Welcome to QFT I!

- **Logistics**: all of the relevant course information can be found in the syllabus posted on Canvas. Here is my **contact info**, and the relevant times and locations for recitation and office hours:

TA	Email	Recitation	Recitation Location	Office Hours	OH Location
Patrick Oare	poare@mit.edu	Fridays, 2 PM	TBD	TBD	6-415A

  I'll try to be very responsive by email, so feel free to send me emails if there's anything you want to discuss or questions you have! Please also try to use the class **Piazza** page, as I'll be monitoring that to answer questions as they crop up.

- **Resources:** There are a lot of different textbooks out there which promise to give an "introduction to QFT"; some of them are decent, but most are indecipherable. I'd recommend trying a few of these out in your first few weeks and seeing which ones fit your learning style better– a lot of these books approach the same topics from different angles, and especially for a field as dense as QFT, it can be very valuable to have a few different ways to see the same problem. The main textbooks for this course are **Peskin & Schroeder** and **Weinberg**, and I also use **Schwartz** quite a bit. I encourage you to sample them all and to find the best ones for you.
  - Peskin and Schroeder, *An Introduction to Quantum Field Theory*. A well-rounded intro to QFT. Lots of detailed worked out examples (very helpful when we get to the calculation-heavy portions of the class), but can be a bit dense in places as a result.
  - Schwartz, *Quantum Field Theory and the Standard Model*. This book is typically the primary source for the MIT QFT sequence. A good, well-written introduction to QFT that's easy to understand and ends up going quite deep into the subject. Could be more rigorous, though.
  - Srednicki, *Quantum Field Theory*. A good introduction book to QFT. Rigorous and precise.
  - Weinberg, *The Quantum Theory of Fields (Vol 1)*. An interesting introduction to the subject, which is very rigorous and can be hard to read on a first pass. The information contained in this book is second to none, but the hard part can be understanding what it's telling you.
  - Zee, *Quantum Field Theory in a Nutshell*. A book which you'll hate during this class but enjoy once you've learned QFT (this was my first QFT book!). Written at a very high level with minimal calculations, but good for understanding the bigger picture.

Why QFT?

This is the first course in the quantum field theory sequence at MIT. It'll be taught at the graduate level, and we'll be assuming that you're familiar with special relativity and quantum mechanics. To get an idea of what QFT is and does and when it's applicable, I want to start heuristically using the

- $\Delta x \Delta p \gtrsim \hbar$

Index notation

- Continuous symmetries are implemented by **Lie groups**. The space of proper (orientation preserving) rotational symmetries in 3 dimensions is $SO(3)$. It can be formally defined as the space of all transformations of $\mathbb{R}^3$ which leaves the norm $r = \sqrt{x^2 + y^2 + z^2}$ invariant for all $(x, y, z) \in \mathbb{R}^3$ and preserves handedness. This defines the group *abstractly*; to work with it, we need a **matrix representation**, which is a way to express an abstract rotation R as a $d \times d$ matrix[1]. For $SO(3)$, $d = 3$ (the dimension of $\mathbb{R}^3$), and we write R_{ij} for the 3×3 matrix representing R. The matrices R_{ij} representing $SO(3)$ are orthogonal and have determinant 1.
- Transformations are implemented using index notation to explicitly write out matrix and products. We will use the **Einstein summation convention**: if an index is repeated, it is assumed to be summed over. A sum on indices is called a **contraction**. For example, the action of a rotation on a vector $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is:

$$x_i \mapsto \sum_j R_{ij} x_j \equiv R_{ij} x_j. \tag{1}$$

- Example: Orthogonal matrices R_{ij} do implement rotations because we can show they leave x^2 invariant:

$$x^2 = x_i x_i \mapsto (R_{ij} x_j)(R_{ik} x_k) = x_j \underbrace{(R^{\mathrm{T}})_{ji} R_{ik}}_{\text{matrix product}} x_k = x_j \delta_{jk} x_k = x^2. \tag{2}$$

- A **metric** is an inner product on a space, and gives us a notation of distance. In physics we represent metrics as symmetric matrices h_{ij}, under which the inner product of two vectors $\vec{x}$ and $\vec{y}$ is $\langle \vec{x} | \vec{y} \rangle = h_{ij} x_i y_j$ (in matrix notation, this is $\vec{x}^{\mathrm{T}} h \vec{y}$). The Euclidean metric on $\mathbb{R}^3$ is the Kronecker delta $h_{ij} = \delta_{ij}$; a rotation can instead be defined as **a transformation which preserves the metric**, which is shown in the following equation:

$$R_{ki} \delta_{kl} R_{lj} = \delta_{ij} \qquad\qquad (R^{\mathrm{T}} I R = I \text{ in matrix notation}) \tag{3}$$

The Lorentz Group

- Special relativity tells us that in any reference frame with coordinates $x^\mu = (t, \vec{x})$, the spacetime interval $s^2 = t^2 - \vec{x}^2$ is invariant. s is the norm of x^μ with respect to the **Minkowski metric**[2]:

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{4}$$

  The dot product between two vectors is $x \cdot y = g_{\mu\nu} x^\mu y^\nu$, so $s^2 = x \cdot x = g_{\mu\nu} x^\mu x^\nu$ is a norm squared. Greek letters $\mu \in \{0, 1, 2, 3\}$ are used for spacetime indices, while Latin letters $i \in \{1, 2, 3\}$ are spatial indices.
- The **Lorentz group** is the set of symmetries of spacetime which preserve the metric $g_{\mu\nu}$. The total Lorentz group has 4 disconnected components: each component contains one of $\{1, P, T, PT\}$, where P is parity and T is time reversal. The component containing 1 is called the **proper orthochronous** subgroup and is denoted by $SO(1, 3)$. An element $\Lambda \in SO(1, 3)$ must satisfy (just like Eq. (3)):

$$\Lambda^\alpha{}_\mu g_{\alpha\beta} \Lambda^\beta{}_\nu = g_{\mu\nu} \tag{5}$$

[1] d is called the **dimension** of the representation.

[2] **WARNING**: Whenever you read a book, check the metric! In particle physics it's conventional to use the "mostly minus" convention, but in GR it's conventional to use the "mostly positive" convention with $\mathrm{diag}(-1, 1, 1, 1)$. This can lead to sign errors if you're not careful.

$SO(1,3)$ is a 6-dimensional Lie group, meaning any Lorentz transformation can be parameterized with 6 parameters: 3 rotation angles θ_i, and 3 boost parameters β_i.

- Rotations or boosts purely along one axis can be written out explicitly:

$$R(\hat{x},\theta_1)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\theta_1&\sin\theta_1\\0&0&-\sin\theta_1&\cos\theta_1\end{pmatrix}\qquad R(\hat{y},\theta_2)=\begin{pmatrix}1&0&0&0\\0&\cos\theta_2&0&-\sin\theta_2\\0&0&1&0\\0&\sin\theta_2&0&\cos\theta_2\end{pmatrix}\qquad R(\hat{z},\theta_3)=\begin{pmatrix}1&0&0&0\\0&\cos\theta_3&\sin\theta_3&0\\0&-\sin\theta_3&\cos\theta_3&0\\0&0&0&1\end{pmatrix}\tag{6}$$

$$B(\hat{x},\beta_1)=\begin{pmatrix}\cosh\beta_1&\sinh\beta_1&0&0\\\sinh\beta_1&\cosh\beta_1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad B(\hat{y},\beta_2)=\begin{pmatrix}\cosh\beta_2&0&\sinh\beta_2&0\\0&1&0&0\\\sinh\beta_2&0&\cosh\beta_2&0\\0&0&0&1\end{pmatrix}\qquad B(\hat{z},\beta_3)=\begin{pmatrix}\cosh\beta_3&0&0&\sinh\beta_3\\0&1&0&0\\0&0&1&0\\\sinh\beta_3&0&0&\cosh\beta_3\end{pmatrix}\tag{7}$$

When multiple boost or rotation parameters are nonzero, one must use a matrix exponential to write Λ down as (we have also included an infinitesimal Lorentz transformation with $\beta_i,\theta_i << 1$):

$$\Lambda=\exp(i\beta_iK_i+i\theta_iJ_i)=\exp\left(\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)\approx 1+\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}+\mathcal{O}(\omega^2)\tag{8}$$

Here K_i and J_i are antisymmetric 4×4 matrices which generate boosts and rotations[3]:

$$(J_i)_{jk}=-i\epsilon_{ijk}\qquad\qquad (K_i)_{0j}=\delta_{ij}=-(K_i)_{j0}\tag{9}$$

and are packaged together covariantly as a *tensor* of 4×4 matrices $\mathcal{J}^{\mu\nu}$. $\omega_{\mu\nu}$ is an antisymmetric tensor which contains the parameters β_i and θ_i:

$$\mathcal{J}^{\mu\nu}=\begin{pmatrix}0&K_1&K_2&K_3\\-K_1&0&J_3&-J_2\\-K_2&-J_3&0&J_1\\-K_3&J_2&-J_1&0\end{pmatrix}\qquad\qquad \omega_{\mu\nu}=\begin{pmatrix}0&\beta_1&\beta_2&\beta_3\\-\beta_1&0&\theta_3&-\theta_2\\-\beta_1&-\theta_3&0&\theta_1\\-\beta_3&\theta_2&-\theta_1&0\end{pmatrix}\tag{10}$$

Tensors

- **Upper and lower indices**: Given a vector V^μ, one can form its dual vector V_μ by using the metric to lower its indices, $V_\mu=g_{\mu\nu}V^\nu$. Vectors with upper indices are called **contravariant**, and vectors with lower indices are **covariant**: under Lorentz transformations V^μ and V_μ transform in a dual way to one another. Some examples we will frequently use are:

$$x^\mu=(t,\vec{x})\qquad\qquad p^\mu=(E,\vec{p})\qquad\qquad \partial_\mu=(\partial_t,\vec{\nabla})\tag{11}$$

Note that as an operator in QM, $p_\mu=i\partial_\mu$, since lowering an index on p^μ makes its spatial components $-\vec{p}$, so we reproduce $\vec{p}=-i\vec{\nabla}$.

- We can form multi-index **tensors** by combining upper and lower indices into one object $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_\ell}$, which transforms under a Lorentz transformation Λ as:

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_\ell}\mapsto\Lambda^{\mu_1}{}_{\alpha_1}\ldots\Lambda^{\mu_k}{}_{\alpha_k}\Lambda^{\beta_1}{}_{\nu_1}\ldots\Lambda^{\beta_\ell}{}_{\nu_\ell}T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_\ell}.\tag{12}$$

The number of indices $k+\ell$ is called the **rank** of the tensor. Covariant and contravariant vectors are rank 1 tensors. Some examples of rank 2 tensors are the metric $g_{\mu\nu}$, the stress-energy tensor $T_{\mu\nu}$, and the field strength $F_{\mu\nu}$.

[3]Explicitly written as matrices in Eqs. (10.14) and (10.15) of Schwartz.

- A quantity is **Lorentz invariant** if it is the same in all reference frames. A general rule is that to form a Lorentz invariant, every upper index you see must be contracted with a lower index, and every lower index with an upper index. Quantities like $x \cdot \partial = x^\mu \partial_\mu$, ∂^2, and $p^\mu \partial^\nu F_{\mu\nu}$ are Lorentz invariant. In particular, any **dot product or square of vectors is invariant**: for example, *for a massive particle* p^2 *will always equal* m^2.
- A quantity that is **Lorentz covariant** will change in different reference frames, but in a way that respects the metric (all indices must be contracted in a Lorentz invariant way), for example $\partial_\mu T^{\mu\nu}$.
- FIeld theorists typically use the **Lagrangian** formulation of quantum mechanics as opposed to the **Hamiltonian** formulation. The reason this approach is nice is because it is manifestly Lorentz invariant: the Lagrangian is a Lorentz scalar, and therefore is the same in all frames. On the other hand, the Hamiltonian is like an energy of a system– it therefore is not a Lorentz invariant, and by itself is *not even Lorentz covariant*, and thus is more difficult to use in QFT when we're working with relativity.

Units