Équations aux dérivées partielles – SOLUTIONS

On va faire des calculs et si c'est nécessaire on va donner des explications additionnelles

1.
$$\begin{aligned}
\nabla\cdot(p\mathbf{v}) &= \frac{\partial}{\partial x}(pv_x)+\frac{\partial}{\partial y}(pv_y)+\frac{\partial}{\partial z}(pv_z)\\
&= v_x\frac{\partial}{\partial x}p+v_y\frac{\partial}{\partial y}p+v_z\frac{\partial}{\partial z}p+p(\frac{\partial}{\partial x}v_x+\frac{\partial}{\partial y}v_y+\frac{\partial}{\partial z}v_z)\\
&= \nabla p\cdot\mathbf{v}+p\nabla\cdot\mathbf{v}.
\end{aligned}$$

$$\begin{aligned}
\nabla\times(p\mathbf{v}) &= (\frac{\partial}{\partial y}(pv_z)-\frac{\partial}{\partial z}(pv_y),\frac{\partial}{\partial z}(pv_x)-\frac{\partial}{\partial x}(pv_z),\frac{\partial}{\partial x}(pv_y)-\frac{\partial}{\partial y}(pv_x))^T\\
&= \begin{pmatrix}\frac{\partial}{\partial y}p\,v_z-\frac{\partial}{\partial z}p\,v_y+p(\frac{\partial}{\partial y}v_z-\frac{\partial}{\partial z}v_y)\\ \frac{\partial}{\partial z}p\,v_x-\frac{\partial}{\partial x}p\,v_x+p(\frac{\partial}{\partial z}v_x-\frac{\partial}{\partial x}v_z)\\ \frac{\partial}{\partial x}p\,v_y-\frac{\partial}{\partial y}p\,v_x+p(\frac{\partial}{\partial x}v_y-\frac{\partial}{\partial y}v_x)\end{pmatrix}\\
&= \nabla p\times\mathbf{v}+p(\nabla\times\mathbf{v}).
\end{aligned}$$

$$\begin{aligned}
\nabla\times(\nabla p) &= \nabla\times(\frac{\partial}{\partial x}p,\frac{\partial}{\partial y}p,\frac{\partial}{\partial z}p)\\
&= (\frac{\partial^2}{\partial y\partial z}p-\frac{\partial^2}{\partial y\partial z}p,\frac{\partial^2}{\partial x\partial z}p-\frac{\partial^2}{\partial x\partial z}p,\frac{\partial^2}{\partial x\partial y}p-\frac{\partial^2}{\partial x\partial y}p)=0\\
\nabla\cdot(\nabla\times\mathbf{v}) &= \nabla\cdot(\frac{\partial}{\partial y}v_z-\frac{\partial}{\partial z}v_y\,,\frac{\partial}{\partial z}v_x-\frac{\partial}{\partial x}v_z\,,\frac{\partial}{\partial x}v_y-\frac{\partial}{\partial y}v_x)\\
&= \frac{\partial^2}{\partial x\partial y}v_z-\frac{\partial^2}{\partial x\partial z}v_y+\frac{\partial^2}{\partial y\partial z}v_x-\frac{\partial^2}{\partial x\partial y}v_z+\frac{\partial^2}{\partial x\partial z}v_y-\frac{\partial^2}{\partial y\partial z}v_x=0
\end{aligned}$$

2. Dans la formule de Green on remplace d'abord u par uv ce qui conduit à
$$\int_\Omega\frac{\partial u}{\partial x_i}v\,d\mathbf{x}=-\int_\Omega u\frac{\partial v}{\partial x_i}\,d\mathbf{x}+\int_{\partial\Omega}uvn_i\,d\sigma.$$
On remplacera ensuite u par $\frac{\partial u}{\partial x_i}$, $i=1,..,N$ et puis on sommera les relations.

3. On va utiliser la formule $\nabla\cdot(p\mathbf{v})=p\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla p$ et la formule de Green.
$$\begin{aligned}
\int_\Omega(\nabla\cdot\mathbf{w})\phi d\mathbf{x} &= \int_\Omega\nabla\cdot(\phi\mathbf{w})d\mathbf{x}-\int_\Omega\mathbf{w}\cdot\nabla\phi d\mathbf{x}\\
&= -\int_\Omega\mathbf{w}\cdot\nabla\phi d\mathbf{x}+\int_\Omega\sum_{i=1}^3\frac{\partial}{\partial x_i}(\phi\mathbf{w})d\mathbf{x}\\
&= -\int_\Omega\mathbf{w}\cdot\nabla\phi d\mathbf{x}+\sum_{i=1}^3\int_{\partial\Omega}(\phi\mathbf{w})\cdot n_i(x)d\sigma\\
&= -\int_\Omega\mathbf{w}\cdot\nabla\phi d\mathbf{x}+\int_{\partial\Omega}(\mathbf{w}\cdot\mathbf{n})\phi d\sigma.
\end{aligned}$$
En prenant $\phi=1$, on a $\nabla\phi=0$ et donc
$$\int_\Omega\nabla\cdot\mathbf{w}d\mathbf{x}=\int_{\partial\Omega}\mathbf{w}\cdot\mathbf{n}d\sigma.$$

4. On utilise la formule $\nabla\times(p\mathbf{v})=p\nabla\times\mathbf{v}+\nabla p\times\mathbf{v}$ et la formule de Green.
$$\begin{aligned}
\int_\Omega(\nabla\times\mathbf{w})\phi d\mathbf{x} &= \int_\Omega\nabla\times(\phi\mathbf{w})d\mathbf{x}-\int_\Omega\nabla\phi\times\mathbf{w}d\mathbf{x}\\
&= \int_\Omega\mathbf{w}\times\nabla\phi d\mathbf{x}+\int_\Omega(\frac{\partial}{\partial y}(\phi w_3)-\frac{\partial}{\partial z}(\phi w_2),\frac{\partial}{\partial z}(\phi w_1)-\frac{\partial}{\partial x}(\phi w_3),\frac{\partial}{\partial x}(\phi w_2)-\frac{\partial}{\partial y}(\phi w_1))^Td\mathbf{x}\\
&= \int_\Omega\mathbf{w}\times\nabla\phi d\mathbf{x}+\int_{\partial\Omega}(\phi(w_3n_2-w_2n_3),\phi(w_1n_3-w_3n_1),\phi(w_2n_1-w_1n_2))^T\\
&= \int_\Omega\mathbf{w}\times\nabla\phi d\mathbf{x}+\int_{\partial\Omega}(\mathbf{n}\times\mathbf{w})\phi d\sigma\\
&= \int_\Omega\mathbf{w}\times\nabla\phi d\mathbf{x}-\int_{\partial\Omega}(\mathbf{w}\times\mathbf{n})\phi d\sigma
\end{aligned}$$

On prend encore une fois $\phi = 1$ et on obtient:

$$\int_\Omega (\nabla \times \mathbf{w})\phi d\mathbf{x} = \int_{\partial\Omega} \mathbf{w} \times \mathbf{n} d\sigma.$$

5. On vérifiera que u satisafait bien l'équation ainsi que les conditions aux limites. Il est évident que $u(0) = u(1) = 0$. On évalue d'abord la dérivée première et ensuite la dérivée seconde

$$\begin{array}{l} u'(x) = \int_0^1 f(s)(1-s)ds - \int_0^x f(s)ds - xf(x) + xf(x) = \int_0^1 f(s)(1-s)ds - \int_0^x f(s)ds \\ u''(x) = (u'(x))' = -f(x) \end{array}$$

donc u est bien solution du problème aux limites.

6. On va calculer d'abord quelques dérivées partielles qui vont être utiles.

$$\begin{array}{l} \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta, \ \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta \\ \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r}, \ \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r} \end{array}$$

Maintenant on calcule le Laplacien

$$\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right) \\
&= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta\right)\frac{\partial r}{\partial x} + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial u}{\partial r}\cos\theta\right)\frac{\partial \theta}{\partial x} - \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\partial r}{\partial x} - \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\partial \theta}{\partial x} \\
&+ \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta\right)\frac{\partial r}{\partial y} + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial u}{\partial r}\sin\theta\right)\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\partial r}{\partial y} + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\partial \theta}{\partial y} \\
&= \frac{\partial^2 u}{\partial r^2}\cos^2\theta - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} \\
&- \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \\
&+ \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} \\
&+ \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}
\end{aligned}$$