Patrick Oare

Physics 105 Midterm 1 Cheat Sheet

Preliminaries:

Polar unit vectors: $\vec{r} = r\hat{r}$, $\dot{\hat{r}} = \dot{\varphi}\hat{\varphi}$

$\hat{r} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$, $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$, $\dot{\hat{\varphi}} = -\dot{\varphi}\hat{r}$

$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$, $\ddot{\vec{r}} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi}$

$\vec{F} = -\nabla U$

$U(\vec{r}) = -\int_c \vec{F}\cdot d\vec{r}$

$T = T_{CM} + T_{rot}$

$\vec{R}_{cm} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$

$(1+z)^n \sim 1 + nz + \frac{n(n-1)}{2!}z^2 + \dots$

$\frac{d\cosh(x)}{dx} = \sinh x$, vice versa.

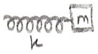
$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ $\quad T = \frac{2\pi}{\omega} = \frac{1}{f}$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

SHO:



$\omega^2 = g/\ell$

$\omega^2 = k/m$

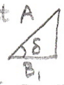
Full width half max:

FWHM = size of interval btwn where A^2 is half its max value. At $\omega \sim \omega_0 \pm \beta$, so:

FWHM $\sim 2\beta$

- **Undamped, undriven:** $\ddot{x} + \omega^2 x = 0$

$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$
$= B_1\cos\omega t + B_2\sin\omega t$
$= A\cos(\omega t - \delta)$

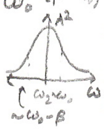


- **Resonance:** As $\omega \to \omega_0$, A^2 gets really big. But, max A at:

$\omega_2 = \sqrt{\omega_0^2 - 2\beta^2} \sim \omega_0$ $(\beta \ll \omega_0)$

- Q-factor:

$Q = \frac{\omega_0}{2\beta} = 2\pi\frac{E}{\Delta E_{dis}}$



- **Damped:** $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$.
- Soln is e^{rt}, w/ $r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$
- **Weak:** $\beta < \omega_0$. Define $\omega_1 := \sqrt{\omega_0^2 - \beta^2}$.

$x(t) = e^{-\beta t}(B_1\cos\omega_1 t + B_2\sin\omega_1 t)$

Motion is oscillatory w/ exp. decay envelope

- **Critical:** $\beta = \omega_0$. Then:

$x(t) = c_1 e^{-\beta t} + c_2 t e^{-\beta t}$

- **Overdamped:** $\beta > \omega_0$

$x(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$

– Motion is exponential

- **General soln to DDO:**

$x = x_p(t) + c_1 e^{r_+ t} + c_2 e^{r_- t}$

– exponentials die off as $t \to \infty$, called **transients**

- Soln: $x(t) = x_h(t) + x_p(t)$.

$x_h(t)$ is a soln of a damped oscillator

- Then for $z(t) = x(t) + iy(t)$, we have $Dz(t) = f_0 e^{i\omega t}$, $x_p = \mathrm{Re}[z(t)]$
- Guess $z(t) = Ce^{i\omega t}$, and get eqns:
- Write $C = Ae^{i\delta}$. Then:

$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$, $\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$ $\quad (x_p = A\cos(\omega t - \delta))$

Driven, Damped Oscillations.

Define $D := \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$

- Solve $Dx(t) = f(t)$
- Find $x_p(t)$, $f(t) = f_0\cos\omega t$.
- Take $y(t)$ to satisfy $Dy(t) = f_0\sin\omega t$

$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$

↑ natural freq ↑ driving freq

Lagrangian Mechanics:

- $S[x(t)]$ stationary when:

$\frac{\delta S}{\delta x(t)} = \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$

- If $\partial f/\partial x = 0$, then:

$\frac{\partial f}{\partial \dot{x}} = \text{const.}$ (indep. of time)

- **Calculus of variations.**

Goal: minimize:

$S[x(t)] = \int_{t_1}^{t_2} f(x, \dot{x}, t)\,dt$

- If $\partial f/\partial t = 0$, then:

$f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{const.}$ (ind. of t)

- **Conservation Laws:** A coord is **ignorable** if $\partial\mathcal{L}/\partial q_i = 0$

– If q_i is ignorable, then $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$ is conserved (ind. of time)

– **Translational invariance**, i.e. $\vec{r}_\alpha \mapsto \vec{r}_\alpha + \vec{\varepsilon}$ leaves physics the same. Then $\sum_\alpha \vec{p}_\alpha$ is conserved.

– **Cons. of energy:** If $\partial\mathcal{L}/\partial t = 0$, then $\mathcal{H}$ is conserved.

For most systems:

$\mathcal{L} = T - U$

- **Hamilton's principle:** Classical path a particle takes makes:

$S = \int_{t_1}^{t_2}\mathcal{L}\,dt$

stationary.

The Hamiltonian

$\mathcal{H} = T + U$ iff the expression for $\vec{r}$, $\vec{r}(q_1, \dots, q_n)$ is time independent

$\mathcal{H} = \sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L}$

- **Generalized coords:** Given N particles, $\alpha = 1, \dots, N$, we say $q_1, \dots, q_n$ are generalized coords if:

$\vec{r}_\alpha = \vec{r}_\alpha(q_1, \dots, q_n)$
$q_i = q_i(\vec{r}_1, \dots, \vec{r}_N)$

Note $n \le 3N$

Lagrange's eqns:

$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$

- To use Lagrange's eqns:

i) Frame inertial
ii) Non-constraint forces are conservative

- **Degrees of freedom:**

# of coords which can be independently varied in a small displacement.

- **Holonomic:** # DoF = # of generalized coords
- **Generalized (conjugate) force:**

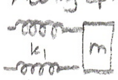
$F_i = \frac{\partial\mathcal{L}}{\partial q_i}$

- **Generalized (conjugate) momentum:**

$p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$

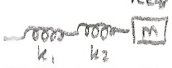
Examples:

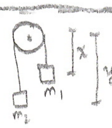
- Adding spring constants:



$k_{eq} = k_1 + k_2$

$k_{eq}^{-1} = k_1^{-1} + k_2^{-1}$





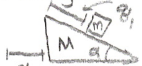
$\ell = x + y + \pi R$
$\Rightarrow \dot{x} = -\dot{y}$
$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$

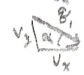
$\mathcal{L} = T - m_1 g x - m_2 g y$
$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx$

↑ Include I. Then: $\dot{x} = R\omega$, so:

$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2$
$\frac{1}{2}I\left(\frac{\dot{x}}{R}\right)^2 - (m_1 - m_2)gx$

Sliding block on wedge.



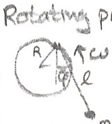


$\vec{V}_m = (\dot{q}_2 + \dot{q}_1\cos\alpha, \dot{q}_1\sin\alpha)$

$T = \frac{1}{2}m\vec{V}_m^2 + \frac{1}{2}M\dot{q}_2^2$

$U = mg(\ell - q_1)\sin\alpha$

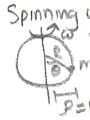
Rotating pendulum:



$\vec{r}(t) = (R\cos\omega t + \ell\sin\varphi)\hat{x} + (R\sin\omega t - \ell\cos\varphi)\hat{y}$

$\mathcal{L} = \frac{1}{2}m(R^2\omega^2 + \ell^2\dot{\varphi}^2 + 2R\omega\ell\dot{\varphi}\sin(\varphi - \omega t))$
$- mg(R\sin\omega t - \ell\cos\varphi)$

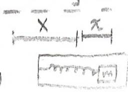
Spinning wire hoop:



$T = \frac{1}{2}m((R\dot{\theta})^2 + (\rho\omega)^2)$

$\rho = R\sin\theta$

$U = mgR(1 - \cos\theta)$

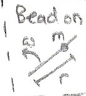


$U = \frac{1}{2}kx^2$

$T = \frac{1}{2}m(\dot{X} - A\omega\sin\omega t)^2$

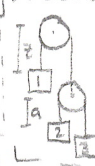
$X(t) = A\cos\omega t$

Bead on rotating rod.



$\mathcal{L} = T = \frac{1}{2}m(\dot{r}^2 + (r\omega)^2)$

Double Atwood: Gen. coords x, y.

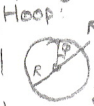


$x + z = \text{const.} \to \dot{x} = -\dot{z}$
$a + y = \text{const.} \to \dot{a} = -\dot{y}$

$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x} - \dot{y})^2 + \frac{1}{2}m_3(\dot{x} + \dot{y})^2$

$U = m_1 g x - m_2 g x + m_2 g y - m_3 g x - m_3 g y$

Spinning Hoop:



$\vec{r} = R(\cos(\omega t) + \cos(\omega t + \varphi))\hat{x} + R(\sin(\omega t) + \sin(\omega t + \varphi))\hat{y}$

$\dot{\vec{r}} = \dots$ (remember $\frac{d}{dt}\cos(\omega t + \varphi) = -(\omega + \dot{\varphi})\sin(\dots)$)

$\mathcal{L} = T = \frac{1}{2}mR^2\left[\omega^2 + (\omega + \dot{\varphi})^2 + 2\omega(\omega + \dot{\varphi})\cos\varphi\right]$

Small q approx: Drop anything $\Theta(q^2)$ or higher:

$\cos q \sim 1 + \frac{1}{2}q^2 \sim 1$, $\sin q \sim q$, $\dot{q}^2 \sim 0$.

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Physics 105 Midterm 2 Cheat Sheet

Central Forces:

m_1, m_2 interacting w/ conservative $U(r)$: $M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{M}$

$\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{R} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$

$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$

$T = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$

CF Lagrangian: $\mathcal{L} = \mathcal{L}_{cm} + \mathcal{L}_{rel}$

$\mathcal{L}_{cm} = \frac{1}{2}M\dot{\vec{R}}^2$

$\mathcal{L}_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$

Gives EoM: $\mu\ddot{\vec{r}} = -\nabla U(r)$, $M\ddot{\vec{R}} = 0$

COM Frame: $M\ddot{\vec{R}} = \vec{0}$ $\Rightarrow \dot{\vec{R}} = $ const, so use reference frame w/ $\vec{R} = \vec{0}$

Two EoM: Use polar (r,φ) in plane of motion. Then $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$, so:

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\varphi} = \mu r^2\dot{\varphi} = \text{const} =: \ell & (\varphi \text{ eqn}) \quad \uparrow \text{ang. momentum} \\ \mu r\dot{\varphi}^2 - \frac{dU}{dr} = \mu\ddot{r} & (r \text{ eqn}) \end{cases}$$

Angular momentum: $\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{r} \times \mu\dot{\vec{r}}$

- Conserved, so $\vec{r} \times \dot{\vec{r}} = $ const $\Rightarrow$ $\vec{r}, \dot{\vec{r}}$ in same plane $\Rightarrow$ motion constrained to plane

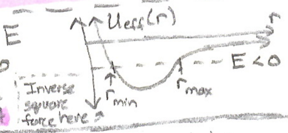
Reduce r eqn to 1D problem: $F_{cf} = \mu r\dot{\varphi}^2 = \frac{\ell^2}{\mu r^3}$

$\mu\ddot{r} = -\frac{dU_{eff}}{dr}$, $U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}$

Energy: U_{eff} contains φ part of kinetic energy.

$\frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = E = $ const.

- $U_{eff}(r_{min}) = E$ b/c then $\dot{r} = 0$. $E<0$ called a bounded orbit

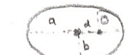


Kepler Orbits: $F(r) = -\frac{\gamma}{r^2} = -\gamma u^2$

- For gravity, $\gamma = Gm_1m_2$. Eqn of orbit: $u'' = -u + \frac{\gamma\mu}{\ell^2} \Rightarrow u(\varphi) = \frac{\gamma\mu}{\ell^2}(1 + \varepsilon\cos\varphi)$ where ε is a constant of integration

- Bounded orbits ($\varepsilon < 1$): orbit is an ellipse w/ eccentricity ε (how elongated ellipse is); $\varepsilon = 0 \to$ circle

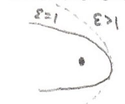
$r_{min/max} = \frac{c}{1 \pm \varepsilon}$



$d = a\varepsilon$, $b = a\sqrt{1-\varepsilon^2}$

$r_{min} \to$ perihelion, $r_{max} \to$ aphelion

- Unbound orbits ($\varepsilon \geq 1$):



$\varepsilon = 1 \Rightarrow$ parabola

$\varepsilon > 1 \Rightarrow$ hyperbola, has $\varepsilon\cos(\varphi_{max}) = -1$

Rewrite soln as: $r(\varphi) = \frac{c}{1 + \varepsilon\cos\varphi}$

$c = \frac{\ell^2}{\gamma\mu}$ ($[c] = $ length)

- Kepler's Laws:
① Planet's orbit has sun at one focus of ellipse
② $dA/dt = \ell/2\mu$
③ $\tau^2 = \frac{4\pi^2}{GM_{sun}}a^3$ $\quad 2a = r_{max} + r_{min}$

ε	Energy	Shape
$\varepsilon = 0$	$E<0$	circle
$0<\varepsilon<1$	$E<0$	ellipse
$\varepsilon = 1$	$E=0$	parabola
$\varepsilon > 1$	$E>0$	hyperbola

ℓ is determined by initial conditions. Use: $\dot{\varphi} = \frac{\ell}{\mu r^2}$

$\varepsilon<1$: Bounded

$\varepsilon \geq 1$: Unbounded (denom. vanishes)

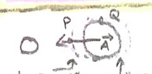
- For bound + unbound orbits, energy E is: $E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)$

Equation of orbit: find $r = r(\varphi)$: $\mu\ddot{r} = F(r) + \frac{\ell^2}{\mu r^3}$, $F = -\frac{dU}{dr}$

Define $u = 1/r$ and $d/dt = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}$ to get: $u''(\varphi) = -u(\varphi) - \frac{\mu}{\ell^2 u(\varphi)^2}F$

Tides: Bulge caused by centrifugal $\vec{a}$ about CM of Earth Moon system.

- Surface is an equipotential: $U(P) = U(Q)$



List of Forces:

$\vec{F}$	U
$m\vec{g}$	mgh
$-\frac{GM_m m}{d^2}\hat{d}$	$-GM_m m\left(\frac{1}{d} + \frac{x}{d_0^2}\right)$
$\vec{F}_{nongrav}$	

Noninertial Frames: S_0 inertial, S noninertial.

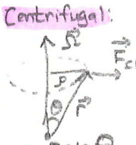
Linear: S accel. relative to S_0 at $\vec{A}$. Then: $\vec{F}_{fict} = -m\vec{A}$

Rotation: Use $\dot{\vec{Q}} = (d\vec{Q}/dt)_S$

$\vec{F}_{cor} = 2m\dot{\vec{r}} \times \vec{\Omega}$

$\vec{F}_{CF} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$

Centrifugal: $\vec{F}_{CF} = m\Omega^2\rho\hat{\rho}$



$\rho = R\sin\theta$

- Free fall: F_{eff} toward center: $\vec{F}_{eff} = m\vec{g}_0 + m\Omega^2 R\sin\theta\hat{\rho}$

$\vec{g} = $ effective acceleration of gravity

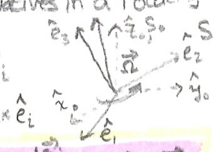
$\vec{g} = \vec{g}_0 + \Omega^2 R\sin\theta\hat{\rho}$

$g_{rad} = g_0 - \Omega^2 R\sin^2\theta$

Time derivatives in a rotating frame:

$\vec{Q} = \sum_i Q_i\hat{e}_i$

$\left(\frac{d\hat{e}_i}{dt}\right)_{S_0} = \vec{\Omega} \times \hat{e}_i$

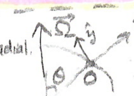


$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\Omega} \times \vec{Q}$

Coriolis: Analogy to $\vec{B}$ field: $q \to 2m$, $\vec{B} \to \vec{\Omega}$

- Always $\perp$ to $\vec{v}$
- 0 if $\vec{v} = \vec{0}$.

Free fall: $\hat{y} \to$ North, $\hat{z} \to$ radial. $\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$



- EoM: $\ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\Omega}$
- Approx in orders of Ω.

Foucault Pendulum: $T \sim mg$

$m\ddot{\vec{r}} = m\vec{g} + \vec{T} + 2m\dot{\vec{r}} \times \vec{\Omega}$

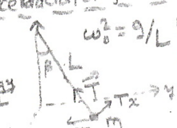
$T_z \sim mg$, $T_x = -mgx/L$, $T_y = -\frac{mgy}{L}$

$\dot{\vec{r}} \times \vec{\Omega} = (\dot{y}\Omega_z, -\dot{x}\Omega_z, 0)$

w/ $\Omega_z = \Omega\cos\theta$. EoM is:

$$\begin{cases} \ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 x = 0 \\ \ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y = 0 \end{cases}$$

$\omega_0^2 = g/L$



To solve: Use $\eta = x + iy$ to get: $\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2\eta = 0$

K3: Exact period of orbit is: $\tau^2 = 4\pi^2\frac{a^3\mu}{\gamma}$

Derive using K2 and saying $\tau = A/(dA/dt)$.

Rigid Body Rotation:

Let $\vec{R} = $ pos. of CoM. $\vec{F}_{ext} = M\ddot{\vec{R}}$

$\vec{L} = \vec{L}(CM) + \vec{L}(\text{relative})$, and $T = T(CM) + T(\text{relative})$

$\vec{L} = \overleftrightarrow{I}\vec{\omega}$ $\quad I_{xx} = \sum_\alpha m_\alpha(y_\alpha^2 + z_\alpha^2)$ $\quad \overleftrightarrow{I} = \overleftrightarrow{I}^T$

$I_{xy} = -\sum m_\alpha x_\alpha y_\alpha$

OR: $I_{ij} = \int\rho(r^2\delta_{ij} - r_ir_j)d^3x$

Miscellaneous: $\square$ = in Chapter summary.

Lagrangian Mechanics: $\mathcal{L} = T - U$.

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$$

q_i ignorable if $\partial\mathcal{L}/\partial q_i = 0$

- q_i ignorable $\Rightarrow$ conj. momenta conserved
- $\partial\mathcal{L}/\partial t = 0 \Rightarrow \mathcal{H}$ conserved.

$\mathcal{H} = \sum_{i=1}^n p_i\dot{q}_i - \mathcal{L}$

Polar unit vectors: $\vec{r} = r\hat{r}$

$\hat{r} = \cos\varphi\hat{x} + \sin\varphi\hat{y}$, $\hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y}$

$\dot{\hat{r}} = \dot{\varphi}\hat{\varphi}$, $\dot{\hat{\varphi}} = -\dot{\varphi}\hat{r}$, $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi}$

Remember: CoM at a focus of the ellipse. For something orbiting a massive body, the body is at a foci of the orbit

Ex: Spinning bucket

$\vec{F} = -mg\hat{z} + m\Omega^2\rho\hat{\rho} = -\nabla U$.

$U = mgz - \frac{1}{2}m\Omega^2\rho^2 = $ const

$\Rightarrow z(\rho) = $ parabola.

$$R(\hat{z},\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $r(\varphi)$: CoM at X (the focus). $r(\varphi)$ gives $|\vec{r}_1 - \vec{r}_2|$ — not a real particle, but the reduced 1D particle. Treat it as a point mass w/ mass μ and trajectory $r(\varphi)$.

Tang. comp of free fall acceleration: $g_{tang} = \Omega^2 R\sin\theta\cos\theta$

$U_g = -G\frac{m_1m_2}{r}$

Use $2a = r_{max} + r_{min}$ to get $a = \frac{c}{1-\varepsilon^2}$

Area of ellipse $= \pi ab$

Changing orbits: Apply an impulse at perigee of orbit 1 — becomes perigee (or apogee) of orbit 2:

$\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2}$ $\quad \lambda = v_2/v_1$, $\ell_2 = \lambda\ell_1$

$\Rightarrow \varepsilon_2 = \lambda^2\varepsilon_1 + (\lambda^2 - 1)$

$\lambda > 1 \to$ forward thrust

$\lambda < 1 \to$ backward thrust

Derivation of $E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)$: At r_{min}, $E = U_{eff}(r_{min})$ as $\dot{r}|_{r_{min}} = 0$, and $r_{min} = \frac{c}{1+\varepsilon}$

Object fired on Earth at velocity $\vec{V}_0$. EoM still the same

$$\begin{cases} \ddot{x} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta) \\ \ddot{y} = -2\Omega\dot{x}\cos\theta \\ \ddot{z} = -g + 2\Omega\dot{x}\sin\theta \end{cases}$$

$\vec{V}_0 = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

For 1st order soln, plug in $\dot{x} = v_x$, $\dot{y} = v_y$, $\dot{z} = v_z - gt$. Then get:

$$\begin{cases} \ddot{x} = 2\Omega(v_y\cos\theta - v_z\sin\theta + gt\sin\theta) \\ \ddot{y} = -2\Omega v_x\sin\theta \\ \ddot{z} = -g + 2\Omega v_x\sin\theta \end{cases}$$

$\to$ integrate up twice and fit $\dot{x}(0) = v_x$, etc. w/ $\ddot{y}, \ddot{z}$.

To solve: First set $\Omega = 0$ in EoM to get: $x = v_xt$, $y = v_yt$, $z = v_zt - \frac{1}{2}gt^2$

This is 0th order soln.

- First order (in Ω) solution is:

$x(t) = v_xt + \Omega(v_y\cos\theta - v_z\sin\theta)t^2 + \frac{1}{3}\Omega gt^3\sin\theta$

$y(t) = v_yt - \Omega v_xt^2\cos\theta$

$z(t) = v_zt - gt + \Omega v_xt^2\sin\theta$

Rigid Body Motion (cont.)

Common inertia tensors:

- Cube (about corner)

$$\overleftrightarrow{I} = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

- Principal axis / value: Eigenvecs + eigen vals of $\overleftrightarrow{I}$. Body frame: $\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t)$ principal axes. Then $\overleftrightarrow{I} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$
- Kinetic: $T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\lambda_i\omega_i^2$
- Find principal vals: solve $\det(\overleftrightarrow{I} - \lambda\,\text{id}) = 0$.

Mirror symmetry about $z = 0$ plane $\Rightarrow I_{xz} = I_{yz} = 0$

- Any body has 3 orthogonal principal axes

Euler's eqns: Dot is a time derivative in body frame. Have:

$(d\vec{L}/dt)_{space} = \vec{\Gamma} = (d\vec{L}/dt)_{body} + \vec{\omega} \times \vec{L}$, so:

$\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}$

- In body frame: $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, $\vec{L} = (\lambda_1\omega_1, \ldots, \lambda_3\omega_3)$

Eqns become:

$\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 = \Gamma_1$

$\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_1\omega_3 = \Gamma_2$

$\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 = \Gamma_3$

Body w/ 2 equal moments $\lambda_1 = \lambda_2$.

Suppose freely rotating, $\vec{\Gamma} = \vec{0}$. Have $\dot{\omega}_3 = 0$, so: $\Omega_b = (\lambda_1 - \lambda_3)\omega_3/\lambda_1$

$\dot{\omega}_1 = \Omega_b\omega_2$, $\dot{\omega}_2 = -\Omega_b\omega_1$ ← const

Use $\eta = \omega_1 + i\omega_2$ to get $\eta = \eta_0e^{-i\Omega_bt}$

$\Rightarrow \vec{\omega} = (\omega_0\cos(\Omega_bt), -\omega_0\sin(\Omega_bt), 0)$

- In body frame, $\vec{\omega}$ traces out body cone.

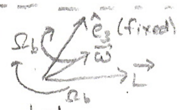
$\Omega_s = L/\lambda_1$.

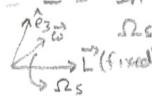
Body w/ 3 different moments:

- Only way to rotate w/ const. $\vec{\omega}$ is for $\vec{\omega}$ to be along a principal axis.
  - How stable is this? Suppose rotating about $\hat{e}_3$. Give it a kick. $\omega_3 \sim$ const, but now:

$$\ddot{\omega}_1 = -\left[\frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]\omega_1 =: A\omega_1$$

from Euler's eqns. If $A<0$, stable; $A>0 \Rightarrow$ unstable. So if $\lambda_3 > \lambda_1, \lambda_2$ or $\lambda_3 < \lambda_1, \lambda_2$ $\Rightarrow$ stable, if $\lambda_1/\lambda_2 < \lambda_3 < \lambda_2/\lambda_1 \Rightarrow$ unstable





Coupled Oscillations:

Linear operators transform as:
$\overleftrightarrow{I}' = F \overleftrightarrow{I} F^{-1}$

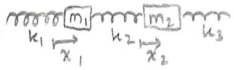
Quadratic forms transform as
$\overleftrightarrow{K} = F^T \overleftrightarrow{K}' F$

$\vec{q} = \begin{pmatrix} q_1 \\ q_n \end{pmatrix}$

Important eqn:
$M\ddot{\vec{q}} = -K\vec{q}$

$M, K = n \times n$ matrices

• Two masses, 3 springs:



$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}$, $K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}$

EoM are $M\ddot{\vec{x}} = -K\vec{x}$

Solving the eqn: Look for normal mode solns $\vec{q} = \mathrm{Re}[\vec{z}(t)]$, where $\vec{z}(t) = \vec{a}e^{i\omega t}$. Then solve:

$(K - \omega^2 M)\vec{a} = 0$

– So the normal modes satisfy

$\det(K - \omega^2 M) = 0$

and the normal mode is the eigenvector

Hamiltonian Mechanics:

$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$, $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$

• Remember to write $\mathcal{H}$ as a fn of q's and p's, not q's and $\dot{q}$'s

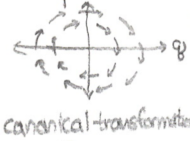
Equations of motion:

$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$

$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$

Phase space: $\vec{q}$ vs. $\vec{p}$
State space: $\vec{q}$ vs $\dot{\vec{q}}$

• SHO:
$\dot{q} = p/m$
$\dot{p} = -kq$



• Hamilton's eqns define a vector field on phase space.
• Lagrange's eqns invariant under coord. change on configuration space $(q_1, \dots, q_n)$. (n-dim).
• Hamilton's eqns invariant under coord change on phase space $(\vec{q}, \vec{p})$ (2n-dim) if it is a canonical transformation

• If q_i ignorable ($\mathcal{H}$ not a fn of q_i) then $\dot{p}_i = 0$, so p_i is a constant of motion.

• Let $\vec{z} = (\vec{q}, \vec{p})$. Hamilton's eqns equiv. to $\dot{\vec{z}} = \vec{h}(\vec{z})$.
– Any point $\vec{z}_0$ defines a unique phase space orbit
– No two distinct orbits in phase space can cross, even at different times (b/c $\dot{\vec{z}} = \vec{h}(\vec{z})$ would force them to be the same).

Double pendulum:

$T = \frac{1}{2}(m_1+m_2)L_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2 L_2 \dot{\varphi}_2^2 + m_2\dot{\varphi}_1\dot{\varphi}_2 L_1 L_2 \cos(\varphi_1 - \varphi_2)$

$U = -(m_1+m_2)gL_1\cos\varphi_1 - m_2 g L_2 \cos\varphi_2$

– Lagrange's eqns are nasty. But Taylor expand to 2nd order, then plug into Lagrange, and get:

$M = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix}$ $\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$

$K = \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2 g L_2 \end{pmatrix}$ $\underbrace{M\ddot{\vec{\varphi}} = -K\vec{\varphi}}_{\text{Lagrange's eqns}}$

• Equal masses + lengths:
$K = mL^2\omega_0^2\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $M = mL^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\omega_0^2 = g/L$

Normal modes ω_1 and ω_2 given by:
$\omega_1^2 = (2-\sqrt{2})\omega_0^2, \vec{a}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ $\omega_2^2 = (2+\sqrt{2})\omega_0^2, \vec{a}_2 = c\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$
– in phase oscillations – out of phase oscillations.

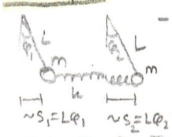
Charged particle: $\mathcal{H} = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + qV$

$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2-x_1)^2 \xRightarrow{\text{EoM}} \begin{cases} m\ddot{x}_1 = k(x_2 - 2x_1) \\ m\ddot{x}_2 = k(x_1 - x_2) \end{cases}$

$\Rightarrow M = mI$, $K = k\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$, solve $\det(K - \omega^2 M) = 0 \Rightarrow \omega^2 = \frac{3 \pm \sqrt{5}}{2}\frac{k}{m}$

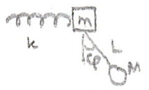
$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$
$\hookrightarrow = \sum_i \left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i\right) + \dot{H}$

Examples:



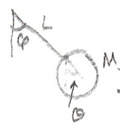
$\mathcal{L} = \frac{1}{2}mL^2\dot{\varphi}_1^2 + \frac{1}{2}mL^2\dot{\varphi}_2^2 - (-mgL\cos\varphi_1 - mgL\cos\varphi_2 + \frac{1}{2}kL^2(\varphi_2 - \varphi_1)^2)$

← small angles



Small angle.
$\vec{r}_M = (x + L\sin\varphi)\hat{x} - L\cos\varphi\hat{y}$

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M(\dot{x} + L\dot{\varphi})^2 - \frac{1}{2}kx^2 - \frac{1}{2}MgL\varphi^2$ $\hookleftarrow \cos\varphi \sim 1 - \frac{1}{2}\varphi^2$



rod is massless. Then want I_{com}:
$I_{com} = \frac{1}{2}MR^2$ (disk)
$T_{rot} = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\Theta}^2$

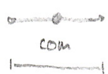
$\vec{r}_{com} = (L\sin\varphi + R\cos\Theta, -L\cos\varphi - R\cos\Theta)$

$T_{trans} = \frac{1}{2}M\dot{\vec{r}}_{com}^2 = \frac{1}{2}M(L^2\dot{\varphi}^2 + 2LR\dot{\varphi}\dot{\Theta}\cos(\Theta - \varphi) + R^2\dot{\Theta}^2)$

$U = Mg(L+R) - MgL\cos\varphi - MgR\cos\Theta$

Double pendulum, but first rod is massless, $m_1 = m_2 = 0$, 2nd rod uniform mass m.

$I_{com} = \frac{m}{L}\int_{-L/2}^{L/2} x^2 dx = \frac{1}{12}mL_2^2$



$T = \frac{1}{2}m\dot{\vec{r}}_{com}^2 + \frac{1}{2}\left(\frac{1}{12}mL_2^2\right)\dot{\varphi}_2^2$

Parallel axis thm: Let $\overleftrightarrow{I}^{cm}$ be about CM of a body of mass M, and $\overleftrightarrow{I}$ be the tensor about a point P displaced from CM by (ξ, η, ζ). Then:

$I_{yz} = I_{yz}^{cm} - M\eta\zeta$

$I_{xx} = I_{xx}^{CM} + M(\eta^2 + \zeta^2)$ and so on.

More Kepler orbit eqns:

$a = \frac{c}{1 - \varepsilon^2}$, $b = \frac{c}{\sqrt{1 - \varepsilon^2}}$

a, b axes of ellipse

Remember that:
$T = T_{trans}^{COM} + T_{rot}^{COM}$
– This means you want to calculate things about the COM

$I = I_{COM} + Mh^2$!

Reason: $T = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2$
$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$
and so if $\dot{\vec{R}} = 0$, then
$T = \frac{1}{2}\mu\dot{\vec{r}}^2$ w/:
$\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

Energy w/ relative distance
$\vec{r}(t)$ (6 DoF → 3 DoF, 2 body → 1
$E = \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{Gm_1m_2}{\vec{r}^2}$

W/ 2 bodies, try to combine to 1 particle motion first.

Parallel axis thm derivation:

– In CoM coords, $\int \vec{r}_{com}\, dm = 0$.

$I_{xx} = \int (y^2 + z^2)dm = \int ((y_{cm} + \eta)^2 + (z_{cm} + \zeta)^2)dm$

$= \int (y_{cm}^2 + z_{cm}^2)dm + 2\eta\overbrace{\int y_{cm}\, dm}^{0} + 2\zeta\overbrace{\int z_{cm}\, dm}^{0} + (\eta^2 + \zeta^2)\int dm = I_{com} + M(\eta^2 + \zeta^2)$