TA Session 2 11/08/2018

An example:

Let Ω be a non-empty set, and let $\mathcal{F}$ be a σ-algebra on Ω. Suppose $B \subseteq \Omega$, $B \in \mathcal{F}$ is a non-empty subset of Ω in $\mathcal{F}$. Define

$$\mathcal{G} = \{ B \cap A : A \in \mathcal{F} \}.$$

Show that $\mathcal{G}$ is a σ-algebra on B.

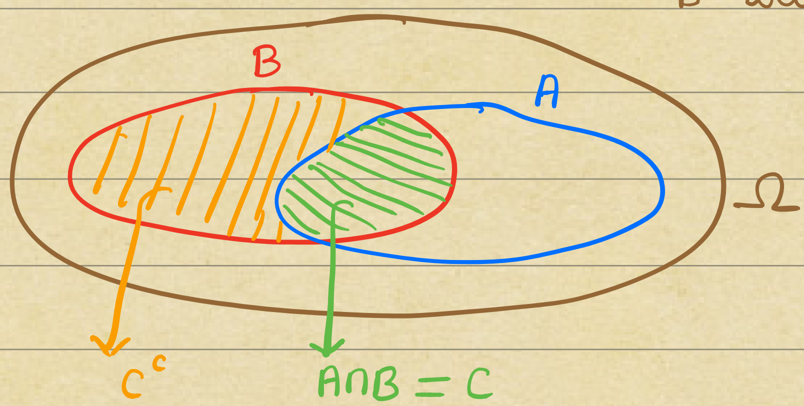
Answer: a) $B = B \cap \Omega$, $\Omega \in \mathcal{F}$

$\Rightarrow B \in \mathcal{G}$

b) Let $C \in \mathcal{G}$

$\Rightarrow \exists A \in \mathcal{F}$ s.t. $C = B \cap A$

We now need to show that C^c as a subset of B belongs to $\mathcal{G}$.



We have

$$C^c = B \cap ((B \cap A)^c) \quad \text{(refer to the above figure)}$$
$$= B \cap A^c, \quad A^c \in \mathcal{F}$$

$\Rightarrow C^c \in \mathcal{G}$.

c) Suppose $C_1, C_2, \ldots \in \mathcal{G}$

$$\Rightarrow \exists\ A_1, A_2, \ldots \in \mathcal{F} \text{ s.t.}$$

$$C_j = B \cap A_j \quad \forall\ j = 1, 2, \ldots$$

$$\Rightarrow \bigcup_{j=1}^{\infty} C_j = \bigcup_{j=1}^{\infty} (B \cap A_j)$$

$$= B \cap \left(\bigcup_{j=1}^{\infty} A_j \right)$$

$$\in \mathcal{G}$$

since $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$.

Thus, $\mathcal{G}$ is a σ-algebra on B.

Probability measure

Consider a measurable space $(\Omega, \mathcal{F})$. We now wish to speak about probabilities of events in $\mathcal{F}$. To do this, we define a non-negative function $\mathbb{P} : \mathcal{F} \to [0,1]$ s.t.

a) $\mathbb{P}(\Omega) = 1$, and

b) for any disjoint collection of sets $A_1, A_2, \ldots$ in $\mathcal{F}$, we have

$$\mathbb{P}\left(\biguplus_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j).$$

Such a function is called a probability measure on $(\Omega, \mathcal{F})$.

Eg: $\Omega = \{1,2,3,4\}$

$\mathcal{F} = \{\phi, \Omega, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$

$$\mathbb{P}(A) = \begin{cases} 0, & \text{if } A = \phi, \{1\}, \{3,4\}, \{1,3,4\} \\ 1, & \text{if } A = \Omega, \{2\}, \{1,2\}, \{2,3,4\}. \end{cases}$$

There can be many probability assignments possible

Eg: $\Omega = \{H, T\}$

$\mathcal{F} = \{\phi, \Omega, \{H\}, \{T\}\}$

$\mathbb{P}(\{H\}) = p \quad , \quad p \in [0,1]$

$\mathbb{P}(\{T\}) = 1-p.$

Some properties of $\mathbb{P}$:

1. $\mathbb{P}(\phi) = 0$

Proof: We use axiom b) in the def^n of $\mathbb{P}$, with

$$A_1 = \phi, \quad A_2 = \Omega, \quad A_j = \phi \quad \forall\ j \geq 3.$$

$$\Rightarrow \biguplus_{j=1}^{\infty} A_j = \Omega$$

$$\Rightarrow 1 = \mathbb{P}\left(\biguplus_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$$

$$= \mathbb{P}(\phi) + \mathbb{P}(\Omega) + \sum_{j=3}^{\infty} \mathbb{P}(\phi)$$

$$\Rightarrow \mathbb{P}(\phi) = 0.$$

2. For any $A \in \mathcal{F}$, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Proof: $A \uplus A^c = \Omega$

$\Rightarrow \mathbb{P}(A) + \mathbb{P}(A^c) = 1$ (by axiom (b) with $A_1 = A$,

$\Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A).$ $A_2 = A^c,\ A_j = \phi\ \forall j \geq 3$).

3. Continuity of probability measure

Suppose $A_1 \subseteq A_2 \subseteq \cdots$ are an increasing sequence of events in $\mathcal{F}$. We saw from (an exercise of) the previous session that

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n.$$

Let us call this set $\lim_{n\to\infty} A_n$, i.e.,

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

Then, we have

$$\mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$$

On similar lines, if $B_1 \supseteq B_2 \supseteq \cdots$, then

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B_k = \bigcap_{n=1}^{\infty} B_n,$$

and we shall call this set $\lim_{n\to\infty} B_n$.

We then have $\mathbb{P}\left(\lim_{n\to\infty} B_n\right) = \lim_{n\to\infty} \mathbb{P}(B_n)$.

So, in the above results, we observe that $\lim_{n\to\infty}$ and $\mathbb{P}$ were "interchanged" in going from the LHS to RHS. Such is the characteristic of continuous functions.

Def^n: Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then, f is said to be continuous at a point $x \in \mathbb{R}$ iff: for every sequence $(x_n)_{n\geq 1}$ s.t. $\lim_{n\to\infty} x_n = x$, we have

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n).$$

Remarks:

a) Suppose $\mathbb{P}(A_n) = 0 \;\forall\, n \geq 1$. What can we say about $\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right)$? :

Since

$$\bigcap_{n=1}^{\infty} A_n \subseteq A_n \quad \forall\, n\geq 1, \text{ we have}$$

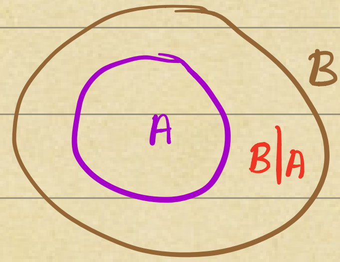
$$0 \leq \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) \leq \mathbb{P}(A_n) = 0 \quad (\text{see below})$$

$$\Rightarrow \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = 0.$$

[Here is a proof of $A \subseteq B$, $A, B \in \mathcal{F} \Rightarrow P(A) \leq P(B)$.

Since $A \subseteq B$,

$$B = A \uplus (B \setminus A)$$



$$\Rightarrow \mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

$$\Rightarrow \mathbb{P}(B) \geq \mathbb{P}(A), \text{ with equality iff } \mathbb{P}(B \setminus A) = 0.$$

Note that $\mathbb{P}(A) \leq \mathbb{P}(B)$ need not mean that $A \subseteq B$. In fact, A and B could be disjoint.

b) If $\mathbb{P}(A_n) = \frac{1}{2} - \frac{1}{n+1}$, $n \geq 1$, what can we say about $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right)$?

Nothing much, except that

$$\bigcup_{n=1}^{\infty} A_n \supseteq A_n \quad \forall\ n \geq 1$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \mathbb{P}(A_n) \quad \forall\ n \geq 1$$

does equality hold here? This is the key question.

$$\Rightarrow \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \lim_{n \to \infty} \mathbb{P}(A_n) = \frac{1}{2},$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \frac{1}{2}$$

If, however, we know that $A_1 \subseteq A_2 \subseteq \cdots$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mathbb{P}(A_n) = \frac{1}{2}.$$

notice the equality.

Similar statements can be made when $\mathbb{P}(A_n) = 1 \ \forall\, n \geq 1$.

<u>Exercises</u>: Assume $(\Omega, \mathcal{F}, \mathbb{P})$ to be given.

1. If $A, B \in \mathcal{F}$, show that
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

2. If $A_n \in \mathcal{F}$ s.t. $\mathbb{P}(A_n) = 1 \ \forall\, n \geq 1$, prove that
$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n \right) = 1.$$

3. Suppose $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$. Show that
$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq 1/3.$$

4. Prove the following result by induction on n:
If $A_1, \ldots, A_n \in \mathcal{F}$ (not necessarily disjoint), then
$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

This is called the property of "finite subadditivity", also referred to as the "union bound".
When does equality hold in the union bound?
Construct an example for which equality does not hold in the union bound.

Independence of events

Assume $(\Omega, \mathcal{F}, \mathbb{P})$ given. Two events $A, B \in \mathcal{F}$ are said to be independent iff

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Exercise: Show that if $A \in \mathcal{F}$ is an event s.t. $\mathbb{P}(A) = 1$, then

a) A is independent of itself.

b) A is independent of every set $B \in \mathcal{F}$.

Def^n (pairwise independence)

A collection of events $A_1, A_2, \ldots, A_n \in \mathcal{F}$ is said to be pairwise independent iff:

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\,\mathbb{P}(A_j) \quad \forall\ 1 \leq i, j \leq n,\ i \neq j.$$

That is, every distinct pair of events is independent.

Def^n (mutual independence)

A collection of events $A_1, \ldots, A_n$ is said to be mutually independent iff:

for every distinct set of indices $i_1, \ldots, i_k$, $k \leq n$, we have that $A_{i_1}, \ldots, A_{i_k}$ are independent.

Exercises:

1. If $A_1, \dots, A_n$ are mutually independent, then they are also pairwise independent.

2. If A and B are independent, then show that

   a) A and B^c are independent

   b) A^c and B are independent

   c) A^c and B^c are independent.

3. Let

$$\Omega = \{1, 2, \dots, p\}, \text{ where } p \text{ is a prime number.}$$

Let $\mathcal{F} = 2^{\Omega}$ be the power set of Ω.

Define

$$\mathbb{P}(A) = \frac{|A|}{p} \quad \text{for all } A \in \mathcal{F},$$

where $|A|$ denotes the number of elements in set A.

Eg: $\mathbb{P}(\{1,2\}) = \frac{2}{p}$, $\mathbb{P}(\{1\}) = \frac{1}{p}$ and so on.

Show that if A and B are 2 independent events in this setting, then at least one of A or B must be either ϕ (empty set) or Ω.

4. Construct an example having 3 events A, B and C such that they are pairwise independent but not mutually independent.