# Stochastic Collocation for Partial Differential Equations with Random Coefficients

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## Contents



Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. We write $\mathrm{d}x = \mathrm{d}\lambda^d(x)$ as an abbreviation for integration with respect to the Lebesgue measure. We abbreviate partial differential equation by PDE. By $\mathscr{P}_k$ we denote the space of polynomials of degree less than or equal to $k$.

## 1. PDEs with Random Coefficients

A boundary value problem (BVP) with random coefficients has the general form

$$\begin{cases} \mathcal{L}(\omega)u = f & \text{on } D \\ \mathcal{B}(\omega)u = g & \text{on } \partial D \end{cases}$$

where we want to find a solution $u = u(x, w)$ which fulfills this system for almost every $\omega \in \Omega$. Here, $D$ is a domain, usually assumed to be simplicial or at least sufficiently regular, $f$ and $g$ are system-dependent functions and $\mathcal{L}$ and $\mathcal{B}$ are partial differential operators. Note that unlike deterministic PDEs, $\mathcal{L}$ and $\mathcal{B}$ depend on a random input.

This input usually comes in the form of a random field $a : \Omega \times D \to \mathbb{R}$. An exemplary model problem is the elliptic problem

$$\begin{cases} -\operatorname{div}(a(\cdot, \omega)\nabla u) = f & \text{on } D \\ u = 0 & \text{on } \partial D \end{cases} \tag{1}$$

which we want to hold for almost all $\omega \in \Omega$, where a usual case of interest is that $a$ is a lognormal random field, that is, $a = e^{b(x)}$ with $b$ being a Gaussian random field [BCM16]. For illustration purposes we will only consider (1). Multiplying with test functions and integrating by parts yields

$$\int_D a(x, \omega)\nabla u(x, \omega) \cdot \nabla v(x) \, \mathrm{d}x = \int_D f(x)v(x) \, \mathrm{d}x \quad \forall v \in H_0^1(D) \tag{2}$$

for almost all $\omega \in \Omega$. Assuming $a > 0$ almost everywhere and $a, a^{-1} \in L^\infty(D)$ almost everywhere as well as $f \in V' = H^{-1}(D)$, by the Lax-Milgram lemma one can show that (2) is a well-posed problem for almost all $\omega \in \Omega$.

Naturally, the solution to (2), respectively (1) is a random field $u$. Instead of the complete mapping $(\omega, x) \mapsto u(\omega, x)$ one is sometimes interested in a so-called *quantity of interest* (QOI), for example the expected value $\mathbb{E}[u(\cdot, x)]$ or the covariance $\operatorname{Cov}(u(x), u(y))$.

## 2. Stochastic Collocation

There are mostly two approaches to compute those QOI: Monte-Carlo based techniques or so-called *spectral approximation*. Stochastic collocation is a form of spectral approximation.

### 2.1. Spectral Approximation

We want to extract information about the mapping $a \mapsto u(a)$. A first step for that cause is to reduce $a$ to a countable collection of random variables

$$a(x,\omega) = \sum_{n\in\mathbb{N}} \varphi_n(x) Y_n(\omega)$$

with uncorrelated random variables $(Y_i)_{i\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ being some orthonormal basis. A common such expansion is the *Karhunen-Loève* expansion [Lo77].

Using these expansions, we can think of $a(x,\omega) = a(x, Y_1(\omega), Y_2(\omega), ...)$ which leads to replacing $\Omega$ in our formulation with $\Sigma := Y_1(\Omega) \times ... \times Y_N(\Omega) \times ...$ . In particular, the weak formulation (2) now reads: Find $u$ such that

$$\int_D a(x,y)\nabla u(x,y) \cdot \nabla v(x)\ \mathrm{d}x = \int_D f(x)v(x)\ \mathrm{d}x \quad \forall v \in H_0^1(D)$$

holds for all $y \in \Sigma$. Note that $\Sigma \subseteq \mathbb{R}^{\mathbb{N}}$ is countably infinite-dimensional.

All in all, spectral approximations are now concerned with approximating the mapping

$$\Sigma \longrightarrow H_0^1(D), \quad y \longmapsto u(y)$$

which potentially is a very high-dimensional problem.

### 2.2. Stochastic Collocation

The following derivation is based on [BNT10] which is a reviewed version of an earlier paper from 2007 [BNT07].

We start by assuming that $a$ depends on a finite number of uncorrelated random variables, $a(x,\omega) = a(x, Y_1(\omega), ..., Y_N(\omega))$ for some $N \in \mathbb{N}$. This is called *finite noise assumption*. Denote the corresponding subspace of $\Sigma$ by $\Sigma_N := Y_1(\Omega) \times ... \times Y_N(\Omega)$. In this case the weak formulation reads: Find $u = u(x, y_1, ..., y_N)$ such that

$$\int_D a(x,y_1,...,y_N)\nabla u(x,y_1,...,y_N) \cdot \nabla v(x)\ \mathrm{d}x = \int_D f(x)v(x)\ \mathrm{d}x \quad \forall v \in H_0^1(D) \qquad (3)$$

holds for almost all $(y_1, ..., y_N) \in \Sigma_N$.

The key to stochastic collocation is that problems like (3) allow a solution for a single point-evaluation $\gamma \in \Sigma_N$, use for example standard finite element methods (FEM). We compute these "samples" repeatedly on some grid and then interpolate on that grid to obtain an approximation for $y \mapsto u(y)$.

That in mind, stochastic collocation is interpolation of vector valued functions - the solutions to the model problem. In the same way this works without the finite noise assumption, however, one has to be more careful with the selection of the finite-dimensional subset $\Sigma_N \subseteq \Sigma$; we refer to the next section for this *sparse interpolation*.

### 2.3. Polynomial Interpolation

As mentioned earlier, we want to interpolate on the tensor product grid $\Gamma_N := \bigotimes_{i=1}^N \Gamma_i$, $\Gamma_i := \{t_{0,i}, ..., t_{N_i,i}\}$. Denote by $\mathscr{P}_N(\Sigma) := \bigotimes_{i=1}^N \mathscr{P}_{N_i}(\Sigma_i)$ the interpolation space given

by the tensor product polynomials. Denote by $I_i$ the interpolation operator on $\Gamma_i$ with the corresponding representation $I_i v(t) = \sum_{j=0}^{N_i} v(t_{j,i})\ell_{j,i}(t)$. Here, the collection of functions $(\ell_{j,i})_{j\leq N_i}$ denotes the *Lagrange basis* for interpolation in $\mathscr{P}_{N_i}(\Sigma_i)$ with respect to $\Gamma_i$, that is, $\ell_{j,i}(t_{k,i}) = \delta_{jk}$, $t_{k,i} \in \Gamma_i$. We want to use these $(I_i)_{i\leq N}$ to define a global interpolation operator on $\mathscr{P}_N(\Sigma)$.

Denote the local indices $1 \leq k_i \leq N_i$ and define $k := k_1 + N_1(k_2-1) + N_1N_2(k_3-1) + ...$ as a global enumeration. Think of it as associating $(k_1, ..., k_N) \cong k$. Correspondingly we associate $t_k = (t_{k_1,1}, ..., t_{k_N,N})$ and the tensor product Lagrange basis $\ell_k(z_1, ..., z_N) = \prod_{i=1}^N \ell_{k_i,i}(z_i)$ as well as the global interpolation operator

$$I_N u(t) = \sum_{k=1}^{K_p} \ell_k(t) u(t_k). \tag{4}$$

Here, $K_p = \prod_{i=1}^N (1 + N_i)$, which shows fast growth of the number of overall points in the size of the $N_i$. This defect is called the *curse of dimensionality* and makes the treatment of many high-dimensional problems practically infeasible.

The interpolation operator is well-defined since the grid $\Gamma_N$ is *unisolvent* for $\mathscr{P}_N$ as long as all gridpoints are pairwise disjoint, that is, each element in $\mathscr{P}_N$ is uniquely defined by its values on the gridpoint; see the next section for more on unisolvency.

Denote by $u_h(x, y)$ the spatial FEM approximation of $u$ with respect to (3), and correspondingly by $u_{h,k}(x)$ the approximation at gridpoint $t_k$. We use the interpolation operator $I_N$ given by (4) to define the eventual approximation of $u$ as

$$u(x, y) \approx u_{h,P}(x, y) := I_N u_h(x, y) = \sum_{i=1}^{K_p} u_{h,k}(x)\ell_k(y). \tag{5}$$

The subscript $P$ denotes the approximation by polynomial interpolation.

The possible error sources in this approximation are twofold: On the one hand the spatial solution by FEM is not exact and on the other hand we interpolate this inexact approximation by something which is not exact either. This phenomenon is similar to the *variance-bias decomposition* from Monte-Carlo-FEM theory; see e.g. [BSZ10] for more on MC-FEM.

### 2.4. Computing Quantities of Interest

With the approximation $u_{h,P}$ as given by (5), we want to compute quantities of interest. We will focus on the expected value as the derivation is very similar for higher moments.

Assume that the random variables $(Y_1, ..., Y_N)$ admit a joint probability density function $\rho : \Sigma \to \mathbb{R}$. Then, the expected value is given by

$$\mathbb{E}[u(x, \cdot)] = \int_\Omega u(x, \omega)\ \mathrm{d}\mathbb{P}(\omega) = \int_\Sigma u(x, y)\rho(y)\ \mathrm{d}y \approx \int_\Sigma u_{h,P}(x, y)\rho(y)\ \mathrm{d}y.$$

To develop a suitable high-dimensional quadrature (also known as cubature) rule, we would like our density $\rho$ to factorise, i.e. $\rho(y_1, ..., y_N) = \prod_{i=1}^N \rho_i(y_i)$. This however is not the case if the random variables are not independent.

To overcome this we introduce the auxiliary function $\hat{\rho} : \Sigma \to \mathbb{R}$ which is the density function of $N$ independent random variables. Hence, $\hat{\rho}(y_1, ..., y_N) = \prod_{i=1}^N \hat{\rho}_i(y_i)$. Assuming $\hat{\rho}$ is such that $\rho/\hat{\rho} \in C^\infty(\Sigma) \cap L^\infty(\Sigma)$ we write

$$\int_\Sigma u_{h,P}(x, y)\rho(y)\ \mathrm{d}y = \int_\Sigma u_{h,P}(x, y)\frac{\rho(y)}{\hat{\rho}(y)}\hat{\rho}(y)\ \mathrm{d}y.$$

If the regularity assumption is not fulfilled we employ a suitable quadrature for singular integrals.

This integral is something we can compute with weighted Gaussian quadrature according to the probability measure $\hat{\rho}\ \mathrm{d}y$. Hence

$$\int_\Sigma u_{h,P}(x,y)\frac{\rho(y)}{\hat{\rho}(y)}\hat{\rho}(y)\ \mathrm{d}y = \sum_{k=1}^{K_p} u_{h,k}(x)\int_\Sigma \ell_k(y)\frac{\rho(y)}{\hat{\rho}(y)}\hat{\rho}(y)\ \mathrm{d}y \approx \sum_{k=1}^{K_p} u_{h,k}(x)\sum_{j=1}^{N_Q} w_j\ell_k(y_j)\frac{\rho(y_j)}{\hat{\rho}(y_j)}$$

where $\{y_j\}_{j\leq N_Q}$ are the roots of the orthogonal polynomials corresponding to $\hat{\rho}\ \mathrm{d}y$ and $w_j$ are the corresponding Gaussian quadrature weights. The number $N_Q$ is the amount of quadrature points. Note that the quadrature sum is a multi-dimensional construction which we simplified here for the sake of readability. The assumption that $\hat{\rho}$ factorises was vital in this step, since by

$$\int_\Sigma \ell_k(y)\frac{\rho(y)}{\hat{\rho}(y)}\hat{\rho}(y)\ \mathrm{d}y = \int_{\Sigma_N}\hat{\rho}_N(y_N)\cdots\int_{\Sigma_1}\hat{\rho}_1(y_1)\ell_k(y_1,...,y_N)\frac{\rho(y_1,...,y_N)}{\hat{\rho}(y_1,...,y_N)}\ \mathrm{d}(y_1,...,y_N)$$

we can combine one-dimensional quadrature rules with respect to $\hat{\rho}_i\mathrm{d}y_i$ to a multi-dimensional cubature for $\ell_k\rho/\hat{\rho}$.

## 3. Sparse Interpolation

This section is mostly based on [CD15, section 6.1, 6.2].

We have seen that the interpolation process can be viewed independently of the process of solving the subproblems. From now on we take an abstract point of view, namely we assume that a function $u$ solves the parametric PDE

$$\mathcal{G}(a,u) = 0$$

with $\mathcal{G} : X \times V \longrightarrow W$ being a partial differential operator acting on a triplet of Banach spaces $(V,X,W)$. In the context of (1), think of $(V,X,W) = (H_0^1(D), L^\infty(D), H^{-1}(D))$ and note $W = V'$. We do not assume finite noise. We want to interpolate in $U \subseteq \mathbb{R}^{\mathbb{N}}$, namely we want to approximate $y \mapsto u(y)$ as a map from $U \to V$.

### 3.1. Construction Sparse Interpolation Operators

We will start by deriving the necessary tools in $U_1 \subseteq \mathbb{R}$, then generalize this to $U \subseteq \mathbb{R}^{\mathbb{N}}$ and eventually introduce sparse operators. Assume that $U$ is built such as $U = \bigotimes_{j\in\mathbb{N}} U_1$. The more general case of arbitrary $U \subseteq \mathbb{R}^{\mathbb{N}}$ is a straightforward extension. Think of $U_1 = [-1,1]$. Denote by $T := (t_k)_{k\in\mathbb{N}} \subseteq U_1$ a sequence of pairwise disjoint points. Define the interpolation grid $\Gamma_k := \{t_0,...,t_k\}$ and the corresponding interpolation operator $I_k$ on $\Gamma_k$. The corresponding interpolation space is given by $\mathscr{P}_k(U_1)$.

Let $\Delta_k$ denote the difference operator on level $k$, i.e. $\Delta_k := I_k - I_{k-1}$. We set $I_{-1} \equiv 0$, this will make the notation easier. Observe

$$\begin{aligned}
I_k u(t) - I_{k-1}u(t) &= \sum_{j=0}^{k} u(t_j)\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{t-t_i}{t_j-t_i} - \sum_{j=0}^{k-1} u(t_j)\prod_{\substack{i=0\\ i\neq j}}^{k-1}\frac{t-t_i}{t_j-t_i}\\
&= u(t_k)\prod_{i=0}^{k-1}\frac{t-t_i}{t_k-t_i} + \sum_{j=0}^{k-1} u(t_j)\left(\frac{t-t_k}{t_j-t_k}-1\right)\prod_{\substack{i=0\\ i\neq j}}^{k-1}\frac{t-t_i}{t_j-t_i}\\
&= u(t_k)\prod_{i=0}^{k-1}\frac{t-t_i}{t_k-t_i} - \sum_{j=0}^{k-1} u(t_j)\left(\frac{t-t_j}{t_k-t_j}\right)\prod_{\substack{i=0\\ i\neq j}}^{k-1}\frac{t-t_i}{t_j-t_i}\frac{t_k-t_i}{t_k-t_i}\\
&= \left(\prod_{i=0}^{k-1}\frac{t-t_i}{t_k-t_i}\right)\left(u(t_k) - \sum_{j=0}^{k-1} u(t_j)\prod_{\substack{i=0\\ i\neq j}}^{k-1}\frac{t_k-t_i}{t_j-t_i}\right) =: h_k(t)\alpha_k
\end{aligned}$$

where in the second equality we implicitly used nestedness of the point set. This allows us to write $\Delta_k u(t) = \alpha_k h(t)$, where $\alpha_k = \alpha_k(u) := u(t_k) - I_{k-1}u(t_k)$ is the interpolation error at $t_k$ and the functions given by $h_k(t) = \prod_{l=1}^{k-1} \frac{t-t_l}{t_k-t_l}$ for $k \geq 1$, $h_0 = 1$ constitute the *hierarchical basis* $\{h_0, ..., h_k\}$ for $\mathscr{P}_k(U_1)$. This hierarchical basis is important for sparse approximation.

As an index set for infinite-dimensional products, we define

$$\mathscr{F} := \{\nu \in \ell^\infty(\mathbb{N}) : \|\nu\|_0 < \infty\}, \quad \|\nu\|_0 = |\{j \in \mathbb{N} : \ \nu_j \neq 0\}|.$$

For any $\nu \in \mathscr{F}$ define the interpolation grid $\Gamma_\nu := \bigotimes_{j\geq 1} \Gamma_{\nu_j}$, the interpolation operator $I_\nu := \bigotimes_{j\geq 1} I_{\nu_j}$ and the interpolation space $\mathscr{P}_\nu := \bigotimes_{j\geq 1} \mathscr{P}_{\nu_j}$.

It is not a priori clear whether $I_\nu$ is well-defined. They key property for well-definedness of an interpolation operator is *unisolvency*, as stated in the following definition.

**Definition 3.1.** Let $\Lambda \subseteq \mathscr{F}$ with cardinality $|\Lambda| = N$. A set $\Gamma \subseteq U$ with cardinality $|\Gamma| = N$ is called *unisolvent* for $\mathscr{P}_\Lambda$ if any element in $\mathscr{P}_\Lambda$ is uniquely determined by its values on $\Gamma$.

To see that $\Gamma_\nu$ is unisolvent for $\mathscr{P}_\nu$, we do induction on the size of

$$\mathscr{F}_k = \{\nu \in \mathscr{F} : \ \nu_j = 0 \text{ for all } j > k\}, \quad k \geq 1.$$

The case for $k = 1$ is clear. Any element in $\mathscr{F}_k$ can be written as an element in $\mathbb{N}_0 \times \mathscr{F}_{k-1}$. We put together the corresponding interpolation operators and see unisolvency; for details see [CD15, section 6.1].

Note that again we have the curse of dimensionality: the dimension of $\mathscr{P}_\nu$ grows exponentially in the size of $\nu$, namely $\dim \mathscr{P}_\nu = \prod_{j\geq 1}(1 + \nu_j)$.

Denote the difference operator $\Delta_\nu = \bigotimes_{j\geq 1} \Delta_{\nu_j}$ together with the hierarchical basis $\{H_{\tilde\nu}, \ \tilde\nu \leq \nu\}$ where $H_{\tilde\nu}(y) = \prod_{\tilde\nu_j \neq 0} h_{\tilde\nu_j}(y_j)$ is the tensor product hierarchical basis.

Now we derive the sparse interpolation. Let $\Lambda \subset \mathscr{F}$ be any index set. Define the interpolation grid $\Gamma_\Lambda := \{y_\nu := (t_{\nu_j})_{j\geq 1}, \ \nu \in \Lambda\}$ and the corresponding interpolation operator $I_\Lambda := \sum_{\nu\in\Lambda} \Delta_\nu$. Note that it is defined using only the difference operators!

Without forcing some additional structure on $\Lambda$ it is not clear whether $I_\Lambda$ is well defined nor how $\mathscr{P}_\Lambda$ looks. It would make sense to require that if some $\nu \in \Lambda$, all $\tilde\nu \leq \nu$ should be in $\Lambda$ as well. This property is called *downward closedness*.

**Definition 3.2.** A generic index set $\Lambda \subseteq \mathscr{F}$ is *downward closed* if for any $\nu \in \Lambda$, the property $\tilde\nu \leq \nu$ implies $\tilde\nu \in \Lambda$.

**Theorem 3.3** ([CD15], Theorem 6.1)**.** *Let $\Lambda \subseteq \mathscr{F}$ be a finite downward closed set. Then, $\Gamma_\Lambda$ is unisolvent for*

$$\mathscr{P}_\Lambda := \operatorname{span}\{y \mapsto y^\nu, \ \nu \in \Lambda\}, \quad y^\nu := \prod_{\nu_j \neq 0} y_j^{\nu_j},$$

*and $I_\Lambda$ is the corresponding interpolation operator.*

*Sketch of the proof.* Downward closedness implies $\mathscr{P}_\nu \subseteq \mathscr{P}_\Lambda$ for all $\nu \in \Lambda$, hence we have $\operatorname{ran} I_\Lambda \subseteq \mathscr{P}_\Lambda$ and by writing $I_\Lambda u = I_\nu + \sum_{\tilde\nu\in\Lambda, \tilde\nu>\nu} \Delta_{\tilde\nu} u$ one can eventually conclude exactness on $\Gamma_\Lambda$. □

### 3.2. Computing Sparse Interpolation

The index set $\Lambda$ has a partial ordering "$\leq$" according to which we order $\Lambda_N := \{\nu_1, ..., \nu_N\}$. If we remove one of the maximal elements $\nu_N$, the remaining set $\Lambda_{N-1} = \{\nu_1, ..., \nu_{N-1}\}$ remains downward closed.

Note that for generic $n \leq N$, we can write the difference operator as $\Delta_{\nu_n} = \alpha_{\nu_n} H_{\nu_n}$, where $\alpha_{\nu_n} = u(y_{\nu_n}) - I_{\Lambda_{n-1}} u(y_{\nu_n})$ is the interpolation error at the $n$-th point $y_{\nu_n}$. Repeating this we compute recursively,

$$I_{\Lambda_n} u = I_{\Lambda_{n-1}} u + \Delta_{\nu_n} u = I_{\Lambda_{n-1}} u + \alpha_{\nu_n} H_{\nu_n}, \quad I_{\Lambda_0} \equiv 0.$$

Thus the algorithm reduces to finding a new $\nu_{N+1}$ and computing the corresponding $I_{\Lambda_{N+1}}$ recursively.

How do we pick the next multi-index? Of course we can always choose an a priori sequence of point sets $(\Lambda_n)_n$. In a non-adaptive method we try to find the $N$ largest $\|\Delta_{\nu_n} u\|_{L^p}$, which is technically an intractable problem for multiple reasons. We have to limit the search to the set of indices for which $I_{\Lambda_N} = \{\nu_N\} \cup \Lambda_{N-1}$ remains downward closed. Hence we only consider the indices in $N(\Lambda) := \{\nu \in \mathscr{F} : \Lambda \cup \{\nu\} \text{ is downward closed}\}$. This set has infinite cardinality and is therefore infeasible for any searching algorithm. A natural restriction to overcome this problem might be

$$\tilde{N}(\Lambda) := \{\nu \in N(\Lambda) : \ \nu_j = 0 \text{ for all } j > j(\Lambda) + 1\}$$

where $j(\Lambda)$ is defined as $j(\Lambda) := \max\{j \in \mathbb{N} : \nu_j > 0 \text{ for some } \nu \in \Lambda\}$.

The more serious defect is that this greedy search might fail to converge. Consider the two-dimensional function $u(x, y) = \sin(x)\exp(y)$ and the Leja point set on $[0, \pi]$: $T = 0, \pi, \pi/2, ...$; more on Leja points later. Starting with $\Lambda_0 = \{0\}$, the greedy search selects sets $\Lambda_n = \{(\nu_i, 0), i = 0, ..., n\}$ since the interpolation error in $x$ always vanishes. The algorithm cannot converge.

One way of modifying this is to "cycle" through the indices, in the sense that for $n$ even we do the classical greedy search and for $n$ odd we pick

$$\nu^n = \arg\min\{k(\nu) : \ \nu \in \tilde{N}(\Lambda_{n-1})\}, \quad k(\nu) = \min\{k : \ \nu \in \tilde{N}(\Lambda_k)\}.$$

This algorithm was introduced in [CCS14] and according to [CD15] it performs well in practice although its convergence analysis is still an open problem [CD15, p. 103]. Therefore we restrict our convergence analysis to the case of a non-adaptive scheme, i.e. an a priori defined sequence of sets $(\Lambda_n)_{n \geq 1}$.

### 3.3. Convergence and Stability of Sparse Interpolation

In this last section we are going to investigate convergence properties of the interpolation. The main thing we will see is that convergence is almost entirely a question of stability.

One can think of $I_\Lambda$ as a projection from the space of bounded uniformly continuous functions to itself. For stability we want it to be a continuous operator, in other words: we want

$$\mathbb{L}_\Lambda := \|I_\Lambda\|_{\text{op}} = \sup_{u \neq 0} \frac{\|I_\Lambda u\|_{L^\infty(U,V)}}{\|u\|_{L^\infty(U,V)}}$$

to be bounded. We call this quantity *Lebesgue constant*. If the interpolation is continuous, we can derive

$$\begin{aligned} \|u - I_\Lambda u\|_{L^\infty(U,V)} &\leq \|u - v_\Lambda\|_{L^\infty(U,V)} + \|I_\Lambda(u - v_\Lambda)\|_{L^\infty(U,V)} \\ &\leq (1 + \mathbb{L}_\Lambda) \inf_{v_\Lambda \in \operatorname{ran} I_\Lambda} \|u - v_\Lambda\|_{L^\infty(U,V)}. \end{aligned} \tag{6}$$

There are two quantities we want to bound now: the Lebesgue constant $\mathbb{L}_\Lambda$ and the error of the best approximation in $\operatorname{ran} I_\Lambda$.

We start by considering the best approximation term. This is a purely functional analytic statement but relies on some strong regularity results with respect to $y \mapsto u(y)$. In particular, in [CD15, section 3] the authors derive regularity results such as the existence of

a holomorphic extension of $y \mapsto u(y)$ which then leads to approximation quality of the, say, Taylor approximation with $N$ terms in an algebraic rate $N^{-s}$.

We take the same point of view, denoting the Taylor approximation $u = \sum_{\nu \in \mathscr{F}} t_\nu T_\nu$, $T_\nu(y) = \prod_{j \geq 1} y_j^{\nu_j}$. As $\operatorname{ran} I_\Lambda \subseteq V \otimes \mathscr{P}_\Lambda$, we can expand

$$\inf_{v \in \operatorname{ran} I_\Lambda} \|u - v\|_{L^\infty(U,V)} \leq \left\| \sum_{\nu \notin \Lambda} t_\nu T_\nu \right\|_{L^\infty(U,V)} \leq \sum_{\nu \notin \Lambda} \|t_\nu\|_V.$$

Due to the regularity of $u(y)$ one can show certain summability properties for the Taylor coefficients $(\|t_\nu\|_V)_\nu$, in particular $(\|t_\nu\|_\nu)_{\nu \in \mathscr{F}} \in \ell^p(\mathbb{N}_0)$ for $p < 1$ in which case

$$\sum_{\nu \notin \Lambda} \|t_\nu\|_V \leq C N^{-s}, \quad s = \frac{1}{p} - 1, \quad N = |\Lambda|$$

for some $C > 0$ [CD15, Theorem 3.9f]. A more general version of this result is known as *Stechkin lemma* [CD15, Lemma 3.6]. Combining this with (6), we obtain

$$\|u - I_\Lambda u\|_{L^\infty(U,V)} \leq C(1 + \mathbb{L}_\Lambda) N^{-s}, \quad s = \frac{1}{p} - 1.$$

It remains to bound the Lebesgue constant. To do that, we introduce univariate Lebesgue constants

$$\lambda_k := \sup \frac{\|I_k u\|_{L^\infty(U,V)}}{\|u\|_{L^\infty(U,V)}}, \quad \delta_k := \sup \frac{\|\Delta_k u\|_{L^\infty(U,V)}}{\|u\|_{L^\infty(U,V)}}, \quad k \geq 0.$$

The key observation is that $\mathbb{L}_\Lambda$ inherits its bound from the bounds on the corresponding univariate Lebesgue constants. It is summed up in the following theorem.

**Theorem 3.4** ([CD15], Theorem 6.3)**.** *If there exists $\theta \geq 1$ such that $\lambda_k \leq (k+1)^\theta$ or $\delta_k \leq (k+1)^\theta$ holds for $k \geq 0$, then the Lebesgue constant $\mathbb{L}_\Lambda$ satisfies $\mathbb{L}_\Lambda \leq N^{\theta+1}$ for any downward closed set $\Lambda$ with $|\Lambda| = N$.*

*Sketch of the proof.* Derive bounds for $\mathbb{L}_\Lambda$ using a collection of $\delta_\cdot$; then one case is clear. The other one will be clear inductively but takes a decent amount of work. □

Which bounds can we obtain for the univariate Lebesgue constants? Recall that we have to compare the bound for $\mathbb{L}_\Lambda$ to the algebraic $N$-term approximation rate $N^{-s}$, hence we want it to be as small as possible.

For point sets such as the Clenshaw-Curtis points one can show $\lambda_k \sim \log(k)$. However, as these are not nested in our sense we cannot use them. Instead, define the Leja points as $t_0 = 0$ and $t_k$ given by

$$t_k := \arg\max \left\{ \prod_{i \neq j} |t_i - t_j| : \ \{t_0, ..., t_k\} \in [-1, 1]^{k+1} \right\}.$$

Simulations show that the bound is supposed to be given by $\lambda_k \leq 1 + k$, however, a rigorous proof is yet to be given - it is an open problem!

As a last point set we mention a point set which uses $t_0 = 1$ and the same iteration as the Leja points above but on the unit disc $\{|z| \leq 1\}$. Projecting onto the real axis we obtain the $\mathfrak{R}$-Leja points for which it can be proven that the univariate Lebesgue constants satisfy $\lambda_k \leq (1+k)^2$ which implies $\mathbb{L}_\Lambda \leq N^3$.

## 4. Conclusion

Let us summarise what we have derived in the past sections. Stochastic collocation is high-dimensional polynomial interpolation of vector-valued functions.

We defined sparse interpolation on $U \subseteq \mathbb{R}^{\mathbb{N}}$ via difference operators and downward closed sets. The convergence analysis boils down to stability estimates. The best $N$-term approximation rate gives error $N^{-s}$ but relies on strong regularity results. The bound for $\mathbb{L}_\Lambda$ is inherited from bounds on the univariate Lebesgue constants and can be derived analytically for e.g. $\mathfrak{R}$-Leja points.

There are natural extensions. It is possible to recover a sharper convergence rate which does not rely on bounding the Lebesgue constant but needs more sophisticated analysis; compare [CD15, section 6.2]. Also the extension to more general sets $U \subseteq \mathbb{R}^{\mathbb{N}}$ instead of $U = \bigotimes_{j\in\mathbb{N}} U_1$ is straightforward.

We also mention that there are other ways to interpolate, for example using reproducing kernel Hilbert spaces [Za15, Chapter 4]. Here one has to be careful however since most of the tools we used in section 3 have to be adapted to the non-polynomial setting.

## References


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