

Pnl Manual

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1 Introduction

1.1 What is Pnl

Pnl is a scientific library written in C and distributed under the Gnu Lesser General Public Licence (LGPL). This manual is divided into four parts.

- Mathematical functions: complex numbers, special functions, standard financial functions for the Black & Scholes model.

- Linear algebra : vectors, matrices (dense and sparse), hypermatrices, tridiagonal matrices, band matrices and the corresponding routines to manipulate them and solve linear systems.
- Probabilistic functions: random number generators and cumulative distribution functions.
- Deterministic toolbox : FFT, Laplace inversion, numerical integration, zero searching, multivariate polynomial regression, ...

1.2 A few helpful conventions

- All header file names are prefixed by `pnl_` and are surrounded by the preprocessor conditionals

```
#ifndef _PNL_MATRIX_H
#define _PNL_MATRIX_H

...

#endif /* _PNL_MATRIX_H
```

All the header files are protected by an `extern "C"` declaration for possible use with a C++ compiler. The header files must be include using

```
#include "pnl/pnl_xxx.h"
```

- All function names are prefixed by `pnl_` except those implementing complex number arithmetic which are named following the *C99* complex library but using a capitalised first letter `C`.
For example, the addition of two complex numbers is performed by the function `Cadd`.
- Function containing `_create` in their names always return a pointer to an object created by one or several calls to dynamic allocation. Once these objects are not used, they must be freed by calling the same function but ending in `_free`. A function `pnl_foo_create_yyy` returns a `PnlFoo * object` (note the “*”) and a function `pnl_foo_bar_create_yyy` returns a `PnlFooBar * object` (note the “*”). These objects must be freed by calling respectively `pnl_foo_free` or `pnl_foo_bar_free`.
- Functions ending in `_clone` take two arguments `src` and `dest` and modify `dest` to make it identical to `src`, ie. they have the same size and data. Note that no new object is allocated, `dest` must exist before calling this function.
- Functions ending in `_copy` create a new object identical (ie. with the same size and content) as its argument but independent (ie. modifying one of them does not alter the other). Calling `A = pnl_xxx_copy(B)` is equivalent to first calling `A = pnl_xxx_new()` function and then `pnl_xxx_clone(A, B)`.
- Every object must implement a `pnl_xxx_new` function which returns a pointer to an empty object with all its elements properly set to 0. This means that the objects returned by the `pnl_xxx_new` functions can be used as output arguments for functions ending in `_inplace` for instance. They are suitable for being resized.

- Functions containing `_wrap_` in their names always return an object, not a pointer to an object, and do not make any use of dynamic allocation. The returned object must not be freed. For instance, a function `pnl_foo_wrap_xxx` returns an object `PnlFoo` and a function `pnl_foo_bar_wrap_xxx` returns an object `PnlFooBar`

```
PnlVectComplex *v1;
PnlVectComplex v2;
v1 = pnl_vect_complex_create_from_scalar (5, Complex(0., 1.))
;
v2 = pnl_vect_complex_wrap_subvect (v1, 1, 2);

...

pnl_vect_complex_free (&v1);
```

The vector `v1` is of size 5 and contains the pure imaginary number i . The vector `v2` only provides a view to `v1(1:1+2)`, which means that modifying `v2` will also modify `v1` and vice-versa because `v1` shares part of its data with `v2`. Note that only `v1` must be freed and **not** `v2`.

- Functions ending in `_init` do not create any object but only perform some internal initialisation.
- Hypermatrices, matrices and vectors are stored using a flat block of memory obtained by concatenating the matrix rows and C-style pointer-to-pointer arrays. Matrices are stored in row-major order, which means that the column index moves continuously. Note that this convention is not *Blas & Lapack* compliant since Fortran expects 2-dimensional arrays to be stored in a column-major order.
- Type names always begin with `Pnl`, they do not contain underscores but instead we use capital letters to separate units in type names.
Examples : `PnlMat`, `PnlMatComplex`.
- Object and function names are intimately linked : an object `PnlFoo` is manipulated by functions starting in `pnl_foo`, an object `PnlFooBar` is manipulated by functions starting in `pnl_foo_bar`. In table 1, we summarise the types and their corresponding prefixes.
- All macro names begin with `PNL_` and are capitalised.
- Differences between **copy** and **clone** methods. The **copy** methods take a single argument and return a pointer to an object of the same type which is an independent copy of its argument. Example:

```
PnlVect *v1, *v2;
v1 = pnl_vect_create_from_scalar (5, 2.5);
v2 = pnl_vect_copy (v1);
```

`v1` and `v2` are two vectors of size 5 with all their elements equal to 2.5. Note that `v2` **must not** have been created by a call to `pnl_vect_create_xxx` because otherwise it will cause a memory leak. `v1` and `v2` are independent in the sense that a modification to one of them does not affect the other.

The **clone** methods take two arguments and fill the first one with the second one. Example:

| Pnl types | Pnl prefix |
|-----------------|--------------------|
| PnlVect | pnl_vect |
| PnlVectComplex | pnl_vect_complex |
| PnlVectInt | pnl_vect_int |
| PnlMat | pnl_mat |
| PnlMatComplex | pnl_mat_complex |
| PnlMatInt | pnl_mat_int |
| PnlSpMat | pnl_sp_mat |
| PnlSpMatComplex | pnl_sp_mat_complex |
| PnlSpMatInt | pnl_sp_mat_int |
| PnlHmat | pnl_hmat |
| PnlHmatComplex | pnl_hmat_complex |
| PnlHmatInt | pnl_hmat_int |
| PnlTridiagMat | pnl_tridiag_mat |
| PnlBandMat | pnl_band_mat |
| PnlList | pnl_list |
| PnlBasis | pnl_basis |
| PnlCgSolver | pnl_cg_solver |
| PnlBicgSolver | pnl_bicg_solver |
| PnlGmresSolver | pnl_gmres_solver |

Figure 1: Pnl types

```

PnlVect *v1, *v2;
v1 = pnl_vect_create_from_scalar (5, 2.5);
v2 = pnl_vect_new ();
pnl_vect_clone (v2, v1);

```

`v1` and `v2` are two vectors of size 5 with all their elements equal to 2.5. Note that `v2` **must** have been created by a call to `pnl_vect_new` because otherwise the function `pnl_vect_clone` will crash. `v1` and `v2` are independent in the sense that a modification to one of them does not modify the other.

- All objects are measured using integers `int` and not `size_t`. Hence, iterations over vectors, matrices, ... should use an index of type `int`.
- In fonctions ending in `inplace`, the output parameter must be different from any of the input parameters.

1.3 Using Pnl

In this section, we assume that the library is installed in the directory `$HOME/pnl-xxx`.

Once installed, the library can be found in the `$HOME/pnl-xxx/lib` directory and the header files in the `$HOME/pnl-xxx/include` directory.

1.3.1 Compiling and Linking

The header files of the library are installed in a root `pnl` directory and should always be included with this `pnl/` prefix. So, for instance to use random number generators you should include

```
#include <pnl/pnl_random.h>
```

Compiling and linking by hand. If `gcc` or `llvm` is used, you should pass the following options

- `-I$HOME/pnl-xxx/include` for compiling
- `-L$HOME/pnl-xxx/lib -lpnl` for linking

This does not work straight away on all OS especially if the library is not installed in a standard directory namely `/usr/` or `/usr/local/` for which you need a privileged writing access. On some systems, you may need to add to the linker flags the dependencies of the library, which can become very tedious. Therefore, we provide a second automatic mechanism which takes care of the dependencies on its own.

Compiling and linking using an automatic Makefile. This mechanism only works under Unix (it has been tested under various Linux distributions and Mac OS X).

First, you need to create a new directory wherever you want, put in all your code and create a Makefile as below

To define your target just add the executable name, say `my-exec`, to the `BINS` list and create an entry `my_exec_SRC` carrying the list of source files needed to create your executable. Note that if dashes '-' may appear in an executable name, the name of the associated variable holding the list of source files is obtained by replacing dashes with underscores '_' and adding the `_SRC` suffix.

Assume you want to create two binaries : `my-exec` based on mixed C and C++ code (`file1.c` and `file2.cpp`) and `mybinary` based on `poo1.cxx` and `poo2.cpp`. You can use the following Makefile.

```
## Flags passed to the linker
LDFLAGS=

## Flags passed to the compiler
CFLAGS=

## list of executables to create
BINS=my-exec mybinary

my_exec_SRC=file1.c file2.cpp
# optional flags for compiling and linking
my_exec_CFLAGS=
my_exec_CXXFLAGS=
```

```

my_exec_LDFLAGS=

mybinary_SRC=poo1.cxx poo2.cpp
# optional flags for compiling and linking
mybinary_CFLAGS=
mybinary_CXXFLAGS=
mybinary_LDFLAGS=

## This line must be the last one
include full_path_to_pnl_build/CMakeuser.incl

```

Let us comment a little the different variables

- CFLAGS: global flags used for creating objects based on C code
- CXXFLAGS: global flags used for creating objects based on C++ code
- LDFLAGS: gobal linker flags.
- `binaryname_CFLAGS`: flags used when creating the objects based on C code and required by `binaryname`
- `binaryname_CXXFLAGS`: flags used when creating the objects based on C++ code and required by `binaryname`
- `binaryname_LDFLAGS`: flags used when linking objects for creating `binaryname`

An example of such a Makefile can be found in `pnl-xxx/perso`.

Warning: if a file appears in the source list of several binairies, the flags used to compile this file are determined by the ones of the first binary involving this file. In the following example `main.cpp` will always be compiled with the flag `-O3` even for generating `bin2`

```

BINS=bin1 bin2

bin1_SRC=main.cpp poo1.c
my_exec_CXXFLAGS=-O3

bin2_SRC=main.cpp poo2.c
mybinary_CXXFLAGS=-g -O0

## This line must be the last one
include full_path_to_pnl_build/CMakeuser.incl

```

Compiling and linking using CMake. If you already use CMake for your new project, just add the following to your toplevel `CMakeLists.txt`

```

find_package(Pnl REQUIRED)
set(LIBS ${LIBS} ${PNL_LIBRARIES})
include_directories(${PNL_INCLUDE_DIRS})
# Deactivate PNL debugging stuff on Release builds
if(${CMAKE_BUILD_TYPE} STREQUAL "Release")
    add_definitions(-DPNL_RANGE_CHECK_OFF)
endif()

```

Then, call cmake with the following extra flag

```
-DCMAKE_PREFIX_PATH=path/to/build-dir
```

or add the variable CMAKE_BUILD_TYPE to the GUI.

Just in case, we give an example of a complete although elementary CMakeLists.txt

```
cmake_minimum_required(VERSION 3.0)
# Declare a project
project(my-project CXX)

# Release or Debug
if (NOT CMAKE_BUILD_TYPE)
    message(STATUS "Setting build type to 'Debug' as none was specified.")
    set(CMAKE_BUILD_TYPE Debug CACHE STRING "Choose the type of build." FORCE)
endif ()

# Detect PNL
find_package(Pnl REQUIRED)
set(LIBS ${LIBS} ${PNL_LIBRARIES})
include_directories(${PNL_INCLUDE_DIRS})
if(${CMAKE_BUILD_TYPE} STREQUAL "Release")
    add_definitions(-DPNL_RANGE_CHECK_OFF)
endif()

# Adding an executable.
add_executable(exec-name list_of_source_files)
target_link_libraries(exec-name ${LIBS})
add_pnl_postbuild(exec-name)
```

1.3.2 Inline Functions and getters

If it is supported by your compiler, getter and setter functions are declared as inline functions. This is automatically detected when running CMake. By default, setter and getter functions check that the required access is valid, basically it boils down to checking whether the index of the access is within an acceptable range. These extra tests can become very expensive when getter and setter functions are intensively called.

Thus, it is possible to alter this default behaviour by defining the macro PNL_RANGE_CHECK_OFF. This macro is automatically defined when the library is compiled in Release mode, ie. with -DCMAKE_BUILD_TYPE=Release passed to CMake.

2 Objects

2.1 The top-level object

The PnlObject structure is used to simulate some inheritance between the objects of Pnl. It must be the first element of all the objects existing in Pnl so that casting any object to a PnlObject is legal

```
typedef unsigned int PnlType;

typedef void (DestroyFunc) (void **);
typedef PnlObject* (CopyFunc) (PnlObject *);
typedef PnlObject* (NewFunc) (PnlObject *);
typedef void (CloneFunc) (PnlObject *dest, const PnlObject *src);
struct _PnlObject
{
    PnlType type; /*!< a unique integer id */
    const char *label; /*!< a string identifier (for the moment not
        useful) */
    PnlType parent_type; /*!< the identifier of the parent object is any
        ,
                               otherwise parent_type=id */
    int nref; /*!< number of references on the object */
    DestroyFunc *destroy; /*!< frees an object */
    NewFunc *constructor; /*!< New function */
    CopyFunc *copy; /*!< Copy function */
    CloneFunc *clone; /*!< Clone function */
};
```

Here is the list of all the types actually defined

We provide several macros for manipulating PnlObjects.

- **PNL_OBJECT** (o)
[Description](#) Cast any object into a PnlObject
- **PNL_VECT_OBJECT** (o)
[Description](#) Cast any object into a PnlVectObject
- **PNL_MAT_OBJECT** (o)
[Description](#) Cast any object into a PnlMatObject
- **PNL_SP_MAT_OBJECT** (o)
[Description](#) Cast any object into a PnlSpMatObject
- **PNL_HMAT_OBJECT** (o)
[Description](#) Cast any object into a PnlHmatObject
- **PNL_BAND_MAT_OBJECT** (o)
[Description](#) Cast any object into a PnlBandMatObject
- **PNL_TRIDIAGMAT_OBJECT** (o)
[Description](#) Cast any object into a PnlTridiagMatObject

| PnlType | Description |
|--------------------------------|------------------------------|
| PNL_TYPE_VECTOR | general vectors |
| PNL_TYPE_VECTOR_DOUBLE | real vectors |
| PNL_TYPE_VECTOR_INT | integer vectors |
| PNL_TYPE_VECTOR_COMPLEX | complex vectors |
| PNL_TYPE_MATRIX | general matrices |
| PNL_TYPE_MATRIX_DOUBLE | real matrices |
| PNL_TYPE_MATRIX_INT | integer matrices |
| PNL_TYPE_MATRIX_COMPLEX | complex matrices |
| PNL_TYPE_TRIDIAG_MATRIX | general tridiagonal matrices |
| PNL_TYPE_TRIDIAG_MATRIX_DOUBLE | real tridiagonal matrices |
| PNL_TYPE_BAND_MATRIX | general band matrices |
| PNL_TYPE_BAND_MATRIX_DOUBLE | real band matrices |
| PNL_TYPE_SP_MATRIX | sparse general matrices |
| PNL_TYPE_SP_MATRIX_DOUBLE | sparse real matrices |
| PNL_TYPE_SP_MATRIX_INT | sparse integer matrices |
| PNL_TYPE_SP_MATRIX_COMPLEX | sparse complex matrices |
| PNL_TYPE_HMATRIX | general hyper matrices |
| PNL_TYPE_HMATRIX_DOUBLE | real hyper matrices |
| PNL_TYPE_HMATRIX_INT | integer hyper matrices |
| PNL_TYPE_HMATRIX_COMPLEX | complex hyper matrices |
| PNL_TYPE_BASIS | bases |
| PNL_TYPE_RNG | random number generators |
| PNL_TYPE_LIST | doubly linked list |
| PNL_TYPE_ARRAY | array |

Table 1: PnlTypes

- **PNL_BASIS_OBJECT** (o)
Description Cast any object into a PnlBasis
- **PNL_RNG_OBJECT** (o)
Description Cast any object into a PnlRng
- **PNL_LIST_OBJECT** (o)
Description Cast any object into a PnlList
- **PNL_LIST_ARRAY** (o)
Description Cast any object into a PnlArray
- **PNL_GET_TYPENAME** (o)
Description Return the name of the type of any object inheriting from PnlObject
- **PNL_GET_TYPE** (o)
Description Return the type of any object inheriting from PnlObject
- **PNL_GET_PARENT_TYPE** (o)
Description Return the parent type of any object inheriting from PnlObject
- **PnlObject * pnl_object_create**(PnlType t)
Description Create an empty PnlObject of type t which can any of the registered types, see Table 1.

2.2 List object

This section describes functions for creating and manipulating lists. Lists are internally stored as doubly linked lists.

The structures and functions related to lists are declared in `pnl/pnl_list.h`.

```
typedef struct _PnlCell PnlCell;
struct _PnlCell
{
    struct _PnlCell *prev; /*!< previous cell or 0 */
    struct _PnlCell *next; /*!< next cell or 0 */
    PnlObject *self; /*!< stored object */
};

typedef struct _PnlList PnlList;
struct _PnlList
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlList pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    PnlCell *first; /*!< first element of the list */
    PnlCell *last; /*!< last element of the list */
};
```

```

PnlCell *curcell; /*!< last accessed element,
                    if never accessed is NULL */
int icurcell; /*!< index of the last accessed element,
                if never accessed is NULLINT */
int len; /*!< length of the list */
};

```

Important note: Lists only store addresses of objects. So when an object is inserted into a list, only its address is stored into the list. This implies that you **must not** free any objects inserted into a list. The deallocation is automatically handled by the function `pnl_list_free`.

- **PnlList * pnl_list_new()**
Description Create an empty list
- **PnlCell * pnl_cell_new()**
Description Create an cell list
- **PnlList * pnl_list_copy(const PnlList *A)**
Description Create a copy of a **PnlList** . Each element of the list **A** is copied by calling the its copy member.
- **void pnl_list_clone(PnlList *dest, const PnlList *src)**
Description Copy the content of **src** into the already existing list **dest**. The list **dest** is automatically resized. This is a hard copy, the contents of both lists are independent after cloning.
- **void pnl_list_free(PnlList **L)**
Description Free a list
- **void pnl_cell_free(PnlCell **c)**
Description Free a list
- **PnlObject * pnl_list_get(PnlList *L, int i)**
Description This function returns the content of the *i*-th cell of the list **L**. This function is optimized for linearly accessing all the elements, so it can be used inside a for loop for instance.
- **void pnl_list_insert_first(PnlList *L, PnlObject *o)**
Description Insert the object **o** on top of the list **L**. Note that **o** is not copied in **L**, so do **not** free **o** yourself, it will be done automatically when calling `pnl_list_free`
- **void pnl_list_insert_last(PnlList *L, PnlObject *o)**
Description Insert the object **o** at the bottom of the list **L**. Note that **o** is not copied in **L**, so do **not** free **o** yourself, it will be done automatically when calling `pnl_list_free`
- **void pnl_list_remove_last(PnlList *L)**
Description Remove the last element of the list **L** and frees it.
- **void pnl_list_remove_first(PnlList *L)**
Description Remove the first element of the list **L** and frees it.
- **void pnl_list_remove_i(PnlList *L, int i)**
Description Remove the *i*-th element of the list **L** and frees it.

- void **pnl_list_concat**(PnlList *L1, PnlList *L2)
Description Concatenate the two lists L1 and L2. The resulting list is store in L1 on exit. Do **not** free L2 since concatenation does not actually copy objects but only manipulates addresses.
- void **pnl_list_resize**(PnlList *L, int n)
Description Change the length of L to become n. If the length of L id increased, the extra elements are set to NULL.
- void **pnl_list_print**(const PnlList *L)
Description Only prints the types of each element. When the PnlObject object has a print member, we will use it.

2.3 Array object

This section describes functions for creating and manipulating arrays of PnlObjects. The structures and functions related to arrays are declared in `pnl/pnl_array.h`.

```
typedef struct _PnlArray PnlArray;
struct _PnlArray
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlArray pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int size;
    PnlObject **array;
    int mem_size;
};
```

Important note: Arrays only store addresses of objects. So when an object is inserted into an array, only its address is stored into the array. This implies that you **must not** free any objects inserted into a array. The deallocation is automatically handled by the function `pnl_array_free`.

- PnlArray * **pnl_array_new**()
Description Create an empty array
- PnlArray * **pnl_array_create**(int n)
Description Create an array of length n.
- PnlArray * **pnl_array_copy**(const PnlArray *A)
Description Create a copy of a PnlArray . Each element of the array A is copied by calling the A[i].object.copy.
- void **pnl_array_clone**(PnlArray *dest, const PnlArray *src)
Description Copy the content of src into the already existing array dest. The array dest is automatically resized. This is a hard copy, the contents of both arrays are independent after cloning.

- void **pnl_array_free**(**PnlArray** **)
Description Free an array and all the objects hold by the array.
- int **pnl_array_resize**(**PnlArray** *T, int size)
Description Resize T to be **size** long. As much as possible of the original data is kept.
- **PnlObject** * **pnl_array_get**(**PnlArray** *T, int i)
Description This function returns the content of the **i**-th cell of the array T. No copy is made.
- **PnlObject** * **pnl_array_set**(**PnlArray** *T, int i, **PnlObject** *O)
Description T[i] = O. No copy is made, so the object O must not be freed manually.
- void **pnl_array_print**(**PnlArray** *)
Description Not yet implemented because it would require that the structure **PnlObject** has a field copy.

3 Mathematical framework

3.1 General tools

The macros and functions of this paragraph are defined in `pnl/pnl_mathtools.h`.

3.1.1 Constants

A few mathematical constants are provided by the library. Most of them are actually already defined in `math.h`, `values.h` or `limits.h` and a few others have been added.

| | |
|--------------------|---|
| M_E | e^1 |
| M_LOG2E | $\log_2 e$ |
| M_LOG10E | $\log_{10} e$ |
| M_LN2 | $\log_e 2$ |
| M_LN10 | $\log_e 10$ |
| M_PI | π |
| M_2PI | 2π |
| M_PI_2 | $\pi/2$ |
| M_PI_4 | $\pi/4$ |
| M_1_PI | $1/\pi$ |
| M_2_PI | $2/\pi$ |
| M_2_SQRTPI | $2/\sqrt{\pi}$ |
| M_SQRT2PI | $\text{sqr}t2\pi$ |
| M_SQRT2 | $\sqrt{2}$ |
| M_EULER | $\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right)$ |
| M_SQRT1_2 | $1/\sqrt{2}$ |
| M_1_SQRT2PI | $1/\sqrt{2\pi}$ |
| M_SQRT2_PI | $\sqrt{2/\pi}$ |
| INT_MAX | 2147483647 |
| MAX_INT | INT_MAX |
| DBL_MAX | $1.79769313486231470e + 308$ |
| DOUBLE_MAX | DBL_MAX |
| DBL_EPSILON | $2.2204460492503131e - 16$ |
| PNL_NEGINF | $-\infty$ |
| PNL_POSINF | $+\infty$ |
| PNL_INF | $+\infty$ |
| NAN | Not a Number |

3.1.2 A few macros

- **PNL_IS_ODD** (int n)
[Description](#) Return 1 if n is odd and 0 otherwise.
- **PNL_IS_EVEN** (int n)
[Description](#) Return 1 if n is even and 0 otherwise.
- **PNL_ALTERNATE** (int n)
[Description](#) Return $(-1)^n$.
- **MIN** (x,y)
[Description](#) Return the minimum of x and y.
- **MAX** (x,y)
[Description](#) Return the maximum of x and y.
- **ABS** (x)
[Description](#) Return the absolute value of x.
- **PNL_SIGN** (x)
[Description](#) Return the sign of x (-1 if x < 0, 0 otheriwse).

- **SQR** (*x*)
[Description](#) Return x^2 .
- **CUB** (*x*)
[Description](#) Return x^3 .

3.1.3 Classifying a floating-point number

- double **pnl_nan**()
[Description](#) Return NaN
- double **pnl_posinf**()
[Description](#) Return + infinity
- double **pnl_neginf**()
[Description](#) Return - infinity
- int **pnl_isnan**(double *x*)
[Description](#) Return +1 if *x*=NaN
- int **pnl_isinf**(double *x*)
[Description](#) Return +1 if *x*=+Inf, -1 if *x*=-Inf and 0 otherwise.
- int **pnl_isfinite**(double *x*)
[Description](#) Return 1 if *x*!=+-Inf

3.1.4 Rounding a floating-point number

- int **pnl_itrunc**(double *s*)
[Description](#) This function is similar to the **trunc** function (provided by the C library) but the result is typed as an integer instead of a double. Digits may be lost if *s* exceeds MAX_INT.
- long int **pnl_ltrunc**(double *s*)
[Description](#) This function is similar to the **trunc** function (provided by the C library) but the result is typed as a long integer instead of a double.
- double **pnl_trunc**(double *s*)
[Description](#) Return the nearest integer not greater than the absolute value of *s*. This function is part of C99 as **trunc**.
- double **pnl_round**(double *s*)
[Description](#) Return the integral value nearest to *x* rounding half-way cases away from zero, regardless of the current rounding direction. This function is part of C99 as **round**.
- int **pnl_iround**(double *s*)
[Description](#) This function is similar to the **round** function (provided by the C library) but the result is typed as an integer instead of a double. Digits may be lost if *s* exceeds MAX_INT.
- long int **pnl_lround**(double *s*)
[Description](#) This function is similar to the **round** function (provided by the C library) but the result is typed as a long integer instead of a double.

3.1.5 Some standard mathematical functions

- double **pnl_fact**(double x)
[Description](#) See `pnl_sf_fact`
- double **pnl_lgamma**(double x)
[Description](#) See `pnl_sf_log_gamma`
- double **pnl_tgamma**(double x)
[Description](#) See `pnl_sf_gamma`
- double **pnl_acosh**(double x)
[Description](#) Compute `acosh(x)`.
- double **pnl_asinh**(double x)
[Description](#) Compute `asinh(x)`.
- double **pnl_atanh**(double x)
[Description](#) Compute `atanh(x)`.
- double **pnl_log1p**(double x)
[Description](#) Compute `log(1+x)` accurately for small values of `x`
- double **pnl_expml**(double x)
[Description](#) Compute `exp(x)-1` accurately for small values of `x`
- double **pnl_cosm1**(double x)
[Description](#) Compute `cos(x)-1` accurately for small values of `x`
- double **pnl_pow_i**(double x, int n)
[Description](#) Compute `x^n` for an integer `n`.

3.1.6 Comparison of floating-point numbers

- int **pnl_isequal_rel**(double x, double y, double relerr)
[Description](#) Compare two floating-point numbers up to a relative precision `relerr`
- int **pnl_isequal_abs**(double x, double y, double abserr)
[Description](#) Compare two floating-point numbers up to an absolute precision `abserr`
- int **pnl_isequal**(double x, double y, double relerr)
[Description](#) Equivalent to `pnl_isequal_abs` if `|x| < 1` and to `pnl_isequal_abs` otherwise.

3.2 Complex numbers

3.2.1 Overview

The complex type and related functions are defined in the header `pnl/pnl_complex.h`.

The first native implementation of complex numbers in the C language appeared in C99, which is unfortunately not available on all platforms. For this reason, we provide here an implementation of complex numbers.

```
typedef struct {
    double r; /*!< real part */
    double i; /*!< imaginary part */
} dcomplex;
```

3.2.2 Constants

CZERO 0 as a complex number
CONE 1 as a complex number
CI I the unit complex number

3.2.3 Functions

- double, double **CMPLX**(dcomplex z)
[Description](#) $z.r, z.i$
- dcomplex **Complex**(double x, double y)
[Description](#) $x + i y$
- dcomplex **Complex__polar**(double r, double theta)
[Description](#) $r \exp(i \theta)$
- double **Creal**(dcomplex z)
[Description](#) $\operatorname{Re}(z)$
- double **Cimag**(dcomplex z)
[Description](#) $\operatorname{Im}(z)$
- dcomplex **Cadd**(dcomplex z, dcomplex b)
[Description](#) $z+b$
- dcomplex **CRadd**(dcomplex z, double b)
[Description](#) $z+b$
- dcomplex **RCadd**(double b, dcomplex z)
[Description](#) $b+z$
- dcomplex **Csub**(dcomplex z, dcomplex b)
[Description](#) $z-b$
- dcomplex **CRsub**(dcomplex z, double b)
[Description](#) $z-b$
- dcomplex **RCsub**(double b, dcomplex z)
[Description](#) $b-z$
- dcomplex **Cminus**(dcomplex z)
[Description](#) $-z$
- dcomplex **Cmul**(dcomplex z, dcomplex b)
[Description](#) $z*b$

- dcomplex **RCmul**(double x, dcomplex z)
Description $x*z$
- dcomplex **CRmul**(dcomplex z, double x)
Description $z * x$
- dcomplex **CRdiv**(dcomplex z, double x)
Description z/x
- dcomplex **RCdiv**(double x, dcomplex z)
Description x/z
- dcomplex **Conj**(dcomplex z)
Description \bar{z}
- dcomplex **Cinv**(dcomplex z)
Description $1/z$
- dcomplex **Cdiv**(dcomplex z, dcomplex w)
Description z/w
- double **Csqr_norm**(dcomplex z)
Description $\text{Re}(z)^2 + \text{Im}(z)^2$
- double **Cabs**(dcomplex z)
Description $|z|$
- dcomplex **Csqrt**(dcomplex z)
Description \sqrt{z} , square root (with positive real part)
- dcomplex **Clog**(dcomplex z)
Description $\log(z)$
- dcomplex **Cexp**(dcomplex z)
Description $\exp(z)$
- dcomplex **CIexp**(double t)
Description $\exp(it)$
- dcomplex **Cpow**(dcomplex z, dcomplex w)
Description z^w , power function
- dcomplex **Cpow_real**(dcomplex z, double x)
Description z^x , power function
- dcomplex **Ccos**(dcomplex z)
Description $\cos(z)$
- dcomplex **Csin**(dcomplex z)
Description $\sin(z)$
- dcomplex **Ctan**(dcomplex z)
Description $\tan(z)$

- dcomplex **Ccotan**(dcomplex z)
Description `cotan(z)`
- dcomplex **Ccosh**(dcomplex z)
Description `cosh(g)`
- dcomplex **Csinh**(dcomplex z)
Description `sinh(g)`
- dcomplex **Ctanh**(dcomplex z)
Description $\tanh(z) = \frac{1-e^{-2z}}{1+e^{-2z}}$
- dcomplex **Ccotanh**(dcomplex z)
Description $\operatorname{cotanh}(z) = \frac{1+e^{-2z}}{1-e^{-2z}}$
- double **Carg**(dcomplex z)
Description `arg(z)`
- dcomplex **Ctgamma**(dcomplex z)
Description `Gamma(z)`, the Gamma function
- dcomplex **Clgamma**(dcomplex z)
Description `log(Gamma (z))`, the logarithm of the Gamma function
- void **Cprintf**(dcomplex z)
Description Print a complex number on the standard output
- int **pnl_complex_isequal_abs**(dcomplex x, dcomplex y, double abserr)
Description Test if two complex numbers are equal up to an absolute error `abserr`.
- int **pnl_complex_isequal_rel**(dcomplex x, dcomplex y, double relerr)
Description Test if two complex numbers are equal up to a relative error `relerr`.
- int **pnl_complex_isequal**(dcomplex x, dcomplex y, double err)
Description Test if two complex numbers using one of the above functions depending on the magnitude of `|y|`.

Most algebraic operations on complex numbers are implemented using the following naming for the functions

- All these function names begin in `C_op_`,
- The small letters `a`, `b` denote two complex numbers whereas `d` is a real number,
- The letter `i` denotes the multiplication by the pure imaginary number i ,
- The letter `c` indicates that the next coming number is conjugated.
- The letters `p`, `m` denote the two standard operations *plus* and *minus* respectively.

For example `C_op_idamcb` is $id(a - \bar{b})$. So functions are :

- dcomplex **C_op_apib**(dcomplex a, dcomplex b)
Description $a + ib$.
- dcomplex **C_op_apcb**(dcomplex a, dcomplex b)
Description $a + \bar{b}$.
- dcomplex **C_op_amcb**(dcomplex a, dcomplex b)
Description $a - \bar{b}$.
- dcomplex **C_op_amib**(dcomplex a, dcomplex b)
Description $a - i b$
- dcomplex **C_op_dapb**(double d, dcomplex a, dcomplex b)
Description $d(a + b)$.
- dcomplex **C_op_damb**(double d, dcomplex a, dcomplex b)
Description $d(a - b)$.
- dcomplex **C_op_dapib**(double d, dcomplex a, dcomplex b)
Description $d(a + ib)$.
- dcomplex **C_op_damib**(double d, dcomplex a, dcomplex b)
Description $d(a - ib)$.
- dcomplex **C_op_dapcb**(double d, dcomplex a, dcomplex b)
Description $d(a + \bar{b})$.
- dcomplex **C_op_damcb**(double d, dcomplex a, dcomplex b)
Description $d(a - \bar{b})$.
- dcomplex **C_op_idapb**(double d, dcomplex a, dcomplex b)
Description $id(a + b)$.
- dcomplex **C_op_idamb**(double d, dcomplex a, dcomplex b)
Description $id(a - b)$.
- dcomplex **C_op_idapcb**(double d, dcomplex a, dcomplex b)
Description $id(a + \bar{b})$.
- dcomplex **C_op_idamcb**(double d, dcomplex a, dcomplex b)
Description $id(a - \bar{b})$.

4 Linear Algebra

4.1 Vectors

4.1.1 Overview

The structures and functions related to vectors are declared in `pnl/pnl_vector.h`. Vectors are declared for several basic types : double, int, and dcomplex. In the following declarations, **BASE** must be replaced by one the previous types and the corresponding vector structures are respectively named PnlVect, PnlVectInt, PnlVectComplex

```

typedef struct _PnlVect {
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlVect pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int size; /*!< size of the vector */
    int mem_size; /*!< size of the memory block allocated for array */
    double *array; /*!< pointer to store the data */
    int owner; /*!< 1 if the object owns its array member, 0 otherwise
    */
} PnlVect;

typedef struct _PnlVectInt {
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlVectInt pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int size; /*!< size of the vector */
    int mem_size; /*!< size of the memory block allocated for array */
    int *array; /*!< pointer to store the data */
    int owner; /*!< 1 if the object owns its array member, 0 otherwise
    */
} PnlVectInt;

typedef struct _PnlVectComplex {
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlVectComplex pointer to be cast
     * to a PnlObject
     */
    PnlObject object;
    int size; /*!< size of the vector */
    int mem_size; /*!< size of the memory block allocated for array */
    dcomplex *array; /*!< pointer to store the data */
    int owner; /*!< 1 if the object owns its array member, 0 otherwise
    */
} PnlVectComplex;

```

`size` is the size of the vector, `array` is a pointer containing the data and `owner` is an integer to know if the vector owns its `array` pointer (`owner=1`) or shares it with another structure (`owner=0`). `mem_size` is the number of elements the vector can hold at most.

4.1.2 Functions

General functions These functions exist for all types of vector no matter what the basic type is. The following conventions are used to name functions operating on vectors. Here is

the table of prefixes used for the different basic types.

| type | prefix | BASE |
|----------|------------------|----------|
| double | pnl_vect | double |
| int | pnl_vect_int | int |
| dcomplex | pnl_vect_complex | dcomplex |

In this paragraph, we present the functions operating on **PnlVect** which exist for all types. To deduce the prototypes of these functions for other basic types, one must replace **pnl_vect** and **double** according the above table.

Constructors and destructors There are no special functions to access the size of a vector, instead the field **size** should be accessed directly.

- **PnlVect * pnl_vect_new()**
Description Create a new **PnlVect** of size 0.
- **PnlVect * pnl_vect_create(int size)**
Description Create a new **PnlVect** pointer.
- **PnlVect * pnl_vect_create_from_zero(int size)**
Description Create a new **PnlVect** pointer and sets it to zero.
- **PnlVect * pnl_vect_create_from_scalar(int size, double x)**
Description Create a new **PnlVect** pointer and sets all elements to **x**.
- **PnlVect * pnl_vect_create_from_ptr(int size, const double *x)**
Description Create a new **PnlVect** pointer and copies **x** to **array**.
- **PnlVect * pnl_vect_create_from_mat(const PnlMat *M)**
Description Create a new **PnlVect** pointer of size **M->mn** and copy the content of **M** row wise.
- **PnlVect * pnl_vect_create_from_list(int size, ...)**
Description Create a new **PnlVect** pointer of length **size** filled with the extra arguments passed to the function. The number of extra arguments passed must be equal to **size** and they must be of the type **BASE**. Example: To create a vector {1., 2.}, you should enter **pnl_vect_create_from_list(2, 1.0, 2.0)** and NOT **pnl_vect_create_from_list(2, 1.0, 2)** or **pnl_vect_create_from_list(2, 1, 2.0)**. Be aware that this cannot be checked inside the function.
- **PnlVect * pnl_vect_create_from_file(const char *file)**
Description Read a vector from a file and creates the corresponding **PnlVect** . The data might be stored as a row or column vector. Entries can be separated by spaces, tabs, commas or semicolons. Anything after a **#** or **%** is ignored up to the end of the line.
- **PnlVect * pnl_vect_copy(const PnlVect *v)**
Description This is a copying constructor. It creates a copy of a **PnlVect** .

- void **pnl_vect_clone**(**PnlVect** *clone, const **PnlVect** *v)
Description Clone a **PnlVect** . **clone** must be an already existing **PnlVect** . It is resized to match the size of **v** and the data are copied. Future modifications to **v** will not affect **clone**.
- **PnlVect** * **pnl_vect_create_subvect_with_ind**(const **PnlVect** *V, const **PnlVectInt** *ind)
Description Create a new vector containing **V(ind(:))**.
- void **pnl_vect_extract_subvect_with_ind**(**PnlVect** *V_sub, const **PnlVect** *V, const **PnlVectInt** *ind)
Description On exit, **V_sub = V(ind(:))**.
- **PnlVect** * **pnl_vect_create_subvect**(const **PnlVect** *V, int i, int len)
Description Create a new vector containing **V(i:i+len-1)**. The elements are copied.
- void **pnl_vect_extract_subvect**(**PnlVect** *V_sub, const **PnlVect** *V, int i, int len)
Description On exit, **V_sub = V(i:i+len-1)**. The elements are copied.
- void **pnl_vect_set_subblock**(**PnlVect** *dest, const **PnlVect** *src, int i)
Description Set **dest[i:] = src**.
- void **pnl_vect_free**(**PnlVect** **v)
Description Free a **PnlVect** pointer and set the data pointer to NULL
- **PnlVect** **pnl_vect_wrap_array**(const double *x, int size)
Description Create a **PnlVect** containing the data **x**. No copy is made. It is just a container.
- **PnlVect** **pnl_vect_wrap_subvect**(const **PnlVect** *x, int i, int s)
Description Create a **PnlVect** containing **x(i:i+s-1)**. No copy is made. It is just a container. The returned **PnlVect** has **size=s** and **owner=0**.
- **PnlVect** **pnl_vect_wrap_subvect_with_last**(const **PnlVect** *x, int i, int j)
Description Create a **PnlVect** containing **x(i:j)**. No copy is made. It is just a container.
- **PnlVect** **pnl_vect_wrap_mat**(const **PnlMat** *M)
Description Return a **PnlVect** (not a pointer) whose array is the row wise array of **M**. The new vector shares its data with the matrix **M**, which means that any modification to one of them will affect the other.

Resizing vectors

- int **pnl_vect_resize**(**PnlVect** *v, int size)
Description Resize a **PnlVect** . It copies as much of the old data to fit in the resized object.
- int **pnl_vect_resize_from_scalar**(**PnlVect** *v, int size, double x)
Description Resize a **PnlVect** . Copy as much of the old data as possible and fill the new cells with **x**.
- int **pnl_vect_resize_from_ptr**(**PnlVect** *v, int size, double *t)
Description Resize a **PnlVect** and uses **t** to fill the vector. **t** must be of size **size**.

Accessing elements If it is supported by the compiler, the following functions are declared inline. To speed up these functions, you can define the macro `PNL_RANGE_CHECK_OFF`, see Section 1.3.2 for an explanation.

Accessing elements of a vector is faster using the following macros

- **GET**(**PnlVect** *v, int i)
Description Return `v[i]` for reading, eg. `x=GET(v,i)`
- **GET_INT**(**PnlVectInt** *v, int i)
Description Same as `GET` but for an integer vector.
- **GET_COMPLEX**(**PnlVectComplex** *v, int i)
Description Same as `GET` but for a complex vector.
- **LET**(**PnlVect** *v, int i)
Description Return `v[i]` as a lvalue for writing, eg. `LET(v,i)=x`
- **LET_INT**(**PnlVectInt** *v, int i)
Description Same as `LET` but for an integer vector.
- **LET_COMPLEX**(**PnlVectComplex** *v, int i)
Description Same as `LET` but for a complex vector.
- void **pnl_vect_set**(**PnlVect** *v, int i, double x)
Description Set `v[i]=x`.
- double **pnl_vect_get**(const **PnlVect** *v, int i)
Description Return the value of `v[i]`.
- void **pnl_vect_lget**(**PnlVect** *v, int i)
Description Return the address of `v[i]`.
- void **pnl_vect_set_all**(**PnlVect** *v, double x)
Description Set all elements to `x`.
- void **pnl_vect_set_zero**(**PnlVect** *v)
Description Set all elements to zero.

Printing vector

- void **pnl_vect_print**(const **PnlVect** *V)
Description Print a **PnlVect** as a column vector
- void **pnl_vect_fprint**(FILE *fic, const **PnlVect** *V)
Description Print a **PnlVect** in file `fic` as a column vector. The file can be read by `pnl_vect_create_from_file`.
- void **pnl_vect_print_asrow**(const **PnlVect** *V)
Description Print a **PnlVect** as a row vector
- void **pnl_vect_fprint_asrow**(FILE *fic, const **PnlVect** *V)
Description Print a **PnlVect** in file `fic` as a row vector. The file can be read by `pnl_vect_create_from_file`.

- void **pnl_vect_print_nsp**(const **PnlVect** *V)
Description Print a vector to the standard output in a format compatible with Nsp.
- void **pnl_vect_fprint_nsp**(FILE *fic, const **PnlVect** *V)
Description Print a vector to a file in a format compatible with Nsp.

Applying external operation to vectors

- void **pnl_vect_minus**(**PnlVect** *lhs)
Description In-place unary minus
- void **pnl_vect_plus_scalar**(**PnlVect** *lhs, double x)
Description In-place vector scalar addition
- void **pnl_vect_minus_scalar**(**PnlVect** *lhs, double x)
Description In-place vector scalar subtraction
- void **pnl_vect_mult_scalar**(**PnlVect** *lhs, double x)
Description In-place vector scalar multiplication
- void **pnl_vect_div_scalar**(**PnlVect** *lhs, double x)
Description In-place vector scalar division

Element wise operations

- void **pnl_vect_plus_vect**(**PnlVect** *lhs, const **PnlVect** *rhs)
Description In-place vector vector addition
- void **pnl_vect_minus_vect**(**PnlVect** *lhs, const **PnlVect** *rhs)
Description In-place vector vector subtraction
- void **pnl_vect_inv_term**(**PnlVect** *lhs)
Description In-place term by term vector inversion
- void **pnl_vect_div_vect_term**(**PnlVect** *lhs, const **PnlVect** *rhs)
Description In-place term by term vector division
- void **pnl_vect_mult_vect_term**(**PnlVect** *lhs, const **PnlVect** *rhs)
Description In-place vector vector term by term multiplication
- void **pnl_vect_map**(**PnlVect** *lhs, const **PnlVect** *rhs, double(*f)(double))
Description lhs = f(rhs)
- void **pnl_vect_map_inplace**(**PnlVect** *lhs, double(*f)(double))
Description lhs = f(lhs)
- void **pnl_vect_map_vect**(**PnlVect** *lhs, const **PnlVect** *rhs1, const **PnlVect** *rhs2, double(*f)(double, double))
Description lhs = f(rhs1, rhs2)
- void **pnl_vect_map_vect_inplace**(**PnlVect** *lhs, **PnlVect** *rhs, double(*f)(double, double))
Description lhs = f(lhs, rhs)

- void **pnl_vect_axpby**(double a, const **PnlVect** *x, double b, **PnlVect** *y)
Description Compute $y : = a x + b y$. When $b=0$, the content of y is not used on input and instead y is resized to match x .
- double **pnl_vect_sum**(const **PnlVect** *lhs)
Description Return the sum of all the elements of a vector
- void **pnl_vect_cumsum**(**PnlVect** *lhs)
Description Compute the cumulative sum of all the elements of a vector. The original vector is modified
- double **pnl_vect_prod**(const **PnlVect** *V)
Description Return the product of all the elements of a vector
- void **pnl_vect_cumprod**(**PnlVect** *lhs)
Description Compute the cumulative product of all the elements of a vector. The original vector is modified

Scalar products and norms

- double **pnl_vect_norm_two**(const **PnlVect** *V)
Description Return the two norm of a vector
- double **pnl_vect_norm_one**(const **PnlVect** *V)
Description Return the one norm of a vector
- double **pnl_vect_norm_infty**(const **PnlVect** *V)
Description Return the infinity norm of a vector
- double **pnl_vect_scalar_prod**(const **PnlVect** *rhs1, const **PnlVect** *rhs2)
Description Compute the scalar product between 2 vectors
- int **pnl_vect_cross**(**PnlVect** *lhs, const **PnlVect** *x, const **PnlVect** *y)
Description Compute the cross product of x and y and store the result in lhs . The vectors x and y must be of size 3 and FAIL is returned otherwise.
- double **pnl_vect_dist**(const **PnlVect** *x, const **PnlVect** *y)
Description Compute the distance between x and y , ie $\sqrt{\sum_i |x_i - y_i|^2}$.

Comparison functions

- int **pnl_vect_isequal**(const **PnlVect** *V1, const **PnlVect** *V2, double err)
Description Test if two vectors are equal up to **err** component-wise. The error **err** is either relative or absolute depending on the magnitude of the components. Return TRUE or FALSE.
- int **pnl_vect_isequal_abs**(const **PnlVect** *V1, const **PnlVect** *V2, double abserr)
Description Test if two vectors are equal up to an absolute error **abserr** component-wise. Return TRUE or FALSE.

- int **pnl_vect_isequal_rel**(const **PnlVect** *V1, const **PnlVect** *V2, double relerr)
Description Test if two vectors are equal up to a relative error **relerr** component–wise. Return TRUE or FALSE.
- int **pnl_vect_eq_all**(const **PnlVect** *v, double x)
Description Test if all the components of **v** are equal to **x**. Return TRUE or FALSE.

Ordering functions The following functions are not defined for **PnlVectComplex** because there is no total ordering on Complex numbers

- double **pnl_vect_max**(const **PnlVect** *V)
Description Return the maximum of a a vector
- double **pnl_vect_min**(const **PnlVect** *V)
Description Return the minimum of a vector
- void **pnl_vect_minmax**(double *m, double *M, const **PnlVect** *)
Description Compute the minimum and maximum of a vector which are returned in **m** and **M** respectively.
- void **pnl_vect_min_index**(double *m, int *im, const **PnlVect** *)
Description Compute the minimum of a vector and its index stored in sets **m** and **im** respectively.
- void **pnl_vect_max_index**(double *M, int *iM, const **PnlVect** *)
Description Compute the maximum of a vector and its index stored in sets **m** and **im** respectively.
- void **pnl_vect_minmax_index**(double *m, double *M, int *im, int *iM, const **PnlVect** *)
Description Compute the minimum and maximum of a vector and the corresponding indices stored respectively in **m**, **M**, **im** and **iM**.
- void **pnl_vect_qsort**(**PnlVect** *, char order)
Description Sort a vector using a quick sort algorithm according to **order** ('i' for increasing or 'd' for decreasing).
- void **pnl_vect_qsort_index**(**PnlVect** *, **PnlVectInt** *index, char order)
Description Sort a vector using a quick sort algorithm according to **order** ('i' for increasing or 'd' for decreasing). On output, **index** contains the permutation used to sort the vector.
- int **pnl_vect_find**(**PnlVectInt** *ind, char *type, int(*f)(double *t), ...)
Description **f** is a function taking a C array as argument and returning an integer. **type** is a string composed by the letters 'r' and 'v' and is used to describe the types of the arguments appearing after **f**. This function aims at simulating Scilab's **find** function. Here are a few examples (capital letters are used for vectors and small letters for real values)

– ind = find (a < X)

```

    int isless ( double *t ) { return t[0] < t[1]; }
    pnl_vect_find ( ind, "rv", isless, a, X );

- ind = find (X <= Y)

    int isless ( double *t ) { return t[0] <= t[1]; }
    pnl_vect_find ( ind, "vv", isless, X, Y );

- ind = find ((a < X) && (X <= Y))

    int cmp ( double *t )
    {
        return (t[0] <= t[1]) && (t[1] <= t[2]);
    }
    pnl_vect_find ( ind, "rvv", cmp, a, X, Y );

```

`ind` contains on exit the indices `i` for which the function `f` returned 1. This function returns OK or FAIL when something went wrong (size mismatch between matrices, invalid string type).

Misc

- void **pnl_vect_swap_elements**(**PnlVect** *v, int i, int j)
[Description](#) Exchange `v[i]` and `v[j]`.
- void **pnl_vect_reverse**(**PnlVect** *v)
[Description](#) Perform a mirror operation on `v`. On output `v[i] = v[n-1-i]` for `i=0, ..., n-1` where `n` is the length of the vector.

Complex vector functions

- void **pnl_vect_complex_mult_double**(**PnlVectComplex** *lhs, double x)
[Description](#) In-place multiplication by a double.
- **PnlVectComplex*** **pnl_vect_complex_create_from_array**(int size, const double *re, const double *im)
[Description](#) Create a **PnlVectComplex** given the arrays of the real parts `re` and imaginary parts `im`.
- void **pnl_vect_complex_split_in_array**(const **PnlVectComplex** *v, double *re, double *im)
[Description](#) Split a complex vector into two C arrays : the real parts of the elements of `v` are stored into `re` and the imaginary parts into `im`.
- void **pnl_vect_complex_split_in_vect**(const **PnlVectComplex** *v, **PnlVect** *re, **PnlVect** *im)
[Description](#) Split a complex vector into two **PnlVect** 's : the real parts of the elements of `v` are stored into `re` and the imaginary parts into `im`.

There exist functions to directly access the real or imaginary parts of an element of a complex vector. These functions also have inlined versions that are used if the variable `HAVE_INLINE` was declared at compilation time.

- `double pnl_vect_complex_get_real(const PnlVectComplex *v, int i)`
Description Return the real part of `v[i]`.
- `double pnl_vect_complex_get_imag(const PnlVectComplex *v, int i)`
Description Return the imaginary part of `v[i]`.
- `double* pnl_vect_complex_lget_real(const PnlVectComplex *v, int i)`
Description Return the real part of `v[i]` as a lvalue.
- `double* pnl_vect_complex_lget_imag(const PnlVectComplex *v, int i)`
Description Return the imaginary part of `v[i]` as a lvalue.
- `void pnl_vect_complex_set_real(const PnlVectComplex *v, int i, double re)`
Description Set the real part of `v[i]` to `re`.
- `void pnl_vect_complex_set_imag(const PnlVectComplex *v, int i, double im)`
Description Set the imaginary part of `v[i]` to `im`.

Equivalently to these functions, there exist macros. When the compiler is able to handle inline code, there is no gain in using macros instead of inlined functions at least in principle.

- `GET_REAL(v, i)`
Description Return the real part of `v[i]`.
- `GET_IMAG(v, i)`
Description Return the imaginary part of `v[i]`.
- `LET_REAL(v, i)`
Description Return the real part of `v[i]` as a lvalue.
- `LET_IMAG(v, i)`
Description Return the imaginary part of `v[i]` as a lvalue.

4.2 Compact Vectors

4.2.1 Short description

```
typedef struct PnlVectCompact {
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlVectCompact pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int size; /* size of the vector */
    double val; /* single value */
    double *array; /* Pointer to double values */
    char convert; /* 'a', 'd' : array, double */
} PnlVectCompact;
```

4.2.2 Functions

- **PnlVectCompact** * **pnl_vect_compact_new**()
Description Create a **PnlVectCompact** of size 0.
- **PnlVectCompact** * **pnl_vect_compact_create**(int n, double x)
Description Create a **PnlVectCompact** filled in with x
- **PnlVectCompact** * **pnl_vect_compact_create_from_ptr**(int n, double *x)
Description Create a **PnlVectCompact** filled in with the content of x. Note that x must have at least n elements.
- int **pnl_vect_compact_resize**(**PnlVectCompact** *v, int size, double x)
Description Resize a **PnlVectCompact** .
- **PnlVectCompact** * **pnl_vect_compact_copy**(const **PnlVectCompact** *v)
Description Copy a **PnlVectCompact**
- void **pnl_vect_compact_free**(**PnlVectCompact** **v)
Description Free a **PnlVectCompact**
- **PnlVect** * **pnl_vect_compact_to_pnl_vect**(const **PnlVectCompact** *C)
Description Convert a **PnlVectCompact** pointer to a **PnlVect** pointer.
- double **pnl_vect_compact_get**(const **PnlVectCompact** *C, int i)
Description Access function
- void **pnl_vect_compact_set_all**(**PnlVectCompact** *C, double x)
Description Set all elements of C to x. C is converted to a compact storage.
- void **pnl_vect_compact_set_ptr**(**PnlVectCompact** *C, double *ptr)
Description Copy the array ptr into C. We assume that the sizes match. C is converted to a non compact storage.

4.3 Matrices

4.3.1 Overview

The structures and functions related to matrices are declared in **pnl/pnl_matrix.h**.

```
typedef struct _PnlMat{  
    /**  
     * Must be the first element in order for the object mechanism to  
     * work  
     * properly. This allows any PnlMat pointer to be cast to a  
     * PnlObject  
     */  
    PnlObject object;  
    int m; /*!< nb rows */  
    int n; /*!< nb columns */  
    int mn; /*!< product m*n */  
    int mem_size; /*!< size of the memory block allocated for array */  
    double *array; /*!< pointer to store the data row-wise */  
};
```

```

    int owner; /*!< 1 if the object owns its array member, 0 otherwise
    */
} PnlMat;

typedef struct _PnlMatInt{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlMatInt pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int m; /*!< nb rows */
    int n; /*!< nb columns */
    int mn; /*!< product m*n */
    int mem_size; /*!< size of the memory block allocated for array */
    int *array; /*!< pointer to store the data row-wise */
    int owner; /*!< 1 if the object owns its array member, 0 otherwise
    */
} PnlMatInt;

typedef struct _PnlMatComplex{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlMatComplex pointer to be cast
     * to a PnlObject
     */
    PnlObject object;
    int m; /*!< nb rows */
    int n; /*!< nb columns */
    int mn; /*!< product m*n */
    int mem_size; /*!< size of the memory block allocated for array */
    dcomplex *array; /*!< pointer to store the data row-wise */
    int owner; /*!< 1 if the object owns its array member, 0 otherwise
    */
} PnlMatComplex;

```

m is the number of rows, n is the number of columns. `array` is a pointer containing the data of the matrix stored line wise, The element (i, j) of the matrix is `array[i*m+j]`. `owner` is an integer to know if the matrix owns its `array` pointer (`owner=1`) or shares it with another structure (`owner=0`). `mem_size` is the number of elements the matrix can hold at most.

The following operations are implemented on matrices and vectors. `alpha` and `beta` are numbers, `A` and `B` are matrices and `x` and `y` are vectors.

| | |
|--|---|
| <code>pnl_mat_axpy</code> | $B := \alpha * A + B$ |
| <code>pnl_mat_scalar_prod</code> | $x' A y$ |
| <code>pnl_mat_dgemm</code> | $C := \alpha * \text{op}(A) * \text{op}(B) + \beta * C$ |
| <code>pnl_mat_mult_vect_transpose_inplace</code> | $y = A' * x$ |
| <code>pnl_mat_mult_vect_inplace</code> | $y = A * x$ |
| <code>pnl_mat_lAxpby</code> | $y := \lambda * A * x + \beta * y$ |
| <code>pnl_mat_dgemv</code> | $y := \alpha * \text{op}(A) * x + \beta * y$ |
| <code>pnl_mat_dger</code> | $A := \alpha x * y' + A$ |

4.3.2 Generic Functions

These functions exist for all types of matrices no matter what the basic type is. The following conventions are used to name functions operating on matrices. Here is the table of prefixes used for the different basic types.

| type | prefix | BASE |
|----------|-----------------|----------|
| double | pnl_mat | double |
| int | pnl_mat_int | int |
| dcomplex | pnl_mat_complex | dcomplex |

In this paragraph we present the functions operating on **PnlMat** which exist for all types. To deduce the prototypes of these functions for other basic types, one must replace **pnl_mat** and **double** according the above table.

Constructors and destructors There are no special functions to access the sizes of a matrix, instead the fields **m**, **n** and **mn** give direct access to the number of rows, columns and the size of the matrix.

- **PnlMat * pnl_mat_new()**
Description Create a **PnlMat** of size 0
- **PnlMat * pnl_mat_create(int m, int n)**
Description Create a **PnlMat** with **m** rows and **n** columns.
- **PnlMat * pnl_mat_create_from_scalar(int m, int n, double x)**
Description Create a **PnlMat** with **m** rows and **n** columns and sets all the elements to **x**.
- **PnlMat * pnl_mat_create_from_zero(int m, int n)**
Description Create a **PnlMat** with **m** rows and **n** columns and sets all elements to 0.
- **PnlMat * pnl_mat_create_from_ptr(int m, int n, const double *x)**
Description Create a **PnlMat** with **m** rows and **n** columns and copies the array **x** to the new vector. Be sure that **x** is long enough to fill all the vector because it cannot be checked inside the function.
- **PnlMat * pnl_mat_create_from_list(int m, int n, ...)**
Description Create a new **PnlMat** pointer of size **m x n** filled with the extra arguments passed to the function. The number of extra arguments passed must be equal to **m x n**, be aware that this cannot be checked inside the function.
- **PnlMat * pnl_mat_copy(const PnlMat *M)**
Description Create a new **PnlMat** which is a copy of **M**.
- **PnlMat * pnl_mat_create_diag_from_ptr(const double *x, int d)**
Description Create a new squared **PnlMat** by specifying its size and diagonal terms as an array.
- **PnlMat * pnl_mat_create_diag(const PnlVect *V)**
Description Create a new squared **PnlMat** by specifying its diagonal terms in a **PnlVect**

- **PnlMat * pnl_mat_create_from_file**(const char *file)
Description Read a matrix from a file and creates the corresponding **PnlMat** . One row of the matrix corresponds to one line of the file and the elements of a row can be separated by spaces, tabs, commas or semicolons. Anything after a # or % is ignored up to the end of the line.
- void **pnl_mat_free**(**PnlMat** **M)
Description Free a **PnlMat** and sets *M to NULL
- **PnlMat pnl_mat_wrap_array**(const double *x, int m, int n)
Description Create a **PnlMat** of size m x n which contains x. No copy is made. It is just a container.
- **PnlMat pnl_mat_wrap_vect**(const **PnlVect** *V)
Description Return a **PnlMat** (not a pointer) whose array is the array of V. The new matrix shares its data with the vector V, which means that any modification to one of them will affect the other.
- void **pnl_mat_clone**(**PnlMat** *clone, const **PnlMat** *M)
Description Clone M into clone. No new **PnlMat** is created.
- int **pnl_mat_resize**(**PnlMat** *M, int m, int n)
Description Resize a **PnlMat** . The new matrix is of size m x n. As much as possible of the old data is kept in the row-wise storage: if the number of columns does not change, the common line values are kept unchanged.
- **PnlVect * pnl_vect_create_submat**(const **PnlMat** *M, const **PnlVectInt** *indi, const **PnlVectInt** *indj)
Description Create a new vector containing the values M(indi(:), indj(:)). indi and indj must be of the same size.
- void **pnl_vect_extract_submat**(**PnlVect** *V_sub, const **PnlMat** *M, const **PnlVectInt** *indi, const **PnlVectInt** *indj)
Description On exit, V_sub = M(indi(:), indj(:)). indi and indj must be of the same size.
- void **pnl_mat_extract_subblock**(**PnlMat** *M_sub, const **PnlMat** *M, int i, int len_i, int j, int len_j)
Description M_sub = M(i:i+len_i-1, j:j+len_j-1). len_i (resp. len_j) is the number of rows (resp. columns) to be extracted.
- void **pnl_mat_set_subblock**(**PnlMat** *M, const **PnlMat** *block, int i, int j)
Description If block is a matrix of size m_block x n_block, the dimensions of M must satisfy that M->m >= i + m_block and M->n >= j + n_block. On output M(i:i+m_block-1, j:j+n_block-1) = block.

Accessing elements. If it is supported by the compiler, the following functions are declared inline. To speed up these functions, you can define the macro PNL_RANGE_CHECK_OFF, see Section 1.3.2 for an explanation.

Accessing elements of a matrix is faster using the following macros

- **MGET**(**PnlMat** *M, int i, int j)
Description Return $M[i, j]$ for reading, eg. $x = \text{MGET}(M, i, j)$
- **MGET_INT**(**PnlMatInt** *M, int i, int j)
Description Same as MGET but for an integer matrix.
- **MGET_COMPLEX**(**PnlMatComplex** *M, int i, int j)
Description Same as MGET but for a complex matrix.
- **MLET**(**PnlMat** *M, int i, int j)
Description Return $M[i, j]$ as a lvalue for writing, eg. $\text{MLET}(M, i, j) = x$
- **MLET_INT**(**PnlMatInt** *M, int i, int j)
Description Same as MLET but for an integer matrix.
- **MLET_COMPLEX**(**PnlMatComplex** *M, int i, int j)
Description Same as MLET but for a complex matrix.
- void **pnl_mat_set**(**PnlMat** *M, int i, int j, double x)
Description Set the value of $M[i, j] = x$
- double **pnl_mat_get**(const **PnlMat** *M, int i, int j)
Description Get the value of $M[i, j]$
- double * **pnl_mat_lget**(**PnlMat** *M, int i, int j)
Description Return the address of $M[i, j]$ for use as a lvalue.
- void **pnl_mat_set_all**(**PnlMat** *M, double x)
Description Set all elements of M to x .
- void **pnl_mat_set_zero**(**PnlMat** *M)
Description Set all elements of M to 0.
- void **pnl_mat_set_id**(**PnlMat** *M)
Description Set the matrix M to the identity matrix. M must be a square matrix.
- void **pnl_mat_set_diag**(**PnlMat** *M, double x, int d)
Description Set the d^{th} diagonal terms of the matrix M to the value x . M must be a square matrix.
- void **pnl_mat_set_from_ptr**(**PnlMat** *M, const double *x)
Description Set M row-wise with the values given by x . The array x must be at least $M \rightarrow mn$ long.
- void **pnl_mat_get_row**(**PnlVect** *V, const **PnlMat** *M, int i)
Description Extract and copies the i -th row of M into V .
- void **pnl_mat_get_col**(**PnlVect** *V, const **PnlMat** *M, int j)
Description Extract and copies the j -th column of M into V .

- **PnlVect** **pnl_vect_wrap_mat_row**(const **PnlMat** *M, int i)
Description Return a **PnlVect** (not a pointer) whose array is the *i*-th row of *M*. The new vector shares its data with the matrix *M*, which means that any modification to one of them will affect the other.
- **PnlMat** **pnl_mat_wrap_mat_rows**(const **PnlMat** *M, int i_start, int i_end)
Description Return a **PnlMat** (not a pointer) holding rows from *i_start* to *i_end* (included) of *M*. The new matrix shares its data with the matrix *M*, which means that any modification to one of them will affect the other.
- void **pnl_mat_swap_rows**(**PnlMat** *M, int i, int j)
Description Swap two rows of a matrix.
- void **pnl_mat_set_col**(**PnlMat** *M, const **PnlVect** *V, int j)
Description Set the *i*-th column of a matrix *M* with the content of *V*
- void **pnl_mat_set_col_from_ptr**(**PnlMat** *M, const double *x, int j)
Description Set the *i*-th column of *M* with the content of *x*.
- void **pnl_mat_set_row**(**PnlMat** *M, const **PnlVect** *V, int i)
Description Set the *i*-th row of *M* with the content of *V*
- void **pnl_mat_set_row_from_ptr**(**PnlMat** *M, const double *x, int i)
Description Set the *i*-th row of *M* with the content of *x*
- void **pnl_mat_add_row**(**PnlMat** *M, int i, const **PnlVect** *r)
Description Add a row in matrix *M* before position *i* and fill it with the content of *r*. If *r* == NULL, row *i* is left uninitialized. The index *i* may vary between 0 — add a row at the top of the matrix — and *M->m* — add a row after all rows.
- void **pnl_mat_del_row**(**PnlMat** *M, int i)
Description Delete the row with index *i* (between 0 and *M->m-1*) of the matrix *M*.

Printing Matrices

- void **pnl_mat_print**(const **PnlMat** *M)
Description Print a matrix to the standard output.
- void **pnl_mat_fprint**(FILE *fic, const **PnlMat** *M)
Description Print a matrix to a file. The saved matrix can be reloaded by the function `pnl_mat_create_from_file`.
- void **pnl_mat_print_csv**(const **PnlMat** *M, char sep)
Description Print a matrix to the standard output in the CSV format using separator *sep*.
- void **pnl_mat_fprint_csv**(FILE *fic, const **PnlMat** *M, char sep)
Description Print a matrix to a CSV file using separator *sep*.
- void **pnl_mat_print_nsp**(const **PnlMat** *M)
Description Print a matrix to the standard output in a format compatible with *Nsp*.

- void **pnl_mat_fprint_nsp**(FILE *fic, const **PnlMat** *M)
[Description](#) Print a matrix to a file in a format compatible with Nsp.

Applying external operations

- void **pnl_mat_plus_scalar**(**PnlMat** *lhs, double x)
[Description](#) In-place matrix scalar addition
- void **pnl_mat_minus_scalar**(**PnlMat** *lhs, double x)
[Description](#) In-place matrix scalar subtraction
- void **pnl_mat_mult_scalar**(**PnlMat** *lhs, double x)
[Description](#) In-place matrix scalar multiplication
- void **pnl_mat_div_scalar**(**PnlMat** *lhs, double x)
[Description](#) In-place matrix scalar division

Element wise operations

- void **pnl_mat_mult_mat_term**(**PnlMat** *lhs, const **PnlMat** *rhs)
[Description](#) In-place matrix matrix term by term product
- void **pnl_mat_div_mat_term**(**PnlMat** *lhs, const **PnlMat** *rhs)
[Description](#) In-place matrix matrix term by term division
- void **pnl_mat_kron_mat_inplace**(**PnlMat** *res, const **PnlMat** *B, const **PnlMat** *A)
[Description](#) In-place Kroenecker product of A and B
- **PnlMat** * **pnl_mat_kron_mat**(const **PnlMat** *B, const **PnlMat** *A)
[Description](#) Return the Kroenecker product of A and B
- void **pnl_mat_map_inplace**(**PnlMat** *lhs, double(*f)(double))
[Description](#) lhs = f(lhs).
- void **pnl_mat_map**(**PnlMat** *lhs, const **PnlMat** *rhs, double(*f)(double))
[Description](#) lhs = f(rhs).
- void **pnl_mat_map_mat_inplace**(**PnlMat** *lhs, const **PnlMat** *rhs, double(*f)(double, double))
[Description](#) lhs = f(lhs, rhs).
- void **pnl_mat_map_mat**(**PnlMat** *lhs, const **PnlMat** *rhs1, const **PnlMat** *rhs2, double(*f)(double, double))
[Description](#) lhs = f(rhs1, rhs2).
- double **pnl_mat_sum**(const **PnlMat** *lhs)
[Description](#) Sum matrix component-wise
- void **pnl_mat_sum_vect**(**PnlVect** *y, const **PnlMat** *A, char c)
[Description](#) Sum matrix column or row wise. Argument c can be either 'r' (to get a row vector) or 'c' (to get a column vector). When c='r', $y(j) = \sum_i A_{ij}$ and when c='c', $y(i) = \sum_j A_{ij}$.

- void **pnl_mat_cumsum**(**PnlMat** *A, char c)
Description Cumulative sum over the rows or columns. Argument c can be either 'r' to sum over the rows or 'c' to sum over the columns. When $c='r'$, $A_{ij} = \sum_{1 \leq k \leq i} A_{kj}$ and when $c='rc'$, $A_{ij} = \sum_{1 \leq k \leq j} A_{ik}$.
- double **pnl_mat_prod**(const **PnlMat** *lhs)
Description Product matrix component-wise
- void **pnl_mat_prod_vect**(**PnlVect** *y, const **PnlMat** *A, char c)
Description Prod matrix column or row wise. Argument c can be either 'r' (to get a row vector) or 'c' (to get a column vector). When $c='r'$, $y(j) = \prod_i A_{ij}$ and when $c='rc'$, $y(i) = \prod_j A_{ij}$.
- void **pnl_mat_cumprod**(**PnlMat** *A, char c)
Description Cumulative prod over the rows or columns. Argument c can be either 'r' to prod over the rows or 'c' to prod over the columns. When $c='r'$, $A_{ij} = \prod_{1 \leq k \leq i} A_{kj}$ and when $c='rc'$, $A_{ij} = \prod_{1 \leq k \leq j} A_{ik}$.

Comparison functions

- int **pnl_mat_isequal**(const **PnlMat** *A, const **PnlMat** *B, double err)
Description Test if two matrices are equal up to **err** component-wise. The error **err** is either relative or absolute depending on the magnitude of the components. Return TRUE or FALSE.
- int **pnl_mat_isequal_abs**(const **PnlMat** *A, const **PnlMat** *B, double abserr)
Description Test if two matrices are equal up to an absolute error **abserr** component-wise. Return TRUE or FALSE.
- int **pnl_mat_isequal_rel**(const **PnlMat** *A, const **PnlMat** *B, double relerr)
Description Test if two matrices are equal up to a relative error **relerr** component-wise. Return TRUE or FALSE.

Ordering operations

- void **pnl_mat_max**(**PnlVect** *M, const **PnlMat** *A, char d)
Description On exit, $M(i) = \max_j(A(i, j))$ when $d='c'$ and $M(i) = \max_j(A(j, i))$ when $d='r'$ and $M(0) = \max_{i,j} A(i, j)$ when $d='*'$.
- void **pnl_mat_min**(**PnlVect** *m, const **PnlMat** *A, char d)
Description On exit, $m(i) = \min_j(A(i, j))$ when $d='c'$ and $m(i) = \min_j(A(j, i))$ when $d='r'$ and $M(0) = \min_{i,j} A(i, j)$ when $d='*'$.
- void **pnl_mat_minmax**(**PnlVect** *m, **PnlVect** *M, const **PnlMat** *A, char d)
Description On exit, $m(i) = \min_j(A(i, j))$ and $M(i) = \max_j(A(i, j))$ when $d='c'$ and $m(i) = \min_j(A(j, i))$ and $M(i) = \max_j(A(j, i))$ when $d='r'$ and $M(0) = \max_{i,j} A(i, j)$ and $m(0) = \min_{i,j} A(i, j)$ when $d='*'$.
- void **pnl_mat_min_index**(**PnlVect** *m, **PnlVectInt** *im, const **PnlMat** *A, char d)
Description Idem as **pnl_mat_min** and **index** contains the indices of the minima. If **index==NULL**, the indices are not computed.

- void **pnl_mat_max_index**(**PnlVect** *M, **PnlVectInt** *iM, const **PnlMat** *A, char d)
Description Idem as **pnl_mat_max** and **index** contains the indices of the maxima. If **index**==NULL, the indices are not computed.
- void **pnl_mat_minmax_index**(**PnlVect** *m, **PnlVect** *M, **PnlVectInt** *im, **PnlVectInt** *iM, const **PnlMat** *A, char d)
Description Idem as **pnl_mat_minmax** and **im** contains the indices of the minima and **iM** contains the indices of the maxima. If **im**==NULL (resp. **iM**==NULL, the indices of the minima (resp. maxima) are not computed.
- void **pnl_mat_qsort**(**PnlMat** *, char dir, char order)
Description Sort a matrix using a quick sort algorithm according to **order** ('i' for increasing or 'd' for decreasing). The parameter **dir** determines whether the matrix is sorted by rows or columns. If **dir**='c', each row is sorted independently of the others whereas if **dir**='r', each column is sorted independently of the others.
- void **pnl_mat_qsort_index**(**PnlMat** *, **PnlMatInt** *index, char dir, char order)
Description Sort a matrix using a quick sort algorithm according to **order** ('i' for increasing or 'd' for decreasing). The parameter **dir** determines whether the matrix is sorted by rows or columns. If **dir**='c', each row is sorted independently of the others whereas if **dir**='r', each column is sorted independently of the others. In addition to the function **pnl_mat_qsort**, the permutation index is computed and stored into **index**.
- int **pnl_mat_find**(**PnlVectInt** *indi, **PnlVectInt** indj, char *type, int(*)(double *t), ...)
Description **f** is a function taking a C array as argument and returning an integer. **type** is a string composed by the letters 'r' and 'm' and is used to describe the types of the arguments appearing after **f** : 'r' for real numbers and 'm' for matrices. This function aims at simulating Scilab's **find** function. Here are a few examples (capital letters are used for matrices and small letters for real values)

```

- [indi, indj] = find ( a < X )

    int isless ( double *t ) { return t[0] < t[1]; }
    pnl_mat_find ( indi, indj, "rm", isless, a, X );

- ind = find (X <= Y)

    int isless ( double *t ) { return t[0] <= t[1]; }
    pnl_mat_find ( ind, "mm", isless, X, Y );

- [indi, indj] = find ((a < X) && (X <= Y))

    int cmp ( double *t )
    {
        return (t[0] <= t[1]) && (t[1] <= t[2]);
    }
    pnl_mat_find ( indi, indj, "rmm", cmp, a, X, Y );

```

(**indi**, **indj**) contains on exit the indices (i,j) for which the function **f** returned 1. Note that if **indj** == NULL on entry, a linear indexing is used for matrices, which means that matrices are seen as large vectors built up by stacking rows. This function returns

OK or FAIL if something went wrong (size mismatch between matrices, invalid string type).

Standard matrix operations

- void **pnl_mat_plus_mat**(**PnlMat** *lhs, const **PnlMat** *rhs)
Description In-place matrix matrix addition
- void **pnl_mat_minus_mat**(**PnlMat** *lhs, const **PnlMat** *rhs)
Description In-place matrix matrix subtraction
- void **pnl_mat_sq_transpose**(**PnlMat** *M)
Description On exit, M is transposed
- **PnlMat** * **pnl_mat_transpose**(const **PnlMat** *M)
Description Create a new matrix which is the transposition of M
- void **pnl_mat_tr**(**PnlMat** *tM, const **PnlMat** *M)
Description On exit, $tM = M'$
- double **pnl_mat_trace**(const **PnlMat** *M)
Description Return the trace of a square matrix.
- void **pnl_mat_axpy**(double alpha, const **PnlMat** *A, **PnlMat** *B)
Description Compute $B := \alpha * A + B$
- void **pnl_mat_dger**(double alpha, const **PnlVect** *x, const **PnlVect** *y, **PnlMat** *A)
Description Compute $A := \alpha x * y' + A$
- **PnlVect** * **pnl_mat_mult_vect**(const **PnlMat** *A, const **PnlVect** *x)
Description Matrix vector multiplication $A * x$
- void **pnl_mat_mult_vect_inplace**(**PnlVect** *y, const **PnlMat** *A, const **PnlVect** *x)
Description In place matrix vector multiplication $y = A * x$. You cannot use the same vector for x and y.
- **PnlVect** * **pnl_mat_mult_vect_transpose**(const **PnlMat** *A, const **PnlVect** *x)
Description Matrix vector multiplication $A' * x$
- void **pnl_mat_mult_vect_transpose_inplace**(**PnlVect** *y, const **PnlMat** *A, const **PnlVect** *x)
Description In place matrix vector multiplication $y = A' * x$. You cannot use the same vector for x and y. The vectors x and y must be different.
- int **pnl_mat_cross**(**PnlMat** *lhs, const **PnlMat** *A, const **PnlMat** *B)
Description Compute the cross products of the vectors given in matrices A and B which must have either 3 rows or 3 columns. A row wise computation is first tried, then a column wise approach is tested. FAIL is returned in case no dimension equals 3.

- void **pnl_mat_lAxpby**(double lambda, const **PnlMat** *A, const **PnlVect** *x, double b, **PnlVect** *y)
Description Compute $y := \text{lambda } A x + b y$. When $b=0$, the content of y is not used on input and instead y is resized to match $A*x$. The vectors x and y must be different.
- void **pnl_mat_dgemv**(char trans, double lambda, const **PnlMat** *A, const **PnlVect** *x, double mu, **PnlVect** *b)
Description Compute $b := \text{lambda } \text{op}(A) x + \text{mu } b$, where $\text{op}(X) = X$ or $\text{op}(X) = X'$. If $\text{trans}='N'$ or $\text{trans}='n'$, $\text{op}(A) = A$, whereas if $\text{trans}='T'$ or $\text{trans}='t'$, $\text{op}(A) = A'$. When $\text{mu}=0$, the content of b is not used and instead b is resized to match $\text{op}(A)*x$. The vectors x and b must be different.
- void **pnl_mat_dgemm**(char transA, char transB, double alpha, const **PnlMat** *A, const **PnlMat** *B, double beta, **PnlMat** *C)
Description Compute $C := \text{alpha} * \text{op}(A) * \text{op}(B) + \text{beta} * C$. When $\text{beta}=0$, the content of C is unused and instead C is resized to store $\text{alpha } A * B$. If $\text{transA}='N'$ or $\text{transA}='n'$, $\text{op}(A) = A$, whereas if $\text{transA}='T'$ or $\text{transA}='t'$, $\text{op}(A) = A'$. The same holds for transB . The matrix C must be different from A and B .
- **PnlMat** * **pnl_mat_mult_mat**(const **PnlMat** *rhs1, const **PnlMat** *rhs2)
Description Matrix multiplication $\text{rhs1} * \text{rhs2}$
- void **pnl_mat_mult_mat_inplace**(**PnlMat** *lhs, const **PnlMat** *rhs1, const **PnlMat** *rhs2)
Description In-place matrix multiplication $\text{lhs} = \text{rhs1} * \text{rhs2}$. The matrix lhs must be different from rhs1 and rhs2 .
- double **pnl_mat_scalar_prod**(const **PnlMat** *A, const **PnlVect** *x, const **PnlVect** *y)
Description Compute $x' * A * y$
- void **pnl_mat_exp**(**PnlMat** *B, const **PnlMat** *A)
Description Compute the matrix exponential $B = \exp(A)$.
- void **pnl_mat_log**(**PnlMat** *B, const **PnlMat** *A)
Description Compute the matrix logarithm $B = \log(A)$. For the moment, this function only works if A is diagonalizable.
- void **pnl_mat_eigen**(**PnlVect** *v, **PnlMat** *P, const **PnlMat** *A, int with_eigenvector)
Description Compute the eigenvalues (stored in v) and optionally the eigenvectors stored column wise in P when $\text{with_eigenvector}=\text{TRUE}$. If A is symmetric or Hermitian in the complex case, P is orthonormal. When $\text{with_eigenvector}=\text{FALSE}$, P can be NULL.

Linear systems and matrix decompositions The following functions are designed to solve linear system of the form $A x = b$ where A is a matrix and b is a vector except in the functions **pnl_mat_syslin_mat**, **pnl_mat_lu_syslin_mat** and **pnl_mat_chol_syslin_mat** which expect the right hand side member to be a matrix too. Whenever the vector b is not needed once the system is solved, you should consider using “inplace” functions. All the functions described in this paragraph return OK if the computations have been carried out successfully and FAIL otherwise.

- `int pnl_mat_chol(PnlMat *M)`
Description Compute the Cholesky decomposition of M . M must be symmetric, the positivity is tested in the algorithm. $M = L * L'$. On exit, the lower part of M contains the Cholesky decomposition L and the upper part is set to zero.
- `int pnl_mat_pchol(PnlMat *M, double tol, int *rank, PnlVectInt *p)`
Description Compute the Cholesky decomposition of M with complete pivoting. $P' * A * P = L * L'$. M must be symmetric positive semi-definite. On exit, the lower part of M contains the Cholesky decomposition L and the upper part is set to zero. The permutation matrix is stored in an integer vector p : the only non zero elements of P are $P(p(k), k) = 1$
- `int pnl_mat_lu(PnlMat *A, PnlPermutation *p)`
Description Compute a $PA = LU$ factorization. P must be an already allocated **PnlPermutation**. On exit the decomposition is stored in A , the lower part of A contains L while the upper part (including the diagonal terms) contains U . Remember that the diagonal elements of L are all 1. Row i of A was interchanged with row $p(i)$.
- `int pnl_mat_upper_syslin(PnlVect *x, const PnlMat *U, const PnlVect *b)`
Description Solve an upper triangular linear system $U x = b$
- `int pnl_mat_lower_syslin(PnlVect *x, const PnlMat *L, const PnlVect *b)`
Description Solve a lower triangular linear system $L x = b$
- `int pnl_mat_chol_syslin(PnlVect *x, const PnlMat *chol, const PnlVect *b)`
Description Solve a symmetric definite positive linear system $A x = b$, in which **chol** is assumed to be the Cholesky decomposition of A computed by `pnl_mat_chol`
- `int pnl_mat_chol_syslin_inplace(const PnlMat *chol, PnlVect *b)`
Description Solve a symmetric definite positive linear system $A x = b$, in which **chol** is assumed to be the Cholesky decomposition of A computed by `pnl_mat_chol`. The solution of the system is stored in b on exit.
- `int pnl_mat_lu_syslin(PnlVect *x, const PnlMat *LU, const PnlPermutation *p, const PnlVect *b)`
Description Solve a linear system $A x = b$ using a LU decomposition. LU and P are assumed to be the $PA = LU$ decomposition as computed by `pnl_mat_lu`. In particular, the structure of the matrix LU is the following : the lower part of A contains L while the upper part (including the diagonal terms) contains U . Remember that the diagonal elements of L are all 1.
- `int pnl_mat_lu_syslin_inplace(const PnlMat *LU, const PnlPermutation *p, PnlVect *b)`
Description Solve a linear system $A x = b$ using a LU decomposition. LU and P are assumed to be the $PA = LU$ decomposition as computed by `pnl_mat_lu`. In particular, the structure of the matrix LU is the following : the lower part of A contains L while the upper part (including the diagonal terms) contains U . Remember that the diagonal elements of L are all 1. The solution of the system is stored in b on exit.

- `int pnl_mat_syslin(PnlVect *x, const PnlMat *A, const PnlVect *b)`
Description Solve a linear system $A x = b$ using a LU factorization which is computed inside this function.
- `int pnl_mat_syslin_inplace(PnlMat *A, PnlVect *b)`
Description Solve a linear system $A x = b$ using a LU factorization which is computed inside this function. The solution of the system is stored in `b` and `A` is overwritten by its LU decomposition.
- `int pnl_mat_syslin_mat(PnlMat *A, PnlMat *B)`
Description Solve a linear system $A X = B$ using a LU factorization which is computed inside this function. `A` and `B` are matrices. `A` must be square. The solution of the system is stored in `B` on exit. On exit, `A` contains the LU decomposition of the input matrix which is lost.
- `int pnl_mat_chol_syslin_mat(const PnlMat *A, PnlMat *B)`
Description Solve a linear system $A X = B$ using a Cholesky factorization of the symmetric positive definite matrix `A`. `A` contains the Cholesky decomposition as computed by `pnl_mat_chol`. `B` is matrix with the same number of rows as `A`. The solution of the system is stored in `B` on exit.
- `int pnl_mat_lu_syslin_mat(const PnlMat *A, const PnlPermutation *p, PnlMat *B)`
Description Solve a linear system $A X = B$ using a $P A = L U$ factorization. `A` contains the `L U` factors and `p` the associated permutation. `A` and `p` must have been computed by `pnl_mat_lu`. `B` is matrix with the same number of rows as `A`. The solution of the system is stored in `B` on exit.

The following functions are designed to invert matrices. The authors provide these functions although they cannot find good reasons to use them. Note that to solve a linear system, one must use the `syslin` functions and not invert the system matrix because it is much longer.

- `int pnl_mat_upper_inverse(PnlMat *A, const PnlMat *B)`
Description Inversion of an upper triangular matrix
- `int pnl_mat_lower_inverse(PnlMat *A, const PnlMat *B)`
Description Inversion of a lower triangular matrix
- `int pnl_mat_inverse(PnlMat *inverse, const PnlMat *A)`
Description Compute the inverse of a matrix `A` and stores the result into `inverse`. A LU factorisation of the matrix `A` is computed inside this function.
- `int pnl_mat_inverse_with_chol(PnlMat *inverse, const PnlMat *A)`
Description Compute the inverse of a symmetric positive definite matrix `A` and stores the result into `inverse`. The Cholesky factorisation of the matrix `A` is computed inside this function.

4.3.3 Functions specific to base type double

Linear systems and matrix decompositions The following functions are designed to solve linear system of the form $A \mathbf{x} = \mathbf{b}$ where A is a matrix and \mathbf{b} is a vector except in the functions `pnl_mat_syslin_mat`, `pnl_mat_lu_syslin_mat` and `pnl_mat_chol_syslin_mat` which expect the right hand side member to be a matrix too. Whenever the vector \mathbf{b} is not needed once the system is solved, you should consider using “inplace” functions. All the functions described in this paragraph return `OK` if the computations have been carried out successfully and `FAIL` otherwise.

- `int pnl_mat_qr(PnlMat *Q, PnlMat *R, PnlPermutation *p, const PnlMat *A)`
Description Compute a $A = PQ$ decomposition. If on entry $P=NULL$, then the decomposition is computed without pivoting, i.e. $A = QR$. When $P \neq NULL$, P must be an already allocated `PnlPermutation`. Q is an orthogonal matrix, i.e. $Q^{-1} = Q^T$ and R is an upper triangular matrix. The use of pivoting improves the numerical stability when A is almost rank deficient, i.e. when the smallest eigenvalue of A is very close to 0.
- `int pnl_mat_qr_syslin(PnlVect *x, const PnlMat *Q, const PnlMat *R, const PnlVectInt *p, const PnlVect *b)`
Description Solve a linear system $A \mathbf{x} = \mathbf{b}$ where A is given by its QR decomposition with column pivoting as computed by the function `pnl_mat_qr`.
- `int pnl_mat_ls(const PnlMat *A, PnlVect *b)`
Description Solve a linear system $A \mathbf{x} = \mathbf{b}$ in the least square sense, i.e. $\mathbf{x} = \arg \min_U \|A * u - b\|^2$. The solution is stored into \mathbf{b} on exit. It internally uses a $AP = QR$ decomposition.
- `int pnl_mat_ls_mat(const PnlMat *A, PnlMat *B)`
Description Solve a linear system $A X = B$ with A and B two matrices in the least square sense, i.e. $X = \arg \min_U \|A * U - B\|^2$. The solution is stored into B on exit. It internally uses a $AP = QR$ decomposition. Same function as `pnl_mat_ls` but handles several r.h.s.

4.3.4 Functions specific to base type dcomplex

- `PnlMatComplex * pnl_mat_complex_create_from_mat(const PnlMat *R)`
Description Create a complex matrix using a real one. The complex parts of the entries of the returned matrix are all set to zero.

4.3.5 Permutations

```
typedef PnlVectInt PnlPermutation;
```

The `PnlPermutation` type is actually nothing else than a vector of integers, i.e. a `PnlVectInt`. It is used to store the partial pivoting with row interchanges transformation needed in the LU decomposition. We use the *Blas* convention for storing permutations. Consider a `PnlPermutation` p generated by a LU decomposition of a matrix A : to compute the decomposition, row i of A was interchanged with row $p(i)$.

- `PnlPermutation * pnl_permutation_new()`
Description Create an empty `PnlPermutation`.

- **PnlPermutation** * **pnl_permutation_create**(int n)
Description Create a **PnlPermutation** of size n.
- void **pnl_permutation_free**(**PnlPermutation** **p)
Description Free a **PnlPermutation** .
- void **pnl_permutation_inverse**(**PnlPermutation** *inv, const **PnlPermutation** *p)
Description Compute in **inv** the inverse of the permutation **p**.
- void **pnl_vect_permute**(**PnlVect** *px, const **PnlVect** *x, const **PnlPermutation** *p)
Description Apply a **PnlPermutation** to a **PnlVect** .
- void **pnl_vect_permute_inplace**(**PnlVect** *x, const **PnlPermutation** *p)
Description Apply a **PnlPermutation** to a **PnlVect** in-place.
- void **pnl_vect_permute_inverse**(**PnlVect** *px, const **PnlVect** *x, const **PnlPermutation** *p)
Description Apply the inverse of **PnlPermutation** to a **PnlVect** .
- void **pnl_vect_permute_inverse_inplace**(**PnlVect** *x, const **PnlPermutation** *p)
Description Apply the inverse of a **PnlPermutation** to a **PnlVect** in-place.
- void **pnl_mat_col_permute**(**PnlMat** *pX, const **PnlMat** *X, const **PnlPermutation** *p)
Description Apply a **PnlPermutation** to the columns of a matrix. **pX** contains the result of the permutation applied to **X**.
- void **pnl_mat_row_permute**(**PnlMat** *pX, const **PnlMat** *X, const **PnlPermutation** *p)
Description Apply a **PnlPermutation** to the rows of a matrix. **pX** contains the result of the permutation applied to **X**.
- void **pnl_permutation_fprint**(FILE *fic, const **PnlPermutation** *p)
Description Print a permutation to a file.
- void **pnl_permutation_print**(const **PnlPermutation** *p)
Description Print a permutation to the standard output.

4.4 Tridiagonal Matrices

4.4.1 Overview

The structures and functions related to tridiagonal matrices are declared in `pnl/pnl_tridiag_matrix.h`.

We only store the three main diagonals as three vectors.

```
typedef struct PnlTridiagMat{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlTridiagMat pointer to be cast to a
     * PnlObject
```

```

    */
    PnlObject object;
    int size; /*!< number of rows, the matrix must be square */
    double *D; /*!< diagonal elements */
    double *DU; /*!< upper diagonal elements */
    double *DL; /*!< lower diagonal elements */
} PnlTridiagMat;

```

`size` is the size of the matrix, `D` is an array of size `size` containing the diagonal terms. `DU`, `DL` are two arrays of size `size-1` containing respectively the upper diagonal ($M_{i,i+1}$) and the lower diagonal ($M_{i-1,i}$).

```

typedef struct PnlTridiagMatLU{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlTridiagMatLU pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int size; /*!< number of rows, the matrix must be square */
    double *D; /*!< diagonal elements */
    double *DU; /*!< upper diagonal elements */
    double *DU2; /*!< second upper diagonal elements */
    double *DL; /*!< lower diagonal elements */
    int *ipiv; /*!< Permutation: row i has been interchanged with row
        ipiv(i) */
};

```

This type is used to store the LU decomposition of a tridiagonal matrix.

4.4.2 Functions

Constructors and destructors

- **PnlTridiagMat** * **pnl_tridiag_mat_new()**
Description Create a **PnlTridiagMat** with size 0
- **PnlTridiagMat** * **pnl_tridiag_mat_create(int size)**
Description Create a **PnlTridiagMat** with size `size`
- **PnlTridiagMat** * **pnl_tridiag_mat_create_from_scalar(int size, double x)**
Description Create a **PnlTridiagMat** with the 3 diagonals filled with `x`
- **PnlTridiagMat** * **pnl_tridiag_mat_create_from_two_scalar(int size, double x, double y)**
Description Create a **PnlTridiagMat** with the diagonal filled with `x` and the upper and lower diagonals filled with `y`
- **PnlTridiagMat** * **pnl_tridiag_mat_create_from_ptr(int size, const double *lower_D, const double *D, const double *upper_D)**
Description Create a **PnlTridiagMat**

- **PnlTridiagMat * pnl_tridiag_mat_create_from_mat**(const **PnlMat** *mat)
Description Create a tridiagonal matrix from a full matrix (all the elements but the 3 diagonal ones are ignored).
- **PnlMat * pnl_tridiag_mat_to_mat**(const **PnlTridiagMat** *T)
Description Create a full matrix from a tridiagonal one.
- **PnlTridiagMat * pnl_tridiag_mat_copy**(const **PnlTridiagMat** *T)
Description Copy a tridiagonal matrix.
- void **pnl_tridiag_mat_clone**(**PnlTridiagMat** *clone, const **PnlTridiagMat** *T)
Description Copy the content of T into clone
- void **pnl_tridiag_mat_free**(**PnlTridiagMat** **v)
Description Free a **PnlTridiagMat**
- int **pnl_tridiag_mat_resize**(**PnlTridiagMat** *v, int size)
Description Resize a **PnlTridiagMat** .

Accessing elements. If it is supported by the compiler, the following functions are declared inline. To speed up these functions, you can use the macro constant `PNL_RANGE_CHECK_OFF`, see Section 1.3.2 for an explanation.

- void **pnl_tridiag_mat_set**(**PnlTridiagMat** *self, int d, int up, double x)
Description Set `self[d, d+up] = x`, up can be $\{-1, 0, 1\}$.
- double **pnl_tridiag_mat_get**(const **PnlTridiagMat** *self, int d, int up)
Description Get `self[d, d+up]`, up can be $\{-1, 0, 1\}$.
- double * **pnl_tridiag_mat_lget**(**PnlTridiagMat** *self, int d, int up)
Description Return the address `self[d, d+up] = x`, up can be $\{-1, 0, 1\}$.

Printing Matrix

- void **pnl_tridiag_mat_fprint**(FILE *fic, const **PnlTridiagMat** *M)
Description Print a tri-diagonal matrix to a file.
- void **pnl_tridiag_mat_print**(const **PnlTridiagMat** *M)
Description Print a tridiagonal matrix to the standard output.

Algebra operations

- void **pnl_tridiag_mat_plus_tridiag_mat**(**PnlTridiagMat** *lhs, const **PnlTridiagMat** *rhs)
Description In-place matrix matrix addition
- void **pnl_tridiag_mat_minus_tridiag_mat**(**PnlTridiagMat** *lhs, const **PnlTridiagMat** *rhs)
Description In-place matrix matrix subtraction
- void **pnl_tridiag_mat_plus_scalar**(**PnlTridiagMat** *lhs, double x)
Description In-place matrix scalar addition

- void **pnl_tridiag_mat_minus_scalar**(**PnlTridiagMat** *lhs, double x)
[Description](#) In-place matrix scalar subtraction
- void **pnl_tridiag_mat_mult_scalar**(**PnlTridiagMat** *lhs, double x)
[Description](#) In-place matrix scalar multiplication
- void **pnl_tridiag_mat_div_scalar**(**PnlTridiagMat** *lhs, double x)
[Description](#) In-place matrix scalar division

Element-wise operations

- void **pnl_tridiag_mat_mult_tridiag_mat_term**(**PnlTridiagMat** *lhs, const **PnlTridiagMat** *rhs)
[Description](#) In-place matrix matrix term by term product
- void **pnl_tridiag_mat_div_tridiag_mat_term**(**PnlTridiagMat** *lhs, const **PnlTridiagMat** *rhs)
[Description](#) In-place matrix matrix term by term division
- void **pnl_tridiag_mat_map_inplace**(**PnlTridiagMat** *lhs, double(*f)(double))
[Description](#) lhs = f(lhs).
- void **pnl_tridiag_mat_map_tridiag_mat_inplace**(**PnlTridiagMat** *lhs, const **PnlTridiagMat** *rhs, double(*f)(double, double))
[Description](#) lhs = f(lhs, rhs).

Standard matrix operations & Linear systems

- void **pnl_tridiag_mat_mult_vect_inplace**(**PnlVect** *lhs, const **PnlTridiagMat** *mat, const **PnlVect** *rhs)
[Description](#) In place matrix multiplication. The vector lhs must be different from rhs.
- **PnlVect** * **pnl_tridiag_mat_mult_vect**(const **PnlTridiagMat** *mat, const **PnlVect** *vec)
[Description](#) Matrix multiplication
- void **pnl_tridiag_mat_lAxpby**(double lambda, const **PnlTridiagMat** *A, const **PnlVect** *x, double mu, **PnlVect** *b)
[Description](#) Compute $b := \lambda A x + \mu b$. When $\mu=0$, the content of b is not used on input and instead b is resized to match $A*x$. Note that the vectors x and b must be different.
- double **pnl_tridiag_mat_scalar_prod**(const **PnlVect** *x, const **PnlTridiagMat** *A, const **PnlVect** *y)
[Description](#) Compute $x' * A * y$
- void **pnl_tridiag_mat_syslin_inplace**(**PnlTridiagMat** *M, **PnlVect** *b)
[Description](#) Solve the linear system $M x = b$. The solution is written into b on exit. On exit, M is modified and becomes unusable.

- void **pnl_tridiag_mat_syslin**(**PnlVect** *x, **PnlTridiagMat** *M, const **PnlVect** *b)
Description Solve the linear system $M x = b$. On exit, M is modified and becomes unusable.
- **PnlTridiagMatLU** * **pnl_tridiag_mat_lu_new**()
Description Create an empty **PnlTridiagMatLU**
- **PnlTridiagMatLU** * **pnl_tridiag_mat_lu_create**(int size)
Description Create a **PnlTridiagMatLU** with size **size**
- **PnlTridiagMatLU** * **pnl_tridiag_mat_lu_copy**(const **PnlTridiagMatLU** *mat)
Description Create a new **PnlTridiagMatLU** which is a copy of **mat**.
- void **pnl_tridiag_mat_lu_clone**(**PnlTridiagMatLU** *clone, const **PnlTridiagMatLU** *mat)
Description Clone a **PnlTridiagMatLU** . clone must already exist, no memory is allocated for the envelope.
- void **pnl_tridiag_mat_lu_free**(**PnlTridiagMatLU** **m)
Description Free a **PnlTridiagMatLU**
- int **pnl_tridiag_mat_lu_resize**(**PnlTridiagMatLU** *v, int size)
Description Resize a **PnlTridiagMatLU**
- int **pnl_tridiag_mat_lu_compute**(**PnlTridiagMatLU** *LU, const **PnlTridiagMat** *A)
Description Compute the LU factorisation of a tridiagonal matrix A. LU must have already been created using **pnl_tridiag_mat_lu_new**. On exit, LU contains the decomposition which is suitable for use in **pnl_tridiag_mat_lu_syslin**.
- int **pnl_tridiag_mat_lu_syslin_inplace**(**PnlTridiagMatLU** *LU, **PnlVect** *b)
Description Solve a linear system $A x = b$ where the matrix LU is given the LU decomposition of A previously computed by **pnl_tridiag_mat_lu_compute**. On exit, b is overwritten by the solution x.
- int **pnl_tridiag_mat_lu_syslin**(**PnlVect** *x, **PnlTridiagMatLU** *LU, const **PnlVect** *b)
Description Solve a linear system $A x = b$ where the matrix LU is given the LU decomposition of A previously computed by **pnl_tridiag_mat_lu_compute**.

4.5 Band Matrices

4.5.1 Overview

```
typedef struct
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlBandMat pointer to be cast to a
     * PnlObject
```

```

    */
    PnlObject object;
    int m; /*!< nb rows */
    int n; /*!< nb columns */
    int nu; /*!< nb of upperdiagonals */
    int nl; /*!< nb of lowerdiagonals */
    int m_band; /*!< nb rows of the band storage */
    int n_band; /*!< nb columns of the band storage */
    double *array; /*!< a block to store the bands */
} PnlBandMat;

```

The structures and functions related to band matrices are declared in `pnl/pnl_band_matrix.h`.

4.5.2 Functions

Constructors and destructors

- **PnlBandMat * pnl_band_mat_new()**
Description Create a band matrix of size 0.
- **PnlBandMat * pnl_band_mat_create(int m, int n, int nl, int nu)**
Description Create a band matrix of size $m \times n$ with nl lower diagonals and nu upper diagonals.
- **PnlBandMat * pnl_band_mat_create_from_mat(const PnlMat *BM, int nl, int nu)**
Description Extract a band matrix from a **PnlMat**.
- **void pnl_band_mat_free(PnlBandMat **)**
Description Free a band matrix.
- **void pnl_band_mat_clone(PnlBandMat *clone, const PnlBandMat *M)**
Description Copy the band matrix M into $clone$. No new **PnlBandMat** is created.
- **PnlBandMat * pnl_band_mat_copy(PnlBandMat *BM)**
Description Create a new band matrix which is a copy of BM . Each band matrix owns its data array.
- **PnlMat * pnl_band_mat_to_mat(PnlBandMat *BM)**
Description Create a full matrix from a band matrix.
- **int pnl_band_mat_resize(PnlBandMat *BM, int m, int n, int nl, int nu)**
Description Resize BM to store a $m \times n$ band matrix with nu upper diagonals and nl lower diagonals.

Accessing elements. If it is supported by the compiler, the following functions are declared inline. To speed up these functions, you can use the macro constant `PNL_RANGE_CHECK_OFF`, see Section 1.3.2 for an explanation.

- **void pnl_band_mat_set(PnlBandMat *M, int i, int j, double x)**
Description $M_{i,j} = x$.

- void **pnl_band_mat_get**(**PnlBandMat** *M, int i, int j)
Description Return $M_{i,j}$.
- void **pnl_band_mat_lget**(**PnlBandMat** *M, int i, int j)
Description Return the address $\&(M_{i,j})$.
- void **pnl_band_mat_set_all**(**PnlBandMat** *M, double x)
Description Set all the elements of M to x.
- void **pnl_band_mat_print_as_full**(**PnlBandMat** *M)
Description Print a band matrix in a full format.

Element wise operations

- void **pnl_band_mat_plus_scalar**(**PnlBandMat** *lhs, double x)
Description In-place addition, $\text{lhs} += x$
- void **pnl_band_mat_minus_scalar**(**PnlBandMat** *lhs, double x)
Description In-place subtraction $\text{lhs} -= x$
- void **pnl_band_mat_div_scalar**(**PnlBandMat** *lhs, double x)
Description $\text{lhs} = \text{lhs} ./ x$
- void **pnl_band_mat_mult_scalar**(**PnlBandMat** *lhs, double x)
Description $\text{lhs} = \text{lhs} * x$
- void **pnl_band_mat_plus_band_mat**(**PnlBandMat** *lhs, const **PnlBandMat** *rhs)
Description In-place addition, $\text{lhs} += \text{rhs}$
- void **pnl_band_mat_minus_band_mat**(**PnlBandMat** *lhs, const **PnlBandMat** *rhs)
Description In-place subtraction $\text{lhs} -= \text{rhs}$
- void **pnl_band_mat_inv_term**(**PnlBandMat** *lhs)
Description In-place term by term inversion $\text{lhs} = 1 ./ \text{rhs}$
- void **pnl_band_mat_div_band_mat_term**(**PnlBandMat** *lhs, const **PnlBandMat** *rhs)
Description In-place term by term division $\text{lhs} = \text{lhs} ./ \text{rhs}$
- void **pnl_band_mat_mult_band_mat_term**(**PnlBandMat** *lhs, const **PnlBandMat** *rhs)
Description In-place term by term multiplication $\text{lhs} = \text{lhs} .* \text{rhs}$
- void **pnl_band_mat_map**(**PnlBandMat** *lhs, const **PnlBandMat** *rhs, double(*f)(double))
Description $\text{lhs} = f(\text{rhs})$
- void **pnl_band_mat_map_inplace**(**PnlBandMat** *lhs, double(*f)(double))
Description $\text{lhs} = f(\text{lhs})$

- void `pnl_band_mat_map_band_mat_inplace`(`PnlBandMat` *lhs, const `PnlBandMat` *rhs, double(*f)(double,double))
Description `lhs = f(lhs,rhs)`

Standard matrix operations & Linear system

- void `pnl_band_mat_lAxpby`(double lambda, const `PnlBandMat` *A, const `PnlVect` *x, double mu, `PnlVect` *b)
Description Compute $b := \text{lambda } A \cdot x + \text{mu } b$. When `mu==0`, the content of `b` is not used on input and instead `b` is resized to match the size of $A \cdot x$.
- void `pnl_band_mat_mult_vect_inplace`(`PnlVect` *y, const `PnlBandMat` *BM, const `PnlVect` *x)
Description `y = BM * x`
- void `pnl_band_mat_syslin_inplace`(`PnlBandMat` *M, `PnlVect` *b)
Description Solve the linear system $M \cdot x = b$ with `M` a `PnlBandMat`. **Note** that `M` is modified on output and becomes unusable. On exit, the solution `x` is stored in `b`.
- void `pnl_band_mat_syslin`(`PnlVect` *x, `PnlBandMat` *M, `PnlVect` *b)
Description Solve the linear system $M \cdot x = b$ with `M` a `PnlBandMat`. **Note** that `M` is modified on output and becomes unusable.
- void `pnl_band_mat_lu`(`PnlBandMat` *BM, `PnlVectInt` *p)
Description Compute the LU decomposition with partial pivoting with row interchanges. On exit, `BM` is enlarged to store the LU decomposition. On exit, `p` stores the permutation applied to the rows. Note that the Lapack format is used to store `p`, this format differs from the one used by `PnlPermutation`.
- void `pnl_band_mat_lu_syslin_inplace`(const `PnlBandMat` *M, `PnlVectInt` *p, `PnlVect` *b)
Description Solve the band linear system $M \cdot x = b$ where `M` is the LU decomposition computed by `pnl_band_mat_lu` and `p` the associated permutation. On exit, the solution `x` is stored in `b`.
- void `pnl_band_mat_lu_syslin`(`PnlVect` *x, const `PnlBandMat` *M, `PnlVectInt` *p, const `PnlVect` *b)
Description Solve the band linear system $M \cdot x = b$ where `M` is the LU decomposition computed by `pnl_band_mat_lu` and `p` the associated permutation.

4.6 Sparse Matrices

4.6.1 Short description

The structures and functions related to matrices are declared in `pnl/pnl_sp_matrix.h`.

```
typedef struct _PnlSpMat
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
```

```

    * properly. This allows a PnlSpMat pointer to be cast to a
      PnlObject
    */
PnlObject object;
int m; /*!< number of rows */
int n; /*!< number of columns */
int nz; /*!< number of non-zero elements */
int *J; /*!< column indices, vector of size nzmax */
int *I; /*!< row offset integer vector,
          array[I[i]] is the first element of row i.
          Vector of size (m+1) */
double *array; /*!< pointer to store the data of size nzmax*/
int nzmax; /*!< size of the memory block allocated for array */
} PnlSpMat;

typedef struct _PnlSpMatInt
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows a PnlSpMat pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int m; /*!< number of rows */
    int n; /*!< number of columns */
    int nz; /*!< number of non-zero elements */
    int *J; /*!< column indices, vector of size nzmax */
    int *I; /*!< row offset integer vector,
              array[I[i]] is the first element of row i.
              Vector of size (m+1) */
    int *array; /*!< pointer to store the data of size nzmax */
    int nzmax; /*!< size of the memory block allocated for array */
} PnlSpMatInt;

typedef struct _PnlSpMatComplex
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows a PnlSpMat pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int m; /*!< number of rows */
    int n; /*!< number of columns */
    int nz; /*!< number of non-zero elements */
    int *J; /*!< column indices, vector of size nzmax */
    int *I; /*!< row offset integer vector,
              array[I[i]] is the first element of row i.
              Vector of size (m+1) */
    dcomplex *array; /*!< pointer to store the data of size nzmax */
    int nzmax; /*!< size of the memory block allocated for array */
}

```

```
} PnlSpMatComplex;
```

The non zero elements of row `i` are stored in `array` between the indices `I[i]` and `I[i+1]-1`. The array `J` contains the column indices of every element of `array`.

Sparse matrices are defined using the internal template approach and can be used for integer, float or complex base data according to the following table

| base type | prefix | type |
|-----------|---------------------------------|------------------------------|
| double | <code>pnl_sp_mat</code> | <code>PnlSpMat</code> |
| int | <code>pnl_sp_mat_int</code> | <code>PnlSpMatInt</code> |
| dcomplex | <code>pnl_sp_mat_complex</code> | <code>PnlSpMatComplex</code> |

4.6.2 Functions

Constructors and destructors

- `PnlSpMat * pnl_sp_mat_new()`
Description Create an empty sparse matrix.
- `PnlSpMat * pnl_sp_mat_create(int m, int n, int nzmax)`
Description Create a sparse matrix with size `m x n` designed to hold at most `nzmax` non zero elements.
- `void pnl_sp_mat_clone(PnlSpMat *dest, const PnlSpMat *src)`
Description Clone `src` into `dest`, which is automatically resized. On output, `dest` and `src` are equal but independent.
- `PnlSpMat * pnl_sp_mat_copy(PnlSpMat *src)`
Description Create an independent copy of `src`.
- `void pnl_sp_mat_free(PnlSpMat **)`
Description Delete a sparse matrix.
- `int pnl_sp_mat_resize(PnlSpMat *M, int m, int n, int nzmax)`
Description Resize an existing `PnlSpMat` to become a `m x n` sparse matrices holding at most `nzmax`. Note that no old data are kept except if `M->m` is left unchanged and we only call this function to increase `M->nzmax`. Return OK or FAIL.
- `PnlMat * pnl_mat_create_from_sp_mat(const PnlSpMat *M)`
Description Create a dense `PnlMat` from a spare one.
- `PnlSpMat * pnl_sp_mat_create_from_mat(const PnlMat *M)`
Description Create a sparse matrix from a dense one.
- `void pnl_sp_mat_create_from_file(char *file)`
Description Read a sparse matrix from the file with name `file`. We use the Matrix Market Exchange Format

```
M  N  L                                | <--- rows , columns ,
    entries                             |
I1  J1  A(I1 , J1)                     | <---+
```

| | | | | | | |
|----|----|-----------|--|------|--|------------|
| I2 | J2 | A(I2, J2) | | | | |
| I3 | J3 | A(I3, J3) | | | | -- L lines |
| . | . | . | | | | |
| IL | JL | A(IL, JL) | | <--- | | |

The format (I1, J1) A(I1, J1) is also accepted. Anything after a # or % is ignored up to the end of the line.

Accessing elements

- void **pnl_sp_mat_set**(**PnlSpMat** *M, int i, int j, double x)
[Description](#) Set $M[i, j] = x$. This function increases $M \rightarrow \text{nzmax}$ if necessary.
- double **pnl_sp_mat_get**(const **PnlSpMat** *M, int i, int j)
[Description](#) Return $M[i, j]$. If M has no entry with such an index, zero is returned.

Applying external operations

- void **pnl_sp_mat_plus_scalar**(**PnlSpMat** *M, double x)
[Description](#) Add x to all non zero entries of M. To apply the operation to all entries including the zero ones, first convert M to a dense matrix and use **pnl_mat_plus_scalar**.
- void **pnl_sp_mat_minus_scalar**(**PnlSpMat** *M, double x)
[Description](#) Subtract x to all non zero entries of M. To apply the operation to all entries including the zero ones, first convert M to a dense matrix and use **pnl_mat_minus_scalar**.
- void **pnl_sp_mat_mult_scalar**(**PnlSpMat** *M, double x)
[Description](#) In-place matrix scalar multiplication
- void **pnl_sp_mat_div_scalar**(**PnlSpMat** *M, double x)
[Description](#) In-place matrix scalar division

Standard matrix operations

- void **pnl_sp_mat_fprint**(FILE *fic, const **PnlSpMat** *M)
[Description](#) Print a sparse matrix to a file descriptor using the format (row, col) -> val. The file can be read by **pnl_sp_mat_create_from_file**.
- void **pnl_sp_mat_print**(const **PnlSpMat** *M)
[Description](#) Same as **pnl_sp_mat_fprint** but print to standard output.
- void **pnl_sp_mat_mult_vect**(**PnlVect** *y, const **PnlSpMat** *A, const **PnlVect** *x)
[Description](#) $y = A x$.
- void **pnl_sp_mat_lAxpby**(double lambda, const **PnlSpMat** *A, const **PnlVect** *x, double b, **PnlVect** *y)
[Description](#) Compute $y := \text{lambda } A x + b y$. When $b=0$, the content of y is not used on input and instead y is resized to match $A \cdot x$. The vectors x and y must be different.

- `void pnl_sp_mat_plus_sp_mat_inplace(PnlSpMat *res, const PnlSpMat *A, const PnlSpMat *B)`
Description In-place addition: `res = A + B`.
- `PnlSpMat *pnl_sp_mat_plus_sp_mat(const PnlSpMat *A, const PnlSpMat *B)`
Description Return the sum of A and B.
- `void pnl_sp_mat_kron_inplace(PnlSpMat *result, const PnlSpMat *A, const PnlSpMat *B)`
Description In-place Kroenecker product of A and B.
- `PnlSpMat *pnl_sp_mat_kron(const PnlSpMat *A, const PnlSpMat *B)`
Description Return the Kroenecker product of A and B.

Comparison functions

- `int pnl_sp_mat_isequal(const PnlSpMat *x, const PnlSpMat *y, double abserr)`
Description Test if two sparse matrices are equal up to `err` component-wise. The error `err` is either relative or absolute depending on the magnitude of the components. Return TRUE or FALSE.
- `int pnl_sp_mat_isequal_abs(const PnlSpMat *x, const PnlSpMat *y, double relerr)`
Description Test if two sparse matrices are equal up to an absolute error `abserr` component-wise. Return TRUE or FALSE.
- `int pnl_sp_mat_isequal_rel(const PnlSpMat *x, const PnlSpMat *y, double err)`
Description Test if two sparse matrices are equal up to a relative error `relerr` component-wise. Return TRUE or FALSE.

4.7 Hyper Matrices

4.7.1 Short description

The Hyper matrix types and related functions are defined in the header `pnl/pnl_matrix.h`.

```
typedef struct PnlHmat{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlHmat pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int ndim; /*!< nb dimensions */
    int *dims; /*!< pointer to store the values of the ndim dimensions
     */
    int mn; /*!< product dim_1 *...*dim_ndim */
    int *pdims; /*!< array of size ndim, s.t. pdims[i] = dims[ndim-1] x
     ... dims[i+1]
     with pdims[ndim - 1] = 1 */
    double *array; /*!< pointer to store */
}
```



```

} PnlHmat;

typedef struct PnlHmatInt{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlHmatInt pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int ndim; /*!< nb dimensions */
    int *dims; /*!< pointer to store the value of the ndim dimensions */
    int mn; /*!< product dim_1 *...*dim_ndim */
    int *pdims; /*!< array of size ndim, s.t. pdims[i] = dims[ndim-1] x
        ... dims[i+1]
                with pdims[ndim - 1] = 1 */
    int *array; /*!< pointer to store */
} PnlHmatInt;

typedef struct PnlHmatComplex{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlHmatComplex pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int ndim; /*!< nb dimensions */
    int *dims; /*!< pointer to store the value of the ndim dimensions */
    int mn; /*!< product dim_1 *...*dim_ndim */
    int *pdims; /*!< array of size ndim, s.t. pdims[i] = dims[ndim-1] x
        ... dims[i+1]
                with pdims[ndim - 1] = 1 */
    dcomplex *array; /*!< pointer to store */
} PnlHmatComplex;

```

ndim is the number of dimensions, **dim** is an array to store the size of each dimension and **mn** contains the product of the sizes of each dimension. **array** is an array of size **mn** containing the data. The integer array **pdims** is used to create the one-to-one map between the natural indexing and the linear indexing used in **array**.

4.7.2 Functions

These functions exist for all types of hypermatrices no matter what the basic type is. The following conventions are used to name functions operating on hypermatrices. Here is the table of prefixes used for the different basic types.

| base type | prefix | type |
|-----------|------------------|----------------|
| double | pnl_hmat | PnlHmat |
| int | pnl_hmat_int | PnlHmatInt |
| dcomplex | pnl_hmat_complex | PnlHmatComplex |

In this paragraph, we present the functions operating on **PnlHmat** which exist for all types. To deduce the prototypes of these functions for other basic types, one must replace **pnl_hmat** and **double** according the above table.

Constructors and destructors

- **PnlHmat** * **pnl_hmat_new**()
Description Create an empty **PnlHmat** .
- **PnlHmat** * **pnl_hmat_create**(int ndim, const int *dims)
Description Create a **PnlHmat** with ndim dimensions and the size of each dimension is given by the entries of the integer array **dims**
- **PnlHmat** * **pnl_hmat_create_from_scalar**(int ndim, const int *dims, double x)
Description Create a **PnlHmat** with ndim dimensions given by $\prod_i \text{dims}[i]$ filled with **x**.
- **PnlHmat** * **pnl_hmat_create_from_ptr**(int ndim, const int *dims, const double *x)
Description Create a **PnlHmat** with ndim dimensions given by $\prod_i \text{dims}[i]$ filled with **x**.
- void **pnl_hmat_free**(**PnlHmat** **H)
Description Free a **PnlHmat**
- **PnlHmat** * **pnl_hmat_copy**(const **PnlHmat** *H)
Description Copy a **PnlHmat** .
- void **pnl_hmat_clone**(**PnlHmat** *clone, const **PnlHmat** *H)
Description Clone a **PnlHmat** .
- int **pnl_hmat_resize**(**PnlHmat** *H, int ndim, const int *dims)
Description Resize a **PnlHmat** .

Accessing elements

- void **pnl_hmat_set**(**PnlHmat** *self, int *tab, double x)
Description Set the element of index **tab** to **x**.
- double **pnl_hmat_get**(const **PnlHmat** *self, int *tab)
Description Return the value of the element of index **tab**
- double* **pnl_hmat_lget**(**PnlHmat** *self, int *tab)
Description Return the address of self[tab] for use as a lvalue.
- **PnlMat** **pnl_mat_wrap_hmat**(**PnlHmat** *H, int *t)
Description Return a true **PnlMat** not a pointer holding the data **H(t, :, :)**. Note that **t** must be of size **ndim-2** and that it cannot be checked within the function. The returned matrix shares its data with **H**, it is only a view not a true copy.
- **PnlVect** **pnl_vect_wrap_hmat**(**PnlHmat** *H, int *t)
Description Return a true **PnlVect** not a pointer holding the data **H(t, :)**. Note that **t** must be of size **ndim-1** and that it cannot be checked within the function. The returned vector shares its data with **H**, it is only a view not a true copy.

Printing hypermatrices

- void **pnl_hmat_print**(const **PnlHmat** *H)
[Description](#) Print an hypermatrix.

Term by term operations

- void **pnl_hmat_plus_hmat**(**PnlHmat** *lhs, const **PnlHmat** *rhs)
[Description](#) Compute lhs += rhs.
- void **pnl_hmat_mult_scalar**(**PnlHmat** *lhs, double x)
[Description](#) Compute lhs *= x where x is a real number.

4.8 Iterative Solvers

4.8.1 Overview

The structures and functions related to solvers are declared in `pnl/pnl_linalgsolver.h`.

```
typedef struct _PnlIterationBase PnlIterationBase;
typedef struct _PnlCgSolver PnlCgSolver;
typedef struct _PnlBicgSolver PnlBicgSolver;
typedef struct _PnlGmresSolver PnlGmresSolver;

struct _PnlIterationBase
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlVectXXX pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int iteration;
    int max_iter;
    double normb;
    double tol_;
    double resid;
    int error;
    /* char * err_msg; */
};

/* When you repeatedly use iterative solvers, do not malloc each time
 */
struct _PnlCgSolver
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlCgSolver pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
```

```

    PnlVect * r;
    PnlVect * z;
    PnlVect * p;
    PnlVect * q;
    double rho;
    double oldrho;
    double beta;
    double alpha;
    PnlIterationBase * iter;
} ;

struct _PnlBicgSolver
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlBicgSolver pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    double rho_1, rho_2, alpha, beta, omega;
    PnlVect * p;
    PnlVect * phat;
    PnlVect * s;
    PnlVect * shat;
    PnlVect * t;
    PnlVect * v;
    PnlVect * r;
    PnlVect * rtilde;
    PnlIterationBase * iter;
} ;

struct _PnlGmresSolver
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
     * properly. This allows any PnlGmresSolver pointer to be cast to a
     * PnlObject
     */
    PnlObject object;
    int restart;
    double beta;
    PnlVect * s;
    PnlVect * cs;
    PnlVect * sn;
    PnlVect * w;
    PnlVect * r;
    PnlMat * H;
    PnlVect * v[MAX_RESTART];
    PnlIterationBase * iter;
    PnlIterationBase * iter_inner;
} ;

```

A Left preconditioner solves the problem :

$$PMx = Pb,$$

and whereas right preconditioner solves

$$MPy = b, \quad Py = x.$$

More information is given in *Saad, Yousef (2003). Iterative methods for sparse linear systems (2nd ed. ed.). SIAM. ISBN 0898715342. OCLC 51266114.* The reader will find in this book some discussion about right or/and left preconditioner and a description of the following algorithms.

These algorithms, we implemented with a left preconditioner. Right preconditioner can be easily computed changing matrix vector multiplication operator from $M x$ to $M P_R x$ and solving $P_R y = x$ at the end of algorithm.

4.8.2 Functions

Three methods are implemented : Conjugate Gradient, BICGstab and GMRES with restart. For each of them a structure is created to store temporary vectors used in the algorithm. In some cases, we have to apply iterative methods more than once : for example to solve at each time step a discrete form of an elliptic problem come from parabolic problem. In the cases, do not call the constructor and destructor at each time, but instead use the initialization and solve procedures.

Formally we have,

```
Create iterative method
For each time step
    Initialisation of iterative method
    Solve linear system link to elliptic problem
end for
free iterative method
```

In these functions, we don't use any particular matrix structure. We give the matrix vector multiplication as a parameter of the solver.

Conjugate Gradient method Only available for symmetric and positive matrices.

- **PnlCgSolver** * **pnl_cg_solver_new()**
Description Create an empty **PnlCgSolver**
- **PnlCgSolver** * **pnl_cg_solver_create**(int Size, int max-iter, double tolerance)
Description Create a new **PnlCgSolver** pointer.
- void **pnl_cg_solver_initialisation**(**PnlCgSolver** *Solver, const **PnlVect** *b)
Description Initialisation of the solver at the beginning of iterative method.
- void **pnl_cg_solver_free**(**PnlCgSolver** **Solver)
Description Destructor of iterative solver

- `int pnl_cg_solver_solve(void(*matrix vector-product)(const void *, const PnlVect *, const double, const double, PnlVect *), const void *Matrix-Data, void(*matrix vector-product-PC)(const void *, const PnlVect *, const double, const double, PnlVect *), const void *PC-Data, PnlVect *x, const PnlVect *b, PnlCgSolver *Solver)`
Description Solve the linear system matrix vector-product is the matrix vector multiplication function matrix vector-product-PC is the preconditionner function Matrix-Data & PC-Data is data to compute matrix vector multiplication.

BICG stab

- `PnlBicgSolver * pnl_bicg_solver_new()`
Description Create an empty PnlBicgSolver.
- `PnlBicgSolver * pnl_bicg_solver_create(int Size, int max-iter, double tolerance)`
Description Create a new PnlBicgSolver pointer.
- `void pnl_bicg_solver_initialisation(PnlBicgSolver *Solver, const PnlVect *b)`
Description Initialisation of the solver at the beginning of iterative method.
- `void pnl_bicg_solver_free(PnlBicgSolver **Solver)`
Description Destructor of iterative solver
- `int pnl_bicg_solver_solve(void(*matrix vector-product)(const void *, const PnlVect *, const double, const double, PnlVect *), const void *Matrix-Data, void(*matrix vector-product-PC)(const void *, const PnlVect *, const double, const double, PnlVect *), const void *PC-Data, PnlVect *x, const PnlVect *b, PnlBicgSolver *Solver)`
Description Solve the linear system matrix vector-product is the matrix vector multiplication function matrix vector-product-PC is the preconditioner function Matrix-Data & PC-Data is data to compute matrix vector multiplication.

GMRES with restart See *Saad, Yousef (2003)* for a discussion about the restart parameter. For GMRES we need to store at the p -th iteration p vectors of the same size of the right and side. It could be very expensive in term of memory allocation. So GMRES with restart algorithm stop if $p = restart$ and restarts the algorithm with the previously computed solution as initial guess.

Note that if restart equals m , we have a classical GMRES algorithm.

- `PnlGmresSolver * pnl_gmres_solver_new()`
Description Create an empty PnlGmresSolver
- `PnlGmresSolver * pnl_gmres_solver_create(int Size, int max-iter, int restart, double tolerance)`
Description Create a new PnlGmresSolver pointer.
- `void pnl_gmres_solver_initialisation(PnlGmresSolver *Solver, const PnlVect *b)`
Description Initialisation of the solver at the beginning of iterative method.
- `void pnl_gmres_solver_free(PnlGmresSolver **Solver)`
Description Destructor of iterative solver

- int **pnl_gmres_solver_solve**(void(*matrix vector-product)(const void *, const **PnlVect** *, const double, const double, **PnlVect** *), const void *Matrix-Data, void(*matrix vector-product-PC)(const void *, const **PnlVect** *, const double, const double, **PnlVect** *), const void *PC-Data, **PnlVect** *x, const **PnlVect** *b, **PnlGmresSolver** *Solver)
Description Solve the linear system matrix vector-product is the matrix vector multiplication function matrix vector-product-PC is the preconditionner function Matrix-Data & PC-Data is data to compute matrix vector multiplication.

In the next paragraph, we write all the solvers for **PnlMat** . This will be done as follows: construct an application matrix vector.

```
static void pnl_mat_mult_vect_applied(const void *mat, const PnlVect *
    vec,
                                     const double a , const double b,
                                     PnlVect *lhs)
{pnl_mat_lAxpby(a, (PnlMat*)mat, vec, b, lhs);}
```

and give it as the parameter of the iterative method

```
int pnl_mat_cg_solver_solve(const PnlMat * Matrix, const PnlMat * PC,
    PnlVect * x, const PnlVect *b, PnlCgSolver
    * Solver)
{ return pnl_cg_solver_solve(pnl_mat_mult_vect_applied,
    Matrix, pnl_mat_mult_vect_applied,
    PC, x, b, Solver);}
```

In practice, we cannot define all iterative methods for all structures. With this implementation, the user can easily :

- implement right preconditioner,
- implement method with sparse matrix and diagonal preconditioner, or special combination of this form ...

Iterative algorithms for **PnlMat**

- int **pnl_mat_cg_solver_solve**(const **PnlMat** *M, const **PnlMat** *PC, **PnlVect** *x, const **PnlVect** *b, **PnlCgSolver** *Solver)
Description Solve the linear system $M x = b$ with preconditionner PC.
- int **pnl_mat_bicg_solver_solve**(const **PnlMat** *M, const **PnlMat** *PC, **PnlVect** *x, const **PnlVect** *b, **PnlBicgSolver** *Solver)
Description Solve the linear system $M x = b$ with preconditionner PC.
- int **pnl_mat_gmres_solver_solve**(const **PnlMat** *M, const **PnlMat** *PC, **PnlVect** *x, **PnlVect** *b, **PnlGmresSolver** *Solver)
Description Solve the linear system $M x = b$ with preconditionner PC.

5 Cumulative distribution Functions

The functions related to this chapter are declared in **pnl/pnl_cdf.h**.

For various distribution functions, we provide functions named **pnl_cdf_xxx** where **xxx** is the abbreviation of the distribution name. All these functions are based on the same prototype

$$p = 1 - q; \quad p = \int^x density(u)du$$

- **which** If **which**=1, it computes **p** and **q**. If **which**=2, it computes **x**. For higher values of **which** it computes one the parameters characterizing the distribution using all the others, **p**, **q**, **x**.
- **p** the probability $\int^x density(u)du$
- **q** = 1 - **p**
- **x** the upper bound of the integral
- **status** an integer which indicates on exit the success of the computation. (0) if calculation completed correctly. (-1) if the input parameter number **I** was out of range. (1) if the answer appears to be lower than the lowest search bound. (2) if the answer appears to be higher than the greatest search bound. (3) if $p + q \neq 1$.
- **bound** is undefined if **STATUS** is 0. Bound exceeded by parameter number **I** if **STATUS** is negative. Lower search bound if **STATUS** is 1. Upper search bound if **STATUS** is 2.
- void **pnl_cdf_bet**(int *which, double *p, double *q, double *x, double *y, double *a, double *b, int *status, double *bound)
[Description](#) Cumulative Distribution Function BETA distribution.
- void **pnl_cdf_bin**(int *which, double *p, double *q, double *x, double *xn, double *pr, double *ompr, int *status, double *bound)
[Description](#) Cumulative Distribution Function BINa distribution.
- void **pnl_cdf_chi**(int *which, double *p, double *q, double *x, double *df, int *status, double *bound)
[Description](#) Cumulative Distribution Function CHI-Square distribution.
- void **pnl_cdf_chn**(int *which, double *p, double *q, double *x, double *df, double *pnonc, int *status, double *bound)
[Description](#) Cumulative Distribution Function Non-central Chi-Square distribution.
- void **pnl_cdf_f**(int *which, double *p, double *q, double *x, double *dfn, double *dfd, int *status, double *bound)
[Description](#) Cumulative Distribution Function F distribution.
- void **pnl_cdf_fnc**(int *which, double *p, double *q, double *x, double *dfn, double *dfd, double *pnonc, int *status, double *bound)
[Description](#) Cumulative Distribution Function Non-central F distribution.
- void **pnl_cdf_gam**(int *which, double *p, double *q, double *x, double *shape, double *rate, int *status, double *bound)
[Description](#) Cumulative Distribution Function GAMma distribution. Note that the parameter **rate** is 1/scale. The density writes $f(x) = 1/(s^a \Gamma(a)) x^{a-1} e^{-x/s}$ with **scale=s** and **shape=1/rate=a**.

- void **pnl_cdf_nbn**(int *which, double *p, double *q, double *x, double *xn, double *pr, double *ompr, int *status, double *bound)
[Description](#) Cumulative Distribution Function Negative BiNomial distribution.
- void **pnl_cdf_nor**(int *which, double *p, double *q, double *x, double *mean, double *sd, int *status, double *bound)
[Description](#) Cumulative Distribution Function NORmal distribution.
- void **pnl_cdf_poi**(int *which, double *p, double *q, double *x, double *xlam, int *status, double *bound)
[Description](#) Cumulative Distribution Function POIsson distribution.
- void **pnl_cdf_t**(int *which, double *p, double *q, double *x, double *df, int *status, double *bound)
[Description](#) Cumulative Distribution Function T distribution.
- double **pnl_cdfchi2n**(double x, double df, double ncparam)
[Description](#) Compute the cumulative density function at **x** of the non central χ^2 distribution with **df** degrees of freedom and non centrality parameter **ncparam**.
- void **pnl_cdfbchi2n**(double x, double df, double ncparam, double beta, double *P)
[Description](#) Store in **P** the cumulative density function at **x** of the random variable **beta *X** where **X** is non central χ^2 random variable with **df** degrees of freedom and non centrality parameter **ncparam**.
- double **pnl_normal_density**(double x)
[Description](#) Normal density function.
- double **pnl_cdfnor**(double x)
[Description](#) Cumulative normal distribution function.
- double **pnl_cdf2nor**(double a, double b, double r)
[Description](#) Cumulative bivariate normal distribution function, returns $\frac{1}{2\pi\sqrt{1-r^2}} \int_{-\infty}^a \int_{-\infty}^b e^{-\frac{x^2-2rxy+y^2}{2(1-r^2)}} dx dy$.
- double **pnl_inv_cdfnor**(double x)
[Description](#) Inverse of the cumulative normal distribution function.

6 Random Number Generators

The functionalities described in this chapter are declared in **pnl/pnl_random.h**.

Random number generators should be called through the new *rng* interface based on the **PnlRng** object. This interface uses reentrant functions and is suitable for multi-threaded applications.

The older *rand* interface is kept for compatibility purposes only and should not be used in new code.

| Random generator | index | Type | Info |
|------------------|----------------------|--------|-----------------------|
| KNUTH | PNL_RNG_KNUTH | pseudo | |
| MRGK3 | PNL_RNG_MRGK3 | pseudo | |
| MRGK5 | PNL_RNG_MRGK5 | pseudo | |
| SHUFL | PNL_RNG_SHUFL | pseudo | |
| L'ECUYER | PNL_RNG_L_ECUYER | pseudo | |
| TAUSWORTHE | PNL_RNG_TAUSWORTHE | pseudo | |
| MERSENNE | PNL_RNG_MERSENNE | pseudo | |
| SQRT | PNL_RNG_SQRT | quasi | |
| HALTON | PNL_RNG_HALTON | quasi | |
| FAURE | PNL_RNG_FAURE | quasi | |
| SOBOL_I4 | PNL_RNG_SOBOL_I4 | quasi | uses 32 bit intergers |
| SOBOL_I8 | PNL_RNG_SOBOL2_I8 | quasi | uses 64 bit intergers |
| NIEDERREITER | PNL_RNG_NIEDERREITER | quasi | |

Table 2: Indices of the random generators

6.1 The rng interface

It is possible to create several random number generators each with its own state variable so that they can evolve independently in a shared memory environment. These generators are suitable for use in multi-threaded programs.

```
typedef struct _PnlRng PnlRng;
struct _PnlRng
{
    PnlObject object;
    int type; /*!< generator type */
    void (*Compute)(PnlRng *g, double *sample); /*!< the function to
        compute the
                                next number in the
                                sequence */
    int rand_or_quasi; /*!< can be PNL_MC or PNL_QMC */
    int dimension; /*!< dimension of the space in which we draw the
        samples */
    int counter; /*!< counter = number of samples already drawn */
    int has_gauss; /*!< Is a gaussian deviate available? */
    double gauss; /*!< If has_gauss==1, gauss a gaussian sample */
    int size_state; /*!< size in bytes of the state variable */
    void *state; /*!< state of the random generator */
};
```

- void **pnl_rng_free**(PnlRng **)
Description Free a PnlRng .
- PnlRng * **pnl_rng_create**(int type)
Description Create a PnlRng corresponding to type which can be any of the values PNL_RNG_XXX listed in Table 2. Once a generator has been created, before using it, you must call
 - pnl_rng_sseed if it is a pseudo random number generator,

- `pnl_rng_sdim` if it is a quasi random number generator,
- `void pnl_rng_sseed(PnlRng *rng, unsigned long int s)`
Description Set the seed of the generator `rng` using `s`. If `s=0`, then a default seed (depending on the generator) is used.
- `int pnl_rng_sdim(PnlRng *rng, int dim)`
Description Set the dimension of the state space for a QMC generator and initializes it accordingly. Returns OK if the generator has been initialized properly and FAIL otherwise.
- `PnlRng *pnl_rng_copy(const PnlRng *rng)`
Description Create a copy of `rng`.
- `void pnl_rng_clone(PnlRng *dest, const PnlRng *src)`
Description Copy the content of `src` into the already existing basis `dest`. On exit, `src` and `dest` are identical but independent.
- `PnlRng *pnl_rng_dcmt_create_id(int id, ulong seed)`
Description Create a generator with type PNL_RNG_DCMT and identifier `id`. Two generators with different `ids` are independent. Note that the returned generator must be initialized with `pnl_rng_sseed` before usage. The identifier `id` can for instance correspond to the thread number or the processor rank in parallel computing.
- `PnlRng **pnl_rng_dcmt_create_array_id(int start_id, int max_id, ulong seed, int *count)`
Description Create an array of generators with types PNL_RNG_DCMT and identifiers linearly varying between `start_id` and `max_id`. The number of generators created is `max_id - start_id + 1`. All the generators are independent. Note that each generator of the returned array must be initialized with `pnl_rng_sseed` before usage.
- `PnlRng **pnl_rng_dcmt_create_array(int n, ulong seed, int *count)`
Description Create an array of `n` independent DCMT. `seed` is the seed used to initialize the Mersenne Twister generator internally used to find new DCMT. On exit, `count` contains the number of generators actually created. Before using the generators, you must initialize each of them by calling the function `pnl_rng_sseed` `count` times.

Some auxiliary functions internally used (to be used with caution)

- `PnlRng *pnl_rng_new()`
Description Create an empty `PnlRng`.
- `void pnl_rng_init(PnlRng *rng, int type)`
Description Initialize an empty `PnlRng` as returned by `pnl_rng_new` to become a generator of type `type` which can be any of the values PNL_RNG_XXX listed in Table 2 which correspond to **pseudo** random number generators. Calling `pnl_rng_create` is equivalent to calling first `pnl_rng_new` and then `pnl_rng_init`.
- `PnlRng *pnl_rng_get_from_id(int id)`
Description Return the global generator described by its macro name. The variable `id` can be any of the values PNL_RNG_XXX listed in Table 2.

The following functions return one sample from the specified distribution.

- int **pnl_rng_bernoulli**(double p, **PnlRng** *rng)
Description Generate a sample from the Bernoulli law on $\{0, 1\}$ with parameter **p**.
- long **pnl_rng_poisson**(double lambda, **PnlRng** *rng)
Description Generate a sample from the Poisson law with parameter **lambda**.
- double **pnl_rng_exp**(double lambda, **PnlRng** *rng)
Description Generate a sample from the Exponential law with parameter **lambda**.
- double **pnl_rng_dblexp**(double lambda_p, double lambda_m, double p, **PnlRng** *rng)
Description Generate a sample from the asymmetric exponential distribution with density
$$p\lambda_p e^{-\lambda_p y} 1_{\{y>0\}} + (1-p)\lambda_m e^{-\lambda_m |y|} 1_{\{y<0\}}$$
where $\lambda_p > 0, \lambda_m > 0$ and $p \in [0, 1]$.
- double **pnl_rng_uni**(**PnlRng** *rng)
Description Generate a sample from the Uniform law on $]0, 1[$.
- double **pnl_rng_uni_ab**(double a, double b, **PnlRng** *rng)
Description Generate a sample from the Uniform law on $[a, b]$.
- double **pnl_rng_normal**(**PnlRng** *rng)
Description Generate a sample from the standard normal distribution.
- double **pnl_rng_lognormal**(double m, double sigma2, **PnlRng** *rng)
Description Generate a sample from the log-normal distribution. The underlying normal distribution has mean **m** and variance **sigma2**.
- double **pnl_rng_invgauss**(double mu, double lambda, **PnlRng** *rng)
Description Generate a sample from the inverse Gaussian distribution with mean **mu** and shape parameter **lambda**.
- long **pnl_rng_poisson1**(double lambda, double t, **PnlRng** *rng)
Description Generate a sample from a Poisson process with intensity **lambda** at time **t**.
- double **pnl_rng_gamma**(double a, double b, **PnlRng** *rng)
Description Generate a sample from the $\Gamma(a, b)$ distribution.
- double **pnl_rng_chi2**(double df, **PnlRng** *rng)
Description Generate a sample from the centered $\chi^2(df)$ distribution.
- double **pnl_rng_ncchi2**(double df, double xnonc, **PnlRng** *rng)
Description Generate a sample from the non central χ^2 distribution with **df** degrees of freedom and non central parameter **xnonc**.
- double **pnl_rng_bessel**(double v, double a, **PnlRng** *rng)
Description Generate a sample from the Bessel distribution with parameters **v** > -1 and **a** > 0 .

- double **pnl_rng_gauss**(int d, int create_or_retrieve, int index, PnlRng *rng)
Description The second argument can be either **CREATE** (to actually draw the sample) or **RETRIEVE** (to retrieve that element of index **index**). With **CREATE**, it draws **d** random normal variables and stores them for future usage. They can be withdrawn using **RETRIEVE** with the index of the number to be retrieved.

The following functions take an already existing **PnlVect** *as first argument and fill each entry of the vector with a sample from the specified distribution. All the entries are independent. The difference between n -samples from a distribution in dimension 1, and one sample from the same distribution in dimension n only matters when using a **quasi** random number generator.

- void **pnl_vect_rng_bernoulli**(PnlVect *V, int samples, double a, double b, double p, PnlRng *rng)
Description Simulate an i.i.d. sample from the Bernoulli distribution with values in **a, b** and parameter **p**. The result is stored in **V**.
- void **pnl_vect_rng_bernoulli_d**(PnlVect *V, int dimension, const PnlVect *a, const PnlVect *b, const PnlVect *p, PnlRng *rng)
Description Simulate a random vector according to the Bernoulli distribution with values in **{a, b}** and parameter **p**. The result is stored in **V**, ie. **V(i)** follows a Bernoulli distribution on **{a(i), b(i)}** with parameter **p(i)**.
- void **pnl_vect_rng_poisson**(PnlVect *V, int samples, double lambda, PnlRng *rng)
Description Simulate an i.i.d. sample from the Poisson distribution with parameter **lambda**. The result is stored in **V**. Note that, we are using double based vectors and not integer based vectors.
- void **pnl_vect_rng_poisson_d**(PnlVect *V, int dimension, const PnlVect *lambda, PnlRng *rng)
Description Simulate a random vector according to the Poisson distribution with **vector** parameter **lambda**. The result is stored in **V**, ie. **V(i)** follows a Poisson distribution with parameter **lambda(i)**. Note that, we are using double based vectors and not integer based vectors.
- void **pnl_vect_rng_uni**(PnlVect *G, int samples, double a, double b, PnlRng *rng)
Description **G** is a vector of independent and identically distributed samples from the uniform distribution on $[a, b]$.
- void **pnl_vect_rng_normal**(PnlVect *G, int samples, PnlRng *rng)
Description **G** is a vector of independent and identically distributed samples from the standard normal distribution.
- void **pnl_vect_rng_uni_d**(PnlVect *G, int d, double a, double b, PnlRng *rng)
Description **G** is a sample from the uniform distribution on $[a, b]^d$.
- void **pnl_vect_rng_normal_d**(PnlVect *G, int d, PnlRng *rng)
Description **G** is a sample from the d -dimensional standard normal distribution.

The following functions take an already existing **PnlMat** *as first argument and fill each entry of the matrix with a sample from the specified distribution. All the entries are independent. On return, the matrix M is of size **samples** x **dimension**. The rows of M are independent and identically distributed. Each row is a sample from the given law in dimension **dimension**.

- void **pnl_mat_rng_uni**(**PnlMat** *M, int samples, int d, const **PnlVect** *a, const **PnlVect** *b, **PnlRng** *rng)
Description M contains **samples** samples from the uniform distribution on $\prod_{i=1}^d [a_i, b_i]$.
- void **pnl_mat_rng_uni2**(**PnlMat** *M, int samples, int d, double a, double b, **PnlRng** *rng)
Description M contains **samples** samples from the uniform distribution on $[a, b]^d$.
- void **pnl_mat_rng_normal**(**PnlMat** *M, int samples, int d, **PnlRng** *rng)
Description M contains **samples** samples from the d-dimensional standard normal distribution.
- void **pnl_mat_rng_bernoulli**(**PnlMat** *M, int samples, int dimension, const **PnlVect** *a, const **PnlVect** *b, const **PnlVect** *p, **PnlRng** *rng)
Description Compute a random matrix with independent rows, each of them having a vector Bernoulli distribution, ie. $M(i, j)$ follows a Bernoulli distribution on $\{a(j), b(j)\}$ with parameter $p(j)$.
- void **pnl_mat_rng_poisson**(**PnlMat** *M, int samples, int dimension, const **PnlVect** *lambda, **PnlRng** *rng)
Description Compute a random matrix with independent rows, each of them having a vector Poisson distribution, ie. $M(i, j)$ follows a Poisson distribution with parameter $p(j)$.

Some examples

```
#include <stdlib.h>
#include "pnl/pnl_random.h"

int main ()
{
    int i, M;
    PnlRng *rng = pnl_rng_create(PNL_RNG_MERSENNE);
    PnlVect *v = pnl_vect_new();
    M = 10000;

    /* rng must be initialized. When sseed=0, a default
       value depending on the generator is used */
    pnl_rng_sseed(rng, 0);

    for (i=0 ; i<M ; i++)
    {
        /* Simulates a normal random vector in  $R^{\{10\}}$  */
        pnl_vect_rng_normal(v, 10, rng);
        /* Do something with v */
    }
}
```

```

    pnl_vect_free(&v);
    pnl_rng_free(&rng); /* Frees the generator */
    exit(0);
}

#include <stdlib.h>
#include <time.h>
#include "pnl/pnl_random.h"

int main ()
{
    int i, M;
    double E;
    PnlRng *rng = pnl_rng_create(PNL_RNG_MERSENNE);
    M = 10000;

    /* rng must be initialized. */
    pnl_rng_sseed(rng, time (NULL));

    for (i=0 ; i<M ; i++)
    {
        /* Simulates an exponential random variable */
        E = pnl_rng_exp(1, rng);
        /* Do something with E */
    }

    pnl_rng_free(&rng); /* Frees the generator */
    exit(0);
}

```

6.2 The *rand* interface (deprecated)

Note: For backward compatibility with older versions of the PNL, we still provide the old *rand* interface to random number generation although we strongly encourage users to use the new *rng* interface (see section 6.1).

Every generator is identified by an integer valued macro. One must **NOT** refer to a generator using directly the value of the macro `PNL_RNG_XXX` because there is no warranty that the order used to store the generators will remain the same in future releases. Instead, one should call generators directly using their macro names.

The initial seeds of all the generators are fixed by the function `pnl_rand_init` but you can change it by calling `pnl_rand_sseed`.

Before starting to use random number generators, you **must** initialize them by calling

- `int pnl_rand_init(int type_generator, int simulation_dim, long samples)`
[Description](#) It resets the sample counter to 0 and checks that the generator described by `type_generator` can actually generate `samples` in dimension `simulation_dim` and fixes the seed.
- `int pnl_rand_or_quasi(int type_generator)`
[Description](#) Return the type the generator of index `type_generator`, `PNL_MC` or `PNL_QMC`

- void **pnl_rand_sseed**((int type_generator, unsigned long int seed))
[Description](#) It sets the seed of the generator **type_generator** with **seed**.
- const char * **pnl_rand_name**(int type_generator)
[Description](#) Return the name of the generator of index **type_generator**

Once a generator is chosen, there are several functions available in the library to draw samples according to a given law.

The following functions return one sample from a specified law.

- int **pnl_rand_bernoulli**(double p, int type_generator)
[Description](#) Generate a sample from the Bernoulli law on $\{0, 1\}$ with parameter **p**.
- long **pnl_rand_poisson**(double lambda, int type_generator)
[Description](#) Generate a sample from the Poisson law with parameter **lambda**.
- double **pnl_rand_exp**(double lambda, int type_generator)
[Description](#) Generate a sample from the Exponential law with parameter **lambda**.
- double **pnl_rand_uni**(int type_generator)
[Description](#) Generate a sample from the Uniform law on $[0, 1]$.
- double **pnl_rand_uni_ab**(double a, double b, int type_generator)
[Description](#) Generate a sample from the Uniform law on $[a, b]$.
- double **pnl_rand_normal**(int type_generator)
[Description](#) Generate a sample from the standard normal distribution.
- long **pnl_rand_poisson1**(double lambda, double t, int type_generator)
[Description](#) Generate a sample from a Poisson process with intensity **lambda** at time **t**.
- double **pnl_rand_gamma**(double a, double b, int type_generator)
[Description](#) Generate a sample from the $\Gamma(a, b)$ distribution.
- double **pnl_rand_chi2**(double n, int type_generator)
[Description](#) Generate a sample from the centered $\chi^2(n)$ distribution.
- double **pnl_rand_bessel**(double v, double a, int generator)
[Description](#) Generate a sample from the Bessel distribution with parameters **v** > -1 and **a** > 0.

The following functions take an already existing **PnlVect** * as its first argument and fill each entry of the vector with a sample from the specified law. All the entries are independent. The difference between n -samples from a distribution in dimension 1, and one sample from the same distribution in dimension n only matters when using a **Quasi** random number generator.

- void **pnl_vect_rand_uni**(**PnlVect** *G, int samples, double a, double b, int type_generator)
[Description](#) **G** is a vector of independent and identically distributed samples from the uniform distribution on $[a, b]$.

- void **pnl_vect_rand_normal**(PnlVect *G, int samples, int generator)
Description G is a vector of independent and identically distributed samples from the standard normal distribution.
- void **pnl_vect_rand_uni_d**(PnlVect *G, int d, double a, double b, int type_generator)
Description G is a sample from the uniform distribution on $[a, b]^d$.
- void **pnl_vect_rand_normal_d**(PnlVect *G, int d, int generator)
Description G is a sample from the d-dimensional standard normal distribution.

The following functions take an already existing **PnlMat** * as first argument and fill each entry of the vector with a sample from the specified law. All the entries are in-dependant. On return, the matrix M is of size **samples** x **dimension**. The rows of M are independently and identically distributed. Each row is a sample from the given law in dimension **dimension**.

- void **pnl_mat_rand_uni**(PnlMat *M, int samples, int d, const PnlVect *a, const PnlVect *b, int type_generator)
Description M contains **samples** samples from the uniform distribution on $\prod_{i=1}^d [a_i, b_i]$.
- void **pnl_mat_rand_uni2**(PnlMat *M, int samples, int d, double a, double b, int type_generator)
Description M contains **samples** samples from the uniform distribution on $[a, b]^d$.
- void **pnl_mat_rand_normal**(PnlMat *M, int samples, int d, int type_generator)
Description M contains **samples** samples from the d-dimensional standard normal distribution.

Because of the use of **Quasi** random number generators, you may need to draw a set of samples at once because they represent one sample from a multi-dimensional distribution. The following function enables to draw one sample from the **dimension**-dimensional standard normal distribution and store it so that you can access the elements individually afterwards.

- double **pnl_rand_gauss**(int d, int create_or_retrieve, int index, int type_generator)
Description The second argument can be either **CREATE** (to actually draw the sample) or **RETRIEVE** (to retrieve that element of index **index**). With **CREATE**, it draws **d** random normal variables and stores them for future usage. They can be withdrawn using **RETRIEVE** with the index of the number to be retrieved.

7 Function bases and regression

7.1 Overview

To use these functionalities, you should include **pnl/pnl_basis.h**.

```
struct _PnlBasis
{
    /**
     * Must be the first element in order for the object mechanism to
     * work
```

```

    * properly. This allows any PnlBasis pointer to be cast to a
      PnlObject
  */
PnlObject      object;
/** The basis type */
int            id;
/** The string to label the basis */
const char     *label;
/** The number of variates */
int            nb_variates;
/** The total number of elements in the basis */
int            nb_func;
/** The sparse Tensor matrix */
PnlSpMatInt    *SpT;
/** The number of functions in the tensor @p SpT */
int            len_T;
/** Compute the i-th element of the one dimensional basis. As a
    convention, (*f)(x, 0) MUST be equal to 1 */
double         (*f)(double x, int i, int dim, void *params);
/** Compute the first derivative of i-th element of the one
    dimensional basis */
double         (*Df)(double x, int i, int dim, void *params);
/** Compute the second derivative of the i-th element of the one
    dimensional basis */
double         (*D2f)(double x, int i, int dim, void *params);
/** PNL_TRUE if the basis is reduced */
int            isreduced;
/** The center of the domain */
double         *center;
/** The inverse of the scaling factor to map the domain to [-1, 1]^
    nb_variates */
double         *scale;
/** An array of additional functions */
PnlRnFuncR     *func_list;
/** The number of functions in @p func_list */
int            len_func_list;
/** Extra parameters to pass to basis functions */
void           *f_params;
/** Size of params in bytes to be passed to malloc */
size_t         f_params_size;
/** Non linear mapping of the data */
double         (*map)(double x, int dim, void *params);
/** First derivative of the non linear mapping */
double         (*Dmap)(double x, int dim, void *params);
/** Second derivate of the linear mapping */
double         (*D2map)(double x, int dim, void *params);
/** Extra parameters for map, Dmap and D2map */
void           *map_params;
/** Size of @p map_params in bytes to be passed to malloc */
size_t         map_params_size;
};

```

A **PnlBasis** is a family of multivariate functions with real values. Two different kinds of

functions can be stored in these families: tensor functions — originally, this was the only possibility — and standard multivariate function typed as **PnlRnFuncR**.

7.1.1 Tensor functions

Tensor functions are built as a tensor product of one dimensional elements. Hence, we only need a tensor matrix **T** to describe a multi-dimensional basis in terms of the one dimensional one. These tensors functions can be easily evaluated and differentiated twice, see `pnl_basis_eval`, `pnl_basis_eval_vect`, `pnl_basis_eval_D`, `pnl_basis_eval_D_vect`, `pnl_basis_eval_D2`, `pnl_basis_eval_D2_vect`, `pnl_basis_eval_derivs`, `pnl_basis_eval_derivs_vect`.

Three bases are already registered as listed in Table 3. A new tensor basis can be registered using the function `pnl_basis_type_register`.

| | |
|-----------------------------|--------------------------------|
| PNL_BASIS_CANONICAL | for the Canonical polynomials |
| PNL_BASIS_HERMITE | for the Hermite polynomials |
| PNL_BASIS_TCHEBYCHEV | for the Tchebychev polynomials |

Table 3: Names of the bases.

The Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

If G is a real valued standard normal random variable, $\mathbb{E}[H_n(G)H_m(G)] = n!1_{\{n=m\}}$.

The two tensors **T** and **SpT** do actually store the same information — **T**(*i*, *j*) is the *degree* w.r.t the *j*-th variable in the *i*-th function. Originally, we were only using the dense representation **T**, which is far more convenient to use when building the basis but it slows down the evaluation of the basis by a great deal. To overcome this lack of efficiency, a sparse storage was added. Such a basis can be created using one of the following functions.

- **PnlBasis** * `pnl_basis_create`(int index, int nb_func, int nb_variates)
Description Create a **PnlBasis** for the family defined by **index** (see Table 3 and `pnl_basis_type_register`) with **nb_variates** variates. The basis will contain **nb_func**.
- **PnlBasis** * `pnl_basis_create_from_degree`(int index, int degree, int nb_variates)
Description Create a **PnlBasis** for the family defined by **index** (see Table 3 and `pnl_basis_type_register`) with total degree less or equal than **degree** and **nb_variates** variates. The total degree is the sum of the partial degrees.
- **PnlBasis** * `pnl_basis_create_from_prod_degree`(int index, int degree, int nb_variates)
Description Create a **PnlBasis** for the family defined by **index** (see Table 3 and `pnl_basis_type_register`) with total degree less or equal than **degree** and **nb_variates** variates. The total degree is the product of **MAX(1, d_i)** where the **d_i** are the partial degrees.
- **PnlBasis** * `pnl_basis_create_from_hyperbolic_degree`(int index, double degree, double q, int n)
Description Create a sparse basis of polynomial with **n** variates. We give the example

of the Canonical basis. A canonical polynomial with n variates writes $X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}$. To be a member of the basis, it must satisfy $(\sum_{i=1}^n \alpha_i^q)^{1/q} \leq \text{degree}$. This kind of basis based on an hyperbolic set of indices gives priority to polynomials associated to low order interaction.

- **PnlBasis** * **pnl_basis_create_from_tensor**(int index, PnlMatInt *T)

Description Create a **PnlBasis** for the polynomial family defined by **index** (see Table 3) using the basis described by the tensor matrix **T**. The number of lines of **T** is the number of functions of the basis whereas the numbers of columns of **T** is the number of variates of the functions.

Here is an example of a tensor matrix. Assume you are working with three variate functions, the basis $\{1, x, y, z, x^2, xy, yz, z^3\}$ is decomposed in the one dimensional canonical basis using the following tensor matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

- **PnlBasis** * **pnl_basis_create_from_sparse_tensor**(int index, PnlSpMatInt *SpT)

Description Create a **PnlBasis** for the polynomial family defined by **index** (see Table 3) using the basis described by the sparse tensor matrix **SpT**. The number of lines of **SpT** is the number of functions of the basis whereas the numbers of columns of **SpT** is the number of variates of the functions.

- void **pnl_basis_set_from_tensor**(**PnlBasis** *b, int index, const **PnlMatInt** *T)

Description Set an already existing basis **b** to a polynomial family defined by **index** (see Table 3) using the basis described by the tensor matrix **T**. The number of lines of **T** is the number of functions of the basis whereas the numbers of columns of **T** is the number of variates of the functions.

Same function as **pnl_basis_create_from_tensor** except that it operates on an already existing basis.

7.1.2 Local basis functions

A local basis is a family of indicator functions of a Cartesian partition of $[-1, 1]^d$. Let $(n_i)_{i \in \{1, \dots, d\}}$ be the number of interval along each direction. An element of the partition write

$$A_k = \prod_{i=1}^d \left[-1 + \frac{k_i}{n_i}, -1 + \frac{k_i + 1}{n_i} \right]$$

where k is the multi-index defined by $k_i \in \{0, \dots, n_i - 1\}$ for all $i \in \{1, \dots, d\}$. These functions are orthogonal for the standard L^2 scalar product; this property is used by **pnl_basis_fit_ls**. Note that they are not differentiable.

We provide the helper function `pnl_basis_local_get_index` to compute the linear representation of the multi-index k .

- **PnlBasis** * `pnl_basis_local_create`(int *n_intervals, int space_dim)
Description Create a local basis with `n_intervals[i - 1]` intervals along dimension i for all $i \in \{1, \dots, \text{space_dim}\}$
- **PnlBasis** * `pnl_basis_local_create_regular`(int n_intervals, int space_dim)
Description Equivalent to calling `pnl_basis_local_create` with an array of size `space_dim` filled with `n_intervals`.
- int `pnl_basis_local_get_index`(const **PnlBasis** *basis, const double *x)
Description Return the linear index of the cell containing \mathbf{x} . It is an integer between 0 and $(\prod_{i=1}^d n_i) - 1$.

If the domain you want to consider is not $[-1, 1]^d$, use the functions `pnl_basis_set_domain`, `pnl_basis_set_reduced` and `pnl_basis_set_map` to map your product space into $[-1, 1]^d$.

7.1.3 Standard multivariate functions

These functions are supposed to be **PnlRnFuncR**.

To make this toolbox more complete, it is now possible to add some extra functions, which are not tensor product. They are stored using an independent mechanism in `func_list`. These additional functions are only taken into account by the methods `pnl_basis_i`, `pnl_basis_i_vect`, `pnl_basis_eval` and `pnl_basis_eval_vect`. Note in particular that it is not possible to differentiate these functions and that they are not sensitive to the `isreduced` attribute. To add an extra function to an existing **PnlBasis**, call the function

- void `pnl_basis_add_function`(**PnlBasis** *b, **PnlRnFuncR** *f)
Description Add the function \mathbf{f} to the already existing basis \mathbf{b} .

7.2 Functions

- **PnlBasis** * `pnl_basis_new`()
Description Create an empty **PnlBasis**.
- void `pnl_basis_clone`(**PnlBasis** *dest, const **PnlBasis** *src)
Description Clone `src` into `dest`. The basis `dest` must already exist before calling this function. On exit, `dest` and `src` are identical and independent.
- **PnlBasis** * `pnl_basis_copy`(const **PnlBasis** *B)
Description Create a copy of \mathbf{B} .
- void `pnl_basis_free`(**PnlBasis** **basis)
Description Free a **PnlBasis** created by `pnl_basis_create`. Beware that `basis` is the address of a **PnlBasis** *.
- void `pnl_basis_del_elt`(**PnlBasis** *B, const **PnlVectInt** *d)
Description Remove the function defined by the tensor product \mathbf{d} from an existing basis \mathbf{B} .

- void **pnl_basis_del_elt_i**(**PnlBasis** *B, int i)
Description Remove the *i*-th element of basis B.
- void **pnl_basis_add_elt**(**PnlBasis** *B, const **PnlVectInt** *d)
Description Add the function defined by the tensor *d* to the Basis B.
- int **pnl_basis_type_register**(const char *name, double (*f)(double, int, int, void), double (*Df)(double, int, int, void), double (*D2f)(double, int, int, void), int is_orthogonal)
Description Register a new basis type and return the index to be passed to **pnl_basis_create**. The variable *name* is a unique string identifier of the family. The variables *f*, *Df*, *D2f* are the one dimensional basis functions, its first and second order derivatives. Each of these functions must return a **double** and take two arguments : the first one is the point at which evaluating the basis functions, the second one is the index of function. The varis_orthogonal if the elements of the basis are orthogonal for the L^2 scalar product. Here is a toy example to show how the canonical basis is registered (this family is actually already available with the id PNL_BASIS_CANONICAL, so the following example may look a little fake)

```
double f(double x, int n, int dim, void *params) { return
    pnl_pow_i(x, n); }
double Df(double x, int n, int dim, void *params) { return n *
    pnl_pow_i(x, n-1); }
double f(double x, int n, int dim, void *params) { return n * (
    n-1) * pnl_pow_i(x, n-2); }

int id = pnl_basis_register ("Canonic", f, Df, D2f, PNL_FALSE);
/*
 * B is the Canonical basis of polynomials with degree less or
 * equal than 2 in dimension 5.
 */
PnlBasis *B = pnl_basis_create_from_degree (id, 2, 5);
```

- void **pnl_basis_print**(const **PnlBasis** *B)
Description Print the characteristics of a basis.

Functional regression based on a least square approach often leads to ill conditioned linear systems. One way of improving the stability of the system is to use centered and renormalized polynomials so that the original domain of interest \mathcal{D} (a subset of \mathbb{R}^d) is mapped to $[-1, 1]^d$. If the domain \mathcal{D} is rectangular and writes $[a, b]$ where $a, b \in \mathbb{R}^d$, the *reduction* mapping is done by

$$x \in \mathcal{D} \mapsto \left(\frac{x_i - (b_i + a_i)/2}{(b_i - a_i)/2} \right)_{i=1, \dots, d} \quad (1)$$

- void **pnl_basis_set_domain**(**PnlBasis** *B, const **PnlVect** *a, const **PnlVect** *b)
Description This function declares B as a centered and normalized basis as defined by Equation 1. Calling this function is equivalent to calling **pnl_basis_set_reduced** with **center**=(b+a)/2 and **scale**=(b-a)/2.
- void **pnl_basis_set_reduced**(**PnlBasis** *B, const **PnlVect** *center, const **PnlVect** *scale)

Description This function declares **B** as a centered and normalized basis using the mapping

$$x \in \mathcal{D} \mapsto \left(\frac{x_i - \text{center}_i}{\text{scale}_i} \right)_{i=1,\dots,d}$$

- void **pnl_basis_reset_reduced**(**PnlBasis** ***B**)

Description Reset the reduction settings.

Note that this renormalization does not apply to the extra functions by **pnl_basis_add_**-function but only to the functions defined by the tensor **T**.

It is also possible to apply a non linear map $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ to the i -th coordinate of the input variable before the reduction operation if any. Then, the input variables are transformed according to

$$x \in \mathcal{D} \mapsto \left(\frac{\varphi_i(x_i) - (b_i + a_i)/2}{(b_i - a_i)/2} \right)_{i=1,\dots,d} \quad (2)$$

- void **pnl_basis_set_map**(**PnlBasis** ***B**, double (***map**)(double x, int i, void*params), double (***Dmap**)(double x, int i, void*params), double (***D2map**)(double x, int i, void*params), void *params, size_t size_params)

Description Define the non linear applied to the input variables before the reduction operation. The parameters **Dmap** and **D2map** can be NULL. The three functions **map**, **Dmap** and **D2map** take as second argument the index of the coordinate, while their third argument is used to passe extra parameters defined by **params**.

- int **pnl_basis_fit_ls**(**PnlBasis** ***P**, **PnlVect** ***coef**, **PnlMat** ***x**, **PnlVect** ***y**)

Description Compute the coefficients **coef** defined by

$$\text{coef} = \arg \min_{\alpha} \sum_{i=1}^n \left(\sum_{j=0}^N \alpha_j P_j(x_i) - y_i \right)^2$$

where **N** is the number of functions to regress upon and n is the number of points at which the values of the original function are known. P_j is the j -th basis function. Each row of the matrix **x** defines the coordinates of one point x_i . The function to be approximated is defined by the vector **y** of the values of the function at the points **x**.

- double **pnl_basis_ik_vect**(const **PnlBasis** ***b**, const **PnlVect** ***x**, int i, int k)

Description An element of a basis writes $\prod_{l=0}^{\text{nb_ariates}} \phi_l(x_l)$ where the ϕ 's are one dimensional polynomials. This functions computes the term ϕ_k of the **i**-th basis function at the point **x**.

- double **pnl_basis_i_vect**(const **PnlBasis** ***b**, const **PnlVect** ***x**, int i)

Description If **b** is composed of $f_0, \dots, f_{\text{nb_func}-1}$, then this function returns $f_i(x)$.

- double **pnl_basis_i_D_vect**(const **PnlBasis** ***b**, const **PnlVect** ***x**, int i, int j)

Description If **b** is composed of $f_0, \dots, f_{\text{nb_func}-1}$, then this function returns $\partial_{x_j} f_i(x)$.

- double **pnl_basis_i_D2_vect**(const **PnlBasis** ***b**, const **PnlVect** ***x**, int i, int j1, int j2)

Description If **b** is composed of $f_0, \dots, f_{\text{nb_func}-1}$, then this function returns $\partial_{x_{j1}, x_{j2}}^2 f_i(x)$.

- void **pnl_basis_eval_derivs_vect**(const **PnlBasis** *b, const **PnlVect** *coef, const **PnlVect** *x, double *fx, **PnlVect** *Dfx, **PnlMat** *D2fx)
Description Compute the function, the gradient and the Hessian matrix of $\sum_{k=0}^n \text{coef}_k P_k(\cdot)$ at the point **x**. On output, **fx** contains the value of the function, **Dfx** its gradient and **D2fx** its Hessian matrix. This function is optimized and performs much better than calling **pnl_basis_eval**, **pnl_basis_eval_D** and **pnl_basis_eval_D2** sequentially.
- double **pnl_basis_eval_vect**(const **PnlBasis** *basis, const **PnlVect** *coef, const **PnlVect** *x)
Description Compute the linear combination of $P_k(\mathbf{x})$ defined by **coef**. Given the coefficients computed by the function **pnl_basis_fit_ls**, this function returns $\sum_{k=0}^n \text{coef}_k P_k(\mathbf{x})$.
- double **pnl_basis_eval_D_vect**(const **PnlBasis** *basis, const **PnlVect** *coef, const **PnlVect** *x, int i)
Description Compute the derivative with respect to **x_i** of the linear combination of $P_k(\mathbf{x})$ defined by **coef**. Given the coefficients computed by the function **pnl_basis_fit_ls**, this function returns $\partial_{x_i} \sum_{k=0}^n \text{coef}_k P_k(\mathbf{x})$. The index **i** may vary between 0 and **P->nb_variates - 1**.
- double **pnl_basis_eval_D2_vect**(const **PnlBasis** *basis, const **PnlVect** *coef, const **PnlVect** *x, int i, int j)
Description Compute the derivative with respect to **x_i** of the linear combination of $P_k(\mathbf{x})$ defined by **coef**. Given the coefficients computed by the function **pnl_basis_fit_ls**, this function returns $\partial_{x_i} \partial_{x_j} \sum_{k=0}^n \text{coef}_k P_k(\mathbf{x})$. The indices **i** and **j** may vary between 0 and **P->nb_variates - 1**.

The following functions are provided for compatibility purposes but are marked as deprecated. Use the functions with the **_vect** extension.

- double **pnl_basis_ik**(const **PnlBasis** *b, const double *x, int i, int k)
Description Same as function **pnl_basis_ik_vect** but takes a C array as the point of evaluation.
- double **pnl_basis_i**(**PnlBasis** *b, double *x, int i)
Description Same as function **pnl_basis_i_vect** but takes a C array as the point of evaluation.
- double **pnl_basis_i_D**(const **PnlBasis** *b, const double *x, int i, int j)
Description Same as function **pnl_basis_i_D_vect** but takes a C array as the point of evaluation.
- double **pnl_basis_i_D2**(const **PnlBasis** *b, const double *x, int i, int j1, int j2)
Description Same as function **pnl_basis_i_D2_vect** but takes a C array as the point of evaluation.
- double **pnl_basis_eval**(**PnlBasis** *P, **PnlVect** *coef, double *x)
Description Same as function **pnl_basis_eval_vect** but takes a C array as the point of evaluation.

- double **pnl_basis_eval_D**(**PnlBasis** *P, **PnlVect** *coef, double *x, int i)
Description Same as function `pnl_basis_eval_D_vect` but takes a C array as the point of evaluation.
- double **pnl_basis_eval_D2**(**PnlBasis** *P, **PnlVect** *coef, double *x, int i, int j)
Description Same as function `pnl_basis_eval_D2_vect` but takes a C array as the point of evaluation.
- void **pnl_basis_eval_derivs**(**PnlBasis** *P, **PnlVect** *coef, double *x, double *fx, **PnlVect** *Dfx, **PnlMat** *D2fx)
Description Same as function `pnl_basis_eval_derivs_vect` but takes a C array as the point of evaluation.

8 Numerical integration

8.1 Overview

To use these functionalities, you should include `pnl/pnl_integration.h`.

Numerical integration methods are designed to numerically evaluate the integral over a finite or non finite interval (resp. over a square) of real valued functions defined on \mathbb{R} (resp. on \mathbb{R}^2).

```
typedef struct {
    double (*function) (double x, void *params);
    void *params;
} PnlFunc;

typedef struct {
    double (*function) (double x, double y, void *params);
    void *params;
} PnlFunc2D;
```

We provide the following two macros to evaluate a **PnlFunc** or **PnlFunc2D** at a given point

```
#define PNL_EVAL_FUNC(F, x) ((*((F)->function))(x, (F)->params))
#define PNL_EVAL_FUNC2D(F, x, y) ((*((F)->function))(x, y, (F)->params))
```

8.2 Functions

- double **pnl_integration**(**PnlFunc** *F, double x0, double x1, int n, char *meth)
Description Evaluate $\int_{x_0}^{x_1} F$ using `n` discretization steps. The method used to discretize the integral is defined by `meth` which can be "rect" (rectangle rule), "trap" (trapezoidal rule), "simpson" (Simpson's rule).
- double **pnl_integration_2d**(**PnlFunc2D** *F, double x0, double x1, double y0, double y1, int nx, int ny, char *meth)
Description Evaluate $\int_{[x_0, x_1] \times [y_0, y_1]} F$ using `nx` (resp. `ny`) discretization steps for `[x0, x1]` (resp. `[y0, y1]`). The method used to discretize the integral is defined by `meth` which can be "rect" (rectangle rule), "trap" (trapezoidal rule), "simpson" (Simpson's rule).

- int **pnl_integration_qng**(**PnlFunc** *F, double x0, double x1, double epsabs, double epsrel, double *result, double *abserr, int *neval)
Description Evaluate $\int_{x_0}^{x_1} F$ with an absolute error less than **epsabs** and a relative error less than **epsrel**. The value of the integral is stored in **result**, while the variables **abserr** and **neval** respectively contain the absolute error and the number of function evaluations. This function returns **OK** if the required accuracy has been reached and **FAIL** otherwise. This function uses a non-adaptive Gauss Konrod procedure (qng routine from *QuadPack*).
- int **pnl_integration_GK**(**PnlFunc** *F, double x0, double x1, double epsabs, double epsrel, double *result, double *abserr, int *neval)
Description This function is a synonymous of **pnl_integration_qng** and is only available for backward compatibility. It is deprecated, please use **pnl_integration_qng** instead.
- int **pnl_integration_qng_2d**(**PnlFunc2D** *F, double x0, double x1, double y0, double y1, double epsabs, double epsrel, double *result, double *abserr, int *neval)
Description Evaluate $\int_{[x_0, x_1] \times [y_0, y_1]} F$ with an absolute error less than **epsabs** and a relative error less than **epsrel**. The value of the integral is stored in **result**, while the variables **abserr** and **neval** respectively contain the absolute error and the number of function evaluations. This function returns **OK** if the required accuracy has been reached and **FAIL** otherwise.
- int **pnl_integration_GK2D**(**PnlFunc** *F, double x0, double x1, double epsabs, double epsrel, double *result, double *abserr, int *neval)
Description This function is a synonymous of **pnl_integration_qng_2d** and is only available for backward compatibility. It is deprecated, please use **pnl_integration_qng_2d** instead.
- int **pnl_integration_qag**(**PnlFunc** *F, double x0, double x1, double epsabs, int limit, double epsrel, double *result, double *abserr, int *neval)
Description Evaluate $\int_{x_0}^{x_1} F$ with an absolute error less than **epsabs** and a relative error less than **epsrel**. **x0** and **x1** can be non finite (i.e. **PNL_NEGINF** or **PNL_POSINF**). The value of the integral is stored in **result**, while the variables **abserr** and **neval** respectively contain the absolute error and the number of iterations. **limit** is the maximum number of subdivisions of the interval (**x0**,**x1**) used during the integration. If on input, **limit** 0, then 750 subdivisions are used. This function returns **OK** if the required accuracy has been reached and **FAIL** otherwise. This function uses some adaptive procedures (qags and qagi routines from *QuadPack*). This function is able to handle functions **F** with integrable singularities on the interval [**x0**,**x1**].
- int **pnl_integration_qagp**(**PnlFunc** *F, double x0, double x1, const PnlVect *singularities, double epsabs, int limit, double epsrel, double *result, double *abserr, int *neval)
Description Evaluate $\int_{x_0}^{x_1} F$ for a function **F** with known singularities listed in **singularities**. **singularities** must be a sorted vector which does not contain **x0** nor **x1**. **x0** and **x1** must be finite. The value of the integral is stored in **result**, while the variables **abserr** and **neval** respectively contain the absolute error and the number of iterations. **limit** is the maximum number of subdivisions of the interval (**x0**,**x1**) used during the integration. If on input, **limit** = 0, then 750 subdivisions are used. This

function returns **OK** if the required accuracy has been reached and **FAIL** otherwise. This function uses some adaptive procedures (qagp routine from *QuadPack*). This function is able to handle functions **F** with integrable singularities on the interval **[x0,x1]**.

9 Fast Fourier Transform

9.1 Overview

This toolbox uses C version of the Fortran FFTPack library available on <http://www.netlib.org/fftpack>.

The forward Fourier transform of a vector c is defined by

$$z_j = \sum_{k=0}^{N-1} c_k e^{-ijk2\pi/N}, \quad j = 0, \dots, N-1$$

The inverse Fourier transform enables to recover c from z and is defined by

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} z_j e^{ijk2\pi/N}, \quad j = 0, \dots, N-1$$

Note that the inverse Fourier transform is scaled by $\frac{1}{N}$, such that the inverse Fourier transform applies to the Fourier transform just yields the original vector.

The coefficients of the Fourier transform of a real function satisfy the following relation

$$z_k = \overline{z_{N-k}}, \quad (3)$$

where N is the number of discretization points.

A few remarks on the FFT of real functions and its inverse transform:

- We only need half of the coefficients.
- When a value is known to be real, its imaginary part is not stored. So the imaginary part of the zero-frequency component is never stored as it is known to be zero.
- For a sequence of even length the imaginary part of the frequency $n/2$ is not stored either, since the symmetry (3) implies that this is purely real too.

FFTPack storage The functions use the fftpack storage convention for half-complex sequences. In this convention, the half-complex transform of a real sequence is stored with frequencies in increasing order, starting from zero, with the real and imaginary parts of each frequency in neighboring locations.

The storage scheme is best shown by some examples. The table below shows the output for an odd-length sequence, $n = 5$. The two columns give the correspondence between the 5 values in the half-complex sequence (stored in a PnlVect V) and the values (PnlVectComplex C) that would be returned if the same real input sequence were passed to `pnl_fft` as a complex

sequence (with imaginary parts set to 0),

$$\begin{aligned}
C(0) &= V(0) + i0, \\
C(1) &= V(1) + iV(2), \\
C(2) &= V(3) + iV(4), \\
C(3) &= V(3) - iV(4) = \overline{C(2)}, \\
C(4) &= V(1) + iV(2) = \overline{C(1)}
\end{aligned} \tag{4}$$

The elements of index greater than $N/2$ of the complex array, as $C(3)$ and $C(4)$ are filled in using the symmetry condition.

The next table shows the output for an even-length sequence, $n = 6$. In the even case, there are two values which are purely real,

$$\begin{aligned}
C(0) &= V(0) + i0, \\
C(1) &= V(1) + iV(2), \\
C(2) &= V(3) + iV(4), \\
C(3) &= V(5) - i0 = \overline{C(0)}, \\
C(4) &= V(3) - iV(4) = \overline{C(2)}, \\
C(5) &= V(1) + iV(2) = \overline{C(1)}
\end{aligned} \tag{5}$$

9.2 Functions

To use the following functions, you should include `pnl/pnl_fft.h`.

9.2.1 Direct call functions

All FFT functions need some extra memory to perform their computations. This is automatically handled by all the functions but you can these repeatedly, for instance inside a Monte Carlo loop, you should allocate a workspace once and for all and use the same at every iteration. In this case, use the functions defined in Section 9.2.2.

- `int pnl_fft_inplace(PnlVectComplex *data)`
Description Compute the FFT of `data` in place. The original content of `data` is lost.
- `int pnl_ifft_inplace(PnlVectComplex *data)`
Description Compute the inverse FFT of `data` in place. The original content of `data` is lost.
- `int pnl_fft(const PnlVectComplex *in, PnlVectComplex *out)`
Description Compute the FFT of `in` and stores it into `out`.
- `int pnl_ifft(const PnlVectComplex *in, PnlVectComplex *out)`
Description Compute the inverse FFT of `in` and stores it into `out`.
- `int pnl_fft2(double *re, double *im, int n)`
Description Compute the FFT of the vector of length `n` whose real (resp. imaginary) parts are given by the arrays `re` (resp. `im`). The real and imaginary parts of the FFT are respectively stored in `re` and `im` on output.

- `int pnl_ifft2(double *re, double *im, int n)`
Description Compute the inverse FFT of the vector of length `n` whose real (resp. imaginary) parts are given by the arrays `re` (resp. `im`). The real and imaginary parts of the inverse FFT are respectively stored in `re` and `im` on output.
- `int pnl_real_fft(const PnlVect *in, PnlVectComplex *out)`
Description Compute the FFT of the real valued sequence `in` and stores it into `out`.
- `int pnl_real_ifft(const PnlVectComplex *in, PnlVect *out)`
Description Compute the inverse FFT of `in` and stores it into `out`. The vector `in` is supposed to be the FFT of a real valued vector.
- `int pnl_real_fft_inplace(double *data, int n)`
Description Compute the FFT of the real valued vector `data` of length `n`. The result is stored in `data` using the FFTPack storage described above, see 9.1.
- `int pnl_real_ifft_inplace(double *data, int n)`
Description Compute the inverse FFT of the vector `data` of length `n`. `data` is supposed to be the FFT coefficients a real valued sequence stored using the FFTPack storage. On output, `data` contains the inverse FFT.
- `int pnl_real_fft2(double *re, double *im, int n)`
Description Compute the FFT of the real vector `re` of length `n`. `im` is only used on output to store the imaginary part the FFT. The real part is stored into `re`
- `int pnl_real_ifft2(double *re, double *im, int n)`
Description Compute the inverse FFT of the vector `re + i * im` of length `n`, which is supposed to be the FFT of a real valued sequence. On exit, `im` is unused.
- `int pnl_fft2d_inplace(PnlMatComplex *data)`
Description Compute the 2D FFT of `data`. This function applies a 1D FFT to each row of the matrix and then a 1D FFT to each column of the modified matrix.
- `int pnl_ifft2d_inplace(PnlMatComplex *data)`
Description Compute the inverse 2D FFT of `data`. This function is the inverse of the function `pnl_fft2d_inplace`.
- `int pnl_fft2d(const PnlMatComplex *in, PnlMatComplex *out)`
Description Compute the 2D FFT of `in` and stores it into `out`.
- `int pnl_ifft2d(const PnlMatComplex *in, PnlMatComplex *out)`
Description Compute the inverse 2D FFT of `in` and stores it into `out`.
- `int pnl_real_fft2d(const PnlMat *in, PnlMatComplex *out)`
Description Compute the 2D FFT of the real matrix `in` and stores it into `out`.
- `int pnl_real_ifft2d(const PnlMatComplex *in, PnlMatComplex *out)`
Description Compute the inverse 2D FFT of the complex matrix `in` which is known to be the forward 2D FFT a real matrix. The result is stored into `out`. Note that this function modifies the input matrix `in`.

9.2.2 Function with workspace

- `double* pnl_fft_alloc_workspace(const char *func, int n)`
Description Return an allocated workspace array ready to use for function `func` and input data of size `n`
- `int pnl_fft_inplace_with_workspace(PnlVectComplex *data, double *wspace)`
Description Compute the FFT of `data` in place. The original content of `data` is lost.
- `int pnl_ifft_inplace_with_workspace(PnlVectComplex *data, double *wspace)`
Description Compute the inverse FFT of `data` in place. The original content of `data` is lost.
- `int pnl_real_fft_inplace_with_workspace(double *data, double *wspace, int n)`
Description Compute the inverse FFT of `in` and stores it into `out`. The vector `in` is supposed to be the FFT of a real valued vector.
- `int pnl_real_ifft_inplace_with_workspace(double *data, double *wspace, int n)`
Description Compute the inverse FFT of the vector `data` of length `n`. `data` is supposed to be the FFT coefficients a real valued sequence stored using the FFTPack storage. On output, `data` contains the inverse FFT.
- `int pnl_real_fft_with_workspace(const PnlVect *in, PnlVectComplex *out, double *wspace)`
Description Compute the FFT of the real valued sequence `in` and stores it into `out`.
- `int pnl_real_ifft_with_workspace(const PnlVectComplex *in, PnlVect *out, double *wspace)`
Description Compute the inverse FFT of `in` and stores it into `out`. The vector `in` is supposed to be the FFT of a real valued vector.

10 Inverse Laplace Transform

For a real valued function f such that $t \mapsto f(t)e^{-\sigma_c t}$ is integrable over \mathbb{R}^+ , we can define its Laplace transform

$$\hat{f}(\lambda) = \int_0^{\infty} f(t) e^{-\lambda t} dt \quad \text{for } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}(\lambda) \geq \sigma_c.$$

To use the following functions, you should include `pnl/pnl_laplace.h`.

```
typedef struct
{
    dcomplex (*F) (dcomplex x, void *params);
    void *params;
} PnlCmplxFunc;
```

- `double pnl_ilap_euler(PnlCmplxFunc *fhat, double t, int N, int M)`
Description Compute $f(t)$ where f is given by its Laplace transform `fhat` by numerically inverting the Laplace transform using Euler's summation. The values `N = M = 15` usually give a very good accuracy. For more details on the accuracy of the method.

- double **pnl_ilap_cdf_euler**(**PnlCmplxFunc** *fhat, double t, double h, int N, int M)
Description Compute the cumulative distribution function $F(\mathbf{t})$ where $F(x) = \int_0^x f(t)dt$ and f is a density function with values on the positive real line given by its Laplace transform **fhat**. The computation is carried out by numerical inversion of the Laplace transform using Euler's summation. The values $N = M = 15$ usually give a very good accuracy. The parameter **h** is the discretization step, the algorithm is very sensitive to the choice of **h**.
- double **pnl_ilap_fft**(**PnlVect** *res, **PnlCmplxFunc** *fhat, double T, double eps)
Description Compute $f(t)$ for $t \in [h, T]$ on a regular grid and stores the values in **res**, where $h = T/\text{size}(\text{res})$. The function f is defined by its Laplace transform **fhat**, which is numerically inverted using a Fast Fourier Transform algorithm. The size of **res** is related to the choice of the relative precision **eps** required on the value of $f(t)$ for all $t \leq T$.
- double **pnl_ilap_gs**(**PnlFunc** *fhat, double t, int n)
Description Compute $f(\mathbf{t})$ where f is given by its Laplace transform **fhat** by numerically inverting the Laplace transform using a weighted combination of different Gaver Stehfest's algorithms. Note that this function does not need the complex valued Laplace transform but only the real valued one. **n** is the number of terms used in the weighted combination.
- double **pnl_ilap_gs_basic**(**PnlFunc** *fhat, double t, int n)
Description Compute $f(\mathbf{t})$ where f is given by its Laplace transform **fhat** by numerically inverting the Laplace transform using Gaver Stehfest's method. Note that this function does not need the complex valued Laplace transform but only the real valued one. **n** is the number of iterations of the algorithm. **Note :** This function is provided for test purposes only. The function **pnl_ilap_gs** gives far more accurate results.

11 Ordinary differential equations

11.1 Overview

To use these functionalities, you should include **pnl/pnl_integration.h**.

These functions are designed for numerically solving n -dimensional first order ordinary differential equation of the general form

$$\frac{dy_i}{dt}(t) = f_i(t, y_1(t), \dots, y_n(t))$$

The system of equations is defined by the following structure

```
typedef struct
{
    void (*F) (int neqn, double t, const double *y, double *yp, void *
        params);
    int neqn;
    void *params;
} PnlODEFunc ;
```

- **int neqn**
Description Number of equations
- **void * params**
Description An untyped structure used to pass extra arguments to the function **f** defining the system
- **void (* F)(int neqn, double t, const double *y, double *yp, void *params)**
Description After calling the function, **yp** should be defined as follows **yp_i = f_i(neqn, t, y, params)**. **y** and **yp** should be both of size **neqn**

We provide the following macro to evaluate a **PnODEFunc** at a given point

```
#define PNL_EVAL_ODEFUNC(Fstruct, t, y, yp) \
    (*((Fstruct)->F))((Fstruct)->neqn, t, y, yp, (Fstruct)->params)
```

11.2 Functions

- **int pnl_ode_rkf45(PnODEFunc *f, double *y, double t, double t_out, double relerr, double abserr, int *flag)**
Description This function computes the solution of the system defined by the **PnODEFunc** **f** at the point **t_out**. On input, **(t,y)** should be the initial condition, **abserr,relerr** are the maximum absolute and relative errors for local error tests (at each step, **abs(local error)** should be less than **relerr * abs(y) + abserr**). Note that if **abserr = 0** or **relerr = 0** on input, an optimal value for these variables is computed inside the function. The function returns an error **OK** or **FAIL**. In case of an **OK** code, the **y** contains the solution computed at **t_out**, in case of a **FAIL** code, **flag** should be examined to determine the reason of the error. Here are the different possible values for **flag**
 - **flag = 2** : integration reached **t_out**, it indicates successful return and is the normal mode for continuing integration.
 - **flag = 3** : integration was not completed because relative error tolerance was too small. **relerr** has been increased appropriately for continuing.
 - **flag = 4** : integration was not completed because more than 3000 derivative evaluations were needed. this is approximately 500 steps.
 - **flag = 5** : integration was not completed because solution vanished making a pure relative error test impossible. must use non-zero **abserr** to continue. using the one-step integration mode for one step is a good way to proceed.
 - **flag = 6** : integration was not completed because requested accuracy could not be achieved using smallest allowable stepsize. user must increase the error tolerance before continued integration can be attempted.
 - **flag = 7** : it is likely that **rkf45** is inefficient for solving this problem. too much output is restricting the natural stepsize choice. use the one-step integrator mode. see **pnl_ode_rkf45_step**.
 - **flag = 8** : invalid input parameters this indicator occurs if any of the following is satisfied - **neqn <= 0**, **t=tout**, **relerr** or **abserr <= 0**.

- `int pnl_ode_rkf45_step(PnlODEFunc *f, double *y, double *t, double t_out, double *relerr, double abserr, double *work, int *iwork, int *flag)`

Description Same as `pnl_ode_rkf45` but it only computes one step of integration in the direction of `t_out`. `work` and `iwork` are working arrays of size $3 + 6 * neqn$ and 5 respectively and should remain untouched between successive calls to the function. On output `t` holds the point at which integration stopped and `y` the value of the solution at that point.

12 Optimization

To use the functions described in this section, you should include `pnl/pnl_optim.h`.

12.1 Linear constrained optimization (linear programming)

12.1.1 Overview

Consider the minimization problem

$$\begin{aligned} \min_x \quad & C^T x \\ \text{s.t.} \quad & A_{\text{ineq}} x \leq B_{\text{ineq}} \\ & A_{\text{eq}} x = B_{\text{eq}} \\ & x_{\min} \leq x \leq x_{\max} \end{aligned}$$

12.1.2 Functions

To solve such a linear problem, we provide a wrapper to the *LPSolve* library (<http://lpsolve.sourceforge.net>).

- `int pnl_optim_linprog(const PnlVect *C, const PnlMat *A_ineq, const PnlVect *B_ineq, const PnlMat *A_eq, const PnlVect *B_eq, const PnlVect *x_min, const PnlVect *x_max, int debug, PnlVect *xopt, double *fobj_opt)`

Description This function has the following arguments:

- `C` The coefficients of the linear objective function.
- `A_ineq` The l.h.s matrix of the inequality constraints. Can be NULL.
- `B_ineq` The r.h.s vector of the inequality constraints. The length of `B_ineq` must match the number of rows of `A_ineq`.
- `A_eq` The l.h.s matrix of the equality constraints. Can be NULL.
- `B_eq` The r.h.s vector of the equality constraints. The length of `B_eq` must match the number of rows of `A_ineq`.
- `x_min` The lower bound on `x`. If NULL, it means all the components of `x` must be non negative.
- `x_max` The upper bound on `x`. If NULL, it means `+Infinity` for all the components.
- `debug` TRUE or FALSE. If TRUE some debugging details are printed.
- `xopt` The argmin of the problem.

- `fobj_opt` The value of the objective function at the optimum `xopt`

The function returns OK or FAIL.

- `int pnl_optim_linprog_sp(const PnlSpMat *C, const PnlSpMat *A_ineq, const PnlVect *B_ineq, const PnlSpMat *A_eq, const PnlVect *B_eq, const PnlVectInt *index_min, const PnlVect *x_min, const PnlVectInt *index_max, const PnlVect *x_max, int debug, PnlVect *xopt, double *fobj_opt)`

Description This function has the following arguments:

- `C` The coefficients of the linear objective function, given as a sparse matrix with a single column.
- `A_ineq` The l.h.s matrix of the inequality constraints. Can be NULL.
- `B_ineq` The r.h.s vector of the inequality constraints. The length of `B_ineq` must match the number of rows of `A_ineq`.
- `A_eq` The l.h.s matrix of the equality constraints. Can be NULL.
- `B_eq` The r.h.s vector of the equality constraints. The length of `B_eq` must match the number of rows of `A_ineq`.
- `index_min` The indices of the variables with a lower bound constraint. The corresponding lower bound is given in `x_min`
- `x_min` The lower bound on `x`. If NULL, it means all the components of `x` must be non negative. Can be NULL. For non specified variables, the default lower bound is 0.
- `index_max` The indices of the variables with an upper bound constraint. The corresponding lower bound is given in `x_max`. Can be NULL.
- `x_max` The upper bound on `x`. If NULL, it means `+Infinity` for all the components. For non specified variables, the default upper bound is `+Infinity`.
- `debug` TRUE or FALSE. If TRUE some debugging details are printed.
- `xopt` The `argmin` of the problem.
- `fobj_opt` The value of the objective function at the optimum `xopt`

The function returns OK or FAIL.

12.2 Nonlinear constrained optimization

12.2.1 Overview

A standard Constrained Nonlinear Optimization problem can be written as:

$$(O) \begin{cases} \min f(x) \\ c^I(x) \geq 0 \\ c^E(x) = 0 \end{cases}$$

where the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective function, $c^I : \mathbb{R}^n \rightarrow \mathbb{R}^{m_I}$ are the inequality constraints and $c^E : \mathbb{R}^n \rightarrow \mathbb{R}^{m_E}$ are the equality constraints. These functions are supposed to be smooth.

In general, the inequality constraints are of the form $c^I(x) = (g(x), x - l, u - x)$. The vector l and u are the lower and upper bounds on the variables x and $g(x)$ and the non linear inequality constraints.

Under some conditions, if $x \in \mathbb{R}^n$ is a solution of problem (O), then there exist a vector $\lambda = (\lambda^I, \lambda^E) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E}$, such that the well known Karush-Kuhn-Tucker (KKT) optimality conditions are satisfied:

$$(P) \quad \left\{ \begin{array}{l} \nabla \ell(x, \lambda^I, \lambda^E) = \nabla f(x) - \nabla c^I(x) \lambda^I - \nabla c^E(x) \lambda^E = 0 \\ c^E(x) = 0 \\ c^I(x) \geq 0 \\ \lambda^I \geq 0 \\ c_i^I(x) \lambda_i^I = 0, \quad i = 1 \dots m_I \end{array} \right.$$

ℓ is known as the Lagrangian of the problem (O), λ^I and λ^E as the dual variables while x is the primal variable.

12.2.2 Functions

To solve an inequality constrained optimization problem, ie $m_E = 0$, we provide the following function.

- `int pnl_optim_intpoints_bfgs_solve(PnlRnFuncR*func, PnlRnFuncRm*grad_func, PnlRnFuncRm*nl_constraints, PnlVect *lower_bounds, PnlVect *upper_bounds, PnlVect *x_input, double tolerance, int iter_max, int print_inner_steps, PnlVect *output)`

Description This function has the following arguments:

- **func** is the function to minimize f .
- **grad** is the gradient of f . If this gradient is not available, then enter **grad=NULL**. In this case, finite difference will be used to estimate the gradient.
- **nl_constraints** is the function $g(x)$, ie the non linear inequality constraints.
- **lower_bounds** are the lower bounds on x . Can be NULL if there is no lower bound.
- **upper_bounds** are the upper bounds on x . Can be NULL if there is no upper bound.
- **x_input** is the initial point where the algorithm starts.
- **tolerance** is the precision required in solving (P).
- **iter_max** is the maximum number of iterations in the algorithm.
- **print_algo_steps** is a flag to decide to print information.
- **x_output** is the point where the algorithm stops.

The algorithm returns an *int*, its value depends on the output status of the algorithm. We have 4 cases:

- 0: Failure: Initial point is not strictly feasible.
- 1: Step is too small, we stop the algorithm.
- 2: Maximum number of iterations reached.

- 3: A solution has been found up to the required accuracy.

The last case is equivalent to the two inequalities:

$$\begin{aligned} \|\nabla \ell(x, \lambda^I)\|_\infty &= \|\nabla f(x) - \nabla c^I(x) \lambda^I\|_\infty < \text{tolerance} \\ \|\nabla c^I(x) \lambda^I\|_\infty &< \text{tolerance} \end{aligned}$$

where $c^I(x) .* \lambda^I$ where ‘.’*’ denotes the term by term multiplication.

The first inequality is known as the optimality condition, the second one as the complementarity condition.

Remarks Our implementation requires the initial point x_0 to be strictly feasible, ie: $c(x_0) > 0$. The algorithm tries to find a pair (x, λ) solving the Equations (P), but this does not guarantee that x is a global minimum of f on the set $\{c(x) \geq 0\}$.

13 Root finding

13.1 Overview

To provide a uniformed framework to root finding functions, we use several structures for storing different kind of functions. The pointer `params` is used to store the extra parameters. These new types come with dedicated macros starting in `PNL_EVAL` to evaluate the function and their Jacobian.

```
/*
 * f: R --> R
 * The function pointer returns f(x)
 */
typedef struct {
    double (*F) (double x, void *params);
    void *params;
} PnlFunc ;
#define PNL_EVAL_FUNC(Fstruct, x) ((*((Fstruct)->F))(x, (Fstruct)->
    params)

/*
 * f: R^2 --> R
 * The function pointer returns f(x)
 */
typedef struct {
    double (*F) (double x, double y, void *params);
    void *params;
} PnlFunc2D ;
#define PNL_EVAL_FUNC2D(Fstruct, x, y) ((*((Fstruct)->F))(x, y, (
    Fstruct)->params)

/*
 * f: R --> R
 * The function pointer computes f(x) and Df(x) and stores them in fx
 * and dfx respectively
```

```

*
typedef struct {
    void (*F) (double x, double *fx, double *dfx, void *params);
    void *params;
} PnlFuncDFunc ;
#define PNL_EVAL_FUNC_FDF(Fstruct, x, fx, dfx) ((*((Fstruct)->F))(x, fx
    , dfx, (Fstruct)->params)

/*
*  $f: R^n \rightarrow R$ 
* The function pointer returns  $f(x)$ 
*
typedef struct {
    double (*F) (const PnlVect *x, void *params);
    void *params;
} PnlRnFuncR ;
#define PNL_EVAL_RNFUNCR(Fstruct, x) ((*((Fstruct)->F))(x, (Fstruct)->
    params)

/*
*  $f: R^n \rightarrow R^m$ 
* The function pointer computes the vector  $f(x)$  and stores it in
* fx (vector of size m)
*
typedef struct {
    void (*F) (const PnlVect *x, PnlVect *fx, void *params);
    void *params;
} PnlRnFuncRm ;
#define PNL_EVAL_RNFUNCRM(Fstruct, x, fx) ((*((Fstruct)->F))(x, fx, (
    Fstruct)->params)

/*
* Synonymous of PnlRnFuncRm for  $f: R^n \rightarrow R^n$ 
*
typedef PnlRnFuncRm PnlRnFuncRn;
#define PNL_EVAL_RNFUNCRN PNL_EVAL_RNFUNCRM

/*
*  $f: R^n \rightarrow R^m$ 
* The function pointer computes the vector  $f(x)$  and stores it in fx
* (vector of size m)
* The Dfunction pointer computes the matrix  $Df(x)$  and stores it in
    dfx
* (matrix of size m x n)
*
typedef struct {
    void (*F) (const PnlVect *x, PnlVect *fx, void *params);
    void (*DF) (const PnlVect *x, PnlMat *dfx, void *params);
    void (*FDF) (const PnlVect *x, PnlVect *fx, PnlMat *dfx, void *
        params);
    void *params;
} PnlRnFuncRmDFunc ;
#define PNL_EVAL_RNFUNCRM_DF(Fstruct, x, dfx) \

```

```

        (*(Fstruct)->Dfunction))(x, dfx, (Fstruct)->params)
#define PNL_EVAL_RNFUNCRM_FDF(Fstruct, x, fx, dfx) \
        (*(Fstruct)->F))(x, fx, dfx, (Fstruct)->params)
#define PNL_EVAL_RNFUNCRM_F_DF(Fstruct, x, fx, dfx) \
        if ( (Fstruct)->FDF != NULL ) \
        { \
            PNL_EVAL_RNFUNCRM_FDF (Fstruct, x, fx, dfx); \
        } \
        else \
        { \
            PNL_EVAL_RNFUNCRM (Fstruct, x, fx); \
            PNL_EVAL_RNFUNCRM_DF (Fstruct, x, dfx); \
        }
/*
 * Synonymous of PnlRnFuncRmDFunc for  $f:R^n \rightarrow R^m$ 
 *
typedef PnlRnFuncRmDFunc PnlRnFuncRnDFunc;
#define PNL_EVAL_RNFUNCRM_DF PNL_EVAL_RNFUNCRM_DF
#define PNL_EVAL_RNFUNCRM_FDF PNL_EVAL_RNFUNCRM_FDF
#define PNL_EVAL_RNFUNCRM_F_DF PNL_EVAL_RNFUNCRM_F_DF

```

13.2 Functions

To use the following functions, you should include `pnl/pnl_root.h`.

Real valued functions of a real argument

- double **pnl_root_brent**(**PnlFunc** *F, double x1, double x2, double *tol)
Description Find the root of F between x1 and x2 with an accuracy of order tol. On exit tol is an upper bound of the error.
- int **pnl_root_newton_bisection**(**PnlFuncDFunc** *Func, double x_min, double x_max, double tol, int N_Max, double *res)
Description Find the root of F between x1 and x2 with an accuracy of order tol and a maximum of N_max iterations. On exit, the root is stored in res. Note that the function Func must also compute the first derivative of the function. This function relies on combining Newton's approach with a bisection technique.
- int **pnl_root_newton**(**PnlFuncDFunc** *Func, double x0, double x_eps, double fx_eps, int max_iter, double *res)
Description Find the root of f starting from x0 using Newton's method with descent direction given by the inverse of the derivative, ie. $d_k = f(x_k)/f'(x_k)$. Armijo's line search is used to make sure $|f|$ decreases along the iterations. $\alpha_k = \max\{\gamma^j ; j \geq 0\}$ such that

$$|f(x_k + \alpha_k d_k)| \leq |f(x_k)|(1 - \omega \alpha_k).$$

In this implementation, $\omega = 10^{-4}$ and $\gamma = 1/2$. The algorithm stops when one of the three following conditions is met:

- the maximum number of iterations **max_iter** is reached;

- the last improvement over \mathbf{x} is smaller than $\mathbf{x} * \mathbf{x_eps}$;
- at the current position $|\mathbf{f}(\mathbf{x})| < \mathbf{fx_eps}$

On exit, the root is stored in **res**.

- `int pnl_root_bisection(PnlFunc *Func, double xmin, double xmax, double epsrel, double epsabs, int N_max, double *res)`
Description Find the root of \mathbf{F} between $\mathbf{x1}$ and $\mathbf{x2}$ with the accuracy $|\mathbf{x2} - \mathbf{x1}| < \mathbf{epsrel} * \mathbf{x1} + \mathbf{epsabs}$ or with the maximum number of iterations $\mathbf{N_max}$. On exit, $\mathbf{res} = (\mathbf{x2} + \mathbf{x1}) / 2$.

Vector valued functions with several arguments

- `int pnl_multiroot_newton(PnlRnFuncRnDFunc *func, const PnlVect *x0, double x_eps, double fx_eps, int max_iter, int verbose, PnlVect *res)`
Description Find the root of \mathbf{func} starting from $\mathbf{x0}$ using Newton's method with descent direction given by the inverse of the Jacobian matrix, ie. $d_k = (\nabla f(x_k))^{-1} f(x_k)$. Armijo's line search is used to make sure $|\mathbf{f}|$ decreases along the iterations. $\alpha_k = \max\{\gamma^j ; j \geq 0\}$ such that

$$|f(x_k + \alpha_k d_k)| \leq |f(x_k)|(1 - \omega \alpha_k).$$

In this implementation, $\omega = 10^{-4}$ and $\gamma = 1/2$. The algorithm stops when one of the three following conditions is met:

- the maximum number of iterations **max_iter** is reached;
- the norm of the last improvement over \mathbf{x} is smaller than $|\mathbf{x}| * \mathbf{x_eps}$;
- at the current position $|\mathbf{f}(\mathbf{x})| < \mathbf{fx_eps}$

On exit, the root is stored in **res**. Note that the function \mathbf{F} must also compute the first derivative of the function. When defining **Func**, you must either define **Func->F** and **Func->DF** or **Func->FDF**.

We provide two wrappers for calling minpack routines.

- `int pnl_root_fsolve(PnlRnFuncRnDFunc *f, PnlVect *x, PnlVect *fx, double xtol, int maxfev, int *nfev, PnlVect *scale, int error_msg)`
Description Compute the root of a function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. Note that the number of components of \mathbf{f} must be equal to the number of variates of \mathbf{f} . This function returns OK or FAIL if something went wrong.
Parameters

- \mathbf{f} is a pointer to a **PnlRnFuncRnDFunc** used to store the function whose root is to be found. \mathbf{f} can also store the Jacobian of the function, if not it is computed using finite differences (see the file `examples/minpack_test.c` for a usage example). $\mathbf{f->FDF}$ can be NULL because it is not used in this function.
- \mathbf{x} contains on input the starting point of the search and an approximation of the root of \mathbf{f} on output,
- **xtol** is the precision required on \mathbf{x} , if set to 0 a default value is used.

- `maxfev` is the maximum number of evaluations of the function `f` before the algorithm returns, if set to 0, a coherent number is determined internally.
 - `nfev` contains on output the number of evaluations of `f` during the algorithm,
 - `scale` is a vector used to rescale `x` in a way that each coordinate of the solution is approximately of order 1 after rescaling. If on input `scale=NULL`, a scaling vector is computed internally by the algorithm.
 - `error_msg` is a boolean (TRUE or FALSE) to specify if an error message should be printed when the algorithm stops before having converged.
 - On output, `fx` contains `f(x)`.
- `int pnl_root_fsolve_lsq(PnlRnFuncRmDFunc *f, PnlVect *x, int m, PnlVect *fx, double xtol, double ftol, double gtol, int maxfev, int *nfev, PnlVect *scale, int error_msg)`

Description Compute the root of $x \in \mathbb{R}^n \mapsto \sum_{i=1}^m f_i(x)^2$, note that there is no reason why `m` should be equal to `n`.

Parameters

- `f` is a pointer to a `PnlRnFuncRmDFunc` used to store the function whose root is to be found. `f` can also store the Jacobian of the function, if not it is computed using finite differences (see the file `examples/minpack_test.c` for a usage example). `f->FDF` can be NULL because it is not used in this function.
- `x` contains on input the starting point of the search and an approximation of the root of `f` on output,
- `m` is the number of components of `f`,
- `xtol` is the precision required on `x`, if set to 0 a default value is used.
- `ftol` is the precision required on `f`, if set to 0 a default value is used.
- `gtol` is the precision required on the Jacobian of `f`, if set to 0 a default value is used.
- `maxfev` is the maximum number of evaluations of the function `f` before the algorithm returns, if set to 0, a coherent number is determined internally.
- `nfev` contains on output the number of evaluations of `f` during the algorithm,
- `scale` is a vector used to rescale `x` in a way that each coordinate of the solution is approximately of order 1 after rescaling. If on input `scale=NULL`, a scaling vector is computed internally by the algorithm.
- `error_msg` is a boolean (TRUE or FALSE) to specify if an error message should be printed when the algorithm stops before having converged.
- On output, `fx` contains `f(x)`.

14 Special functions

The special function approximations are defined in the header `pnl/pnl_specfun.h`.

Most of these functions rely on the *Cephes* library which uses its own error mechanism which can be activated or deactivated using the two following functions

- void **pnl_deactivate_mtherr()**
[Description](#) Deactivate Cephes error mechanism
- void **pnl_activate_mtherr()**
[Description](#) Activate Cephes error mechanism

14.1 Real Bessel functions

- double **pnl_bessel_i**(double v, double x)
[Description](#) Modified Bessel function of the first kind of order v.
- double **pnl_bessel_i_scaled**(double v, double x)
[Description](#) Modified Bessel function of the first kind of order v divided by $e^{|x|}$.
- double **pnl_bessel_rati**(double v, double x)
[Description](#) Ratio of modified Bessel functions of the first kind : $I_{v+1}(x)/I_v(x)$.
- double **pnl_bessel_j**(double v, double x)
[Description](#) Bessel function of the first kind of order v.
- double **pnl_bessel_j_scaled**(double v, double x)
[Description](#) Bessel function of the first kind of order v. Same function as `pnl_bessel_j`.
- double **pnl_bessel_y**(double v, double x)
[Description](#) Modified Bessel function of the second kind of order v.
- double **pnl_bessel_y_scaled**(double v, double x)
[Description](#) Modified Bessel function of the second kind of order v. Same function as `pnl_bessel_y`.
- double **pnl_bessel_k**(double v, double x)
[Description](#) Bessel function of the third kind of order v.
- double **pnl_bessel_k_scaled**(double v, double x)
[Description](#) Bessel function of the third kind of order v multiplied by e^x .
- dcomplex **pnl_bessel_h1**(double v, double x)
[Description](#) Hankel function of the first kind of order v.
- dcomplex **pnl_bessel_h1_scaled**(double v, double x)
[Description](#) Hankel function of the first kind of order v and divided by e^{Ix} .
- dcomplex **pnl_bessel_h2**(double v, double x)
[Description](#) Hankel function of the second kind of order v.
- dcomplex **pnl_bessel_h2_scaled**(double v, double x)
[Description](#) Hankel function of the second kind of order v and multiplied by e^{Ix} .

14.2 Complex Bessel functions

- dcomplex **pnl_complex_bessel_i**(double v, dcomplex z)
[Description](#) Complex Modified Bessel function of the first kind of order v.
- dcomplex **pnl_complex_bessel_i_scaled**(double v, dcomplex z)
[Description](#) Complex Modified Bessel function of the first kind of order v divided by $e^{|Creal(z)|}$.
- dcomplex **pnl_complex_bessel_rati**(double v, dcomplex x)
[Description](#) Ratio of complex modified Bessel functions of the first kind : $I_{v+1}(x)/I_v(x)$.
- dcomplex **pnl_complex_bessel_j**(double v, dcomplex z)
[Description](#) Complex Bessel function of the first kind of order v.
- dcomplex **pnl_complex_bessel_j_scaled**(double v, dcomplex z)
[Description](#) Complex Bessel function of the first kind of order v divided by $e^{|Cimag(z)|}$.
- dcomplex **pnl_complex_bessel_y**(double v, dcomplex z)
[Description](#) Complex Modified Bessel function of the second kind of order v.
- dcomplex **pnl_complex_bessel_y_scaled**(double v, dcomplex z)
[Description](#) Complex Modified Bessel function of the second kind of order v divided by $e^{|Cimag(z)|}$.
- dcomplex **pnl_complex_bessel_k**(double v, dcomplex z)
[Description](#) Complex Bessel function of the third kind of order v.
- dcomplex **pnl_complex_bessel_k_scaled**(double v, dcomplex z)
[Description](#) Complex Bessel function of the third kind of order v multiplied by e^z .
- dcomplex **pnl_complex_bessel_h1**(double v, dcomplex z)
[Description](#) Complex Hankel function of the first kind of order v.
- dcomplex **pnl_complex_bessel_h1_scaled**(double v, dcomplex z)
[Description](#) Complex Hankel function of the first kind of order v and divided by e^{Iz} .
- dcomplex **pnl_complex_bessel_h2**(double v, dcomplex z)
[Description](#) Complex Hankel function of the second kind of order v.
- dcomplex **pnl_complex_bessel_h2_scaled**(double v, dcomplex z)
[Description](#) Complex Hankel function of the second kind of order v and multiplied by e^{Iz} .

14.3 Error functions

- double **pnl_sf_erf**(double x)
[Description](#) Compute the error function $\frac{2}{\pi} \int_0^x e^{-t^2} dt$.
- dcomplex **pnl_sf_complex_erf**(dcomplex z)
[Description](#) Same as pnl_sf_erf for complex arguments.

- double **pnl_sf_erfc**(double x)
[Description](#) Compute the complementary error function $1 - \text{erf}(x)$.
- dcomplex **pnl_sf_complex_erfc**(dcomplex x)
[Description](#) Same as pnl_sf_erfc for complex arguments.
- double **pnl_sf_erfcx**(double x)
[Description](#) Compute the scaled complementary error function of x, defined by $e^{x^2} \text{erfc}(x)$.
- dcomplex **pnl_sf_complex_erfcx**(dcomplex z)
[Description](#) Same as pnl_sf_erfcx for complex arguments. Note that $\text{erfcx}(-i x) = w(x)$.
- dcomplex **pnl_sf_w**(dcomplex z)
[Description](#) Compute $e^{-z^2} \text{erfc}(-i z)$.
- double **pnl_sf_w_im**(double x)
[Description](#) Compute $2 \text{Dawson}(x) / \sqrt{\pi}$
- double **pnl_sf_erfi**(double x)
[Description](#) Compute $-i \text{erf}(i x)$
- dcomplex **pnl_sf_complex_erfi**(dcomplex z)
[Description](#) Same as pnl_sf_erfi for complex arguments.
- double **pnl_sf_dawson**(double x)
[Description](#) Compute $\sqrt{\pi}/2 e^{-x^2} \text{erfi}(x)$.
- dcomplex **pnl_sf_complex_dawson**(dcomplex z)
[Description](#) Same as pnl_sf_dawson for complex arguments.
- double **pnl_sf_log_erf**(double x)
[Description](#) Compute $\log \text{pnl_sf_erf}(x)$
- double **pnl_sf_log_erfc**(double x)
[Description](#) Compute $\log \text{pnl_sf_erfc}(x)$

14.4 Gamma functions

For $x > 0$, the Gamma Function is defined by

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du.$$

- double **pnl_sf_fact**(int n)
[Description](#) Computes factorial of n $\Gamma(n + 1)$.
- double **pnl_sf_gamma**(double x)
[Description](#) Computes $\Gamma(x)$, $x \geq 0$
- double **pnl_sf_log_gamma**(double x)
[Description](#) Computes $\log(\Gamma(x))$, $x \geq 0$

- int **pnl_sf_log_gamma_sgn**(double x, double *y, int *sgn)
[Description](#) Computes $y = \log(|\Gamma(x)|)$ for $x > 0$ **sgn** contains the sign of $\Gamma(x)$ (-1 or +1).
- double **pnl_sf_choose**(int n, int k)
[Description](#) Computes the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \leq k \leq n$ in double precision.

14.5 Digamma function

For $x > 0$, the digamma function ψ is defined as the logarithmic derivative of the Gamma function Γ

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The function ψ admits the following integral representation

$$\psi(x) = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-xu}}{1 - e^{-u}} \right) du.$$

- double **pnl_sf_psi**(double x)
[Description](#) Return $\psi(x)$.

14.6 Incomplete Gamma functions

For $a \in \mathbb{R}$ and $x > 0$, the Incomplete Gamma Function is defined by

$$\Gamma(a, x) = \int_x^\infty e^{-u} u^{a-1} du.$$

A relation similar to the one existing for the standard Gamma function holds

$$\Gamma(a, x) = \frac{-x^a e^{-x} + \Gamma(a+1, x)}{a}.$$

$$\begin{aligned} \Gamma(a) &= \int_0^\infty u^{a-1} e^{-u} du \\ P(a, x) &= \frac{\Gamma(a) - \Gamma(a, x)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^x u^{a-1} e^{-u} du \\ Q(a, x) &= 1 - P(a, x) = \frac{\Gamma(a, x)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_x^\infty e^{-u} u^{a-1} du. \end{aligned}$$

- double **pnl_sf_gamma_inc**(double a, double x)
[Description](#) Computes $\Gamma(a, x)$, $a \in \mathbb{R}, x \geq 0$
- void **pnl_sf_gamma_inc_P**(double a, double x)
[Description](#) Computes $P(a, x)$, $a > 0, x \geq 0$
- void **pnl_sf_gamma_inc_Q**(double a, double x)
[Description](#) Computes $Q(a, x)$, $a > 0, x \geq 0$

14.7 Exponential integrals

For $x > 0$ and $n \in \mathbb{N}$, the function E_n is defined by

$$E_n(x) = \int_1^\infty e^{-xu} u^{-n} du$$

This function is linked to the Incomplete Gamma function by

$$E_n(x) = \int_x^\infty e^{-xu} (xu)^{-n} x^{n-1} d(xu) = x^{n-1} \int_x^\infty e^{-t} t^{-n} dt = x^{n-1} \Gamma(1-n, x),$$

from which we can deduce

$$nE_{n+1}(x) = e^{-x} - xE_n(x).$$

For $n > 1$, the series expansion is given by

$$E_n(x) = x^{n-1} \Gamma(1-n) + \left[-\frac{1}{1-n} + \frac{x}{2-n} - \frac{x^2}{2(3-n)} + \frac{x^3}{6(4-n)} - \dots \right].$$

The asymptotic behaviour is given by

$$E_n(x) = \frac{e^{-x}}{x} \left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} + \dots \right].$$

The special case $n = 1$ gives

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u} du, \quad |\text{Arg}(x)| \geq \pi.$$

For any complex number x with positive real part, this can be written

$$E_1(x) = \int_1^\infty \frac{e^{-ux}}{u} du, \quad \Re(x) \geq 0.$$

By integrating the Taylor expansion of e^{-t}/t , and extracting the logarithmic singularity, we can derive the following series representation for $E_1(x)$,

$$E_1(x) = -\gamma - \ln x - \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k k!} \quad |\text{Arg}(x)| < \pi.$$

The function E_1 is linked to the exponential integral Ei

$$Ei(x) = \int_{-\infty}^x \frac{e^u}{u} du = - \int_{-x}^\infty \frac{e^{-u}}{u} du \quad \forall x \neq 0.$$

The above definition can be used for positive values of x , but the integral has to be understood in terms of its Cauchy principal value, due to the singularity of the integrand at zero.

$$Ei(-x) = -E_1(x), \quad \Re(x) \geq 0.$$

We deduce,

$$Ei(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{k k!}, \quad x > 0.$$

For $x \in \mathbb{R}$

$$\Gamma(0, x) = \begin{cases} -Ei(-x) - i\pi & x < 0, \\ -Ei(-x) & x > 0. \end{cases}$$

- double **pnl_sf_expint_En**(int n, double x)
[Description](#) Computes $E_n(x)$ for $n \geq 0, x \geq 0$, or $x > 0$ when $n = 0$ or 1 .

14.8 Hypergeometric functions

- double **pnl_sf_hyperg_2F1**(double a, double b, double c, double x)
Description Compute the Gauss hypergeometric function ${}_2F_1(a, b, c, x)$ for $|x| < 1$ and for $x < -1$ when $b, a, c, (b-a), (c-a), (c-b)$ are not integers
- double **pnl_sf_hyperg_1F1**(double a, double b, double x)
Description Compute the hypergeometric function ${}_1F_1(a, b, x)$
- double **pnl_sf_hyperg_2F0**(double a, double b, double x)
Description Compute the hypergeometric function ${}_2F_0(a, b, x)$ for $x < 0$ using the relation ${}_2F_0(a, b, x) = (-x)^{-a} U(a, 1 + a - b, -\frac{1}{x})$.
- double **pnl_sf_hyperg_0F1**(double c, double x)
Description Compute the hypergeometric function ${}_0F_1(c, x)$
- double **pnl_sf_hyperg_U**(double a, double b, double x)
Description Compute the confluent hypergeometric function $U(a, b, x)$ with $x > 0$

15 Some bindings

15.1 MPI bindings

15.1.1 Overview

We provide some bindings for the MPI library to natively handle *PnlObjects*. The functionalities described in this chapter are declared in `pnl/pnl_mpi.h`.

15.1.2 Functions

All the following functions return an error code as an integer value. This returned value should be tested against `MPI_SUCCESS` to check that no error occurred.

- int **pnl_object_mpi_pack_size**(const **PnlObject** *Obj, MPI_Comm comm, int *size)
Description Compute in `size` the amount of space needed to pack `Obj`.
- int **pnl_object_mpi_pack**(const **PnlObject** *Obj, void *buf, int bufsize, int *pos, MPI_Comm comm)
Description Pack `Obj` into `buf` which must be at least of length `size`. `size` must be at least equal to the value returned by `pnl_object_mpi_pack_size`.
- int **pnl_object_mpi_unpack**(**PnlObject** *Obj, void *buf, int bufsize, int *pos, MPI_Comm comm)
Description Unpack the content of `buf` starting at position `pos` (unless several objects have been packed contiguously, `*pos` should be equal to 0). `buf` is a contiguous memory area of length `bufsize` (note that the size is counted in bytes). `pos` is incremented and is on output the first location in the input buffer after the locations occupied by the message that was unpacked. `pos` is properly set for a future call to `MPI_Unpack` if any.

- int **pnl_object_mpi_send**(const **PnlObject** *Obj, int dest, int tag, MPI_Comm comm)
Description Perform a standard-mode blocking send of **Obj**. The object is sent to the process with rank **dest**.
- int **pnl_object_mpi_ssend**(const **PnlObject** *Obj, int dest, int tag, MPI_Comm comm)
Description Perform a standard-mode synchronous blocking send of **Obj**. The object is sent to the process with rank **dest**.
- int **pnl_object_mpi_recv**(**PnlObject** *Obj, int src, int tag, MPI_Comm comm, MPI_Status *status)
Description Perform a standard-mode blocking receive of **Obj**. The object is sent to the process with rank **dest**. Note that **Obj** should be an already allocated object and that its type should match the true type of the object to be received. **src** is the rank of the process who sent the object.
- int **pnl_object_mpi_bcast**(**PnlObject** *Obj, int root, MPI_Comm comm)
Description Broadcast the object **Obj** from the process with rank **root** to all other processes of the group **comm**.
- int **pnl_object_mpi_reduce**(**PnlObject** *Sendbuf, **PnlObject** *Recvbuf, MPI_Op op, int root, MPI_Comm comm)
Description Perform the reduction described by **op** on the objects **Sendbuf** and stores the result into **Recvbuf**. Note that **Recvbuf** and **Sendbuf** must be of the same type. The argument **root** is the index of the root process and **comm** is a communicator. Not all reductions are implemented for all types. Here is the list of compatible reductions

| | |
|-------------------------------|--|
| MPI_SUM | PnlVect, PnlVectInt, PnlVectComplex, PnlMat, PnlMatInt, PnlMatComplex |
| MPI_PROD, MPI_MAX, MPI_MIN | PnlVect, PnlVectInt, PnlMat, PnlMatInt |

For more expert users, we provide the following nonblocking functions.

- int **pnl_object_mpi_isend**(const **PnlObject** *Obj, int dest, int tag, MPI_Comm comm, MPI_Request *request)
Description Start a standard-mode, nonblocking send of object **Obj** to the process with rank **dest**.
- int **pnl_object_mpi_irecv**(void **buf, int *size, int src, int tag, MPI_Comm comm, int *flag, MPI_Request *request)
Description Start a standard-mode, nonblocking receive of object **Obj** from the process with rank **root**. On output **flag** equals to **TRUE** if the object can be received and **FALSE** otherwise (this is the same as for *MPI_Iprobe*).

15.2 The save/load interface

The interface is only accessible when the MPI bindings are compiled since it is based on the Packing/Unpacking facilities of MPI.

The functionalities described in this chapter are declared in `pnl/pnl_mpi.h`.

- **PnlRng ** pnl_rng_create_from_file**(char *str, int n)
Description Load `n` rng from the file of name `str` and returns an array of `n` **PnlRng**.
- int **pnl_rng_save_to_file**(**PnlRng** **rngtab, int n, char *str)
Description Save `n` rng stored in `rngtab` into the file of name `str`.
- int **pnl_object_save**(PnlObject *O, FILE *stream)
Description Save the object `O` into the stream `stream`. `stream` is typically created by calling `fopen` with `mode="wb"`. This function can be called several times to save several objects in the same stream.
- PnlObject* **pnl_object_load**(FILE *stream)
Description Load an object from the stream `stream`. `stream` is typically created by calling `fopen` with `mode="rb"`. This function can be called several times to load several objects from the same stream. If `stream` was empty or it did not contain any PnlObjects, the function returns NULL.
- PnlList* **pnl_object_load_into_list**(FILE *stream)
Description Load as many objects as possible from the stream `stream` and stores them into a **PnlList**. `stream` is typically created by calling `fopen` with `mode="rb"`. If `stream` was empty or it did not contain any PnlObjects, the function returns NULL.

16 Financial functions

The financial functions are defined in the header `pnl/pnl_finance.h`.

- int **pnl_cf_call_bs**(double s, double k, double T, double r, double divid, double sigma, double *ptprice, double *ptdelta)
Description Compute the price and delta of a call option $(s - k)_+$ in the Black-Scholes model with volatility `sigma`, instantaneous interest rate `r`, maturity `T` and dividend rate `divid`. The parameters `ptprice` and `ptdelta` are respectively set to the price and delta on output.
- int **pnl_cf_put_bs**(double s, double k, double T, double r, double divid, double sigma, double *ptprice, double *ptdelta)
Description Compute the price and delta of a put option $(k - s)_+$ in the Black-Scholes model with volatility `sigma`, instantaneous interest rate `r`, maturity `T` and dividend rate `divid`. The parameters `ptprice` and `ptdelta` are respectively set to the price and delta on output.
- double **pnl_bs_call**(double s, double k, double T, double r, double divid, double sigma)

Description Compute the price of a call option with spot s and strike k in the Black-Scholes model with volatility σ , instantaneous interest rate r , maturity T and dividend rate $divid$.

- double **pnl_bs_put**(double s , double k , double T , double r , double $divid$, double σ)

Description Compute the price a put option with spot s and strike k in the Black-Scholes model with volatility σ , instantaneous interest rate r , maturity T and dividend rate $divid$.

- double **pnl_bs_call_put**(int $iscall$, double s , double k , double T , double r , double $divid$, double σ)

Description Compute the price of a put option if $iscall=0$ or a call option if $iscall=1$ with spot s and strike k in the Black-Scholes model with volatility σ , instantaneous interest rate r , maturity T and dividend rate $divid$.

- double **pnl_bs_vega**(double s , double k , double T , double r , double $divid$, double σ)

Description Compute the vega of a put or call option with spot s and strike k in the Black-Scholes model with volatility σ , instantaneous interest rate r , maturity T and dividend rate $divid$.

- double **pnl_bs_gamma**(double s , double k , double T , double r , double $divid$, double σ)

Description Compute the gamma of a put or call option with spot s and strike k in the Black-Scholes model with volatility σ , instantaneous interest rate r , maturity T and dividend rate $divid$.

Practitioners do not speak in terms of option prices, but rather compare prices in terms of their implied Black & Scholes volatilities. So this parameter is very useful in practice. Here, we propose two functions to compute σ_{impl} : the first one is for one up-let, maturity, strike, option price. the second function is for a list of strikes and maturities, a matrix of prices (with strikes varying row-wise).

- double **pnl_bs_implicit_vol**(int is_call , double $Price$, double s , double K , double T , double r , double $divid$, int $*error$)

Description Compute the implied volatility of a put option if $iscall=0$ or a call option if $iscall=1$ with spot s and strike K in the Black-Scholes model with instantaneous interest rate r , maturity T and dividend rate $divid$. On output $error$ is OK if the computation of the implied volatility succeeded or FAIL if it failed.

- int **pnl_bs_matrix_implicit_vol**(const **PnlMatInt** $*iscall$, const **PnlMat** $*Price$, double s , double r , double $divid$, const **PnlVect** $*K$, const **PnlVect** $*T$, **PnlMat** $*Vol$)

Description Compute the matrix of implied volatilities $Vol(i,j)$ of a put option if $iscall(i,j)=0$ or a call option if $iscall(i,j)=1$ with spot s and strike $K(j)$ in the Black-Scholes model with instantaneous interest rate r , maturity $T(j)$ and dividend rate $divid$. This function returns the number of failures, when everything succeeded it returns 0.

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