Patrick Oare | $\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$ $Q=\int_V \rho\, d^3x$ $\nabla\cdot\vec{S}+\frac{\partial u}{\partial t}=0$

Physics 110B Midterm I Cheat Sheet

Energy / Momentum

Energy density $u(\vec{r})$ w/ $\mathcal{U}=\int_V u(\vec{r})d^3x$

$u(\vec{r})=\frac{1}{2}(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2)$ $\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}=u(\vec{r})c\hat{k}$ $\langle\vec{S}\rangle=\frac{1}{T}\int_T\vec{S}(t)dt$

- Momentum: $\vec{E}/\vec{B}$ fields have momentum, $\overleftrightarrow{T}$. $\vec{F}=\frac{d}{dt}\vec{P}_{mech}=\int_V\nabla\cdot\overleftrightarrow{T}d^3r=\oint_{\partial V}\overleftrightarrow{T}\cdot d\vec{a}$ $\vec{f}=\nabla\cdot\overleftrightarrow{T}-\frac{1}{c^2}\frac{\partial\vec{S}}{\partial t}$ (vacuum ↑ →) $\vec{f}=F$ / unit vol

$T_{ij}=\varepsilon_0(E_iE_j-\frac{1}{2}\delta_{ij}E^2)+\frac{1}{\mu_0}(B_iB_j-\frac{1}{2}\delta_{ij}B^2)$ $(\vec{a}\cdot\overleftrightarrow{T})_j=\sum_i a_iT_{ij}$, $(\overleftrightarrow{T}\cdot\vec{a})_j=\sum_i T_{ji}a_i$

- If $\vec{k}=k\hat{z}$, $\hat{n}=\hat{x}$, then $\vec{E}=E_0\cos(kz-\omega t)\hat{x}$ and $\vec{B}=\frac{E_0}{c}\cos(kz-\omega t)\hat{y}$, so: $\vec{S}=\frac{1}{\mu_0}\hat{z}\frac{E_0^2}{c}\cos^2(kz-\omega t)$

If $\vec{A}=\mathrm{Re}[\vec{A}_0e^{i(\vec{k}\cdot\vec{r}-\omega t)}]$, and same w/ $\vec{B}$, then: $\langle\vec{A}\times\vec{B}\rangle=\frac{1}{2}\mathrm{Re}[\vec{A}_0\times\vec{B}_0^*]$

$\Rightarrow\langle\vec{S}\rangle=\frac{1}{2\mu_0c}\hat{k}|\vec{E}_0|^2$ ← for wave in vacuum w/ $\hat{E}_0=\hat{n}$.

$\langle S\rangle=\frac{1}{2}\frac{|E_0|^2}{Z_0}$, $Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}}=377\,\Omega$

- Wave sharply peaked: $u=\omega-\omega_0$

$f(z,t)=\underbrace{e^{i\omega_0(z/v-t)}}_{\text{carrier}}\underbrace{\int_{\mathbb{R}}A(u)e^{iu(z/v-t)}du}_{\text{envelope}}$

Waves

$\omega=\frac{2\pi}{T}=2\pi f$, $k=\frac{2\pi}{\lambda}$. $V_\varphi=\frac{\omega}{k}$ = rate of phase advance, $V_g=\frac{d\omega}{dk}$ = envelope speed

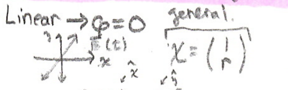
- Polarization: $\hat{k}=\hat{z}$, so $\vec{k}=k\hat{z}$. $r=|E_{0y}|/|E_{0x}|$, $\varphi=\varphi_y-\varphi_x$

$\vec{E}(t)=|E_{0x}|e^{i\varphi_x}(\hat{x}+re^{i\varphi}\hat{y})e^{-i\omega t}$

$\mathrm{Re}[\vec{E}(t)]=\hat{x}\cos(\omega t)+\hat{y}r\cos(\omega t-\varphi)$ (elliptical)

- Linear → $\varphi=0$ general: $\chi=\binom{1}{r}$ basis: $\{\binom{1}{0},\binom{0}{1}\}$

- Circular: $r=1$, $\varphi=\pm\pi/2$, $\chi=\binom{1}{\pm i}$ basis: $\{\binom{1}{i},\binom{1}{-i}\}$



- $\vec{E},\vec{B}$ in vacuum. Recall $\nabla\times\vec{c}e^{\vec{A}\cdot\vec{x}}=\vec{A}\times\vec{c}e^{\vec{A}\cdot\vec{x}}$

– Solns to wave eqn are $\vec{E}=\mathrm{Re}[\vec{E}_0e^{i(kz-\omega t)}]$, same w/ B.

- From $\nabla\times\vec{E}$ eqn, get: $\hat{b}=\hat{k}\times\hat{n}$.

$\vec{k}\times\vec{E}_0=\omega\vec{B}_0\Rightarrow\vec{B}_0=\frac{1}{c}\hat{k}\times\vec{E}_0\Rightarrow B_0=E_0/c$.

$\vec{E}=E_0\cos(\vec{k}\cdot\vec{r}-\omega t+\delta)\hat{n}$

$\vec{B}=\frac{E_0}{c}\cos(\vec{k}\cdot\vec{r}-\omega t+\delta)(\hat{k}\times\hat{n})$

- Any soln is linear superpos of these modes.

Wave eqns: $\nabla^2\vec{E}=\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$, $\nabla^2\vec{B}=\mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$

$(ik-\frac{\omega^2}{c^2k})\vec{B}_T=\nabla_\rho B_z$

Maxwell's eqns: $\nabla\cdot\vec{E}=\rho/\varepsilon_0$, $\nabla\cdot\vec{B}=0$, $\nabla\times\vec{E}=-\partial\vec{B}/\partial t$, $\nabla\times\vec{B}=\mu_0\vec{J}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

Guided Waves

Preserve translational symmetry in 1D. Separate to longitudinal / transverse components: $\vec{r}=\vec{\rho}+z\hat{z}$, $\vec{E}_0=\hat{z}E_z(x,y)+\vec{E}_T(x,y)$, $\vec{E}(\vec{r},t)=\vec{E}_0(x,y)e^{ik_zz}e^{-i\omega t}$, $\nabla_\rho=\hat{x}\partial_x+\hat{y}\partial_y$, $\nabla=\nabla_\rho+\hat{z}\partial_z$. Plug into ME, separate L and T:

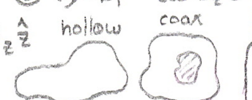
① $\nabla_\rho\cdot\vec{E}_T+ikE_z=0$ ② $\nabla_\rho\cdot\vec{B}_T+ikB_z=0$ ③ $\nabla_\rho\times\vec{E}_T=i\omega B_z\hat{z}$

④ $\hat{z}\times[ik\vec{E}_T-\nabla_\rho E_z]=i\omega\vec{B}_T$ ⑤ $\nabla_\rho\times\vec{B}_T=-\frac{i\omega}{c^2}B_z\hat{z}$

⑥ $\hat{z}\times[ik\vec{B}_T-\nabla_\rho B_z]=-\frac{i\omega}{c^2}\vec{E}_T$

Guided wave eqns

hollow coax



- Purely transverse solns? ⇔ $E_z=B_z=0$. ③ ⇒ $\nabla_\rho\times\vec{E}_T=0\Rightarrow\vec{E}_T=-\nabla_\rho\Phi$, and ① ⇒ $\nabla_\rho\cdot\vec{E}_T=0\Rightarrow\nabla_\rho^2\Phi=0$, Laplace's eqn. For hollow waveguide, B.C. ⇒ $\Phi=0\Rightarrow\vec{E}_T=\vec{0}$, so hollow waveguide cannot support purely T modes. For coax, can have $\Phi_{in}\neq\Phi_{out}$, so it can support purely T modes. ← grif says this is TEM

- Purely T → T mode waves, $E_z=0\leftrightarrow$ TE waves, $B_z=0\leftrightarrow$ TM waves.

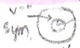
$E_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right)$ $E_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right)$

$B_x=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$ $B_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$

⇒ we see E_z,B_z completely determine $\vec{E}_T,\vec{B}_T$

- E_z,B_z satisfy $[\nabla_\rho^2+(\omega/c)^2-k^2]E_z=0$, $[\nabla_\rho^2+(\omega/c)^2-k^2]B_z=0$

- Circular guide T modes



$\varphi=\varphi(\rho)$ by sym.

$\nabla_\rho^2\varphi=\frac{1}{\rho^2}\frac{d}{d\rho}(\rho\frac{d\varphi}{d\rho})=0$

$\Rightarrow\rho\frac{d\varphi}{d\rho}=\text{const}=A$

$\Rightarrow V_0=\int_a^b d\varphi=\int_a^b A\frac{d\rho}{\rho}$

$\Rightarrow A=\frac{V_0}{\ln(b/a)}$, and

$\varphi(\rho)=A\ln(\rho/a)$,

$\vec{E}_T=-\nabla_\rho\varphi=-\frac{A}{\rho}\hat{\rho}$

$\vec{B}_T=\frac{1}{c}\hat{z}\times\vec{E}_T=-\frac{A\hat{\theta}}{c\rho}$

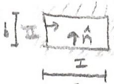
- If $\omega<\omega_{nm}$, $k_{nm}(\omega)\notin\mathbb{R}$, so get evanescent nm mode – For the mode to propagate, need $\omega>\omega_{nm}$

- Rectangular guide: TEM wave ($E_z=0$, not B_z)

$\hat{n}\times(\hat{n}\times\vec{A})=\hat{n}\cdot(\hat{n}\cdot\vec{A})-\vec{A}\cdot(\hat{n}\cdot\hat{n})$

$\Rightarrow\hat{z}\times(\hat{z}\times\vec{E}_T)=-\vec{E}_T$

⇒ ④ implies $\vec{E}_T=-\frac{\omega}{k}\hat{z}\times\vec{B}_T$



$\Rightarrow\nabla_\rho^2B_z=-\gamma^2B_z$, $\gamma^2=\frac{\omega^2}{c^2}-k^2$ (*)

– Separate $B_z=F(x)G(y)$, use B.C.'s: $\hat{n}\cdot\nabla_\rho B_z=0$ at boundary (I ⇒ C=0, II ⇒ A=0)

Solns F, G sinusoids, and after applying B.C.'s:

$k_{x_n}=\frac{n\pi}{a}$, $k_{y_m}=\frac{m\pi}{b}$ $F=A\sin(k_xx)+B\cos(k_xx)$, $G=C\sin(k_yy)+D\cos$

$B_z(x,y)=B_{z_0}\cos(\frac{\pi n}{a}x)\cos(\frac{\pi m}{b}y)$

Solve for $\vec{B}_T$ w/ $(ik-\frac{\omega^2}{c^2k})\vec{B}_T=\nabla_\rho B_z$. Disp. rln

$\gamma^2=(\frac{\pi n}{a})^2+(\frac{\pi m}{b})^2=\frac{\omega^2}{c^2}-k^2$ (from *)

Cutoff frequency ω_{nm} is:

$\omega_{nm}:=c\sqrt{(\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2}$ w/

$k_{nm}(\omega)=\frac{1}{c}\sqrt{\omega^2-\omega_{nm}^2}$

- Group velocity in rectangular waveguide:

$V_g=\frac{d\omega}{dk}=\frac{d}{dk}(\sqrt{c^2k^2+\omega_{nm}^2})$

$=\frac{c^2k}{\sqrt{c^2k^2+\omega_{nm}^2}}=c\sqrt{1-(\omega_{nm}/\omega)^2}$

EM Waves in Matter

Sources have time dependence $\vec{J}(\vec{r},t)=\vec{J}(\vec{r})e^{-i\omega t}$, $\rho(\vec{r},t)=\rho(\vec{r})e^{-i\omega t}$

- Continuity eqn: $\nabla\cdot\vec{J}(\vec{r})-i\omega\rho(\vec{r})=0$. - Isotropic: Symmetric, no favored direction, $\vec{C}(\vec{r})\propto e^{i\vec{k}\cdot\vec{r}}$

- L/T comps → $\rho(\vec{r})=\frac{k}{\omega}J_L(\vec{r})$, $\vec{J}=\vec{J}_L+\vec{J}_T$. $\vec{B}$ will have $B_L=0$, and can separate M.E.'s:

$(\frac{\omega^2}{c^2}-k^2)\vec{E}_T=-i\omega\mu_0\vec{J}_T$, $\vec{E}_L=-\frac{i\mu_0c^2}{\omega}\vec{J}_L$, $\vec{B}=\frac{1}{\omega}\vec{k}\times\vec{E}_T$, $\vec{B}_L=\vec{0}$.

- Conductors have $\vec{J}=\sigma\vec{E}$; other materials more complicated. ($\vec{J}=\vec{J}_f$)

- Dfns: $k^2(\omega)=\frac{\omega^2}{c^2}\tilde{n}^2(\omega)$, for T modes, $\tilde{n}(\omega)=\sqrt{1+\frac{i}{\varepsilon_0\omega}\sigma(\omega)}$

– Complex dielectric: $\tilde{n}(\omega)=n(\omega)+iK(\omega)$, $k(\omega)=k_0(n+iK)$

$\varepsilon(\omega)=\varepsilon_0\tilde{n}^2(\omega)$, $1+\chi(\omega)=\varepsilon(\omega)/\varepsilon_0$, $k_0=\omega/c$ = vac wave #

$\mathrm{Im}(k)=k_0K$

$\langle\vec{S}\rangle=\frac{\hat{z}}{2\mu_0}\frac{|E_0|^2}{c}e^{-2\mathrm{Im}(k)z}\mathrm{Re}[\tilde{n}(\omega)]$

$Y_0:=1/Z_0$ and $Y(\omega)=Y_0\tilde{n}(\omega)$

$=\frac{\hat{z}}{2\mu_0}\frac{|E_0|^2}{c}n(\omega)e^{-2k_0Kz}=\frac{1}{2}\hat{z}Y(\omega)|E_0|^2e^{-2k_0Kz}$

- Linear response: $J(t)=K(t)*E(t)$ (probably anything he gives us will use this), $k(t)=\mathfrak{F}\{\sigma(\omega)\}$, and so:

$\tilde{J}(\omega)=\sigma(\omega)\tilde{E}(\omega)$ $\sigma_1(\omega)=\mathrm{Re}[\sigma]$, $\sigma_2(\omega)=\mathrm{Im}[\sigma]$.

- Dispersion rln for T modes: $k^2(\omega_T)=\frac{\omega_T^2}{c^2}+i\omega_T\mu_0\sigma(\omega_T)$

L modes: $\sigma(\omega_L)=\frac{i\omega}{\mu_0c^2}$ will only have L modes if this is satisfied (we get these from eqns for $\vec{E}_L,\vec{E}_T,\vec{B}_L,\vec{B}_T$)

- EM waves in linear isotropic media:

i) T waves: $\{\vec{E},\vec{B},\vec{k}\}$ orthogonal, $\vec{B}=\frac{1}{\omega}\vec{k}\times\vec{E}_T$

ii) If $\vec{k}=k\hat{z}$, $\vec{E}=E_0\hat{x}e^{i(kz-\omega t)}$, $\vec{B}=B_0\hat{y}e^{i(kz-\omega t)}$, $B_0=\tilde{n}(\omega)E_0/c$

iii) L modes iff $\varepsilon(\omega_L)=0$. In vac, $\varepsilon(\omega)=\varepsilon_0$ = const ⇒ no L mode

- Skin depth $\delta=1/K=\mathrm{Im}[k]$ → distance to reduce amplitude of $\vec{E},\vec{B}$ by $1/e$

$\vec{E}=\vec{E}_0e^{-Kz}e^{i(kz-\omega t)}$ for $\vec{k}=k+ik_0$ the complex wavevector.

- If $\tilde{n}$ imaginary, $\langle\vec{S}\rangle=\vec{0}$. → evanescent wave, no energy loss, and ⇒ $\sigma(\omega)$ imaginary ⇒ J, E out of phase.

- Free e^- gas: $\vec{P}$ = momentum / volume $=nm_e\langle\vec{v}\rangle$, $n=N/V$. $E(t)=\mathrm{Re}[E_0e^{-i\omega t}]$

$\frac{d\vec{P}}{dt}=-ne\vec{E}$ (Newton's laws) → "momentum memory" τ: use $\dot{\vec{P}}=-ne\vec{E}-\frac{1}{\tau}\vec{P}$.

Set $\vec{P}=\mathrm{Re}[P_0e^{-i\omega t}]\Rightarrow P_0=\frac{-ne}{1/\tau-i\omega}E_0$, $J_0=-ne\langle v\rangle$, $\sigma(\omega)=J_0/E_0\Rightarrow\sigma(\omega)=\frac{ne^2/m}{1/\tau-i\omega}$

$\sigma_{DC}=\lim_{\omega\to0}\sigma(\omega)$, $\sigma(\omega)=\sigma_1+i\sigma_2$, then $\varepsilon_1(\omega)=\varepsilon_0-\frac{\sigma_2(\omega)}{\omega}$

and $\varepsilon_2(\omega)=\frac{\sigma_1(\omega)}{\omega}$. Plasma frequency $\Omega_p^2=\frac{ne^2}{\varepsilon_0m}$ $k(\omega)=\frac{\omega}{c}\tilde{n}(\omega)=(1+i)\sqrt{\frac{\mu_0\omega\sigma_R}{2}}$

- Dissipative regime: $\langle\vec{S}\rangle\neq\vec{0}$, have $|\varepsilon_2(\omega)|\gg|\varepsilon_1(\omega)|$ and $\omega\ll1/\tau$. $\tilde{n}(\omega)=\sqrt{\varepsilon(\omega)/\varepsilon_0}\sim\sqrt{i\varepsilon_2/\varepsilon_0}\sim\frac{1+i}{\sqrt{2}}\sqrt{\sigma_{dc}/\varepsilon_0\omega}$

Then plug in for $\langle\vec{S}\rangle\neq\vec{0}$ ($R<1$)

$\vec{\mathcal{P}}$ = dipole moment / unit vol $=-\frac{e}{V}\sum_{i=1}^{N}\vec{x}_i=-ne\vec{x}$

- Polarizability: $\chi(\omega)$

$\vec{\mathcal{P}}=+\frac{1}{V}\sum_i q\vec{x}_i=\varepsilon_0\chi(\omega)\vec{E}$

• B.C.'s for incidence $\nabla\cdot\vec{D}=0$ (no ρ_f) $\Rightarrow \hat{n}\cdot\vec{D}, \hat{n}\cdot\vec{B}$ continuous. $\nabla\times\vec{E}$ can $\Rightarrow$ continuity of tangential $\vec{E}$ and $\vec{B}$

TE vs TEM:
- Ovensteein: TEM $\Rightarrow E_z=0$
- Grif: TE $\Rightarrow E_z=0$
- Grif: TEM $\Rightarrow E_z=B_z=0$.

• Define $r:=E_r/E_i \quad t:=E_t/E_i$

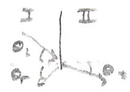
• Normal incidence:

$$r=\frac{n_1-n_2}{n_1+n_2} \quad t=\frac{2n_1}{n_1+n_2}$$

• Non-normal: $\alpha=\cos(\theta_t)/\cos(\theta_i)$

$$r=\frac{\alpha n_1-n_2}{\alpha n_1+n_2} \quad t=\frac{2n_1}{\alpha n_1+n_2},$$

$1+r=\alpha t$, $Y_\alpha=n_\alpha Y_0$



$\theta_i=\theta_r$

$n_1\sin\theta_i=n_2\sin\theta_t$

• Energy reflected/trans.
$R=S_r/S_i,\ T=S_t/S_i$

$R=|r|^2$

• Brewster's angle: $R(\theta_B)=0$
- E_r, E_T at $90°$
Need $\cos\theta_t/\cos\theta_i=n_2/n_1$

• Energy in TEM mode travels at v_g

• EM in conductors: $\vec{J}_f=\sigma\vec{E}$,
$\rho_f(t)=e^{-(\sigma/\varepsilon)t}\rho_f(0)$ ← from continuity → wait for charge to dissipate so $\rho_f=0$, then plug into M.E.'s.

• Wavenumber $\tilde{k}$ is complex:
$\tilde{k}^2=\mu\varepsilon\omega^2+i\mu\sigma\omega$, $\tilde{k}=k+iK$

$$k=\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

$$K=\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

• Circular coax TE modes: $E_z=0$.
Use $\nabla_\rho^2 B_z=-\gamma^2 B_z$

$$\Rightarrow\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)B_z=-\gamma^2 B_z$$

0 w/ circ symmetry

$B_z(r)=AJ_0(\gamma r)+BN_0(\gamma r)$ — Bessel fn, neuman fn

• Get Bessel fn solutions $\rightarrow \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\gamma^2\right]B_z=0$.

• B.C.'s: $\frac{\partial B_z}{\partial r}=0$ $(\hat{n}\cdot\nabla_\rho)$.

Bound e^- in metal: acts like a spring.

$$\frac{d\vec{P}}{dt}=-ne\vec{E}-\frac{\vec{P}}{\tau}-nkx$$

$P=P_0e^{-i\omega t} \quad x=x_0e^{-i\omega t}$,

$P=nm\dot{x}$. Get: $x_0=\dfrac{-eE_0/m}{-\omega_0^2-\omega^2-i\omega/\tau}$, $\omega_0:=k/m$

$J=-\frac{e}{m}P$

$$\sigma_1=\frac{\varepsilon_0\Omega_p^2\omega^2/\tau}{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}, \quad \sigma_2=-\frac{\varepsilon_0\Omega_p^2\omega(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}$$

$R(\hat{z},\theta)=\begin{pmatrix}\cos\theta & +\sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ for polarization. Some Jones matrices (note convention differs, so may not be the same)

$W(\varphi)=\begin{pmatrix}e^{i\varphi_H} & 0\\ 0 & e^{-i\varphi_V}\end{pmatrix}$ (along x,y axes) $\quad P_H=\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad P_V=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

• HW problem: Rotated waveplate by θ:

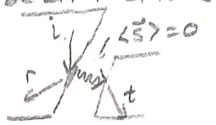
$\begin{pmatrix}\cos^2\theta+e^{i\varphi}\sin^2\theta & -2\sin\theta\cos\theta\\ -2\sin\theta\cos\theta & \sin^2\theta-\cos^2\theta\end{pmatrix}$ $\quad P_\theta=\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}$

$P_{\hat{e}}=\hat{e}\hat{e}^\dagger$ ← we used $(\ ^{l}_{r})$ as general polarization state in class.

Evanescent waves: Occurs when $\langle\vec{S}\rangle=\vec{0}$. Occurs when $\tilde{n}(\omega)$ purely imaginary, we have seen this in the collisionless regime of the free e^- gas. Problem will probably be on interfaces:

TIR: Critical angle at:

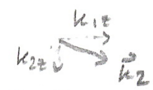
$\sin\theta_c=n_2/n_1$, $\sin\theta_1>\frac{n_2}{n_1}\Rightarrow$ evanescent in n_2.



We can calculate the wavenumbers in n_2, for

$$k_{2z}^2=\frac{\omega^2}{c^2}\left(n_2^2-n_1^2\sin^2\theta_1\right)$$

(B.C.'s imply $k_{1x}=k_{2x}$ $=\frac{\omega}{c}n_1\sin\theta_1$)



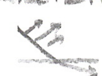
• For TIR, k_{2z} imaginary. Since $\tilde{n}=\frac{c}{\omega}\tilde{k}$, we have $\tilde{n}^2=n_2^2-n_1^2\sin^2\theta_1$, and this will be imaginary for $\theta_1>\theta_c$.

Also $k_2^2=k_{2x}^2+k_{2z}^2$

$K:=\mathrm{Im}[k_{2z}]=\frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_1-n_2^2}$

$k:=k_{2x}=\frac{\omega}{c}n_1\sin\theta_1$

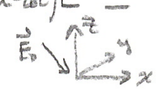
Non-normal incidence: In plane of incidence, like for Brewster's angle



The evanescent waves have:

$$\vec{E}_2(\vec{r},t)=\vec{E}_0\exp(-\mathrm{Im}[k_z]z)\exp(i(k_xx-\omega t))$$
$$=\vec{E}_0\exp\left(-\frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_1-n_2^2}\,z\right)\cos\left(\frac{\omega}{c}n_1\sin\theta_1 x-\omega t\right)$$

(propogates like a normal wave w/ $\exp(i(\vec{k}\cdot\vec{r}-\omega t))$).



Now suppose $\vec{E}_0$ is in the $\hat{y}$ direction. Then:

$\vec{E}=E_0e^{-Kz}\cos(kx-\omega t)\hat{y}$ (use $\nabla\times\vec{E}=-\partial\vec{B}/\partial t$)

$$\vec{B}=\frac{E_0}{\omega}e^{-Kz}\left[K\sin(kx-\omega t)\hat{x}+k\cos(kx-\omega t)\hat{z}\right]$$ ← Kappa

$$\vec{S}=\frac{E_0^2}{\mu_2\omega}e^{-2Kz}\left[k\cos^2(kx-\omega t)\hat{x}-K\sin(kx-\omega t)\cos(kx-\omega t)\hat{z}\right]$$

$$\langle\vec{S}\rangle=\frac{E_0^2k}{2\mu_2\omega}e^{-2Kz}\hat{x}$$

→ no energy transmitted in $\hat{z}$ b/c $\hat{z}\cdot\langle\vec{S}\rangle=0$.

Patrick Oare | $\vec{J}(\vec{r},t)=\vec{J}(\vec{r})e^{-i\omega t}$, same w/ ρ. Continuity: $\nabla\cdot\vec{J}(\vec{r})-i\omega\rho(\vec{r})=0$. $\vec{J}=\vec{J}_L+\vec{J}_T$, $\vec{B}=\frac{1}{\omega}\vec{k}\times\vec{E}_T$

Physics 110B Midterm 2 Cheat Sheet

EM Waves in Metal

$\rho(\vec{r})=\frac{k}{\omega}J_L(\vec{r})$, $\left(\frac{\omega^2}{c^2}-k^2\right)\vec{E}_T=-i\omega\mu_0\vec{J}_T$, $\vec{E}_L=-\frac{i\mu_0c^2}{\omega}\vec{J}_L$, $\vec{B}_L=\vec{0}$

Conductors: $\vec{J}=\sigma\vec{E}$; Linear response: $\tilde{J}(\omega)=\sigma(\omega)\tilde{E}(\omega)$

$B_0=\tilde{n}(\omega)E_0/c$, $\tilde{n}$ complex $\Rightarrow\vec{B},\vec{E}$ out of phase

$Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}}=377\,\Omega$

Free e^- gas: $\vec{P}$ = momentum/volume $=nm_e\langle\vec{v}\rangle$, $n=N/V$, $\vec{E}(t)=\vec{E}_0e^{-i\omega t}$. Newton's Laws $\Rightarrow\frac{d\vec{P}}{dt}=-ne\vec{E}(t)\rightarrow$ "momentum memory time" τ so $\dot{\vec{P}}=-ne\vec{E}-\vec{P}/\tau$. $\vec{P}=\mathrm{Re}[\vec{P}_0e^{-i\omega t}]\Rightarrow P_0=\frac{-ne}{1/\tau-i\omega}E_0$

$J_0=-ne\langle\vec{v}\rangle=-\frac{e}{m_e}P_0=\frac{ne^2/m}{1/\tau-i\omega}E_0\Rightarrow\sigma(\omega)=\frac{ne^2/m}{1/\tau-i\omega}=\sigma_1+i\sigma_2$

$\sigma_{DC}=\lim_{\omega\to0}\sigma(\omega)$, $\sigma_1=\frac{\sigma_{DC}}{1+\omega^2\tau^2}$, $\sigma_2=\omega\tau\sigma_1$, $\varepsilon_1(\omega)=\varepsilon_0-\sigma_2/\omega$, $\varepsilon_2=\sigma_1/\omega$.

Plasma freq $\Omega_p^2=\frac{ne^2}{\varepsilon_0m}$, $\sigma_{DC}\tau=\varepsilon_0(\Omega_p\tau)^2$. In metal, $\tau\sim10\,fs$, $\Omega_p\sim10^{15}s^{-1}$, so $(\Omega_p\tau)^2>>\varepsilon_0$ $\therefore\lim_{\omega\to0}\varepsilon_1=\varepsilon_0-\sigma_{DC}\tau$; $\lim_{\omega\to\infty}\varepsilon_1=\varepsilon_0$, $\lim_{\omega\to\infty}\varepsilon_2=\sigma_{DC}/\omega^3\tau^2$

I: Dissipative, $\omega<<1/\tau$, so $|\varepsilon_2|>>|\varepsilon_1|$. Then $\varepsilon_2=\frac{\sigma_{DC}/\omega}{\tau^2(1/\tau^2+\omega^2)}\sim\sigma_{DC}/\omega$, $\tilde{n}=\sqrt{\varepsilon(\omega)/\varepsilon_0}\sim\sqrt{i\varepsilon_2/\varepsilon_0}=\frac{1+i}{\sqrt2}\sqrt{\frac{\sigma_{DC}}{\varepsilon_0\omega}}$, $k=\frac{\omega}{c}\tilde{n}=(1+i)\sqrt{\frac{\mu_0\omega\sigma_{DC}}{2}}$, $\delta=\sqrt{\frac{2}{\mu_0\omega\sigma_{DC}}}$, B_0 out of phase w/ E_0 by $<90°$, and $\langle\vec{S}\rangle=\frac{\hat{z}|E_0|^2}{2\mu_0c}\sqrt{\frac{\sigma_{DC}}{2\varepsilon_0\omega}}e^{-2z/\delta}$

II: Collisionless, $|\varepsilon_2|<<|\varepsilon_1|$. Condition is $1/\tau<\omega<\Omega_p$. Get $\varepsilon(\omega)\sim\varepsilon_1(\omega)\sim-\frac{\sigma_{DC}\tau}{\omega^2\tau^2}=-\frac{\varepsilon_0\Omega_p^2}{\omega^2}<0$. Then have $\tilde{n}(\omega)=\sqrt{\varepsilon(\omega)/\varepsilon_0}=i\Omega_p/\omega$, and $k(\omega)=i\Omega_p/c$, so $\delta=c/\Omega_p$; B_0 90° out of phase w/ E_0, same w/ $\vec{J}$ and $\vec{E}$ as σ imaginary. $\langle\vec{S}\rangle=\vec{0}$ b/c $\tilde{n}\notin\mathbb{R}\Rightarrow$ evanescent

Oscillations in metal: Displacement of negative charge COM by δz gives $\sigma=-ne\delta z$, $E=-\sigma/\varepsilon_0=ne\delta z/\varepsilon_0$, $\vec{F}=-ne\vec{E}=-n^2e^2\delta z/\varepsilon_0=\dot{\vec{P}}=\frac{d}{dt}(nmv)$ $\Rightarrow\ddot{z}=-\frac{ne^2}{m\varepsilon_0}z\Rightarrow\omega^2=\Omega_p^2$, ω = freq of oscillations of sheet of charge.

Bound e^- in metal: Model as springs, displacement x_i, so $\langle x\rangle=\frac1N\sum_ix_i=:x$. Now have $\dot{\vec{P}}=-ne\vec{E}-\vec{P}/\tau-nkx$, and $P=nm\dot{x}$, so using $e^{-i\omega t}$ time dep gives one $nm\ddot{x}=\dot{\vec{P}}\Rightarrow-nm\omega^2x_0=-neE_0+i\omega nmx_0/\tau-nkx_0\Rightarrow x_0=\frac{-eE_0/m}{\omega_0^2-\omega^2-i\omega/\tau}$, $\omega_0^2:=k/m$. Then $J_0=-\frac{e}{m}P_0=-ne\dot{x}=i\omega nex_0$. This gives us σ_{bound} as $J_0=\sigma_bE_0$. Get $\lim_{\omega\to0}\sigma_b\omega_{2b}=0$, so $\sigma_{DC_b}=0$. We have $\vec{p}=-ne\langle x\rangle$ which is polarization, and $\vec{p}=\varepsilon_0\chi(\omega)\vec{E}$, $\vec{J}=d\vec{p}/dt$, $\Rightarrow J_0=-i\omega p_0=\sigma(\omega)E_0$

Dfns: $k^2(\omega)=\frac{\omega^2}{c^2}\tilde{n}^2(\omega)$, $\tilde{n}(\omega)=\sqrt{1+\frac{i}{\varepsilon_0\omega}\sigma(\omega)}$

$\tilde{n}(\omega)=n(\omega)+iK(\omega)$, $k(\omega)=k_0(n+iK)$

$\varepsilon(\omega)=\varepsilon_0\tilde{n}^2(\omega)$, $1+\chi(\omega)=\varepsilon(\omega)/\varepsilon_0$, $Y_0=1/Z_0$

$K_0=\omega/c$ = vac. wavevec, $\mathrm{Im}(k)=k_0K$, $Y(\omega)=Y_0\tilde{n}$

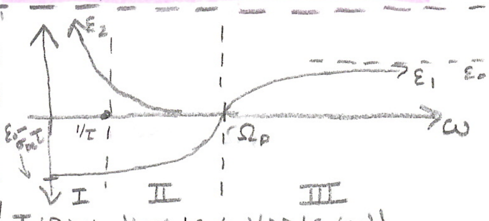
$$\langle\vec{S}\rangle=\frac{\hat{z}}{2\mu_0}\frac{|E_0|^2}{c}e^{-2\mathrm{Im}[k]z}\,\mathrm{Re}[\tilde{n}(\omega)]$$

Dispersion rlns:

L mode: $\sigma(\omega_L)=\frac{i\omega}{\mu_0c^2}\iff\varepsilon(\omega_L)=0$

T mode: $k^2(\omega_T)=\frac{\omega_T^2}{c^2}+i\omega_T\mu_0\sigma(\omega_T)$

III: Transparent. Condition is $\left|\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right|>>|\mu_0\vec{J}|$ (recall $\nabla\times\vec{B}$ eqn). $\iff\frac{i\omega}{c^2}>>\mu_0\sigma(\omega)$. But at high ω, $\sigma(\omega)\to\frac{i\sigma_{DC}}{\omega\tau}$, so $\omega^2>>\Omega_p^2$. For zero, use $\sigma_{DC}\tau-\varepsilon_0\sim\sigma_{DC}\tau$ to show zero crossing is Ω_p. For $\omega>\Omega_p$, e^- gas becomes transparent



I: Dissipative; $|\varepsilon_2(\omega)|>>|\varepsilon_1(\omega)|$

II: Collisionless; $|\varepsilon_2(\omega)|<<|\varepsilon_1(\omega)|$

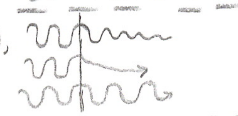
III: Transparent: $\omega^2>>\Omega_p^2$

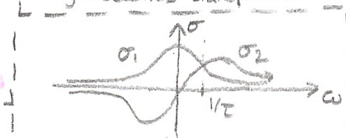
Penetration depth: $\delta:=1/\mathrm{Im}[k]$

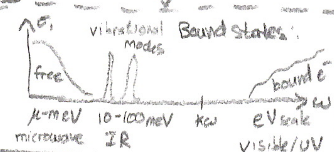
I: $\langle\vec{S}\rangle$ exp. decay

II: $\langle\vec{S}\rangle=0$

III: $\langle\vec{S}\rangle\neq0$, $R\sim1$







L modes iff $\varepsilon(\omega_L)=0$ iff $\omega_L=\Omega_p$ (take $1/\tau<<-i\omega$)

Radiation

Rotating dipole. Start w/ coordinate free expressions for electric dipole $\vec{p}=\vec{p}_0e^{-i\omega t}$, so:

$\vec{E}=\frac{i\omega^2\mu_0}{4\pi r}e^{ikr}(\hat{r}\times(\hat{r}\times\vec{p}(t)))$ $\vec{B}=\frac1c\hat{r}\times\vec{E}=\frac{i\omega^2\mu_0}{4\pi rc}e^{ikr}\hat{r}\times\vec{p}(t)$

For V, take for z axis $V(\vec{r},t)=-\frac{p_0\omega}{4\pi\varepsilon_0c}\frac{z}{x^2+y^2+z^2}\sin[\ldots]$. Make one on x axis, and for the one on y axis, give it a phase shift of $\pi/2$ so $\sin[\ldots+\pi/2]=-\cos[\ldots]$, and $\therefore$ we have:

$V=-\frac{p_0\omega}{4\pi\varepsilon_0cr^2}(x\sin[\ldots]-y\cos[\ldots])=-\frac{p_0\omega}{4\pi\varepsilon_0c}\frac{\sin\theta}{r}(\cos\varphi\sin[\ldots]-\sin\varphi\cos[\ldots])$

$\vec{A}=-\frac{\mu_0p_0\omega}{4\pi r}(\hat{x}\sin[\ldots]-\hat{y}\cos[\ldots])$ $\quad\ldots=\omega(t-r/c)$

For $\vec{E},\vec{B}$ fields, can write $-\hat{\theta}\sin\theta=\hat{z}-\frac{z}{r}\hat{r}$. Replace w/ x, y, + phase to get:

$\vec{E}=\frac{\mu_0p_0\omega^2}{4\pi r}\left(\cos[\ldots]\left(\hat{x}-\frac{x}{r}\hat{r}\right)+\sin[\ldots]\left(\hat{y}-\frac{y}{r}\hat{r}\right)\right)$ now $\cos[\ldots+\pi/2]=\sin[\ldots]$

$\vec{B}=\frac1c\hat{r}\times\vec{E}$, $\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}=\frac{E^2}{\mu_0c}\hat{r}$ as $\vec{E}\cdot\hat{r}=0$. Get:

$E^2=\left(\frac{\mu_0p_0\omega^2}{4\pi r}\right)^2[1-(\sin\theta\cos[\ldots-\varphi])^2]$, and so we get:

$\vec{S}=\frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2[1-(\sin\theta\cos[\omega(t-r/c)-\varphi])^2]\hat{r}$

$\langle\vec{S}\rangle=\frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2[1-\frac12\sin^2\theta]\hat{r}$, $P=\int\langle\vec{S}\rangle\cdot d\vec{a}=\frac{\mu_0p_0^2\omega^4}{6\pi c}$

Get double power b/c cross terms of field 1 and 2 are 90° out of phase

Larmor: $P=\frac{\mu_0q^2a^2}{6\pi c}$. Radiation fields $\propto1/r$. Point charge has:

$\vec{E}(\vec{r},t)=\frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}[(c^2-v^2)\vec{u}+\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})]$, $\vec{B}=\frac1c\hat{\mathscr{r}}\times\vec{E}$

velocity field — acceleration field

$\vec{u}=c\hat{\mathscr{r}}-\vec{v}$

Approximations to general solns: take $R>>d$, $\lambda>>d$ (d = size of distribution). Then $|\vec{r}-\vec{r}'|\sim r$, so:

$\vec{A}(\vec{r},\omega)=\frac{\mu_0}{4\pi}e^{i\frac{\omega}{c}r}\int\vec{J}(\vec{r}',t_r)d^3r'$

$V(\vec{r},\omega)=\frac{1}{4\pi\varepsilon_0}e^{i\frac{\omega}{c}r}\int\rho(\vec{r}',t_r)d^3r'$

Dipole moments:

$\vec{p}=\int\vec{r}'\rho(\vec{r}')d^3r'=q\vec{d}$

$\vec{m}=I\vec{a}=\pi a^2I\hat{z}$ (loop)

Electric dipole: $\vec{p}=p\hat{z}$

$\vec{A}=-i\frac{\mu_0}{4\pi r}e^{i\frac{\omega}{c}r}\omega p(\omega)e^{-i\omega t}$

$\vec{E}=-\frac{i\omega^2\mu_0}{4\pi r}pe^{ikr}\sin\theta\hat{\theta}$, $k=\frac{\omega}{c}$

$\vec{B}=-\frac{i\omega^2\mu_0}{4\pi rc}pe^{ikr}\sin\theta\hat{\varphi}$

Take real part as physical

$\langle\vec{S}\rangle=\frac{1}{2\mu_0}\mathrm{Re}[\vec{E}\times\vec{B}^*]=\frac{1}{2\mu_0c}\left(\frac{\omega^2\mu_0p}{4\pi r}\right)^2\sin^2\theta\,\hat{r}$

Radiation gauge: $\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}=0$ to get:

$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=\mu_0\vec{J}=\Box^2\vec{A}$

$\nabla^2V-\frac{1}{c^2}\frac{\partial^2V}{\partial t^2}=-\rho/\varepsilon_0=\Box^2V$

Solutions in radiation gauge:

$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}d^3r'$ $\quad\mathscr{r}=|\vec{r}-\vec{r}'|$

$V=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d^3r'$ $\quad t_r=t-\frac{\mathscr{r}}{c}$

Frequency solutions:

$\vec{A}(\vec{r},\omega)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',\omega)}{\mathscr{r}}e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}d^3r'$

$V(\vec{r},\omega)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',\omega)}{\mathscr{r}}e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}d^3r'$

Magnetic dipole: $I(t)=I_0e^{-i\omega t}$. $V=0$,

$\vec{A}=-i\frac{\mu_0}{4\pi r}\frac{\omega}{c}e^{ikr}\vec{m}\times\hat{r}$, $\vec{m}\times\hat{r}=m\hat{\varphi}$

$\vec{E}=i\omega\vec{A}=\frac{\mu_0\omega^2m}{4\pi rc}e^{ikr}\sin\theta\hat{\varphi}$

$\vec{B}=\frac{\mu_0\omega^2m}{4\pi rc^2}e^{ikr}\sin\theta\hat{\theta}=\frac1c\hat{r}\times\vec{E}$

$\hat{r}\times\hat{\theta}=\hat{\varphi}$, $\hat{r}\times\hat{\varphi}=-\hat{\theta}$, $\hat{r}\times(\hat{r}\times\hat{z})=-\hat{\theta}\sin\theta$

$\hookrightarrow$ Griffiths gives (Electric dipole): $\rightarrow -ie^{i\omega r/c}$

$V(\vec{r},t)=-\frac{p_0\omega}{4\pi\varepsilon_0c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]$

$\vec{E}_{ed}=-\frac{\mu_0p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\hat{\theta}$

$\vec{B}_{ed}=-\frac{\mu_0p_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\hat{\varphi}$

(electric dipole fields from Griff).

$\hat{\varphi}=-\sin\varphi\hat{x}+\cos\varphi\hat{y}$

$\hat{\theta}=\hat{x}\cos\varphi\cos\theta+\hat{y}\cos\theta\sin\varphi-\hat{z}\sin\theta$

$x=r\sin\theta\cos\varphi$

$y=r\sin\theta\sin\varphi$

$z=r\cos\theta$

$\hat{r}=\sin\theta\cos\varphi\hat{x}+\sin\theta\sin\varphi\hat{y}+\cos\theta\hat{z}$

$\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}$

$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$

$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad \nabla \times \vec{E} = -\frac{\partial B}{\partial t}$

$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

$\nabla \cdot \vec{S} + \frac{\partial u}{\partial t} = 0$

$\mathcal{U} = \int u(\vec{r})\, d^3 r$

$P = \frac{d\mathcal{U}}{dt} = \oint \vec{S} \cdot d\vec{a}$

Bound e^-: $\varepsilon(\omega)/\varepsilon_0 = 1 + \frac{i\sigma(\omega)}{\varepsilon_0 \omega} + \chi_b(\omega)$

$\sigma_{1b}(\omega) = \frac{\varepsilon_0 \Omega_p^2 \omega^2/\tau}{(\omega_0^2 - \omega^2)^2 + (\omega/\tau)^2} \qquad \sigma_{2b}(\omega) = -\frac{\varepsilon_0 \Omega_p^2 \omega(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\omega/\tau)^2}$

$\chi(\omega) = \frac{i\sigma_b(\omega)}{\varepsilon_0 \omega} = \frac{\Omega_p^2}{\omega_0^2 - \omega^2 - i\omega/\tau}$

$\begin{cases} \vec{E} = -\nabla V - \partial \vec{A}/\partial t \\ \vec{B} = \nabla \times \vec{A} \end{cases}$

$\langle \cos^2(\varphi(t)) \rangle = 1/2$

Radiation reaction force:

$\vec{F} = \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}}$

Magnetic dipoles:

To derive, we use $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{I(\vec{\ell}', t_r)}{\mathscr{r}} d\ell'$.

We consider a current loop w/ current $I(t) = I_0 \cos(\omega t)$. We have:

$\vec{m} = \int I\, d\vec{a} = I\vec{a}$

- For rotating $\vec{m}$, will probably want to use coordinate free $\vec{A}$.

Radiation from accelerating charge: The acceleration field is what gives us radiation. This is:

$\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}} \cdot \vec{u})^3}(\vec{\mathscr{r}} \times (\vec{u} \times \vec{a}))$

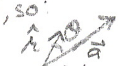
$\vec{S}_{rad} = \frac{1}{\mu_0 c} E_{rad}^2 \hat{\mathscr{r}}$

- **Nonrelativistic limit:** $\vec{u} \sim c\hat{\mathscr{r}}$, and in this limit $P = \oint \vec{S} \cdot d\vec{a}$ reduces to Larmor formula. These simplify:

$* \begin{cases} \vec{E}_{rad} = \frac{\mu_0 q}{4\pi \mathscr{r}}[(\hat{\mathscr{r}} \cdot \vec{a})\hat{\mathscr{r}} - \vec{a}] \\ \vec{S}_{rad} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi \mathscr{r}}\right)^2 [a^2 - (\hat{\mathscr{r}} \cdot \vec{a})^2]\hat{\mathscr{r}} \end{cases}$

We let Θ be the angle btwn $\vec{\mathscr{r}}$ and $\vec{a}$, so:

$\vec{S}_{rad} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\Theta}{\mathscr{r}^2}\right)\hat{\mathscr{r}}$ ← simplified version of *



Griffiths magnetic dipole: $V(\vec{r},t) = 0$

m = dipole moment.

$\vec{A}(\vec{r},t) = -\frac{\mu_0 m \omega}{4\pi c}\left(\frac{\sin\Theta}{r}\right)\sin[\omega(t - r/c)]\,\hat{\varphi}$

$\vec{E}(\vec{r},t) = \frac{\mu_0 m \omega^2}{4\pi c}\left(\frac{\sin\Theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\varphi}$

$\vec{B} = -\frac{\mu_0 m \omega^2}{4\pi c^2}\left(\frac{\sin\Theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\Theta}$

$\vec{S} = \frac{\mu_0}{c}\left[\frac{m\omega^2}{4\pi c}\left(\frac{\sin\Theta}{r}\right)\cos[\omega(t - r/c)]\right]^2 \hat{r}$

$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \left(\frac{1}{c}\hat{\mathscr{r}} \times \vec{E}\right) = \frac{1}{\mu_0 c}\left[E^2 \hat{\mathscr{r}} - (\hat{\mathscr{r}} \cdot \vec{E})\vec{E}\right]$

(total $\vec{S}$, including velocity fields). For $\vec{S}_{rad}$, 2nd term goes away.

- Probably use Larmor formula for this question, unless it asks for $\vec{E}, \vec{S}$.

- To get total power radiated from anything, integrate $\oint \vec{S} \cdot d\vec{a}$ over a sphere, and use $d\Omega = \sin\Theta\, d\Theta\, d\varphi$.

Power from oscillating magnetic dipole:

$P = \frac{\mu_0 \ddot{m}}{6\pi c^3} \xrightarrow{m_0 \cos(\omega t)} \langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$

Patrick Oare | $\beta = v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $d\tau = dt/\gamma$

Physics 110B Final Cheat Sheet

Relativity: $\gamma \geq 1$ always.

$\vec{p} = \gamma m\vec{v}$, $E = \gamma mc^2$, $T = (\gamma - 1)mc^2$

$\vec{f} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{v})$ $\quad \vec{v} = d\vec{x}/dt$

spacelike: $a_\mu a^\mu < 0$
light: $a_\mu a^\mu = 0$
time: $a_\mu a^\mu > 0$

A **4-vector** is a quadruple a^μ that transforms as:

$(a^\mu)' = \Lambda^\mu_{\ \nu} a^\nu \quad a^\mu = g^{\mu\nu} a_\nu,\ a_\mu = g_{\mu\nu} a^\nu$

$$\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix},\quad (\Lambda^\mu_{\ \nu})^{-1} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix},\quad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$a_\mu a^\mu$ is a Lorentz invariant.

$p_i^\mu = p_f^\mu$ for collision

$$x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix},\ v^\mu = \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma\begin{pmatrix} c \\ \vec{v} \end{pmatrix},\ p^\mu = mv^\mu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$$

$$K^\mu = \frac{dp^\mu}{d\tau} = \begin{pmatrix} \gamma \vec{f}\cdot\vec{v}/c \\ \gamma\vec{f} \end{pmatrix},\ J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix},\ A^\mu = \begin{pmatrix} V/c \\ \vec{A} \end{pmatrix},\ \partial_\mu = \frac{\partial}{\partial x^\mu}$$

$ds^2 = dx_\mu dx^\mu = c^2dt^2 - (d\vec{x})^2$

$v_\mu v^\mu = c^2$

$p_\mu p^\mu = m^2c^2 \Rightarrow E^2 = m^2c^4 + c^2p^2$

Lorentz force law:

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}v_\nu = K^\mu$$

Continuity eqn:

$\partial_\mu J^\mu = 0$

Four tensors:

$G^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

(get $G^{\mu\nu}$ from $F^{\mu\nu}$ by $\vec{E}/c \mapsto \vec{B}$, $\vec{B} \mapsto -\vec{E}/c$

Transformation rules:

$F^{\mu\nu\,\prime} = \Lambda^\mu_{\ \lambda}\Lambda^\nu_{\ \sigma}F^{\lambda\sigma}$ (*)

$F^{\mu\nu} = g^{\mu\sigma}F_{\sigma\rho}g^{\rho\nu}$

M.E.'s:

$$\begin{cases} \partial_\mu F^{\mu\nu} = \mu_0 J^\nu \\ \partial_\mu G^{\mu\nu} = 0 \end{cases}$$

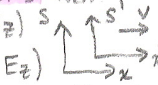
(OR):

$$\partial_\mu\partial^\mu A^\nu = \mu_0 J^\nu$$

(*) means $(F^{\mu\nu})' = \Lambda^T F^{\mu\nu}\Lambda$

· Suppose **S' moves at $v\hat{x}$ w.r.t. S**. Then:

$$\begin{cases} E_x' = E_x & E_y' = \gamma(E_y - vB_z) \\ B_x' = B_x & B_y' = \gamma(B_y + \frac{v}{c^2}E_z) \end{cases}$$



$E_z' = \gamma(E_z + vB_y) \quad B_z' = \gamma(B_z - \frac{v}{c^2}E_y)$

can cyclically permute coords, i.e. use $(x,y,z) \mapsto (y,z,x)$ or (z,x,y)

· If $\vec{B} = \vec{0}$ in S, then:
$\vec{B}' = -\frac{1}{c^2}\vec{v}\times\vec{E}'$

· **Transformation of I, ρ:** In 1D, we use:

$I^\mu = \begin{pmatrix} c\lambda \\ I \end{pmatrix}$, $I'^\mu = \Lambda^\mu_{\ \nu}I^\nu$

· If $\lambda = 0$ in S, all force comes from $\vec{B}$ as $|q\vec{v}\times\vec{B}| = \frac{qv\mu_0 I}{2\pi r}$. Assume charges move at $\vec{v}$.

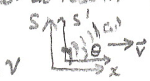
$\lambda' = -\frac{\gamma v}{c^2}I$ (w/ $\lambda = 0$) $\Rightarrow$ in S', $F = qE = \frac{qv\mu_0 I}{2\pi r}\gamma$

then in S', $\lambda' \neq 0$ but $I = 0$. This gives $E = \frac{\lambda'}{2\pi\varepsilon_0 r}$

$\vec{K}$ = surface current

$K^\mu = \begin{pmatrix} c\sigma \\ \vec{K} \end{pmatrix}$

Doppler Shift: Light source w/ freq ω' at rest in S' moving at $v\hat{x}$ w.r.t. S. Define:



$k^\mu := \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix}$, so $k^\mu = (\Lambda^{-1})^\mu_{\ \nu}k'^\nu$

Label eqns ①②③④, let $\omega = ck$, $\omega' = ck'$. Let:

$\cos\Theta = k_x/k$, $\cos\Theta' = k_x'/k'$

$$\omega = \gamma\omega'\left[1 + \beta\left(\frac{\cos\Theta - \beta}{1 - \beta\cos\Theta}\right)\right]$$

$k_\mu k^\mu = \omega^2/c^2 - k^2$ invariant

$k_\mu x^\mu = \omega\Delta t - \vec{k}\cdot\Delta\vec{x} = \Delta\varphi$

$$\cos\Theta = \frac{\beta + \cos\Theta'}{1 + \beta\cos\Theta'}$$

· Something receding away has $\Theta = \pi$, so:

$\omega = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega'$ $\quad$ · If $\beta \sim 1$, $\gamma^2 \sim \frac{1}{2(1-\beta)}$

For Lorentz force law, note:

$v_\mu = g_{\mu\nu}v^\nu = \gamma\begin{pmatrix} c \\ -\vec{v} \end{pmatrix}$

Note $F^{\mu\nu}$ is row μ, column ν of F.

· The spacelike components take the same form:

$K^1 = qF^{1\nu}v_\nu = q\left[(E_x/c)(\gamma c) + 0 + (-B_z)(-\gamma v_y) + (B_y)(-\gamma v_z)\right]$

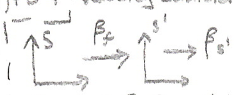
$= q\gamma(\vec{E} + \vec{v}\times\vec{B})_x = \frac{dp_x}{d\tau}$

· The timelike component gives:

$K^0 = qF^{0\nu}v_\nu \Rightarrow \frac{dE}{d\tau} = \gamma q\vec{E}\cdot\vec{v}$

(we get an eqn for power).

Velocity addition:



· β_s = vel. in S, $\beta_{s'}$ = vel. in S', β_f = vel. of S' w.r.t. S. Then:

$$\beta_s = \frac{\beta_f + \beta_{s'}}{1 + \beta_f\beta_{s'}}$$

$\vec{\beta} = c\vec{p}/E$

Force largest in object's rest frame, $F = F'/\gamma$

Object longest in its rest frame.

As you move faster, your clock runs slower