AI 5090: STOCHASTIC PROCESSES
HOMEWORK 4

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BASICS OF MARKOV CHAINS, MARKOV PROPERTY, STRONG MARKOV PROPERTY

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing below are defined with respect to this probability space.

1. Let $\{X_n\}_{n=0}^{\infty}$ be a DTMC on a discrete state space $\mathcal{X}$.
   Show that for all $n \in \mathbb{N}$ and for all $x_0, \ldots, x_n \in \mathcal{X}$,
   $$\mathbb{P}(X_0 = x_0 \mid X_1 = x_1, \ldots, X_n = x_n) = \mathbb{P}(X_0 = x_0 \mid X_1 = x_1).$$
   Remark: This exercise demonstrates that the *reverse* chain is also Markov.

2. Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ with TPM P.
   Let $\{Y_n\}_{n=0}^{\infty}$ be another process defined as
   $$Y_n = X_{kn}, \qquad n \in \{0, 1, 2, \ldots\},$$
   where $k \in \mathbb{N}$ is a fixed constant.
   Prove that $\{Y_n\}_{n=0}^{\infty}$ is a time-homogeneous DTMC, and identify its TPM.

3. Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous DTMC with state space $\mathcal{X} = \{0, 1\}$, and having the TPM as shown below.
   $$\begin{array}{cc} & \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$
   Assume that $\mathbb{P}(X_0 = 0) = 0.5$. Evaluate the following.
   (a) $\mathbb{P}(X_1 = 0 \mid X_0 = 0, X_2 = 0)$.
   (b) $\mathbb{P}(X_1 \neq X_2)$.
   (c) For $p = 0.3$ and $q = 0.4$, compute $\mathbb{P}(X_4 = 0 \mid X_0 = 0)$.

4. Suppose that the numbers of families that check in to a hotel on successive days are independent Poisson random variables with mean $\lambda > 0$. Also, suppose that the number of days that any given family stays in the hotel is a Geometric random variable with parameter $p \in (0, 1)$. (Thus, a family who spent the previous night in the hotel will, independently of how long they have already spent in the hotel, check out the next day with probability p.) Assume that all families act independently of one another.
   (a) If X_n denotes the number of families present in the hotel at the end of day n, show that $\{X_n\}_{n=0}^{\infty}$ is a time-homogeneous DTMC. You may assume that $X_0 = 0$ (no families in the hotel on day 0).
   (b) Identify the TPM of the above DTMC.

5. Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous DTMC on a discrete state space $\mathcal{X}$ with TPM P.
   Fix a state $y \in \mathcal{X}$, and let $\tau_y^{(1)}$ denote the first hitting time of y, i.e.,
   $$\tau_y^{(1)} := \inf\{n \geq 1 : X_n = y\}.$$
   For a fixed $n \in \mathbb{N}$, compute the value of
   $$\mathbb{P}(X_{\tau_y^{(1)}+n} = y \mid X_0 = y,\ \tau_y^{(1)} < +\infty).$$

6. **(Eigenvalues and eigenvectors of a row stochastic matrix)**

Let P be a row-stochastic matrix of size $d \times d$ for some fixed $d \in \mathbb{N}$. For any $i \in \{1, \ldots, d\}$, let

$$R_i := \sum_{j \neq i} P_{i,j}.$$

(a) Let λ be an eigenvalue of P with an associated eigenvector $\mathbf{x} = [x_1, \ldots, x_d]^\top$. Let

$$i^\star := \arg\max_i |x_i|$$

denote the coordinate of $\mathbf{x}$ with the largest absolute value. If $i^\star$ is not unique, simply pick $i^\star$ at random from the set of all indices which attain the maximum in the expression above. Show that

$$|\lambda - P_{i^\star,i^\star}| \leq R_{i^\star}.$$

Hint: Consider the equation

$$(P - \lambda I)\, \mathbf{x} = \mathbf{0}.$$

Rearrange terms in the above equation by having $i^\star$ term on the left-hand side and remaining $i \neq i^\star$ terms on the right-hand side, apply triangle inequality, and use the fact that $|x_i|/|x_{i^\star}| \leq 1$ for all $i \neq i^\star$.

(b) Conclude from part (a) that if λ is any eigenvalue of P, then $|\lambda| \leq 1$.

(c) If P has strictly positive entries ($P_{i,j} > 0$ for all $i, j \in \mathcal{X}$), then show that $\lambda = 1$ is a simple eigenvalue (i.e., its eigenspace has dimension 1).

Hint:
We know that $\mathbf{1} = [1, \ldots, 1]^\top$ is an eigenvector corresponding to eigenvalue $\lambda = 1$. We need to demonstrate that any other eigenvector corresponding to $\lambda = 1$ is only a scaled version of $\mathbf{1}$, of the form $c\,\mathbf{1}$. Assume that there exists an eigenvector $\mathbf{x} = [x_1, \ldots, x_d]^\top$ in which $x_i \neq x_j$ for some $i \neq j$, and try come up with a contradiction.

(d) Continuing on part (c), assuming that P has all entries strictly positive, show that if $\lambda' \neq 1$ is any other eigenvalue of P, then $|\lambda'| < 1$.