भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 05: Construction of (Borel) σ-Algebra for $\{0, 1\}^{\mathbb{N}}$ and $[0, 1]$

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σ-Algebra

Definition (σ-Algebra)

Let Ω be a sample space.
A collection $\mathscr{F}$ of subsets of Ω is called a σ-algebra if it satisfies the following properties:

- $\Omega \in \mathscr{F}$.
- $A \in \mathscr{F} \implies A^{\complement} \in \mathscr{F}$ (closed under complements).
- $A_1, A_2, \ldots \in \mathscr{F} \implies \bigcup_{i \in \mathbb{N}} A_i \in \mathscr{F}$ (closure under countably infinite unions).

Remark: The symbol σ in σ-algebra connotes countably infinite unions.

Remarks:

- Elements of a σ-algebra are called events
- An event $A \in \mathscr{F}$ is also referred to as an $\mathscr{F}$-measurable set
- The pair $(\Omega, \mathscr{F})$ is called a measurable space

Examples of σ-Algebra

- Given a sample space Ω, the smallest σ-algebra is $\mathscr{F}_{\text{smallest}} = \{\emptyset, \Omega\}$
- Given a sample space Ω, the largest σ-algebra is $\mathscr{F}_{\text{largest}} = 2^{\Omega}$
- For $\Omega = \{1, \ldots, 6\}$, complete the following collection to make it a σ-algebra:

$$\mathscr{F} = \left\{ \emptyset, \Omega, \{1,2\}, \{3,4\}, \qquad\qquad\qquad\qquad \right\}$$

- Consider the experiment of tossing a coin till first head is observed
  - $\Omega = \{H, TH, TTH, TTTH, \cdots\}$
  - Let
    $\mathscr{C} = \left\{ \emptyset, \Omega, \{H\}, \{TH\}, \{TTH\}, \{TTTH\}, \cdots \right\} = \left\{ \emptyset, \Omega, \text{ all singleton subsets of } \Omega \right\}$.
  - Is $\mathscr{C}$ a σ-algebra? **No!**
  - Can we convert $\mathscr{C}$ to a σ-algebra by including more subsets of Ω? **Yes!**
  - Let $\sigma(\mathscr{C})$ denote the smallest σ-algebra constructed starting from $\mathscr{C}$

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Algebra vs σ-Algebra

- Consider the experiment of tossing a coin till first head is observed
  - $\Omega = \{H, TH, TTH, TTTH, \cdots\}$
  - $\mathscr{C} = \Big\{\emptyset, \Omega, \{H\}, \{TH\}, \{TTH\}, \{TTTH\}, \cdots\Big\}$
  - Let $\mathscr{A} = \alpha(\mathscr{C})$
  - Let $\mathscr{F} = \sigma(\mathscr{C})$

Observe the Following Properties

- $A^* = \{TH, TTTH, TTTTTH, \cdots\} \notin \mathscr{A}$, $\qquad A^* = \{TH, TTTH, TTTTTH, \cdots\} \in \mathscr{F}$
- $\mathscr{A} \subseteq \mathscr{F}$, i.e., a σ-algebra is a larger collection than its precursor algebra
- A σ-algebra satisfies all the properties of an algebra, but an algebra may not satisfy the properties of a σ-algebra

Construction of σ-Algebra for $\{0, 1\}^{\mathbb{N}}$

- Let $\Omega = \{0, 1\}^{\mathbb{N}} = \{0, 1\} \times \{0, 1\} \times \cdots$
- Each $\omega \in \Omega$ may be expressed as

$$\omega = (\omega_1, \omega_2, \ldots), \qquad \omega_i \in \{0, 1\} \text{ for all } i \in \mathbb{N}.$$

- For each $n \in \mathbb{N}$, let $\mathscr{D}_n$ denote the set of all binary strings of length n, i.e.,

$$\mathscr{D}_n = \{(b_1, \ldots, b_n) : \quad b_i \in \{0, 1\} \text{ for all } i \in \{1, \ldots, n\}\}.$$

- Let $\mathscr{D}$ denote the set of all finite-length binary strings, i.e.,

$$\mathscr{D} = \bigcup_{n \in \mathbb{N}} \mathscr{D}_n.$$

Cylinder Sets

Consider $\Omega = \{0,1\}^{\mathbb{N}}$

Definition (Cylinder Set)

Given a finite-length binary string $\mathbf{b} \in \mathscr{D}_m$ of length m, the **cylinder set** $[\mathbf{b}]$ is defined as

$$[\mathbf{b}] := \{\omega \in \{0,1\}^{\mathbb{N}} : (\omega_1, \ldots, \omega_m) = \mathbf{b}\}$$

The set $[\mathbf{b}]$ may be expressed as

$$\mathbf{b} = \underbrace{\{b_1\} \times \cdots \times \{b_m\}}_{\text{base}} \times \underbrace{\{0,1\} \times \{0,1\} \times \cdots}_{\text{axis}}$$

The name "cylinder" comes from the fact that the base is fixed and axis is freely chosen.

Working with Cylinder Sets

- $[1] \cap [10] =$
- $[1] \cap [01] =$
- $[10]^{\complement} =$
- If $A = \{\omega \in \{0,1\}^{\mathbb{N}} : \omega_5 = 1\}$, express A in terms of cylinder sets
- If A is the set

$$A = \{\omega \in \{0,1\}^{\mathbb{N}} : \omega \text{ contains at least 2 ones in the first 10 bits}\},$$

express A in terms of cylinder sets

Construction of Algebra from Cylinder Sets

- Consider the collection

$$\mathscr{C} = \left\{ [\mathbf{b}] : \mathbf{b} \in \mathscr{D} \right\}.$$

- Is $\mathscr{C}$ an algebra? **No!**
- Can we convert $\mathscr{C}$ to an algebra? **Yes!**
- Denote by $\alpha(\mathscr{C})$ the smallest algebra constructed starting from $\mathscr{C}$

Interpretation of Sets in $\alpha(\mathscr{C})$

The set $\alpha(\mathscr{C})$ consists of those sets whose occurrence or non-occurrence can be completely determined by looking only at the first finitely many bits.

- $A = \{\omega \in \{0,1\}^{\mathbb{N}} : \quad \omega_{10} = 0\} \in \alpha(\mathscr{C})$
- $B = \{\omega \in \{0,1\}^{\mathbb{N}} : \quad \sum_{i=1}^{25} \omega_i \geq 15\} \in \alpha(\mathscr{C})$

Is $\mathscr{A} = \alpha(\mathscr{C})$ a σ-Algebra?

- Consider the collection $\mathscr{C} = \left\{ [\mathbf{b}] : \mathbf{b} \in \mathscr{D} \right\}$.
- Let $\mathscr{A} = \alpha(\mathscr{C})$ denote the smallest algebra constructed starting from $\mathscr{C}$
- Let A^* denote the set

$$A^* = \{\omega \in \{0,1\}^{\mathbb{N}} : \quad \omega_i = 1 \text{ for all } i \in \{2,4,6,8,\ldots\}\}$$

- Clearly, $A^* \notin \mathscr{A}$, as its occurrence/non-occurrence cannot be determined from only observing the first finitely many bits of any infinite binary string
- This shows that $\mathscr{A}$ is **not a σ-algebra**
- Let $\sigma(\mathscr{A})$ denote the smallest σ-algebra constructed starting from $\mathscr{A}$

The Borel σ-Algebra

The σ-algebra $\sigma(\mathscr{A})$ so constructed is called the **Borel σ-algebra** of subsets of $\{0,1\}^{\mathbb{N}}$. Henceforth, we shall denote the same by $\mathscr{B}(\{0,1\}^{\mathbb{N}})$.

Construction of σ-Algebra for $(0, 1)$

Construction of σ-Algebra for $(0, 1)$

- Consider the experiment of throwing a dart on the unit interval $(0, 1)$
- An outcome: $\omega \in (0, 1)$
- Sample space: $\Omega = (0, 1)$
- Consider the collection

$$\mathscr{P} = \left\{ (a, b) : \quad a, b \in \mathbb{R}, \ 0 \leq a \leq b \leq 1 \right\}.$$

- Is $\mathscr{P}$ a σ-algebra? **No!**
- Let $\sigma(\mathscr{P})$ denote the smallest σ-algebra constructed starting from $\mathscr{P}$

The Borel σ-Algebra

The σ-algebra $\sigma(\mathscr{P})$ so constructed is called the **Borel σ-algebra** of subsets of $(0, 1)$. Henceforth, we shall denote the same by $\mathscr{B}(0, 1)$.