# AI5030 / EE5817: Probability and Stochastic Processes
## Homework 02

### Algebras, $\sigma$-Algebras

1. (a) Let $\Omega = \{1, \ldots, 6\}$. For each $i \in \{1, 2, 3, 4\}$, construct a $\sigma$-algebra $\mathscr{F}_i$ of subsets of $\Omega$ such that $|\mathscr{F}_i| = 2^i$.
   (b) Let $\Omega$ be a finite sample space with $|\Omega| = n$ for some $n \in \mathbb{N}$. Let $\mathscr{F}$ be a $\sigma$-algebra of subsets of $\Omega$.
   Show that $|\mathscr{F}| = 2^k$ for some $1 \leq k \leq n$.

2. Let $\Omega$ be an arbitrary set (finite, countably infinite, or uncountable).
   (a) Let $\mathscr{A}$ be a collection of subsets of $\Omega$ satisfying the property that if $A, B \in \mathscr{A}$, then $A \cap B^{\complement} \in \mathscr{A}$.
   Show that $\mathscr{A}$ must be an algebra (of subsets of $\Omega$).
   (b) Suppose $\mathscr{F}$ is a collection of subsets of $\Omega$ satisfying the following properties:
   - If $A \in \mathscr{F}$, then $A^{\complement} \in \mathscr{F}$ (closure under complements).
   - If $A, B$ are two **disjoint** subsets of $\Omega$, then $A \cup B \in \mathscr{F}$ (closure under finite **disjoint** unions).

   Construct an explicit example to demonstrate that $\mathscr{F}$ need not be an algebra.

3. Let $\Omega$ be an arbitrary set (finite, countably infinite, or uncountable).
   (a) Let $\mathscr{F}_1$ denote the collection of all finite subsets of $\Omega$, i.e.,
   $$\mathscr{F}_1 := \Big\{ A \subseteq \Omega : \ |A| \in \mathbb{N} \Big\}.$$
   Is $\mathscr{F}_1$ an algebra?
   (b) Let $\mathscr{F}_2$ denote the collection of all finite subsets of $\Omega$, plus all subsets of $\Omega$ whose complement is finite, i.e.,
   $$\mathscr{F}_2 := \Big\{ A \subseteq \Omega : \ A \text{ is finite or } (\Omega \setminus A) \text{ is finite or both} \Big\}.$$
   Show that $\mathscr{F}_2$ is an algebra.
   Construct an example to demonstrate that $\mathscr{F}_2$ need not necessarily be a $\sigma$-algebra.
   (c) Let $\mathscr{F}_3$ denote the collection of all countable subsets of $\Omega$, plus all subsets of $\Omega$ whose complement is countable, i.e.,
   $$\mathscr{F}_3 := \Big\{ A \subseteq \Omega : \ A \text{ is countable or } (\Omega \setminus A) \text{ is countable or both} \Big\}.$$
   Show that $\mathscr{F}_3$ is a $\sigma$-algebra.
   **Note:** Countable means finite or countably infinite.

4. Let $\Omega = \mathbb{R}$. Let $\mathscr{P}$ denote the collection
   $$\mathscr{P} := \Big\{ [a, b) : \ a, b \in \mathbb{R}, \ a < b \Big\}.$$
   Clearly, $\mathscr{P}$ consists of uncountably infinitely many subsets of $\Omega$.
   In Lecture 6, we saw that $\sigma(\mathscr{P}) = \mathscr{B}(\mathbb{R})$, i.e., $\mathscr{P}$ is a generating class for $\mathscr{B}(\mathbb{R})$.
   In this exercise, we will see an alternative construction of $\mathscr{B}(\mathbb{R})$ starting from a **countably infinite** collection of subsets of $\Omega$.

   Consider the collection $\mathscr{C}$ given by
   $$\mathscr{C} := \Big\{ [a, b) : a \leq b, \ a, b \text{ are dyadic rational numbers} \Big\}.$$
   **Note:** A dyadic rational number is of the form $m/2^n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$.

(a) Given $x \in \mathbb{R}$, express $\{x\}$ in terms of sets in $\mathscr{C}$ using countable set operations.
**Hint:** Note that $\lfloor 2^n x \rfloor \le 2^n x \le \lceil 2^n x \rceil$ for all $n \in \mathbb{N}$. Therefore,

$$\frac{\lfloor 2^n x \rfloor}{2^n} \le x \le \frac{\lceil 2^n x \rceil}{2^n} \qquad \forall n \in \mathbb{N}.$$

(b) Given $a, b \in \mathbb{R}$ with $a < b$, express $[a, b)$ in terms of sets in $\mathscr{C}$ using countable set operations.

(c) Using the result in part (b), what can you say about the relationship between $\mathscr{P}$ and $\sigma(\mathscr{C})$?

(d) What can you say about the relationship between $\mathscr{C}$ and $\sigma(\mathscr{P})$?

(e) Using the results of parts (c), (d), what can you say about the relationship between $\sigma(\mathscr{C})$ and $\sigma(\mathscr{P})$?

5. Let $\Omega$ be an arbitrary set (finite, countably infinite, or uncountable).

   (a) Let $\mathscr{C}$ denote the collection of all singleton subsets of $\Omega$. What is $\sigma(\mathscr{C})$?
   **Hint:** See Question 3c.

   (b) Fix two elementary outcomes $a, b \in \Omega$.
   Let $\mathscr{E}_{a,b}$ denote the collection of all those subsets of $\Omega$ which either contain both $a$ and $b$ or do not contain both.
   Let $\mathscr{F} = \sigma(\mathscr{E}_{a,b})$. Show that every set in $\mathscr{F}$ has the same property as the sets in $\mathscr{E}_{a,b}$.

6. Consider the collection

$$\mathscr{D} := \Big\{ (a, b] \cup [-b, -a) : \; a, b \in \mathbb{R}, \; a \le b \Big\}.$$

   Show that $\sigma(\mathscr{D}) \subsetneq \mathscr{B}(\mathbb{R})$ by constructing a non-empty set $B \in \mathscr{B}(\mathbb{R}) \setminus \sigma(\mathscr{D})$.