Data 605 - Assignment 13

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Introduction

There is a table of contents at the front of the document. For grading, if you do not need to see my reasoning you can jump right to the answers.

Problem 1

Use integration by substitution to solve the integral below.

$$\int 4e^{-7x} dx$$

Let $u = -7x$, $u'(x) = -7$

$$\frac{du}{dx} = -7 \iff \frac{-1}{7} du = dx$$

$$\int 4e^{-7x} dx = 4 \int e^u \frac{-1}{7} du = \frac{-4}{7} \int e^u du$$

$$= \frac{-4}{7}(e^u + C)$$

Answer

$$\frac{-4}{7} e^{-7x} + C$$

Problem 2

Biologists are treating a pond contaminated with bacteria. The level of contamination is changing at a rate of $\frac{dN}{dt} = -\frac{3150}{t^4} - 220$ bacteria per cubic centimeter per day, where t is the number of days since treatment began. Find a function $N(t)$ to estimate the level of contamination if the level after 1 day was 6530 bacteria per cubic centimeter.

Approach

We're given a rate of change. We're given a value from the domain of contamination levels. We're asked for a function that returns the level of contamination.

- To do that, we'll integrate the provided derivative, producing the function for contamination.
- Express the contamination at time $t = 1$ using the provided level.
- Solve for the integration constant C.
- Provide the general function for contamination level.

We're given:

$$\frac{dN}{dt} = -\frac{3150}{t^4} - 220$$

So,

$$N(t) = \int (\frac{-3150}{t^4} - 220) dt$$

$$= -3150 \int t^{-4} - 220 \int dt$$

$$= -3150(\frac{t^{-3}}{-3}) - 220t + C$$

$$= \frac{1050}{t^3} - 220t + C$$

We're given:

$$N(1) = 6530$$

Substituting $t = 1$ into our antiderivative:

$$1050 - 220 + C = 6530$$

$$\iff C = 6530 - 830 = 5700$$

Answer

Therefore:

$$N(t) = \frac{1050}{t^3} - 220t + 5700$$

Problem 3

Find the total area of the red rectangles in the figure below, where the equation of the line is $f(x) = 2x - 9$.

```
x <- c(5:8)
y <- 2 * x - 9
width <- 1

ans <- sum(width * y)
```

Answer

16

Problem 4

Find the area of the region bounded by the graphs of the given equations.

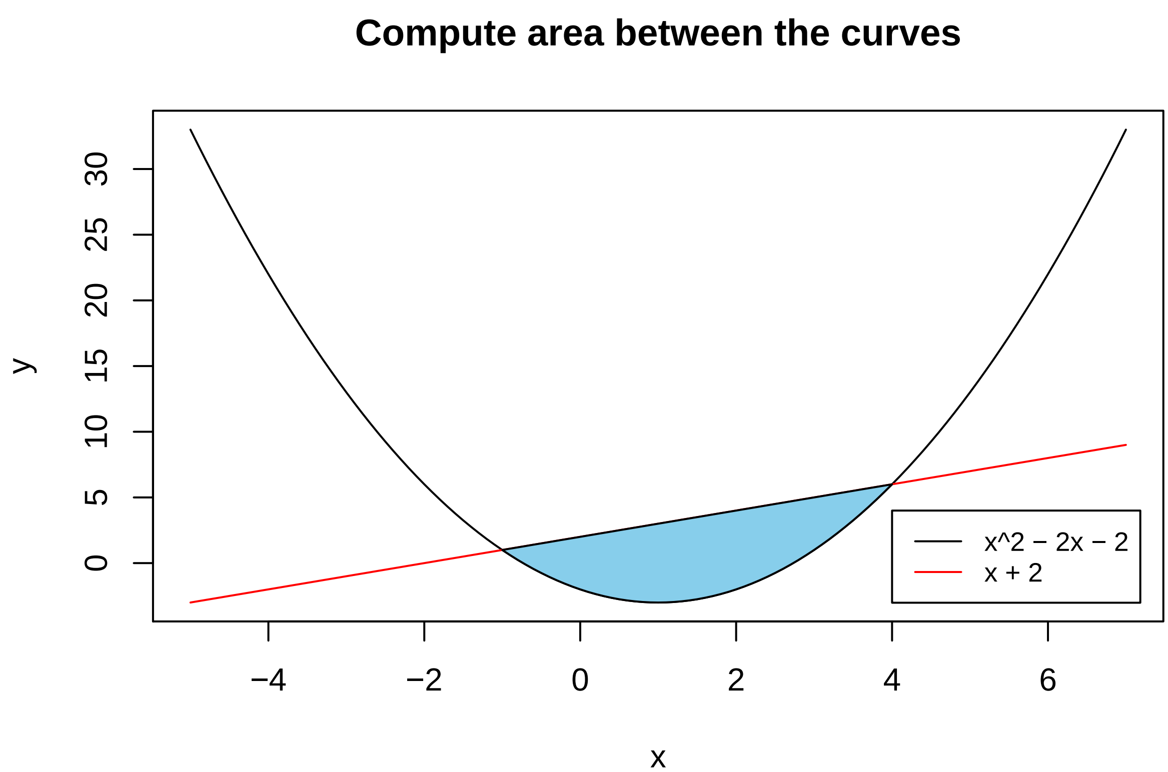
$y = x^2 - 2x - 2$, $y = x + 2$

Approach

- Plot the curves. Shade the area to be computed.
- Find the end points.
- Compute the definite integral under each curve and calculate their difference.

```
x <-  seq(from = -5, to = 7, by = 0.1)
y.1 <-  x^2 - 2 * x - 2
y.2 <-  x + 2
shade <- x >= -1 & x <= 4

plot(x, y.1, type = "l", col = "black",
     main = "Compute area between the curves",
     ylab = "y")
```

```
lines(x, y.2, col = "red")
polygon(c(x[shade], rev(x[shade])), c(y.2[shade], rev(y.1[shade])),col="skyblue")
legend(4, 4, legend=c("x^2 - 2x - 2", "x + 2"),
       col=c("black", "red") , lty=c(1, 1), cex=0.8)
```



Find the end points for the area to compute

It's easy to discern in the plot, but let's make this honest. The end points are where the two functions are equal.

$$
\begin{aligned}
x^2 - 2x - 2 &= x + 2 \\
\iff x^2 - 3x - 4 &= 0 \\
&= (x+1)(x-4) \\
\iff x &= \{-1, 4\}
\end{aligned}
$$

Guessing these roots isn't hard, but it's really easy when you've drawn the picture first.

Integrate

$$
\begin{aligned}
&\int_{-1}^{4} (x+2)dx - \int_{-1}^{4} (x^2 - 2x - 2)dx \\
&= \int_{-1}^{4} [(x+2) - (x^2 - 2x - 2)]dx \\
&= \int_{-1}^{4} (-(x^2) + 3x + 4)dx
\end{aligned}
$$

$$= (\frac{-1}{3})x^3 + (\frac{3}{2})x^2 + 4x\Big|_{x=-1}^{4}$$

$$= (\frac{-1}{3})4^3 + (\frac{3}{2})4^2 + 4 \cdot 4 - \frac{1}{3} - \frac{3}{2} + 4$$

```
ans <- (-1 / 3) * 4^3 + (3 / 2) * 4^2 + 4 * 4 - (1 / 3) - (3 / 2) + 4
```

Answer

20.83

I guess you can integrate in `R`, too. Who knew? Let me try this to double check my work

```
integrand <- function(x) {-(x^2) + 3 * x + 4}
integrate(integrand, lower = -1, upper = 4)
```

```
## 20.83333 with absolute error < 2.3e-13
```

That is pretty cool. Like magic.

Reference:

- Shading between curves in `R`

Problem 5

A beauty supply store expects to sell 110 flat irons during the next year. It costs \$3.75 to store one flat iron for one year. There is a fixed cost of \$8.25 for each order. Find the lot size and the number of orders per year that will minimize inventory costs.

Analysis

First, I needed to figure out what "lot size" meant. I thought it had something to do with storage and I tried to figure out if enough was given about inventory costs. No, so I looked it up.

> Definition of 'Lot Size': Lot size refers to the quantity of an item ordered for delivery on a specific date or manufactured in a single production run... A simple example of lot size is: when we buy a pack of six chocolates, it refers to buying a single lot of chocolate.

Now I remember the term "odd lots." Sellers don't want to break up a lot and sell piecemeal.

Calculation

- Let x equal lot size. Expected sales are 110 irons, so the number of orders will be $110/x$.
- $Inventory\ Cost = Order\ cost + Storage\ cost = f(x)$
- $Order\ cost = 8.25 \times \frac{110}{x}$
- We assume our inventory is x upon receipt of order, it runs down at a constant rate, and we deplete our stock before receiving another order. Therefore, we estimate the average inventory balance as the quantity ordered divided by 2. $Storage\ cost = 3.75 \times \frac{x}{2}$
- $Inventory\ cost = 8.25 \times \frac{110}{x} + 3.75 \times \frac{x}{2}$

Simplified,

$$Inventory\ cost = f(x) = 907.50(x^{-1}) + 1.875(x)$$

To optimize the function, we'll find the value of x where the cost function's first derivative = 0, indicating a local extremum, and the second derivative is positive, indicating the curve is concave up and the value of x under consideration is a local minimum.

$$f'(x) = \frac{-907.50}{x^2} + 1.875$$

$$f''(x) = \frac{-2 \cdot -907.50}{x^3} = \frac{1815.00}{x^3}$$

Set the first derivative to 0.

$$0 = \frac{-907.50}{x^2} + 1.875$$

$$\iff -1.875 = \frac{-907.50}{x^2}$$

$$\iff x^2 = \frac{-907.50}{-1.875}$$

$$\iff x = \sqrt{\frac{907.50}{1.875}} = \pm 22$$

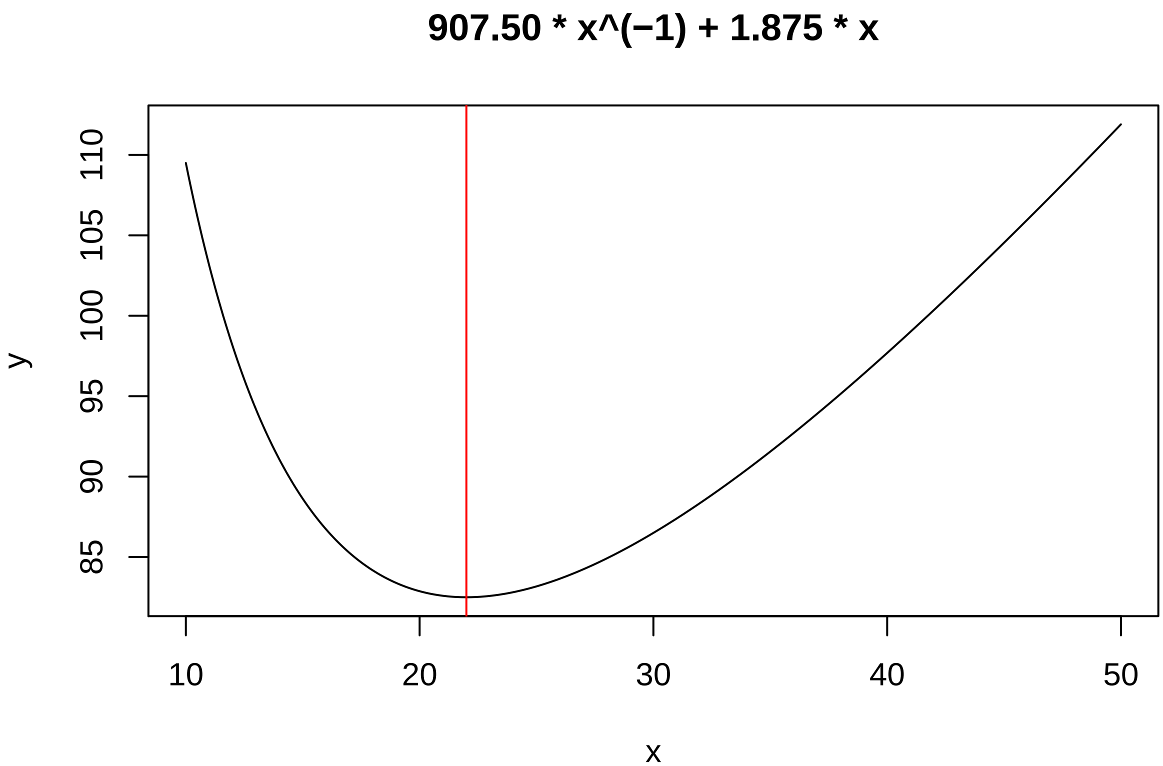
I don't really need to consider the negative value, since you can't order negative lot sizes, but for fun, consider the second derivative for evaluating concavity.

$$f''(x) = \frac{1815.00}{x^3}$$

This calculation is negative for a negative x and positive for a positive x. As expected, $x = 22$ is a local minimum.

I'd like to plot this and see that it behaves as expected.

```r
x <-  seq(from = 10, to = 50, by = 0.1)
y <- 907.50 * x^(-1) + 1.875 * x

plot(x, y, type = "l", col = "black",
     main = "907.50 * x^(-1) + 1.875 * x")
abline(v = 22, col = "red")
```



Reference:

This is an accounting calculation for inventory management called Economic Order Quantity (EOQ)

Answer

5 orders of 22 flat irons per order minimize inventory costs.

Problem 6

Use integration by parts to solve the integral below.

$$\int ln(9x) \cdot x^6 dx$$

Variable names

We're going to use two integration methods, both of which traditionally assign the variable name u to a function. We'll call them u_1 and u_2.

Set up parts

Integration by parts is given by:

$$\int u\ dv = uv - \int v\ du$$

Let $u_1 = ln(9x)$, $dv = x^6 dx$

Derive remaining terms

Find the anti-derivative of dv

$$v = \int x^6 dx = \frac{x^7}{7} + C$$

Take the derivative of u_1. We apply the chain rule, designating the inner function u_2.

$$\frac{d}{dx}(ln(9x))$$

Let $u_2 = 9x$

$$\frac{d}{dx}(ln(9x)) = \frac{d}{du_2}(ln(u_2))\frac{du_2}{dx} = \frac{1}{u_2} \cdot \frac{du_2}{dx}$$

Substitute for u_2.

$$\frac{1}{9x} \cdot 9 = \frac{1}{x}$$

Therefore,

$$du_2 = \frac{1}{x}dx$$

Substitute into the formula for integration by parts and complete

$$\int u\ dv = uf - \int v\ du = \frac{1}{7}ln(9x)x^7 - \int \frac{x^7}{7} \cdot \frac{1}{x}dx$$
$$= \frac{1}{7}x^7 ln(9x) - \int \frac{x^6}{7}dx = \frac{1}{7}x^7 ln(9x) - \frac{x^7}{49} + C$$

Answer

$$\frac{1}{7}x^7 ln(9x) - \frac{x^7}{49} + C$$

Problem 7

Determine whether $f(x)$ is a probability density function on the interval $[1, e^6]$. If not, determine the value of the definite integral.

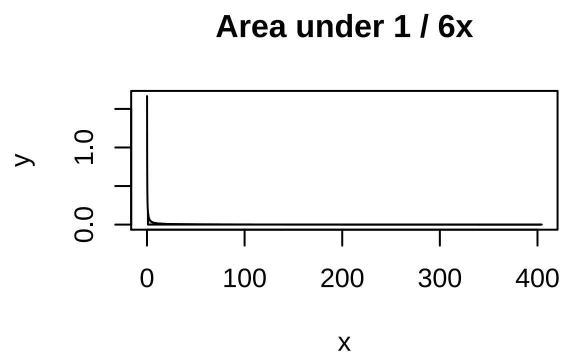
$$f(x) = \frac{1}{6x}$$

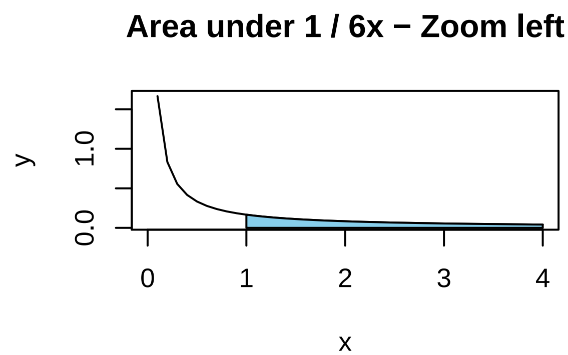
Analysis

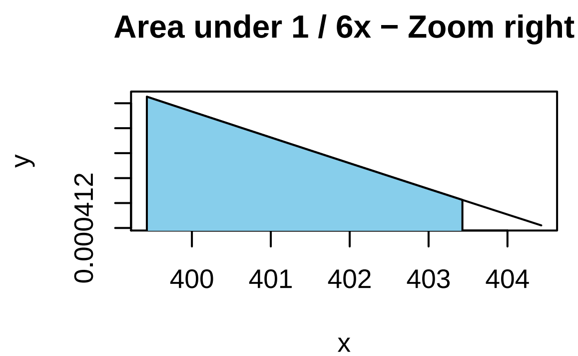
A PDF must have an area under the curve for the specified interval equal to 1. Let's plot this first.

```
x <-  seq(from = 0, to = exp(6) + 1, by = 0.1)
y.1 <- 1 / (6 * x)
y.2 <- 0 * x
shade <- x >= 1 & x <= exp(6)

par(mfrow = c(2,2))
plot(x, y.1, type = "l", col = "black",
     main = "Area under 1 / 6x",
```

```
     ylab = "y")
polygon(c(x[shade], rev(x[shade])), c(y.2[shade], rev(y.1[shade])),col="skyblue")

x <-  seq(from = 0, to = 4, by = 0.1)
y.1 <- 1 / (6 * x)
y.2 <- 0 * x
shade <- x >= 1 & x <= exp(6)

plot(x, y.1, type = "l", col = "black",
     main = "Area under 1 / 6x - Zoom left",
     ylab = "y")
polygon(c(x[shade], rev(x[shade])), c(y.2[shade], rev(y.1[shade])),col="skyblue")

x <-  seq(from = exp(6) - 4, to = exp(6) + 1, by = 0.1)
y.1 <- 1 / (6 * x)
y.2 <- 0 * x
shade <- x >= 1 & x <= exp(6)

plot(x, y.1, type = "l", col = "black",
     main = "Area under 1 / 6x - Zoom right",
     ylab = "y")
polygon(c(x[shade], rev(x[shade])), c(y.2[shade], rev(y.1[shade])),col="skyblue")
```







Integrate

$$\int_1^{e^6} \frac{1}{6x} dx$$
$$= \frac{1}{6} \int_1^{e^6} \frac{1}{x} dx$$
$$= \frac{1}{6} ln|x| \Big|_{x=1}^{e^6}$$
$$= \frac{1}{6} ln|e^6| - \frac{1}{6} ln|1|$$
$$= \frac{1}{6} \cdot 6| - \frac{1}{6} \cdot 0 = 1$$

Simple, but I like double checking.

```
(1 / 6) * log(abs(exp(6))) - (1 / 6) * log(abs(1))
```

```
## [1] 1
```

Answer

Since the area under the function is equal to 1 on the interval $[1, e^6]$, $f(x)$ is a probability density function.