Solution TD5

Probleme 1

1.) $$\varepsilon_j^n = \underbrace{\frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t}}_{(1)} + V \cdot \underbrace{\frac{u(x_{j+1},t_n) - u(x_{j-1},t_n)}{2\Delta x}}_{(2)} - \frac{V^2 \Delta t}{2} \underbrace{\frac{u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_{j-1},t_n)}{\Delta x^2}}_{=\ \frac{\partial^2 u}{\partial x^2} + O(\Delta x^2)\ (3)}$$

$$u(x_j,t_{n+1}) = u(x_j,t_n) + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + O(\Delta t^4)$$

$$(1) = \boxed{\frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + O(\Delta t^3)}$$

$$u(x_{j\pm 1},t_n) = u(x_j,t_n) \pm \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)$$

$$(2) = \boxed{\frac{\partial u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^3)}$$

$$= \underbrace{\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x}}_{0} - \frac{\Delta t}{2}\underbrace{\left(\frac{\partial^2 u}{\partial t^2} - V^2\frac{\partial^2 u}{\partial x^2}\right)}_{=0}$$

$$+ \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + V\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} +$$

$$+ O(\Delta t^3) + O(\Delta x^3) + O(\Delta x^2 \Delta t)$$

$$\frac{\partial u}{\partial t} = -V\frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial t^2} = V^2\frac{\partial^2 u}{\partial x^2},$$

$$\frac{\partial^3 u}{\partial t^3} = -V^3\frac{\partial^3 u}{\partial x^3}$$

$$\Rightarrow \varepsilon_j^n = \frac{V\Delta x^2}{6}\left(1 - V^2\frac{\Delta t^2}{\Delta x^2}\right)\frac{\partial^3 u}{\partial x^3}$$

$$+ O(\Delta t^3) + O(\Delta x^3) + O(\Delta x^2 \Delta t)$$

Eq equivalente

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} - \frac{V\Delta x^2}{6}\left(1 - V^2\frac{\Delta t^2}{\Delta x^3}\right)\frac{\partial^3 u}{\partial x^3}$$

$$= 0$$

Precision? si on discrétise

2) Crank-Nicolson

$$\varepsilon_j^n = \frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t} \quad (1)$$

— voir l'exercice précédent

$$+ V \frac{u(x_{j+1},t_n) - u(x_{j-1},t_n)}{4\Delta x} \quad (2)$$

$$+ V. \frac{u(x_{j+1},t_{n+1}) - u(x_{j-1},t_{n+1})}{4\Delta x} \quad (3)$$

$$(1) \quad \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + O(\Delta t^3)$$

$$(2) \quad \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^3)\right)$$

$$(3) \quad \frac{1}{2}\left(\underline{\frac{\partial u}{\partial x}(x_j,t_{n+1})} + \frac{\Delta x^2}{6}\underline{\frac{\partial^3 u}{\partial x^3}(x_j,t_{n+1})} + O(\Delta x^3)\right)$$

$$\frac{\partial u}{\partial x}(x_j,t_n) + \frac{\partial^2 u}{\partial x \partial t}\Delta t + O(\Delta t^2)$$

$$\frac{\partial^3 u}{\partial x^3}(x_j,t_n) + \frac{\partial^4 u}{\partial x^3 \partial t}\Delta t + O(\Delta t^2)$$

(1) + (2) + (3) $\longrightarrow$ les termes d'ordre 1 et 2 disparaissent

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = 0 \quad u \text{ sol de l'eq}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^3 u}{\partial t^3} = -V^3\frac{\partial^3 u}{\partial x^3}$$

$$\varepsilon_j^n = \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{V\Delta x^2}{12}\frac{\partial^3 u}{\partial x^3} + O(\Delta t^3) + O(\Delta x^3) + O(\Delta x^2 \Delta t)$$

$$\varepsilon_j^n = \left(\frac{V\Delta x^2}{12} - \frac{V^3\Delta t^2}{6}\right)\frac{\partial^3 u}{\partial x^3} + O(\Delta t^3) + O(\Delta x^3) + O(\Delta x^2 \Delta t)$$

L'eq équivalente

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} - \left(\frac{V^3\Delta t^2}{6} - \frac{V\Delta x^2}{12}\right)\frac{\partial^3 u}{\partial x^3} = 0$$

En discrétisant l'eq équivalente

on obtient un meilleur schéma seulement si Δt est du même ordre que Δx $\longrightarrow$ schéma d'ordre 3 en espace en temps

Stabilité Crank-Nicolson

↓ Méthode de Von Neumann

$u_j^n = A(k)^n e^{2\pi i j k \Delta x}$ on remplace dans le schéma et on simplifie le terme $A(k)^n e^{2\pi i j k \Delta x}$

$$\frac{A(k)-1}{\Delta t} + V \cdot A(k) \cdot \frac{\overbrace{e^{2i\pi k \Delta x} - e^{-2i\pi k \Delta x}}^{2i\sin(2\pi k \Delta x)}}{4\Delta x} + V \cdot \frac{e^{2i\pi k \Delta x} - e^{-2i\pi k \Delta x}}{4\Delta x} = 0$$

On multiplie par Δt et on sépare les termes avec $A(k)$ (à gauche) et sans $A(k)$ à droite.

$$A(k)\left(2 + i\frac{V\Delta t}{\Delta x}\sin(2\pi k \Delta x)\right) = 2 - i\frac{V\Delta t}{\Delta x}\sin(2\pi k \Delta x)$$

$|A(k)| = 1$ inconditionnellement stable

Ex 2

$$\frac{\partial u}{\partial t} + v\,\frac{\partial u}{\partial x} - \gamma\,\frac{\partial^2 u}{\partial x^2} = 0 \qquad (*)$$

advection-difusion (ou convection-diffusion)

$$\varepsilon_j^n = \underbrace{\frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t}}_{1} + v\,\underbrace{\frac{u(x_j,t_n) - u(x_{j-1},t_n)}{\Delta x}}_{2}$$

$$- \gamma\,\underbrace{\frac{u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_{j-1},t_n)}{\Delta x^2}}_{3}$$

$$= \text{①}\ \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^2)$$

$$\text{②}\ v\,\frac{\partial u}{\partial x} - v\,\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2)$$

$$\text{③}\ -\gamma\,\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2)$$

$$= \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - v\,\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + O(\Delta t^2) + O(\Delta x^2)$$

terme dominant

$$(*)\quad \frac{\partial^2 u}{\partial t^2} = v^2\,\frac{\partial^2 u}{\partial x^2} - 2\gamma v\,\frac{\partial^3 u}{\partial x^3} + \gamma^2\,\frac{\partial^4 u}{\partial x^4}$$

$$\varepsilon_j^n = \underbrace{\frac{v^2\Delta t - v\Delta x}{2}\frac{\partial^2 u}{\partial x^2}}_{\text{terme dom}} - \gamma v^2 \Delta t \frac{\partial^3 u}{\partial x^3} + \frac{\gamma \Delta t}{2}\frac{\partial^4 u}{\partial x^4} + O(\Delta t^2) + O(\Delta x^2)$$

Même en discrétisant l'eq equivalente on va pas améliorer la précision en temps, seulement la précision en espace.

(Problème 2)

$\checkmark$ $1-\cos 2x = 2\sin^2 x$
$\checkmark$ $\sin 2x = 2\sin x\cos x$

(schéma initial)

1) Condition CFL

$u_j^n = A(k)^n e^{2i\pi k j\Delta x}$ on remplace dans le schéma et on simplifie $A(k)^n e^{2i\pi k j\Delta x}$

$$\checkmark \quad \frac{A(k)-1}{\Delta t} + V\cdot\frac{1-e^{-2i\pi k\Delta x}}{\Delta x} - \gamma\cdot\frac{e^{2i\pi k\Delta x}-2+e^{-2i\pi k\Delta x}}{\Delta x^2} = 0$$

$\Rightarrow$ le facteur d'amplification est (on multiplie par Δt)

$$\checkmark \quad A(k) = 1 - \frac{V\Delta t}{\Delta x}\left(1-e^{-2i\pi k\Delta x}\right) + \frac{\gamma\Delta t}{\Delta x^2}\left(\underbrace{2\cos(2\pi k\Delta x)-2}_{-4\sin^2(k\pi\Delta x)}\right) \checkmark$$

$$(*) = 1-\cos 2\pi k\Delta x + i\sin(2\pi k\Delta x)$$
$$= 2i\sin(k\pi\Delta x)\left(\cos(k\pi\Delta x) - i\sin(k\pi\Delta x)\right)$$

Notation pour simplifier

$S_k = \sin(k\pi\Delta x)$ $\quad C_k = \cos(k\pi\Delta x)$

$$\boxed{A(k) = 1 - \frac{V\Delta t}{\Delta x}2iS_k(C_k - iS_k) - 4\frac{\gamma\Delta t}{\Delta x^2}S_k^2}$$

$$|A(k)|^2 = \left(1 - 2\frac{V\Delta t}{\Delta x}S_k^2 - 4\frac{\gamma\Delta t}{\Delta x^2}S_k^2\right)^2 + \left(\frac{V\Delta t}{\Delta x}\right)^2 4S_k^2C_k^2 \checkmark$$

$$= 1 + 4\left(\frac{v\Delta t}{\Delta x}\right)^2 s_k^2 + 16\left(\frac{\gamma\Delta t}{\Delta x^2}\right)^2 s_k^4 \Big] \rightarrow \text{carré}$$

$$- 4\,\frac{v\Delta t}{\Delta x}\, s_k^2 \quad - 8\,\frac{\gamma\Delta t}{\Delta x^2}\, s_k^2$$

$$+ 16\,\frac{v\Delta t}{\Delta x}\cdot\frac{\gamma\Delta t}{\Delta x^2}\, s_k^4 \overset{?}{\leq} 1$$

si $v \to 0$ il faut $\dfrac{\gamma\Delta t}{\Delta x^2} \leq \dfrac{1}{2}$ CFL chaleur

$\gamma \to 0$ —— $\dfrac{v\Delta t}{\Delta x} \leq 1$ CFL advection

$\Leftrightarrow$ $\Big[$ $\alpha_1 = \dfrac{v\Delta t}{\Delta x}$ $\quad \alpha_2 = \dfrac{\gamma\Delta t}{\Delta x^2}$ $\rightsquigarrow$ CFL advection et chaleur

simplifier par $4 s_k^2$ $\rfloor$

$\forall s_k$

$$\boxed{\alpha_1^2 + 4\alpha_2^2 s_k^2 - \alpha_1 - 2\alpha_2 + 4\alpha_1\alpha_2 s_k^2 \leq 0}$$

Il suffit d'avoir (car $s_k \leq 1$) $\Uparrow$

cond suff. $\boxed{\alpha_1^2 + 4\alpha_2^2 - \alpha_1 - 2\alpha_2 + 4\alpha_1\alpha_2 \leq 0}$

$$(\alpha_1 + 2\alpha_2)^2 \leq \alpha_1 + 2\alpha_2$$

$\alpha_1 + 2\alpha_2 \leq 1$ (CFL ou cond suffisante)

$$\Leftrightarrow \quad \boxed{\frac{v\Delta t}{\Delta x} + 2\,\frac{\gamma\Delta t}{\Delta x^2} \leq 1}$$

Schéma centré + Crank-Nicolson (ordre 2 espace et temps)

2) On refait les mêmes calculs.

Après la première simplification on a :

$$A(k) - 1 + \frac{V\Delta t}{2\Delta x}\overbrace{\left(e^{2i\pi k\Delta x} - e^{-2i\pi k\Delta x}\right)}^{2i\sin(k\pi\Delta x)} + 2\frac{\gamma\Delta t}{\Delta x^2}\cdot\sin^2(k\pi\Delta x)$$

conv diff schéma centré

Crank-Nicolson diffusif

$$+ 2\frac{\gamma\Delta t}{\Delta x^2}\sin^2(k\pi\Delta x)\cdot A(k) = 0$$

$$\Rightarrow \boxed{A(k)\left(1 + 2\frac{\gamma\Delta t}{\Delta x^2}\sin^2(k\pi\Delta x)\right) = 1 - i\underbrace{\frac{V\Delta t}{\Delta x}}_{\alpha_1}\sin(k\pi\Delta x) - 2\underbrace{\frac{\gamma\Delta t}{\Delta x^2}}_{\alpha_2}\sin^2(k\pi\Delta x)}$$

$S_k = \sin(k\pi\Delta x)$

$$\Leftrightarrow |A(k)|^2 = \frac{(1 - 2\alpha_2 S_k^2)^2 + \alpha_1^2 S_k^2}{(1 + 2\alpha_2 S_k^2)^2} \overset{?}{\leq} 1.$$

(mêmes notations qu'au point 1)

$$\Leftrightarrow \quad \alpha_1^2 S_k^2 - 4\alpha_2 S_k^2 \leq 4\alpha_2 S_k^2$$

$$\Leftrightarrow \boxed{\alpha_1^2 \leq 8\alpha_2} \quad \text{cond CFL}$$

$$\left(\frac{V\Delta t}{\Delta x}\right)^2 \leq 8\gamma\frac{\Delta t}{\Delta x^2}$$

Si $\gamma \to 0$ pas possible donc le schéma sera inconditionnellement instable