3 Variational formulation of elliptic problems

3.1 Generalities

3.1.1 Introduction

In this chapter we are interested in the mathematical analysis of **elliptic partial differential equations** (PDEs) (see definition 1.5.5). In general, these elliptic equations correspond to stationary physical models, that is, models which are independent of time. We shall see that boundary value problems are well-posed for these elliptic PDEs, that is, they have a solution which is unique and depends continuously on the data. The approach that we shall follow is called the **variational approach**. First we should say that the interest of this approach goes far beyond the framework of elliptic PDEs and even the framework of the 'pure' mathematical analysis to which we restrict ourselves. Indeed, we shall return to this variational approach for problems of evolution in time (parabolic or hyperbolic PDEs), and it will be crucial for understanding the finite element method that we develop in Chapter 6. Additionally, this approach has a very natural physical or mechanical interpretation. The reader should make the effort to study this variational approach carefully!

In this chapter and the following, the prototype example of elliptic PDEs will be the Laplacian for which we shall study the following boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where we impose Dirichlet boundary conditions (we refer to Section 1.3.3 for a presentation of this model). In (3.1), Ω is an open set of the space $\mathbb{R}^N$, $\partial\Omega$ is its boundary, f is a right-hand side data for the problem, and u is the unknown. Of course, in Chapter 5 we shall give many other examples of elliptic PDEs which can be studied, thanks to the variational approach.

The plan of this chapter is the following. In Section 3.2 we recall some integration by parts formulas, called **Green's formulas**, then we define the **variational formulation**. Section 3.3 is dedicated to **Lax–Milgram theorem** which will be the essential tool allowing us to show existence and uniqueness of the solutions of the variational formulation. We shall see that, to apply this theorem, it is inescapable that we must give up the space $C^1(\overline{\Omega})$ of continuously differentiable functions and use its 'generalization', the Sobolev space $H^1(\Omega)$.

We conclude this introduction by mentioning other methods to solve PDEs which are less powerful or more complicated than the variational approach (we refer the curious, and courageous, reader to the encyclopaedia [14]).

3.1.2 Classical formulation

The 'classical' formulation of (3.1), which might appear 'natural' at first sight, is to assume sufficient regularity for the solution u so that equations (3.1) have a meaning at every point of Ω or of $\partial\Omega$. First we recall some notation related to spaces of regular functions.

Definition 3.1.1 *Let Ω be an open set of $\mathbb{R}^N$, and $\overline{\Omega}$ its closure. We denote by $C(\Omega)$ (respectively, $C(\overline{\Omega})$) the space of continuous function in Ω (respectively, in $\overline{\Omega}$). Let $k \geq 0$ be an integer. We denote by $C^k(\Omega)$ (respectively, $C^k(\overline{\Omega})$) the space of functions k times continuously differentiable in Ω (respectively, in $\overline{\Omega}$).*

A **classical solution** (we also say **strong solution**) of (3.1) is a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, which implies that the right-hand side f must be in $C(\Omega)$. This classical formulation, unfortunately, has a number of problems! Without going into detail, we note that, under the single hypothesis $f \in C(\overline{\Omega})$, there is not in general a solution of class C^2 for (3.1) if the dimension of the space is greater than two ($N \geq 2$). In fact, a solution does exist, as we shall see later, but it is not of class C^2 (it is a little less regular except if the data f is more regular than $C(\overline{\Omega})$). The case of a space with dimension one ($N = 1$) is particular as it is easy to find classical solutions (see exercise 3.1.1), but we shall, nevertheless, see that, even in this successful case, the classical formulation is inconvenient.

In what follows, to study (3.1), we shall replace its classical formulation by a so-called variational formulation, which is much more advantageous.

3.1.3 The case of a space of one dimension

In one space dimension ($N = 1$), if $\Omega = (0,1)$, the boundary value problem (3.1) becomes

$$\begin{cases} -\dfrac{d^2u}{dx^2} = f & \text{for } 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{3.2}$$

This problem is so simple it has an explicit solution!

Exercise 3.1.1 If f is a continuous function in $[0,1]$, show that (3.2) has a unique solution in $C^2([0,1])$ given by the formula

$$u(x) = x\int_0^1 f(s)(1-s)ds - \int_0^x f(s)(x-s)ds \quad \text{for } x \in [0,1]. \tag{3.3}$$

For the remainder of this section we shall forget the explicit formula (3.3) which does not have any equivalent for more complicated problems.

In one space dimension, ‘partial differential equation’ loses its meaning as, since we only have one variable, we can more simply say ‘ordinary differential equation’. However, equation (3.2) is not a ‘normal’ ordinary differential equation in the sense that the solution must satisfy conditions ‘at both ends’ rather than an initial condition at one end of the interval $[0,1]$. This is exactly the difference between a boundary value problem (with conditions ‘at both ends’) and a Cauchy problem (with an initial condition ‘at one end’).

It is, however, interesting to see that, even in one dimension, the classical methods of ordinary differential equations are not useful to study (3.2) (and are completely useless in higher dimensions). For a parameter $m \in \mathbb{R}$, we consider the Cauchy problem for the Laplacian with initial data at 0

$$\begin{cases} -\dfrac{d^2u}{dx^2} = f & \text{for } 0 < x < 1 \\ u(0) = 0, & \dfrac{du}{dx}(0) = m. \end{cases} \tag{3.4}$$

Obviously there is a unique solution of (3.4): it is enough to integrate this linear equation (or more generally to use the Cauchy–Lipschitz existence theorem). It is not at all clear, on the other hand, that the solution of (3.4) coincides with that of (3.2) (if it exists). We ask the question if there exists a parameter m such that the solution of (3.4) also satisfies $u(1) = 0$ and therefore is a solution of (3.2). This is the principle of the **shooting method** which allows us to solve, both theoretically and numerically, the boundary value problem (3.2). Iteratively, we predict a value of m (shooting from the point 0), we integrate the Cauchy problem (3.4) (we calculate the trajectory of the shot), then depending on the result $u(1)$ we correct the value of m. In practice, this is not a very effective method which has the major problem that it cannot be generalized to higher dimensions.

The conclusion is that we need methods specific to boundary value problems which have nothing to do with those related to Cauchy problems.

3.2 Variational approach

The principle of the variational approach for the solution of PDEs is to replace the equation by an equivalent so-called variational formulation obtained by integrating the equation multiplied by an arbitrary function, called a test function. As we need to carry out integration by parts when establishing the variational formulation, we start by giving some essential results on this subject.

3.2.1 Green's formulas

In this section Ω is an open set of the space $\mathbb{R}^N$ (which may be bounded or unbounded), whose boundary is denoted by $\partial\Omega$. We also assume that Ω is a **regular** open set of class $\mathcal{C}^1$. The precise definition of a regular open set is given below in definition 3.2.5, but it is not necessary to understand this absolutely to follow the rest of this course. It is enough to know that an open regular set is *roughly speaking* an open set whose boundary is a regular hypersurface (a manifold of dimension $N-1$), and this open set is locally situated on one side of its boundary. We then define the **outward normal** at the boundary $\partial\Omega$ as being the unit vector $n = (n_i)_{1\leq i\leq N}$ normal at every point to the tangent plane of Ω and pointing to the exterior of Ω (see Figure 1.1). In $\Omega \subset \mathbb{R}^N$ we denote by dx the volume measure, or Lebesgue measure of dimension N. On $\partial\Omega$, we denote by ds the surface measure, or Lebesgue measure of dimension $N-1$ on the manifold $\partial\Omega$. The principal result of this section is the following theorem (see [4], [38]).

Theorem 3.2.1 (Green's formula) *Let Ω be a regular open set of class $\mathcal{C}^1$. Let w be a $C^1(\overline{\Omega})$ function with bounded support in the closure $\overline{\Omega}$. Then w satisfies Green's formula*

$$\int_\Omega \frac{\partial w}{\partial x_i}(x)\,dx = \int_{\partial\Omega} w(x)n_i(x)\,ds, \tag{3.5}$$

where n_i is the ith component of the unit outward normal to Ω.

Remark 3.2.2 To say that a regular function w has bounded support in the closed set $\overline{\Omega}$ is the same as saying that it is zero at infinity if the closed set is unbounded. We also say that the function w has compact support in $\overline{\Omega}$ (take care: this does not imply that w is zero on the boundary $\partial\Omega$). In particular, the hypothesis of theorem 3.2.1 in connection with the bounded support of the function w in $\overline{\Omega}$ is pointless if the open set Ω is bounded. If Ω is unbounded, this hypothesis ensures that the integrals in (3.5) are finite •

Theorem 3.2.1 has many corollaries which are all immediate consequences of Green's formula (3.5). The reader who wants to save his memory need only remember Green's formula (3.5)!

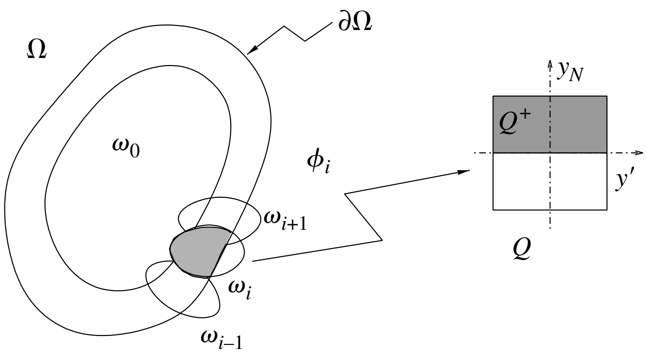

Figure 3.1. Definition of the regularity of an open set.

Corollary 3.2.3 (Integration by parts formula) *Let Ω be a regular open set of class $\mathcal{C}^1$. Let u and v be two $C^1(\overline{\Omega})$ functions with bounded support in the closed set $\overline{\Omega}$. Then they satisfy the integration by parts formula*

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)\,dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)\,dx + \int_{\partial\Omega} u(x)v(x)n_i(x)\,ds. \qquad (3.6)$$

Proof. It is enough to take $w = uv$ in theorem 3.2.1. □

Corollary 3.2.4 *Let Ω be a regular open set of class $\mathcal{C}^1$. Let u be a function of $C^2(\overline{\Omega})$ and v a function of $C^1(\overline{\Omega})$, both with bounded support in the closed set $\overline{\Omega}$. Then they satisfy the integration by parts formula*

$$\int_\Omega \Delta u(x)v(x)\,dx = -\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial n}(x)v(x)\,ds, \qquad (3.7)$$

where $\nabla u = \left(\frac{\partial u}{\partial x_i}\right)_{1\le i\le N}$ is the gradient vector of u, and $\frac{\partial u}{\partial n} = \nabla u\cdot n$.

Proof. We apply corollary 3.2.3 to v and $\frac{\partial u}{\partial x_i}$ and we sum in i. □

Definition 3.2.5 *We say that an open set Ω of $\mathbb{R}^N$ is regular of class $\mathcal{C}^k$ (for an integer $k \ge 1$) if there exist a finite number of open sets $(\omega_i)_{0\le i\le I}$ such that*

$$\overline{\omega_0} \subset \Omega, \qquad \overline{\Omega} \subset \cup_{i=0}^I \omega_i, \qquad \partial\Omega \subset \cup_{i=1}^I \omega_i,$$

and that, for every $i \in \{1,\dots,I\}$ (see Figure 3.1), there exists a bijective mapping ϕ_i of class C^k, from ω_i into the set

$$Q = \left\{y = (y', y_N) \in \mathbb{R}^{N-1}\times\mathbb{R},\ |y'| < 1,\ |y_N| < 1\right\},$$

whose inverse is also of class C^k, and such that

$$\begin{array}{l}\phi_i(\omega_i\cap\Omega)=Q\cap\left\{y=(y',y_N)\in\mathbb{R}^{N-1}\times\mathbb{R},y_N>0\right\}=Q^+,\\ \phi_i(\omega_i\cap\partial\Omega)=Q\cap\left\{y=(y',y_N)\in\mathbb{R}^{N-1}\times\mathbb{R},y_N=0\right\}.\end{array}$$

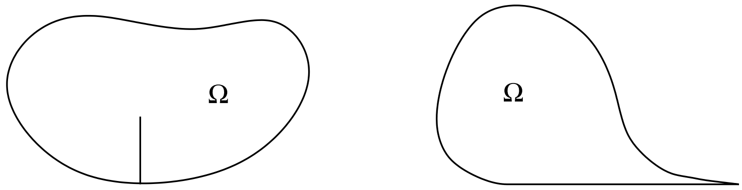

Figure 3.2. Two examples of a nonregular open set: open set with a crack on the left, open set with a cusp on the right.

Remark 3.2.6 Even though Figure 3.1 represents a bounded regular open set, the definition 3.2.5 also applies to unbounded open sets. Definition 3.2.5 does not only exclude open sets whose boundary is not a regular surface, but it also excludes open sets which do not lie locally on one side of their boundary. Figure 3.2 contains two typical examples of a nonregular open set which give an irremovable singularity, a crack, and a cusp. These examples are not 'mathematical inventions': the cracked set is used to study crack problems in structural mechanics. We can, nevertheless, generalize the class of regular open set a little to open sets which are 'piecewise regular', provided that the pieces of the boundary are 'joined' by angles different from either 0 (a cusp) or from 2π (a crack). All of these details are largely outside the scope of this course, and we refer the reader to remark 4.3.7 for another explanation of regularity problems. •

Exercise 3.2.1 From Green's formula (3.5) deduce the Stokes formula

$$\int_\Omega \mathrm{div}\sigma(x)\phi(x)\,dx=-\int_\Omega \sigma(x)\cdot\nabla\phi(x)\,dx+\int_{\partial\Omega}\sigma(x)\cdot n(x)\,\phi(x)\,ds,$$

where ϕ is a scalar function of $C^1(\overline{\Omega})$ and σ a vector valued function of $C^1(\overline{\Omega})$, with bounded supports in the closed set $\overline{\Omega}$.

Exercise 3.2.2 In $N=3$ dimensions we define the curl of a function of Ω in $\mathbb{R}^3$, $\phi=(\phi_1,\phi_2,\phi_3)$, as the function of Ω in $\mathbb{R}^3$ defined by

$$\nabla\times\phi=\left(\frac{\partial\phi_3}{\partial x_2}-\frac{\partial\phi_2}{\partial x_3},\frac{\partial\phi_1}{\partial x_3}-\frac{\partial\phi_3}{\partial x_1},\frac{\partial\phi_2}{\partial x_1}-\frac{\partial\phi_1}{\partial x_2}\right).$$

For ϕ and ψ, vector valued functions of $C^1(\overline{\Omega})$, with bounded supports in the closed set $\overline{\Omega}$, deduce Green's formula (3.5)

$$\int_\Omega \nabla\times\phi\cdot\psi\,dx-\int_\Omega\phi\cdot\nabla\times\psi\,dx=-\int_{\partial\Omega}(\phi\times n)\cdot\psi\,ds.$$