Renormalization on the Lattice

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1 Regularization-Independent Momentum Subtraction (RI-MOM)

The RI-MOM scheme (also known as Rome-Southampton) is a renormalization scheme well equipped to deal with the lattice, namely because we can calculate the relevant quantities that define the scheme easily on the lattice. The lattice spacing a provides us a with a natural UV regulator, and RI-MOM will tell us how to go from such a regulated result to a physical observable quantity.

The RI-MOM has a relatively simple renormalization condition. We will define it here for an arbitrary Green's function, for example a three point function. For a renormalization scale μ and working in a fixed gauge, **we define the amputated, renormalized Green's function at momentum $p^2 = -\mu^2$ to be equal to its tree level value**.

We will denote bare quantities as $a^{(0)}$, and renormalized quantities with a R subscript, when appropriate. Note that any quantities computed directly on the lattice are bare, and the entire point of using RI-MOM is to extract a sensible definition for these bare quantities.

In practice on the lattice, there are two things that we must compute. We will work with a specific example here for a given quark field $q(x)$. Suppose the operator we are trying to compute is $\mathcal{O}(z)$. We will renormalize the three point function:

$$G(p) = \frac{1}{V} \sum_{x,y,z} e^{ip\cdot(x-y)} \langle q(x) \mathcal{O}(z) \overline{q}(y) \rangle \tag{1}$$

i.e. we are projecting the source and sink to a definite momentum and projecting the operator $\mathcal{O}(z)$ to zero momentum.

We also will need to compute the momentum projected propagator:

$$S(p) = \frac{1}{V} \sum_{x,y} e^{ip\cdot(x-y)} S(x,y) \tag{2}$$

where $S_{ij}^{ab}(x,y) = \langle q_i^a(x) \bar{q}_j^b(y) \rangle$ is the standard propagator. On the lattice, the two objects that we must compute directly are $S(p)$ and $G(p)$, and everything else follows once we have these quantities.

To explain the method, our goal is to compute the operator renormalization:

$$\mathcal{O}_R(\mu) = \mathcal{Z}(\mu)\, \mathcal{O}^{(0)} \tag{3}$$

where $\mathcal{O}_R(\mu)$ is our renormalized operator, $\mathcal{Z}(\mu)$ is the renormalization coefficient of interest, and $\mathcal{O}^{(0)}$ is the lattice (bare) operator. We will also assume the quark fields have been renormalized by some quark field renormalization $\mathcal{Z}_q$ (note this is the opposite convention studied in many QFT classes):

$$q_R(\mu) = \sqrt{\mathcal{Z}_q}\, q^{(0)} \tag{4}$$

There is an analytical expression for $\mathcal{Z}_q$ on the lattice in the RI-MOM scheme, and it has been determined to be:

$$\mathcal{Z}_q(p)|_{p^2=-\mu_R^2} = \left[\frac{tr\left\{-i\sum_{\nu=1}^4 \gamma_\nu \sin(ap_\nu) aS(p)^{-1}\right\}}{12\sum_{\nu=1}^4 \sin^2(ap_\nu)} \right]_{p^2=-\mu^2} \tag{5}$$

The twelve on the bottom is a normalization $12 = 3 \times 4$ for the number of color and number of spin indices, which we will see in many of the expressions.

Let $\Gamma(p)$ be the **amputated three point function**, bare or renormalized. We can relate Γ to the other quantities we have already computed by using the inverse propagator to manually cut the legs off the full three point function:

$$\Gamma(p) = S(p)^{-1}G(p)S(p)^{-1} \tag{6}$$

We will denote the tree level version of this by Γ_B, where the B subscript stands for "Born".

Using our quantities already computed on the lattice, we can compute the bare $\Gamma^{(0)}(p)$ directly in terms of the renormalized quantities. In the continuum limit, the renormalized Green's function will be:

$$G_R(p;\mu) = \int d^4x\, d^4y\, d^4z\, e^{ip\cdot(x-y)}\langle q_R(x;\mu)\mathcal{O}_R(z;\mu)\overline{q}_R(y;\mu)\rangle \tag{7}$$

$$= \mathcal{Z}_q(\mu)\mathcal{Z}(\mu)\int d^4x\, d^4y\, d^4z\, e^{ip\cdot(x-y)}\langle q^{(0)}(x)\mathcal{O}^{(0)}(z)q^{(0)}(y)\rangle \tag{8}$$

$$= \mathcal{Z}_q(\mu)\mathcal{Z}(\mu)G^{(0)}(p) \tag{9}$$

Similarly, the bare and renormalized propagators are related as:

$$S_R(p;\mu) = \int d^4xd^4y\, e^{ip\cdot(x-y)}\langle q_R(x;\mu)\overline{q}_R(y;\mu)\rangle = \mathcal{Z}_q(\mu)S^{(0)}(p) \tag{10}$$

These relation translates immediately to the amputated Green's function $\Gamma(p)$, as $S_R^{-1}(p;\mu) = \mathcal{Z}_q^{-1}(\mu)(S^{(0)})^{-1}$ we see that:

$$\Gamma_R(p;\mu) = \mathcal{Z}_q(\mu)^{-1}\mathcal{Z}(\mu)\Gamma^{(0)}(p) \tag{11}$$

We are now in a position to apply the renormalization condition. We must equate the renormalized, amputated Green's function to the tree level Green's function $\Gamma_B(p)$. This gives us:

$$\mathcal{Z}_q(\mu)^{-1}\mathcal{Z}(\mu)\Gamma(p) = \Gamma_B(p) \tag{12}$$

Dividing by a conventional factor of 12 as a normalization and inverting, we can clean this expression up into a simple equation for $\mathcal{Z}(\mu)$:

$$\mathcal{Z}(p^2 = -\mu^2) = \left[\frac{12\mathcal{Z}_q(p)}{tr\left\{\Gamma(p)\Gamma_B(p)^{-1}\right\}}\right]_{p^2=-\mu^2} \tag{13}$$

2 Isospin

We are interested here in computing matrix elements of the operator:

$$\mathcal{O}(z) = \mathcal{O}_u(z) - \mathcal{O}_d(z) \tag{14}$$

where the quark operators $\mathcal{O}_q$ are given by

$$\mathcal{O}_q(z) = \frac{1}{\sqrt{2}}(\mathcal{T}^q_{33}(z) - \mathcal{T}^q_{44}(z)) \tag{15}$$

and the irreducible tensor operators $\mathcal{T}^q_{\mu\nu}$ are defined as

$$\mathcal{T}^q_{\mu\nu} = \overline{q}(z)\,\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}\,q(z) \tag{16}$$

with $\overleftrightarrow{D} = \vec{D} - \overleftarrow{D}$ the symmetrized covariant derivative. Note we define the symmetric and traceless component of a tensor to be:

$$a_{\{\mu}b_{\nu\}} = \frac{1}{2}(a_\mu b_\nu + a_\nu b_\mu) - \frac{1}{4}a_\alpha b^\alpha g_{\mu\nu} \tag{17}$$

We are inserting the operator $\mathcal{O}_q$ with momentum $\vec{p} = 0$. We will focus our analysis on the tensor operator $\mathcal{T}^q_{\mu\mu}$ (note there is no sum on μ here), and note that $\mathcal{O}(z)$ can be obtained through linearity. We may write:

$$\sum_z \mathcal{T}^q_{\mu\mu}(z) = \sum_{z,z'} \overline{q}(z)\,J_\mu(z,z')\,q(z') \tag{18}$$

Plugging in the definition of the derivatives:

$$\vec{D}\psi(z) = \frac{1}{2}\left(U_\mu(z)\psi(z+\hat{\mu}) - U_\mu(n-\hat{\mu})^\dagger\psi(z-\hat{\mu})\right) \tag{19}$$

$$\overline{\psi}(z)\overleftarrow{D} = \frac{1}{2}\left(\overline{\psi}(z+\hat{\mu})U_\mu(z)^\dagger - \overline{\psi}(z-\hat{\mu})U_\mu(z-\hat{\mu})\right) \tag{20}$$

we find the current $J_\mu(z,z')$ is:

$$J_\mu(z,z') = \left[U_\mu(z)\delta_{z+\hat{\mu},z'} - U_\mu(z')^\dagger\delta_{z-\hat{\mu},z'}\right]\gamma_\mu \tag{21}$$

We may now use this expansion to compute the three point function for the operator $\mathcal{T}_\mu = \mathcal{T}^u_{\mu\mu} - \mathcal{T}^d_{\mu\mu}$ (we can simply take $\mathcal{T}_3 - \mathcal{T}_4$ to get the operator of interest in Equation 14). Using Equation 18, we write

$$\sum_z \mathcal{T}_\mu(z) = \sum_{z,z'}\left[\overline{u}(z)\,J_\mu(z,z')\,u(z') - \overline{d}(z)\,J_\mu(z,z')\,d(z')\right] \tag{22}$$

Plugging this into Equation 1, we find that we can expand the total up quark Green's function (here α,β are Dirac indices) as:

$$G^{\alpha\beta}(p) = \frac{1}{\sqrt{2}}\left(G^{\alpha\beta}_3(p) - G^{\alpha\beta}_4(p)\right) \tag{23}$$

where:

$$\begin{aligned} G^{\alpha\beta}_\mu(p) &= \frac{1}{V}\sum_{x,y,z} e^{-ip(x-y)}\langle u^\alpha(x)\mathcal{T}_\mu(z)\overline{u}^\beta(y)\rangle & (24)\\ &= \frac{1}{V}\sum_{x,y,z,z'} e^{-ip(x-y)}\left[\langle u^\alpha(x)\overline{u}^\sigma(z)J^{\sigma\rho}_\mu(z,z')u(z')^\rho\overline{u}^\beta(y)\rangle - \langle u^\alpha(x)\overline{d}^\sigma(z)J^{\sigma\rho}_\mu(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle\right] & (25)\end{aligned}$$

Now we perform all possible Wick contractions on the matrix elements to write them as propagators:

$$
\begin{aligned}
\langle u^\alpha(x)\overline{u}^\sigma(z)J_\mu^{\sigma\rho}(z,z')u^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{u}Ju\overline{u}\rangle + \langle u\overline{u}Ju\overline{u}\rangle \\
&= S^{\alpha\sigma}(x,z)J_\mu^{\sigma\rho}(z,z')S^{\rho\beta}(z',y) + (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\langle u^\alpha(x)\overline{d}^\sigma(z)J_\mu^{\sigma\rho}(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{d}Jd\overline{u}\rangle \\
&= (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)
\end{aligned} \tag{27}
$$

where the factors of (-1) come from rearranging the contraction so that the contracted pieces are of the form $\langle u\overline{u}\rangle$. The vacuum pieces cancel because the up and down quark propagators are degenerate, so the final result is very clean:

$$
G_\mu(p) = \frac{1}{V}\sum_{x,y,z,z'} e^{ip(x-y)} S(x,z)J_\mu(z,z')S(z',y) \tag{28}
$$

This is our central equation, but note that computing this directly on the lattice would involve computing the two point propagator $S(x,y)$ at each two points on the lattice. This is much too computationally intensive, so we must resort to other techniques to accomplish this.

There are two primary ways to compute this on the lattice. We can compute this directly using momentum sources, or we can use the sequential source technique. Momentum sources work specifically for Equation 28, but produce a significantly better signal on a small number of configurations. On the other hand, sequential source is much more general, but produces more noise. We will discuss each method below.

2.1 Momentum sources

Observe that we can rewrite Equation 28 as:

$$
\begin{aligned}
G_\mu(p) &= \frac{1}{V}\sum_{x,y,z,z'} e^{ipx} S(x,z)J_\mu(z,z')e^{-ipy}S(z',y) \\
&= \frac{1}{V}\sum_{z,z'} \gamma_5 \left(\sum_x S(z,x)e^{-ipx}\right)^\dagger \gamma_5 J_\mu(z,z') \left(\sum_y S(z',y)e^{-ipy}\right) \\
&= \frac{1}{V}\sum_{z,z'} \gamma_5 \tilde{S}_p(z)^\dagger \gamma_5 J_\mu(z,z')\tilde{S}_p(z')
\end{aligned} \tag{29}
$$

where we have defined $\tilde{S}_p(z)$ as:

$$
\tilde{S}_p(z) = \sum_x S(z,x)e^{-ipx} \tag{30}
$$

The advantage of casting the equation in this form is that we can solve for $\tilde{S}_p(z)$ directly by inverting the Dirac equation with a momentum source, i.e. we have:

$$
\sum_z D(x,z)\tilde{S}_p(z) = e^{-ipx} \tag{31}
$$

where $D(x,z)$ is the Dirac operator. This means that upon solving for $\tilde{S}_p(z)$ and plugging this into Equation 29, we can solve directly for $G_\mu(p)$.

We can also use the propagator we get from the momentum source inversion to directly compute the propagator in Equation 2 as follows:

$$S(p) = \frac{1}{V}\sum_{x,y} e^{ip\cdot(x-y)} S(x,y) = \frac{1}{V}\sum_{x} e^{ip\cdot x} \tilde{S}_p(x) \tag{32}$$

This is an exact equation and it does not rely on translational invariance in the infinite statistics limit. Therefore, this method will give much better signal and can be run efficiently on a small number of configurations. The downside to this is that we require a propagator inversion for each choice of sink momentum. To compute $G(p)$ for a large number of sink momenta, as we need to do to extrapolate $\mathcal{Z}(\mu)$ in the continuum limit, a propagator inversion at each sink momenta is not feasible. We must instead choose the sink momentum wisely to be able to extract the discretization artifacts and extrapolate $\mathcal{Z}(\mu)$ to the continuum, which we will describe in Section 4.

2.2 Sequential source method

In practice we will use the sequential source method, which if implemented correctly does not force us to invert a propagator at every sink momenta. This technique is also much more general than the one previously described, but it suffers from more noise because it relies on the translational invariance of the lattice, which only exists in the infinite statistics limit. The idea of the sequential source method is that if we have an equation involving the full propagator $S(x,y)$, we can invert a source which depends on the propagator $S(x)$. For example, in this problem we wish to evaluate Equation 28, but we cannot simply evaluate $S(x,y)$ for every x and y. To get around this, consider using a source

$$b(z) = \sum_{z'} J_\mu(z,z') S(z',0) \tag{33}$$

to invert the Dirac equation, which will solve for $M(x)$ in this equation:

$$\sum_x D(z,x) M(x) = b(z) \tag{34}$$

where $D(x,z)$ is the Dirac operator. Upon inversion, using that $\sum_z S(y,z)D(z,x) = \delta(y-x)$, we can move the Dirac operator to the other side as $D^{-1}(y,z) = S(y,z)$ and obtain:

$$M(x) = \sum_z D^{-1}(x,z) b(z) = \sum_{z,z'} S(x,z) J_\mu(z,z') S(z',0) \tag{35}$$

Note that we have summed the full propagator $S(x,z)$ for the price of a single inversion of the source $b(z)$. We can then reconstruct Equations 2 and 28 for the momentum-projected Green's function and propagator in the infinite statistics limit when translational invariance is restored:

$$G_\mu(p) \to \frac{1}{V}\sum_x e^{ipx} M(x) = \frac{1}{V}\sum_{x,z,z'} e^{ipx} S(x,z) J_\mu(z,z') S(z',0) \tag{36}$$

$$S(p) \to \frac{1}{V}\sum_x e^{ipx} S(x,0) \tag{37}$$

To check an implementation of this method, we can invert a sequential propagator which depends on momentum. If we replace $S(z',0)$ in Equation 33 and invert, we find that:

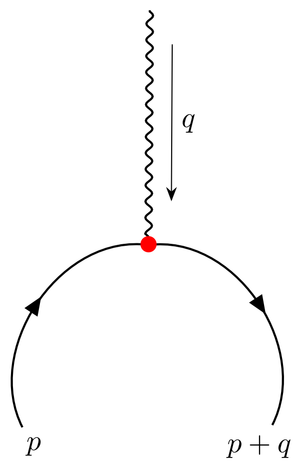
$$b_\mu^{(p)}(z) = \sum_{z'} J_\mu(z,z') S_p(z') = \sum_{z',y} e^{-ipy} J_\mu(z,z') S(z',y) \tag{38}$$

$$M_\mu^{(p)}(x) = \sum_z S(x,z) b_\mu^{(p)}(z) = \sum_{z,z',y} e^{-ipy} S(x,z) J_\mu(z,z') S(z',y) \tag{39}$$

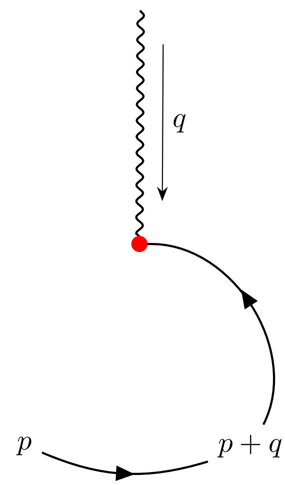
When we momentum project to find the Green's function $G(p)$, this no longer relies on translation invariance and should match our result with the momentum inversion exactly:

$$G_\mu(p) = \sum_x e^{ipx} M_\mu^{(p)}(x) = \sum_{x,y,z,z'} e^{ip(x-y)} S(x,z) J_\mu(z,z') S(z',y) \tag{40}$$

In our case with a large amount of sink momenta, this method is much more robust than inverting a momentum source because we one inversion can give us $G(p)$ at every value of the sink momentum. We will call this construction going **through the operator**, because the inversion in Equation 35 projects the current insertion onto $q = 0$ momentum. If we had been interested in projecting the operator onto different momentum values, then we would need to use a new sequential source (modify Equation 33) for each value of the operator momentum. Pictorially, we are inverting at the operator momentum, then tying up at the sink momentum. On the other hand, we reverse the direction of inversion and invert our propagator at each sink momentum first, then tie up the line at the operator. This method is called going **through the sink**. We can represent these different methods below, where in our case $q = 0$.



(a) Inversion through the operator



(b) Inversion through the sink

In this problem inversion through the sink would require too many propagator inversions like in the previous momentum source method, and it would also be noisy like inversion through the operator. As such, there is no reason to consider it, and I included it here mainly for generality.

3 Matching to a continuum scheme

We must now discuss how to convert between schemes, because typically we want our results to be in a common scheme used by many other researchers. For our case, we will convert the renormalization constant $Z_{RI-MOM}(p)$ which we have calculated in the previous sections to the $\overline{MS}$ scheme.

4 Hypercubic Artifacts

When we compute observables at a finite lattice spacing a, we suffer discretization artifacts which are relics of the explicit symmetry breaking $SO(1,3) \to H(4)$ suffered by putting the theory on a lattice. Namely, because we have less symmetry, there are more invariant quantities of p^μ in a lattice theory than in the continuum. In the continuum, the basic invariant that we can create is $p^2 = p_\mu p^\mu$.

On the lattice, we can find other invariants because the orbits of $H(4)$ are strictly smaller than the orbits of $O(4)$, the Euclidean isometry group in $d = 4$. For example, the vectors $(2, 0, 0, 0)$ and $(1, 1, 1, 1)$ have the same value of $p^2 = 4$, yet there is no element $g \in H(4)$ such that $g \cdot (1, 1, 1, 1) = (2, 0, 0, 0)$, i.e. they cannot be rotated into one another by hypercubic symmetry. This is because we can define *other hypercubic invariants than just* p^2. The functions:

$$p^{[n]} := \sum_\mu p^n \tag{41}$$

for n even are also invariants, and these dictate the orbits of momenta under $H(4)$. Since $(1, 1, 1, 1)$ has $p^{[4]} = 4$ and $(2, 0, 0, 0)$ has $p^{[4]} = 16$, we can conclude they are distinct momenta under hypercubic symmetry and thus cannot live in the same orbit of $H(4)$.

Any function which is invariant under the action of $H(4)$ must be a function of these hypercubic invariants, much like how in the continuum any function which was invariant under Lorentz symmetry was a function of Lorentz scalars like p^2 or $\not{p}$. Because we are computing out quantities on the lattice which only has $H(4)$ symmetry, these extra terms like $p^{[4]}$ can come into play when form factors or renormalization constants are computed, and we must take this into account. These are a type of **discretization artifact** which we will seek to remove when we extrapolate to the continuum.

Another artifact that we must consider when performing calculations on the lattice is that lattice momenta is quantized. The possible values that the momenta can take are:

$$p_\mu = \frac{2\pi}{aL_\mu} n_\mu \tag{42}$$

where $n_\mu \in \mathbb{Z}$ and L_μ is the size of the lattice in direction μ.