

Lie Groups and Algebras

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1 Basic Definitions

In physics, we use Lie groups to describe sets of continuous symmetries. Although we will attempt to make definitions as general as possible, we will often restrict ourselves to the case of specific examples which are of interest for physics, namely the Lie groups $SU(N)$, $SO(N)$, and occasionally symplectic groups. We define Lie groups and algebras, and then will discuss the interplay between the two.

Definition 1.1 (Lie group). A **Lie group** (G, \cdot) is a group which is also a differentiable manifold, in which the group operation respects the structure of the manifold. Namely, we require that the maps $\cdot : G^2 \rightarrow G$ and $\cdot^{-1} : G \rightarrow G$ be smooth.

Definition 1.2 (Lie algebra). A **Lie algebra** \mathfrak{g} over a field k is a k -valued vector space equipped with a map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called a **Lie bracket**, such that the following hold:

1. $[\cdot, \cdot]$ is bilinear.
2. $[\cdot, \cdot]$ is antisymmetric.
3. $[\cdot, \cdot]$ satisfies the **Jacobi identity**, i.e. for $A, B, C \in \mathfrak{g}$:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \quad (1)$$

We will generally be considering matrix Lie groups and algebras, in which case the Lie bracket is simply the commutator, $[A, B] = AB - BA$. In the case of matrix groups, our Lie algebras will also be matrix-valued, so \mathfrak{g} is a matrix-valued vector space. We call it an algebra because the map $[\cdot, \cdot]$ gives the vector space an algebra-like structure¹.

The essential idea behind Lie groups is this: Lie groups act on a vector space V as the symmetry operation (for example, the group $SO(3)$ of orthogonal real valued 3×3 matrices with determinant 1 act as rotations in $V = \mathbb{R}^3$). Lie algebras generate the Lie group via the exponential map in the following way: suppose $U \in G$ is an arbitrary element (assume G is path-connected, or at least that U is in the path-component of $1 \in G$). Then, there exists $X \in \mathfrak{g}$ such that:

$$U = \exp(iX) \quad (2)$$

The proof of this existence is one of the fundamental theorems of Lie theory, and is why we care about Lie algebras. In essence, the Lie algebra parameterizes the Lie group. Let $n = \dim(\mathfrak{g})$. If we pick a basis $\{T^a\}_{a=1}^n$ for \mathfrak{g} , the elements of this basis are called **generators** of the Lie group G because an arbitrary element of G can be represented as:

$$\exp(iX^a T^a) \quad (3)$$

¹An algebra is simply a vector space with a ring structure, i.e. with a multiplication $\cdot : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. However, the axioms that ring multiplication must satisfy are different than those that $[\cdot, \cdot]$ must satisfy.

because T^a spans \mathfrak{g} . This means the coordinates X^a parameterize the Lie group G , and so we can specify an element of G by specifying a set of n coordinates.

Every Lie algebra is defined by its Lie bracket. Since $[\cdot, \cdot]$ maps \mathfrak{g}^2 into \mathfrak{g} , we must be able to expand:

$$[T^a, T^b] = if^{abc}T^c \quad (4)$$

where f^{abc} is an antisymmetric tensor of numbers known as the **structure constants** of the Lie algebra. Specifying the structure constants of an algebra exactly define the algebra and its Lie bracket.

The associated Lie algebra with a Lie group can be defined by taking its tangent space at the identity, and in this way we have an correspondence from Lie groups to Lie algebras, and back. Geometrically, I like to view the Lie group as a “smoothly deformed version” of the flat Lie algebra. The exponential map does the deformation, and to me this emphasizes the fact that a Lie group is first and foremost a manifold.

To act Lie groups on spaces, we must discuss representations. Note that the endomorphism ring $End(V)$ of a vector space is also a Lie algebra, by allowing the Lie bracket to equal the commutator.

Definition 1.3 (Representation). A **representation** of a Lie algebra \mathfrak{g} on a vector space V is a Lie algebra homomorphism² $\mathfrak{g} \rightarrow End(V)$. A representation of a Lie group G is a Lie group homomorphism $G \rightarrow Aut(V)$, where $Aut(V)$ denotes the space of vector space automorphisms on V . Every Lie algebra homomorphism can be extended in the obvious manner to a Lie group homomorphism.

Definition 1.4. The **dimension** of a representation $\pi : G \rightarrow Aut(V)$ is the dimension of V . If $\pi : G \rightarrow GL(n, \mathbb{F})$ is a representation of a Lie group with $n \times n$ matrices, then n is the dimension of π .

Definition 1.5 (Faithful). A representation $\Pi : G \rightarrow V$ is **faithful** if Π is injective.

A representation can likewise be defined as an action of G or \mathfrak{g} on V , as given a representation $\pi : \mathfrak{g} \rightarrow V$, we have a \mathfrak{g} -action on V defined by $X \cdot v := \pi(X)(v)$. The easiest way to define a representation is to give an explicit embedding of the generators of \mathfrak{g} into $End(V)$ as matrices, as every endomorphism can be realized as a matrix. Once the generators are embedded into matrices, then the rest of the representation can be embedded by linearity.

Representations are more than just a Lie group and a vector space: they also implicitly contain the map from $G \rightarrow Aut(V)$. Because of this, we have been labelling a representation $\Pi : G \rightarrow Aut(V)$ as Π WLOG. However, we also may label the representation by the triple (Π, G, V) (and similarly for an algebra representation we may label it as (π, \mathfrak{g}, V)). Since each G -representation induces a canonical \mathfrak{g} -representation (and vice versa), it is often useful to abuse notation and to let V denote the G -representation and the \mathfrak{g} -representation, since V is the common element of the triple denoting Π and π . This is often used in physics, where for example we may refer to the octet of $SU(3)$ as **8** to denote the representation $\pi : \mathfrak{su}(3) \rightarrow \mathbb{R}^8$ which has action $\pi(T^a)^{bc} = -if^{abc}$ (the adjoint representation).

We now consider what a morphism between representations looks like. This is called an intertwining map, and is essentially a vector space map which respects the action induced by the representation.

Definition 1.6 (Intertwining map). Let $\Pi : G \rightarrow Aut(V)$ and $\Sigma : G \rightarrow Aut(W)$ be two representations of a Lie group G , and let $\phi : V \rightarrow W$ be a linear map. We call ϕ an **intertwining map** if

²Meaning it preserves $+$ and $[\cdot, \cdot]$

for each $g \in G$ and $v \in V$ we have:

$$\phi(\Pi(g)v) = \Sigma(g)(\phi v) \tag{5}$$

In other words, the canonical diagram commutes for each element $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\Pi(g)} & V \\ \downarrow \phi & & \downarrow \phi \\ W & \xrightarrow{\Sigma(g)} & W \end{array}$$

An intertwining map which is invertible is a **representation isomorphism**.

An intertwining map is simply a morphism in the category of representations: it preserves both the linear structure of the spaces it acts on and the action of the representation. A more obvious way to view such a map is as follows: denote by \cdot the action of G on V or W , i.e. $g \cdot v := \Pi(g)v$. Then the definition of ϕ being an intertwining map can equivalently be written as:

$$\phi(g \cdot v) = g \cdot \phi(v) \tag{6}$$

which makes it completely obvious that such an intertwining map is a morphism in the category of linear G -representations.

2 Irreducible Representations

Irreducible representations (irreps) are the simplest representations that one can create— they are akin to simple groups in standard group theory, or prime ideals in ring theory. They will give us a way to decompose complicated Lie groups into sums of irreps, which will allow us to study these groups based off of studying these simpler substructures. Any representation we are interested in will be able to be decomposed into irreps, which makes it quite important to study these irreps in the context of physics.

Definition 2.1 (Irreducible representation). If $\Pi : G \rightarrow \text{Aut}(V)$ is a representation of G , we say that a subspace $W \subset V$ is an **invariant subspace** if $\pi(g)(W) \subseteq W$ for each $g \in G$, i.e. that the group *always* goes into itself under symmetry transformations. We say that W is **nontrivial** if W is a nonempty proper subspace. If the representation π has no nontrivial invariant subspaces, then we call π an **irreducible representation**.

Definition 2.2 (Irreducible subspace). Let $\Pi : G \rightarrow \text{Aut}(V)$ be a representation of G . If $W \subseteq V$ is invariant and if it contains no proper invariant subspaces, then we call W an **irreducible subspace** of Π .

Note that saying V is an irreducible subspace of itself is the same as saying that the representation V is irreducible.

Because we are studying Lie theory in the context of physics, we are interested in representations of symmetry groups. In particular, these groups are unitary, and so are the representations of these groups.

Definition 2.3 (Unitary). Let $\Pi : G \rightarrow \text{Aut}(V)$ be a representation. Then Π is a **unitary representation** if $\Pi(g)$ is a unitary operator on V for each $g \in G$.

Luckily for us, unitary representations are quite simple and can always be decomposed into the direct sum of irreps. We will make this precise with a few definitions.

Definition 2.4 (Completely reducible). A finite dimensional representation of a Lie group or algebra is **completely reducible** if it is isomorphic to a direct sum of a finite number of irreps.

Prop 2.1. Let $\Pi : G \rightarrow \text{Aut}(V)$ be a completely reducible representation. Then:

1. For each invariant subspace $U \leq V$, V splits as a direct product $V = U \oplus W$ with W also invariant.
2. Every invariant subspace of V is completely reducible.

The last part of this proposition will allow us to prove that unitary representations are completely reducible, hence when studying representations in physics we can build up our representations from irreps without loss of generality.

Theorem 2.1. Any finite dimensional unitary representation of a Lie group or algebra is completely reducible.

Proof. Let $\Pi : G \rightarrow \text{Aut}(V)$ be unitary. If Π is an irrep, then we are done. So, suppose it is not, and pick a nontrivial invariant subspace $W \leq V$. Then V splits as $V = W \oplus W^\perp$. We show that W^\perp is invariant as well. Let $v \in W^\perp$. Then for each $g \in G$ and $w \in W$, we have:

$$\langle \Pi(g)v | w \rangle = \langle v | \Pi(g^{-1}) | w \rangle = 0 \quad (7)$$

because $\Pi(g^{-1})|w\rangle \in W$ as W is invariant. So, $\Pi(g)v \in W^\perp$ for each $g \in G$. We can inductively repeat this process on both W and W^\perp a finite number of times until this terminates with each summand being an irrep, and the process must terminate because V is finite dimensional. \square

In addition, finite representations of compact Lie groups are also completely reducible. We will not prove this, but it is certainly a nice fact to know.

Theorem 2.2. If G is compact, then every finite dimensional representation of G is completely reducible.

2.1 Schur's Lemma

3 Constructions

3.1 Complexification

It is often useful to study a real Lie algebra by complexifying it and studying the algebra as a complex vector space instead of a real vector space.

Definition 3.1 (Complex Lie group). A Lie group G is **complex** if its Lie algebra \mathfrak{g} is a vector space over \mathbb{C} , i.e. if $iX \in \mathfrak{g}$ for each $X \in \mathfrak{g}$.

We will next define how to complexify a real vector space and treat it as a \mathbb{C} -vector space. This will extend naturally to any Lie algebra (as a Lie algebra has a vector space structure in addition to its Lie bracket) in a functorial manner.

Definition 3.2 (Complexification). Let V be a vector space over \mathbb{R} . Then the **complexification** of V , denoted $V_{\mathbb{C}}$, is the set of formal linear combinations:

$$V_{\mathbb{C}} = \{v_1 + iv_2 : v_1, v_2 \in V\} \quad (8)$$

We endow $V_{\mathbb{C}}$ with a vector space structure by defining for $v_1 + iv_2 \in V_{\mathbb{C}}$:

$$i(v_1 + iv_2) := -v_2 + iv_1 \quad (9)$$

This extends by linearity to all of $V_{\mathbb{C}}$, making it into a vector space over \mathbb{C} .

The complexification of a vector space is in itself also a real vector space of dimension $2\dim(V)$, since as a set it is isomorphic to the direct sum $V \oplus V$. For a Lie algebra, we can use this construction to complexify the structure of the algebra as a vector space, and also extend the bracket to this new complex space. The complexification is also the “best” way we can embed a real Lie algebra into a complex Lie algebra, in the sense that it is a universal attractor in the category of maps $\mathfrak{g} \rightarrow h$, where h is a complex Lie algebra.

Theorem 3.1. Let \mathfrak{g} be a finite dimensional real Lie algebra. Let $\mathfrak{g}_{\mathbb{C}}$ be its complexification as a vector space. Then the bracket $[\cdot, \cdot]$ extends uniquely to $\mathfrak{g}_{\mathbb{C}}$ by:

$$[X_1 + iX_2, Y_1 + iY_2] := ([X_1, Y_1] - [X_2, Y_2]) + i([X_2, Y_1] + [X_1, Y_2]) \quad (10)$$

making $\mathfrak{g}_{\mathbb{C}}$ into a Lie algebra.

Prop 3.1. Let $\mathfrak{g} \subset M_n(\mathbb{C})$ be a real Lie algebra, and suppose that $iX \notin \mathfrak{g}$ for $X \in \mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to the set of matrices in $M_n(\mathbb{C})$ of the form $X + iY$ with $X, Y \in \mathfrak{g}$.

Theorem 3.2. Let \mathfrak{g} be a real Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$, and let $\iota : \mathfrak{g} \rightarrow \mathfrak{g}_{\mathbb{C}}$ be the canonical inclusion. Then \mathfrak{g} is universal in the sense that if \mathfrak{h} is any complex Lie algebra and $f : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism, then we can factor f through $\mathfrak{g}_{\mathbb{C}}$:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\iota} & \mathfrak{g}_{\mathbb{C}} \\ & \searrow f & \downarrow \bar{f} \\ & & \mathfrak{h} \end{array}$$

Some common complexifications which are useful are:

- $\mathfrak{u}(N)_\mathbb{C} \cong \mathfrak{gl}(N; \mathbb{C})$
- $\mathfrak{su}(N)_\mathbb{C} \cong \mathfrak{sl}(N; \mathbb{C})$

Complexification leaves irreps invariant, so studying the irreps of a complexified Lie algebra is equivalent to studying the irreps of the original algebra. This will be very useful as it will allow us to study the irreps of $\mathfrak{su}(N)$ through the irreps of $\mathfrak{sl}(n; \mathbb{C})$.

Theorem 3.3. Let \mathfrak{g} be a real Lie algebra with complexification $\mathfrak{g}_\mathbb{C}$. Then every finite dimensional complex representation π of \mathfrak{g} has a unique extension to a complex representation of $\mathfrak{g}_\mathbb{C}$ by linearity, also denoted π . Furthermore, π is irreducible as a representation of $\mathfrak{g}_\mathbb{C}$ iff it is irreducible as a representation of \mathfrak{g} .

3.2 Sums

Recall the direct sum $V_1 \oplus V_2$ of two vector spaces V_1 and V_2 is simply a fancy way of writing the Cartesian product $V_1 \times V_2$ after we give it a vector space structure. Let G be a Lie group with two representations $\pi_1 : G \rightarrow V_1$ and $\pi_2 : G \rightarrow V_2$. Then we can form a representation of G on $V_1 \oplus V_2$ in the obvious way:

$$\pi_1 \oplus \pi_2 : G \rightarrow \text{Aut}(V_1 \oplus V_2) \quad (11)$$

where $\pi_1 \oplus \pi_2(g)$ acts on elements $(v_1, v_2) \in V_1 \oplus V_2$ by:

$$[(\pi_1 \oplus \pi_2)g](v_1, v_2) := ((\pi_1 g)v_1, (\pi_2 g)v_2) \quad (12)$$

We define this in an analogous way for a representation of a Lie algebra.

When we sum together two representations, their dimensions are additive because the direct sum adds dimensions of vector spaces:

$$\dim(\pi_1 \oplus \pi_2) = \dim(\pi_1) + \dim(\pi_2) \quad (13)$$

3.3 Products

Suppose that we have representations of groups G and H , namely $\pi_1 : G \rightarrow \text{Aut}(V)$ and $\pi_2 : H \rightarrow \text{Aut}(W)$. Then we can define a canonical representation of $G \times H$ by:

$$\pi_1 \otimes \pi_2 : G \times H \rightarrow \text{Aut}(V \otimes W) \quad (g, h) \mapsto (\pi_1 g) \otimes (\pi_2 h) \quad (14)$$

where we see that $(\pi_1 g) \otimes (\pi_2 h)$ is a morphism on the space $V \otimes W$ by its definition.

Now, further suppose that $H = G$, and so we have two representations of G , $\pi_1 : G \rightarrow V$ and $\pi_2 : G \rightarrow W$. Then using the previous construction, we can embed G into $G \times G$:

$$G \hookrightarrow G \times G \xrightarrow{\pi_1 \otimes \pi_2} \text{Aut}(V \otimes W) \quad (15)$$

to form a new representation of G , one which acts on $V \otimes W$ as $(\pi_1 \otimes \pi_2)(g) = (\pi_1 g) \otimes (\pi_2 g)$. Note the dimensionality of the new representation is:

$$\dim(\pi_1 \otimes \pi_2) = \dim(\pi_1)\dim(\pi_2) \quad (16)$$

In general, the product of irreducible representations is not reducible, and the factorization of the product of irreps into a sum of irreps is the basis for the Clebsch-Gordan theory often studied in the context of adding angular momentum in quantum mechanics. For example, consider the irreps of $SU(3)$. Then $3 \otimes \bar{3}$ (the fundamental times the antifundamental) is reducible, and in fact $3 \otimes \bar{3} = 8 \oplus 1$, i.e. it splits as the sum of the adjoint 8 and the singlet 1.

3.4 Conjugate (dual) representations

4 Common Irreducible Representations

Because irreps can be combined together to form new representations, we will now discuss some of the irreps that are seen very often in physics.

4.1 The Fundamental Representation

4.2 The Adjoint Representation

The adjoint representation is perhaps the most canonical of representations because every Lie algebra admits an adjoint representation on itself. For $X \in \mathfrak{g}$, we define the adjoint operator $ad_X : \mathfrak{g} \rightarrow \mathfrak{g}$ by:

$$ad_X(Y) := [X, Y] \quad (17)$$

It is easy to show that the map $\mathfrak{g} \rightarrow End(\mathfrak{g})$, $X \mapsto ad_X$ is a Lie algebra homomorphism. Thus this must define a representation, called the **adjoint representation** of \mathfrak{g} , in which the elements of \mathfrak{g} can act on the algebra that they span. Suppose that our Lie algebra is d dimensional, so it has d generators T^a . We can explicitly embed the T^a into the adjoint representation by matrices T_A^a . To compute these matrices, suppose that we have an element $X = X^a T^a \in \mathfrak{g}$. Since the matrices T_A^a act on the column vector X^a , they must be $d \times d$ dimensional. Additionally, when T_A^a is multiplied against the vector X^a , it must give the components of $ad_{T^a}(X)$ (because T_A^a represents the operator ad_{T^a}), so:

$$(ad_{T^a}(X))^b = (T_A^a)^{bc} X^c \iff ([T^a, X^c T^c])^b = i X^c f^{acb} = (T_A^a)^{bc} X^c \quad (18)$$

And thus we see that the components of the generators in the adjoint representation must be given by:

$$(T_A^a)^{bc} = -i f^{abc} \quad (19)$$

Consider what the transformation law looks like in the adjoint representation. Of course, this is a matrix multiplication of X^a by the group element. Let $V = \exp(i\alpha^a T_A^a)$ be an element of G in the adjoint representation. Then:

$$X^a \mapsto (X^a)' = V^{ab} X^b \quad (20)$$

by definition. Infinitesimally, this is:

$$X^a \mapsto (1 + i\alpha^c T_A^c)^{ab} X^b = X^a + i\alpha^c (-i f^{cab}) X^b = X^a - f^{abc} \alpha^b X^c \quad (21)$$

Now for the interesting question: what does the transformation law for the element $X = X^a T^a$ look like? We claim that the matrix X actually transforms under conjugation by V :

$$X \mapsto X' = V X V^\dagger \quad (22)$$

We can check this by expanding out this transformation law for an infinitesimal α :

$$\begin{aligned} X^a T^a \mapsto (1 + i\alpha^a T^a)(X^b T^b)(1 - i\alpha^c T^c) &= X + i\alpha^a X^b [T^a, T^b] = X - \alpha^a X^b f^{abc} T^c \\ &= (X^a - f^{abc} \alpha^b X^c) T^a \end{aligned}$$

which is exactly the transformation law we were looking for. Thus, we can view transformation in the adjoint representation in two different ways. First, as how the vector of component fields transforms:

$$X^a \mapsto V^{ab} X^b \quad (23)$$

And secondly, how the entire matrix X transforms:

$$X \mapsto V X V^\dagger \quad (24)$$

5 Invariants