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Stochastic Processes

Markov Chains

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07 March 2025

Markov Chains

Markov Chain

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Markov Chain)

A process $\{X_t : t \in \mathcal{T}\}$ is called a Markov chain if for any $t \in \mathcal{T}$,

$$(X_s : s < t) \perp\!\!\!\perp (X_s : s > t) \mid X_t,$$

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i.e., for any $m, n \in \mathbb{N}$, $s_1 < \cdots < s_m < t$, $t < t_1 < \cdots < t_n$, $x_1, \ldots, x_m \in \mathbb{R}$, $y_1, \ldots, y_n \in \mathbb{R}$, and $x \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{P}(\underbrace{X_{s_1} \leq x_1, \ldots, X_{s_m} \leq x_m,}_{\text{before } t} \underbrace{X_{t_1} \leq y_1, \ldots, X_{t_n} \leq y_n}_{\text{after } t} \mid X_t \leq x) \\ &\quad = \mathbb{P}(X_{s_1} \leq x_1, \ldots, X_{s_m} \leq x_m \mid X_t \leq x) \cdot \mathbb{P}(X_{t_1} \leq y_1, \ldots, X_{t_n} \leq y_n \mid X_t \leq x). \end{aligned}$$

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$$= \mathbb{P}(X_{s_1} \le x_1, \ldots, X_{s_m} \le x_m \mid X_t \le x) \cdot \mathbb{P}(X_{t_1} \le y_1, \ldots, X_{t_n} \le y_n \mid X_t \le x).$$

In words: given the present value, the past is independent of future.

Discrete-Time Markov Chain Taking Finitely Many Values

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (DTMC taking Finitely Many Values)

Consider a process $\{X_n\}_{n=1}^{\infty}$ taking values in a finite set $\mathcal{X}$.
Then, $\{X_n\}_{n=1}^{\infty}$ is called a discrete time Markov chain (DTMC) on $\mathcal{X}$ if

$$(X_1, \ldots, X_{n-1}) \perp\!\!\!\perp (X_{n+1}, X_{n+2}, \ldots) \mid X_n \qquad \text{for any } n \in \mathbb{N},$$

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i.e., for any $n, L \in \mathbb{N}$, $\;n < t_1 < \cdots < t_L$,
$x_1, \ldots, x_{n-1} \in \mathbb{R}$, $y_1, \ldots, y_L \in \mathbb{R}$, and $x \in \mathbb{R}$,

$$\mathbb{P}(\underbrace{X_1 = x_1, \ldots, X_{n-1} = x_{n-1},}_{\text{before } n} \underbrace{X_{t_1} = y_1, \ldots, X_{t_L} = y_L}_{\text{after } n} \mid X_n = x)$$

$$= \mathbb{P}(X_1 = x_1, \ldots, X_{n-1} = x_{n-1} \mid X_n = x) \cdot \mathbb{P}(X_{t_1} = y_1, \ldots, X_{t_L} = y_L \mid X_n = x).$$

Alternate Viewpoint of DTMC

$$\mathbb{P}(\underbrace{X_{t_1} = y_1, \ldots, X_{t_L} = y_L}_{\text{after } n} \mid X_n = x, \underbrace{X_1 = x_1, \ldots, X_{n-1} = x_{n-1}}_{\text{before } n})$$
$$= \mathbb{P}(X_{t_1} = y_1, \ldots, X_{t_L} = y_L \mid X_n = x).$$

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$$= \mathbb{P}(X_{t_1} = y_1, \ldots, X_{t_L} = y_L \mid X_n = x).$$

Important

To determine X_{n+1} given the history $(X_1, \ldots, X_n)$ up to time n, it suffices to retain only X_n and discard $(X_1, \ldots, X_{n-1})$.

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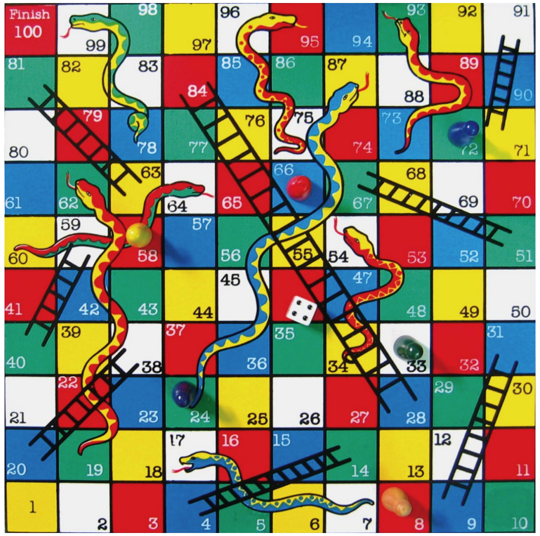
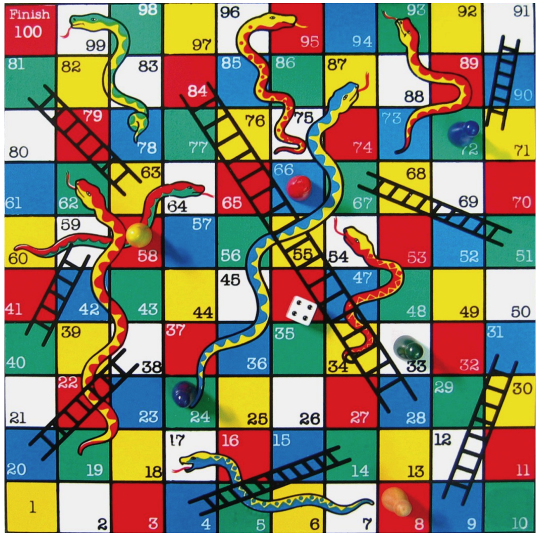
Example



- $x =$

Example



- $\mathcal{X} = \{1, \ldots, 100\}$
- Suppose $X_n = 6$ at some time n
- Then, $X_{n+1} \in$

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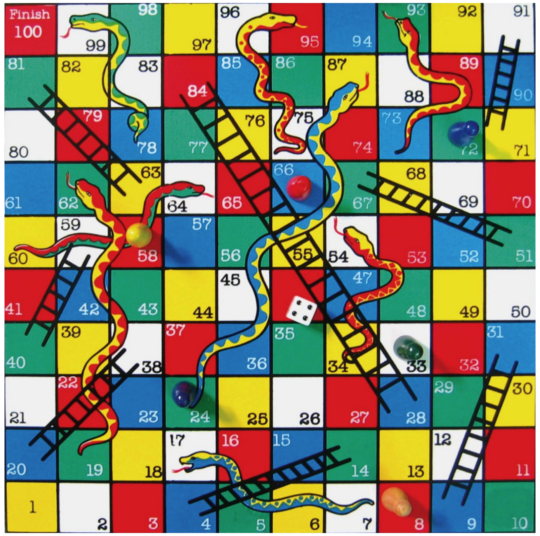
Example

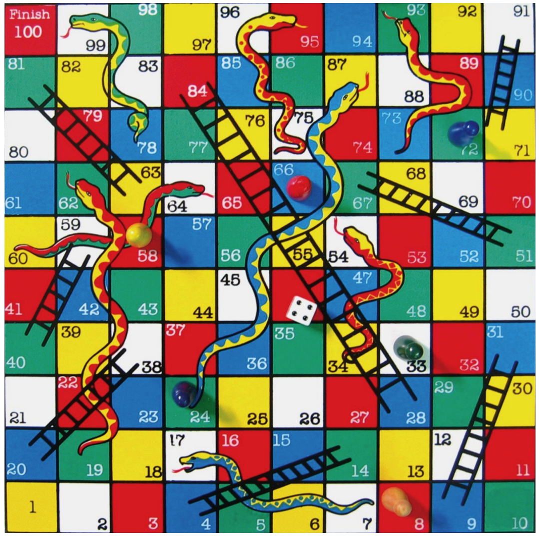


- $\mathcal{X} = \{1, \ldots, 100\}$
- Suppose $X_n = 6$ at some time n
- Then, $X_{n+1} \in \{7, 8, 31, 10, 11, 12\}$
- For all $x, x_1, \ldots, x_{n-1} \in \mathcal{X}$, we have

$$\mathbb{P}(X_{n+1} = x | X_n = 6, \underbrace{X_{n-1} = x_{n-1}, \ldots, X_1 = x_1}_{\text{trajectory before } n})$$

$$= \mathbb{P}(X_{n+1} = x | X_n = 6)$$

Example



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Remark

The set $\mathcal{X}$ is called the state space of the Markov chain.

Example

Let $\{X_n\}_{n=1}^{\infty}$ be an $\mathbb{N}$-valued i.i.d. process.
For each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} X_i$.

For any $\mathbf{s} = (s_1, \ldots, s_{n-1})$,

$$\mathbb{P}(S_{n+1} = s_{n+1} \mid S_n = s_n,\ S_{1:n-1} = \mathbf{s}) =$$

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$$\mathbb{P}(S_{n+1} = s_{n+1} \mid S_n = s_n, \; S_{1:n-1} = \mathbf{s}) = \mathbb{P}(X_{n+1} = s_{n+1} - s_n)$$

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Let $\{X_n\}_{n=1}^{\infty}$ be an $\mathbb{N}$-valued i.i.d. process.
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For any $\mathbf{s} = (s_1, \ldots, s_{n-1})$,

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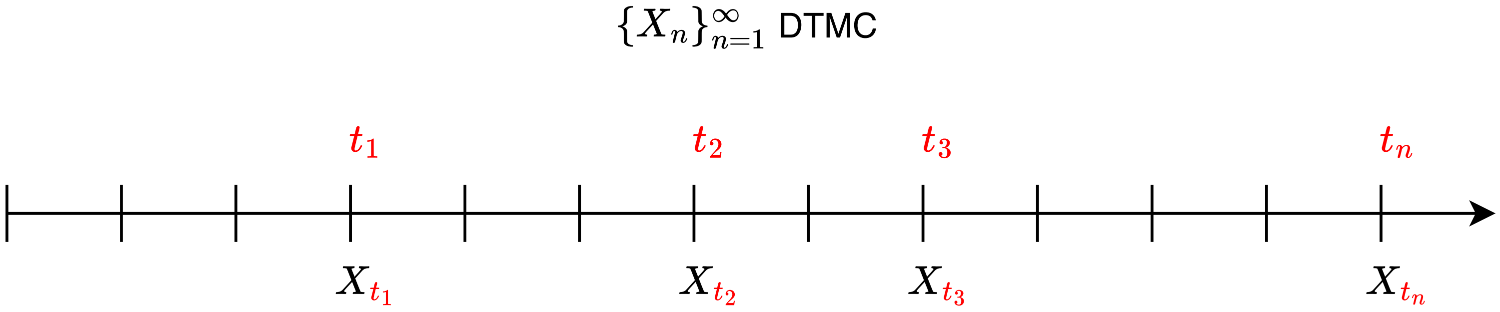
Markov Property for Deterministic Sampling Times



Lemma (Markov Property for Deterministic Sampling Times)

Suppose that $\{X_n\}_{n=1}^{\infty}$ is a DTMC with a finite state space $\mathcal{X}$.
For all $n \in \mathbb{N}$, deterministic $t_1 < t_2 < \cdots < t_n$, and $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$\mathbb{P}(X_{t_n} = x_n \mid X_{t_{n-1}} = x_{n-1}, \ldots, X_{t_1} = x_1) = \mathbb{P}(X_{t_n} = x_n \mid X_{t_{n-1}} = x_{n-1}).$$

In words: Suffices to retain the most recent information and discard the history.

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Transition Probability Matrix

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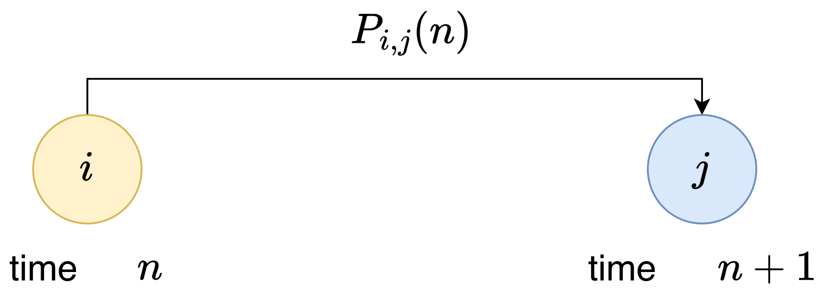
Transition Probability Matrix

Definition (Transition Probability Matrix)

Let $\{X_n\}_{n=1}^{\infty}$ be a DTMC with discrete state space $\mathcal{X}$.
The transition probability matrix (TPM) of the Markov chain at any time $n \in \mathbb{N}$ is a matrix $P(n) = [P_{i,j}(n)]_{i,j\in\mathcal{X}}$ defined as

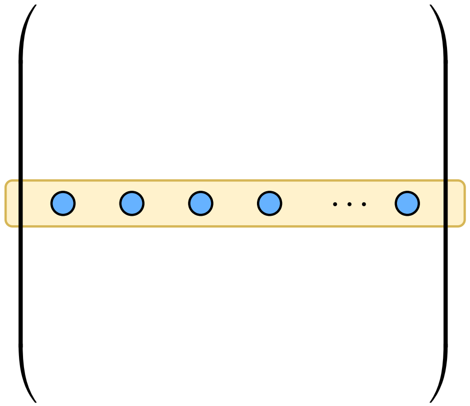
$$P_{i,j}(n) = \mathbb{P}(X_{n+1} = j \mid X_n = i), \qquad i,j \in \mathcal{X}.$$



Transition Probability Matrix

- For each $i \in \mathcal{X}$, $\sum_{j \in \mathcal{X}} P_{i,j}(n) =$

i



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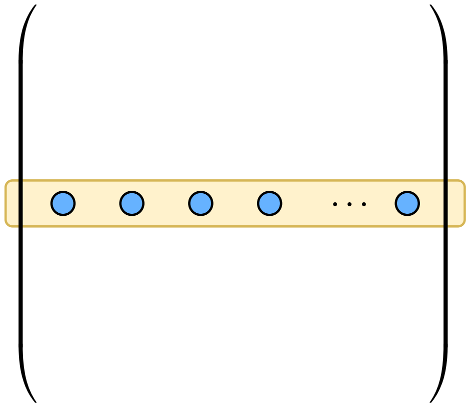

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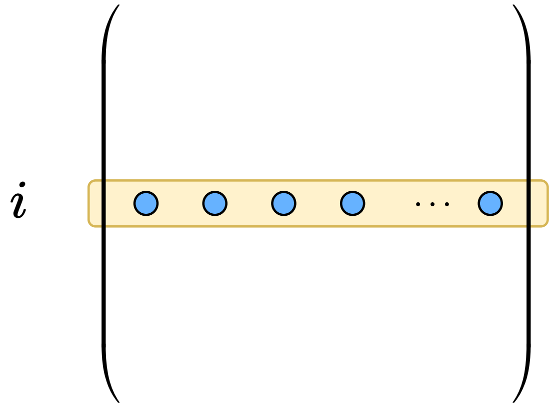
- For each $i \in \mathcal{X}$, $\quad \sum_{j \in \mathcal{X}} P_{i,j}(n) = 1.$
- A matrix with non-negative entries and row sums equal to 1 is called a row stochastic matrix



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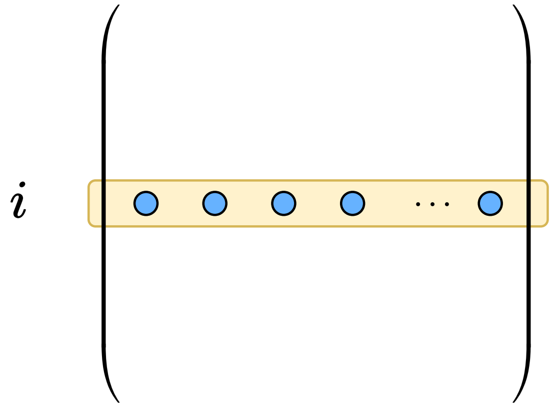

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- $P(n)$ is a row stochastic matrix for every n

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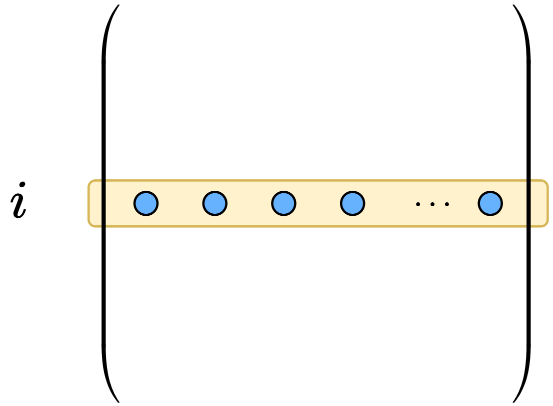

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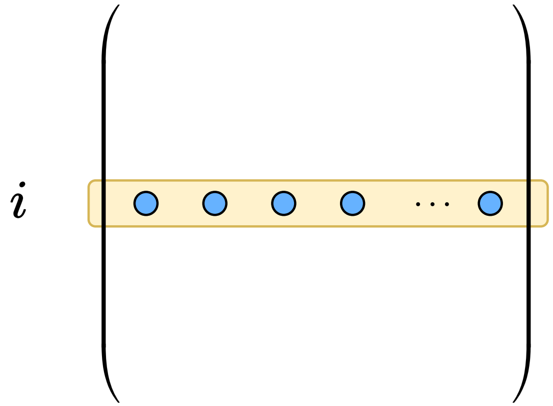

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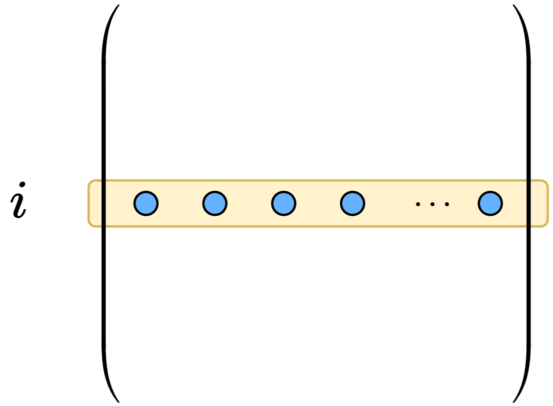
- Each row of $P(n)$ = PMF on $\mathcal{X}$
- For instance,

$$\sum_{j \in \mathcal{X}} j^2 \, P_{i,j}(n) = \mathbb{E}[X_{n+1}^2 \mid X_n = i],$$

where $\mathbb{E}$ above is w.r.t. row i of $P(n)$

Transition Probability Matrix



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- $P(n)$ has a right eigenvector with eigenvalue 1

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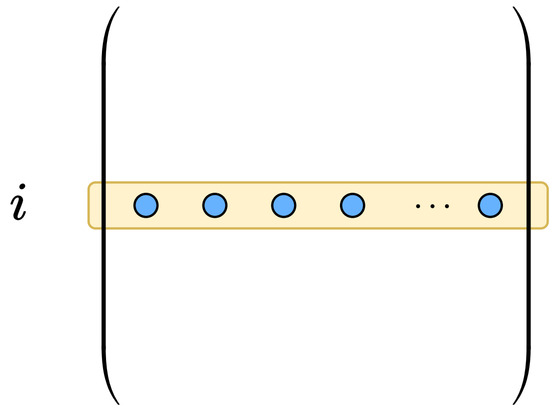

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- $P(n)$ has a right eigenvector with eigenvalue 1
- $P(n) \cdot \mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the all-ones vector

Time Homogeneous DTMC

Definition (Time Homogeneous DTMC

A DTMC with discrete state space $\mathcal{X}$ and TPMs $\{P(n)\}_{n=1}^{\infty}$ is called time homogeneous if

$$P(n) = P(n+1) \qquad \forall n \in \mathbb{N}.$$

In this case, we simply write P to denote the common TPM.

Example

Let $X_1, X_2, \ldots$ be i.i.d. on $\{-1, +1\}$, with $\mathbb{P}(X_1 = 1) = p$.
For each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} X_i$.
What is the TPM of $\{S_n\}_{n=1}^{\infty}$?

$\{X_n\}_{n=1}^{\infty}$ DTMC ; state space $\mathcal{X}$.

$\{X_n\}_{n=1}^{\infty}$ time homogeneous; TPM P.

$$\mathbb{P}(X_{n+1}=j \mid X_n = i) = P_{i,j} \quad \forall\, n \in \mathbb{N}.$$

$$\mathbb{P}(X_2 = j \mid X_1 = i) = P_{i,j}.$$

$$\mathbb{P}(X_3=j \mid X_1 = i) = \sum_{k\in\mathcal{X}} \mathbb{P}(X_3=j \mid X_2=k,\, X_1=i)\, \mathbb{P}(X_2=k \mid X_1=i)$$

$$= \sum_{k\in\mathcal{X}} \mathbb{P}(X_3=j \mid X_2=k) \cdot \mathbb{P}(X_2=k \mid X_1=i)$$

$$= \sum_{k\in\mathcal{X}} P_{k,j} \cdot P_{i,k} \quad = P^2_{i,j}$$

$$= \underset{P_{i,\cdot}}{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}} \overset{P_{\cdot,j}}{\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}}$$

Chapman – Kolmogorov Equation

Consider a time-homogeneous DTMC $\{X_n\}_{n=1}^{\infty}$ with discrete state space $\mathcal{X}$.

For any $n \in \mathbb{N}$, $i, j \in \mathcal{X}$, let

$$p_{i,j}^{(n)} := \mathbb{P}\left(X_{n+1} = j \mid X_1 = i \right).$$

Chapman - Kolmogorov:

Let $P^{(n)} = \left[p_{i,j}^{(n)} \right]_{i,j \in \mathcal{X}}$. Then

$$\boxed{P^{(n)} = P^n \qquad \forall\, n \in \mathbb{N}}$$

Proof: Exercise (induction!)