TA Session 13 03/11/2018

Agenda:

- Problems on stationarity, wide-sense stationarity and basics of DTMCs.

Recall the definitions of stationarity and wide-sense stationarity.

Def^n (stationary process)

Let T be an index set, and let $(\Omega, \mathcal{F}, P)$ be a probability space. The collection $\{X_t : t \in T\}$ of $\mathcal{F}$-measurable random variables is said to be a stationary random process if:

for all $n \geq 1$, for all $t_1, t_2, \dots, t_n \in T$ and for all $x_1, \dots, x_n \in \mathbb{R}$, the following relation holds for any shift $\tau \in T$:

$$F_{X_{t_1}, \dots, X_{t_n}}(x_1, x_n) = F_{X_{t_1+\tau}, \dots, X_{t_n+\tau}}(x_1, \dots, x_n).$$

That is, every finite dimensional distrubution is shift invariant.

Def^n (wide-sense stationary process)

Let T be an index set, and let $(\Omega, \mathcal{F}, P)$ be a probability space. The collection $\{X_t : t \in T\}$ of $\mathcal{F}$-measurable random variables is said to be a wide-sense stationary (WSS) random process if:

a) for all $t, s \in T$, $E[X_t] = E[X_s]$

b) for all $t, s, u \in T$, $E[X_s X_t] = E[X_{s+u}, X_{t+u}]$.

Problem 1:

Let $\{Z_t : t \in \mathbb{R}\}$ be a Gaussian process with mean function $m_x(t) = E[Z_t] = 0$ for all $t \in \mathbb{R}$. Show that $\{Z_t : t \in \mathbb{R}\}$ is stationary if and only if for all $t_1, t_2 \in \mathbb{R}$, we have

$$K_Z(t_1, t_2) = K_Z(t_1 - t_2, 0), \text{ where}$$

$$K_Z(t,s) := \text{Covariance matrix of the Gaussian vector } \begin{bmatrix} Z_t \\ Z_s \end{bmatrix}.$$

Sol

Soln: (Only if)

Suppose $\{Z_t : t \in \mathbb{R}\}$ is stationary. Then, for any $t_1, t_2 \in \mathbb{R}$,

$$K_Z(t_1, t_2) = \begin{bmatrix} E[Z_{t_1}^2] & E[Z_{t_1} Z_{t_2}] \\ E[Z_{t_2} Z_{t_1}] & E[Z_{t_2}^2] \end{bmatrix}.$$

Since the process is Stationary, it is also WSS. Hence, $E[Z_t^2] = R_Z(t,t)$ is Constant wrt time. We thus do the following replacements:

$$E[Z_{t_1}^2] = E[Z_{t_1 - t_2}^2]$$
$$E[Z_{t_1} Z_{t_2}] = E[Z_{t_1 - t_2} Z_0]$$
$$E[Z_{t_2}^2] = E[Z_0^2].$$

With these substitutions, we get $K_Z(t_1, t_2) = K_Z(t_1 - t_2, 0)$.

(If part)

Suppose $K_Z(t_1, t_2) = K_Z(t_1 - t_2, 0) \quad \forall \; t_1, t_2 \in \mathbb{R}$.

Then, for any $n \geq 1$, $s_1, \dots, s_n \in \mathbb{R}$ and $z_1, \dots, z_n \in \mathbb{R}$, we have that the joint distribution

$$F_{Z_{s_1}, \dots, Z_{s_n}}(z_1, \dots, z_n)$$

is specified completely by the covariance matrix of the Gaussian vector $(Z_{s_1}, \dots, Z_{s_n})^T$. Similarly, for any $\tau \in \mathbb{R}$, the joint distribution

$$F_{Z_{s_1+\tau}, \dots, Z_{s_n+\tau}}(z_1, \dots, z_n)$$

is specified completely by the covariance matrix of the Gaussian vector $(Z_{s_1+\tau}, \dots, Z_{s_n+\tau})$. However, we notice that

$$\begin{aligned} K_Z(s_m, s_l) &= K_Z(s_m - s_l, 0) \\ &= K_Z((s_m+\tau) - (s_l+\tau), 0) \\ &= K_Z(s_m+\tau, s_l+\tau) \end{aligned}$$

for all $1 \leq m, l \leq n$. This proves that $\{Z_t : t \in \mathbb{R}\}$ is stationary.

Corollary: A Gaussian process with non-zero mean is stationary if and only if the mean is constant and covariance matrix satisfies the property in the above question.

Remark: The above problem says that for a Gaussian process,

$$\text{WSS} \Leftrightarrow \text{Stationarity}.$$

Stopping times

Let $\{X_n : n \in \mathbb{N}\}$ be a random sequence. A random variable T taking values in $\mathbb{N}$ is called a stopping time wrt $\{X_n : n \in \mathbb{N}\}$ if :

$$\forall\, n \in \mathbb{N}, \quad \{T \leq n\} \in \sigma(X_1, \dots, X_n).$$

OR

$$\{T = n\} \in \sigma(X_1, \dots, X_n).$$

Interpretation: Given the history upto time n of the process $\{X_m : m \in \mathbb{N}\}$, we can tell whether or not $T=n$ (or $T \leq n$).

In other words,

$$P\Big(T=n \,\Big|\, \sigma(X_1, \dots, X_n)\Big) = 0 \text{ or } 1.$$

Example:

1. $N = \inf\{n \geq 1 : X_n \in A\}$ = first hitting time of set A.

   This is a stopping time since

   $$\{N > n\} = \{X_1 \notin A, X_2 \notin A, \dots, X_n \notin A\}.$$

   Thus, by knowing $X_1, \dots, X_n$, we can tell whether $\{N > n\}$ has occured or not

   $$\Rightarrow \{N > n\} \in \sigma(X_1, \dots, X_n) \quad \forall\, n \geq 1$$

   $$\Rightarrow \{N \leq n\} \in \sigma(X_1, \dots, X_n) \quad \forall\, n \geq 1.$$

   Hence, N is a stopping time.

2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-negative random variables. Let

$$S_n = \sum_{k=1}^{n} X_k , \quad n \geq 1.$$

Define

$$N_t = \sup\{n\geq 1 : S_n \leq t\}.$$

(That is, N_t is the last time index n such that $S_n \leq t$). Then,

$$\{N_t = n\} = \{S_1 \leq t,\ S_2 \leq t, \dots, S_n \leq t,\ S_{n+1} > t,\ S_{n+2} > t, \dots\}$$

$$= \bigcap_{k=1}^{n} \{S_k \leq t\} \cap \underbrace{\bigcap_{k=n+1}^{\infty} \{S_k > t\}}_{\text{depends on } X_{n+1},\, X_{n+2},\, \dots}$$

Hence, N_t is not a stopping time

3. Let $\{X_n : n\in\mathbb{N}\}$ be an iid sequence of Bernoulli random variables. Let

$$N_n := \sum_{k=1}^{n} 1_{\{X_k = 1\}} = \#\text{ successes upto time } n.$$

Define

$$T_m := \inf\{n\geq 1 : N_n = m\}.$$

Then,

$$\{T_m \leq n\} = \underbrace{\{N_n \geq m\}}_{\substack{\text{By looking at } X_1,\dots,X_n,\text{ we can say} \\ \text{whether or not atleast } m \text{ of these are } 1\text{'s.}}} \quad \forall\, n\geq 1.$$

Hence, T_m is a stopping time.

Remark: For a general index set T and a random process $\{X_t : t \in T\}$, a random variable τ taking values in T is called a stopping time wrt $\{X_t : t \in T\}$ if:

$$\{\tau \leq t\} \in \sigma(X_s : s \leq t), \quad \forall\ t \in T.$$

Lemma: Let $\{X_t : t \in T\}$ be a random process, and let τ_1 and τ_2 be two stopping times with respect to $\{X_t : t \in T\}$. Then, $\min\{\tau_1, \tau_2\}$ and $\max\{\tau_1, \tau_2\}$ are also stopping times.

Proof:

$$\{\min\{\tau_1, \tau_2\} \leq t\} = \underbrace{\{\tau_1 \leq t\}}_{\in \sigma(X_s : s \leq t)} \cup \underbrace{\{\tau_2 \leq t\}}_{\in \sigma(X_s : s \leq t)}$$

$$\in \sigma(X_s : s \leq t) \quad \forall\ t \in T.$$

$$\{\max\{\tau_1, \tau_2\} \leq t\} = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\}$$

$$\in \sigma(X_s : s \leq t) \quad \forall\ t \in T.$$

Remark: It follows from the Lemma that $\min\{\tau_1, 2\}$ and $\max\{\tau_2, 5\}$ are stopping times. In general, $\min\{\tau_1, t\}$ and $\max\{\tau_2, t\}$ are stopping times for all $t \in T$.

Some important results on Markov Chains

1. Let $\{X_n : n \in \mathbb{N}\}$ be a Markov process on state space $\mathcal{X}$ which is countable. If T is a stopping time wrt $\{X_n : n \in \mathbb{N}\}$ such that $P(T < \infty) = 1$, then

$$P\left(X_{T+1} = y \mid X_T = x, \dots, X_0 = x_0\right) = P\left(X_{T+1} = y \mid X_T = x\right).$$

Further, if $\{X_n : n \in \mathbb{N}\}$ is time homogeneous, then

$$P\left(X_{T+1} = y \mid X_T = x\right) = P\left(X_1 = y \mid X_0 = x\right).$$

This result is called strong Markov property. It says that a time homogeneous DTMC renews itself by resetting the clock to 0.

<u>Problem 2</u>:

Suppose $\{X_n : n \geq 0\}$ is a time homogeneous DTMC on a countable state space $\mathcal{X}$. Let $(p_{xy})_{x,y \in \mathcal{X}}$ denote its transition probabilities. Assume $p_{xx} < 1$ for all $x \in \mathcal{X}$. Define

$T_1 = \inf\{n \geq 1 : X_n \neq X_0\}$, and for $m \geq 2$, define

$$T_m = \inf\left\{n > T_{m-1} : X_n \neq X_{T_{m-1}}\right\}.$$

Thus, T_m's are the random times when DTMC changes its state.

a) Show that $P(T_m < \infty) = 1 \quad \forall\, m \geq 1$.

b) Define $Z_0 = X_0$ and $Z_m = X_{T_m}$. Prove that $(Z_m)_{m \geq 0}$

is also time homogeneous, with transition probabilities $\left(\tilde{p}_{xy}\right)_{x,y\in\mathcal{X}}$ given by

$$\tilde{p}_{xy} = P\left(Z_{m+1} = y \mid Z_m = x\right) = \begin{cases} 0, & \text{if } y = x \\ \dfrac{p_{xy}}{1-p_{xx}}, & \text{if } y \neq x. \end{cases}$$

Solution :

(a) $\{T_1 = n\} = \{X_1 = X_0, X_2 = X_0, \dots, X_{n-1} = X_0, X_n \neq X_0\}$

Thus, $P(T_1 = n) = \sum_{x\in\mathcal{X}} P(T_1 = n \mid X_0 = x) \cdot P(X_0 = x)$

$$= \sum_{x\in\mathcal{X}} P\left(X_1 = X_0, X_2 = X_0, \dots, X_{n-1} = X_0, X_n \neq X_0 \mid X_0 = x\right) P(X_0 = x)$$

$$= \sum_{x\in\mathcal{X}} P\left(X_1 = x, X_2 = x, \dots, X_{n-1} = x, X_n \neq x \mid X_0 = x\right) P(X_0 = x)$$

$$= \sum_{x\in\mathcal{X}} p_{xx}^{n-1} \left(1 - p_{xx}\right) P\left(X_0 = x\right).$$

Now, $P(T_1 < \infty) = \sum_{n=1}^{\infty} P(T_1 = n)$

$$= \sum_{n=1}^{\infty} \sum_{x\in\mathcal{X}} p_{xx}^{n-1} (1 - p_{xx}) P(X_0 = x)$$

$$\overset{(a)}{=} \sum_{x\in\mathcal{X}} \sum_{n=1}^{\infty} p_{xx}^{n-1} (1 - p_{xx}) P(X_0 = x)$$

$$= \sum_{x\in\mathcal{X}} P(X_0 = x)$$

$= 1,$

where (a) above follows by noting that all terms inside the summation are positive, and hence $\sum_{x \in \mathcal{X}}$ and $\sum_{n=1}^{\infty}$ may be interchanged.

Thus, we have showed that T_1 is a stopping time with the property that $P(T_1 < \infty) = 1$.

We now show by induction that for any $m \geq 2$, T_m is a stopping time. Towards this, note that for $m=2$,

$$\{T_2 = n\} = \biguplus_{m=1}^{n-1} \{T_2 = n, T_1 = m\}$$

$$= \biguplus_{m=2}^{n-1} \{X_1 = X_0, \ldots, X_{m-1} = X_0, X_m \neq X_0, X_{m+1} = X_m, \ldots, X_{n-1} = X_m, X_n \neq X_m\}$$

$$\in \sigma(X_0, \ldots, X_n) \quad \forall \; n \geq 2.$$

Thus, T_2 is a stopping time.

Let us assume that T_m is a stopping time for some $m > 2$. Then,

$$\{T_{m+1} = n\} = \biguplus_{k=m}^{n-1} \{T_{m+1} = n, T_m = k\} \quad (\because T_m \geq m \text{ a.s.})$$

$$= \biguplus_{k=m}^{n-1} \{\underbrace{T_m = k}_{\in \sigma(X_0, \ldots, X_k)}, X_{k+1} = X_k, \ldots, X_{n-1} = X_k, X_n \neq X_k\}$$

$\in \sigma(X_0, \ldots, X_n) \quad \forall \; n \geq m+1.$

Thus, T_{m+1} is a stopping time.

We now show by induction that for any $m \geq 2$,

$$P(T_m - T_{m-1} < \infty) = 1.$$

Towards this, note that for $m = 2$,

$$P(T_2 - T_1 = n) = P(T_2 - T_1 = n, \; T_1 < \infty) \quad \left(\because \text{ we have shown that } P(T_1 < \infty) = 1\right)$$

$$= \sum_{m=1}^{\infty} P(T_2 - T_1 = n, \; T_1 = m).$$

Now, $\{T_2 - T_1 = n, \; T_1 = m\} = \{X_1 = X_0, \ldots, X_{m-1} = X_0, \; X_m \neq X_0,$
$X_{m+1} = X_m, \ldots, X_{m+n-1} = X_m, \; X_{m+n} \neq X_m\}.$

Thus,

$$P(T_2 - T_1 = n, \; T_1 = m) = \sum_{x \in \mathcal{X}} P(T_2 - T_1 = n, \; T_1 = m \mid X_0 = x) \, P(X_0 = x)$$

$$= \sum_{x \in \mathcal{X}} P(X_1 = x, \ldots, X_{m-1} = x, \; X_m \neq x, \; X_{m+1} = X_m, \ldots, X_{m+n-1} = X_m, \; X_{m+n} \neq X_m \mid X_0 = x) \, P(X_0 = x)$$

$$= \sum_{x \in \mathcal{X}} \sum_{y \neq x} P_1(X = x, \ldots, X_{m-1} = x, \; X_m = y, \; X_{m+1} = y, \ldots, X_{m+n-1} = y, \; X_{m+n} \neq y \mid X_0 = x) \, P(X_0 = x)$$

$$\doteq \sum_{x \in \mathcal{X}} \sum_{y \neq x} p_{xx}^{m-1} \, p_{xy} \cdot p_{yy}^{n-1} (1 - p_{yy}) \, P(X_0 = x)$$

Therefore, we get

$$P(T_2 - T_1 = n) = \sum_{m=1}^{\infty} P(T_2 - T_1 = n,\ T_1 = m)$$

$$= \sum_{m=1}^{\infty} \sum_{x \in \mathcal{X}} \sum_{y \neq x} p_{xx}^{m-1}\, p_{xy}\, p_{yy}^{n-1} (1 - p_{yy})\, P(X_0 = x)$$

$$= \sum_{x \in \mathcal{X}} \frac{1}{1 - p_{xx}} \sum_{y \neq x} p_{xy}\, p_{yy}^{n-1} (1 - p_{yy})\, P(X_0 = x).$$

Thus,

$$P(T_2 - T_1 < \infty) = \sum_{n=1}^{\infty} P(T_2 - T_1 = n)$$

$$= \sum_{n=1}^{\infty} \sum_{x \in \mathcal{X}} \frac{1}{1 - p_{xx}} \sum_{y \neq x} p_{xy}\, p_{yy}^{n-1} (1 - p_{yy})\, P(X_0 = x)$$

$$= \sum_{x \in \mathcal{X}} \frac{1}{1 - p_{xx}} \underbrace{\sum_{y \neq x} p_{xy}}_{1 - p_{xx}}\, P(X_0 = x)$$

$$= \sum_{x \in \mathcal{X}} P(X_0 = x)$$

$$= 1.$$

Thus, we have $P(T_2 - T_1 < \infty) = 1 = P(T_1 < \infty)$

$$\Rightarrow P(T_2 < \infty) = 1.$$

We now assume that $P(T_m - T_{m-1} < \infty) = 1$ and $P(T_m < \infty) = 1$ for some $m > 2$. Then,

$$P(T_{m+1} - T_m = n) = P(T_{m+1} - T_m = n,\ T_m < \infty)$$

$$= \sum_{k=1}^{\infty} P(T_{m+1} - T_m = n,\ T_m = k)$$

$$= \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}} P(T_{m+1} - T_m = n,\ T_m = k \mid X_k = x)\ P(X_k = x)$$

$$= \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}} P(X_{k+1} = x,\ X_{k+2} = x,\ \dots,\ X_{k+n-1} = x,\ X_{k+n} \neq x,\ T_m = k \mid X_k = x) \cdot P(X_k = x)$$

$$= \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}} P(T_m = k \mid X_k = x) \cdot P(X_{k+1} = x, \dots, X_{k+n-1} = x,\ X_{k+n} \neq x \mid T_m = k,\ X_k = x)\ P(X_k = x)$$

$$= \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}} P(T_m = k \mid X_k = x) \cdot P(X_{k+1} = x, \dots, X_{k+n-1} = x,\ X_{k+n} \neq x \mid X_k = x)\ P(X_k = x)$$

$$= \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}} P(T_m = k \mid X_k = x) \cdot p_{xx}^{n-1} (1 - p_{xx})\ P(X_k = x).$$

Thus,

$$P(T_{m+1} - T_m < \infty) = \sum_{n=1}^{\infty} P(T_{m+1} - T_m = n)$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}} P(T_m = k \mid X_k = x) \cdot p_{xx}^{n-1} (1 - p_{xx})\ P(X_k = x)$$

$$= \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}} P(T_m = k \mid X_k = x)\ P(X_k = x)$$

$$= \sum_{k=1}^{\infty} P(T_m = k) = P(T_m < \infty) = 1.$$

Thus, $P(T_{m+1} - T_m < \infty) = 1$ and $P(T_m < \infty) = 1$

$$\Rightarrow P(T_{m+1} < \infty) = 1.$$

(b) We now note that for any $x_0, \cdots, x_{m-1}, x, y \in \mathcal{X}$,

$$P(Z_{m+1} = y, Z_m = x, Z_{m-1} = x_{m-1}, \dots, Z_0 = x_0)$$

$$= P(X_{T_{m+1}} = y, X_{T_m} = x, X_{T_{m-1}} = x_{m-1}, \dots, X_{T_1} = x_1, X_0 = x_0) \mathbb{1}_{\{y \neq x\}}$$

$$= P(T_m < \infty, T_{m+1} - T_m < \infty, X_0 = x_0, \dots, X_{T_m} = x, X_{T_{m+1}} = y) \mathbb{1}_{\{y \neq x\}}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} P(T_m = k, T_{m+1} - T_m = n, X_0 = x_0, \dots, X_{T_m} = x, X_{T_{m+1}} = y) \mathbb{1}_{\{y \neq x\}}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} P(T_m = k, T_{m+1} = k+n, X_0 = x_0, \dots, X_k = x, X_{k+n} = y) \mathbb{1}_{\{y \neq x\}}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} P(T_m = k, X_0 = x_0, \dots, X_k = x, X_{k+1} = x, \dots, X_{k+n-1} = x, X_{k+n} = y). \mathbb{1}_{\{y \neq x\}}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} P(T_m = k, X_0 = x_0, \dots, X_k = x). \, p_{xx}^{n-1} \cdot p_{xy} \mathbb{1}_{\{y \neq x\}}$$

$$= \sum_{k=1}^{\infty} P(T_m = k, X_0 = x_0, \dots, X_k = x). \frac{p_{xy}}{1 - p_{xx}} \mathbb{1}_{\{y \neq x\}}$$

$$= P(X_0 = x_0, \dots, X_{T_m} = x) \cdot \frac{p_{xy}}{1 - p_{xx}} \mathbb{1}_{\{y \neq x\}}$$

$$= P\left(Z_0 = x_0, \ldots, Z_{m-1} = x_{m-1}, Z_m = x\right) \cdot \frac{p_{xy}}{1-p_{xx}} \mathbb{1}_{\{y \neq x\}}.$$

Summing over all $x_0, \ldots, x_{m-1}$, we get

$$P\left(Z_{m+1} = y, Z_m = x\right) = P\left(Z_m = x\right) \frac{p_{xy}}{1-p_{xx}} \mathbb{1}_{\{y \neq x\}}$$

$$\Rightarrow P\left(Z_{m+1} = y \mid Z_m = x\right) = \begin{cases} 0, & y = x \\ \dfrac{p_{xy}}{1-p_{xx}}, & y = x. \end{cases}$$