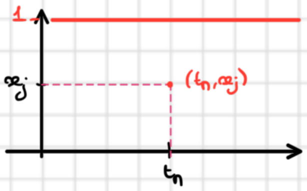
TD2 : Étude de la consistance :

Considérons l'équat° de la chaleur :

$$\begin{cases} \frac{\partial u}{\partial t} - V\frac{\partial^2 u}{\partial x^2} = 0, \quad t>0, \ x \in \Omega :=]0,1[\\ CL \\ CI \end{cases}$$



$x_j = j\Delta x$, $\forall j \in$

$t_n = n\Delta t$

θ-schéma, $\theta \in [0,1]$: $\theta = \frac{1}{2}$: Krank-Nicolson

explicite

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - (1-\theta)V.\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \theta.V.\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} = 0$$

implicite

Combinaison Convexe

Rq : $\theta > 0 \Rightarrow$ schéma implicite
$\theta = 0$: Euler (explicite)
$\theta = 1$: Euler implicite

On se place en $(t,x) = (t_n, x_j)$ et on applique Taylor pour u solut° :

DL à l'ordre 2 en temps :

$$u(t_{n+1}, x_j) = u(t_n, x_j) + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}.\Delta t^2 + O(\Delta t^3)$$

$$\Rightarrow \frac{u(t_{n+1}, x_j) - u(t_n, x_j)}{\Delta t} = \frac{\partial u}{\partial t}(t_n, x_j) + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + O(\Delta t^3)$$

- à l'ordre 4 en espace :

$$u(t_n, x_{j+1}) = u(t_n, x_j) + \frac{\partial u}{\partial x}(t_n, x_j)\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t_n, x_j) + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(t_n, x_j)\Delta x^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(t_n, x_j)\Delta x^4 + O(\Delta x^5)$$

$$\Rightarrow \frac{u(t_n, x_{j+1}) - 2u(t_n, x_j) + u(t_n, x_{j-1})}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2}(t_n, x_j) + \frac{1}{12}\frac{\partial^4 u}{\partial x^4} + O(\Delta x^3)$$

idem pour t_{n+1} (pour le schéma implicite)

pour t_{n+1}

$$= \underbrace{\frac{\partial^2 u}{\partial x^2}(t_{n+1}, x_j)}_{\frac{\partial^2 u}{\partial x^2}(t_n, x_j) + \frac{\partial^3 u}{\partial t \partial x^2}.\Delta t} + \frac{1}{12}\frac{\partial^4 u}{\partial x^4} + O(\Delta x^3)$$

Or $\frac{\partial^2 u}{\partial x^2} = \frac{1}{V}\frac{\partial u}{\partial t}$ car u solut°

$$\Rightarrow (*) = \frac{1}{V}\frac{\partial u}{\partial t}(t_{n+1}, x_j) + \frac{1}{12}.\frac{\partial^4 u}{\partial x^4}\Delta x^2 + O(\Delta x^3)$$

$$= \frac{1}{V}\frac{\partial u}{\partial t}(t_n, x_j) + \frac{1}{V}\frac{\partial^2 u}{\partial t^2}.\Delta t + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(t_n, x_j)\Delta x^2 + \frac{1}{12}\frac{\partial^5 u}{\partial t \partial x^4}.\Delta x^2.\Delta t + O(\Delta x^3)$$

$\Rightarrow$ On remplace par u solut° dans le schéma :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - (1-\theta)V.\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \theta.V.\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} = \frac{\partial u}{\partial t}(t_n, x_j) + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\boxed{\Delta t} + O(\Delta t^2)$$

$$- (1-\theta).V\left(\frac{1}{V}\frac{\partial u}{\partial x^2} + \frac{1}{12}\frac{\partial^4 x}{\partial x^4}.\boxed{\Delta x^2} + O(\Delta x^3)\right)$$

$$- \theta V\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{V}\frac{\partial^2 u}{\partial t^2}\Delta t + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}\boxed{\Delta x^2} + \dots\right)$$

$$= 0 + E_j^n$$

$= □ + □ + ⬠ + ◇$

$$= \left(\frac{1}{2} - \theta\right)\frac{\partial^2 u}{\partial t^2}(t_n, x_j)\Delta t - \frac{V}{6}\frac{\partial^4 u}{\partial x^4} + \dots$$

$\Rightarrow \theta \neq \frac{1}{2}$ ordre 1 en t
ordre ≥ 2 sinon
(= 2 : à démontrer !)