Bài Tập 1: cho

$$G_n(z) = \sum_{k=0}^{\infty} p_{n,k} z^k$$

Tính $G_n(Z)$ bằng phương pháp hàm sinh của dãy xác suất

Bài Làm:

$$G_n(z) = \sum_{k=0}^{\infty} p_{n,k} z^k$$

$$= p_{n,0} + \sum_{k=0}^{\infty} p_{n,k+1} z^{k+1}$$

$$= p_{n,0} + \sum_{k=0}^{\infty} (\frac{1}{n} p_{n-1,k} + (\frac{n-1}{n}) p_{n-1,k-1})\, z^{k+1}$$

$$= \frac{1}{n} + \frac{z}{n} \sum_{k=0}^{\infty} p_{n-1,k} z^k + \frac{n-1}{n} \sum_{k=0}^{\infty} p_{n-1,k+1} z^{k+1}$$

$$= \frac{1}{n} + \frac{z}{n} G_{n-1}(Z) + \frac{n-1}{n} \sum_{k=0}^{\infty} p_{n-1,i} z^i \quad (i = k+1)$$

$$= \frac{1}{n} + \frac{z}{n} G_{n-1}(z) + \frac{n-1}{n} \left(G_{n-1}(z) - \frac{1}{n-1} \right)$$

$$= \frac{z+n-1}{n} G_{n-1}(z)$$

Với

$$G_n(z) = \frac{z+n-1}{n} G_{n-1}(z)$$

Tiếp tục triển khai:

$$G_{n-1}(z) = \frac{z+n-2}{n-1} G_{n-2}(z)$$

$$G_{n-2}(z) = \frac{z+n-3}{n-2} G_{n-3}(z)$$

$$\dots \dots \dots$$

$$G_3(z) = \frac{z+2}{3} G_2(z)$$

$$G_n(z) = \prod_{i=2}^{n} \frac{z+i-1}{i}$$

Đặt

$$a_n = \left. \frac{dG_n(z)}{dz} \right|_{z=1}$$

Tính đạo hàm ta được :

$$a_n = \left. \frac{dG_n(z)}{dz} \right|_{z=1} = \left. \frac{d(\prod_{i=2}^{n} \frac{z+i-1}{i})}{dz} \right|_{z=1} = \sum_{k=2}^{n} \frac{1}{i}$$

Vậy

$$a_n = \sum_{k=2}^{n} \frac{1}{i}$$