Synchrony and Desynchrony in Integrate-and-Fire Oscillators

Shannon Campbell[†] and DeLiang Wang[‡]
†Department of Physics, shannon@cis.ohio-state.edu
‡Department of Computer and Information Science and Center for Cognitive Science
dwang@cis.ohio-state.edu
The Ohio State University, Columbus, Ohio 43210, USA

Abstract

Due to many experimental reports of synchronized neural activity in the brain, there is much interest in understanding synchronization in networks of oscillators and using these systems for perceptual organization. We examine locally coupled networks of integrate-and-fire oscillators and, contrary to a recent report by Hopfield and Herz [2], we find that they can quickly synchrony. Moreover, we find that time to achieve synchrony increases logarithmically with the system size. Inspired by LEGION dynamics with relaxation oscillators, we are able to reliably desynchronize different oscillator groups in a network of integrate-and-fire oscillators using a global inhibitor.

1. Introduction

Different features of visual objects appear to be processed in different cortical areas [13]. How these features are linked to form perceptually coherent objects is known as the feature binding problem. Theoreticians have proposed that correlations in the firing times of neurons may encode the binding between these neurons [4], [3]. A considerable amount of neurophysiological evidence supports this conjecture of temporal correlation (for a review see [8]).

Based on the experimental findings, many oscillator networks have been proposed that use synchronous oscillations to link features together. This particular form of temporal correlation was called *oscillatory correlation* [12]. In oscillatory correlation there are two issues to be addressed. The first is the need to achieve synchrony quickly in locally coupled networks. Our usage of the word "quickly" refers to a time of 2-3 periods. This is based on biological data, which show that synchronous neural activity begins 2-3 periods after the onset of stimulus [8]. Locally coupled networks are emphasized because all-to-all coupled networks do not maintain pertinent spatial relationships that are critical for information processing (for further explanation see [9]). The second issue is how to desynchronize the phases of different oscillator groups rapidly and robustly so that segmentation occurs.

Integrate-and-fire oscillators have frequently been used as simplified models of neuronal behavior [6] and networks of such oscillators have been widely studied (for example, [5], [10], [11], [1]). Most of the literature on integrate-and-fire oscillators deals with globally coupled networks and relatively little is known about locally connected networks. In simulations with locally coupled networks of integrate-and-fire oscillators, a number of authors have noted their ability to achieve synchrony [5], [1], [2]. Hopfield and Herz [2] tested several different types of integrate-and-fire oscillators and noted that they quickly attain local synchrony (periodic states in which groups of connected oscillators fire simultaneously) and then slowly (on the time scale of several 100 periods) reorganize to a synchronous state in which all oscillators are firing at the same time. To our knowledge, no other information regarding the rate of synchronization in networks of integrate-and-fire oscillators is available.

Given their relevance to neurobiology, and their ability to achieve synchrony, we perform a systematic numerical study of synchronization in locally coupled networks of integrate-and-fire oscillators. In our examination, we find that the time needed to achieve synchrony is proportional to the logarithm of the network size. This observed scaling relationship holds in both one- and two-dimensional systems and for all parameters tested. We also indicate how the rate of synchronization is related to the parameters of the system.

Because integrate-and-fire oscillators can achieve synchrony quickly, we also investigate how they may be used to create an oscillator network for feature binding using oscillatory correlation. We use the architecture of the locally excitatory globally inhibitory oscillator networks

0-7803-4859-1/98 $10.00©1998 IEEE

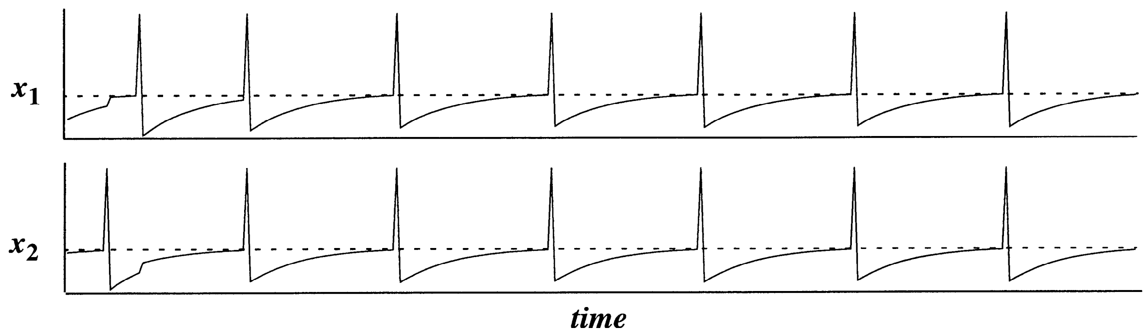

Figure 1. A diagram of a pair of integrate-and-fire oscillators with pulsatile coupling. The solid curves represent the potentials of the two coupled oscillators and the dashed lines represent the threshold. The initial potentials of the oscillators are chosen randomly. The oscillator labeled x_2 fires first and the potential of x_1 increases at that time. Similarly, when x_1 fires, the potential of x_2 increases. The phase shifts caused by the pulsatile interaction cause the oscillators to fire synchronously by the second cycle. The spikes shown when an oscillator fires are for illustration only.

(LEGION) proposed by Terman and Wang [9] and we create a global inhibitory unit which actively desynchronizes different oscillator groups. We then demonstrate how this network segregates multiple objects of various sizes.

2. Model Definition

We define a network of locally coupled integrate-and-fire oscillators as follows,

$$\dot{x}_i = -x_i + I + \sum_{j \in N(i)} J_{ij} P_j(t) \tag{1}$$

where the sum is over the oscillators in a neighborhood, $N(i)$, about oscillator i with $i = 1, \ldots, n$. The network contains n oscillators. The variable x_i represents the potential of oscillator i. The parameter I is taken to be an intrinsic property of the oscillator and controls the period of an uncoupled oscillator. The threshold of an oscillator is 1. When $x_i = 1$ the oscillator is said to "fire"; its value is instantly reset to 0 and it sends excitation to its neighbors.

The interaction between oscillators, $P_j(t)$, is given by,

$$P_j(t) = \sum_m \delta(t - t_j^m) \tag{2}$$

where t_j^m represents the m firing times of oscillator j and $\delta(t)$ is the Dirac Delta function. When oscillator j fires at time t, oscillator i receives an instantaneous pulse. This pulse increases the value of x_i by an amount J_{ij}. If the value of x_i is increased above the threshold, then it will fire. Note that information is transmitted between two oscillators instantaneously, thus the propagation speed is infinite.

A single oscillator can be described as a function $f(\phi)$, where ϕ is a phase variable. Note that $f(\phi)$ has the following properties; it increases monotonically, $f'(\phi) > 0$ and it is concave down, $f''(\phi) < 0$ (see Figure 1). We conjecture that our results generalize to all integrate-and-fire oscillators with these properties.

The coupling is nearest-neighbor only, i.e. two neighbors in the one-dimensional case and four neighbors in the two-dimensional case. The connection strength from oscillator j to oscillator i is normalized as follows

$$J_{ij} = \frac{\alpha}{Z_i} \tag{3}$$

where Z_i is the number of nearest neighbors that oscillator i has, e.g. $Z_i = 2$ for an oscillator i at the corner of a two-dimensional network. The constant α is the coupling strength. This normalization ensures that all oscillators receive the same total stimulus, and therefore, have the same trajectory in phase space when synchronous.

We also define a reset rule to describe how oscillators not at the threshold respond when they are induced to fire. Let oscillator i attain its threshold at time t. It will fire, and its value will be reset to zero. Oscillator i sends an instantaneous impulse to a neighboring oscillator j. If oscillator j is induced to fire, then its potential just after firing (at time t^+) is related to its potential just before it fires (at time t^-) in the following manner,

$$x_j(t^+) = x_j(t^-) + J_{ji} - 1 \tag{4}$$

Since oscillator j fires, oscillator i receives excitation and thus $x_i(t^+) = J_{ij}$. Because the two oscillators have fired at the same time, we refer to them as *synchronous*. The period of the synchronous system is shorter than the period of a single uncoupled oscillator. The synchronous period of the system is given by

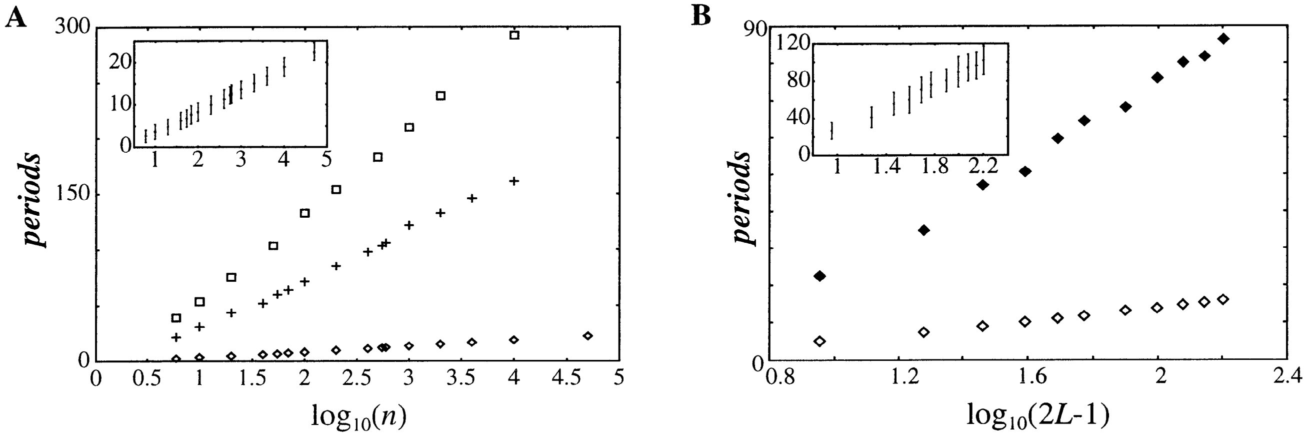

Figure 2. (A) The average time (in periods) needed for a chain of n oscillators to synchronize as a function of $\log_{10}(n)$. Three different symbols represent different pairs of parameters; the squares, $\alpha = 0.48, I = 10$, the plus signs $\alpha = 0.025, I = 1.1$, and the small diamonds $\alpha = 0.2, I = 1.11$. The inset indicates the standard deviation for the data represented by the diamonds. (B) The average times for a two-dimensional network of oscillators to synchronize are plotted as a function of $\log_{10}(2L-1)$, where L is the length of the square network. The solid diamonds are for the parameters $\alpha = 0.2, I = 2.0$ and the open diamonds are for $\alpha = 0.2, I = 1.11$. The inset indicates the standard deviation for the data represented by the solid diamonds. Each average is based on approximately 300 trials with random initial conditions.

$$\log\left(\frac{I-\alpha}{I-1}\right) \tag{5}$$

This particular realization of a network of integrate-and-fire oscillators is called "Model A" by Hopfield and Herz [2].

3. Time to Synchrony

Our data indicate that the average time to synchrony increases as the logarithm of the system size for one- and two-dimensional networks without noise. We first note that all trials with locally coupled integrate-and-fire networks in which $f(\phi) > 0$, $f'(\phi) < 0$, and with positive coupling resulted in synchrony. Approximately 10^5 trials with random initial conditions chosen uniformly and randomly from the range [0,1] resulted in synchrony. More correlated initial conditions were also tested. Spin waves of all tested wavelengths resulted in synchronous solutions (spin waves typically disrupt long range order in Hamiltonian systems). We could not find any initial conditions such that the time needed for synchrony was an order of magnitude larger than the average time to synchrony. Also, traveling waves never arose, even in one-dimensional systems with periodic boundary conditions. Similar tests in two-dimensional networks produced similar results. Based on these observations, we conjecture that locally coupled networks of integrate-and-fire oscillators *always* synchronize (given $f(\phi) > 0$, $f'(\phi) < 0$, and positive coupling).

In Figure 2A we display the average time needed to synchronize a chain of n oscillators as a function of $\log_{10}(n)$. The averages are based on several hundred trials with random initial conditions. Our analysis indicated that only a few hundred trials were needed to accurately calculate an average. The averages appear to lie on a straight line for each set of parameters tested. Data for only three different pairs of parameters is displayed, although we have tested more parameters and the only change is the slope of the line. The data represented by the square shaped symbols in Figure 2A arise from the exact same parameters used in Hopfield and Herz [2]. The average time to synchrony we obtained for these parameters is consistent with the reported time to synchrony in [2]. We discuss how to choose parameters for fast synchronization next.

The slope of the lines in Figure 2A are, to a good approximation, inversely proportional to

$$\frac{\alpha_S}{I}\left[\frac{\alpha_S + \sqrt{\alpha_S^2 + 4I(I-1)}}{2(I-1)}\right] \tag{6}$$

where α_S represents the single connection strength between oscillators in the bulk of the network (α is the total input a single oscillator receives). This single connection strengths represent the effect that one oscillator has on another for oscillators that are not on the boundary. Equation (6) is the derivative at the unstable fixed point of the return map for a pair of oscillators - note that it is greater than one for all positive values of α_S and I. Thus the degree of repulsion from the unstable fixed point is a good indication of the rate of synchrony. One can obtain fast synchronization for $\alpha < 0.5$ so long as I is chosen to be near 1. As shown in Figure 2A, one can choose parameters ($\alpha = 0.2, I = 1.11$) so that a chain of 10^4 oscillators synchronizes in approximately 19 periods. As α approaches 1, the oscillators fire frequently and exhibit a less rapid rate of synchronization. For $\alpha \geq 1$ the system becomes meaningless and one would expect the above approximation to fail for large values of α. Our data indicate that (6) becomes invalid for values of $\alpha > 0.8$. Our data also indicate that the above approximation becomes invalid for $I < 1.05$. We tested various combinations of α and I in the ranges $\alpha \in [0.002, 0.99]$ and $I \in [1.001, 20]$.

In Figure 2B we show our data for two-dimensional systems. The data also indicate that the time to synchrony scales as the logarithm of the system size. We used the quantity $2L$-1 to represent the size of the system in this case because it is the largest distance between any two oscillators on a square grid with couplings between the nearest four neighbors. We have also tested other pairs of parameters with similar results. As in one-dimensional systems the rate of synchrony is inversely proportional to (6). Also, periodic boundary conditions did not significantly alter the time to synchrony. Travelling waves, rotating waves, or other desynchronous solutions were never observed in either one- or two-dimensional networks.

Only several hundred trials were needed to compute the averages because simulations indicated that the distribution of the synchronization times did not appear to have a long tail. Analysis of the data from 20000 trials with random initial conditions of a chain of 500 oscillators suggested that the tail has an exponential decay with an exponent between 1 and 2. Also, we never observed a time to synchrony that was an order of magnitude greater than the average time to synchrony.

In one-dimensional networks, we have a heuristic explanation for why the time for synchrony has a logarithmic relation with the size of the network. In one dimension, we can show that a block of oscillators does not become smaller as the system evolves, i.e. it only increases in size by merging with other blocks of oscillators. Because of the infinite propagation speed, two blocks of oscillators can merge in one cycle, independent of their size. Thus, we deduce that the rate at which blocks merge is a constant which is dependent only on the system parameters, and not on the size of the blocks. From this assumption it follows that the average size of a block in a one-dimensional network increases exponentially in time.

4. Desynchronization

The integrate-and-fire oscillator systems examined so far have the property that synchrony is quickly achieved. But synchrony alone is not useful for information processing, since the system is dissipative and almost all information is lost. In oscillatory correlation, the different phases between oscillators are used to encode information regarding binding and segregation of features. Thus, we must be able to desynchronize different groups of oscillators quickly and robustly, but without destroying synchrony within each group.

To create a network of integrate-and-fire oscillators for oscillatory correlation, we use the architecture of LEGION proposed by Terman and Wang [9]. Figure 3A displays a diagram of this architecture. In LEGION, the local excitatory connections result in synchrony between a connected groups of oscillators. An inhibitory unit connected to every oscillator serves to control or regulate the phases of different oscillator groups. This unit is connected to all oscillators so that it can respond to groups of oscillators regardless of their position in the network. The global inhibitor sends an inhibitory signal to the entire network when one oscillator, or one group of oscillators fires. This signal lowers the potential of all other oscillators and ensures that there is a finite amount of time between firings of the different oscillator groups.

We now define a LEGION network which uses integrate-and-fire oscillators as its basic units. A single oscillator in the network is given by,

$$\dot{x}_i = -x_i + I_i + \sum_{j \in N(i)} J_{ij} P_j(t) - G(t) \tag{7}$$

where n is the number of oscillators and $i = 1, \ldots, n$. $N(i)$ represents the four nearest neighbors of oscillator i. The stimulus, I_i, for each oscillator is now dependent on the input image. If an oscillator receives input, then it is called excited, and if it does not receive input, it does not oscillate and is called quiet. As before the threshold for each oscillator is 1. The interaction term $P_j(t)$, is the same as in (2). The connections between the oscillators are also input dependent. Only neighboring excited oscillators will be coupled. This is an implicit encoding of the Gestalt principle of connectedness [7]. As before, the connection strengths are normalized so all stimulated oscillators receive the same total input, and thus have the same frequency. However, (3) needs to be modified so that oscillator i is normalized by the number of stimulated neighbors that it is coupled with. This is called dynamic normalization [12].

The global inhibitor, $G(t)$, sends an instantaneous inhibitory pulse (weighted by Γ) to the entire network when any oscillator in the network fires. The constant Γ is

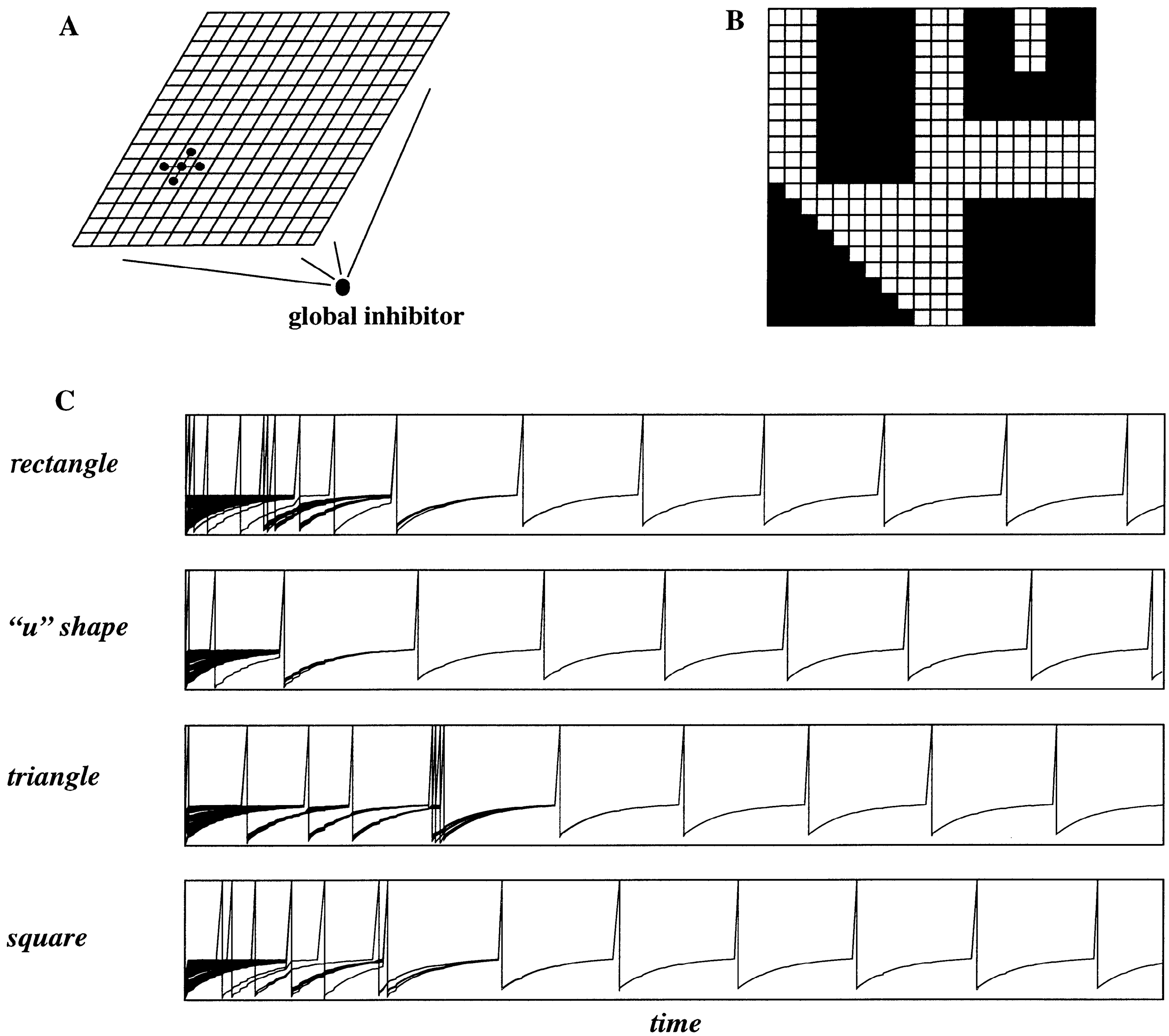

Figure 3. (A) A diagram of a locally excitatory globally inhibitory oscillator network. Each oscillator has local excitatory connections. The global inhibitor is coupled with every oscillator in the network and serves to desynchronize different groups of oscillators. (B) The input we use to demonstrate the behavior of our network. The black squares represent those oscillators which receive stimulus and the unfilled squares receive no stimulus. (C) We display the temporal activities of all oscillators comprising each of the four objects in Figure 3B. The parameters used are $I_i = 1.05$ for those oscillators receiving stimulus, $I_i = 0$ for those oscillators not receiving stimulus, $\alpha = 0.2$, and $\Gamma = 0.01$.

less than the smallest nonzero coupling strength. When an oscillator fires, the global inhibitor serves to lower the potential of all oscillators, but because this impulse is not as large as the excitatory signal between neighboring stimulated oscillators, it does not destroy the synchronizing effect of the local couplings (see [9]). In this fashion, a connected region of oscillators receiving input synchronizes as the network evolves in time. This region has no direct excitatory connection with other spatially separate regions in the input. It will however, interact with other groups of oscillators through the global inhibitor. This interaction inhibits other blocks of oscillators from firing at the same time and causes desynchrony among the different oscillator groups.

In Figure 3C we display the network response to the input shown in Figure 3B. The four graphs in Figure 3C display the potentials of every oscillator comprising each of the four objects in Figure 3B. The oscillators have random initial conditions varying uniformly from 0 to 1. Many oscillators fire during the first period, and the effect of the global inhibitor can be seen in the jitter, or lack of smoothness, in the potentials of the oscillators during this time. As the system evolves, clusters of oscillators begin to form and the curves become smoother because the global inhibitor does not send inhibitory impulses as often. By the third cycle each group of oscillators comprising a distinct object is almost perfectly synchronous, and the different oscillator groups have distinct phases. Oscillators that do not receive excitation (not shown) experience an exponential decay towards 0 and are periodically perturbed by the small inhibitory signals from the global inhibitor.

The architecture and dynamics of this network allow for segregation multiple objects regardless of their position on the network, or their size. We also note that the segmentation capacity of this network can be rather large - we have repeatedly desynchronized several hundred groups of oscillators in simulations.

5. Summary

We have examined locally coupled networks of integrate-and-fire oscillators and our data indicate that the time they need to achieve synchrony scales logarithmically with the size of the system in both one- and two-dimensional systems. We have also indicated how the rate of synchronization is related to the system parameters for a wide range of parameters based on the analysis of two oscillators. Through appropriate parameter choices one can synchronize 10^5 oscillators in five periods (on average).

With this property of rapid synchrony we have used the LEGION architecture proposed by Terman and Wang [9] to construct a network of integrate-and-fire oscillators that is able to perform oscillatory correlation. We have demonstrated the ability of this network to segment multiple objects of various sizes.

Acknowledgments

We would like to thank Dr. C. Jayaprakash for insightful discussions. This work was supported by an ONR grant (N00014-93-10335), an NSF grant (IRI-9423313), and an ONR Young Investigator Award (N00014-96-1-0676) to DLW.

References

[1] Corral, A., Perez, C.J., Diaz-Guilera, A., and Arenas, A. (1995) Self-organized criticality and synchronization in a lattice model of integrate-and-fire neurons. *Phys. Rev. Let.* **74**:118-121.

[2] Hopfield, J.J. and Herz, A.V.M. (1995) Rapid local synchronization of action potentials: Toward computation with coupled integrate-and-fire oscillator neurons. *Proc. Natl. Acad. Sci. USA* **92**:6655-6662.

[3] von der Malsburg, C. (1981) The correlation theory of brain functions. Internal Report 81-2. Max-Planck-Institute for Biophysical Chemistry, Göttingen, FRG.

[4] Milner, P.M. (1974) A model for visual shape recognition. *Psych. Rev.* **81**:521-535.

[5] Mirollo, R.E. and Strogatz, S.H. (1990) Synchronization of pulse-coupled biological oscillators. *SIAM J. Appl. Math.* **50**:1645-1662.

[6] Peskin, C.S. (1975) *Mathematical Aspects of Heart Physiology.* Courant Institute of Mathematical Sciences, New York University, New York. pp. 268-275.

[7] Rock, I. and Palmer, S. (1990) The legacy of Gestalt psychology. *Sci. Am.* **263**(12):84-90.

[8] Singer, W. and Gray, C.M. (1995) Visual feature integration and the temporal correlation hypothesis. *Ann. Rev. Neurosci.* **18**:555-586.

[9] Terman, D. and Wang, D.L. (1995) Global competition and local cooperation in a network of neural oscillators. *Physica D* **81**:148-176.

[10] Tsodyks, M., Mitkov, I., and Sompolinsky, H. (1993) Pattern of synchrony in inhomogeneous networks of oscillators with pulse interactions. *Phys. Rev. Let.* **71**:1280-1283.

[11] van Vreeswijk, C., Abbott, L.F., and Ermentrout, B. (1994) When inhibition not excitation synchronizes neural firing. *J. Comp. Neurosci.* **1**:313-321.

[12] Wang, D.L. and Terman, D. (1995) Locally excitatory globally inhibitory oscillator networks. *IEEE Trans. Neural Net.* **6**:283-286.

[13] Zeki, S. (1993) *A Vision of the Brain.* Blackwell Scientific Publications, Oxford