TD 6 ∇u, Δu, $\nabla \cdot \underline{w}$, $\nabla \times \underline{w}$

champs scalaires champs de vecteurs

$$\int_\Omega \frac{\partial u}{\partial x_i}\, dx = \int_{\partial\Omega} u\, n_i\, ds$$

1) $\Delta u = \nabla \cdot (\nabla u)$ ✓

$\to \nabla \cdot (u \underline{v}) = u(\nabla \cdot v) + \underline{v} \cdot \nabla u$ ✓

$\to \nabla \times (u \underline{v}) = u(\nabla \times \underline{v}) + \nabla u \times \underline{v}$ ✓

a) $$\nabla \cdot (\nabla u) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)$$
$$= \Delta u$$

b) $$\nabla \cdot (u \underline{v}) = \frac{\partial}{\partial x}(u \cdot v_x) + \frac{\partial}{\partial y}(u \cdot v_y) + \frac{\partial}{\partial z}(u v_z)$$

où $\underline{v} = (v_x, v_y, v_z)$

formule dérivation du produit

$$= \frac{\partial u}{\partial x} v_x + u \frac{\partial v_x}{\partial x} + \frac{\partial u}{\partial y} v_y + u \cdot \frac{\partial v_y}{\partial y}$$
$$+ \frac{\partial u}{\partial z} \cdot v_z + u \frac{\partial v_z}{\partial z} = \nabla u \cdot \underline{v} + u \cdot \nabla \cdot \underline{v}$$

c) $$\nabla \times (u \underline{v}) = \begin{pmatrix} \frac{\partial}{\partial y}(u v_z) - \frac{\partial}{\partial z}(u v_y) \\ \frac{\partial}{\partial z}(u v_x) - \frac{\partial}{\partial x}(u v_z) \\ \frac{\partial}{\partial x}(u v_y) - \frac{\partial}{\partial y}(u v_x) \end{pmatrix}$$

$w = u \underline{v}$

$= (u v_x, u v_y, u v_z)$

$w_x = u v_x$, $w_y = u v_y$

$$= \left[\begin{array}{l} \frac{\partial u}{\partial y} v_z - \frac{\partial u}{\partial z} \cdot v_y + u\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) \\ \frac{\partial u}{\partial z} v_x - \frac{\partial u}{\partial x} v_z + u\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) \\ \frac{\partial u}{\partial x} v_y - \frac{\partial u}{\partial y} v_x + u\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{\nabla u \times \underline{v}} \qquad \underbrace{\qquad\qquad}_{u \cdot \nabla \times w}$$

- $\nabla \times (\nabla u) = \nabla \times \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$

$$= \left(\cancel{\frac{\partial^2 u}{\partial z \partial y}} - \frac{\partial^2 u}{\partial y \partial z}, \text{-----} \quad \right) = \underline{0}$$

- $\nabla \cdot (\nabla \times \underline{v}) = \nabla \cdot \left(\overbrace{\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}}^{(1)}, \overbrace{\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}}^{(2)}, \overbrace{\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}}^{(3)}\right)$

$$= \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(3) = 0$$

Important: $\forall u, \underline{v}$ $\quad \nabla \times (\nabla u) = 0$

$\nabla \cdot (\nabla \times \underline{v}) = 0$

2) $$\int_\Omega \frac{\partial w}{\partial x_i}\,dx = \int_{\partial\Omega} w \cdot n_i(x)\,ds \quad \text{(Green)}$$

Integration par parties, $w = u \cdot v$ et on remplace dans la formule de Green

$$\int_\Omega \frac{\partial}{\partial x_i}(u v)\,dx = \int_{\partial\Omega} u v\, n_i(x)\,ds$$

$$\int_\Omega \left(\frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \right) dx = \int_{\partial\Omega} u v\, n_i(x)\,dx$$

$$\boxed{\int_\Omega \frac{\partial u}{\partial x_i} v\,dx = -\int_\Omega u \frac{\partial v}{\partial x_i}\,dx + \int_{\partial\Omega} u v\, n_i\,ds}$$

↙ première formule d'intégration par parties

. On utilise cette formule : on remplace u par $\frac{\partial u}{\partial x_i}$ $\Rightarrow$

$$\int_\Omega \frac{\partial^2 u}{\partial x_i^2} v\,dx = -\int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial x_i} \cdot v \cdot n_i\,ds$$

$i = 1, \dots, N$

On va sommer toutes les relations

$$\int_\Omega \Delta u \cdot v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx$$

$$+ \int_{\partial\Omega} (\nabla u \cdot \underline{n}) \, v \, ds$$

$\frac{\partial u}{\partial n}$

$$\Rightarrow \boxed{\int_\Omega \Delta u \cdot v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds}$$

3) Formule de Stokes

$$\nabla \cdot (\varphi \, \underline{w}) = \varphi \cdot \nabla \cdot \underline{w} + \underline{w} \cdot \nabla \varphi$$

(voir l'exercice 1))

On intègre sur le domaine Ω

$$\boxed{\int_\Omega (\nabla \cdot \underline{w}) \varphi \, dx} = \int_\Omega \nabla \cdot (\varphi \underline{w}) \, dx - \int_\Omega \underline{w} \cdot \nabla \varphi \, dx$$

$$= -\int_\Omega \underline{w} \cdot \nabla \varphi \, dx + \int_\Omega \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\varphi w_i) \, dx$$

$$= -\int_\Omega \underline{w} \cdot \nabla \varphi \, dx + \sum_{i=1}^{3} \int_{\partial\Omega} \varphi (w_i \cdot n_i(x)) \, dx$$

$$\boxed{= -\int_\Omega \underline{w} \cdot \nabla \varphi \, dx + \int_{\partial\Omega} (\underline{w} \cdot \underline{n}) \varphi \, ds}$$

si jamais $\varphi = 1 \Rightarrow \nabla \varphi = 0$

$$\boxed{\int_\Omega \nabla \cdot \underline{w} \, dx = \int_{\partial\Omega} \underline{w} \cdot \underline{n} \, ds}$$

formule de la divergence (Gauss)

4)

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{vmatrix} =$$

$$= \underline{i} (\partial_y u_z - \partial_z u_y) - \underline{j} (\partial_x u_z - \partial_z u_x)$$

$$+ \underline{k} (\partial_x u_y - \partial_y u_x)$$

$$\nabla \times (\varphi \underline{w}) = \varphi \nabla \times \underline{w} + \nabla \varphi \times \underline{w}$$

(exercice 1) $\rightsquigarrow$ on intègre et on applique la formule de Green

$$\boxed{\int_\Omega (\nabla \times \underline{w})\varphi \, dx} = \int_\Omega \nabla \times (\varphi \underline{w})\, dx - \int_\Omega \nabla\varphi \times \underline{w}$$

$$= \int_\Omega \underline{w} \times \nabla\varphi \, dx$$

$$+ \int_\Omega \Big[\frac{\partial}{\partial y}(\varphi w_z) - \frac{\partial}{\partial z}(\varphi w_y),$$

$$\frac{\partial}{\partial z}(\varphi w_x) - \frac{\partial}{\partial x}(\varphi w_z),$$

$$\frac{\partial}{\partial x}(\varphi w_y) - \frac{\partial}{\partial y}(\varphi w_x) \Big]^T dx$$

$$\overset{\uparrow}{\underset{\text{Green}}{=}} \int_\Omega \underline{w} \times \nabla\varphi \, dx + \int_{\partial\Omega} \Big[\varphi . (w_z \cdot n_y - w_y n_z),$$

$$\varphi (w_x n_z - w_z n_x),$$

$$\varphi (w_y n_x - w_x n_y) \Big]^T ds$$

$$= \int_\Omega \underline{w} \times \nabla\varphi \, dx + \int_{\partial\Omega} (\underline{n} \times \underline{w})\varphi \, ds$$

$$= \boxed{\int_\Omega \underline{w} \times \nabla\varphi \, dx - \int_{\partial\Omega} (\underline{w} \times \underline{n})\varphi \, ds}$$

Si $\varphi = 1$ $\quad \displaystyle\int_\Omega (\nabla \times \underline{w})\, dx = - \int_{\partial\Omega} (\underline{w} \times \underline{n})\, ds$

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$$\begin{cases} -u''(x) = f(x), & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$$u(x) = x\int_0^1 f(s)(1-s)\,ds - \int_0^x f(s)(x-s)\,ds$$

On vérifie si u est solution du problème aux limites.

Conditions aux limites :

✓ $u(0) = 0 \cdot \int \quad - \int_0^0 \text{---} \quad = 0$

✓ $u(1) = \int_0^1 f(s)(1-s)\,dx - \int_0 f(s)(1-s)\,ds = 0$

$$u'(x) = \int_0^1 f(s)(1-s)\,ds -$$

$$-\left(x\int_0^x f(s)\,ds - \int_0^x f(s)\cdot s\,ds\right)'$$

$$= \int_0^1 f(s)(1-s)\,ds - \int_0^x f(s)\,ds - x\cancel{f(x)} + \cancel{x}f(x)$$

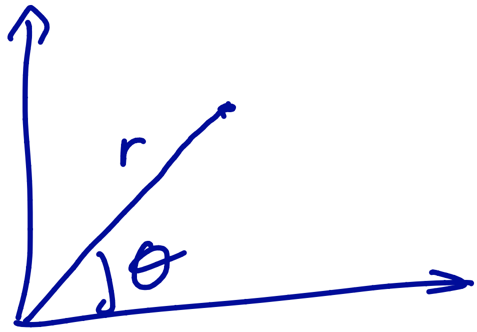
$$u'(x) = \int_0^1 f(s)(1-s)\,ds - \int_0^x f(s)\,ds$$

$u''(x) = -f(x) \rightsquigarrow u$ est solution du problème

6) $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

Ecrire en coordonnées polaires

$(x, y) \longrightarrow (r, \theta)$



$$\begin{cases} x = r\cos\theta, & x = x(r,\theta) \\ y = r\sin\theta, & y = y(r,\theta) \end{cases}$$

$\rightsquigarrow \boxed{r = \sqrt{x^2+y^2}}$ $\qquad \theta = \arctan \dfrac{y}{x}$

$$\begin{cases} \rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial x} \\ \rightarrow \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial y} \end{cases}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial u}{\partial x} = \cos\theta \cdot \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos\theta \cdot \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial r}\left(\cos\theta \cdot \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) \cdot \underbrace{\frac{\partial r}{\partial x}}_{=\cos\theta}$$

$$+ \frac{\partial}{\partial \theta}\left(\cos\theta \cdot \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) \cdot \underbrace{\frac{\partial \theta}{\partial x}}_{-\frac{\sin\theta}{r}}$$

$$= \left(\cos\theta \cdot \frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2} \cdot \frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta}\right)\cos\theta$$

$$+ \left(-\sin\theta \frac{\partial u}{\partial r} + \cos\theta \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial \theta^2}\right) \cdot \left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) \cdot \underbrace{\frac{\partial r}{\partial y}}_{=\sin\theta}$$

$$+ \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial u}{\partial \theta}\right) \cdot \underbrace{\frac{\partial \theta}{\partial y}}_{\frac{\cos\theta}{r}}$$

$$= \left(\sin\theta \frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta}{r^2} \cdot \frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} \right) \sin\theta$$

$$+ \left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r} \cdot \frac{\partial^2 u}{\partial \theta^2} \right) \frac{\cos\theta}{r}.$$

On somme les 2 :

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r}}$$

très utile quand la géométrie du domaine s'y prête !

<u>Remarque</u> : Il $\exists$ aussi des coordonnées sphériques ou cylindriques, etc...