On Exotic Spheres

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1 Introduction

Seminal work done by Milnor and others starting in the 1950s ushered in the golden age of what we now call *differential topology*. One of these works was the discovery of *exotic spheres*, manifolds of dimension n that are homeomorphic but not diffeomorphic to the standard Euclidean sphere S^n. Before this era it was thought that smooth structures on manifolds were more of a formality. It was known that every 2-manifold had a unique smooth structure, and in 1952, Edwin Moise showed the same result applied to 3-manifolds (a result now known as *Moise's Theorem*). It was thus thought that this would hold true in higher dimensions. In 1956, the first example of an exotic sphere was discovered by Milnor in dimension 7, earning him the Fields Medal and Abel prize. Quoting Sletsjøe on the awarding of Milnor's Abel prize,

> "Milnor's discovery of exotic smooth spheres in seven dimensions was completely unexpected. It signaled the arrival of differential topology and an explosion of work by a generation of brilliant mathematicians; this explosion has lasted for decades and changed the landscape of mathematics. With Michel Kervaire, Milnor went on to give a complete inventory of all the distinct differentiable structures on spheres of all dimensions; in particular they showed that the 7-dimensional sphere carries exactly 28 distinct differentiable structures. They were among the first to identify the special nature of four-dimensional manifolds, foreshadowing fundamental developments in topology."

In this paper we present a detailed introduction to the theory and background needed to construct the first exotic $7-$sphere, aimed at advanced undergraduates. As this was my first time learning many of the tools covered here, I try to be very explicit with my definitions and exposition.

In Section 2.1, we cover the classification of vector bundles over S^n. We show that S^3 bundles over S^4 are uniquely characterized by their behaviour on the boundary of S^4, given by *clutching functions*. We then note such clutching functions are classified by $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, giving us a robust classification of our bundles of choice. We construct our exotic sphere by choosing appropriate pairs of integers.

In Section 2.2, we compute the homology of our candidate manifold $\mathcal{M}$. $\mathcal{M}$ is simply connected, and we show that it has the same homotopy type as S^7, and so $\mathcal{M}$ is homotopy equivalent to S^7. The topological Poincaré conjecture then implies our manifold is a topological $7-$sphere.

In Section 2.3, we present another approach to showing that $\mathcal{M}$ is a topological sphere through the use of Morse theory. Indeed this was Milnor's original strategy. The main tool here is *Reeb's Theorem*. Namely, we show there exists a smooth function on $\mathcal{M}$ with only two critial points both non-degenerate, giving us that it is homeomorphic to S^7.

The second half of the paper deals with showing that $\mathcal{M}$ is not diffeomorphic to S^7. This is done via constructing a diffeomorphism invariant on closed manifolds homeomorphic to S^7. We show that this invariant is 0 on S^7, and with an appropriate choice of clutching function, is non-zero on $\mathcal{M}$. Milnor's invariant is defined using the theory of characteristic classes, and a special case of the Hirzebruch signature theorem.

In Section 3.1 we introduce the theory of characteristic classes and define the Euler, Chern, and Pontryagin classes. In Section 3.2, we define the Pontryagin number in terms of the Pontryagin class. Milnor's invariant is defined in Section 3.3 in terms of the signature and Pontryagin number of a manifold B that bounds our manifold $\mathcal{M}$. A key result is that this invariant is independent of the manifold B. This is proven using the Hirzebruch signature theorem.

More precisely, the idea comes from the fact that any 8-manifold B which bounds our 7-manifold $\mathcal{M}$ has non-trivial $H_4(B)$, combined with the fact that $H_4(D^8) = 0$. Thus we wish to find some invariant that captures this property i.e., doesn't change on 8-manifolds with diffeomorphic boundary. Our investigation involves degree 4 homology groups, which for 8-manifolds are isomorphic to their degree 4 cohomology groups by Poincaré homology, and it is exactly these observations that incentivizes the use of Pontryagin classes, invariants on real vector bundles defined by cohomology classes of degree of multiple 4. In particular, they are defined in terms of cohomology classes of the *tangent bundle* of the space, and so are diffeomorphism invariant.

All this to say, the main reason why our manifold $\mathcal{M}$ is not diffeomorphic to S^7 is due to the non-triviality of any manifold B that bounds $\mathcal{M}$, which does not occur with S^7.

We end with a brief overview of some particularly interesting results.

2 Sphere Bundles

2.1 Classification

The motivation for this construction came from the *Hopf fibration*, special fiber bundles whose total space, base space and fibers are all spheres, and can be defined using quaternions. The Hopf fibration locally describes spheres, and was used to prove higher homotopy groups of spheres can be non-trivial. The main takeaway is we can extract much information by locally describing spaces using vector bundles. In particular, S^7 is an S^3 bundle over S^4. We briefly review this construction. Quaternionic projective space $\mathbb{HP}^1$ is diffeomorphic to S^4. We then take the restriction of the canonical bundle $\pi : \mathbb{H}^2 \setminus \{0\} \to \mathbb{HP}^1$ over S^7, giving us an S^3 bundle over $\mathbb{HP}^1 = S^4$ with total space S^7. The idea then is to construct a similar vector bundle in such a way that gives rise to a total space homeomorphic to S^7.

Any vector bundle gives a unique sphere bundle by taking a unit sphere over each fiber. Our goal then is to construct vector bundles over spheres. This gives us a sphere bundle where the base and total space are also spheres, a construction analogous to the ones previously discussed. Any vector bundle $\pi : E \to S^k$ is also a vector bundle over the hemispheres. Since the hemispheres are contractible, we have $\pi^{-1}(D^k_\pm) \cong D^k_\pm \times F$. Thus any vector bundle over S^k is trivial on the upper and lower hemispheres. So the only data that distinguishes bundles over S^k is what occurs at the intersection of the hemispheres i.e., how the hemispheres are glued together via transition maps.

Given two local trivialization charts $(U_i, \varphi_i), (U_j, \varphi_j)$, we have the following map $\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ where $(x, \ell) \mapsto (x, \ell_x)$. Fixing x and varying ℓ, we obtain for each x a homeomorphism $f(x) : F \to F$. These are called the bundle's *transition maps.* f is a function $f : U_i \cap U_j \to \text{Diff}(F)$.

Consider S^n as the union of the upper and lower hemispheres, $S^n = D^n_+ \cup D^n_-$. Given a map $f : S^{n-1} \to \text{GL}_k(\mathbb{R})$, there always exists a vector bundle with transition maps given by f.

Proposition 2.1.1. *Let $f : S^{n-1} \to GL_k(\mathbb{R})$. Let E_f be the quotient space defined by the disjoint union $(D^n_+ \times \mathbb{R}^k) \sqcup (D^n_- \times \mathbb{R}^k)$ quotiented by the relation $\sim$ defined by $(x, v) \sim (x, f(x)(v))$. Then the natural projection $p : E_f \to S^n$ defined by $(x, v) \mapsto (x, 0)$ is a $k-$dimensional vector bundle.*

Proof. Since $\mathbb{R}^k$ is a $k-$dimensional vector space, it is clear that for every $x \in S^n, p^{-1}(x) \approx \mathbb{R}^k$ is a $k-$dimensional vector space. To find local trivializations, let U_+ and U_- be the two hemispheres of S^n but slightly enlarged by ϵ. The preimage under p of the intersection of $U_\pm$ is diffeomorphic to $(-\epsilon, \epsilon) \times S^{n-1} \times \mathbb{R}^k$ (it is a band around the center of the sphere e.g., $p(\{0\} \times S^{n-1} \times \mathbb{R}^k) = S^{n-1}$ the equator). Define $(t, x, v) \mapsto (t, x, f_x^{-1}(v))$. This is clearly a linear isomorphism that takes fibers to fibers. Thus we have a $k-$dimensional vector bundle. $\square$

In other words, given a map f into the homeomorphisms of the fiber, we have a corresponding vector bundle whose transitions maps are defined by f. The essence of this result is that we are gluing two bundles along the boundary using f, giving us another bundle. We now show that if the fibers are taken to be $\mathbb{C}^n$, this correspondence is an isomorphism up to homotopy.

Proposition 2.1.2. *Let $[S^{k-1}, GL_n(\mathbb{C})]$ denote the space of homotopy classes of maps $S^{k-1} \to GL_n(\mathbb{C})$. The map $\Phi : [S^{k-1}, GL_n(\mathbb{C})] \to Vect^n_{\mathbb{C}}(S^k)$ sending a clutching function f to the vector bundle E_f is bijective.*

Proof. We construct an inverse $\Psi : \mathrm{Vect}^n_{\mathbb{C}}(S^k) \to [S^{k-1}, \mathrm{GL}_n(\mathbb{C})]$. Consider an $n-$dimensional vector bundle $p : E \to S^k$ over S^k i.e., $p \in \mathrm{Vect}^n_{\mathbb{C}}(S^k)$. The hemispheres are contractible, so the restriction of p over the hemispheres of S^k are trivial. Denote the restrictions $E_+ = p^{-1}(D^k_+), E_- = p^{-1}(D^k_-)$. Consider trivializations $h_+ : E_+ \to D^k_+ \times \mathbb{C}^n$ and $h_- : E_- \to D^k_- \times \mathbb{C}^n$. Recall that $D^k_- \cap D^k_+ = \partial D^k_\pm = S^{k-1}$. In particular we have the composition

$$h_+h_-^{-1} : D^k_- \cap D^k_+ \times \mathbb{C}^n = S^{k-1} \times \mathbb{C}^n \to E_\pm \to D^k_- \cap D^k_+ \times \mathbb{C}^n = S^{k-1} \times \mathbb{C}^n$$

i.e., it defines a clutching map $h : S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$. Define $\Psi(E)$ to be the homotopy class of this clutching map. To show this is well-defined we need to show this homotopy class is unique up to local trivializations. Choose different local trivializations g_+, g_-. Then $g := g_+g_-^{-1}$ is another clutching map. We must show that g is homotopic to h. In other words, there is a homotopy $F : S^{k-1} \times I \to \mathrm{GL}_n(\mathbb{C})$ such that $F_0 = h$ and $F_1 = g$. It is known that $\mathrm{GL}_n(\mathbb{C})$ is path-connected, which immediately gives us a homotopy i.e., the clutching map is unique up to homotopy. It is evident that Ψ is an inverse of Φ. $\square$

We wish to classify real vector bundles. To do so we must place greater restrictions on our bundles: the above proof uses the fact that $\mathrm{GL}_n(\mathbb{C})$ is path-connected, but if we attempt to use the same strategy for real bundles, we run into the problem that $\mathrm{GL}_n(\mathbb{R})$ is *not* path-connected. This can be remedied by taking the path-connected components of $\mathrm{GL}_n(\mathbb{R})$. It turns out that this classifies real bundles of positive orientation.

Definition 2.1.3. An *orientation* of a real vector bundle $p : E \to B$ is a function assigning an orientation to each fiber such that near each point of B there is a local trivialization $h : p^{-1}(U) \to U \times \mathbb{R}^n$ carrying orientations of fibers in $p^{-1}(U)$ to the standard orientation of $\mathbb{R}^n$ in the fibers of $U \times \mathbb{R}^n$.

Proposition 2.1.4. *Let $GL_n^+(\mathbb{R})$ denote the subgroup of $GL_n(\mathbb{R})$ with positive determinant and $Vect_+^n(S^k)$ the isomorphism classes of real vector bundles with positive orientation. The map $\Phi : [S^{k-1}, GL_n^+(\mathbb{R})] \to Vect_+^n(S^k)$ sending a clutching function f to the vector bundle E_f is bijective.*

Proof. This follows the same argument as the complex case above. Local trivializations define clutching maps $h : S^{k-1} \to \mathrm{GL}_n^+(\mathbb{R})$. $\mathrm{GL}_n^+(\mathbb{R})$ is path-connected, so these clutching maps are unique up to homotopy, and this defines an inverse map $\Psi : \mathrm{Vect}_+^n(S^k) \to [S^{k-1}, \mathrm{GL}_n^+(\mathbb{R})]$ $\square$

Proposition 2.1.5. *$SO(n)$ is a deformation retract of $GL_n^+(\mathbb{R})$.*

Proof. Gram-Schmidt orthogonalization can be realized by a homotopy, providing a deformation retract of $\mathrm{GL}_n(\mathbb{R})$ onto $\mathrm{O}(n)$. Restricting to $\mathrm{GL}_n^+(\mathbb{R})$ provides a deformation retract onto $\mathrm{SO}(n)$. $\square$

Corollary 2.1.6. *The natural map $[S^{k-1}, SO(n)] \to [S^{k-1}, GL_n^+(\mathbb{R})]$ is a bijection.*

Proof. Any map $S^{k-1} \to \mathrm{GL}_n^+(\mathbb{R})$ is homotopic to a map $S^{k-1} \to \mathrm{SO}(n)$ via the deformation retraction above. If we have two maps $S^{k-1} \to \mathrm{GL}_n^+(\mathbb{R})$ both homotopic to a map $S^{k-1} \to \mathrm{SO}(n)$, then the two maps must be homotopic themselves, also through the deformation retraction $\mathrm{GL}_n^+(\mathbb{R}) \to \mathrm{SO}(n)$. $\square$

Corollary 2.1.7. *The groups $\pi_{k-1}(SO(n)) = [S^{k-1}, SO(n)]$ and $Vect_+^n(S^k)$ are bijective.*

Proof. $\mathrm{Vect}_+^n(S^k) = [S^{k-1}, \mathrm{GL}_n^+(\mathbb{R})] = [S^{k-1}, \mathrm{SO}(n)]$ $\square$

By noting that $\mathrm{SO}(4)$ is a deformation retract of $\mathrm{GL}_4^+(\mathbb{R})$ and that the homotopy classes of the maps S^3 into these spaces were naturally isomorphic, we now have a natural isomorphism between real bundles of positive orientation, and bundles with fiber S^3. This allows us to classify S^3 bundles over S^4; all we need to know is the structure of $\pi_3(\mathrm{SO}(4))$.

Proposition 2.1.8. $\pi_3(\mathrm{SO}(4))$ *is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}$.

Proof. We invoke Proposition 4.1 of [1], a special case of which is that a covering space projection $p : S^3 \times S^3 \to \mathrm{SO}(4)$ induces an isomorphism $\pi_3(S^3 \times S^3) \to \pi_3(\mathrm{SO}(4))$. The map $p : S^3 \times S^3 \to \mathrm{SO}(4)$ is given by $(u, v) \mapsto (x \mapsto uxv^{-1})$, where we treat $\mathbb{R}^4$ as the quaternions and S^3 as the unit quaternions. This is a group homomorphism with kernel $\{(1, 1), (-1, -1)\}$, so is a double covering of SO(4), and so $\pi_3(S^3 \times S^3) = \pi_3(\mathrm{SO}(4))$. Since S^3 is path-connected, we have $\pi_3(S^3 \times S^3) = \pi_3(S^3) \times \pi_3(S^3)$ (Proposition 4.2 of [1]). We also have $\pi_3(S^3) = \mathbb{Z}$ (the calculation is given in the next section). Thus $\pi_3(\mathrm{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$. $\square$

The preceding exposition gives us an explicit isomorphism $(i, j) \mapsto (u \mapsto (v \mapsto u^i v u^j))$, where we have chosen to identify bundles with clutching maps. To construct his sphere, Milnor considered S^3 bundles over S^4 with structure group SO(4), which we now know are classified by $\pi_3(\mathrm{SO}(4))$. He chose this specific structure group because $\pi_3(\mathrm{SO}(4))$ is unusually large, inducing many more bundles than other groups. For example $\pi_3(\mathrm{SO}(4))$ is the largest homotopy group out of all the groups $\pi_n(\mathrm{SO}(k))$ where $1 \leq n \leq 9, 2 \leq k \leq 6$, which are either $0, \mathbb{Z}, \mathbb{Z}_i$, or $\mathbb{Z}_i \oplus \mathbb{Z}_i$. This result is summarized below.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
SO(2)	$\mathbb{Z}$	0	0	0	0	0	0	0	0
SO(3)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
SO(4)	$\mathbb{Z}_2$	0	$(\mathbb{Z})^2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_{12})^2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_3)^2$
SO(5)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	0	0
SO(6)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$

The hope was that one of these many options in $\pi_3(\mathrm{SO}(4))$ would be an exotic one. We now define our candidate manifolds as the total spaces of the S^3 bundles over S^4, and identify them by pairs $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$. Denote these bundles as ξ_{ij}, and their total space as $\mathcal{M}_{ij}$.

For appropriate values of i and j, we present two methods of proving that $\mathcal{M}_{ij}$ is homeomorphic to S^7. The first approach is done via explicitly computing the homology groups of $\mathcal{M}_{ij}$. The second approach is done via tools from Morse theory, which is the original proof presented by Milnor. The former approach uses tools that weren't available to Milnor at the time of his discovery, but were proven just a few years later.

> *"When I came upon such an example in the mid-50s, I was very puzzled and didn't know what to make of it. At first, I thought I'd found a counterexample to the generalized Poincaré conjecture in dimension seven. But careful study showed that the manifold really was homeomorphic to S^7."* – John Milnor

2.2 Computing Homology

The first strategy we consider invokes the generalized Poincaré conjecture. Indeed the two approaches are not disjoint; the Morse theoretic approach done by Milnor can be considered a special case of a more general framework; Morse theory is used to prove the *h-cobordism theorem*, which the generalized Poincaré conjecture follows as a corollary. The details are beyond the scope of this paper.

Definition. A *homotopy* $n-$*sphere* is an $n-$dimensional manifold that is homotopy equivalent to S^n.

Theorem 2.2.1. An $n-$dimensional manifold $\mathcal{M}$ is a *homotopy sphere* if $\pi_1(\mathcal{M}) = 0$ and $H_n(\mathcal{M}) = H_n(S^n)$ for all n.

Theorem 2.2.2. (Generalized Poincaré Conjecture). An $n-$dimensional homotopy sphere $n \geq 6$ is homeomorphic to S^n.

Indeed this has been proven for all n. The commonly known "Poincaré conjecture" proved by Grigori Perelman in 2003 is the case where $n = 3$. The proof for dimensions $n \geq 6$ was done by Stephen Smale in the early 1960s, just a few years after Milnor's discovery.

Theorem 2.2.3. *If $i + j = 1$, then $\mathcal{M}_{ij}$ is homeomorphic to S^7.*

Proof. We first need to compute the homology of spaces of the form S^k and $S^k \times S^l$. We then use these results with the Mayer-Vietoris sequence to compute the homology of $\mathcal{M}$.

We may decompose S^k into the upper and lower hemispheres $S^k = D_+ \cup D_-$. Notice that $D_+ \cap D_- = S^{k-1}$. The Mayer-Vietoris sequence is then

$$\cdots \longrightarrow H_n(D_-) \oplus H_n(D_+) \longrightarrow H_n(S^k) \longrightarrow H_{n-1}(S^{k-1}) \longrightarrow H_{n-1}(D_-) \oplus H_{n-1}(D_+) \longrightarrow \cdots$$

Since the hemispheres are contractible, their homology groups are 0. The sequence is then just

$$\cdots 0 \longrightarrow H_n(S^k) \longrightarrow H_{n-1}(S^{k-1}) \longrightarrow 0 \longrightarrow \cdots$$

which tells us $H_n(S^k) = H_{n-1}(S^{k-1})$. By induction we have

$$H_n(S^k) = \begin{cases} \mathbb{Z}, & n = k \\ 0, & n \neq k \end{cases}$$

To compute homology groups of $S^3 \times S^3$ we can decompose $S^3 \times S^3 = (D_+ \times S^3) \cup (D_- \times S^3)$. Then their intersection is $S^2 \times S^3$. Mayer-Vietoris gives us

$$\cdots \longrightarrow H_n(S^2 \times S^3) \longrightarrow H_n(S^2) \oplus H_n(S^3) \longrightarrow H_n(S^3 \times S^3) \longrightarrow H_{n-1}(S^2 \times S^3) \longrightarrow \cdots$$

We repeat the same process for $S^2 \times S^3$ and $S^1 \times S^3$. With induction this gives us the following result

$$H_n(S^k \times S^l) = \begin{cases} \mathbb{Z}, & n = 0, k, l \\ \mathbb{Z} \oplus \mathbb{Z}, & n = k = l \\ 0, & \text{otherwise} \end{cases}$$

We now compute the fundamental group. By Theorem A.2, our bundle $S^3 \to M^7 \to S^4$ induces a long exact sequence of homotopy groups,

$$\cdots \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(M^7) \longrightarrow \pi_n(S^4) \longrightarrow \cdots$$

In particular we have the following sequence

$$\cdots \longrightarrow \pi_1(S^3) \longrightarrow \pi_1(M^7) \longrightarrow \pi_1(S^4) \longrightarrow \cdots$$

Since $\pi_1(S^3) = \pi_1(S^4) = 0$, exactness implies that $\pi_1(M^7) = 0$.

If $n = 0$ or $n \geq 7$ then since M^7 is path-connected, Theorem A.3 tells us $H_0(M^7) = \mathbb{Z}$. Since M^7 is path-connected, simply connected and $\mathbb{Z}-$orientable, Theorem A.4 tells us $H_7(M^7) = \mathbb{Z}$ and $H_n(M^7) = 0$ for all $n > 7$.

If $n = 1$, we know that $H_1(M^7)$ is the abelianization of $\pi_1(M^7)$. Since $\pi_1(M^7) = 0$, its abelianization, or $H_1(M^7)$ is 0.

If $n = 2, 5, 6$, notice that M^7 can be decomposed into $M^7 = (D^4_+ \times S^3) \cup (D^4_- \times S^3)$, with the equator identified. In particular,

$$(D^4_+ \times S^3) \cap (D^4_- \times S^3) = (D^4_+ \cap D^4_-) \times S^3 = S^3 \times S^3$$

Additionally, since D^4 is contractible, $H_n(D^4 \times S^3) = H_n(\{d_0\} \times S^3) = H_n(S^3)$. Thus the Mayer-Vietoris sequence is then given by

$$\cdots \longrightarrow H_n(S^3) \oplus H_n(S^3) \longrightarrow H_n(M^7) \longrightarrow H_{n-1}(S^3 \times S^3) \longrightarrow \cdots$$

In particular, $n = 2, 5, 6$ gives us the following sequences

$$\cdots \longrightarrow 0 \longrightarrow H_2(M^7) \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow H_5(M^7) \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow H_6(M^7) \longrightarrow 0 \longrightarrow \cdots$$

Which immediately tells us $H_2(M^7) = H_5(M^7) = M_6(M^7) = 0$. This strategy doesn't work for $n = 3, 4$ because their part of the sequence is given by

$$\cdots \longrightarrow H_3(S^3) \oplus H_3(S^3) \longrightarrow H_3(M^7) \longrightarrow H_2(S^3 \times S^3) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_4(S^3) \oplus H_4(S^3) \longrightarrow H_4(M^7) \longrightarrow H_3(S^3 \times S^3) \longrightarrow \cdots$$

or

$$\cdots \longrightarrow 0 \longrightarrow H_4(M^7) \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow H_3(M^7) \longrightarrow 0 \longrightarrow \cdots$$

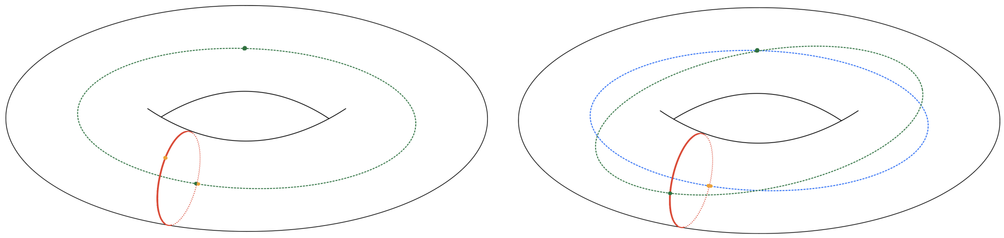
If $n = 3, 4$ then what we do have though is this sequence,

$$\cdots \longrightarrow 0 \xrightarrow{\varphi} H_4(M^7) \xrightarrow{\eta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\psi} H_3(M^7) \xrightarrow{\zeta} 0 \longrightarrow \cdots$$

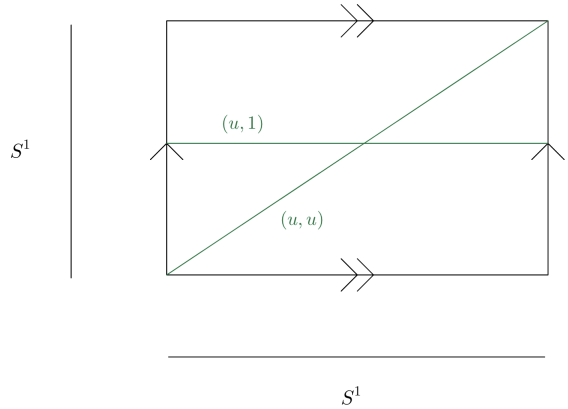
If we prove ϕ is an isomorphism, then $H_4(M^7) = H_3(M^7)$, by the following argument. If ϕ is injective then the kernel of ϕ is 0. Exactness then tells us $\mathrm{im}(\eta) = \ker(\phi) = 0$. This tells us $\ker(\eta) = 0$, so η is injective. So all elements in $H_4(M^7)$ get sent to 0 and η is injective, so $H_4(M^7) = 0$. If ϕ is surjective, then we have $\mathbb{Z}\oplus\mathbb{Z} = \mathrm{im}(\phi) = \ker(\psi)$. This tells us ψ sends all of $\mathbb{Z}\oplus\mathbb{Z}$ to 0, or $\mathrm{im}(\psi) = 0 = \ker(\zeta)$. Since the kernel of ζ is 0, ζ is injective, but the codomain is 0, so $H_3(M^7) = 0$. $\square$

Let $[F], [S]$ be a basis for $H_3(D^4_\pm \times S^3)$. To show that ϕ is an isomorphism we will explicitly compute $\phi([F], [S])$. This is trickier than the previous computations, so we will first introduce a motivating example.

To gain intuition, it helps to work in a lower-dimensional example. Consider an S^1 bundle over S^2 constructed by gluing the upper and lower hemispheres of S^2 under an analogous clutching function.



We may view the first torus as $[S] = \{(u,1)\}$. The green is the equator of the lower hemisphere. The green dot(s) is the generator 1 of S^3. The orange dot is the location on the equator. This section need only be described by $[S]$. The second torus is after inclusion into the upper hemisphere. $(u,1)$ is mapped to (u,u) by the clutching function. This twists the section in the first torus, and we now need both $[S]$ and $[F]$ to describe the new section. In two dimensions this can be visualized as the following,



Lemma 2.2.4. $\phi([F],[S]) = ([F],[S]+[F])$ and thus an isomorphism.

Proof. Let $[F],[S]$ be a basis for $H_3(D^4_\pm \times S^3)$, where $[S]$ generates the homology class of the equator and $[F]$ generates the class of the fiber. We assume that $[S]$ belongs to the lower hemisphere. Thus the inclusion into $H_3(D^4_- \times S^3) = H_3(S^3)$ need only be described by $[F]$. Under this assumption the inclusion into $H_3(D^4_+ \times S^3)$ undergoes the clutching function. To see how this inclusion behaves, we pick a generator 1 of S^3. $[S]$ can be described as $(u,1)$ where u ranges in S^3. When this is included into $D^4_+ \times S^3$, $(u,1)$ is mapped to (u,u). In particular this must be described with both $[S]$ and $[F]$ i.e, we have $\phi([S],[F]) = ([F],[S]+[F])$, an isomorphism.

To be more explicit, let $i : D^4_+ \cap D^4_- \times S^3 \to D^4_+ \times S^3$ be the inclusion. The group $H_3(D^4_- \cap D^4_+ \times S^3)$ is generated by the basis $[S],[F]$, where $[S]$ generates the homology of $D^4_+ \cap D^4_-$ and $[F]$ generates the homology of S^3. Under inclusion, we will first treat $D^4_\pm$ as belonging to D^4_-, and so the homology class $i_*[S] = [i(S_-)]$ generates the boundary of D^4_+ i.e., so $i_*[S] = 0$. Inclusion of the class $[F]$ is just inclusion into the same homology, so the first component of the Mayer-Vietoris map is just $[F]$.

The second component is a bit trickier. In this case, the inclusion into the other hemisphere makes use of the clutching function i.e., the inclusion is given by $j : (u,v) \mapsto (u, f_u(v))$, and the induced homology map is $j_*[\sigma] = [j\sigma]$. Similar to before, the generator $[F]$ is still included as $[F]$. To see what happens to $[S]$, we may pick an element of the fiber, denoted 1. The resulting map is then $(u,1) \mapsto (u, u^i 1 u^j) = (u, u^{i+j}) = (u,u)$, picking up an additional class. Thus the induced homology map is $([S],[F]) = ([F],[S]+[F])$. $\square$

2.3 Morse Theory

Morse theory as a general framework allows us to extract topological information from a space through functions defined on them. Here it provides an additional method of proving the homeomorphicity of $\mathcal{M}_{ij}$ and S^7. The main Morse-theoretic tool invoke is Reeb's Theorem, used to detect spheres. We begin with a brief introduction to the tools of Morse theory, followed by the application of Reeb's Theorem.

Recall that if f is a smooth function $f : M \to \mathbb{R}$, a point $p \in M$ is a *critical point* if the differential at p is 0. A crticial point p is *non-degenerate* if the *Hessian*, a matrix defined by

$$\left[\frac{\partial^2 f}{\partial x^i \partial^j}(p)\right]$$

is non-invertible.

Lemma 2.3.1. (Lemma of Morse). *Let p be a non-degenerate critical point of f. Then there is a local coordinate system $(y^1, \ldots, y^n)$ in a neighbourhood U of p with $y^i(p) = 0$ for all i such that the following identity holds on U, where λ is the index of f at p.*

$$f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$$

Proof. We first show such a coordinate system exists. In an appropriate coordinate system, assume that $p = 0$ and $f(p) = f(0) = 0$. Recalling Lemma A.1, we may write

f as

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j g_j(x_1, \ldots, x_n)$$

for $(x_1, \ldots, x_n)$ in some neighbourhood of 0. 0 is a critical point so this gives

$$g_j(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i h_{ij}(x_1, \ldots, x_n)$$

and

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, \ldots, x_n)$$

Assume $h_{ij} = h_{ji}$ by symmetrization. The result follows from diagonalizing the matrix $[h_{ij}]$ and applying the inverse function theorem. We now show if f satisfies such an expression, then λ must be the index of f at p. We have

$$\frac{\partial^2 f}{\partial z^i \partial z^j}(p) = \begin{cases} -2 & \text{if } i = j \leq \lambda \\ 2 & \text{if } i = j > \lambda \\ 0 & \text{otherwise} \end{cases}$$

which defines a diagonal matrix. There is a subspace of dimension λ where f_{**} is negative definite, and one of dimension $n - \lambda$ on which f_{**} is positive definite. If λ is not the index of f_{**} then the maximal subspace where f_{**} is negative definite would intersect the positive definite subspace, a contradiction. $\square$

The Lemma of Morse allows us to easily describe a smooth function around such a point p. The requirement that p is critical and non-degenerate is not too restrictive i.e., this result describes a very large class of functions, and is particularly useful in the proof of Reeb's Theorem.

Theorem 2.3.2. *Let f be a smooth real valued function on a manifold M. Let $a < b$ and suppose that $f^{-1}[a,b]$ is compact and has no critical points of f. Then M^a is diffeomorphic to M^b. Here we define $M^a = f^{-1}(\infty, a]$*

Proof. Let g be any Riemannian metric on M. Let $\text{grad}(f)$ denote the vector field defined by $g(X, \text{grad}(f)) = X(f)$. Let $\rho : M \to \mathbb{R}$ be a smooth function equal to $1/g(\text{grad}(f), \text{grad}(f))$ on $f^{-1}[a,b]$ and vanishes outside a compact neighbourhood of $f^{-1}[a,b]$. Consider the vector field X defined by $X_q = \rho(q)(\text{grad}(f))_q$. It is known that a smooth vector field on a manifold M that vanishes outside a compact set $K \subset M$ generates a unique $1-$parameter group of diffeomorphisms of M. In particular, notice that X satisfies these conditions because $f^{-1}[a,b]$ is compact. Denote our $1-$parameter group of diffeomorphisms as $\varphi_t : M \to M$. For fixed q, consider the function $t \mapsto f(\phi_t(q))$. If $\phi_t(q) \in f^{-1}[a,b]$, then $g(\phi_t(q)/dt, \text{grad}(f)) = g(X, \text{grad}(f)) = 1$. Thus $f(\phi_t(q)) = f(q) + t$ when $f(\phi_t(q)) \in [a,b]$. The diffeomorphism ϕ_{b-a} is now a diffeomorphism $M^a \to M^b$. $\square$

Theorem 2.3.3. (Reeb's Theorem) *If M is a compact manifold and f is a differentiable function on M with only two critical points, both non-degenerate, then M is homeomorphic to a sphere.*

Proof. Since M is compact, the two critical points p, q correspond to the minimum and maximum of f. By normalizing f we may make the maximum 1 and minimum 0. By Lemma 2.3.1, there exists local coordinates systems satisfying the result. If we shrink the associated neighbourhoods around the points small enough, we obtain that $f^{-1}[0, \epsilon]$ and $f^{-1}[1-\epsilon, 1]$ are closed $n-$cells (since coordinate systems are homeomorphisms). By Theorem 2.3.2, these preimages are diffeomorphic. It follows that M is just two closed $n-$cells glued via f along their boundary, which we know is homeomorphic to a sphere. □

Unlike our previous "intrinsic" approach with computing homology groups, the Morse theoretic approach requires us to work with an explicit construction of $\mathcal{M}_{ij}$.

Theorem 2.3.4. *If $i + j = 1$, then $\mathcal{M}_{ij}$ is homeomorphic to S^7.*

Proof. The reasoning here is similar to the first approach, except instead of taking the upper and lower hemisphere here, we take the entire sphere minus the north or south pole. This is still contractible and thus any bundle over them is trivial, and like before, the only data that matters is what happens on their intersection. It is clear that this approach is equivalent to the previous approach. Since the intersection is homotopic to the equator S^3, it is characterized by a map $f_{ij} : S^3 \to \mathrm{SO}(4)$. In this case their intersection is $S^4 \setminus \{N, S\}$. We take two trivial bundles: one S^3 bundle over $S^4 \setminus \{N\}$ and one S^3 bundle over $S^4 \setminus \{S\}$. Since $S^4 \setminus \{p\}$ is homeomorphic to $\mathbb{R}^4$, we may treat both bundles as just S^3 bundles over $\mathbb{R}^4$ i.e., $\mathbb{R}^4 \times S^3$. The intersection of $S^4 \setminus \{N\}$ and $S^4 \setminus \{S\}$ is $S^4 \setminus \{N, S\}$, which we know is homeomorphic to $\mathbb{R}^4 \setminus \{0\}$. We wish to glue the subsets $\mathbb{R}^4 \setminus \{0\} \times S^3$ together using an appropriate f_{ij}, creating an S^3 bundle over S^4. We will identify S^3 with the unit quaternions. The gluing diffeomorphism is then given by $(u, v) \to (u', v') = (u/\|u\|^2, u^i v u^j/\|u\|)$ i.e.,

$$(u, v) \sim \left(\frac{u}{\|u\|^2}, \frac{u^i v u^j}{\|u\|} \right)$$

We first describe coordinates on our new bundle, and then define a differentiable function f. Let $(u'', v') = (u'(v')^{-1}, v')$ and define

$$f(u, v) = \frac{R(v)}{\sqrt{1 + \|u\|}} = \frac{R(u'')}{\sqrt{1 + \|u''\|}}$$

The equality is given by the fact that $R(u'')\|u\| = R(v)$, and that

$$\frac{R(u'')}{\sqrt{1 + \|u''\|}} = \frac{R(u'')\|u\|}{\sqrt{1 + \|u\|}}$$

This is just done by quaternionic algebraic manipulation (calculation below). The assumption that $i+j = 1$ is used here. We first note that the real part of a quaternion is given by $R(q) = q/2 + \bar{q}/2$. Additionally, recall that $q^{-1} = \bar{q}/\|q\|^2$, so we may rewrite

$R(q) = q + \|q\|^2 q^{-1}$. We then have

$$2R(u'') = u'' + \|u''\|^2(u'')^{-1} = u'v'^{-1} + \|u''\|^2(u'v'^{-1})^{-1}$$

Expanding this expression gives

$$= \frac{u}{\|u\|^2}\frac{\|u\|}{u^i v u^j} + \left\|\frac{u}{\|u\|^2}\frac{\|u\|}{u^i v u^j}\right\|^2 \frac{\|u\| u^i v u^j}{u}$$

Cancelling terms and simplifying this expression yields $2R(u'') = 2/\|u\|R(v)$ i.e., $R(u'')\|u\| = R(v)$. The proven equality shows these charts cover our entire manifold, and thus f is well-defined everywhere. In the first chart, differentiating f allows us to see critical points can only occur here when $v = \pm 1$. Further manipulation confirms that the only critical points in this chart are $(0, \pm 1)$ (one can explicitly see this by setting $v = (v^1, v^2, v^3, v^4)$ and $u = (u^1, u^2, u^3, u^4)$). Additionally, there are no critical points in the other chart. Checking the Hessian confirms these critical points are both non-degenerate. By Reeb's Theorem, our manifold is homeomorphic to S^7. $\square$

3 Diffeomorphism Invariants

We now show our manifold is not diffeomorphic to S^7. We do this by constructing diffeomorphism invariants, and show that these invariants differ on $\mathcal{M}_{ij}$ and S^7. More precisely, we construct an invariant on manifolds homeomorphic to S^7. We first introduce basic cohomology theory and *characteristic classes*, special cohomology classes that measure how twisted the fibers of a vector bundle are. It should be clear that the fibers of a trivial bundle are not "twisted", and indeed the characteristic classes of any trivial bundle are 0. Characteristic classes can be defined to be invariant on various classes of vector bundles, so non-trivial characteristic classes automatically tells us the fibers of such a bundle are twisted. A detailed exposition can be found in more advanced texts. These classes are then used to define *characteristic numbers*, integer invariants on smooth manifolds defined using the characteristic classes of their tangent bundles. The characteristic number and the *signature* of a manifold come together to define Milnor's invariant.

3.1 Cohomology

The *cochain group* $C^n(X)$ is the dual of the chain group $\{C_n(X) \to \mathbb{Z}\}$ i.e., all $\mathbb{Z}-$linear maps $C_n(X) \to \mathbb{Z}$. Elements of the cochain group are called *cochains*. The value of the cochain c evaluated at a chain γ is denoted $\langle c, \gamma \rangle$. The *coboundary map* is a homomorphism $\delta : C^n(X) \to C^{n+1}(X)$ defined like so. Given a cochain $c \in C^n(X)$, the cochain $\delta c \in C^{n+1}(X)$ is defined $\langle \delta c, \alpha \rangle = (-1)^{n+1} \langle c, \partial \alpha \rangle$. This is well-defined;

δc is a map $C_{n+1} \to \mathbb{Z}$, defined by taking values of C_n on a map $c : C_n \to \mathbb{Z}$. The *singular cohomology group* $H^n(X)$ is defined as the kernel of the nth coboundary map modded by the image of the $(n-1)$th coboundary map. We have the following diagram

$$\begin{array}{ccccccccc} \cdots & \xleftarrow{\delta} & C^{n+1} & \xleftarrow{\delta} & C^n & \xleftarrow{\delta} & C^{n-1} & \xleftarrow{\delta} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longleftarrow & H^{n+1} & \longleftarrow & H^n & \longleftarrow & H^{n-1} & \longleftarrow & \cdots \end{array}$$

It is easy to see that homology groups are covariant functors while cohomology is contravariant. While our method of defining cohomology by dualizing homology may seem like it doesn't add much intrinsically, the contravariance of cohomology naturally defines a product operation between cohomology groups of different degree called the *cup product*, which turns the cohomology groups of a space into a ring. This extra structure on cohomology is useful in that it allows us to further distinguish between different spaces; two spaces may have the same cohomology whose cup product structures differ.

Definition 3.1.1. Let $\sigma : \Delta^{m+n} \to X$ be a singular simplex. The *front m-face* of σ is the composition $\sigma \circ \alpha_m : \Delta^m \to X$ where $\alpha_m(t_0, \ldots, t_m) = (t_0, \ldots, t_m, 0, \ldots, 0)$. The *back n-face* of σ is the composition $\sigma \circ \beta_n : \Delta^n \to X$ where $\beta_n(t_m, \ldots, t_{m+n}) = (0, \ldots, 0, t_m, \ldots, t_{m+n})$.

Definition 3.1.2. Given cochains $c \in C^m X, c' \in C^n X$, we define the *cup product* $cc' := c \smile c' \in C^{m+n} X$ by the identity

$$\langle c \smile c', [\sigma] \rangle = (-1)^{mn} \langle c, [\sigma \circ a_m] \rangle \cdot \langle c', [\sigma \circ \beta_n] \rangle \in \mathbb{Z}$$

It's important to keep track of what's happening here. Cochains are maps from chains to $\mathbb{Z}$. The cup product takes two cochains and produces another cochain. The cup product is defined by its action on chains, denoted by $\langle c \smile c', [\sigma] \rangle$. We only need to define its action on the generators $[\sigma]$, because this defines the rest of the map. The cup product naturally induces a product on cohomology,

$$\smile : H^m(X) \otimes H^n(X) \to H^{m+n}(X)$$

We may also define *relative cohomology* in an analogous way. Relative cochain groups $C^n(X, A)$ are the dual of the relative chain groups $\{C_n(X, A) \to \mathbb{Z}\}$ i.e., all $\mathbb{Z}-$linear maps $C_n(X, A) \to \mathbb{Z}$. One may also view relative cochains as cochains that vanish on A. Relative coboundary maps are defined as just the restriction of coboundary maps.

3.2 Characteristic Classes

Given a vector bundle $E \to F$, characteristic classes are special cohomology classes of the bundle E. One can think of characteristic classes as one way of measuring how "twisted" the fibers of a bundle are. For sake of clarity, we now assume all coefficients are in $\mathbb{Z}$. $H^*(X, A)$ denotes relative homology. Given a vector bundle E, E_0 denotes E minus the zero section. Given a fiber F, F_0 denotes F minus 0.

Definition 3.2.1. Let $\xi : E \to X$ be a vector bundle of rank n. An *orientation* for ξ is a function which assigns an orientation to each fiber F of ξ (in the vector space sense), subject to the following local compatibility condition. For every point b_0 in the base space X there exists a local coordinate system (N, h), with $b_0 \in N$ and $h : N \times \mathbb{R}^n \to \pi^{-1}(N)$, so that for each fiber $F = \pi^{-1}(b)$ over N the homomorphism $x \mapsto h(b, x)$ from $\mathbb{R}^n$ to F is orientation preserving. (Or equivalently there should exist sections $s_1, \ldots, s_n : N \to \pi^{-1}(N)$ so that the basis $s_1(b), \ldots, s_n(b)$ determines the required orientation of $\pi^{-1}(b)$ for each b in N.)

We may recast this construction in terms of cohomology. Suppose we've chosen an orientation for our vector space F. We may then pick an orientation preserving $n-$cycle $\sigma : \Delta^n \to V$ that maps the boundary of Δ^n into the 0 vector. We now say the homology class of σ denoted μ_F is the preferred generator for $H_n(V, V_0)$. Then the cohomology group $H^n(V, V_0)$ also has a preferred generator u_V defined by $\langle u_F, \mu_F \rangle = +1$. In terms of vector bundle orientation, our previous definition means that each fiber F has a preferred generator $u_F \in H^n(F, F_0)$, chosen in a continuous way i.e., for every point in the base space there exists a neighbourhood N and cohomology class $u \in H^n(\pi^{-1}(N), \pi^{-1}(N)_0)$ such that for every fiber F over N the restriction $u|_{(F,F_0)} \in H^n(F, F_0)$ is equal to u_F.

Theorem 3.2.2. (Thom Isomorphism). *Let ξ be a real oriented rank n bundle with total space E. Then the cohomology group $H^i(E, E_0; \mathbb{Z})$ is zero for $i < n$, and $H^n(E, E_0)$ contains one and only one cohomology class u whose restriction*

$$u|_{(F,F_0)} \in H^n(F, F_0)$$

is equal to the preferred generator u_F for every fiber F of ξ. Furthermore the correspondence $y \mapsto y \smile u$ maps $H^k(E)$ isomorphically onto $H^{k+n}(E, E_0)$ for every integer k.

We call u the *fundamental cohomology class* or the *Thom class.* To be even more explicit, we have

$$\cdots \to C^{n-1}(E, E_0) \xrightarrow{\delta} C^n(E, E_0) \to \cdots$$

where $C^i(E, E_0) = \{u : C_i(E, E_0) \to \mathbb{Z}\}$. Then

$$H^n(E, E_0) = \{u \in C^n(E, E_0) : \delta u = 0\}/\{\delta v \in C^n(E, E_0) : v \in C^{n-1}(E, E_0)\}$$

Then consider $u \in H^n(E, E_0)$. E is composed of fibers F, and thus $u \in H^n(F, F_0) \subset H^n(E, E_0)$. In particular one can think of u as a map $C_n(E, E_0) \to \mathbb{Z}$, and elements of $C_n(E, E_0)$ as maps $\Delta^n \to E$, which naturally induce maps $\Delta^n \to F \subset E$, and thus induce maps $C_n(F, F_0) \to \mathbb{Z}$. Restricting the Thom class u to this cohomology class in $H^n(F, F_0)$ gives us the preferred generator u_F.

Now given a real oriented bundle ξ of rank n, the inclusion $(E, \varnothing) \hookrightarrow (E, E_0)$ gives a homomorphism $H^*(E, E_0) \to H^*(E)$. We apply this homomorphism to the fundamental class $u \in H^n(E, E_0)$, denoted $u|_E \in H^n(E)$. $H^n(E)$ is isomorphic to the cohomology group $H^n(B)$ through the canonical isomorphism π^* induced by the projection map π, so we can further identify $u|_E$ as an element of $H^n(B)$. This particular cohomology class turns out to be an invariant on real, oriented vector bundles, called the *Euler class*.

Definition 3.2.3. The *Euler class* of an oriented bundle ξ of rank n is the cohomology class $e(\xi) \in H^n(B)$ corresponding to $u|_E$ under the canonical isomorphism $H^n(B) \to H^n(E)$.

It is easy to get lost in the details here. Given a rank n vector bundle $E \to X$ with fibers F, an orientation amounts to a continuous choice of generator for the cohomology group $H^n(F, F_0)$. We also know that there is a distinguished element of $H^n(E, E_0)$, called the fundamental class. We then recall that the inclusions

$(B, \varnothing) \hookrightarrow (E, \varnothing) \hookrightarrow (E, E_0)$ induce a map

$$H^n(E_0) \to H^n(E) \to H^n(B)$$

The Euler class is defined as the image of the fundamental class under this map; an element of $H^n(B)$, and is an invariant on real oriented vector bundles of rank n. The Chern classes, an invariant on complex vector bundles, are then defined in terms of the Euler class. Let ξ be a real bundle of rank n with projection $\pi : E \to B$. Let $\pi_0 : E_0 \to B$ denote the projection of non-zero vectors of E.

Example 3.2.4. To illuminate why one would even care about characteristic classes, we will give an informal example with the Euler class. Given a vector bundle $E \to X$, a *section* is a continuous map from an open subset $U \subset X$ to E, such that every point in U is mapped to a point in its corresponding fiber. Intuitively, we are quite literally taking a cross-section of all the fibers. Whether we can do this in a continuous manner depends on how "twisted" the fibers are. Clearly there is a trivial section by taking all the 0 vectors. Thus a natural question is the existence of non-trivial sections. It is not hard to prove that the existence of non-trivial sections is equivalent to the existence of any section at all on the corresponding unit sphere bundle. It turns out that the Euler class measures the first obstruction for a real, oriented vector bundle to have a non-trivial section.

Theorem 3.2.5. *For any oriented real bundle ξ of rank n, there is an exact se-*

quence

$$\cdots \to H^i(B) \overset{\smile e}{\to} H^{i+n}(B) \overset{\pi_0^*}{\to} H^{i+n}(E_0) \to H^{i+1}(B) \overset{\smile e}{\to} \cdots$$

where $\smile e$ *denotes the map* $a \mapsto a \smile e(\xi)$.

Proof. [1] $\square$

Consider any real oriented bundle of rank $2n$. We then have the sequence (called a *Gysin sequence*)

$$\cdots \to H^{i-2n}(B) \overset{\smile e}{\to} H^i(B) \overset{\pi_0^*}{\to} H^i(E_0) \to H^{i-2n+1}(B) \overset{\smile e}{\to} \cdots$$

For $i < 2n-1$, the left and right groups are 0, so $\pi_0^* : H^i(B) \to H^i(E_0)$ is an isomorphism.

Lemma 3.2.6. *If* ω *is a complex vector bundle, then the underlying real vector bundle* $\omega_{\mathbb{R}}$ *has a canonical preferred orientation. In particular, because every orientation for the tangent bundle of a manifold induces a unique orientation of the manifold, we have that every complex manifold has a canonical preferred orientation.*

Proof. Let V be a complex vector space of dimension n, and choose a basis $a_1, \ldots, a_n$ for V over $\mathbb{C}$. Notice that the vectors $a_1, ia_1, a_2, ia_2, \ldots, a_n, ia_n$ are a basis for the underlying real vector space $V_{\mathbb{R}}$, giving us an orientation for $V_{\mathbb{R}}$. Since $\mathrm{GL}_n(\mathbb{C})$ is connected, this orientation does not depend on the choice of complex basis. Applying this to every fiber of a complex vector bundle ω gives us the orientation for the underlying real vector bundle $\omega_{\mathbb{R}}$. $\square$

Let ω be a complex vector bundle of rank n. Denote by ω_0 the canonical complex vector bundle of rank $(n-1)$ over the total space E_0 defined by the following. E_0 is the space specified by a fiber F of ω and a non-zero vector v in the fiber. We endow ω with a Hermitian metric. Define the fibers of ω_0 to be the orthogonal complement of nonzero vectors v in F, a complex vector space of dimension $(n-1)$.

Definition 3.2.7. Given a complex vector bundle ω, we inductively define the *Chern classes* $c_i(\omega) \in H^{2i}(B)$

$$c_i(\omega) = \begin{cases} e(\omega_{\mathbb{R}}) & i = n \\ c_i(\omega) = \pi_0^{*-1} c_i(\omega_0) & i < n \\ 0 & i > n \end{cases}$$

Thus we obtain an invariant on complex vector bundles. More precisely, every complex vector bundle has an underlying real vector bundle with a canonical preferred orientation. The Chern classes are then defined to inductively depend on the Euler class, an invariant on real oriented vector bundles. Once again, the Chern classes are elements of the cohomology of the base space $H^i(B)$.

Definition 3.2.8. Given a real rank n bundle ξ with fibers F, its *complexifica-*

tion $\xi \otimes \mathbb{C}$ is the complex rank n bundle with fibers $F \otimes \mathbb{C}$ (tensor product).

We now have all the tools to define the *Pontryagin classes*, the most important tool in our quest to construct Milnor's invariant. These classes are invariants of real vector bundles, defined by taking the Chern class of their complexification.

Definition 3.2.9. The ith *Pontryagin class* $p_i(\xi) \in H^{4i}(B)$ is defined to be $p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C})$.

The reason why we only define the Pontryagin class in terms of the $2i$th Chern class is that the odd Chern classes turn out to be expressible in terms of the Stiefel-Whitney classes (a simpler characteristic class whose introduction we have omitted) of the original bundle – not its complexification – so this gives no new information.

Lemma 3.2.10. Pontryagin classes are independent of orientation.

3.3 Characteristic Numbers and Signature

We now introduce the final couple of tools used to define Milnor's invariant. Given compact, connected, oriented manifolds M and N, we wish to compare their tangent bundles using the Pontryagin classes $p_i(TM) \in H^{4i}(M), p_i(TN) \in H^{4i}(N)$. We run into the problem where the Pontryagin classes are elements of cohomology groups that may be wildly different, but we can remedy this problem by reducing the Pontryagin classes to an integer-valued invariant.

Given a compact, connected and oriented manifold M^n, recall the fundamental class $[M] \in H_n(M)$. As an interesting side note, if we work in de Rham cohomology, then the fundamental class satisfies

$$\langle \omega, [M] \rangle = \int_M \omega$$

Definition. Let M be a compact, oriented manifold of dimension $4n$ and $I = (i_1, \ldots, i_k)$ a partition of n. The Ith *Pontryagin number* is defined

$$p_I[M] = p_{i_1} \cdots p_{i_k}[M] = (p_{i_1}(TM) \smile \cdots \smile p_{i_k}(TM))[M] \in \mathbb{Z}$$

For manifolds of dimension not divisible by 4, the Pontryagin number is defined to be 0.

In other words, the Pontryagin number of a manifold M is the cup product of the i_1th, i_2th, $\ldots$, i_kth Pontryagin classes applied to the fundamental class $[M] \in H_n(M)$, giving us an integer. This is well-defined; the Pontryagin class of the bundle TM is an element of the cohomology of the base space M, thus it is defined on the fundamental class $[M]$.

Lemma 3.3.1. If a Pontryagin number $p_I[M]$ is non-zero, then there is no orientation reversing diffeomorphism on M.

Proof. If there exists an orientation reversing diffeomorphism on M, then the Pontryagin classes will not change since they are independent of orientation. However the sign of the fundamental class *does* change, so the Pontryagin number $p_I[M]$ will also change sign i.e., $p_I[M] = -p_I[M]$, so it must be that $p_I[M] = 0$. $\square$

Pontryagin numbers are also a *bordism* invariant.

Definition 3.3.2. Given a compact, oriented manifold M of dimension $4n$, the *signature* of M denoted $\sigma(M)$ is the signature of the quadratic form $Q : H^{2n}(M; \mathbb{Q}) \to \mathbb{Z}$ defined by $c \mapsto \langle c \smile c, \mu \rangle$.

The signature is an invariant on compact, oriented manifolds of dimension $4n$. More precisely, we define a map $H^{2n}(M; \mathbb{Q}) \times H^{2n}(M; \mathbb{Q}) \to \mathbb{Q}$ by $(a, b) = \langle a \smile b, \mu \rangle$. Since the degree of both cohomology elements is $2n$, unravelling the definition reveals that this map is symmetric, and also bilinear. In particular, this induces a quadratic form on $H^{2n}(M; \mathbb{Q})$, and the signature of M is defined to be the signature of this quadratic form Q. Recall that the signature of a quadratic form Q is given by the number of positive diagonal entries minus the number of negative entries in the corresponding diagonal matrix of Q. Equivalently, we choose a basis $a_1, \ldots, a_r$ for $H^{2n}(M;\mathbb{Q})$ such that the symmetric matrix $[\langle a_i \smile a_j, \mu\rangle]$ is diagonal. The signature is additive with respect to disjoint unions of manifolds, and multiplicative with respect to Cartesian products of manifolds. If a manifold is an oriented boundary, then its signature is 0. Detailed proofs are found in [3]. There is a close relationship between the signature of a manifold and its Pontryagin numbers, encoded in the *Hirzebruch Signature Theorem.* We now introduce some definitions needed to state the theorem.

A sequence of polynomials $P_1, P_2, \ldots$ with variables $p_1, p_2, \ldots$ is *multiplicative* if the identity $1 + p_1 z + p_2 z^2 + \cdots = (1 + q_1 z + q_2 z^2 + \cdots)(1 + r_1 z + r_2 z^2 + \cdots)$ implies that

$$\sum P_i(p_1, p_2, \ldots) z^i = \sum P_i(q_1, q_2, \ldots) z^i \sum P_j(r_1, r_2, \ldots) z^j$$

In other words, the sum of the polynomials can be decomposed as a product.

Definition 3.3.3. The *K-genus* of a multiplicative sequence K with rational coefficients is a ring homomorphism from the cobordism ring of compact, oriented manifolds of dimension $4n$ to $\mathbb{Q}$ given by $\Phi(M) = K(p_1, p_2, p_3, \ldots) = \langle K_n(p_1, \ldots, p_n), \mu_{4n}\rangle$, where p_i are the Pontryagin classes of M.

Lemma 3.3.4. Given a formal power series $f(z)$ with coefficients in $\mathbb{Q}$, there exists a unique multiplicative sequence $\{K_n\}$ with coefficients in $\mathbb{Q}$ satisfying $K(1+z) = f(z)$. We call $\{K_n\}$ the multiplicative sequence belonging to the power series $f(z)$. $\square$

Theorem 3.3.5. (Hirzebruch Signature Theorem). The signature $\sigma(M)$ of a compact, oriented manifold M of dimension $4n$ is equal to the $L-$genus $L[M]$ where $\{L_n\}$ is the multiplicative sequence of polynomials with coefficients in $\mathbb{Q}$ belonging to the power series

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + \frac{(-1)^{n-1}2^{2n}B_n t^n}{(2n)!} + \cdots$$

where B_n is the nth Bernoulli number. The first few L polynomials are given below

$$\begin{aligned} L_1 &= \frac{1}{3}p_1 \\ L_2 &= \frac{1}{45}(7p_2 - p_1^2) \\ L_3 &= \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3) \end{aligned}$$

Proof. [1] $\square$

The proof of this theorem is beyond the scope of the paper. The significance of this theorem is that it gives a highly non-trivial relationship between the signature of a manifold and its Pontryagin numbers. Since the signature is an integer, Hirzebruch's theorem forces rich arithmetic relationships on the Pontryagin numbers. We will in particular use the L_2 polynomial; our candidate manifold is 7-dimensional and bounds an $8-$manifold that we will take the signature of. Since $8 = 4(2)$, we use the L_2 polynomial.

3.4 Milnor's Invariant

We are now at a place to define Milnor's invariant λ, an integer invariant on manifolds homeomorphic to S^7. In particular, λ is defined by taking an $8-$manifold that bounds our $7-$manifold, and computing its signature and Pontryagin number. The natural question is what happens if we take a different $8-$manifold? It turns out, quite surprisingly, that our invariant is independent of the $8-$manifold. This fact is shown using Hirzebruch's Signature Theorem.

Definition 3.4.1. (Milnor's Invariant). If M is a manifold homeomorphic to S^7 and B is an $8-$manifold that M bounds, then define $\lambda(M) = 2q(B) - \sigma(B)$, where $q(B) = (i^{-1}p_1)^2[B]$ where $i^{-1}p_1 \in H^4(B, M)$, and $\sigma(B)$ is the signature of B.

Let M be homeomorphic to S^7. It is known that any compact 7-manifold M bounds an 8-manifold B [11]. Consider this manifold B i.e., $M = \partial B$. Consider the long exact sequence

$$\cdots \longrightarrow H^3(M) \longrightarrow H^4(B, M) \longrightarrow H^4(B) \longrightarrow H^4(M) \longrightarrow \cdots$$

Recalling that $H^3(S^7) = H^4(S^7) = 0$ we have that the inclusion $i : H^4(B, M) \to H^4(B)$ is an isomorphism. We then pull the first Pontryagin class of TB, an element of $H^4(B)$, back to $H^4(B, M)$, and define a new integer invariant q by evaluating the pulled back Pontryagin class squared (by the cup product) on the fundamental class

$[B]$ of B, and denote it $q(B)$.

Theorem 3.4.2. $\lambda(M) \mod 7$ is independent of B.

Proof. Let B_1, B_2 be two manifolds that bound M. Let C be the manifold defined by the union of B_1, B_2 glued at their boundary M. Choose an orientation for C by letting B_1 and B_2 have opposite orientation. Since C is closed, we invoke Hirzebruch's signature theorem, $\sigma(C) = \frac{1}{45}(7p_2 - p_1^2)([C])$. Rearranging and reducing mod 7 gives us that $\lambda(C) = 0 \mod 7$ (we are abusing notation here; $\lambda(C) = 2q(C) - \sigma(C)$). We now show that $\sigma(C) = \sigma(B_1) - \sigma(B_2)$ and $q(C) = q(B_1) - q(B_2)$. It follows that $\lambda(B_1) = \lambda(B_2)$.

As before, we note that $H^4(C, M)$ is isomorphic to $H^4(M)$. Additionally, the Mayer-Vietoris sequence gives the following,

$$\begin{tikzcd}
H^4(C,M) \arrow[r] & H^4(B_1,M) \oplus H^4(B_2,M) \arrow[d] \\
H^3(B_1 \cap B_2, \varnothing) \arrow[u] & H^4(B_1 \cap B_2, \varnothing)
\end{tikzcd}$$

which implies $H^4(C, M)$ is isomorphic to $H^4(B_1, M) \oplus H^4(B_2, M)$, and like before, $H^4(B_i, M)$ is isomorphic to $H^4(B_i)$. Altogether this gives the following diagram

$$\begin{tikzcd}
H^4(C,M) \arrow[r] \arrow[d] & H^4(B_1,M) \oplus H^4(B_2,M) \arrow[d] \\
H^4(C) \arrow[r] & H^4(B_1) \oplus H^4(B_2)
\end{tikzcd}$$

We now show $\sigma(C) = \sigma(B_1) - \sigma(B_2)$. Let $\alpha \in H^4(C)$. Pull this back into $i^{-1}\alpha \in H^4(C, M)$. By our construction above, identify $i^{-1}\alpha$ with some $\alpha_1 \oplus \alpha_2 \in H^4(B_1, M) \oplus$

$H^4(B_2, M)$. In particular, evaluating $(i^{-1}\alpha)^2[C]$ is equivalent to evaluating $(\alpha_1^2 \oplus \alpha_2^2)([B_1] \oplus -[B_2])$, which is just $\alpha_1^2[B_1] - \alpha_2^2[B_2]$. The quadratic form defined on C is then just the sum of quadratic forms on B_1 and $-B_2$, which implies the matrix of this form is block diagonal;

$$\begin{pmatrix} B_1 & 0 \\ 0 & -B_2 \end{pmatrix}$$

and hence its signature is the sum of the signatures of B_1 and B_2 i.e., $\sigma(C) = \sigma(B_1) + \sigma(-B_2) = \sigma(B_1) - \sigma(B_2)$.

An analogous argument holds for $q(C) = q(B_1) - q(B_2)$. Let f_i be the embedding of B_i into C, which embeds $TB_i \to TC$. The Mayer-Vietoris map gives us that the Pontryagin class of TC is sent to the Pontryagin class of TB_i under the induced cohomology map f_i^*. More explicitly, $p_1(TC) = -c_2(TC \otimes \mathbb{C})$ and $p_1(TB_i) = -c_2(TB_i \otimes \mathbb{C})$. Then $f_i^* p_1(TC) = f_i^*(-c_2(TC \otimes \mathbb{C})) = -c_2(TB_i \otimes \mathbb{C}) = p_1(TB_i)$. One can see this by noting that $C_n(TB)$ naturally embeds into $C_n(TC)$ by f. Thus the set of maps $C_n(TC) \to \mathbb{Z}$ naturally embed into the set of maps $C_n(TB) \to \mathbb{Z}$ by $C_n(TB) \to_f C_n(TC) \to \mathbb{Z}$, and hence $H^4(B_i)$ naturally embeds into $H^4(C)$ under f i.e., the embedding under f of the Pontryagin class of TC should give us the Pontryagin class of TB. In particular, the induced map in the Mayer-Vietoris sequence is exactly the inclusion (f_1, f_2). By definition of the Pontryagin number and the relationship $\alpha^2[C] = \alpha_1^2[B_1] - \alpha_2^2[B_2]$ shown before, it follows that $q(C) = q(B_1) - q(B_2)$. This part is just diagram chasing, so we will omit it.

Thus $0 = \lambda(C) = 2q(C) - \sigma(C) = 2(q(B_1) - q(B_2)) - (\sigma(B_1) - \sigma(B_2)) =$

$2q(B_1) - \sigma(B_1) - (2q(B_2) - \sigma(B_2)) = \lambda(M_{B_1}) - \lambda(M_{B_2})$. So the invariant is constant for whichever manifold B we use. $\square$

Lemma 3.4.3. $\lambda(S^7) = 0$.

Proof. The standard 7-sphere is bounded by the 8−disc D^8. We know $H^4(D^8, S^7) = H^4(D^8) = 0$, so the Pontryagin classes and the resulting quadratic form are both 0 and so $\lambda(M) = 0$. $\square$

Theorem 3.4.4. $\lambda(\mathcal{M}_{ij}) = (i-j)^2 - 1 \mod 7$.

Proof. We first note that $p_1(\mathcal{M}_{ij}) = \pm 2(i-j)\mu$ where μ generates $H^4(S^4)$. This can be seen by the fact that the manifolds $\mathcal{M}_{ij}$ and $\mathcal{M}_{-i,-j}$ only differ by opposite orientations. Since the Pontryagin class is independent of orientation, we have $p_1(\mathcal{M}_{ij}) = p_1(\mathcal{M}_{-i,-j})$. Linearity implies that $p_1(\mathcal{M}_{ij}) = c(i-j)\mu$ for some constant c, which is calculated to be 2 by evaluating p_1 on $\mathcal{M}_{1,0}$.

Now consider the corresponding disc bundle $\pi : B_{ij} \to S^4$ with fibers D^4, where $\mathcal{M}_{ij}$ bounds B_{ij}. $\mathcal{M}_{ij}$ embeds into B_{ij}, so $T\mathcal{M}_{ij}$ naturally embeds into $T\mathcal{B}_{ij}$. Similar to the proof of Theorem 3.3.1, we have that the Pontryagin class is preserved under embedding. Consider the bundles $\mathcal{M}_{ij} \to S^4$ and $TS^4 \to S^4$. Now consider the pullback bundles $\pi^*\mathcal{M}_{ij} : \mathcal{M}_{ij} \to B_{ij}$ and $\pi^*TS^4 : TS^4 \to B_{ij}$. In particular we may split TB_{ij} into the Whitney sum of these two bundles,

$$TB_{ij} = \pi^*\mathcal{M}_{ij} \oplus \pi^*TS^4$$

Taking the Pontryagin class on both sides and noting that it commutes with pullbacks gives us

$$p_1(TB_{ij}) = \pi^*(p_1(\mathcal{M}_{ij} \oplus TS^4))$$

Which equals $\pi^*(p_1(\mathcal{M}_{ij})) = \pi^*(2(i-j)\mu)$. Since S^4 is a deformation retract of B_{ij} and $H^4(S^4) = \mathbb{Z}$, π^* must map μ to a generator of $H^4(B_{ij})$, denoted ν. Thus $p_1(B_{ij}) = 2(i-j)\nu$. To compute $q(B_{ij})$ we note the definition of the fundamental class;

$$q(B_{ij}) = [i^{-1}(2(i-j)\nu) \smile i^{-1}(2(i-j)\nu)][B] = 2(i-j)\langle\nu,[B]\rangle 2(i-j)\langle\nu,[B]\rangle = 4(i-j)^2$$

The signature $\sigma(B_{ij})$ is calculated by just picking our orientation so that $\sigma(B_{ij}) = 1$. We have freedom to do this since Milnor's invariant is independent of B.

Then $\lambda(\mathcal{M}_{ij}) = 2q(B) - \sigma(B) = 2(4(i-j)^2) - 1 = (i-j)^2 - 1 \mod 7$. □

Corollary 3.4.5. If $i + j = 1$ and $i - j \not\equiv \pm 1 \mod 7$, then $\mathcal{M}_{ij}$ is homeomorphic, but not diffeomorphic to S^7.

Proof. By Theorem 2.3.4., $\mathcal{M}_{ij}$ is homeomorphic to S^7 if $i + j = 1$. By Theorem 3.4.3., $\lambda(\mathcal{M}_{ij}) \neq 0$ if $i - j \not\equiv \pm 1 \mod 7$, and hence is not diffeomorphic to S^7. □

4 Further Developments

Milnor's discovery was only the first of a massive successful program. Here we collect a microscopic portion of these results of note to the author.

Soon after the discovery of an exotic 7-sphere, Stephen Smale proved the topological Poincaré conjecture in dimensions 5 or greater; that every homotopy sphere is in fact a topological sphere. Smale further proved that for homotopy spheres, *h-cobordism* is equivalent to diffeomorphism, a corollary of his celebrated *h-cobordism theorem*. For this work Smale was awarded the 1966 Fields Medal. This equivalence between $h-$cobordism and diffeomorphism was extremely significant in the theory of exotic spheres – it follows that the group of h-cobordism classes of homotopy n-spheres denoted Θ_n is also the set of all smooth structures on the topological $n-$sphere. In particular, the manifold we studied in this paper is a single element of Θ_n. This opens a whole new set of tools to investigate the classification of exotic spheres. Indeed Kervaire and Milnor published their landmark paper *Groups of Homotopy Spheres* in 1962, where they described a vast array of the groups Θ_n and related results. For $n = 7$, the order of Θ_n is 28, thus corresponding to 28 distinct smooth structures on the 7-sphere.

More precisely, two closed, compact, oriented smooth manifolds M, N of same dimension are *h-cobordant* if there exists another manifold W such that $\partial W = M \sqcup N$ and the inclusion map(s) into W is a homotopy equivalence. The connected sum of two manifolds is the result of removing an $n-$sphere inside both manifolds and gluing the boundary together. Kervaire and Milnor showed that the h-cobordism classes of homotopy spheres form an abelian group under the connected sum operation, and found the order of Θ_n for several $n \neq 3$ along with numerous results concerning their structure.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	1	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16

The order of Θ_3 immediately follows from the Poincaré conjecture in dimension 3, proven by Perelman in 2003.

In 1966, Brieskorn described Θ_7 by taking the complex manifold in $\mathbb{C}^5$ defined by $a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0$ and intersecting it with a small sphere around the origin. It turns out this gives rise to all 28 possible exotic structures in dimension 7, each corresponding to $k = 1, 2, \ldots, 28$.

One who is physics-inclined may ask about the physical interpretation of exotic spheres. Consider the group of orientation-preserving diffeomorphisms of S^{n-1}, denoted $\text{Diff}(S^{n-1})$. One can obtain an $n-$sphere with any differentiable structure by gluing two $n-1$ spheres along their boundary with an orientation-preserving diffeomorphism. Indeed the group of smooth structures on S^n is isomorphic to the group of connected components of $\text{Diff}(S^{n-1})$. Note that string theory takes place in dimension 10, and (as is commonly the case in physics) we wish for certain quantities to be conserved i.e., invariant under diffeomorphisms of spacetime. Thus when spacetime is a 10 dimensional sphere, one must check that our quantities are invariant under diffeomorphisms for the 991 components of $\text{Diff}(S^{10})$, which happen to correspond to exotic 11-spheres. More information can be found in the paper *Global gravitational anomalies* by Edward Witten. Here he argues that relevant gravitational instantons - Riemannian manifolds satisfying Einstein's equations, are realized as exotic spheres. This was quite a nice find, as the author was also taking a class on general relativity at the time of writing.

There is still very little known about the diffeomorphism group of S^4; the existence of an exotic structure on the 4-sphere is still an open problem. A more recent result in the theory of exotic spheres was published by Guozhen Wang and Zhouli Xu in 2016, who showed that the only odd-dimensional spheres with a unique smooth structure are S^1, S^3, S^5 and S^{61}.

More generally,the early work on exotic spheres triggered an explosion of research in the structure and classification of smooth manifolds and the invention of tools for the purpose of this goal including surgery and cobordism theory, intersection forms, characteristic classes, elliptic operator theory, and the development of algebraic K-theory. This paradigm and the resulting techniques greatly influenced many other areas of mathematics, and their study is what we now call *differential topology.*

Appendix

Lemma A.1. Let f be a smooth function in a convex neighbourhood V of 0 in $\mathbb{R}^n$ and $f(0) = 0$. Then

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n)$$

for some suitable smooth functions g_i defined on V such that $g_i(0) = \partial_i f(0)$

Singular Homology

A *n-simplex* in a space X is a continuous map $\sigma : \Delta^n \to X$. The ith *face* of an $n-$simplex σ is the $(n-1)-$simplex $\sigma \circ \phi_i$, where ϕ_i inserts 0 for the ith coordinate. The chain group $C_n(X)$ is the free $\mathbb{Z}-$module with one generator $[\sigma]$ for each $n-$simplex in X. The *boundary map* is a homomorphism $\partial : C_n(X) \to C_{n-1}(X)$ defind by $\partial[\sigma] = \sum_{i=0}^{n}(-1)^i[\sigma \circ \phi_i]$. Note that $\partial \circ \partial = 0$. The nth singular homology group $H_n(X)$ is defined as the kernel of the nth boundary map modded by the image of the $(n+1)$th boundary map. We have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1} \arrow[r] & H_n \arrow[r] & H_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

A continuous map $f : X \to Y$ naturally induces a homomorphism $f_\# : C_n(X) \to C_n(Y)$, i.e., $f_\#[\sigma] = [f \circ \sigma]$, which naturally extends to homomorphisms $f_* : H_n(X) \to H_n(Y)$ i.e., $f_*[\sigma] = [f \circ \sigma]$. The following equality holds $\partial f_\# = f_\# \partial$.

Singular Cohomology

The *cochain group* $C^n(X)$ is the dual of the chain group $\{C_n(X) \to\}$. The cochain c evaluated at a chain γ is denoted $\langle c, \gamma\rangle$. The *coboundary map* is a homomorphism $\delta : C^n(X) \to C^{n+1}(X)$. Given a cochain $c \in C^n(X)$, the cochain $\delta c \in C^{n+1}(X)$ is defined $\langle \delta c, \alpha\rangle = (-1)^{n+1}\langle c, \partial\alpha\rangle$. This is well-defined; δc is a map $C_{n+1} \to \mathbb{Z}$, defined by taking values of C_n on a map $c : C_n \to \mathbb{Z}$. The *singular cohomology group* $H^n(X)$ is defined as the kernel of the nth coboundary map modded by the image of the $(n-1)$th coboundary map. We have the following diagram

$$\begin{tikzcd}
\cdots & C^{n+1} \arrow[l, "\delta"'] \arrow[d] & C^n \arrow[l, "\delta"'] \arrow[d] & C^{n-1} \arrow[l, "\delta"'] \arrow[d] & \cdots \arrow[l, "\delta"'] \\
\cdots & H^{n+1} \arrow[l] & H^n \arrow[l] & H^{n-1} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Relative (Co)homology

Definition. Given topological spaces (X, A) with inclusion $\iota : A \hookrightarrow X$, we have an induced map $\iota_* : C_n(A) \to C_n(X)$. Treating $C_n(A)$ as a submodule of $C_n(X)$, we may define the *relative chain group* of (X, A) as $C_n(X, A) = C_n(X)/C_n(A)$. In particular, the boundary map ∂ induces a map $\bar{\partial} : C_{n+1}(X, A) \to C_n(X, A)$.

Definition. The *relative homology groups* are defined as

$$H_n(X, A) = Z_n(X, A)/B_n(X, A)$$

where $Z_n(X,A) = \ker(\bar{\partial})$ are the *relative cycles* and $B_n(X,A) = \bar{\partial}(C_{n+1}(X,A))$ are the *relative boundaries*.

Relative cohomology groups are defined in an analogous manner.

Theorem A.2. Suppose $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Then if B is path-connected, there is a long exact sequence

$$\cdots \longrightarrow \pi_n(F,x_0) \longrightarrow \pi_n(E,x_0) \xrightarrow{p_*} \pi_n(B,b_0) \longrightarrow \pi_{n-1}(F,x_0) \longrightarrow \cdots \longrightarrow \pi_0(E,x_0) \longrightarrow 0$$

Theorem A.3. If $X \neq \varnothing$ and path-connected, then $H_0(X) = \mathbb{Z}$.

Theorem A.4. Let M be a closed connected $n-$manifold then if M is $R-$orientiable, $H_n(M;R) = R$ and $H_i(M;R) = 0, i > n$.

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