

1. Given: $a_n \xrightarrow[n \rightarrow \infty]{} L$, $L \in \mathbb{R}$

$\Rightarrow \forall \varepsilon > 0, \exists N_\varepsilon$ (possibly depending on ε) such that

$$|a_n - L| < \varepsilon \quad \forall n \geq N_\varepsilon$$

Choosing $\varepsilon = 1$, we get that $\exists N_1 \in \mathbb{N}$ s.t.

$$|a_n - L| < 1 \quad \forall n \geq N_1$$

$$\Rightarrow L-1 < a_n < L+1 \quad \forall n \geq N_1. \rightarrow \textcircled{1}$$

Thus, $\{a_n\}_{n=N_1}^\infty$ is bounded. Furthermore, let

$$\bar{a} := \max\{a_1, \dots, a_{N_1-1}\}, \quad \underline{a} := \min\{a_1, \dots, a_{N_1-1}\}.$$

Then,

$$\underline{a} \leq a_n \leq \bar{a} \quad \forall n \in \{1, \dots, N_1-1\} \rightarrow \textcircled{2}$$

Combining \textcircled{1} and \textcircled{2}, we get

$$a_n \geq \underline{a} \geq \min\{L-1, \underline{a}\} \quad \forall n \in \{1, \dots, N_1-1\},$$

$$a_n > L-1 \geq \min\{L-1, \underline{a}\} \quad \forall n \geq N_1$$

$$a_n \leq \bar{a} \leq \max\{\bar{a}, L+1\} \quad \forall n \in \{1, \dots, N_1-1\},$$

$$a_n < L+1 \leq \max\{\bar{a}, L+1\} \quad \forall n \geq N_1$$

$$\Rightarrow \min\{\underline{a}, L-1\} \leq a_n \leq \max\{\bar{a}, L+1\} \quad \forall n \in \mathbb{N}.$$

Setting

$$M = \max\{\min\{\underline{a}, L-1\}, \max\{\bar{a}, L+1\}\},$$

we see that

$$-M \leq a_n \leq M \quad \forall n \in \mathbb{N}.$$

2. Given: $a_n \rightarrow L$, $L \in \mathbb{R}$, $L \neq 0$.

Without loss of generality, let $L > 0$.

We have

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} - 1 \right| &= \frac{|a_{n+1} - a_n|}{|a_n|} \\ &= \frac{|(a_{n+1} - L) - (a_n - L)|}{|a_n|} \end{aligned}$$

triangle inequality $\leq \frac{|a_{n+1} - L| + |a_n - L|}{|a_n|}$

Now, $a_n \rightarrow L \Rightarrow \forall \varepsilon' > 0, \exists N_{\varepsilon'}, \text{ s.t. } |a_n - L| < \varepsilon' \forall n \geq N_{\varepsilon'}$.

Fix $\varepsilon > 0$ arbitrarily. Set $\varepsilon' = \varepsilon L / 4$. We know that

$$\left| \frac{a_{n+1}}{a_n} - 1 \right| \leq \frac{2\varepsilon'}{|a_n|}. \quad \forall n \geq N_{\varepsilon'}$$

Furthermore, because $a_n \rightarrow L$, $\exists N \in \mathbb{N}$ s.t. $a_n > \frac{L}{2} \forall n \geq N$.
Thus,

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} - 1 \right| &\leq \frac{2\varepsilon'}{\frac{L}{2}} \\ &= \frac{4\varepsilon'}{L} \\ &= \varepsilon \quad \forall n \geq \max \{N_{\varepsilon'}, N\} \\ &\quad \underbrace{\quad}_{:= N_{\varepsilon} (\text{depends on } \varepsilon)} \end{aligned}$$

$\Rightarrow \forall \varepsilon > 0, \exists N_{\varepsilon} \text{ s.t. } \left| \frac{a_{n+1}}{a_n} - 1 \right| < \varepsilon \quad \forall n \geq N_{\varepsilon}$

$$\Rightarrow \frac{a_{n+1}}{a_n} \rightarrow 1.$$

$$3. \text{ a) } a_n = n \sin\left(\frac{n\pi}{2}\right).$$

$$\begin{aligned} \liminf_{n \rightarrow \infty} a_n &= \sup_{n \geq 1} \inf_{k \geq n} a_k \\ &= \sup_{n \geq 1} \inf_{k \geq n} k \sin\left(\frac{k\pi}{2}\right). \end{aligned}$$

Observe that

$$k \sin\left(\frac{k\pi}{2}\right) = \begin{cases} k, & k \in \{1, 5, 9, 13, \dots\}, \\ -k, & k \in \{3, 7, 11, 15, \dots\}, \\ 0, & k \in \{2, 4, 6, 8, \dots\} \end{cases}$$

Thus,

$$\begin{aligned} \inf_{k \geq n} a_k &= -\infty \quad \forall n \in \mathbb{N} \\ \Rightarrow \sup_{n \geq 1} \inf_{k \geq n} a_k &= -\infty. \end{aligned}$$

Also,

$$\sup_{k \geq n} a_k = +\infty \quad \forall n \in \mathbb{N} \Rightarrow \inf_{n \geq 1} \sup_{k \geq n} a_k = \limsup_{n \rightarrow \infty} a_n = +\infty.$$

b) Fix $n \in \mathbb{N}$. Observe that

$$\inf_{k \geq n} \{a_k : k \geq n\} \leq \inf_{k \geq n} \left\{ \frac{1}{k} : k \geq n \right\} \quad \begin{array}{l} \text{(the subsequence } \left\{ \frac{1}{k} \right\}_{k=2}^{\infty} \text{ is} \\ \text{bigger the set,} \\ \text{smaller its inf value} \end{array} \quad \begin{array}{l} \text{part of the sequence} \\ \text{)} \end{array}$$

$$= 0.$$

Thus, we have

$$\inf_{k \geq n} a_k \leq 0 \quad \forall n \geq N$$

$$\Rightarrow \sup_{n \geq 1} \inf_{k \geq n} a_k \leq 0.$$

However, every term in the sequence is non-negative, and therefore

$$\sup_{n \geq 1} \inf_{k \geq n} a_k \geq 0.$$

Combining, we get

$$\liminf_{n \rightarrow \infty} a_n = 0.$$

Similarly, for each $n \geq 1$,

$$\begin{aligned} \sup_{k \geq n} a_k &= \sup \{a_k : k \geq n\} \\ &\geq \sup \left\{ \frac{k}{k+1} : k \geq n \right\} \quad \begin{array}{l} \text{(the subsequence } \left\{ \frac{k}{k+1} \right\}_{k=n}^{\infty} \text{ is part of the sequence)} \\ \text{smaller the set, smaller its sup value} \end{array} \\ &= 1. \end{aligned}$$

$$\Rightarrow \inf_{n \geq 1} \sup_{k \geq n} a_k \geq 1.$$

But every term in the sequence is < 1

$$\Rightarrow \inf_{n \geq 1} \sup_{k \geq n} a_k \leq 1$$

Combining, we get

$$\limsup_{n \rightarrow \infty} a_n = 1.$$

4.

Observe that

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$$

$$C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$$

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \dots$$

Therefore,

$$\bigcap_{n=1}^{\infty} A_n = A_1 = (-\infty, a-1)$$

$$\bigcup_{n=1}^{\infty} B_n = B_1 = (-\infty, a+1)$$

$$\liminf_{n \rightarrow \infty} C_n = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} C_k}_{=C_n} = \bigcup_{n=1}^{\infty} C_n = (-\infty, a)$$

open because $a \notin C_n$ for any n .

$$\limsup_{n \rightarrow \infty} D_n = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} D_k}_{=D_n} = \bigcap_{n=1}^{\infty} D_n = (-\infty, a]$$

Closed because $a \in D_n$ for every n .

5.

$$a) (x^*)^{-1}((x, \infty)) = \left\{ \inf_{n \geq 1} \sup_{k \geq n} X_k > x \right\}$$

$$= \bigcap_{n=1}^{\infty} \left\{ \sup_{k \geq n} X_k > x \right\}$$



$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{ X_k > x \right\}.$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} x_k^{-1}((x, \infty))$$

$\in \mathcal{F}$ as X_k is a RV w.r.t. \mathcal{F}

$$b) (x^*)^{-1}(\{-\infty\}) = \left\{ x^* = -\infty \right\}$$

$$= \left\{ \inf_{n \geq 1} \sup_{k \geq n} X_k = -\infty \right\}$$

$\inf_{n \geq 1} y_n = -\infty \Leftrightarrow$ sequence not bounded from below

$$\Leftrightarrow \forall q \in \mathbb{Q}_+, \exists N_q \text{ s.t. } y_{N_q} < -q$$

$$= \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=1}^{\infty} \left\{ X_N < -q \right\}$$

$$= \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=1}^{\infty} x_N^{-1}((- \infty, -q))$$

$\in \mathcal{F}$ because X_N is a RV for each N

$$c) (x^*)^{-1}(\{+\infty\}) = \left\{ x^* = +\infty \right\}$$

$$= \left\{ \inf_{n \geq 1} \sup_{k \geq n} X_k = +\infty \right\}$$

$$= \bigcap_{n=1}^{\infty} \left\{ \sup_{k \geq n} X_k = +\infty \right\}$$

$\sup_{k \geq n} y_n = +\infty \Leftrightarrow \{y_k\}_{k=n}^{\infty}$ is not bounded from above

$$\Leftrightarrow \forall q \in \mathbb{Q}_+, \exists N_q \geq n \text{ s.t. } x_{N_q} > q$$

$$\begin{aligned}
 &= \bigcap_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=n}^{\infty} \{X_N > q\} \\
 &= \bigcap_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=n}^{\infty} X_N^{-1}((q, \infty)) \\
 &\quad \in \mathcal{F} \text{ as } X_n \text{ is a RV w.r.t. } \mathcal{F} \text{ for every } N
 \end{aligned}$$

6.

a) $X_n(\omega) = n\omega - \lfloor n\omega \rfloor$, $n \in \mathbb{N}$.

* Case 1: $\omega = 1$.

In this case, $X_n(\omega) = 0 \ \forall n \Rightarrow \lim_{n \rightarrow \infty} X_n(\omega) = 0$.

* Case 2: $\omega = \frac{p}{q}$, where $q \neq 0$, and $p, q > 0$ are co-prime (e.g: $\omega = \frac{4}{5}$) integers

In this case,

$$\begin{aligned}
 X_n(\omega) &= \frac{np}{q} - \left\lfloor \frac{np}{q} \right\rfloor = \frac{\text{remainder of dividing } np \text{ with } q}{q} \\
 &= \frac{np \bmod q}{q}.
 \end{aligned}$$

$$\in \left\{ 0, \frac{1}{q}, \dots, \frac{q-1}{q} \right\}.$$

Thus, $X_n(\omega)$ takes values periodically in the set $\left\{ 0, \frac{1}{q}, \dots, \frac{q-1}{q} \right\}$, and hence does not admit any limit.

* Case 3: ω is irrational.

Suppose that $\lim_{n \rightarrow \infty} X_n(\omega) = L$.

Clearly, because $X_n(\omega) \in [0, 1] \ \forall n$, we must have $L \in [0, 1]$.
In this case,

$$\lim_{n \rightarrow \infty} X_{n+1}(\omega) - X_n(\omega) = L - L = 0. \rightarrow ①$$

However, we notice that

$$\begin{aligned}
 X_{n+1}(\omega) - X_n(\omega) &= ((n+1)\omega - \lfloor (n+1)\omega \rfloor) - (n\omega - \lfloor n\omega \rfloor) \\
 &= \omega - (\lfloor (n+1)\omega \rfloor - \lfloor n\omega \rfloor) \\
 &= \omega - (\lfloor nw + \omega \rfloor - \lfloor nw \rfloor).
 \end{aligned}$$

i) If $n\omega + \omega$ is s.t. $\lfloor nw \rfloor + \omega < 1$, then \rightarrow

think of $n\omega = 32.56$
 $\omega = 0.11$

$$\lfloor nw + \omega \rfloor = \lfloor nw \rfloor + \lfloor \omega \rfloor = \lfloor nw \rfloor.$$

$$\Rightarrow X_{n+1}(\omega) - X_n(\omega) = \omega.$$

ii) If $n\omega + \omega$ is s.t. $\lfloor nw \rfloor + \omega \geq 1$, think of $n\omega = 32.56$
 $\omega = 0.68$
then

$$\lfloor nw + \omega \rfloor = \lfloor nw \rfloor + \lfloor \omega \rfloor + 1 = \lfloor nw \rfloor + 1$$

$$\Rightarrow X_{n+1}(\omega) - X_n(\omega) = \omega - 1.$$

Thus, we have proved that

$$X_{n+1}(\omega) - X_n(\omega) \in \{\omega, \omega - 1\} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \lim_{n \rightarrow \infty} X_{n+1}(\omega) - X_n(\omega) \neq 0 \rightarrow \textcircled{2}$$

\textcircled{1} and \textcircled{2} are contradictory.

So, our assumption that $\lim_{n \rightarrow \infty} X_n(\omega) = L$ should be wrong.

$\Rightarrow \lim_{n \rightarrow \infty} X_n(\omega)$ does not exist.

Summary:

$$\lim_{n \rightarrow \infty} X_n(\omega) = \begin{cases} 0, & \omega = 1, \\ \text{does not exist,} & \omega \neq 1 \end{cases} \rightarrow \begin{array}{l} \text{No pointwise limit} \\ \text{No almost-sure limit} \end{array}$$

b) $y_n(\omega) = n^2 \omega \mathbf{1}_{(0, y_n)}(\omega)$

For any $\omega \in (0, 1]$, there exists N sufficiently large s.t. $\frac{1}{N} < \omega$

$$\Rightarrow y_n(\omega) = 0 \quad \forall n \geq N$$

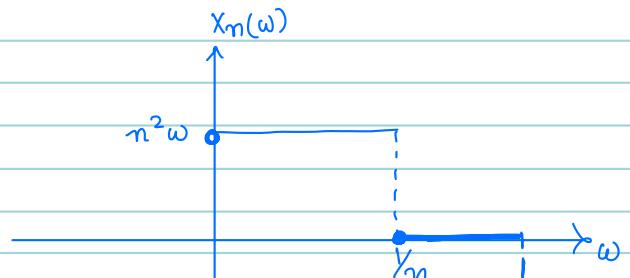
$$\Rightarrow \lim_{n \rightarrow \infty} y_n(\omega) = 0 \quad \forall \omega \in (0, 1]$$

Thus,

$$A_{\lim} = \{\omega : \lim_{n \rightarrow \infty} y_n(\omega) = 0\} = \Omega.$$

Pointwise convergence ✓

Almost-sure convergence ✓



c) $Z_n(\omega) = \sin(2\pi n\omega)$, $n \in \mathbb{N}$.

If $\omega \in \{\frac{1}{2}, 1\}$, then $Z_n(\omega) = 0 \quad \forall n \Rightarrow \lim_{n \rightarrow \infty} Z_n(\omega) = 0$.

If $\omega \notin \{\frac{1}{2}, 1\}$:

without loss of generality, let $\omega \in (0, \frac{1}{2})$.

Let $t > 0$.

Consider numbers of the form

$$\frac{t+1}{\omega}, \frac{t+2}{\omega}, \frac{t+3}{\omega}, \dots$$

Let n_k be the smallest integer s.t.

the values of $\{n_k\}$ might depend on choice of t

$$n_k \leq \frac{t+k}{\omega} < n_k + 1 \quad \forall k \in \mathbb{N}$$

but it's ok! $\Rightarrow 2\pi n_k \omega \leq 2\pi t + 2\pi k < 2\pi n_k \omega + 2\pi \omega \quad \forall k \in \mathbb{N}$

$$\Rightarrow 2\pi(t - \omega) < 2\pi n_k \omega - 2\pi k \leq 2\pi t \quad \forall k \in \mathbb{N}$$

The above relation holds for any $t > 0$.

In particular, for $t = \frac{\omega + 1}{2}$, we get

$$\frac{\pi}{2} - \pi\omega \leq 2\pi n_k \omega - 2\pi k \leq \frac{\pi}{2} + \pi\omega \quad \forall k \in \mathbb{N}$$

$\underbrace{\text{lies in } (0, \pi/2)}$

$\underbrace{\text{lies in } (\pi/2, \pi)}$

Case 1: n_k is s.t.

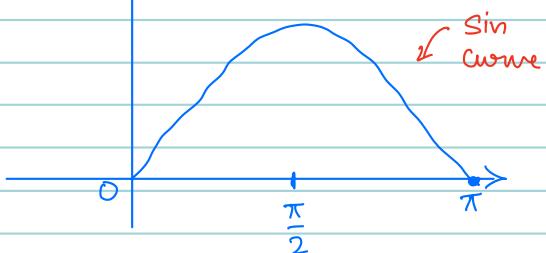
$$2\pi n_k \omega - 2\pi k \in (0, \pi/2)$$

In this case, because sin curve is increasing in $(0, \pi/2)$, we have

$$\sin(2\pi n_k \omega - 2\pi k) \geq \sin(\pi/2 - \pi\omega)$$

$$\Rightarrow \sin(2\pi n_k \omega) \geq \cos(\pi\omega) > 0 \rightarrow \textcircled{1}$$

\downarrow
lies in $(0, \pi/2)$



Case 2: n_k is s.t.

$$2\pi n_k \omega - 2\pi k \in [\pi/2, \pi)$$

In this case, because sin curve is decreasing in $[\pi/2, \pi)$,

$$\begin{aligned} \sin(2\pi n_k \omega - 2\pi k) &\geq \sin\left(\frac{\pi}{2} + \omega\pi\right) \\ \Rightarrow \sin(2\pi n_k \omega) &\geq \cos(\omega\pi) > 0. \end{aligned} \quad \rightarrow \textcircled{2}$$

Combining ① and ②, we get

$$\begin{aligned} \sin(2\pi n_k \omega) &> \cos(\omega\pi) > 0 \quad \forall k \in \mathbb{N} \\ \Rightarrow z_{n_k}(\omega) &> \cos(\omega\pi) > 0 \quad \forall k \in \mathbb{N} \\ \Rightarrow z_n(\omega) &> \cos(\omega\pi) > 0 \quad \text{for infinitely many values of } n \\ \Rightarrow \limsup_{n \rightarrow \infty} z_n(\omega) &\geq \cos(\omega\pi) > 0 \end{aligned} \quad \rightarrow \textcircled{a}$$

Choosing $t = \frac{\omega - 1/2}{2}$, we get

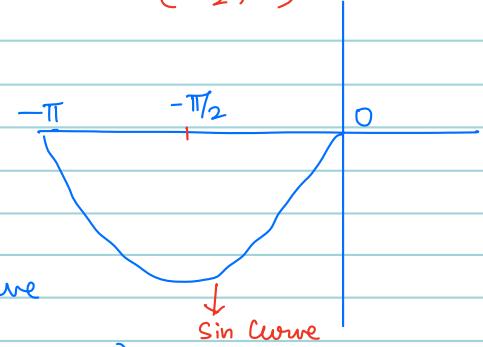
$$-\pi\left(\omega + \frac{1}{2}\right) < 2\pi n_k \omega - 2\pi k \leq \pi\left(\omega - \frac{1}{2}\right)$$

$\underbrace{-\pi\left(\omega + \frac{1}{2}\right)}_{\text{lies in } (-\pi, -\pi/2)}$ $\underbrace{\pi\left(\omega - \frac{1}{2}\right)}_{\text{lies in } (-\pi/2, 0)}$

Case 1: n_k is s.t.

$$2\pi n_k \omega - 2\pi k \in (-\pi, -\pi/2)$$

In this case, because sin curve is decreasing in $(-\pi, -\pi/2)$, we have



$$\begin{aligned} \sin(2\pi n_k \omega - 2\pi k) &\leq \sin(-\pi(\omega + 1/2)) \\ \Rightarrow \sin(2\pi n_k \omega) &\leq -\cos(\omega\pi) < 0. \end{aligned} \quad \rightarrow \textcircled{3}$$

Case 2: n_k is s.t.

$$2\pi n_k \omega - 2\pi k \in [-\pi/2, 0).$$

Because sin curve is increasing in $[-\pi/2, 0)$, we have

$$\begin{aligned} \sin(2\pi n_k \omega - 2\pi k) &\leq \sin(\pi\omega - \pi/2) \\ \Rightarrow \sin(2\pi n_k \omega) &\leq -\cos(\omega\pi). \end{aligned} \quad \rightarrow \textcircled{4}$$

Combining ③ and ④, we get

$$\sin(2\pi n_k \omega) \leq -\cos(\omega\pi) < 0 \quad \forall k \in \mathbb{N}$$

$$\Rightarrow Z_{n_k}(\omega) \leq -\cos(\pi\omega) < 0 \quad \forall k \in \mathbb{N}$$

$\Rightarrow Z_n(\omega) \leq -\cos(\pi\omega) < 0$ for infinitely many values of n

$$\Rightarrow \liminf_{n \rightarrow \infty} Z_n(\omega) \leq -\cos(\pi\omega) < 0. \rightarrow \text{(b)}$$

From (a) and (b), we have

$$\liminf_{n \rightarrow \infty} Z_n(\omega) \leq -\cos(\pi\omega) < 0$$

$$\limsup_{n \rightarrow \infty} Z_n(\omega) \geq \cos(\pi\omega) > 0.$$

} true for any $\omega \in (0, \frac{1}{2})$

$\Rightarrow \lim_{n \rightarrow \infty} Z_n(\omega)$ does not exist for any $\omega \in (0, \frac{1}{2})$

Similar arguments (with ω replaced by $1-\omega$) can be used to show that $\lim_{n \rightarrow \infty} Z_n(\omega)$ does not exist for any $\omega \in (\frac{1}{2}, 1)$.

Conclusion :

No pointwise convergence

No almost-sure convergence