भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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Probability and Stochastic Processes

Lecture 01: Functions, Cardinality, Countability

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Functions

Definition (Function)

Given two sets A, B, a function $f : A \to B$ is a rule that maps each element of A to a unique element of B.

- For every $x \in A$,

$$f : x \mapsto f(x) \in B$$

- A is called the domain of f
- B is called the co-domain of f

Note

While every element of A is mapped to some element of B, the converse may not always be true.

Range of a Function

Definition (Range)

The range of a function $f : A \to B$, denoted by $R(f)$, is the subset of B defined as

$$R(f) = \left\{ y \in B : y = f(x) \text{ for some } x \in A \right\}.$$

- Given $x \in A$, if $f(x) = y$, then y is called the image of x (under f)
- Given $y \in B$, the set $f^{-1}(y) := \{x \in A : f(x) = y\}$ is called the pre-image of y

Image and Pre-Image

- A function $f : A \to B$ is said to be injective if f is *one-one*, i.e., each element of $R(f)$ has a unique pre-image
- A function $f : A \to B$ is said to be surjective if it is *onto*, i.e., range $=$ codomain
- A function $f : A \to B$ is said to be bijective if it is both injective and surjective

Note

- If $f : A \to B$ is bijective, then for each $y \in B$, there exists a unique element $x \in A$ such that $f^{-1}(y) = \{x\}$. In this case, we simply write $f^{-1}(y) = x$.
- Alternatively, if $f : A \to B$ is bijective, we have $f^{-1} : B \to A$.

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Cardinality

Definition (Cardinality)

Notation: $|A|$ = cardinality of set A

- Two sets A and B are said to be equicardinal ($|A| = |B|$) if there exists $f : A \to B$ bijective.
- $|B| \geq |A|$ if there exists $f : A \to B$ injective
- $|B| > |A|$ if there exists $f : A \to B$ injective, and A and B are not equicardinal (i.e., no bijective function mapping A to B exists)

Note

$|A|$ is representative of the number of elements in A.

Countability

- A set A is said to be finite if A is empty or $|A| = |\{1, \ldots, n\}| = n$ for some $n \in \mathbb{N}$
- A set A is said to be countably infinite if $|A| = |\mathbb{N}|$, where $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers
- A set A is countable if either $|A| < +\infty$ or $|A| = |\mathbb{N}|$

Remark

If A is countably infinite, then its elements may be listed as $A = \{a_1, a_2, \ldots\}$.

Examples of Countable Sets

- Set of odd natural numbers, set of even natural numbers
- Set of integers, $\mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$
- Set of prime numbers
- Set of rational numbers, $\mathbb{Q}$

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$\mathbb{Q}$ is Countable – Proof

Step 1: $\mathbb{Q} \cap [0,1]$ is countable. Indeed, note that

$$\mathbb{Q} \cap [0,1] = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \ldots\right\}.$$

Step 2: "Countable union of countable sets is countable."

Lemma

Let $\mathcal{I}$ be a countable index set, and let $\{A_i : i \in \mathcal{I}\}$ be a countable collection of countable sets. Then, $\bigcup_{i\in\mathcal{I}} A_i$ is countable.

Step 3: Complete the proof using the above lemma.