Characteristic Classes

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Characteristic classes provide a way to study manifolds based on looking at specific cohomology classes. We will show these cohomology classes characterize specific aspects of the topology of a given manifold, and that they are universal in the sense that any two manifolds which are diffeomorphic will share the same characteristic classes. Typically, characteristic classes give a measure of how twisted a vector bundle is: all the classes of a trivial bundle vanish.

Generally, a characteristic class will give an **obstruction** to putting some form of a section on a vector bundle. The first example of this we will see is the Euler class, which is a characteristic class of a real vector bundle. We will see that if a nowhere-vanishing section of a vector bundle exists, then the Euler class of the bundle must vanish. Chern classes will provide a generalization of the Euler class, and then we will also examine Pontryagin classes and Steifel-Whitney classes.

This subject can be studied from both an algebraic perspective and a differential one. We will study the Euler class and Chern classes predominantly from an algebraic lens, and the other cohomology classes from a differential point of view.

1 The Euler Class

The Euler class is a characteristic cohomology class of a real vector bundle. Let $\pi : E \to M$ be a rank n vector bundle. Then the Euler class is

2 Chern Classes