Complex Analysis

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Introduction

Complex analysis is one of the most basic tools in every physicist's toolkit, but unfortunately physics courses rarely teach one how to use it. My rule of thumb for deciding if I'm sitting in on an upper level physics course is if the professor shouts out "close the contour!" or "analytic continuation" every five minutes or so, seemingly at random with little to no explanation whatsoever. I'm hoping these notes will provide you with a resource on what these phrases actually mean and make the rambling professor a bit clearer from here on out.

Unfortunately I don't find the mathematical theory of complex analysis to be that interesting, so I likely won't do many proofs unless I find them to be particularly illuminating. From a math perspective, this subject is sort of like the drier aspects of real analysis, until you put it on manifolds (which I hope to learn at some point). However, the main results of complex analysis are not only extremely useful but also very, very beautiful– the nature of the complex plane significantly constrains the type of functions we wish to study, and all these functions are quite well behaved.

1 Differentiation

We will start with some basic notions, and soon move differentiation over to the complex plane. A **function** of a complex variable is a map $f : \mathbb{C} \to \mathbb{C}$, denoted by $f(z)$ for $z \in \mathbb{C}$. We can decompose $f(z)$ into a real part and an imaginary part, as well as decomposing $z = x + iy$ into a real part and an imaginary part. This means every complex function f can be written as a function of x and $y \in \mathbb{R}$ as follows:

$$f(z) = f(x, y) = u(x, y) + iv(x, y) \tag{1}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are real valued functions. For example, the function $f(z) = z^2$ can be written as $f(x, y) = (x + iy)^2 = (x^2 - y^2) + i(2xy)$, so $f = u + iv$ with $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Keep in mind the definitions of u and v, as we will soon use them to impose a strong condition on when a function is complex-differentiable.

Limits of complex variable functions work the exact same way as in multivariable calculus. For example, the limit $\lim_{z \to z_0} f(z)$ only exists if $f(z)$ approaches *the same value* no matter which direction you approach z_0 from.

After two paragraphs, we're now able to define differentiability. In complex analysis, a differentiable function is called **holomorphic**. More precisely, we call a function $f : \mathbb{C} \to \mathbb{C}$ **holomorphic at** $z_0 \in \mathbb{C}$ if the limit:

$$f'(z_0) := \lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w} \tag{2}$$

exists. If f is holomorphic at z_0, $f'(z_0)$ is called its **derivative**. If f is holomorphic at every $z_0 \in \mathbb{C}$, then f is sometimes called **entire**.

The basic rules of derivatives still hold in complex analysis. Complex functions satisfy the product rule, chain rule, power rule, and essentially every other derivative identity you learned in single variable calculus. So, if $f(z) = \sin(z^2)$, then $f'(z) = 2z\cos(z^2)$.

Complex analysis captures all the nice properties of real-valued differentiation, and a lot more. To see this, we will quickly develop the **Cauchy Riemann equations**. Suppose that a complex function $f(z) = f(x, y) = u(x, y) + iv(x, y)$ is holomorphic at $z \in \mathbb{C}$. Then the limit in Equation 2 must hold if $w \to 0$ on the real line, i.e. if $w = h \in \mathbb{R}$. So, we must have:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim \frac{u(x+h, y) + iv(x+h, y) - (u(x, y) + iv(x, y))}{h} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

We can also have approach z on the imaginary axis, if $w = ih$ with $h \in \mathbb{R}$. Then:

$$f'(z) = \lim_{h\to 0} \frac{f(z+ih) - f(z)}{ih} = \lim \frac{u(x, y+h) + iv(x, y+h) - (u(x, y) + iv(x, y))}{ih} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

These two equations must be equal for the limit to exist by definition, so we can equate the real and imaginary parts. This gives the **Cauchy-Riemann equations**:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \tag{3}$$

In these equations is the inherent difference between differentiability in $\mathbb{R}$ (or even $\mathbb{R}^2$) and in $\mathbb{C}$. A complex function $f(z) = u(x, y) + iv(x, y)$ is holomorphic if and only if both u and v are differentiable as real valued functions, and if they satisfy the Cauchy Riemann relations. So, holomorphic functions are a subset of real-valued differentiable functions. The fact that a holomorphic function must satisfy 3 *in addition* to having u and v being differentiable means that being holomorphic is a much stronger constraint than simply being differentiable in $\mathbb{R}^2$.

This simple relation leads to some very nice properties that holomorphic functions satisfy, which we will now explore. The immediate corollary of this is that **any holomorphic function can be expanded in a Taylor series**, i.e. if $f(z)$ is holomorphic at $z_0 \in \mathbb{C}$[1], then we can always expand it as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{4}$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$, and this will converge. This is in contrast to real-valued differentiation, where for example the Taylor series of the function e^{-x^2} is identically 0 everywhere, even though e^{-x^2} is infinitely differentiable.

The ability to write any holomorphic function $f(z)$ as a power series has an immediate corollary; it means that **holomorphic functions are infinitely differentiable**, i.e. if $f'(z)$ exists, then $f^{(n)}(z)$ exists for each $n \in \mathbb{N}$. This is clearly not true in the real-differentiable case, because for example the function $f(x) = x^{4/3}$ is differentiable at 0, but not twice differentiable at 0.

2 Integration

Integration in the complex plane all boils down to one master theorem: the residue theorem. If you ever decide to take an introductory complex analysis course, likely the course will be building up

[1]Specifically if f is holomorphic on a disk $D \subseteq \mathbb{C}^2$, which allows us to consider functions which may have singularities elsewhere in $\mathbb{C}$.

to this entire theorem for the entire semester. There is a lot of machinery that one needs to prove this, but we will take all the machinery for granted because if we didn't, it would take much more than a page to explain why integrals are so well behaved. The heuristic idea is because holomorphic functions are analytic, and we can expand them in a Laurent series. If we can thus determine how to integrate a polynomial or rational function, we can extend this by linearity and integrate any holomorphic function.

The complex integral is defined exactly as you would expect it to be in terms of Lebesgue (or Riemann) integration. For a function $f(z) = u(x, y) + iv(x, y)$, we define its **integral** over a curve $C \subseteq \mathbb{C}$ to be:

$$\int_C f(z)\,dz := \int_C (u(x,y) + iv(x,y))\,(dx + idy) = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx) \tag{5}$$

where in the right hand side it is understood that we are integrating the real valued functions $u(x, y)$ and $v(x, y)$. The interesting piece here will come when we discuss how to evaluate these integrals, because there are some very nice tricks.

2.1 Poles and residues

In this section, we will build up some machinery we need to state the residue theorem, namely we will examine the behavior of singularities in complex functions. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function. A **zero** of f is a point $z_0 \in \mathbb{C}$ where $f(z_0) = 0$. A **pole** of f is a point $z_0 \in \mathbb{C}$ which is a zero of the function $\frac{1}{f}$. Essentially, a pole is a type of "removable singularity" of f, and will always look like $f \sim (z - z_0)^{-n}$ where n is a positive integer[2]. In general, if $f(z)$ has a pole at $z_0 \in \mathbb{C}$, then locally around z_0 we may write f as[3]:

$$f(z) = \frac{g(z)}{(z - z_0)^n} \tag{6}$$

where $g(z)$ is holomorphic and n is as small as possible. n is called the **order** of the pole at z_0, and $g(z_0)$ is called the **residue** at z_0.

As an example of this, consider the function:

$$f(z) = \frac{e^{ikz}}{z^2 + 1} \tag{7}$$

This function has poles where $z^2 = -1$, i.e. at $z = \pm i$. An easy way to determine the residues at each point is to factor f as:

$$f(z) = \frac{e^{ikz}}{(z + i)(z - i)} \tag{8}$$

and then to "cover up" the singularity at each z_0 and evaluate the remainder of the function z_0. We can thus read off the residues as:

$$\begin{aligned} res_{z=i}(f) &= \left.\frac{e^{ikz}}{z + i}\right|_{z=i} = \frac{e^{-k}}{2i} \\ res_{z=-i}(f) &= \left.\frac{e^{ikz}}{z - i}\right|_{z=-i} = -\frac{e^{k}}{2i} \end{aligned} \tag{9}$$

[2]There are also singularities which are not so nicely behaved called essential singularities, for example at $z = 0$ in the function $e^{\frac{1}{z}}$. We will not discuss these here as the singularities that one encounters in physics are typically poles.

[3]To be precise, by "locally" I mean there is $r > 0$ such that on a disk of radius r centered at z_0, f equals this function.

These are both order 1 poles, which we call **simple poles**. Most of the functions you deal with will resemble this, only they may be multivariate. It is occasionally more complex to compute residues, but I doubt you'd ever see a malicious function like this in physics. For example, the function:

$$\frac{\sin(z)}{z} \tag{10}$$

looks like it has a pole of order 1 at $z = 0$. However, $\lim_{z\to 0}\sin(z) = 0$, so we must qualitatively determine how the numerator goes to 0 as well. We can Taylor expand $\sin(z) = z - z^3/3! + ...$ and see that this approaches 1 as $z \to 0$, and therefore this is not a pole; it is instead called a **removable singularity**, and we can treat it just like the function is continuous at $z = 0$.

2.2 The residue theorem

Before introducing the residue theorem, we will begin with Cauchy's theorem. This may look familiar: it is quite similar to a statement in multivariable calculus about conservative vector fields.

Theorem 2.1 (Cauchy)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then for any closed curve $C \subset \mathbb{C}$, we have:

$$\oint_C f(z)\,dz = 0 \tag{11}$$

However, not every function we wish to integrate is holomorphic. We wish to consider functions which are holomorphic almost everywhere, with the exception of possibly having some poles somewhere in the complex plane. These functions are called **meromorphic**, and these are the types of functions which will give us interesting physics. For functions like this, we must use the full residue theorem.

Theorem 2.2 (Residue)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Let C be a closed curve in the complex plane, and suppose that f has poles at[4] $\{z_1, ..., z_n\} \subset \mathbb{C}$ within the boundary of C. Then:

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{res}_{z_j}(f) \tag{12}$$

i.e. the integral of f around C is proportional to the sum of its residues within C.

This theorem makes evaluating integrals extremely easy, and leads to some nice properties which we will take advantage of. We can see by example how this works for the case of $f(z) = 1/z$. We will evaluate this by explicit parameterization in Equation 16, but for now we can do the easy way and use the residue theorem. The function $f(z) = 1/z$ has a pole at $z = 0$ with residue 1 (cover up the singularity and evaluate it at $z = 0$; here, we can write $f(z) = 1 \cdot \frac{1}{z}$, so covering up the singularity of $\frac{1}{z}$ and evaluating the rest of this at $z = 0$, we see the residue is 1. So, since the pole at $z = 0$ is contained within the path $C = S^1 = \{\cos(t) + i\sin(t) : t \in [0, 2\pi]\}$, the residue theorem tells us that:

$$\oint_C \frac{dz}{z} = 2\pi i\,\mathrm{res}_0\left(\frac{1}{z}\right) = 2\pi i \tag{13}$$

[4] f can also have a countably infinite set of poles and the theorem remains true, but we won't discuss that here.

2.3 Example: Green's functions in Quantum Mechanics

3 Odds and Ends

3.1 Branch cuts and Riemann sheets

The other type of singularity you will deal with in complex analysis is called a **branch cut**, which occur when we promote specific types of real functions to being complex. For example, consider $f(z) = \log z$. What do we mean by the logarithm of a complex function? When we are working with real numbers, $log(1) = 0$ certainly, since the only real number $x \in \mathbb{R}$ with $e^x = 1$ is 0. However, when we allow x to be a complex number, there are an infinite number of solutions to $e^z = 1$ which are given by $z = (2\pi i)n$ with $n \in \mathbb{Z}$, since $e^{2\pi i n} = \cos(2\pi n) + i\,\sin(2\pi n) = 1$. Evidently, we must do some extra work to define the function $\log(z)$ on the complex plane in a nice manner.

Heuristically, the idea of "branch cuts" is to pick a value for the function at a point, and then extend it by continuity. For the case of $\log(z)$, it is easy to determine how to do this by parameterizing a complex number as $z = re^{i\theta}$ with $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Then the possible values of the logarithm acting on z are precisely:

$$\log(z) = \log(r) + i\theta + (2\pi i)n \tag{14}$$

with $n \in \mathbb{Z}$. Note in particular that if we fix n to be a specific integer value, then we get a specific value of $\log(z)$ almost everywhere in the complex plane, and it is no longer a multivalued function! This is called **choosing a branch** of the logarithm.

An intuitive picture of what the logarithm looks like in the complex plane is this:

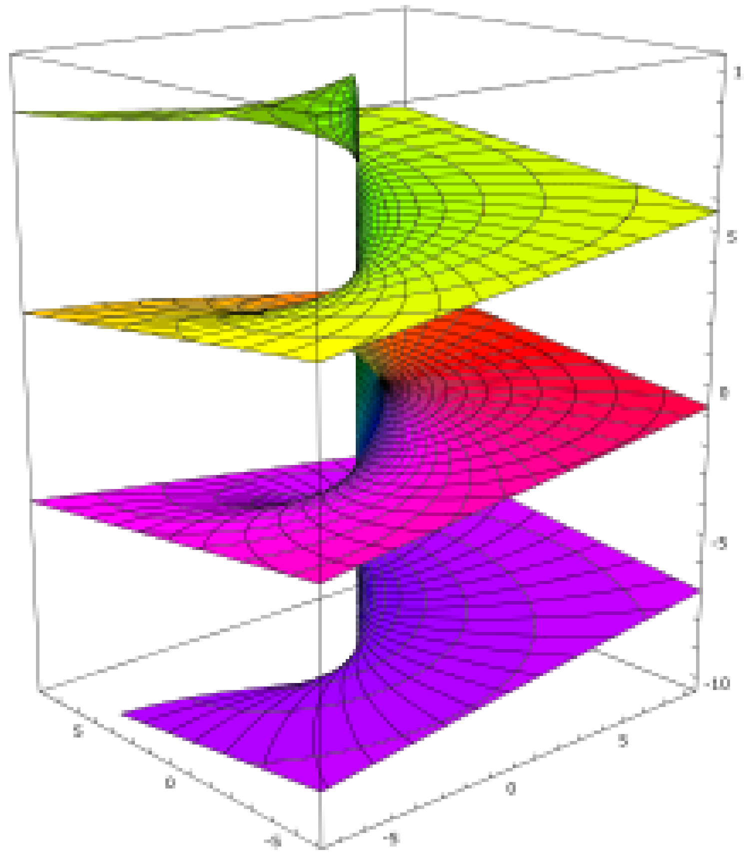

Figure 1: The complex logarithm with multiple branches.

The function forms a spiral around 0; as we wrap around 0, the logarithm gets larger. This can be seen intuitively because as $d\log(z)/dz = 1/z$, so if we want to evaluate the change in the logarithm from z_2 to z_1, we can integrate $1/z$ like so:

$$\log(z_1) - \log(z_2) = \int_{z_2}^{z_1} \frac{dz}{z} \tag{15}$$

But, notice that this does not specify the path we are taking. Namely, we can take a closed loop C around the origin, parameterized by $C = \{\cos(t) + i\sin(t) : t \in [0, 2\pi]\}$, to evaluate this integral. Starting this integral at $z_1 = z_2 = 1$, we plug in $z = \cos(t) + i\sin(t)$ to see:

$$\log(1_+) - \log(1_-) = \oint_C \frac{dz}{z} = \int_0^{2\pi} dt \frac{-sin(t) + i\cos(t)}{\cos(t) + i\sin(t)} = \int_0^{2\pi} dt \frac{i(\sin^2(t) + \cos^2(t))}{\cos^2(t) + \sin^2(t)} = 2\pi i \tag{16}$$

which is non-zero. As we wrap around 0, the value of the function $\log(z)$ can change: it is multiple valued, and there is nothing saying we cannot get to another possible value for $\log(1)$ if we move along a curve surrounding the origin.

To make this more precise, we choose a branch by *defining* the value of $\log(z_0)$ for some $z_0 \in \mathbb{C}$ as one of its possible values. Consider defining $\log((r+\epsilon)e^{i\theta})$ for ϵ infinitesimal. The possible values of this we can choose from are $\log(r+\epsilon) + i\theta + 2\pi i n$ for $n \in \mathbb{Z}$, but here's the key point: only one specific value of n works if we want this definition of the logarithm to be continuous! In this way, we can build up the definition of the logarithm on this branch (this is called the **principal branch** of the logarithm) until we have defined it on almost the entire complex plane.

Branch cuts come in when we have defined the logarithm almost everywhere. We *cannot* wrap the logarithm around the entire plane, or else we will have to deal with the fact that it is multi-valued when we wrap around the origin. We can extend it almost everywhere, but there must be at least a ray in the complex plane starting from 0 for where we cannot define $\log(z)$, or else we could continue this "wrapping around 0" nonsense and destroy the single-valuedness of our branch.

can extend $\log(z)$ by integrating its derivative $\frac{1}{z}$. We

3.2 Analytic Continuation

Analytic continuation is perhaps one

3.3 Laurent Series

3.4 Example: Wick rotation

The canonical example of this that you've likely seen in a physics class is a **Wick rotation** in which time t is replaced with imaginary time τ via:

$$t \mapsto i\tau \tag{17}$$

This exact example occurs as well in the correspondence between statistical mechanics and quantum mechanics, as the replacement $t \mapsto i\beta$ maps time evolution to the statistical Boltzmann factor:

$$e^{iHt} \mapsto e^{-\beta E} \tag{18}$$

where we identify H with E.

In Minkowski spacetime, the advantage of a Wick rotation is apparent when one considers the metric:

$$ds^2 = dt^2 - d\vec{x}^2 = (id\tau)^2 - d\vec{x}^2 = -(d\tau^2 + d\vec{x}^2) \tag{19}$$

After Wick rotation, the Minkowski metric becomes Euclidean. This means that if we have an integral over Minkowski spacetime, Wick rotation takes this into an ordinary Euclidean space integral and we can use all the formulas we know and love. Note the measure $d^4k = dk_t\, d\vec{k} = idk_\tau\, d\vec{k} = d^4k_E$ where k_E is the Euclidean momenta, and the invariant square $k^2 = k_t^2 - \vec{k}^2 = (ik_\tau)^2 - \vec{k}^2 = -k_E^2$, so for example:

$$\int \frac{d^4k}{k^2} = \int \frac{id^4k_E}{-k_E^2} = -\int d\Omega_3 \int dk_E\, \frac{k_E^3}{k_E^2} = -\Omega_3 \int dk_E\, k_E \tag{20}$$

where Ω_3 is the measure on the shell S^3 (this is in contrast to $d^3\vec{k} = k^2\, dk\, d\Omega_2 = d^2\, dk\, \sin\theta\, d\theta\, d\phi$ in 3 dimensional space, since k_E is a four dimensional Euclidean vector).

We are discussing Wick rotation because when asked when it works, the usual response by a physicist is to simply say "analytic continuation!" and move on. In fact, analytic continuation is a rather small part of why the Wick rotation works.