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Indian Institute of Technology Hyderabad

Stochastic Processes

DTMCs, TPM, Transition Graph, Chapman–Kolmogorov Equation, Strong Markov Property, Hitting and Recurrence Times

Karthik P. N.
Assistant Professor, Department of AI
Email: pnkarthik@ai.iith.ac.in

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Discrete-Time Markov Chain Taking Finitely Many Values

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (DTMC)

Consider a process $\{X_n\}_{n=1}^{\infty}$ taking values in a discrete set $\mathcal{X}$.
Then, $\{X_n\}_{n=1}^{\infty}$ is called a discrete time Markov chain (DTMC) on $\mathcal{X}$ if

$$(X_1, \ldots, X_{n-1}) \perp\!\!\!\perp (X_{n+1}, X_{n+2}, \ldots) \mid X_n \qquad \text{for any } n \in \mathbb{N},$$

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i.e., for any $n, L \in \mathbb{N}$, $\;n < t_1 < \cdots < t_L$,
$x_1, \ldots, x_{n-1} \in \mathbb{R}$, $y_1, \ldots, y_L \in \mathbb{R}$, and $x \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}(\underbrace{X_1 = x_1, \ldots, X_{n-1} = x_{n-1}}_{\text{before } n}, \underbrace{X_{t_1} = y_1, \ldots, X_{t_L} = y_L}_{\text{after } n} \mid X_n = x) \\
&\quad = \mathbb{P}(X_1 = x_1, \ldots, X_{n-1} = x_{n-1} \mid X_n = x) \cdot \mathbb{P}(X_{t_1} = y_1, \ldots, X_{t_L} = y_L \mid X_n = x).
\end{aligned}$$

Consistency of FDDs for a DTMC

Lemma (Consistency of FDDs for a DTMC)

Consider a DTMC $\{X_n\}_{n=1}^{\infty}$ on a discrete state space $\mathcal{X}$.
Then, for all $m, n \in \mathbb{N}$ with $m < n$, integral time instants $t_1 < t_2 < \cdots < t_m < t_{m+1} < \cdots < t_n$, and $x_1, \ldots, x_m \in \mathbb{R}$,

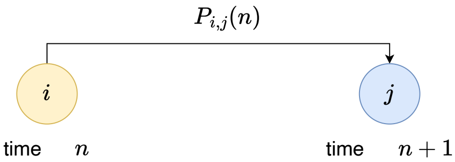
$$F_{X_{t_1},\ldots,X_{t_m}}(x_1, \ldots, x_m) = F_{X_{t_1},\ldots,X_{t_m},X_{t_{m+1}},\ldots,X_{t_n}}(x_1, \ldots, x_m, \infty, \ldots, \infty).$$

Transition Probability Matrix

Definition (Transition Probability Matrix)

Let $\{X_n\}_{n=1}^{\infty}$ be a DTMC with discrete state space $\mathcal{X}$.
The transition probability matrix (TPM) of the Markov chain at any time $n \in \mathbb{N}$ is a matrix $P(n) = [P_{i,j}(n)]_{i,j \in \mathcal{X}}$ defined as

$$P_{i,j}(n) = \mathbb{P}(X_{n+1} = j \mid X_n = i), \qquad i,j \in \mathcal{X}.$$



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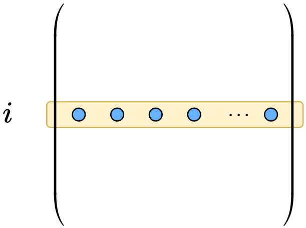
Transition Probability Matrix



- For each $i \in \mathcal{X}$, $\quad \sum_{j \in \mathcal{X}} P_{i,j}(n) = 1.$
- A matrix with non-negative entries and row sums equal to 1 is called a row stochastic matrix
- $P(n)$ is a row stochastic matrix for every n
- Each row of $P(n)$ = PMF on $\mathcal{X}$
- For instance,

$$\sum_{j \in \mathcal{X}} j^2 \, P_{i,j}(n) = \mathbb{E}[X_{n+1}^2 \mid X_n = i],$$

  where $\mathbb{E}$ above is w.r.t. row i of $P(n)$
- $P(n)$ has a right eigenvector with eigenvalue 1
- $P(n) \cdot \mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the all-ones vector

Time Homogeneous DTMC

Definition (Time Homogeneous DTMC

A DTMC with discrete state space $\mathcal{X}$ and TPMs $\{P(n)\}_{n=1}^{\infty}$ is called time homogeneous if

$$P(n) = P(n+1) \qquad \forall n \in \mathbb{N}.$$

In this case, we simply write P to denote the common TPM.

Stationarity of Conditional FDDs for Time Homogeneous DTMCs

Proposition (Stationarity of Conditional FDDs for Time Homogeneous DTMCs)

Let $\{X_n\}_{n=1}^{\infty}$ be a time-homogeneous DTMC on a discrete state space $\mathcal{X}$.
Conditioned on the initial state, any finite dimensional joint PMF of $\{X_n\}_{n=1}^{\infty}$ is stationary, i.e., for all $m, n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{X}$,

$$\mathbb{P}\left(\bigcap_{i=2}^{n}\{X_i = x_i\} \,\middle|\, \{X_1 = x_1\}\right) = \mathbb{P}\left(\bigcap_{i=2}^{n}\{X_{m+i} = x_i\} \,\middle|\, \{X_{m+1} = x_1\}\right).$$

As a consequence, all conditional FDDs, conditioned on the initial state, are stationary.

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As a consequence, all conditional FDDs, conditioned on the initial state, are stationary.

Corollary

k-step transition probabilities of a time-homogeneous DTMC are stationary for all $k \in \mathbb{N}$, i.e.,

$$\mathbb{P}(X_k = y \mid X_1 = x) = \mathbb{P}(X_{k+n} = y | X_n = x) \qquad \forall x, y \in \mathcal{X}, \;\; k, n \in \mathbb{N}.$$

Transition Graph of a Time-Homogeneous DTMC

Definition (Transition Graph)

Consider a time-homogeneous DTMC $\{X_n\}_{n=1}^{\infty}$ on a discrete state space $\mathcal{X}$ and TPM P. The transition graph of the DTMC is a weighted, directed graph, say $G = (\mathcal{X}, \mathcal{E}, \mathbf{w})$, with

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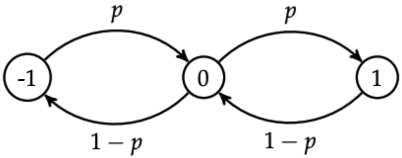

$$\mathcal{X} = \{-1, 0, 1\}$$

$$P = \begin{pmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 0 & 1-p & p \end{pmatrix}$$

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Chapman–Kolmogorov Equation – 1

Theorem (Chapman–Kolmogorov, Part 1)

Let $\{X_n\}_{n=1}^{\infty}$ be a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ and TPM P. For any $i, j \in \mathcal{X}$ and $k \in \mathbb{N}$, let

$$p_{i,j}^{(k)} := \mathbb{P}(X_{k+1} = j \mid X_1 = i).$$

Further, let $P^{(k)} = [p_{i,j}^{(k)}]_{i,j \in \mathcal{X}}$. Then,

1. $P^{(k+\ell)} = P^{(k)} \cdot P^{(\ell)}$ for all $k, \ell \in \mathbb{N}$.
2. $P^{(k)} = P^k$ for all $k \in \mathbb{N}$.

Chapman–Kolmogorov Equation – 2

Theorem (Chapman–Kolmogorov, Part 2)

Let $\{X_n\}_{n=1}^{\infty}$ be a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ and TPM P.
For each $n \in \mathbb{N}$, let π_n denote the unconditional PMF of X_n. Then,

$$\pi_{n+1} = \pi_n \cdot P \qquad \forall n \in \mathbb{N}.$$

Strong Markov Property

Strong Markov Property

- Suppose that $\{X_n\}_{n=1}^{\infty}$ is a time-homogeneous DTMC
- Markov property: For any $n \in \mathbb{N}$,

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n, \cap_{\ell=1}^{n-1}\{X_\ell = x_\ell\}) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n).$$

  That is, $X_{n+1} \perp\!\!\!\perp (X_1, \ldots, X_{n-1}) \mid X_n$
- Does the above Markov property hold when n is replaced with a random variable τ?

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  That is, $X_{n+1} \perp\!\!\!\perp (X_1, \ldots, X_{n-1}) \mid X_n$
- Does the above Markov property hold when n is replaced with a random variable τ?

YES, when τ is a stopping time! This is captured by the strong Markov property.

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Strong Markov Property

Lemma (Strong Markov Property)

Let $\{X_n\}_{n=1}^{\infty}$ be a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ with TPM P. Let τ be an $\mathbb{N}$-valued stopping time w.r.t the process $\{X_n\}_{n=1}^{\infty}$. Then, for all $i, j \in \mathcal{X}$

$$\mathbb{P}(X_{\tau+1} = j \mid X_\tau = i, \underbrace{\cap_{\ell=1}^{\tau-1}\{X_\ell = x_\ell\}}_{\text{before } \tau}) = \mathbb{P}(X_{\tau+1} = j \mid X_\tau = i) = P_{i,j}.$$

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Corollary

For any $k \in \mathbb{N}$,

$$\mathbb{P}(X_{\tau+k} = j \mid X_\tau = i) = P^k_{i,j}.$$

Proof of Lemma

$$\mathbb{P}(\underbrace{X_1 = i_1, \ldots, X_{\tau-1} = i_{\tau-1}}_{\text{before } \tau},\ X_\tau = i_\tau,\ X_{\tau+1} = i_{\tau+1})$$

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Proof of Lemma

$$\mathbb{P}(\underbrace{X_1 = i_1, \ldots, X_{\tau-1} = i_{\tau-1}}_{\text{before } \tau},\ X_\tau = i_\tau,\ X_{\tau+1} = i_{\tau+1})$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(X_1 = i_1, \ldots, X_{\tau-1} = i_{\tau-1},\ X_\tau = i,\ X_{\tau+1} = j,\ \tau = n)$$

Proof of Lemma

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$$= \sum_{n=1}^{\infty} \mathbb{P}(X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i, \tau = n) \cdot \mathbb{P}(X_{n+1} = j | X_n = i)$$

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Proof of Lemma

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\begin{aligned}
&\mathbb{P}(\underbrace{X_1 = i_1, \ldots, X_{\tau-1} = i_{\tau-1},}_{\text{before } \tau} X_\tau = i_\tau,\, X_{\tau+1} = i_{\tau+1}) \\
=\quad & \sum_{n=1}^{\infty} \mathbb{P}(X_1 = i_1, \ldots, X_{\tau-1} = i_{\tau-1},\, X_\tau = i,\, X_{\tau+1} = j,\, \tau = n) \\
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=\quad & \sum_{n=1}^{\infty} \mathbb{P}(X_1 = i_1, \ldots, X_{n-1} = i_{n-1},\, X_n = i,\, \tau = n) \cdot \mathbb{P}(X_{n+1} = j | X_n = i) \\
=\quad & \mathbb{P}(\underbrace{X_1 = i_1, \ldots, X_{\tau-1} = i_{\tau-1},}_{\text{before } \tau} X_\tau = i_\tau) \cdot P_{i,j}.
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Hitting and Recurrence Times

Hitting Times

Definition (Hitting Times)

Let $\{X_n\}_{n=1}^{\infty}$ be DTMC on a discrete state space $\mathcal{X}$ with TPM P.
Fix $y \in \mathcal{X}$.
Let $\tau_y^{(0)} := 0$, and

$$\tau_y^{(k)} = \inf\{n > \tau_y^{(k-1)} : X_n = y\}, \qquad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, the random variable $\tau_y^{(k)}$ is called the "kth hitting time of state y".

Exercise:
For each $k \in \mathbb{N}$, verify that $\{\tau_y^k = n\} \in \sigma(X_1, \ldots, X_n)$ for all n.

An Important Observation Regarding $\tau_y^{(k)}$)

Lemma (An Important Observation Regarding $\tau_y^{(k)}$)

For each $k \in \mathbb{N}$, suppose that $\mathbb{P}(\tau_y^{(k)} < +\infty) = 1$.
Then, the history up to $\tau_y^{(k)}$ is independent of the future unconditionally, i.e.,

$$(X_1, \ldots, X_{\tau_y^{(k)}-1}) \perp\!\!\!\perp (X_{\tau_y^{(k)}+1}, X_{\tau_y^{(k)}+2}, \ldots).$$