Stochastic Collocation for Partial Differential Equations with Random Coefficients

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May 28, 2018

Outline

PDEs with random coefficients

Stochastic collocation

Sparse interpolation operators

Convergence and stability of sparse interpolation

Conclusion

PDEs with random coefficients

What is a PDE with random coefficients?

- **It looks like:**

  Find u such that for almost every $\omega \in \Omega$:

$$\begin{cases} \mathcal{L}(\omega)u = f & \text{on } D \\ \mathcal{B}(\omega)u = g & \text{on } \partial D. \end{cases}$$

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- **For illustration purposes we treat:**

  Find u such that for almost every $\omega \in \Omega$:

$$\begin{cases} -\operatorname{div}(a(\cdot,\omega)\nabla u) = f & \text{on } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

Weak formulation

- Find u s.t. for almost all $\omega \in \Omega$ and for all $v \in H_0^1(D)$:

$$\int_D a(x,\omega)\nabla u(x,\omega) \cdot \nabla v(x) \; \mathrm{d}x = \int_D f(x) v(x) \; \mathrm{d}x$$

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- Monte Carlo methods or **spectral approximation** ($\rightarrow$ SC)

Spectral approximation

- Reduce random field to **countable** collection of r.v.:

$$a(x,\omega) = \sum_{n\in\mathbb{N}} \varphi_n(x) Y_n(\omega)$$

(Karhunen-Loève, Polynomial Chaos, ...)

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- **Can be a very high-dimensional problem!**

Stochastic collocation

(A special form of spectral approximation)

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- For single evaluations $\gamma = (\gamma_1, \gamma_2)$ we can solve/approx. for u_γ

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Important:

Also works for countably many variables $\rightarrow$ **sparse** interpolation

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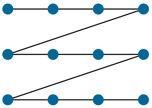
- Global enumeration of local indices:

$$k := k_1 + N_1 k_2 \in \{0, ..., \underbrace{(N_1+1)(N_2+1)}_{=:N_P}\}$$



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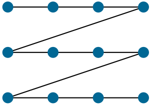

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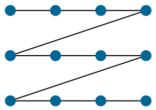

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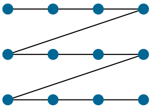

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- Solve equation **at each gridpoint** with FEM:

$$u_{h,k}(x) \approx u(x, t_{k_1,1}, t_{k_2,2})$$

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Error sources (think of: "variance-bias decomposition"):

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What does the more general case look like?

Computing a quantity of interest

- Assume joint probability density function ρ for the r.v.:

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- Introduce auxiliary density $\hat{\rho}$ for weighted Gaussian quadrature

$$\int_\Sigma u_{h,p}(x,y)\rho(y)\, \mathrm{d}y = \sum_{k=1}^{N_P} u_{h,k}(x) \int_\Sigma \ell_k(y) \frac{\rho(y)}{\hat{\rho}(y)} \hat{\rho}(y)\, \mathrm{d}y$$

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- Compute (possibly high-dimensional) quadrature

$$\int_\Sigma \ell_k(y)\frac{\rho(y)}{\hat{\rho}(y)}\hat{\rho}(y)\ \mathrm{d}y \approx \sum_{i=1}^{N_Q} w_i \ell_k(x_i)\frac{\rho(x_i)}{\hat{\rho}(x_i)}$$

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Analogy between SC and SG:

- For certain interpolation points they give the same result

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6. SC decouples system of equations

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Sparse interpolation operators

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- Approach: interpolation tools in $[-1, 1] \rightarrow [-1, 1]^{\mathbb{N}} \rightarrow$ sparse

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- "Hierarchical basis": $\{h_0, ..., h_k\}$ for $\mathcal{P}_k$

Interpolation in infinite dimensions 1/2

- Use as an index-set if working on $\mathbb{R}^{\mathbb{N}}$:

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Definition

Let $\Lambda \subset \mathscr{F}$ be a set with cardinality $|\Lambda| = N$. A set $\Gamma \subseteq [-1, 1]^{\mathbb{N}}$ with cardinality $|\Gamma| = N$ is called *unisolvent* for $\mathcal{P}_\Lambda$ if any element in $\mathcal{P}_\Lambda$ is uniquely determined by its values on Γ.

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Curse of dimensionality: $\dim \mathcal{P}_\nu = \prod_{j\geq 1}(1 + \nu_j)$

Sparse interpolation 1/3

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- Demand Λ to be "downward closed"

Sparse interpolation 2/3

Definition

A generic index set $\Lambda \subseteq \mathscr{F}$ is *downward closed* if for any $\nu \in \Lambda$, the property $\tilde{\nu} \leq \nu$ implies $\tilde{\nu} \in \Lambda$.

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A generic index set $\Lambda \subseteq \mathscr{F}$ is *downward closed* if for any $\nu \in \Lambda$, the property $\tilde{\nu} \leq \nu$ implies $\tilde{\nu} \in \Lambda$.

Theorem

Let $\Lambda \subseteq \mathscr{F}$ be a finite downward closed set. Then, Γ_Λ is unisolvent for

$$\mathcal{P}_\Lambda := \mathrm{span}\{y \mapsto y^\nu,\ \nu \in \Lambda\}, \quad y^\nu := \prod_{\nu_j \neq 0} y_j^{\nu_j},$$

and $\mathcal{I}_\Lambda$ is the corresponding interpolation operator.

Sparse interpolation 3/3

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- How well does it approximate in the infinite-dim. $[-1,1]^{\mathbb{N}}$?

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- Compute recursively:

$$\mathcal{I}_{\Lambda_n} u = \mathcal{I}_{\Lambda_{n-1}} u + \Delta_{\nu_n} u = \mathcal{I}_{\Lambda_{n-1}} u + \alpha_{\nu_n} H_{\nu_n}, \quad \mathcal{I}_{\Lambda_0} \equiv 0$$

Computing sparse interpolation 2/2

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  - In practice good behaviour; proof of convergence is **open**

Convergence and stability of sparse interpolation

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- Hopefully $\mathbb{L}_\Lambda$ smaller than bound for $\inf_{v_\Lambda \in \operatorname{ran}\mathcal{I}_\Lambda} \|u - v_\Lambda\|$

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- "Stechkin": if $(\|t_\nu\|_\nu)_{\nu \in \mathscr{F}} \in \ell^p(\mathbb{N}_0)$ for $p < 1$:

$$\sum_{\nu \notin \Lambda} \|t_\nu\|_V \leq N^{-s}, \quad s = \frac{1}{p} - 1, \quad N = |\Lambda|$$

Bounding the Lebesgue constant

Introduce **univariate** Lebesgue constants for $k \geq 0$:

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If there exists $\theta \geq 1$ such that $\lambda_k \leq (k+1)^\theta$ or $\delta_k \leq (k+1)^\theta$ holds for $k \geq 0$, then the Lebesgue constant $\mathbb{L}_\Lambda$ satisfies $\mathbb{L}_\Lambda \leq N^{\theta+1}$ for any downward closed set Λ with $|\Lambda| = N$.

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Sketch of the proof.

Derive bounds for $\mathbb{L}_\Lambda$ using a collection of $\delta_\cdot$; then one case is clear, the other one will be clear inductively. $\square$

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- So-called $\mathfrak{R}$-Leja points: $\lambda_k \leq (1+k)^2 \Rightarrow \mathbb{L}_\Lambda \leq N^3$

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Extensions:

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- Non-polynomial interpolation: RKHS, piecewise linear functions ($\to$ sparse grids), ...

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7. There is more to learn

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4. Analysis: N-term approx. & Lebesgue const.

Main references

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