 13: First 10 samples: [0 0 0 0 0 1 1 0 1 0]
Experiment 13: Sample mean = 0.6013
Experiment 14: First 10 samples: [1 0 0 1 1 1 1 1 1 1]
Experiment 14: Sample mean = 0.5989
Experiment 15: First 10 samples: [1 0 0 0 0 0 0 1 0 0]
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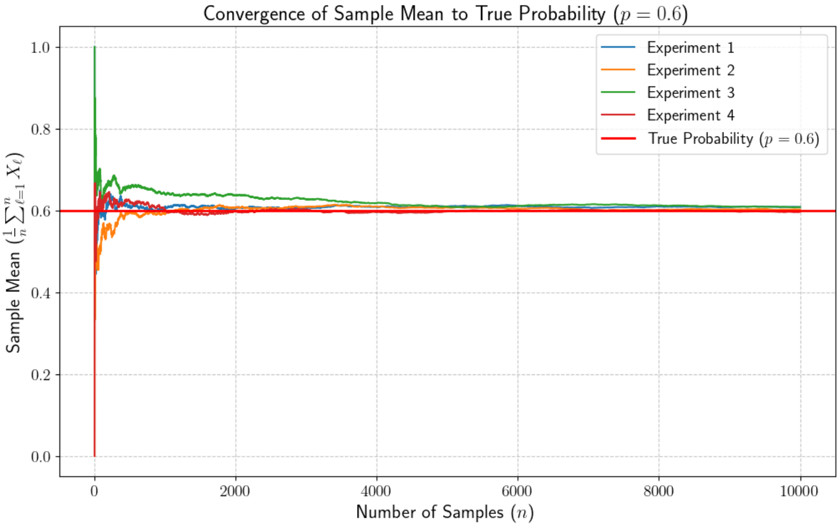
SLLN Example: Estimating the Unknown Bias of a Coin



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Lecture 08

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For a proof, see [KT08].

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- For instance,

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  Suppose that $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \text{Ber}(p)$, where p is **unknown**
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  Suppose $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, 1)$, where μ is **unknown**
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  $$\frac{1}{n}\sum_{\ell=1}^{n} X_\ell \xrightarrow{\text{a.s.}} \mu.$$

- **Linear regression in LLMs:**

  Suppose $Y_i = \beta X_i + \varepsilon_i$, where $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, $\varepsilon_1, \varepsilon_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$ are independent of $\{X_n\}_{n\in\mathbb{N}}$. Here, $\beta \in \mathbb{R}$ is **unknown**, while $\sigma > 0$ is **known**. According to SLLN, we have

  $$\frac{\sum_{\ell=1}^{n} X_\ell Y_\ell}{\sum_{\ell=1}^{n} X_\ell^2} \xrightarrow{\text{a.s.}} \frac{\mathbb{E}[X_1 Y_1]}{\mathbb{E}[X_1^2]} = \beta.$$

Proof of SLLN Under Regularity Assumptions

- We will present a proof of SLLN under finite fourth moment assumption: $\mathbb{E}[X_1^4] < +\infty$
- Without loss of generality, let $\mathbb{E}[X_1] = 0$
- We want to show

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} 0.$$

- Fixing an arbitrary $\varepsilon > 0$, we note that

$$\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{S_n}{n}\right| > \varepsilon\right\}\right) &= \mathbb{P}\left(\{|S_n| > n\varepsilon\}\right) \\
&\leq \frac{\mathbb{E}[(S_n)^4]}{n^4\,\varepsilon^4} \\
&= \frac{\mathbb{E}[(X_1 + \cdots + X_n)^4]}{n^4\,\varepsilon^4} \\
&= \frac{n\,\mathbb{E}[X_1^4] + 6\binom{n}{2}\left(\mathrm{Var}(X_1)\right)^2}{n^4\,\varepsilon^4} \\
&\leq \frac{1}{\varepsilon^4}\left(\frac{\mathbb{E}[X_1^4]}{n^3} + 3\left(\mathrm{Var}(X_1)\right)^2\frac{1}{n^2}\right)
\end{aligned}$$

Proof of SLLN Under Regularity Assumptions

- Summing both sides over n, we get

$$\forall\, \varepsilon > 0, \qquad \sum_{n\in\mathbb{N}} \mathbb{P}\left(\left\{\left|\frac{S_n}{n}\right| > \varepsilon\right\}\right) \le \frac{\mathbb{E}[X_1^4]}{\varepsilon^4}\left(\sum_{n\in\mathbb{N}} \frac{1}{n^3}\right) + \frac{3\mathrm{Var}(X_1))^2}{\varepsilon^4}\left(\sum_{n\in\mathbb{N}} \frac{1}{n^2}\right) < +\infty.$$

- The result follows from an application of Borel–Cantelli lemma

Central Limit Theorem (CLT)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Theorem (Strong Law of Large Numbers)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of **independent and identically distributed (i.i.d.)** real-valued RVs. Let $\mathbb{E}[X_1] = \mu \in \mathbb{R}$ and $\mathrm{Var}(X_1) = \sigma^2 < +\infty$. For each $n \in \mathbb{N}$, let

$$S_n := \sum_{i=1}^{n} X_i$$

denote the partial sum of random variables up to time n. Then,

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{\text{d.}} \mathcal{N}(0,1).$$

More formally,

$$\forall x \in \mathbb{R}, \qquad \lim_{n\to\infty} \mathbb{P}\left(\left\{\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} \leq x\right\}\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t.$$

Proof of CLT

$$Z_i = \frac{X_i - \mathbb{E}[X_i]}{\sqrt{\mathrm{Var}(X_1)}}, \qquad U_n = \frac{\sum_{i=1}^{n} Z_i}{\sqrt{n}}$$

$$\begin{aligned}
C_{Z_i}(s) &= 1 - \frac{s^2}{2} + o(s^2), \qquad s \in \mathbb{R}, \\
C_{U_n}(s) &= \left(C_{Z_1}\left(\frac{s}{\sqrt{n}}\right)\right)^n = \left(1 - \frac{s^2}{2n} + o\left(\frac{s^2}{n}\right)\right)^n
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} C_{U_n}(s) &= \lim_{n\to\infty} \left(1 - \frac{s^2}{2n} + o\left(\frac{s^2}{n}\right)\right)^n \\
&= \exp\left(\frac{-s^2}{2}\right) \qquad \forall\, s \in \mathbb{R}.
\end{aligned}$$

Some Notes about CLT

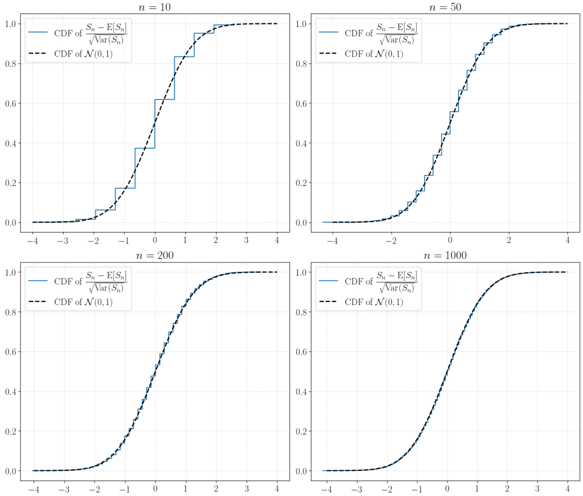
- From SLLN, we know that

$$\frac{S_n - \mathbb{E}[S_n]}{n} \xrightarrow{\text{a.s.}} 0.$$

- Thus, the random variable $S_n - \mathbb{E}[S_n]$, when **divided by n**, is degenerate for large n
- **CLT**: the random variable $S_n - \mathbb{E}[S_n]$, when **divided by $\sqrt{n}$**, is non-degenerate for large n
- According to CLT, for large n,

$$\mathbb{P}\left(\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} > t\right) = \mathbb{P}\left(\frac{S_n - \mathbb{E}[S_n]}{\sigma\sqrt{n}} > t\right) \approx \mathbb{P}(X > t), \qquad X \sim \mathcal{N}(0,1)$$

Demonstration of CLT



What CLT is Not

- CLT is not a statement about convergence of the PDFs of $\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}}$ to Gaussian PDF
- If $X_1, X_2, \ldots$ are discrete random variables, then $\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}}$ is a discrete random variable, and hence does not admit any PDF
- Even if $X_1, X_2, \ldots$ are continuous random variables, and $\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}}$ admits a PDF, **CLT does not make any claim about the convergence of these PDFs to the Gaussian PDF**
- In many practical examples where $X_1, X_2, \ldots$ are continuous, one may observe convergence of PDFs of $\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}}$ to the Gaussian PDF, but this is NOT to be interpreted as a consequence of the CLT
  This may be a consequence of some stronger property playing in hindsight

CLT $\neq$ Convergence of PDFs to Standard Normal PDF

- [GK68] Suppose that $X_1, X_2, \ldots$ are i.i.d. with common PDF

$$f(x) = \begin{cases} \dfrac{1}{2|x| \left(\log(1/|x|)\right)^2}, & 0 < |x| < e^{-1}, \\ 0, & |x| \geq e^{-1}. \end{cases}$$

- It can be shown that $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}(X_1) = \sigma^2 < +\infty$
- Also, with a bit of cumbersome calculations, one can show that

$$\lim_{n\to\infty} f_{\frac{S_n}{\sigma\sqrt{n}}}(0) = +\infty \neq \frac{1}{\sqrt{2\pi}} \qquad \text{(value of standard normal PDF evaluated at 0).}$$

Local CLT – Convergence of PDFs to Standard Normal PDF

Theorem (Local Central Limit Theorem)

Suppose that $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} f$.
Without loss of generality, let $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}(X_1) = 1$.
Suppose that there exists $r \in \mathbb{N}$ such that

$$\int_{-\infty}^{\infty} |C_{X_1}(s)|^r \, \mathrm{d}s \; < \; +\infty.$$

Then,

$$\lim_{n \to \infty} f_{\frac{S_n}{\sqrt{n}}}(x) \; = \; \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad \forall\, x \in \mathbb{R}.$$

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Fatou's Lemma,
Dominated Convergence Theorem,
Monotone Convergence Theorem

Interchange of Limit/Limsup/Liminf and Expectations

Monotone Convergence Theorem

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Theorem (Monotone Convergence Theorem)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let X, Y be two other extended real-valued RVs.

1. *Let $X_n \geq Y$ for all $n \in \mathbb{N}$ with $\mathbb{E}[Y] > -\infty$.*
   Furthermore, suppose that

   $$Y \leq X_1 \leq X_2 \leq X_3 \leq \cdots \leq X, \qquad X_n \xrightarrow{\text{pointwise}} X \qquad \left(X = \lim_{n\to\infty} X_n\right).$$

   Then, we have

   $$\mathbb{E}[Y] \leq \mathbb{E}[X_1] \leq \mathbb{E}[X_2] \leq \mathbb{E}[X_3] \leq \cdots \leq \mathbb{E}[X], \qquad \lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X] = \mathbb{E}\left[\lim_{n\to\infty} X_n\right].$$

2. *An exact analogue of the above statement holds when $X_1, X_2, \ldots$ are non-increasing and $X_n \leq Y$ for all $n \in \mathbb{N}$, with $\mathbb{E}[Y] < +\infty$*

Fatou's Lemma

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables are assumed to be defined on this space.

Lemma (Fatou's Lemma)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued RVs, and let Y be another real-valued RV.

1. *If $X_n \geq Y$ for all $n \in \mathbb{N}$ and $\mathbb{E}[Y] > -\infty$, then*
$$\mathbb{E}[Y] \;\leq\; \mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \;\leq\; \liminf_{n\to\infty}\; \mathbb{E}[X_n].$$
2. *If $X_n \leq Y$ for all $n \in \mathbb{N}$ and $\mathbb{E}[Y] < +\infty$, then*
$$\mathbb{E}[Y] \;\geq\; \mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \;\geq\; \limsup_{n\to\infty}\; \mathbb{E}[X_n].$$
3. *If $|X_n| \leq Y$ for all $n \in \mathbb{N}$, with $|\mathbb{E}[Y]| < +\infty$, then*
$$-\mathbb{E}[Y] \;\leq\; \mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \;\leq\; \liminf_{n\to\infty}\; \mathbb{E}[X_n] \;\leq\; \limsup_{n\to\infty}\; \mathbb{E}[X_n] \;\leq\; \mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \;\leq\; \mathbb{E}[Y].$$

Proof of Lemma 2, Part 1

- For each $n \in \mathbb{N}$, let

$$Y_n = \inf_{k \geq n} X_k.$$

- Because $X_k \geq Y$ for all k, it follows that

$$Y_n \geq Y \qquad \forall\, n \in \mathbb{N}$$

- Furthermore, note that

$$Y \leq Y_1 \leq Y_2 \leq Y_3 \leq \cdots, \qquad \lim_{n\to\infty} Y_n = \sup_{n\geq 1} Y_n = \liminf_{n\to\infty} X_n.$$

- Applying MCT, we get

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] = \mathbb{E}\left[\lim_{n\to\infty} Y_n\right] \overset{\text{MCT}}{=} \lim_{n\to\infty} \mathbb{E}[Y_n] = \liminf_{n\to\infty} \mathbb{E}[Y_n] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

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