Ex 1 SUJET A

1.) $\varepsilon_j^n = \dfrac{u(x_j, t^{n+1}) - u(x_j, t^{n-1})}{2\Delta t}_{(1)} - \gamma \dfrac{u(x_{j+1}, t^{n+1}) - 2u(x_j, t^{n+1}) + u(x_{j-1}, t^{n+1})}{\Delta x^2}_{(2)}$ 1

On développe en temps autour de t^{n-1} et en espace autour de x_j

On obtient

(1) $= \dfrac{\partial u}{\partial t}(x_j, t^{n+1}) + O(\Delta t)$ 1

(2) $= -\gamma \dfrac{\partial^2 u}{\partial x^2}(x_j, t^{n+1}) + O(\Delta x^2)$ 1

$\Rightarrow \varepsilon_j^n = \underbrace{\dfrac{\partial u}{\partial t}(x_j, t^{n+1}) - \gamma \dfrac{\partial^2 u}{\partial x^2}(x_j, t^{n+1})}_{=0} + O(\Delta t) + O(\Delta x^2)$ (u sol de l'équation

$\Rightarrow$ le schéma est d'ordre 1 en temps et 2 en espace 1

(erreur de troncature + conclusion)

2) On remplace le mode de Fourier dans le schéma

$u_j^n = A(k)^n e^{2i\pi k j \Delta x}$ 2 pour avoir simplifié 1

$\dfrac{A(k)^2 - 1}{2\Delta t} - \gamma A^2(k) \cdot \dfrac{(2\cos(2k\pi\Delta x) - 2)}{\Delta x^2} = 0 \iff A(k^2)\left(1 + 8\dfrac{\gamma\Delta t}{\Delta x^2}\sin^2(k\pi\Delta x)\right) = 1$

$A^2(k) = \dfrac{1}{1 + 8\dfrac{\gamma\Delta t}{\Delta x^2}\sin^2(k\pi\Delta x)} \leq 1 \quad \forall k \in \mathbb{Z}$) 1

$\Rightarrow$ le schéma est inconditionnellement stable

Ex 2 1. On remplace le mode de Fourier

$\dfrac{A(k) - 1}{\Delta t} + V A(k) \cdot \dfrac{2i\sin(2k\pi\Delta x)}{2\Delta x} - \dfrac{V^2\Delta t}{2\Delta x^2}(2\cos(2k\pi\Delta x) - 2) = 0$ 1

$A(k)\left(1 + i\dfrac{V\Delta t}{\Delta x}\sin(2k\pi\Delta x)\right) = 1 - 2\dfrac{V^2\Delta t^2}{\Delta x^2}\sin^2(k\pi\Delta x)$

Notation $\alpha = \dfrac{V\Delta t}{\Delta x}$

$s_k = \sin(k\pi\Delta x)$

$c_k = \cos(k\pi\Delta x)$

$A(k) = \dfrac{1 - 2\alpha^2 s_k^2}{1 + 2i\alpha s_k c_k}$, $|A(k)|^2 = \dfrac{1 - 4\alpha^2 s_k^2 + 4\alpha^4 s_k^4}{1 + 4\alpha^2 s_k^2 c_k^2} \overset{?}{\leq} 1$ 1

$\cancel{1} - 4\alpha^2 s_k^2 + 4\alpha^4 s_k^4 \leq \cancel{1} + 4\alpha^2 s_k^2 c_k^2 \iff -1 + \alpha^2 s_k^2 \leq 1 - s_k^2$ 1

$s_k^2(1 + \alpha^2) \leq 2$ mais si $\alpha \leq 1$ ou bien $\dfrac{V\Delta t}{\Delta x} \leq 1$ 1

2) $\varepsilon_j^n = \dfrac{\partial u}{\partial t}(x_j, t^n) + O(\Delta t) + V\left(\dfrac{\partial u}{\partial x}(x_j, t^{n+1}) + O(\Delta x^2)\right) - \dfrac{V^2\Delta t}{\Delta x^2}\left(\dfrac{\partial^2 u}{\partial x^2}(x_j, t^n) + O(\Delta x^2)\right)$ 2

$\dfrac{\partial u}{\partial x}(x_j, t^n) + O(\Delta t)$

$= \underbrace{\dfrac{\partial u}{\partial t}(x_j, t^n) + V\dfrac{\partial u}{\partial x}(x_j, t^n)}_{=0} + O(\Delta t) + O(\Delta x^2) = O(\Delta t) + O(\Delta x^2)$ 2

$=0$ (u sol de l'équation

consistant avec l'ep d'advection ordre 1 en temps, 2 en espace

Ex 1 1.

$$\varepsilon_j^n = \underbrace{\frac{2u(x_j,t^{n+1}) - 3u(x_j,t^n) + u(x_j,t^{n-1})}{\Delta t}}_{①} - \gamma \underbrace{\frac{u(x_{j+1},t^{n+1}) - 2u(x_j,t^{n+1}) + u(x_{j-1},t^{n+1})}{\Delta x^2}}_{②}$$ (1)

Developpement autour de t^{n+1} :

$$u(x_j,t^n) = u(x_j,t^{n+1}) - \Delta t \frac{\partial u}{\partial t}(x_j,t^{n+1}) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x_j,t^{n+1}) + O(\Delta t^3)$$ (1)

$$u(x_j,t^{n-1}) = u(x_j,t^{n+1}) - 2\Delta t \frac{\partial u}{\partial t}(x_j,t^{n+1}) + 2\Delta t^2 \frac{\partial^2 u}{\partial t^2}(x_j,t^{n+1}) + O(\Delta t^3)$$ (1)

$$\left.\begin{array}{l} ① = \frac{\partial u}{\partial t}(x_j,t^{n+1}) + O(\Delta t) \\ ② = -\gamma \frac{\partial^2 u}{\partial x^2}(x_j,t^{n+1}) + O(\Delta x^2) \end{array}\right\} \Rightarrow \varepsilon_j^n = O(\Delta t) + O(\Delta x^2)$$ (1)

schéma d'ordre 1 en temps et 2 en espace

2. On remplace le mode de Fourier et on simplifie :

$$2A^2(k) - 3A(k) + 1 - \alpha A^2(k)\underbrace{(2\cos(2k\pi\Delta x) - 2)}_{-4\sin^2(k\pi\Delta x)} = 0, \quad \alpha = \frac{\gamma\Delta t}{\Delta x^2}, \quad S_k = \sin(k\pi\Delta x)$$ (1)

$$(2 + 4\alpha)A^2(k) - 3A(k) + 1 = 0$$

$$A_{1/2}(k) = \frac{3 \pm \sqrt{9 - 4(2+4\alpha)}}{2(2+4\alpha)} = \frac{3 \pm \sqrt{1-16\alpha}}{4(1+2\alpha)}$$ (1).

si $1 - 16\alpha < 0 \rightarrow$ racines complexes conjuguées (1)

$$|A_1(k)| = |A_2(k)| = \sqrt{\frac{1}{2+4\alpha}} < 1 \Rightarrow \text{schéma stable}$$

si $1 - 16\alpha \geq 0 \rightarrow$ racines réelles (et positives) (1)

$$3 \pm \sqrt{1-16\alpha} \leq 4 + 8\alpha \iff \pm\sqrt{1-16\alpha} \leq 1 + 8\alpha$$ (evident) $\Rightarrow$ schéma stable

Ex 2 1. Mode de Fourier (1)

$$2A(k) - 2\cos(2k\pi\Delta x) + \alpha A(k) \cdot 2i\sin(k\pi\Delta x) = 0$$

$$A(k)(1 + i\alpha\sin(k\pi\Delta x)) = \cos(2k\pi\Delta x) \Rightarrow |A(k)|^2 = \frac{\cos^2(2k\pi\Delta x)}{1 + \alpha^2\sin^2(2k\pi\Delta x)} \leq 1$$ $\rightsquigarrow$ inconditionnellement stable

(1) (1) (1)

2. Erreur de troncature

$$\varepsilon_j^n = \underbrace{\frac{2u(x_j,t^{n+1}) - u(x_{j+1},t^n) - u(x_{j-1},t^n)}{2\Delta t}}_{①} + V\underbrace{\frac{u(x_{j+1},t^{n+1}) - u(x_{j-1},t^{n+1})}{2\Delta x}}_{②}$$ (1)

Développement autour de t^{n+1} en temps et x_j en espace

$$u(x_{j+1},t^n) + u(x_{j-1},t^n) = 2u(x_j,t^n) + \Delta x^2\frac{\partial^2 u}{\partial x^2}(x_j,t^n) + O(\Delta x^4)$$ (1)

$$= 2u(x_j,t^{n+1}) - 2\Delta t\frac{\partial u}{\partial t}(x_j,t^n) + O(\Delta t^2) + O(\Delta x^2)$$

$$\Rightarrow ① = \frac{\partial u}{\partial t}(x_j,t^n) + O(\Delta t) + O\left(\frac{\Delta x^2}{\Delta t}\right)$$ (1)

$$② = V\frac{\partial u}{\partial x}(x_j,t^n) + O(\Delta x^2)$$

$$\Rightarrow \varepsilon_j^n = O(\Delta t) + O\left(\frac{\Delta x^2}{\Delta t}\right) + O(\Delta x^2) \Rightarrow \text{si } \frac{\Delta x}{\Delta t} = \text{constant} \rightarrow$$ (1)

le schéma est consistant d'ordre 1 en temps et espace