Option Pricing in Discrete Time

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In this note, I shall outline the fundamentals of option pricing in discrete time.

The Put-Call Parity for European Options

With simpler financial derivatives, we derived the fair value of the derivative by utilizing no arbitrage arguments. As option require dynamic replication, we cannot utilize such a simple framework here. We can however, describe the relationship between the fair value of European put and call options with an arbitrage argument.
We can essentially draw any arbitrary payoff strategy that mimics an options payoff in order to derive this relationship. The most common approach is to consider combining:

- A long call option with strike price K and borrowing the present value of the strike price
- A long put option with strike price K and holding the underlying asset.

Thus we find the relationship:

$$C_0 + PV(K) = P_0 + S_0$$

Or

$$C_0 + Ke^{-rT} = P_0 + S_0$$

Parity Bounds for American Options

It is trivial that we must have

$$C_0^A \geq C_0^E$$

and

$$P_0^A \geq P_0^E$$

We can then prove, that it is **never** optimal to exercise an American call option early, if there are no dividends and interest is positive. Using the Put-Call parity for European options we can write:

$$C_0^A \geq C_0^E = P_0 + S_0 - PV(K) - PV(D)$$

Exercising early yields $S_0 - K$ so we can write

$$S_0 - K \geq P_0 + S_0 - PV(K) - PV(D) + K - K$$

<=>

$$0 \geq P_0 + (K - PV(K)) - PV(D)$$

<=>

$$PV(D) \geq (K - PV(K) + P_0$$

Thus, early exercise is worth it only if the present value of the dividends exceeds the time value of money factor plus the value of the European Put. As the RHS is positive, this trivially proves that early exercise is never worth it for non-dividend paying stocks with positive interest rates.
Note also, that the above gives us a presentation of what drives option prices:

$$C_0 = (S_0 - K) + (K - PV(K)) + P_0 - PV(D)$$

- The first term is the intrinsic value of the call option i.e. how deep in the money it is.
- The second term is the time value of money
- Third term is the value of the put option (insurance value)
- Fourth term is dividends

We can follow the same procedure for a put option, but reach a slightly different result!

$$P_0^E = C_0 + PV(K) - S_0 + PV(D) = (K - S_0) - (K - PV(K)) + C_0 + PV(D)$$

Notably, time value of money works in the opposite way here.
And for the American option:

$$P_0^A \geq K - S_0 \geq K - S_0 - (K - PV(K)) + C_0 + PV(D)$$

<=>

$$(K - PV(K)) \geq C_0 + PV(D)$$

Thus, early exercise **may** be worth it for put options, even in the absence of dividends. This depends on the magnitude of the present value of the strike price comparatively to the insurance value.

Dynamic Replication

As mentioned previously, pricing options requires dynamic replication arguments. In a simple two-period binomial tree setup, we can replicate the payoff of an option by combining the underlying asset and the risk-free asset in a way such that:

$$\Delta S_U + B(1+R) = C_u$$

$$\Delta S_D + B(1+R) = C_d$$

Subtract (2) from (1):

$$\Delta = \frac{C_U - C_D}{S_U - S_D}$$

$$B = \frac{C_D - \Delta S_D}{1+R}$$

Such that

$$C_0 = \Delta S_0 + B$$

Risk Neutral Pricing

A risk neutral probability q satisfies that the expected return of an asset is equal to the risk free rate:

$$\frac{qS_u + (1-q)S_d}{S} = 1 + r$$

Such that

$$q = \frac{1 + r - d}{u - d}$$

and

$$S = \frac{1}{1+r} \cdot \left(qS_u + (1-q)S_d\right)$$

Thus, the price of any type of option V is

$$V = \frac{1}{1+r}(qV_u + (1-q)V_d)$$

For two periods, it is trivially

$$V = \frac{1}{(1+r)^2}(q^2V_{uu} + 2q(1-q)V_{ud} + (1-q)^2V_{dd})$$

For N periods, the return of the asset is

$$S_n = u^i d^{n-i} S_0$$

Such that the value of any option in a n period binomial tree is

$$V_0^n = \frac{1}{(1+r)^n} \sum_{i=0}^{n} u^i d^{n-i} \cdot \max\{S_n^i - K, 0\}$$

A Note on Option Returns

We should note, that an option is essentially just a levered derivative of the underlying. Define the option elasticity

$$\Omega = \frac{\Delta S}{V}$$

The value of any option is

$$V = \Delta S + B$$

The excess return of an option is

$$ER_{Option} - R_f = E\frac{V_1}{V_0} - R_f - 1 = \frac{E(\Delta S_0(1 + R_{Stock})) + B(1 + R_f) - V_0(1 + R_f)}{V_0}$$

<=>

$$\frac{\Delta S_0 + E\Delta S_0 R_{Stock} + (B - V_0)(1 + R_f)}{V_0}$$

Use that $B - V_0 = -\Delta S_0$

$$\frac{\Delta S_0 + \Delta S_0 ER_{Stock} - \Delta S_0(1 + R_f)}{V_0}$$

<=>

$$\frac{\Delta S_0}{V_0}(ER_{Stock} - R_f) = \Omega(ER_{Stock} - R_f)$$

What is U and D?

How do we pick the up and down factors in the binomial tree? We shall assume:

$$\ln\left(\frac{S_t}{S_0}\right) \sim (\mu T, \sigma T)$$

For any time period, we then have

$$\ln\left(\frac{S_{t+\Delta t}}{S_0}\right)$$

Equal $\ln(u)$ or $\ln(d)$ with probability p and 1-p:

$$E\ln\left(\frac{S_{t+\Delta t}}{S_0}\right) = p\ln(u) + (1-p)\ln(d) = \mu\Delta t$$

$$V\ln\left(\frac{S_{t+\Delta t}}{S_0}\right) = p\ln(u)^2 + (1-p)\ln(d)^2 - (p\ln(u) + (1-p)\ln(d))^2 = p(1-p)(\ln(u) - \ln(d))^2 = \sigma\Delta t$$

Solve for p in the first equation:

$$p = \frac{\mu\Delta t - \ln(d)}{\ln(u) - \ln(d)}$$

$$1 - p = \frac{\ln(u) - \mu\Delta t}{\ln(u) - \ln(d)}$$

Such that

$$V\ln\left(\frac{S_{t+\Delta t}}{S_0}\right) = (\mu\Delta t - \ln(d))(\ln(u) - \mu\Delta t) = \sigma\Delta t$$

<=>

$$\mu\Delta t\ln(u) - \mu^2\Delta t^2 - \ln(d)\ln(u) + \ln(d)\mu\Delta t$$

For $t \to 0$:

$$\mu\Delta t\ln(u) - \ln(d)\ln(u) + \ln(d)\mu\Delta t = \sigma\Delta t$$

This is one equation with two unknowns (u and d). By Cox et al:

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$

Which implies

$$p = \frac{\mu\Delta t - \ln(d)}{\ln(u) - \ln(d)} = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t}$$

Note that

$$\frac{u}{d} = e^{2\sigma\sqrt{\Delta t}}$$

Which measures the volatility of the underlying!