

Renormalization

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1 Regularization-Independent Momentum Subtraction (RI-MOM)

The RI-MOM scheme (also known as Rome-Southampton) is a renormalization scheme well equipped to deal with the lattice, namely because we can calculate the relevant quantities that define the scheme easily on the lattice. The lattice spacing a provides us with a natural UV regulator, and RI-MOM will tell us how to go from such a regulated result to a physical observable quantity.

The RI-MOM has a relatively simple renormalization condition. We will define it here for an arbitrary Green's function, for example a three point function. For a renormalization scale μ and working in a fixed gauge, **we define the amputated, renormalized Green's function at momentum $p^2 = -\mu^2$ to be equal to its tree level value.**

In practice on the lattice, there are two things that we must compute. We will work with a specific example here for a given quark field $q(x)$. Suppose the operator we are trying to compute is $\mathcal{O}(z)$. We will renormalize the three point function:

$$G(p) = \frac{1}{V} \sum_{x,y,z} e^{-ip \cdot (x-y)} \langle q(x) \mathcal{O}(z) q(y) \rangle \quad (1)$$

i.e. we are projecting the source and sink to a definite momentum and projecting the operator $\mathcal{O}(z)$ to zero momentum.

We also will need to compute the momentum projected propagator:

$$S(p) = \frac{1}{V} \sum_{x,y} e^{-ip \cdot (x-y)} S(x, y) \quad (2)$$

where $S_{ij}^{ab}(x, y) = \langle q_i^a(x) \bar{q}_j^b(y) \rangle$ is the standard propagator. On the lattice, the two objects that we must compute directly are $S(p)$ and $G(p)$, and everything else follows once we have these quantities.

To explain the method, our goal is to compute the operator renormalization:

$$\mathcal{O}_R(\mu) = \mathcal{Z}(\mu) \mathcal{O}_{lat} \quad (3)$$

where $\mathcal{O}_R(\mu)$ is our renormalized operator, $\mathcal{Z}(\mu)$ is the renormalization coefficient of interest, and \mathcal{O}_{lat} is the lattice (bare) operator. We will also assume the quark fields have been renormalized by some quark field renormalization \mathcal{Z}_q :

$$q_{lat} = \sqrt{\mathcal{Z}_q} q_R(\mu) \quad (4)$$

There is an analytical expression for \mathcal{Z}_q on the lattice in the RI-MOM scheme, and it has been determined to be:

$$\mathcal{Z}_q(p)|_{p^2=-\mu_R^2} = \left[\frac{\text{tr} \left\{ -i \sum_{\nu=1}^4 \gamma_\nu \sin(ap_\nu) a S(p)^{-1} \right\}}{12 \sum_{\nu=1}^4 \sin^2(ap_\nu)} \right]_{p^2=-\mu^2} \quad (5)$$

The twelve on the bottom is a normalization $12 = 3 \times 4$ for the number of color and number of spin indices, which we will see in many of the expressions.

Let $\Gamma(p)$ be the amputated three point function, bare or renormalized. We can relate Γ to the other quantities we have already computed by using the inverse of the propagator to manually cut the legs off the full three point function:

$$\Gamma(p) = S(p)^{-1} G(p) S(p)^{-1} \quad (6)$$

We will denote the tree level version of this by Γ_B , where the B subscript stands for Born term.

Using our quantities already computed on the lattice, we can compute the bare $\Gamma_{lat}(p)$ directly in terms of the renormalized quantities. In the continuum limit, the renormalized Green's function will be:

$$G_R(p; \mu) = \int d^4x d^4y d^4z e^{-ip \cdot (x-y)} \langle q_R(x; \mu) \mathcal{O}_R(z; \mu) q_R(y; \mu) \rangle \quad (7)$$

$$= \mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) \int d^4x d^4y d^4z e^{-ip \cdot (x-y)} \langle q_{lat}(x) \mathcal{O}_{lat}(z) q_{lat}(y) \rangle \quad (8)$$

$$= \mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) G_{lat}(p) \quad (9)$$

This relation translates immediately to the amputated Green's function $\Gamma(p)$:

$$\Gamma_R(p; \mu) = \mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) \Gamma_{lat}(p) \quad (10)$$

We are now in a position to apply the renormalization condition. We must equate the renormalized, amputated Green's function to the tree level Green's function $\Gamma_B(p)$. This gives us:

$$\mathcal{Z}_q(\mu)^{-1} \mathcal{Z}(\mu) \Gamma(p) = \Gamma_B(p) \quad (11)$$

Dividing by a conventional factor of 12 as a normalization and inverting, we can clean this expression up into a simple equation for $\mathcal{Z}(\mu)$:

$$\mathcal{Z}(p^2 = -\mu^2) = \left[\frac{12 \mathcal{Z}_q(p)}{\text{tr} \{ \Gamma(p) \Gamma_B(p)^{-1} \}} \right]_{p^2 = -\mu^2} \quad (12)$$

2 Continuum Schemes: \overline{MS} , on-shell, and off-shell

3 Matching between schemes

4 Example: Isospin