2/ $\begin{cases} a_0 = 1\,;\; a_1 = 2 \\ a_{n+2} = 5a_{n+1} - 4a_n \end{cases} \forall n \geq 0$

$$G(x) = \sum_{i=0}^{\infty} a_i\, x^i$$

$$= 1 + x + \sum_{i=0}^{\infty} a_{n+2} x^{n+2}$$

$$= 1 + x + \sum_{i=0}^{\infty} (5a_{n+1} - 4a_n) x^{n+2}$$

$$= 1 + x + \sum_{i=0}^{\infty} (5a_{n+1}) x^{n+2} - \sum_{i=0}^{\infty} (4a_n) x^{n+2}$$

$$= 1 + x + 5x \sum_{i=0}^{\infty} (a_{n+1}) x^{n+1} - 4x^2 \sum_{i=0}^{\infty} (a_n) x^n$$

$$= 1 + x + 5x \sum_{i=1}^{\infty} (a_i) x^i - 4x^2 \sum_{i=0}^{\infty} (a_n) x^n$$

$$= 1 + x + 5x[G(x) - a_0] - 4x^2 G(x)$$

$$= 1 + x + 5x[G(x) - 1] - 4x^2 G(x)$$

$$= 1 + x + 5xG(x) - 5x - 4x^2 G(x)$$

$$= 1 - 4x + 5xG(x) - 4x^2 G(x)$$

$$= 1 - 4x + G(x)(5x - 4x^2)$$

$$\Longrightarrow G(x) = \frac{1 - 4x}{1 - 5x + 4x^2}$$

Ta được:

$$\frac{A}{1 - x} + \frac{B}{1 - 4x}$$

Quy đồng:

$$\frac{A(1 - 4x) + B(1 - x)}{(1 - x)(1 - 4x)} = \frac{A - 4Ax + B - Bx}{(1 - x)(1 - 4x)}$$

$$\begin{cases} A + B & = 1 \\ -4A - B & = -4 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 0 \end{cases}$$

$$\Longrightarrow G(x) = \frac{1}{1-x} + \frac{0}{1-4x}$$

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

Xét $x \leq \min\left\{|1|; \left|\frac{1}{4}\right|\right\}$

$$G(x) = \sum_{i=0}^{\infty} x^i$$

3/ $\begin{cases} a_0 = 1\,;\ a_1 = 1 \\ a_{n+2} = a_{n+1} + 2a_n \end{cases} \forall n \geq 0$

$$G(x) = \sum_{i=0}^{\infty} a_i\, x^i$$

$$= 1 + x + \sum_{i=0}^{\infty} a_{n+2} x^{n+2}$$

$$= 1 + x + \sum_{i=0}^{\infty} (a_{n+1} + 2a_n) x^{n+2}$$

$$= 1 + x + \sum_{i=0}^{\infty} (a_{n+1}) x^{n+2} + \sum_{i=0}^{\infty} (2a_n) x^{n+2}$$

$$= 1 + x + x\sum_{i=0}^{\infty} (a_{n+1}) x^{n+1} + 2x^2 \sum_{i=0}^{\infty} (a_n) x^n$$

$$= 1 + x + x\sum_{i=1}^{\infty} (a_i) x^i + 2x^2 \sum_{i=0}^{\infty} (a_n) x^n$$

$$= 1 + x + x[G(x) - a_0] + 2x^2 G(x)$$

$$= 1 + x + x[G(x) - 1] + 2x^2 G(x)$$

$$= 1 + x + xG(x) - x + 2x^2 G(x)$$

$$= 1 + xG(x) + 2x^2(x)$$

$$= 1 + G(x)(x + 2x^2)$$

$$\Longrightarrow G(x) = \frac{1}{1 - x - 2x^2}$$

Ta được:

$$\frac{A}{1-x} + \frac{B}{1+2x}$$

Quy đồng:

$$\frac{A(1+2x) + B(1-x)}{(1-x)(1+2x)} = \frac{A + 2Ax + B - Bx}{(1-x)(1+2x)}$$

$$\begin{cases} A + B &= 1 \\ 2A - B &= 0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{1}{3} \\ B = \dfrac{2}{3} \end{cases}$$

$$\Longrightarrow G(x) = \frac{1}{3(1-x)} + \frac{2}{3(1+2x)}$$

$$\frac{1}{(1-x)} = \frac{1}{3}\sum_{i=0}^{\infty} x^i$$

$$\frac{2}{3(1+2x)} = \frac{2}{3}\sum_{i=0}^{\infty} (-2x)^i$$

Xét $x \le \min\left\{|-1|; \left|\frac{1}{2}\right|\right\}$

$$G(x) = \frac{1}{3}\sum_{i=0}^{\infty} x^i + \frac{2}{3}\sum_{i=0}^{\infty} (-2x)^i$$

4/ $\begin{cases} a_0 = 1\ ;\ a_1 = 3 \\ a_{n+2} = 7a_{n+1} - 12a_n \end{cases} \forall n \ge 0$

$$G(x) = \sum_{i=0}^{\infty} a_i\, x^i$$

$$= 1 + x + \sum_{i=0}^{\infty} a_{n+2} x^{n+2}$$

$$= 1 + x + \sum_{i=0}^{\infty} (7a_{n+1} - 12a_n) x^{n+2}$$

$$= 1 + x + \sum_{i=0}^{\infty}(7a_{n+1})x^{n+2} - \sum_{i=0}^{\infty}(12a_n)x^{n+2}$$

$$= 1 + x + 7x\sum_{i=0}^{\infty}(a_{n+1})x^{n+1} - 12x^2\sum_{i=0}^{\infty}(a_n)x^n$$

$$= 1 + x + 7x\sum_{i=1}^{\infty}(a_i)x^i - 12x^2\sum_{i=0}^{\infty}(a_n)x^n$$

$$= 1 + x + 7x[G(x) - a_0] - 12x^2G(x)$$

$$= 1 + x + 7x[G(x) - 1] - 12x^2G(x)$$

$$= 1 + x + 7xG(x) - 7x - 12x^2G(x)$$

$$= 1 - 6x + 7xG(x) - 12x^2(x)$$

$$= 1 - 6x + G(x)(7x - 12x^2)$$

$$\Longrightarrow G(x) = \frac{1 - 6x}{1 - 7x + 12x^2}$$

Ta được:

$$\frac{A}{1 - 3x} + \frac{B}{1 - 4x}$$

Quy đồng:

$$\frac{A(1 - 4x) + B(1 - 3x)}{(1 - 3x)(1 - 4x)} = \frac{A - 4Ax + B - 3Bx}{(1 - 3x)(1 - 4x)}$$

$$\begin{cases} A + B & = 1 \\ -4A - 3B & = -6 \end{cases} \Rightarrow \begin{cases} A = 3 \\ B\ = -2 \end{cases}$$

$$\Longrightarrow G(x) = \frac{3}{(1 - 3x)} + \frac{-2}{(1 - 4x)}$$

$$\frac{1}{(1 - 3x)} = 3\sum_{i=0}^{\infty}(3x)^i$$

$$\frac{-2}{(1 - 4x)} = -2\sum_{i=0}^{\infty}(4x)^i$$

Xét $x \leq \min\left\{\left|\frac{1}{3}\right|; \left|\frac{1}{4}\right|\right\}$

$$G(x) = 3\sum_{i=0}^{\infty}(3x)^i - 2\sum_{i=0}^{\infty}(4x)^i$$