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Indian Institute of Technology Hyderabad

Stochastic Processes

Stopping Times, Wald’s Lemma, Strong Independence Property, Properties of Stopping Times

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28 February 2025

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Filtrations

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $\mathcal{T}$ be an ordered index set.

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Definition (Filtration)

Consider a collection of σ-algebras $\mathscr{G}_\bullet = \{\mathscr{G}_t : t \in \mathcal{T}\}$ such that $\mathscr{G}_t \subseteq \mathscr{F}$ for all t.

The above collection is called a filtration if

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Example:

Let $\{X_t : t \in \mathcal{T}\}$ be a stochastic process defined w.r.t. $\mathscr{F}$. Then,

$$\mathscr{G}_t = \sigma(X_s : s \leq t)$$

is called the natural filtration associated with the process $\{X_t : t \in \mathcal{T}\}$.

Stopping Time

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Fix a filtration $\mathscr{G}_\bullet = \{\mathscr{G}_t : t \in \mathcal{T}\}$.

Definition (Stopping Time)

A random variable τ is called a stopping time w.r.t. the filtration $\mathscr{G}_\bullet$ if:

- $\mathbb{P}(\tau < +\infty) = 1$.
- For each $t \in \mathcal{T}$,

$$\{\tau \leq t\} \in \mathscr{G}_t.$$

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If $\mathcal{G}_t = \sigma(X_s : s \leq t)$, then the question "is $\tau \leq t$?" can be answered by simply looking at the process up to time t.

Stopping Time w.r.t. a Process

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\mathcal{T}$ be an ordered index set.
Let $\{X_t : t \in \mathcal{T}\}$ be a process w.r.t. $\mathscr{F}$.

Definition (Stopping Time w.r.t. a Process)

A random variable τ is called a stopping time w.r.t. the process $\{X_t : t \in \mathcal{T}\}$ if:

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- For each $t \in \mathcal{T}$,
$$\{\tau \le t\} \in \sigma(X_s : s \le t).$$
That is, the question "is $\tau \le t$?" can be answered by simply looking at the process up to time t.

Examples

- Let $\{X_n : n \in \mathbb{N}\}$ be a process.
  Fix a set $A \subseteq \mathbb{R}$.
  Let τ_X^A be defined as
  $$\tau_X^A := \inf\{n \in \mathbb{N} : X_n \in A\}.$$
  Is τ_X^A a stopping time w.r.t. the process $\{X_n : n \in \mathbb{N}\}$?

Discrete Stopping Times

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Lemma (Discrete Stopping Times)

A discrete random variable τ is a stopping time w.r.t. $\{\mathscr{G}_n\}_{n=1}^{\infty}$ if and only if

$$\mathbb{P}(\tau < +\infty) = 1, \qquad \{\tau = n\} \in \mathscr{G}_n \quad \forall n \in \mathbb{N}.$$

$$\Updownarrow$$

$$\mathbb{P}(\tau < +\infty) = 1, \qquad \{\tau \leq n\} \in \mathscr{G}_n \quad \forall n \in \mathbb{N}.$$

Proof of Lemma

Assume $\mathbb{P}(\tau < +\infty) = 1$.

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  - Observe that for any $n \in \mathbb{N}$,

$$\{\tau \leq n\} = \bigcup_{k=1}^{n} \{\tau = k\}$$

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$$\{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n - 1\}$$

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  - Observe that

$$\{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n-1\}$$

  - $\{\tau = n-1\} \in \mathscr{G}_{n-1} \subset \mathscr{G}_n$

Example

- Let $\{X_n\}_{n=1}^{\infty}$ be an $\mathbb{N}$-valued process.
  Fix $y \in \mathbb{N}$.
  Let $\tau_y^{(0)} := 0$, and

$$\tau_y^{(k)} = \inf\{n > \tau_y^{(k-1)} : X_n = y\}, \qquad k \in \mathbb{N}.$$

  For each $k \in \mathbb{N}$, prove that $\tau_y^{(k)}$ is a stopping time w.r.t. the process $\{X_n\}_{n=1}^{\infty}$.

Wald's Lemma

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Wald's Lemma

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Lemma (Wald's Lemma [Wal44, Wal45])

Let $\{X_n\}_{n=1}^{\infty}$ be an IID process w.r.t. $\mathscr{F}$, with $\mathbb{E}|X_1| < +\infty$.
For each $n \in \mathbb{N}$, let

$$S_n = \sum_{i=1}^{n} X_i.$$

If τ is a stopping time w.r.t. the process $\{X_n\}_{n=1}^{\infty}$, with $\mathbb{E}|\tau| < +n\infty$, then

$$\mathbb{E}[S_\tau] = \mathbb{E}\left[\sum_{i=1}^{\tau} X_i\right] = \mathbb{E}[\tau] \cdot \mathbb{E}[X_1].$$

Example

- Suppose $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim} \text{Geometric}(0.5)$.
  For each $n \in \mathbb{N}$, let $S_n = \sum_{i=1}^{n} X_i$.
  Let τ be defined as

$$\tau := \inf\left\{n \geq 1 : S_n = 33\right\}.$$

  Determine $\mathbb{E}[\tau]$.

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