Assignment 1
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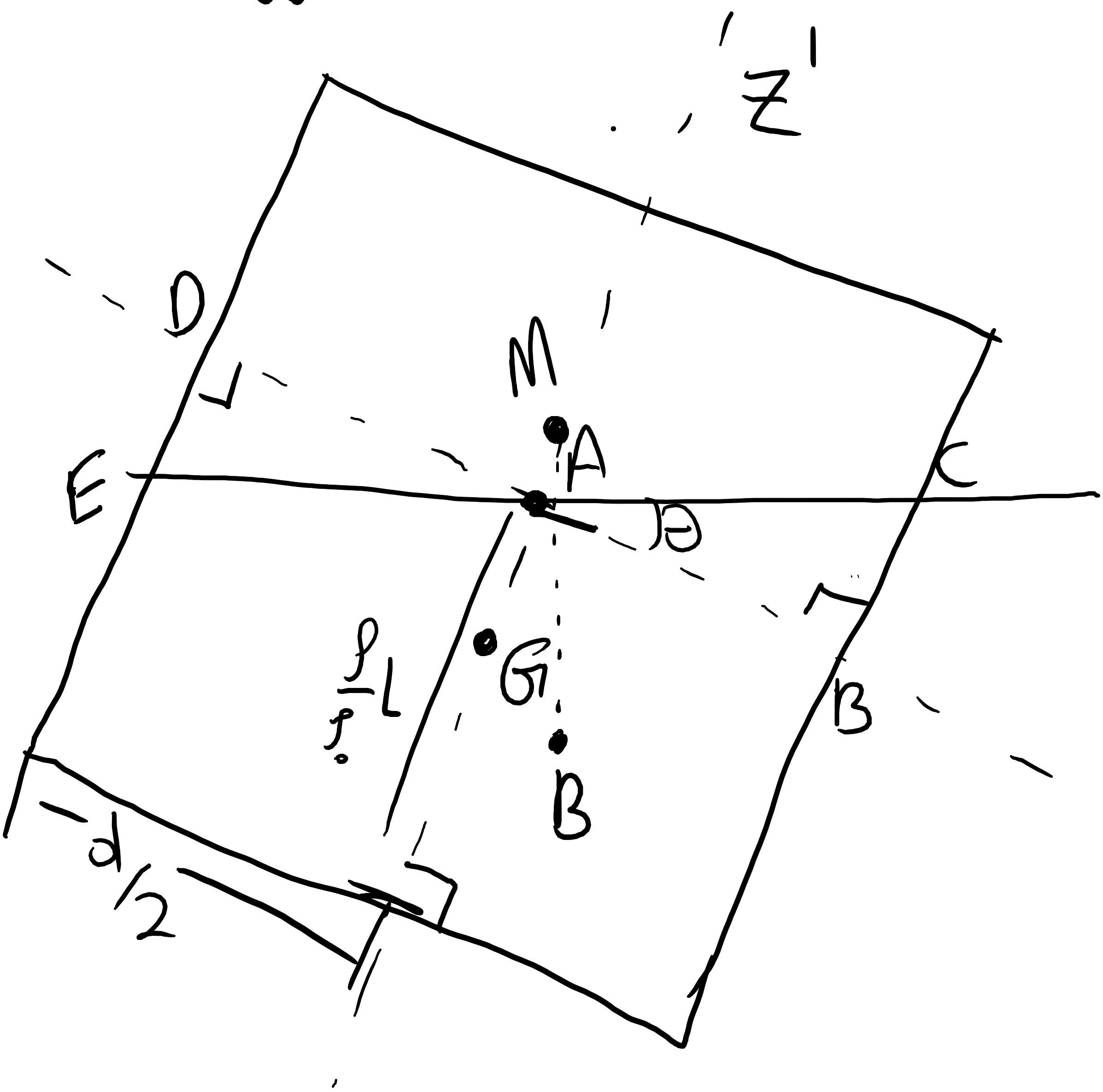
1) if the block has density ρ then the depth the block sinks into the fluid is given by

$$h = \frac{\rho}{\rho_0} L$$



we know that the body will be stable to pertebation if the metacenter of the body M (defined as the point on the z' axis directly above the center of Buoancy) is above the center of gravity on the axis z'

the limiting condition for this is when the sum of moments is 0

so $\sum_i M_i > 0 \Rightarrow$ stability

note: for small θ we can approximate length $\underset{\frown}{CB}$ and $\underset{\frown}{DE}$ as $\frac{d}{2}\theta$ and the centroid of these two triangles is a distance $r = \frac{2}{3}\left(\frac{d}{2}\right) = \frac{d}{3}$ from pivot point A

So

$$0 > \frac{1}{2}\frac{d}{2}(CB)\left(\frac{d}{3}\right) + \frac{1}{2}\frac{d}{2}(DE)\left(\frac{d}{3}\right) - \frac{\rho}{\rho_0}Ld(Gb)$$

$$0 > \frac{1}{12}d^2\theta - \frac{\rho}{\rho_0}L(Gb)$$

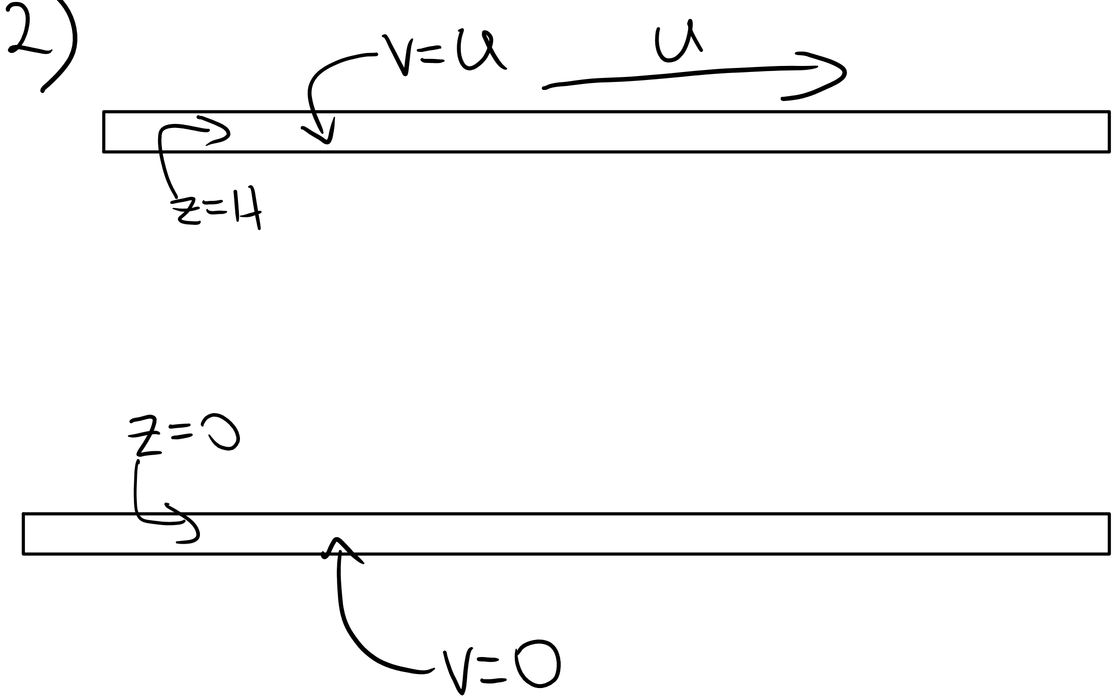
$$Gb = \frac{L}{2}\left(1 - \frac{\rho}{\rho_0}\right)\theta$$

$$\frac{1}{12}d^2 > \frac{\rho}{\rho_0}\frac{L^2}{2}\left(1 - \frac{\rho}{\rho_0}\right)$$

$$\boxed{\frac{d^2}{L^2} > 6\frac{\rho}{\rho_0}\left(1 - \frac{\rho}{\rho_0}\right)}$$

b) if an unstable block is rotated $90°$ it will result in a stable block because if it was not a block which was unstable on both sides would become a perpetual motion machine from a small pertebation θ.

2)



the equation of motion for a packet of fluid is eqtn (4.45)

$$\rho \frac{D\underset{\sim}{u}}{Dt} = -\underset{\sim}{\nabla P} + \rho \underset{\sim}{g} + \mu \nabla^2 \underset{\sim}{u}$$

for the steady state $\frac{Du}{Dt} = 0$

with no pressure forces $\nabla P = 0$

Since there is no z velocity $\rho \underset{\sim}{g} = 0$

rather is exactly cancelled by pressure gradients

So

$$0 = \mu \nabla^2 \underset{\sim}{u}$$

$$0 = \mu \frac{d^2 u}{dz^2}$$

integrating we get

$$C = \mu \frac{du}{dz}$$

$$Cz = \mu u(z) + C_2$$

Now we want to apply Boundry Conditions

$$u(z=0) = 0 \Rightarrow 0 = 0 + C_2 \Rightarrow C_2 = 0$$

$$u(z=H) = U \Rightarrow CH = \mu U \Rightarrow C = \frac{\mu U}{H}$$

$$\frac{\mu U}{H} z = \mu U(z)$$

$$\boxed{U(z) = \frac{U}{H} z}$$

b) for the transient flow $\frac{Du}{Dt}$ no longer equals 0, but all other terms remain identical.

$$\frac{Du}{Dt} \equiv \frac{du}{dt} + \underbrace{u \cdot \nabla u}$$

This term is 0 for this problem because ∇u only has a z component and velocity u has only an x component

So the navier-stokes equation becomes

$$\rho \frac{du}{dt} = \mu \frac{d^2u}{dz^2}$$

since the RHS stays the same as part a as non of that changes in time

let $w(z,t) = u(z,t) - u_0(z)$

then

$$\boxed{\rho \frac{dw}{dt} = \mu \frac{d^2w}{dz^2}}$$

Boundary conditions

$u(0,t) = 0 \; ; \; u(H,t) = U$

$u(z,0) = 0 \; ; \; u(z,\infty) = \frac{U}{H} z$

Sin

$\Rightarrow u_0(z) = \frac{u}{H} z$

$$\boxed{\begin{array}{ll} w(0,t) = 0 & w(H,t) = 0 \\ w(z,0) = -\frac{u}{H} z & w(z,\infty) = 0 \end{array}}$$

c) so we have the heat equation

given as

$$\rho \frac{d\omega}{dt} = \mu \frac{\partial^2 \omega}{\partial z^2}$$

this has solutions of the form

$$\omega(z,t) = \phi(z)G(t)$$

where $\frac{dG}{dt} = -k\lambda G$

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

Solving and applying boundary conditions $\phi(0) = 0$ and $\phi(H) = 0$ we get that:

$$\phi(z) = c_2 \sin\left(\frac{n\pi}{H} z\right) \; ; \; \frac{n\pi}{H} = \sqrt{\lambda} \text{ and } n \in \mathbb{N}$$

Solving for $G(t)$ we get

$$G(t) = ce^{-k\lambda t} = ce^{-k\left(\frac{n\pi}{H}\right)^2 t}$$

So the solution to the equation

is given by

$$w(z,t) = C\sin\left(\frac{n\pi}{H} z\right) e^{-k\left(\frac{n\pi}{H}\right)^2 t}$$

note that this is actually infinite solutions with $n \in \mathbb{N}$ so we write

$$\boxed{w(z,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{H} z\right) e^{-k\left(\frac{n\pi}{H}\right)^2 t}}$$

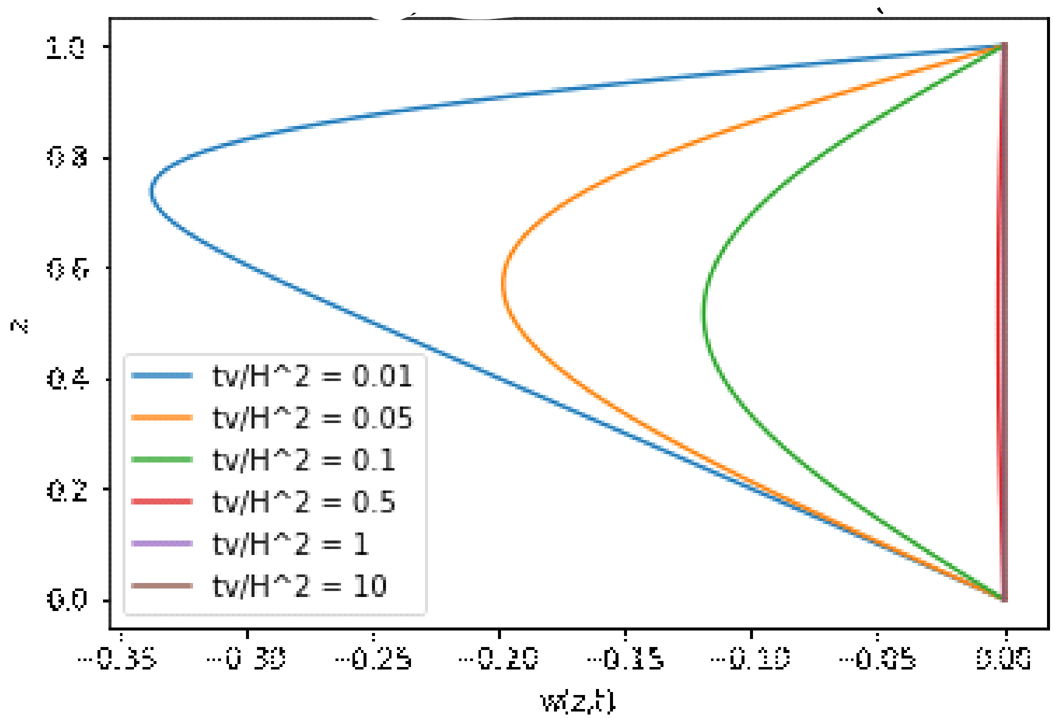
This is a fourier series with $k = \frac{\mu}{\rho}$

$$\boxed{C_n = \frac{1}{H}\int_0^H \left(-U_0(z)\right) \sin\left(\frac{n\pi}{H} z\right) dz = -\frac{U}{H^2}\int_0^H z \sin\left(\frac{n\pi z}{H}\right) dz}$$

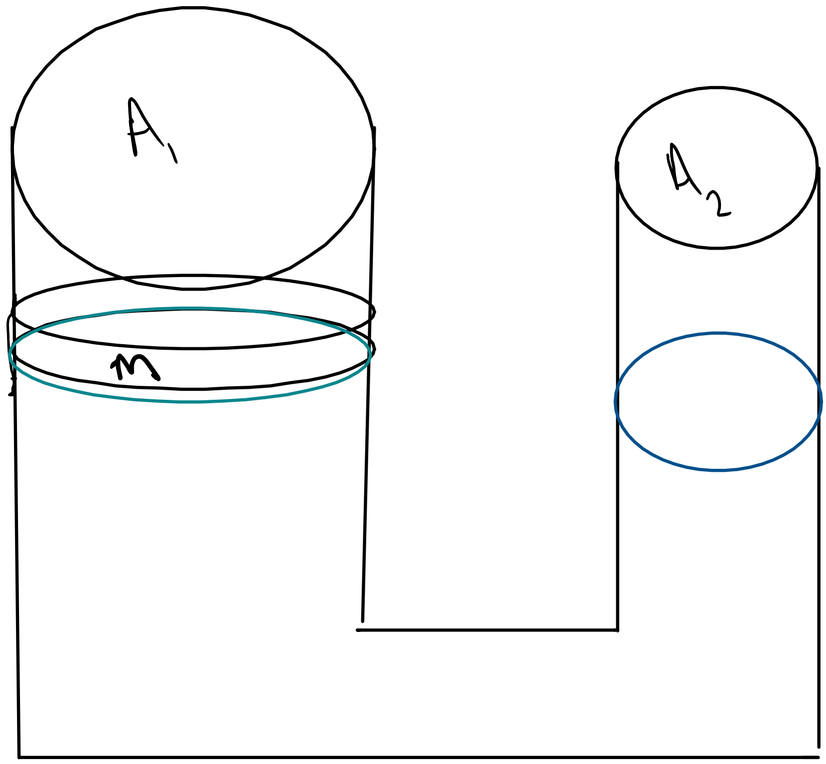
d)

Plot of $w(z,t)$ vs height z

max height at $z = 1$



3)



At equilibrium $P_1 = P_2 \Rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2}$

with a plunger of mass m added to the A_1 side we get

$$P_1 = \frac{mg}{A_1} - \frac{\cancel{A_1} \Delta h_1 \rho g}{\cancel{A_1}} = \frac{\cancel{A_2} \Delta h_2 \rho g}{\cancel{A_2}}$$

$$\frac{m\cancel{g}}{A_1} = (\Delta h_2 + \Delta h_1) \rho \cancel{g}$$

$$m = (\Delta h_2 + \Delta h_1) \rho A_1$$

the total volume change of water on either side must be equal so.

$$\Delta h_2 A_2 = \Delta h_1 A_1$$

$$\Rightarrow \Delta h_2 = \frac{\Delta h_1 A_1}{A_2}$$

So

$$m = \Delta h_1 \left(\frac{A_1}{A_2} + 1\right) \rho A_1$$

then

$$\boxed{\Delta h_1 = \frac{m}{\rho A_1} * \frac{1}{\left(1 + \frac{A_1}{A_2}\right)}}$$

units deeper than initially

b) if we exert a force on area A_2 we need the change in pressures to be equal on each side

$$\Delta P_1 = \Delta P_2$$

$$\Rightarrow \frac{F_2}{A_2} = \frac{F_1}{A_1}$$

So

$$\boxed{F_1 = F_2\left(\frac{A_1}{A_2}\right)}$$

c) The force on the plunger has a magnitude F_1 as calculated above. So the work done is simply given by

$$\boxed{F \cdot d = F_2\left(\frac{A_1}{A_2}\right)h_1}$$

D) This does not violate conservation of energy since

$$P = \frac{F}{A} = \frac{F \cdot d}{A \cdot d} = \frac{W}{V} = \frac{\text{Energy}}{\text{Volume}}$$

So

$$P \cdot V = \text{Energy} \; ; \; P = \frac{F}{A} \text{ and } V = A \cdot h$$

Setting the total energy on each side of the system equal we see

$$P_1V_1 = P_2V_2 \Rightarrow \frac{F_1}{\cancel{A_1}}\cancel{A_1}h_1 = \frac{F_2}{\cancel{A_2}}\cancel{A_2}h_2$$

we know from b) that $F_1 = \frac{A_1}{A_2}F_2$

So

$$\frac{F_2 A_1 h_1}{A_2} = F_2 h_2$$

$$\Rightarrow A_1h_1 = A_2h_2$$

we know that the change in volume of water on each side must be equal by conservation of mass so

$$A_1h_1 = A_2h_2$$

which is what we derived by conservation assuming energy was true. So even though the forces are unequal the total energy is still conserved

4) $$(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = \omega \times \underset{\sim}{u} + \nabla\left(\frac{\underset{\sim}{u}^2}{2}\right)$$

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\underset{\sim}{u} = \omega \times \underset{\sim}{u} + \nabla\left(\frac{\underset{\sim}{u}^2}{2}\right)$$

$$\left(u\frac{d\underset{\sim}{u}}{dx} + v\frac{\partial \underset{\sim}{u}}{\partial y} + w\frac{\partial \underset{\sim}{u}}{dz}\right) = \omega \times \underset{\sim}{u} + \nabla\left(\frac{\underset{\sim}{u}^2}{2}\right)$$

$$\left(u\frac{\partial \underset{\sim}{u}}{\partial x} + v\frac{\partial \underset{\sim}{u}}{\partial y} + w\frac{\partial \underset{\sim}{u}}{\partial z}\right) = \left((\underset{\sim}{\nabla} \times \underset{\sim}{u}) \times \underset{\sim}{u}\right) + \nabla\left(\frac{\underset{\sim}{u}^2}{2}\right)$$

Vector triple product $\longrightarrow$ $\left(\underset{\sim}{u}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u}) - \underset{\sim}{u}(\underset{\sim}{\nabla}\cdot\underset{\sim}{u})\right) = 0$

So

$$\left(u\frac{\partial \underset{\sim}{u}}{\partial x}+v\frac{\partial \underset{\sim}{u}}{\partial y}+w\frac{\partial \underset{\sim}{u}}{\partial z}\right)=\nabla\left(\frac{\underset{\sim}{u}^2}{2}\right)$$

$$\left(u\frac{\partial \underset{\sim}{u}}{\partial x}+v\frac{\partial \underset{\sim}{u}}{\partial y}+w\frac{\partial \underset{\sim}{u}}{\partial z}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x}\underset{\sim}{u}^2+\frac{\partial}{\partial y}\underset{\sim}{u}^2+\frac{\partial}{\partial z}\underset{\sim}{u}^2\right)$$

$$\left(u\frac{\partial \underset{\sim}{u}}{\partial x}+v\frac{\partial \underset{\sim}{u}}{\partial y}+w\frac{\partial \underset{\sim}{u}}{\partial z}\right)=\frac{1}{2}\left(\frac{\partial \underset{\sim}{u}^2}{\partial \underset{\sim}{u}}\frac{\partial \underset{\sim}{u}}{\partial x}+\frac{\partial \underset{\sim}{u}^2}{\partial \underset{\sim}{u}}\frac{\partial \underset{\sim}{u}}{\partial y}+\frac{\partial \underset{\sim}{u}^2}{\partial \underset{\sim}{u}}\frac{\partial \underset{\sim}{u}}{\partial z}\right)$$

$$u\frac{\partial \underset{\sim}{u}}{\partial x}+v\frac{\partial \underset{\sim}{u}}{\partial y}+w\frac{\partial \underset{\sim}{u}}{\partial z}=\underset{\sim}{u}\cdot\left(\frac{\partial \underset{\sim}{u}}{\partial x}+\frac{\partial \underset{\sim}{u}}{\partial y}+\frac{\partial \underset{\sim}{u}}{\partial z}\right)$$

$$\underset{\sim}{u}\cdot\left(\frac{\partial \underset{\sim}{u}}{\partial x}+\frac{\partial \underset{\sim}{u}}{\partial y}+\frac{\partial \underset{\sim}{u}}{\partial z}\right)=u\cdot\left(\frac{\partial \underset{\sim}{u}}{\partial x}+\frac{\partial \underset{\sim}{u}}{\partial y}+\frac{\partial \underset{\sim}{u}}{\partial z}\right)$$