Best Arm Identification with Limited Precision Sampling

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Joint Work With

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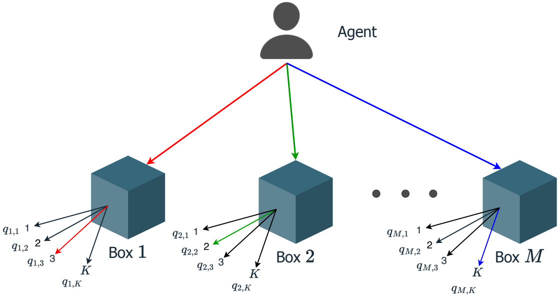
Problem Setup



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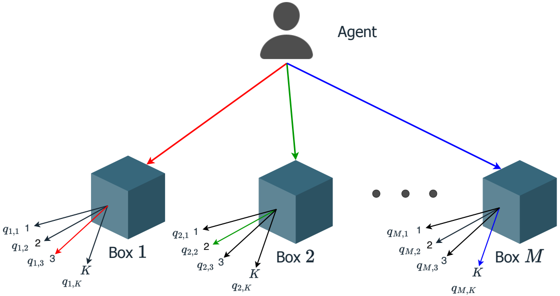

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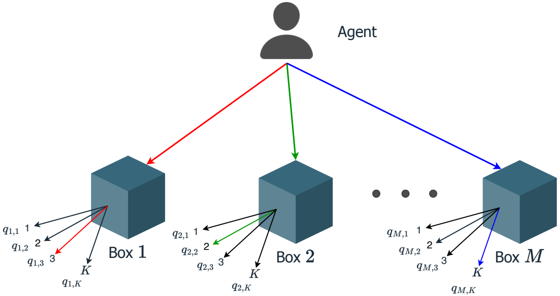

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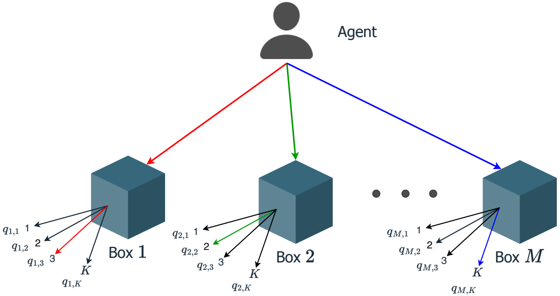

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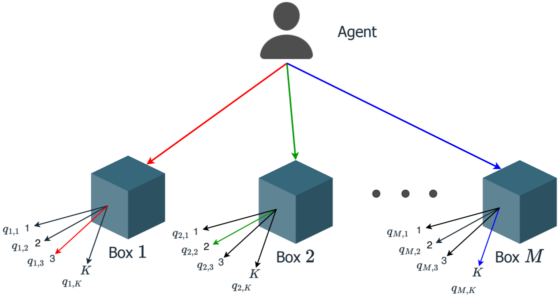

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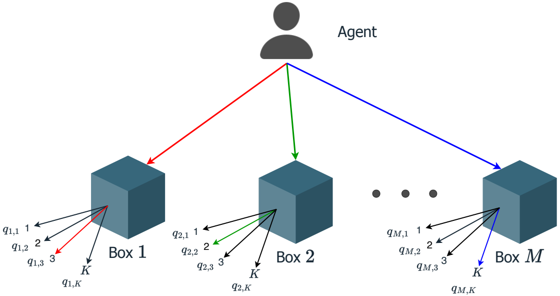

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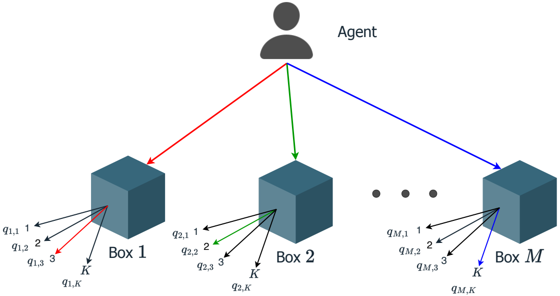

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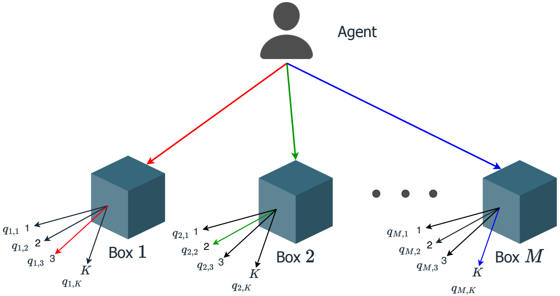

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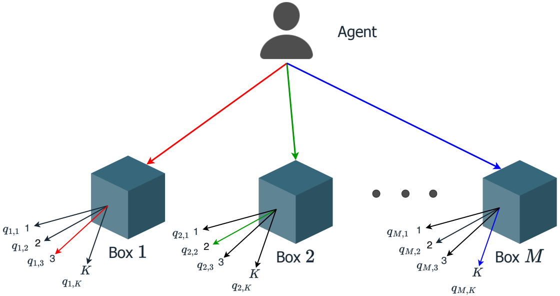

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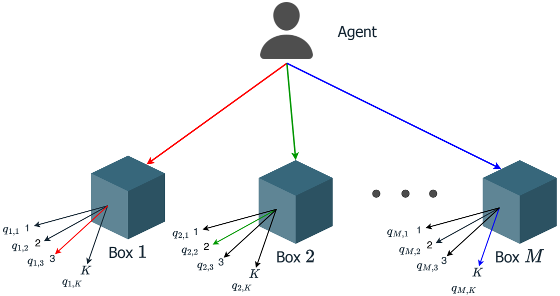

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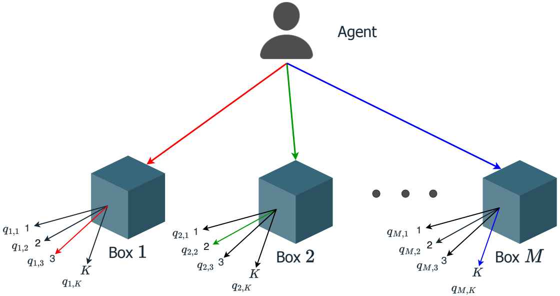

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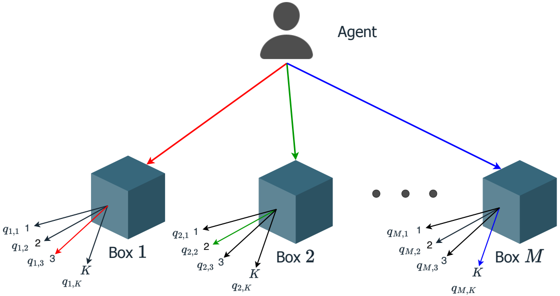

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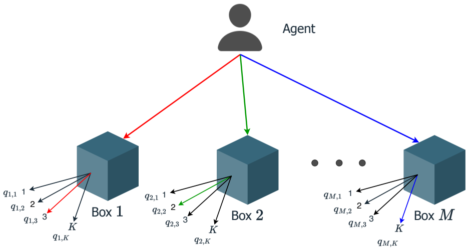

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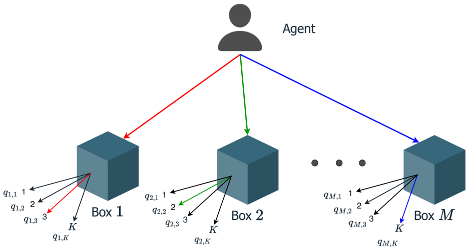

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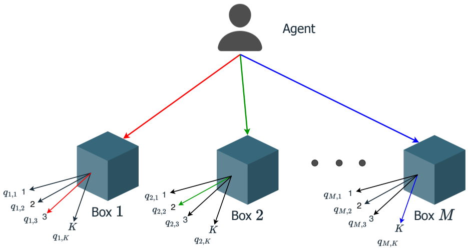

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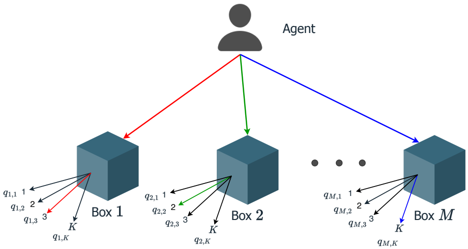

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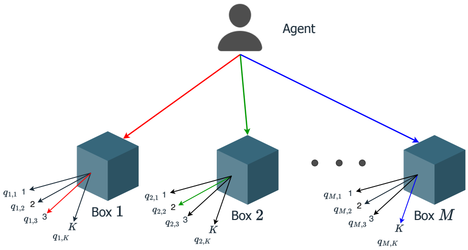

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  - If $\boldsymbol{q} = \{q_{m,k}\}_{m,k}$ is unknown, then?
    If $\{q_{m,k}\}_k = \{q_{m'k}\}_k$ for all m, m', then every box weight is optimal

Outline

Asymptotic Analysis

Converse

Fix a problem instance $\boldsymbol{q}_0 = \{q_{m,k}^0\}_{m,k}$, $\boldsymbol{\mu}_0 = \{\mu_k^0\}_k$

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Fix a problem instance $\boldsymbol{q}_0 = \{q^0_{m,k}\}_{m,k}$, $\boldsymbol{\mu}_0 = \{\mu^0_k\}_k$

Theorem

$$\liminf_{\delta \downarrow 0} \inf_{\pi \ \delta\text{-}PC} \frac{\mathbb{E}[\tau_\pi]}{\log(1/\delta)} \geq \frac{1}{T^*(\boldsymbol{q}_0, \boldsymbol{\mu}_0)},$$

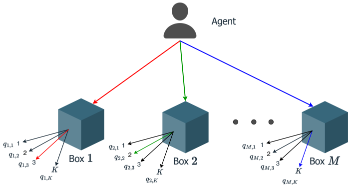
where $T^*(\boldsymbol{q}_0, \boldsymbol{\mu}_0)$ is given by

$$T^*(\boldsymbol{q}_0, \boldsymbol{\mu}_0) = \sup_{\mathbf{w} \in \Sigma_M} \inf_{\boldsymbol{\lambda} \in ALT(\boldsymbol{\mu}_0)} \sum_{m=1}^{M} \sum_{k=1}^{K} w_m \, q^0_{m,k} \, \frac{(\mu^0_k - \lambda_k)^2}{2}.$$

The supremum is over $\Sigma_M = \{\mathbf{w} = (w_1, \ldots, w_M) : \; w_m \geq 0 \;\; \forall m, \;\; \sum_{m=1}^{M} w_m = 1\}$.

- From transportation Lemma 1 of Kaufmann et al. [2016],

$$\pi\ \delta\text{-PC} \implies \inf_{\boldsymbol{\lambda} \in \mathsf{ALT}(\boldsymbol{\mu}_0)} \sum_{k=1}^{K} \underbrace{\mathbb{E}[N_k(\tau_\pi)]}_{\#\text{ pulls of arm } k} \frac{(\mu_k^0 - \lambda_k)^2}{2} \geq D_{\mathsf{KL}}(\mathsf{Ber}(\delta) \| \mathsf{Ber}(1-\delta)).$$



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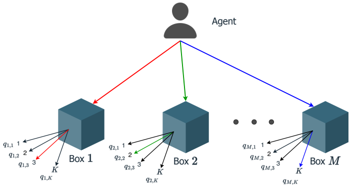
- For each $k\in[K]$,

$$\mathbb{E}[N_k(\tau_\pi)] = \sum_{m=1}^{M} q_{m,k}^0 \underbrace{\mathbb{E}[N(\tau_\pi,m)]}_{\#\text{ selections of box } m}.$$



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Non-Uniqueness of Optimal Box Weights

Example (1)

$\{q^0_{m,k}\}_k$ independent of m, i.e., $\{q^0_{m,k}\}_k = \{q^0_{m',k}\}_k = \{\alpha_k\}_k$ for all m, m'. In this case,

$$\sum_{m=1}^{M} w_m \, q^0_{m,k} = \sum_{m=1}^{M} w_m \, \alpha_k = \alpha_k \quad \forall k \in [K], \; \mathbf{w} \in \Sigma_M.$$

Example (2)

$M = 2$, $K = 4$, $\boldsymbol{\mu}_0 = \{0.5, 0.4, 0.3, 0.3\}$, $\boldsymbol{q}_0 = \begin{pmatrix} 0.3 & 0.3 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.1 & 0.3 \end{pmatrix}$

For the above examples, every $\mathbf{w} \in \Sigma_M$ is optimal

Achievability: Preliminaries

- Set of optimal box weights under $(\boldsymbol{q}, \boldsymbol{\mu})$:

$$\mathcal{W}^\star(\boldsymbol{q}, \boldsymbol{\mu}) = \arg\sup_{w \in \Sigma_M} \inf_{\boldsymbol{\lambda} \in \textsc{Alt}(\boldsymbol{\mu})} \sum_{m=1}^{M} \sum_{k=1}^{K} w_m \, q_{m,k} \, \frac{(\mu_k - \lambda_k)^2}{2}$$

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- $(\boldsymbol{q}, \boldsymbol{\mu}) \mapsto \mathcal{W}^\star(\boldsymbol{q}, \boldsymbol{\mu})$ is compact-valued and upper hemicontinuous
- $\mathcal{W}^\star(\boldsymbol{q}, \boldsymbol{\mu})$ is convex for each $(\boldsymbol{q}, \boldsymbol{\mu})$

Achievability: Modified D-Tracking (1)

- Parameter estimates at time t:

$$\hat{q}_{m,k}(t) = \frac{\text{\# times box } m \text{ selected and arm } k \text{ pulled}}{N(t,m)}, \quad \hat{\mu}_k(t) = \frac{1}{N_k(t)} \sum_{s=1}^{t} \mathbf{1}_{\{A_s=k\}} X_s$$

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- $\underbrace{i_0 = 0, \quad i_{t+1} = (i_t \bmod M) + \mathbf{1}_{\{\min_{m \in [M]} N(t,m) < f(t)\}} \quad \text{for all } t \geq 0}_{\text{round-robin box selection counter}}$

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- Let $\{\mathbf{w}(t) : t \geq 1\}$ be such that $\mathbf{w}(t+1) \in \mathcal{W}^\star(\hat{\boldsymbol{q}}(t), \hat{\boldsymbol{\mu}}(t))$ for all t

Achievability: Modified D-Tracking (2)

- The modified D-Tracking rule:

$$B_{t+1} = \begin{cases} i_t, & \min_{m \in [M]} N(t,m) < f(t), \\ b_t, & \text{otherwise}, \end{cases}$$

where $\{b_t : t \geq 1\}$ is specified by

$$b_t = \arg \min_{m \in \text{supp}(\sum_{s=1}^{t} w(s))} N(t,m) - \sum_{s=1}^{t} w_m(s).$$

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- Inspired from Jedra and Proutiere [2020]

Tracking the Optimal Set

- Define $d_\infty(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i|, \quad d_\infty(\mathbf{x}, C) = \min_{y \in C} d_\infty(x, y)$

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Under the modified D-tracking rule,

$$\lim_{t \to \infty} d_\infty\left((N(t,m)/t)_{m \in [M]},\ \mathcal{W}^\star(\boldsymbol{q}_0, \boldsymbol{\mu}_0)\right) = 0 \quad a.s..$$

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- Inspired by Degenne and Koolen [2019], the key idea in the proof is to track the behaviour of $\bar{\mathbf{w}}(t) = \frac{1}{t}\sum_{s=1}^{t} \mathbf{w}(s) \in \mathcal{W}^\star(\hat{\boldsymbol{q}}(t-1), \hat{\boldsymbol{\mu}}(t-1))$

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- When $\mathcal{W}^\star(\boldsymbol{q}_0,\boldsymbol{\mu}_0) = \{\mathbf{w}^\star\}$, we recover the classical tracking result

$$\frac{N(t,m)}{t} \xrightarrow{t\to\infty} w_m^\star \quad \forall m, \ \text{a.s.}.$$

Stopping & Recommendation Rules

- The GLLR statistic between arms $a, b \in [K]$ at time t is

$$Z_{a,b}(t) = \begin{cases} N_a(t)\frac{\left(\hat{\mu}_a(t)-\hat{\mu}_{a,b}(t)\right)^2}{2} + N_b(t)\frac{\left(\hat{\mu}_b(t)-\hat{\mu}_{a,b}(t)\right)^2}{2}, & \hat{\mu}_a(t) \geq \hat{\mu}_b(t), \\ -Z_{b,a}(t), & \text{otherwise,} \end{cases}$$

where $\hat{\mu}_{a,b}(t) = \frac{N_a(t)}{N_a(t)+N_b(t)}\,\hat{\mu}_a(t) + \frac{N_b(t)}{N_a(t)+N_b(t)}\,\hat{\mu}_b(t)$.

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- Let $Z(t) = \max_a \min_{b \neq a} Z_{a,b}(t)$
- Given $\delta \in (0,1)$, let $\beta(t,\delta,\rho) = \log \frac{C\,t^{1+\rho}}{\delta}$, where C is a predetermined constant

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$$Z_{a,b}(t) = \begin{cases} N_a(t)\frac{\left(\hat{\mu}_a(t)-\hat{\mu}_{a,b}(t)\right)^2}{2} + N_b(t)\frac{\left(\hat{\mu}_b(t)-\hat{\mu}_{a,b}(t)\right)^2}{2}, & \hat{\mu}_a(t) \geq \hat{\mu}_b(t), \\ -Z_{b,a}(t), & \text{otherwise}, \end{cases}$$

  where $\hat{\mu}_{a,b}(t) = \frac{N_a(t)}{N_a(t)+N_b(t)}\,\hat{\mu}_a(t) + \frac{N_b(t)}{N_a(t)+N_b(t)}\,\hat{\mu}_b(t)$.
- Let $Z(t) = \max_a \min_{b \neq a} Z_{a,b}(t)$
- Given $\delta \in (0,1)$, let $\beta(t,\delta,\rho) = \log \frac{C\,t^{1+\rho}}{\delta}$, where C is a predetermined constant
- Stopping rule: $\tau_{\delta,\rho} = \min\{t \geq 1 : Z(t) \geq \beta(t,\delta,\rho)$ and $\min_{k\in[K]} N_k(t) > 0\}$

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- Recommendation rule: $\hat{k} = \arg\max_k \hat{\mu}_k(\tau_{\delta,\rho})$

Results

Under the box sampling, stopping, and recommendation rules stated before:

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$$\tau_{\delta,\rho} \leq \frac{1+\rho}{T^\star(\boldsymbol{q}_0, \boldsymbol{\mu}_0)} \log(1/\delta) + o(\log(1/\delta)) \quad a.s..$$

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- *Asymptotic upper bound on* $\mathbb{E}[\tau_{\delta,\rho}]$*:*

$$\limsup_{\delta \downarrow 0} \frac{\mathbb{E}[\tau_{\delta,\rho}]}{\log(1/\delta)} \leq \frac{1+\rho}{T^*(\boldsymbol{q}_0, \boldsymbol{\mu}_0)}.$$

Non-Asymptotic Analysis:

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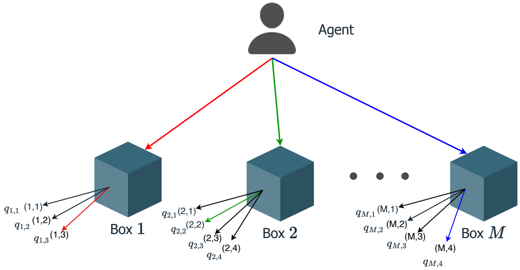
- Simplified setting: arms partitioned across boxes

Simplified Problem Setup: Partition



Goal: fixed-confidence BAI

- Arms partitioned across boxes
- Arm k of box m indexed as $A_{m,k}$ or simply as (m,k)
- $\mathcal{A}_m$: set of arms in box m
- $\sum_{m=1}^{M} |\mathcal{A}_m| = K$
- Agent knows $\mathcal{A}_1, \ldots, \mathcal{A}_M$
- Unknowns:
  - $\boldsymbol{q}_0 = \{q_{m,k}^0\}_{m,k}$
  - $\boldsymbol{\mu}_0 = \{\mu_{m,k}^0\}_{m,k}$
- Best arm: $(m^*, k^*) = \arg\max_{m,k} \mu_{m,k}^0$

Converse

- WLOG, let $(1,1)$ be the best arm
- Let $\Delta_{m,k} = \mu^0_{1,1} - \mu^0_{m,k}$ for all $(m,k) \neq (1,1)$, and $\Delta_{1,1} = \min_{(m,k)\neq(1,1)} \Delta_{m,k}$

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- Let $\Delta_{m,k} = \mu^0_{1,1} - \mu^0_{m,k}$ for all $(m,k) \neq (1,1)$, and $\Delta_{1,1} = \min_{(m,k)\neq(1,1)} \Delta_{m,k}$

Theorem

Under any δ-PC algorithm,

$$\mathbb{E}[\tau_\pi] \geq \log\left(\frac{1}{2.4\,\delta}\right) \cdot \sum_{m=1}^{M} \max_{k\in\mathcal{A}_m} \frac{1}{q^0_{m,k}\,\Delta^2_{m,k}}.$$

- Technique: change-of-measure arguments of Garivier and Kaufmann [2016]

Achievability: Successive Elimination

Notations:

- $t_{m,k}(n)$: # pulls of arm $A_{m,k}$ up to round n
- $\alpha_\delta(x) = \sqrt{\frac{2\ln(8Kx^2/\delta)}{x}}$
- $\mathrm{UCB}_{m,k}(n) = \hat{\mu}_{m,k}(n) + \alpha_\delta(t_{m,k}(n))$
- $\mathrm{LCB}_{m,k}(n) = \hat{\mu}_{m,k}(n) - \alpha_\delta(t_{m,k}(n))$

Select box until each active arm is pulled n times in round n

---

Algorithm 1 Successive Elimination

Input: $K, M, \delta > 0, \mathcal{A}_m$ for $m \in [M]$
Output: $\hat{a} \in [K]$ (best arm).
Initialization: $S = [K], B = [M], S_m = [a_m], n = 0,$
$\hat{\mu}_{m,k}(n) = 0\ \forall k, m,\ S_m = \mathcal{A}_m\ \forall m, t = 0.$

```
while |S| > 1 do
    n ← n + 1
    For each m ∈ B, select box m until every active arm A_{m,k} in box m is pulled at least n times.
    For every box selection, increment t by 1.
    Update t_{m,k}(n), μ̂_{m,k}(n), UCB_{m,k}(n) and LCB_{m,k}(n) for all the active arms.
    if ∃A_{m',k'} ∈ S such that UCB_{m,k}(n) < LCB_{m',k'}(n) then
        S_m ← S_m \ A_{m,k},   S ← ∪_{m∈[M]} S_m,
        B ← {m : S_m ≠ ∅}.
    end if
    if |S| = 1 then
        â ← a ∈ S,   S ← ∅,   B ← ∅.
    end if
end while
return â.
```

Results

Theorem

Fix $\delta \in (0,1)$. With probability greater than $1-\delta$:

- *The SE algorithm outputs the correct best arm*
- *The SE algorithm stops at time $\leq \sum_{m=1}^{M} U_m$, where U_m is a random variable with*

$$P\left(U_m = \max_{k \in \mathcal{A}_m} O\left(\frac{\ln\left(\frac{K}{\delta \Delta_{m,k}}\right)}{q_{m,k}^0 \, \Delta_{m,k}^2}\right)\right) \geq 1 - \frac{\delta |\mathcal{A}_m|}{K}.$$

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- Lower bound $= \Omega\left(\sum_{m=1}^{M} \max_{k \in \mathcal{A}_m} \frac{1}{q_{m,k}^0 \, \Delta_{m,k}^2}\right)$ (order-wise matching in problem unknowns)

In Summary

- Problem studied: BAI with limited precision sampling
- Modified D-tracking algorithm to handle non-unique optimal box weights
- Partition setting: SE algorithm that selects each box until each active arm is pulled n times in round n
- Non-partition setting: SE/LUCB-type algorithm design is an open question

Thank You!

Questions? Hit me up!
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