Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD   (Tick one)

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

AI5090/EE5817: Probability and Stochastic Processes

Quiz 02

Date: 26 August 2025

Question	1	2	Total
Marks Scored			

1. **(3 Marks)**

Fix a sample space Ω and an associated σ-algebra $\mathscr{F}$.
Fix a non-empty set $A \in \mathscr{F}$, and consider the collection

$$\mathscr{F}_A := \left\{ B \subseteq A : \; B \in \mathscr{F} \right\}.$$

Show that $\mathscr{F}_A$ is a σ-algebra of subsets of A. Justify every point clearly.

Solution to Q1.

Let $(\Omega, \mathscr{F})$ be a measurable space and let $A \in \mathscr{F} \setminus \{\varnothing\}$. Define

$$\mathscr{F}_A := \{ B \subseteq A : B \in \mathscr{F} \}.$$

We need to show that $\mathscr{F}_A$ is a σ-algebra on the base set A. That is, we need to show the following three properties:

- $A \in \mathscr{F}_A$.
- If $E \in \mathscr{F}_A$, then $A \setminus E \in \mathscr{F}_A$ (closure under complements).
- If $E_1, E_2, \ldots \in \mathscr{F}_A$, then $\bigcup_{n\in\mathbb{N}} E_n \in \mathscr{F}_A$ (closure under countable unions).

(i) Containment of A:

Because $A \in \mathscr{F}$ and $A \subseteq A$, it follows immediately that $A \in \mathscr{F}_A$.

(ii) Closure under complements:

Let $E \in \mathscr{F}_A$. Then, by definition, $E \subseteq A$ and $E \in \mathscr{F}$.
Because $\mathscr{F}$ is a σ-algebra of subsets of Ω, it follows that $E^{\complement} = \Omega \setminus E \in \mathscr{F}$.
The result then follows by noting that $A \setminus E = A \cap (\Omega \setminus E) \subseteq A.$.

(iii) Closure under countable unions.

For each $n \in \mathbb{N}$, let $E_n \in \mathscr{F}_A$. By definition, Then $E_n \subseteq A$ and $E_n \in \mathscr{F}$.
Because $\mathscr{F}$ is a σ-algebra,

$$\bigcup_{n\in\mathbb{N}} E_n \in \mathscr{F}.$$

Moreover, because $E_n \subseteq A$ for all $n \in \mathbb{N}$, we have

$$\bigcup_{n\in\mathbb{N}} E_n \subseteq A.$$

Thus $\bigcup\limits_{n\in\mathbb{N}} E_n \in \mathscr{F}_A$.

Having verified (i), (ii), and (iii), we conclude that $\mathscr{F}_A$ is a σ-algebra of subsets of A.

2. Consider the collection

$$\mathscr{D} := \left\{[-b, -a) \cup (a, b] : \ a, b \in \mathbb{R}, \ a \leq b\right\}.$$

(a) **(1 Mark)**
Give an example of a set that belongs to $\sigma(\mathscr{D}) \setminus \mathscr{D}$.

(b) **(1 Mark)**
Give an example of a set belongs to $\mathscr{B}(\mathbb{R}) \setminus \sigma(\mathscr{D})$.

Solution: We examine the structure of the generator class

$$\mathscr{D} = \{\, [-b, -a) \cup (a, b] : a, b \in \mathbb{R}, \ a \leq b \,\}.$$

Every set in $\mathscr{D}$ is symmetric with respect to the origin: for any $E \in \mathscr{D}$, if $x \in E$, then $-x \in E$.

Symmetric closed intervals. Take $a < 0$ and $b > 0$ with $|a| < |b|$. Then

$$[-b, -a) \cup (a, b] = [-b, b].$$

Hence all symmetric closed intervals of the form $[-b, \ b]$, $b \in \mathbb{R}$, are present in $\mathscr{D}$.

Symmetric open intervals. Let $a < 0$ and $b > 0$ with $|a| > |b|$. Then

$$[-b, -a) \cup (a, b] \ = \ (a, -a).$$

Thus all symmetric open intervals of the form $(-a, \ a)$, $a \in \mathbb{R}$, are present in $\mathscr{D}$.

Valid examples of sets in $\sigma(\mathscr{D}) \setminus \mathscr{D}$.

1) For any $a, b \in \mathbb{R}$ with $ab > 0$, the set $A = (-b, -a) \cup (a, b) \in \sigma(\mathscr{D})$. Indeed, A can be expressed as

$$A = \bigcup_{n \in \mathbb{N}} \left[-b + \frac{1}{n}, \ -a\right) \cup \left(a, \ b - \frac{1}{n}\right].$$

2) For any $a, b \in \mathbb{R}$ with $ab > 0$, the set $B = [-b, -a] \cup [a, b] \in \sigma(\mathscr{D})$. Indeed, B can be expressed as

$$B = \bigcap_{n \in \mathbb{N}} \left[-b, \ -a + \frac{1}{n}\right) \cup \left(a - \frac{1}{n}, \ b\right],$$

3) For any $x \in \mathbb{R}$, the set $C = \{-x, x\} \in \sigma(\mathscr{D})$. Indeed, using the result of part 2) above, C can be expressed as

$$C = \bigcap_{n \in \mathbb{N}} \left[-x - \frac{1}{n}, \ -x + \frac{1}{n}\right] \cup \left[x - \frac{1}{n}, \ x + \frac{1}{n}\right].$$

4) For any $a, b \in \mathbb{R}$, the set $D = (-\infty, \ -b) \cup [-a, \ a] \cup (b, \ +\infty) \in \sigma(\mathscr{D})$. Indeed, the set D can be expressed as

$$D = ([-b, -a) \cup (a, b])^{\complement}.$$

(b) An example of a set in $\mathscr{B}(\mathbb{R}) \setminus \sigma(\mathscr{D})$.

If $E \in \mathscr{D}$ and $x \in E$, then by construction $-x \in E$; thus all sets in $\mathscr{D}$ are symmetric.
The class of symmetric sets is closed under complements and under countable unions (hence also countable intersections), so $\sigma(\mathscr{D})$ consists entirely of *symmetric* Borel sets.
Consequently, any Borel set that is not symmetric cannot lie in $\sigma(\mathscr{D})$.
A simple choice is the interval $(0, 1)$, which is Borel but not symmetric.