Patrick Oare | Physics 137B Midterm | Cheat Sheet

• Exchange force: $\langle (x_1-x_2)^2\rangle_\pm = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b \mp 2|\langle x\rangle_{ab}|^2$, $\langle x\rangle_{ab} := \int_{\mathbb{R}} \psi_a^*(x)\hat{x}\psi_b(x)dx$

– Bosons group together, fermions pull apart (but not always if we take spin into account)

• Ex: Total (space + spin) ψ must (anti) symmetrize

Identical Particles:

• Symmetrize (bosons, integer spin) or antisymmetrize (fermions, half int) $|\psi\rangle$:

$$\psi_{ab}(\vec{x}_1,\vec{x}_2) = \frac{1}{\sqrt{2}}\left[\psi_a(\vec{x}_1)\psi_b(\vec{x}_2) \pm \psi_a(\vec{x}_2)\psi_b(\vec{x}_1)\right]$$

• COM coordinates: For $\vec{r} := \vec{x}_1 - \vec{x}_2$, $\vec{R} := \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1+m_2}$, $V = V(\vec{r})$:

$E_r\psi_r(\vec{r}) = -\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r(\vec{r}) + V(\vec{r})\psi_r(\vec{r})$

$E_R\psi_R(\vec{R}) = -\frac{\hbar^2}{m_1+m_2}\nabla_R^2\psi_R(\vec{R})$

• Pauli Exclusion: Two fermions cannot occupy the same state (can be in same degenerate space though)

• Para/orthohelium:
- Para → Singlet → Sym space → attraction → higher energy
- Ortho → Triplet → Antisym space → repulsion → lower E

• Hydrogen: $|\psi\rangle = |\text{space}\rangle \otimes |\text{spin}\rangle$

• Singlet $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ antisym spin, so 2 e^-s will have sym spacial ψ in singlet ⇒ attracted

• Triplet $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\}$ for 2 e^-s has sym. spin ⇒ antisym space ⇒ repulsion

• This means two H atoms only bond in the singlet state

• Spin Symbols: $L = 0, 1, 2, 3, \ldots$ is $S, P, D, F, \ldots$ and we write $^{2S+1}L_J$

• Singlet has $j=0$, triplet has $j=1$

Perturbation Theory:

• Original Hamiltonian $\hat{\mathcal{H}}^0$, perturbation $\hat{V}$

• Unperturbed $\{|\psi_n^0\rangle\}$, $\{E_n^0\}$:

$$E_n^1 = \langle \hat{V}\rangle = \langle \psi_n^0|\hat{V}|\psi_n^0\rangle$$

$$|\psi_n^1\rangle = \sum_{m\neq n}\frac{|\psi_m^0\rangle\langle\psi_m^0|\hat{V}|\psi_n^0\rangle}{E_n^0 - E_m^0}$$

$$E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|\hat{V}|\psi_n^0\rangle|^2}{E_n^0 - E_m^0}$$

• Stark effect: $\hat{V} = -eE\hat{z}$, $E_n^1 = 0$, $E_n^2 = -\frac{9}{2}\varepsilon_0 a_0^3 E^2$

• For a system in the ground state, $E_{n=0}^2 < 0$ b/c $E_{n=0}^0 = E_{min}^0$

• Feynman Hellman Theorem:

$$\frac{\partial E_n}{\partial\lambda} = \left\langle \psi_n \left| \frac{\partial\hat{\mathcal{H}}}{\partial\lambda}\right|\psi_n\right\rangle$$

• Degenerate: Let $\{|\psi_\alpha\rangle\}$ s.t. $\hat{\mathcal{H}}^0|\psi_\alpha\rangle = E^0|\psi_\alpha\rangle$

• Find $[\hat{V}]_\beta$, $\beta := \{|\psi_\alpha\rangle\}$
- Eigenvalues are energy corrections
- Eigenvectors are good states

• Lifting the degeneracy: when $\hat{V}$ is applied, degenerate eigenspace will break up b/c only "good states" stay eigenstates

• Find "good states" in degenerate space that diagonalize $\hat{V}$. For $|n\rangle := \sum_\alpha c_\alpha|\psi_\alpha\rangle$, need $\langle m|\hat{V}|n\rangle = 0$ if $n\neq m$. Then use nondegenerate on $\{|n\rangle\}$

• Theorem: Let $\hat{A}$ be Hermitian. If $[\hat{A},\hat{V}] = 0$ and $\hat{A}|\psi_a\rangle = \alpha_a|\psi_a\rangle$ w/ $\alpha_a \neq \alpha_b$ (distinct eigenvalues), then nondegenerate may be used
- Also need $[\hat{A},\hat{\mathcal{H}}^0] = 0$ so we can form a simultaneous eigenbasis

Hydrogen:

$\hat{\mathcal{H}}^0 = \frac{\hat{p}^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}$ $\quad a_0 := \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}$ $\quad \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$

• Fine Structure: $\ell \le n-1$

$\hat{V}_r = -\frac{\hat{p}^4}{8m^3c^2}$, $[\hat{L}^2,\hat{V}_r] = 0$ and $\langle n\ell m'|\hat{V}_r|n\ell m\rangle = 0$ if $m\neq m'$, so we use the uncoupled basis

$\hat{V}_{LS} = -\vec{\mu}_e\cdot\vec{B} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{m^2c^2r^3}\hat{\vec{S}}\cdot\hat{\vec{L}}$ • $[\hat{J}^2,\hat{V}_{LS}] = [\hat{J}_z,\hat{V}_{LS}] = 0$ so I use coupled basis

$E_r^1 = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{\ell+1/2} - 3\right]$ $\quad E_{LS}^1 = \frac{(E_n)^2 n}{mc^2}\,\frac{j(j+1) - \ell(\ell+1) - s(s+1)}{\ell(\ell+1)(\ell+1/2)}$, $s(s+1) = 3/4$

$$E_{fs}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

$n=2$: $j = 3/2$, $j = 1/2$; $n=1$: $j=1/2$ • Lifts degeneracy in j!

Hyperfine Structure: Spin-Spin coupling, p^+ spin is $\vec{I}$, use $\ell = 0$, $g_p = 5.581$

$\hat{V}_{HF} = -\vec{\mu}_e\cdot\vec{B} = \frac{\mu_0 g_p e^2}{8\pi m_p m_e}\frac{3(\vec{I}\cdot\hat{r})(\vec{S}\cdot\hat{r}) - \vec{I}\cdot\vec{S}}{r^3} + \frac{\mu_0 g_p e^2}{3m_p m_e}\vec{I}\cdot\vec{S}\,\delta^3(\vec{r})$

– w/ $\ell = 0$, first term $= 0$, and $\langle\delta^3(\vec{r})\rangle = |\psi(\vec{0})|^2$

$$E_{HF}^1 = \frac{4g_p\hbar^2}{3m_p m_e^2c^2a_0^4}\begin{cases}1/4 & \text{triplet}\\ -3/4 & \text{singlet}\end{cases}$$

(triplet/singlet refers to p^+ and e^- spins)

$\lambda = 21\text{cm}$

$E_n^0 = \frac{-13.6\text{eV}}{n^2}$

• Zeeman effect: $\hat{V}_Z = \frac{e}{2m}(\vec{J}+\vec{S})\cdot\vec{B} = \frac{eB_z}{2m}(\hat{J}_z + \hat{S}_z)$
- Use $\vec{B} = B_z\hat{z}$ to use z components

• Weak field: $\hat{V}_Z$ perturbation on fine structure, since $[\hat{J}_z,\hat{V}_Z] = 0$ (no degeneracy in j anymore) use $|n\ell s j m_j\rangle$
- Sketchy $\langle S_z\rangle = \frac{\vec{S}\cdot\vec{J}}{J^2}\vec{J}$ thing to get $\langle S\rangle$

$$E_{WFZ}^1 = \mu_B g_J B_{ext} m_j$$ (lifts m_j degeneracy)

• Strong field: fs perturbation on Zeeman $[\hat{L}^2,\hat{V}_Z] = [\hat{L}_z,\hat{V}_Z] = 0$ (b/c no fs, still have j degeneracy and $[\hat{J}^2,\hat{V}_Z]\neq 0$) use $|n\ell s m_\ell m_s\rangle$

$$E_{SFZ}^1 = \mu_B B_{ext}(m_\ell + 2m_s)$$

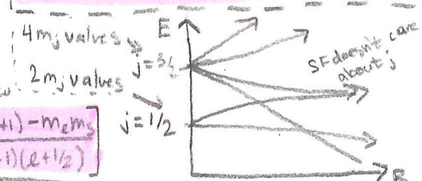
- Use $|n\ell s m_\ell m_s\rangle$ for fs b/c $[\hat{L}^2,\hat{V}_{fs}] = [\hat{S}^2,\hat{V}_{fs}] = 0$ and no more m_ℓ, m_s degeneracy
- E_r^1 stays same, E_{LS}^1 changes b/c $\langle\vec{S}\cdot\vec{L}\rangle = \hbar^2 m_\ell m_s$ now
- After Strong Field: $E_{z,fs}^1 = E_n\alpha^2\left[\frac{3}{4n} - \frac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1)(\ell+1/2)}\right]$

$\left\langle\frac{1}{r}\right\rangle = \frac{1}{n^2a_0}$ $\quad \left\langle\frac{1}{r^2}\right\rangle = \frac{1}{(\ell+1/2)n^3a_0^2}$

$\left\langle\frac{1}{r^3}\right\rangle = \frac{1}{\ell(\ell+1/2)(\ell+1)n^3a_0^3}$

$\mu_B := \frac{\hbar e}{2m_e}$ $\quad g_J = \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}$

$$\hat{\vec{S}}\cdot\hat{\vec{L}} = \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$$



Miscellaneous:

$[\hat{a},\hat{a}^\dagger] = \hat{I}$

• SHO: $E_n = \hbar\omega(n+1/2)$, $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$ eigenbasis $\{|n\rangle\}$, $n\ge 0$ $\quad \langle n|\hat{V}|n\rangle = \frac{1}{2}E_n$

$\hat{a} := \frac{1}{\sqrt{2\hbar m\omega}}(\hat{x} + i\hat{p})$ $\quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(\hat{x} - i\hat{p})$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ $\quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger) \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

$[\hat{L}_z,\vec{r}] = i\hbar\hat{z}\times\vec{r} \neq 0$, $[\hat{L}_z, r] = 0$,

$[\hat{L}^2,\hat{r}^2] = [\hat{L}^2,\hat{p}^2] = 0 = [\hat{L}^2, r]$

– Systems w/ only a radial DoF will preserve angular momentum, so for $V(r)$ you can use $|n\ell m\rangle$ $\quad [\hat{L}_i, r] = 0$

$[\hat{J}^2,\vec{S}\cdot\vec{L}] = [\hat{J}_z,\vec{S}\cdot\vec{L}] = 0$

$[\hat{L}_z,\hat{z}] = [\hat{L}_z,\hat{p}_z] = [\hat{L}_z,\hat{r}^2] = [\hat{L}_z,\hat{p}^2] = 0$

• If $V = V(r)$, then all components of $\vec{L}$ commute w/ the Hamiltonian

Norm. gaussian $= \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$

$R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{3/2}e^{-Zr/a_0}$

$\vec{\mu} = \gamma\vec{S}$

$V_{eff}(r) = \begin{cases}\infty & r<0\\ V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2} & r\ge 0\end{cases}$ $\quad u(r) := rR(r)$

$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$ $\quad \hat{L}^2|\ell m\rangle = \hbar^2\ell(\ell+1)|\ell m\rangle$

$\hat{L}_z|\ell m\rangle = \hbar m|\ell m\rangle$

$m_j = m_\ell + m_s$ $\quad |\ell - s| \le j \le \ell + s$

$\ell\times s$	j / m_j
$m_\ell\ m_s$	

(block for each m_j)
- Sqrt each entry
- If $s > \ell$, replace them
- $m_\ell + m_s$ always constant in each block

$\hat{S}_i \doteq \frac{\hbar}{2}\sigma_i$

• If an operator $\hat{V}$ commutes w/ another operator w/ eigenket $|a\rangle$, $\hat{V}$ leaves the quantum number a unchanged
- Ex: If $[\hat{V},\hat{L}_z] = 0$, $\hat{V}|n\ell m\rangle = \sum_{n'\ell'}c_{n'\ell'}|n'\ell' m\rangle$, so m is undisturbed

Don't forget spin when finding degeneracy of uncoupled/coupled basis
- Ex: for $\ell = 1$, $s = 1/2$, eigenspace is $\text{span}\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}\otimes\text{span}\{|1/2,1/2\rangle, |1/2,-1/2\rangle\}$ has dim 6 (with $\ell\ m_\ell$ labelling the first factor)
- But the $n=2$ level has 8 fold degeneracy b/c also has $\ell = 0$

$$\int\sqrt{A - Bx^2}\,dx = \frac{1}{2}x\sqrt{A - Bx^2} + \frac{A}{2\sqrt{B}}\arctan\left(\frac{\sqrt{B}x}{\sqrt{A+Bx^2}}\right)$$

• Angular Momentum/Spin: $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$

$[\hat{L}_i,\hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k$ $\quad [\hat{L}^2,\hat{L}_i] = 0$ $\quad \hat{L}^2 = -\hbar^2\nabla^2$

$\hat{L}_\pm := \hat{L}_x \pm i\hat{L}_y$ $\quad \hat{L}_\pm|\ell m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\,|\ell, m\pm 1\rangle$

$[\hat{L}_+,\hat{L}_-] = 2\hbar\hat{L}_z$ $\quad \hat{L}^2 = \hat{L}_+\hat{L}_- - \hbar\hat{L}_z + \hat{L}_z^2$

$m_\ell \in \mathbb{Z}$, but not always m_s

• Mutually commuting sets:

$\{\hat{L}^2, \hat{S}^2, \hat{J}^2, \hat{J}_z\} \to |n\ell s j m_j\rangle$

$\{\hat{L}^2, \hat{S}^2, \hat{L}_z, \hat{S}_z\} \to |n\ell s m_\ell m_s\rangle$

• Pauli matrices:

$\{\sigma_i,\sigma_j\} = 2\hat{I}\delta_{ij}$

$[\sigma_i,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k$

$d\Omega = \sin\theta\, d\theta\, d\varphi$ $\quad \theta\in[0,\pi]$, $\varphi\in[0,2\pi]$

$d^3x = r^2 dr\, d\Omega = r^2\sin\theta\, dr\, d\theta\, d\varphi$

$$\iint_{\text{Sphere}} Y_\ell^m(\theta,\varphi)^* Y_{\ell'}^{m'}(\theta,\varphi)\,d\Omega = \delta_{m'}^m\delta_{\ell'}^\ell$$

$Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$ $\quad W_{ij} = \langle\psi_i^0|\hat{V}|\psi_j^0\rangle$

$$\sum_{m=-\ell}^{\ell}|Y_\ell^m|^2 = \frac{2\ell+1}{4\pi}$$

$$\vec{L}\cdot\vec{S} = \frac{1}{2}(\hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+) + \hat{L}_z\hat{S}_z$$

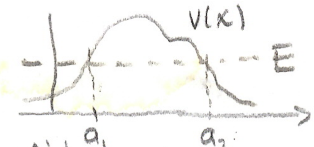
$$\frac{d\langle\hat{Q}\rangle}{dt} = \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle + \left\langle\frac{i}{\hbar}[\hat{\mathcal{H}},\hat{Q}]\right\rangle$$

Variational and WKB Methods

- Variational: For any state $|\psi\rangle$:
  $\langle\psi|\hat{\mathcal{H}}|\psi\rangle \geq E_{gs}$
  - If $\langle\psi|\hat{\mathcal{H}}|\psi\rangle = E_{gs}$, $|\psi_{ground}\rangle = |\psi\rangle$
- Guesses should be normalizable, continuous, odd/even, correct asymptotic behavior
- Method: Take $\psi_T(x)$ and vary a parameter
  - Normalize $\psi_T(x)$
  - Calculate $\langle\psi_T|\hat{\mathcal{H}}|\psi_T\rangle =: f(a)$
  - Take $\frac{\partial f}{\partial a} = 0$ (or $\nabla f = \vec{0}$) for a_0
  - $f(a_0)$ is your approx for E_{GS}

- WKB: Slowly varying ψ:
  $\psi(x) \sim A(x) e^{\pm i\varphi(x)}$
- Assume debroglie $\lambda << 1$, p large, $A''/A << (\varphi')^2$ and $A''/A << p^2/\hbar^2$ $(A'' \to 0)$ | Semiclassical: best for large E
- Classical region $(E > V(x))$:
  $p(x) = \sqrt{2m(E - V(x))}$
- Tunneling $(E < V(x))$:
  $|p(x)| = \sqrt{2m(V(x) - E)}$
- The wavefunction in one region is:
  $\psi(x) \sim \frac{C}{\sqrt{p(x)}} e^{\pm\frac{i}{\hbar}\int p(x)dx}$
- In tunneling region, no i in exp

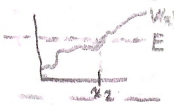
- Tunneling: When $E < V(x)$:
  $\psi(x) \sim \frac{C}{\sqrt{|p(x)|}} e^{\pm\frac{1}{\hbar}\int|p(x)|dx}$
  - No i in phase
- P of transmission is T:
  $T = e^{-2\gamma}$ $\quad \gamma := \frac{1}{\hbar}\int_{a_1}^{a_2}|p(x)|dx$



- Alpha decay:
  $\gamma \sim K_1 \frac{Z}{\sqrt{E}} - K_2\sqrt{Z r_1}$
  $K_1 \sim 2\sqrt{MeV}$, $K_2 \sim 1.5\, 1/\sqrt{fm}$

- Deriving Connection formulas: WKB blows up at turning points (when $E = V(x_2)$), so we linearize:
  $V_p(x) \sim E + V'(0)x \quad (x_2 = 0)$
  - Solve TISE w/ $V_p(x)$, define $\alpha := \left[\frac{2m}{\hbar^2}V'(0)\right]^{1/3}$, $z := \alpha x$, to get $\frac{d^2\psi_p}{dz^2} = z\psi_p$
  - Solns to this are $\{Ai(z), Bi(z)\}$
  - Fit B.C.'s ($Bi(z) \to \infty$ as $z \to \infty$) to get connection formulas for T.P. at x_2:

$$\psi(x) \sim \begin{cases} \frac{2D}{\sqrt{p(x)}}\sin\left[\frac{1}{\hbar}\int_x^{x_2}p(x')dx' + \frac{\pi}{4}\right] & x < x_2 \\ \frac{D}{\sqrt{p(x)}}\exp\left[\frac{-1}{\hbar}\int_{x_2}^{x}|p(x')|dx'\right] & x > x_2 \end{cases}$$



Time Dependent Perturbation Theory: Two state systems, $\{|a\rangle\, |b\rangle\}$

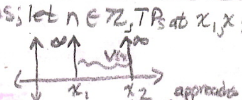
- Find solution: $|\psi\rangle = c_a(t)e^{-iE_at/\hbar}|a\rangle + c_b(t)e^{-iE_bt/\hbar}|b\rangle$ when $\hat{\mathcal{H}} = \hat{\mathcal{H}}^0 + \hat{\mathcal{H}}'(t)$
- Generate eqns:

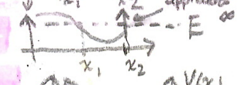
$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar}\mathcal{H}'_{ab}e^{-i\omega_0 t}c_b \\ \dot{c}_b = -\frac{i}{\hbar}\mathcal{H}'_{ba}e^{i\omega_0 t}c_a \end{cases} \qquad \omega_0 := \frac{E_b - E_a}{\hbar} \qquad \mathcal{H}'_{ij} = \langle i|\hat{\mathcal{H}}'(t)|j\rangle$$

- Can solve exactly when $\mathcal{H}'_{ab} \propto e^{i\omega t}$
- We assume $\mathcal{H}'_{aa} = \mathcal{H}'_{bb} = 0$

- 0^{th} order approximation: $c_a^{(0)}(0) = 1$, $c_b^{(0)}(0) = 0$
- 1^{st} order approximation: Plug in $c_a^{(0)}$ and $c_b^{(0)}$ to get $\dot{c}_a^{(1)} = 0 \Rightarrow c_a^{(1)}(t) = 1$, and:
  $\dot{c}_b^{(1)}(t) = -\frac{i}{\hbar}\mathcal{H}'_{ba}e^{i\omega_0 t}$ ($c_a^{(0)} = 1$) $\Rightarrow c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t \mathcal{H}'_{ba}(t')e^{i\omega_0 t'}dt'$
  - Now, the state is no longer normalized → this approx only works when $|c_b(t)| << 1$.
- 2^{nd} order approximation: $c_a^{(1)}(t) = 1 \Rightarrow c_b^{(2)}(t) = c_b^{(1)}(t)$, and:
  $\dot{c}_a^{(2)} = -\frac{i}{\hbar}\mathcal{H}'_{ab}e^{-i\omega_0 t}c_b^{(1)}(t) \Rightarrow c_a^{(2)}(t) = 1 - \frac{1}{\hbar^2}\int_0^t \mathcal{H}'_{ab}(t')e^{-i\omega_0 t'}\left[\int_0^{t'}\mathcal{H}'_{ba}(t'')e^{i\omega_0 t''}dt''\right]dt'$

- Potential Wells: Different Quantization conditions; let $n \in \mathbb{Z}$, TPs at x_1, x_2:
  - Two infinite walls: $\frac{1}{\hbar}\int_{x_1}^{x_2}p(x)dx = n\pi$
  - One infinite wall: $\frac{1}{\hbar}\int_{x_1}^{x_2}p(x)dx + \frac{\pi}{4} = n\pi$
  - No infinite walls: $\frac{1}{\hbar}\int_{x_1}^{x_2}p(x)dx + \frac{\pi}{2} = n\pi$





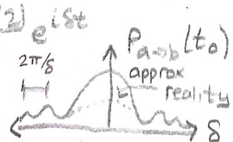
$\int = (n - \frac{1}{2})\pi\hbar$

  - Requires interior ψ's to be identical: From left and right, have different ψ's:
  $\psi_R(x) \sim \frac{2D}{\sqrt{p}}\sin(\theta_2(x))$, $\theta_2(x) := \frac{1}{\hbar}\int_x^{x_2}p(x')dx' + \frac{\pi}{4}$
  $\psi_L(x) \sim -\frac{2D}{\sqrt{p}}\sin(\theta_1(x))$, $\theta_1(x) := \frac{1}{\hbar}\int_{x_1}^{x}p(x')dx' + \frac{\pi}{4}$
  - $\theta_1(x) \equiv \theta_2(x) \mod \pi$ to be equal ($\psi_R = \pm\psi_L$)

POOR

- Transition probability is $P_{a\to b}(t) = |c_b(t)|^2$
  - w/ sinusoidal driving, oscillates
  - $P_{a\to b}(t) = P_{b\to a}(t)$ if we start w/ $c_b(0) = 1, c_a(0) = 0$

- Sinusoidal Driving: $\mathcal{H}'_{ab}(t) = V_{ab}\cos(\omega t)$
  $c_b(t) = -\frac{i}{\hbar}V_{ba}\int_0^t \cos(\omega t')e^{i\omega_0 t'}dt' = -\frac{i}{\hbar}\frac{\sin(\delta t/2)}{\delta}e^{i\delta t}$
  $\delta := \omega_0 - \omega$ $\qquad P_{a\to b}(t) = \frac{|V_{ba}|^2}{\hbar^2}\frac{\sin^2(\delta t/2)}{\delta^2}$



  - w/ $\delta \to 0$, $P_{a\to b}$ becomes large and we can't trust the approximation.

- Spontaneous Emission: $\dot{N}_b = -AN_b - N_bB_{ba}\rho(\omega_0) + N_aB_{ab}\rho(\omega_0)$
  - In eq, $\dot{N}_b = 0$, $\rho(\omega_0) = \frac{A}{\frac{N_a}{N_b}B_{ab} - B_{ba}}$ $\quad$ • $\frac{N_a}{N_b} = e^{\hbar\omega_0/k_bT}$ $\quad$ • $\rho(\omega) = \frac{\hbar}{\pi^2c^3}\frac{\omega^3}{e^{\hbar\omega/k_bT} - 1}$
  - Conclude $B_{ab} = B_{ba}$, and confirmed A → $A = \frac{\omega_0^3|d|^2}{3\pi\epsilon_0\hbar c^3}$
- Larger energy splits have larger ω_0 → A is greater → fast decay

- Radiation: $\vec{E} = \vec{E}_0\cos(\omega t)$ $\quad \hat{\mathcal{H}}'(t) = -qE_0\hat{z}\cos(\omega t)$
  - $\langle z\rangle = 0 \Rightarrow \mathcal{H}'_{aa} = \mathcal{H}'_{bb} = 0$ $\quad d := q\langle b|\hat{z}|a\rangle$
  $\mathcal{H}'_{ba} = -dE_0\cos(\omega t)$ $\quad$ (plug into sinusoidal formula):
  $P_{b\to a} = \frac{2u}{\epsilon_0\hbar^2}|d|^2\frac{\sin^2(\delta t/2)}{\delta^2} = \frac{2|d|^2}{\epsilon_0\hbar^2}\int_0^\infty \rho(\omega)\frac{\sin^2(\delta t/2)}{\delta^2}d\omega$
  $\rho(\omega) := \frac{\partial u}{\partial\omega}$ (energy density at given ω)
  - Since $\sin^2(\delta t/2)/\delta^2$ is large at ω_0, the integral $\sim \rho(\omega_0)$ ($\int$ is π)
  $P_{b\to a}(t) = \frac{\pi|d|^2}{\epsilon_0\hbar^2}\rho(\omega_0)t$ $\qquad R_{b\to a}(t) := \frac{\pi|d|^2}{3\epsilon_0\hbar^2}\rho(\omega_0)$ ← 3 dimensions

- w/o assuming $\mathcal{H}'_{aa} = \mathcal{H}'_{bb} = 0$

$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar}\left[c_a\mathcal{H}'_{aa} + c_b\mathcal{H}'_{ab}e^{-i\omega_0 t}\right] \\ \dot{c}_b = -\frac{i}{\hbar}\left[c_b\mathcal{H}'_{bb} + c_a\mathcal{H}'_{ba}e^{i\omega_0 t}\right] \end{cases}$$

$\delta \to 0$, $P \to t^2/4$

E/M Stuff

Selection Rules

$\Delta\ell = \pm 1$, $\Delta m = \pm 1, 0$

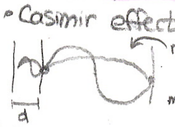
- $\Delta\ell = \ell - \ell'$, $\Delta m = m - m'$ for a transition between $|n\ell m\rangle$ and $|n'\ell' m'\rangle$

$\vec{A} = \sum_{k,r}\sqrt{\frac{\hbar c^2}{2V\omega_k}}\vec{\varepsilon}_r(\vec{k})\left[a_r e^{i(\vec{k}\cdot\vec{x} - \omega t)} + a_r^* e^{-i(\vec{k}\cdot\vec{x} - \omega t)}\right]$

$\vec{B} = \nabla\times\vec{A}$ $\quad \Box^2\vec{A} = \vec{0}$ $\quad \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$\hat{\mathcal{H}} = \sum_{k,r}\hbar\omega_k(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ $\quad \hat{N}_r(\vec{k}) = \hat{a}_r^\dagger\hat{a}$ $\quad \vec{k} = \frac{2\pi}{L}\langle \ell, m, n\rangle$

- Matrix elements of $\mathcal{H}$ can be nonzero when not in the state ⇒ background vacuum energy.
- Casimir effect



$E = f(d) = \sum_{modes}\hbar\omega_k(0 + \frac{1}{2})$

- Renormalization: Cutting off serves to force them to converge
- Eigenkets: $|\ldots, n_r(\vec{k}), \ldots\rangle$
  - $n_r(\vec{k})$ is # of photons w/ polarization r, wavevector $\vec{k}$

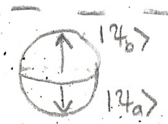
$H = \frac{1}{2}\int(|\vec{E}|^2 + |\vec{B}|^2)d^3x$

- Bloch Sphere: 2 level system

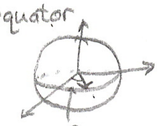
$E_a = -\frac{\hbar\omega_a}{2}$, $E_b = \frac{\hbar\omega_a}{2}$, $|\psi(t)\rangle = a(t)|\psi_a\rangle + b(t)|\psi_b\rangle$

$\mathcal{H}'_{ab} = \frac{\hbar\Omega_R}{2}$ $\qquad \dot{\vec{r}} = \vec{\Omega}\times\vec{r}$ $\Rightarrow |\vec{r}|$ const.

- $\vec{r}$ precesses about $\vec{\Omega}$ at rate $|\vec{\Omega}|$, and $|\vec{\Omega}| = \sqrt{\Omega_R^2 + \omega_a^2}$



- If $|a| = |b|$, $\vec{r}$ on equator
  - Ex: $a = 1$, $b = e^{i\varphi}$



$\vec{r} := \begin{pmatrix} ab^* + a^*b \\ i(ab^* - a^*b) \\ bb^* - aa^* \end{pmatrix}$ $\qquad \vec{\Omega} := \begin{pmatrix} -\mathrm{Re}(\Omega_R) \\ -\mathrm{Im}(\Omega_R) \\ -\omega_a \end{pmatrix}$

- Let $\Omega_R^{old} \mapsto e^{i\omega t}\Omega_R$ (sinusoidal driving) then: $\delta := \omega - \omega_0$
  $\vec{\Omega} = \begin{pmatrix} -\mathrm{Re}(\Omega_R) \\ -\mathrm{Im}(\Omega_R) \\ -\delta \end{pmatrix}$
- Real Ω_R ⇒ $\vec{\Omega}$ on x axis
- π pulse rotates $\vec{r}$ π radians, π/2 pulse rotates π/2
- $\Omega_R = 0 \Rightarrow \vec{\Omega} = \hat{z}$

Patrick Oare | Physics 137B Final Cheat Sheet

Adiabatic Approximation:

- **Theorem**: If $\hat{\mathcal{H}}(t)$ varies slowly:

$$\psi_n \mapsto e^{i\theta_n(t)} e^{i\gamma_n(t)} \psi_n$$

- θ_n dynamic phase, γ_n Berry phase

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt'$$

$$\gamma_n(t) := i\int_0^t \langle \psi_n(t') | \frac{\partial \psi_n}{\partial t'}(t')\rangle dt'$$

- Berry Phase is 0 if: ← doesn't matter if no cycle
  - i) ψ_n is real (up to a phase)
    $\frac{d}{dt}\langle \psi_n|\psi_n\rangle = \underset{\text{normalized}}{0} = \langle \dot\psi_n|\psi_n\rangle + \langle\psi_n|\dot\psi_n\rangle$
  - ii) $R_i = R_f$, $\hat{\mathcal{H}} = \hat{\mathcal{H}}(R(t))$
- Can be nonzero if $\hat{\mathcal{H}} = \hat{\mathcal{H}}(R_1, \dots, R_n)$ and $|\psi_n\rangle$ nontrivially complex
- Sudden approximation: Opposite of adiabatic — if $\mathcal{H}(t)$ changes suddenly, $|\psi\rangle$ stays in its initial state (not energy level)

- Closed loop:
$$\gamma_n = i\oint_c \langle \psi_n | \nabla_R \psi_n\rangle \cdot d\vec{R}$$
- Generally:
$$\gamma_n = i\int_{\vec{R}_i}^{\vec{R}_f} \langle \psi_n | \nabla_R \psi_n\rangle \cdot d\vec{R}$$
- Aharonov – Bohm: Potentials are fundamental, not forces $\vec{p} \mapsto \vec{p} - \frac{e}{m}\vec{A}$

$$\vec{A} = \frac{\Phi}{2\pi r}\hat{\varphi}$$

Scattering: Want to calculate $d\sigma/d\Omega$

- $\frac{dP_{out}}{dP_{in}} = \frac{d\sigma}{d\Omega}$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

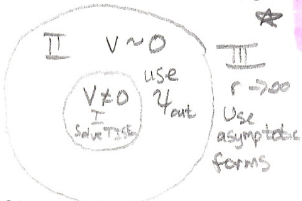
- Total cross section: Probability the particle will scatter:
$$\sigma = \iint_S \frac{d\sigma}{d\Omega} d\Omega$$
$k = \sqrt{2mE}/\hbar$

- Ansatz for solution:
$$\psi(\vec{r}) = A\left(e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right)$$

- No φ dependence, so $Y_\ell^0 = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$ PWA

- Asymptotically comparing $u(r)$ w/ $V=0$ and using $h_\ell^\pm$ instead of j_ℓ, n_ℓ, we have:
$$\psi(\vec{r}) = A\left[e^{ikz} + k\sum_{\ell=0}^\infty i^{\ell+1}(2\ell+1)a_\ell h_\ell^+(kr)P_\ell(\cos\theta)\right]$$
- Using the Rayleigh formula, $e^{ikz} = \sum_{\ell=0}^\infty i^\ell(2\ell+1)j_\ell(kr)P_\ell(\cos\theta)$
- Total wavefunction is:

$$\star\ \psi_{out} = A\sum_{\ell=0}^\infty i^\ell(2\ell+1)\left[j_\ell(kr) + ika_\ell h_\ell^+(kr)\right]P_\ell(\cos\theta)$$



$$f(\theta) = \sum_{\ell=0}^\infty (2\ell+1)a_\ell P_\ell(\cos\theta)$$

Exploit orthonormality of $P_\ell(\cos\theta)$ to find σ – $d\sigma/d\Omega$ not so obvious

$$\sigma = 4\pi\sum_{\ell=0}^\infty(2\ell+1)|a_\ell|^2$$

- Solutions to Schrödinger eqn for $V(r)$:
$u(r) := rR(r)$ $\quad \psi = R(r)Y_\ell^m(\theta,\varphi)$
$$-\frac{\hbar^2}{2m}u'' + \left[V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u(r) = Eu$$
- No potential: $k = \sqrt{2mE}$
- Potential V_0: $k = \sqrt{2m(E-V_0)}$

$r \to \infty$, $V_{(r)} \to 0$: $u \sim Ce^{ikr} + De^{-ikr}$

$$u(r) = Arj_\ell(kr) + Bn_\ell(kr)\cdot r$$

- Neumann fns blow up at origin. For $x \ll 1$:
$$j_\ell \to \frac{2^\ell \ell!}{(2\ell+1)!}x^\ell \qquad n_\ell \to -\frac{(2\ell)!}{2^\ell \ell!}x^{-(\ell+1)}$$
  - $B = 0$ in interior b/c of this

$n_1 = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$

$j_0 = \frac{\sin x}{x}$ $\quad j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$ $\quad n_0 = \frac{\cos x}{x}$

$$h_\ell^\pm = j_\ell(x) \pm i n_\ell(x) \qquad h_0^\pm = \mp i\frac{e^{\pm ix}}{x}$$

$$h_\ell^\pm \xrightarrow{x\to\infty} (\mp i)^{\ell+1}\frac{e^{\pm ix}}{x}$$
($\psi \to h_\ell^+$ as $r \to \infty$)

$$L = R'/R$$

- Strategy: ① Solve TISE in region I ② Fit solution in interior to $\star$ w/ B.C.'s ③ Read off a_ℓ and find $f(\theta)$
- PWA works well w/ low energy, because most higher a_ℓ become negligible
- Scattering phases: Can reduce 2 DoF needed for a_ℓ to 1 w/ δ_ℓ
- Scattering preserves probability for each ℓ b/c of $V(r)$, so all it does is a phase shift
- Writing $\star$ w/ j_ℓ as $h_\ell^\pm$ gives:

$$\psi = A\sum_{\ell=0}^\infty \frac{2\ell+1}{2ikr}\left[e^{ikr + 2i\delta_\ell} - (-1)^\ell e^{-ikr}\right]P_\ell(\cos\theta) \quad (r\to\infty)$$

$$a_\ell = \frac{1}{k}e^{i\delta_\ell}\sin(\delta_\ell) \qquad f(\theta) = \frac{1}{k}\sum_{\ell=0}^\infty(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta)$$

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^\infty(2\ell+1)\sin^2(\delta_\ell)$$

- Optical theorem
$$\sigma = \frac{4\pi}{k}\operatorname{Im}(f(0))$$

- Total ψ is now (region II):
$$\psi = A\sum_{\ell=0}^\infty i^\ell\frac{2\ell+1}{kr}e^{i\delta_\ell}\left[\cos\delta_\ell\sin\left(kr - \frac{\ell\pi}{2}\right) + \sin\delta_\ell\cos\left(kr - \frac{\ell\pi}{2}\right)\right]P_\ell(\cos\theta)$$

- Strategy: ① Solve TISE in region I ② Find $R(r)$ in II by matching logarithmic derivative, $L = \frac{d}{dr}(\ln(R))$ at $r=a$ ③ Solve for $\tan\delta_\ell$

- Born Approximation: $\vec{k}$ wavenumber, $\vec{k}' := k\hat{z}$, $k = |\vec{k}|$
$$f(\theta) \sim -\frac{m}{2\pi\hbar^2}\int_{\mathbb{R}^3}e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}V(\vec{r})d^3r$$
- Low energy ($k', k \ll 1$):
$$f(\theta) \sim -\frac{m}{2\pi\hbar^2}\int_{\mathbb{R}^3}V(\vec{r})d^3r$$
- Approximation: ψ not significantly changed by scattering, so $\psi \sim \psi_0$

- Spherical Symmetry ($V(r)$): $\kappa := 2k\sin(\theta/2)$
$$f(\theta) \sim -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)\sin(\kappa r)dr$$

- Derivation: $k := \sqrt{2mE}/\hbar$, $Q := \frac{2m}{\hbar^2}V(\vec{r})\psi \Rightarrow (\nabla^2 + k^2)\psi = Q$
  - Want to find G s.t. $(\nabla^2 + k^2)G = \delta^3(\vec{r})$, so $\psi = \int_{\mathbb{R}^3}G(\vec{r}-\vec{r}_0)Q(\vec{r}_0)d^3r_0$
  - Ansatz for $G(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{i\vec{s}\cdot\vec{r}}g(\vec{s})d^3s$, use $\nabla^2 e^{i\vec{s}\cdot\vec{r}} = -s^2e^{i\vec{s}\cdot\vec{r}}$ to get $g(\vec{s})$, then find $G(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{i\vec{s}\cdot\vec{r}}\frac{1}{k^2-s^2}d^3s$, $\vec{s}\cdot\vec{r} = sr\cos\theta$, then do integral

- As $x\to\infty$:
$$j_\ell \to \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)$$
$$n_\ell \to -\frac{1}{x}\cos\left(x - \frac{\ell\pi}{2}\right)$$
- For phase shifts, compare logarithmic derivative

- Note: If $R_\ell = \alpha_\ell j_\ell(ka) + \beta_\ell n_\ell(ka)$ in region II (write like this), then $\tan\delta_\ell = -\frac{\beta_\ell}{\alpha_\ell}$
- Resonance: Occurs when $\tan\delta_\ell$ diverges ($\to\infty$) for some k, so $\sin^2\delta_\ell = 1$. This will make the ℓ^{th} partial wave dominant and so $\sigma \sim \frac{4\pi}{k^2}(2\ell+1)$ and we can neglect the other a_ℓ and δ_ℓ

• Log derivative shortcut: For the outer wavefunction, $\underset{R=}{\psi} = \alpha_\ell j_\ell(ka) + \beta_\ell n_\ell(ka)$, $L = \frac{R'}{R} = k\frac{\alpha_\ell j_\ell'(ka) + \beta_\ell n_\ell'(ka)}{\alpha_\ell j_\ell(ka) + \beta_\ell n_\ell(ka)}$ (can also insert $\beta_\ell = -\alpha_\ell \tan\delta_\ell$)

$$\Rightarrow \tan\delta_\ell = \frac{k j_\ell'(ka) - L j_\ell(ka)}{k n_\ell'(ka) - L n_\ell(ka)}$$

• So we can just solve for the L from $V(r)$ at $r=a$ and plug in to get $\tan\delta_\ell$

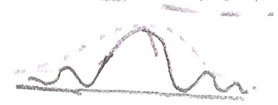
• For a wavefunction $R_{in}(r)$, $L = \frac{R'(a)}{R(a)}$

$$\sin^2\delta_\ell = \frac{\tan^2\delta_\ell}{1+\tan^2\delta_\ell}$$

Clocks: Perturb twice, once from $[0,\tau]$ and again from $[T, T+\tau]$. c_b's add:

$$c_b = -\frac{i}{\hbar}\left[\int_0^\tau H'_{ba} e^{i\omega t'}dt' + \int_T^{T+\tau} H'_{ba} e^{i\omega t'}dt'\right]$$

$$\text{FWHM} = \sin^2(\delta t/2)/\delta^2$$



• Selection rules: $\langle \ell' m' | \hat{z} | \ell m\rangle = 0$ unless $\Delta m = \pm 1, 0$, and $\Delta\ell = \pm 1$.

$[\hat{L}_z, \hat{x}] = i\hbar\hat{y}$ $[\hat{L}_z, \hat{y}] = -i\hbar\hat{x}$

• For Δm, apply $\langle \ell' m' | [\hat{L}_z, \hat{x}_i] | \ell m\rangle$

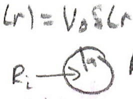
- Ex: $\langle \ell' m' | [\hat{L}_z, \hat{z}] | \ell m\rangle = \langle \ell' m' | \hat{0} | \ell m\rangle = 0$
  $= \langle \ell' m' | \hat{L}_z \hat{z} - \hat{z}\hat{L}_z | \ell m\rangle = \hbar(m'-m)\langle \ell' m' | \hat{z} | \ell m\rangle$
- Use x, y commutators in tandem

$$[\hat{L}^2, [\hat{L}^2, \vec{r}]] = 2\hbar^2(\vec{r}\hat{L}^2 - \hat{L}^2\vec{r})$$

- Sandwich and use Hermiticity

• Spherical shell: $V(r) = V_0\delta(r-a)$

$R_i(a) = R_0(a)$



$$-\frac{\hbar^2}{2m}(rR)'' + \left[\frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} + V_0\delta(r-a)\right](rR) = ErR$$

• Integrate that from $a-\varepsilon$ to $a+\varepsilon$ to get:

$$V_0 a R(r) = \frac{\hbar^2}{2m}\Delta((rR)')$$

$$\Delta((rR)') = \Delta(rR' + R) = R_0'(a) - R_i'(a)$$

$$\Rightarrow R_i'(a) + \frac{2mV_0}{\hbar^2}R(a) = R_0'(a)$$

• Now divide by $R_i(a) = R_0(a)$:

$$\frac{R_i'(a)}{R_i(a)} + \frac{2mV_0}{\hbar^2} = \frac{R_0'(a)}{R_0(a)}$$

• Using $R_0 = \alpha j_\ell(kr) + \beta n_\ell(kr)$ w/ $\tan\delta_\ell = -\beta_\ell/\alpha_\ell$ gives $\tan\delta_\ell$

• S wave $\rightarrow \ell = 0$

• Log derivative w/ $u(r)$ vs $R(r)$:

- Not true that $\frac{u_I'}{u_I} = \frac{R_{II}'}{R_{II}}$, however $\frac{u_I'}{u_I} = \frac{u_{II}'}{u_{II}}$ yields same eqn as $\frac{R_I'}{R_I} = \frac{R_{II}'}{R_{II}}$ ☆ Be consistent

$$\frac{u'}{u} = \frac{R'}{r} - \frac{1}{r}$$

☆ In above: $L = R'/R$

$$\frac{2\ell!!}{(2\ell+1)!} = \frac{1}{(2\ell+1)!!}$$

$$2^\ell \ell! = (2\ell)!!$$

• Resonance in Born — probably not b/c if you get a resonance, ψ differs vastly from ψ_0 so Born approx doesn't hold

• For radial scattering w/ Born, $\vec{k} = k\hat{r}$

• Rabi: Energy gap changes sign, so switch labels on a and b and switch sign on δ

• Partial waves: match ☆ eqn w/ B.C.'s from interior soln (solve TISE). Can fit log derivative (of u, or of R, whichever is easier)

• To derive $\tan\delta_\ell$ eqn, use: $(\tan\delta_\ell = -\beta_\ell/\alpha_\ell)$

$$R_{outside} = \alpha_\ell j_\ell(ka) + \beta_\ell n_\ell(ka)$$

$$L = \frac{R'}{R} = k\frac{\alpha_\ell j_\ell'(ka) + \beta_\ell n_\ell'(ka)}{\alpha_\ell j_\ell(ka) + \beta_\ell n_\ell(ka)}$$

$$L(j_\ell - \tan\delta_\ell n_\ell) = k(j_\ell' - \tan\delta_\ell n_\ell')$$

$$\tan\delta_\ell = \frac{k j_\ell'(ka) - L j_\ell(ka)}{k n_\ell'(ka) - L n_\ell(ka)}$$