DATA605 - Assignment 8

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Contents

p. 303, ex. 11

A company buys 100 lightbulbs, each of which has an exponential lifetime of 1000 hours. What is the expected time for the first of these bulbs to burn out?

We know:

1. $n = 100$. (Problem definition)
2. The expected value for each bulb's lifetime is $\mu = 1000$. (Problem definition)
3. The distribution of lifetime is exponential. (Problem definition)
4. M is the minimum of the distribution of the life of a single bulb, x_j. (Notation)
5. $E(M) = \frac{\mu}{n}$ (Given in problem 10)
6. $E(M) = \frac{1000}{100} = 10$ (1, 2, 5)

Note: This question was all about the answers hiding in plain sight. It will be worth returning to the section to learn more about exponential distributions.

Answer

The expected time for the first bulb to burn out is 10 hours.

p. 303, ex. 14

Assume that $X1$ and $X2$ are independent random variables, each having an exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density

$f_Z(z) = (1/2)\lambda e^{-\lambda|z|}$

Derivation

I base my approach on the example shown in [Grinstead], p.292. However, since I want to get to an anti-derivative with respect to x, I tailor the formula by reversing the variable names.

First, suppose we wish to sum $Z = X + Y$. From the convolution formula, its density function would be,

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)f_X(x)dx$$

Here's the trick. Substitute $(-y)$ for y, since we will want to evaluate $X_1 - X_2$.

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{-Y}(z-x)f(x)dx$$

Observe this equivalence.

$$f_{-Y}(z-x) = f_Y(x-z)$$

So,

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(x-z)f_X(x)dx$$

This integral expresses the density function for the difference between two random variables. Now, substitute the exponential function requested in this problem.

The exponential density function is,

- $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$
- 0, otherwise.

Substitute the exponential distribution into the definite integral. Note that the exponential distribution equals 0 for the range below 0. Since the integral evaluates to 0 in that range, we change the lower boundary of the integral.

$$\int_0^{\infty} \lambda e^{-\lambda(x-z)}\lambda e^{-\lambda x}dx$$

Simplify the exponents

$$= \int_0^{\infty} \lambda e^{-\lambda x+\lambda z}\lambda e^{-\lambda x}dx$$

$$= \int_0^{\infty} \lambda e^{\lambda z}\lambda e^{-2\lambda x}dx$$

The first term is a constant. Pull it out.

$$= \lambda e^{\lambda z}\int_0^{\infty} \lambda e^{-2\lambda x}dx$$

To find the anti-derivative, apply u substitution.

Let $u = -2\lambda x$. So, $du = -2\lambda dx$. The integral has λdx, but not -2, so in order to substitute du into the integral we must divide by -2 to cancel it out. The boundaries of the integral are the same, even though we're rescaling them to u.

$$= \lambda e^{\lambda z}(\frac{-1}{2})\int_0^\infty e^u du$$

$$= \lambda e^{\lambda z}(\frac{-1}{2})e^u\Big|_{u=0}^{\infty}$$

Substitute for u.

$$= \lambda e^{\lambda z}(\frac{-1}{2})e^{-2\lambda x}\Big|_{x=0}^{\infty}$$

The limit as x approaches infinity of e^x is 0. So the integral computes to,

$$= 0 - (\frac{-1}{2}\lambda e^{\lambda z} \cdot 1)$$

$$= \frac{1}{2}\lambda e^{\lambda z}$$

The rest of this relies on a proof I viewed on YouTube *here*. The instructor said that the place I got up to only considers the range of the function $f_Z(z) < 0$. You can infer the other side, since $f_Z(z$ is symmetric around $z = 0$, $f_Z(z) thus,

- $= \frac{1}{2}\lambda e^{\lambda z}$, $z < 0$
- $= \frac{1}{2}\lambda e^{-\lambda z}$, $z \geq 0$

Therefore,

$$f_Z(z) = \frac{\lambda}{2}e^{-\lambda|z|}$$

Since I don't really get why I need to partition the range of Z, or how I would know I need to do that beforehand, the wrap up of my answer feels like a bit of hand-waving.

Answer

$$f_Z(z) = \frac{\lambda}{2}e^{-\lambda|z|}$$

p. 320, ex. 1

Let X be a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 100/3$. Using Chebyshev's Inequality, find an upper bound for the following probabilities.

Note: *REA's Problem Solvers: Statistics* saved the day. I just didn't even understand what Chebyshev's Inequality was trying to do. The REA book was a GREAT suggestion. I pored over that and, once I saw it I cried, "Is THAT all it is?!"

Answer

(a) $P(|X - 10| \geq 2) <= \frac{V(X)}{k^2} = \frac{100}{3} \cdot \frac{1}{2^2} = \frac{25}{3} = 8.3333$. The most a probability can be is 1, so the answer is 1.
(b) $P(|X - 10| \geq 5) <= \frac{V(X)}{k^2} = \frac{100}{3} \cdot \frac{1}{5^2} = \frac{4}{3} = 1.3333$. The most a probability can be is 1, so the answer is 1.
(c) $P(|X - 10| \geq 9) <= \frac{V(X)}{k^2} = \frac{100}{3} \cdot \frac{1}{9^2} = \frac{100}{243} = 0.4115$
(d) $P(|X - 10| \geq 20) <= \frac{V(X)}{k^2} = \frac{100}{3} \cdot \frac{1}{20^2} = \frac{1}{12} = 0.0833$

Extra study

References

- L12.2 The Sum of Independent Discrete Random Variables. This is the clearest video I found explaining convolution.
- Convolution Theorem: Application & Examples. And this is the clearest article.

Grinstead exercises

Ex. 7.1, p. 289

A die is rolled three times. Find the probability that the sum of the outcomes is

(a) greater than 9.
(b) an odd number.

I'll use a solution that counts the probabilities that I found on the internet and adapted in `R`. This helped me understand the summation when I added traces.

```r
# PMF
P <- function(x) {
    if (x %in% c(1:6)) {
        return(1 / 6)
    } else {
        return(0)
    }
}

# S(2). Include some traces.
S2 <- function(x, verbose = F) {
    total <- 0

    if (verbose) {
        print(paste0("x=", x))
    }
    for (k in 0:x) {
        total <- total + (P(k) * P(x - k))

        if (verbose) {
            print(paste0("k=", k, ", x-k=", x-k))
            print(paste0("    P(k)=", P(k), ", P(x-k)=", P(x-k)))
            print(paste0("    P(k)P(x-k)=", P(k) * P(x - k)))
            print(paste0("    total=", total))
        }
    }
    return(total)
}

S3 <- function(x, verbose = F) {
    total <- 0
    for (k in 0:x) {
        prob <- S2(k, verbose) * P(x - k)
```

```r
        total <- total + prob
    }
    return(total)
}
```

Check out the PMF

```r
P(0)
```

```
## [1] 0
```

```r
P(3)
```

```
## [1] 0.1667
```

```r
P(7)
```

```
## [1] 0
```

Check out $P(S_2 = 2)$ and compare with p. 286.

```r
S2(2, verbose = T)
```

```
## [1] "x=2"
## [1] "k=0, x-k=2"
## [1] "    P(k)=0, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0"
## [1] "k=1, x-k=1"
## [1] "    P(k)=0.166666666666667, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0.0277777777777778"
## [1] "    total=0.0277777777777778"
## [1] "k=2, x-k=0"
## [1] "    P(k)=0.166666666666667, P(x-k)=0"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0.0277777777777778"
```

```
## [1] 0.02778
```

```r
S2(3, verbose = T)
```

```
## [1] "x=3"
## [1] "k=0, x-k=3"
## [1] "    P(k)=0, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0"
## [1] "k=1, x-k=2"
## [1] "    P(k)=0.166666666666667, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0.0277777777777778"
## [1] "    total=0.0277777777777778"
```

```
## [1] "k=2, x-k=1"
## [1] "    P(k)=0.166666666666667, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0.0277777777777778"
## [1] "    total=0.0555555555555556"
## [1] "k=3, x-k=0"
## [1] "    P(k)=0.166666666666667, P(x-k)=0"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0.0555555555555556"
```

```
## [1] 0.05556
```

Check out $P(S_2 = 2)$ and compare with p. 287.

```
S3(3, verbose = T)
```

```
## [1] "x=0"
## [1] "k=0, x-k=0"
## [1] "    P(k)=0, P(x-k)=0"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0"
## [1] "x=1"
## [1] "k=0, x-k=1"
## [1] "    P(k)=0, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0"
## [1] "k=1, x-k=0"
## [1] "    P(k)=0.166666666666667, P(x-k)=0"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0"
## [1] "x=2"
## [1] "k=0, x-k=2"
## [1] "    P(k)=0, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0"
## [1] "k=1, x-k=1"
## [1] "    P(k)=0.166666666666667, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0.0277777777777778"
## [1] "    total=0.0277777777777778"
## [1] "k=2, x-k=0"
## [1] "    P(k)=0.166666666666667, P(x-k)=0"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0.0277777777777778"
## [1] "x=3"
## [1] "k=0, x-k=3"
## [1] "    P(k)=0, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0"
## [1] "k=1, x-k=2"
## [1] "    P(k)=0.166666666666667, P(x-k)=0.166666666666667"
## [1] "    P(k)P(x-k)=0.0277777777777778"
## [1] "    total=0.0277777777777778"
## [1] "k=2, x-k=1"
## [1] "    P(k)=0.166666666666667, P(x-k)=0.166666666666667"
```

```
## [1] "    P(k)P(x-k)=0.0277777777777778"
## [1] "    total=0.0555555555555556"
## [1] "k=3, x-k=0"
## [1] "    P(k)=0.166666666666667, P(x-k)=0"
## [1] "    P(k)P(x-k)=0"
## [1] "    total=0.0555555555555556"
```

```
## [1] 0.00463
```

Answer

- Part a

```
total <- 0
for (i in 1:9) {
    total <- total + S3(i)
}

1 - total
```

```
## [1] 0.625
```

- Part b

```
total <- 0
for (i in 0:((3 * 6) - 1)) {
    if (i %% 2 == 1) {
        total <- total + S3(i)
    }
}

1 - total
```

```
## [1] 0.5
```

Ex. 8.1.1

A fair coin is tossed 100 times. The expected number of heads is 50, and the standard deviation for the number of heads is (100 · 1/2 · 1/2)1/2 = 5. What does Chebyshev's Inequality tell you about the probability that the number of heads that turn up deviates from the expected number 50 by three or more standard deviations (i.e., by at least 15)?

Note:

- What's given about standard deviation? We know the number of trials is 100 and the probability of the outcome heads. The formula for calculating standard deviation for the binomial distribution is $\sigma = \sqrt{npq}$.
- The question asks for *at least* 3 standard deviations. Use the tail form of the theorem.

$P(|X - \mu| \geq k\sigma) \geq \frac{1}{k^2}$

So, $P(|X - 50| \geq 3 \cdot 5) >= \frac{1}{9} = 0.1111$.

Answer

The probability that the number of heads is at least 15 is, at most, 0.1111.

Ex. 8.1.2

Write a program that uses the function binomial(n, p, x) to compute the exact probability that you estimated in Exercise 1. Compare the two results.

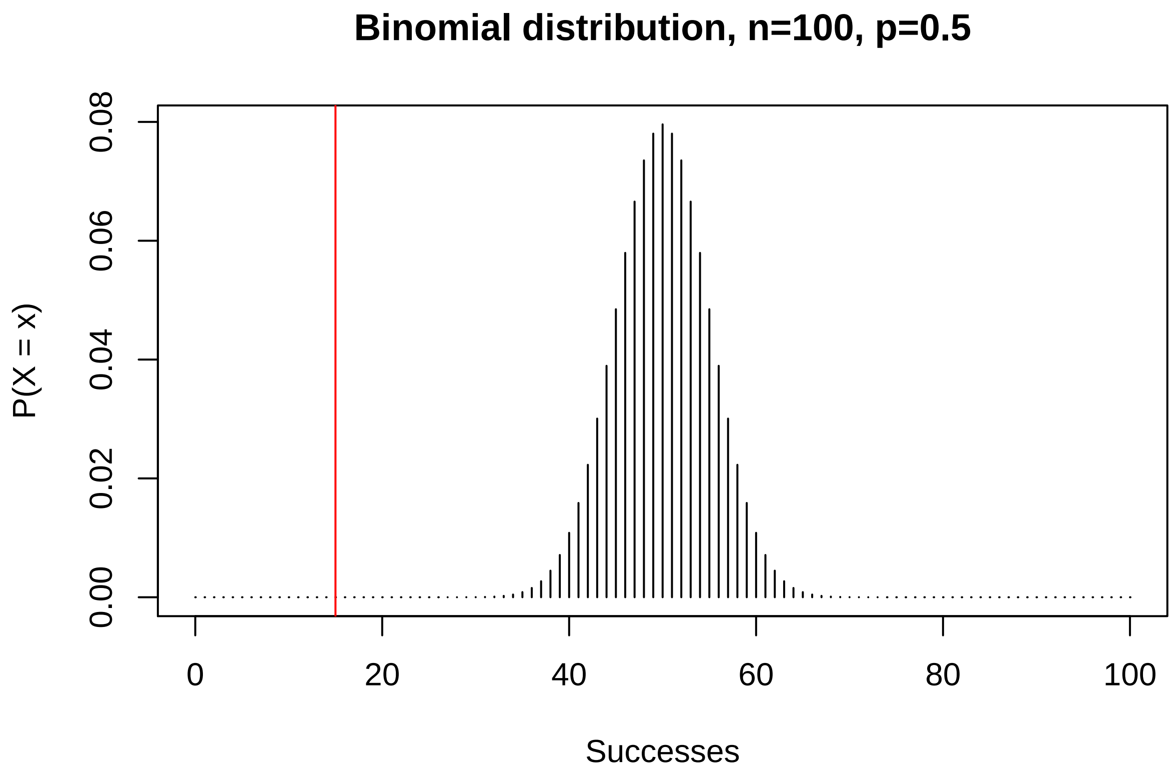
I wrote this function to visualize and calculate all the binomial intervals when I reviewed the binomial distribution section for DATA606. *Note:* there still might be some work to do on some of the boundaries.

```
binom_summary <- function(successes, trials, prob,
                          main = NULL, xlab = "Successes", ylab = "P(X = x)") {
  # Annotate plot.
  param_str <- paste0(", n=", trials, ", p=", prob)
  main <- ifelse(is.null(main), paste0("Binomial distribution", param_str),
                 paste0(main, param_str))

  plot(0:trials, dbinom(0:trials, trials, prob = prob),
     xlab = xlab,
     ylab = ylab,
     type = "h",
     main = main)
  abline(v = successes, col = "red")

  mu <- trials * prob
  sigma <- sqrt(trials * prob * (1 - prob))
  is_normal_approx <- trials * prob >= 10 & trials * (1 - prob) >= 10

  print(paste0("mu: ", mu))
  print(paste0("sigma: ", sigma))

  print(paste0("Trials: ", trials))
  print(paste0("Successes: ", successes))

  prob.exact <- dbinom(successes, trials, prob)

  prob.cumulative.lt <- pbinom(successes - 1, trials, prob)
  # Normal approximation
  prob.cumulative.lt.approx <- pnorm(successes, mu, sigma)
  # Normal approximation with continuity correction.
  prob.cumulative.lt.approx.corr <- pnorm(successes - 0.5, mu, sigma)

  prob.cumulative.le <- pbinom(successes, trials, prob)
  # Normal approximation
  prob.cumulative.le.approx <- pnorm(successes, mu, sigma)
  # Normal approximation with continuity correction.
  prob.cumulative.le.approx.corr <- pnorm(successes + 0.5, mu, sigma)

  prob.cumulative.gt <- pbinom(successes, trials, prob, lower.tail = F)
  # Normal approximation
  prob.cumulative.gt.approx <- pnorm(successes, mu, sigma, lower.tail = F)
  # Normal approximation with continuity correction.
  prob.cumulative.gt.approx.corr <- pnorm(successes + 0.5, mu, sigma, lower.tail = F)
```

```r
  prob.cumulative.ge <- pbinom(successes - 1, trials, prob, lower.tail = F)
  # Normal approximation
  prob.cumulative.ge.approx <- pnorm(successes, mu, sigma, lower.tail = F)
  # Normal approximation with continuity correction.
  prob.cumulative.ge.approx.corr <- pnorm(successes - 0.5, mu, sigma, lower.tail = F)

  print(paste0("Binomial probability. P(X=x): ", round(prob.exact, 4)))

  print(paste0("Cumulative probability. P(X<x): ", round(prob.cumulative.lt, 4)))
  if (is_normal_approx) {
    print(paste0("    Normal approximation. P(X<x): ",
                 round(prob.cumulative.lt.approx, 4)))
    print(paste0("    Normal approximation w/ continuity correction. P(<x): ",
                 round(prob.cumulative.lt.approx.corr, 4)))
  }

  print(paste0("Cumulative probability. P(X<=x): ", round(prob.cumulative.le, 4)))
  if (is_normal_approx) {
    print(paste0("    Normal approximation. P(X<=x): ",
                 round(prob.cumulative.le.approx, 4)))
    print(paste0("    Normal approximation w/ continuity correction. P(<=x): ",
                 round(prob.cumulative.le.approx.corr, 4)))
  }

  print(paste0("Cumulative probability. P(X>x): ", round(prob.cumulative.gt, 4)))
  if (is_normal_approx) {
    print(paste0("    Normal approximation. P(X>x): ",
                 round(prob.cumulative.gt.approx, 4)))
    print(paste0("    Normal approximation w/ continuity correction. P(X>x): ",
                 round(prob.cumulative.gt.approx.corr, 4)))
  }

    print(paste0("Cumulative probability. P(X>=x): ", round(prob.cumulative.ge, 4)))
  if (is_normal_approx) {
    print(paste0("    Normal approximation. P(X>=x): ",
                 round(prob.cumulative.ge.approx, 4)))
    print(paste0("    Normal approximation w/ continuity correction. P(X>=x): ",
                 round(prob.cumulative.ge.approx.corr, 4)))
  }
}
```

I ran it for this problem (incorrectly).

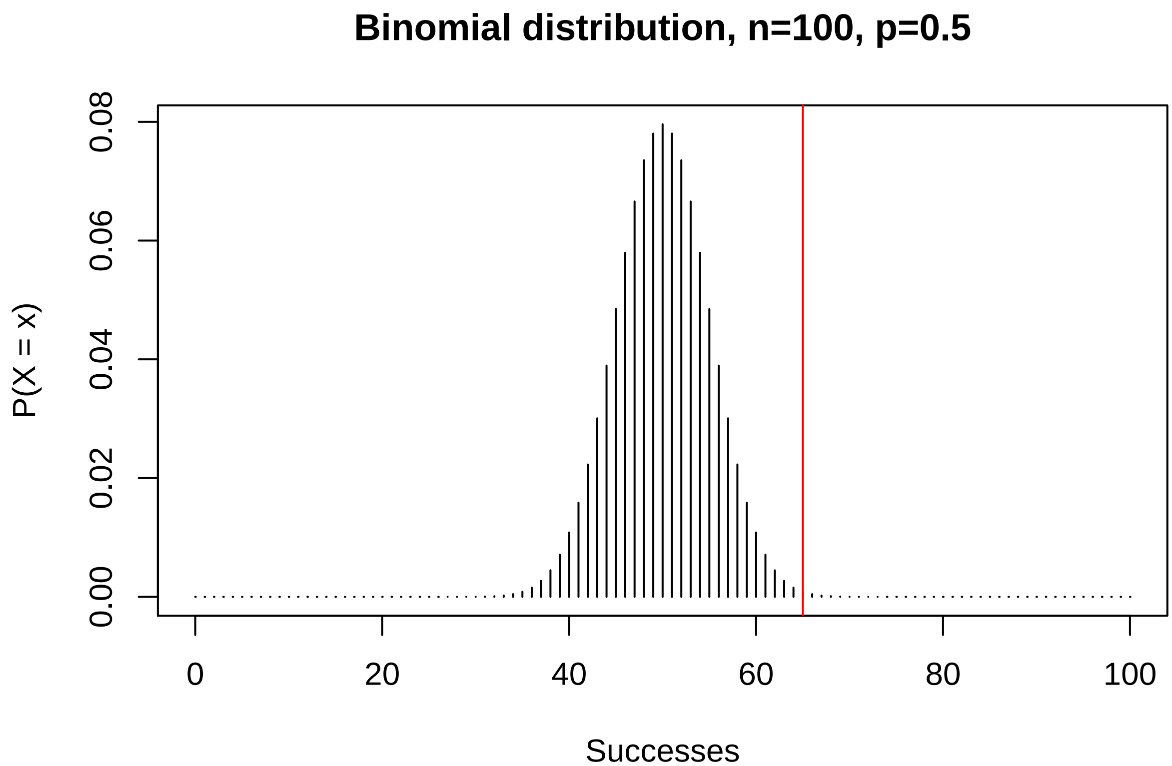
```r
binom_summary(15, 100, 0.5)
```




```
## [1] "mu: 50"
## [1] "sigma: 5"
## [1] "Trials: 100"
## [1] "Successes: 15"
## [1] "Binomial probability. P(X=x): 0"
## [1] "Cumulative probability. P(X<x): 0"
## [1] "    Normal approximation. P(X<x): 0"
## [1] "    Normal approximation w/ continuity correction. P(<x): 0"
## [1] "Cumulative probability. P(X<=x): 0"
## [1] "    Normal approximation. P(X<=x): 0"
## [1] "    Normal approximation w/ continuity correction. P(<=x): 0"
## [1] "Cumulative probability. P(X>x): 1"
## [1] "    Normal approximation. P(X>x): 1"
## [1] "    Normal approximation w/ continuity correction. P(X>x): 1"
## [1] "Cumulative probability. P(X>=x): 1"
## [1] "    Normal approximation. P(X>=x): 1"
## [1] "    Normal approximation w/ continuity correction. P(X>=x): 1"
```

How great are pictures?! The successes I input, 15, is plotted with the red line. Instantly, I see my mistake. The successes I'm supposed to check don't amount to 15, they are the mean PLUS 3 standard deviations. Let's try again.

```
binom_summary(50 + 15, 100, 0.5)
```




```
## [1] "mu: 50"
## [1] "sigma: 5"
## [1] "Trials: 100"
## [1] "Successes: 65"
## [1] "Binomial probability. P(X=x): 9e-04"
## [1] "Cumulative probability. P(X<x): 0.9982"
## [1] "    Normal approximation. P(X<x): 0.9987"
## [1] "    Normal approximation w/ continuity correction. P(<x): 0.9981"
## [1] "Cumulative probability. P(X<=x): 0.9991"
## [1] "    Normal approximation. P(X<=x): 0.9987"
## [1] "    Normal approximation w/ continuity correction. P(<=x): 0.999"
## [1] "Cumulative probability. P(X>x): 9e-04"
## [1] "    Normal approximation. P(X>x): 0.0013"
## [1] "    Normal approximation w/ continuity correction. P(X>x): 0.001"
## [1] "Cumulative probability. P(X>=x): 0.0018"
## [1] "    Normal approximation. P(X>=x): 0.0013"
## [1] "    Normal approximation w/ continuity correction. P(X>=x): 0.0019"
```

Answer

We want both tails, so twice the tail probability is 0.0036, rounded to 4 places. I'll calculate that again without rounding. The answer, 0.0035. That is consistent with my answer for the previous problem which said that the probability *at most* is 0.1111. There can be a big margin around the results we calculate using Chebyshev's Inequality.