Hypercubic Symmetry

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These notes will briefly discuss the representation theory of the hypercubic group $H(4)$. We will begin by discussing the cubic group $H(3)$, as it is easier to visualize and many of its properties are shared by its larger cousin $H(4)$. We will then characterize $H(4)$ as a group, and classify its irreps via studying its characters. Finally, we will focus on some of the more important representations and show how a rank 2 tensor decomposes into the irreps of $H(4)$.

1 The Cubic Group

An easier case to begin studying this subject is by studying the cubic group, which is the group of symmetries of the cube.

2 The Hypercubic Group $H(4)$

The generalization from $H(3) \to H(4)$ is not too difficult to make, but there are a few subtleties that make this group's structure slightly harder to unravel. In 3 dimensions, reflection is not a rotation, and thus the structure of the total group $H(3)$ is a direct product of the group of proper symmetries, $H(3)^+$, with the group of reflections, $\mathbb{Z}/2\mathbb{Z}$:

$$H(3) = SH(3) \times (\mathbb{Z}/2\mathbb{Z}) \tag{1}$$

In 4 dimensions, reflection $R = diag(-1, -1, -1, -1)$ is a proper rotation, and thus there is no decomposition like this. There are still improper symmetries in $H(4)$, and those are characterized to have negative determinant, for example spatial inversion $diag(1, -1, -1, -1)$. So, $H(4)$ still has a strict subgroup of proper transformation $SH(4)$, but there is no direct or semidirect product structure relating the two. In fact, $H(4)^+$ is not even a quotient of $H(4)$, as can be seen directly by examining the orders of the two groups.

3 Representations of $H(4)$

3.1 Young Diagrams