



# Probability and Stochastic Processes

Open Quiz 01

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## Question

Suppose  $X, Y, Z$  are IID continuous random variables with a common PDF  $f$  and common CDF  $F$ .

Determine  $\mathbb{P}(\{X < Y\})$ .

Show your working clearly.

**Note:** Do not assume any specific continuous distribution in your working.

## Question

A stick of unit length is broken uniformly at random.

**Evaluate**

$$\mathbb{P}\left(\text{larger piece has length} > \frac{3}{4} \mid \text{smaller piece has length} < \frac{1}{3}\right)$$

Show your working clearly.

## Question

Let  $Z \sim \mathcal{N}(0, 1)$ . Let  $c \in \mathbb{R}$  be a fixed constant.

Suppose that

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{c^2}{2}\right) - c(1 - \Phi(c)) = \mathbb{E}[f(Z)].$$

Express the function  $f$  in closed form. Show your working clearly.

**Note:**  $\Phi(\cdot)$  denotes the CDF of a standard Normal distribution.

## Question

Which of the following is larger?

$$\mathbb{E} \left[ \max_{\ell \in \{1, \dots, n\}} X_\ell \right] \quad \text{versus} \quad \max_{\ell \in \{1, \dots, n\}} \mathbb{E}[X_\ell].$$

Justify clearly.

## Question

Fix numbers  $a_1, a_2, \dots \in (0, 1)$ .

**Determine**

$$\sum_{i=1}^{\infty} a_i \prod_{j=1}^{i-1} (1 - a_j) + \prod_{i=1}^{\infty} (1 - a_i).$$

Justify your answer clearly.

## Error in ChatGPT's Solution

Identify the error in the following sequence of steps. Clearly explain what is the error.

Suppose that  $X_1, X_2, X_3 \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 1)$ . Then,

$$\begin{aligned}\mathbb{P}(X_1 > X_2 > X_3) &= \int_0^1 \mathbb{P}(\{X_1 > x > X_3\} \cdot f_{X_2}(x) \, dx \\ &= \int_0^1 \mathbb{P}(\{X_1 > x\} \cap \{X_3 < x\}) \cdot f_{X_2}(x) \, dx \\ &= \int_0^1 \mathbb{P}(\{X_1 > x\}) \cdot \mathbb{P}(\{X_3 < x\}) \cdot f_{X_2}(x) \, dx \\ &= \int_0^1 \mathbb{P}(\{X_1 > x\} \mid \{X_2 = x\}) \cdot \mathbb{P}(\{X_3 < x\} \mid \{X_2 = x\}) \cdot f_{X_2}(x) \, dx \\ &= \int_0^1 \mathbb{P}(\{X_1 > X_2\} \mid \{X_2 = x\}) \cdot \mathbb{P}(\{X_3 < X_2\} \mid \{X_2 = x\}) \cdot f_{X_2}(x) \, dx \\ &= \mathbb{P}(\{X_1 > X_2\}) \cdot \mathbb{P}(\{X_2 > X_3\}).\end{aligned}$$

## Saddle Point

The **saddle point** of a matrix is an entry that is both the **smallest in its row** and **largest in its column**. For example, the  $(1, 1)$  entry of the below matrix is a saddle point.

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -2 & 6 \\ 0.5 & 12 & 3 \end{pmatrix}.$$

Given an  $m \times n$  matrix whose entries are chosen IID from an arbitrary **continuous distribution**, what is the probability that the  $(1, 1)$  entry is a saddle point?

Show your working clearly.

## Stopping Times

Fix  $c \in (0, 1)$ . Let  $X_1, X_2, \dots \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 1)$ , and let

$$N := \min\{n \geq 1 : X_n > c\}.$$

Compute  $\mathbb{E}[X_N]$ .

Show your working clearly.

## Bayes Classification

Let  $f_1$  and  $f_2$  be two PDFs defined as follows:

$$f_1(y) = \begin{cases} 1, & y \in (0, 1), \\ 0, & \text{otherwise,} \end{cases} \quad f_2(y) = \begin{cases} \frac{2}{3}(y+1), & y \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

A **binary classifier**  $T : (0, 1) \rightarrow \{1, 2\}$  is designed as follows:

$$T(y) = \begin{cases} 1, & \frac{f_1(y)}{f_2(y)} > \tau, \\ 2, & \text{otherwise,} \end{cases} \quad \tau : \text{hyperparameter.}$$

A data sample  $Y \in (0, 1)$  is observed, and needs to be labeled 1 or 2.

If  $f_1$  is twice as likely as  $f_2$ , what value of  $\tau$  yields the smallest probability of error in labeling?



## Randomised Permutation

$n$  students of AI@IITH use the DGX server for resource-intensive AI research.

Each student's account on the server is protected by a unique username and password.

During one of the campus-wide power shutdown, the passwords of the  $n$  students are randomly permuted such that each of the  $n!$  permutations is equally likely.

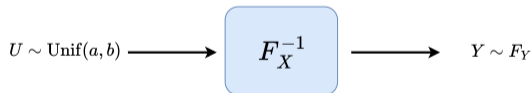
Only those students lucky enough to have had their passwords unchanged in the permutation can continue to use the server.

What is the expected number of students who can continue to use the server?

## Restricted Sampling

Suppose that  $X \sim \text{Exponential}(1)$ . Let  $Y$  be a random variable whose CDF is given by

$$F_Y(y) = \begin{cases} 0, & y \leq 2, \\ \frac{F_X(y) - F_X(2)}{F_X(3) - F_X(2)}, & 2 < y < 3, \\ 1, & y \geq 3. \end{cases}$$



What should be the values of  $a$  and  $b$  so that the output random variable  $Y$  has its desired CDF.

Clearly show in your working why the output random variable  $Y$  will have the desired PDF.

## Expectation of Inverse is Expectation

Construct a non-constant random variable  $X$  for which

$$\mathbb{E} \left[ \frac{1}{X} \right] = \mathbb{E}[X].$$

Show your working clearly