(a function of Ω in $\mathbb{R}$) the resultant normal of the forces, then the normal component of the displacement $u(x)$ (a scalar) is the solution of the thin plate equation

$$\begin{cases} \Delta(\Delta u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.24}$$

where we denote by $\frac{\partial u}{\partial n} = \nabla u \cdot n$ with n the outward unit normal vector to $\partial\Omega$. We remark that this is a partial differential equation which is fourth order in space (also called the bi-Laplacian). This is why it is necessary to have two boundary conditions. These boundary conditions represent the clamping of the plate (there is neither displacement nor rotation of the edge of the plate).

We remark that it is possible to justify the plate equation (1.24) asymptotically from the Lamé system (1.22) by letting the thickness of the plate to tend to zero. This is an example of mathematical modelling.

1.4 Numerical calculation by finite differences

1.4.1 Principles of the method

Apart from some very particular cases, it is impossible to calculate explicitly the solutions of the different models presented above. It is therefore necessary to have recourse to numerical calculation on a computer to estimate these solutions both qualitatively and quantitatively. The principle of all methods for the numerical solution of PDEs is to obtain discrete numerical values (that is, a finite number) which **'approximate'** (in a suitable sense, to be made precise) the exact solution. In this process we must be aware of two fundamental points: first, we do not calculate exact solutions but approximate ones; second, we **discretize** the problem by representing functions by a finite number of values, that is, **we move from the 'continuous' to the 'discrete'.**

There are numerous methods for the numerical approximation of PDEs. We present one of the oldest and simplest, called the finite difference method (later we shall see another method, called the finite element method). For simplicity, we limit ourselves to one space dimension (see Section 2.2.6 for higher dimensions). For the moment, we shall only consider the practical principles of this method, that is, the construction of what we call the **numerical schemes**. We reserve the theoretical justification of these schemes for Chapter 2, that is, the study of their convergence (in what way the approximate discrete solutions are close to the exact continuous solutions).

To discretise the spatio-temporal continuum, we introduce a **space step** $\Delta x > 0$ and a **time step** $\Delta t > 0$ which will be the smallest scales represented by the numerical method. We define a mesh or discrete coordinates in space and time (see Figure 1.4)

$$(t_n, x_j) = (n\Delta t, j\Delta x) \quad \text{for } n \geq 0,\ \ j \in \mathbb{Z}.$$

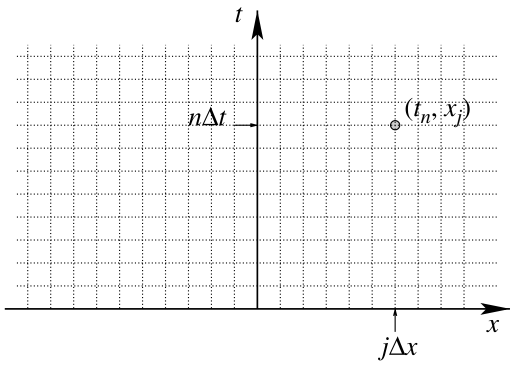


Figure 1.4. Finite difference mesh.

We denote by u_j^n the value of the discrete solution at (t_n, x_j), and $u(t,x)$ the (unknown) exact solution. The principle of the finite difference method is to replace the derivatives by finite differences by using Taylor series in which we neglect the remainders. For example, we approximate the second space derivative (the Laplacian in one dimension) by

$$-\frac{\partial^2 u}{\partial x^2}(t_n, x_j) \approx \frac{-u_{j-1}^n + 2u_j^n - u_{j+1}^n}{(\Delta x)^2} \tag{1.25}$$

where we recall the Taylor formula

$$\begin{aligned} -u(t,x-\Delta x) + 2u(t,x) - u(t,x+\Delta x) = \quad & -(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}(t,x) \\ & - \frac{(\Delta x)^4}{12} \frac{\partial^4 u}{\partial x^4}(t,x) + \mathcal{O}\Big((\Delta x)^6\Big) \end{aligned} \tag{1.26}$$

If Δx is 'small', formula (1.25) is a 'good' approximation (it is natural, but not unique). The formula (1.25) is called **centred** since it is symmetric in j.

To discretize the convection–diffusion equation

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.27}$$

we must also discretize the convection term. A centred formula gives

$$V\frac{\partial u}{\partial x}(t_n, x_j) \approx V\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

It only remains to do the same thing for the time derivative. Again we have a choice between finite difference schemes: centred or one sided. Let us look at three 'natural' formulas.

1. As a first choice, the centred finite difference

$$\frac{\partial u}{\partial t}(t_n, x_j) \approx \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}$$

leads to a scheme which is completely symmetric with respect to n and j (called the centred scheme or **Richardson's scheme**)

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + V\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \nu\frac{-u_{j-1}^n + 2u_j^n - u_{j+1}^n}{(\Delta x)^2} = 0. \tag{1.28}$$

Even though it is 'natural' **this scheme cannot calculate approximate solutions** of the convection–diffusion equation (1.27) (see the numerical example of Figure 1.5)! We shall justify the inability of this scheme to approximate the exact solution in Lemma 2.2.23. For the moment, we shall simply say that the difficulty comes from the centred character of the finite difference which approximates the time derivative.

2. A second choice is the one-sided upwind scheme (we go back in time) which gives the **backward Euler scheme**

$$\frac{\partial u}{\partial t}(t_n, x_j) \approx \frac{u_j^n - u_j^{n-1}}{\Delta t}$$

which leads to

$$\frac{u_j^n - u_j^{n-1}}{\Delta t} + V\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \nu\frac{-u_{j-1}^n + 2u_j^n - u_{j+1}^n}{(\Delta x)^2} = 0. \tag{1.29}$$

3. The third choice is the opposite of the preceding: the downwind one-sided finite difference (we go forward in time; we also talk of the **forward Euler scheme**)

$$\frac{\partial u}{\partial t}(t_n, x_j) \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

which leads to

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + V\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \nu\frac{-u_{j-1}^n + 2u_j^n - u_{j+1}^n}{(\Delta x)^2} = 0. \tag{1.30}$$

The principal difference between these last two schemes is that (1.29) is called **implicit** since we must solve a system of linear equations to calculate the values $(u_j^n)_{j\in\mathbb{Z}}$ as functions of the preceding values $(u_j^{n-1})_{j\in\mathbb{Z}}$, while (1.30) is called **explicit** since it immediately gives the values $(u_j^{n+1})_{j\in\mathbb{Z}}$ as a function of $(u_j^n)_{j\in\mathbb{Z}}$. The shift of 1 in the index n between the schemes (1.29) and (1.30) is only evident when we rewrite (1.30) in the form

$$\frac{u_j^n - u_j^{n-1}}{\Delta t} + V\frac{u_{j+1}^{n-1} - u_{j-1}^{n-1}}{2\Delta x} + \nu\frac{-u_{j-1}^{n-1} + 2u_j^{n-1} - u_{j+1}^{n-1}}{(\Delta x)^2} = 0.$$

In the three schemes which we have defined, there must be initial data to start the iterations in n: the initial values $(u_j^0)_{j\in\mathbb{Z}}$ are defined by, for example, $u_j^0 = u_0(j\Delta x)$ where u_0 is the initial data of the convection–diffusion equation (1.27). We remark that the 'bad' centred scheme (1.28) has an additional difficulty in starting: for $n = 1$ we also have to know the values $(u_j^1)_{j\in\mathbb{Z}}$ which, therefore, must be calculated in another way (for example, by applying one of the two other schemes).

1.4.2 Numerical results for the heat flow equation

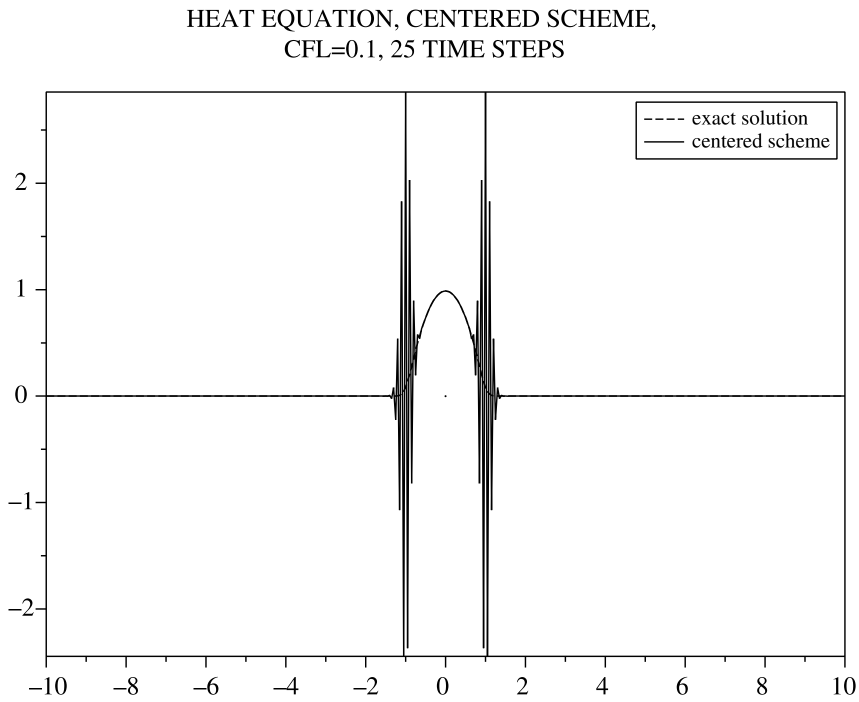


Figure 1.5. Unstable centred scheme with $\nu\Delta t = 0.1(\Delta x)^2$.

We start by making some simple numerical tests in the case where $V = 0$ and $\nu = 1$, that is, **we solve the heat flow equation numerically.** As initial condition, we choose the function

$$u_0(x) = \max(1 - x^2, 0).$$

To be able to compare the numerical solutions with the exact (1.14), we want to work on the infinite domain $\Omega = \mathbb{R}$, that is, calculate, for each $n \geq 0$, an infinite number of values $(u_j^n)_{j\in\mathbb{Z}}$, but the computer will not allow this as the memory is finite! To a first approximation, we therefore replace $\mathbb{R}$ by the 'large' domain $\Omega = (-10, +10)$ equipped with Dirichlet boundary conditions. The validity of this approximation is confirmed by the numerical calculations below. We fix the space step at $\Delta x = 0.05$: there are therefore 401 values $(u_j^n)_{-200\leq j\leq +200}$ to calculate. We should remember that the values u_j^n calculated by the computer are subject to rounding errors and are therefore not the exact values of the difference scheme: nevertheless, in the calculations presented here, these rounding errors are completely negligible and are in no way responsible for the phenomena which we shall observe. On all the figures we show the exact solution, calculated by the explicit formula (1.14), and the approximate numerical solution under consideration.

Let us first look at the outcome of the centred scheme (1.28): since as we have said, this scheme is not able to calculate approximate solutions of the heat flow equation. Whatever the choice of the time step Δt, this scheme is **unstable**, that is the numerical solution oscillates unboundedly if we decrease the step sizes Δx and Δt. This highly characteristic phenomenon (which appears rapidly) is illustrated by Figure 1.5. We emphasize that **whatever the choice** of steps Δt and Δx, we see these oscillations (which are nonphysical). We say that the scheme is unconditionally unstable. A rigorous justification will be given in the following chapter (see lemma 2.2.23).

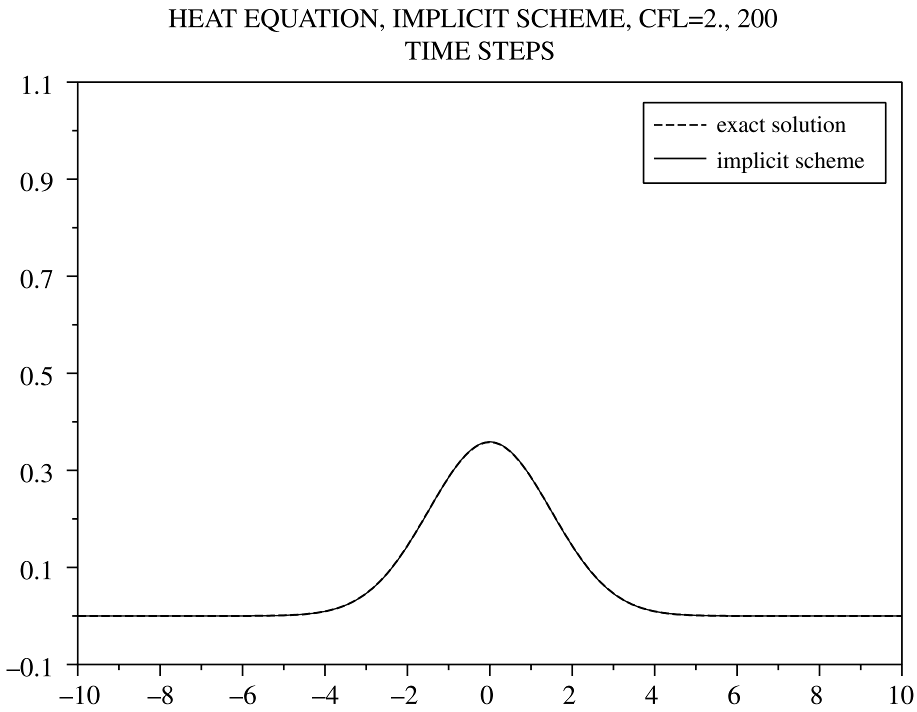


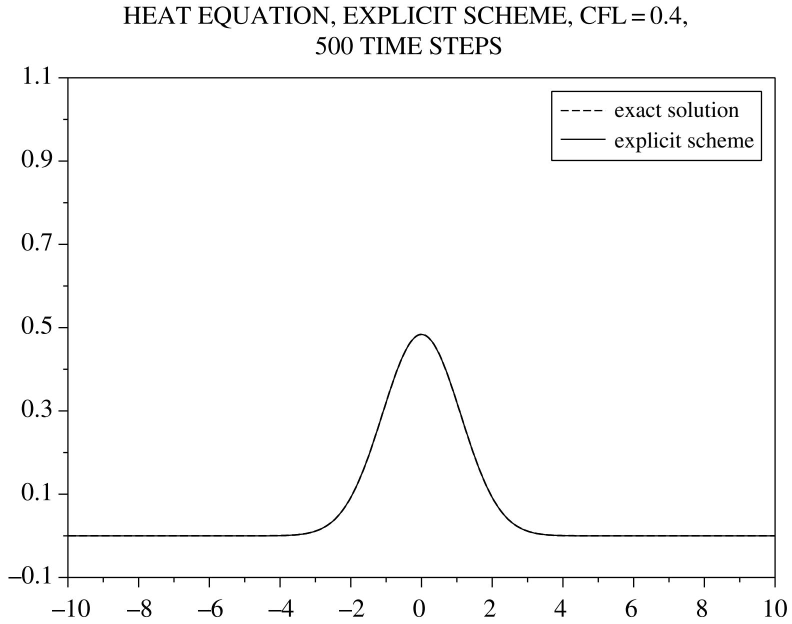
Figure 1.6. Implicit scheme with $\nu\Delta t = 2(\Delta x)^2$.

Contrary to the preceding scheme, the implicit scheme (1.29) calculates 'good' approximate solutions of the heat flow equation **whatever** the time step Δt (see Figure 1.6). In particular, we never see numerical oscillation for any choice of steps Δt and Δx. We say that the implicit scheme is unconditionally stable.

Let us now consider the explicit scheme (1.30): numerical experiments show that we obtain numerical oscillations depending on the time step Δt (see Figure 1.7). The stability limit is easy to find experimentally: if the choice of steps Δt and Δx **satisfy** the condition

$$2\nu\Delta t \leq (\Delta x)^2 \tag{1.31}$$

the scheme is stable, while if (1.31) is not satisfied, then the scheme is unstable. We say that the explicit scheme is conditionally stable. The stability condition (1.31) is **one of the simplest but most profound observations in numerical analysis**. It was discovered in 1928 (before the appearance of the first computers!) by Courant, Friedrichs, and Lewy. It takes the name **CFL condition or the Courant, Friedrichs, Lewy condition**.



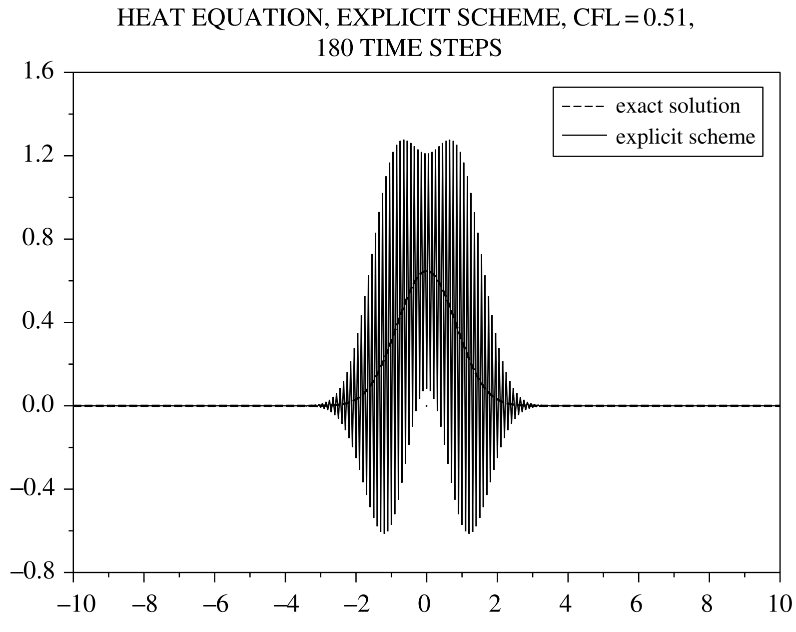


Figure 1.7. Explicit scheme with $\nu\Delta t = 0.4(\Delta x)^2$ (top) and $\nu\Delta t = 0.51(\Delta x)^2$ (bottom).

We shall briefly justify this stability condition (a more through analysis will be carried out in the next chapter). Rewriting the explicit scheme in the form

$$u_j^{n+1} = \frac{\nu\Delta t}{(\Delta x)^2} u_{j-1}^n + \left(1 - 2\frac{\nu\Delta t}{(\Delta x)^2}\right) u_j^n + \frac{\nu\Delta t}{(\Delta x)^2} u_{j+1}^n. \tag{1.32}$$

If the CFL condition is satisfied, then (1.32) shows that u_j^{n+1} is a convex combination of the values at the preceding time $u_{j-1}^n, u_j^n, u_{j+1}^n$ (all of the coefficients on the right-hand side of (1.32) are positive and their sum is 1). In particular, if the initial data u_0 is bounded by two constants m and M such that

$$m \leq u_j^0 \leq M \quad \text{for all } j \in \mathbb{Z},$$

then a recurrence easily shows that the same inequalities remain true for all time

$$m \leq u_j^n \leq M \quad \text{for all } j \in \mathbb{Z} \text{ and for all } n \geq 0. \tag{1.33}$$

Property (1.33) prevents the scheme from oscillating unboundedly: it is therefore stable subject to the CFL condition. Property (1.33) is called a discrete maximum principle: it is the **discrete** equivalent of the **continuous** maximum principle for exact solutions which we have seen in remark 1.2.10.

Suppose, on the other hand, the CFL condition is not satisfied, that is,

$$2\nu\Delta t > (\Delta x)^2,$$

then, for certain initial data the scheme is unstable (it may be stable for certain 'exceptional' initial data: for example, if $u_0 \equiv 0$!). Let us take the initial data defined by

$$u_j^0 = (-1)^j$$

which is uniformly bounded. A simple calculation shows that

$$u_j^n = (-1)^j \left(1 - 4\frac{\nu\Delta t}{(\Delta x)^2}\right)^n$$

which tends, in modulus, to infinity as n tends to infinity since $1 - 4\nu\Delta t/(\Delta x)^2 < -1$. The explicit scheme is therefore unstable if the CFL condition is not satisfied.

Exercise 1.4.1 The aim of this exercise is to show that the implicit scheme (1.29), with $V = 0$, also satisfies the discrete maximum principle. We impose Dirichlet boundary conditions, that is, formula (1.29) is valid for $1 \leq j \leq J$ and we fix $u_0^n = u_{J+1}^n = 0$ for all $n \in \mathbb{N}$. Take two constants $m \leq 0 \leq M$ such that $m \leq u_j^0 \leq M$ for $1 \leq j \leq J$. Verify that we can uniquely calculate the u_j^{n+1} as a function of u_j^n. Show that for all time $n \geq 0$ we again have the inequalities $m \leq u_j^n \leq M$ for $1 \leq j \leq J$ (without any condition on Δt and Δx).

If we have illuminated the question of the stability of the explicit scheme a little, we have not said anything about its convergence, that is, its capacity to approximate the exact solution. We shall answer this question rigorously in the following chapter. We remark that stability is a necessary condition for convergence, but it is not sufficient. We shall be content for the moment with experimentally verifying the convergence of the scheme, that is, when the space and time steps become smaller and smaller,

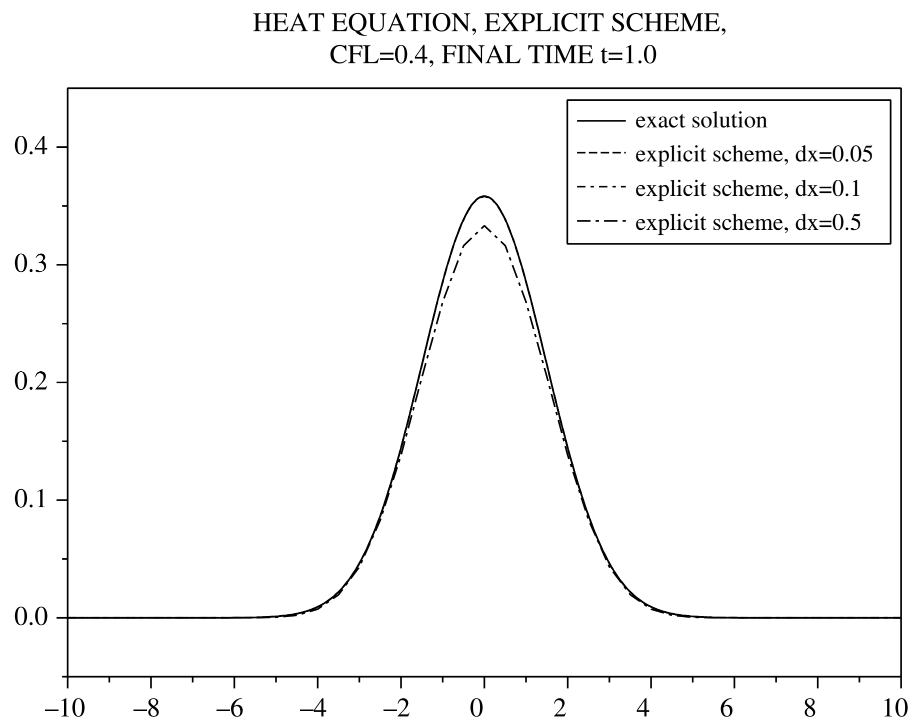


Figure 1.8. Explicit scheme with $\nu\Delta t = 0.4(\Delta x)^2$ for various values of Δx.

the corresponding numerical solutions converge and their limit is the exact solution (we can check this as the exact solution is available). In Figure 1.8 we numerically verify that if we reduce the space step Δx (which has values 0.5, 0.1, and 0.05) and the time step Δt by keeping the ratio $\nu\Delta t/(\Delta x)^2$ (the CFL number) constant, then the numerical solution becomes closer and closer to the exact solution. (The comparison is carried out at the same final time $t = 1$, therefore the number of time steps grows as the time step Δt decreases.) This process of **'numerical verification of convergence'** is very simple and we should never hesitate to use it if nothing better is available (that is, if the theoretical convergence analysis is impossible or too difficult).

1.4.3 Numerical results for the advection equation

We shall carry out a second series of numerical experiments on the **convection–diffusion equation** (1.27) with a nonzero velocity $V = 1$. We take the same data as before and we choose the explicit scheme with $\nu\Delta t = 0.4(\Delta x)^2$. We look at the influence of the diffusion constant ν (or the inverse of the Péclet number) on the stability of the scheme. Figure 1.9 shows that the scheme is stable when $\nu = 1$, unstable for $\nu = 0.01$, and that for the intermediate value $\nu = 0.1$, the scheme seems stable but the approximate solution is slightly different from the exact solution. Clearly, the smaller the inverse of the Péclet number ν is, the more the convective term dominates the diffusive term. Consequently, the CFL condition (1.31), obtained when the velocity V is zero, is less and less valid as ν decreases.

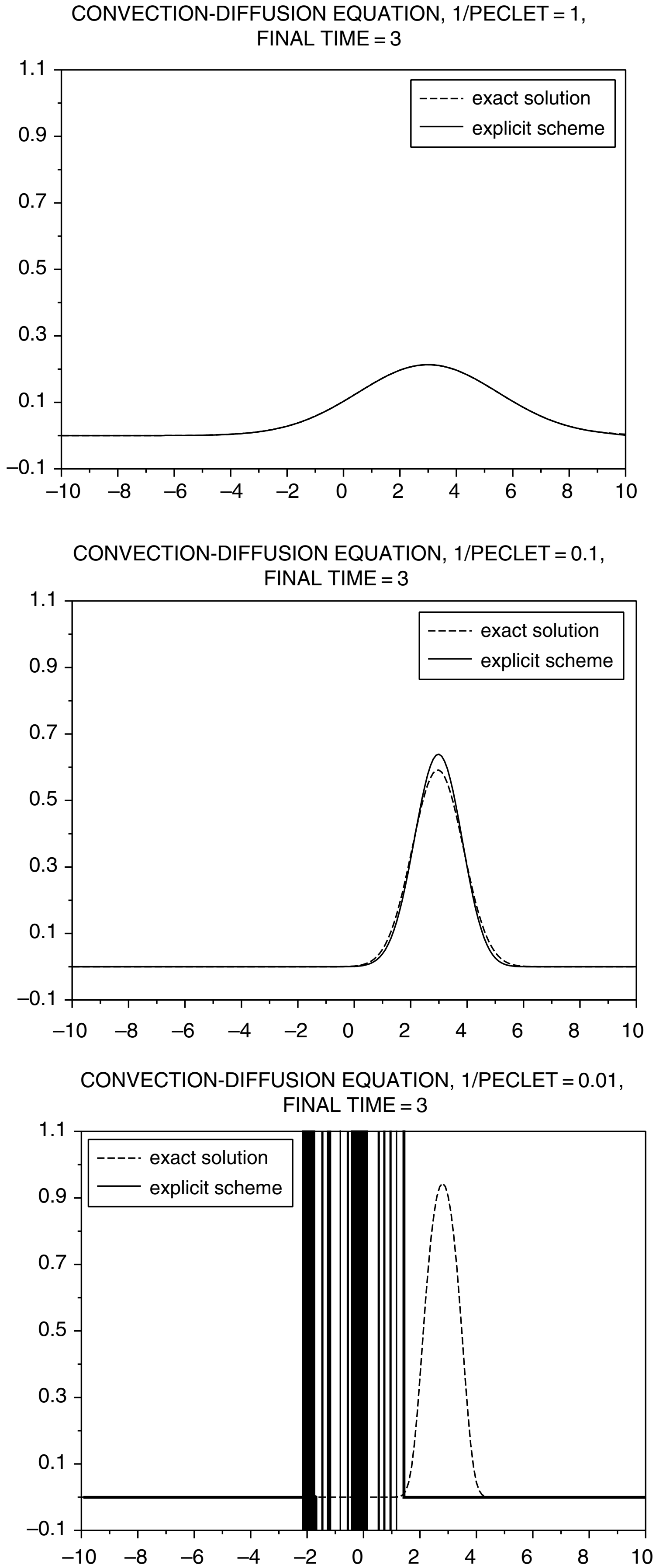


Figure 1.9. Explicit scheme for the convection–diffusion equation with $\nu\Delta t = 0.4(\Delta x)^2$ and $V = 1$. At the top, $\nu = 1$, in the middle $\nu = 0.1$, and at the bottom $\nu = 0.01$.

To understand this phenomenon, we examine **the advection equation** which is obtained in the limit $\nu = 0$. We remark first that the CFL condition (1.31) is automatically satisfied when $\nu = 0$ (whatever Δt and Δx), which seems to contradict the experimental result at the bottom of Figure 1.9.

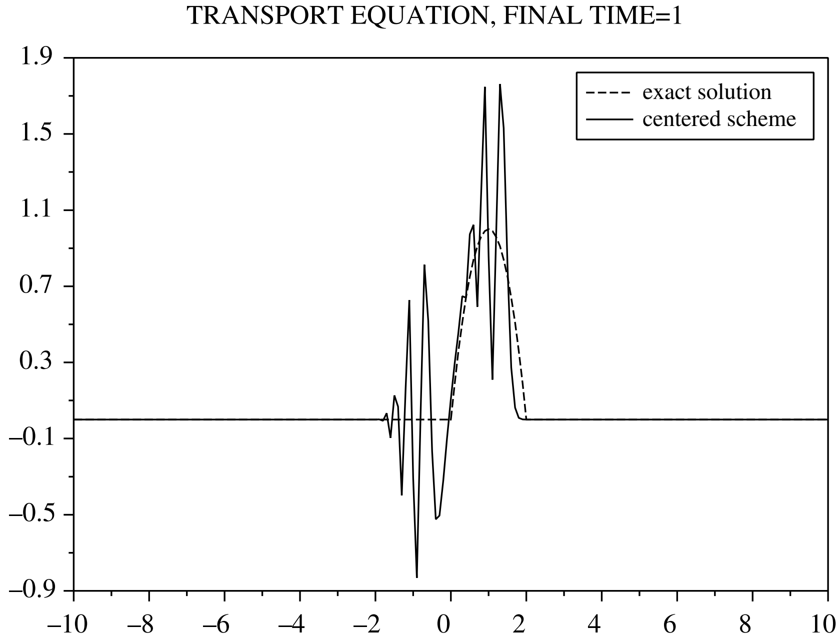


Figure 1.10. Explicit centred scheme for the advection equation with $\Delta t = 0.9\Delta x$, $V = 1$, $\nu = 0$.

For the advection equation (that is, (1.27) with $\nu = 0$), the explicit scheme (1.30) may be rewritten

$$u_j^{n+1} = \frac{V\Delta t}{2\Delta x}u_{j-1}^n + u_j^n - \frac{V\Delta t}{2\Delta x}u_{j+1}^n. \tag{1.34}$$

This scheme leads to the oscillations in Figure 1.10 under the same experimental conditions as the bottom of Figure 1.9. We see that u_j^{n+1} is never (no matter what Δt) a convex combination of u_{j-1}^n, u_j^n, and u_{j+1}^n. Therefore, there cannot be a discrete maximum principle for this scheme, which is an additional indication of its instability (a rigorous proof will be given in lemma 2.3.1). This instability occurs because, in the explicit scheme (1.34), we have chosen to use a centred approximation to the convective term. We can, however, make this term one-sided as we have done for the time derivative. Two choices are possible: weighting to the right or left. The sign of the velocity V is crucial: here we assume that $V > 0$ (a symmetric argument is possible if $V < 0$). For $V > 0$, the weighting to the right is called **downwinding**: we obtain

$$V\frac{\partial u}{\partial x}(t_n, x_j) \approx V\frac{u_{j+1}^n - u_j^n}{\Delta x}$$

we try to find 'information' by following the current. This leads to a 'disastrous' downwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + V \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0 \tag{1.35}$$

which is as unstable as the centred scheme. On the other hand, the **upwinding** (which is to the left if $V > 0$), looks for 'information' by going against the current

$$V \frac{\partial u}{\partial x}(t_n, x_j) \approx V \frac{u_j^n - u_{j-1}^n}{\Delta x}$$

leading to an explicit upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + V \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \tag{1.36}$$

which gives the results of Figure 1.11. We verify easily that the scheme (1.36) is stable

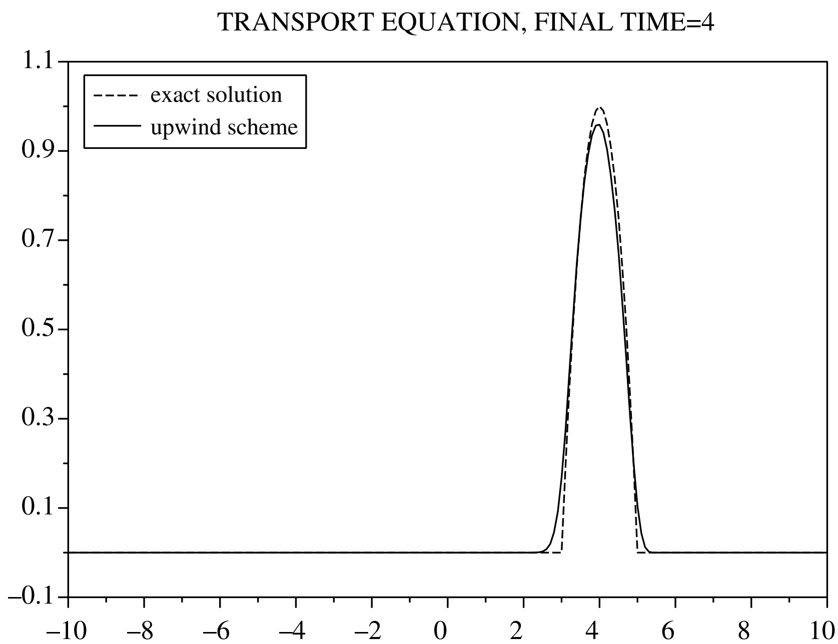


Figure 1.11. Explicit upwind scheme for the advection equation with $\Delta t = 0.9 \Delta x$, $V = 1$.

under a new CFL condition (different from the preceding CFL condition (1.31))

$$|V|\Delta t \leq \Delta x. \tag{1.37}$$

Indeed, we can rewrite (1.36) in the form

$$u_j^{n+1} = \frac{V\Delta t}{\Delta x} u_{j-1}^n + \left(1 - \frac{V\Delta t}{\Delta x}\right) u_j^n,$$

which shows that, if condition (1.37) is satisfied, u_j^{n+1} is a convex combination of u_{j-1}^n and u_j^n. Consequently, the one-sided upwind scheme (1.36) satisfies a discrete maximum principle, which implies conditional stability. The idea of **upwinded methods is another major idea in numerical analysis**. It is particularly important in all fluid mechanics problems where it was first discovered, but it appears in many other models.

The conclusion of this study on the advection equation is that for the convection–diffusion model with a small diffusion constant ν, we must upwind the convective term and obey the CFL condition (1.37) rather than (1.31). With this price we can improve the results of Figure 1.9.

Exercise 1.4.2 Show that if (1.37) is not satisfied, the upwind scheme (1.36) for the advection equation is unstable for the initial data $u_j^0 = (-1)^j$.

Exercise 1.4.3 Write an explicit scheme centred in space for the wave equation (1.18) in one space dimension and without source term. Specify how to start the iterations in time. Verify the existence of a discrete cone of dependence analogous to the continuous one shown in figure 1.3. Deduce that, if this scheme converges, the time and space steps must satisfy the (CFL-like) condition $\Delta t \leq \Delta x$.

The conclusions of this section are numerous and will feed the reflections of the subsequent chapter. First of all, all 'reasonable' numerical schemes do not work, far from it. We meet stability problems (without even considering convergence) which require us to analyse these schemes: this is the *raison d'être* of numerical analysis which reconciles practical objectives and theoretical studies. Finally, the 'good' numerical schemes must have a certain number of properties (for example, the discrete maximum principle, or upwinding) which are the expression (at the discrete level) of the physical properties or the mathematics of the PDE. **We cannot therefore skimp on a good understanding of the physical modelling and of the mathematical properties of the models if we want to have good numerical simulations.**

1.5 Remarks on mathematical models

We finish this chapter with a number of definitions which allow the reader to understand the terms in classical works on numerical analysis.

1.5.1 The idea of a well-posed problem

Definition 1.5.1 *We use the term* **boundary value problem** *to refer to a PDE equipped with boundary conditions on the entire boundary of the domain in which it is posed.*

For example, the Laplacian (1.20) is a boundary value problem. Conversely, the ordinary differential equation

$$\begin{cases} \dfrac{dy}{dt} = f(t,y) & \text{for } 0 < t < T \\ y(t=0) = y_0 \end{cases} \tag{1.38}$$

is not a boundary value problem as it is posed on an interval $(0,T)$, with $0 < T \le +\infty$, it only has 'boundary' conditions at $t = 0$ (and not at $t = T$).

Definition 1.5.2 *We say* **Cauchy problem** *to mean a PDE where, for at least one variable (usually time t), the 'boundary' conditions are initial conditions (that is, only hold at the boundary $t = 0$, and not at $t = T$).*

For example, the ordinary differential equation (1.38) is a Cauchy problem, but the Laplacian (1.20) is not (no matter which component of the space variable x we make to play the role of time).

Numerous models are, at the same time, boundary value problems and Cauchy problems. Thus, the heat flow equation (1.8) is a Cauchy problem with respect to the time variable t and a boundary value problem with respect to the space variable x. All the models we shall study in this course belong to one of these two categories of problem.

The fact that a mathematical model is a Cauchy problem or a boundary value problem does not automatically imply that it is a 'good' model. The expression **good model** is not used here in the sense of the physical relevance of the model and of its results, but in the sense of its mathematical coherence. As we shall see, this mathematical coherence is a necessary condition before we can consider numerical simulations and physical interpretations. The mathematician Jacques Hadamard gave a definition of what is a 'good' model, while speaking about **well-posed problems** (an ill-posed problem is the opposite of a well-posed problem). We denote by f the data (the right-hand side, the initial conditions, the domain, etc.), u the solution sought, and $\mathcal{A}$ 'the operator' which acts on u. We are using abstract notation, $\mathcal{A}$ denotes simultaneously the PDE and the type of initial or boundary conditions. The problem is therefore to find u, the solution of

$$\mathcal{A}(u) = f. \tag{1.39}$$

Definition 1.5.3 *We say that problem (1.39) is* **well-posed** *if for all data f it has a unique solution u, and if this solution u depends continuously on the data f.*

Let us examine Hadamard's definition in detail: it contains, in fact, three conditions for the problem to be well-posed. First, a solution must at least exist: this is the least we can ask of a model supposed to represent reality! Second, the solution must be unique: this is more delicate since, while it is clear that, if we want to predict tomorrow's weather, it is better to have 'sun' or 'rain' (with an exclusive 'or') but not both with equal chance, there are other problems which 'reasonably' have several or an infinity of solutions. For example, problems involving finding the best route often have several solutions: to travel from the South to the North Pole then any meridian will do, likewise, to travel by plane from Paris to New York, your travel agency sometimes makes you go via Brussels or London, rather than directly, because it can be more economic. Hadamard excludes this type of problem from his definition since the multiplicity of solutions means that the model is indeterminate: to make the final choice between all of those that are best, we use another criterion (which has been 'forgotten' until now), for example, the most practical or most comfortable journey. This is a situation of current interest in applied mathematics: when a model has many solutions, we must add a selection criterion to obtain the 'good' solution (see, for a typical example, problems in gas dynamics [23]). Third, and this is the least obvious condition *a priori*, the solution must depend continuously on the data. At first sight, this seems a mathematical fantasy, but it is crucial from the perspective of **numerical approximation**. Indeed, numerically calculating an approximate solution of (1.39) amounts to perturbing the data (when continuous becomes discrete) and solving (1.39) for the perturbed data. If small perturbations of the data lead to large perturbations of the solution, there is no chance that the numerical approximation will be close to reality (or at least to the exact solution). Consequently, this continuous dependence of the solution on the data is an absolutely necessary condition for accurate numerical simulations. We note that this condition is also very important from the physical point of view since measuring apparatus will not give us absolute precision: if we are unable to distinguish between two close sets of data which can lead to very different phenomena, the model represented by (1.39) has no predictive value, and therefore is of almost no practical interest.

We finish by acknowledging that, at this level of generality, the definition (1.5.3) is a little fuzzy, and that to give it a precise mathematical sense we should say in which function spaces we put the data or look for the solution, and which norms or topologies we use for the continuity. It is not uncommon that changing the space (which can appear anodyne) implies very different properties of existence or uniqueness!

Exercise 1.5.1 The point of this exercise is to show that the Cauchy problem for the Laplacian is ill-posed. Take a two-dimensional domain $\Omega = (0,1) \times (0, 2\pi)$. We consider the following Cauchy problem in x and boundary value problem in y

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{in } \Omega \\ u(x,0) = u(x,2\pi) = 0 & \text{for } 0 < x < 1 \\ u(0,y) = 0, \ \dfrac{\partial u}{\partial x}(0,y) = -e^{-\sqrt{n}} \sin(ny) & \text{for } 0 < y < 2\pi \end{cases}$$

Verify that $u(x,y) = (e^{-\sqrt{n}}/n) \sin(ny) \mathrm{sh}(nx)$ is a solution. Show that the initial condition and all its derivatives at $x = 0$ converge uniformly to 0, while, for all $x > 0$, the solution $u(x,y)$ and all its derivatives are unbounded as n tends to infinity.

1.5.2 Classification of PDEs

Definition 1.5.4 *The* **order** *of a partial differential equation is the order of the highest derivative in the equation.*

For example, the Laplacian (1.20) is a second order equation, while the plate equation (1.24) is a fourth order equation. We often distinguish between the order with respect to the time variable t and with respect to the space variable x. Therefore, we say that heat flow equation (1.8) is first order in time and second order in space; likewise, the wave equation (1.18) is second order in space-time.

In order to understand the vocabulary often used with PDEs, that is, **elliptic, parabolic, or hyperbolic**, we shall briefly classify linear, second order PDEs acting on real functions of two real variables $u(x, y)$ (we shall not carry out a systematic classification for all PDEs). Such an equation is written

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g. \tag{1.40}$$

For simplicity we assume that the coefficients a, b, c, d, e, f are constant.

Definition 1.5.5 *We say that the equation (1.40) is elliptic if $b^2 - 4ac < 0$, parabolic if $b^2 - 4ac = 0$, and hyperbolic if $b^2 - 4ac > 0$.*

The origin of this vocabulary is in the classification of conic sections, from which Definition 1.5.5 is copied. Indeed, it is well-known that the second degree equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

defines a plane curve which is (except in some degenerate cases) an ellipse if $b^2 - 4ac < 0$, a parabola if $b^2 - 4ac = 0$, and a hyperbola if $b^2 - 4ac > 0$.

If we apply Definition 1.5.5 to the various second order models we have stated in this chapter (replacing the variables (x, y) by the variables (t, x) in one space dimension), we conclude that **the heat flow equation is parabolic** (as is the convection–diffusion equation), that **the Laplacian is elliptic**, and that **the wave equation is hyperbolic**. A suitable generalisation of this definition allows us to check that the advection equation is hyperbolic, and that the Stokes, elasticity, and plate equations are elliptic. In general, stationary problems (independent of time) are modelled by elliptic PDEs, while evolution problems are modelled by parabolic or hyperbolic PDEs.

We shall see later that boundary value problems are well posed for elliptic PDEs, while problems which are Cauchy in time and boundary value problems in space are well-posed for parabolic or hyperbolic PDEs. There are therefore important differences in behaviour between these two types of equation.

Remark 1.5.6 The elliptic, hyperbolic or parabolic character of the equations (1.40) is not modified by a change of variable. Let $(x, y) \to (X, Y)$ be such a change of variable which is nonsingular, that is, its Jacobian $J = X_x Y_y - X_y Y_x$ is not zero (denoting by Z_z the derivative of Z with respect to z). A simple but tedious calculation shows that (1.40) becomes

$$A\frac{\partial^2 u}{\partial X^2} + B\frac{\partial^2 u}{\partial X \partial Y} + C\frac{\partial^2 u}{\partial Y^2} + D\frac{\partial u}{\partial X} + E\frac{\partial u}{\partial Y} + Fu = G,$$

with $A = aX_x^2 + bX_xX_y + cX_y^2$, $B = 2aX_xY_x + b(X_xY_y + X_yY_x) + 2cX_yY_y$, $C = aY_x^2 + bY_xY_y + cY_y^2$, and we verify that $B^2 - 4AC = J^2(b^2 - 4ac)$. In particular, a suitable change of variables allows us to simplify the PDE (1.40) and return it to its 'canonical' form. Thus, any elliptic equation can be reduced to the Laplacian $\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$, any parabolic equation to the heat flow equation $\frac{\partial}{\partial X} - \frac{\partial^2}{\partial Y^2}$, and any hyperbolic equation to the wave equation $\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2}$. •

Remark 1.5.7 It is well-known that the general conic equation has a number of degenerate cases when it no longer describes a cone but a set of lines. The same situation can hold with the PDE (1.40). For example, the equation $\frac{\partial^2 u}{\partial x^2} = 1$ with $a = 1$ and $b = c = d = e = f = 0$ is not parabolic in two dimensions (even though $b^2 - 4ac = 0$) but elliptic in one dimension (the variable y plays no role here). It is therefore necessary to be careful before deciding on the type of these 'degenerate' equations. •