Patrick Oare | Physics 137A Midterm | Cheat Sheet

Postulates:

① The state of a quantum system at any time t_0 is given by a complex normalized wavefunction $\Psi(x,t_0)$.

② Operators: Observable (measurable) quantities are represented in QM by linear Hermitian operators

③ Possibilities: A precise measurement of an observable Q can only yield one of the eigenvalues of $\hat{Q}$ (where: $\hat{Q}\psi_{q_i} = q_i\psi_{q_i}$)

④ Probabilities: Given a state $\Psi(x,t)$, a measurement of Q at time t yields eigenvalue q_i w/ probability $P(Q=q_i) = |\langle q_i|\Psi(t)\rangle|^2 = \left|\int_{\mathbb{R}} \psi^*_{q_i}(x)\Psi(x,t)dx\right|^2$ ★ Remember the squared! ★ ψ_{q_i}, Ψ normalized

⑤ Collapse: If a measurement of Q yields eigenvalue q_i, immediately after measurement the wavefunction collapses into the normalized eigenfunction belonging to q_i

⑥ Time evolution: The wavefunction evolves in time according to the Schrödinger eqn: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{\mathcal{H}}\Psi = \frac{\hat{p}^2}{2m}\Psi + V(\hat{x})\Psi$ (TDSE)

$\Psi(x,t)$ and Operators:

- The probability density is $|\Psi(x,t)|^2$ ($|\Psi|^2 = \Psi^*\Psi$)
- $dP = |\Psi(x,t)|^2dx \to P(x\in[a,b]) = \int_a^b|\Psi|^2dx$
- Expectation value: $\langle f(n)\rangle = \sum f(n)P(n)$, $\langle f(x)\rangle = \int_{\mathbb{R}} f(x)dP$

$\sigma_Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$

$\langle Q\rangle = \langle\Psi|\hat{Q}|\Psi\rangle = \int_{\mathbb{R}}\Psi^*\hat{Q}\Psi dx$

- $\Psi(x,t)$ must be normalized: $\langle\Psi|\Psi\rangle = \int_{\mathbb{R}}\Psi^*(x,t)\Psi(x,t)dx = 1$
- Unique up to a phase $e^{i\phi}$, $\phi\in\mathbb{R}$

$\hat{x}\Psi(x,t) = x\Psi(x,t)$

$\hat{p}\Psi(x,t) = \left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)$

- Operators: $\langle x|p\rangle = \psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}$

$\hat{x}\Phi(p,t) = \left(i\hbar\frac{\partial}{\partial p}\right)\Phi(p,t)$

$\hat{p}\Phi(p,t) = p\Phi(p,t)$

- Dirac Delta / Kronecker Delta: $\delta(x) := \begin{cases}\infty & x=0\\ 0 & \text{else}\end{cases}$, $\delta_{ij} := \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$\int_{\mathbb{R}}\delta(x-x_0)f(x)dx = f(x_0)$, $\sum_i f(a_i)\delta_{ij} = f(a_j)$

$\delta(x-x_0) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik(x-x_0)}dk$

Orthonormality: The eigenfunctions of the Hamiltonian form a complete orthonormal basis. For an orthonormal set $\{|i\rangle\}$, we have: $\langle n|m\rangle = \int_{\mathbb{R}}\psi_n^*(x)\psi_m(x)dx = \delta_{mn}$

- Completeness: $\forall\Psi(x,t_0)\in L^2(\mathbb{R})$, $\exists!\ \{c_i\}\in\mathbb{C}$ s.t. $\Psi(x,t_0) = \sum_n c_n\psi_n(x)$
  - Where ψ_n are the vectors in the basis
  - Generally, ψ_n are eigenfns
- Ehrenfest Theorem: Expectation values obey classical laws ($\langle p\rangle = m\frac{d\langle x\rangle}{dt}$)

Schrödinger Eqn:

- Solve by separation of variables, let $\Psi(x,t) = \psi(x)T(t)$
- If ψ_1 and ψ_2 are solutions, then $\forall\psi\in\text{span}\{\psi_1,\psi_2\}$ are as well
- Separable solutions are ($\Psi_n(x,t)$):
  ① Stationary states: All probabilities / expected values constant in time
  ② Any solution $\Psi(x,t)$ to the TDSE is a linear combo of separable solutions

TISE: $\psi'' = -\frac{2m}{\hbar^2}(E-V(x))\psi$

- Plugging in, we see that we have 2 ODEs:

$i\hbar\frac{1}{T}\frac{dT}{dt} = E_n \longrightarrow T(t) = e^{-iE_nt/\hbar}$ (time dependence)

$-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V(x) = E_n \longrightarrow E_n\psi_n(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + V(x)\psi_n(x)$ (TISE)

- General solution: Solve TISE for specific $\psi_n(x)$, and the soln is: $\Psi(x,t) = \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}$
  - c_n can be determined from boundary conditions
- Find c_n's using Fourier's trick — Apply $\int_{\mathbb{R}}dx\,\psi_n^*$ to both sides of $\Psi(x,0) = \sum c_m\psi_m(x)$, use the fact that $\int_{\mathbb{R}}\psi_n^*\psi_m dx = \delta_{mn}$ to kill the sum and see that:

$c_n = \int_{\mathbb{R}}\psi_n^*(x)\Psi(x,0)dx$ (If know Ψ at t_0, use to and change time dependence to $e^{-iE(t-t_0)/\hbar}$) ($c_n = \langle n|\Psi\rangle$)

$P(E=E_n) = |\langle n|\Psi\rangle|^2 = |c_n|^2$

- Can view $\Psi(x,t) = \sum_n c_n\Psi_n(x,t)$ or $\Psi(x,t) = \sum_n c_n(t)\psi_n(x)$ w/ $c_n(t) = c_ne^{-iE_nt/\hbar}$
- $\langle E\rangle$ is const. in time, b/c $|c_n(t)|^2 = c_n^*e^{iE_nt/\hbar}c_ne^{-iE_nt/\hbar} = |c_n(0)|^2$

$\langle E\rangle = \sum_n|c_n|^2E_n$ (Use orthonormality)

- $\sum_n|c_n|^2 = 1$
- Continuous spectra: $\langle\alpha|\beta\rangle = \delta(\alpha-\beta)$, $\Psi(x,t) = \int_{\mathbb{R}}c(\alpha)\psi_\alpha(x)d\alpha$, $c(\alpha) = \int_{\mathbb{R}}\psi_\alpha^*(x)\Psi(x,0)dx$

Momentum Space / Sketching:

Let $\varphi(p)$ be expansion coefficients for a continuous p spectrum ($\varphi(p) = \langle p|\Psi\rangle$)

$\psi(x) = \mathcal{F}^{-1}\{\varphi(p)\} = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}}\varphi(p)e^{ipx/\hbar}dp$

$\varphi(p) = \mathcal{F}\{\psi(x)\} = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}}\psi(x)e^{-ipx/\hbar}dx$

If $\varphi(p)$ is normalized, then: $dP = |\varphi(p)|^2dp$

$P(p\in[a,b]) = \int_a^b\varphi^*(p)\varphi(p)dp$

- Free particle: $\hat{\mathcal{H}} = \hat{p}^2/2m$
- Begin w/ $\Psi(x,0) \to \Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}}\Psi(x,0)e^{-ipx/\hbar}dx$
- This is an eigenfn of $\hat{\mathcal{H}}$ w/ $E = p^2/2m$, so we have: $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}}\Psi(x',0)e^{-ipx'/\hbar}dx'\,e^{-\frac{ip^2t}{2m\hbar}}$
- To find $\Psi(x,t)$, we inverse F.T. $\Phi(p,t)$: $\Psi(x,t) = \frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2}\Psi(x',0)e^{-ipx'/\hbar}e^{-\frac{ip^2t}{2m\hbar}}e^{\frac{ipx}{\hbar}}dx'dp$

Good luck, pal!

$\langle E\rangle$ either ① $\sum|c_n|^2E_n$ ② Directly $\int\psi^*\hat{\mathcal{H}}\psi$ $\langle\psi|\hat{\mathcal{H}}|\psi\rangle$

- Bound States: Energy eigenfn w/ energy E s.t. no CAR extends to ∞ → quantized energies
- Scattering State: ... s.t. at least one CAR extends to $\pm\infty$ → continuous energies

① Draw potential and determine spectrum ② Choose energy — Bound (high or low), tunneling, scattering
③ Low energy bound states, determine loc and # of nodes; high energy bound states use classical P envelope; scattering use guess of T to determine relative amplitude ④ CARs: Oscillation, $k(x) := \sqrt{\frac{2m(E-V(x))}{\hbar^2}}$
  - Larger $k(x)$ → faster oscillation, smaller amplitude ⑤ CFRs: Exp. behavior, use $K(x) := \sqrt{\frac{2m(V(x)-E)}{\hbar^2}}$, larger K → faster decay, more curvature
⑥ Boundary conditions: ψ continuous everywhere, ψ' continuous everywhere $V\neq\pm\infty$. If → ∞, only decay behavior
  - $V=\infty$ in region → $\psi=0$; $V=\infty$ at point → kink in graph of ψ w/ $\Delta\psi' = \frac{2m\alpha}{\hbar}\psi(x_0)$ (For $V = \alpha\delta(x-x_0)$)
⑦ Bound states: If $V(x)$ is symmetric, ψ_n alternates sym/antisym
⑧ Nodes: Ground state usually has 0, w/ each subsequent excited state has 1 more than the last
  - No nodes in CFR that → ∞; at most 1 node in a CFR

- Probability current: $J(x) := \frac{i\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right)$

$T := \frac{|J_{trans}|}{|J_{incident}|} = \frac{|\text{amp of transmitted}|^2}{|\text{amp of incident}|^2}$, $R := \frac{|J_{ref.}|}{|J_{inc}|}$

- $1+R=T$
- Determine incident / trans/reflected waves

$e^{+ipx/\hbar}$ is right moving, $e^{-ipx/\hbar}$ is left moving

Common Potentials:

Infinite Square Well: $V(x) = \begin{cases}0 & x\in[0,a]\\ \infty & \text{else}\end{cases}$

$E_n = \frac{\hbar^2\pi^2n^2}{2ma^2}$, $n\in\mathbb{Z}^+$

$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

Ground state is $n=1$

Finite Square Well: $V(x) = \begin{cases}-V_0 & x\in[-a,a]\\ 0 & \text{else}\end{cases}$

- Split into region I, II, III and solve TISE in each
- Apply boundary conditions: ψ,ψ' cont at $x=\pm a$
- Should have 4 eqns, solve and divide through to get transcendental eqn

SHO: $V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$

- Let $\xi := \sqrt{\frac{m\omega}{\hbar}}x$, so TISE is: $\psi''(\xi) = (\xi^2-K)\psi(\xi)$ ($K = \frac{2E}{\hbar\omega}$) I make up your mind dude it's K or k
- Solve by guessing $\psi(\xi) = h(\xi)e^{-\xi^2/2}$ and using power series for $h(\xi)$
- Try to use even/odd before integrating

$E_n = \left(n+\frac{1}{2}\right)\hbar\omega$ ★ Remember $\psi\in L^2$ B.C.

$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^nn!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega x^2}{2\hbar}}$

$H_n(\xi) = \sum_{n=0}^{\infty}a_n\xi^n$, $a_n = 2^n$, $a_{n-1} = 0$, $a_{j-2} = -\frac{j(j-1)}{2(n+2-j)}a_j$

$H_0 = 1$, $H_1 = 2\xi$, $H_2 = 4\xi^2 - 1$, $H_3 = 8\xi^3 - 12\xi$

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Patrick Oare

Physics 137A Midterm 2 Cheat Sheet

Postulates:

① A quantum system is described by a Hilbert space. The state is described (up to a phase) by a normalized ket.

② **Observables**: Physically measurable quantities (observables) are represented by Hermitian operators in the Hilbert Space.

③ **Possibilities**: Given an observable Q w/ associated Hermitian operator $\hat{Q}$, the only possible results of a measurement of Q are the eigenvalues of $\hat{Q}$.

$\Delta E \Delta t \geq \hbar/2$ $\quad \frac{\sigma_Q}{\Delta t} \sim \frac{d\langle Q\rangle}{dt}$

④ **Probabilities**: Let Q be an observable w/ associated operator $\hat{Q}$ that has distinct eigenvalues $\{\lambda_r\}$, their degeneracies $\{N_r\}$, and eigenbasis $\{|\lambda_r, n\rangle\}$. Let $\{\hat{P}_r\}$ be the associated projection operator onto the λ_r eigenspace ($\hat{P}_r = \sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|$). Suppose a wavefunction is in state $|\Psi\rangle$ at $t = t_0$ w/ components $c_{rn} = \langle\lambda_r, n|\Psi\rangle$. Then, the probability a measurement of Q will produce λ_r is: $\mathbb{P}(Q=\lambda_r) = \|\hat{P}_r|\Psi\rangle\|^2 = \langle\Psi|\hat{P}_r|\Psi\rangle = \sum_{n=1}^{N_r} |\langle\lambda_r, n|\Psi\rangle|^2 = \sum_{n=1}^{N_r} |c_{rn}|^2$

⑤ **Collapse**: Let $|\Psi(t_0^-)\rangle$ be the state of a system before a measurement is made and $|\Psi(t_0^+)\rangle$ be the state just after measurement. If the measurement yielded λ_r, then $|\Psi(t_0^+)\rangle = N \cdot \hat{P}_r|\Psi(t_0^-)\rangle$. N is a normalization constant: $1 = |N|^2 \|\hat{P}_r|\Psi(t_0^-)\rangle\|^2 = |N|^2 \cdot \langle\Psi(t_0^-)|\hat{P}_r^\dagger \hat{P}_r|\Psi(t_0^-)\rangle = \langle\Psi(t_0^-)|\hat{P}_r|\Psi(t_0^-)\rangle \cdot |N|^2 = |N|^2 \cdot \mathbb{P}(Q=\lambda_r)$

⑥ **Time evolution**: The wavefunction evolves in time according to the time-dependent Schrödinger Equation: $i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$

Kets and Operators:

- A **Hilbert Space** is a complete (every limit that appears to be converging does converge) inner product space over $\mathbb{C}$.
- Vectors in the Hilbert Space are called **kets** and denoted $|\ldots\rangle$
- Bra vectors are linear functionals on H: A bra $\langle\beta|: H \to \mathbb{C}$
- Given a basis of kets $\{|e_i\rangle\}$, we can form its dual basis $\{\langle e^i|\}$ s.t. $\langle e^i|e_j\rangle = \delta^i_j$
- **Inner Product**: Binary function on H that maps to $\mathbb{C} \to \langle\cdot|\cdot\rangle: H\times H \to \mathbb{C}$. Must have:
  ① Skew Symmetry: $\langle u|v\rangle = \langle v|u\rangle^*$
  ② Linearity in 2nd argument: $\langle\alpha|(b|\beta_1\rangle + c|\beta_2\rangle) = b\langle\alpha|\beta_1\rangle + c\langle\alpha|\beta_2\rangle$
  ③ Positive semidefinite: $\langle\alpha|\alpha\rangle \geq 0$
  - If $\langle\alpha|\alpha\rangle = 0 \Longrightarrow |\alpha\rangle = \vec{0}$
- The norm of a ket is defined by $\|\alpha\|^2 = \langle\alpha|\alpha\rangle$
- If $|\beta\rangle = \sum \beta^i |e_i\rangle$, $|\alpha\rangle = \sum \alpha^i|e_i\rangle$, then $\langle\beta|\alpha\rangle = \sum_i \beta_i \alpha^i = \sum_i (\beta^i)^* \alpha^i$
- The i^{th} component of a ket is $\alpha^i = \langle e^i|\alpha\rangle$, and likewise of a bra is $\beta_i = \langle\beta|e_i\rangle$
- Schwarz Inequality: $|\langle\alpha|\beta\rangle|^2 \leq \|\alpha\|^2 \cdot \|\beta\|^2$

Basis and Coordinates:

- A **basis** is a set of linearly independent vectors that generate the space
  - Linear independence: If $\sum c^i|e_i\rangle = 0$, $\forall c^i$ must be 0
  - Generating set: $\forall|v\rangle \in H$, $\exists! c^i \in \mathbb{C}$ s.t. $|v\rangle = \sum_i c^i|e_i\rangle$
- **Coordinates**: Let $|\Psi\rangle \in H$, $\{|e_i\rangle\} =: e$ a basis for H. If $|\Psi\rangle = \sum_i c^i|e_i\rangle$, the coordinates of $|\Psi\rangle$ are: $|\Psi\rangle \doteq_e \begin{pmatrix} c^1 \\ c^2 \\ \vdots \\ c^N \end{pmatrix} = [\Psi]_e$ $\quad \begin{pmatrix} a \\ b \end{pmatrix}^\perp = \begin{pmatrix} -b^* \\ a^* \end{pmatrix}$
- **Normalization**: For a physical state, $\langle\Psi|\Psi\rangle = 1$
- **Adjoint**: $|\alpha\rangle^\dagger = \langle\alpha|$
  $(a|\alpha\rangle + b|\beta\rangle)^\dagger = \langle\alpha|a^* + \langle\beta|b^*$
- If $|\alpha\rangle \doteq (\alpha^1 \ldots \alpha^N)^T$, then $\langle\alpha| \doteq (\alpha^{1*} \ldots \alpha^{N*})$

- **Operator**: Linear map from H onto H
- Expectation value: $\langle Q\rangle = \langle\Psi|\hat{Q}|\Psi\rangle$
- Operators have a diagonal matrix representation in their eigenbasis
- $\hat{Q} \doteq_e \begin{pmatrix} Q^1_1 & Q^1_2 & \ldots & Q^1_N \\ Q^2_1 & Q^2_2 & \ldots & Q^2_N \\ \vdots & & & \\ Q^N_1 & Q^N_2 & \ldots & Q^N_N \end{pmatrix}$
- $Q^i_j = \langle e^i|\hat{Q}|e_j\rangle$ $\quad \hat{Q}^m_n \sim \hat{Q}_{mn}$
- Adjoint: $I(|\alpha\rangle, \hat{Q}|\beta\rangle) = I(\hat{Q}^\dagger|\alpha\rangle, |\beta\rangle)$
  $\langle\alpha|\hat{Q}|\beta\rangle = \langle\beta|\hat{Q}^\dagger|\alpha\rangle^*$
- To take the adjoint of an equation, flip order and $\dagger$ each element
- $[\hat{Q}^\dagger] = [\hat{Q}]^\dagger = ([\hat{Q}]^T)^*$

Ket-bras: A ket times a bra gives an operator

- $|\alpha\rangle\langle\beta|: H \to H$ $\quad \hat{I}^i_j = \delta^i_j$
- For $\forall\hat{Q}: H \to H$ linear, $\hat{Q} = \sum_{i,j} Q^i_j |e_i\rangle\langle e^j|$ $\quad |1\rangle\langle 3| \doteq \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$ $\quad |e_i\rangle\langle e^j| \doteq_e \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 0 \end{pmatrix}$ — 1 in i^{th} row, j^{th} column
- Resolution of the identity: $\hat{I} = \sum_i |e_i\rangle\langle e^i| = \int |x\rangle\langle x| dx$
  $\langle\alpha|\beta\rangle = \langle\alpha|\hat{I}|\beta\rangle = \sum_i \langle\alpha|e_i\rangle\langle e^i|\beta\rangle = \sum_i \alpha_i\beta^i$

Hermitian / Unitary Operators:

- **Hermitian**: $\hat{Q}^\dagger = \hat{Q}$
  - All eigenvalues are real
  - Can form an orthonormal basis of eigenkets
  - Spectral decomposition: $\hat{Q} = \sum_r \sum_{n=1}^{N_r} \lambda_r |\lambda_r, n\rangle\langle\lambda_r, n|$
- **Unitary**: $\hat{U}^\dagger = \hat{U}^{-1}$
  - Preserves norms: $\|\Psi\| = \|\hat{U}\Psi\|$
  - Preserves inner products: $\langle\alpha|\beta\rangle = \langle\hat{U}\alpha|\hat{U}\beta\rangle$
    $|\hat{U}\beta\rangle := \hat{U}|\beta\rangle$, $\langle\hat{U}\alpha| = (\hat{U}|\alpha\rangle)^\dagger$

Position/Momentum:

$\hat{x}|x\rangle = x|x\rangle$ $\quad \hat{p}|p\rangle = p|p\rangle$

$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ $\quad \langle x|\hat{x}|x'\rangle = x\delta(x-x')$ $\quad \langle x|\hat{p}|x'\rangle = -i\hbar\delta(x-x')\frac{\partial}{\partial x'}$

$\langle x|\hat{p}|\Psi\rangle = -i\hbar\frac{\partial\Psi}{\partial x}$ $\quad \langle p|\hat{x}|\Psi\rangle = i\hbar\frac{\partial\Phi}{\partial p}$

Finding eigenvalues / eigenkets:

① Solve $\text{char}_Q(\lambda) = \det([Q] - \lambda I) = 0$ in a basis

② For each λ, solve $\text{nul}([Q] - \lambda I)$ for the eigenkets

Projection operators: $\hat{P}_r = \sum_{n=1}^{N_r} |\lambda_r, n\rangle\langle\lambda_r, n|$

① Hermitian ② Idempotent ($\hat{P}^2 = \hat{P}$)

- Expectation value evolution: $\frac{d\langle Q\rangle}{dt} = \left\langle \frac{i}{\hbar}[\hat{H}, \hat{Q}]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$

Compatibility and Transformations:

- **Commutator**: $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ $\quad [\hat{x}, \hat{p}] = i\hbar\hat{I}$
  $[\hat{A}, f(\hat{A})] = 0$; $[c\hat{A}, \hat{B}] = [\hat{A}, c\hat{B}] = c[\hat{A}, \hat{B}]$ $\quad [\hat{A}+\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$
  $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$ $\quad [\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$
- If $[\hat{A}, \hat{B}] = 0$, $\hat{A}$ and $\hat{B}$ are compatible and simultaneously diagonalizable
- Time-translation operator: $\hat{U}(t) := \exp(-i\hat{H}t/\hbar)$
  $|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$
  - Diagonalize and apply $\exp(-it\cdot/\hbar)$ to each diagonal element
  - Unitary — $\hat{U}(t_1)\hat{U}(t_2) = \hat{U}(t_1+t_2)$
- **Active transformations**: Acting unitary $\hat{U}$ on everything in a system:
  $|\tilde{\Psi}\rangle = \hat{U}|\Psi\rangle$ $\quad \tilde{\hat{Q}} = \hat{U}\hat{Q}\hat{U}^\dagger$
  - Brackets are conserved: $\langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle\alpha|\beta\rangle$
- Space translation: $\hat{\mathcal{T}}(a) = \exp(-i\hat{p}a/\hbar)$
- Parity: $\hat{\pi}|x\rangle = |-x\rangle$, $\hat{\pi}|p\rangle = |-p\rangle$
  $\hat{\pi}\hat{x}\hat{\pi}^\dagger = -\hat{x}$ $\quad \hat{\pi}\hat{p}\hat{\pi}^\dagger = -\hat{p}$
  $\hat{\pi} = \hat{\pi}^\dagger = \hat{\pi}^{-1}$, so $\hat{\pi}$ is Hermitian and Unitary
  - Reflection around origin of system
  - $\lambda = \pm 1$ — Symmetric $V(x)$ invariant under $\hat{\pi}$

SHO and DSW:

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$ $\quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$

- Creation/Annihilation:
  $\hat{a}_- := \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$
  $\hat{a}_+ := \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p}) = \hat{a}_-^\dagger$
  $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$
  $\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$
  $\hat{a}_-|0\rangle = \vec{0}$
  $\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})$ $\quad \Psi_n(x) = \frac{1}{\sqrt{n!}}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)^n \Psi_0(x)$
  $[\hat{a}_-, \hat{a}_+] = \hat{I}$
  $\langle n|\hat{a}_+|m\rangle = \sqrt{m+1}\,\delta_{n,m+1}$
  $\langle n|\hat{a}_-|m\rangle = \sqrt{m}\,\delta_{n,m-1}$
  - Raise/lower energy level
  $[\hat{H}, \hat{a}_+] = \hbar\omega\hat{a}_+$
  $[\hat{H}, \hat{a}_-] = -\hbar\omega\hat{a}_-$
- To find $[\hat{p}]_e$, $[\hat{x}]_e$ or any other $\hat{Q}$, write as sum of $\hat{a}_+$ and $\hat{a}_-$
- $\hat{N}|n\rangle = n|n\rangle$ $\quad \hat{N} = \hat{a}_+\hat{a}_-$

DSW:

$e := \{|-\rangle, |+\rangle\}$, $w := \{|L\rangle, |R\rangle\}$

$\hat{H}|-\rangle = (E_0 - \varepsilon)|-\rangle$ $\quad \hat{H}|+\rangle = (E_0 + \varepsilon)|+\rangle$

$|L\rangle \doteq_e \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad |R\rangle \doteq_e \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$|-\rangle \doteq_w \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad |+\rangle \doteq_w \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$[\hat{H}]_e = \begin{pmatrix} E-\varepsilon & 0 \\ 0 & E+\varepsilon \end{pmatrix}$ $\quad [\hat{H}]_w = \begin{pmatrix} E_0 & -\varepsilon \\ -\varepsilon & E_0 \end{pmatrix}$

- Theoretical lower bound on $\sigma_A\sigma_B$: $\sigma_A\sigma_B \geq \left|\left\langle\frac{1}{2i}[\hat{A}, \hat{B}]\right\rangle\right|$

Passive Transformations: $e := \{|e_i\rangle\}$, $f := \{|f_i\rangle\}$

$C_{f\leftarrow e} = [\hat{I}]^f_e = ([e_1]_f \ldots [e_N]_f)$

- If bases ONB, C is unitary

$[\alpha]_f = C[\alpha]_e$ $\quad [\langle\beta|]_f = [\langle\beta|]_e C^\dagger$

$Q_{[f]} = C\, Q_{[e]}\, C^\dagger$

- Trace, char_Q, det, Hermitivity, and unitarity invariant under change of basis

Patrick Oare | $dP = |\Psi(\vec{x})|^2 d^3x$

Physics 137A Final Cheat Sheet

3D Quantum Mechanics:

- **Position/momentum:** $\hat{x}_i \Psi(\vec{x}) = x_i \Psi(\vec{x})$, $\hat{p}_i \Psi(\vec{x}) = -i\hbar \frac{\partial}{\partial x_i}\Psi(\vec{x})$
- $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$, $[\hat{x}_i, \hat{p}_j] = i\hbar \hat{I}\delta_{ij}$

- **Combining Systems:** $|\alpha,\beta\rangle := |\alpha\rangle \otimes |\beta\rangle$, $\mathcal{H}_1 \otimes \mathcal{H}_2$ Hilbert Space
- If $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as $|\alpha\rangle \otimes |\beta\rangle$, $|\alpha\rangle \in \mathcal{H}_1$, $|\beta\rangle \in \mathcal{H}_2$, $|\Psi\rangle$ is a separable state
 - Linear combinations of separable states create entangled states
- $\langle \vec{x}|\vec{p}\rangle = \left(\frac{1}{2\pi}\right)^{3/2} \times e^{i\vec{p}\cdot\vec{x}/\hbar}$
- Form a basis: Let $\{|e_i\rangle\}$ be a basis for $\mathcal{H}_1$ and $\{|f_j\rangle\}$ a basis for $\mathcal{H}_2$. Then, $\{|e_i\rangle \otimes |f_j\rangle\}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$
- $\otimes$ associates, distributes, commutes w/ scalar mult $((\lambda|\alpha\rangle)\otimes|\beta\rangle = |\alpha\rangle \otimes (\lambda|\beta\rangle))$
- Inner Product: $\langle \alpha_1, \beta_1 | \alpha_2, \beta_2\rangle = (\langle\alpha_1| \otimes \langle\beta_1|)(|\alpha_2\rangle \otimes |\beta_2\rangle) = \langle\alpha_1|\alpha_2\rangle\langle\beta_1|\beta_2\rangle$
- Operators: $(\hat{S}^{(1)} \otimes \hat{T}^{(2)})|\alpha,\beta\rangle := (\hat{S}^{(1)}|\alpha\rangle) \otimes (\hat{T}^{(2)}|\beta\rangle)$
- If $\hat{S}^{(1)}: \mathcal{H}_1 \to \mathcal{H}_1$, $\hat{S}^{(2)}: \mathcal{H}_2 \to \mathcal{H}_2$, $\hat{S}^{(1)} + \hat{S}^{(2)} := \hat{S}^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{S}^{(2)}$

Infinite Cubical Well: $E_n = E_{n_x} + E_{n_y} + E_{n_z}$

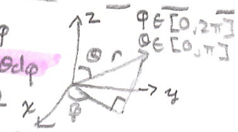
$V(\vec{x}) = \begin{cases} 0 & x \in [0,a], y \in [0,a], z \in [0,a] \\ \infty & \text{else} \end{cases}$

$V(\vec{x}) = V_{ISW}(x) + V_{ISW}(y) + V_{ISW}(z)$

- If $\Psi_n(x_i)$ is the n^{th} eigenfn for the x_i coord: $\Psi_{n_x n_y n_z}(\vec{x}) = \left(\frac{2}{a}\right)^{3/2} \Psi_{n_x}(x) \Psi_{n_y}(y) \Psi_{n_z}(z)$
- Separation of variables will work whenever our potential is $V_1(x) + V_2(y) + V_3(z)$

- **TDSE/TISE:** $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi(\vec{x}) + V(\vec{x})\Psi$, $E\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$
- $\hat{I} = \int |\vec{x}\rangle\langle\vec{x}| d^3x$

- **Spherical Coords:** $\begin{pmatrix} x \\ y \\ z\end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}$, $d\Omega = \sin\theta\, d\theta\, d\varphi$, $d^3x = r^2\sin\theta\, dr\, d\theta\, d\varphi = r^2 dr\, d\Omega$, $\varphi \in [0, 2\pi]$, $\theta \in [0,\pi]$



$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$\delta^3(\vec{x} - \vec{x}') = \frac{1}{r^2}\delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi')$

Spherically Symmetric Potentials: $V(\vec{x}) = V(r)$

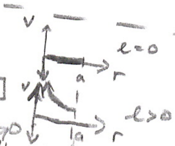
- Ansatz: $\Psi(\vec{x}) = R(r) Y(\theta,\varphi)$
- Solve the TISE by separation of variables
- Normalization: $\int_{\mathbb{R}^3} |\Psi(\vec{x})|^2 d^3x = 1 = \int_0^\infty r^2 |R|^2 dr \iint |Y|^2 d\Omega$
- Require both integrals to be equally 1
- Radial: $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2\mu r^2}{\hbar^2}[V(r) - E]R = \ell(\ell+1)R$
- Define $u(r) := rR(r)$ to get: $-\frac{\hbar^2}{2\mu}\frac{d^2 u_{n\ell}}{dr^2} + V_{eff}(r) u(r) = E_{n\ell} u_{n\ell}(r)$

$V_{eff}(r) := \begin{cases} \infty & r < 0 \\ V(r) + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} & r \geq 0 \end{cases}$

- **Angular Eqn:** Guess $Y_\ell^m = \Theta(\theta)\Phi(\varphi)$, $-\ell \leq m \leq \ell$
 $\frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right) = -\ell(\ell+1)$
 - Azimuthal: $\Phi'' + m^2\Phi = 0 \iff \Phi(\varphi) = e^{im\varphi}$
 - BC: $\Phi(\varphi) = \Phi(\varphi + 2\pi) \iff m$ is an integer
 - Polar: $\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta = m^2$
 - Solution: $\Theta_\ell^m(\theta) = A P_\ell^m(\cos\theta)$
- Full solution for Y_ℓ^m gives spherical harmonics
 $Y_\ell^{-m}(\theta,\varphi) = (-1)^m [Y_\ell^m(\theta,\varphi)]^*$, $\iint_{sphere} Y_\ell^{m*} Y_{\ell'}^{m'} d\Omega = \delta_{mm'}\delta_{\ell\ell'}$
 $\sum_{m=-\ell}^{\ell} |Y_\ell^m|^2 = \frac{2\ell+1}{4\pi}$, $Y_\ell^0 = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$

- $Y_\ell^m(\theta,\varphi) = \langle\theta,\varphi|\ell,m\rangle$
- $f(\theta,\varphi) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} c_{m\ell} Y_\ell^m(\theta,\varphi)$
- $c_{m\ell} = \int (Y_\ell^m)^* f(\theta,\varphi) d\Omega$

- **Infinite Spherical Well:** $V_{eff}(r) = \begin{cases} \infty & \text{else} \\ \frac{\hbar^2\ell(\ell+1)}{2\mu r^2} & r \in [0,a] \end{cases}$
- $\ell = 0$ solutions just those of ISW, $\Psi_n(r) = u(r)$; $\Psi_{n,0,0} = \frac{\Psi_n(r)}{r} Y_0^0(\theta,\varphi)$
- Solns for $\ell > 0$ are given by: $u_{n\ell}(r) = A r j_\ell(k_{n\ell} r) + B r n_\ell(k_{n\ell} r)$, $k_{n\ell} := \sqrt{\frac{2\mu E_{n\ell}}{\hbar^2}}$
- j_ℓ are spherical Bessel fns, n_ℓ are Neumann fns
- Need $u(0) = 0$; small x has $n_\ell(x) \to \infty$, so $B = 0$
- Need $u(a) = 0$, so $k_{n\ell} a = \beta_{n\ell}$ where $\beta_{n\ell}$ is the nth root of the ℓth Bessel fn.



- Full soln becomes: $\Psi_{n\ell m} = A j_\ell\left(\frac{\beta_{n\ell} r}{a}\right) Y_\ell^m(\theta,\varphi)$

- **Hydrogen:** $V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}$
- Bound states $\to E < 0$
- Define: $k := \sqrt{\frac{-2\mu E}{\hbar^2}}$, $\rho := kr$, $\rho_0 := \frac{\mu e^2}{2\pi\varepsilon_0\hbar^2 k}$
- Termination of power series leads to: $\ell \leq n - 1$
- $\rho_0 = 2n$ b/c of termination, and we can plug in to see: $E_n = \frac{-13.6\text{eV}}{n^2} =: \frac{E_1}{n^2}$ Bohr energy
- E_n depends on n, not ℓ, so we get a large degeneracy w/ ℓ and m states since $\ell \leq n - 1$ and $-\ell \leq m \leq \ell$
- Bohr radius: $a := \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu}$, $k = \frac{1}{an}$, $\Psi_{n\ell m}(\vec{x}) = R_{n\ell}(r) Y_\ell^m(\theta,\varphi)$
- n is the Principle quantum #, ℓ is orbital, m is magnetic

Angular Momentum: $\vec{L} = \vec{x} \times \vec{p}$

$\hat{L}_i = \hat{x}_j \hat{p}_k \varepsilon_{ijk}$, $\hat{L}^2 = \hat{L}_i \hat{L}_i$

$\vec{L}\cdot\vec{S} = \hat{L}_z\hat{S}_z + \frac{1}{2}(\hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+)$

$\hat{L}_x = -i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$

- Summation over repeated indices is assumed $(a_i a_i = \sum a_i a_i)$
- $\hat{\vec{x}}\Psi(\vec{x}) = r\Psi\hat{r}$, $\hat{\vec{p}}\Psi(\vec{x}) = -i\hbar\nabla\Psi(\vec{x})$
- $\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$, $\hat{L}^2 = -\hbar^2\nabla^2$, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki}$

- Eigenkets: $Y_\ell^m(\theta,\varphi) = \langle\theta,\varphi|\ell,m\rangle$ *half integers allowed for spin
 $\hat{L}^2|\ell,m\rangle = \hbar^2\ell(\ell+1)|\ell,m\rangle$, $\ell = 0, 1, 2, \ldots$
 $\hat{L}_z|\ell,m\rangle = \hbar m|\ell,m\rangle$, $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$
- Commutation: $[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k$, $[\hat{L}_i, \hat{L}^2] = 0$
- Ladder operators: $\hat{L}_\pm := \hat{L}_x \pm i\hat{L}_y$, $\hat{L}_\pm^\dagger = \hat{L}_\mp$
 $\hat{L}_\pm|\ell,m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}\,|\ell,m\pm1\rangle$
 - $\hat{L}_+|\ell, m=\ell\rangle = 0 = \hat{L}_-|\ell, m=-\ell\rangle$, $[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$
 $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$, $\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$, $\hat{L}^2 = \hat{L}_+\hat{L}_- - \hbar\hat{L}_z + \hat{L}_z^2$

Spin: Internal angular momentum a particle has

- State fully described in $\mathcal{H}_{space} \otimes \mathcal{H}_{spin}$
- Same commutation relations: $[\hat{S}_i, \hat{S}_j] = i\hbar\varepsilon_{ijk}\hat{S}_k$, $[\hat{S}^2, \hat{S}_i] = 0$
- Everything same as angular momentum: $\ell \mapsto s$ and $m \mapsto m_s$
- s can now take half integer values
- $-s \leq m_s \leq s$ in integer steps
- Spin states have fixed value of s, while m_s can vary
- General spin state: $|\Psi\rangle \doteq \begin{pmatrix} \Psi_s(\vec{x}) \\ \Psi_{s-1}(\vec{x}) \\ \vdots \\ \Psi_{-s}(\vec{x})\end{pmatrix}$
- Angular translation: $\hat{\mathcal{U}}(\hat{n},\theta) = \exp(-i\theta\hat{n}\cdot\vec{L}/\hbar)$

- **Spin 1/2:** $|\uparrow\rangle := |\frac{1}{2},\frac{1}{2}\rangle$, $|\downarrow\rangle := |\frac{1}{2},-\frac{1}{2}\rangle$
 - $\beta := \{|\uparrow\rangle, |\downarrow\rangle\}$ is a basis for the space
 - If not otherwise said, will work in this basis
 $\vec{\sigma} := \hat{x}\sigma_x + \hat{y}\sigma_y + \hat{z}\sigma_z$, $\hat{S}_i \doteq \frac{\hbar}{2}\sigma_i$, $\vec{S} \doteq \frac{\hbar}{2}\vec{\sigma}$
 $\sigma_n := \hat{n}\cdot\vec{\sigma} = \begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix}$ for $\hat{n} = \begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}$

- **Pauli Matrices:** $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\hat{I} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$
 $\sigma_i\sigma_j = \hat{I}\delta_{ij} + i\varepsilon_{ijk}\sigma_k$, $\sigma_n^2 = \hat{I}$
 $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$, $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\hat{I}$
- $\langle\vec{S}\rangle := \begin{pmatrix}\langle\hat{S}_x\rangle \\ \langle\hat{S}_y\rangle \\ \langle\hat{S}_z\rangle\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\chi^\dagger\sigma_x\chi \\ \chi^\dagger\sigma_y\chi \\ \chi^\dagger\sigma_z\chi\end{pmatrix}$, $|\hat{n},+\rangle$ has $\langle\vec{S}\rangle = \frac{\hbar}{2}\hat{n}$

- **Magnetic Field:** $\vec{\mu} := \gamma\vec{S}$
- Bohr Magneton: $\mu_B := \gamma\hbar/2$
- Same eigenvalues as $\hat{S}_n$: $E_\pm = \mp\frac{\gamma B_0\hbar}{2} = \mp\mu_B B_0$
- $\hat{\mathcal{H}} = -\vec{\mu}\cdot\vec{B} = -\gamma B_0\hat{S}_n \doteq -\mu_B B_0\sigma_n$, $\vec{B} = B_0\hat{n}$
- Where the eigenket is $|\hat{n},\pm\rangle$
- w.l.o.g., take $\hat{n} = \hat{z}$: $\hat{\mathcal{H}} \doteq \begin{pmatrix}-\mu_B B_0 & 0 \\ 0 & \mu_B B_0\end{pmatrix} \Rightarrow \hat{\mathcal{U}}(t) = \exp(-i\hat{\mathcal{H}}t/\hbar)$
- Spinor w/ $\langle\vec{S}\rangle = \begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}$ is $\chi_{\hat{n}} \doteq \begin{pmatrix}\cos\theta/2 \\ e^{i\varphi}\sin\theta/2\end{pmatrix}$
- **Larmor Precession:** If $\chi_{\hat{n}}$ placed in field, evolves with $\theta(t) = \theta$ and $\varphi(t) = \varphi - \gamma B_0 t \implies \omega = \gamma B_0$; $\hat{\mathcal{U}} \doteq \begin{pmatrix} e^{i\mu_B B_0 t/\hbar} & 0 \\ 0 & e^{-i\mu_B B_0 t/\hbar}\end{pmatrix}$

- **Eigenspinors:** $|\to\rangle \doteq$
 $\hat{S}_x$: $+\hbar/2 \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $-\hbar/2 \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \doteq |\leftarrow\rangle$
 $\hat{S}_y$: $+\hbar/2 \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \doteq |\otimes\rangle$, $-\hbar/2 \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \doteq |\odot\rangle$
 - To find, collapse into matrix rep. w/ ladder operators
 $\hat{S}_+ \doteq \begin{pmatrix}0 & \hbar \\ 0 & 0\end{pmatrix}$, $\hat{S}_- \doteq \begin{pmatrix}0 & 0 \\ \hbar & 0\end{pmatrix}$, $|\hat{x},-\rangle := |-\hat{n}_x,+\rangle$
 $\hat{S}_n$: $+\hbar/2 \leftrightarrow |\hat{n}_y,+\rangle = (\cos\theta/2 \;\; \sin\theta/2\, e^{i\varphi})^T$
 $-\hbar/2 \leftrightarrow |\hat{n}_y,-\rangle = (\sin\theta/2\, e^{-i\varphi} \;\; -\cos\theta/2)^T$

Addition of Angular Momenta

$\vec{L}\cdot\vec{S} = \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$

$\vec{J} := \vec{L} + \vec{S}$, $\hat{J}^2 := \vec{J}\cdot\vec{J}$, $[\hat{J}^2, \hat{L}_z]$ and $[\hat{J}^2, \hat{S}_z] \neq 0$ b/c $\vec{L}\cdot\vec{S}$

$\hat{J}_i = \hat{L}_i + \hat{S}_i$, $m_j = m_\ell + m_s$, $|\ell - s| \leq j \leq \ell + s$

- To create a Clebsch Gordan table, use ladder operators on a j ladder ($\hat{J}_\pm = \hat{L}_\pm + \hat{S}_\pm$) and start w/ stretch states ($m_\ell = \pm\ell$, $m_s = \pm s$)

Mutually Commuting Sets:
$\{\hat{L}^2, \hat{S}^2, \hat{L}_z, \hat{S}_z\} \leftrightarrow \{|\ell, s, m_\ell, m_s\rangle\}$
$\{\hat{L}^2, \hat{S}^2, \hat{J}^2, \hat{J}_z\} \leftrightarrow \{|\ell, s; j, m_j\rangle\}$

- **Clebsch-Gordan:** m_s must be $m_j - m_\ell$
 $|\ell, s; j, m_j\rangle = \sum_{m_\ell=-\ell}^{\ell} C^{\ell s j}_{m_\ell m_s m_j}|\ell, s, m_\ell, m_s\rangle$
- Pick either m_s or m_ℓ to sum over, depending on if s or ℓ is smaller
 $|\ell, s, m_\ell, m_s\rangle = \sum_{j=|\ell-s|}^{\ell+s} C^{\ell s j}_{m_\ell m_s m_j}|\ell, s; j, m_j\rangle$ w/ $m_j = m_\ell + m_s$

$\ell \times s$ | j / m_j | m_ℓ m_s — Block for each m_j
- If $s > \ell$, replace the 2 (so $s \times \ell$)
- Note $m_\ell + m_s$ must equal m_j, and the # of combos that can do this will be the size

$\Psi_{n\ell m} = R_{n\ell}(r) \times Y_\ell^m(\theta,\varphi)$

$\ell \otimes s = \bigoplus_{j=|\ell-s|}^{\ell+s} j$

$\hat{\pi} f(r,\theta,\varphi) = f(r, \pi - \theta, \varphi + \pi)$

- Ex: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$
 $\frac{1}{2}\otimes\frac{1}{2}$: $\{|\frac{1}{2},\frac{1}{2}\rangle, |\frac{1}{2},-\frac{1}{2}\rangle, |-\frac{1}{2},\frac{1}{2}\rangle, |-\frac{1}{2},-\frac{1}{2}\rangle\}$
 or
 $0 \oplus 1$: $\{|0,0\rangle, |1,-1\rangle, |1,0\rangle, |1,1\rangle\}$