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Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 19: Multiple Random Variables, Transformations (Part 1)

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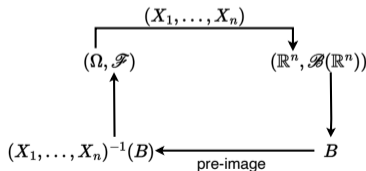**Email: pnkarthik@ai.iith.ac.in**

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Definition (Multivariate Random Vector)

Fix a measurable space $(\Omega, \mathscr{F})$.

Let $\quad X_1 : \Omega \to \mathbb{R}, \quad X_2 : \Omega \to \mathbb{R}, \quad \ldots \quad X_n : \Omega \to \mathbb{R} \quad$ be random variables with respect to $\mathscr{F}$.

We say $(X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$ is a **multivariate random vector** if

$$\forall \, B \in \mathscr{B}(\mathbb{R}^n), \qquad (X_1, \ldots, X_n)^{-1}(B) = \underbrace{\{\omega \in \Omega : \big(X_1(\omega), \ldots, X_n(\omega)\big) \in B\}}_{\text{pre-image of } B} = \{(X_1, \ldots, X_n) \in B\} \in \mathscr{F}.$$

$$\forall B \in \mathscr{B}(\mathbb{R}^n), \quad (X_1, \ldots, X_n)^{-1}(B) \in \mathscr{F}$$

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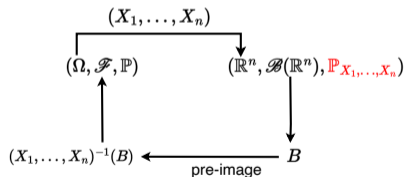Indian Institute of Technology Hyderabad

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Definition (Joint Probability Law)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $(X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$ be a multivariate random vector.

The **joint probability law of** $X_1, \ldots, X_n$ is a function $\mathbb{P}_{X_1, \ldots, X_n} : \mathscr{B}(\mathbb{R}^n) \to [0, 1]$, defined as

$$\forall\, B \in \mathscr{B}(\mathbb{R}^n), \qquad \mathbb{P}_{X_1, \ldots, X_n}(B) = \mathbb{P}((X_1, \ldots, X_n)^{-1}(B)) = \mathbb{P}(\{(X_1, \ldots, X_n) \in B\}).$$

$$\mathbb{P}_{X_1, \ldots, X_n}(B) = \mathbb{P}((X_1, \ldots, X_n)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^n)$$

On $\mathbb{P}_{X_1, \ldots, X_n}$

$\mathbb{P}_{X_1, \ldots, X_n}$ is a **probability measure** on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$.

Marginals from Joint

CDF	Probability Law
$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n), \qquad (x_1,\ldots,x_n) \in \mathbb{R}^n$	$\mathbb{P}_{X_1,\ldots,X_n}(B), \qquad B \in \mathscr{B}(\mathbb{R}^n)$
$F_{X_1}(x_1) = \lim_{\substack{x_2 \to +\infty \\ x_3 \to +\infty \\ \vdots \\ x_n \to +\infty}} F_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n)$	$\mathbb{P}_{X_1}(A) = \mathbb{P}_{X_1,\ldots,X_n}(A \times \mathbb{R} \times \cdots \times \mathbb{R})$
$F_{X_2}(x_2) = \lim_{\substack{x_1 \to +\infty \\ x_3 \to +\infty \\ \vdots \\ x_n \to +\infty}} F_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n)$	$\mathbb{P}_{X_2}(A) = \mathbb{P}_{X_1,\ldots,X_n}(\mathbb{R} \times A \times \mathbb{R} \times \cdots \times \mathbb{R})$

Remark

Given joint CDF/law, we may extract the marginal CDFs/laws.
The converse is not possible in general.

Independence of Multiple Random Variables

Proposition (Independence and Joint CDFs)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $(X_1, \ldots, X_n) : \Omega \to \mathbb{R}^n$ be a multivariate random vector.

We say that $X_1, \ldots, X_n$ are **mutually independent** if any one of the following equivalent conditions holds.

1. $\mathbb{P}_{X_1, \ldots, X_n}(B_1 \times \cdots \times B_n) = \prod_{i=1}^{n} \mathbb{P}_{X_i}(B_i)$ for all $B_1, \ldots, B_n \in \mathscr{B}(\mathbb{R})$.

2. $F_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i)$ for all $x_1, \ldots, x_n \in \mathbb{R}$.

3. $\sigma(X_1), \ldots, \sigma(X_n)$ are independent.

Independent and Identically Distributed (IID) Random Variables

We say that $X_1, \ldots, X_n$ are **independent and identically distributed (IID)** if:

- $X_1, \ldots, X_n$ are **mutually independent**.

- $X_1, \ldots, X_n$ **have the same CDF**,   i.e., $F_{X_1} = \cdots = F_{X_n}$.

Transformations of Random Variables

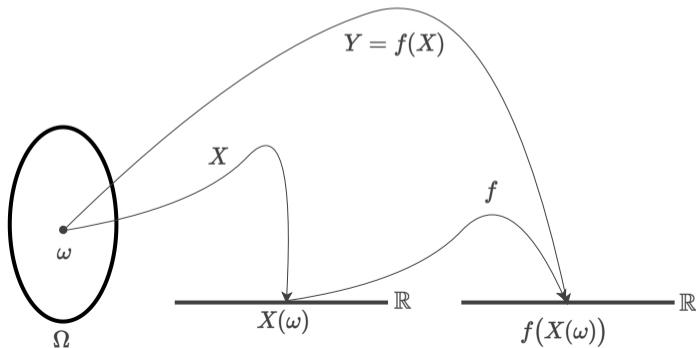# Transformations of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a **random variable** defined with respect to $\mathscr{F}$.
Let $f : \mathbb{R} \to \mathbb{R}$ be **measurable**.
Define $Y : \Omega \to \mathbb{R}$ as

$$Y = f(X), \qquad Y(\omega) = f(X(\omega)) = f \circ X(\omega), \qquad \omega \in \Omega.$$

Measurable Functions and Random Variables

- Y is a random variable with respect to $\mathscr{F}$
- Indeed, note that

$$\forall\, B \in \mathscr{B}(\mathbb{R}), \qquad Y^{-1}(B) = X^{-1}\Big(f^{-1}(B)\Big)$$

- $f^{-1}(B) \in \mathscr{B}(\mathbb{R})$ because f is measurable
- $X^{-1}\Big(f^{-1}(B)\Big) \in \mathscr{F}$ because X is a random variable

The Key Question

Given the CDF/PMF/PDF of X, what is the CDF/PMF/PDF of $Y = f(X)$?

Definition (Measurable Function)

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **measurable** if

$$f^{-1}(B) = \{x \in \mathbb{R} : f(x) \in B\} \in \mathscr{B}(\mathbb{R}) \qquad \forall B \in \mathscr{B}(\mathbb{R}).$$

Remarks:

- **Every continuous function is measurable**. Thus,

$$f(x) = |x|, \quad f(x) = x^2, \quad f(x) = e^x, \quad f(x) = \log x,$$

are measurable

- $X : \Omega \to \mathbb{R}$ **random variable** $\qquad f : \mathbb{R} \to \mathbb{R}$ **measurable**
  $\implies \quad f(X) : \Omega \to \mathbb{R}$ random variable

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Definition (Measurable Function)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **measurable** if

$$f^{-1}(B) = \{x \in \mathbb{R}^n : f(x) \in B\} \in \mathscr{B}(\mathbb{R}^n) \qquad \forall B \in \mathscr{B}(\mathbb{R}^m).$$

Implication for $m = 1$:

- $X_1, \ldots, X_n$ **random variables**, $f : \mathbb{R}^n \to \mathbb{R}$ **measurable**

  $\implies f(X_1, \ldots, X_n) : \Omega \to \mathbb{R}$ random variable with respect to $\mathscr{F}$

- Every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is measurable. Thus, for instance,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$$

  is measurable. Hence, $\sum_{i=1}^{n} X_i$ is a random variable.

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X_1, \ldots, X_n$ be random variables defined with respect to $\mathscr{F}$, with joint CDF $F_{X_1,\ldots,X_n}$.

- Show that $Y_n = \max\{X_1, \ldots, X_n\}$ is a random variable with respect to $\mathscr{F}$.

- Derive the CDF of Y_n.

- Simplify the CDF of Y_n when $X_1, \ldots, X_n$ are IID.

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X_1, \ldots, X_n$ be random variables defined with respect to $\mathscr{F}$, with joint CDF $F_{X_1, \ldots, X_n}$.

- Show that $Z_n = \min\{X_1, \ldots, X_n\}$ is a random variable with respect to $\mathscr{F}$.

- Derive the CDF of Z_n.

- Simplify the CDF of Z_n when $X_1, \ldots, X_n$ are IID.

Fix $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\lambda_1, \ldots, \lambda_n > 0$ be fixed constants.
Let $X_1, \ldots, X_n$ be independent, with $X_i \sim \text{Exponential}(\lambda_i)$ for each $i \in \{1, \ldots, n\}$.
Find the distribution of $Z = \min\{X_1, \ldots, X_n\}$.

Sums of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables with respect to $\mathscr{F}$.

- Show that $X + Y$ is a random variable with respect to $\mathscr{F}$.

- In the cases when X and Y are jointly discrete/continuous, derive the PMF/PDF of $X + Y$.

- Simplify the PMF/PDF when X and Y are independent.

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X \sim \mathrm{Poisson}(\lambda_1)$ and $Y \sim \mathrm{Poisson}(\lambda_2)$. Assume $X \perp\!\!\!\perp Y$.
Determine the distribution of $Z = X + Y$.

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X \sim \text{Exponential}(\mu_1)$ and $Y \sim \text{Exponential}(\mu_2)$. Assume $X \perp\!\!\!\perp Y$.
Determine the distribution of $Z = X + Y$.

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables with respect to $\mathscr{F}$. Assume $X \perp\!\!\!\perp Y$.
Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be Borel-measurable.
Prove that $g(X) \perp\!\!\!\perp h(Y)$.

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $\{X_i : i \in \mathbb{N}\}$ be a collection of i.i.d. random variables defined with respect to $\mathscr{F}$ and having a common CDF F.

Let N be a positive integer-valued random variable defined with respect to $\mathscr{F}$ and having the PMF p_N.

Let N be independent of $\{X_i : i \in \mathbb{N}\}$.

Consider the sum

$$S_N := \sum_{i=1}^{N} X_i; \qquad\qquad S_N(\omega) = \sum_{i=1}^{N(\omega)} X_i(\omega), \quad \omega \in \Omega.$$

- Show that $S_N : \Omega \to \mathbb{R}$ is a random variable with respect to $\mathscr{F}$.

- Determine the CDF of S_N.

In the previous example, let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda)$. Let $N \sim \text{Geom}(p)$. Determine the distribution of S_N.