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Indian Institute of Technology Hyderabad

Mathematical Foundations for Data Science (Probability)

Conditional Expectations – Examples, Law of Iterated Expectation, Gaussian Random Vectors

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Conditional Expectation – Examples

X and Y Jointly Continuous

- Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx(y-x)e^{-y}, & 0 \le x \le y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

What is $\mathbb{E}[Y|X]$?

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Y Continuous, X Discrete

- Let $Y \sim \mathcal{N}(0, 1)$. Suppose that the conditional PMF of X, conditioned on the event $\{Y = y\}$, is

$$p_{X|Y=y}(x) = \frac{1}{2}\mathbf{1}_{|x-\mathrm{sgn}(y)|=1},$$

where $\mathrm{sgn}(y)$ denotes the sign of y, and is defined as

$$\mathrm{sgn}(y) = \begin{cases} 1, & y > 0, \\ 0, & y = 0, \\ -1, & y < 0. \end{cases}$$

Compute $\mathbb{E}[X|Y]$?

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Law of Iterated Expectations

Law of Iterated Expectations – 1

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables w.r.t. $\mathscr{F}$.

Theorem (Law of Iterated Expectations)

Suppose that $\mathbb{E}[X]$ is well defined, i.e., not of the form $\infty - \infty$. Then,

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big].$$

More generally, if $g : \mathbb{R} \to \mathbb{R}$ is a function such that $\mathbb{E}[g(X)]$ is well defined, then

$$\mathbb{E}[g(X)] = \mathbb{E}\big[\mathbb{E}[g(X)|Y]\big].$$

Law of Iterated Expectations – 2

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables w.r.t. $\mathscr{F}$.

Theorem (Law of Iterated Expectations)

If Y is a discrete random variable with PMF p_Y, then

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y=y] \cdot p_Y(y), \qquad \mathbb{E}[g(X)] = \sum_y \mathbb{E}[g(X)|Y=y] \cdot p_Y(y)$$

and Y is a continuous random variable with PDF f_Y, then

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \mathbb{E}[X|Y=y] \cdot f_Y(y)\, dy, \qquad \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} \mathbb{E}[g(X)|Y=y] \cdot f_Y(y)\, dy$$

Examples

- Let $X \sim \mathrm{Geometric}(p)$ for some $p \in (0, 1)$.
  Determine $\mathbb{E}[X]$ using the law of iterated expectations.

Miscellaneous Examples on Conditional Expectations

- Let $X \sim \text{Exponential}(1)$. Compute $\mathbb{E}[X|\{X > 1\}]$.

Miscellaneous Examples on Conditional Expectations

- Let X, Y be jointly distributed uniformly over the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 2)$.
  Compute $\mathbb{E}[X|\{Y > 1\}]$.

Miscellaneous Examples on Conditional Expectations

- Suppose that X and Y are independent random variables. What is $\mathbb{E}[X|Y]$?
- Given a random variable Y and a function g, what is $\mathbb{E}[g(Y)|Y]$?
- Given $X \perp\!\!\!\perp Y$, what is $\mathbb{E}[X\,g(Y)|Y]$?

Gaussian Random Vectors

Gaussian Random Vectors

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Fix $n \in \mathbb{N}$. Let $X_1, X_2, \ldots, X_n$ be random variables defined w.r.t. $\mathscr{F}$.

Definition

We say the vector $X = (X_1, \ldots, X_n)^\top$ is a Gaussian random vector if

1. $X_1, \ldots, X_n$ are jointly continuous, and

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We say the vector $X = (X_1, \dots, X_n)^\top$ is a Gaussian random vector if

1. $X_1, \dots, X_n$ are jointly continuous, and
2. The joint PDF of $X_1, \dots, X_n$ at any point $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ may be expressed as

$$f_{X_1,\dots,X_n}(\mathbf{x}) = \frac{1}{\sqrt{2\pi \det(K)}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top K^{-1} (\mathbf{x} - \boldsymbol{\mu})\right),$$

for some $\boldsymbol{\mu} \in \mathbb{R}^n$ and positive definite matrix K.

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$\boldsymbol{\mu}$ – mean vector, K – covariance matrix

Remarks

- If $X = (X_1, \ldots, X_n)^\top$ is a Gaussian random vector, we say that $X_1, \ldots, X_n$ are jointly Gaussian

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- If X and Y are individually Gaussian, and $X \perp\!\!\!\perp Y$, then X and Y are jointly Gaussian

Important Results

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X_1, \ldots, X_n$ be random variables defined w.r.t. $\mathscr{F}$.

Important Results

1. $X_1, \ldots, X_n$ are jointly Gaussian if and only if

$$a_1 X_1 + \cdots + a_n X_n \quad \text{is Gaussian} \quad \forall a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}.$$

   is a Gaussian random variable
2. If X and Y are jointly Gaussian, and $\mathrm{Cov}(X, Y) = 0$, then

$$X \perp\!\!\!\perp Y.$$

The Case $\det(K) = 0$

Proposition (The Case $\det(K) = 0$)

If $\det(K) = 0$, then $X_1, \ldots, X_n$ are not jointly continuous.
In other words, there exist constants $a_1, \ldots, a_n \in \mathbb{R}^n \setminus \{0\}$ such that the random variable

$$a_1 X_1 + \cdots + a_n X_n$$

is not Gaussian.