AI 5090: STOCHASTIC PROCESSES

LECTURE 06

SCRIBE: RAJKUMAR T, ABHILASH S

24 JANUARY 2025

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Contents

The lecture will touch upon the need for almost-sure convergence, the Borel–Cantelli lemma, and discuss these concepts in detail with examples.

Topics Covered: almost-sure convergence, Borel–Cantelli lemma.

Note: This scribe considers $(\Omega, \mathscr{F}, \mathbb{P})$ as the underlying probability space for all definitions.

1 Almost-Sure Convergence of Sequence of Random Variables

In previous lecture, the concept of almost-sure convergence was introduced. This section provides further details and highlights the importance of the same.

1.1 Need for Almost-Sure Convergence

- Every time we will be not having access to $(\Omega, \mathscr{F}, \mathbb{P})$ or to the exact mapping $\omega \mapsto X_n(\omega)$, but we may be given the statistics of X_n (e.g., PMF/PDF). Based on the statistics, we start with a guess for the limit random variable X.
- If for every $\omega \in \Omega$, the sample path corresponding to ω converges to $X(\omega)$, then we say that $X_n \xrightarrow{\text{pointwise}} X$.
- Suppose, even for one ω, the corresponding sample path does not converge to $X(\omega)$, then there is no pointwise convergence. In such cases, is it good to discard the guess? Answer is NO! In such cases, we can come up with other notions of convergence. One such notion of convergence of sequence is that of almost-sure convergence.

Fix a probability space $(\Omega, \mathscr{F}, P)$. Let $\{X_n\}_{n=1}^{\infty}$ and X be defined with respect to $\mathscr{F}$.

Definition (Almost-Sure Convergence:)
We say that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to X almost-surely (a.s.) if

$$P(A_{\text{lim}}) = 1,$$

where

$$A_{\text{lim}} = \left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}.$$

Notation:

$$X_n \xrightarrow{\text{a.s.}} X.$$

Some remarks are in order.

1. If $A_{\text{lim}} = \Omega$, then clearly $\mathbb{P}(A_{\text{lim}}) = 1$. Thus, $X_n \xrightarrow{\text{pointwise}} X$, then $X_n \xrightarrow{\text{a.s.}} X$.
2. A sequence may converge almost-surely, but not necessarily pointwise. It can so happen that the set A_{lim} is a strict subset of Ω, yet satisfy $\mathbb{P}(A_{\text{lim}}) = 1$, as demonstrated next.

Example 1.1. *Suppose that $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$. For each $n \in \mathbb{N}$, let*

$$X_n(\omega) = \begin{cases} 1, & \omega \in [0, \frac{1}{n}) \\ 0, & \textit{otherwise}. \end{cases}.$$

Identify the pointwise and almost-sure limits.
The pointwise limit of the above sequence of random variables, as worked out in the previous lecture is,

$$X_n(\omega) = \begin{cases} 1, & \textit{if } \omega = 0, \\ 0, & \textit{otherwise}. \end{cases}$$

This can be concisely written as $\mathbf{1}_{\{0\}}(\omega)$.

If pointwise limit exists then naturally almost-sure limit also exists and the pointwise limit will be an almost-sure limit, i.e., $X_n \xrightarrow{\text{p.w.}} X$ *implies* $X_n \xrightarrow{\text{a.s.}} X$. *Hence, we may write*

$$X_n \xrightarrow{\text{a.s.}} X = \mathbf{1}_{\{0\}}.$$

Let us now check if $\{X_n\}_{n=1}^{\infty}$ *has any other almost-sure limits. Let* $Y = 2 \cdot \mathbf{1}_{\{0\}}$. *That is,*

$$Y(\omega) = \begin{cases} 2, & \text{if } \omega = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly for all $\omega \in (0,1]$, *the sequence* $\{X_n(\omega)\}_{n=1}^{\infty}$ *converges to* $Y(\omega)$, *except for* $\omega = 0$ *where there is no convergence. Notice that* $X_n(0) \to 1$, *but* $Y(0) = 2$. *Hence* $\{X_n\}_{n=1}^{\infty}$ *does not converge to* Y *pointwise.*
However, for all $\omega \in (0,1]$, *the sequence* $\{X_n(\omega)\}_{n=1}^{\infty}$ *converges to* $Y(\omega)$, *and* $\mathbb{P}((0,1]) = 1$. *That is,* $A_{\text{lim}} = (0,1]$, *and* $\mathbb{P}(A_{\textit{lim}}) = 1$, *thereby implying that* Y *is also an almost-sure limit.*

Building further on the above example, suppose that $Z = 2 \cdot \mathbf{1}_{\{0,0.5,0.75\}}$. *That is,*

$$Z(\omega) = \begin{cases} 2, & \text{if } \omega \in \{0, 0.5, 0.75\}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly $\{X_n\}_{n=1}^{\infty}$ *does not converge pointwise to* Z. *However,* Z *is also an almost-sure limit of* $\{X_n\}_{n=1}^{\infty}$, *because* $A_{\text{lim}} = [0,1] \setminus \{0, 0.5, 0.75\}$, *and* $\mathbb{P}(A_{\text{lim}}) = 1$.

Extending the above example, we note that the random variable W *given by* $W = \mathbf{1}_{\mathbb{Q}\cap[0,1]}(\omega)$ *is also a valid almost-sure limit. For this example, we have* $A_{\text{lim}} = \{0\} \cup (\Omega \setminus \mathbb{Q})$, *and* $\mathbb{P}(A_{\text{lim}}) = \mathbb{P}(\{0\}) + \mathbb{P}((0,1] \setminus \mathbb{Q}) = 1$. *Observe that* $A_{\text{lim}} \subset \Omega$ *here, yet* $\mathbb{P}(A_{\text{lim}}) = 1$.

The above example demonstrates that a sequence of random variables can have multiple almost-sure limits.

Some remarks are in order.

- Pointwise limit, if it exists, is always unique.
- While the pointwise limit is trivially an almost-sure limit, there can exist multiple almost-sure limits, other than the pointwise limit. However, all the almost-sure limits must agree on with probability 1. Thus, for instance, if $X_n \xrightarrow{\text{a.s.}} X$ and $X_n \xrightarrow{\text{a.s.}} Y$, then $\mathbb{P}(X = Y) = 1$. Indeed, let $A_{\text{lim}}^X := \{\lim_{n\to\infty} X_n = X\}$ and $A_{\text{lim}}^Y := \{\lim_{n\to\infty} X_n = Y\}$. Then, from the almost-sure convergences $X_n \xrightarrow{\text{a.s.}} X$ and $X_n \xrightarrow{\text{a.s.}} Y$, we have $\mathbb{P}(A_{\text{lim}}^X) = 1 = \mathbb{P}(A_{\text{lim}}^Y)$. Observe that $\{X = Y\} \supseteq A_{\text{lim}}^X \cap A_{\text{lim}}^Y$, and therefore $\mathbb{P}(X = Y) \geq \mathbb{P}(A_{\text{lim}}^X \cap A_{\text{lim}}^Y) = 1$.

2 Borel–Cantelli Lemma(BCL) and Almost-Sure Convergence

- Before looking at BCL, we need to understand the lim inf and lim sup events.
- We have seen the lim inf and lim sup of sequence of real numbers and sequence of random variables before.
- Now we will see the lim inf and lim sup of sequence of sets.

2.1 The lim inf and lim sup Events:

Definition 1. *Let* $A_1, A_2, \ldots$ *be events in* $\mathscr{F}$. *The* **limit infimum** *of the sequence* $\{A_n\}_{n=1}^{\infty}$ *is defined as the set*

$$A_\star := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Clearly, $A_\star \in \mathscr{F}$.

Interpretation:

$$\omega \in A_\star \Rightarrow \exists n \in \mathbb{N} \text{ such that } \omega \in A_k \text{ for all } k \geq n$$

$$\Rightarrow \omega \text{ belongs to all but finitely many } A_n.$$

Simply put, possibly except for finitely many sets, ω belongs to every set after some index n.

Definition 2.

Let $A_1, A_2, \ldots$ be events in $\mathscr{F}$. The **limit supremum** of the sequence $\{A_n\}_{n=1}^{\infty}$ is defined as the set

$$A^\star := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Clearly, $A^\star \in \mathscr{F}$.
Interpretation:

$$\omega \in A^\star \Rightarrow \forall n \in \mathbb{N}, \exists k \geq n \text{ such that } \omega \in A_k$$

$$\Rightarrow \omega \text{ belongs to infinitely many of the } A_n.$$

- There is a subtle difference between the liminf event ($A_\star$) and the limsup event lim sup ($A^\star$).
- If $\omega \in A_\star$, then there exists an index n such that $\omega \in A_k$ for all $k \geq n$ (and therefore ω belongs to A_k for infinitely many k).
- However, if $\omega \in A^\star$, then ω belongs to A_k for infinitely many k, but this does not mean that ω must belong to all A_k, $k \geq n$, for some index n.
- For instance, if ω belongs to only odd numbered sets, then ω belongs to infinitely many **BUT NOT ALL** sets after some stage.
- It is then clear that if $\omega \in A_\star$, then $\omega \in A^\star$, but the converse is not true in general.

Properties of Lim Inf and Lim Sup

- We have
$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n \quad (A_\star \subseteq A^\star).$$
- Some texts use the phrase "A_n infinitely often" or "A_n i.o." to refer to $\limsup_{n\to\infty} A_n$.

2.2 Borel–Cantelli Lemma

Fix a probability space $(\Omega, \mathscr{F}, P)$.

Lemma 1. *The Borel–Cantelli lemma may be stated in two parts as follows.*

1. *Suppose $A_1, A_2, \cdots \in \mathscr{F}$ are such that $\sum_{i\in\mathbb{N}} P(A_i) < +\infty$. Then,*
$$\mathbb{P}\left(A_n \text{ i.o.}\right) = 0.$$
2. *Suppose $A_1, A_2, \cdots \in \mathscr{F}$ are independent and satisfy $\sum_{i\in\mathbb{N}} P(A_i) = +\infty$. Then,*
$$\mathbb{P}\left(A_n \text{ i.o.}\right) = 1.$$

Interpretation:

- BCL say that if we have countably many events/sets $A_1, A_2, A_3, \cdots \in \mathscr{F}$, whose individual probabilities sum up to a finite number, then infinitely many such events cannot occur. That is, $\mathbb{P}(\limsup_{n\to\infty} A_n) = 0$, thereby implying that $\mathbb{P}(\liminf_{n\to\infty} A_n^c) = 1$. Thus, only finitely many of the A_n will occur with probability 1.

- Suppose we conduct an experiment and observe $\omega \in \Omega$ as the outcome. We can then count how many of the A_n's contain the point ω. The Borel–Cantelli lemma states that if $\sum_{i=1}^{\infty} P(A_i) < +\infty$, then ω cannot belong to infinitely many events($A_n s$).
- On the contrary, if $\sum_{i=1}^{\infty} P(A_i) = +\infty.$, and if $A_i s$ are mutually independent, then ω belongs to infinitely many of the A_n with probability equal to ONE.
- The above lemma can be used to verify almost-sure convergence property in some scenarios.

Example 2.1. *For each $n \in \mathbb{N}$, let*

$$P(X_n = 1) = \frac{1}{n^2} = 1 - P(X_n = 0).$$

Identifying an Almost-Sure Limit:
Observe that we are not provided with information about $(\Omega, \mathscr{F}, \mathbb{P})$ here. Furthermore, information about the mapping $\omega \mapsto X_n(\omega)$ is not given either.

Let $A_n = \{X_n = 1\}$ for each $n \in \mathbb{N}$. To determine the almost-sure (a.s.) limit of X_n, we analyze whether X_n converges to a random variable X with probability 1.
Because

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 < +\infty,$$

by part 1 of the Borel–Cantelli lemma, it follows that

$$P(X_n = 1 \textit{ infinitely often}) = 0.$$

Hence, infinitely many $X_n s$ can't be 1. Only finitely many $X_n s$ can be 1. Which means after some stage all the $X_n s$ shall take the value 0.
Thus, with probability 1, X_n takes the value 1 only finitely many times, meaning that for large n, we have $X_n = 0$ almost-surely. It thus follows that

$$X_n \xrightarrow{\text{a.s.}} 0.$$