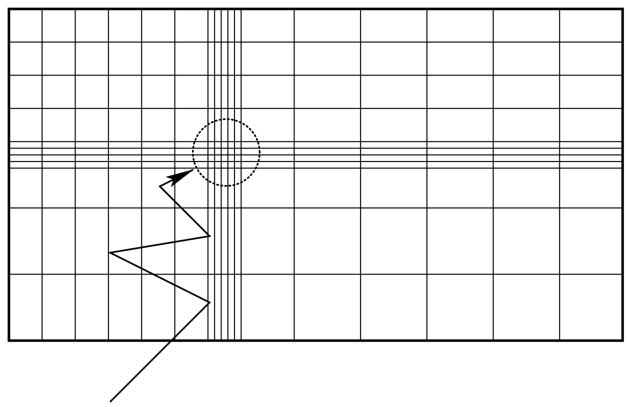

Figure 2.3. Refinement of a finite difference mesh: the encircled zone is that where we want more accuracy.

for regular, or **rectangular**, meshes. It is not always easy to discretize an arbitrary space domain by rectangular meshes! Additionally, it is not possible to locally refine the mesh to have better accuracy at a particular point of the domain. It is possible to vary the space step in each direction but this variation is uniform in perpendicular directions (Δx and Δy can change along the x and y axes, respectively, but this variation is uniform in orthogonal directions; see Figure 2.3). Such a refinement of a finite difference mesh therefore has effects far outside the zone of interest. Moreover, the theory and practice of the finite differences become much more complicated when the coefficients in the partial differential equations are variables and when the problems are nonlinear.

2.3 Other models

2.3.1 Advection equation

We consider the advection equation in one space dimension in the bounded domain $(0,1)$ with a constant velocity $V > 0$ and with the periodic boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} + V\dfrac{\partial u}{\partial x} = 0 & \text{for } (x,t) \in (0,1) \times \mathbb{R}_*^+ \\ u(t, x+1) = u(t,x) & \text{for } (x,t) \in (0,1) \times \mathbb{R}_*^+ \\ u(0,x) = u_0(x) & \text{for } x \in (0,1). \end{cases} \tag{2.32}$$

We always discretize space with a step $\Delta x = 1/(N+1) > 0$ (N a positive integer) and the time with $\Delta t > 0$, and we denote by $(t_n, x_j) = (n\Delta t, j\Delta x)$ for $n \geq 0, j \in \{0, 1, \ldots, N+1\}$, u_j^n the value of an approximate discrete solution at the point (t_n, x_j), and $u(t,x)$ the exact solution of (2.32). The periodic boundary conditions lead to equations $u_0^n = u_{N+1}^n$ for all $n \geq 0$, and more generally $u_j^n = u_{N+1+j}^n$. Consequently, the discrete unknown at each time step is a vector $u^n = (u_j^n)_{0\leq j\leq N} \in \mathbb{R}^{N+1}$. We give some possible schemes for the advection equation (2.32). In Chapter 1 we have already noted the bad numerical behaviour of the **explicit centred scheme**

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + V\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \tag{2.33}$$

for $n \geq 0$ and $j \in \{0, \dots, N\}$. The unstable character of this scheme is confirmed by the following lemma.

Lemma 2.3.1 *The explicit centred scheme (2.33) is consistent with the advection equation (2.32), accurate with order 1 in time and 2 in space, but unconditionally unstable in the L^2 norm.*

Proof. With the help of a Taylor expansion around the point (t_n, x_j), we easily see that the scheme is consistent, accurate with order 1 in time and 2 in space. By Fourier analysis, we study the L^2 stability. With the notation of Section 2.2.3, the Fourier components $\hat{u}^n(k)$ of u^n satisfy

$$\hat{u}^{n+1}(k) = \left(1 - i\frac{V\Delta t}{\Delta x}\sin(2\pi k\Delta x)\right)\hat{u}^n(k) = A(k)\hat{u}^n(k).$$

We see that the amplification factor is always greater than 1,

$$|A(k)|^2 = 1 + \left(\frac{V\Delta t}{\Delta x}\sin(2\pi k\Delta x)\right)^2 \geq 1,$$

with strict inequality when $2k\Delta x$ is not an integer. Therefore the scheme is unstable. $\square$

We can write an implicit version of the preceding scheme which is stable: it is the **implicit centred scheme**

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + V\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = 0. \tag{2.34}$$

Exercise 2.3.1 Show that the implicit centred scheme (2.34) is consistent with the advection equation (2.32), accurate with order 1 in time and 2 in space, unconditionally stable in the L^2 norm, and therefore convergent.

If we absolutely must stay centred and explicit, the **Lax–Friedrichs scheme**

$$\frac{2u_j^{n+1} - u_{j+1}^n - u_{j-1}^n}{2\Delta t} + V\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \tag{2.35}$$

is a scheme which is simple, robust but not very accurate.

Lemma 2.3.2 *The Lax–Friedrichs scheme (2.35) is stable in the L^2 norm under the CFL condition*

$$|V|\Delta t \leq \Delta x.$$

If the ratio $\Delta t/\Delta x$ is held constant as Δt and Δx tend to zero, it is consistent with the advection equation (2.32) and accurate with order 1 in space and time. Consequently, it is conditionally convergent.

Proof. By Fourier analysis we have

$$\hat{u}^{n+1}(k) = \left(\cos(2\pi k \Delta x) - i \frac{V\Delta t}{\Delta x} \sin(2\pi k \Delta x) \right) \hat{u}^n(k) = A(k)\hat{u}^n(k).$$

The modulus of the amplification factor is given by

$$|A(k)|^2 = \cos^2(2\pi k \Delta x) + \left(\frac{V\Delta t}{\Delta x} \right)^2 \sin^2(2\pi k \Delta x).$$

We see, therefore, that $|A(k)| \leq 1$ for all k if the condition $|V|\Delta t \leq \Delta x$ is satisfied, while if not there exist unstable modes k such that $|A(k)| > 1$. The scheme is therefore conditionally stable. To study the consistency, we make a Taylor expansion around (t_n, x_j) for the solution u:

$$\frac{2u(t_{n+1}, x_j) - u(t_n, x_{j+1}) - u(t_n, x_{j-1})}{2\Delta t} + V \frac{u(t_n, x_{j+1}) - u(t_n, x_{j-1})}{2\Delta x} =$$

$$(u_t + V u_x)(t_n, x_j) - \frac{(\Delta x)^2}{2\Delta t} \left(1 - \frac{(V\Delta t)^2}{(\Delta x)^2} \right) u_{xx}(t_n, x_j) + \mathcal{O}\Big((\Delta x)^2 + \frac{(\Delta x)^4}{\Delta t} \Big). \quad (2.36)$$

Since the truncation error contains a term in $\mathcal{O}\Big((\Delta x)^2/\Delta t\Big)$, the scheme is not consistent if Δt tends to zero more quickly than $(\Delta x)^2$. Conversely, it is consistent and is accurate with order 1 if the ratio $\Delta t/\Delta x$ is constant. To obtain the convergence we recall the proof of the Lax Theorem 2.2.20. The error e^n is always bounded by the truncation error, and therefore

$$\|e^n\| \leq \Delta t n K C \left(\frac{(\Delta x)^2}{\Delta t} + \Delta t \right).$$

If we keep the ratio $\Delta x/\Delta t$ fixed, the error is therefore bounded by a constant times Δt which tends to zero, from which we have the convergence. □

Remark 2.3.3 The Lax–Friedrichs scheme is not (in the strict sense of the definition 2.2.4) consistent. Nevertheless, it is conditionally consistent and convergent. We must, however, pay attention to the fact that if we take a much smaller time step Δt than is permitted by the CFL stability condition, the convergence will be very slow. In practice, the Lax–Friedrichs scheme is not recommended. •

An explicit centred scheme which is more accurate is **Lax–Wendroff scheme**

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+V\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}-\left(\frac{V^2\Delta t}{2}\right)\frac{u_{j-1}^n-2u_j^n+u_{j+1}^n}{(\Delta x)^2}=0. \tag{2.37}$$

The derivation is not immediate, we present it in detail. We start by writing an expansion of order 2 in time of the exact solution

$$u(t_{n+1},x_j)=u(t_n,x_j)+(\Delta t)u_t(t_n,x_j)+\frac{(\Delta t)^2}{2}u_{tt}(t_n,x_j)+\mathcal{O}\Big((\Delta t)^3\Big).$$

By using the advection equation we replace the time derivatives by space derivatives

$$u(t_{n+1},x_j)=u(t_n,x_j)-(V\Delta t)u_x(t_n,x_j)+\frac{(V\Delta t)^2}{2}u_{xx}(t_n,x_j)+\mathcal{O}\Big((\Delta t)^3\Big).$$

Finally, we replace the space derivatives by a centred formula of order 2

$$\begin{aligned}u(t_{n+1},x_j)=&\;u(t_n,x_j)-V\Delta t\frac{u(t_n,x_{j+1})-u(t_n,x_{j-1})}{2\Delta x}\\&+\frac{(V\Delta t)^2}{2}\frac{u(t_n,x_{j+1})-2u(t_n,x_j)+u(t_n,x_{j-1})}{(\Delta x)^2}+\mathcal{O}\Big((\Delta t)^3+\Delta t(\Delta x)^2\Big).\end{aligned}$$

We recover the Lax–Wendroff scheme by neglecting the third order terms and replacing $u(t_n,x_j)$ by u_j^n. We remark that, compared to the previous schemes, we have 'simultaneously' discretized the space and time derivatives of the advection equation. By design, the Lax–Wendroff scheme is accurate with order 2 in time and in space. We can show that it does not satisfy the discrete maximum principle (see exercise 2.3.3). Conversely, it is stable in the L^2 norm and therefore convergent under the CFL condition $|V|\Delta t\le\Delta x$.

Exercise 2.3.2 Show that the Lax–Wendroff scheme is stable and convergent in the L^2 norm if $|V|\Delta t\le\Delta x$.

Exercise 2.3.3 Show that the Lax–Friedrichs scheme satisfies the discrete maximum principle if the CFL condition $|V|\Delta t\le\Delta x$ is satisfied, while the Lax–Wendroff scheme does not satisfy it except if $V\Delta t/\Delta x$ is $-1, 0,$ or 1.

Exercise 2.3.4 Show that the Lax–Wendroff scheme (2.37) is the only scheme which is accurate with order 2 in space and time of the type

$$u_j^{n+1}=\alpha u_{j-1}^n+\beta u_j^n+\gamma u_{j+1}^n,$$

where α,β,γ depend only on $V\Delta t/\Delta x$.

As we have already seen in Chapter 1, a fundamental idea to obtain 'good' schemes for the advection equation (2.32) is **upwinding**. We give the general form of the **upwinded scheme**

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} + V \frac{u_j^n - u_{j-1}^n}{\Delta x} &= 0 \qquad \text{if} \qquad V > 0 \\ \frac{u_j^{n+1} - u_j^n}{\Delta t} + V \frac{u_{j+1}^n - u_j^n}{\Delta x} &= 0 \qquad \text{if} \qquad V < 0. \end{aligned} \tag{2.38}$$

We have already seen in Chapter 1 that the upwinded scheme is stable in the L^∞ norm if the CFL condition, $|V|\Delta t \le \Delta x$, is satisfied. As it is consistent and accurate with order 1 in space and time, it converges in the L^∞ norm from the Lax theorem. The same result is true in the L^2 norm with the same CFL condition.

Exercise 2.3.5 Show that the explicit upwinded scheme (2.38) is consistent with the advection equation (2.32), accurate with order 1 in space and time, stable and convergent in the L^2 norm if the CFL condition $|V|\Delta t \le \Delta x$ is satisfied.

Remark 2.3.4 For nonlinear problems (where the velocity V itself depends on the unknown u), and particularly for fluid flow models, the upwinded scheme is clearly superior to the others. It is the source of many generalizations, much more complex than the original (see [23]). In particular, even though the original scheme is only of order 1, it has variants of order 2. •

Scheme	Stability	Truncation error
Explicit centred (2.33)	Unstable	$\mathcal{O}\Big(\Delta t + (\Delta x)^2\Big)$
Implicit centred (2.34)	L^2 stable	$\mathcal{O}\Big(\Delta t + (\Delta x)^2\Big)$
Lax–Friedrichs (2.35)	L^2 and L^∞ stable for the CFL condition $\lvert V\rvert\Delta t \le \Delta x$	$\mathcal{O}\Big(\Delta t + (\Delta x)^2/\Delta t\Big)$
Lax–Wendroff (2.37)	L^2 stable for the CFL condition $\lvert V\rvert\Delta t \le \Delta x$	$\mathcal{O}\Big((\Delta t)^2 + (\Delta x)^2\Big)$
Upwinded (2.38)	L^2 and L^∞ stable for the CFL condition $\lvert V\rvert\Delta t \le \Delta x$	$\mathcal{O}\Big(\Delta t + \Delta x\Big)$

Table 2.2. Summary of properties of various schemes for the advection equation

To compare these various schemes (see Table 2.2) from a practical viewpoint, a pertinent (though formal) concept is that of the equivalent equation.

Definition 2.3.5 *We call the* **equivalent equation** *of a scheme the equation obtained by adding the principal part (that is, the term with dominant order) of the truncation error to the model studied.*

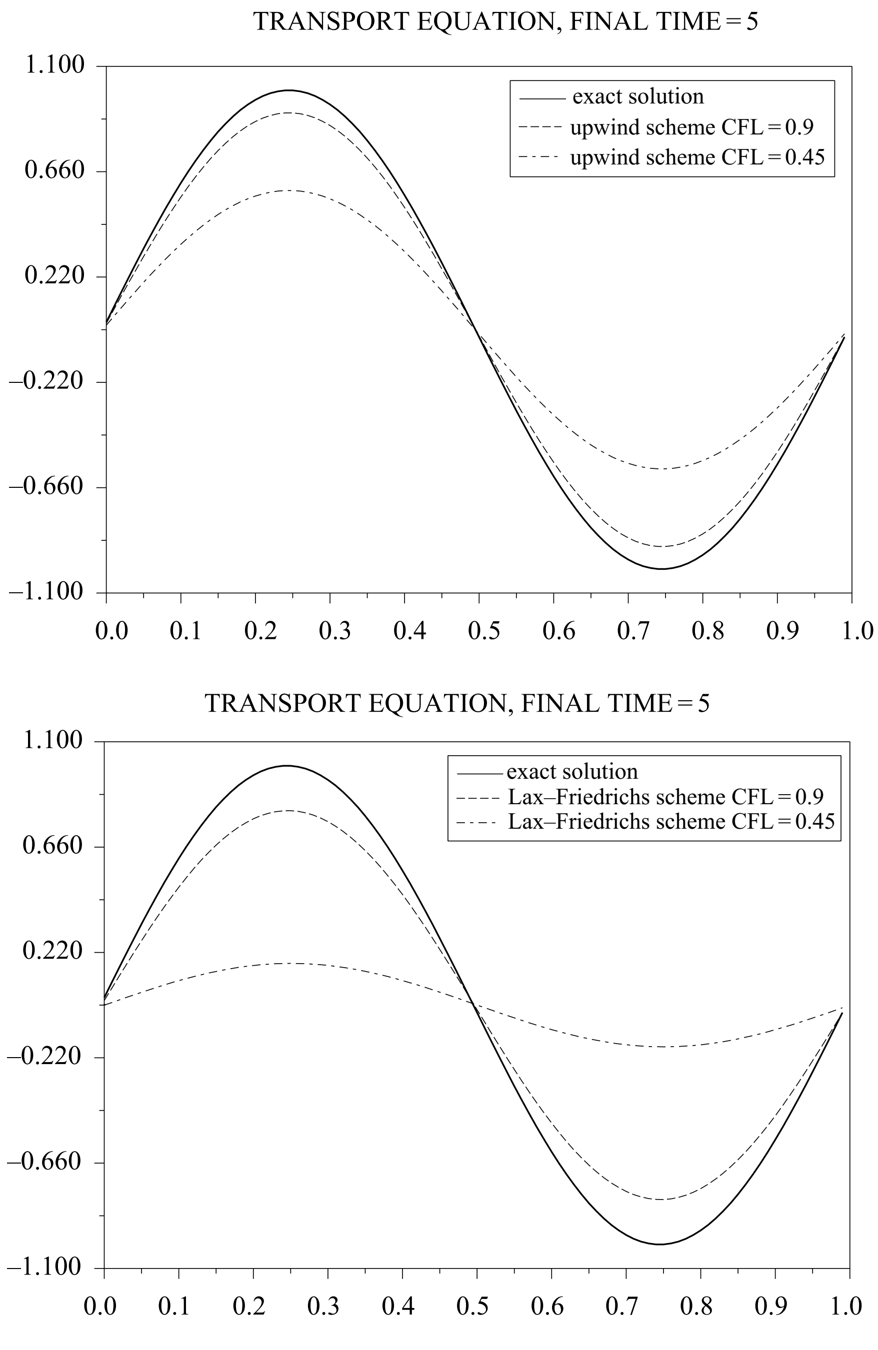

Figure 2.4. Influence of the CFL condition on the numerical diffusion of the Lax–Friedrichs scheme (top) and of the upwinded scheme (bottom).

All of the schemes which we have seen are consistent. However, if we add the principal part of the truncation error of a scheme to the equation, then this scheme is not only consistent with this new 'equivalent' equation, but is also strictly more accurate for this equivalent equation. In other words, the scheme is 'more consistent' with the equivalent equation than with the original equation. Let us take the example of the Lax–Friedrichs scheme (2.35) for the advection equation: from (2.36), the principal part of its truncation error is $-\frac{(\Delta x)^2}{2\Delta t}\left(1-\frac{(V\Delta t)^2}{(\Delta x)^2}\right)u_{xx}$. Consequently, the equivalent equation of the Lax–Friedrichs scheme is

$$\frac{\partial u}{\partial t}+V\frac{\partial u}{\partial x}-\nu\frac{\partial^2 u}{\partial x^2}=0 \quad \text{with } \nu=\frac{(\Delta x)^2}{2\Delta t}\left(1-\frac{(V\Delta t)^2}{(\Delta x)^2}\right). \tag{2.39}$$

This equivalent equation will give us invaluable information on numerical behaviour of the scheme. Indeed, the Lax–Friedrichs scheme is a good approximation (of order 2) of the convection–diffusion equation (2.39) where the coefficient of diffusion ν is small (even zero if the CFL condition is exactly satisfied, that is, $\Delta x=|V|\Delta t$). We remark that if the time step is taken to be very small, the coefficient of diffusion ν may be very large and the scheme is bad as it is too weighted to the diffusion (see Figure 2.4). The coefficient of diffusion ν of the equivalent equation is called **numerical diffusion**. If it is large, we say that the scheme is **diffusive** (or dissipative). The typical behaviour of a diffusive scheme is its tendency to artificially spread out the initial data in the course of time. The schemes which are too diffusive are therefore 'bad' schemes.

Exercise 2.3.6 Show that the equivalent equation of the upwinded scheme (2.38) is

$$\frac{\partial u}{\partial t}+V\frac{\partial u}{\partial x}-\frac{|V|}{2}\left(\Delta x-|V|\Delta t\right)\frac{\partial^2 u}{\partial x^2}=0.$$

The upwinded scheme is also diffusive (except if the CFL condition is exactly satisfied, that is, $\Delta x=|V|\Delta t$). In any case, the diffusion coefficient of the equivalent equation does not tend to infinity as the time step tends to zero (for Δx fixed), which is a clear improvement with respect to the Lax–Friedrichs scheme (see Figure 2.4). This numerical diffusion effect is illustrated by the Figure 2.4 where we solve the advection equation on an interval of length 1 with periodic boundary conditions, sinusoidal initial data, space step $\Delta x=0.01$, velocity $V=1$ and final time $T=5$. We compare two values of the time step $\Delta t=0.9\Delta x$ and $\Delta t=0.45\Delta x$.

Exercise 2.3.7 Show that the equivalent equation of the Lax–Wendroff scheme (2.37) is

$$\frac{\partial u}{\partial t}+V\frac{\partial u}{\partial x}+\frac{V(\Delta x)^2}{6}\left(1-\frac{(V\Delta t)^2}{(\Delta x)^2}\right)\frac{\partial^3 u}{\partial x^3}=0.$$

As the Lax–Wendroff scheme is accurate with order 2, the equivalent equation does not contain a diffusion term but a third order term, called **dispersive**. Let us remark that the coefficient of this dispersive term is of much smaller order than the coefficient of diffusion of the equivalent equations for the diffusive schemes. This is why this dispersive effect can only, in general, be seen on a nondiffusive scheme. The typical behaviour of a dispersive scheme is that it produces oscillations when the solution is discontinuous (see Figure 2.5). In effect, the dispersive term modifies the velocity of propagation of the plane waves or Fourier modes of the solution (particularly of these modes with high frequency), whereas a diffusive term only attenuates its amplitude (see Exercise 2.3.8).

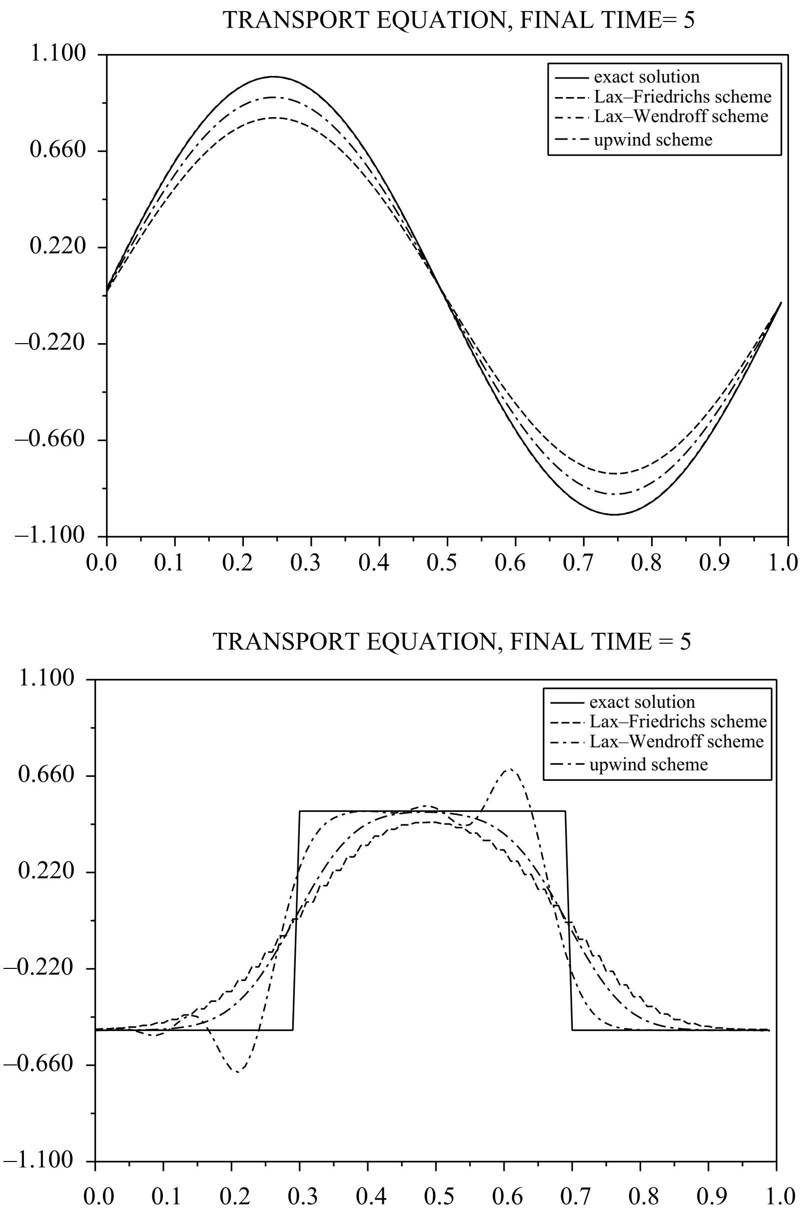

Figure 2.5. Comparison of the Lax–Friedrichs, Lax–Wendroff, and upwinded schemes for sinusoidal initial data (top) and square wave (bottom).

To illustrate this, we show calculations made on an interval of length 1 with periodic boundary conditions, space step $\Delta x = 0.01$, time step $\Delta t = 0.9 * \Delta x$, velocity $V = 1$ and final time $T = 5$. Two types of initial conditions are tested: first a very regular initial condition, a sine wave, then a discontinuous initial condition, a square wave (see Figure 2.5). The schemes accurate with order 1 are clearly diffusive: they destroy the solution. The Lax–Wendroff scheme which is accurate with order 2 is very good for a regular solution but oscillates for the square wave since it is dispersive. The concept of the equivalent equation allows us to understand these numerical phenomena.

Exercise 2.3.8 Take the equation

$$\begin{cases} \frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial^3 u}{\partial x^3} = 0 & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_*^+ \\ u(t=0,x) = \sin(\omega x + \phi) & \text{for } x \in \mathbb{R}, \end{cases}$$

with $V, \nu, \mu, \omega, \phi \in \mathbb{R}$. Show that its solution is

$$u(t,x) = \exp(-\nu\omega^2 t)\sin\left(\omega(x - (V + \mu\omega^2)t) + \phi\right)$$

(we shall assume uniqueness). Deduce that the diffusion attenuates the amplitude of the solution, while the dispersion modifies the velocity of propagation.

Exercise 2.3.9 Define the 'leapfrog' scheme

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + V\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0.$$

Study the consistency and the truncation error of this scheme. Show by Fourier analysis that it is stable under the condition CFL $|V|\Delta t \leq M\Delta x$ with $M < 1$.

Exercise 2.3.10 Define the Crank–Nicolson scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + V\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{4\Delta x} + V\frac{u_{j+1}^n - u_{j-1}^n}{4\Delta x} = 0.$$

Study the consistency and the truncation error of this scheme. Show by Fourier analysis that it is unconditionally stable.

2.3.2 Wave equation

We consider the wave equation in the bounded domain $(0,1)$ with periodic boundary conditions

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{for } (x,t) \in (0,1) \times \mathbb{R}_*^+ \\ u(t, x+1) = u(t,x) & \text{for } (x,t) \in (0,1) \times \mathbb{R}_*^+ \\ u(t=0,x) = u_0(x) & \text{for } x \in (0,1) \\ \frac{\partial u}{\partial t}(t=0,x) = u_1(x) & \text{for } x \in (0,1). \end{cases} \tag{2.40}$$