1. Given: $a_n \xrightarrow[n\to\infty]{} L$, $L \in \mathbb{R}$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N_\varepsilon$ (possibly depending on ε) such that

$$|a_n - L| < \varepsilon \quad \forall\ n \geq N_\varepsilon$$

Choosing $\varepsilon = 1$, we get that $\exists\ N_1 \in \mathbb{N}$ s.t.

$$|a_n - L| < 1 \quad \forall\ n \geq N_1$$

$$\Rightarrow \quad L-1 < a_n < L+1 \quad \forall\ n \in N_1. \quad \rightarrow \text{①}$$

Thus, $\{a_n\}_{n=N_1}^{\infty}$ is bounded. Furthermore, let

$$\bar{a} := \max\{a_1, \dots, a_{N_1-1}\}, \quad \underline{a} := \min\{a_1, \dots, a_{N_1-1}\}.$$

Then,

$$\underline{a} \leq a_n \leq \bar{a} \quad \forall\ n \in \{1, \dots, N_1-1\} \rightarrow \text{②}$$

Combining ① and ②, we get

$$a_n \geq \underline{a} \geq \min\{L-1, \underline{a}\} \quad \forall\ n \in \{1, \dots, N_1-1\},$$

$$a_n > L-1 \geq \min\{L-1, \underline{a}\} \quad \forall\ n \geq N_1$$

$$a_n \leq \bar{a} \leq \max\{\bar{a}, L+1\} \quad \forall\ n \in \{1, \dots, N_1-1\},$$

$$a_n < L+1 \leq \max\{\bar{a}, L+1\} \quad \forall\ n \geq N_1$$

$$\Rightarrow \min\{\underline{a}, L-1\} \leq a_n \leq \max\{\bar{a}, L+1\} \quad \forall\ n \in \mathbb{N}.$$

Setting

$$M = \max\{\min\{\underline{a}, L-1\}, \max\{\bar{a}, L+1\}\},$$

we see that

$$-M \leq a_n \leq M \quad \forall\ n \in \mathbb{N}.$$

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2. <u>Given</u>: $a_n \to L$, $L \in \mathbb{R}$, $L \neq 0$.

Without loss of generality, let $L > 0$.

We have

$$\left| \frac{a_{n+1}}{a_n} - 1 \right| = \frac{|a_{n+1} - a_n|}{|a_n|}$$

$$= \frac{|(a_{n+1} - L) - (a_n - L)|}{|a_n|}$$

triangle inequality

$$\leq \frac{|a_{n+1} - L| + |a_n - L|}{|a_n|}$$

Now, $a_n \to L \Rightarrow \forall\, \varepsilon' > 0$, $\exists\, N_{\varepsilon'}$ s.t. $|a_n - L| < \varepsilon'$ $\forall\, n \geq N_{\varepsilon'}$.

Fix $\varepsilon > 0$ arbitrarily. Set $\varepsilon' = \varepsilon L / 4$. We know that

$$\left| \frac{a_{n+1}}{a_n} - 1 \right| \leq \frac{2\varepsilon'}{|a_n|}. \quad \forall\, n \geq N_{\varepsilon'}.$$

Furthermore, because $a_n \to L$, $\exists\, N \in \mathbb{N}$ s.t. $a_n > \frac{L}{2}$ $\forall\, n \geq N$.
Thus,

$$\left| \frac{a_{n+1}}{a_n} - 1 \right| \leq \frac{2\varepsilon'}{L/2}$$

$$= \frac{4\varepsilon'}{L}$$

$$= \varepsilon \qquad \forall\, n \geq \underbrace{\max\{N_{\varepsilon'}, N\}}_{:= N_\varepsilon \text{ (depends on } \varepsilon)}$$

$\Rightarrow \forall\, \varepsilon > 0$, $\exists\, N_\varepsilon$ s.t. $\left| \frac{a_{n+1}}{a_n} - 1 \right| < \varepsilon$ $\forall\, n \geq N_\varepsilon$

$$\Rightarrow \frac{a_{n+1}}{a_n} \to 1.$$

---

3. a) $a_n = n \sin\left(\frac{n\pi}{2}\right)$.

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{k\geq n} a_k$$

$$= \sup_{n\geq 1} \inf_{k\geq n} k \sin\left(\frac{k\pi}{2}\right).$$

Observe that

$$k \sin\left(\frac{k\pi}{2}\right) = \begin{cases} k, & k \in \{1, 5, 9, 13, \dots\}, \\ -k, & k \in \{3, 7, 11, 15, \dots\}, \\ 0, & k \in \{2, 4, 6, 8, \dots\} \end{cases}$$

Thus,

$$\inf_{k\geq n} a_k = -\infty \quad \forall\ n \in \mathbb{N}$$

$$\Rightarrow \quad \sup_{n\geq 1} \inf_{k\geq n} a_k = -\infty.$$

Also,

$$\sup_{k\geq n} a_k = +\infty \quad \forall\ n\in\mathbb{N} \quad \Rightarrow \quad \inf_{n\geq 1} \sup_{k\geq n} a_k = \limsup_{n\to\infty} a_n = +\infty.$$

b) Fix $n \in \mathbb{N}$. Observe that

$$\inf\{a_k : k \geq n\} \leq \inf\left\{\frac{1}{k} : k \geq n\right\}$$ (the subsequence $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$ is part of the sequence)

bigger the set, smaller its inf value

$= 0.$

Thus, we have

$$\inf_{k\geq n} a_k \leq 0 \quad \forall\ n \geq N$$

$$\Rightarrow \quad \sup_{n\geq 1} \inf_{k\geq n} a_k \leq 0.$$

However, every term in the sequence is non-negative, and therefore

$$\sup_{n\geq 1} \inf_{k\geq n} a_k \geq 0.$$

Combining, we get

$$\liminf_{n\to\infty} a_n = 0.$$

Similarly, for each $n \geq 1$,

$$\sup_{k \geq n} a_k = \sup\{a_k : k \geq n\}$$

$$\underset{\substack{\swarrow \\ \text{smaller the set, smaller its sup value}}}{\geq} \sup\left\{\frac{k}{k+1} : k \geq n\right\}$$ (the subsequence $\left\{\frac{k}{k+1}\right\}_{k=1}^{\infty}$ is part of the sequence)

$$= 1.$$

$$\Rightarrow \inf_{n \geq 1} \sup_{k \geq n} a_k \geq 1.$$

But every term in the sequence is <1

$$\Rightarrow \inf_{n \geq 1} \sup_{k \geq n} a_k \leq 1$$

Combining, we get

$$\limsup_{n \to \infty} a_n = 1.$$

4.

Observe that

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$$

$$C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$$

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$$

Therefore,

$$\bigcap_{n=1}^{\infty} A_n = A_1 = (-\infty, a-1)$$

$$\bigcup_{n=1}^{\infty} B_n = B_1 = (-\infty, a+1)$$

$$\liminf_{n \to \infty} C_n = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} C_k}_{= C_n} = \bigcup_{n=1}^{\infty} C_n = (-\infty, a)$$

↳ open because $a \notin C_n$ for any n.

$$\limsup_{n \to \infty} D_n = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} D_k}_{= D_n} = \bigcap_{n=1}^{\infty} D_n = (-\infty, a]$$

↳ Closed because $a \in D_n$ for every n.

5.

a) $$(X^*)^{-1}((x,\infty)) = \left\{ \inf_{h\geq 1} \sup_{k\geq n} X_k > x \right\}$$

$$= \bigcap_{n=1}^{\infty} \left\{ \sup_{k\geq n} X_k > x \right\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ X_k > x \}.$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \underbrace{X_k^{-1}((x,\infty))}_{\in \mathcal{F} \text{ as } X_k \text{ is a RV w.r.t. } \mathcal{F}}$$

b) $$(X^*)^{-1}(\{-\infty\}) = \{ X^* = -\infty \}$$

$$= \left\{ \inf_{n\geq 1} \sup_{k\geq n} X_k = -\infty \right\}$$

$\inf_{n\geq 1} y_n = -\infty \iff$ sequence not bounded from below

$\iff \forall q \in \mathbb{Q}_+, \ \exists N_q$ s.t. $y_{N_q} < -q$

$$= \bigcap_{q\in\mathbb{Q}_+} \bigcup_{N=1}^{\infty} \{ X_N < -q \}$$

$$= \bigcap_{q\in\mathbb{Q}_+} \bigcup_{N=1}^{\infty} \underbrace{X_N^{-1}((-\infty,-q))}_{\in \mathcal{F} \text{ because } X_N \text{ is a RV for each } N}$$

c) $$(X^*)^{-1}(\{+\infty\}) = \{ X^* = +\infty \}$$

$$= \left\{ \inf_{n\geq 1} \sup_{k\geq n} X_k = +\infty \right\}$$

$$= \bigcap_{n=1}^{\infty} \left\{ \sup_{k\geq n} X_k = +\infty \right\}$$

$\sup_{k\geq n} y_n = +\infty \iff \{y_k\}_{k=n}^{\infty}$ is not bounded from above

$\iff \forall q \in \mathbb{Q}_+, \ \exists N_q \geq n$ s.t. $X_{N_q} > q$

$$= \bigcap_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=n}^{\infty} \{X_N > q\}$$

$$= \bigcap_{n=1}^{\infty} \bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=n}^{\infty} \underbrace{X_N^{-1}\big((q,\infty)\big)}_{\in \mathcal{F}}$$

$\in \mathcal{F}$ as X_N is a RV w.r.t. $\mathcal{F}$ for every N

6.

a) $X_n(\omega) = n\omega - \lfloor n\omega \rfloor, \quad n \in \mathbb{N}.$

★ Case 1: $\omega = 1$.

In this case, $X_n(\omega) = 0 \;\; \forall\, n \Rightarrow \lim_{n\to\infty} X_n(\omega) = 0$.

★ Case 2: $\omega = \frac{p}{q}$, where $q \neq 0$, and $p, q > 0$ are co-prime integers (e.g. $\omega = 4/5$)

In this case,

$$X_n(\omega) = \frac{np}{q} - \left\lfloor \frac{np}{q} \right\rfloor = \frac{\text{remainder of dividing } np \text{ with } q}{q}$$

$$= \frac{np \bmod q}{q}.$$

$$\in \left\{0, \frac{1}{q}, \dots, \frac{q-1}{q}\right\}.$$

Thus, $X_n(\omega)$ takes values periodically in the set $\left\{0, \frac{1}{q}, \dots, \frac{q-1}{q}\right\}$, and hence does not admit any limit.

★ Case 3: ω is irrational.

Suppose that $\lim_{n\to\infty} X_n(\omega) = L$.

Clearly, because $X_n(\omega) \in [0,1) \;\forall\, n$, we must have $L \in [0,1)$.
In this case,

$$\lim_{n\to\infty} X_{n+1}(\omega) - X_n(\omega) = L - L = 0. \quad \rightarrow (1)$$

However, we notice that

$$X_{n+1}(\omega) - X_n(\omega) = \big((n+1)\omega - \lfloor (n+1)\omega \rfloor\big) - \big(n\omega - \lfloor n\omega \rfloor\big)$$

$$= \omega - \big(\lfloor (n+1)\omega \rfloor - \lfloor n\omega \rfloor\big)$$

$$= \omega - \big(\lfloor n\omega + \omega \rfloor - \lfloor n\omega \rfloor\big).$$

i) If $n\omega + \omega$ is s.t. $\lfloor n\omega \rfloor + \omega < 1$, then $\rightarrow$ think of $n\omega = 32.56$, $\omega = 0.11$

$$\lfloor n\omega + \omega \rfloor = \lfloor n\omega \rfloor + \lfloor \omega \rfloor = \lfloor n\omega \rfloor .$$

$$\Rightarrow \quad X_{n+1}(\omega) - X_n(\omega) = \omega .$$

ii) If $n\omega + \omega$ is s.t. $\lfloor n\omega \rfloor + \omega \geq 1$, then → think of $n\omega = 32.56$, $\omega = 0.68$

$$\lfloor n\omega + \omega \rfloor = \lfloor n\omega \rfloor + \lfloor \omega \rfloor + 1 = \lfloor n\omega \rfloor + 1$$

$$\Rightarrow \quad X_{n+1}(\omega) - X_n(\omega) = \omega - 1 .$$

Thus, we have proved that

$$X_{n+1}(\omega) - X_n(\omega) \in \{\omega, \omega - 1\} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad \lim_{n\to\infty} X_{n+1}(\omega) - X_n(\omega) \neq 0 \quad \rightarrow \quad (2)$$

(1) and (2) are contradictory.
So, our assumption that $\lim_{n\to\infty} X_n(\omega) = L$ should be wrong.

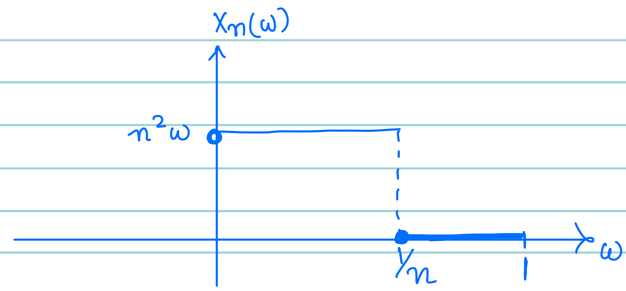
$\Rightarrow$ $\lim_{n\to\infty} X_n(\omega)$ does not exist.

Summary:

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0, & \omega = 1, \\ \text{does not exist}, & \omega \neq 1 \end{cases}$$

→ No pointwise limit
No almost-sure limit

b) $$Y_n(\omega) = n^2 \omega \, \mathbb{1}_{(0, 1/n)}(\omega)$$



For any $\omega \in (0,1]$, there exists N sufficiently large s.t. $\frac{1}{N} < \omega$

$$\Rightarrow \quad Y_n(\omega) = 0 \quad \forall \, n \geq N$$

$$\Rightarrow \quad \lim_{n\to\infty} Y_n(\omega) = 0 \quad \forall \, \omega \in (0,1]$$

Thus,

$$A_{\lim} = \{\omega : \lim_{n\to\infty} Y_n(\omega) = 0\} = \Omega .$$

Pointwise convergence ✓
Almost-sure convergence ✓

c) $Z_n(\omega) = \sin(2\pi n\omega), \quad n \in \mathbb{N}.$

If $\omega \in \{1/2, 1\}$, then $Z_n(\omega) = 0 \quad \forall\, n \Rightarrow \lim_{n\to\infty} Z_n(\omega) = 0.$

<u>If $\omega \notin \{1/2, 1\}$:</u>

Without loss of generality, let $\omega \in (0, 1/2)$.
Let $t > 0$.
Consider numbers of the form

$$\frac{t+1}{\omega}, \quad \frac{t+2}{\omega}, \quad \frac{t+3}{\omega}, \quad \ldots$$

Let n_k be the smallest integer s.t.

the values of $\{n_k\}$ might depend on choice of t but it's ok!

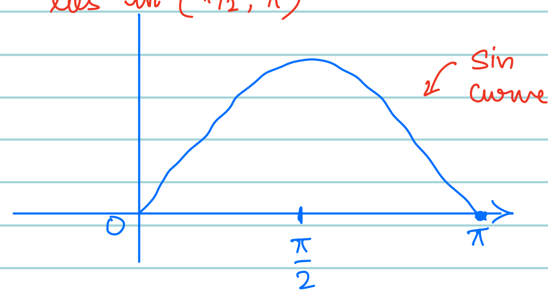
$$n_k \leq \frac{t+k}{\omega} < n_k + 1 \qquad \forall\ k \in \mathbb{N}.$$

$$\Rightarrow \quad 2\pi n_k \omega \leq 2\pi t + 2\pi k < 2\pi n_k \omega + 2\pi\omega \qquad \forall\ k \in \mathbb{N}$$

$$\Rightarrow \quad 2\pi(t-\omega) < 2\pi n_k \omega - 2\pi k \leq 2\pi t \qquad \forall\ k \in \mathbb{N}$$

The above relation holds for any $t > 0$.
In particular, for $t = \frac{\omega + 1/2}{2}$, we get

$$\underbrace{\frac{\pi}{2} - \frac{\omega\pi}{2}}_{\text{lies in } (0, \pi/2)} \leq 2\pi n_k \omega - 2\pi k \leq \underbrace{\frac{\pi}{2} + \frac{\omega\pi}{2}}_{\text{lies in } (\pi/2, \pi)} \qquad \forall\ k \in \mathbb{N}$$



<u>Case 1</u>: n_k is s.t.
$$2\pi n_k(\omega) - 2\pi k \in (0, \pi/2)$$

In this case, because sin curve is increasing in $(0, \pi/2)$, we have

$$\sin(2\pi n_k \omega - 2\pi k) \geq \sin(\pi/2 - \pi\omega)$$

$$\Rightarrow \quad \sin(2\pi n_k \omega) \geq \cos(\underbrace{\pi\omega}_{\text{lies in } (0, \pi/2)}) > 0 \quad \rightarrow (1)$$

<u>Case 2</u>: n_k is s.t.
$$2\pi n_k \omega - 2\pi k \in [\pi/2, \pi)$$

In this case, because sin curve is decreasing in $[\pi/2, \pi)$,

$$\sin(2\pi n_k \omega - 2\pi k) \geq \sin\left(\frac{\pi}{2} + \omega\pi\right)$$

$$\Rightarrow \quad \sin(2\pi n_k \omega) \geq \cos(\omega\pi) > 0. \quad \rightarrow (2)$$

Combining (1) and (2), we get

$$\sin(2\pi n_k \omega) > \cos(\pi\omega) > 0 \qquad \forall\, k \in \mathbb{N}$$

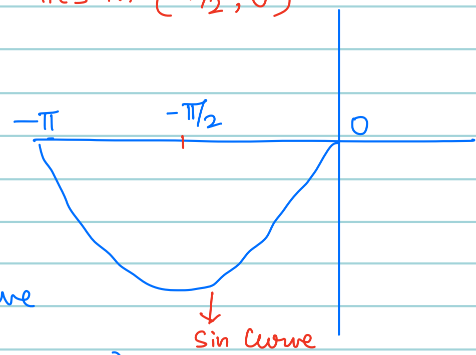
$$\Rightarrow \quad Z_{n_k}(\omega) > \cos(\pi\omega) > 0 \qquad \forall\, k \in \mathbb{N}$$

$$\Rightarrow \quad Z_n(\omega) > \cos(\pi\omega) > 0 \quad \text{for infinitely many values of } n$$

$$\Rightarrow \quad \limsup_{n\to\infty} Z_n(\omega) \geq \cos(\pi\omega) > 0 \longrightarrow (a)$$

Choosing $t = \frac{\omega - 1/2}{2}$, we get

$$\underbrace{-\pi\left(\omega + \frac{1}{2}\right)}_{\text{lies in } (-\pi, -\pi/2)} < 2\pi n_k \omega - 2\pi k \leq \underbrace{\pi\left(\omega - \frac{1}{2}\right)}_{\text{lies in } (-\pi/2, 0)}$$



Case 1 : n_k is s.t.

$$2\pi n_k \omega - 2\pi k \in (-\pi, -\pi/2)$$

In this case, because sin curve is decreasing in $(-\pi, -\pi/2)$, we have

$$\sin(2\pi n_k \omega - 2\pi k) \leq \sin(-\pi(\omega + 1/2))$$

$$\Rightarrow \quad \sin(2\pi n_k \omega) \leq -\cos(\pi\omega) < 0. \quad \rightarrow (3)$$

Case 2 : n_k is s.t.

$$2\pi n_k \omega - 2\pi k \in [-\pi/2, 0).$$

Because sin curve is increasing in $[-\pi/2, 0)$, we have

$$\sin(2\pi n_k \omega - 2\pi k) \leq \sin(\pi\omega - \pi/2)$$

$$\Rightarrow \quad \sin(2\pi n_k \omega) \leq -\cos(\pi\omega). \quad \rightarrow (4)$$

Combining (3) and (4), we get

$$\sin(2\pi n_k \omega) \leq -\cos(\pi\omega) < 0 \qquad \forall\, k \in \mathbb{N}$$

$\Rightarrow \quad Z_{n_k}(\omega) \leq -\cos(\pi\omega) < 0 \qquad \forall\, k \in \mathbb{N}$

$\Rightarrow \quad Z_n(\omega) \leq -\cos(\pi\omega) < 0$ for infinitely many values of n

$\Rightarrow \quad \liminf\limits_{n\to\infty} Z_n(\omega) \leq -\cos(\pi\omega) < 0. \rightarrow$ (b)

From (a) and (b), we have

$$\left.\begin{aligned} &\liminf_{n\to\infty} Z_n(\omega) \leq -\cos(\pi\omega) < 0 \\ &\limsup_{n\to\infty} Z_n(\omega) \geq \cos(\pi\omega) > 0. \end{aligned}\right\} \text{ true for any } \omega \in (0, 1/2)$$

$\Rightarrow \quad \lim\limits_{n\to\infty} Z_n(\omega)$ does not exist for any $\omega \in (0, 1/2)$

Similar arguments (with ω replaced by $1-\omega$) can be used to show that $\lim\limits_{n\to\infty} Z_n(\omega)$ does not exist for any $\omega \in (1/2, 1)$.

Conclusion:

No pointwise convergence
No almost-sure convergence