8.513 Term Paper: The Index Theorem in Lattice QCD

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1 Introduction

Quantum Chromodynamics (QCD) is the theory of the strong nuclear force. It is a $SU(3)$ gauge theory coupled to 6 flavors of fermions, which are known as the quarks. At low energies, the coupling of QCD is reasonably large and perturbation theory cannot be applied to the theory, which means physics must be extracted by non-perturbative means. One such way to do this is to formulate QCD as a **lattice gauge theory** by discretizing spacetime– the advantage of this is that the path integral becomes a finite (albeit large) dimensional integral which can be evaluated numerically using computers.

One of the key properties of QCD in the continuum is an approximate chiral symmetry; when we take the light quarks (u, d, and sometimes s) to be massless, the QCD Lagrangian decouples into left and right handed fields of definite chirality. This allows us to study QCD in many ways which do not require perturbation theory, and allows for a definition of topological charge in QCD. However, when QCD is put on the lattice, chiral symmetry appears broken and the original continuum definition of a topological charge is no longer a good quantity to consider. In this paper, I will discuss how chiral symmetry can be modified to generate a new definition of topological charge that is valid on the lattice.

2 QCD on the Lattice

This section will outline the background we need to study symmetries on the lattice; specifically, we will define our theory and the operators of relevance to us. We begin by making some initial definitions: denote our spacetime lattice with spacing a by Λ, and Wick rotate to imaginary time so that Λ is a Euclidean lattice[1]. In the full theory of QCD, the dynamical fields in the path integral are the quark fields ψ_f and gluon fields A_μ. When we discretize QCD, we will still work with the quark fields, but instead of directly working with the gauge fields A_μ we will work **link fields** $U_\mu(n)$ which transform in the following way under a gauge transformation $\Omega : \Lambda \to SU(3)$:

$$U_\mu(n) \xrightarrow{\Omega(n)} \Omega(n)U_\mu(n)\Omega(n+\hat{\mu})^\dagger \tag{1}$$

Here $\hat{\mu}$ is the unit vector in the μ direction, and $n \in \Lambda$ denotes a site in the lattice. The link fields can be taken to be $U_\mu(n) = \exp(iaA_\mu(n))$, so are intimately related to the gauge field, and take values (as they must) in $SU(3)$. With this transformation law, the link fields act as a connection between the fibers at different points in Λ:

$$U_\mu(n)\psi(n+\hat{\mu}) \xrightarrow{\Omega} \Omega(n)U_\mu(n)\psi(n+\hat{\mu}) \tag{2}$$

This transformation means that $U_\mu(n)\psi(n+\hat{\mu})$ is valued in the fiber at point n, and so can directly be compared with $\psi(n)$. This allows us to add and subtract fermion fields at different points in a gauge invariant way, and so define a covariant derivative.

[1] We must Wick rotate so that the Boltzmann factor of e^{iS} in the path integral becomes a valid probability density e^{-S}, in order to perform any computations at all.

We may now write down a first pass at a fermion action, which will be equivalent to $\not{D}+m$ upon taking the continuum limit. The direct discretization of this action is thus:

$$S_f^0[\psi_f,\bar{\psi}_f,U] = a^4\sum_{n\in\Lambda}\sum_f \bar{\psi}_f(n)\left(\gamma^\mu\frac{U_\mu(n)\psi_f(n+\hat{\mu})-U_{-\mu}(n)\psi_f(n-\hat{\mu})}{2a} - m_f\psi_f(n)\right) \tag{3}$$

$$= a^4\sum_{n,m\in\Lambda}\sum_f \bar{\psi}_f(n)^a_\alpha D_f^0(n|m)^{ab}_{\alpha\beta}\psi_f(m)^b_\beta \tag{4}$$

where we have defined the **Dirac operator** $D^{ab}_{\alpha\beta}(n|m)$ to be:

$$D_f^0(n|m)^{ab}_{\alpha\beta} := (\gamma^\mu)_{\alpha\beta}\left(\frac{U_\mu(n)^{ab}\delta_{n+\hat{\mu},m}-U_{-\mu}(n)^{ab}\delta_{n-\hat{\mu},m}}{2a}\right) + m_f\delta_{\alpha\beta}\delta^{ab}\delta_{nm} \tag{5}$$

Note the Greek indices α,β are in Dirac space, and the Latin indices a,b are in color space. The Dirac operator is one of the fundamental objects we will study later when considering the role of topology in lattice QCD, and is the discretized version of $i\not{D}-m$ in Euclidean space.

However, there is a slight problem with the action in Equation 4. Because we are working on a lattice, the Fourier transform $\tilde{D}^0(p)$ of the Dirac operator $D^0(n|m)$ has extra unphysical poles at the edges of the Brilliuon zone. These extra poles are known in the literature as **doublers**, and must be eliminated by adjusting the action. We do this by adding a corresponding Wilson term to the Dirac operator, so that the full Dirac operator and action now become:

$$D_f^W(n|m)^{ab}_{\alpha\beta} := \left(m_f+\frac{4}{a}\right)\delta_{\alpha\beta}\delta^{ab}\delta_{nm} - \frac{1}{2a}\sum_{\mu=\pm1}^{\pm4}(1-\gamma^\mu)_{\alpha\beta}U_\mu(n)^{ab}\delta_{n+\hat{\mu},m} \tag{6}$$

$$S_f[\psi,\bar{\psi},U] = a^4\sum_{n,m\in\Lambda}\sum_f \bar{\psi}_f(n)^a_\alpha D_f^W(n|m)^{ab}_{\alpha\beta}\psi_f(m)^b_\beta \tag{7}$$

We will soon see that the Wilson term makes it much more difficult to deal with chiral symmetry on the lattice than in the continuum, and is responsible for many of the interesting topological properties of lattice gauge theories as compared to their continuum counterparts.

For completeness we will record the glue action as well:

$$S_g[U] = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu}\mathbb{R}e\ tr\{1-U_{\mu\nu}(n)\} \tag{8}$$

where $U_{\mu\nu}(n) := U_\mu(n)U_\nu(n+\hat{\mu})U_{-\mu}(n+\hat{\mu}+\hat{\nu})U_{-\nu}(n+\hat{\nu})$ is known as a **plaquette**, and performs parallel transport around a closed loop. The partition function for the full theory of lattice QCD is thus:

$$Z = \int D\psi D\bar{\psi}DUe^{-S_f-S_g} \tag{9}$$

where the measures $D\psi, D\bar{\psi}, DU$ now contain only a finite amount of of sites to be integrated over.

3 Chiral Symmetry and the Ginsparg-Wilson Relation

Chiral symmetry in standard QCD is realized by taking the approximations that the light quarks are massless. Let $D_{cont} := \gamma^\mu(\partial_\mu + iA_\mu)$ be the continuum Dirac operator in Euclidean space for a massless field. The key equation for chiral symmetry in the continuum is the anticommutation of D_{cont} with γ_5:

$$\{D,\gamma_5\} = 0 \tag{10}$$

Using Equation 10, it is immediate to show that for a fermion described by $\mathcal{L} = \bar{\psi}D_{cont}\psi$, the Lagrangian is invariant under:

$$\psi \mapsto \exp(i\theta\gamma_5)\psi \tag{11}$$

Furthermore, using the projector $P_\pm = \frac{1\pm\gamma_5}{2}$, we can split a fermion field ψ into two pieces with definite chirality $\psi_L = P_-\psi$, $\psi_R = P_+\psi$ which rotate into themselves under chiral rotations. Note $P_\pm P_\mp = 0$, and that $DP_\pm = P_\mp D$ due to Equation 10. Using these relations, we see that the chiral fields decouple in the Dirac Lagrangian and make chiral symmetry manifest:

$$\mathcal{L} = \bar{\psi}_L D_{cont}\psi_L + \bar{\psi}_R D_{cont}\psi_R \tag{12}$$

Because of this decoupling, rotating ψ_L and ψ_R separately in flavor space[2] leaves the Lagrangian invariant and gives us the chiral symmetry $SU(N_f)_L \times SU(N_f)_R$, where N_f is the number of (approximately) massless quarks.

Upon discretization, the relation $\{D, \gamma_5\} = 0$ falls apart because of the Wilson term. Even if we assume our lattice quarks are massless, the piece proportional to $\delta_{\alpha\beta}$ does not anticommute with γ_5. More generally, it was shown by Nielson and Ninomiya [1] that *any attempt* to remove the doublers from a lattice regularized theory would result in such a breaking of this anticommutation relation, and thus the essence of chiral symmetry in lattice theories must be reformulated.

Ginsparg and Wilson [2] proposed an alternative symmetry on the lattice that acts as chiral symmetry, and indeed goes into chiral symmetry in the continuum limit $a \to 0$. The modified anticommutation relation is known as the **Ginsparg-Wilson equation**:

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \tag{13}$$

Although this is not satisfied by the Wilson-Dirac operator in Equation 6, it is satisfied by a different class of Dirac operators on the lattice. One in particular was defined by Neuberger [3] and is known as the **overlap operator**:

$$D_{over} = \frac{1}{a}\left(1 - A(A^\dagger A)^{-1/2}\right) \qquad\qquad A = 1 - aD^W \tag{14}$$

For the duration of this paper, we will take $D_f(n|m)^{ab}_{\alpha\beta}$ to be a lattice Dirac operator which satisfies Equation 13.

Luscher [4] showed that because D satisfies the Ginsparg-Wilson equation, the Lagrangian density is invariant under a modified chiral rotation of the field ψ:

$$\psi \mapsto \exp\left(ia\gamma_5\left(1 - \frac{1}{2}aD\right)\right)\psi \qquad\qquad \bar{\psi} \mapsto \bar{\psi}\exp\left(ia\left(1 - \frac{1}{2}aD\right)\gamma_5\right) \tag{15}$$

because taking a to be an small parameter and expanding in powers of a, we find that the extra terms obey the Ginsparg-Wilson equation order by order and cancel:

$$\mathcal{L} \mapsto \bar{\psi}D\psi + ia\bar{\psi}\left[\gamma_5 D + D\gamma_5 - aD\gamma_5 D\right]\psi + O(a^3) = \mathcal{L} + O(a^3) \tag{16}$$

hence we can take the Noether current generated by this symmetry to be our new definition of a chiral current. Note that the transformation in Equation 15 reduces to the standard chiral transformation in the limit $a \to 0$, and so this symmetry become chiral symmetry in the continuum limit.

To make manifest the modified chiral symmetry of the Lagrangian, we follow an argument by Niedermayer [5] and introduce modified chiral projectors which act as the standard projectors $\frac{1\pm\gamma_5}{2}$ in the continuum limit:

$$\hat{\gamma}_5 := \gamma_5(1 - aD) \qquad\qquad \hat{P}_\pm := \frac{1 \pm \hat{\gamma}_5}{2} \tag{17}$$

Because D satisfies Equation 13, these new projectors satisfy $D\hat{P}_\pm = P_\mp D$, which is a very similar algebra to the standard chiral projectors. This means that if we introduce modified chiral fields:

$$\psi_L = \hat{P}_-\psi; \qquad \psi_L = \hat{P}_+\psi; \qquad \bar{\psi}_L = \bar{\psi}P_+; \qquad \bar{\psi}_R = \bar{\psi}P_- \tag{18}$$

then our Lagrangian splits into chiral components $\mathcal{L} = \bar{\psi}_L D\psi_L + \bar{\psi}_R D\psi_R$ as in the continuum case.

While this composition successfully decouples the Lagrangian into chiral components, there is a major difference between this decomposition and the continuum case in Equation 12. In the continuum, *chirality is local* since chiral fields at one point are direct projections of the original fields at the same point. In other words, $\psi_L(x)$ only depends on the value of $\psi(x)$. On the lattice, this is no longer true– *chirality is non-local* because the projectors $\hat{P}_\pm$ contain a copy of D; this means that $\psi_L(x)$ will contain information from **other spacetime points** $\psi_L(y)$ for which $D(x|y)$ is nonzero. Thus, we have chiral symmetry on the lattice, but it comes at a cost.

[2]Flavor space is where we view ψ as a vector of different species of quarks. For example, we will generally either view $\psi_i = (u_i\ d_i)^T$ or $\psi_i = (u_i\ d_i\ s_i)^T$ when we examine the massless quarks in the theory, depending on whether or not we approximate the strange quark as massless.

4 The Index Theorem and Topological Charge

We now turn to other implications the Ginsparg-Wilson equation has for fermions on the lattice.

References

[1] A No-Go Theorem For Regularizing Chiral Fermions TODO

[2] Ginsparg Wilson

[3] Neuberger

[4] Luscher

[5] Neidermayer