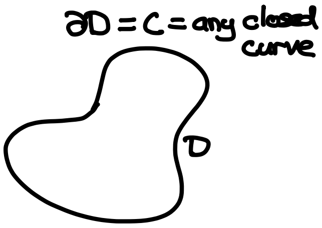
Contour Integration

- Consider a function f, holomorphic on D, w/ $C = \partial D$.
  - Cauchy's theorem says:

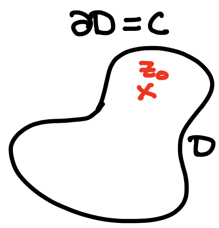
$$\oint_C dz\, f(z) = 0$$

  where C is any closed curve.



  - Now consider $g(z) := \dfrac{f(z)}{z - z_0}$.
    i) If $z_0 \notin D$, g is holomorphic on D, so $\oint_C dz\, g(z) = 0$.
    ii) $z_0 \in D$ is the interesting case. Then:

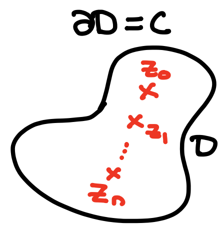
$$\oint_C dz\, \frac{f(z)}{z - z_0} = 2\pi i f(z_0) \equiv 2\pi i\, \mathrm{Res}_g(z_0)$$



This is the residue theorem. This generalizes if there are $n+1$ poles $z_0, \dots, z_n \in D$. For:

$$h(z) = \frac{f(z)}{(z - z_0)(z - z_1)\dots(z - z_n)}$$

Each pole has a residue



We can write:

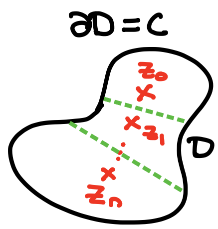
$$h(z) = \frac{1}{z - z_0} \underbrace{\mathrm{Res}_h(z_0)}_{\frac{f(z)}{(z-z_1)\dots(z-z_n)}} = \dots = \frac{1}{z_0 - z_i} \mathrm{Res}_h(z_i)$$

no sum on i

- The residue @ z_i is what you get when you block out the pole @ z_i and evaluate the rest @ $z = z_i$
- I can split up the domain of integration. This gives the full residue theorem:

$$\oint_C dz\, h(z) = 2\pi i \sum_j \mathrm{Res}_h(z_j)$$

sum on poles

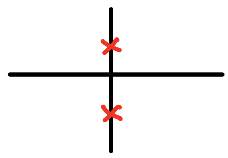


- Evaluate:

$$\mathcal{I} = \int_{\mathbb{R}} dx\, \overbrace{\frac{e^{ix}}{x^2 + 1}}^{f} = \int_{\mathbb{R}} dx\, \frac{e^{ix}}{(x+i)(x-i)}$$

Idea: write this out as a complex integral

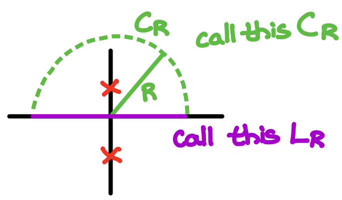
- Poles: $x = +i$, $\mathrm{Res} = \dfrac{e^{-1}}{2i}$; $x = -i$, $\mathrm{Res} = \dfrac{e}{}$



- "Close the contour": Consider the contribution from the arc C_R. We know:

$$\oint_{L_R \cup C_R} dz\, \frac{e^{iz}}{z^2 + 1} = 2\pi i\, \mathrm{Res}(i) = \frac{\pi}{e}$$

from the residue thm (ind. of R). As $R \to \infty$:



$$\lim_{R\to\infty} \oint_{L_R \cup C_R} dz\, f(z) = \int_{\mathbb{R}} dz\, f(z) + \lim_{R\to\infty} \int_{C_R} dz\, f(z) = \frac{\pi}{e}$$

- Now the catch: the second integral is exp. suppressed and vanishes:

$$\lim_{R\to\infty} \int_{C_R} dz\, f(z) \sim \lim_{R\to\infty} (\pi R) \frac{e^{-R}}{R^2} = 0$$

- This is why we closed the contour up! Closing down $\to \int_{C_R}$ doesn't vanish.
- Thus:

$$\mathcal{I} = \frac{\pi}{e}$$

names for this: "Closing / "Pushing / "Deforming the contour"

- Steps for this technique for an $\int_{\mathbb{R}}$:
  ① Identify the poles and residues
  ② Identify if closing up or down will give a contribution which $\to 0$
  ③ Close in the corresponding direction and use the residue theorem.

• We will use this to do integrals of the form:

$$\Pi(p) = -i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip\cdot(x-y)}}{p^2+m^2-i\epsilon}\,, \quad \epsilon \text{ infinitesimal}$$

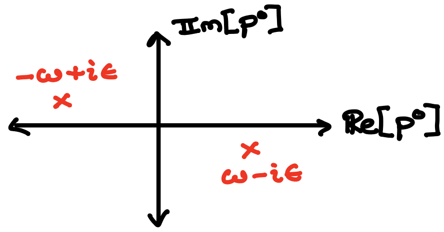
Expand as $p^2+m^2-i\epsilon = -(p^0)^2+\vec{p}^{\,2}+m^2-i\epsilon = -(p^0)^2+\omega_{\vec{p}}^2-i\epsilon = (p^0+\omega_{\vec{p}}-i\epsilon)(-p^0+\omega_{\vec{p}}-i\epsilon)$, so:

really $\tilde{\epsilon}$, where $\epsilon = 2\omega$

$$\Pi(p) = -i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip^0(x^0-y^0)}\,e^{i\vec{p}(\vec{x}-\vec{y})}}{(p^0+\omega_{\vec{p}}-i\epsilon)(-p^0+\omega_{\vec{p}}-i\epsilon)}$$

$$= -i\int \frac{d^3\vec{p}}{(2\pi)^3}\,e^{i\vec{p}(\vec{x}-\vec{y})}\underbrace{\int_{\mathbb{R}} \frac{dp^0}{2\pi}\,\frac{e^{-ip^0(x^0-y^0)}}{(p^0+\omega_{\vec{p}}-i\epsilon)(-p^0+\omega_{\vec{p}}-i\epsilon)}}_{\mathcal{I}}$$

	1	2
Poles	$p^0 = -\omega_{\vec{p}}+i\epsilon$	$p^0 = \omega_{\vec{p}}-i\epsilon$
Res	$\frac{1}{2\pi}\,\frac{e^{i\omega_{\vec{p}}(x^0-y^0)}}{2\omega_{\vec{p}}}$	$\frac{1}{2\pi}\,\frac{e^{-i\omega_{\vec{p}}(x^0-y^0)}}{2\omega_{\vec{p}}}$

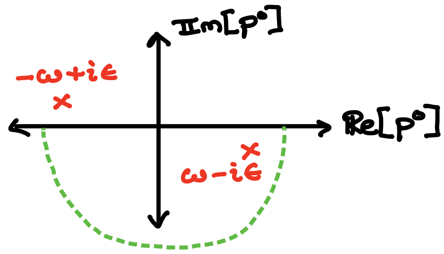


– Close up or close down? We want exp. decay in $e^{-ip^0(x^0-y^0)}$

$\boxed{x^0>y^0}$

x^0-y^0 is positive $\Rightarrow$ close down. This gives:

$$\mathcal{I} = (2\pi i)\frac{1}{2\pi}\,\frac{e^{-i\omega_{\vec{p}}(x^0-y^0)}}{2\omega_{\vec{p}}} = \frac{i}{2\omega_{\vec{p}}}\,e^{-i\omega_{\vec{p}}(x^0-y^0)}$$



$$\Rightarrow \Pi(p) = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,e^{ip\cdot(x-y)}\Big|_{p^0=\omega_{\vec{p}}}$$

$\boxed{x^0<y^0}$

x^0-y^0 is negative, so we close down.

$$\mathcal{I} = \ldots = \frac{i}{2\omega_{\vec{p}}}\,e^{i\omega_{\vec{p}}(x^0-y^0)}$$

$$\Rightarrow \Pi(p) = \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,e^{-i\omega_{\vec{p}}(y^0-x^0)}\,e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$= (\vec{p}\mapsto-\vec{p})\int \ldots e^{-i\omega_{\vec{p}}(y^0-x^0)}\,e^{i\vec{p}(\vec{y}-\vec{x})}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,e^{ip\cdot(y-x)}\Big|_{p^0=\omega_{\vec{p}}}$$

Putting them together:

$$\Pi(p) = \begin{cases} \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,e^{ip\cdot(x-y)}\Big|_{p^0=\omega_{\vec{p}}} & x^0>y^0 \\ \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,e^{ip\cdot(y-x)}\Big|_{p^0=\omega_{\vec{p}}} & y^0>x^0 \end{cases}$$

You'll show in class this equals the time-ordered 2-point function $\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle$.

Contour integration

See handwritten notes.

Symmetries

- A **Lie group** G is a manifold with a group structure. A **Lie algebra** $\mathfrak{g}$ is a vector space equipped with a Lie bracket $[\cdot,\cdot]$ (a commutator). The Lie algebra $\mathfrak{g}$ implements infinitesimal symmetry transformations, and the Lie group G implements finite symmetry transformations.
- The **exponential map** relates elements of the Lie algebra to the Lie group. If I have an element $A \in \mathfrak{g}$ of the Lie algebra, I can get a corresponding element of the Lie group by:

$$g = e^{iA} \in G \tag{1}$$

  The nice part about this is that the Lie algebra *parameterizes* the Lie group. Since $\mathfrak{g}$ is a vector space, we can write out $A = A^a t^a$ where $\{t^a\}$ is a basis for the Lie algebra. Then if I specify the coordinates A^a (just an n-tuple of numbers), I can write down any symmetry operator I want.
- Ex: Rotations in quantum mechanics. For a spin 1/2 system, rotations are implemented with the exponential of the angular momentum. The rotation operator is:

$$U_{1/2}(\hat{n},\theta) = \exp(-i\theta\hat{n}\cdot\boldsymbol{\sigma}/2) \tag{2}$$

  were $\boldsymbol{\sigma}$ is the vector of Pauli matrices, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Here the Lie algebra is $\mathfrak{su}(2)$, and spanned by the Pauli matrices– a basis for $\mathfrak{su}(2)$ is $\{\sigma_x, \sigma_y, \sigma_z\}$, and as such we can write *any* element of $\mathfrak{su}(2)$ as $-\theta\hat{n}\cdot\boldsymbol{\sigma}/2$.
- Ex: Lorentz group. The Lorentz algebra $\mathfrak{so}(1,3)$ is spanned by the tensor $\mathcal{J}^{\mu\nu}$, or equivalently the boost or rotation generators K_i, J_i. We saw in the first recitation that we can write down any Lorentz transformation as:

$$\Lambda = \exp\left(\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \tag{3}$$

  where $\omega_{\mu\nu}$ contains 6 independent real parameters (boost and rotation angles) that parameterize the Lie algebra. In the context of what we're doing here, we see that $\mathfrak{so}(1,3)$ is spanned by $\{\mathcal{J}^{01}, \mathcal{J}^{02}, \mathcal{J}^{03}, \mathcal{J}^{12}, \mathcal{J}^{13}, \mathcal{J}^{23}\}$, and so any element of the Lie algebra is written as a linear combination of these $\mathcal{J}^{\mu\nu}$, i.e. $\omega_{\mu\nu}\mathcal{J}^{\mu\nu}$.
- Another example: Conserved charges H and P^i in QFT. These are **generators** of time and spatial translation, respectively. They generate the Lie algebra for translation, and an arbitrary element of the algebra can be written as $Ht - P^i x^i = -P_\mu x^\mu$. We exponentiate this to get finite symmetry transformations:

$$U_x = \exp\left(iHt - iP^i x^i\right) = e^{-iP_\mu x^\mu} \tag{4}$$

- A Lie group (algebra) is typically pretty abstractly defined. To act it on a vector space, we need a **representation** of the group or algebra. For a vector space V, a representation of G is a map $D : G \to GL(V)$, where $GL(V)$ is the space of invertible linear transformations on V. Likewise, a representation of $\mathfrak{g}$ is a map $d : \mathfrak{g} \to gl(V)$, where $gl(V)$ is the space of *all* linear transformations on V. All a representation does is it gives us a way to act a symmetry group on a vector space.

- Given a representation of the algebra, we can *induce* a representation of the group by:

$$D(g) = e^{id(A)} \tag{5}$$

where $g = e^{iA}$ as in Eq. (1).

- Let's consider our Fock space:

$$H = |0\rangle \oplus \{a^\dagger_{\vec{k}}|0\rangle : \vec{k} \in \mathbb{R}^3\} \oplus \{a^\dagger_{\vec{k}} a^\dagger_{\vec{k}'}|0\rangle : \vec{k}, \vec{k}' \in \mathbb{R}^3\} \oplus ... \tag{6}$$

To act a Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)$ on an element $|\psi\rangle \in H$, we need a representation of $SO(1,3)$ on H. To specify the representation, we can simply show us where it sends the generators to, i.e. we can specify $d(\mathcal{J}^{\mu\nu})$, since:

$$U_\Lambda = D(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\, d(\mathcal{J}^{\mu\nu})\right). \tag{7}$$

This equation is important! Fields of different spin in QFT all correspond to different representations $d(\mathcal{J}^{\mu\nu})$.

- Conserved charges $M_{\mu\nu}$: The representation that we use to act symmetries on our Fock space H is:

$$d_{\text{Fock}}(J^{\mu\nu}) = M^{\mu\nu} = -\frac{i}{2}\int \frac{d^3\vec{k}}{(2\pi)^3} k^\mu \left(a^\dagger_{\vec{k}} \partial_{k_\nu} a_{\vec{k}} - (\partial_{k_\nu} a^\dagger_{\vec{k}}) a_{\vec{k}}\right) - (\mu \leftrightarrow \nu) \tag{8}$$

This means that we want to act a Lorentz transformation on a state, let's say a one-particle state $|\vec{k}\rangle$ for concreteness, we just need to act U_Λ on it:

$$|\vec{k}\rangle \mapsto U_\Lambda|\vec{k}\rangle = \exp\left(\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right)|\vec{k}\rangle. \tag{9}$$

- Invariance of a state under symmetries. For a state $|\Omega\rangle$ which is **conserved** under a symmetry U, we require that it is unchanged by the symmetry, $U|\Omega\rangle = |\Omega\rangle$. In the language we've been using, since $U = e^{id(A)}$ for generators A, invariance of $|\Omega\rangle$ implies that **generators map the state to 0**. For example, the vacuum in QFT is invariant under time-translation, spatial translation, and Lorentz transformations. This implies that:

$$H|0\rangle = P^i|0\rangle = M^{\mu\nu}|0\rangle = 0 \tag{10}$$

Note that $|0\rangle$ is the vacuum state, and 0 is the zero vector– these are different things!

Complex fields

- Complex fields: For a scalar field theory with complex fields, ϕ and ϕ^* are now independent degrees of freedoms. They can both be expanded with two types of creation and annihilation operators:

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} (a_{\vec{k}} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} + b^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}}) \tag{11}$$

$$\phi^*(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} (b_{\vec{k}} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}}). \tag{12}$$

  Here $a_{\vec{k}}, a^\dagger_{\vec{k}}$ create and destroy *particles*, and $b_{\vec{k}}, b^\dagger_{\vec{k}}$ create and destroy *anti-particles*. When ϕ was a real field, the reality constraint $\phi = \phi^*$ removed the second set of creation / annihilation operators from the equation, but when ϕ is complex you must include the second set of operators $b_{\vec{k}}$. Note that:

$$[a_{\vec{k}}, b_{\vec{k}}] = [a^\dagger_{\vec{k}}, b_{\vec{k}}] = [a_{\vec{k}}, b^\dagger_{\vec{k}}] = 0. \tag{13}$$

- Physical intuition: $a^\dagger_{\vec{k}}|0\rangle$ creates a particle from the vacuum with momentum $\vec{k}$. How do you create a particle localized at position $\vec{x}$? Let's see what $\phi(\vec{x}, t)$ does when it acts on the vacuum:

$$\phi(\vec{x}, 0)|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} e^{-i\vec{k}\cdot\vec{x}} b^\dagger_{\vec{k}}|0\rangle = \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{-i\vec{k}\cdot\vec{x}} |\vec{k}\rangle_b}_{\text{Antiparticle wave packet at } \vec{x}} \tag{14}$$

  So, this creates an antiparticle with momentum $\vec{k}$. If we want to create a particle, we instead need to act with ϕ^*:

$$\phi^*(\vec{x}, 0)|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{-i\vec{k}\cdot\vec{x}} |\vec{k}\rangle_a \tag{15}$$