AI5090: Stochastic Processes [Lecture 01]   03 Jan 2025

Agenda:

Primer on real analysis:

- Supremum, infimum of sets on $\mathbb{R}$
- Limsup, liminf of $\mathbb{R}$-valued sequences
- Limits of $\mathbb{R}$-valued sequences

Andrey Nikolaevich Kolmogorov (1903-1987)

Kolmogorov during his visit to ISI Kolkata (Feb 1962)

Kolmogorov preparing for a talk

$\mathbb{N}$ - set of natural numbers $= \{1, 2, 3, \dots\}$

$\mathbb{Z}$ - set of integers $= \{0, \pm 1, \pm 2, \pm 3, \dots\}$

$\mathbb{Q}$ - set of rational numbers

$= \{ p/q : p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0 \}$

$\mathbb{R}$ - set of real numbers

<u>Sequences of real numbers</u>

A sequence of real numbers is a mapping $f : \mathbb{N} \to \mathbb{R}$.

$$\left.\begin{array}{l} 1 \mapsto f(1) = a_1 \\ 2 \mapsto f(2) = a_2 \\ 3 \mapsto f(3) = a_3 \\ \vdots \end{array}\right\} \quad \{a_n\}_{n=1}^{\infty}$$

<u>Supremum of a set of real numbers</u>

Let $A \subseteq \mathbb{R}$. The supremum of A is an element x^* such that :

i) x^* is an upper bound for A, i.e.,

$\forall\ x \in A, \quad x \leq x^*$

for all

ii) x^* is the least upper bound, i.e.,

$\forall \varepsilon > 0$, $x^* - \varepsilon$ is NOT an upper bound

$\Rightarrow \forall\ \varepsilon > 0,\ \exists\ x \in A$ s.t. $x > x^* - \varepsilon$.

<u>Examples</u> :

i) $A = (0, 1)$

$\boxed{\sup A = 1}$

ii) $A = \{1,2,3\}$

$\boxed{\sup A = 3}$

(iv) $A = \phi$

$\boxed{\sup A = -\infty}$

(iii) $A = \mathbb{N}$

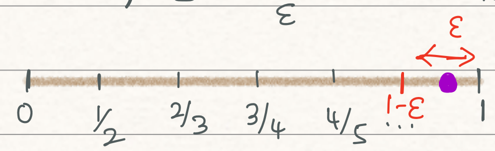
$\boxed{\sup A = +\infty}$

(v) $A = \left\{ 1 - \frac{1}{n} : n \geq 1 \right\}$

★ 1 is an upper bound

$1 - \frac{1}{n} \leq 1 \quad \forall \; n \geq 1$

★ $\forall \; \varepsilon > 0, \; \exists \; n > \frac{1}{\varepsilon}$ s.t. $1 - \frac{1}{n} > 1 - \varepsilon$.



$1 - \frac{1}{n} > 1 - \varepsilon$

$\Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$

$\boxed{\sup A = 1}$

<u>Infimum of a set of real numbers</u>

Let $A \subseteq \mathbb{R}$. The infimum of A is a number x_* s.t.

i) x_* is a <u>lower bound</u> for A, i.e.,

$\forall x \in A, \quad x \geq x_*$

ii) x_* is the greatest lower bound for A

$\Rightarrow \forall \; \varepsilon > 0, \; x_* + \varepsilon$ is NOT a lower bound for A

$\Rightarrow \forall \; \varepsilon > 0, \; \exists \; x \in A$ s.t. $x < x_* + \varepsilon$.

Examples:

i) $A = (0,1)$

$\boxed{\inf A = 0}$

ii) $A = \{1, 2, 3\}$

$\boxed{\inf A = 1}$

iii) $A = \mathbb{N}$

$\boxed{\inf \mathbb{N} = 1}$

iv) $\boxed{\inf \phi = +\infty}$

v) $A = \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}$

$\boxed{\inf A = 0}$

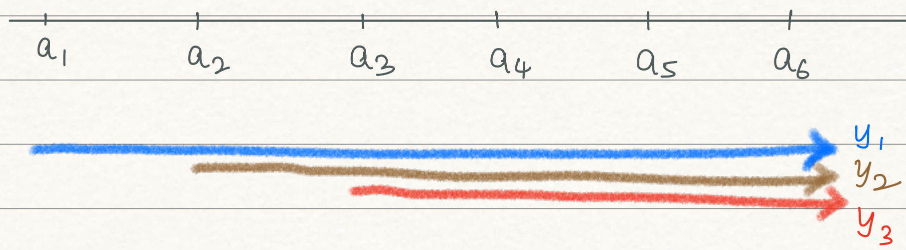
<u>Limit supremum & Limit Infimum of sequences</u>

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

<u>Def^n (Limit supremum)</u>

Limit supremum of $\{a_n\}_{n=1}^{\infty}$ is a number $\underline{L}$ s.t.

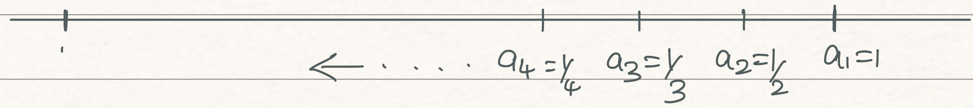
$$\boxed{\overline{L} = \inf_{n \geq 1} \sup_{k \geq n} a_k}$$

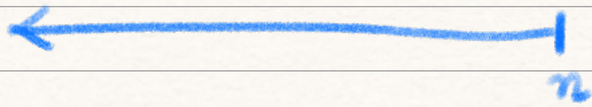


$$y_n = \sup_{k \geq n} a_k$$

$$\boxed{\underline{\text{Notation}}: \quad \limsup_{n \to \infty} a_n}$$

<u>Eg</u>: $a_n = \frac{1}{n} \quad \forall \; n \in \mathbb{N}$





$$y_n = \sup_{k \geq n} a_k = \frac{1}{n}$$

$$\limsup_{n \to \infty} a_n = \inf_{n \geq 1} y_n = \inf \left\{ \frac{1}{n} : n \geq 1 \right\} = 0.$$

Formally, we say that

$$\limsup_{n\to\infty} a_n = \bar{L}$$

if:

$\forall \varepsilon > 0, \ \exists N_\varepsilon \in \mathbb{N}$ s.t.

$$a_n < \bar{L} + \varepsilon \quad \forall \ n \geq N_\varepsilon .$$

$$y_n = \sup_{k \geq n} a_k$$

$$\limsup_{n\to\infty} a_n = \inf_{n \geq 1} y_n$$

$$\inf_{n \geq 1} y_n = \bar{L}$$

i) $y_n \geq \bar{L} \quad \forall \ n \geq 1$

ii) $\forall \varepsilon > 0, \ \exists N_\varepsilon$ s.t.

$$y_N < \bar{L} + \varepsilon$$

$$\sup_{k \geq N_\varepsilon} a_k$$

$$\Rightarrow a_k < \bar{L} + \varepsilon \quad \forall \ k \geq N_\varepsilon$$

Def^n (Limit infimum)

The limit infimum of a sequence $\{a_n\}_{n=1}^{\infty}$ is a number $\underline{L}$ s.t.

$$\boxed{\underline{L} = \sup_{n\geq 1} \inf_{k\geq n} a_k}$$

Formally, $\liminf\limits_{n\to\infty} a_n = \underline{L}$ if

$\forall \varepsilon > 0, \ \exists N_\varepsilon \in \mathbb{N}$ s.t.

$$a_n > \underline{L} - \varepsilon \quad \forall n \geq N_\varepsilon$$

Result:

$$\boxed{\begin{array}{l} \inf A \leq \sup A \quad \forall A \neq \emptyset \\ \liminf\limits_{n\to\infty} a_n \leq \limsup\limits_{n\to\infty} a_n \end{array}}$$

Limit:

We say that $\{a_n\}_{n=1}^{\infty}$ admits a limit, say L, if

$$\boxed{\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L}$$

Mathematically, we say

$$\lim_{n\to\infty} a_n = L$$

if:

$\forall \varepsilon > 0, \ \exists N_\varepsilon \in \mathbb{N}$ s.t.

$|a_n - L| < \varepsilon$

$$L - \varepsilon < a_n < L + \varepsilon \quad \forall n \geq N_\varepsilon .$$