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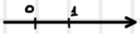
# TD1 : EDP1

**Exercice 1 :** $\partial\Omega = \{0,1\}$ $\quad u(t,0) = u(t,1) = 0 \longleftarrow$ CL Dirichlet homogène



On considère le problème de la chaleur en une dimension d'espace posé dans le domaine $\Omega = (0,1)$ :

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, & t > 0, \quad x \in (0,1), \; (*) \\ u(t,x) = 0, & t > 0, \quad x \in \{0,1\}, \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases}$$

1.1) Soit $v \in \mathcal{C}^1([0,1])$ tq $v(0)=0$, mq $v$ vérifie l'inégalité de Poincaré

$$\int_0^1 v^2(x)\,dx \leqslant \int_0^1 v'^2(x)\,dx$$

$$(\Leftrightarrow \|v\|_2 \leqslant \|v'\|_2)$$

On a pour $x \in [0,1]$ :

$$v(x) = \underbrace{\cancel{v(0)}}_{=0} + \int_0^x v'(y)\,dy$$

$$\Rightarrow |v(x)| \leqslant \int_0^x |v'|\,dy$$

Il suffit d'appliquer l'inégalité de Cauchy-Schwarz :

$$|v.1|\,dy \leqslant \left(\int_0^x |v'|^2 dy\right)^{1/2} \left(\int_0^x |1|^2 dy\right)^{1/2}$$

$$\leqslant \left(\int_0^x |v'|^2 dy\right)^{1/2} \underbrace{\sqrt{x}}_{\in[0,1]} \leqslant 1$$

$$\leqslant \sqrt{\int_0^1 |v'|^2 dy}$$

$$\leqslant \int_0^1 |v'|^2 dy$$

$$\Rightarrow \int_0^1 |v(x)|^2 dx \leqslant \int_0^1 \left(\int_0^1 |v(y)|^2 dy\right) dx \quad \Rightarrow \quad \boxed{\int_0^1 |v(x)|^2 dx \leqslant \int_0^1 |v'(y)|^2 dy}$$

Rq : $|v(x)| \leqslant \int_0^1 |v'|\,dy \leqslant 1 \cdot \|v'\|_2$ (CS sur $L^2([0,1])$)

$\Rightarrow \|v\|_\infty \leqslant \|v\|_2$

ET (cf MI1), $\mu_L([0,1]) < \infty \Rightarrow C\|v\|_\infty \leqslant C\|v'\|_2$

et $C=1$ cf. $\int_0^1 |v|^2 dx \leqslant \|v\|_\infty^2 \int_0^1 1\,dx$

1.2) Soit $E(t) := \int_0^1 |u(t,x)|^2 dx$, avec $u$ solut° de $(*)$

Mq $E$ dérivable et que :

$$\frac{1}{2}E'(t) = -\int_0^1 \left(\frac{\partial u}{\partial x}(t,x)\right)^2 dx$$

Rq : $\left|\frac{\partial}{\partial t}\left(|u(t,x)|^2\right)\right| \leqslant v(x)$, $v \in L^1([0,1])$ (cf. MI1)

$$\Rightarrow E'(t) = \frac{d}{dt}\int_0^1 |u|^2 dx$$

$$= \int_0^1 \frac{\partial}{\partial t}(u(t,x))^2 dx$$

$$= 2\int_0^1 \frac{\partial u}{\partial t}(t,x) \times u(t,x)\,dx$$

$= \frac{\partial^2 u}{\partial x^2}$, cf solut° de la chaleur : $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$

$$= 2\int_0^1 \frac{\partial^2 u}{\partial x^2}(t,x) \times u(t,x)\,dx$$

Rq : $u(t,0) = 0, t>0 \Longrightarrow \frac{\partial u}{\partial t}(t,0) = 0, t>0$

De m̂, $\frac{\partial u}{\partial t}(t,1) = 0$

$$= 2\left(\underbrace{\cancel{\left[u\frac{\partial u}{\partial x}\right]_0^1}}_{=0,\text{ car condit° de Dirichlet}} - \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\right)$$

$r = u \quad | \quad r' = \frac{\partial u}{\partial x}$

$q' = \frac{\partial^2 u}{\partial x^2} \quad | \quad q = \frac{\partial u}{\partial x}$

$$\Rightarrow \boxed{\frac{1}{2}E'(t) = -\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx}$$

1.3) En déduire que : $E(t) \leqslant E(0)e^{-2t}$

On a : $E(t) = \int_0^1 |u(t,x)|^2 dx$

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int |u|^2 dx$$

$$E'(t) = \int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 dx \leqslant \int_0^1 v(x)\, dx$$

On a : $E'(t) = -2\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx$

$$\leqslant -2\int_0^1 u^2\, dx \qquad \text{(cf. MI1)}$$

Supposons que $E(t)>0$, on a :

$$\frac{E'(t)}{E(t)} \leqslant -2 \Rightarrow \int_0^t \frac{E'(s)}{E(s)} ds \leqslant -2t, \ t>0.$$

$$\Rightarrow \ln\left(\frac{E(t)}{E(0)}\right) \leqslant -2t, \text{ car } E>0$$

$$E(t) \leqslant e^{-2t}E(0) : (**)$$

→ démo. complète (sans hypothèse $E(t)>0$) :

on a

$$E'(t) \leqslant -2E(t)$$

$$\Rightarrow \underbrace{(E'(t)+2E(t)).e^{2t}}_{(E(t).e^{2t})'} \leqslant 0$$

$\Rightarrow E e^{2t}$ est ↘, donc que

$E(t).e^{2t} \leqslant E(0)$, d'où la conclusion.

Rq. : pour ce type de comparaison, plus généralement voir le Lemme de Gronwall pour les EDO.

**Exercice 2 :**

On se propose de caractériser la solution régulière du problème des ondes en une dimension d'espace dans le domaine $\Omega \subset \mathbb{R}$ :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, & t>0, \quad x \in \Omega, \\ u(t,x) = 0, & t>0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \\ \frac{\partial u}{\partial t}(0,x) = u_1(x), & x \in \Omega, \end{cases}$$

où $u_0$ et $u_1$ sont des fonctions régulières et $U_1$ une primitive de $u_1$.

2.1) Mq $E(t) = \int_0^1 \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) dx$ = cste

Il faut donc montrer que $\frac{dE}{dt}(t) = 0 \Rightarrow \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2}\right)\frac{\partial u}{\partial t} = 0$

et mq $\int_0^1 \frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] dx = 0$

On a : $E'(t) = 2\int_0^1 \left(\frac{\partial u}{\partial t}\cdot\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\frac{\partial^2 u}{\partial x^2} \text{ (cf. sol)}} + \frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial x^2}\right) dx$

or $\int_0^1 \frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial x^2} dx = \underbrace{\cancel{\left[\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x}\right]_0^1}}_{=0} - \int_0^1 \frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x \partial t} dx$

$\frac{\partial u}{\partial t}(t,0) = \frac{\partial u}{\partial t}(t,1) = 0$ Neumann

cf CL $u(t,0)=0$, $t>0$

$\Rightarrow \frac{\partial u}{\partial t}(t,0) = 0$, $t>0$

de m̂ en $x=1$

$\Rightarrow E'(t) = 0$

2.2) En déduire l'unicité de solut° pour cette EDP.

Supposons que $u$ et $v$ soient 2 solut° de (*) ; par linéarité $w := u-v$ vérifie cette nouvelle EDP :

$$(**) : \begin{cases} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0, & t>0, x \in \Omega \\ w(t,0) = w(t,1) = 0, & t>0 \\ w(0,x) = u_0(x) - u_0(x) = 0 \\ \frac{\partial w}{\partial t}(0,x) = u_1(x) - u_0(x) = 0 \end{cases}$$

Or l'énergie associée à $w$ (sol. de (**)) est également cste (cf. 2.2)

$E(t) = \int_0^1 \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dx = E(0)$ et $\frac{\partial w}{\partial t}(0,x)$, ainsi que $w(0,x) = 0$, $x \in \Omega$

$\Rightarrow \frac{\partial w}{\partial t}(0,x) = 0$

$\Rightarrow E(t) = 0$, $t>0$

$\Rightarrow \exists C \in \mathbb{R}$, $w(t,x) = C$

$\Rightarrow w(t,x) = 0$, car $w(0,x) = 0$ (CL. Dirichlet)

2.3) Soit $U_1$ une primitive de $U_1$.

Mq $u(t,x) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\left(U_1(x+t) - U_1(x-t)\right)$

est une solut° si $\Omega = \mathbb{R}$, ie solut° de :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, & t>0, x \in \Omega = \mathbb{R}. \\ u(0,x) = u_0(x) & (\partial\Omega = \emptyset) \\ \frac{\partial u}{\partial t}(0,x) = u_1(x) & (U_1' = u_1) \end{cases}$$

- $\frac{\partial u}{\partial t} = \frac{1}{2}\left(u_0'(x+t) - u_0'(x-t)\right) + \frac{1}{2}\left(u_1(x+t) + u_1(x-t)\right)$
- $\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\left(u_0''(x+t) + u_0''(x-t)\right) + \frac{1}{2}\left(u_1'(x+t) - u_1'(x-t)\right)$
- $\frac{\partial u}{\partial x} = \frac{1}{2}\left(u_0'(x+t) + u_0'(x-t)\right) + \frac{1}{2}\left(u_1(x+t) - u_1(x-t)\right)$
- $\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(u_0''(x+t) + u_0''(x-t)\right) + \frac{1}{2}\left(u_1'(x+t) - u_1'(x-t)\right)$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

+ CI évident

EXERCICE 3 : Schrödinger $u : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$

$$\begin{cases} i\dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial^2 u}{\partial x^2}(t,x) - V(x).u(t,x) = 0, & t>0, x\in\mathbb{R} \\ u(t,x) \xrightarrow[|x|\to\infty]{} 0, & t>0 \\ u(0,x) = u_0(x), & u\in\Omega \end{cases}$$

$\overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$

$u(t,x) = 0,\ x\in\underbrace{\partial\Omega}_{=\{\pm\infty\}}$

3.1) $v : \mathbb{R}^n \longrightarrow \mathbb{C}$ dérivable
$y \longmapsto v(y)$

Mq $\mathcal{R}e\left(\dfrac{\partial v}{\partial y_i}\right) = \dfrac{1}{2}\dfrac{\partial}{\partial y_i}\left(|v|^2\right)$

on sait que $|v(y)|^2 = v(y).\bar{v}(y)$, $v(y) = \overbrace{v_1(y)}^{\mathcal{R}e} + i\overbrace{v_2(y)}^{\mathcal{I}m} \in \mathbb{C}$

bilinéaire

$\simeq \begin{bmatrix} v_1(y) \\ v_2(y) \end{bmatrix} \in \mathbb{R}^2$, car $\mathbb{C} \cong \mathbb{R}^2$

Par bilinéarité, $v$ étant dérivable

$\Rightarrow |v|^2$ dérivable aussi et pour tout $i\in[\![1,n]\!]$

$$\frac{\partial}{\partial y_i}(|v|^2)(y) = \frac{\partial v}{\partial y_i}(y).\bar{v}(y) + \underbrace{v(y).\frac{\partial \bar{v}}{\partial y_i}(y)}_{\bar{v}(y).\frac{\partial v}{\partial y_i}(y)}$$

$$= 2\,\mathcal{R}e\left(\frac{\partial v}{\partial y_i}\langle y\,|\,\bar{v}(y)\rangle\right)$$

3.2) En multipliant l'EDP par $\bar{u}$ (et aussi en intégrant sur $\Omega = \mathbb{R}$), mq $\underbrace{\int_{\mathbb{R}} |u(t,x)|^2 dx}_{\text{Energie}} = \text{cte} = \int |u_0(x)|^2 dx$ (cf. $t=0 \Rightarrow u(0,x) = u_0(x)$)

$u$ sol $\Rightarrow \underbrace{i\int_{\mathbb{R}} \frac{\partial u}{\partial t}.\bar{u}\,dx}_{=A} + \underbrace{\int_{\mathbb{R}}\left(\frac{\partial^2 u}{\partial x^2}\bar{u} - \underbrace{V|u|^2}_{\geq 0}\right)dx}_{\text{et mq } \in\mathbb{R}} = 0$ IPP $\Rightarrow \mathcal{R}e|A| = 0$

IPP : $\displaystyle\int \frac{\partial^2 u}{\partial x^2}\bar{u}\,dx = \underbrace{\cancel{\left[\frac{\partial u}{\partial x}\bar{u}\right]_{\mathbb{R}}}}_{=0,\ CL} - \int_{\mathbb{R}} \underbrace{\frac{\partial u}{\partial x}.\frac{\partial \bar{u}}{\partial x}}_{\left|\frac{\partial u}{\partial x}\right|^2\in\mathbb{R}} dx$

$\Rightarrow \mathcal{R}e(A) = 0$

$\Rightarrow \mathcal{R}e\left(\int \frac{\partial u}{\partial x}\bar{u}\,dx\right) = 0$

$\Rightarrow \int \mathcal{R}e\left(\frac{\partial u}{\partial x}\bar{u}\right)dx = 0 \quad \overset{\text{cf 3.1}}{\Longrightarrow} \int_{\mathbb{R}} \mathcal{R}e\left(\frac{\partial u}{\partial t}.\bar{u}\right)dx = \int_{\mathbb{R}} \frac{1}{2}\frac{\partial}{\partial t}\left(|u|^2\right)dx.$

$\Rightarrow \dfrac{d}{dt}\displaystyle\int_{\mathbb{R}} \frac{1}{2}|u|^2\,dx$

$\Rightarrow \displaystyle\int_{\mathbb{R}} \underbrace{|u(t,x)|^2}_{} dx = \text{cst} = \int_{\mathbb{R}} |u(0,x)|^2 dx$

densité de la probabilité
de la présence d'un électron

**Autre méthode :**

$$\frac{d}{dt}\int_{\mathbb{R}} |u(t,x)|\,dx = \int_{\mathbb{R}} \frac{\partial}{\partial t}|u(t,x)|^2 dx \quad : (*)$$

3.1 $\Rightarrow (*) = 2\,\mathcal{R}e\displaystyle\int_{\mathbb{R}} \frac{\partial u}{\partial t}(t,x)\,\bar{u}(t,x)\,dx$

en multipliant par (1) par $-i$ on a :

$$-\frac{\partial u}{\partial t}(t,x) + i\frac{\partial^2 u}{\partial x^2} - iv\,u(t,x) = 0$$

donc $(*) = 2\,\mathcal{R}e\displaystyle\int_{\mathbb{R}}\left(i\frac{\partial^2 u}{\partial x^2}(t,x)\bar{u}(t,x) - \underbrace{\cancel{iv\,u(t,x)\bar{u}(t,x)}}_{iV|u|^2 \in i\mathbb{R}}\right)dx$ $\in i\mathbb{R}$

$$= 2\,\mathcal{R}e\left(i\int \frac{\partial^2 u}{\partial x^2}\bar{u}\,dx\right)$$

$$= 2\,\mathcal{R}e\left(\underbrace{i\left[\frac{\partial u}{\partial x}\bar{u}\right]_{-\infty}^{+\infty} - i\int_{\mathbb{R}}\frac{\partial u}{\partial x}.\frac{\partial \bar{u}}{\partial x}\,dx}_{\in i\mathbb{R}}\right)$$

$= 0.$

d'où $E = \text{cst} = \displaystyle\int_{\mathbb{R}} |u_0(x)|^2 dx$

3.3) En multipliant l'EDP par $\frac{\partial \bar{u}}{\partial t}$, mq : $\int_{\mathbb{R}} \left( \left|\frac{\partial u}{\partial x}\right|^2 + V|u|^2 \right) dx = cst$

$$= \int_{t=0} \int_{\mathbb{R}} \left( \left|\frac{\partial u_0}{\partial x}(x)\right|^2 + V(x)\left|u_0(x)\right|^2 \right) dx$$

On a : $\int_{\mathbb{R}} \left( i \underbrace{\frac{\partial u}{\partial t} \cdot \frac{\partial \bar{u}}{\partial t}}_{\frac{\partial u}{\partial t} \cdot \frac{\partial \bar{u}}{\partial t} = \left|\frac{\partial u}{\partial t}\right|^2} + \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial \bar{u}}{\partial t} - Vu \cdot \frac{\partial \bar{u}}{\partial t} \right) dx = 0$

$\in i\mathbb{R}$

$\Rightarrow \mathcal{R}e \left( \int_{\mathbb{R}} \left( i \frac{\partial u}{\partial t} \cdot \frac{\partial \bar{u}}{\partial t} + \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial \bar{u}}{\partial t} - Vu \cdot \frac{\partial \bar{u}}{\partial t} \right) dx = 0 \right.$

$\mathcal{R}e \left( \int \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial \bar{u}}{\partial t} - Vu \frac{\partial \bar{u}}{\partial t} \right) dx = 0$

or $\int \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial \bar{u}}{\partial t} dx = \underbrace{\cancel{\left[ \frac{\partial u}{\partial x} \cdot \frac{\partial \bar{u}}{\partial t} \right]}}_{=0} - \int \frac{\partial u}{\partial x} \cdot \underbrace{\frac{\partial^2 \bar{u}}{\partial x \partial t}}_{\frac{\partial^2 \bar{u}}{\partial t \partial x} = \frac{\partial}{\partial t}\left(\frac{\partial \bar{u}}{\partial x}\right)} dx$

$\Rightarrow - \int \mathcal{R}e \left( \underbrace{\frac{\partial u}{\partial x} \cdot \underbrace{\frac{\partial^2 \bar{u}}{\partial t \partial x}}_{\frac{\partial}{\partial t}\left(\frac{\partial \bar{u}}{\partial x}\right)}}_{= \frac{1}{2} \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial x}\right|^2\right)} - \underbrace{V u \frac{\partial \bar{u}}{\partial t}}_{= \frac{V}{2} \frac{\partial}{\partial t} |u|^2} \right) dx = 0$

Donc : $\int_{\mathbb{R}} \left( \frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial x}\right|^2\right) + V\left|\frac{\partial u}{\partial t}\right|^2 \right) dx = 0$

$\Rightarrow \frac{d}{dt} \int_{\mathbb{R}} \left( \left|\frac{\partial u}{\partial x}\right|^2 + V|u|^2 \right) dx = 0$

$\Rightarrow \int_{\mathbb{R}} \left( \left|\frac{\partial u}{\partial x}\right|^2 + V|u|^2 \right) dx = cst$