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Stochastic Processes

Borel–Cantelli Lemma, Almost-Sure Convergence, Mean-Squared Convergence, Convergence in Probability, Convergence in Distribution, Examples

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Dedication

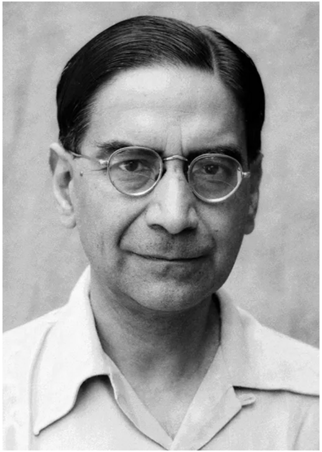

Figure: Dr. Prasanta Chandra Mahalanobis, FNA, FASc, FRS (1893-1972).

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Borel–Cantelli Lemma and Almost-Sure Convergence

Borel–Cantelli Lemma

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma (Borel–Cantelli Lemma)

1. Suppose $A_1, A_2, \ldots \in \mathscr{F}$ are such that $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < +\infty$. Then,

$$\mathbb{P}\left(A_n \text{ i.o. }\right) = 0.$$

2. Suppose $A_1, A_2, \ldots \in \mathscr{F}$ are independent and satisfy $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = +\infty$. Then,

$$\mathbb{P}\left(A_n \text{ i.o. }\right) = 1.$$

For the proof, see [Grimmett and Stirzaker, 2020, Ch. 7, Sec. 7.3].

Almost-Sure Convergence and A_n i.o.

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Proposition

The following statements are equivalent.

1. $X_n \xrightarrow{\text{a.s.}} X$.
2. For every $\varepsilon > 0$,
$$\mathbb{P}(|X_n - X| \geq \varepsilon \ \text{ i.o. }) = 0.$$

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Proof of Proposition

$$X_n \xrightarrow{\text{a.s.}} X \quad \implies \quad \mathbb{P}\left(\bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < q\} \right) = 1$$

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$$X_n \xrightarrow{\text{a.s.}} X \implies \mathbb{P}\left(\bigcap_{q\in\mathbb{Q}_+} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < q\}\right) = 1$$

$$\implies \mathbb{P}\left(\bigcup_{q\in\mathbb{Q}_+} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{|X_n - X| \geq q\}\right) = 0$$

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Proof of Proposition

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X_n \xrightarrow{\text{a.s.}} X \quad &\implies \quad \mathbb{P}\left(\bigcap_{q\in\mathbb{Q}_+} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < q\}\right) = 1 \\
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X_n \xrightarrow{\text{a.s.}} X \quad &\iff \quad \mathbb{P}\left(\bigcap_{q \in \mathbb{Q}_+} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{|X_n - X| < q\} \right) = 1 \\
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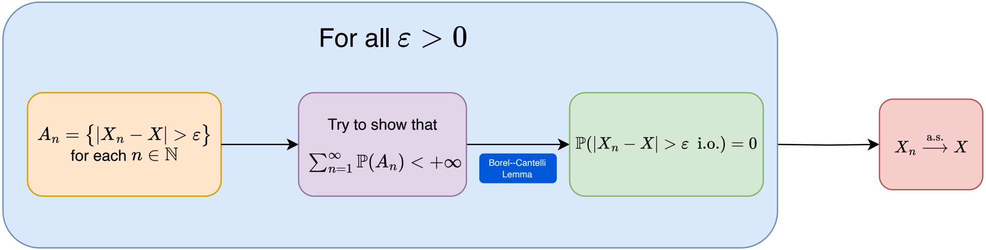
Borel–Cantelli Lemma and Almost-Sure Convergence

A Generic Template



Borel–Cantelli Lemma and Almost-Sure Convergence

- For each $n \in \mathbb{N}$, let
$$\mathbb{P}(X_n = 1) = \frac{1}{n^2} = 1 - \mathbb{P}(X_n = 0).$$
Identify an almost-sure limit.
- For each $n \in \mathbb{N}$, let
$$\mathbb{P}(X_n = 1) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0).$$
Furthermore, suppose that $X_1, X_2, \ldots$ are mutually independent.
What can we say about the convergence of the above sequence?

Borel–Cantelli Lemma and Almost-Sure Convergence

Determining Bias of Coin

- Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathrm{Ber}(p)$ for a fixed $p \in (0, 1)$.
  For each $n \in \mathbb{N}$, let S_n be defined as

$$S_n = \sum_{i=1}^{n} X_i$$

Borel–Cantelli Lemma and Almost-Sure Convergence

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  For each $n \in \mathbb{N}$, let S_n be defined as

$$S_n = \sum_{i=1}^{n} X_i = \#\text{ heads in the first } n \text{ tosses.}$$

Show that $\frac{S_n}{n} \xrightarrow{\text{a.s.}} p$ (the constant random variable which takes the value p).

Examples

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.

- Moving Rectangles
  Let $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.
  $X_1 = \mathbf{1}_{[0,1]}$
  $X_2 = \mathbf{1}_{[0,\frac{1}{2}]}, \quad X_3 = \mathbf{1}_{[\frac{1}{2},1]}$
  $X_4 = \mathbf{1}_{[0,\frac{1}{4}]}, \quad X_5 = \mathbf{1}_{[\frac{1}{4},\frac{1}{2}]}, \quad X_6 = \mathbf{1}_{[\frac{1}{2},\frac{3}{4}]}, \quad X_7 = \mathbf{1}_{[\frac{3}{4},1]},$ and so on.

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Note

- There is no pointwise limit or almost-sure limit for the above sequence.

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Note

- There is no pointwise limit or almost-sure limit for the above sequence.
- However, we observe that $\mathbb{P}(X_n = 0) \approx 1$ for large n.
  In what sense is the constant RV 0 a limit here?

Other Notions of Convergence

Convergence in Probability

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Convergence in Probability)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X in probability (p.) if

$$\forall \varepsilon > 0, \qquad \lim_{n\to\infty} \mathbb{P}(\{|X_n - X| > \varepsilon\}) = 0.$$

Notation:

$$X_n \xrightarrow{\text{p.}} X.$$

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Note

The in-probability limit is only specified up to sets of zero probability. That is,

$$X_n \xrightarrow{\text{p.}} X, \;\; X_n \xrightarrow{\text{p.}} Y \quad \Longrightarrow \quad \mathbb{P}(X = Y) = 1.$$

Mean-Squared Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Mean-Squared Convergence)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X in mean-squared (m.s.) sense if

- $\mathbb{E}[X_n^2] < +\infty$ for all $n \in \mathbb{N}$.
- We have

$$\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0.$$

Notation:

$$X_n \xrightarrow{\text{m.s.}} X.$$

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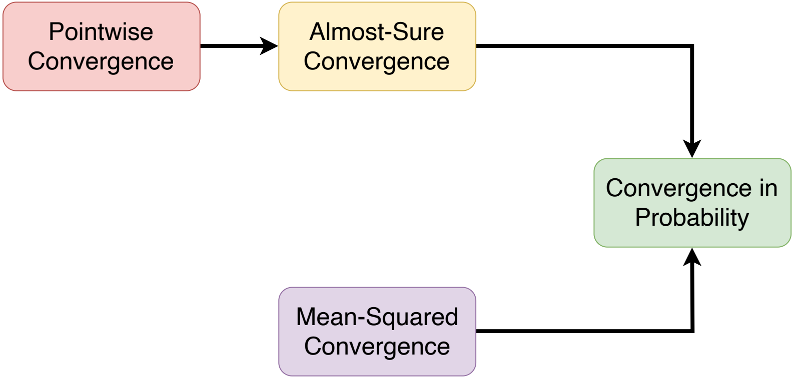
$$X_n \xrightarrow{\text{m.s.}} X, \quad X_n \xrightarrow{\text{m.s.}} Y \qquad \Longrightarrow \qquad \mathbb{P}(X = Y) = 1.$$

A Picture to Have in Mind



(proof of the implications to come later)

Example

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.
Fix a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$.
For each $n \in \mathbb{N}$, let

$$X_n(\omega) = \begin{cases} a_n, & \omega \in \left[0, \frac{1}{n}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Identify the forms of convergence and corresponding limit RVs.

Some Remarks on the Previous Example

- If $a_n = n$, then

$$X_n \xrightarrow{\text{a.s.}} 0, \quad \text{but} \quad X_n \overset{\text{m.s.}}{\not\longrightarrow} 0.$$

Some Remarks on the Previous Example

- If $a_n = n$, then
$$X_n \xrightarrow{\text{a.s.}} 0, \quad \text{but} \quad X_n \overset{\text{m.s.}}{\not\to} 0.$$
- Although $X_n \xrightarrow{\text{a.s.}} 0$, the value of $|X_n(\omega) - 0|$ can be arbitrarily large for some ω. This can lead to a large value for $\mathbb{E}[(X_n - X)^2]$ as $n \to \infty$.

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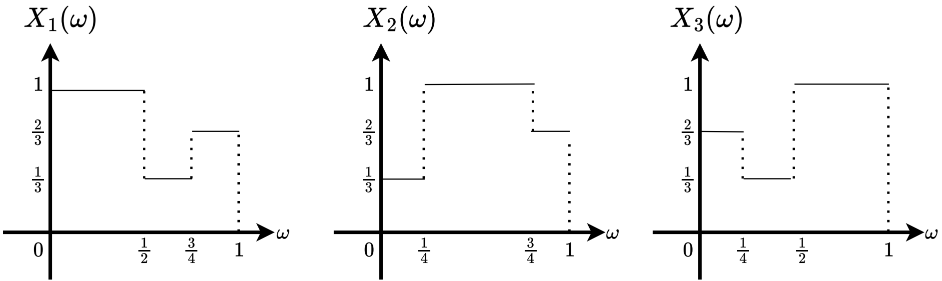
Almost-Sure Convergence and Mean-Squared Convergence

In general,
$$X_n \xrightarrow{\text{a.s.}} 0 \quad \not\Rightarrow \quad X_n \xrightarrow{\text{m.s.}} 0, \qquad X_n \xrightarrow{\text{m.s.}} 0 \quad \not\Rightarrow \quad X_n \xrightarrow{\text{a.s.}} 0.$$

Example

Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \mathrm{Unif})$.



Let $X_n = X_{n+3}$ for all $n \in \mathbb{N}$.
Identify forms of convergence and their corresponding limits.

Remarks on Previous Example

- The sequence of RVs do not converge pointwise, almost-surely, in mean-squared sense, or in probability
- However, the PMFs (hence CDFs) of X_1, X_2, X_3 are identical, hence there is convergence of CDFs

Convergence in Distribution

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_n\}_{n=1}^{\infty}$ and X be defined w.r.t. $\mathscr{F}$.

Definition (Convergence in Distribution)

We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X in distribution (d.) if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \qquad \forall x \in C_{F_X},$$

where C_{F_X} denotes the points of continuity of F_X.

Notation:

$$X_n \xrightarrow{\text{d.}} X.$$

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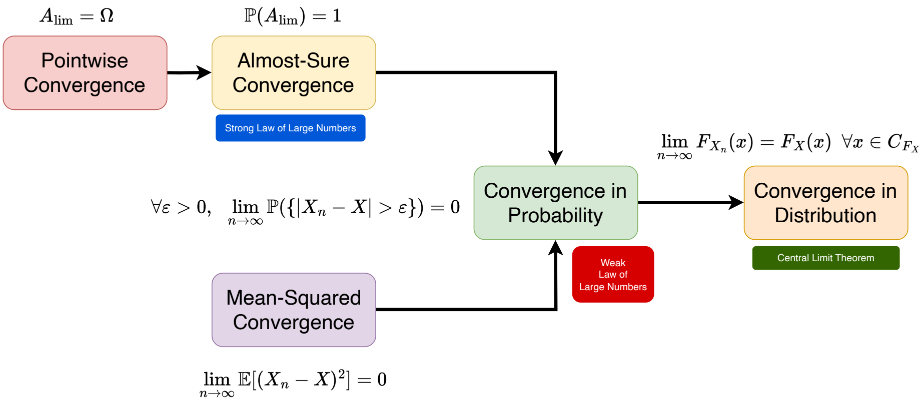
Convergence – The Full Picture

$$A_{\text{lim}} = \left\{ \omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega) \right\}$$



References

Grimmett, G. and Stirzaker, D. (2020). *Probability and random processes*. Oxford university press.