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Probability and Stochastic Processes

Lecture 13: CDFs (contd.), Probability Mass Function (PMF), Discrete Random Variable, Examples of Discrete RVs

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Probability Law of a Random Variable

Definition (Probability Law)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable (with respect to $\mathscr{F}$).
The **probability law** of X is a function $\mathbb{P}_X : \mathscr{B}(\mathbb{R}) \to [0, 1]$ defined as

$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)), \qquad B \in \mathscr{B}(\mathbb{R}).$$

Remarks:

- $\mathbb{P}_X$ is sometimes referred to the **pushforward** of $\mathbb{P}$ under the random variable X
- $\mathbb{P}_X$ is sometimes denoted as $\mathbb{P} \circ X^{-1}$

Proposition (Probability Law)

$\mathbb{P}_X$ is a **probability measure** on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Completing the Picture

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R}), {\color{red}\mathbb{P}_X}) \arrow[d] \\
X^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}
$$

$$
{\color{red}\mathbb{P}_X}(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})
$$

Figure: Pictorial representation of probability law

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Cumulative Distribution Function (CDF)

Definition (Cumulative Distribution Function (CDF))

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.
The function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(x) = \mathbb{P}_X\big((-\infty, x]\big) = \mathbb{P}\big(\{\omega \in \Omega : X(\omega) \leq x\}\big) = \mathbb{P}\big(\{X \leq x\}\big), \qquad x \in \mathbb{R},$$

is called the **cumulative distribution function (CDF)** of X.

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & & (-\infty, x] \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}$$

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Properties of CDF

Lemma (Properties of CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable with CDF F_X.
Then, F_X satisfies the following properties.

1. **(Monotonicity)** If $x \leq y$, then $F_X(x) \leq F_X(y)$.
2. If $x_1, x_2, \ldots$ is any sequence such that $\lim_{n\to\infty} x_n = -\infty$, then $\lim_{n\to\infty} F_X(x_n) = 0$.
3. If $x_1, x_2, \ldots$ is any sequence such that $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} F_X(x_n) = 1$.
4. **(Right-Continuity)**

   F_X is right-continuous at every point in its domain.
   More formally, for each $x \in \mathbb{R}$,

$$x_n > x \ \forall\, n \in \mathbb{N}, \qquad \lim_{n\to\infty} x_n = x \qquad \Longrightarrow \qquad \lim_{n\to\infty} F_X(x_n) = F_X(x).$$

Example

- Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Fix $A \in \mathscr{F}$, and suppose that $\mathbb{P}(A) = p$ for some $p \in [0, 1]$. Plot the CDF of the random variable $\mathbf{1}_A$.

Tidbits

- CDF can have jumps
- At any jump, the value of the CDF is equal to the value obtained by approaching from the right of jump point **(right-continuity)**
- CDF is right-continuous at every domain point. **What about left-continuity?**
- Fix $x \in \mathbb{R}$. Let $x_1, x_2, \ldots$ be a sequence such that

$$x_1 \leq x_2 \leq \cdots, \qquad x_n < x \;\forall\, n \in \mathbb{N}, \qquad \lim_{n\to\infty} x_n = x.$$

- Observe that

$$(-\infty, x_1] \subseteq (-\infty, x_2] \subseteq \cdots, \qquad \bigcup_{n\in\mathbb{N}} (-\infty, x_n] = (-\infty, x),$$

$$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbb{P}_X((-\infty, x_n]) = \mathbb{P}_X\left(\bigcup_{n\in\mathbb{N}} (-\infty, x_n]\right) = \mathbb{P}_X((-\infty, x)) \neq F_X(x).$$

- CDF **need not be left-continuous** at a domain point $x \in \mathbb{R}$

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Tidbits

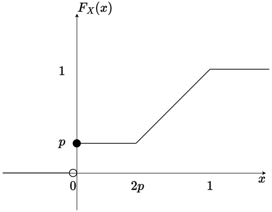
- If for some domain point $x \in \mathbb{R}$,
$$\mathbb{P}_X\big((-\infty, x)\big) = F_X(x),$$
then F_X is said to be **left-continuous at the point x**
- If a CDF is both right-continuous and left-continuous at a domain point $x \in \mathbb{R}$, it is said to be **continuous** at point x
- If a CDF is continuous at a domain point $x \in \mathbb{R}$, then
$$\mathbb{P}_X\big((-\infty, x]\big) = \mathbb{P}_X\big((-\infty, x)\big) \implies \mathbb{P}_X(\{x\}) = 0 \implies \mathbb{P}(\{X = x\}) = 0,$$
$$\mathbb{P}(\{X = x\}) = 0 \implies \mathbb{P}_X(\{x\}) = 0 \implies \mathbb{P}_X\big((-\infty, x]\big) = \mathbb{P}_X\big((-\infty, x)\big)$$

A Point to Always Remember

F_X is continuous at a point $x \in \mathbb{R}$ $\iff$ $\mathbb{P}\big(\{X = x\}\big) = 0.$

Example

- Suppose that X has the below CDF with $p = \frac{1}{4}$.



1. What is $\mathbb{P}(\{X = 0\})$?
2. What is $\mathbb{P}_X\left(\left(\frac{1}{4}, \frac{1}{2}\right] \right)$?
3. What is $\mathbb{P}_X(\mathbb{Q} \cap [0, 1])$?

Another Important Function

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

- Taking $B = (-\infty, x]$, and varying x, we get a mapping

$$x \quad \mapsto \quad \mathbb{P}_X\big((-\infty, x]\big)$$

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Another Important Function

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$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

- Taking $B = (-\infty, x]$, and varying x, we get a mapping

$$x \quad \mapsto \quad \mathbb{P}_X\big((-\infty, x]\big)$$

- Taking $B = \{x\}$, and varying x, we get a mapping

$$x \quad \mapsto \quad \mathbb{P}_X\big(\{x\}\big)$$

Another Important Function

$$\begin{tikzcd} (\Omega,\mathscr{F},\mathbb{P}) \arrow[r, "X"] & (\mathbb{R},\mathscr{B}(\mathbb{R}),{\color{red}\mathbb{P}_X}) \arrow[d] \\ X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"'] \end{tikzcd}$$

$${\color{red}\mathbb{P}_X}(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

- Taking $B = (-\infty, x]$, and varying x, we get a mapping

$$x \quad \mapsto \quad \mathbb{P}_X\big((-\infty, x]\big)$$

- The above map is called the **cumulative density function (CDF)**, denoted F_X

- Taking $B = \{x\}$, and varying x, we get a mapping

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$$ {\color{red}\mathbb{P}_X}(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}) $$

- Taking $B = (-\infty, x]$, and varying x, we get a mapping

$$ x \quad \mapsto \quad \mathbb{P}_X\big((-\infty, x]\big) $$

- The above map is called the **cumulative density function (CDF)**, denoted F_X

- Taking $B = \{x\}$, and varying x, we get a mapping

$$ x \quad \mapsto \quad \mathbb{P}_X\big(\{x\}\big) $$

- The above map is called the **probability mass function (PMF)**, denoted p_X

Another Important Function

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
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$$\mathbb{P}_X(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})$$

- Taking $B = (-\infty, x]$, and varying x, we get a mapping

$$x \quad \mapsto \quad \mathbb{P}_X\big((-\infty, x]\big)$$

- The above map is called the **cumulative density function (CDF)**, denoted F_X
- $F_X(x) = \mathbb{P}_X\big((-\infty, x]\big) = \mathbb{P}(\{X \leq x\})$

- Taking $B = \{x\}$, and varying x, we get a mapping

$$x \quad \mapsto \quad \mathbb{P}_X(\{x\})$$

- The above map is called the **probability mass function (PMF)**, denoted p_X

Another Important Function

$$\begin{tikzcd} (\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\ X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"'] \end{tikzcd}$$

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- The above map is called the **probability mass function (PMF)**, denoted p_X
- $p_X(x) = \mathbb{P}_X(\{x\}) = \mathbb{P}(\{X = x\})$

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Probability Mass Function

Definition (Probability Mass Function)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable.
Let $\mathbb{P}_X$ denote the probability law of X.
The **probability mass function** of X is a function $p_X : \mathbb{R} \to [0, 1]$ defined as

$$p_X(x) = \mathbb{P}_X(\{x\}) = \mathbb{P}(\{X = x\}), \qquad x \in \mathbb{R}.$$

- **CDF (F_X) and PMF (p_X) are always defined for any random variable X**
- It is **always possible** to go from $\mathrm{CDF} \longrightarrow \mathrm{PMF}$:

  In terms of $\mathbb{P}_X$: $\quad p_X(x) = \mathbb{P}_X(\{x\}) = \mathbb{P}_X\big((-\infty, x]\big) - \mathbb{P}_X\big((-\infty, x)\big) = F_X(x) - \mathbb{P}_X\big((-\infty, x)\big).$

  In terms of $\mathbb{P}$: $\quad p_X(x) = \mathbb{P}(\{X = x\}) = \mathbb{P}\big(\{X \leq x\}\big) - \mathbb{P}_X\big(\{X < x\}\big) = F_X(x) - \mathbb{P}\big(\{X < x\}\big).$

- F_X is continuous at point x if and only if $p_X(x) = 0$
- In general, $\quad \mathrm{PMF} \not\longrightarrow \mathrm{CDF}$

Types of Random Variables

Types of Random Variables

- Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
  Let $X : \Omega \to \mathbb{R}$ be a random variable (RV)

- Depending on the nature of the probability law $\mathbb{P}_X$, the RV X may be categorized majorly into one of following three types:

  1. **Discrete** RV
  2. **Continuous** RV
  3. **Singular** RV

Discrete Random Variables

Discrete Random Variable

Definition (Discrete Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **discrete** if there exists a **countable** set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that

$$\mathbb{P}_X(E) = 1.$$

- By countable additivity,

$$1 = \mathbb{P}_X(E) = \mathbb{P}_X\left(\bigsqcup_{i \in \mathbb{N}} \{e_i\}\right) = \sum_{i \in \mathbb{N}} \mathbb{P}_X(\{e_i\}) = \sum_{i \in \mathbb{N}} p_X(e_i).$$

- For any Borel set $B \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}_X(B) = \mathbb{P}_X(B \cap E) = \sum_{i:\ e_i \in B} \mathbb{P}_X(\{e_i\}) = \sum_{i:\ e_i \in B} p_X(e_i).$$

Discrete Random Variable

Definition (Discrete Random Variable)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable (RV).
Let $\mathbb{P}_X$ denote the probability law of X.
The RV X is said to be **discrete** if there exists a **countable** set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that

$$\mathbb{P}_X(E) = 1.$$

PMF $\longrightarrow$ CDF for a Discrete RV

The following implications are noteworthy:

$$p_X \quad \underset{\text{any } X}{\overset{X \text{ discrete}}{\rightleftarrows}} \quad \mathbb{P}_X \quad \underset{\text{any } X}{\overset{\text{any } X}{\rightleftarrows}} \quad F_X.$$

PMF $=$ complete probabilistic description for discrete RV.

Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $p \in [0, 1]$.
  X is said to be **Bernoulli distributed** with parameter p if:

$$E = \{0, 1\}, \qquad p_X(x) = \begin{cases} 1 - p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

  In this case, the CDF F_X is given by

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - p, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

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Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $n \in \mathbb{N}$.
  X is said to be **Uniformly distributed** on $E = \{e_1, \ldots, e_n\}$ if

$$E = \{e_1, \ldots, e_n\}, \qquad p_X(x) = \frac{1}{n} \quad \forall x \in E.$$

In this case, the CDF F_X is given by (assuming WLOG $e_1 < e_2 < \cdots < e_n$)

$$F_X(x) = \begin{cases} 0, & x < e_1, \\ \frac{1}{n}, & e_1 \leq x < e_2, \\ \frac{2}{n}, & e_2 \leq x < e_3, \\ \vdots & \\ 1, & x \geq e_n. \end{cases}$$

Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $p \in [0, 1]$.
  X is said to be **Geometric** with parameter p if

$$E = \{1, 2, \ldots\}, \qquad p_X(k) = (1-p)^{k-1}\, p \quad \forall k \in E.$$

  In this case, the CDF F_X is given by

$$\forall\, x \in \mathbb{R}, \qquad F_X(x) = \sum_{\substack{k \in \mathbb{N}: \\ k \leq x}} p_X(k).$$

Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $\lambda > 0$.
  X is said to be **Poisson distributed** with parameter λ if

$$E = \{0, 1, 2, \ldots\}, \qquad p_X(k) = \exp(-\lambda)\,\frac{\lambda^k}{k!} \quad \forall k \in E.$$

  In this case, the CDF F_X is given by

$$\forall\, x \in \mathbb{R}, \qquad F_X(x) = \sum_{\substack{k \in E: \\ k \leq x}} p_X(k).$$

Examples

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.

- Fix $n \in \mathbb{N}$ and $p \in [0, 1]$.
  X is said to be **Binomial distributed** with parameters (n, p) if:

$$E = \{0, 1, \ldots, n\}, \qquad p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \forall k \in E.$$

  In this case, the CDF F_X is given by

$$\forall\, x \in \mathbb{R}, \qquad F_X(x) = \sum_{\substack{k \in E: \\ k \leq x}} p_X(k).$$