

# TOPOLOGICAL MANIFOLDS

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Topological manifolds are an interesting topic of study. In these notes, we focus on the algebraic properties of these manifolds rather than the differential properties of the manifolds. The main goal will be to classify all 0, 1, and 2 manifolds up to homeomorphism.

## 1. DEFINITIONS

We begin with the definition of a manifold; however, we need to define the notion of a second-countable topological space first:

**Definition 1.1** (Second-Countable). A space  $X$  is second-countable if it has a countable base.

Now we may define a manifold:

**Definition 1.2** (Manifold). A **manifold of dimension  $n$**  is a second countable Hausdorff space  $X$  such that  $\forall x \in X$ , there is an open neighborhood  $U_x \ni x$  and a homeomorphism  $\phi : U_x \rightarrow \mathbb{R}^n$ . Such a subset and map  $(U_x, \phi)$  is called a **chart**.

We include second-countable and Hausdorff in the definition to exclude strange spaces from being manifolds, including the “long line” and the line with two origins.

This definition of a manifold only includes spaces which we would consider open, i.e. they have no boundary. We now generalize this to include spaces with boundary.

**Definition 1.3** (Manifold with Boundary). An  **$n$ -manifold with boundary** is a topological space  $X$  such that:

- (1)  $X$  is Hausdorff and second-countable.
- (2)  $\forall x \in X$ , there is an open neighborhood  $U_x \ni x$  and a homeomorphism  $\phi : U_x \rightarrow \mathbb{R}^n$  or a homeomorphism  $\phi : U_x \rightarrow \mathbb{R}_+^n$ , where  $\mathbb{R}_+^n = \{(x_1, \dots, x_n) : x_n \geq 0\}$ .

Now we may talk about two subsets of a manifold  $X$ , the **interior** and the **boundary** of  $X$ .

**Definition 1.4** (Interior). The **interior** of  $X$  is:

$$\text{int}(X) = \{x \in X : \exists \text{ a chart } (U_x, \phi) \text{ around } x \text{ with } U_x \cong \mathbb{R}^n.\}$$

**Definition 1.5** (Boundary). The **boundary** of  $X$  is:

$$\partial X = X \setminus \text{int}(X) = \{x \in X : \exists \text{ a chart } (U_x, \phi) \text{ around } x \text{ and } \phi(x) \in \mathbb{R}_+^n.\}$$

**Definition 1.6** (Closed Manifold). A manifold  $X$  is **closed** if it is compact and  $\partial X = \emptyset$ .

For some examples of manifolds:

- $\mathbb{R}^n$  and  $S^n$  are  $n$ -manifolds.
- Any open subset  $U \subseteq X$  of an  $n$ -manifold is an  $n$ -manifold.
- If  $X$  and  $Y$  are  $n$ -manifolds, then  $X \times Y$  is an  $n + m$  manifold.
- If  $X$  is an  $n$ -manifold with boundary, then  $\partial X$  is an  $(n - 1)$ -manifold. For example,  $B^n$  is an  $n$ -manifold with boundary  $S^{n-1} = \partial B^n$ , which is an  $n - 1$  manifold.
- We can attach manifolds along a boundary. If  $X, Y$  are  $n$ -manifolds with boundary and  $f : \partial X \rightarrow \partial Y$  is a homeomorphism, then  $X \cup_f Y$  is an  $n$ -manifold.

This next example is quite important, so we give it a definition. It is an operation that joins two manifolds together called the **connected sum**.

**Definition 1.7** (Connected Sum). Let  $X, Y$  be two  $n$ -manifolds. Choose  $x \in X, y \in Y$  and charts  $(U_x, \phi)$  and  $(V_y, \psi)$  of  $x$  and  $y$ , and pick  $U \subseteq U_x, V \subseteq V_y$  such that  $U \cong_\phi B^n$  and  $V \cong_\psi B^n$ . Let  $X' = X \setminus \text{int}(U)$  and  $Y' = Y \setminus \text{int}(V)$ , and choose a homeomorphism  $f : \partial Y' \rightarrow \partial X'$ . We define the **connected sum** of  $X$  and  $Y$  to be the manifold  $X \# Y$  given by:

$$X \# Y := X' \cup_f Y'$$

If  $X$  and  $Y$  are path-connected, then the connected sum is a well defined manifold, i.e. it does not depend on the choice of  $x$  and  $y$ .

Next, we will turn to the classification of 0, 1, and 2 manifolds. We will aim to classify all closed and connected 0, 1, and 2 manifolds up to homeomorphism. This will ultimately allow us to prove the following theorem for  $n = 0, 1, 2$  dimensions:

**Theorem 1.1** (Generalized Poincare Conjecture). If  $X$  is a closed, connected  $n$ -manifold and  $X$  is homotopy equivalent to  $S^n$ , then  $X$  is homeomorphic to  $S^n$ .

## 2. 0-MANIFOLDS

If  $X$  is a 0 manifold, there is not much interesting we can say about it. Given  $x \in X$ , we have a chart  $(\phi, U_x)$  where  $\phi : U_x \rightarrow \mathbb{R}^0 = \{0\}$  is a homeomorphism, so because  $\phi$  is a bijection,  $U_x = \{x\}$ . But then every subset of  $X$  is open, so  $X$  must be a discrete space. Conversely, every discrete space which is second countable (discrete spaces are automatically Hausdorff) is a 0 manifold, so  **$X$  is a 0 manifold if and only if it is a second countable discrete topological space**. This implies that connected 0 manifolds are homeomorphic to the one-point space  $\{*\}$ , which can never be homotopy equivalent to  $S^0$ , and thus proves the Generalized Poincare Conjecture in  $n = 0$  dimensions.

## 3. 1-MANIFOLDS

We will begin with the classification, and then cite the tools used to prove the theorem.

**Theorem 3.1** (Classification of connected 1-manifolds). If  $X$  is a connected 1-manifold (perhaps with boundary), then  $X$  is homeomorphic to one of:

- (1)  $\mathbb{R}$
- (2)  $S^1$
- (3)  $[0, 1]$
- (4)  $[0, 1)$

**Corollary 3.1.1** (Generalized Poincare,  $n = 1$ ). If  $X$  is a connected and closed 1-manifold, then  $X \cong S^1$ .

The corollary is immediate from the classification theorem. The easiest way to prove this classification is to prove a lemma, which we will state.

**Lemma 3.2.** Let  $X$  be a connected 1-manifold. If  $(U, \phi)$  and  $(V, \psi)$  are charts on  $X$  and  $U, V \cong \mathbb{R}$ , then  $U \cap V$  has at most 2 connected components.

## 4. FREE PRODUCTS AND ABELIANIZATION

Before we proceed with the classification of 2-manifolds, it is important to develop some group theory which will help us to classify 2-manifolds (especially when we consider the uniqueness of the decomposition). Let  $G = \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1 \rangle$  and  $G' = \langle b_1, \dots, b_{n'} \mid s_1 = 1, \dots, s_{m'} = 1 \rangle$  be two groups. We define:

**Definition 4.1** (Free Product). The **free product** of  $G$  and  $G'$  is the group  $G * G'$  given by:

$$G * G' := \langle a_1, \dots, a_n, b_1, \dots, b_{n'} | r_1 = 1, \dots, r_m = 1, s_1 = 1, \dots, s_{m'} = 1 \rangle$$

The best way to visualize the free product is to be the group on all letters from  $G$  and  $G'$  in which the group operation is concatenation and all words are reduced via inverses and the relations on the group. We don't impose any additional relations on the structure of  $G * G'$  by taking the free product; in particular, we cannot commute letters from different groups because we don't have any relations that say they commute.

**Definition 4.2** (Free Group). Let  $n \in \mathbb{N}$ . The **free group on  $n$  generators** is the group:

$$F_n := \langle a_1, \dots, a_n \rangle$$

This is the group on  $n$  generators with no relations, and it can also be constructed by  $F_1 = \mathbb{Z}$  and  $F_{n+1} = F_n * \mathbb{Z}$  for all  $n \in \mathbb{N}$ .

Free groups will be very helpful for describing fundamental groups of surfaces, but they are difficult to work with, primarily because they and their quotients are not necessarily Abelian. In particular, it is hard to tell when two presentations of non-Abelian groups actually correspond to different groups, so we wish to force these groups to be Abelian. This process is known as **Abelianization**.

**Definition 4.3** (Abelianization). Let  $G$  be a group, and let  $N$  be its **commutator subgroup**, i.e.  $N$  is the normal closure of  $\{ghg^{-1}h^{-1} : g, h \in G\}$ . Then we define its **Abelianization** to be the group  $Ab(G) := G/N$ .

By quotienting out the commutator subgroup of  $G$ , we have necessarily forced an additional relation on it; that of  $gh = hg$ . This is best seen as what Abelianizing does to a presentation of  $G$ . If we have a presentation of  $G$  as before, then a presentation of  $Ab(G)$  is given by:

$$Ab(G) = \langle a_1, \dots, a_n | r_1 = 1, \dots, r_m = 1, a_1a_2 = a_2a_1, \dots, a_1a_n = a_na_1, \dots, a_{n-1}a_n = a_na_{n-1} \rangle$$

We essentially keep the original presentation but add additional relations to the group forcing all its elements to commute. There is a nice relation between free products and direct products when groups are Abelianized; it is easiest to think of direct products as the “abelian” version of free products. This is reflected in the Abelianization of the free group on  $n$  generators, which is:

$$Ab(F_n) \cong \mathbb{Z}^n$$

It is also evident for two arbitrary groups  $G, H$ , in that:

$$Ab(G * H) \cong Ab(G) \times Ab(H)$$

As one final fact for this section, if  $G \cong G'$ , then  $Ab(G) \cong Ab(G')$ . However, the converse is not true at all; this is apparent because  $Ab(F_2) \cong \mathbb{Z}^2 = Ab(\mathbb{Z}^2)$ , but obviously  $F_2$  and  $\mathbb{Z}^2$  are not isomorphic groups.

## 5. 2-MANIFOLDS (SURFACES)

2-manifolds are the most interesting types that we will study; the classification is long and quite involved, but the proofs are not as tedious as those for the 1-manifolds, so I will include them whenever possible.

**Definition 5.1** (Surface). A **surface** is a 2-manifold with or without boundary.

Note that if  $S$  is a compact surface with non-zero boundary, then  $\partial S$  is a compact 1-manifold without boundary, and thus by our previous classification  $\partial S \cong S^1 \sqcup \dots \sqcup S^1$ . So, given any compact surface  $S$ , we may glue on copies of  $D^2$  via an attaching map to each copy of  $S^1$  in the boundary of  $S$ , and we may form a 2-manifold  $\tilde{S}$  which is compact and has no boundary. Thus, we will restrict

our study of 2-manifolds to closed manifolds, i.e. 2-manifolds which are compact and connected. We will now begin our classification, and we will start with the theorem:

**Theorem 5.1** (Classification of 2-manifolds). Let  $S$  be a closed, connected surface. Then:

$$S \cong S^2 \# T^2 \# \dots \# T^2 \# \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$$

where there are  $n, m \in \mathbb{Z}_{\geq 0}$  copies of  $T^2$  and  $\mathbb{R}P^2$ , respectively.

**5.1. Existence and Cellular Decompositions.** We will now attempt to simplify our study of surfaces by restricting it to studying polygons which correspond to the surface in some way, called a **cellular decomposition** of  $S$ . For a rigorous definition, check the class notes. It can be shown that every cellular decomposition is equivalent to one with a single polygon, so we may restrict our cellular decomposition to one polygon with identified edges. Given a cellular decomposition of a surface, we may describe it by a **word** with letters representing the orientation of each edge. We will say the surface is equal to this word by abuse of notation. **Words are equivalent up to cyclic permutation**, i.e. if a surface  $S$  is described by  $abcb^{-1}a^{-1}c^{-1}$ , then it can also be described by the word  $cb^{-1}a^{-1}c^{-1}ab$ . Some simple examples of this are:

- $S^2 = aa^{-1}$ .
- $\mathbb{R}P^2 = aa$ .
- $T^2 = aba^{-1}b^{-1}$ .

These become apparent upon drawing out the cellular decompositions of these surfaces, which I will include when I have time. We will now state a collection of lemmas which will allow us to reduce a cellular decomposition into something we recognize. These lemmas will essentially allow us to reduce a word by representing the corresponding space as a connect sum of two simpler spaces which are described by subwords of the original.

**Lemma 5.2** (S). If  $S \cong Xaa^{-1}$  and  $X \neq \emptyset$ , then  $S \cong X$ .

This lemma allows us to simplify inverse letters that are next to one another, assuming the rest of the word is non-trivial. If there are no other letters in the word, then we cannot do this, and the surface is instead homeomorphic to  $S^2$ .

**Lemma 5.3** (P1). If  $S = Xaa$ , then  $S \cong \mathbb{R}P^2 \# X$ .

**Lemma 5.4** (T1). If  $S = Xaba^{-1}b^{-1}$ , then  $S \cong T^2 \# X$ .

**Lemma 5.5** (P2). If  $S = XaYa$  and  $X, Y \neq \emptyset$ , then  $S = bbXY^{-1}$ .

**Lemma 5.6** (T2). If  $S = WaXbYa^{-1}Zb^{-1}$ , then  $S \cong T^2 \# S_1$ , where  $S_1 = ZYXW$ .

A brief note on inverse words: you can take an inverse of a word by flipping the order and taking individual inverses of letters. This is also equivalent to reading the word in the other direction starting at the same base point; either way will give you the same inverse word. We may use these lemmas to prove that

$$T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$$

and that:

$$K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$$

where  $K$  is the Klein bottle. Now, the existence of a decomposition of  $S$  into a connect sum of a sphere, projective planes, and tori can be proven by just stripping off the correct words from its cellular decomposition.

**5.2. Uniqueness and Seifert-van Kampen.** The next question to ask is if the  $n$  and  $m$  values in the decomposition are unique, which we will eventually prove. We define:

**Definition 5.2** (Genus). Let:

$$S_g := T^2 \# \dots \# T^2 \qquad N_g := \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$$

where in each case,  $g$  copies of each surface are connect summed together.  $g$  is called the **genus** of the surface.

We will now state the Seifert-van Kampen theorem, which we will use to finish the classification of surfaces theorem.

**Theorem 5.7** (Seifert-van Kampen). Let  $X = A \cup B$  with  $A, B$  open and path-connected,  $A \cap B$  path-connected, and  $p \in A \cap B$ . Let  $\iota_A : A \cap B \rightarrow A$  and  $\iota_B : A \cap B \rightarrow B$  be the inclusion maps. Then:

$$\pi_1(X, p) \cong \pi_1(A, p) * \pi_1(B, p) / N$$

where  $N \triangleleft \pi_1(A, p) * \pi_1(B, p)$  is the normal closure of  $\{(\iota_A)_*(g)[(\iota_B)_*(g)]^{-1} : g \in \pi_1(A \cap B, p)\}$ .

We need to quotient by  $N$  to enforce relations on the new fundamental group and avoid double counting loops in the intersection  $A \cap B$ . The Seifert-van Kampen theorem can be used to compute the fundamental group of  $X_n$ , the one-point union on  $n$  circles. It is easiest to use when  $A \cap B$  is contractible, because then we don't need to enforce any relations on  $\pi_1(A, p) * \pi_1(B, p)$ . We can do something similar to compute the fundamental group of  $S_g$  and  $N_g$ . These are:

$$\begin{aligned} \pi_1(S_g) &\cong \langle a_1, \dots, a_{2g} | a_1 a_2 a_1^{-1} a_2^{-1} \dots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} = 1 \rangle \\ \pi_1(N_g) &\cong \langle a_1, \dots, a_g | a_1^2 a_2^2 \dots a_g^2 = 1 \rangle \end{aligned}$$

To show these are distinct, we can Abelianize them. For  $S_g$ , this allows us to commute the  $a_i$  past one another in the single relation, clearing that relation and leaving us with an abelian group on  $2g$  generators:

$$H_1(S_g) = Ab(\pi_1(S_g)) = \mathbb{Z}^{2g}$$

For  $N_g$ , we can change the presentation to (where we assume the presentation includes commutation relations)  $Ab(N_g) = \langle a_1, \dots, a_1 \dots a_g | (a_1 \dots a_g)^2 = 1 \rangle = \langle a_1, \dots, a_{g-1}, b | b^2 = 1 \rangle$ , and so we see that:

$$H_1(N_g) = Ab(\pi_1(N_g)) = \mathbb{Z}^{g-1} \times (\mathbb{Z}/2\mathbb{Z})$$

Thus  $S_g$  and  $N_{g'}$  have different abelianized fundamental groups and therefore different fundamental groups, so they are not homotopy equivalent.