Midterm Exam

AI 75853 Biometric Information Processing, Fall 2020
Dept. of Artificial Intelligence, PNU
Oct. 22. 00:00 - 23:59

I. REMARK

- This is an open book exam.
- There are a total of 100 points in the exam. Each problem specifies its point total.
- You must SHOW YOUR WORK to get full credit.

II. PROBLEM SET

1) [10 points] Suppose that $x[n] = x[n+L]$, and its samples are arranged column-wise into a matrix $A \in \mathbb{C}^{p\times q}$ so that
$$A_{mn} = x[m + (n-1)p], 1 \le m \le p, 1 \le n \le q.$$
Show that regardless of the size of p and q, $rank(A) \le L$.

2) [5 points] A square matrix $P \in \mathbb{C}^{n\times n}$ is called an oblique projection if $P^2 = P$. What are the eigenvalues of an oblique projection? Show that $I - P$ is an oblique projection as well.

3) [10 points] Let $A \in \mathbb{C}^{n\times n}$ be a square self-adjoint complex matrix, namely, $A = A^*$.
   a) Prove the all eigenvalues of A are real.
   b) Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two eigenvectors of A corresponding to two distinct eigenvalues λ_1 and λ_2. Prove that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal.
   c) Let $B \in \mathbb{C}^{m\times n}$. Prove that $R(B) = R(BB^*)$.

4) [10 points] Suppose that random variables x and y have the joint pdf:
$$f_{xy}(u, v) = \begin{cases} 4u^2, & \text{if } 0 < u < v < 1 \\ 0, & \text{elsewhere} \end{cases}$$
   a) Find $E[xy]$.
   b) Find $f_y(v)$. Be sure to specify it for all values of v.
   c) Find $f_{x|y}(u|v)$. Be sure to specify where it is undefined, and where it is zero.

5) [10 points] An LTI system is characterized by the impulse response
$$h[n] = \sqrt{3}\frac{\sin(\frac{1}{3}\pi n)}{\pi n}$$
   a) Compute the DTFT $H(w)$. What kind of a filter is that?
   b) Let $x[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$ and $y = h * x$. Compute and sketch the DTFT Y(w) of the output.

6) [10 points] Let $x = \exp(y)$, where y has the standard normal distribution. Find linear MMSE estimator $\hat{E}[x|y]$ and calculate the MSE, $E(x - \hat{E}[x|y])^2$. Verify your answer using Matlab.

7) [15 points] Let x be a sequence. Consider the deterministic autocorrelation sequence defined by
$$a_x[k] = \sum_{n\in Z} x[n]x[n+k]^*$$
   a) Prove or find a counter examples for the following statements: $|a_x[k]| \le a_x[0]$ for every $k \in Z$.
   b) A signal x is received by two receivers (with separate antennas), such that the received signals x_1 and x_2 are
$$x_1[n] = \alpha_1 x[n - n_1], x_2[n] = \alpha_2 x[n - n_2]$$
   where the constants $\alpha_1, \alpha_2 \in R$ are unknown gain coefficients and $n_1, n_2 \in Z$ are unknown time delays. Derive an algorithm to determine the time delay $\delta = n_2 - n_1$ and the gain ratio $\rho = \frac{\alpha_1}{\alpha_2}$ given the sequences x_1 and x_2 as inputs.
   c) Implement the algorithm you suggested in b). As input, use a signal $x \in \mathbb{R}^{100}$ where $x[n] = 0$ for $30 < n \le 100$. Set random shifts $n_1, n_2 > 0$, random gains α_1 and α_2. The output will be the estimated delay δ and gain ration ρ.

8) [15 points] Use Matlab.
   a) Write two Matlab programs to implement the downsampling and upsampling operators. Each function should take two inputs: a vector as the input signal, and an integer as the sampling factor.
   b) Obtain a lowpass filter of length 30 and cut-off frequency $w_c = \pi/3$ by using the Matlab command $fir1$. Using command $freqz$, plot the frequency response of this filter together with its downsampled and upsampled versions by 2. Verify that these plots match with the theory of spectrum change by downsampling and upsampling.
   c) Create a test input signal by adding three sinusoids with the frequencies 800 Hz, 1600 Hz and 2400 Hz, where the underlying sampling frequency is 8192 Hz, and amplitudes 1, 0.2, and 0.4, respectively. Show the signal by downsampling the input by 3. Then, filter the downsampled signal

with the above lowpass filter with cutoff frequency $w_c = \pi/3$. Plot the Fourier transforms of all signals and clearly identify any aliased components.

9) [15 points] Define a set of vectors $p_0,\ p_1,\ \ldots,\ p_{N-1} \in \mathbb{R}^N$ by

$$p_k = [1^k\ 2^k\ 3^k\ \ \ldots\ \ n^k]^T,\ \ k = 0, 1, \ldots, N-1$$

S_d is defined as the span of the first d of those vectors:

$$S_d = span\{p_0, p_1, \ldots, p_{d-1}\},\ \ 1 \leq d \leq N$$

a) Write a MATLAB function that applies the Gram-Schmidt orthogonalization on the first d vectors $\{p_0, \ldots, p_{d-1}\}$ to compute an orthonormal basis for S_d.

b) With the orthonormal bases computed above, given an input signal $x \in \mathbb{R}^N$, compute the successive orthogonal projections of x onto the subspaces S_1, $S_2,\ \ldots,\ S_N$. Plot the orthogonal projections and the error signals. (Pick x to be one line of your favorite image)