3.2.2 Variational formulation

To simplify the presentation, we assume that the open set Ω is bounded and regular, and that the right-hand side f of (3.1) is continuous on $\overline{\Omega}$. The principal result of this section is the following proposition.

Proposition 3.2.7 *Let u be a function of $C^2(\overline{\Omega})$. Let X be the space defined by*

$$X = \left\{ \phi \in C^1(\overline{\Omega}) \text{ such that } \phi = 0 \text{ on } \partial\Omega \right\}.$$

Then u is a solution of the boundary value problem (3.1) if and only if u belongs to X and satisfies the equation

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx \quad \text{for every } v \in X. \tag{3.8}$$

Equation (3.8) is called the **variational formulation** *of the boundary value problem (3.1).*

Remark 3.2.8 An immediate consequence of the variational formulation (3.8) is that it is meaningful if the solution u is only a function of $C^1(\overline{\Omega})$, as opposed to the 'classical' formulation (3.1) which requires u to belong to $C^2(\overline{\Omega})$. We therefore already suspect that it is easier to solve (3.8) than (3.1) since it is less demanding on the regularity of the solution.

In the variational formulation (3.8), the function v is called the **test function**. The variational formulation is also sometimes called the weak form of the boundary value problem (3.1). In mechanics, the variational formulation is known as the 'principle of virtual work'. In physics, we also talk of the balance equation or the reciprocity formula.

When we take $v = u$ in (3.8), we obtain what is called an **energy equality**, which in general expresses the equality between the stored energy in the domain Ω (the left-hand term of (3.8)) and a potential energy associated with f (the right-hand term of (3.8)). •

Proof. If u is a solution of the boundary value problem (3.1), we multiply the equation by $v \in X$ and we use the integration by parts formula of corollary 3.2.4.

$$\int_\Omega \Delta u(x) v(x)\, dx = -\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(x) v(x)\, ds,$$

where $v = 0$ on $\partial\Omega$ since $v \in X$, therefore

$$\int_\Omega f(x) v(x)\, dx = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx,$$

which is nothing other than the formula (3.8). Conversely, if $u \in X$ satisfies (3.8), by using the integration by parts formula 'in reverse' we obtain

$$\int_\Omega \Big(\Delta u(x) + f(x) \Big) v(x)\, dx = 0 \qquad \text{for every } v \in X.$$

As $(\Delta u + f)$ is a continuous function, thanks to lemma 3.2.9 we conclude that $-\Delta u(x) = f(x)$ for all $x \in \Omega$. In addition, since $u \in X$, we recover the boundary condition $u = 0$ on $\partial\Omega$, that is, u is a solution of the boundary value problem (3.1). □

Lemma 3.2.9 *Let Ω be an open set of $\mathbb{R}^N$. Let $g(x)$ be a continuous function in Ω. If for every function ϕ of $C^\infty(\Omega)$ with compact support in Ω, we have*

$$\int_\Omega g(x)\phi(x)\,dx = 0,$$

then the function g is zero in Ω.

Proof. Assume that there exists a point $x_0 \in \Omega$ such that $g(x_0) \neq 0$. Without loss of generality, we can assume that $g(x_0) > 0$ (otherwise we take $-g$). By continuity, there exists a small open neighbourhood $\omega \subset \Omega$ of x_0 such that $g(x) > 0$ for all $x \in \omega$. Let ϕ be a nonzero positive test function with support in ω. We have

$$\int_\Omega g(x)\phi(x)\,dx = \int_\omega g(x)\phi(x)\,dx = 0,$$

which contradicts the hypothesis on g. Therefore $g(x) = 0$ for all $x \in \Omega$. □

Remark 3.2.10 We can rewrite the variational formulation (3.8) in compact notation: find $u \in X$ such that

$$a(u, v) = L(v) \quad \text{for every } v \in X,$$

with

$$a(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx$$

and

$$L(v) = \int_\Omega f(x)v(x)\,dx,$$

where $a(\cdot,\cdot)$ is a bilinear form on X and $L(\cdot)$ is a linear form on X. It is in this abstract form that we solve (with some hypotheses) the variational formulation in the next section. •

The principle idea of **the variational approach** is to show the existence and uniqueness of the solution of the variational formulation (3.8), which implies the same result for the equation (3.1) because of proposition 3.2.7. Indeed, we shall see that there is a theory, both simple and powerful, for analysing variational formulations. Nonetheless, this theory only works if the space in which we look for the solution and in which we take the test functions (in the preceding notation, the space X) is a Hilbert space, which is not the case for $X = \{v \in C^1(\overline{\Omega}),\ v = 0 \text{ on } \partial\Omega\}$ equipped with the 'natural' scalar product for this problem. The main difficulty in the application of the variational approach will therefore be that we must use a space other than X, that is the Sobolev space $H_0^1(\Omega)$ which is indeed a Hilbert space (see Chapter 4).

Exercise 3.2.3 In a bounded open set Ω we consider the Laplacian with Neumann boundary condition

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Let u be a function of $C^2(\overline{\Omega})$. Show that u is a solution of the boundary value problem (3.9) if and only if u satisfies the equation

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x)v(x)\, dx \quad \text{for every } v \in C^1(\overline{\Omega}). \tag{3.10}$$

Deduce from this that a necessary condition for the existence of a solution in $C^2(\overline{\Omega})$ of (3.9) is that $\int_\Omega f(x)dx = 0$.

Exercise 3.2.4 In a bounded open set Ω we consider the plate equation

$$\begin{cases} \Delta(\Delta u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.11}$$

We denote by X the space of functions v of $C^2(\overline{\Omega})$ such that v and $\frac{\partial v}{\partial n}$ are zero on $\partial\Omega$. Let u be a function of $C^4(\overline{\Omega})$. Show that u is a solution of the boundary value problem (3.11) if and only if u belongs to X and satisfies the equation

$$\int_\Omega \Delta u(x)\Delta v(x)\, dx = \int_\Omega f(x)v(x)\, dx \quad \text{for every } v \in X. \tag{3.12}$$

3.3 Lax–Milgram theory

3.3.1 Abstract framework

We describe an abstract theory to obtain the existence and the uniqueness of the solution of a variational formulation in a Hilbert space. We denote by V a real Hilbert space with scalar product $\langle , \rangle$ and norm $\| \ \|$. Following remark 3.2.10 we consider a variational formulation of the type:

$$\text{find } u \in V \text{ such that } a(u, v) = L(v) \text{ for every } v \in V. \tag{3.13}$$

The hypotheses on a and L are

(1) $L(\cdot)$ is a continuous linear form on V, that is, $v \to L(v)$ is linear from V into $\mathbb{R}$ and there exists $C > 0$ such that

$$|L(v)| \le C\|v\| \quad \text{for all } v \in V;$$

(2) $a(\cdot, \cdot)$ is a bilinear form on V, that is, $w \to a(w, v)$ is a linear form from V into $\mathbb{R}$ for all $v \in V$; and $v \to a(w, v)$ is a linear form from V into $\mathbb{R}$ for all $w \in V$;

(3) $a(\cdot,\cdot)$ is continuous, that is, there exists $M > 0$ such that

$$|a(w,v)| \leq M\|w\|\,\|v\| \quad \text{for all } w,v \in V; \tag{3.14}$$

(4) $a(\cdot,\cdot)$ is **coercive** (or elliptic), that is, there exists $\nu > 0$ such that

$$a(v,v) \geq \nu\|v\|^2 \quad \text{for all } v \in V. \tag{3.15}$$

As we shall see in this section, all the hypotheses above are necessary to solve (3.13). In particular, the coercivity of $a(\cdot,\cdot)$ is essential.

Theorem 3.3.1 (Lax–Milgram) *Let V be a real Hilbert space, $L(\cdot)$ a continuous linear form on V, $a(\cdot,\cdot)$ a continuous coercive bilinear form on V. Then the variational formulation (3.13) has a unique solution. Further, this solution depends continuously on the linear form L.*

Proof. For all $w \in V$, the mapping $v \to a(w,v)$ is a continuous linear form on V: consequently, the Riesz representation theorem 12.1.18 implies that there exists an element of V, denoted $A(w)$, such that

$$a(w,v) = \langle A(w), v\rangle \quad \text{for all } v \in V.$$

Moreover, the bilinearity of $a(w,v)$ obviously implies the linearity of the mapping $w \to A(w)$. Further, by taking $v = A(w)$, the continuity (3.14) of $a(w,v)$ shows that

$$\|A(w)\|^2 = a(w, A(w)) \leq M\|w\|\|A(w)\|,$$

that is, $\|A(w)\| \leq M\|w\|$ and therefore $w \to A(w)$ is continuous. Another application of the Riesz representation theorem 12.1.18 implies that there exists an element of V, denoted f, such that $\|f\|_V = \|L\|_{V'}$ and

$$L(v) = \langle f, v\rangle \quad \text{for all } v \in V.$$

Finally, the variational problem (3.13) is equivalent to:

$$\text{find } u \in V \text{ such that } A(u) = f. \tag{3.16}$$

To prove the theorem we must therefore show that the operator A is bijective from V to V (which implies the existence and the uniqueness of u) and that its inverse is continuous (which proves the continuous dependence of u with respect to L).

The coercivity (3.15) of $a(w,v)$ shows that

$$\nu\|w\|^2 \leq a(w,w) = \langle A(w), w\rangle \leq \|A(w)\|\|w\|,$$

which gives

$$\nu\|w\| \leq \|A(w)\| \quad \text{for all } w \in V, \tag{3.17}$$

that is, A is injective. To show that A is surjective, that is, $\mathrm{Im}(A) = V$ (which is not obvious if V is infinite dimensional), it is enough to show that $\mathrm{Im}(A)$ is closed in V and that $\mathrm{Im}(A)^\perp = \{0\}$. Indeed, in this case we see that $V = \{0\}^\perp = (\mathrm{Im}(A)^\perp)^\perp = \overline{\mathrm{Im}(A)} = \mathrm{Im}(A)$, which proves that A is surjective. Let $A(w_n)$ be a sequence in $\mathrm{Im}(A)$ which converges to b in V. By virtue of (3.17) we have

$$\nu\|w_n - w_p\| \leq \|A(w_n) - A(w_p)\|$$

which tends to zero as n and p tend to infinity. Therefore w_n is a Cauchy sequence in the Hilbert space V, that is, it converges to a limit $w \in V$. Then, by continuity of A we deduce that $A(w_n)$ converges to $A(w) = b$, that is, $b \in \mathrm{Im}(A)$ and $\mathrm{Im}(A)$ is therefore closed. On the other hand, let $v \in \mathrm{Im}(A)^\perp$; the coercivity (3.15) of $a(w,v)$ implies that

$$\nu\|v\|^2 \leq a(v,v) = \langle A(v), v\rangle = 0,$$

that is, $v = 0$ and $\mathrm{Im}(A)^\perp = \{0\}$, which proves that A is bijective. Let A^{-1} be its inverse: the inequality (3.17) with $w = A^{-1}(v)$ proves A^{-1} is continuous, therefore the solution u depends continuously on f. □

Remark 3.3.2 If the Hilbert space V is finite dimensional (which is however never the case for the applications we shall see), the proof of the Lax–Milgram theorem 3.3.1 simplifies considerably. Indeed, in finite dimensions all linear mappings are continuous and the injectivity (3.17) of A is equivalent to its invertibility. We see, in this case, (as in the general case) that the coercivity hypothesis on the bilinear form $a(w,v)$ is indispensable since it is this that gives the injectivity of A. Finally we remark that, if $V = \mathbb{R}^N$, a variational formulation is only the statement, $\langle Au, v\rangle = \langle f, v\rangle$ for all $v \in \mathbb{R}^N$, of a simple linear system $Au = f$. •

Remark 3.3.3 Another proof (a little less technical but which disguises some of the essential arguments) of the Lax–Milgram theorem 3.3.1 is the following. We begin as before until we reach the formulation (3.16) of the problem. To show the existence and uniqueness of the solution u of (3.16), we introduce an affine mapping T from V into V, defined by

$$T(w) = w - \mu\Big(A(w) - f\Big) \quad \text{with } \mu = \frac{\nu}{M^2},$$

which we shall show is a strict contraction, which proves the existence and the uniqueness of $u \in V$ such that $T(u) = u$ (from which we have the result). Indeed, we have

$$\begin{aligned}
\|T(v) - T(w)\|^2 \quad &= \|v - w - \mu A(v-w)\|^2 \\
&= \|v-w\|^2 - 2\mu\langle A(v-w), v-w\rangle + \mu^2\|A(v-w)\|^2 \\
&= \|v-w\|^2 - 2\mu a(v-w, v-w) + \mu^2\|A(v-w)\|^2 \\
&\leq (1 - 2\mu\nu + \mu^2 M^2)\|v-w\|^2 \\
&\leq (1 - \nu^2/M^2)\|v-w\|^2.
\end{aligned}$$

•

A variational formulation often has a physical interpretation, in particular if the bilinear form is symmetric. Indeed in this case, the solution of the variational formulation (3.13) attains the **minimum of an energy** (very natural in physics or mechanics).

Proposition 3.3.4 *We take the hypotheses of the Lax–Milgram theorem 3.3.1. We further assume that the bilinear form is symmetric $a(w,v) = a(v,w)$ for all $v,w \in V$. Let $J(v)$ be the energy defined for $v \in V$ be*

$$J(v) = \frac{1}{2}a(v,v) - L(v). \tag{3.18}$$

Let $u \in V$ be the unique solution of the variational formulation (3.13). Then u is also the unique point of the minimum of energy, that is,

$$J(u) = \min_{v \in V} J(v).$$

Conversely, if $u \in V$ is a point giving an energy minimum $J(v)$, then u is the unique solution of the variational formulation (3.13).

Proof. If u is the solution of the variational formulation (3.13), we can write (thanks to the symmetry of a)

$$J(u+v) = J(u) + \frac{1}{2}a(v,v) + a(u,v) - L(v) = J(u) + \frac{1}{2}a(v,v) \geq J(u).$$

As $u+v$ is arbitrary in V, u minimizes the energy J in V. Conversely, let $u \in V$ be such that

$$J(u) = \min_{v \in V} J(v).$$

For $v \in V$ we define a function $j(t) = J(u+tv)$ from $\mathbb{R}$ into $\mathbb{R}$ (it is just a polynomial of second degree in t). Since $t = 0$ is a minimum of j, we deduce that $j'(0) = 0$ which, by a simple calculation, is exactly the variational formulation (3.13). □

Remark 3.3.5 We see later in Chapter 9 that, when the bilinear form a is symmetric, there is an argument other than the Lax–Milgram 3.3.1 theorem to prove the existence and the uniqueness of a solution of (3.13). Indeed, we shall demonstrate directly the existence of a unique minimum of the energy $J(v)$. By virtue of proposition 3.3.4, this shows the existence and the uniqueness of the solution of the variational formulation. •

3.3.2 Application to the Laplacian

We now try to apply the Lax–Milgram theorem 3.3.1 to the variational formulation (3.8) of the Laplacian with Dirichlet boundary conditions. This is written in the form (3.13) with

$$a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx$$

and

$$L(v) = \int_\Omega f(x)v(x)\,dx,$$

where clearly $a(\cdot,\cdot)$ is a bilinear form, and $L(\cdot)$ a linear form. The space V (called before as X) is

$$V = \left\{ v \in C^1(\overline{\Omega}),\ v = 0 \text{ on } \partial\Omega \right\}. \tag{3.19}$$

As a scalar product on V we shall choose

$$\langle w, v\rangle = \int_\Omega \nabla w(x)\cdot\nabla v(x)\,dx, \tag{3.20}$$

which has for associated norm

$$\|v\| = \left(\int_\Omega |\nabla v(x)|^2 dx \right)^{1/2}.$$

We verify easily that (3.20) defines a scalar product on V: the only point which slows us is the property $\|v\| = 0 \Rightarrow v = 0$. Indeed, from the equality

$$\int_\Omega |\nabla v(x)|^2 dx = 0$$

we deduce that v is a constant on Ω, and as $v = 0$ on $\partial\Omega$ we have $v = 0$. The motivation of the choice of (3.20) as scalar product is above all the fact that the bilinear form $a(\cdot,\cdot)$ is **automatically coercive** for (3.20). In addition, we can easily check that a is continuous. To show that L is continuous, we must rely on the Poincaré inequality of lemma 3.3.6: we then have

$$\left| \int_\Omega f(x)v(x)\,dx \right| \le \left(\int_\Omega |f(x)|^2 dx \right)^{1/2} \left(\int_\Omega |v(x)|^2 dx \right)^{1/2} \le C\|v\|,$$

where C is a constant which depends on f but not on v. Therefore, L is continuous over V. All the of the hypotheses of the Lax–Milgram theorem 3.3.1 seem satisfied, however, we have missed one which prevents its application: the space V is not a Hilbert space since it is not complete for the norm induced by (3.20)! This obstruction does not come so much from the choice of the scalar product as the C^1 regularity requirement on functions of the space V. An immediate way, which can be clarified, to solve the difficulty is to replace V by $\overline{V}$, its closure for the scalar product (3.20). Obviously, we have only moved the problem: what is the space $\overline{V}$? The answer will be given in Chapter 4: $\overline{V}$ is the Sobolev space $H_0^1(\Omega)$ whose elements are no longer regular functions but only measurable. Another difficulty will be to see in what sense proposition 3.2.7 (which expresses the equivalence between the boundary value problem (3.1) and its variational formulation (3.8)) remains true when we replace the space V by $\overline{V}$.

We hope that we have therefore convinced the reader of the **natural and inescapable character of Sobolev spaces in the solution of variational formulations** of elliptic PDEs. We finish this chapter with a technical lemma, called the Poincaré inequality, which we used above.

Lemma 3.3.6 *Let Ω be an open set of $\mathbb{R}^N$ bounded in at least one space direction. There exists a constant $C > 0$ such that, for every function $v \in C^1(\overline{\Omega})$ which is zero on the boundary $\partial\Omega$,*

$$\int_\Omega |v(x)|^2 dx \leq C \int_\Omega |\nabla v(x)|^2 dx.$$

Proof. The hypothesis on the bounded character of Ω says (after a possible rotation) that for all $x \in \Omega$ the first component of x_1 is bounded, $-\infty < a \leq x_1 \leq b < +\infty$. Let v be a function of $C^1(\overline{\Omega})$ which is zero on $\partial\Omega$. We can extend it continuously by zero outside of Ω (v is then a continuous function which is piecewise of class C^1 in $\mathbb{R}^N$) and write, for $x \in \Omega$,

$$v(x) = \int_a^{x_1} \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_N)\, dt,$$

from which we deduce by the Cauchy–Schwarz inequality

$$|v(x)|^2 \leq (x_1 - a) \int_a^{x_1} \left| \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_N) \right|^2 dt \leq (b-a) \int_a^b \left| \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_N) \right|^2 dt.$$

Integrating over Ω we obtain

$$\int_\Omega |v(x)|^2 dx \leq (b-a) \int_\Omega \int_a^b \left| \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_N) \right|^2 dt\, dx,$$

and permuting the two integrations with respect to t and x, we conclude

$$\int_\Omega |v(x)|^2 dx \leq (b-a)^2 \int_\Omega \left| \frac{\partial v}{\partial x_1}(x) \right|^2 dx \leq (b-a)^2 \int_\Omega |\nabla v(x)|^2 dx.$$

□

Exercise 3.3.1 The aim of this exercise is to show that the space V, defined by (3.19) and equipped with the scalar product (3.20), is not complete. Let Ω be the open unit ball in $\mathbb{R}^N$. If $N = 1$, we define

$$u_n(x) = \begin{cases} -x - 1 & \text{if } -1 < x < -n^{-1}, \\ (n/2)x^2 - 1 + 1/(2n) & \text{if } -n^{-1} \leq x \leq n^{-1}, \\ x - 1 & \text{if } n^{-1} < x < 1. \end{cases}$$

If $N = 2$, for $0 < \alpha < 1/2$, we define

$$u_n(x) = |\log(|x|^2/2 + n^{-1})|^\alpha - |\log(1/2 + n^{-1})|^\alpha.$$

If $N \geq 3$, for $0 < \beta < (N-2)/2$, we define

$$u_n(x) = \frac{1}{(|x|^2 + n^{-1})^{\beta/2}} - \frac{1}{(1 + n^{-1})^{\beta/2}}.$$

Show that the sequence u_n is Cauchy in V but it does not converge in V as n tends to infinity.