Homework I

Due: Sep. 25. (Fri) 23:59 PM

I. REMARK

- Reading materials: PPT materials, book (Foundations of Signal Processing, Ch 2.1-2.5)
- Please write whole processes to justify your answer.

II. PROBLEM SET

1) Two vectors are given as

$$\mathbf{h} = \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

Let $y[n] = x[n] * h[n]$ where $*$ denotes the convolution operator. What is $y[n] \in \mathbb{R}^{7\times 1}$? The expression can be converted to the matrix equation, $\mathbf{y} = H\mathbf{x}$, $H \in \mathbb{R}^{7\times 3}$. What is H? What is the rank of H, rank(H)? Now, assume that $\mathbf{y}$ and H are priorly known but $\mathbf{x}$ is unknown. Use pseudo-inverse to obtain $\mathbf{x}$. Check everything in MATLAB.

2) In Problem 1, the columns of H are independent? Justify your answer.

3) In Problem 1, $\mathbf{y}$ is in the range of H, $\mathbf{y} \in R(H)$. Assume that $\mathbf{y}$ is data and H is system. The data are distorted due to noise $\mathbf{e}$ as $\tilde{\mathbf{y}} = \mathbf{y} + \mathbf{e}$, where $\tilde{\mathbf{y}}$ denotes the distorted data. The noise is given as

$$\mathbf{e} = \begin{bmatrix} e[0] \\ e[1] \\ e[2] \\ e[3] \\ e[4] \\ e[5] \\ e[6] \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.2 \\ 0.1 \\ 0.3 \\ 0.1 \\ -0.2 \\ 0.1 \end{bmatrix}.$$

Find the estimate $\tilde{\mathbf{x}}$ from $\tilde{\mathbf{y}}$ using MATLAB.

4) In Problem 3, note that estimate $\tilde{\mathbf{x}}$ and ground-truth $\mathbf{x}$ are in vector space $\mathbb{R}^7$. Define any norm in this space, and compute the distance (metric) between two vectors, $d = ||\tilde{\mathbf{x}} - \mathbf{x}||$.

5) In Problem 1, assume that inner product is defined as $< \mathbf{a}, \mathbf{b} >= \mathbf{a}^T\mathbf{b}$. Then, derive that H^T is the adjoint of the H.

6) Assume that $\mathbf{y} = A\mathbf{x}$, $A \in \mathbb{R}^{M\times N}$. When rank($A$)$<N$, the system A is called an underdetermined system. Given $\mathbf{y}$, explain why there must be an infinite number of solution $\mathbf{x}$. In order to get one (unique) solution, it needs a constraint. The standard one is minimizing $||\mathbf{x}||$ subject to $A\mathbf{x} = \mathbf{y}$. When $||\mathbf{x}||= \sqrt{\mathbf{x}^T\mathbf{x}}$, derive that the solution $\hat{\mathbf{x}}$ is

$$\hat{\mathbf{x}} = A^T(AA^T)^{-1}\mathbf{y}.$$

7) Download 'database1.mat' file. Load the file in MATLAB. The file contains real brain (activation) signal recorded over time using a Near-infrared Spectroscopy (NIRS) instrument. The variable $\mathbf{y} \in R^{1024\times 1}$ corresponds to the (one-channel) data. However, the data are distorted by noise and artifact. Here, let us focus on artifact associated with DC-component. We can simply model the NIRS data as

$$\mathbf{y} = A\mathbf{x} = [\mathbf{h}, \mathbf{d}] \begin{bmatrix} a \\ b \end{bmatrix}.$$

where $\mathbf{h}$ is hemodynamic response function and $\mathbf{d}$ is $[1, 1, 1, \cdots, 1, 1]^T$. a is activation strength and b is DC-component.

- Plot hemodynamic response function (A(:,1))
- Plot DC-function (A(:,2))
- Find an estimation $\hat{\mathbf{x}}$=$[\hat{a}, \hat{b}]^T$ using pseudo-inverse.
- Plot brain activation estimate $\hat{\mathbf{z}} = \mathbf{h}\hat{a}$. Note that DC component is discarded (filtered).
- Plot distorted data $\mathbf{y}$, estimate $\hat{\mathbf{z}}$, and ground-truth $\mathbf{z}$ together using "hold on". The estimate will be different from the ground-truth because we do not define other artifacts in the simple model.