Operator Renormalization

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1 Local Operator Renormalization

Renormalization of a theory is often seen as a program that replaces the bare couplings and fields in a Lagrangian by renormalized quantities to make the observable quantities we can compute from a theory finite in the UV or the IR. However, renormalization can be viewed from a much more general perspective in terms of local composite operators. A **local composite operator** is an operator $\mathcal{O}(x)$ which is built up from the fields and derivatives in a theory at a single spacetime point x. Schematically for a theory with fermion fields $\{\Psi_i(x)\}$ and a gauge field A_μ, such an operator looks like:

$$\mathcal{O}_{\mu...\nu\rho...\sigma}(x) = \mathcal{Z}_{\mathcal{O}} D_\mu ... D_\nu A_\sigma ... A_\rho \Gamma_{\ell_1} ... \Gamma_{\ell_k} \Psi_{i_1} ... \Psi_{i_k} \overline{\Psi}_{j_1} ... \overline{\Psi}_{j_k} \tag{1}$$

where $\mathcal{Z}_{\mathcal{O}}$ is a normalization constant which we will use to renormalize the operator $\mathcal{O}$, and Γ is any matrix with a Dirac structure. The operator can have any number of spin indices that one wants, but often we will consider a relatively small value for its spin because the operators which are dominant in the OPE are those with the smallest twist[1].

Even after renormalizing all the fields and couplings in the Lagrangian, matrix elements of composite operators are often divergent, which is why we need to renormalize them separately. Take for example the renormalization of the operator $\mathcal{O} := \phi^2$ in the ϕ^4 scalar field theory. When we renormalize the bare field $\phi_0 = \mathcal{Z}_\phi(\mu)\phi(\mu)$, this does not immediately renormalize $\mathcal{O}$. This is because as an operator, ϕ^2 is implicitly the normal ordering $:\phi^2:$ of the actual ϕ^2 operator. Because of this normal ordering, renormalization constants are not multiplicative[2], i.e. we cannot just take $\mathcal{O} = \mathcal{Z}(\mu)^2$.

To renormalize a composite operator, it suffices to compute a (divergent) Green's function and impose renormalization conditions on that. Typically the renormalization condition on this Green's function will be setting it equal to its tree level value.

When we study composite operator renormalization, we do not need to even insert the operator into the Lagrangian. Instead, we will view the operator $\mathcal{O}$ we wish to renormalize as an **external operator** which is separate from the theory's Lagrangian. The goal of renormalizing $\mathcal{O}$ is to make any Green's function of $\mathcal{O}$ with another operator finite, and we can do this without introducing a coupling. Later we will see what happens when we renormalize an external operator then add it to a Lagrangian; based on the operator renormalization, we will immediately be able to read off how its Wilson coefficient flows.

For any of these computations, we will need to understand how to diagrammatically compute Green's functions. We can use what Schwartz calls **off-shell Feynman rules** to diagrammatically compute this Green's function. Off-shell Feynman rules are the same as the typical Feynman rules for a theory, with some extra caveats:

[1] The twist of an operator is its spin s (number of Lorentz indices) minus the mass dimension d. Operators must have twist ≥ 2, so we will often consider twist 2 operators.

[2] More on that here: https://www.physicsoverflow.org/27963/renormalization-determined-renormalization-elementary

- We replace external polarizations with propagators. Because of this, we will often want to consider amputated Green's functions without the propagators as the objects to renormalize.
- Current insertions have their own set of rules, and are taken at non-zero momenta. This means currents inject momentum into our diagrams, and this must be considered when labeling all the momenta.
- We do not always "work backwards" from the tip of the arrow to its bottom when computing diagrams with fermion lines (as we will see in the example).

The main tool we will use to expand these diagrams in perturbation theory is the following relation between operators in the interacting theory and operators in the free theory:

$$\langle 0|T\left\{\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\right\}|0\rangle = \frac{1}{\mathcal{A}}\langle 0|T\left\{\exp\left[i\int d^4z\,\mathcal{L}^0_{int}(z)\right]\mathcal{O}^0_1(x_1)...\mathcal{O}^0_n(x_n))\right\}|0\rangle \tag{2}$$

where $\mathcal{O}^0$ denotes an operator in the free theory, and $\mathcal{L}^0_{int}$ is the interaction Lagrangian in the free theory. $\mathcal{A}$ is a normalization given by:

$$\mathcal{A} = \langle 0|T\left\{\exp\left(i\int d^4z\,\mathcal{L}^0_{int}(z)\right)\right\}|0\rangle \tag{3}$$

With this expansion, we can draw Feynman diagrams. Intuitively it helps to compare this to a derivation of Feynman diagrams in Schwartz, where he shows how to compute n-point functions diagrammatically using Equation 7.64 to relate them to their free field values:

$$\langle 0|T\left\{\phi(x_1)...\phi(x_n)\right\}|0\rangle = \frac{1}{\mathcal{A}}\langle 0|T\left\{\exp\left[i\int d^4z\,\mathcal{L}^0_{int}(z)\right]\phi^0(x_1)...\phi^0(x_n)\right\}|0\rangle \tag{4}$$

The key point in this equation (as in Equation 2) is that we can Taylor expand the exponential and contract the fields on the right hand side of the equation to give ourselves propagators and derive Feynman rules for the theory.

Once we renormalize an operator, an important quantity to compute is the operator's **anomalous dimension** $\gamma_{\mathcal{O}}$, which describes how the operator flows under renormalization:

$$\gamma_{\mathcal{O}} := -\frac{\mu}{Z_{\mathcal{O}}}\frac{d\mathcal{Z}_{\mathcal{O}}}{d\mu} = -\mu\frac{d}{d\mu}\log\mathcal{Z}_{\mathcal{O}} \tag{5}$$

Once we have determined the counterterm for the operator $\mathcal{Z}_{\mathcal{O}}$, we can immediately get the anomalous dimension. This quantity acts similarly to the β function for a coupling, and also appears in the Callan-Symanzik equation on equal terms.

TODO discuss Callan-Symanzik.

1.1 Example: QED current

We will do an example in QED, and compute the renormalization of $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$. The simplest non-vanishing Green's function involving $j^\mu(x)$ is the three point function, which we will expand in momentum space:

$$\langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle = \int d\!\!{}^-{}^4p\,d\!\!{}^-{}^4q_1\,d\!\!{}^-{}^4q_2\,e^{-ipx}e^{-ip_1x_1}e^{iq_2x_2}i\mathcal{M}^\mu(p,q_1,q_2)(2\pi)^4\delta^4(p+q_1-q_2) \tag{6}$$

We have chosen the signs on the momenta to replicate a current insertion of momentum p; namely, if a particle is propagating initially with momentum q_1, the current insertion knocks the momentum of the particle so that:

$$q_2 = p + q_1 \tag{7}$$

We will compute this in perturbation theory to one loop and associate a diagram with each term. Using the expansion in Equation 2, we can expand the exponential to one loop:

$$\exp\left(i\int d^4z\,\mathcal{L}^0_{int}(z)\right) \sim 1 + i\int d^4z\,\mathcal{L}^0_{int} + \frac{i^2}{2}\int d^4z\,d^4z'\,\mathcal{L}^0_{int}(z)\mathcal{L}^0_{int}(z') + ... \tag{8}$$

When inserted into the equation, the first piece gives the tree level vertex and the piece with two integrals gives the first loop correction. Plugging in at first order, we can use Wick's theorem to evaluate the free field correlators (here $\alpha, \beta, \rho, \sigma$ are Dirac indices):

$$\langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle_{\text{tree}} = \langle j_0^\mu(x)\psi_0(x_1)\overline{\psi}_0(x_2)\rangle \tag{9}$$

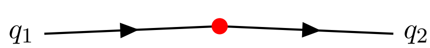
$$= \langle\overline{\psi}^\alpha_0(x)\gamma^\mu_{\alpha\beta}\psi^\beta_0(x)\psi^\rho_0(x_1)\overline{\psi}^\sigma_0(x_2)\rangle \tag{10}$$

$$= S(x_1, x)\gamma^\mu S(x, x_2) \tag{11}$$

as $\langle\psi(x)\overline{\psi}(y)\rangle = S(x,y) = i\int đ^4k\,\frac{e^{-ik(x-y)}}{\not{k}-m}$. We can take this tree level result to momentum space, where we see:

$$\begin{aligned}
\mathcal{M}^\mu(p,q_1,q_2)_{\text{tree}} &= \int d^4x\,d^4x_1\,d^4x_2\,e^{ipx}e^{iq_1x_1}e^{-iq_2x_2}\langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle_{tree} \\
&= \int d^4x\,d^4x_1\,d^4x_2\,e^{ipx}e^{iq_1x_1}e^{-iq_2x_2}\int đ^4k\int đ^4k'\,\frac{ie^{-ik(x_1-x)}}{\not{k}-m}\gamma^\mu\frac{ie^{-ik'(x-x_2)}}{\not{k}'-m} \\
&= \int đ^4k\,đ^4k'\,d^4x\,d^4x_1\,d^4x_2 e^{i(p+k-k')x}e^{i(q_1-k)x_1}e^{i(k'-q_2)x_2}\frac{i}{\not{k}-m}\gamma^\mu\frac{i}{\not{k}'-m} \\
&= \frac{i}{\not{q}_1-m}\gamma^\mu\frac{i}{\not{q}_2-m}\delta^4(p+q_1-q_2)
\end{aligned} \tag{12}$$

where we leave the delta function in to emphasize that we **must enforce momentum conservation from current insertion**. Diagrammatically, this corresponds to the diagram



(13)

where the red dot denotes a current insertion of momentum p.

Now we move towards the actual process of renormalization: we can expand Equation 2 to one loop and compute the one loop 1PI correction, then renormalize the operator. This is going to get very tiring very soon, but I think it is instructive to get familiar with where the Feynman rules come from for operator insertions. At one loop, we pick up the $(\mathcal{L}^0_{int})^2$ piece in our free theory Green's function (recall $\mathcal{L}^0_{int}(z) = -e\overline{\psi}_0(z)\not{A}^0(z)\psi_0(z)$). This gives us (also switching the Dirac indices to be Latin indices because there are a lot of them and I don't know the entire Greek alphabet):

$$\begin{aligned}
\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} = -\frac{e^2}{2}\int d^4z\,d^4z'\,\langle\overline{\psi}^c_0(z)\gamma^\nu_{cd}A^0_\nu(z)\psi^d_0(z)\times \\
\overline{\psi}^e_0(z')\gamma^\lambda_{ef}A^0_\lambda(z')\psi^f_0(z')\overline{\psi}^g_0(x)\gamma^\mu_{gh}\psi^h_0(x)\psi^a_0(x_1)\overline{\psi}^b_0(x_2)\rangle
\end{aligned} \tag{14}$$

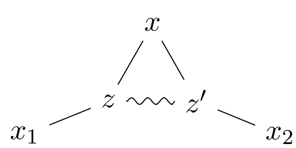
We now have to sum on all contractions of each of the fields. However, we can simplify this by remembering that we are only dealing with connected, 1PI diagrams and throw away many of the contractions. Namely, the general structure of the correlator will look like this:

$$\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} \sim \frac{(-ie)^2}{2}\int \langle A(z)A(z')\rangle\langle\overline{\psi}(z)\psi(z)\overline{\psi}(z')\psi(z')\overline{\psi}(x)\psi(x)\overline{\psi}(x_1)\psi(x_2)\rangle$$

where we sum over contractions for the fermionic correlator. Because the photon propagator $S^{(\gamma)}_{\nu\lambda}(z,z') = \langle A^0_\nu(z)A^0_\lambda(z')\rangle$ already connects the z and z' points, this limits the structure of this to two specific contractions:

$$\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} \sim \frac{(-ie)^2}{2}\int S^{(\gamma)}_{z,z'}\left[\langle\overline{\psi}_z\psi_z\overline{\psi}_{z'}\psi_{z'}\overline{\psi}_x\psi_x\psi_{x_1}\overline{\psi}_{x_2}\rangle + \langle\overline{\psi}_z\psi_z\overline{\psi}_{z'}\psi_{z'}\overline{\psi}_x\psi_x\psi_{x_1}\overline{\psi}_{x_2}\rangle\right]$$

Diagrammatically, two diagrams which contribute look like this:



(15)

where the second term has the z and z' vertices flipped. These will end up being the same and canceling the factor of 1/2.

We only get these two contractions for a few reasons. First, because of the photon propagator connecting z and z', we cannot contract the fermion fields at z and z' together. If we did, we would get a graph which is not 1P1, because we could cut the bubble out of the diagram. Second, we cannot contract the x_1 and x_2 points together, or else we would have a disconnected diagram, hence we must contract x_1 and x_2 to one of z or z' (therefore we have two diagrams). This is easiest to see pictorially– in the manner of the previous diagram.

So, we have reduced this computation to calculating (now restoring the Dirac structure):

$$\begin{aligned}\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} &= (-ie)^2\int d^4z\, d^4z'\, S^{(\gamma)}_{\nu\lambda}(z,z')\langle\overline{\psi}^c_z\gamma^\nu_{cd}\psi^d_z\overline{\psi}^e_{z'}\gamma^\lambda_{ef}\psi^f_{z'}\overline{\psi}^g_x\gamma^\mu_{gh}\psi^h_x\psi^a_{x_1}\overline{\psi}^b_{x_2}\rangle \\ &= (-ie)^2\int d^4z\, d^4z'\, S^{(\gamma)}_{\nu\lambda}(z,z')S^{ac}(x_1,z)\gamma^\nu_{cd}S^{dg}(z,x)\gamma^\lambda_{ef}S^{he}(x,z')\gamma^\mu_{gh}S_{fb}(z',x_2) \\ &= (-ie)^2\int d^4z\int d^4z'\, S^{(\gamma)}_{\nu\lambda}(z,z')S(x_1,z)\gamma^\nu S(z,x)\gamma^\mu S(x,z')\gamma^\lambda S(z',x_2)\end{aligned} \tag{16}$$

where:

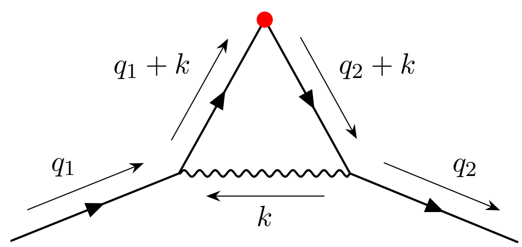
$$S^{(\gamma)}_{\mu\nu}(x,y) = \int d^4k\frac{-ie^{ik(x-y)}g_{\mu\nu}}{k^2+i\epsilon} \tag{17}$$

is the photon propagator. We can now plug this into our expression for $\mathcal{M}^\mu(p,q_1,q_2)$ and associate a one-loop diagram to this computation (where the measure $d^{20}x = d^4x\, d^4x_1\, d^4x_2\, d^4z\, d^4z'$):

$$\begin{aligned}
\mathcal{M}^\mu(p,q_1,q_2)_{\text{1-loop}} &= (-ie)^2 \int d^{20}x\, e^{i(px+q_1x_1-q_2x_2)} S^{(\gamma)}_{\nu\lambda}(z,z') S(x_1,z)\gamma^\nu S(z,x)\gamma^\mu S(x,z')\gamma^\lambda S(z',x_2) \\
&= (-ie)^2 \int d^{20}x\bar{d}^{20}k\, e^{i(px+q_1x_1-q_2x_2)} \frac{-ie^{ik(z-z')}}{k^2}\frac{ie^{-ik_1(x_1-z)}}{\not{k}_1-m}\gamma^\nu\frac{ie^{-ik_2(z-x)}}{\not{k}_2-m}\gamma^\mu\frac{ie^{-ik_3(x-z')}}{\not{k}_3-m}\gamma_\nu\frac{ie^{-ik_4(z'-x_2)}}{\not{k}_4-m} \\
&= (-ie)^2 \int d^{20}x\bar{d}^{20}k\, e^{i(p+k_2-k_3)x} e^{i(k+k_1-k_2)z} e^{i(k_3-k_4-k)z'} e^{i(q_1-k_1)x_1} e^{-i(q_2-k_4)x_2} \times \\
&\qquad \left(\frac{-i}{k^2}\frac{i}{\not{k}_1-m}\gamma^\nu\frac{i}{\not{k}_2-m}\gamma^\mu\frac{i}{\not{k}_3-m}\gamma_\nu\frac{i}{\not{k}_4-m}\right) \\
&= (-ie)^2\frac{i}{\not{q}_1-m}\left[\int \bar{d}^4k\frac{i}{k^2}\gamma^\nu\frac{i}{\not{q}_1+\not{k}-m}\gamma^\mu\frac{i}{\not{q}_2+\not{k}-m}\gamma_\nu\right]\frac{i}{\not{q}_2-m} \qquad (18)
\end{aligned}$$

The delta functions enforce momentum conservation at each vertex in the following diagram, and we can thus encode this amplitude using the off-shell Feynman rules on the following diagram, where the current insertion injects momentum p into the diagram, as well as a γ^μ factor. Our result is thus:

$$i\mathcal{M}^\mu(p,q_1,q_2)_{\text{1-loop}} = \qquad (19)$$



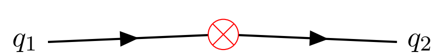
$$= (-ie)^2\frac{i}{\not{q}_1-m}\left[\int \bar{d}^4k\frac{i}{k^2}\gamma^\nu\frac{i}{\not{q}_1+\not{k}-m}\gamma^\mu\frac{i}{\not{q}_2+\not{k}-m}\gamma_\nu\right]\frac{i}{\not{q}_2-m} \qquad (20)$$

where $q_2 = p + q_1$ is the result of current insertion at the red dot in the diagram. At this point, we can evaluate this loop integral in dimensional regularization. Stripping off the mass dimensions of the electron coupling with $e \mapsto \mu^\epsilon e$ in $d = 4-\epsilon$ dimensions, we have:

$$i\mathcal{M}^\mu(p,q_1,q_2) = i\mathcal{M}^\mu_{\text{tree}}(p,q_1,q_2)\frac{e^2}{16\pi^2}\frac{2}{\epsilon} \qquad (21)$$

To finish our renormalization process, we must include the counterterm diagram as well by expanding $\mathcal{Z}_j = 1 + \delta_j$ in the Green's function. We have:

$$\delta_j\mathcal{M}^\mu(p,q_1,q_2) = \qquad (22)$$



$$= i\mathcal{M}^\mu_{\text{tree}}(p,q_1,q_2)\delta_j \qquad (23)$$

where the red $\otimes$ denotes a counterterm insertion. We can thus read off the counterterm:

$$\delta_j = -\frac{e^2}{16\pi^2}\frac{2}{\epsilon} \qquad (24)$$

Now that we have read off the counterterm, the bulk of the work is done! Notice that this calculation is much easier to do diagrammatically with Feynman rules than from first principles; however, I wanted to illustrate how to do build up and motivate Feynman rules for operator insertions, i.e. the off-shell Feynman rules that Schwartz discusses. Note that in this case, we read the fermion arrow from backwards to forwards because of the position of the propagators. In general, I would suggest doing Wick contractions at tree level first to determine the order of the propagators in the diagram, and then using this to move to the loop diagrams.

To compute the anomalous dimension, the bare current can be expressed as:

$$J^{\mu}_{bare} = \overline{\psi}_0 \gamma^{\mu} \psi_0 = \mathcal{Z}_J^{-1} \mathcal{Z}_{\psi} J^{\mu}_R \tag{25}$$

where ψ_0 is the bare field. Since this is purely a function of the bare parameters, it cannot run with μ. Using $\mathcal{Z}_J = \mathcal{Z}_{\psi}$, we can differentiate this to find:

$$\mu \frac{dJ^{\mu}}{d\mu} = 0 \tag{26}$$

where J^{μ} is the renormalized current.

2 Regularization-Independent Momentum Subtraction (RI-MOM)

The RI-MOM scheme (also known as Rome-Southampton) is a renormalization scheme well equipped to deal with the lattice, namely because we can calculate the relevant quantities that define the scheme easily on the lattice. The lattice spacing a provides us a with a natural UV regulator, and RI-MOM will tell us how to go from such a regulated result to a physical observable quantity.

The RI-MOM has a relatively simple renormalization condition. We will define it here for an arbitrary Green's function, for example a three point function. For a renormalization scale μ and working in a fixed gauge, **we define the amputated, renormalized Green's function at momentum $p^2 = -\mu^2$ to be equal to its tree level value**.

We will denote bare quantities as $a^{(0)}$, and renormalized quantities with a R subscript, when appropriate. Note that any quantities computed directly on the lattice are bare, and the entire point of using RI-MOM is to extract a sensible definition for these bare quantities.

In practice on the lattice, there are two things that we must compute. We will work with a specific example here for a given quark field $q(x)$. Suppose the operator we are trying to compute is $\mathcal{O}(z)$. We will renormalize the three point function:

$$G(p) = \frac{1}{V} \sum_{x,y,z} e^{ip\cdot(x-y)} \langle q(x) \mathcal{O}(z) \overline{q}(y) \rangle \tag{27}$$

i.e. we are projecting the source and sink to a definite momentum and projecting the operator $\mathcal{O}(z)$ to zero momentum.

We also will need to compute the momentum projected propagator:

$$S(p) = \frac{1}{V} \sum_{x,y} e^{ip\cdot(x-y)} S(x,y) \tag{28}$$

where $S^{ab}_{ij}(x,y) = \langle q^a_i(x) \bar{q}^b_j(y) \rangle$ is the standard propagator. On the lattice, the two objects that we must compute directly are $S(p)$ and $G(p)$, and everything else follows once we have these quantities.

To explain the method, our goal is to compute the operator renormalization:

$$\mathcal{O}_R(\mu) = \mathcal{Z}(\mu)\, \mathcal{O}^{(0)} \tag{29}$$

where $\mathcal{O}_R(\mu)$ is our renormalized operator, $\mathcal{Z}(\mu)$ is the renormalization coefficient of interest, and $\mathcal{O}_{lat}$ is the lattice (bare) operator. We will also assume the quark fields have been renormalized by some quark field renormalization $\mathcal{Z}_q$ (note this is the opposite convention studied in many QFT classes):

$$q_R(\mu) = \sqrt{\mathcal{Z}_q}\, q^{(0)} \tag{30}$$

There is an analytical expression for $\mathcal{Z}_q$ on the lattice in the RI-MOM scheme, and it has been determined to be:

$$\mathcal{Z}_q(p)|_{p^2=-\mu_R^2} = \left[\frac{tr\left\{-i\sum_{\nu=1}^4 \gamma_\nu \sin(ap_\nu) aS(p)^{-1}\right\}}{12\sum_{\nu=1}^4 \sin^2(ap_\nu)}\right]_{p^2=-\mu^2} \tag{31}$$

The twelve on the bottom is a normalization $12 = 3 \times 4$ for the number of color and number of spin indices, which we will see in many of the expressions.

Let $\Gamma(p)$ be the **amputated three point function**, bare or renormalized. We can relate Γ to the other quantities we have already computed by using the inverse propagator to manually cut the legs off the full three point function:

$$\Gamma(p) = S(p)^{-1} G(p) S(p)^{-1} \tag{32}$$

We will denote the tree level version of this by Γ_B, where the B subscript stands for "Born".

Using our quantities already computed on the lattice, we can compute the bare $\Gamma^{(0)}(p)$ directly in terms of the renormalized quantities. In the continuum limit, the renormalized Green's function will be:

$$G_R(p;\mu) = \int d^4x\, d^4y\, d^4z\, e^{ip\cdot(x-y)} \langle q_R(x;\mu)\mathcal{O}_R(z;\mu)\overline{q}_R(y;\mu)\rangle \tag{33}$$

$$= \mathcal{Z}_q(\mu)\mathcal{Z}(\mu) \int d^4x\, d^4y\, d^4z\, e^{ip\cdot(x-y)} \langle q^{(0)}(x)\mathcal{O}^{(0)}(z) q^{(0)}(y)\rangle \tag{34}$$

$$= \mathcal{Z}_q(\mu)\mathcal{Z}(\mu) G^{(0)}(p) \tag{35}$$

Similarly, the bare and renormalized propagators are related as:

$$S_R(p;\mu) = \int d^4x d^4y\, e^{ip\cdot(x-y)} \langle q_R(x;\mu)\overline{q}_R(y;\mu)\rangle = \mathcal{Z}_q(\mu) S^{(0)}(p) \tag{36}$$

These relation translates immediately to the amputated Green's function $\Gamma(p)$, as $S_R^{-1}(p;\mu) = \mathcal{Z}_q^{-1}(\mu)(S^{(0)})^{-1}$ we see that:

$$\Gamma_R(p;\mu) = \mathcal{Z}_q(\mu)^{-1}\mathcal{Z}(\mu)\Gamma^{(0)}(p) \tag{37}$$

We are now in a position to apply the renormalization condition. We must equate the renormalized, amputated Green's function to the tree level Green's function $\Gamma_B(p)$. This gives us:

$$\mathcal{Z}_q(\mu)^{-1}\mathcal{Z}(\mu)\Gamma(p) = \Gamma_B(p) \tag{38}$$

Dividing by a conventional factor of 12 as a normalization and inverting, we can clean this expression up into a simple equation for $\mathcal{Z}(\mu)$:

$$\mathcal{Z}(p^2 = -\mu^2) = \left[\frac{12\mathcal{Z}_q(p)}{tr\left\{\Gamma(p)\Gamma_B(p)^{-1}\right\}}\right]_{p^2=-\mu^2} \tag{39}$$

3 Matching to a continuum scheme

4 Example: Isospin

We are interested here in computing matrix elements of the operator:

$$\mathcal{O}(z) = \mathcal{O}_u(z) - \mathcal{O}_d(z) \tag{40}$$

where the quark operators $\mathcal{O}_q$ are given by

$$\mathcal{O}_q(z) = \frac{1}{\sqrt{2}}(\mathcal{T}^q_{33}(z) - \mathcal{T}^q_{44}(z)) \tag{41}$$

and the irreducible tensor operators $\mathcal{T}^q_{\mu\nu}$ are defined as

$$\mathcal{T}^q_{\mu\nu} = \overline{q}(z)\, \gamma_{\{\mu} \overleftrightarrow{D}_{\nu\}}\, q(z) \tag{42}$$

with $\overleftrightarrow{D} = \vec{D} - \overleftarrow{D}$ the symmetrized covariant derivative. Note we define the symmetric and traceless component of a tensor to be:

$$a_{\{\mu} b_{\nu\}} = \frac{1}{2}(a_\mu b_\nu + a_\nu b_\mu) - \frac{1}{4} a_\alpha b^\alpha g_{\mu\nu} \tag{43}$$

We are inserting the operator $\mathcal{O}_q$ with momentum $\vec{p} = 0$. We will focus our analysis on the tensor operator $\mathcal{T}^q_{\mu\mu}$ (note there is no sum on μ here), and note that $\mathcal{O}(z)$ can be obtained through linearity. We may write:

$$\sum_z \mathcal{T}^q_{\mu\mu}(z) = \sum_{z,z'} \overline{q}(z)\, J_\mu(z,z')\, q(z') \tag{44}$$

Plugging in the definition of the derivatives:

$$\vec{D}\psi(z) = \frac{1}{2}\Big(U_\mu(z)\psi(z+\hat{\mu}) - U_\mu(n-\hat{\mu})^\dagger \psi(z-\hat{\mu})\Big) \tag{45}$$

$$\overline{\psi}(z)\overleftarrow{D} = \frac{1}{2}\Big(\overline{\psi}(z+\hat{\mu})U_\mu(z)^\dagger - \overline{\psi}(z-\hat{\mu})U_\mu(z-\hat{\mu})\Big) \tag{46}$$

we find the current $J_\mu(z,z')$ is:

$$J_\mu(z,z') = \Big[U_\mu(z)\delta_{z+\hat{\mu},z'} - U_\mu(z')^\dagger \delta_{z-\hat{\mu},z'}\Big]\gamma_\mu \tag{47}$$

We may now use this expansion to compute the three point function for the operator $\mathcal{T}_\mu = \mathcal{T}^u_{\mu\mu} - \mathcal{T}^d_{\mu\mu}$ (we can simply take $\mathcal{T}_3 - \mathcal{T}_4$ to get the operator of interest in Equation 40). Using Equation 44, we write

$$\sum_z \mathcal{T}_\mu(z) = \sum_{z,z'} \left[\overline{u}(z)\, J_\mu(z,z')\, u(z') - \overline{d}(z)\, J_\mu(z,z')\, d(z')\right] \tag{48}$$

Plugging this into Equation 27, we find that we can expand the total up quark Green's function (here α,β are Dirac indices) as:

$$G^{\alpha\beta}(p) = \frac{1}{\sqrt{2}}\Big(G_3^{\alpha\beta}(p) - G_4^{\alpha\beta}(p)\Big) \tag{49}$$

where:

$$G_\mu^{\alpha\beta}(p) = \frac{1}{V}\sum_{x,y,z} e^{-ip(x-y)}\langle u^\alpha(x)\mathcal{T}_\mu(z)\overline{u}^\beta(y)\rangle \tag{50}$$

$$= \frac{1}{V}\sum_{x,y,z,z'} e^{-ip(x-y)}\left[\langle u^\alpha(x)\overline{u}^\sigma(z)J_\mu^{\sigma\rho}(z,z')u(z')^\rho\overline{u}^\beta(y)\rangle - \langle u^\alpha(x)\overline{d}^\sigma(z)J_\mu^{\sigma\rho}(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle\right] \tag{51}$$

Now we perform all possible Wick contractions on the matrix elements to write them as propagators:

$$\begin{aligned}\langle u^\alpha(x)\overline{u}^\sigma(z)J_\mu^{\sigma\rho}(z,z')u^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{u}Ju\overline{u}\rangle + \langle u\overline{u}Ju\overline{u}\rangle\\ &= S^{\alpha\sigma}(x,z)J_\mu^{\sigma\rho}(z,z')S^{\rho\beta}(z',y) + (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)\end{aligned} \tag{52}$$

$$\begin{aligned}\langle u^\alpha(x)\overline{d}^\sigma(z)J_\mu^{\sigma\rho}(z,z')d^\rho(z')\overline{u}^\beta(y)\rangle &= \langle u\overline{d}Jd\overline{u}\rangle\\ &= (-1)^3 S^{\alpha\beta}(x,y)J_\mu^{\sigma\rho}(z,z')S^{\rho\sigma}(z',z)\end{aligned} \tag{53}$$

where the factors of (-1) come from rearranging the contraction so that the contracted pieces are of the form $\langle u\overline{u}\rangle$. The vacuum pieces cancel because the up and down quark propagators are degenerate, so the final result is very clean:

$$G_\mu(p) = \frac{1}{V}\sum_{x,y,z,z'} e^{ip(x-y)}S(x,z)J_\mu(z,z')S(z',y) \tag{54}$$

This is our central equation, but note that initially there is a difficulty with

There are two primary ways to compute this on the lattice. We can compute this directly using momentum sources, or we can use the sequential source technique. Momentum sources work specifically for Equation 54, but produce a significantly better signal on a small number of configurations. On the other hand, sequential source is much more general, but produces more noise. We will discuss each method below.

4.1 Momentum sources

Observe that we can rewrite Equation 54 as:

$$\begin{aligned}G_\mu(p) &= \frac{1}{V}\sum_{x,y,z,z'} e^{ipx}S(x,z)J_\mu(z,z')e^{-ipy}S(z',y)\\ &= \frac{1}{V}\sum_{z,z'}\gamma_5\left(\sum_x S(z,x)e^{ipx}\right)^\dagger\gamma_5 J_\mu(z,z')\left(\sum_y S(z',y)e^{ipy}\right)\\ &= \frac{1}{V}\sum_{z,z'}\gamma_5\tilde{S}_p(z)^\dagger\gamma_5 J_\mu(z,z')\tilde{S}_p(z')\end{aligned} \tag{55}$$

where we have defined $\tilde{S}_p(z)$ as:

$$\tilde{S}_p(z) = \sum_x S(z,x)e^{ipx} \tag{56}$$

The advantage of casting the equation in this form is that we can solve for $\tilde{S}_p(z)$ directly by inverting the Dirac equation with a momentum source, i.e. we have:

$$\sum_z D(x,z)\tilde{S}_p(z) = e^{ipx} \tag{57}$$

where $D(x,z)$ is the Dirac operator. This means that upon solving for $\tilde{S}_p(z)$ and plugging this into Equation 55, we can solve directly for $G_\mu(p)$.

This is an exact equation and it does not rely on translational invariance in the infinite statistics limit. Therefore, this method will give much better signal and can be run efficiently on a small number of configurations. The downside to this is that we require a propagator inversion for each choice of sink momentum. To compute $G(p)$ for a large number of sink momenta, as we need to do to extrapolate $\mathcal{Z}(\mu)$ in the continuum limit, a propagator inversion at each sink momenta is not feasible. We must instead choose the sink momentum wisely to be able to extract the discretization artifacts and extrapolate $\mathcal{Z}(\mu)$ to the continuum (we will describe these discretization artifacts in Section 5.

4.2 Sequential source method

In practice we will use the sequential source method, which if implemented correctly does not force us to invert a propagator at every sink momenta. This technique is also much more general than the one previously described, but it suffers from more noise because it relies on the translational invariance of the lattice, which only exists in the infinite statistics limit. The idea of the sequential source method is that if we have an equation involving the full propagator $S(x,y)$, we can invert a source which depends on the propagator $S(x)$. For example, in this problem we wish to evaluate Equation 54, but we cannot simply evaluate $S(x,y)$ for every x and y. To get around this, consider using a source

$$b(z) = \sum_{z'} J_\mu(z,z')S(z',0) \tag{58}$$

to invert the Dirac equation, which will solve for $M(x)$ in this equation:

$$\sum_x D(z,x)M(x) = b(z) \tag{59}$$

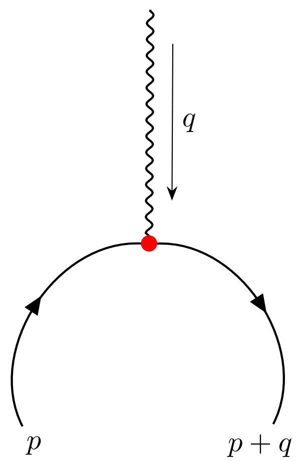
where $D(x,z)$ is the Dirac operator. Upon inversion, using that $\sum_z S(y,z)D(z,x) = \delta(y-x)$, we can move the Dirac operator to the other side as $D^{-1}(y,z) = S(y,z)$ and obtain:

$$M(x) = \sum_z D^{-1}(x,z)b(z) = \sum_{z,z'} S(x,z)J_\mu(z,z')S(z',0) \tag{60}$$

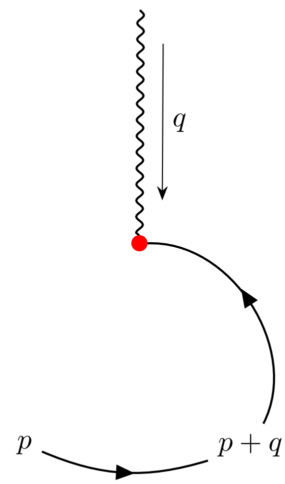
Note that we have summed the full propagator $S(x,z)$ for the price of a single inversion of the source $b(z)$. We can then reconstruct Equation 54 in the infinite statistics limit when translational invariance is restored:

$$G_\mu(p) \to \frac{1}{V}\sum_x e^{ipx}M(x) = \frac{1}{V}\sum_{x,z,z'} e^{ipx}S(x,z)J_\mu(z,z')S(z',0) \tag{61}$$

In our case with a large amount of sink momenta, this method is much more robust than inverting a momentum source because we one inversion can give us $G(p)$ at every value of the sink momentum. We will also call this construction going **through the operator**, because the inversion in Equation 60 projects the current insertion onto $q = 0$ momentum. If we had been interested in projecting the operator onto different momentum values, then we would need to use a new sequential source (modify Equation 58) for each value of the operator momentum. Pictorially, we are inverting at the operator momentum, then tying up at the sink momentum. On the other hand, we reverse the direction of inversion and invert our propagator at each sink momentum first, then tie up the line at the operator. This method is called going **through the sink**. We can represent these different methods below, where in our case $q = 0$.



(a) Inversion through the operator



(b) Inversion through the sink

In this problem inversion through the sink would require too many propagator inversions like in the previous momentum source method, and it would also be noisy like inversion through the operator. As such, there is no reason to consider it, and I included it here mainly for generality.

5 Hypercubic Artifacts

When we compute observables at a finite lattice spacing a, we suffer discretization artifacts which are relics of the explicit symmetry breaking $SO(1,3) \to H(4)$ suffered by putting the theory on a lattice.