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Mathematical Foundations for Data Science (Probability)

Independence of Random Variables, Jointly Discrete Random Variables, Joint PMF, Conditional PMF, Joint PDF, Conditional PDF, Transformations of Random Variables

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Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Two Random Variables)

Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables with respect to $\mathscr{F}$.

$$
\begin{aligned}
X \perp\!\!\!\perp Y \quad &\iff \quad F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y) \quad \forall x, y \in \mathbb{R} \\
&\iff \quad \mathbb{P}(\{X \le x\} \cap \{Y \le y\}) = \mathbb{P}(\{X \le x\}) \cdot \mathbb{P}(\{Y \le y\}) \quad \forall x, y \in \mathbb{R} \\
&\iff \quad \{X \le x\} \perp\!\!\!\perp \{Y \le y\} \quad \forall x, y \in \mathbb{R}.
\end{aligned}
$$

Implications:

- $\{X \le x\} \perp\!\!\!\perp \{Y > y\}$ for all $x, y \in \mathbb{R}$, i.e.,

$$\mathbb{P}(\{X \le x, Y > y\}) = \mathbb{P}(\{X \le x\}) \cdot \mathbb{P}(\{Y > y\}) \qquad \forall x, y \in \mathbb{R}$$

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& \Longleftrightarrow \quad \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}) = \mathbb{P}(\{X \leq x\}) \cdot \mathbb{P}(\{Y \leq y\}) \quad \forall x, y \in \mathbb{R} \\
& \Longleftrightarrow \quad \{X \leq x\} \perp\!\!\!\perp \{Y \leq y\} \quad \forall x, y \in \mathbb{R}.
\end{aligned}
$$

Implications:

- $\{X > x\} \perp\!\!\!\perp \{Y > y\}$ for all $x, y \in \mathbb{R}$, i.e.,

$$
\mathbb{P}(\{X > x, Y > y\}) = \mathbb{P}(\{X > x\}) \cdot \mathbb{P}(\{Y > y\}) \qquad \forall x, y \in \mathbb{R}
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Implications:

- $\{X = x\} \perp\!\!\!\perp \{Y = y\}$ for all $x, y \in \mathbb{R}$, i.e.,

$$\mathbb{P}(\{X = x, Y = y\}) = \mathbb{P}(\{X = x\}) \cdot \mathbb{P}(\{Y = y\}) \qquad \forall x, y \in \mathbb{R}$$

Example

Let X_1 and X_2 be distributed exponentially with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively. Determine the distribution of $Z = \min\{X_1, X_2\}$.

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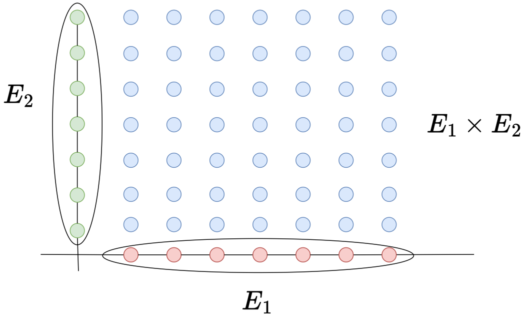
Jointly Discrete Random Variables

Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Jointly Discrete Random Variables)

Random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$ are said to be jointly discrete if X and Y are individually discrete random variables.



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Joint PMF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint PMF)

The joint PMF of jointly discrete random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined on $\mathscr{F}$ is a function $p_{X,Y} : \mathbb{R}^2 \to [0, 1]$ defined as

$$p_{X,Y}(x, y) = \mathbb{P}(\{X = x\} \cap \{Y = y\}), \qquad x, y \in \mathbb{R}.$$

Note:

$$\mathbb{P}(\{(X, Y) \in E_1 \times E_2\}) = \sum_{x \in E_1} \sum_{y \in E_2} p_{X,Y}(x, y) = 1,$$

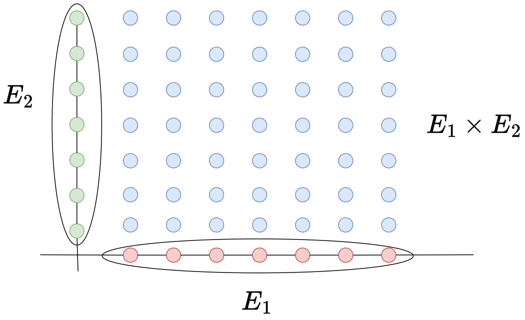
$$\mathbb{P}(\{(X, Y) \in B\}) = \sum_{(x,y) \in B \cap (E_1 \times E_2)} p_{X,Y}(x, y), \quad B \subseteq \mathbb{R}^2.$$

Properties of Joint PMF

- $\sum_{x \in E_1} \sum_{y \in E_2} p_{X,Y}(x,y) = 1.$

- $p_X(x) = \sum_{y \in E_2} p_{X,Y}(x,y), \quad x \in \mathbb{R}$

- $p_Y(y) = \sum_{x \in E_1} p_{X,Y}(x,y), \quad y \in \mathbb{R}$



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Conditional PMF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

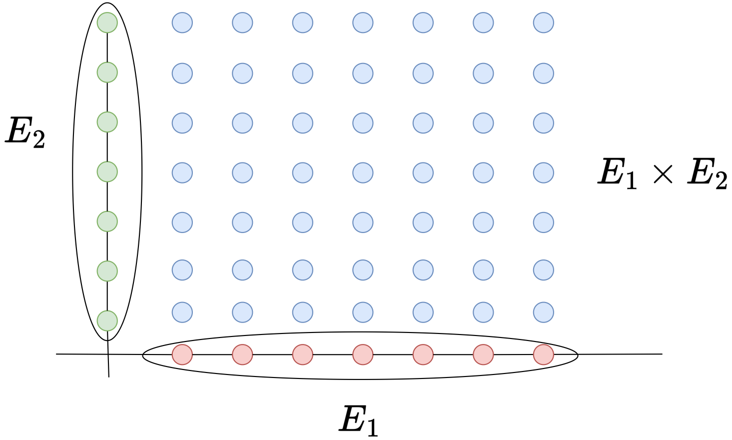
Definition (Conditional PMF)

Let X, Y be jointly discrete random variables defined with respect to $\mathscr{F}$. Fix $y \in \mathbb{R}$ such that $p_Y(y) = \mathbb{P}(\{Y = y\}) > 0$. The conditional PMF of X, conditioned on the event $\{Y = y\}$, is a function $p_{X|Y=y} : \mathbb{R} \to [0, 1]$ defined as

$$p_{X|Y=y}(x) = \frac{\mathbb{P}(\{X = x\} \cap \{Y = y\})}{\mathbb{P}(\{Y = y\})} = \frac{p_{X,Y}(x, y)}{p_Y(y)}, \qquad x \in \mathbb{R},$$

defined for all $y \in \mathbb{R}$ such that $p_Y(y) = \mathbb{P}(\{Y = y\}) > 0$.

Conditional PMF



Independence of Two Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Theorem

Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be discrete random variables with respect to $\mathscr{F}$.
The following statements are equivalent.

1. $X \perp\!\!\!\perp Y$.
2. $\{X = x\} \perp\!\!\!\perp \{Y = y\}$ for all $x, y \in \mathbb{R}$.
3. $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$ for all $x, y \in \mathbb{R}$.
4. For all $y \in \mathbb{R}$ such that $p_Y(y) > 0$,

$$p_{X|Y=y}(x) = p_X(x) \qquad \forall x \in \mathbb{R}.$$

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Jointly Continuous Random Variables

Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables defined with respect to $\mathscr{F}$.

Definition (Jointly Continuous Random Variables)

X and Y are said to be jointly continuous if there exists a function $f_{X,Y} : \mathbb{R}^2 \to [0, +\infty)$ such that the joint CDF of X and Y may be expressed as

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du \qquad \forall x, y \in \mathbb{R}.$$

The function $f_{X,Y}$ is called the joint PDF of X and Y.

Remark:
If X and Y are individually continuous, then they need not be jointly continuous.

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Properties of Joint PDF

- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(u, v)\, dv\, du = 1.$

Properties of Joint PDF

- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(u,v)\,dv\,du = 1.$

- $\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\,dv = f_X(u)$ for all $u \in \mathbb{R}$.

  This says if X and Y are jointly continuous, then X is a continuous RV

Properties of Joint PDF

- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(u,v)\, dv\, du = 1.$
- $\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\, dv = f_X(u)$ for all $u \in \mathbb{R}$.
  This says if X and Y are jointly continuous, then X is a continuous RV
- $\int_{-\infty}^{+\infty} f_{X,Y}(u,v)\, du = f_Y(v)$ for all $v \in \mathbb{R}$.
  This says if X and Y are jointly continuous, then Y is a continuous RV

Conditional CDF and Conditional PDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Conditional CDF and Conditional PDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Conditional CDF of X conditioned on $\{Y = y\}$: $\mathbb{P}(\{X \leq x\} | \{Y = y\})$.
However, this conditional probability is not defined because $\mathbb{P}(\{Y = y\}) = 0$.

Conditional CDF and Conditional PDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Conditional CDF of X conditioned on $\{Y = y\}$: $\quad \mathbb{P}(\{X \leq x\} | \{Y = y\})$.
However, this conditional probability is not defined because $\mathbb{P}(\{Y = y\}) = 0$.

Remedy:
Fix $y \in \mathbb{R}$ and $\varepsilon > 0$ such that $\mathbb{P}\big(\{Y \in (y - \varepsilon, y + \varepsilon)\}\big) > 0$.
Define conditional probability with respect to the event $\{Y \in (y - \varepsilon, y + \varepsilon)\}$, and let $\varepsilon \downarrow 0$.

Conditional CDF and Conditional PDF

$$\mathbb{P}(\{X \leq x\}|\{Y \in (y-\varepsilon, y+\varepsilon\}) = \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})}$$

Conditional CDF and Conditional PDF

$$\mathbb{P}(\{X \leq x\} | \{Y \in (y-\varepsilon, y+\varepsilon)\}) = \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})}$$

$$= \frac{\int\limits_{-\infty}^{x} \int\limits_{y-\varepsilon}^{y+\varepsilon} f_{X,Y}(u,v)\, dv\, du}{\int\limits_{y-\varepsilon}^{y+\varepsilon} f_Y(v)\, dv}$$

Conditional CDF and Conditional PDF

$$\mathbb{P}(\{X \leq x\} | \{Y \in (y-\varepsilon, y+\varepsilon\}) = \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})}$$

$$= \frac{\int\limits_{-\infty}^{x} \int\limits_{y-\varepsilon}^{y+\varepsilon} f_{X,Y}(u,v)\, dv\, du}{\int\limits_{y-\varepsilon}^{y+\varepsilon} f_Y(v)\, dv}$$

$$\approx \frac{\int\limits_{-\infty}^{x} f_{X,Y}(u,y)\, du \cdot 2\varepsilon}{f_Y(y) \cdot 2\varepsilon}$$

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Conditional CDF and Conditional PDF

$$\begin{aligned}
\mathbb{P}(\{X \leq x\} | \{Y \in (y-\varepsilon, y+\varepsilon\}) &= \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})} \\
&= \frac{\int\limits_{-\infty}^{x} \int\limits_{y-\varepsilon}^{y+\varepsilon} f_{X,Y}(u,v)\, dv\, du}{\int\limits_{y-\varepsilon}^{y+\varepsilon} f_Y(v)\, dv} \\
&\approx \frac{\int\limits_{-\infty}^{x} f_{X,Y}(u,y)\, du \cdot 2\varepsilon}{f_Y(y) \cdot 2\varepsilon} \\
&= \int_{-\infty}^{x} \underbrace{\frac{f_{X,Y}(u,y)}{f_Y(y)}}_{\text{conditional PDF}} du
\end{aligned}$$

Conditional CDF for Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Definition (Conditional CDF for Jointly Continuous Random Variables)

The conditional CDF of X, conditioned on the event $\{Y = y\}$, is the function $F_{X|Y=y} : \mathbb{R} \to [0, 1]$ defined as

$$F_{X|Y=y}(x) = \int_{-\infty}^{x} \frac{f_{X,Y}(x, y)}{f_Y(y)}\, du, \qquad x \in \mathbb{R},$$

defined for all $y \in \mathbb{R}$ such that $f_Y(y) > 0$.

Conditional PDF for Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Definition (Conditional PDF for Jointly Continuous Random Variables)

The conditional PDF of X, conditioned on the event $\{Y = y\}$, is the function $f_{X|Y=y} : \mathbb{R} \to [0, +\infty)$ defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \qquad x \in \mathbb{R},$$

defined for all $y \in \mathbb{R}$ such that $f_Y(y) > 0$.

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Independence and Joint Continuity

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Definition (Joint Continuity and Independence)

X and Y are independent if

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \qquad \forall x, y \in \mathbb{R}.$$

Remark:

- $X \perp\!\!\!\perp Y \quad \Longleftrightarrow \quad f_{X|Y=y} = f_X$ for all y such that $f_Y(y) > 0$

Conditional PMF and Conditional PDF – 1

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables defined with respect to $\mathscr{F}$.

- If X and Y are jointly discrete,

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \qquad x \in \mathbb{R},\ p_Y(y) > 0.$$

  Furthermore, for any event $A \in \mathscr{F}$,

$$\mathbb{P}(\{X \in A\}|Y = y) = \sum_{x \in A} p_{X|Y=y}(x).$$

Conditional PMF and Conditional PDF – 2

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables defined with respect to $\mathscr{F}$.

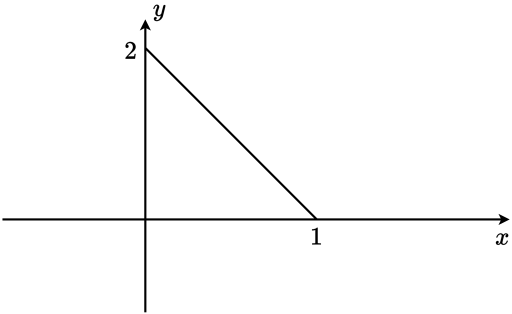
- If X and Y are jointly continuous,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad x \in \mathbb{R},\ f_Y(y) > 0.$$

  Furthermore, for any event $A \in \mathscr{F}$,

$$\mathbb{P}(\{X \in A\}|Y = y) = \int_A f_{X|Y=y}(u)\, du.$$

Example



Let $f_{X,Y}(x,y) = 1$ inside the triangle, and 0 elsewhere.
Compute the marginal PDFs of X and Y, and the conditional PDF of X conditioned on $\{Y = y\}$ for various values of y. Argue if X and Y are independent.

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Transformations of Random Variables

Transformations of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathscr{F}$.

Given a function $f : \mathbb{R} \to \mathbb{R}$, our interest is to characterise the CDF/PMF/PDF of the random variable $Y = f(X)$.

For ease of analysis, we shall consider functions f which are continuous and/or differentiable.

Examples

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathscr{F}$, with CDF F_X. Determine the CDF of $Y = aX + b$ for some $a, b \in \mathbb{R}$.

Examples

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathscr{F}$, with CDF F_X. Determine the CDF of $Y = X^2$.