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Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 04: Probability Basics (Sample Space, Algebra, σ-Algebra)

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Sample Space

We begin with two universally accepted entities:

- Random experiment
- Outcome (denoted by ω) – source of randomness

Definition (Sample Space)

The sample space (denoted by Ω) of a random experiment is the set of all possible outcomes of the random experiment.

Example: Tossing a coin once

- If our interest is in the face that shows, then $\Omega = \{H, T\}$
- If our interest is in the velocity with which the coin lands on ground, then $\Omega = [0, \infty) = \mathbb{R}_+$
- If our interest is in the number of times coin flips in air, then $\Omega = \mathbb{N}$

Example: Toss a coin n times, for some finite n

Interest: faces that show up

$\Omega = \{H, T\}^n$

Example: Toss a coin countably infinitely many times

Interest: faces that show up

$\Omega = \{H, T\}^{\mathbb{N}}$

Remark

Often times, we are not interested in a particular outcome $\omega \in \Omega$ occurred or not. We are often interested in whether a **subset** of outcomes occurred or not.

- For $\Omega = \{H, T\}^{10}$, we may be interested in "Did we get $>$ 5 heads"?

Event

Informal Definition (Event)

Informally,[a] an event is a subset of outcomes “of interest” to us.

[a] We shall give a more formal definition of an event later.

Example: Toss a coin 3 times; interest is in the faces that show up
$\Omega = \{H, T\}^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Event A of interest: at least 2 heads show up
$A = \{HHH, THH, HTH, HHT\}$

Note

If an outcome $\omega \in A$ occurs, we say that the event A occurs.

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Algebra

Definition (Algebra)

Let Ω be a sample space.
A collection $\mathscr{A}$ of subsets of Ω is called an algebra if it satisfies the following properties:

1. $\Omega \in \mathscr{A}$.
2. $A \in \mathscr{A} \implies A^{\complement} \in \mathscr{A}$ (closure under complements).
3. $A_1, A_2 \in \mathscr{A} \implies A_1 \cup A_2 \in \mathscr{A}$.

Note: An algebra is a collection of all events of interest to us.

Note:

$$A_1, \ldots, A_n \in \mathscr{A} \implies \bigcup_{i=1}^{n} A_i \in \mathscr{A} \quad \text{for all } n \in \mathbb{N} \quad \text{(closure under finite unions)},$$

$$A_1, \ldots, A_n \in \mathscr{A} \implies \bigcap_{i=1}^{n} A_i \in \mathscr{A} \quad \text{for all } n \in \mathbb{N} \quad \text{(closure under finite intersections)}.$$

Examples of Algebra

- Given a sample space Ω, the smallest algebra is $\mathscr{A}_{\text{smallest}} = \{\emptyset, \Omega\}$
- Given a sample space Ω, the largest algebra is $\mathscr{A}_{\text{largest}} = 2^{\Omega}$
- For $\Omega = \{1, \ldots, 6\}$, complete the following collection to make it an algebra:

$$\mathscr{A} = \left\{ \emptyset, \Omega, \{1, 2\}, \{3, 4\}, \qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

- Consider the experiment of tossing a coin till first head is observed
  - $\Omega = \{H, TH, TTH, TTTH, \cdots\}$
  - Let $\mathscr{C} = \left\{ \emptyset, \Omega, \{H\}, \{TH\}, \{TTH\}, \{TTTH\}, \cdots \right\} = \left\{ \emptyset, \Omega, \text{ all singleton subsets of } \Omega \right\}.$
  - Is $\mathscr{C}$ an algebra? **No!**
  - Can we convert $\mathscr{C}$ to an algebra by including more subsets of Ω? **Yes!**
  - Let $\alpha(\mathscr{C})$ denote the smallest algebra constructed starting from $\mathscr{C}$

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Does Algebra Suffice?

- Consider the experiment of tossing a coin till first head is observed
  - $\Omega = \{H, TH, TTH, TTTH, \cdots\}$
  - $\mathscr{C} = \left\{\emptyset, \Omega, \{H\}, \{TH\}, \{TTH\}, \{TTTH\}, \cdots\right\}$
  - Let $\mathscr{A} = \alpha(\mathscr{C})$
- Consider the event

$$A^* = \{\#\text{ of tosses is even}\} = \{TH, TTTH, TTTTTH, \cdots\}$$

- Does $A^* \in \mathscr{A}$? **No!**

What went wrong?

A^* cannot be expressed as a union of finite number of elements of $\mathscr{C}$!

σ-Algebra

Definition (σ-Algebra)

Let Ω be a sample space.
A collection $\mathscr{F}$ of subsets of Ω is called a σ-algebra if it satisfies the following properties:

- $\Omega \in \mathscr{F}$.
- $A \in \mathscr{F} \implies A^{\complement} \in \mathscr{F}$ (closed under complements).
- $A_1, A_2, \ldots \in \mathscr{F} \implies \bigcup_{i \in \mathbb{N}} A_i \in \mathscr{F}$ (closure under countably infinite unions).

Remark: The symbol σ in σ-algebra connotes countably infinite unions.

Remarks:

- Elements of a σ-algebra are called events
- An event $A \in \mathscr{F}$ is also referred to as an $\mathscr{F}$-measurable set
- The pair $(\Omega, \mathscr{F})$ is called a measurable space

Examples of σ-Algebra

- Given a sample space Ω, the smallest σ-algebra is $\mathscr{F}_{\text{smallest}} = \{\emptyset, \Omega\}$
- Given a sample space Ω, the largest σ-algebra is $\mathscr{F}_{\text{largest}} = 2^{\Omega}$
- For $\Omega = \{1, \ldots, 6\}$, complete the following collection to make it a σ-algebra:

$$\mathscr{F} = \Big\{ \emptyset, \Omega, \{1,2\}, \{3,4\}, \qquad\qquad\qquad\qquad\qquad\qquad \Big\}$$

- Consider the experiment of tossing a coin till first head is observed
  - $\Omega = \{H, TH, TTH, TTTH, \cdots\}$
  - Let
    $\mathscr{C} = \Big\{ \emptyset, \Omega, \{H\}, \{TH\}, \{TTH\}, \{TTTH\}, \cdots \Big\} = \Big\{ \emptyset, \Omega, \text{ all singleton subsets of } \Omega \Big\}.$
  - Is $\mathscr{C}$ a σ-algebra? **No!**
  - Can we convert $\mathscr{C}$ to a σ-algebra by including more subsets of Ω? **Yes!**
  - Let $\sigma(\mathscr{C})$ denote the smallest σ-algebra constructed starting from $\mathscr{C}$

Algebra vs σ-Algebra

- Consider the experiment of tossing a coin till first head is observed
  - $\Omega = \{H, TH, TTH, TTTH, \cdots\}$
  - $\mathscr{C} = \Big\{\emptyset, \Omega, \{H\}, \{TH\}, \{TTH\}, \{TTTH\}, \cdots\Big\}$
  - Let $\mathscr{A} = \alpha(\mathscr{C})$
  - Let $\mathscr{F} = \sigma(\mathscr{C})$

Observe the Following Properties

- $A^* = \{TH, TTTH, TTTTTH, \cdots\} \notin \mathscr{A}$, $\qquad A^* = \{TH, TTTH, TTTTTH, \cdots\} \in \mathscr{F}$
- $\mathscr{A} \subseteq \mathscr{F}$, i.e., a σ-algebra is a larger collection than its precursor algebra
- A σ-algebra satisfies all the properties of an algebra, but an algebra may not satisfy the properties of a σ-algebra