EDAP30 Advanced Applied Machine Learning

Lecture 6: Recurrent Networks

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Recurrent Neural Networks

In feed-forward networks, predictions in a sequence of classifications are independent.

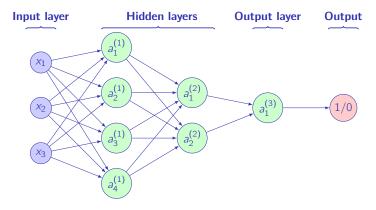
In many cases, given an input, the prediction also depends on the previous decision.

For instance, in weather forecast, if the input is the temperature and the output is rain/not rain, for a same temperature, it the previous output was rain, the next one is likely to be rain.

Recurrent neural networks (RNN) try to model these dependencies In this lecture, we will examine recurrent neural networks for:

- Categorization, i.e. given a sentence, predict its category, one output
- Sequence-to-sequence prediction, i.e. given a sentence, predict a sequence of symbols, a sequence of outputs

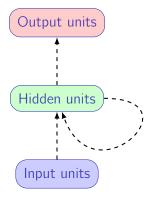
Feed Forward (Reminder)



For the first layer, we have:

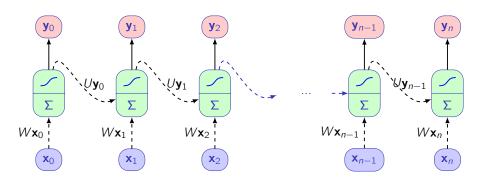
activation
$$(W^{(1)}\mathbf{x} + \mathbf{b}^{(1)})$$
.

The RNN Architecture



A simple recurrent neural network; the dashed lines represent trainable connections.

The Unfolded RNN Architecture



The network unfolded in time. Equation used by implementations¹.

$$\mathbf{y}_{(t)} = \tanh(W\mathbf{x}_{(t)} + U\mathbf{y}_{(t-1)} + \mathbf{b})$$

¹https://pytorch.org/docs/stable/generated/torch.nngRNN.html = ->

LSTMs

Simple RNNs use the previous output as input. They have then a very limited feature context.

Long short-term memory units (LSTM) are an extension to RNNs that can remember, possibly forget, information from longer or more distant sequences.

Given an input at index t, \mathbf{x}_t , a LSTM unit produces:

- ullet A short term state, called $oldsymbol{h}_t$ and
- ullet A long-term state, called ${f c}_t$ or memory cell.

The short-term state, \mathbf{h}_t , is the unit output, i.e. \mathbf{y}_t ; but both the long-term and short-term states are reused as inputs to the next unit.

LSTM Equations

A LSTM unit starts from a core equation that is identical to that of a RNN:

$$\mathbf{g}_t = \tanh(W_g \mathbf{x}_t + U_g \mathbf{h}_{t-1} + \mathbf{b}_g).$$

From the previous output and current input, we compute three kinds of filters, or gates, that will control how much information is passed through the LSTM cell

The two first gates, \mathbf{i} and \mathbf{f} , defined as:

$$\mathbf{i}_t = \arctan(W_i \mathbf{x}_t + U_i \mathbf{h}_{t-1} + \mathbf{b}_i),$$

 $\mathbf{f}_t = \arctan(W_f \mathbf{x}_t + U_f \mathbf{h}_{t-1} + \mathbf{b}_f),$

model respectively how much we will keep from the base equation and how much we will forget from the long-term state.

LSTM Equations (II)

To implement this selective memory, we apply the two gates to the base equation and to the previous long-term state with the element-wise product (Hadamard product), denoted \circ , and we sum the resulting terms to get the current long-term state:

$$\mathbf{c}_t = \mathbf{i}_t \circ \mathbf{g}_t + \mathbf{f}_t \circ \mathbf{c}_{t-1}.$$

The third gate:

$$\mathbf{o}_t = \operatorname{activation}(W_o \mathbf{x}_t + U_o \mathbf{h}_{t-1} + \mathbf{b}_o)$$

modulates the current long-term state to produce the output:

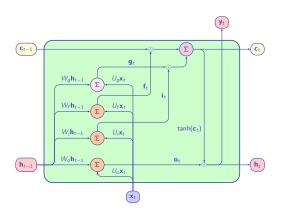
$$\mathbf{h}_t = \mathbf{o}_t \circ \mathrm{tanh}(\mathbf{c}_t).$$

The LSTM parameters are determined by a gradient descent.

See also:

https://pytorch.org/docs/stable/generated/torch.nn.LSTM_html

The LSTM Architecture



An LSTM unit showing the data flow, where \mathbf{g}_t is the unit input, \mathbf{i}_t , the input gate, \mathbf{f}_t , the forget gate, and \mathbf{o}_t , the output gate. The activation functions have been omitted

Recurrent Networks for Classification

- We will now use a recurrent network to classify texts
- We will use the IMDB dataset of movie reviews annotated as positive or negative
- In a feed-forward network, we build a representation of the document using the vector space concepts (with tfidf parameters for instance)
- In a recurrent architecture, the words will go through the network and we will use the last output to classify a text
- As vectorization, we will first use a one-hot encoding and then GloVe

Building a LSTM with PyTorch

```
def __init__(self, lstm_units, nbr_classes, num_layers=1,
           bidi lstm=False):
    super().__init__()
    self.dropout = nn.Dropout(DROPOUT)
    self.lstm = nn.LSTM(MAX_TOKENS + 2, lstm_units,
          num_layers=num_layers,
          dropout=DROPOUT, batch_first=True,
          bidirectional=bidi_lstm)
    if not bidi lstm:
      self.fc = nn.Linear(lstm_units, nbr_classes)
    else:
      # twice the units if bidirectional
      self.fc = nn.Linear(2*lstm_units, nbr_classes)
```

Building a LSTM with PyTorch

Code Example

```
Experiment: Jupyter Notebook:
https://github.com/pnugues/edap30/blob/main/programs/
1-text-classification_lstm_one_hot.ipynb
The IMDB corpus with a LSTM and unit vectors
```

Using Embeddings

- Embeddings encapsulate knowledge: If the training procedure has seen *Denmark*, but not *Sweden*, the embeddings will be close and the prediction will be possible
- With one-hot vectors, Denmark and Sweden are orthogonal
- In addition, replacing one-hot vectors with embeddings will save space
- Training embeddings is unsupervised

Using GloVe Embeddings

We create a dictionary, where the keys are the words and the value, the embedding vector

Initializing the Matrix

We create the embeddings matrix by using the GloVe embedding or a random vector, if not in GloVe

```
vocabulary_words = sorted(list(set(embedded_words + voc_train)))
embedding_matrix =\
   torch.rand((len(vocabulary_words) + 2, EMBEDDING_DIM))/10 - 0.05
out_of_embeddings = []
for word in vocabulary_words:
    if word in embeddings_dict:
        # If the words are in the embeddings, we fill them with a v
        embedding_matrix[word2idx[word]] = embeddings_dict[word]
    else:
        # Otherwise, it keeps a random value in the matrix
        # We store the out of vocabulary words
        out_of_embeddings += [word]
```

Building the Network

```
def __init__(self, embedding_matrix, lstm_units, nbr_classes,
    freeze_embs=True, num_layers=1, bidi_lstm=False):
        super().__init__()
        embedding_dim = embedding_matrix.size()[-1]
        self.embeddings = nn.Embedding.from_pretrained(
          embedding_matrix, freeze=freeze_embs, padding_idx=0
        self.dropout = nn.Dropout(DROPOUT)
        self.lstm = nn.LSTM(embedding_dim, lstm_units,
                            num_layers=num_layers,
                            dropout=DROPOUT, batch_first=True
                            bidirectional=bidi_lstm)
        if not bidi_lstm:
            self.fc = nn.Linear(lstm_units, nbr_classes)
        else:
            # twice the units if bidirectional
            self.fc = nn.Linear(2*1stm_units, nbr_classes)
```

Structure of a Network

```
def forward(self, sentence):
    embeds = self.embeddings(sentence)
    embeds = self.dropout(embeds)
    lstm_out, (h_n, c_n) = self.lstm(embeds)
    lstm_last = F.relu(h_n[-1])
    lstm_last = self.dropout(lstm_last)
    logits = self.fc(lstm_last)
    return logits
```

Code Example

Experiment: Jupyter Notebook,
https://github.com/pnugues/edap30/blob/main/programs/
2-text-classification_lstm_embs.ipynb
The IMDB corpus with a LSTM and embeddings

Outline

So far, we used networks to produce one output y per input vector \mathbf{x} , for instance one category per sentence.

Given an input sequence \mathbf{x} , we will now produce an output sequence: \mathbf{y} . We will see four kinds of neural networks:

- Feed forward
- 2 Recurrent
- Street
 LSTM
- Transformers

In the laboratory assignment, you will experiment with items 2. and 3.

Motivation

The analysis of sentences often involves the analysis of words.

We can divide it in three main tasks:

- Identify the type of word, for instance noun or verb using the classical grammar;
- 2 Identify a group or segment, for instance are these three words, *Kjell Olof Andersson*,

the name of a person;

Identify the relations between two words: for instance is this group the subject of a verb? This corresponds to parsing, semantic analysis, or information extraction.

We will consider the two first tasks.

This lecture will show you how to solve the first one, part-of-speech tagging, and you will write a program for the second one, named entity recognition (NER), in the next laboratory assignment.

Word Categorization: The Parts of Speech

Sentence:

That round table might collapse

Annotation:

Words	Parts of speech	POS tags		
that	Determiner	DET		
round	Adjective	ADJ		
table	Noun	NOUN		
might	Modal verb	AUX		
collapse	Verb	VERB		

The automatic annotation uses predefined POS tagsets such as the Penn Treebank tagset for English

Ambiguity

Words	Possible tags	Example of use			
that	Subordinating conjunction	That he can swim is good			
	Determiner	That white table			
	Adverb	It is not that easy			
	Pronoun	That is the table			
	Relative pronoun	The table that collapsed			
round	Verb	Round up the usual suspects			
	Preposition	Turn round the corner			
	Noun	A big round			
	Adjective	A round box			
	Adverb	He went round			
table	Noun	That white table			
	Verb	I table that			
might	Noun	The might of the wind			
	Modal verb	She might come			
collapse	Noun	The collapse of the empire			
	Verb	The empire can collapse			

Training Sets: The CoNLL Format

The CoNLL format is a tabular format to distribute annotated texts.

This format was created for evaluations carried out by the Conference in natural language learning

The CoNLL annotation has varied much across the years. We use CoNLL-U, the latest iteration.

Annotation of the Spanish sentence:

La reestructuración de los otros bancos checos se está acompañando por la reducción del personal

'The restructuring of Czech banks is accompanied by the reduction of personnel'

Example of Annotation (CoNLL-U)

La reestructuración de los otros bancos checos se está acompañando por la reducción del personal

ID	FORM	LEMMA	UPOS	FEATS
1	La	el	DET	Definite=Def Gender=Fem Number=Sing PronType=Art
2	reestructuración	reestructuración	NOUN	Gender=Fem Number=Sing
3	de	de	ADP	AdpType=Prep
4	los	el	DET	Definite=Def Gender=Masc Number=Plur PronType=Art
5	otros	otro	DET	Gender=Masc Number=Plur PronType=Ind
6	bancos	banco	NOUN	Gender=Masc Number=Plur
7	checos	checo	ADJ	Gender=Masc Number=Plur
8	se	se	PRON	Case=Acc Person=3 PrepCase=Npr PronType=Prs Reflex=Yes
9	está	estar	AUX	Mood=Ind Number=Sing Person=3 Tense=Pres VerbForm=Fin
10	acompañando	acompañar	VERB	VerbForm=Ger
11	por	por	ADP	AdpType=Prep
12	la	el	DET	Definite=Def Gender=Fem Number=Sing PronType=Art
13	reducción	reducción	NOUN	Gender=Fem Number=Sing
14	del	del	ADP	AdpType=Preppron
15	personal	personal	NOUN	Gender=Masc Number=Sing
16			PUNCT	PunctType=Peri

Another Example

ID	FORM	LEMMA	PLEMMA	POS	PPOS	FEAT	PFEAT
1	Battle	battle	battle	NN	NN		
2	-	-	_	HYPH	HYPH	_	_
3	tested	tested	tested	NN	NN	_	_
4	Japanese	japanese	japanese	JJ	JJ	_	_
5	industrial	industrial	industrial	JJ	JJ	_	_
6	managers	manager	manager	NNS	NNS	_	_
7	here	here	here	RB	RB	_	_
8	always	always	always	RB	RB	_	_
9	buck	buck	buck	VBP	VB	_	_
10	up	up	up	RP	RP	_	_
11	nervous	nervous	nervous	JJ	JJ	_	_
12	newcomers	newcomer	newcomer	NNS	NNS	_	_
13	with	with	with	IN	IN	_	_
14	the	the	the	DT	DT	_	_
15	tale	tale	tale	NN	NN	_	_
16	of	of	of	IN	IN	_	_
17	the	the	the	DT	DT	_	_
18	first	first	first	JJ	JJ	_	_
19	of	of	of	IN	IN	_	_
20	their	their	their	PRP\$	PRP\$	_	_
21	countrymen	countryman	countryman	NNS	NNS	_	_
22	to	to	to	TO	TO	_	_
23	visit	visit	visit	VB	VB	_	_
24	Mexico	mexico	mexico	NNP	NNP	_	_
25		,	,	,	,	_	_
26	a	a	a	DT	DT	_	_
27	boatload	boatload	boatload	NN	NN	_	_
28	of	of	of	IN	IN	_	_
29	samurai	samurai	samurai	NN	NN	_	_
30	warriors	warrior	warrior	NNS	NNS	_	_
31	blown	blow	blow	VBN	VBN□►	4 🗇 ▶	< ≧ <u>→</u> < ⊵ →

Designing a Part-of-Speech Tagger

We will now create part-of-speech taggers, where we will examine three architectures:

- A feed-forward pipeline with a one-hot encoding of the words;
- A feed-forward pipeline with word embeddings: We will replace the one-hot vectors with GloVe embeddings;
- A recurrent neural network, either a simple RNN or a LSTM, with word embeddings.

Features for Part-of-Speech Tagging

The word *visit* is ambiguous in English:

I paid a **visit** to a friend -> noun

I went to visit a friend -> verb

The context of the word enables us to tell, here an article or the infinitive marker

To train and apply the model, the tagger extracts a set of features from the surrounding words, for example, a sliding window spanning five words and centered on the current word.

We then associate the feature vector $(w_{i-2}, w_{i-1}, w_i, w_{i+1}, w_{i+2})$ with the part-of-speech tag t_i at index i.

Part-of-Speech Tagging

ID	FORM	PPOS	
	BOS	BOS	Padding
	BOS	BOS	
1	Battle	NN	
2	=	HYPH	
3	tested	NN	
17	the	DT	
18	first	JJ	
19	of	IN	
20	their	PRP\$	
21	countrymen	NNS	Input features
22	to	TO	
23	visit	VB	Predicted tag
24	Mexico		↓
25	,		
26	a		
27	boatload		
34	years		
35	ago		
36			
	EOS		Padding
	EOS		

Feature Vectors

ID	Feature vectors							PPOS
	W_{i-2}	w_{i-1}	w_i	w_{i+1}	W_{i+2}	t_{i-2}	t_{i-1}	
1	BOS	BOS	Battle	-	tested	BOS	BOS	NN
2	BOS	Battle	-	tested	Japanese	BOS	NN	HYPH
3	Battle	-	tested	Japanese	industrial	NN	HYPH	JJ
19	the	first	of	their	countrymen	DT	JJ	IN
20	first	of	their	countrymen	to	JJ	IN	PRP\$
21	of	their	countrymen	to	visit	IN	PRP\$	NNS
22	their	countrymen	to	visit	Mexico	PRP\$	NNS	TO
23	countrymen	to	visit	Mexico	,	NNS	TO	VB
24	to	visit	Mexico	,	a	TO	VB	NNP
25	visit	Mexico	,	a	boatload	VB	NNP	
34	ashore	375	years	ago		RB	CD	NNS
35	375	years	ago		EOS	CD	NNS	RB
36	years	ago		EOS	EOS	NNS	RB	

Architecture 1: A Feed-Forward Neural Network

We first use a feed-forward architecture corresponding to a logistic regression:

Preprocessing

Preprocessing is more complex though: Four steps:

column_names = load_ud_en_ewt()

Read the corpus train_sentences, dev_sentences, test_sentences, \

Store the rows of the CoNLL corpus in dictionaries
conll_dict = CoNLLDictorizer(column_names, col_sep='\t')
train_dict = conll_dict.transform(train_sentences)
test_dict = conll_dict.transform(test_sentences)

- Extract the features and store them in dictionaries
 context_dictorizer = ContextDictorizer()
 context_dictorizer.fit(train_dict)
 X_dict, y_cat = context_dictorizer.transform(train_dict)
- Vectorize the symbols

```
# We transform the X symbols into numbers
dict_vectorizer = DictVectorizer()
X_num = dict_vectorizer.fit_transform(X_dict)
```

Code Example

```
Jupyter Notebook: https://github.com/pnugues/edap30/blob/
main/programs/3-pos-tagger_ff.ipynb
```

Architecture 2: Using Embeddings

We replace the one-hot vectors with embeddings, the rest being the same Word embeddings are dense vectors obtained by a principal component analysis or another method.

They can be trained by the neural network or pretained In this implementation:

- We use pretrained embeddings from the GloVe project;
- Our version of GloVe is lowercased, so we set all the characters in lowercase;
- We add the embeddings as an Embedding layer at the start of the network;
- We initialize the embedding layer with GloVe and make it trainable or not.

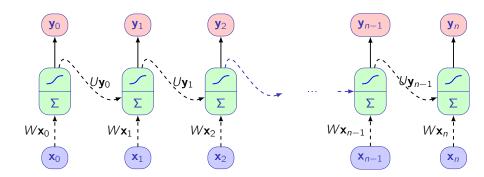
It would be possible to use a randomly initialized matrix as embeddings instead

The Model with Embeddings

Code Example

```
Jupyter Notebook: https://github.com/pnugues/edap30/blob/
main/programs/4-pos-tagger_ff_embs.ipynb
```

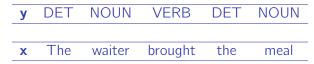
The RNN Architecture



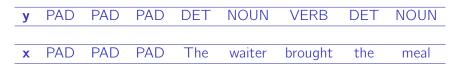
Input Format for RNNs

The input format is different from feed forward networks.

We need to build two lists: one for the input and the other for the output



All the vectors in a same batch must have the same length. We pad them:



We could apply the padding after

Building the Sequences

```
def build_sequences(corpus_dict, key_x='form', key_y='pos',
                  tolower=True):
   X, Y = [], []
    for sentence in corpus_dict:
        x, y = [], []
        for word in sentence:
            x += [word[key_x]]
            y += [word[key_y]]
        if tolower:
            x = list(map(str.lower, x))
        X += [x]
        Y += [V]
    return X, Y
```

At this point, we have \mathbf{x} and \mathbf{y} vectors of symbols

Building Index Sequences

0 is for the padding symbol and 1 for the unknown words

```
idx_word = dict(enumerate(vocabulary_words, start=2))
idx_pos = dict(enumerate(pos, start=2))
word_idx = {v: k for k, v in idx_word.items()}
pos_idx = {v: k for k, v in idx_pos.items()}
```

At this point, we have **x** and **y** vectors of numbers

Padding the Index Sequences

We build the complete X_idx and Y_idx matrices for the whole corpus And we pad the matrices:

```
X_train_padded = pad_sequence(X_train_idx, batch_first=True)
Y_train_padded = pad_sequence(Y_train_idx, batch_first=True)
```

```
X_val_padded = pad_sequence(X_val_idx, batch_first=True)
Y_val_padded = pad_sequence(Y_val_idx, batch_first=True)
```

See: https://pytorch.org/docs/stable/generated/torch.nn.utils.rnn.pad_sequence.html pad_sequences can have an argument that specifies the padding value padding_value

The padded sentences must have the same length in a batch. This is automatically computed by PyTorch

Recurrent Neural Networks (RNN)

```
def __init__(self, embedding_matrix, lstm_units, nbr_classes,
       freeze_embs=True, num_layers=1, bidi_lstm=False):
    super().__init__()
    embedding_dim = embedding_matrix.size()[-1]
    self.embeddings = nn.Embedding.from_pretrained(
      embedding_matrix, freeze=freeze_embs, padding_idx=0)
    self.dropout = nn.Dropout(DROPOUT)
    self.lstm = nn.LSTM(embedding_dim, lstm_units,
                num_layers=num_layers, dropout=DROPOUT,
                batch_first=True, bidirectional=bidi_lstm)
    if not bidi_lstm:
        self.fc = nn.Linear(lstm_units, nbr_classes)
    else:
        # twice the units if bidirectional
        self.fc = nn.Linear(2*1stm_units, nbr_classes)
```

Recurrent Neural Networks (RNN)

```
def forward(self, sentence):
    embeds = self.embeddings(sentence)
    embeds = self.dropout(embeds)
    lstm_out, _ = self.lstm(embeds)
    lstm_out = F.relu(lstm_out)
    lstm_out = self.dropout(lstm_out)
    logits = self.fc(lstm_out)
    return logits
```

Code Example

```
Jupyter Notebook: https://github.com/pnugues/edap30/blob/
main/programs/5-pos-tagger_lstm.ipynb
```

Segment Recognition

```
Group detection – chunking –:
```

```
Brackets: [NG] The government NG] has [NG] other agencies and instruments NG] for pursuing [NG] these other objectives NG].
```

- Tags: The/I government/I has/O other/I agencies/I and/I instruments/I for/O pursuing/O these/I other/I objectives/I ./O
- Brackets: Even [N_G Mao Tse-tung N_G] [N_G 's China N_G] began in [N_G 1949 N_G] with [N_G a partnership N_G] between [N_G the communists N_G] and [N_G a number N_G] of [N_G smaller, non-communists parties N_G].
 - Tags: Even/O Mao/I Tse-tung/I 's/B China/I began/O in/O 1949/I with/O a/I partnership/I between/O the/I communists/I and/O a/I number/I of/O smaller/I ,/I non-communists/I parties/I ./O

Segment Categorization

Tages extendible to any type of chunks: nominal, verbal, etc. For the IOB scheme, this means tags such as I.Type, O.Type, and B.Type, Types being NG, VG, PG, etc. In CoNLL 2000, ten types of chunks

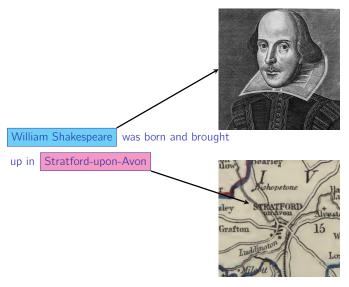
Word	POS	Group	Word	POS	Group
Не	PRP	B-NP	to	TO	B-PP
reckons	VBZ	B-VP	only	RB	B-NP
the	DT	B-NP	£	#	I-NP
current	JJ	I-NP	1.8	CD	I-NP
account	NN	I-NP	billion	CD	I-NP
deficit	NN	I-NP	in	IN	B-PP
will	MD	B-VP	September	NNP	B-NP
narrow	VB	I-VP			О

Noun groups (NP) are in red and verb groups (VP) are in blue.

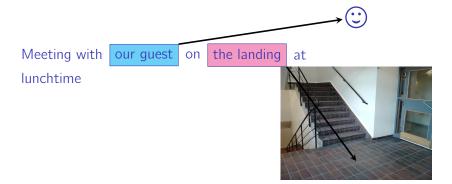
IOB Annotation for Named Entities

Co	NLL 2002	CoNLL 2003					
Words	Named entities	Words	POS	Groups	Named entities		
Wolff	B-PER	U.N.	NNP	I-NP	I-ORG		
	0	official	NN	I-NP	0		
currently	0	Ekeus	NNP	I-NP	I-PER		
a	0	heads	VBZ	I-VP	0		
journalist	0	for	IN	I-PP	0		
in	0	Baghdad	NNP	I-NP	I-LOC		
Argentina	B-LOC			O	0		
	0						
played	0						
with	0						
Del	B-PER						
Bosque	I-PER						
in	0						
the	O						
final	0						
years	0						
of	0						
the	0						
seventies	0						
in	0						
Real	B-ORG						
Madrid	I-ORG						
	0						

Named Entities: Proper Nouns



Others Entities: Common Nouns



Evaluation

There are different kinds of measures to evaluate the performance of machine learning techniques, for instance:

- Precision and recall in information retrieval and natural language processing;
- The receiver operating characteristic (ROC) in medicine.

	Positive examples: <i>P</i>	Negative examples: N
Classified as P	True positives: A	False positives: B
Classified as N	False negatives: C	True negatives: D

More on the receiver operating characteristic here: http: //en.wikipedia.org/wiki/Receiver_operating_characteristic

Recall, Precision, and the F-Measure

The **accuracy** is $\frac{|A \cup D|}{|P \cup N|}$.

Recall measures how much relevant examples the system has classified correctly, for P:

$$Recall = \frac{|A|}{|A \cup C|}.$$

Precision is the accuracy of what has been returned, for P:

$$Precision = \frac{|A|}{|A \cup B|}.$$

Recall and precision are combined into the **F-measure**, which is defined as the harmonic mean of both numbers:

$$F = \frac{2 \cdot \text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}}.$$

Evaluation: Accuracy, precision, and recall

For noun groups with the predicted output:

Word	POS	Group	Predicted		Word		POS	Group	Predicted	
Не	PRP	B-NP	B-NP			to	TO	B-PP	B-PP	
reckons	VBZ	B-VP	B-VP			only	RB	B-NP	B-NP	X
the	DT	B-NP	B-NP	Х		£	#	I-NP	I-NP	X
current	JJ	I-NP	B-NP	Χ		1.8	CD	I-NP	B-NP	Χ
account	NN	I-NP	I-NP	Χ		billion	CD	I-NP	I-NP	X
deficit	NN	I-NP	I-NP	X		in	IN	B-PP	B-PP	
will	MD	B-VP	B-VP			September	NNP	B-NP	B-NP	
narrow	VB	I-VP	I-VP					0	O	

There are 16 chunk tags, 14 are correct: Accuracy = $\frac{14}{16}$ = 0.875 There are 4 noun groups, the system retrieved 2 of them: Recall = $\frac{2}{4}$ = 0.5

The system identified 6 noun groups, two are correct: Precision = $\frac{2}{6}$ = 0.33

Harmonic mean = $2 \times \frac{0.33 \times 0.5}{0.33 + 0.5} = 0.4$