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Probability and Stochastic Processes

Lecture 16: Conditional CDFs, σ-Algebra Generated by a Random Variable, Independence of Two, Random Variables, Jointly Discrete Random Variables

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Two Random Variables (Bivariate Random Vector)

Definition (Bivariate Random Vector)

Fix a measurable space $(\Omega, \mathscr{F})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables (with respect to $\mathscr{F}$).
We say $(X, Y) : \Omega \to \mathbb{R}^2$ is a **bivariate random vector** with respect to $\mathscr{F}$ if

$$\forall\, B \in \mathscr{B}(\mathbb{R}^2), \qquad (X,Y)^{-1}(B) = \underbrace{\{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\}}_{\text{pre-image of } B} = \{(X,Y) \in B\} \in \mathscr{F}.$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2)) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

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Bivariate Random Vector

Theorem (Equivalent Characterization of Bivariate Random Vector)

Fix a measurable space $(\Omega, \mathscr{F})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables (with respect to $\mathscr{F}$).
Then,

$$(X, Y) \text{ random vector} \qquad \Longleftrightarrow \qquad (X, Y)^{-1}(B) \in \mathscr{F} \quad \forall\, B \in \mathscr{P},$$

where $\mathscr{P}$ is the collection $\mathscr{P} = \Big\{(-\infty, x] \times (-\infty, y] : \ x, y \in \mathbb{R}\Big\}$.

Bivariate Random Vector Simplified

Fix a measurable space $(\Omega, \mathscr{F})$, and let X, Y be random variables.
$(X, Y) : \Omega \to \mathbb{R}^2$ is a bivariate random vector **if and only if** for all $x, y \in \mathbb{R}$,

$$(X, Y)^{-1}\big((\infty,\ x] \times (-\infty,\ y]\big) = \underbrace{\{\omega \in \Omega : X(\omega) \le x\} \cap \{\omega \in \Omega : Y(\omega) \le y\}}_{\text{pre-image of } (-\infty,\ x] \times (-\infty,\ y]} \in \mathscr{F}.$$

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Joint Probability Law

Definition (Joint Probability Law)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
The **joint probability law of X and Y** is a function $\mathbb{P}_{X,Y} : \mathscr{B}(\mathbb{R}^2) \to [0, 1]$, defined as

$$\forall\, B \in \mathscr{B}(\mathbb{R}^2), \qquad \mathbb{P}_{X,Y}(B) = \mathbb{P}((X, Y)^{-1}(B)) = \mathbb{P}(\{(X, Y) \in B\}).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "{(X,Y)}"] & & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

On $\mathbb{P}_{X,Y}$

$\mathbb{P}_{X,Y}$ is a **probability measure** on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$.

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Joint CDF

Definition (Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector.
The **joint CDF of X and Y (or CDF of the vector (X, Y))** is a function $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ defined as

$$\forall\, x, y \in \mathbb{R}, \qquad F_{X,Y}(x, y) = \mathbb{P}_{X,Y}\Big((-\infty, x] \times (-\infty, y]\Big) = \mathbb{P}\Big(\{X \leq x\} \cap \{Y \leq y\}\Big).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "{(X,Y)}"] & & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
\{X \leq x\} \cap \{Y \leq y\} \arrow[u] & & (-\infty, x] \times (-\infty, y] \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}((-\infty, x] \times (-\infty, y]) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}), \quad x, y \in \mathbb{R}$$

Properties of Joint CDF

Lemma (Properties of Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector with CDF $F_{X,Y}$.
Then, $F_{X,Y}$ satisfies the following properties.

1. **(Monotonicity)** If $x_1 \le x_2$ and $y_1 \le y_2$, then $F_{X,Y}(x_1, y_1) \le F_{X,Y}(x_2, y_2)$.
2. If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are any two sequences such that $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} F_{X,Y}(x_n, y_n) = 0$.
3. If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are any two sequences such that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = +\infty$, then $\lim_{n\to\infty} F_{X,Y}(x_n, y_n) = 1$.
4. **(Continuity from the Top-Right Quadrant)**

   $F_{X,Y}$ is continuous from the top-right quadrant at each point in its domain.
   More formally, for each $(x, y) \in \mathbb{R}^2$,

$$x_n > x \ \forall n \in \mathbb{N}, \quad y_n > y \ \forall n \in \mathbb{N}, \quad \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y \implies \lim_{n\to\infty} F_{X,Y}(x_n, y_n) = F_{X,Y}(x, y).$$

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Another Important Function

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), {\color{red}\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$ {\color{red}\mathbb{P}_{X,Y}}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2) $$

- Taking $B = (-\infty,\ x] \times (-\infty,\ y]$, and varying x, y, we get a mapping

$$ (x, y) \quad \mapsto \quad \mathbb{P}_{X,Y}\big((-\infty,\ x] \times (-\infty,\ y]\big) $$

- The above map is called the **joint CDF**, denoted $F_{X,Y}$
- $F_{X,Y}(x, y) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\})$

- Taking $B = \{x\} \times \{y\}$, and varying x, y, we get a mapping

$$ x \quad \mapsto \quad \mathbb{P}_{X,Y}(\{x\} \times \{y\}) $$

- The above map is called the **joint PMF**, denoted $p_{X,Y}$
- $p_{X,Y}(x, y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})$

Joint PMF

Definition (Joint PMF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector.
Let $\mathbb{P}_{X,Y}$ denote the joint probability law of X and Y.
The **joint PMF of X and Y (or PMF of the vector (X, Y))** is a function $p_{X,Y} : \mathbb{R}^2 \to [0, 1]$ defined as

$$\forall\, x, y \in \mathbb{R}, \qquad p_{X,Y}(x, y) = \mathbb{P}_{X,Y}(\{x\} \times \{y\}) = \mathbb{P}(\{X = x\} \cap \{Y = y\}).$$

- **Joint CDF ($F_{X,Y}$) and joint PMF ($p_{X,Y}$) are always defined for any two RVs X and Y**

Marginal CDFs from Joint CDF

Theorem (Marginal CDFs from Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector. Let $F_{X,Y}$ denote the joint CDF of X and Y.
Then, the following properties hold.

1. **(Marginalization of Y)**

   If $y_1, y_2, \ldots$ is any sequence of real numbers such that $\lim_{n\to\infty} y_n = +\infty$, then

$$\forall\, x \in \mathbb{R}, \qquad \lim_{n\to\infty} F_{X,Y}(x,\ y_n) = F_X(x).$$

2. **(Marginalization of X)**

   If $x_1, x_2, \ldots$ is any sequence of real numbers such that $\lim_{n\to\infty} x_n = +\infty$, then

$$\forall\, y \in \mathbb{R}, \qquad \lim_{n\to\infty} F_{X,Y}(x_n,\ y) = F_Y(y).$$

Conditional CDF

Definition (Conditional CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a random vector.

1. Fix $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$.
   The **conditional CDF of X, conditioned on A**, is defined as

$$F_{X|A} : \mathbb{R} \to [0, 1], \qquad F_{X|A}(x) := \frac{\mathbb{P}(\{X \leq x\} \cap A)}{\mathbb{P}(A)}, \qquad x \in \mathbb{R}.$$

2. The **conditional CDF of X, conditioned on Y**, is defined as

$$\forall\, x, y \in \mathbb{R}, \qquad F_{X|Y}(x|y) := \frac{F_{X,Y}(x, y)}{F_Y(y)} = \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \leq y\})}{\mathbb{P}(\{Y \leq y\})},$$

   whenever denominator is non-zero.

Independence of Random Variables

σ-Algebra Generated by a Random Variable

Definition (σ-Algebra Generated by a Random Variable)

Fix a measurable space $(\Omega, \mathscr{F})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable.
The **σ-algebra generated by X**, denoted $\sigma(X)$, is defined as

$$\sigma(X) := \left\{ X^{-1}(B) : \;\; B \in \mathscr{B}(\mathbb{R}) \right\}.$$

Interpretation

$\sigma(X)$ is the collection of all events whose occurrence or non-occurrence may be decided purely based on the realization of X.

- $\sigma(X)$ is a σ-algebra of subsets of Ω (see Homework 4, Question 3)

Independence of Two Random Variables

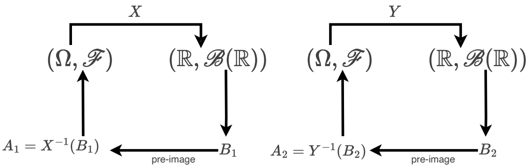
Definition (Independence of Random Variables)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
We say that **X and Y are independent random variables** (in the language of $\mathbb{P}$) if

$$\sigma(X) \perp\!\!\!\perp \sigma(Y), \qquad \text{i.e.,} \qquad \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\,\mathbb{P}(A_2) \quad \forall A_1 \in \sigma(X),\ A_2 \in \sigma(Y).$$

Equivalently, in the language of probability laws,

$$\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_X(B_1)\,\mathbb{P}_Y(B_2) \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$



Independence Simplified

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
\{X \leq x\} \arrow[u] & (-\infty,\, x] \arrow[l, "\text{pre-image}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "Y"] & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
\{X \leq y\} \arrow[u] & (-\infty,\, y] \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

Proposition (Independence Simplified)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
X and Y are independent random variables **if and only if**

$$\forall\, x, y \in \mathbb{R}, \qquad \mathbb{P}_{X,Y}\big((-\infty,\, x] \times (-\infty,\, y]\big) = \mathbb{P}_X\big((-\infty,\, x]\big)\, \mathbb{P}_Y\big((-\infty,\, y]\big).$$

Equivalently, in the language of $\mathbb{P}$,

$$\forall\, x, y \in \mathbb{R}, \quad \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}) = \mathbb{P}(\{X \leq x\})\, \mathbb{P}(\{Y \leq y\}) \iff F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y).$$

Jointly Discrete Random Variables

Jointly Discrete Random Variables

Definition (Jointly Discrete Random Variables)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
X and Y are said to be **jointly discrete** if the vector (X, Y) is a discrete random variable, i.e., there exists a **countable** set $E \subseteq \mathbb{R}^2$ such that

$$\mathbb{P}_{X,Y}(E) = \mathbb{P}(\{(X, Y) \in E\}) = 1.$$

- Define E_1 and E_2 as

$$E_1 := \{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } (x, y) \in E\}, \qquad E_2 = \{y \in \mathbb{R} : \exists x \in \mathbb{R} \text{ such that } (x, y) \in E\}.$$

- If (X, Y) is discrete, then X and Y are discrete with

$$\mathbb{P}_X(E_1) = 1, \qquad \mathbb{P}_Y(E_2) = 1.$$

- (X, Y) discrete $\quad\Longrightarrow\quad$ X discrete, $\;Y$ discrete.

Jointly Discrete Random Variables

Definition (Jointly Discrete Random Variables)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a bivariate random vector.
X and Y are said to be **jointly discrete** if the vector (X, Y) is a discrete random variable, i.e., there exists a **countable** set $E \subseteq \mathbb{R}^2$ such that

$$\mathbb{P}_{X,Y}(E) = \mathbb{P}(\{(X, Y) \in E\}) = 1.$$

- Suppose X and Y are individually discrete with

$$\mathbb{P}_X(E_1) = 1, \qquad \mathbb{P}_Y(E_2) = 1,$$

  for some countable sets $E_1, E_2 \subset \mathbb{R}$
- Then, (X, Y) is discrete: $\quad E_1 \times E_2$ countable, $\quad \mathbb{P}_{X,Y}(E_1 \times E_2) = 1.$
- X discrete, $\;Y$ discrete $\qquad \Longrightarrow \qquad (X, Y)$ discrete .

Marginal PMFs and Conditional PMFs

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Marginal PMFs from Joint PMF

For Jointly Discrete Random Variables

Theorem (Marginal PMFs from Joint PMF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a discrete random variable with a countable range $E \subset \mathbb{R}^2$. Then, the following properties hold.

1. The joint PMF on the range must sum to 1, i.e.,

$$\sum_{x,y:\ (x,y) \in E} p_{X,Y}(x,y) = 1.$$

2. **(Marginalization Property)**

$$\forall\, x \in \mathbb{R}, \qquad \sum_{y:\ (x,y) \in E} p_{X,Y}(x,y) = p_X(x),$$

$$\forall\, y \in \mathbb{R}, \qquad \sum_{x:\ (x,y) \in E} p_{X,Y}(x,y) = p_Y(y).$$

Independence for Jointly Discrete Random Variables

Proposition (Independence for Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $(X, Y) : \Omega \to \mathbb{R}^2$ be a discrete random variable with a countable range $E \subset \mathbb{R}^2$. Then,

$$X \perp\!\!\!\perp Y \qquad \Longleftrightarrow \qquad p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y) \quad \forall x, y \in \mathbb{R}.$$

- If $X \perp\!\!\!\perp Y$, then by definition,

$$\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_X(B_1) \cdot \mathbb{P}_Y(B_2) \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

- Taking $B_1 = \{x\}$ and $B_2 = \{y\}$, we get

$$\underbrace{\mathbb{P}_{X,Y}(\{x\} \times \{y\})}_{p_{X,Y}(x,y)} = \underbrace{\mathbb{P}_X(\{x\})}_{p_X(x)} \cdot \underbrace{\mathbb{P}_Y(\{y\})}_{p_Y(y)}$$

- This proves the $\implies$ direction