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Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 11: Measurable Function, Random Variable, Probability Law, Cumulative Distribution Function (CDF)

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Measurable Function

Definition (Measurable Function)

Let $(E, \mathscr{E})$ and $(G, \mathscr{G})$ be two measurable spaces. Consider a function $f : E \to G$.
The function f is said to be **measurable** if

$$\forall\, B \in \mathscr{G}, \qquad \underbrace{f^{-1}(B)}_{\text{pre-image of } B} = \{e \in E : f(e) \in B\} \in \mathscr{E}.$$

$$\begin{tikzcd}
(E, \mathscr{E}) \arrow[rr, "f"] & & (G, \mathscr{G}) \arrow[d] \\
f^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

Random Variable

Definition (Random Variable)

Fix a measurable space $(\Omega, \mathscr{F})$.
A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if it is measurable, i.e.,

$$\forall\, B \in \mathscr{B}(\mathbb{R}), \qquad \underbrace{X^{-1}(B)}_{\text{pre-image of } B} = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}.$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[rr, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

Random Variable

- A random variable is neither random nor a variable; it is a **deterministic function**
- A random variable assigns numerical values to outcomes
- The definition of a random variable is closely tied to the underlying σ-algebra $\mathscr{F}$
- If X is a random variable with respect to $\mathscr{F}$, it is said to be **$\mathscr{F}$-measurable**
- The definition of a random variable does not involve $\mathbb{P}$

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Properties of a Random Variable

Proposition (Random Variable Properties)

Let $(\Omega, \mathscr{F})$ be a measurable space, and let $X : \Omega \to \mathbb{R}$ be a random variable.

1. For any $B \subseteq \mathbb{R}$, $\quad X^{-1}(B^{\complement}) = \left(X^{-1}(B)\right)^{\complement}$.
2. For any $B_1 \subseteq \mathbb{R}, B_2 \subseteq \mathbb{R}, \ldots,$
$$X^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \bigcup_{n\in\mathbb{N}} X^{-1}(B_n).$$
3. Let $\mathscr{B}_1$ denote the collection
$$\mathscr{B}_1 := \left\{B \subseteq \mathbb{R} : X^{-1}(B) \in \mathscr{F}\right\}. \tag{1}$$

   Then, $\mathscr{B}_1$ is a σ-algebra of subsets of $\mathbb{R}$. Furthermore, $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{B}_1$.

Proof of Proposition – 1

$$
\begin{aligned}
\omega' \in X^{-1}(B^{\mathsf{C}}) \quad &\iff \quad && \iff \quad X(\omega') \notin B \\
&\iff \quad \omega' \notin X^{-1}(B) \\
&\iff \quad \omega' \in \Omega \setminus X^{-1}(B) \\
&\iff \quad \omega' \in \left(X^{-1}(B)\right)^{\mathsf{C}}.
\end{aligned}
$$

Proof of Proposition – 2

$$
\begin{aligned}
\omega' \in X^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right) &\iff \exists\, n \in \mathbb{N} : \; X(\omega') \in B_n \\
&\iff \exists\, n \in \mathbb{N} : \; \omega' \in X^{-1}(B_n) \\
&\iff \omega' \in \bigcup_{n\in\mathbb{N}} X^{-1}(B_n).
\end{aligned}
$$

Proof of Proposition – 3

- To show that $\emptyset \in \mathscr{B}_1$, note that
$$X^{-1}(\emptyset) = \emptyset \in \mathscr{F}.$$
- Suppose $B \in \mathscr{B}_1$. That is, by definition, $X^{-1}(B) \in \mathscr{F}$.
Then, note that
$$X^{-1}(B^{\complement}) = \left(X^{-1}(B)\right)^{\complement} \in \mathscr{F}.$$
This proves that $B^{\complement} \in \mathscr{B}_1$.
- For any $B_1, B_2, \ldots \in \mathscr{B}_1$, we have
$$X^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \bigcup_{n\in\mathbb{N}} X^{-1}(B_n) \in \mathscr{F}.$$
This proves that $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}_1$.

Generating Classes for $\mathscr{B}(\mathbb{R})$

$\mathscr{B}(\mathbb{R})$

$\mathscr{P}_1 = \left\{(a,b) : \ a, b \in \mathbb{R}, \ a \leq b\right\}$

$\mathscr{P}_2 = \left\{[a,b] : \ a, b \in \mathbb{R}, \ a \leq b\right\}$

$\mathscr{P}_3 = \left\{[a,b) : \ a, b \in \mathbb{R}, \ a \leq b\right\}$

$\mathscr{P}_4 = \left\{(a,b] : \ a, b \in \mathbb{R}, \ a \leq b\right\}$

$\mathscr{P}_5 = \left\{(-\infty, \ x) : \ x \in \mathbb{R}\right\}$

$\mathscr{P}_6 = \left\{(-\infty, \ x] : \ x \in \mathbb{R}\right\}$

$\mathscr{P}_7 = \left\{(x, \ +\infty) : \ x \in \mathbb{R}\right\}$

$\mathscr{P}_8 = \left\{[x, \ +\infty) : \ x \in \mathbb{R}\right\}$

Equivalent Definitions of Random Variable

Fix a measurable space $(\Omega, \mathscr{F})$.

Theorem (Equivalent Definitions of Random Variable)

$X : \Omega \to \mathbb{R}$ is a random variable if and only if:

1. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_1$.
2. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_2$.
3. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_3$.
4. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_4$.
5. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_5$.
6. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_6$.
7. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_7$.
8. $X^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{P}_8$.

Proof of Theorem (Considering $\mathscr{P}_6$)

- Recall that

$$\mathscr{P}_6 = \Big\{(-\infty,\, x] \,:\, x \in \mathbb{R}\Big\}.$$

- If X is a random variable, then by definition,

$$\forall\, B \in \mathscr{B}(\mathbb{R}), \qquad X^{-1}(B) \in \mathscr{F}.$$

  Because $\mathscr{P}_6 \subseteq \mathscr{B}(\mathbb{R})$, it follows that

$$X \text{ random variable} \quad \Longrightarrow \quad X^{-1}\Big((-\infty, x]\Big) \in \mathscr{F} \quad \forall\, x \in \mathbb{R}.$$

Proof of Theorem (Considering $\mathscr{P}_6$)

- Suppose now that $X^{-1}\Big((-\infty,\ x]\Big) \in \mathscr{F}$ for all $x \in \mathbb{R}$.

  In other words,
  $$X^{-1}(B) \in \mathscr{F} \qquad \forall\, B \in \mathscr{P}_6.$$

- This implies that
  $$\mathscr{P}_6 \subseteq \mathscr{B}_1 \qquad \text{(defined in (1))}.$$

- In turn, this implies that
  $$\sigma(\mathscr{P}_6) \subseteq \sigma(\mathscr{B}_1), \qquad \text{i.e.,} \qquad \mathscr{B}(\mathbb{R}) \subseteq \mathscr{B}_1.$$

- This verifies that
  $$X^{-1}(B) \in \mathscr{F}\ \ \forall\, B \in \mathscr{P}_6 \quad \Longrightarrow \quad X^{-1}(B) \in \mathscr{F}\ \ \forall\, B \in \mathscr{B}(\mathbb{R}) \quad \Longrightarrow \quad X \text{ random variable.}$$

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Random Variable Simplified

Definition (Random Variable)

Fix a measurable space $(\Omega, \mathscr{F})$.
A function $X : \Omega \to \mathbb{R}$ is called a random variable with respect to $\mathscr{F}$ if and only if

$$\forall x \in \mathbb{R}, \qquad \underbrace{X^{-1}((-\infty, x])}_{\text{pre-image of } (-\infty,\, x]} = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathscr{F}.$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[rr, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & & (-\infty, x] \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

Examples

- $\Omega = \{1, 2, \ldots, 6\}, \qquad \mathscr{F} = \Big\{\emptyset, \Omega\Big\}, \qquad X(\omega) = \omega$

  Is X a random variable with respect to $\mathscr{F}$?

- What functions X are random variables with respect to $\mathscr{F}$?

Examples

- $\Omega = [0,1], \qquad \mathscr{F} = \left\{\emptyset, \Omega, A, A^c\right\}$ for a fixed $A \subseteq \Omega$

  What functions X are random variables with respect to $\mathscr{F}$?

Examples

- $\Omega = \{1, 2, 3, 4, 5\}, \qquad \mathscr{F} = \sigma\left(\left\{\{1\}, \{2, 3\}\right\}\right)$
  What functions X are random variables with respect to $\mathscr{F}$?

Examples

- $\Omega = \mathbb{N}, \qquad \mathscr{F} = 2^{\Omega}$
  What functions X are random variables with respect to $\mathscr{F}$?

Examples

- Provide an example construction of $(\Omega, \mathscr{F})$ and a function $X : \Omega \to \mathbb{R}$ that is NOT a random variable (with respect to $\mathscr{F}$).

Indicator Functions

Fix a sample space Ω.
Fix a subset $A \subseteq \Omega$.

Definition (Indicator Function)

The **indicator function** of set A is the function $\mathbf{1}_A : \Omega \to \mathbb{R}$ defined as

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

Exercise

Fix a measurable space $(\Omega, \mathscr{F})$. Show that

$$\mathbf{1}_A \text{ is a random variable} \quad \Longleftrightarrow \quad A \in \mathscr{F}.$$