Momentum Fraction NPR

Patrick Oare

1 Renormalization in the RI'-MOM scheme

We will renormalize a local operator $\mathcal{O}(z)$ in the RI'-MOM scheme [1]. To renormalize $\mathcal{O}$ at scale μ^2 in this scheme, we impose that in a fixed gauge, the amputated three point function Γ at incoming quark momentum $p^2 = \mu^2$ and current momentum $q = 0$ equals its tree level value at $p^2 = \mu^2$.

Our notation will denote bare quantities as $A^{(0)}$, and renormalized quantities with a R subscript, when appropriate. The lattice parameter will be denoted a and the size of the lattice in the μ direction is L_μ. We will work with momentum scales:

$$\tilde{p}_\mu \equiv \frac{2}{a}\sin\left(\frac{\pi a}{L_\mu}k_\mu\right) \qquad\qquad p_\mu \equiv \frac{2\pi}{L_\mu}k_\mu \tag{1}$$

where $\tilde{p}$ is the lattice momentum and p is the linear momentum for mode $k \in \mathbb{Z}^4$. We will work in Landau gauge, where the gauge parameter is set to $\tau = 0$.

On the lattice we must directly compute the propagator $S^{(0)}(p^2)$ and the bare three point function $G^{(0)}(p^2)$, defined as:

$$S^{(0)}(p^2) \equiv \frac{1}{V}\sum_{x,y} e^{ip\cdot(x-y)}\langle q(x)\overline{q}(y)\rangle \tag{2}$$

$$G^{(0)}(p^2) \equiv \frac{1}{V}\sum_{x,y,z} e^{ip\cdot(x-y)}\langle q(x)\mathcal{O}(z)\overline{q}(y)\rangle \tag{3}$$

i.e. we are projecting the source and sink to a definite momentum and projecting the operator $\mathcal{O}(z)$ to zero momentum.

We define the renormalized operator and quark fields to be:

$$\mathcal{O}_R(\mu) \equiv \mathcal{Z}(\mu)\,\mathcal{O}^{(0)} \tag{4}$$

$$q_R(\mu) \equiv \sqrt{\mathcal{Z}_q(\mu)}q^{(0)} \tag{5}$$

where $\mathcal{O}_R(\mu)$ is our renormalized operator, $\mathcal{Z}(\mu)$ is the renormalization coefficient of interest, and $\mathcal{O}^{(0)}$ is the lattice (bare) operator.

In the RI'-MOM scheme the quark field renormalization $\mathcal{Z}_q$ is defined to be:

$$\mathcal{Z}_q(p^2)|_{\tilde{p}^2=-\mu_R^2} = i\frac{1}{12\tilde{p}^2}tr\left\{S^{-1}(\tilde{p}^2)\not{\tilde{p}}\right\}\Big|_{p^2=\mu^2} = \left[\frac{tr\left\{i\sum_{\nu=1}^4\gamma_\nu\sin(ap_\nu)aS(p^2)^{-1}\right\}}{12\sum_{\nu=1}^4\sin^2(ap_\nu)}\right]_{p^2=\mu^2} \tag{6}$$

The twelve is a normalization for the number of color and number of spin indices, which we will see in many of the expressions. Note that this may differ by a negative sign from other references, as we are using the phase convention $e^{ip\cdot(x-y)}$ for our momentum projection in Equations 2 and 3.

Let Γ be the **amputated three point function**, bare or renormalized. Γ can be defined in terms of the propagator $S(p^2)$ and three point function $G(p^2)$ as:

$$\Gamma(p^2) = S(p^2)^{-1} G(p^2) S(p^2)^{-1} \tag{7}$$

By using the definitions 4 and 5 we can see that $G_R(p^2) = \mathcal{Z}_q \mathcal{Z}_{\mathcal{O}} G^{(0)}(p^2)$ and $S_R(p^2) = \mathcal{Z}_q S^{(0)}(p^2)$, hence we can relate the renormalized Γ_R to the bare $\Gamma^{(0)}$ which is computed on the lattice:

$$\Gamma_R(p^2) = \mathcal{Z}_q(\mu^2)^{-1} \mathcal{Z}_{\mathcal{O}}(\mu^2) \Gamma^{(0)}(p^2) \tag{8}$$

We can now apply the renormalization condition and define that $\Gamma_R(\tilde{p}^2)$ equals its tree level value $\Gamma_B(\tilde{p}^2)$ at the renormalization point $\tilde{p}^2 = \mu^2$:

$$\left[\frac{\mathcal{Z}_{\mathcal{O}}(\tilde{p}^2)}{\mathcal{Z}_q(\tilde{p}^2)} \Gamma^{(0)}(\tilde{p}^2) \right]_{\tilde{p}^2 = \mu^2} = \Gamma_B(\tilde{p}^2 = \mu^2) \tag{9}$$

2 Isospin operator

We will now specify the RI'-MOM method to our operator $\mathcal{O}^i(z) = \mathcal{O}^i_u(z) - \mathcal{O}^i_d(z)$, where the quark operators $\{\mathcal{O}^i_q\}_{i=1}^3$ are basis elements of the tensor operator

$$\mathcal{T}^q_{\mu\nu}(z) \equiv \overline{q}(z) \gamma_{\{\mu} \overleftrightarrow{D}_{\nu\}} q(z) \tag{10}$$

in the irreducible representation $\tau_1^{(3)}$ of the hypercubic group $H(4)$. Here $\overleftrightarrow{D} = \vec{D} - \overleftarrow{D}$ is the antisymmetrized covariant derivative and the symmetric and traceless component of a tensor is defined to be:

$$a_{\{\mu} b_{\nu\}} \equiv \frac{1}{2}(a_\mu b_\nu + a_\nu b_\mu) - \frac{1}{4} a_\alpha b^\alpha g_{\mu\nu} \tag{11}$$

We now must determine the tree level structure of this operator, and then perform a lattice computation to determine $\Gamma^{(0)}(\tilde{p}^2$ and $\mathcal{Z}_q(\tilde{p}^2)$.

2.1 Tree level structure of $\mathcal{T}$

We begin by determining the amputated three point function for this operator at tree level $\Gamma_B(p^2)$, closely following the method in [2]. When $\mathcal{T}_{\mu\nu}$ is put on the lattice, its tree level value is proportional to two tensor structures:

$$i\Lambda^1_{\mu\nu}(\tilde{p}) \equiv \frac{1}{2}(\tilde{p}_\mu \gamma_\nu + p_\nu \gamma_\mu) - \frac{1}{4} \not{\tilde{p}} \delta_{\mu\nu} \tag{12}$$

$$i\Lambda^2_{\mu\nu}(\tilde{p}) \equiv \frac{\tilde{p}_\mu \tilde{p}_\nu}{p^2} \not{\tilde{p}} - \frac{1}{4} \not{\tilde{p}} \delta_{\mu\nu} \tag{13}$$

In the continuum, $\mathcal{T}_{\mu\nu}$ is directly propotional to just $\Lambda^{(1)}_{\mu\nu}$, and we would not have to consider any mixing between these tensor structures. On the lattice, $\Lambda^{(2)}_{\mu\nu}$ comes in as a $\mathcal{O}(a)$ correction. To account for this mixing directly, we must redefine the renormalization condition in Equation 9. We expand the amputated Green's function $\Gamma(p)$. If we let $\Gamma^D(p) := \frac{1}{3} tr_C\{\Gamma(p)\}$ be the Dirac components of $\Gamma(p)$, then this expansion amounts to imposing the renormalization condition:

$$\left[\Gamma^D_{\mu\nu}(\tilde{p}^2) \right]_{\tilde{p}^2 = \mu^2} = \Pi^1(\tilde{p}^2) \Lambda^1_{\mu\nu}(\tilde{p}^2) + \Pi^2(\tilde{p}^2) \Lambda^2_{\mu\nu}(\tilde{p}^2) \tag{14}$$

where $\Pi^1(\tilde{p}^2)$ and $\Pi^2(\tilde{p}^2)$ are coefficients which are related to the renormalization coefficients by:

$$\Pi^i(\tilde{p}^2 = \mu^2) = \frac{Z_q}{Z_i(\tilde{p}^2 = \mu^2)} \tag{15}$$

We will extract the renormalization coefficient proportional to the continuum tree level piece, $\mathcal{Z}_{\mathcal{O}} \equiv \mathcal{Z}_1$.

To solve Equation 14 for the Π^i, we construct a linear functional on the space of Dirac and Lorentz matrices by defining a form $\langle \cdot, \cdot \rangle$ whose action is:

$$\langle \lambda^1, \lambda^2 \rangle := \sum_{i \in \mathfrak{R}} tr_D \left\{ \lambda_i^1 \lambda_i^2 \right\} \tag{16}$$

where λ^1, λ^2 are objects with two Dirac and two Lorentz indices. Here, $\mathfrak{R}$ denotes the irrep of $H(4)$ that we are working in, which in our case is $\mathfrak{R} = \tau_1^{(3)}$. This form therefore sums over the normalized basis elements of the irrep:

$$\tau_1^{(3)} = span \Bigg\{ \frac{1}{\sqrt{2}} \left(\mathcal{O}_{\{33\}} - \mathcal{O}_{\{44\}} \right), \tag{17}$$

$$\frac{1}{\sqrt{2}} \left(\mathcal{O}_{\{11\}} - \mathcal{O}_{\{22\}} \right), \tag{18}$$

$$\frac{1}{2} \left(\mathcal{O}_{\{11\}} + \mathcal{O}_{\{22\}} - \mathcal{O}_{\{33\}} - \mathcal{O}_{\{44\}} \right) \Bigg\} \tag{19}$$

We apply the functionals $\langle \Lambda^1, \cdot \rangle$ and $\langle \Lambda^2, \cdot \rangle$ to Equation 14 to obtain the following system of equations which we can use to solve for the coefficients Π^1 and Π^2:

$$\mathcal{A}^{ab} \Pi^b = \begin{pmatrix} \langle \Lambda^1, \Gamma(p) \rangle \\ \langle \Lambda^2, \Gamma(p) \rangle \end{pmatrix} \tag{20}$$

where $\mathcal{A}^{ab}$ is the matrix of inner products:

$$\mathcal{A}^{ab} \equiv \langle \Lambda^a, \Lambda^b \rangle \tag{21}$$

where a, b range over 1 and 2. Using the explicit form for Λ^1 and Λ^2, the matrix $\mathcal{A}^{ab}$ can be computed directly using the identities $tr\{\gamma_\mu \gamma_\nu\} = 4\delta_{\mu\nu}$, $(\not{k})^2 = k^2$, and $tr\{1\} = 4$. The result is:

$$\mathcal{A}^{ab}(\tilde{p}) = \begin{pmatrix} 3\tilde{p}^2 & \frac{1}{\tilde{p}^2} \left(-3\tilde{p}^{[4]} + 2 \sum_{i<j} \tilde{p}_i^2 \tilde{p}_j^2 \right) \\ \frac{1}{\tilde{p}^2} \left(-3\tilde{p}^{[4]} + 2 \sum_{i<j} \tilde{p}_i^2 \tilde{p}_j^2 \right) & \frac{1}{\tilde{p}^2} \left(-3\tilde{p}^{[4]} + 2 \sum_{i<j} \tilde{p}_i^2 \tilde{p}_j^2 \right) \end{pmatrix} \tag{22}$$

where $p^{[4]}$ is the hypercubic invariant $\sum_i p_i^4$. We invert this matrix at each lattice momentum $\tilde{p}_\mu$ used in our computation to solve for the Π^i coefficients, and hence $\mathcal{Z}_{\mathcal{O}}$.

2.2 Lattice computation

We now turn our attention to computing the bare Green's functions $S^{(0)}(\tilde{p}^2)$ and $G^{(0)}(\tilde{p}^2)$. As in [3], we expand the diagonal components of our operator as:

$$\sum_z \mathcal{T}_{\mu\mu}^q(z) = \sum_{z,z'} \overline{q}(z)\, J_\mu(z, z')\, q(z') \tag{23}$$

Using the definition of the derivatives:

$$\vec{D}\psi(z) = \frac{1}{2}\left(U_\mu(z)\psi(z+\hat{\mu}) - U_\mu(n-\hat{\mu})^\dagger\psi(z-\hat{\mu})\right) \tag{24}$$

$$\overline{\psi}(z)\overleftarrow{D} = \frac{1}{2}\left(\overline{\psi}(z+\hat{\mu})U_\mu(z)^\dagger - \overline{\psi}(z-\hat{\mu})U_\mu(z-\hat{\mu})\right) \tag{25}$$

we find the current $J_\mu(z,z')$ is:

$$J_\mu(z,z') = \left[U_\mu(z)\delta_{z+\hat{\mu},z'} - U_\mu(z')^\dagger\delta_{z-\hat{\mu},z'}\right]\gamma_\mu \tag{26}$$

For the up-down quark operator difference $\mathcal{T}_\mu \equiv \mathcal{T}^u_\mu - \mathcal{T}^d_{\mu\mu}$, we use Equation 23 to expand:

$$\sum_z \mathcal{T}_\mu(z) = \sum_{z,z'}\left[\overline{u}(z)\,J_\mu(z,z')\,u(z') - \overline{d}(z)\,J_\mu(z,z')\,d(z')\right] \tag{27}$$

Plugging this into Equation 3, we find that we can expand the total up quark Green's function (here α,β are Dirac indices) as:

$$G^{\alpha\beta}(p) = \frac{1}{\sqrt{2}}\left(G_3^{\alpha\beta}(p) - G_4^{\alpha\beta}(p)\right) \tag{28}$$

where:

$$G_\mu^{\alpha\beta}(p) = \frac{1}{V}\sum_{x,y,z} e^{-ip(x-y)}\langle u^\alpha(x)\mathcal{T}_\mu(z)\overline{u}^\beta(y)\rangle \tag{29}$$

$$= \frac{1}{V}\sum_{x,y,z,z'} e^{-ip(x-y)}\left[\langle u^\alpha(x)\overline{u}^\sigma(z)J_\mu^{\sigma\rho}(z,z')u(z')^\rho\overline{u}^\beta(y)\rangle - \langle u^\alpha(x)\overline{d}^\sigma(z)J_\mu^{\sigma\rho}(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle\right] \tag{30}$$

Now we perform all possible Wick contractions on the matrix elements to write them as propagators:

$$\begin{aligned}\langle u^\alpha(x)\overline{u}^\sigma(z)J_\mu^{\sigma\rho}(z,z')u^\rho(z')\overline{u}^\beta(y)\rangle &= \langle \overbrace{u\overline{u}J}\,\overbrace{u\overline{u}}\rangle + \langle \overbrace{u\overline{u}J\,u\overline{u}}\rangle \\ &= S^{\alpha\sigma}(x,z)J_\mu^{\sigma\rho}(z,z')S^{\rho\beta}(z',y) + (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)\end{aligned} \tag{31}$$

$$\begin{aligned}\langle u^\alpha(x)\overline{d}^\sigma(z)J_\mu^{\sigma\rho}(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle &= \langle \overbrace{u\overline{d}Jd\overline{u}}\rangle \\ &= (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)\end{aligned} \tag{32}$$

where the factors of (-1) come from rearranging the contraction so that the contracted pieces are of the form $\langle u\overline{u}\rangle$. The vacuum pieces cancel because the up and down quark propagators are degenerate, so the final result is very clean:

$$G_\mu(p) = \frac{1}{V}\sum_{x,y,z,z'} e^{ip(x-y)}S(x,z)J_\mu(z,z')S(z',y) \tag{33}$$

This is our central equation, but note that computing this directly on the lattice would involve computing the two point propagator $S(x,y)$ at each two points on the lattice. This is much too computationally intensive, so we must resort to other techniques to accomplish this.

There are two primary ways to compute this on the lattice. We can compute this directly using momentum sources, or we can use the sequential source technique. Momentum sources work specifically for Equation 33, but produce a significantly better signal on a small number of configurations. On the other hand, sequential source is much more general, but produces more noise. We will discuss each method below.

2.3 Momentum sources

Observe that we can rewrite Equation 33 as:

$$\begin{aligned} G_\mu(p) &= \frac{1}{V}\sum_{x,y,z,z'} e^{ipx} S(x,z) J_\mu(z,z') e^{-ipy} S(z',y) \\ &= \frac{1}{V}\sum_{z,z'} \gamma_5 \left(\sum_x S(z,x) e^{-ipx}\right)^\dagger \gamma_5 J_\mu(z,z') \left(\sum_y S(z',y) e^{-ipy}\right) \\ &= \frac{1}{V}\sum_{z,z'} \gamma_5 \tilde{S}_p(z)^\dagger \gamma_5 J_\mu(z,z') \tilde{S}_p(z') \end{aligned} \tag{34}$$

where we have defined $\tilde{S}_p(z)$ as:

$$\tilde{S}_p(z) = \sum_x S(z,x) e^{-ipx} \tag{35}$$

The advantage of casting the equation in this form is that we can solve for $\tilde{S}_p(z)$ directly by inverting the Dirac equation with a momentum source, i.e. we have:

$$\sum_z D(x,z)\tilde{S}_p(z) = e^{-ipx} \tag{36}$$

where $D(x,z)$ is the Dirac operator. This means that upon solving for $\tilde{S}_p(z)$ and plugging this into Equation 34, we can solve directly for $G_\mu(p)$.

We can also use the propagator we get from the momentum source inversion to directly compute the propagator in Equation **??** as follows:

$$S(p) = \frac{1}{V}\sum_{x,y} e^{ip\cdot(x-y)} S(x,y) = \frac{1}{V}\sum_x e^{ip\cdot x}\tilde{S}_p(x) \tag{37}$$

This is an exact equation and it does not rely on translational invariance in the infinite statistics limit. Therefore, this method will give much better signal and can be run efficiently on a small number of configurations. The downside to this is that we require a propagator inversion for each choice of sink momentum. To compute $G(p)$ for a large number of sink momenta, as we need to do to extrapolate $\mathcal{Z}(\mu)$ in the continuum limit, a propagator inversion at each sink momenta is not feasible. We must instead choose the sink momentum wisely to be able to extract the discretization artifacts and extrapolate $\mathcal{Z}(\mu)$ to the continuum, which we will describe in Section 3.2.

2.4 Sequential source method

In practice we will use the sequential source method, which if implemented correctly does not force us to invert a propagator at every sink momenta. This technique is also much more general than the one previously described, but it suffers from more noise because it relies on the translational invariance of the lattice, which only exists in the infinite statistics limit. The idea of the sequential source method is that if we have an equation involving the full propagator $S(x,y)$, we can invert a source which depends on the propagator $S(x)$. For example, in this problem we wish to evaluate Equation 33, but we cannot simply evaluate $S(x,y)$ for every x and y. To get around this, consider using a source

$$b(z) = \sum_{z'} J_\mu(z,z') S(z',0) \tag{38}$$

to invert the Dirac equation, which will solve for $M(x)$ in this equation:

$$\sum_x D(z,x)M(x) = b(z) \tag{39}$$

where $D(x,z)$ is the Dirac operator. Upon inversion, using that $\sum_z S(y,z)D(z,x) = \delta(y-x)$, we can move the Dirac operator to the other side as $D^{-1}(y,z) = S(y,z)$ and obtain:

$$M(x) = \sum_z D^{-1}(x,z)b(z) = \sum_{z,z'} S(x,z)J_\mu(z,z')S(z',0) \tag{40}$$

Note that we have summed the full propagator $S(x,z)$ for the price of a single inversion of the source $b(z)$. We can then reconstruct Equations **??** and 33 for the momentum-projected Green's function and propagator in the infinite statistics limit when translational invariance is restored:

$$G_\mu(p) \approx \frac{1}{V}\sum_x e^{ipx}M(x) = \frac{1}{V}\sum_{x,z,z'} e^{ipx}S(x,z)J_\mu(z,z')S(z',0) \tag{41}$$

$$S(p) \approx \frac{1}{V}\sum_x e^{ipx}S(x,0) \tag{42}$$

We can improve the performance of this by randomly sampling N lattice points to approximate the sums over y in our equations for $S(p)$ and $G(p)$. If we sample the points $\{y_i\}_{j=1}^N$, then we can make the replacement $\sum_y \mapsto \frac{V}{N}\sum_j$, and so we have:

$$G_\mu(p) \approx \frac{1}{N}\sum_{x,z,z'}\sum_j e^{ip(x-y_j)}S(x,z)J_\mu(z,z')S(z',y_j) \tag{43}$$

$$S(p) \approx \frac{1}{N}\sum_x\sum_j e^{ip\cdot(x-y_j)}S(x,y_j) \tag{44}$$

Since we are interested in renormalizing the operators in each irrep, we must take the operators we are measuring to be:

$$\mathcal{O}_1 = \frac{1}{\sqrt{2}}\left(\mathcal{T}_{33} - T_{44}\right) \tag{45}$$

$$\mathcal{O}_2 = \frac{1}{\sqrt{2}}\left(\mathcal{T}_{11} - T_{22}\right) \tag{46}$$

$$\mathcal{O}_3 = \frac{1}{2}\left(\mathcal{T}_{11} + \mathcal{T}_{22} - \mathcal{T}_{33} - T_{44}\right) \tag{47}$$

which give us currents:

$$\mathcal{J}_1 = \frac{1}{\sqrt{2}}(J_3 - J_4) \tag{48}$$

$$\mathcal{J}_2 = \frac{1}{\sqrt{2}}(J_1 - J_2) \tag{49}$$

$$\mathcal{J}_3 = \frac{1}{2}(J_1 + J_2 - J_3 - J_4) \tag{50}$$

and we must compute the 3 propagators using $\mathcal{J}_i$ QLUA.

To check an implementation of this method, we can invert a sequential propagator which depends on momentum. If we replace $S(z', 0)$ in Equation 38 and invert, we find that:

$$b_\mu^{(p)}(z) = \sum_{z'} J_\mu(z, z')\tilde{S}_p(z') = \sum_{z',y} e^{-ipy} J_\mu(z, z')S(z', y) \tag{51}$$

$$M_\mu^{(p)}(x) = \sum_{z} S(x, z)b_\mu^{(p)}(z) = \sum_{z,z',y} e^{-ipy} S(x, z)J_\mu(z, z')S(z', y) \tag{52}$$

When we momentum project to find the Green's function $G(p)$, this no longer relies on translation invariance and should match our result with the momentum inversion exactly:

$$G_\mu(p) = \sum_{x} e^{ipx} M_\mu^{(p)}(x) = \sum_{x,y,z,z'} e^{ip(x-y)} S(x, z)J_\mu(z, z')S(z', y) \tag{53}$$

In our case with a large amount of sink momenta, this method is much more robust than inverting a momentum source because we one inversion can give us $G(p)$ at every value of the sink momentum. We will call this construction going **through the operator**, because the inversion in Equation 40 projects the current insertion onto $q = 0$ momentum. If we had been interested in projecting the operator onto different momentum values, then we would need to use a new sequential source (modify Equation 38) for each value of the operator momentum. Pictorially, we are inverting at the operator momentum, then tying up at the sink momentum. On the other hand, we reverse the direction of inversion and invert our propagator at each sink momentum first, then tie up the line at the operator. This method is called going **through the sink**. We can represent these different methods below, where in our case $q = 0$.

In this problem inversion through the sink would require too many propagator inversions like in the previous momentum source method, and it would also be noisy like inversion through the operator. As such, there is no reason to consider it, and I included it here mainly for generality.

Our result for the coefficient in the RI'-MOM scheme is plotted below.

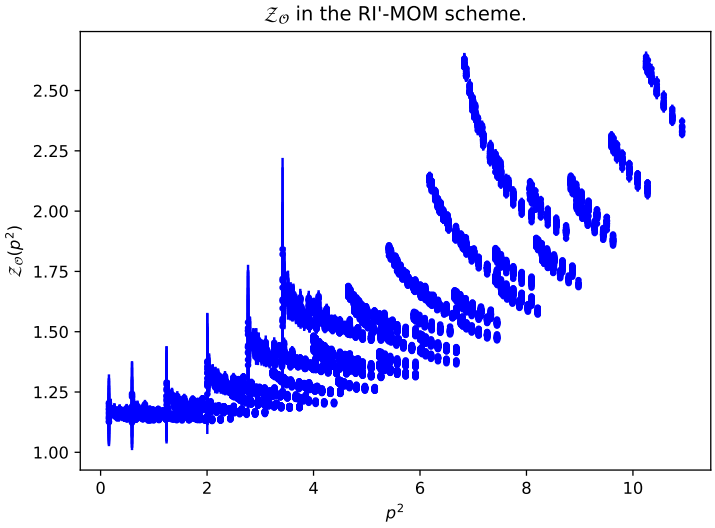

Figure 1: Renormalization coefficient for $\mathcal{Z}_\mathcal{O}$ in the RI'-MOM scheme.

3 Analysis

3.1 Matching to $\overline{MS}$

To convert our results from RI'-MOM to $\overline{MS}$, we will use J.A. Gracey's results for the renormalization coefficients $\mathcal{Z}_{\mathcal{O}}$ [4] and $\mathcal{Z}_q$ [5], computed using lattice perturbation theory both in the RI'-MOM and the $\overline{MS}$ schemes.

We will match our results to $\overline{MS}$ by converting them multiplicatively with the corresponding matching coefficient:

$$C(\mu_0) \equiv \frac{\mathcal{Z}^{\overline{MS}}(\mu_0)}{\mathcal{Z}^{RI'-MOM}(\mu_0)} \tag{54}$$

evaluated at the scale $\mu_0 \equiv 2\,GeV$. This is done in two steps:

1. Use the RI'-MOM anomalous dimension γ to run $Z(\mu)$ to the matching scale μ_0.
2. Convert to $\overline{MS}$ by multiplying with C.

We will work through these two steps in detail for the renormalization coefficient $\mathcal{Z}_{\mathcal{O}}$. We begin by running the strong coupling $\alpha(\mu)$ to each renormalization point, as it needed to determine γ. This running is performed using the RunDec package [6] to integrate the 3 loop beta function. We initialize the running at the scale of the Z boson to be

$$\alpha(m_Z) = 0.119 \tag{55}$$

which has been determined by the Particle Data Group (PDG) [7]. We will also use the PDG for the mass of the Z-boson and the quark masses:

$$m_z = 91.187 \pm 0.0021 \text{ GeV} \qquad m_b = 4.18^{+0.03}_{-0.02} \text{ GeV} \tag{56}$$

$$m_c = 1.27 \pm 0.02 \text{ GeV} \qquad m_s = 93^{+11}_{-5} \text{ MeV} \tag{57}$$

We use the 5-flavor QCD beta function to flow $\alpha(m_Z)$ to m_b, then the 4-flavor beta function to flow $\alpha(m_b)$ to m_c, and finally use the 3 flavor beta function to determine $\alpha(\mu_0)$. For any renormalization scales $\mu < m_s$, we flow to m_s using the 3 flavor beta function, then flow $\alpha(m_s)$ to the corresponding scale μ with 2 flavors. Our result for $\mu_0 = 2$ GeV is:

$$\alpha(\mu_0 = 2 \text{ GeV}) = 0.3069 \tag{58}$$

We are now in position to run $\mathcal{Z}_{\mathcal{O}}^{RI'-MOM}(\mu)$ to μ_0 for each renormalization point μ. We can integrate the running equation for $\mathcal{Z}(\mu)$ from mu_0 to μ to see that in lattice perturbation theory:

$$\frac{d\log\mathcal{Z}}{d\log\mu} = -\gamma \implies \left(\frac{\mathcal{Z}(\mu)}{\mathcal{Z}(\mu_0)}\right)_{P.T.} = \exp\left(-\int_{\mu_0}^{\mu_f}\frac{d\mu}{\mu}\gamma(\mu)\right) \tag{59}$$

where $\mathcal{Z}$ and γ are evaluated in RI'-MOM, and $\gamma(\mu)$ is explicitly given by Equation 2.13 in [4]. Note that in Gracey's notation, the strong coupling is $a(\mu) = \frac{g^2}{16\pi^2} = \frac{\alpha(\mu)}{4\pi}$. We have plotted the solution for this ratio in the following figure.

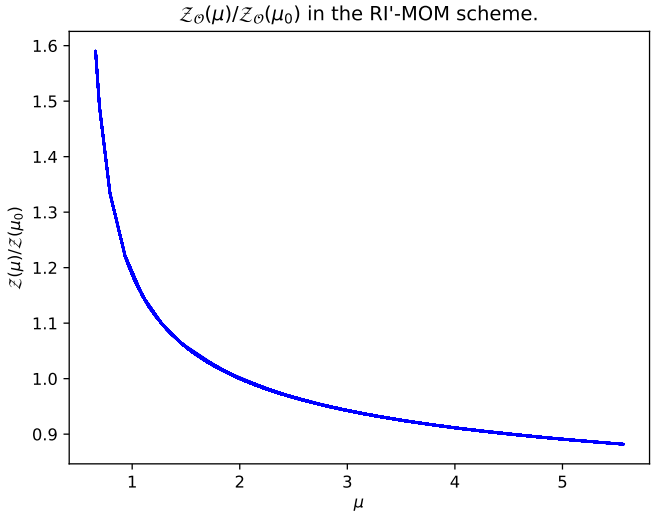

Figure 2: Ratio $\mathcal{Z}(\mu)/\mathcal{Z}(\mu_0)$ in the RI'-MOM scheme.

To convert a R.C. given in RI'-MOM to $\overline{MS}$, we use Gracey's result for the matching coefficients, Equation 4.3 of [4]. This explicitly gives us:

$$C_{\mathcal{O}}(a,0) = \frac{\mathcal{Z}_{RI'-MOM}(\mu_0)}{\mathcal{Z}_{\overline{MS}}(\mu_0)} \tag{60}$$

and we take $C_{\mathcal{O}}^{-1}$ to pass from the RI'-MOM scheme to the $\overline{MS}$ scheme.

Using these pieces, we can explicitly convert the renormalization coefficients computed at scale μ to $\overline{MS}$:

$$\mathcal{Z}_{\mathcal{O}}^{\overline{MS}}(\mu) = C_{\mathcal{O}}^{-1}\left(\frac{\mathcal{Z}(\mu_0)}{\mathcal{Z}(\mu)}\right)_{P.T.} \mathcal{Z}_{\mathcal{O}}^{RI'-MOM}(\mu) \tag{61}$$

Our result for $\mathcal{Z}_{\mathcal{O}}^{\overline{MS}}$ is compiled in the following figure.

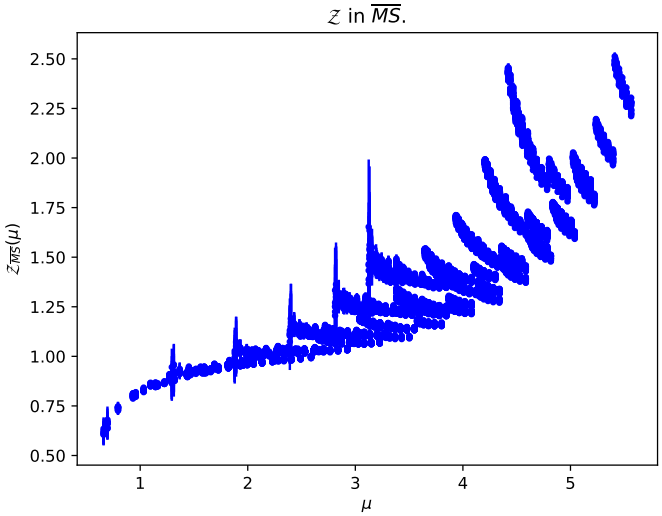

Figure 3: $\overline{MS}$ renormalization coefficient.

3.2 Fitting discretization artifacts

When we compute observables at a finite lattice spacing a, we suffer **discretization artifacts** which are relics of the explicit symmetry breaking $SO(1,3) \to H(4)$ suffered by putting the theory on a lattice. Namely, because we have less symmetry, there are more invariant quantities of p^μ in a lattice theory than in the continuum. In the continuum, the basic invariant that we can create is $p^2 = p_\mu p^\mu$.

There are two basic types of discretization artifacts which will show up: artifacts from the breaking of Lorentz invariance $O(4) \to H(4)$, and artifacts from running. The symmetry breaking artifacts will appear in the data as "fans". When these are removed, the data will appear to have a smooth structure, at which point one can solve for the running artifacts by expanding in all possible terms consistent with symmetry.

To fit the artifacts from symmetry breaking, we may use the $\mathbf{p^{[2n]}}$ **extrapolation method**. This amounts to expanding the artifacts in a Taylor series in hypercubic invariants.

On the lattice, we can find other invariants because the orbits of $H(4)$ are strictly smaller than the orbits of $O(4)$, the Euclidean isometry group in $d = 4$. For example, the vectors $(2,0,0,0)$ and $(1,1,1,1)$ have the same value of $p^2 = 4$, yet there is no element $g \in H(4)$ such that $g \cdot (1,1,1,1) = (2,0,0,0)$, i.e. they cannot be rotated into one another by hypercubic symmetry. This is because we can define *other hypercubic invariants than just* p^2. The functions:

$$p^{[2n]} := \sum_\mu p^{2n} \tag{62}$$

for $n \in \mathbb{N}$ are also invariants, and these dictate the orbits of momenta under $H(4)$. Since $(1,1,1,1)$ has $p^{[4]} = 4$ and $(2,0,0,0)$ has $p^{[4]} = 16$, we can conclude they are distinct momenta under hypercubic symmetry and thus cannot live in the same orbit of $H(4)$.

Any function which is invariant under the action of $H(4)$ must be a function of these hypercubic invariants, much like how in the continuum any function which was invariant under Lorentz symmetry was a function of Lorentz scalars like p^2 or $\not{p}$. Because we are computing out quantities on the lattice which only has $H(4)$ symmetry, these extra terms like $p^{[4]}$ can come into play when form factors or renormalization constants are computed, and this extrapolation method will account for these.

Another source of error that we must consider when performing calculations on the lattice is that lattice momenta is quantized. The possible values that the momenta can take are:

$$p_\mu = \frac{2\pi}{aL_\mu} k_\mu \tag{63}$$

where $k_\mu \in \mathbb{Z}$ and L_μ is the size of the lattice in direction μ. In the lattice action, the momentum is modified to become:

$$\tilde{p}_\mu = \frac{1}{a} \sin(a p_\mu) \tag{64}$$

and in the small a limit, notice that this reduces to the standard momentum values p_μ. As a result, our renormalization coefficient computed on the lattice will be a function of $\tilde{p}_\mu$, *not* a function of p_μ. We wish to account for this and for our final result to be a function of p_μ, so we must take this into account when fitting the hypercubic artifacts.

TODO finish this section

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