it satisfies $\|A(k)\|_2 = \rho(A(k))$ and $\|A(k)^n\|_2 = \|A(k)\|_2^n$ (see lemma 13.1.6), therefore the Von Neumann condition (2.27) is necessary and sufficient (we had the 'luck' in the proof of lemma 2.2.23 to be in this favourable case). However, if $A(k)$ is not normal, then in general, the Von Neumann stability condition is **not sufficient** and we must make a more delicate analysis of $A(k)$ (and in particular of its diagonalization). •

Remark 2.2.25 The Lax theorem 2.2.20 generalizes without difficulty to multilevel schemes if we choose the L^2 norm. The method of proof is unchanged: it uses Fourier analysis and vector notation (2.24). •

Remark 2.2.26 Everything which we have said about the stability and the convergence of multilevel schemes generalizes immediately to schemes for systems of equations. In this case, we must also write a vectorial version of the recurrence relation (2.25) and of the amplification matrix (instead of a scalar factor). •

Exercise 2.2.6 Show that the Gear scheme (2.8) is unconditionally stable and therefore convergent in the L^2 norm.

Exercise 2.2.7 Show that the DuFort–Frankel scheme (2.7) is stable in the L^2 norm and therefore convergent, if the ratio $\Delta t/(\Delta x)^2$ remains bounded as we let Δt and Δx tend to 0.

2.2.6 The multidimensional case

The finite difference method extends without difficulty to problems in several space dimensions. Let us consider, for example, the heat equation in two space dimensions (the case of there or more space dimensions is not more complicated, at least in theory) in the rectangular domain $\Omega = (0,1) \times (0,L)$ with the Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} - \nu \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{for } (x,y,t) \in \Omega \times \mathbb{R}_*^+ \\ u(t=0,x,y) = u_0(x,y) & \text{for } (x,y) \in \Omega \\ u(t,x,y) = 0 & \text{for } t \in \mathbb{R}_*^+, (x,y) \in \partial\Omega. \end{cases} \tag{2.28}$$

To discretize the domain Ω, we introduce two space steps $\Delta x = 1/(N_x+1) > 0$ and $\Delta y = L/(N_y+1) > 0$ (with N_x and N_y being two positive integers). With the time step $\Delta t > 0$, we define the nodes of a regular mesh (see Figure 2.1)

$$(t_n, x_j, y_k) = (n\Delta t, j\Delta x, k\Delta y) \quad \text{for } n \geq 0, 0 \leq j \leq N_x+1, 0 \leq k \leq N_y+1.$$

We denote by $u_{j,k}^n$ the value of an approximate discrete solution at the point (t_n, x_j, y_k), and $u(t,x,y)$ the exact solution of (2.28).

The Dirichlet boundary conditions are expressed, for $n > 0$, as

$$u_{0,k}^n = u_{N_x+1,k}^n = 0, \quad \forall\, k, \quad \text{and} \quad u_{j,0}^n = u_{j,N_y+1}^n = 0, \ \forall\, j.$$

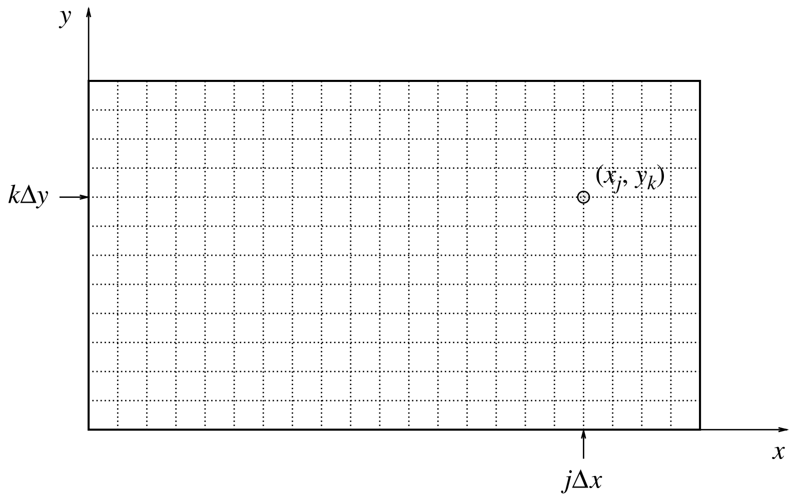

Figure 2.1. Rectangular finite difference mesh.

The initial data is discretized by

$$u_{j,k}^0 = u_0(x_j, y_k) \quad \forall\, j, k.$$

The generalization to the two-dimensional case of the **explicit scheme** is obvious

$$\frac{u_{j,k}^{n+1} - u_{j,k}^n}{\Delta t} + \nu \frac{-u_{j-1,k}^n + 2u_{j,k}^n - u_{j+1,k}^n}{(\Delta x)^2} + \nu \frac{-u_{j,k-1}^n + 2u_{j,k}^n - u_{j,k+1}^n}{(\Delta y)^2} = 0 \quad (2.29)$$

for $n \geq 0$, $j \in \{1, \ldots, N_x\}$ and $k \in \{1, \ldots, N_y\}$. The only notable difference with the one-dimensional case is the extra severity of the CFL condition.

Exercise 2.2.8 Show that the explicit scheme (2.29) is stable in the L^∞ norm (and that it satisfies the maximum principle) under the CFL condition

$$\frac{\nu \Delta t}{(\Delta x)^2} + \frac{\nu \Delta t}{(\Delta y)^2} \leq \frac{1}{2}.$$

We illustrate the explicit scheme (2.29) (to which we add a convection term) by Figure 2.2 which represents convection–diffusion of a 'hump' (the coefficient of diffusion is 0.01 and the velocity $(1., 0.)$).

Likewise, we have the **implicit scheme**

$$\frac{u_{j,k}^{n+1} - u_{j,k}^n}{\Delta t} + \nu \frac{-u_{j-1,k}^{n+1} + 2u_{j,k}^{n+1} - u_{j+1,k}^{n+1}}{(\Delta x)^2} + \nu \frac{-u_{j,k-1}^{n+1} + 2u_{j,k}^{n+1} - u_{j,k+1}^{n+1}}{(\Delta y)^2} = 0. \quad (2.30)$$

We remark that the implicit scheme needs, to calculate u^{n+1} as a function of u^n, the solution of a linear system significantly more complicated than that in one space

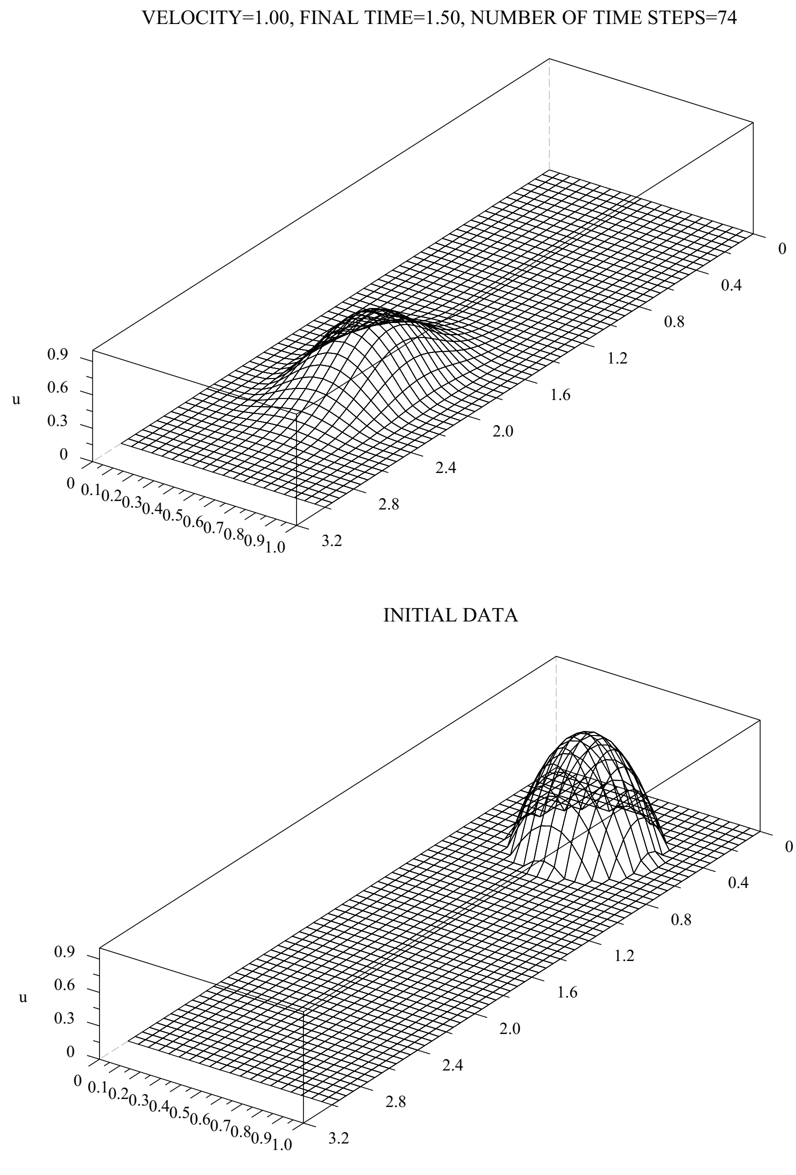

Figure 2.2. Explicit scheme for the convection–diffusion equation in two dimensions: initial data (top) and solution (bottom).

dimension (the situation will be even worse in three dimensions). Recall that in one dimension, it is sufficient to invert a tridiagonal matrix. We shall see that in two dimensions the matrix has a less simple structure. The discrete unknown $u_{j,k}^n$ is indexed by two integers j and k, but in practice we use only one index to store u^n in the form of a vector in the computer. A (simple and efficient) way of putting the unknowns $u_{j,k}^n$ into a single vector is to write

$$u^n = (u_{1,1}^n, \dots, u_{1,N_y}^n, u_{2,1}^n, \dots, u_{2,N_y}^n, \dots, u_{N_x,1}^n, \dots, u_{N_x,N_y}^n).$$

Note that we have arranged the unknowns 'column by column', but we could equally well have done it 'row by row' by using the j index first instead of the k index (N_x is the number of columns and N_y the number of rows). With this convention, the implicit scheme (2.30) requires the inversion of the matrix, which is 'block' symmetric tridiagonal,

$$M = \begin{pmatrix} D_1 & E_1 & & & 0 \\ E_1 & D_2 & E_2 & & \\ & \ddots & \ddots & \ddots & \\ & & E_{N_x-2} & D_{N_x-1} & E_{N_x-1} \\ 0 & & & E_{N_x-1} & D_{N_x} \end{pmatrix}$$

where the diagonal blocks D_j are square matrices of dimension N_y

$$D_j = \begin{pmatrix} 1+2(c_y+c_x) & -c_y & & & 0 \\ -c_y & 1+2(c_y+c_x) & -c_y & & \\ & \ddots & \ddots & \ddots & \\ & & -c_y & 1+2(c_y+c_x) & -c_y \\ 0 & & & -c_y & 1+2(c_y+c_x) \end{pmatrix}$$

with $c_x = \nu\Delta t/(\Delta x)^2$ and $c_y = \nu\Delta t/(\Delta y)^2$, and the extra-diagonal blocks $E_j = (E_j)^*$ are square matrices of dimension N_y

$$E_j = \begin{pmatrix} -c_x & 0 & & & 0 \\ 0 & -c_x & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & -c_x & 0 \\ 0 & & & 0 & -c_x \end{pmatrix}.$$

In summary, the matrix M is symmetric and pentadiagonal. However, the five diagonals are not contiguous, this implies a considerable extra cost to solve a linear system associated with M (see the appendix on numerical linear algebra and particularly the remarks 13.1.21 and 13.1.41). The situation will be even worse in three dimensions.

Exercise 2.2.9 Show that the Peaceman–Rachford scheme

$$\frac{u_{j,k}^{n+1/2}-u_{j,k}^{n}}{\Delta t}+\nu\frac{-u_{j-1,k}^{n+1/2}+2u_{j,k}^{n+1/2}-u_{j+1,k}^{n+1/2}}{2(\Delta x)^2}+\nu\frac{-u_{j,k-1}^{n}+2u_{j,k}^{n}-u_{j,k+1}^{n}}{2(\Delta y)^2}=0$$

$$\frac{u_{j,k}^{n+1}-u_{j,k}^{n+1/2}}{\Delta t}+\nu\frac{-u_{j-1,k}^{n+1/2}+2u_{j,k}^{n+1/2}-u_{j+1,k}^{n+1/2}}{2(\Delta x)^2}+\nu\frac{-u_{j,k-1}^{n+1}+2u_{j,k}^{n+1}-u_{j,k+1}^{n+1}}{2(\Delta y)^2}=0.$$

has accuracy of order 2 in space and time and is unconditionally stable in the L^2 norm (for periodic boundary conditions in each direction).

Because of the heightened cost of calculation, we often replace the implicit scheme by a generalization to several space dimensions of the one-dimensional scheme, obtained by a technique called **alternating directions**, also called operator splitting, or **splitting**. The idea is, instead of solving the two-dimensional equation (2.28), we solve alternatively the two one-dimensional equations

$$\frac{\partial u}{\partial t}-2\nu\frac{\partial^2 u}{\partial x^2}=0 \quad \text{and} \quad \frac{\partial u}{\partial t}-2\nu\frac{\partial^2 u}{\partial y^2}=0$$

whose average again gives (2.28). For example, by using a Crank–Nicolson scheme in each direction for a half time step $\Delta t/2$, we obtain an **alternating direction scheme**

$$\frac{u_{j,k}^{n+1/2}-u_{j,k}^{n}}{\Delta t}+\nu\frac{-u_{j-1,k}^{n+1/2}+2u_{j,k}^{n+1/2}-u_{j+1,k}^{n+1/2}}{2(\Delta x)^2}+\nu\frac{-u_{j-1,k}^{n}+2u_{j,k}^{n}-u_{j+1,k}^{n}}{2(\Delta x)^2}=0$$

$$\frac{u_{j,k}^{n+1}-u_{j,k}^{n+1/2}}{\Delta t}+\nu\frac{-u_{j,k-1}^{n+1}+2u_{j,k}^{n+1}-u_{j,k+1}^{n+1}}{2(\Delta y)^2}+\nu\frac{-u_{j,k-1}^{n+1/2}+2u_{j,k}^{n+1/2}-u_{j,k+1}^{n+1/2}}{2(\Delta y)^2}=0 \tag{2.31}$$

The advantage of this type of scheme is that it is enough, at each half time step, to invert a 'one-dimensional' tridiagonal matrix (therefore an inexpensive calculation). In three dimensions, it is enough to take three one-third time steps and the properties of the scheme are unchanged. This scheme is not only stable but also consistent with the two-dimensional equation (2.28).

Exercise 2.2.10 Show that the alternating direction scheme (2.31) has accuracy of order 2 in space and time and is unconditionally stable in the L^2 norm (for periodic boundary conditions in each direction).

Let us conclude this section with some practical considerations for the finite difference method. Its principal advantage is its simplicity as well as its computational implementation. However, it has a certain number of defects which, for many complex problems, leads us to prefer other methods such as the finite element method (see Chapters 6 and 8). One of the principal limitations of the method is that it only works

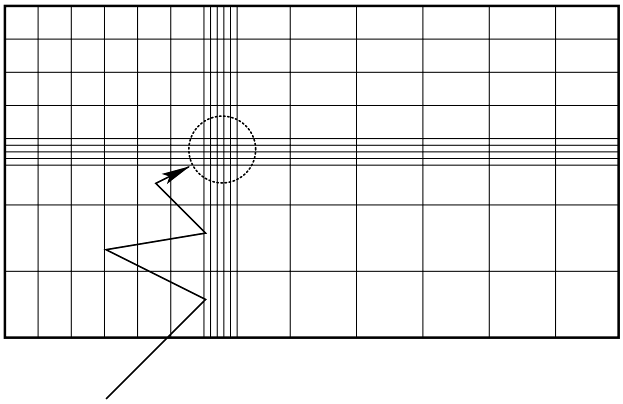

Figure 2.3. Refinement of a finite difference mesh: the encircled zone is that where we want more accuracy.

for regular, or **rectangular**, meshes. It is not always easy to discretize an arbitrary space domain by rectangular meshes! Additionally, it is not possible to locally refine the mesh to have better accuracy at a particular point of the domain. It is possible to vary the space step in each direction but this variation is uniform in perpendicular directions (Δx and Δy can change along the x and y axes, respectively, but this variation is uniform in orthogonal directions; see Figure 2.3). Such a refinement of a finite difference mesh therefore has effects far outside the zone of interest. Moreover, the theory and practice of the finite differences become much more complicated when the coefficients in the partial differential equations are variables and when the problems are nonlinear.

2.3 Other models

2.3.1 Advection equation

We consider the advection equation in one space dimension in the bounded domain $(0, 1)$ with a constant velocity $V > 0$ and with the periodic boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} + V\dfrac{\partial u}{\partial x} = 0 & \text{for } (x,t) \in (0,1) \times \mathbb{R}_*^+ \\ u(t, x+1) = u(t,x) & \text{for } (x,t) \in (0,1) \times \mathbb{R}_*^+ \\ u(0,x) = u_0(x) & \text{for } x \in (0,1). \end{cases} \tag{2.32}$$

We always discretize space with a step $\Delta x = 1/(N+1) > 0$ (N a positive integer) and the time with $\Delta t > 0$, and we denote by $(t_n, x_j) = (n\Delta t, j\Delta x)$ for $n \geq 0, j \in \{0, 1, \ldots, N+1\}$, u_j^n the value of an approximate discrete solution at the point (t_n, x_j), and $u(t, x)$ the exact solution of (2.32). The periodic boundary conditions lead to