Patrick Dave

Physics H7C Midterm I Cheat Sheet

Waves:

- Phase velocity: $V_\phi := \frac{\omega}{k}$
  - Speed at which phase front moves
- Group velocity: Speed at which wave packets move
  $V_g := \frac{d\omega}{dk}$
- Dispersion relation: $\omega = \Omega(k)$
  - ω varies as a fn of k
- Dispersion free: $\omega = \pm V_\phi k$
  - No dispersion, all phase fronts move at V_ϕ

- The wave eqn: Waves satisfy this (no matter the $\omega(k)$) if there are no external forces (driving, damping, etc):

$$\frac{1}{V_\phi^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}$$

- Use linearity, local physics, invariance
- Plane waves solve the wave equation:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

$$k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi f = \frac{2\pi}{T}$$

- Any fn of this form

$$c_1 \underbrace{f(x-vt)}_{\text{Right Moving}} + c_2 \underbrace{g(x+vt)}_{\text{Left Moving}}$$

- Waves on string: $\mu := \frac{m}{L}$

$\sin\theta \sim \theta \sim \tan\theta \sim \frac{\partial \psi}{\partial x}$

$T_1 \approx T_2 \approx T_x$ $\quad F_{net} = T\sin(\theta + d\theta) - T\sin\theta$

$\sim T\left(\frac{\partial \psi}{\partial x}(x+dx) - \frac{\partial \psi}{\partial x}(x)\right)$

$\rightarrow m\frac{\partial^2 \psi}{\partial t^2} = T\left(\frac{\partial \psi}{\partial x}(x+dx) - \frac{\partial \psi}{\partial x}\right)$

$\frac{\partial \psi}{\partial x}(x+dx) \sim \frac{\partial \psi}{\partial x}(x) + \frac{\partial^2 \psi}{\partial x^2}(x)\,dx + \dots$

$\rightarrow \mu\, dx\, \ddot{\psi} = T\frac{\partial^2 \psi}{\partial x^2}dx$

$\Rightarrow \frac{\mu}{T}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} \rightarrow V_\phi = \sqrt{\frac{T}{\mu}}$

- Energy: $KE = \frac{\mu}{2}\int_0^L \left(\frac{\partial \psi}{\partial t}\right)^2 dx$

$$V = \frac{T}{2}\int_0^L \left(\frac{\partial \psi}{\partial x}\right)^2 dx$$

- Power: $J(x,t) = -T\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial t}$

- Fourier transform: $t \longleftrightarrow \omega$

$$\Psi(x,t) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} \Psi(k,\omega)e^{i(kx-\omega t)}dk\,d\omega$$

$$\psi(\omega) = \hat{\mathcal{F}}^*\{\psi(t)\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \psi(t)e^{i\omega t}dt$$

$$\psi(t) = (\hat{\mathcal{F}}^{-1})^*\{\psi(\omega)\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \psi(\omega)e^{-i\omega t}d\omega$$

$$\psi(k) = \hat{\mathcal{F}}\{\psi(x)\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \psi(x)e^{-ikx}dx$$

$$\psi(x) = \hat{\mathcal{F}}^{-1}\{\psi(k)\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \psi(k)e^{ikx}dk$$

- Time dependence:
  1. Real space: Solve for $\psi(x,0)$ as a linear combo of e^{ikx}'s and attach a $e^{-i\omega(k)t}$ to each
  2. Fourier: $\psi(x,0) \xrightarrow{\hat{\mathcal{F}}} \psi(k,0) \rightarrow \psi(k,t) = \psi(k,0)e^{-i\omega(k)t} \xrightarrow{\hat{\mathcal{F}}^{-1}} \psi(x,t)$

- Alternatively, if a string is plucked, will turn into 2 opp. direction traveling waves w/ A/2 — x will turn to $x \pm vt$

- Standing Waves: Normal modes of wave eqn
  - Plug in $\Psi(x,t) = \psi(x)\varphi(t)$ and solve for $\psi(x)$
  - Use boundary conditions to discretize ψ_n
  - Any wave can be expressed as a linear combo:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x,t)\,;\; c_n = \int_0^L \psi_n^*(x)\Psi(x,0)dx$$

- For a string: $\omega_n = Vk_n = V\frac{n\pi}{L}$

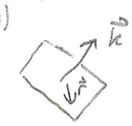
$$\psi_n(x,t) = (a_n\cos(\omega_n t) + b_n\sin(\omega_n t))\sqrt{\frac{2}{L}}\sin(k_n t)$$

- Orthonormality: $\langle \psi_m | \psi_n \rangle = \int_0^L \psi_m^*(x)\psi_n(x)dx = \delta_{mn}$

- 3D: $\frac{1}{V^2}\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi(\vec{r},t)$

$\vec{\psi}(\vec{r},t) = \hat{e}Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}$

- Has surfaces of const. phase where $\vec{k}\cdot\vec{r} = 0$



$\vec{V}_\phi = \frac{\omega}{|\vec{k}|}\hat{k} \qquad V_g = \frac{\partial\omega}{\partial\vec{k}} = \sum \partial_{k_i}\omega\,\hat{x}_i$

- Boundaries btwn media: Write all incident/reflected/transmitted, then use boundary conds for $\psi, \psi', \dot{\psi}$

- E&M waves:

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

- Use these in a vacuum along w/: $\nabla\times\nabla\times\vec{f} = \nabla(\nabla\cdot\vec{f}) - \nabla^2\vec{f}$ to show $\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$
- $E_0 = cB_0$
- $\vec{E}\times\vec{B}$ is direction of propagation
- Energy flow: $\frac{\partial U}{\partial t}\Big|_V = \oint_{\partial V}\vec{S}\cdot d\vec{A}$

- Energy: $U = \frac{1}{2}\iiint_{\mathbb{R}^3}\left(\varepsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right)d^3\vec{x}$

- Poynting vector: $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ $\quad I = P/A = \langle\vec{S}\rangle$

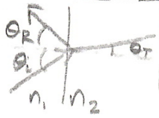
- Average energy: $\langle u\rangle = \frac{1}{2}\varepsilon_0|E_0|^2$ $\quad$ • $\langle\vec{S}\rangle = \frac{1}{2}\varepsilon_0|E_0|^2c\hat{k}$ — Magnitude of $\vec{S}$ is irradiance/intensity

- Wave Impedence: $F_{local} = -Z\frac{\partial\psi}{\partial t}$
  - For a string, $F_L = T\frac{\partial\psi}{\partial x}$ (use wave eqn for $\partial\psi$)

  $\rightarrow Z = T\frac{k}{V_\phi} = \sqrt{\mu T}$
  - Match impedances btwn boundaries for no reflection

Optics:

- At a planar boundary:



  1. All $\vec{k}$'s stay in same plane
  2. Law of Reflection: $\theta_R = \theta_i$
  3. Snell's Law: $n_1\sin\theta_i = n_2\sin\theta_T$
  - θ's are measured w.r.t. normal

- Index of refraction: In a medium where light moves at V_ϕ: $\quad n_{air} \sim 1$

$$n := c/V_\phi$$

- Bigger index, slower wave speed

- Total internal reflection: All light reflected when $\theta_T > 90°$, so:

$$\sin\theta_c = \frac{n_2}{n_1}$$

- Huygens-Fresnel Principle: Waves are local, so we can imagine treating a wavefront as each point making its own "wavelets"

- Evanescent waves: Result of continuity of $\vec{E}, \vec{E}'$ at boundary
  - Complex $\vec{k}$, decay exponentially, transport no power

- Hamilton's Ray Eqns: $\omega = \Omega(\vec{k},\vec{x},t)$

$\frac{d\vec{x}}{dt} = \frac{\partial\Omega}{\partial\vec{k}}$ (velocity of ray is group velocity)

$\frac{d\vec{k}}{dt} = -\frac{\partial\Omega}{\partial\vec{x}}$ (Generalization of Snell's Law)

$\frac{d\omega}{dt} = \frac{\partial\Omega}{\partial t}$ (ω is const. in time if dispersion has not dep.)

- Consider $\omega = \frac{c|\vec{k}|}{n(\vec{x})}$

$$\frac{d}{ds}\left(n\frac{d\vec{x}}{ds}\right) = \nabla n(\vec{x})$$

- s is path length
- If you plug into 2nd eqn, see that $\vec{k} \propto \nabla n(\vec{x})$

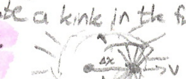
- Rays:
  - Move along $\vec{k}$ vector
  - Propagate at V_g, not V_ϕ

- $\psi(x,t) = A(x,t)e^{iH(x,t)}$

$k_{loc} = \frac{\partial H}{\partial x} \quad \omega_{loc} := -\frac{\partial H}{\partial \omega}$

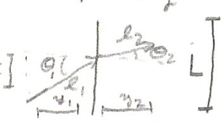
- Radiation:
  - Arises from accelerating charges
  - Accelerated charges create a kink in the field

$$P = \frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{c^3}|\vec{a}|^2$$



- Fermat's Principle: Light travels the path to make the quantity $\frac{1}{c}\int_C n\,ds$ stationary

- Derive Snell's Law:



$$\Delta t = \frac{n_1\ell_1}{c} + \frac{n_2\ell_2}{c}$$

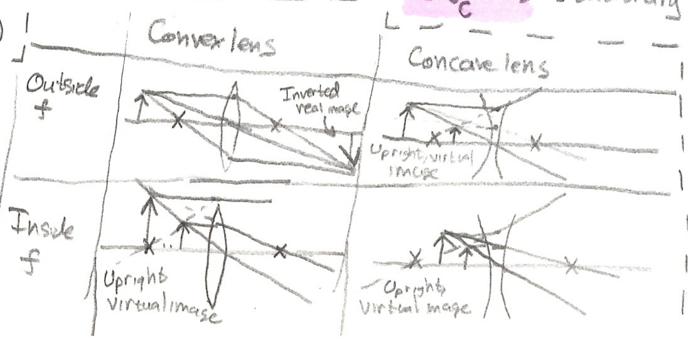
$$\Delta t = \frac{1}{c}\left(n_1\sqrt{x^2+y_1^2} + n_2\sqrt{(L-x)^2+y_2^2}\right)$$

$$\frac{d\Delta t}{dx} = 0 = \frac{n_1 x}{\sqrt{x^2+y_1^2}} + \frac{n_2(-1)(L-x)}{\sqrt{(L-x)^2+y_2^2}}$$

$$0 = n_1\sin\theta_1 - n_2\sin\theta_2$$

$$\Rightarrow n_1\sin\theta_1 = n_2\sin\theta_2$$

- Ray Tracing:
  - Examine ≥ 2 rays and find intersection
    1. Ray that starts parallel
    2. Ray through center
    3. Ray through focus



• Focal Point:

$$\frac{1}{f}=\frac{1}{S_o}+\frac{1}{S_i}$$

- f is focal point, S_o is object dist, S_i is image dist
- If negative S_i, virtual image
- Negative $f \Rightarrow$ diverging lens

• Thin Lens Equation:

$$\frac{1}{f}=(n_{lens}-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$=\left(\frac{n_{lens}}{n_{medium}}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

Matrix Ray Tracing:

$$\begin{pmatrix} n\theta' \\ x' \end{pmatrix} = A\begin{pmatrix} n\theta \\ x \end{pmatrix}$$

- x is transverse displacement

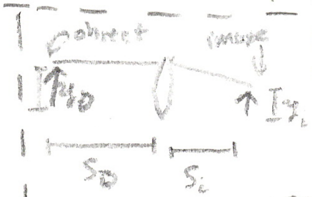
• Drift by distance s:

$$A=\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

- Composition of mappings gives you $\begin{pmatrix} n\theta \\ x \end{pmatrix}_f$

• Thin converging lens:

$$A=\begin{pmatrix} 1 & -1/f \\ 0 & 1 \end{pmatrix}$$



$$\frac{y_o}{|y_i|}=\frac{S_o}{S_i}$$

$$x_o x_i = f^2$$

- x is dist from f
- Both x_o, x_i must have same sign

$$M_T=\frac{y_i}{y_o}=-\frac{S_i}{S_o}$$

- $M_T>0 \Rightarrow$ real image
- $M_T<0 \Rightarrow$ virtual

• R and T $\perp$ to plane of incidence

r = reflected, t = transmitted

$$r_\perp=\left(\frac{E_{or}}{E_{oi}}\right)_\perp=\frac{n_i\cos\theta_i-n_t\cos\theta_t}{n_i\cos\theta_i+n_t\cos\theta_t}$$

Ratio of reflected amplitude to incident

$$t_\perp=\left(\frac{E_{ot}}{E_{oi}}\right)_\perp=\frac{2n_i\cos\theta_i}{n_i\cos\theta_i+n_t\cos\theta_t}$$

$t_\perp = 1 + r_\perp$

Ratio of transmitted amplitude to incident

• R, T // to plane of incidence

$$r_{\parallel}=\frac{n_t\cos\theta_i-n_i\cos\theta_t}{n_i\cos\theta_t+n_t\cos\theta_i}$$

$$t_{\parallel}=\frac{2n_i\cos\theta_i}{n_i\cos\theta_t+n_t\cos\theta_i}$$

$$r_{\parallel}+t_{\parallel}=1$$

Total R and T (Ratio of power):

$$R=r^2 \qquad (R_{\parallel}=r_{\parallel}^2,\ \text{etc})$$

$$T=\left(\frac{n_t\cos\theta_t}{n_i\cos\theta_i}\right)t^2$$

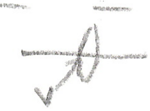
Effective focal length:

$$\frac{1}{f_{eq}}=\sum_{i=1}^{N}\frac{1}{f_i}$$

- If separated by distance d:

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}$$

Sign conventions:

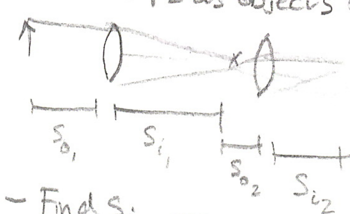


S_o, f_o	+ to left of V
x_o	+ to left of F_o
S_i, f_i	+ to right of V
x_i	+ to right of F_i
R	if center of sphere to right of V
y_o, y_i	+ above optical axis

Meanings of Signs of thin lens vars:

Quantity	Sign +	−
S_o	real obj.	virtual obj.
S_i	real image	virtual image
f	conv. lens	div. lens
y_o	erect obj.	inverted obj.
y_i	erect im.	inverted im.
M_T	erect im.	inverted im.

• Compound lens − ~~[illegible]~~ Treat images for first lens as objects for next lens:



- Find S_{i_1}, assuming second lens isn't there
- Use this to find S_{o_2}, and then find S_{i_2}

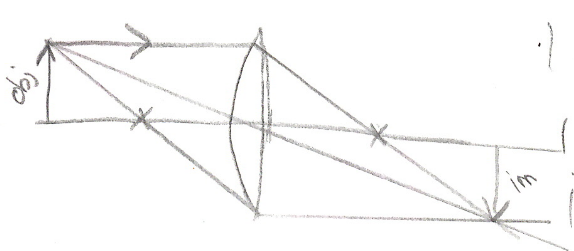
• Denser substance, less lateral scattering
- Destructive interference

$$\begin{matrix}\sin u \sin v \\ \cos u \cos v\end{matrix}=\frac{1}{2}\left[\cos(u-v)\mp\cos(u+v)\right]$$

$$-M_L=\frac{f^2}{x_o^2}=M_T^2$$

Longitudinal magnification

- Waves on string: write general $Ae^{ik_ix}+Be^{-ik_ix}$ each region, then apply B.C.'s btwn regions



Convex

Location	Type	location	orientation	rel. size
$\infty > S_o > 2f$	Real	$f < S_i < 2f$	inverted	minified
$S_o = 2f$	Real	$S_i = 2f$	inverted	same
$f < S_o < 2f$	Real	$\infty > S_i > 2f$	inverted	magnified
$S_o = f$	—	$\pm\infty$		
$S_o < f$	Virtual	$\lvert S_i \rvert > S_o$	erect	magnified

Concave

Anywhere	Virtual	$\lvert S_i \rvert < \lvert f \rvert$, $\lvert S_i \rvert < S_o$	erect	minified

Patrick Oare

• Reflection of light off a surface polarizes light in direction of plane of surface

Polarization:

• Malus' Law:

$$I \propto \cos^2\theta$$

– I is intensity through polarizer, θ is angle btwn $\vec{E}$ and transmission axis

• Unit vectors:

$\vec{H} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{R} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\vec{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\vec{L} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\vec{D} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\vec{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

• Polarizer:

$$\hat{P}_\theta = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

– Idempotent ($\hat{P}^2 = \hat{P}$)

– Hermitian ($\hat{P}^\dagger = \hat{P}$)

• Waveplates:

– Phase shift ϕ_H and ϕ_V:

$$\hat{\omega} = \begin{pmatrix} e^{i\phi_H} & 0 \\ 0 & e^{i\phi_V} \end{pmatrix} \qquad \hat{H} = \hat{Q}^2$$

– Ex:

$$\underset{\text{fast}=V}{\hat{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \qquad \underset{\text{fast}=H}{\hat{Q}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

• Circular polarizers: Differ from waveplates b/c they project and don't preserve norm

$$\hat{P}_R = \vec{R}\vec{R}^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

• Filters:

$$\hat{F} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

• Dichroic:

$$\hat{D} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

• $\hat{P}_\theta$ will have $\lambda = 0, 1$

$E_1 = \text{span}\left\{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right\}$

$E_0 = \text{span}\left\{\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\right\}$

• Coherence matrix:

$$\Gamma := \begin{pmatrix} \langle E_x^* E_x \rangle & \langle E_x E_y^* \rangle \\ \langle E_x^* E_y \rangle & \langle E_y E_y^* \rangle \end{pmatrix}$$

– Γ is Hermitian and positive semidefinite ($\forall |x\rangle \in \mathcal{H}, \langle x|\Gamma|x\rangle \geq 0$)

– $\Gamma \therefore$ has $\forall \lambda_i \geq 0$

– Completely unpolarized light has $\Gamma = kI$

– Can view $\Gamma = \langle \vec{e}\vec{e}^\dagger \rangle$

$$\Gamma_{out} = \hat{J}_n \dots \hat{J}_2 \hat{J}_1 \Gamma \hat{J}_1^\dagger \hat{J}_2^\dagger \dots \hat{J}_n^\dagger$$

• Intensity:

$$I = \text{trace}(\Gamma)$$

• Spectral Decomposition:

$$\Gamma = \lambda_1 |1\rangle\langle 1| + \lambda_2 |2\rangle\langle 2|$$

• Recall: $\text{trace}(AB) = \text{trace}(BA)$

• If $\lambda = 0$ for Γ, then it is completely polarized

• Let $\lambda_1 \geq \lambda_2$:

$$\Gamma = \lambda_1 |1\rangle\langle 1| + \lambda_2 |2\rangle\langle 2|$$

$$= \lambda_2 (|1\rangle\langle 1| + |2\rangle\langle 2|) + (\lambda_1 - \lambda_2)|1\rangle\langle 1|$$

$$= \underbrace{\lambda_2 \hat{I}}_{\text{Unpolarized}} + \underbrace{(\lambda_1 - \lambda_2)|1\rangle\langle 1|}_{\text{Polarized in } |1\rangle}$$

• Degree of polarization:

$$\text{dop} := \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$$

– dop = 0 $\Rightarrow$ unpolarized

– dop = 1 $\Rightarrow$ completely polarized

• Coherence properties of light: How close light is to a plane wave

• Unpolarized light can be seen as an equal mixture of $\perp$ polarization states

• Stoke's Parameters:

$S_0 = I = \langle |E_x|^2 \rangle + \langle |E_y|^2 \rangle$

$S_1 = Q = \langle |E_x|^2 \rangle - \langle |E_y|^2 \rangle$

– difference in intensity

$S_2 = U = \langle |E_A|^2 \rangle - \langle |E_D|^2 \rangle = 2\,\text{Re}\langle E_x E_y^* \rangle$

$S_3 = V = \langle |E_R|^2 \rangle - \langle |E_L|^2 \rangle = 2\,\text{Im}\langle E_x E_y^* \rangle$

• Polarization Intensity: $I_P^2 := Q^2 + U^2 + V^2$

$$\text{dop} = I_P / I$$

• Density of modes:

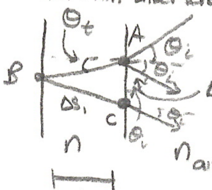
$$dk\,dx = 2\pi$$

$$\#\text{modes} = \int \frac{d^3k\,d^3x}{(2\pi)^3}$$

Box of size Δz, #modes change as: $\Delta k = \frac{2\pi}{\Delta z}$

Interference/Diffraction:

• Thin film interference:



– Phase difference given by $\Delta s = n\Delta l$

$$\Delta l = 2\overline{AB} = 2\frac{d}{\cos\theta_t}$$

$$\cos\theta_t = \frac{d}{AB} \qquad \Delta s_1 = \frac{2nd}{\cos\theta_t}$$

$$\sin\theta_i = n\sin\theta_t \qquad \sin\theta_i = \frac{\Delta s_2}{AC}$$

$\Delta s_2 = AC\sin\theta_i \quad AC = 2d\tan\theta_t \quad \Delta s = \Delta s_1 - \Delta s_2$

$$\Delta s_2 = 2d\tan\theta_t \sin\theta_i = \frac{2nd\sin^2\theta_t}{\cos\theta_t}$$

$$\Delta s = \frac{2nd}{\cos\theta_t}(1 - \sin^2\theta_t) = \boxed{2nd\cos\theta_t}$$

To get $\Delta\phi$, we add in a π phase shift when the wave goes from low n to high n

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta s + \pi = \boxed{\frac{4\pi nd\cos\theta_t}{\lambda} + \pi}$$

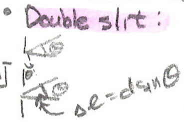
Maxima @ $\Delta\phi = 2k\pi$, minima @ $\Delta\phi = \pi + 2k\pi$

• Interference controlled by Δs, optical path length, $\Delta s = n\Delta l$

– Maxima when $\Delta s = m\lambda$

– Minima when $\Delta s = (m+\frac{1}{2})\lambda$

• Double slit:



$m \in \mathbb{Z}$

• Intensity maxima:

$\Delta l = m\lambda$

$d\sin\theta = m\lambda$

• Intensity minima

$\Delta l = (m+\frac{1}{2})\lambda$

$d\sin\theta = (m+\frac{1}{2})\lambda$

• Fourier Transform:

$$\hat{\mathcal{F}}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ikx}dx$$

$$\int_{\mathbb{R}} \delta(x-a)f(x)dx = f(a)$$

• Very close to the aperture:

$$I \sim |f(x,y)|^2$$

• Fresnel / Fraunhofer:

– Fresnel number: L^2/λ

– Fresnel diffraction near field

– Fraunhofer far field

– Large z: $z \gg \omega L^2/v$

• Fraunhofer diffraction:

$$I \propto \frac{1}{R^2}\left|F\left(k\frac{x}{R}, k\frac{y}{R}\right)\right|^2$$

$\cos\theta = x/R$, $\sin\theta = y/R$

$$\mathcal{H}(\vec{x}) = \frac{e^{ikR}}{2\pi R}\,x$$

$$\int f(x,y)e^{-ik(xX+yY)/R}dxdy$$

• Single slit diffraction:

– Pair off at the edge w/ a ray from the middle

$\sin\theta = \frac{\Delta l}{a/2} \Rightarrow \Delta l = \frac{a}{2}\sin\theta$

Max: $\frac{a}{2}\sin\theta = m\lambda$

Min: $\frac{a}{2}\sin\theta = (m+\frac{1}{2})\lambda$

• As we increase width of the transmission fn, we decrease the width of the diffraction pattern

• N slits separated by d:

$$I(x) \propto \frac{\sin^2\left(\frac{N\pi d}{\lambda R}x\right)}{\sin^2\left(\frac{\pi d}{\lambda R}x\right)}$$

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

$$\gamma := \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \beta := v/c \qquad \Lambda := \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

Special Relativity:

• Postulates:

① The laws of physics are the same in all reference frames

② The speed of light is constant in all frames

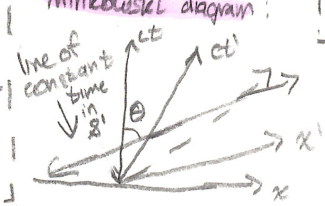
• Spacetime interval:

$$ds^2 = c^2dt^2 - \|d\vec{x}\|^2$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

• Minkowski diagram:



$\tan\theta = \beta$

– Events are points

• Spacelike events: $\exists$ ref frame that they happen at same time and $ds^2 < 0$

• Timelike: $\exists$ ref. frame that they happen at the same location and $ds^2 > 0$

• Lightlike: Move at c, and $ds^2 = 0$

• Length contraction: If an object has length $\mathcal{L}$ in its rest frame:

$$L' = \mathcal{L}/\gamma$$

• Time dilation: If an interval has $\Delta\tau$ in its rest frame:

$$\Delta t' = \gamma\Delta\tau$$

$\Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}$

Lorentz Inner Product:

$\underline{X}_1 = \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\underline{X}_2 = \begin{pmatrix} ct_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$\underline{X}_1 \cdot \underline{X}_2 = c^2 t_1 t_2 - \vec{x}_1 \cdot \vec{x}_2$

- Proper time: $d\tau := \frac{dt}{\gamma(v)}$

Velocity addition: $u' = \frac{u-v}{1-vu/c^2}$ $\beta' = \frac{\beta - \beta_{frame}}{1-\beta\beta_{frame}}$

- Four velocity:

$\underline{u} := \frac{d}{d\tau}\underline{X} = \gamma \frac{d}{dt}\underline{X} = \gamma \begin{pmatrix} c \\ d\vec{x}/dt \end{pmatrix}$

$\underline{u} \cdot \underline{u} = c^2$ $\underline{u}' = \Lambda \underline{u}$

The Lorentz inner product of a 4-vector is a Lorentz invariant

- Doppler Shift: λ in $F_{emitter}$

$\lambda' = \lambda \sqrt{\frac{1+\beta}{1-\beta}}$

Energy/momentum:

$\underline{P} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = m\gamma \begin{pmatrix} c \\ d\vec{x}/dt \end{pmatrix} = m\underline{u}$

$\vec{p} = m\gamma\vec{v}$ $E_0 = mc^2$

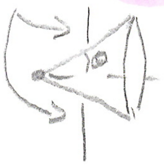
$E = \gamma mc^2$ $T = (\gamma - 1)mc^2$

$E^2 = (cp)^2 + (mc^2)^2$ $\vec{\beta} = \frac{c\vec{p}}{E}$

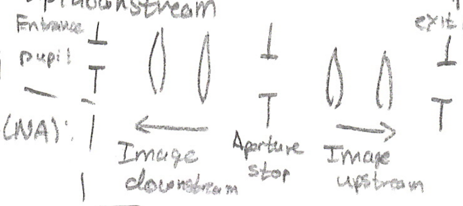
$\underline{P} \cdot \underline{P} = m^2c^2$

Miscellaneous:

- Aperture stop: Limits angle of acceptance
  - Numerical Aperture (NA): $NA = n\sin\theta$
  - $(f/\#) := \frac{1}{2\,NA}$



- Entrance/exit pupil: Image of aperture stop up/downstream



- Chief ray: Ray from off-axis object that goes through center of aperture stop
- Marginal ray: Same as chief but goes through edge
  - Chief also goes through center of both pupils
  - Marginal ... edge ...

- Field stop: Limits angular acceptance of chief rays

- Wave 4-vector:

$\underline{K} = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix}$ $\underline{K} \cdot \underline{X} = \omega t - \vec{k} \cdot \vec{x}$ $\underline{K} \cdot \underline{K} = 0$

- Doppler derivation: k' in frame of emitter, k in frame of observer:

$\underline{K}' = \begin{pmatrix} \omega'/c \\ k' \end{pmatrix} = \begin{pmatrix} k' \\ -k' \end{pmatrix}$ b/c moving backward

$\begin{pmatrix} \omega/c \\ -k \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} k' \\ -k' \end{pmatrix} \rightarrow -k = \beta\gamma k' - \gamma k' \rightarrow k = (1-\beta)\gamma k'$

$-k$ b/c moving left; Moving left

$\rightarrow k = \frac{\sqrt{(1-\beta)^2}}{\sqrt{(1-\beta)(1+\beta)}} k' \rightarrow k = \sqrt{\frac{1-\beta}{1+\beta}}\, k'$

- Notice we have the opposite defs of k and k' in this and the formula in the top

- Convolution:

$f(t) * g(t) := \int_{\mathbb{R}} f(\tau) g(t-\tau)\, d\tau$

$f * g = g * f$

$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$

$f(x) * \delta(x-a) = f(x-a)$

- Phase shift of waveplate

$\Delta\varphi = \frac{2\pi d}{\lambda} |n_f - n_s|$

$n_f \rightarrow n$ of fast axis

$n_s \rightarrow n$ of slow axis

Patrick Cave • All four vectors transform by the Lorentz transformation, $\underline{A}' = \Lambda \underline{A}$

Physics H7C Final Cheat Sheet

Relativistic Dynamics:

$\vec{\beta} = \frac{c\vec{p}}{E}$

• Collisions: Conservation of 4-momentum, 3-momentum, energy, but not mass

• CoM frame: $\underline{P}_{total} = \sum_i \underline{P}_i$ – Generally, switch to frame w/ 0 3-momentum

• Momentum:

$$\underline{P} = m\underline{u} = m\frac{d}{d\tau}\underline{x} = m\gamma\begin{pmatrix} c \\ d\vec{x}/dt \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$$

– From this, we see that we must have:

$$E = \gamma mc^2 \qquad \vec{p} = \gamma m\vec{v} \qquad \underline{P}\cdot\underline{P} = m^2c^2$$

• Energy:

– Kinetic energy is $E - E_0 = E - mc^2$

$$T = (\gamma - 1)mc^2$$

$$E^2 = (cp)^2 + (mc^2)^2$$

• Acceleration:

$$\underline{a} := \frac{d}{d\tau}\underline{u} = \gamma c\begin{pmatrix} \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}} \\ (\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}})\vec{\beta} + \gamma\dot{\vec{\beta}} \end{pmatrix}$$

– Proper acceleration: $\underline{a}_0 \cdot \underline{a}_0 = \underline{a}\cdot\underline{a}$ (in rest frame):

$$\underline{a}_0 = \begin{pmatrix} 0 \\ c\dot{\vec{\beta}} \end{pmatrix}$$

– $\underline{u}$ and $\underline{a}$ are orthogonal:

$$\underline{u}\cdot\underline{a} = 0$$

Quantum Mechanics:

$$E = h\nu = \hbar\omega \qquad \vec{p} = \hbar\vec{k}$$

• Compton shift:

$$\Delta\lambda = \frac{4\pi\hbar}{m_e c}\sin^2(\theta/2) \qquad \lambda_c := \frac{h}{m_e c}$$

• Force:

– 3-force is: $\vec{f} = \frac{d\vec{p}}{dt}$ – Notice it is a t derivative, not a τ derivative

– 4-force is: $\underline{K} := \gamma\begin{pmatrix} \vec{\beta}\cdot\vec{f} \\ \vec{f} \end{pmatrix} = \frac{d}{d\tau}\underline{P} = m\underline{a}$

$$\frac{dE}{dt} = (c\vec{\beta})\cdot\vec{f} \qquad E' = \underline{e}_0'\cdot\underline{P} \quad (\underline{e}_0' \text{ is the timelike unit vector})$$

• Bohr Model: If matter is a wave, then the wave must have completed an integer number of oscillations in one full orbit: $2\pi r = n\lambda = n\,h/p$

$\Longrightarrow mvr = n\hbar$ – Using velocity from centripetal force, we can determine the allowed radii

$$a_0 := \frac{4\pi\varepsilon_0\hbar^2}{me^2} \qquad r_n = \frac{a_0}{Z}n^2 \qquad E = \frac{Z^2E_1}{n^2} \qquad E_1 = -\frac{e^4 m_e}{(4\pi\varepsilon_0)^2 2\hbar^2}$$

• Bohr-Sommerfeld Quantization:

– Phase difference around an orbit should be an integer multiple of 2π

$$\oint p(x)\,dx = (n+\alpha)h \qquad -n \in \mathbb{Z},\ \alpha \text{ is a correction}$$

$$\oint k\,dx = 2\pi(n+\alpha) \qquad \alpha = 0 + \tfrac{1}{4} \text{ for every soft wall } (V<\infty)$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi \quad \text{(TDSE)}$$

$$E\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi \quad \text{(TISE)}$$

$$\Delta x\Delta p \geq \hbar/2$$

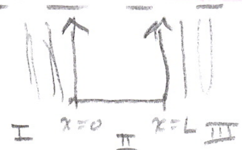
• Born's Rule:

$$d\mathbb{P} = |\psi(x,t)|^2 dx$$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}}\Psi(x,t)e^{-ipx/\hbar}\,dx$$

• Fourier transform takes you to momentum space

• Infinite Square well: $V(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{else} \end{cases}$



– $V = \infty$ for I, III, so $\psi(0) = \psi(L) = 0$

– In region II: $E\psi = -\frac{\hbar^2}{2m}\psi''$, $k := \sqrt{\frac{2mE}{\hbar^2}}$

$\Longrightarrow \psi'' + k^2\psi = 0 \Longrightarrow \psi(x) = A\sin kx + \cancel{B\cos kx}$

$\psi(L) = A\sin(kL) = 0$

– Fit boundary conditions: $\psi(0) = 0 = B\cos 0 = B$ $\Longrightarrow kL = n\pi, n = 1, 2, \ldots$

$$k_n = \frac{n\pi}{L} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \qquad \psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

SHO: $E_n = (n+\frac{1}{2})\hbar\omega$, $V(x) = \frac{1}{2}m\omega^2x^2$

• If $\psi_n(x) := \langle x|n\rangle$ are energy eigenfns:

$$\Psi(x,0) = \sum_{n=0}^{\infty} c_n\psi_n(x)$$

$$c_n = \int_{\mathbb{R}}\psi_n^*(x)\Psi(x,0)\,dx$$

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar}$$

Sketching:

• Bound states: Eigenfn w/ E s.t. no CAR extends to $\infty \to$ quantized

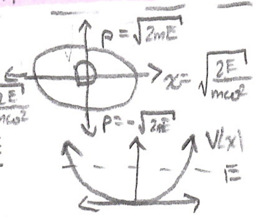
• Scattering: Eigenfn w/ E s.t. at least 1 CAR extends to $\infty \to$ continuous

1. Draw potential and determine spectrum
2. Choose energy (bound/scattering)
3. Low energy bound states, determine location and # nodes; high energy use classical envelope $k(x) := \sqrt{\frac{2m(E-V(x))}{\hbar^2}}$ $K(x) := \sqrt{\frac{2m(V(x)-E)}{\hbar^2}}$
4. CARs: Oscillation, larger k $\to$ faster oscillations, more curvature
5. CFRs: Decay behavior; larger K $\to$ faster decay, more curvature
6. Boundary conditions: ψ always continuous; ψ' continuous if $V \neq \infty$; $V = \infty$ in region $\to \psi = 0$; $V = \infty$ at point $\to$ kink in graph
7. Bound states: If $V(x)$ symmetric, ψ_n alternates sym/antisym
8. Nodes: Ground state usually 0, each subsequent excited state has 1 more than last – No nodes in CFR $\to \infty$; each CFR has at most 1 node

Bohr-Sommerfeld $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$V(x) = \frac{1}{2}m\omega^2x^2$

– In phase space, this becomes an ellipse

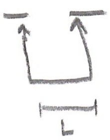


$$\oint p\,dx = A(O) = \pi\left(\sqrt{2mE}\right)\left(\sqrt{\frac{2E}{m\omega^2}}\right) = \frac{2\pi E}{\omega}$$

– 2 soft walls $\Longrightarrow \alpha = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\oint p\,dx = \frac{2\pi E}{\omega} = (n+\tfrac{1}{2})h \Longrightarrow E_n = (n+\tfrac{1}{2})\hbar\omega$$

• Infinite Square Well: $E = \frac{p^2}{2m}$



$\alpha = 0$ b/c 2 hard walls

$$\oint p\,dx = \oint\sqrt{2mE}\,dx = \sqrt{2mE}(2L) = nh$$

$$E = \frac{n^2h^2}{8mL^2}\left(\frac{2\pi}{2\pi}\right)^2 \Longrightarrow E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$$

• Collisions example:

$\pi \to \mu + \nu$ $\quad \underline{P}_\pi = \begin{pmatrix} m_\pi c \\ 0 \end{pmatrix}$

$\underline{P}_\pi = \underline{P}_\mu + \underline{P}_\nu$ $\quad \underline{P}_\nu = \underline{P}_\pi - \underline{P}_\mu$

– Find E_μ by going to rest frame of π

$$\underline{P}_\nu\cdot\underline{P}_\nu = m_\nu^2c^2 = (\underline{P}_\pi - \underline{P}_\mu)\cdot(\underline{P}_\pi - \underline{P}_\mu) \qquad \text{Let } m_\nu \sim 0$$

$$= \underline{P}_\pi\cdot\underline{P}_\pi - 2\underline{P}_\mu\cdot\underline{P}_\pi + \underline{P}_\mu\cdot\underline{P}_\mu$$

$$= m_\pi^2c^2 - 2(m_\pi c)\left(\frac{E_\mu}{c}\right) + m_\mu^2c^2$$

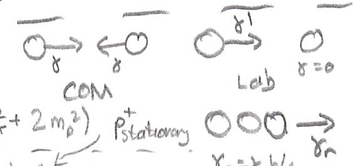
$$E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}$$

OR: COM frame: $E_{products} = (2m_p^2 + m_\pi^2)c^2$ (Products stationary)

$E_{products} = 2E_p = 2\gamma m_pc^2 \Rightarrow \gamma = \frac{1}{2m_p}(m_\pi^2 + 2m_p^2)$

Lab: $E_p' + m_pc^2 = \gamma_r(m_\pi^2 + 2m_p^2)c^2$

$\Rightarrow E_p' = \gamma(m_\pi^2 + 2m_p^2)c^2 - m_pc^2$



$p^+ + p^+ \to p^+ + n^0 + \pi^+$ $\quad \gamma_r = \gamma$ b/c