AI 5090: STOCHASTIC PROCESSES

LECTURE 01

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Contents

We begin with a review of some basic concepts from real analysis such as supremum and infimum of a set of real numbers, limit supremum, limit infimum, and limit of a sequence of real numbers.

1 Notation

We will use the following notations throughout the course.

- $\mathbb{N} = \{1, 2, 3, \ldots\}$: the set of **natural** numbers.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$: the set of **integers**. Historically, the usage of the symbol $\mathbb{Z}$ (rather than the more natural symbol $\mathbb{I}$) to denote the set of integers has its roots in German literature, where integers (or "numbers" more generally) are referred to as *Zahlen*.
- $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\right\}$: the set of **rational** numbers.
- $\mathbb{R}$: the set of **real** numbers.
- $\mathbb{R} \cup \{\pm\infty\}$: the set of **extended real** numbers.

2 Supremum and Infimum of a Set of Real Numbers

We begin this section with a formal definition of a **sequence** of real numbers.

Definition 1 (Sequence of Real Numbers)**.** *A sequence of real numbers is mapping (function) $f : \mathbb{N} \to \mathbb{R}$ from the set of natural numbers $\mathbb{N}$ to the set of real numbers $\mathbb{R}$. That is, a sequence defines a list of real numbers, one corresponding to every natural number.*

Consider a real sequence in which the element of the sequence corresponding to the natural number n is denoted by $f(n) = a_n$. Such a sequence is represented in shorthand notation as $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n\in\mathbb{N}}$ or $\{a_n : n \geq 1\}$ or $\{a_n : n \in \mathbb{N}\}$. In this sense, a real sequence may be perceived as a countably infinite set of real numbers.

Let $A \subseteq \mathbb{R}$ be any subset of real numbers. A number $l \in \mathbb{R} \cup \{\pm\infty\}$ is said to be a **lower bound** for the set A if $x \geq l$ for all $x \in A$. In other words, l is a lower bound for every element of A. Likewise, a number $u \in \mathbb{R} \cup \{\pm\infty\}$ is said to be an **upper bound** for the set A if $x \leq u$ for all $x \in A$. That is, u is an upper bound for every element of A.

We now present the formal definition of the supremum of a subset of real numbers.

Definition 2 (Supremum)**.** *The **supremum** of a set of real numbers, denoted by* $\sup A$*, is an element $x^\star \in \mathbb{R} \cup \{\pm\infty\}$ such that:*

1. *$x^\star$ is an upper bound for the set A, and*

2. *$x^\star$ is the **least upper bound** for the set A. That is, there exists no upper bound for A lesser than $x^\star$. Formally, for every choice of $\varepsilon > 0$, there exists an element $x \in A$ (potentially depending on ε) such that $x > x^\star - \varepsilon$.*

Note that the supremum of a set A may not necessarily belong to A. Below, we present some examples.

- If $A = (0, 1)$, then $\sup A = 1$. To see this formally, note that $x \leq 1$ for all $x \in A$. Furthermore, for any $\varepsilon > 0$, the number $1 - \varepsilon$ is not an upper bound for A, as the number $x = 1 - \frac{\varepsilon}{2}$ is an element of A and is greater than $1 - \varepsilon$. Therefore, 1 is the least upper bound for A. Note that the supremum is not an element of A in this example.

- If $A = \{1, 2, 3\}$, then $\sup A = 3$. In this case, because the supremum belongs to A, it is referred to as the **maximum**.

- If $A = \mathbb{N}$, then $\sup A = +\infty$.

- If $A = \emptyset$, then by convention, $\sup A = -\infty$.

- If $A = \{a_n\}_{n=1}^{\infty}$, where $a_n = 1 - \frac{1}{n}$ for every $n \in \mathbb{N}$, then $\sup A = 1$. To see this formally, note that $a_n \leq 1$ for all $n \in \mathbb{N}$. Furthermore, for any $\varepsilon > 0$, the number $1 - \varepsilon$ is not an upper bound for A. Clearly,

$$1 - \frac{1}{n} > 1 - \varepsilon \qquad \Longleftrightarrow \qquad n > \frac{1}{\varepsilon}.$$

  In particular, the element $1 - \frac{1}{\lceil 1/\varepsilon \rceil + 1}$ is an element of A, and is larger than $1 - \varepsilon$, thereby proving that $1 - \varepsilon$ is **NOT** an upper bound for A. Here, $\lceil x \rceil$ denotes the **ceil** of x, i.e., the smallest integer greater than or equal to x.

We now present the definition of the **infimum** of a subset of real numbers.

Definition 3 (Infimum). *The **infimum** of a set of real numbers, denoted by* $\inf A$, *is an element $x_\star \in \mathbb{R} \cup \{\pm\infty\}$ such that:*

1. *$x_\star$ is a lower bound for the set A, and*

2. *$x_\star$ is the **greatest lower bound** for the set A. That is, there exists no lower bound for A greater than $x_\star$. Formally, for every choice of $\varepsilon > 0$, there exists an element $x \in A$ (potentially depending on ε) such that $x < x_\star + \varepsilon$.*

Just as in the case of supremum, the infimum of a set A may not belong to A. We now present some examples.

- If $A = (0, 1)$, then $\inf A = 0$. To see this formally, note that $x \geq 0$ for all $x \in A$. Furthermore, for any $\varepsilon > 0$, the number $0 + \varepsilon = \varepsilon$ is not an upper bound for A, as the number $x = \frac{\varepsilon}{2}$ is an element of A and is lesser than ε. Therefore, 0 is the least upper bound for A. Note that the supremum is not an element of A in this example.

- If $A = \{1, 2, 3\}$, then $\inf A = 1$. In this case, because the infimum belongs to A, it is referred to as the **minimum**.

- If $A = \mathbb{N}$, then $\inf A = 1$.

- If $A = \emptyset$, then by convention, $\inf A = +\infty$.

- If $A = \{a_n\}_{n=1}^{\infty}$, where $a_n = 1 - \frac{1}{n}$ for every $n \in \mathbb{N}$, then $\inf A = 0$.

Remark 1. *It is customary to denote the supremum of a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ as $\sup_{n \geq 1} a_n$. Similarly, the infimum of the sequence $\{a_n\}_{n=1}^{\infty}$ is typically denoted by $\inf_{n \geq 1} a_n$.*

It is an easy exercise to show that the infimum of a non-empty subset of real numbers is lesser than or equal to its supremum. We state this result formally below, leaving the proof as exercise.

Theorem 4. *For any non-empty set $A \subseteq \mathbb{R}$, we have $\inf A \leq \sup A$.*
As a consequence, for any real sequence $\{a_n\}_{n=1}^{\infty}$, we have $\inf_{n \geq 1} a_n \leq \sup_{n \geq 1} a_n$.

Question 1. *When does equality hold in the inequalities of Theorem 4?*

3 Limit Infimum, Limit Supremum, and Limit of a Real Sequence

In this section, we define the notions of limit supremum, limit infimum, and limit of a sequence of real numbers.

Definition 5 (Limit Infimum)**.** *The* **limit infimum** *of a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers, denoted as* $\liminf_{n\to\infty} a_n$*, is defined as*

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1}\ \inf_{k\geq n} a_k.$$

For any fixed $n \in \mathbb{N}$, let y_n denote the inner infimum on the right-hand side of the above equation, i.e., $y_n = \inf_{k\geq n} a_k$. To compute the lim inf of the sequence $\{a_n\}_{n=1}^{\infty}$, we first compute y_n for each $n \in \mathbb{N}$, and then take the supremum of the sequence $\{y_n\}_{n=1}^{\infty}$.

We note the following lemma in connection with the limit infimum of a sequence.

Lemma 1. *Fix* $\underline{L} \in \mathbb{R}$*, and suppose that* $\liminf_{n\to\infty} a_n = \underline{L}$*. Then, for every choice of* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *(possibly depending on the choice of* ε*) such that*

$$a_k > \underline{L} - \varepsilon \qquad \forall k \geq N.$$

Proof of Lemma 1. For any $n \in \mathbb{N}$, let $y_n = \inf_{k\geq n} a_k$. Then, as per the definition of lim inf, we have $\underline{L} = \sup_{n\geq 1} y_n$. This implies that $\underline{L}$ is an upper bound for $\{y_n\}_{n=1}^{\infty}$, and for every choice of $\varepsilon > 0$, the number $\underline{L} - \varepsilon$ is not an upper bound for $\{y_n\}_{n=1}^{\infty}$. That is, for every choice of $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (possibly depending on the choice of ε) such that $y_N > \underline{L} - \varepsilon$, or equivalently $\inf_{k\geq N} a_k > \underline{L} - \varepsilon$. Noting that

$$\inf_{k\geq N} a_k > \underline{L} - \varepsilon \qquad \text{implies} \qquad a_k > \underline{L} - \varepsilon \qquad \forall k \geq N$$

completes the desired proof. □

The limit supremum of a sequence is defined as below.

Definition 6 (Limit Supremum)**.** *The* **limit supremum** *of a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers, denoted as* $\limsup_{n\to\infty} a_n$*, is defined as*

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1}\ \sup_{k\geq n} a_k.$$

For any fixed $n \in \mathbb{N}$, let z_n denote the inner supremum on the right-hand side of the above equation, i.e., $z_n = \sup_{k\geq n} a_k$. To compute the lim sup of the sequence $\{a_n\}_{n=1}^{\infty}$, we first compute z_n for each $n \in \mathbb{N}$, and then take the infimum of the sequence $\{z_n\}_{n=1}^{\infty}$.

We note the following lemma in connection with the limit supremum of a sequence.

Lemma 2. *Fix* $\bar{L} \in \mathbb{R}$*, and suppose that* $\limsup_{n\to\infty} a_n = \bar{L}$*. Then, for every choice of* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *(possibly depending on the choice of* ε*) such that*

$$a_k < \bar{L} + \varepsilon \qquad \forall k \geq N.$$

Proof of Lemma 2. For any $n \in \mathbb{N}$, let $z_n = \sup_{k\geq n} a_k$. Then, as per the definition of lim sup, we have $\bar{L} = \inf_{n\geq 1} z_n$. This implies that $\bar{L}$ is a lower bound for $\{z_n\}_{n=1}^{\infty}$, and for every choice of $\varepsilon > 0$, the number $\bar{L} + \varepsilon$ is not a lower bound for $\{z_n\}_{n=1}^{\infty}$. That is, for every choice of $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (possibly depending on the choice of ε) such that $z_N < \bar{L} + \varepsilon$, or equivalently $\sup_{k\geq N} a_k > \bar{L} + \varepsilon$. Noting that

$$\sup_{k\geq N} a_k < \bar{L} + \varepsilon \qquad \text{implies} \qquad a_k < \bar{L} + \varepsilon \qquad \forall k \geq N$$

completes the desired proof. □

With the above ingredients in place, we now formally define the limit of a real sequence.

Definition 7 (Limit)**.** *We say that a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers has a* **limit** *if*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Mathematically, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ *has a (finite) limit* $L \in \mathbb{R}$ *if for every choice of* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *(possibly depending on the choice of* ε*) such that*

$$L - \varepsilon < a_k < L + \varepsilon \qquad \forall k \geq N.$$

In such a case, we say that the sequence **converges** *to* L*, and write* $a_n \xrightarrow{n\to\infty} L$ *or* $\lim_{n\to\infty} a_n = L$*.*

It is important to note that a sequence may not have always have a limit, but will always have lim inf and lim sup. When the latter are equal, then the limit exists. If a sequence has an infinite limit (i.e., $L = \pm\infty$), we say that the sequence **diverges**.