Renormalization

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1 Regularization-Independent Momentum Subtraction (RI-MOM)

The RI-MOM scheme (also known as Rome-Southampton) is a renormalization scheme well equipped to deal with the lattice, namely because we can calculate the relevant quantities that define the scheme easily on the lattice. The lattice spacing a provides us a with a natural UV regulator, and RI-MOM will tell us how to go from such a regulated result to a physical observable quantity.

The RI-MOM has a relatively simple renormalization condition. We will define it here for an arbitrary Green's function, for example a three point function. For a renormalization scale μ and working in a fixed gauge, **we define the amputated, renormalized Green's function at momentum $p^2 = -\mu^2$ to be equal to its tree level value**.

In practice on the lattice, there are two things that we must compute. We will work with a specific example here for a given quark field $q(x)$. Suppose the operator we are trying to compute is $\mathcal{O}(z)$. We will renormalize the three point function:

$$G(p) = \frac{1}{V} \sum_{x,y,z} e^{-ip\cdot(x-y)} \langle q(x)\mathcal{O}(z)\overline{q}(y)\rangle \tag{1}$$

i.e. we are projecting the source and sink to a definite momentum and projecting the operator $\mathcal{O}(z)$ to zero momentum.

We also will need to compute the momentum projected propagator:

$$S(p) = \frac{1}{V} \sum_{x,y} e^{-ip\cdot(x-y)} S(x,y) \tag{2}$$

where $S_{ij}^{ab}(x,y) = \langle q_i^a(x)\bar{q}_j^b(y)\rangle$ is the standard propagator. On the lattice, the two objects that we must compute directly are $S(p)$ and $G(p)$, and everything else follows once we have these quantities.

To explain the method, our goal is to compute the operator renormalization:

$$\mathcal{O}_R(\mu) = \mathcal{Z}(\mu)\mathcal{O}_{lat} \tag{3}$$

where $\mathcal{O}_R(\mu)$ is our renormalized operator, $\mathcal{Z}(\mu)$ is the renormalization coefficient of interest, and $\mathcal{O}_{lat}$ is the lattice (bare) operator. We will also assume the quark fields have been renormalized by some quark field renormalization $\mathcal{Z}_q$:

$$q_{lat} = \sqrt{\mathcal{Z}_q}\, q_R(\mu) \tag{4}$$

There is an analytical expression for $\mathcal{Z}_q$ on the lattice in the RI-MOM scheme, and it has been determined to be:

$$\mathcal{Z}_q(p)|_{p^2=-\mu_R^2} = \left[\frac{tr\left\{-i\sum_{\nu=1}^4 \gamma_\nu \sin(ap_\nu) a S(p)^{-1}\right\}}{12\sum_{\nu=1}^4 \sin^2(ap_\nu)}\right]_{p^2=-\mu^2} \tag{5}$$

The twelve on the bottom is a normalization $12 = 3 \times 4$ for the number of color and number of spin indices, which we will see in many of the expressions.

Let $\Gamma(p)$ be the amputated three point function, bare or renormalized. We can relate Γ to the other quantities we have already computed by using the inverse of the propagator to manually cut the legs off the full three point function:

$$\Gamma(p) = S(p)^{-1} G(p) S(p)^{-1} \tag{6}$$

We will denote the tree level version of this by Γ_B, where the B subscript stands for Born term.

Using our quantities already computed on the lattice, we can compute the bare $\Gamma_{lat}(p)$ directly in terms of the renormalized quantities. In the continuum limit, the renormalized Green's function will be:

$$G_R(p;\mu) = \int d^4x\, d^4y\, d^4z\, e^{-ip\cdot(x-y)} \langle q_R(x;\mu) \mathcal{O}_R(z;\mu) q_R(y;\mu) \rangle \tag{7}$$

$$= \mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) \int d^4x\, d^4y\, d^4z\, e^{-ip\cdot(x-y)} \langle q_{lat}(x) \mathcal{O}_{lat}(z) q_{lat}(y) \rangle \tag{8}$$

$$= \mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) G_{lat}(p) \tag{9}$$

This relation translates immediately to the amputated Green's function $\Gamma(p)$:

$$\Gamma_R(p;\mu) = \mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) \Gamma_{lat}(p) \tag{10}$$

We are now in a position to apply the renormalization condition. We must equate the renormalized, amputated Green's function to the tree level Green's function $\Gamma_B(p)$. This gives us:

$$\mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) \Gamma(p) = \Gamma_B(p) \tag{11}$$

Dividing by a conventional factor of 12 as a normalization and inverting, we can clean this expression up into a simple equation for $\mathcal{Z}(\mu)$:

$$\mathcal{Z}(p^2 = -\mu^2) = \left[\frac{12 \mathcal{Z}_q(p)}{tr\, \{\Gamma(p) \Gamma_B(p)^{-1}\}} \right]_{p^2=-\mu^2} \tag{12}$$

2 Continuum Schemes: $\overline{MS}$, on-shell, and off-shell

3 Matching between schemes

4 Example: Isospin

We are interested here in computing matrix elements of the operator:

$$\mathcal{O}(z) = \mathcal{O}_u(z) - \mathcal{O}_d(z) \tag{13}$$

where the quark operators $\mathcal{O}_q$ are given by

$$\mathcal{O}_q(z) = \frac{1}{\sqrt{2}} (\mathcal{T}^q_{33}(z) - \mathcal{T}^q_{44}(z)) \tag{14}$$

and the irreducible tensor operators $\mathcal{T}^q_{\mu\nu}$ are defined as

$$\mathcal{T}^q_{\mu\nu} = \overline{q}(z)\, \gamma_{\{\mu} \overleftrightarrow{D}_{\nu\}}\, q(z) \tag{15}$$

with $\overleftrightarrow{D} = \vec{D} - \overleftarrow{D}$ the symmetrized covariant derivative. Note we define the symmetric and traceless component of a tensor to be:

$$a_{\{\mu}b_{\nu\}} = \frac{1}{2}(a_\mu b_\nu + a_\nu b_\mu) - \frac{1}{4}a_\alpha b^\alpha g_{\mu\nu} \tag{16}$$

We are inserting the operator $\mathcal{O}_q$ with momentum $\vec{p} = 0$. We will focus our analysis on the tensor operator $\mathcal{T}^q_{\mu\mu}$ (note there is no sum on μ here), and note that $\mathcal{O}(z)$ can be obtained through linearity. We may write:

$$\sum_z \mathcal{T}^q_{\mu\mu}(z) = \sum_{z,z'} \overline{q}(z)\, J_\mu(z,z')\, q(z') \tag{17}$$

Plugging in the definition of the derivatives:

$$\vec{D}\psi(z) = \frac{1}{2}\Big(U_\mu(z)\psi(z+\hat{\mu}) - U_\mu(n-\hat{\mu})^\dagger \psi(z-\hat{\mu})\Big) \tag{18}$$

$$\overline{\psi}(z)\overleftarrow{D} = \frac{1}{2}\Big(\overline{\psi}(z+\hat{\mu})U_\mu(z)^\dagger - \overline{\psi}(z-\hat{\mu})U_\mu(z-\hat{\mu})\Big) \tag{19}$$

we find the current $J_\mu(z,z')$ is:

$$J_\mu(z,z') = \Big[U_\mu(z)\delta_{z+\hat{\mu},z'} - U_\mu(z')^\dagger \delta_{z-\hat{\mu},z'}\Big]\gamma_\mu \tag{20}$$

We may now use this expansion to compute the three point function for the operator $\mathcal{T}_\mu = \mathcal{T}^u_{\mu\mu} - \mathcal{T}^d_{\mu\mu}$ (we can simply take $\mathcal{T}_3 - \mathcal{T}_4$ to get the operator of interest in Equation 13). Using Equation 17, we write

$$\sum_z T_\mu(z) = \sum_{z,z'} \left[\overline{u}(z)\, J_\mu(z,z')\, u(z') - \overline{d}(z)\, J_\mu(z,z')\, d(z')\right] \tag{21}$$

Plugging this in, we find that for the up quark Green's function (here α, β are Dirac indices):

$$\begin{aligned} G^{\alpha\beta}_\mu(p) &= \sum_{x,y,z} e^{-ip(x-y)} \langle u^\alpha(x)\mathcal{T}_\mu(z)\overline{u}^\beta(y)\rangle \qquad (22)\\ &= \sum_{x,y,z,z'} e^{-ip(x-y)} \left[\langle u^\alpha(x)\overline{u}^\sigma(z)J^{\sigma\rho}_\mu(z,z')u(z')^\rho \overline{u}^\beta(y)\rangle - \langle u^\alpha(x)\overline{d}^\sigma(z)J^{\sigma\rho}_\mu(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle\right] \qquad (23) \end{aligned}$$

Now we perform all possible Wick contractions on the matrix elements to write them as propagators:

$$\begin{aligned} \langle u^\alpha(x)\overline{u}^\sigma(z)J^{\sigma\rho}_\mu(z,z')u^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{u}Ju\overline{u}\rangle + \langle u\overline{u}Ju\overline{u}\rangle \\ &= S^{\alpha\sigma}(x,z)J^{\sigma\rho}_\mu(z,z')S^{\rho\beta}(z',y) + (-1)^3 S^{\alpha\beta}(x,y)J^{\sigma\rho}_\mu(z,z')S^{\rho\sigma}(z',z) \qquad (24) \end{aligned}$$

$$\begin{aligned} \langle u^\alpha(x)\overline{d}^\sigma(z)J^{\sigma\rho}_\mu(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{d}Jd\overline{u}\rangle \\ &= (-1)^3 S^{\alpha\beta}(x,y)J^{\sigma\rho}_\mu(z,z')S^{\rho\sigma}(z',z) \qquad (25) \end{aligned}$$

where the factors of (-1) come from rearranging the contraction so that the contracted pieces are of the form $\langle u\overline{u}\rangle$. The vacuum pieces cancel because the up and down quark propagators are degenerate, so the final result is very clean:

$$G_\mu(p) = \sum_{x,y,z,z'} e^{-ip(x-y)} S(x,z)J(z,z')S(z',y) \tag{26}$$