- A Riemannian metric on a manifold M is an assignment of an inner product to each tangent space T_pM of M, denoted $\langle \cdot,\cdot \rangle_p : T_pM \times T_pM \to \mathbb{R}$. Moreover, if $X, Y \in \mathfrak{X}(M)$ are C^∞ vector fields on M, we require the function $M \to \mathbb{R}$, $p \mapsto \langle X_p, Y_p \rangle_p \in \mathbb{R}$ to be in $C^\infty(M)$. A manifold M endowed w/ a Riemannian metric $\langle \cdot,\cdot \rangle$ is called a Riemannian metric.
- A C^∞ map $F:(M,\langle \cdot,\cdot \rangle) \to (N, \langle \cdot,\cdot \rangle')$ is metric preserving if $\forall p \in M$

$$\langle F_*(X_p), F_*(Y_p) \rangle'_{F(p)} = \langle X_p, Y_p \rangle_p \qquad \forall X_p, Y_p \in T_pM$$

If F is also a diffeomorphism then we call F an isometry.

- A metric is equivalently a smoothly varying rank 2 symmetric tensor field $g \in \Gamma(T^*M \otimes T^*M)$. The correspondence is:

$$\langle X_p, Y_p \rangle_p = g_p(X_p, Y_p)$$

Note in a coordinate chart (U, x^i), we may write:

$$g = g_{ij}\, dx^i \otimes dx^j \qquad g_{ij} = \langle \partial_i, \partial_j \rangle$$

where g_{ij} is symmetric and positive definite.

$X^i Y^j \langle \partial_i, \partial_j \rangle$
$= \langle X, Y \rangle = g(X,Y)$
$= g_{ij} dx^i(X) dx^j(Y)$
$= X^i Y^j g_{ij}$
$\Rightarrow g_{ij} = \langle \partial_i, \partial_j \rangle$

- We can put a metric on any manifold M via a partition of unity.
- The space of vector fields $\mathfrak{X}(M) := \Gamma(TM)$ is a vector space over $\mathbb{R}$ and a module over $C^\infty(M)$.
  - The Lie bracket of vector fields is a map $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by its action on $f \in C^\infty(M)$:

$$[X,Y]_p f := X_p(Yf) - Y_p(\underset{C^\infty(M)}{Xf})$$

since $[X,Y]_p \in T_pM$ means it is a map $C^\infty(M) \to \mathbb{R}$

$f \in C^\infty(M)$
$X_p f \in \mathbb{R}$
$Xf \in C^\infty(M)$ by $(Xf)(p) = X_p f$
- Note Xf is essentially the function $X^i \partial_i f \sim \frac{df}{dx}$

  - Recall the Lie derivative is $\mathcal{L}_X Y = [X,Y]$

$= \nabla_X Y - \nabla_Y X - T(X,Y)$

Geometry in Euclidean Space

- Let $M \subset \mathbb{R}^3$ be a surface. A normal section of M at $p \in M$ is a curve in M which is the intersection $P \cap M$ of M with a plane P. Each unit tangent vector X_p determines a normal section.
  - Given $X_p \in T_pM$, let $\gamma(s)$ be the arc length parameterization of its normal section. The normal curvature is:

$$K(X_p) := \langle \gamma''(s=0), N_p \rangle$$

where N_p is the unit normal vector @ p.

  - The map K drops to $K: S^1 \longrightarrow \mathbb{R}$ and is called the Gauss map.
    (S^1: unit tangent vectors @ p.)

    Principal directions X_1/X_2 satisfy $K_1 = K(X_1)$, $K_2 = K(X_2)$

  - The principal curvatures K_1/K_2 are the max/min values of K. The mean curvature is $H := \frac{K_1+K_2}{2}$ and their product $K := K_1K_2$ is the Gaussian curvature and K is invariant under isometries. The Gauss-Bonnet theorem states that:

$$\int_M K\,dS = 2\pi\chi(M) \qquad \chi(M) = \text{Euler characteristic}$$

- Directional derivatives: For $X_p \in T_pM$ and $Y = b^i\partial_i \in \mathfrak{X}(M)$, the derivative of Y @ $p \in M$ is:

$$D_{X_p}Y = (X_p b^i)\,\partial_i$$

(If $X \in \mathfrak{X}(M)$, define $(D_XY) \in \mathfrak{X}(M)$ by $(D_XY)_p := D_{X_p}Y$)

and D defines a map:

$$D: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M) \qquad X, Y \longmapsto D_XY$$

such that:

i) D is $\mathcal{F} := C^\infty(\mathbb{R}^n)$ linear in X and $\mathbb{R}$-linear in Y.

ii) For $f \in C^\infty(\mathbb{R}^n)$, $D_X(fY) = (D_Xf)Y + f(D_XY)$

Note $\mathrm{End}(V)$ is a Lie algebra under $[A,B] = A \circ B - B \circ A$

- In $\mathbb{R}^n$, the map $\mathfrak{X}(M) \longrightarrow \mathrm{End}(\mathfrak{X}(M))$, $X \longmapsto D_X$ is a Lie algebra homomorphism since $D_{[X,Y]} = [D_X, D_Y]$. This is not true in general, and the difference is known as the curvature

$$R(X,Y) = [D_X, D_Y] - D_{[X,Y]}$$

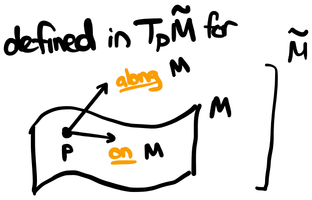
- The torsion of a derivative is proportional to its failure to be symmetric, w/ the caveat that in $\mathbb{R}^n$ D has no torsion. In flat space, $D_X Y - D_Y X = [X,Y]$ by the dfn of $[\cdot,\cdot]$, but in curved space this need not hold and we can have nonzero torsion:

$$T(X,Y) := D_X Y - D_Y X - [X,Y]$$

- Other property: Compatibility w/ the metric. For $Y, Z \in \mathfrak{X}(M)$, $\langle Y,Z\rangle \in C^\infty(M)$ by the map $p \mapsto \langle X_p, Y_p\rangle$. So, we can differentiate it. As $X_p f = D_{X_p} f$ on functions, we get a type of Leibniz rule we can use:

$$X\langle Y,Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$$

- Subvector fields: Two notions. If $M \subset \tilde{M}$, a vector field X is along M if it is defined in $T_p\tilde{M}$ for each $p \in M$. X is on M if $X_p \in T_p M$, $\forall p \in M$. We denote the $C^\infty(M)$ module of vector fields along M by $\Gamma(T\tilde{M}|_M)$



  - D is formally a map:

$$D : \mathfrak{X}(M) \times \Gamma(T\tilde{M}|_M) \longrightarrow \Gamma(T\tilde{M}|_M) \qquad D_X(b^i\partial_i) := (Xb^i)\,\partial_i$$

basis for $TM \subset T\tilde{M}$

$D_X Y$ stays along M.

where we can take D_X w/ any $X \in \mathfrak{X}(M)$ whether $X_p \in T_p M$

  - Vector fields on M are just $\mathfrak{X}(M)$.

- The shape operator is a map $L : TM \to \mathbb{R}$ which describes how the unit normal to $M \subset \mathbb{R}^n$ changes:

$$L(X_p) := -D_{X_p} N$$

  - This satisfies $\langle L(X), Y\rangle = \langle D_X Y, N\rangle$ and is self adjoint, $\langle L(X), Y\rangle = \langle X, L(Y)\rangle$. Thus in an ONB for T_pM, the matrix L_{ij} is symmetric and L has real eigenvalues
  - The 2nd fundamental form $\mathrm{II} : T_pM \times T_pM \to \mathbb{R}$ is defined as:

$$\mathrm{II}(X_p, Y_p) := \langle L(X_p), Y_p\rangle$$

describes the curvature in direction X_p, as one can show:

$$K(X_p) = \mathrm{II}(X_p, X_p)$$

  - The principal directions @ $p \in M$ are thus the eigenvectors of L @ p, and the principal curvatures are the eigenvalues of L.

This implies the Gaussian curvature is:

$$K = \det(L) \qquad \text{(take } \det[L] \text{ where } [L] \text{ is matrix of } L \text{ w.r.t. } \{e_1, e_2\})$$

- The first fundamental form is simply the metric $\mathrm{I} : T_pM \times T_pM \longrightarrow \mathbb{R}$:

$$\mathrm{I}(X_p, Y_p) := \langle X_p, Y_p\rangle_p$$

  - Both I and II are symmetric and bilinear, so have symmetric matrix representations in any ONB of T_pM
  - $\varphi : M \longrightarrow M'$ is an isometry iff the matrix of I is the same for $\{e_1, e_2\} \subset M$ and $\{\varphi_* e_1, \varphi_* e_2\} \subset M'$

Affine Connections

- Defining the derivative on $\mathbb{R}^n$ implicitly uses a choice of basis $\{e_1,\dots,e_n\}$ for each tangent space. For an arbitrary manifold M, we <u>don't have</u> a canonical basis.
  - We can still define the directional derivative of $f \in C^\infty(M)$ as:

    $$\nabla_{X_p} f := X_p f$$

    but we cannot define $D_{X_p} Y$ for $Y \in \mathfrak{X}(M)$ w/o a connection.
    - Instead we take the properties of D in Euclidean space as axioms and later construct a connection.
- An affine connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ denoted $\nabla_X Y \in \mathfrak{X}(M)$ satisfying (let $\mathcal{F} := C^\infty(M)$):

  i) $\nabla_X Y$ is $\mathcal{F}$-linear in X.

  ii) $\nabla_X Y$ satisfies the Leibniz rule in Y: (note $Xf = \nabla_X f$)

  $$\nabla_X (fY) = (Xf)Y + f(\nabla_X Y)$$

  Given a connection ∇, define its curvature and torsion by:

  $$R(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(M))$$

  $$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \in \mathfrak{X}(M)$$

  - For the Euclidean connection on $\mathbb{R}^n$, there is no torsion or curvature
  - The curvature $R(X,Y)$ measures the deviation of the map $\mathfrak{X}(M) \longrightarrow \mathrm{End}(\mathfrak{X}(M))$, $X \mapsto \nabla_X$ from being a Lie algebra homomorphism.
- The torsion $T(X,Y)$ is $\mathcal{F}$-linear in X and Y, and the curvature $R(X,Y)Z \in \mathfrak{X}(M)$ is $\mathcal{F}$-linear in X, Y, and Z.
  - To show this use $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$.
- A connection is compatible with the metric if $\forall X,Y,Z \in \mathfrak{X}(M)$:

  $$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

- A Riemannian connection (also called a Levi-Civita connection) is a torsion-free connection which is compatible w/ the metric.

- On any Riemannian manifold M, there exists a unique Riemannian connection defined by $\forall X, Y, Z \in \mathfrak{X}(M)$:

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}\{ X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle$$

since knowledge of $\langle X,Z\rangle$ for all $Z \in \mathfrak{X}(M)$ uniquely defines $X \in \mathfrak{X}(M)$.

- For a submanifold $M \subset \mathbb{R}^3$, we can define a projection map $pr : \Gamma(T\mathbb{R}^3|_M) \longrightarrow \mathfrak{X}(M)$ projecting vector fields along M to vector fields on M. The Riemannian connection on M is the projection of the Euclidean connection. For $X, Y \in \mathfrak{X}(M)$:

$$\nabla_X Y = pr(\underbrace{D_X Y}_{\text{vector field along } M \text{, but not necessarily on } M.})$$

- The set $\mathfrak{X}(M)$ of C^∞ vector fields on M is a:
  i) $\mathbb{R}$-vector space under $(cX)_p := cX_p$
  ii) $\mathcal{F} = C^\infty(M)$ module under $(fX)_p := f(p)X_p$
  - Note $\mathfrak{X}(M) = \Gamma(TM)$ can be realized as the space of sections of the tangent bundle TM.
- For any vector bundle $\pi : E \to M$, the space of sections $\Gamma(M)$ is a:
  i) $\mathbb{R}$ vector space under $(cs)(p) := cs(p) \in \pi^{-1}(\{p\})$
  ii) $\mathcal{F} = C^\infty(M)$ module under $(fs)(p) := f(p)s(p) \in \pi^{-1}(\{p\})$
- Let $E \to M$, $F \to M$ be two vector bundles over M. A bundle map $\varphi : E \to F$ over M induces a map on the space of sections:

$$\varphi_\# : \Gamma(E) \longrightarrow \Gamma(E)$$
$$s \longmapsto \varphi \circ s$$

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi"] & F \arrow[d, "\pi'"] \\ M \arrow[r, "id_M"] \arrow[u, bend left, "s"] & M \end{tikzcd}$$

where "over M" means φ restricts to id_M on M. This, along w/ the fact that φ restricts to each fiber, means $\varphi_\#(s)$ is a section of F.

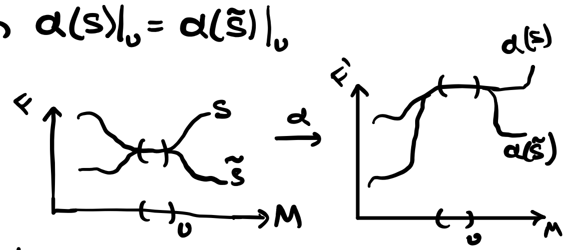
- The easiest way to think of sections $\Gamma(E)$ is as fields on the base manifold M valued in the fibers $F \cong \pi^{-1}(\{x\})$.
  - A map of sections is essentially just mapping fields to fields
- $E \longrightarrow M \longleftarrow F$ two vector bundles. A map $\alpha : \Gamma(E) \longrightarrow \Gamma(F)$ is local if whenever $s \in \Gamma(E)$ is identically zero on some open $U \subset M$, $\alpha(s)$ is also identically zero on U.
  - Ex: View $C^\infty(M)$ as $\Gamma(\mathbb{R} \times \mathbb{R})$. Then $\frac{d}{dt} : \Gamma(\mathbb{R}\times\mathbb{R}) \longrightarrow \Gamma(\mathbb{R}\times\mathbb{R})$ is local.
  - Likewise, $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is local b/c if $\omega|_U \equiv 0$, $(d\omega)|_U \equiv 0$ as well.

  α is a point operator if whenever $s \in \Gamma(E)$ vanishes @ $p \in M$, $\alpha(s)(p) = 0$ also.

and also a point operator

- Theorem: $E \to M \leftarrow F$ vector bundles. Any map $\alpha : \Gamma(E) \to \Gamma(F)$ which is $\mathcal{F} = C^\infty(M)$ linear is local.
  - It can also be shown if α_X is $\mathbb{R}$-linear + satisfies Leibniz $\alpha_X(fY) = (Xf)Y + f\alpha_X(Y)$, α_X is local $\Longrightarrow \nabla_X$ is local.

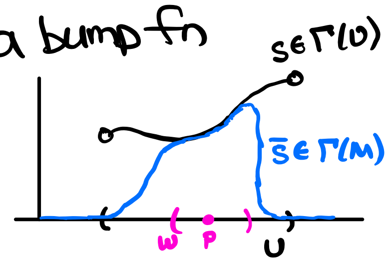
- Another way to think about local operators: α is local iff given any two sections $s, \tilde{s} \in \Gamma(U)$, if $s|_U = \tilde{s}|_U$ for some open $U|_M$, then $\alpha(s)|_U = \alpha(\tilde{s})|_U$
  - If this is not satisfied then α must know more info about s than is available globally.



- If $E \to M \leftarrow F$ vector bundles over M and $\alpha: \Gamma(E) \to \Gamma(F)$ is a local operator, we define its restriction to open $U \subset M$ as the unique map $\alpha|_U: \Gamma(U,E) \to \Gamma(U,F)$ s.t.:

$$\alpha_U(t|_U) = \alpha(t)|_U \qquad \forall t \in \Gamma(M,E)$$

  - We can only restrict local operators b/c w/o locality, we can't make the dfn well defined. We define α_U on $s \in \Gamma(U,E)$ @ p by locally extending s to $\bar{s} \in \Gamma(M,E)$ w/ $\bar{s}|_W = s|_W$, $p \in W \subset U$, and if α is not local, defining $\alpha_U(s)(p) := \alpha(\bar{s})(p)$ isn't well defined.
    - Restriction is fundamentally a local property $\Rightarrow$ need local operators.
      If α is not local, $\exists\, s \neq t \in \Gamma(M)$ w/ $s|_U = t|_U$ w/ $\alpha(s)|_U \neq \alpha(t)|_U$ by dfn, so you can't even ask for $\alpha_U(s|_U) = \alpha(s)|_U$ and $\alpha_U(t|_U) = \alpha(t)|_U$ b/c $s|_U = t|_U$ w/ $\alpha(s)|_U \neq \alpha(t)|_U$
- If E, F are over M and $\alpha: \Gamma(E) \to \Gamma(F)$ is $\mathcal{F}$-linear, then $\alpha|_U: \Gamma(U,E) \to \Gamma(U,F)$ is also $\mathcal{F}$-linear for any open $U \subset M$.
- Important theorem used in these proofs: If $U \subset M$ is open, $E \to M$ a vector bundle, given $s \in \Gamma(U,E)$, for any $p \in U$ $\exists$ a neighborhood $p \in W \subset U$ and a global section $\bar{s}: M \to E$ s.t. $\bar{s}|_W = s|_W$
  - $\bar{s}$ is constructed by multiplying s w/ a bump fn
  - We can extend $s: U \to E$ locally @ $p \in U$, but the extension only equals s on $W \subset U$.



- A frame for a vector bundle $E \to M$ of rank n is a set $\{e_i\}_{i=1}^n$ of sections $e_i: M \to E$ s.t. $\{e_i(p)\}_{i=1}^n$ is lin. independent $\forall p \in M$.
- Theorem: There is a one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{bundle maps } E \to F \text{ over } M \\ \varphi \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{F}\text{-linear maps } \Gamma(E) \to \Gamma(F) \\ \varphi_\# \end{array} \right\}$$

- So every $\mathcal{F}$-linear map of sections is uniquely induced by a corresponding bundle map.

- Any $\mathcal{F}$-multilinear k-alternating multilinear map $\omega : \mathfrak{X}(M)^k \to C^\infty(M)$ induces a corresponding k-form on M, and vice versa.
  - Thus differential forms can be uniquely identified as k-alternating $\mathcal{F}$-multilinear maps $\mathfrak{X}(M)^k \to C^\infty(M)$

- On a surface $M \subset \mathbb{R}^3$, the curvature is intimately related to the Riemannian connection $\nabla_X = \mathrm{pr}(D_X)$ and the shape operator L:

$$R(X,Y)Z = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y)$$

$$\nabla_X L(Y) - \nabla_Y L(X) - L([X,Y]) = 0$$

This allows us to express the Gaussian curvature K @ $p \in M$ as:

$$K_p = \langle R_p(e_1,e_2)e_2, e_1\rangle \qquad \text{(dfn of } K \text{ for a 2-manifold)}$$

where $\{e_1,e_2\}$ is an ONB for T_pM. More generally for $X,Y \in \mathfrak{X}(M)$, we have:

$$\langle R(X,Y)Y, X\rangle = \mathrm{II}(X,X)\,\mathrm{II}(Y,Y) - \mathrm{II}(X,Y)^2$$

so we see the interplay of the curvature R with II, which governs the curvature @ $p \in M$.

Generalizations to Vector Bundles and Differential Forms

- An affine connection is specific to the case of the tangent bundle: its domain/codomain is $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$.
  - We will next define connections on an arbitrary vector bundle. However, the lack of symmetry will mean we can't define torsion, since $\nabla_X Y - \nabla_Y X$ is not defined here. So, the notion of a Riemannian connection is unique to affine connections

Affine connection	$\rightarrow TM$
connection	$\rightarrow$ arbitrary vect. bundle.

- $E \to M$ a C^∞ vector bundle. A connection on E is a map:

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

such that:

i) $\nabla_X s$ is $\mathcal{F}$-linear in X, $\mathbb{R}$-linear in s.

ii) (Leibniz) For $f \in C^\infty(M)$:

$$\nabla_X(fs) = (Xf)s + f\nabla_X s$$

- Note $Xf = (df)X$ by dfn of the 1-form df, so $\nabla(fs) = (df)\cdot s + f\nabla s$

A section $s \in \Gamma(E)$ is flat if $\nabla_X s = 0$ $\forall X \in \mathfrak{X}(M)$.

- Given connections $\{\nabla^i\}$, any convex combination of the ∇^i is also a connection, i.e. any sum $\sum_i t^i \nabla^i$ satisfying $\sum_i t^i = 1$.
- Any vector bundle has a connection.

$\{v_1, \dots, v_n\}$ ONB for $\mathbb{R}^n$, $\varphi_\alpha : E_\alpha \to U_\alpha \times \mathbb{R}^n$, $e_i(p) := \varphi^{-1}(p, v_i)$ is an orthonormal frame on E_α

i) Take a local triv $\{(\varphi_\alpha, U_\alpha)\}$ and an orthonormal frame $\{e_1, \dots, e_n\}$ on U_α. Then for $s \in \Gamma(U_\alpha)$, write $s = \sum_{i=1}^n h_i e_i$ w/ $h_i \in C^\infty(U_\alpha)$. For $X \in \mathfrak{X}(M)$:

$$\nabla_X^\alpha s := \sum_{i=1}^n (Xh_i) e^i$$

defines a connection on the trivial bundle $E|_{U_\alpha}$.

ii) Stitch the connections together. Take a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. Then for $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$, we define:

$$\nabla_X s := \sum_\alpha \rho_\alpha \nabla_X^\alpha s|_{U_\alpha}$$

This sum is convex b/c on each $p \in M$, local finiteness implies $\exists U \ni p$ w/ only finitely many $\rho_\alpha|_U$ nonzero. $\sum_\alpha \rho_\alpha = 1$ implies this is a convex sum, so ∇_X is a connection.

- For a connection on an arbitrary vector bundle, we can still define curvature:

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

$$R(X,Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

$R(X,Y)Z$ is $\mathcal{F}$-linear and hence a local / point operator

- Note $R(X,Y) = -R(Y,X)$, so $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathrm{End}(\Gamma(E))$ is an alternating bilinear map:

$$R_p : T_pM \times T_pM \longrightarrow \mathrm{End}(E_p) \qquad E_p = \pi^{-1}(\{p\})$$

which gives rise to the curvature tensor.

- Given a vector bundle $E \to M$, a Riemannian metric is an inner product $\langle \cdot,\cdot \rangle_p : E_p \times E_p \to \mathbb{R}$ on each fiber which is C^∞ in the sense that if s,t are C^∞ sections in $\Gamma(E)$, then $\langle s,t\rangle \in C^\infty(M)$. A Riemannian bundle is a vector bundle w/ a metric.
  - Any vector bundle has a Riemannian connection.
- A connection ∇ on a Riemannian bundle is a metric connection if:

  (∇ is metric compatible)

$$X\langle s,t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$$

  - The connection induced on the trivial bundle $\nabla_X(h^i e_i) = (Xh^i)e_i$ is a metric connection
- Suppose $\{\nabla^i\}_{i=1}^m$ are metric connections on M. Then if $a_1,\dots,a_m$ are C^∞ functions on M which sum to 1, $\sum_{i=1}^m a^i = 1$, then $\nabla = \sum_{i=1}^m a^i \nabla^i$ is also a metric connection.
- On any Riemannian bundle, there is a metric connection $\nabla_X = \sum_\alpha \rho_\alpha \nabla_X^\alpha$, where $\{U_\alpha, \varphi_\alpha\}$ is a local trivialization, $\{\rho_\alpha\}$ is subordinate to $\{U_\alpha\}$, and ∇_X^α is the metric connection induced on U_α.
- Any connection ∇ is local in the sense that $\forall U \subset M$ open, $X \equiv 0$ on U or $s \equiv 0$ on U implies $\nabla_X s \equiv 0$ on U as well. This is sufficient for us to restrict ∇ to U to a map:

$$\nabla^U : \mathfrak{X}(U) \times \Gamma(U,E) \to \Gamma(U,E)$$

  such that for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$ we have:

$$\nabla^U_{X|_U}(s|_U) = (\nabla_X s)\big|_U$$

- We move towards a coordinate description of ∇, R, and T (for tangent bundles). Given a rank r vector bundle $E \to M$, a trivializing open set U, and a frame $\{e_1,\dots,e_r\} \subset \Gamma(U)$ for $E|_U$, any section $s \in \Gamma(U)$ can be written $s = a^i e_i$ w/ $a^i \in C^\infty(U)$. The action of ∇_X on s is then determined completely through the Leibniz rule and knowing $\nabla_X e_i$. We can expand:

$$\nabla_X e_i := \omega_i^{\ j}(X) e_j \qquad\qquad \omega^i_{\ j} \in \Omega^1(U)$$

  where b/c $\nabla_{fX} e_i = f\nabla_X e_i \Rightarrow \omega_i^{\ j}(fX)e_j = f\omega_i^{\ j}(X)e_j$, we see $\omega^i_{\ j}$ is a matrix of 1-forms on U. $\omega^i_{\ j}$ are called connection forms and ω the connection matrix.
- We can also expand $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\Gamma(M))$ in the basis $\{e_i\}_{i=1}^n$ to define curvature forms

$$R(X,Y)e_i := \Omega_i^{\ j}(X,Y)e_j \qquad\qquad \Omega^i_{\ j} \in \Omega^2(U)$$

  where $\mathcal{F}$-bilinearity and antisymmetry of $R(X,Y)$ implies $\Omega^i_{\ j} \in \Omega^2(U)$. Ω is the curvature matrix.
- Second structural equation: On any triv. open set U w/ frame $\{e_i\}_{i=1}^n$ of $E_U \to U$, we have:

$$\Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} \qquad\qquad \Omega = \begin{pmatrix} \Omega^1_1 & \Omega^1_2 & \dots \\ \Omega^2_1 & \Omega^2_2 & \dots \\ \vdots & \vdots & \end{pmatrix}$$

  - Note in QFT, F is Ω and A is ω.
  - This follows from, for $\alpha \in \Omega^1(U)$:

$$(d\alpha)(X,Y) = X\alpha(Y) - Y\alpha(X) - [X,Y]$$

  which can be worked out from the Cartan homotopy formula $\mathcal{L}_X = \iota_X d + d\iota_X$
- An open set $U \subset M$ s.t. $\exists$ a C^∞ frame $\{e_i\}_{i=1}^n$ on E_U is called a framed open set.
- Any connection on a framed open set determines a unique connection matrix ω, and likewise any matrix of 1-forms $\omega^i_{\ j} \in \Omega^1(U)$ determines a connection $\forall X,Y \in \mathfrak{X}(U)$, as $Y = h^i e_i$, so $\nabla_X Y := (Xh^i)e_i + h^i \underbrace{\omega_i^{\ j}(X)e_j}_{\nabla_X e_j}$

- Given a framed open set U, we can always take the C^∞ frame $\{e_1,\dots,e_n\}$ to be orthonormal by Gram-Schmidt.
- Summary: On a framed open set U, a connection $\nabla : \mathfrak{X}(U)\times\Gamma(U)\to\Gamma(U)$ is exactly determined by the $n\times n$ curvature matrix $\omega_i^j \in \Omega^1(U)$, defined on a frame $\{e_1,\dots,e_n\}$ for $E_U \to U$ as $\nabla_X e_i = \omega_i^j(X)e_j$.
- $E\to M$ a Riemannian bundle, ∇ a connection on E.

i) If ∇ is metric compatible, then on any trivialization U w.r.t. ONF $\{e_i\}_{i=1}^n$, ω_i^j is antisymmetric:

$$\omega_i^j = -\omega_j^i$$

ii) If $\forall p\in M$ has a trivializing n'borhood w/ ω_i^j antisymmetric, then ∇ is metric compatible

– Furthermore if ω_i^j is antisymmetric, so is Ω_i^j. Thus for a metric compatible connection ∇, its connection matrix ω_i^j and curvature matrix Ω_i^j are antisymmetric w.r.t. an ONF

- Affine connections (connections on the tangent bundle) have a defined torsion $T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]$.

This gives rise on any framed open set U w/ C^∞ frame $\{e_i\}_{i=1}^n$ to torsion forms as $T(X,Y)\in\mathfrak{X}(U)$:

$$T(X,Y)=\tau^i(X,Y)e_j \qquad \tau^i\in\Omega^2(M) \text{ b/c } T \text{ is } \mathcal{F}\text{-bilinear and antisymmetric.}$$

Let $\{e_i\}_{i=1}^n$ be a frame for the tangent bundle of $U\subset M$, $\{\theta^i\}_{i=1}^n$ the dual basis to $\{e_i\}$. The first structural equation relates the torsion forms to these:

$$\tau^i = d\theta^i + \omega^i_j\wedge\theta^j$$

- If M is a Riemannian manifold, $U\subset M$ a framed open subset w/ ONF $\{e_i\}_{i=1}^n$, dual basis $\{\theta^i\}_{i=1}^n$, $\exists!$ antisym matrix of one-forms $\omega_i^j\in\Omega^1(U)$ s.t.:

$$d\theta^i + \omega^i_j\wedge\theta^j = 0$$

– This connection matrix corresponds to the Riemannian connection ∇ as it is torsion free/metric compatible.

- For a surface $M\subset\mathbb{R}^3$, things are particularly simple. Recall $\nabla_X Y = \mathrm{pr}_M(D_X Y)$ for D_X the Euclidean connection on $\mathbb{R}^3$, so take an ONF $\{e_1,e_2\}$ of a neighborhood $U\ni p$ of M. Then for $e_3 := \{e_1,e_2,e_3\}$ an ONF of $T\mathbb{R}^3|_M$, we have:

$$\begin{cases} D_X e_1 = 0 + \omega_{12}(X)e_2 + \omega_{13}(X)e_3 \\ D_X e_2 = \omega_{21}(X)e_1 + 0 + \omega_{23}(X)e_3 \\ D_X e_3 = \omega_{31}(X)e_1 + \omega_{32}(X)e_2 + 0 \end{cases}$$

$\omega_{ij}\in\Omega^1(\mathbb{R}^3)$ connection matrix for D_X w.r.t. $\{e_1,e_2,e_3\}$

$] = \pm L(X)$ as $\pm D_{X_p}N := L(X_p)$ is the shape operator, $N = \pm e_3$

– Since $\nabla = \mathrm{pr}(D)$, the connection matrix for ∇ w.r.t. $\{e_1,e_2\}$ is $\begin{pmatrix}0 & \omega_{12}\\ -\omega_{12} & 0\end{pmatrix}$. Note for 2d, $\omega_i^k\wedge\omega_k^i = 0$ by the antisymmetry, so for the Riemannian connection ∇ on M:

$$\Omega = d\omega + \cancel{\omega\wedge\omega} = \begin{pmatrix}0 & d\omega_{12}\\ -d\omega_{12} & 0\end{pmatrix} = \begin{pmatrix}0 & \Omega_2^1\\ -\Omega_2^1 & 0\end{pmatrix}$$

(only for 2 manifolds embeddable in $\mathbb{R}^3$)

we see ω and Ω only have one nonzero component, and everything about the Riemannian connection is determined by ω_{12}.

– Using the 2nd structural eqn + curvature tensor for D on $\mathbb{R}^3$ being 0, we get $d\omega_{12} + \omega_{13}\wedge\omega_{32} = 0$, so:

$$\Omega_2^1 = \omega_1^2\wedge\omega_3^2$$

which is the Gauss curvature eqn, as can show $K = (\omega_1^3\wedge\omega_3^2)(e_1,e_2)$ so the Gaussian curvature is:

$$K = \Omega_2^1(e_1,e_2)$$

- Aside: A tensor is a point operator. The map $\langle R(X,Y)Z,W\rangle$ is $\mathcal{F}$-linear $\Rightarrow$ point operator $\Rightarrow$ tensor. It is antisymmetric in X/Y, and in Z/W. This is the curvature tensor $R_{\mu\nu\alpha\beta}$ in GR, defined as: $\swarrow R(e_i,e_j)e_k = R_{ijk}{}^{\ell} e_\ell$

$$R_{\mu\nu\alpha}{}^{\rho} := dx^{\rho}(R(\partial_\mu,\partial_\nu)\partial_\alpha) = \Omega(\partial_\mu,\partial_\nu)_\alpha^{\ \beta} dx^{\rho}(\partial_\beta) = \Omega(\partial_\mu,\partial_\nu)_\alpha{}^{\rho}$$

(antisym in $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \rho$)

- So for a 2d subspace of T_pM, $K = R^1{}_{122}$.

- To determine the curvature/connection forms for the Riemannian connection ∇ on a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$:
  i) Find an ONF $\{e_i\}_{i=1}^n$ for the tangent bundle TM.
  ii) Find the dual coframe $\{\theta^i\}_{i=1}^n$ to $\{e_i\}_{i=1}^n$ by applying $\theta^i(e_j) = \delta$. Note the $\{\theta^i\}$ should provide an ONF for the bundle T^*M, i.e. each $\theta^i \in \Omega^1(M)$.
  iii) That ∇ is the Riemannian connection implies it is torsion free, so use 1st structural eqn:

$$0 = \tau^i = d\theta^i + \omega_k^i \wedge \theta^k$$

  Using the fact that ∇ is metric compatible and $\{e_i\}$ is orthonormal, we can also apply the fact that $\omega_j{}^i$ is antisymmetric, $\omega_j{}^i = -\omega_i{}^j$, to solve for ω.
  iv) Once ω is known, apply 2nd structural eqn:

$$\Omega_i{}^j = d\omega_i{}^j + \omega_i{}^k \wedge \omega_k{}^j$$

  v) For a surface the Gaussian curvature is simply:

$$K = \Omega_2{}^1(e_1,e_2)$$

Geodesics

- A velocity vector field along a curve $c:[a,b]\to M$ is a map $V:[a,b]\to \coprod_{t\in[a,b]} T_{c(t)}M$ s.t. $V(t)\in T_{c(t)}M$ ($V(t)$ along $c(t)$, not necessarily proportional to $c(t)$). V is C^∞ iff $V(t)f$ is C^∞ $\forall\, t\in[a,b]$, $f\in C^\infty(M)$.
  The space of C^∞ velocity fields along c is:

  $$\Gamma(TM|_{c(t)})$$

  – For $V(t)=v^i(t)\partial_i\in\Gamma(TM|_{c(t)})$, its derivative is:

  $$\frac{dV}{dt}:=\left(\frac{dv^i(t)}{dt}\right)\partial_i$$

  and $\frac{d}{dt}:\Gamma(TM|_{c(t)})\to\Gamma(TM|_{c(t)})$ is $\mathbb{R}$-linear in V and satisfies Leibniz, $\frac{d}{dt}(fV)=\frac{df}{dt}V+f\frac{dV}{dt}$. $\frac{d}{dt}(f(c(t)))$

  Furthermore if $M\subset\mathbb{R}^n$ and $V(t)$ $\underbrace{\text{induced}}_{\text{i.e. } V(t)=\tilde{V}_{c(t)}}$ by a vector field $\tilde{V}\in\Gamma(T\mathbb{R}^n|_M)$ along M, then:

  $$\frac{dV}{dt}=D_{c'(t)}\tilde{V}$$

- The properties above are extendable to an arbitrary Riemannian manifold w/ affine connection ∇. For a curve $c:[a,b]\to M$, there exists a unique map

  $$\frac{D}{dt}:\Gamma(TM|_{c(t)})\to\Gamma(TM|_{c(t)})$$

  called the covariant derivative along $c(t)$, such that:

  f can be $g\circ c$, where g is defined on $\mathrm{im}(c)$.

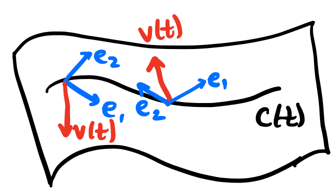
  i) DV/dt is $\mathbb{R}$-linear
  ii) D/dt satisfies the Leibniz rule, i.e. $\frac{D(fV)}{dt}=\frac{df}{dt}V+f\frac{DV}{dt}$ for any $f\in C^\infty([a,b])$
  iii) D/dt is compatible w/ the connection:

  $$\frac{DV}{dt}=\nabla_{c'(t)}\tilde{V}$$

  for any vector field V induced by $\tilde{V}$, i.e. $V(t)=\tilde{V}|_{c(t)}$ w/ $\tilde{V}\in\mathfrak{X}(M)$

  – On a framed open set $(U,\{e_i\}_{i=1}^n)$, D/dt can be uniquely defined as:

  $$\frac{D}{dt}\left(v^i(t)e_{i,c(t)}\right)=\frac{dv^i}{dt}e_{i,c(t)}+v^i\nabla_{c'(t)}\overbrace{e_i}^{e_{i,c(t)}}$$



  – If ∇ is metric compatible, so is D/dt, i.e.:

  $$\frac{d}{dt}\langle V,W\rangle=\left\langle\frac{DV}{dt},W\right\rangle+\left\langle V,\frac{DW}{dt}\right\rangle$$

- We can push forward vector fields along a curve w/ a map $f:M\to\tilde{M}$. W/ an arbitrary vector field in $\mathfrak{X}(M)$ we can only do this if f is injective, but w/ vector fields along a curve f need not be injective since the time $t\in[a,b]$ @ which the field takes a vector value matters too. Define:

  $$f_*:\Gamma(TM|_{c(t)})\to\Gamma(T\tilde{M}|_{f(c(t))})$$

  $$(f_*V)(t):=f_{*,c(t)}(V(t))$$

  induced differential $f_{*,p}:T_pM\to T_{f(p)}\tilde{M}$

- A connection preserving diffeomorphism is a diffeomorphism $f: M \to \tilde{M}$ s.t. $f_*(\nabla_X Y) = \tilde{\nabla}_{f_*X} f_*Y$. (we can define f_* on $\mathfrak{X}(M) \to \mathfrak{X}(\tilde{M})$ b/c f is a diffeo). Under such a diffeomorphism, for a curve $c:[a,b] \to M$ w/ covariant derivative D/dt, $\tilde{c} = f \circ c : [a,b] \to \tilde{M}$ w/ derivative $\tilde{D}/dt$, the covariant derivative is preserved in the following sense:

$$f_*\left(\frac{DV}{dt}\right) = \frac{\tilde{D}}{dt}(f_* V)$$

- Christoffel symbols: Allow us to define and work with ∇_X in a coordinate chart.
  - This is similar to the connection forms $\omega_i^j \in \Omega^1(M)$, except connection forms describe the data of ∇_X relative to a frame for TM, and are more general in that they are defined for a frame on an arbitrary connection on a vector bundle, whereas Christoffel symbols are used for affine connections.
  - Note every chart (U, x^i) determines a frame $\{\partial_i\}$ on the tangent bundle.
  - For a chart (U, x^i) on M of dimension n, $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ an affine connection, the Christoffel symbols Γ_{ij}^k are defined as:

    $$\nabla_{\partial_i} \partial_j := \Gamma_{ij}^k \partial_k$$

    since $\nabla_X \partial_j \in \mathfrak{X}(M) = \text{span}\{\partial_k\}$. These symbols tell us how the frame $\{\partial_i\}$ changes w.r.t. the connection.
  - An affine connection ∇ is torsion free iff Γ_{ij}^k is symmetric in ij:

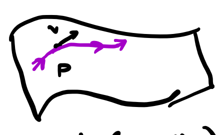
    $$\Gamma_{ij}^k = \Gamma_{ji}^k$$

- Note that for connection forms we consider an ONF for the bundle $E \to M$, while the coordinate vectors $\{\partial_i\}$ need not be orthonormal.

- A parameterized curve $c:[a,b] \to M$ is a geodesic if the covariant derivative of its velocity vector field along $c(t)$ vanishes, i.e. if $T(t) = c'(t) \in T_{c(t)}M$, then:

  $$\frac{DT}{dt} = 0$$

  The geodesic is maximal if its domain cannot be extended
- Defining the speed of a curve as $\|c'(t)\| = \sqrt{\langle c'(t), c'(t) \rangle}$, a geodesic must have constant speed.
- For a surface $M \subset \mathbb{R}^n$, since $DV/dt = pr(dV/dt)$, a geodesic is defined by having no variation when its the derivative dT/dt is projected onto $\coprod_{t \in [a,b]} T_{c(t)}M$.
- A geodesic $\gamma(t)$ can be reparameterized as $\tilde{\gamma}(t) := \gamma(u(t))$ for $u : [c,d] \to [a,b]$. $\tilde{\gamma}$ is a geodesic iff $u(t)$ is linear in t, i.e. $u(t) = \alpha + \beta t$.
- Existence/uniqueness: Let M have affine connection ∇, $p \in M$ and $v \in T_pM$. Then there exists a geodesic $\gamma(t)$ with $\gamma(0) = p$ and $\gamma'(0) = v$. γ is unique in the sense that if $\tilde{\gamma}$ is another geodesic w/ initial point p/velocity v, then $\tilde{\gamma} \equiv \gamma$ on the intersection of their domains.



  - This is a generalization of straight lines in $\mathbb{R}^n$.

(Note $\Rightarrow$ geodesics exist locally)

- Any connection preserving diffeomorphism $f:(M,\nabla)\to(\tilde{M},\tilde{\nabla})$ takes geodesics to geodesics, hence any isometry takes geodesics to geodesics.
- **Geodesic equations**: A parameterized curve $\gamma(t)$ is a geodesic iff for any chart $(U,\varphi)=(U,x^i)$ on M, the following system of equations for $y^i(t):=x^i\circ\gamma(t)$ is satisfied:

  $\uparrow$ $\gamma(t)=(y^1(t),\dots,y^n(t))$ in coordinates

$$\ddot{y}^k+\dot{y}^i\Gamma^k_{ij}\dot{y}^j=0$$

  - Note in flat space w/ $\Gamma\equiv 0$, this reduces to $\ddot{y}=0$.
  - In coordinates, note $y^i(t)=(x^i\circ c)(t)$, and c is given in coordinates as $(y^1(t),\dots,y^n(t))$.
    The derivative of $c(t)$ is:

$$T(t)=c'(t)=\dot{y}^i(t)\partial_{i,c(t)}=\dot{y}^i\partial_i$$

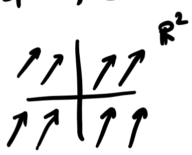
    where note the dfn of $c'(t)$ is the pushforward of d/dt:

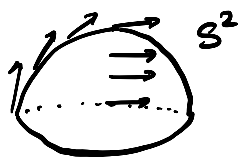
$$c'(t):=c_{*,c(t)}\left(\frac{d}{dt}\Big|_t\right)\in T_{c(t)}M$$

- **Parallel transport**: A vector field V along a curve $c:[a,b]\to M$ is parallel if:

$$\frac{DV}{dt}=0$$

  - A geodesic is simply a curve which parallel transports its tangent vector $T(t)=c'(t)$.
  - A parallel vector field "doesn't change" w.r.t. the connection on M as we move along c.
    - For flat space, this is literally just parallel lines

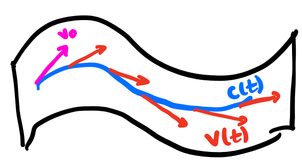




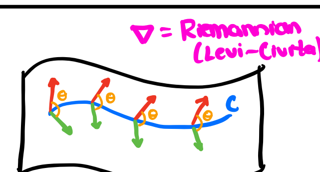
- A vector field $V(t)=v^i(t)e_{i,c(t)}$ on a framed open set containing $\mathrm{im}(c)$ is parallel iff $DV/dt=\dot{v}^ie_i+v^iDe_i/dt=\dot{v}^ie_i+v^i\overbrace{\nabla_{c'(t)}e_i}^{\omega(c'(t))^j_i e_j}$,
  so v^ie_i is parallel along c iff:

$$\dot{v}^i(t)+\omega_j{}^i(c'(t))v^j(t)=0$$

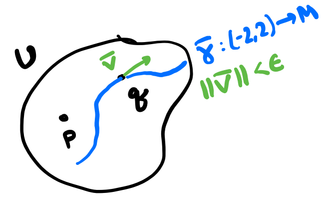
  for each t.
- If $V(t)$ is parallel along the entire curve $c:[a,b]\to M$, we say $V(b)$ is the **parallel transport** of $V(a)$ along c.
  - Uniqueness of the soln to parallel ODE implies $V(b)$ is unique given c and $V(a)$, if it exists.
- **Existence**: Given a curve $c:[a,b]\to M$ and a vector $v_0\in T_{c(a)}M$, $\exists!$ parallel vector field $V(t)$ along c s.t. $V(a)=v_0$.



- Note: Geodesics exist locally on a n'borhood of $p\in M$, while a parallel vector field exists globally for any curve c.
- For a Riemannian connection ∇ on a Riemannian manifold M, parallel translation preserves lengths/angles: If $V(t)$ and $W(t)$ are both parallel vector fields on $c(t)$, then $\langle V(t),W(t)\rangle$ and $\|V(t)\|$ are constant.



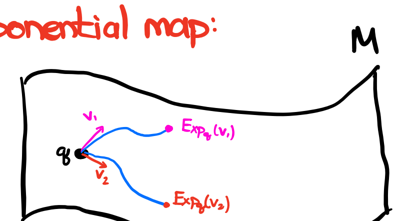
- Given $p \in M$, $v \in T_pM$, $\exists$ a geodesic $\gamma : (-a,a) \to M$ w/ $\gamma(0) = p$, $\gamma'(0) = v$. However, the domain size of γ, a, may be very small. We can lengthen the domain of γ by rescaling $\tilde{\gamma}(t) := \gamma(t/k)$ for $k > 1$, but this shortens the norm $\|\tilde{\gamma}'(0)\|$ of the initial vector.
- **Theorem**: Given $p \in M$, $\exists$ a nborhood $U \ni p$ and $\epsilon > 0$ s.t. $\forall q \in U$, $\forall \bar{v} \in T_q$ w/ $\|\bar{v}\| < \epsilon$, there is a unique geodesic $\bar{\gamma} : (-2,2) \to M$ w/ $\bar{\gamma}(0) = q$, $\bar{\gamma}'(0) = \bar{v}$.
  - This is a rigorous way of saying we can find a bound ϵ s.t. if $\|v\| < \epsilon$, we can define a geodesic w/ velocity v that has an arbitrarily large domain.
  - This ϵ is also a uniform bound on the nborhood U.
  - This only holds for a Riemannian connection.



- Let $q \in M$, $v \in T_qM$, and $\gamma_{q,v} : [a,b] \to M$ the unique maximal geodesic w/ $\gamma_{q,v}(0) = q$ and $\gamma_{q,v}'(0) = v$. If $\gamma_{q,v}(1)$ is defined, we define the **exponential map**:

$$\mathrm{Exp}_q : T_qM \to M$$
$$\mathrm{Exp}_q(v) := \gamma_{q,v}(1)$$

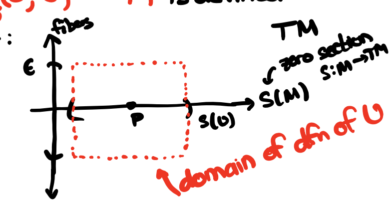
Generally $\mathrm{Exp}_p : T_pM \to M$ is only defined for subset $V \subset T_pM$, since if $\|v\|$ is too large the geodesic $\gamma_{p,v}$ will only be defined on $(-\delta, \delta)$ w/ $\delta < 1$.



  - This map moves us along the geodesic in the direction v.
  - On a manifold w/ a Riemannian connection, previous theorem implies for any $p \in M$, there is a neighborhood $U \ni p$ and $\epsilon > 0$ s.t. for each $q \in U$, $\mathrm{Exp}_q : B_q(0,\epsilon) \to M$ is defined on the ball $B_q(0,\epsilon) \subset T_qM$. This means on U, Exp is defined on the cylinder:

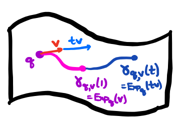
$$\mathrm{Exp} : U \times \coprod_{q \in U} B_q(0,\epsilon) \to M$$

and formally Exp is defined on the ϵ-tube around $s(U) \subset TM$, where $s : M \to TM$ is the zero section.



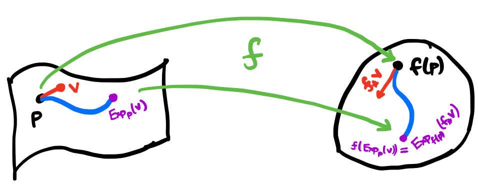
- The unique maximal geodesic about $q \in M$ in direction $v \in T_qM$ is:

$$\gamma_{q,v} : [a,b] \to M$$
$$\gamma_{q,v}(t) := \mathrm{Exp}_q(tv)$$



  - The parameter t increases the speed of the geodesic, and this smoothly lets us increase the domain of γ.
- Let $f : M \to N$ be an isometry, $p \in M$, $V \subset T_pM$ and $U \subset T_{f(p)}N$ subsets on which the exponential is defined. Then the diagram commutes:

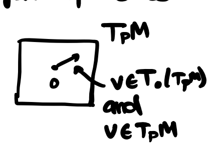
$$\begin{tikzcd} V \subset T_pM \arrow[r, "f_{*,p}"] \arrow[d, "\mathrm{Exp}_p"] & U \subset T_{f(p)}N \arrow[d, "\mathrm{Exp}_{f(p)}"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$



i.e. Exp_p is a natural transformation between two (complicated) functors.

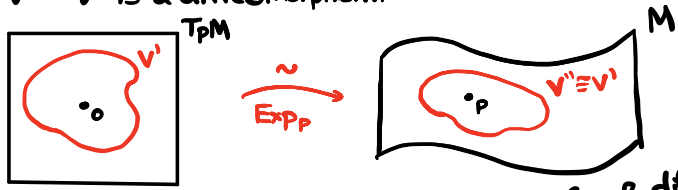
- Since $\mathrm{Exp}_p : T_pM \to M$ and $T_pM \cong \mathbb{R}^n$, we can compute the pushforward $(\mathrm{Exp}_p)_{*,0}$ on the tangent space to 0 in T_pM, $T_0(T_pM)$, which is $\cong T_pM$, $V \subset T_pM$ open s.t. Exp_p is defined on V. Thus we have a map:

$$(\mathrm{Exp}_p)_{*,0} : T_0(T_pM) = T_pM \to T_{\mathrm{Exp}_p(0)}M = T_pM$$



and it can be shown this map is the identity, $(\mathrm{Exp}_p)_{*,0} = \mathrm{id}_p : T_pM \to T_pM$. The inverse function theorem implies that $\mathrm{Exp}_p : V \subset T_pM \to M$ is a local diffeomorphism, i.e. $\exists$ neighborhoods $0 \in V' \subset T_pM$ and $p \in V'' \subset M$ s.t.

$\mathrm{Exp}_p : V' \to V''$ is a diffeomorphism.



- **Normal coordinates**: Since $\mathrm{Exp}_p : T_pM \to M$ is a local diffeomorphism, we can use it to locally transfer coordinates from T_pM to M. Let $\{e_1, \dots, e_n\} \subset T_pM$ be an ONB w/ corresponding coordinate functions $\{r^1, \dots, r^n\}$. Take $V \subset T_pM$, $p \in U \subset M$ s.t. $\mathrm{Exp}_p : V \to U$ is a diffeomorphism. Then $x^i := r^i \circ \mathrm{Exp}_p^{-1} : U \to \mathbb{R}$ are the normal coordinates on U, and the chart (U, x^i) is a normal neighborhood.
  - Normal coordinates are locally flat in that the Christoffel symbols and all derivatives of the metric vanish at p:

$$\nabla_{\partial_i} \partial_j \big|_p = 0 \qquad\qquad \partial_k g_{ij}\big|_p = 0$$

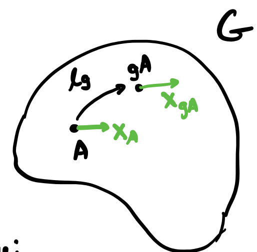
  - Geodesics in normal coordinates are lines. Suppose $\gamma(t)$ is the geodesic through $p \in M$ w/ initial direction $a = a^i e_i \in T_pM$, and γ is on a normal n'borhood (U, x^i) of p. Then:

$$x^i(\gamma(t)) = a^i t$$

  so $\gamma(t)$ is locally a straight line through p.

- G a Lie group w/ algebra $\mathfrak{g} := T_eG$. Left multiplication is the operation:

$$\ell_g : G \to G$$
$$\ell_g(A) := gA$$



A vector field $X \in \mathfrak{X}(G)$ is left-invariant if $\forall g, A \in G$, we have:

$$(\ell_g)_* X_A = X_{gA}$$

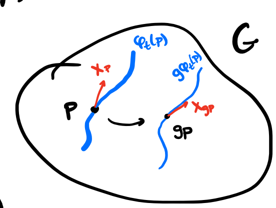
For any $X_e \in \mathfrak{g} = T_eG$, we can define a left invariant vector field $\tilde{X} \in \mathfrak{X}(G)$ by:

$$\tilde{X}_g := (\ell_g)_* X_e$$

and this defines a bijection between $\mathfrak{g}$ and the $\overbrace{\text{LI}}^{\text{any LI field } X \text{ is uniquely determined by } X_e, \text{ as } (\ell_g)_* X_e \text{ must equal } X_g.}$ vector fields on G.

- Recall an integral curve of $X \in \mathfrak{X}(M)$ is a curve $c : (a,b) \to M$ s.t. $c'(t) = X_{c(t)}$. Left translation preserves integral curves, i.e. if $p \in G$ and $\varphi_t(p) : (a,b) \to G$ is an integral curve of a left invariant vector field X starting @ p (so $\varphi_0(p) = p$), then $g\varphi_t(p)$ is the integral curve of X starting @ $gp \in G$. Thus we have:

$$\varphi_t \circ \ell_g = \ell_g \circ \varphi_t$$

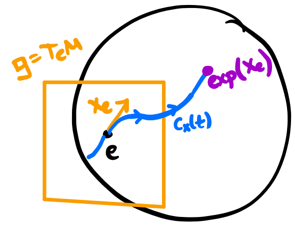


for a LI vector field X.

- A global flow of $X \in \mathfrak{X}(G)$ is a map $\varphi : \mathbb{R} \times G \to G$ s.t. $\frac{d}{dt}\varphi_t(p) = X_{\varphi_t(p)}$ (i.e. a maximal integral curve for X starting @ p). Any LI vector field has a global flow.

- To rigorously define the exponential map $\exp: \mathfrak{g} \to G$, we let $X_e \in \mathfrak{g} = T_eG$ and let $X \in \mathfrak{X}(M)$ be the LI vector field extending X_e. We let $c_X(t) = \varphi_t(e)$ be the global integral curve of X, and we define:

$$\exp(X_e) := c_X(1)$$



for $X_e \in \mathfrak{g}$. This coincides w/ the normal exponential map on a Lie group.

- Switching gears: A Riemannian manifold is geodesically complete if the domain of every geodesic can be extended to $\mathbb{R}$.
  - Ex: $\mathbb{R}^n$, S^n. $\mathbb{R}^2 \setminus \{\vec{0}\}$ is <u>not</u> b/c $\{(t,t) \in \mathbb{R}^2 \setminus \{\vec{0}\} : t \geq 1\}$ cannot be extended
  - Given a Riemannian manifold, we can define a metric d on M by:

$$d(x,y) = \inf_c \operatorname{len}(c) \qquad \operatorname{len}(c) := \int_{\operatorname{dom}(c)} \|c'(t)\|\,dt$$

where c is any curve $[a,b] \to M$ w/ $c(a) = x, c(b) = y$.

- M is geodesically complete iff (M,d) is complete as a metric space.
- Change of frame: Suppose $\{e_i\}_{i=1}^n$ and $\{\bar{e}_j\}_{j=1}^n$ are two frames on $U \subset M$ open, related by:

$$\bar{e}_j = \sum_i A^i_j e_i$$

(if $\{\bar{e}_j\}$ and $\{e_i\}$ are orthonormal, then A is unitary. Let $\{\theta^i\}$ and $\{\bar{\theta}^j\}$ be the dual basis of one-forms, $\theta^i(e_j) = \bar{\theta}^i(\bar{e}_j) = \delta^i_j$. Writing $e := (e_1 \dots e_n)$ and $\theta := (\theta^1 \dots \theta^n)^T$, this is expressed as $\theta(e) = \mathbb{1} \in M_{n \times n}(\mathbb{R})$. Then:

$$\bar{e} = eA \qquad \bar{\theta} = A^{-1}\theta \qquad A = \begin{pmatrix} A^1_1 & A^1_2 & A^1_3 & \dots \\ A^2_1 & A^2_2 & & \\ A^3_1 & & & \\ \vdots & & & \end{pmatrix}$$

Furthermore, if $\bar{\theta}^i = B^i_j \theta^j$ are two dual coframes, then:

$$\bar{\theta}^1 \wedge \dots \wedge \bar{\theta}^n = \det(B)\, \theta^1 \wedge \dots \wedge \theta^n$$

- Volume forms: On an oriented manifold M, no way to single out a unique volume form. But w/ a metric, we can. Let (U, x^i) be a chart on M, and apply Graham-Schmidt to $\{\partial_i\}_{i=1}^n$ over U to get an ONF $\{e_i\}_{i=1}^n$. If $\{\theta^i\}_{i=1}^n$ is the dual coframe, we define the volume form:

$$\operatorname{vol}_M := \theta^1 \wedge \dots \wedge \theta^n$$

  - This dfn is independent of ONF $\{\bar{e}_i\}$ (assuming it is in the same orientation class as $\{e_i\}$) b/c if $\{\bar{e}_i\}$ is another ONF, $\bar{e} = eA$ w/ $A \in SO(n)$, hence $\overline{\operatorname{vol}} = \det(A^{-1})\operatorname{vol} = \operatorname{vol}$.
  - Note the assumption that M has a metric is necessary for us to have an ONF.
- Ex: In $\mathbb{R}^2$, $\operatorname{vol}_{\mathbb{R}^2} = dx \wedge dy$. On $\mathbb{H}$ ($\mathbb{R}^2_{y>0}$ w/ $\langle \cdot,\cdot \rangle = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$), an ONF is $e_1 = y\partial_x$, $e_2 = y\partial_y$, so $\theta^1 = \frac{1}{y}dx$, $\theta^2 = \frac{1}{y}dy$, and:

$$\operatorname{vol}_{\mathbb{H}} = \frac{1}{y^2}\, dx \wedge dy$$

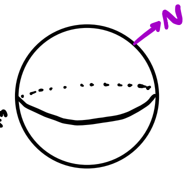
- Recall for $X \in \mathfrak{X}(M)$, we define interior multiplication $\iota_X : \Omega^q \to \Omega^{q-1}$ by:

$$(\iota_X \omega)(X_2, \dots, X_n) := \omega(X, X_2, \dots, X_n)$$

and explicitly $\iota_X(\alpha^1\wedge\dots\wedge\alpha^n)=\sum_{i=1}^{n}(-1)^{i-1}\alpha^i(X)\,\alpha^1\wedge\dots\wedge\hat{\alpha}^i\wedge\alpha^n$ and $\iota_X\circ\iota_X=0$.

- If M has boundary ∂M and N is an outward pointed normal vector field (so if the ordered basis $[X_2,\dots,X_n]$ is the orientation class for ∂M, then $[N,X_2,\dots,X_n]$ determines the class $[M]$), then the volume forms of M and ∂M are related:

  $vol_{\partial M}=\iota_N(vol_M)$

  - Using this w/ $S^n(a)\subset\mathbb{R}^{n+1}$, the volume form for $S^n(a)$ is:

  $$vol_{S^n}=\iota_N(vol_{\mathbb{R}^{n+1}})=\iota_N(dx^1\wedge\dots\wedge dx^{n+1})=\sum_{i=1}^{n+1}(-1)^{i-1}dx^i\left(\frac{x^j\partial_j}{\|\vec{x}\|}\right)dx^1\wedge\dots\wedge\hat{dx^i}\wedge\dots\wedge dx^{n+1}$$

  $$=\frac{1}{a}\sum_{i=1}^{n+1}(-1)^{i-1}x^i\,dx^1\wedge\dots\wedge\hat{dx^i}\wedge\dots\wedge dx^{n+1}$$

  $$N=\frac{x^i\partial_i}{\sqrt{x^ix_i}}$$

    - For S^1, this gives $vol_{S^1}=x\,dy-y\,dx$

- Let (U,x^i) be a chart on M with $g_{ij}=\langle\partial_i,\partial_j\rangle$ the standard components of the metric on U. Then explicitly on U:

  $$vol_M=\sqrt{\det(g)}\,dx^1\wedge\dots\wedge dx^n$$

  - We see $\sqrt{\det(g)}$ is the amount $\{\partial_i\}$ fails to be an orthonormal frame.

- The **Hodge Star**: Defines an isomorphism btwn Λ^kV and $\Lambda^{n-k}V$ for any inner product space.
  - On a Riemannian manifold, this will pointwise extend to an operation on $\Omega^k(M)$.
  - Natural and functorial way to define this isomorphism independent of ordered basis.
    - Without an inner product on V (metric on M) there is no way to make $*$ functorial, i.e. independent of basis.
    - This can also be defined if V has a symmetric nondegenerate bilinear form $g\in V^*\otimes V^*$ in the case of the Minkowski metric (which is not a Riemannian metric). A manifold w/ such a form is called pseudo-Riemannian.
    - The $*$ will be defined w.r.t. an ONB $\{e_1,\dots,e_n\}$ of V, and just as in the volume form the orthonormality allows for a unique dfn up to orientation.
      - When in Minkowski space, an ONB satisfies $\langle e_i,e_j\rangle=\pm\delta_{ij}$.

- We define $*:\Lambda^kV\to\Lambda^{n-k}V$ to satisfy:

  $$\omega\wedge\rho=\langle *\omega,\rho\rangle\,Vol$$

  for any $\omega\in\Lambda^kV$, $\rho\in\Lambda^{n-k}V$, and where $\langle\cdot,\cdot\rangle$ is the form on Λ^mV given by:

  $$\langle v_1\wedge\dots\wedge v_m,\,w_1\wedge\dots\wedge w_m\rangle=\det(\langle v_i,w_j\rangle)$$

  - This implies:

  $$\omega\wedge(*\omega)=\langle *\omega,*\omega\rangle\,Vol$$

- To compute $*\omega$, we define its action on an ONB and extend this by linearity, let $1 \le i_1 < \dots < i_k \le n$ be a multi-index. Then

$$*(e_{i_1} \wedge \dots \wedge e_{i_k}) = (-1)^{\sigma} e_{j_1} \wedge \dots \wedge e_{j_{n-k}}$$

where $1 \le j_1 < \dots < j_{n-k} \le n$ is the complement of $\{i_1, \dots, i_k\}$ in $\{1, \dots, n\}$, and $(-1)^{\sigma}$ is a sign chosen so:

$$e_{i_1} \wedge \dots \wedge e_{i_k} \wedge e_{j_1} \wedge \dots \wedge e_{j_{n-k}} = (-1)^{\sigma} \langle e_{i_1} \wedge \dots \wedge e_{i_k}, e_{j_1} \wedge \dots \wedge e_{j_{n-k}} \rangle \mathrm{Vol}$$

$$(-1)^{\sigma} = \mathrm{sgn}(\tau) \prod_{\ell=1}^{n-k} \langle e_{j_\ell}, e_{j_\ell} \rangle$$ where $\tau \in S^n$ is $\tau(i_1, \dots, i_k, j_1, \dots, j_{n-k}) = (1, \dots, n)$

- Note on an ONB, $\langle e_{i_1} \wedge \dots \wedge e_{i_k}, e_{i_1} \wedge \dots \wedge e_{i_k} \rangle = \prod_{j=1}^{k} \langle e_{i_j}, e_{i_j} \rangle$
- The $\prod \langle e_{j_i}, e_{j_i} \rangle$ is there for the case of a pseudo-Riemannian manifold, i.e. for Minkowski space it equals -1.

- $*$ is almost its own inverse; for $\omega \in \Lambda^k V$, it satisfies:

$$** \omega = (-1)^{k(n-k)} \omega \qquad \text{(Riemannian manifold)}$$

- In 3 dimensions: The cross and triple product are special cases of $*$ for $V = \mathbb{R}^3$:

$$u \times v = *(u \wedge v)$$
$$u \cdot (v \times w) = *(u \wedge v \wedge w)$$

for any vectors $u, v, w \in \mathbb{R}^3$. In $\mathbb{R}^3$, we also have:

$*dx^1 = dx^2 \wedge dx^3$ $\quad *dx^2 = dx^3 \wedge dx^1$ $\quad *dx^3 = dx^1 \wedge dx^2$

Here we get the complex:

$$\begin{tikzcd} \Omega^0(\mathbb{R}^3) \arrow[r, "d"', "\text{grad}"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"', "\text{curl}"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"', "\text{div}"] \arrow[l, bend left, "*"] & \Omega^3(\mathbb{R}^3) \arrow[lll, bend left, "*"] \end{tikzcd}$$

$*$ can be applied after curl to get a 1-form, and after div to get an honest to goodness scalar.

Vector-valued forms

- Recall the exterior power of a vector space V is the space:

$$\Lambda^k(V) := \bigotimes^k V / \mathcal{I}(V) \qquad [v_1]\wedge\ldots\wedge[v_k] := [v_1\otimes\ldots\otimes v_k]$$

where $\mathcal{I}(V)$ is generated by elements of the form $v_1\otimes\ldots\otimes v\otimes\ldots\otimes v\otimes\ldots\otimes v_k$. There is an inclusion:

$$V^k \longrightarrow \Lambda^k V, \quad (v_1,\ldots,v_k) \longmapsto v_1\wedge\ldots\wedge v_k$$

which satisfies the universal property for any k-alternating map:

$$\begin{tikzcd} V^k \arrow[r, "\iota"] \arrow[dr, "f\ k\text{-alternating}"'] & \Lambda^k V \arrow[d, dotted, "\bar{f}\ \text{linear}"] \\ & W \end{tikzcd}$$

- There is a natural isomorphism btwn the space $A_k(V)$ of k-alternating multilinear fns $V^k \to \mathbb{R}$ and linear maps $\Lambda^k V \to \mathbb{R}$, so:

$$A_k(V) \cong (\Lambda^k V)^* = \Lambda^k(V^*)$$

which means we can identify a k-covector either as a k-alternating map $(T_pM)^k \to \mathbb{R}$, or as an element of the exterior power $\Lambda^k(T_p^*M)$, and of course:

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*M))$$

where $\Lambda^k(T^*M)$ is a vector bundle w/ fiber $\Lambda^k(T^*M)_p = \Lambda^k(T_p^*M)$.

- Let T, V be vector spaces ($T \sim T_pM$). A V-valued k-covector is a k-alternating map:

$$f : T^k \longrightarrow V$$

$$\begin{tikzcd} T^k \arrow[r, "\iota"] \arrow[dr, "f"'] & \Lambda^k T \arrow[d, dotted, "\bar{f}"] \\ & V \end{tikzcd}$$

which can be identified w/ a linear map $\bar{f} : \Lambda^k T \to V$, i.e. $\bar{f} \in \mathrm{Hom}_{\mathbb{R}}(\Lambda^k T, V)$. We see we have a correspondence:

$$A_k(T,V) \cong \mathrm{Hom}_{\mathbb{R}}(\Lambda^k T, V) \cong (\Lambda^k T)^* \otimes V$$

where $U^*\otimes W \underset{f}{\overset{\sim}{\to}} \mathrm{Hom}(U,W)$ by the map $f(\theta\otimes w)(v) := \theta(v)w$. Practically, this means any map $U \xrightarrow{\xi} W$ can be decomposed into $n = \dim W$ components $\xi^i \in U^*$ w/ $\xi = \xi^i e_i$ and $\xi^i : U \to \mathbb{R}$ real valued fns on U. So, a V-valued k-covector is an element of $(\Lambda^k T^*)\otimes V$. (note $(\Lambda^k T^*)\otimes V$ is defined fiberwise).

- A V-valued differential form on M assigns to each point a V-valued k-covector, i.e. it is a section of $\Lambda^k(T^*M)\otimes V$:

$$\Omega^k(M,V) := \Gamma((\Lambda^k T^*M)\otimes V)$$

- If $\omega \in \Omega^k(M,V)$ is a form and $\{v_i\}_{i=1}^n$ is a basis for V, then we may write $\omega = \omega^i \otimes v_i$ for forms $\omega^i \in \Omega^k(M)$, i.e. ω^i_p is a k-alternating map $\omega^i_p : (T_pM)^k \to \mathbb{R}$.

- The form ω is smooth if each component form $\omega^i \in \Omega^k(M)$ is C^∞.

- **Wedge products**: Let $\alpha \in A_k(T,V)$ and $\beta \in A_\ell(T,W)$. Since V and W are arbitrary vector spaces, we can't form $\alpha(\vec{t}_1)\beta(\vec{t}_2)$, but need an additional bilinear map $\mu: V\times W \to Z$. Given such a map, we define $\alpha\cdot\beta \in A_{k+\ell}(T,Z)$ by:

$$(\alpha\cdot\beta)(t_1,\dots,t_{k+\ell}) := \frac{1}{k!\,\ell!}\sum_{\sigma\in S_{k+\ell}} \mu\left(\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)}),\ \beta(v_{\sigma(k+1)},\dots,v_{\sigma(\ell)})\right)$$

  - It can be shown that:

$$\alpha\cdot\beta = \sum_{i,j}(\alpha^i\wedge\beta^j)\,\mu(v_i,w_j)$$

  - The simplest case here is $\mu: V\times W \to V\otimes W$, which gives:

$$\alpha\cdot\beta = (\alpha^i\wedge\beta^j)\,v_i\otimes w_j \in A_{k+\ell}(T, V\otimes W)$$

  - We can thus use this wedge product as a map:

$$\Omega^k(M,V)\times\Omega^\ell(M,W) \longrightarrow \Omega^{k+\ell}(M, V\otimes W)$$

$$\alpha\cdot\beta = (\alpha^i\wedge\beta^j)\,v_i\otimes w_j$$

  where $\alpha^i\wedge\beta^j$ is the usual wedge product of $\mathbb{R}$-valued forms.

- We can extend most of our dfns pointwise to vector valued functions. For $X_p\in T_pM$ and $f: M\to V$ a vector valued function, write $f = f^i v_i$ w/ $\{v_i\}$ a basis for V and each $f^i\in C^\infty(M)$. Then define:

$$X_p f := (X_p f^i)\,v_i \in V$$

  and now $Xf \in C^\infty(M,V)$ as opposed to $C^\infty(M)$.

- Similarly, we extend the exterior derivative $d: \Omega^k(M,V) \to \Omega^{k+1}(M,V)$ by linearity:

$$d(\omega^i v_i) := (d\omega^i)\,v_i$$

  and we have our usual property:

$$d(\alpha\cdot\beta) = d\alpha\cdot\beta + (-1)^{\deg(\alpha)}\alpha\cdot d\beta$$

- We will often consider the case when $V=\mathfrak{g}$ is a Lie algebra. In this case, we can use the bracket $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ as our bilinear map, so if $\{T_a\}$ is a basis for $\mathfrak{g}$, then:

$$[\alpha,\beta] := \alpha\cdot\beta = (\alpha^a\wedge\beta^b)[T_a,T_b] \in \Omega^{k+\ell}(M,\mathfrak{g})$$

  where $\alpha = \alpha^a T_a$, $\beta = \beta^b T_b$. If $\alpha\in\Omega^k(M,\mathfrak{g})$ and $\beta\in\Omega^\ell(M,\mathfrak{g})$, then:

$$[\alpha,\beta] = (-1)^{k\ell+1}[\beta,\alpha]$$

  We note that d is an antiderivation on $[\cdot,\cdot]$:

$$d[\alpha,\beta] = [d\alpha,\beta] + (-1)^{\deg\alpha}[\alpha,d\beta]$$

- For a matrix Lie algebra $\mathfrak{g} = \mathfrak{gl}(\mathbb{R}^n)$, $[A,B] = AB - BA$. Let $\{e_{ij}\} \subset \mathfrak{gl}(\mathbb{R}^n)$ be the basis:

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl} \qquad e_{ij} = \begin{pmatrix} 0 & 0 & \\ 0 & & \\ & 1 \cdots & \end{pmatrix} \text{ only nonzero in the } (i,j) \text{ spot}$$

Any form $\alpha \in \Omega^k(M, \mathfrak{gl}(\mathbb{R}^n))$ can then be expressed as $\alpha = \alpha^{ij} e_{ij}$ w/ $\alpha^{ij} \in \Omega^k(M)$, and we define $\alpha \wedge \beta$ as $\alpha \cdot \beta$ w/ $\mu =$ matrix mult, and $[\alpha, \beta]$ as $\alpha \cdot \beta$ w/ $\mu = [\cdot,\cdot]$. This gives:

$$\alpha \wedge \beta = \alpha^{ij} \wedge \beta^{kl} e_{ij} e_{kl} = \alpha^{ik} \wedge \beta^{kj} e_{ij}$$

$$[\alpha, \beta] = \alpha^{ij} \wedge \beta^{kl} [e_{ij}, e_{kl}]$$

which implies:

$$[\alpha, \beta] = \alpha \wedge \beta - (-1)^{\deg(\alpha)\deg(\beta)} \beta \wedge \alpha$$

- Pullbacks of vector valued forms are defined in the same way:

$$(f^*\alpha)_p(u_1, \dots, u_k) := \alpha_p(f_* u_1, \dots, f_* u_k) \qquad f: M \to N, \ \alpha \in \Omega^k(N, V)$$

For any set of vectors $\{v_i\} \subset V$, if we can expand $\alpha \in \Omega^k(M, V)$ as $\alpha = \alpha^i v_i$, then:

$$f^*\alpha = (f^*\alpha^i) v_i$$

The pullback still respects the wedge product and is a cochain map:

$$f^*(\alpha \cdot \beta) = (f^*\alpha) \cdot (f^*\beta) \qquad df^* = f^* d \qquad (\text{so } f^*[\alpha, \beta] = [f^*\alpha, f^*\beta])$$

- Vector bundle valued forms: For E a vector bundle, an E-valued k-covector at $p \in M$ is an alternating map ω_p, or equivalently a member of the exterior power of T_pM times E_p:

$$\omega_p : (T_pM)^k \to E_p \iff \omega_p \in \Lambda^k(T_p^*M) \otimes E_p$$

  - This lets us study vector valued forms where the vector spaces are twisted.
  - As usual, the space of smooth E-valued k-forms is:

$$\Omega^k(M, E) := \Gamma\left(\Lambda^k(T^*M) \otimes E\right)$$

  - Ex: We associated a matrix $\Omega^i_j \in \Omega^2(M)$, the curvature form, to a connection via $R(X,Y)s_j := \Omega^i_j(X,Y)s_i$. Equivalently, we may view the curvature form Ω as an $\mathrm{End}(E)$ valued 2 form:

$$\Omega_p : T_pM \times T_pM \to \mathrm{End}(E_p) \implies \Omega \in \Omega^2(M, \mathrm{End}(E)) = \Gamma\left(\Lambda^2(T^*M) \otimes \mathrm{End}(E)\right)$$

    - For an affine connection, note E is the tangent bundle TM. (note $\mathfrak{X}(M) = \Gamma(TM)$)

- A rank (a,b) tensor field on M is a section of:

$$T^{a,b}(M) := \left(\bigotimes^a TM\right) \otimes \left(\bigotimes^b T^*M\right)$$

  - The metric is a $(0,2)$ tensor, a vector field is a $(1,0)$ tensor, and a rank r differential form is an alternating $(0,r)$ tensor. For such a field, $T(\omega_1, \dots, \omega_a, Y_1, \dots, Y_b) \in C^\infty(M)$.
  - A rank (a,b) E-valued tensor is a section of:

$$\left(\bigotimes^a TM\right) \otimes \left(\bigotimes^b T^*M\right) \otimes E$$

where E is a vector bundle.

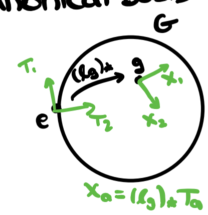
- There is a one-to-one correspondence between the space of tensor fields $\Gamma(T^{a,b}(M))$ and the space:

  $\{T : \Omega^1(M)^a \times \mathfrak{X}(M)^b \to \mathcal{F} \mid T \text{ is } \mathcal{F}\text{-multilinear}\}$

  where $\mathcal{F} = C^\infty(M)$. For example, a $(0,2)$ field g is a section of $\bigotimes^2 T^*M$, and can equivalently be viewed as a map $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}$, $g(X,Y)(p) := g_p(X_p, Y_p)$.
  - Likewise, a vector $X \in \mathfrak{X}(M)$ is a $(1,0)$ tensor field and can be seen as a map $X : \Omega^1(M) \to \mathcal{F}$, $X(\omega)(p) := \omega_p(X_p)$.
- The Maurer-Cartan form: Let G be a Lie group w/ Lie algebra $\mathfrak{g} = T_eG$. The Maurer-Cartan form on G is the unique $\mathfrak{g}$-valued 1-form Θ which is the identity on $\mathfrak{g} = T_eG$, i.e. for $X_e \in T_eG$:

  $$\Theta(X_e) = X_e$$

  Let $\{T_a\}$ be a basis for $\mathfrak{g}$. As we saw previously, we can left translate T_a to form $(\ell_g)_* T_a \in T_gG$, and the set of vector fields $\{(\ell_g)_* T_a\}_{a=1}^n \subset \mathfrak{X}(G)$ form a frame for TG, i.e. @ each $g \in G$, $\{(\ell_g)_* T_a\}_{a=1}^n$ is a canonical basis of T_gG.

  - The left multiplication map ℓ_g thus lets us create n left invariant vector fields on G naturally from a basis for $\mathfrak{g}$, and essentially gives us a way to parallel transport $\{T_a\} \subset \mathfrak{g}$ to any tangent space T_gG.
  - Let $\{X_a\} \subset \mathfrak{X}(G)$ be this frame, i.e. $(X_a)_g := (\ell_g)_* T_a$, and let $\{\Theta^a\}$ be the set of dual 1-forms, so $\Theta^a(X_b) = \delta^a_b \in C^\infty(G)$. Then the Maurer Cartan form can be expressed as:

    $$\Theta = T_a \otimes \Theta^a \quad \in \Omega^1(G, \mathfrak{g})$$

    - At each $g \in G$, $\Theta_g \in \mathrm{Hom}(T_gG, \mathfrak{g})$ sends a vector $A^a X_a \in T_gG$ to $\Theta_g(A^a X_a) = A^a T_a$ b/c $\Theta(X_a) = T_a$, so Θ essentially flows X_a back by $(\ell_{g^{-1}})^*$.
- Θ can be shown to be unique, and for $g \in G$ corresponds to $(\ell_{g^{-1}})_*$:

  $$\Theta_g(X_g) = (\ell_{g^{-1}})_* X_g \in \mathfrak{g} \qquad \forall g \in G,\ X_g \in T_gG$$

  - Because of these properties, if $A \in \mathfrak{g}$ and $X_A \in \mathfrak{X}(G)$ is the LI vector field generated by A, i.e. $(X_A)_g = (\ell_g)_* A$, then:

    $$\Theta(X_A) = A$$

    - Note here $\Theta(X_A)$ is a map $M \to \mathfrak{g}$, so really we mean $\Theta X_A = A\,\mathrm{id}$.
- Let (U, g^a) be a coordinate patch on G. Then we can pick $T_a = \partial/\partial g^a|_e$, which gives $X_a = \partial/\partial g^a$, $\Theta^a = dg^a$. Thus:

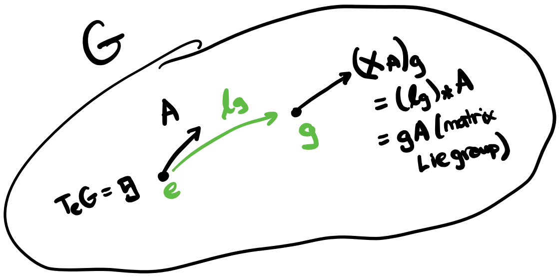
  $$\Theta = T_a \otimes \Theta^a = \frac{\partial}{\partial g^a}\Big|_e \otimes dg^a = (\ell_{g^{-1}})_* \frac{\partial}{\partial g^a} \otimes dg^a$$

  - For a matrix Lie group we have, for $g \in G$ and $A \in \mathfrak{g}$:

    $$(\ell_g)_* A = gA$$

thus we can write this as:

$$\Theta = g^{-1}\frac{\partial}{\partial g^a} \otimes dg^a =: g^{-1}dg$$



where here dg is defined as the operator:

$$dg := \frac{\partial}{\partial g^a} \otimes dg^a = id|_{T_gG}$$

$$dg: \mathfrak{X}(G) \longrightarrow T_gG, \ X \mapsto X_g$$

– Now explicitly if we let $A \in \mathfrak{g}$ and $(X_A)_g := (l_g)_* A = gA$ be its LI flow, then we see:

$$\Theta(X_A) = (g^{-1}dg)X_A = g^{-1}\overbrace{(X_A)|_g}^{\text{action of } dg} = g^{-1}(gA) = A$$

$c_g: h \mapsto ghg^{-1}$

• The adjoint representation is the differential of conjugation at $e \in G$:

$$C_g: G \to G \qquad h \mapsto ghg^{-1}$$

$$Ad(g) := (C_g)_{*,e}: \mathfrak{g} \to \mathfrak{g}$$

$$Ad: G \longrightarrow Aut(\mathfrak{g})$$

$Ad(g) = (c_g)_{*,e}: \mathfrak{g} \to \mathfrak{g}$, so $Ad: G \to GL(\mathfrak{g})$, and $ad = Ad_{*,e}$

Since $c_g(h) = ghg^{-1} = (l_g \circ r_{g^{-1}})(h)$, we see:

$$Ad(g) = (l_g)_* \circ (r_{g^{-1}})_*$$

and for a matrix Lie group, its action is $Ad_g(V) = gVg^{-1}$! The differential of Ad at e is a representation of $\mathfrak{g}$ known as ad:

$$ad := Ad_{*,e}: \mathfrak{g} \to End(\mathfrak{g})$$

and has action:

$$ad_A B = [A, B]$$

• The Maurer-Cartan form satisfies the equations:

$$d\Theta + \tfrac{1}{2}[\Theta, \Theta] = 0$$

$$(r_g)^* \Theta = Ad_{g^{-1}} \Theta \qquad \text{(show on LI vector fields)}$$

which will be important when we study principal bundles. The form also preserves the bracket between LI vector fields $X_A, X_B \in \mathfrak{X}(G)$:

$$\Theta([X_A, X_B]) = [\Theta(X_A), \Theta(X_B)] = [A, B]$$

- Recap of some notation: For a frame $\{e_1,\dots,e_r\}$ of a vector bundle $E \to M$ w/ a connection ∇:

$$e=(e_1 \dots e_n) \qquad \omega = \begin{pmatrix} \omega^1_1 & \omega^1_2 & \cdots \\ \omega^2_1 & \ddots & \\ \vdots & & \end{pmatrix} \qquad \Omega = \begin{pmatrix} \Omega^1_1 & \Omega^1_2 & \cdots \\ \Omega^2_1 & \ddots & \\ \vdots & & \end{pmatrix}$$

  - The dfn of ω is $\nabla_X e_i = \omega^j_i(X) e_j$, which is:

$$\nabla e = e\omega$$

- If we are given another frame $\{\bar{e}_i\}$ w/ $\bar{e}_i = a^j_i e_j$, we have $\bar{e} = ea$ as row vectors. Note that the components a^i_j are C^∞ functions on M. Under the change in frame if $\omega, \bar{\omega}$ and $\Omega, \bar{\Omega}$ are the curvature/connection matrices for ∇ w.r.t. e and $\bar{e}$, then:

$$\bar{\omega} = a^{-1}\omega a + a^{-1}da$$

$$\bar{\Omega} = a^{-1}\Omega a$$

  - These should be familiar, since they are the transformation laws for $\underset{\substack{\text{one form} \\ \swarrow A_\mu dx^\mu}}{A_\mu}$ and $\underset{\substack{\text{two form} \\ \swarrow F_{\mu\nu} dx^\mu dx^\nu}}{F_{\mu\nu}}$ under gauge transformation

- For $E = TM$ the tangent bundle, ∇ is an affine connection and torsion is defined as $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and has torsion form $T(X,Y) = \tau^i(X,Y)e_i$, w/ $\tau = (\tau^1 \dots \tau^n)^T$ being a column vector of two forms. Let $\theta = (\theta^1 \dots \theta^n)^T$ be the column vector of the dual coframe. Then we have the Bianchi identities

  i) $d\tau = \Omega \wedge \theta - \omega \wedge \tau$

  ii) $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$

  - ii) generalizes to $d(\Omega^k) = \Omega^k \wedge \omega - \omega \wedge \Omega^k$, where $\Omega^k = \Omega \wedge \dots \wedge \Omega$.
  - In the absence of torsion, we get $\Omega \wedge \theta = 0$. This implies a symmetry for the curvature: For a torsionless connection on TM (i.e. the Riemannian connection) $\forall X,Y,Z \in \mathfrak{X}(M)$, we have:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

  - Combined with $R(X,Y) = -R(Y,X)$ and $\langle R(X,Y)W, Z\rangle = -\langle R(X,Y)Z, W\rangle$, this implies:

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$$

  i.e. $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$.

- Let $T^{a,b}(M) := (\otimes^a TM) \otimes (\otimes^b T^*M)$ be the space of rank (a,b) tensors. We can define:

$$\nabla_X : T^{a,b}(M) \to T^{a,b}(M)$$

by defining $\nabla_X \omega \in \Omega^1(M)$ for a 1-form ω:

$$(\nabla_X \omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y)$$

and for $T \in T^{a,b}(M)$, defining:

$$(\nabla_X T)(\omega_1,\dots,\omega_a, Y_1,\dots,Y_b) := X(T(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b)) - \sum_{i=1}^{a} T(\omega_1,\dots,\nabla_X\omega_i,\dots) - \sum_{j=1}^{b} T(\omega_1,\dots,\nabla_X Y_j,\dots)$$

- The subtractions are to make this satisfy Leibniz.

- The Riemann curvature tensor is:

$$Rm(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$$

If ∇ is metric compatible, then the Bianchi identity can be recast:

$$\sum_{\substack{\text{cyclic} \\ X,Y,Z}} \nabla_X Rm(Y,Z,W,V) = 0$$

- Generally out of the Riemann tensor, other objects of interest can be constructed. The Ricci curvature, which associates to $u,v \in T_pM$ the trace of the map $T_pM \to T_pM$, $w \mapsto R(w,u)v$. If $U \subset M$ and $\{E_1, \dots, E_n\}$ is an orthonormal frame on U, then:

$$Ric(X,Y) := tr\{R(\cdot, X)Y\} = \sum_{i=1}^{n} \langle R(E_i, X)Y, E_i\rangle$$

  - The Ricci tensor is a symmetric rank $(0,2)$-tensor, $R(X,Y) = R(Y,X)$.
  - The scalar curvature $S: M \to \mathbb{R}$ is defined as the trace of the Ricci tensor. For $\{e_1, \dots, e_n\}$ an orthonormal basis of T_pM, this is:

$$S(p) := \sum_i Ric(e_i, e_i) = \sum_{i,j} \langle R(e_i, e_j)e_j, e_i\rangle$$

- A connection ∇ is completely determined by a framed open cover $\{(U, e_1, \dots, e_n)\}$ of M, and a set of connection forms $\{\omega_e\}$ for each framed set s.t. on each intersection $U \cap V$ with frames $\{e_i\}$ and $\{\bar{e}_i\}$, we have:

$$\omega_{\bar{e}} = a^{-1}\omega_e a + a^{-1}da$$

- Let $E \to M$ be a vector bundle with connection ∇, $\{(U, e_1, \dots, e_n)\}$ a framed open cover w/ $\{\omega_e\}$ determining the connection. Any map $f: N \to M$ induces a connection on the pullback bundle f^*E, determined by $\{(U, f^*e_1, \dots, f^*e_n)\}$ and $\{f^*\omega_e\}$.

Characteristic Classes

- Idea: The curvature form Ω transforms under change of frame as $\Omega \mapsto a^{-1}\Omega a$. So, if we have an invariant polynomial $P(X)=P(AXA^{-1})\ \forall A\in GL(\mathbb{R}^r)$, i.e. a polynomial $P(X)=P(\{x^i_j\}_{i,j})$, then $P(\Omega)$ will define a global differential form $P(\Omega)\in\Omega^{2\cdot\deg(P)}(M)$ on the base M. This will be shown to be closed and $\therefore$ define a global cohomology class.

- A polynomial $P(X)$ is on $\mathfrak{gl}(F^r)$ (rxr matrices in F) for a field F if it is an element of the algebra $F[x^i_j]$, where $X=(x^i_j)^r_{i,j=1}$. X is invariant if for all $A\in GL(\mathbb{R}^r)$, $P(X)=P(A^{-1}XA)$. Note x^i_j are indeterminates, but an identity which holds for all real matrices X also holds for indeterminants.
  - Examples: $tr(X)$, $det(X)$.
  - Characteristic polynomial: For X an $r\times r$ matrix of indeterminates, λ another, we can expand:
    $\det(X+\lambda I)=\lambda^r+f_1(X)\lambda^{r-1}+\dots+f_{r-1}(X)\lambda+f_r(X)$
    Since $\det(A^{-1}XA+\lambda I)=\det(X+\lambda I)$ and $\{1,\lambda,\lambda^2,\dots,\lambda^r\}$ are lin. ind., we see $f_k(A^{-1}XA)=f_k(X)$. The invariants $f_k(X)$ are called the coefficients of the characteristic polynomial of $-X$.
  - The k^{th} trace polynomial is $\Sigma_k(X):=tr(X^k)$.
- Let $\mathcal{A}$ be a commutative $\mathbb{R}$-algebra with identity. Then there is a canonical map $\mathbb{R}\to\mathcal{A}$, and this can be used to transfer an invariant polynomial $P(X)$ on $\mathfrak{gl}(\mathbb{R}^r)$ to $\mathfrak{gl}(\mathcal{A}^r)$, i.e. $P(X)=P(A^{-1}XA)$ for $A\in GL(\mathcal{A}^r)$
- Let $\mathcal{A}:=\bigoplus_{i=0}^{\infty}\Lambda^{2i}T_p^*M$ be the algebra of even degree covectors at $p\in M$, $P(X)$ on $\mathfrak{gl}(\mathbb{R}^r)$ a homogeneous invariant polynomial of degree k. Let $e,\bar{e}$ be two frames on $U\subset M$ w/ $\bar{e}=ae$, and curvature matrices $\Omega,\bar{\Omega}$. Then $P(\Omega_p)=P(\bar{\Omega}_p)$ b/c $\bar{\Omega}_p=a(p)^{-1}\Omega_p a(p)$, so $P(\Omega)\in\Lambda^{2k}(T^*M)$ is a well defined form on U, ind. of frame. $P(\Omega)$ also stitches together at the intersections of a framed open cover b/c $\Omega_\alpha=a^{-1}\Omega_\beta a$ for $\Omega_\alpha/\Omega_\beta$ on U_α/U_β, thus $P(\Omega)$ defines a global $2k$ form on M.
- Theorem: The form $P(\Omega)\in\Omega^{2k}(M)$ is closed and independent of the connection ∇ on E.
  - Note $\Omega^i_j\in\Omega^2(M)$, although it describes the vector bundle $E\to M$.
  - The map:
    $$C_E:\mathrm{Inv}(\mathfrak{gl}(\mathbb{R}^r))\longrightarrow H^*(M) \qquad P(X)Q(X)\mapsto[P(\Omega)\wedge Q(\Omega)]$$
    $$P(X)\mapsto[P(\Omega)]$$
    is a well defined algebra homomorphism, independent of connection on E, called the Chern-Weil homomorphism.
- Newton's identity relates the characteristic polynomials to the trace polynomials:
  $\Sigma_k-f_1\Sigma_{k-1}+f_2\Sigma_{k-2}+\dots+(-1)^{k-1}f_{k-1}\Sigma_1+(-1)^k k f_k=0$
  The ring of invariant polynomials on $\mathfrak{gl}(\mathbb{R}^r)$ is generated both by the characteristic polynomials $\{f_k(X)\}$ and the trace polynomials $\{\Sigma_k\}$:
  $\mathrm{Inv}(\mathfrak{gl}(\mathbb{R}^r))=\mathbb{R}[f_1,\dots,f_r]=\mathbb{R}[\Sigma_1,\dots,\Sigma_r]$
  Recall for matrices of forms A^i_j and B^i_j, $(A\wedge B)^i_j=A^i_k\wedge A^k_j$ and $d(A)^i_j=d(A^i_j)$. If A^i_j and B^i_j have dimensions (as forms) a and b, then:
  $$(A\wedge B)^T=(-1)^{ab}B^T\wedge A^T \qquad tr(A\wedge B)=(-1)^{ab}tr(B\wedge A) \qquad d\,tr(A)=tr(dA)$$
  - Using these identities + 2nd Bianchi, one can show $d\Sigma_k=tr(d\Omega^k)=0$.

- A family $\{\omega_t\}_{t\in J}$ of differential forms varies smoothly w/ t if for each (U,x^i) in an atlas, $\omega_t = a_I(x,t)\,dx^I$ for smooth functions $a_I(x,t)$. Here $J \subset \mathbb{R}$ is an open interval. We define the derivative/integral of ω_t to be:

$$\frac{d\omega_t}{dt} := \left(\frac{\partial a_I(x,t)}{\partial t}\right)dx^I \qquad \int_a^b dt\,\omega_t := \left(\int_a^b dt\, a_I(x,t)\right)dx^I$$

d/dt satisfies everything you might want:

$$\frac{d}{dt}\operatorname{tr}(\omega) = \operatorname{tr}\left(\frac{d\omega}{dt}\right) \qquad \frac{d}{dt}(\omega\wedge\tau) = \dot\omega\wedge\tau + \omega\wedge\dot\tau \qquad d\dot\omega = \frac{d}{dt}(d\omega) \qquad d\int_a^b dt\,\omega = \int_a^b dt\, d\omega$$

- Characteristic classes are natural in the sense that for a map $f: N \to M$, $c(f^*E) = f^*c(E)$, where $c(E) = [P(\Omega)]$ for some $P \in \mathrm{Inv}(\mathfrak{gl}(\mathbb{R}^r))$.
- The Pontryagin classes of a real vector bundle $E \to M$ are the characteristic classes arising from the invariant characteristic polynomials, $P(X) = f_k(X)$.
- Let $E \to M$ be a rank r vector bundle and ∇ a metric connection. Recall this implies $\Omega^i{}_j$ is antisymmetric. Then if we are working in a ring s.t. 2 is not a zero divisor, the diagonal of $\Omega^i{}_j$ is 0. So, the Σ_k polynomials for odd k vanish:

$$\Sigma_k(\Omega) = \operatorname{tr}(\Omega^k) = 0,\ k \text{ odd}$$

  - Since we can compute $[\Sigma_k(\Omega)]$ w.r.t. any connection and frame, $[\Sigma_k(\Omega)] = 0$ for any connection.
  - Because $\Sigma_k(X)$ generates $\mathrm{Inv}(\mathfrak{gl}(\mathbb{R}^r))$, any polynomial $P(X) \in \mathrm{Inv}(\mathfrak{gl}(\mathbb{R}^r))$ of odd degree k has vanishing characteristic class $[P(\Omega)] = 0$:

$$C_E(\text{any odd degree } P(X)) = 0 \in H^{2k}(M)$$

- A cohomology class $[\omega] \in H^q(M)$ is integral if $\int_S \omega \in \mathbb{Z}$ for any closed dim q submanifold $S \subset M$.
- The k^{th} Pontryagin class of E is the cohomology class:

$$p_k(E) := \left[f_{2k}\left(\tfrac{i}{2\pi}\Omega\right)\right] \in H^{4k}(M)$$

  - The i is a convention which disappears in $p_k(E)$ b/c it is even degree, and the $\frac{1}{2\pi}$ makes $p_k(E)$ integral.
  - By using the dfn of $f_k(X)$ as the coefficients of λ^{r-k} for $\det(X+\lambda I)$, we obtain the total Pontryagin class:

$$p(E) := \det\left(I + \tfrac{i}{2\pi}\Omega\right) = 1 + p_1(E) + \dots + p_{\lfloor r/2\rfloor}(E)$$

  where $r = \operatorname{rank}(E)$ (since Ω is an $r\times r$ matrix, there are only at maximum $\lfloor r/2 \rfloor$ Pontryagin classes).
- Let $E \to M$ be a vector bundle of rank r w/ $\dim(M) = 4m$. A monomial $p_1^{a_1}\cdots p_{\lfloor r/2\rfloor}^{a_{\lfloor r/2\rfloor}}$ with $4(a_1 + 2a_2 + \dots + \lfloor r/2 \rfloor) = 4m$ represents a cohomology class in $H^{4m}(M)$, so can be integrated over M. The resulting integral

$$\int_M p_1^{a_1}\cdots p_{\lfloor r/2\rfloor}^{a_{\lfloor r/2\rfloor}}$$

  is called a Pontryagin number. The Pontryagin numbers of a dim $4m$ manifold M are those of its tangent bundle $TM \to M$.
  - For $\dim(M) = 4$, there is 1 Pontryagin number, $\int_M p_1$. For dim 8, we have $\int_M p_1^2$ and $\int_M p_2$.
- Whitney product formula: The Pontryagin class of a sum splits:

$$p(E\oplus E') = p(E)\,p(E')$$

  - The reason for this is $E\oplus E'$ inherits the connection $\nabla_X\begin{pmatrix} s \\ s'\end{pmatrix} = \begin{pmatrix}\nabla_X^E s \\ \nabla_X^{E'} s'\end{pmatrix} \Rightarrow \omega = \begin{pmatrix}\omega_E & 0 \\ 0 & \omega_{E'}\end{pmatrix}$ and $\Omega = \begin{pmatrix}\Omega_E & \\ & \Omega_{E'}\end{pmatrix}$
- Recall a vector bundle E of rank r is orientable iff it has a nowhere vanishing r form $v_1\wedge\dots\wedge v_r$, equivalently iff $\Lambda^r E$ is the trivial bundle. For an oriented vector bundle, if e and $\bar e$ are positively oriented ONFs on $U \subset M$, $\bar e = ea$ with $a: U \to SO(r)$. So on an oriented vector bundle, $P(\Omega)$ is a global form iff $P(X)$ is invariant under $SO(r)$, not all of $GL(\mathbb{R}^r)$.
  - Any polynomial invariant under $SO(r)$ but not $GL(\mathbb{R}^r)$ will give us a new characteristic class that an unoriented bundle won't have.
  - Since $\Omega^i{}_j$ is antisym, $P(X)$ only needs to be defined on $\mathfrak{so}(r)$, antisym matrices in $\mathbb{R}^r$.
    - For r odd, the $SO(r)$ invariant polynomials on $\mathfrak{so}(r)$ are exactly the $GL(\mathbb{R}^r)$ invariant polys on $\mathfrak{gl}(\mathbb{R}^r)$, so r odd $\Rightarrow$ no new characteristic classes for an orientable vector bundle.

- For even r, there is a new generator for $SO(r)$ invariant polys: the Pfaffian, invariant under $\forall A \in SO(r)$.
  - For an antisymmetric $2m\times 2m$ matrix A, $\det(A)$ is a perfect square. The Pfaffian is defined to be that square root:
    $$Pf(X)^2 := \det(A)$$
    subject to the normalization convention:
    $$Pf\begin{pmatrix} S & & \\ & S & \\ & & \ddots \\ & & & S \end{pmatrix} = 1, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
    For a $r\times r = 2m \times 2m$ matrix $X^i{}_j$ of indeterminates, $Pf(X)$ is a polynomial of degree m.
- For a real, oriented vector bundle $E \to M$ of even rank $r = 2m$, its Euler class is the cohomology class of $Pf(\Omega/2\pi)$:
  $$e(E) := \left[Pf\left(\tfrac{1}{2\pi}\Omega\right)\right] \in H^r(M)$$
  as $Pf(X)$ is degree m and Ω is a 2-form.
  - The Euler class of an even dimensional manifold $\dim(M) = 2m$, when integrated, gives $\chi(M)$:
    $$\int_M e(TM) = \chi(M)$$
- A complex vector bundle is Hermitian if it assigns to each fiber a Hermitian metric $\langle\cdot,\cdot\rangle_p : E_p \times E_p \to \mathbb{C}$, $\langle z, w\rangle := z_i \overline{w_i}$. One can analagously define a connection and curvature. For a connection ∇_X compatible w/ the metric, in this case $\omega^i{}_j$ is not just antisymmetric:
  $$0 = X\langle e_i, e_j\rangle = \langle \nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle = \langle \omega(X)^k_i e_k, e_j\rangle + \langle e_i, \omega(X)^k_j e_k\rangle$$
  $$= \omega^j_i(X) + \overline{\langle \omega^k_j(X) e_k, e_i\rangle} = \omega^j_i(X) + \overline{\omega^i_j(X)}$$
  $$\Longrightarrow \omega^i_j = -\overline{\omega^j_i} \qquad \omega^i{}_j \in H^1(M,\mathbb{C}),\ \Omega^i{}_j \in H^2(M,\mathbb{C})$$
  This drops to Ω, meaning the diagonal of Ω must be imaginary. This implies $tr(\Omega)^k$ for k odd doesn't need to vanish, so there are nonvanishing characteristic classes from odd degree polynomials in the complex case. For a Hermitian bundle, the Chern classes are the cohomology classes $[f_k(\Omega)]$:
  $$c_k(E) := \left[f_k\left(\tfrac{i}{2\pi}\Omega\right)\right] \in H^{2k}(M,\mathbb{C})$$
  $$c(E) := \left[\det\left(I + \tfrac{i}{2\pi}\Omega\right)\right] = 1 + c_1(E) + \dots + c_r(E)$$

Principal Bundles

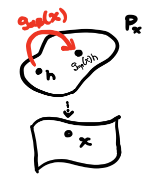
- Associated to every vector bundle $E \to M$ is its frame bundle $Fr(E)$ whose fiber at $p \in M$ is the set of ordered bases $[e_{1,p} \ldots e_{r,p}]$. A frame over $U \subset M$ is just a section $s : U \to Fr(E)$.
  - The frame bundle is a principal bundle over M w/ structure group $GL(\mathbb{R}^r)$.
- A manifold M equipped w/ a smooth right action $\cdot : M \times G \to M$ from a Lie group G is a G-manifold. Recall the stabilizer of $x \in M$ is the set of points which fix x, $Stab(x) := \{g \in G : x \cdot g = x\}$, and the orbit-stabilizer theorem implies:

$$Orb(x) \xleftrightarrow{\sim} Stab(x) \backslash G$$
$$x \cdot g \longleftrightarrow [g]$$

  where $Stab(x)\backslash G$ is the set of right cosets and $Orb(x) = \{x \cdot g \in M : g \in G\}$.
  - Any right G-action induces a left G-action by $g \cdot x := x \cdot g^{-1}$! A map $f : N \to M$ between two G-manifolds is G-equivariant if it respects the action, $f(x \cdot g) = f(x) g$.
- A fiber bundle $\pi : P \to M$ is a principal G-bundle if it has fibers $\pi^{-1}(\{x\}) \cong G$, G acts smoothly and freely on P fiberwise on the right, and the local trivializations $\varphi_U : P|_U \to U \times G$ are G-equivariant w.r.t. the action on $U \times G$ of $(x,g)h := (x, gh)$.
  - A free action is one with no nontrivial stabilizers, i.e. $x \cdot g = x$ implies $g = 1$.
  - Ex: For a closed subgroup $H < G$, G/H is a principal H bundle
  - For any principal bundle, G acts transitively (only one orbit) on each fiber.
  - Lemma: Any right equivariant map $f : G \to G$ is necessarily left translation. This is determined by $f(1)$, i.e. $f(g) = ag$, where $a = f(1)$.
- For the local triv. $\{(U_\alpha, \varphi_\alpha)\}$ w/ $\varphi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$, the transition functions are $(\varphi_\alpha \circ \varphi_\beta^{-1}) : U_\alpha \cap U_\beta \times G \to Aut(G)$.

These transition functions are all G-equivariant under the right action of G, thus we can regard $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ as the left translation corresponding to this:

$$(\varphi_\alpha \circ \varphi_\beta^{-1})(x, h) := (x, g_{\alpha\beta}(x) h)$$



  - Thus the structure group of P is G, and $g_{\alpha\beta}(x)$ acts on each fiber by left mult. $g_{\alpha\beta}$ satisfy the cocycle condition, $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, for $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Furthermore, $g_{\alpha\alpha}(x) = 1$ and $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$.
  - Note G acts on P on the right, but the transition functions $g_{\alpha\beta}(x)$ multiply each fiber on the left.
- The frame manifold: Let V be a vector space w/ $\dim(V) = r$. There is a transitive and free action of $GL(\mathbb{R}^r)$ on V by $[v_1 \ldots v_n] A = [v_i A^i_1 \ldots v_i A^i_n] \in Fr(V)$, so for $v \in Fr(V)$, orbit stabilizer implies:

$$\varphi_v : GL(\mathbb{R}^r) = Stab(v)\backslash GL(\mathbb{R}^r) \xleftrightarrow{\sim} Orb(v) = Fr(V)$$

This bijection can be used to transfer the topology of $GL(\mathbb{R}^r)$ to $Fr(V)$, turning $Fr(V)$ into an r-manifold and φ_v a diffeomorphism.
  - The manifold structure is independent of $v \in Fr(V)$.
  - So, to summarize, because of the free and transitive action on V, there is a diffeomorphism:

$$Fr(V) \cong GL(\mathbb{R}^r) \qquad r = \dim V$$

- Let $E \to M$ be a rank r vector bundle. Then E has structure group $GL(\mathbb{R}^r)$, and E has an associated principal bundle, called the frame bundle, constructed as follows. The frame bundle $Fr(E) \to M$ has fibers:

$$Fr(E)_p := Fr(E_p)$$

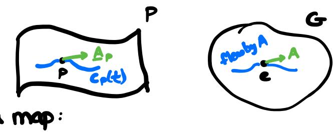
  and $Fr(E)$ inherits the local trivializations $\tilde{\varphi}_\alpha : Fr(E)|_{U_\alpha} \to U_\alpha \times Fr(\mathbb{R}^r)$:

$$[v_1, \ldots, v_r] \in Fr(E_p) \longmapsto (x, [\varphi_\alpha(x, v_1), \ldots, \varphi_\alpha(x, v_r)])$$

  where $(U_\alpha, \varphi_\alpha)$ is a trivialization of $E \to M$.
  - The transition functions $g_{\alpha\beta}(x)$ and $\tilde{g}_{\alpha\beta}(x)$ are the same, just one acts on E_p and one on $Fr(E)_p \cong GL(\mathbb{R}^r)$

• Let G act on the right of a manifold P, and $A\in\mathfrak{g}$. Then the fundamental vector field on P associated to A is:

$$\underline{A}_p := \frac{d}{dt}\left[p\cdot e^{tA}\right]_{t=0}\in T_pP$$

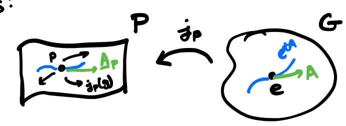


If we let $c_p: t\mapsto p\cdot e^{tA}$, then $\underline{A}_p$ is the tangent vector @ 0 associated w/ c_p, i.e. $\underline{A}_p = c_p'(0)$. On a function $f\in C^\infty(P)$, $\underline{A}_p(f) = \frac{d}{dt}\left[f(pe^{tA})\right]_{t=0}$. This defines a map:

$$\sigma:\mathfrak{g}\to\mathfrak{X}(P) \qquad\qquad \sigma(A):=\underline{A}$$

which is in fact a Lie algebra homomorphism. Alternatively, consider $j_p: G\to P$, $g\mapsto p\cdot g$. Then taking the derivative of $j_p(g)$ along the curve e^{tA} generated by $A\in\mathfrak{g}$ at e gives:

$$(j_p)_{*,e}(A) = \frac{d}{dt}\left[j_p(e^{tA})\right]_{t=0} = \underline{A}_p$$



– Let $P=G$ and G act on P by right multiplication, $p\cdot g := pg = \ell_p(g)$. Then $j_p(g)=\ell_p(g)$, so $\underline{A}_p = (\ell_p)_{*,e}A$ and thus for a group acting on itself, $\underline{A}$ is the canonical left invariant vector field generated by A.

 –The arbitraryness of the action lets us generalize this LI flow to other manifolds.

• Let G act on P from the right, $r_g(p):=p\cdot g$. Then the fund. vector field associated to A satisfies:

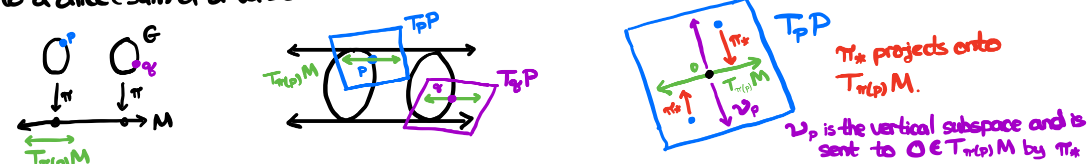
$$(r_g)_*\underline{A} = \underline{\mathrm{Ad}(g^{-1})A}$$

where $\mathrm{Ad}(g)=(c_g)_{*,e}:\mathfrak{g}\to\mathfrak{g}$ is the adjoint map w/ c_g conjugation $x\mapsto gxg^{-1}$.

• The curve $c_p(t):=pe^{tA}$ is the integral curve of $\underline{A}_p$ at p. Furthermore, $\underline{A}_p$ vanishes iff $A\in\mathfrak{g}$ is in the Lie algebra of the stabilizer $\mathrm{Stab}(p)\subset G$, i.e. $e^{tA}\in\mathrm{Stab}(p)$. We denote this by $\underline{A}_p=0 \Leftrightarrow A\in\mathrm{Lie}(\mathrm{Stab}(p))\subset\mathfrak{g}$.

Furthermore, $\mathrm{Lie}(\mathrm{Stab}(g))$ equals $\ker((j_p)_{*,e})\subset\mathfrak{g}$, where $(j_p)_{*,e}:\mathfrak{g}\to T_pP$.

• Now we will dive into the notion of a connection on a principal bundle P. This will amount to a separation of T_pP into a direct sum of a "vertical" and a "horizontal" subspace. Consider the projection $\pi: P\to M$, ex for a circle bundle:



Using a local trivialization at $p\in U$, $\varphi: P|_U\to U\times G$, we see $T_pP = T_p(P|_U) = T_p(U\times G)\supset T_{\pi(p)}M$. This is the projection:

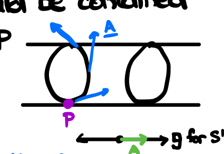
$$\pi_*: T_pP \twoheadrightarrow T_{\pi(p)}M$$

which, as in the picture, projects a $\dim(M)+\dim(G)$ vector space T_pP onto the $\dim(M)$ vector space $T_{\pi(p)}M$. Intuitively, T_pP has $\dim(M)$ dimensions for the base manifold and $\dim(G)$ extra dimensions for the fiber G, which is locally Euclidean.

The vertical tangent space $\mathcal{V}_p$ at $p\in P$ is the kernel of the map $\pi_*: T_pP\to T_{\pi(p)}M$:

$$\mathcal{V}_p := \ker(\pi_{*,p})\subset T_pP \qquad\qquad 0\to\mathcal{V}_p\to T_pP\to T_{\pi(p)}M\to 0 \text{ exact, } \dim\mathcal{V}_p=\dim G$$

• For $A\in\mathfrak{g}$, the fundamental vector field $\underline{A}$ is composed of vertical tangent vectors, $\underline{A}_p\in\mathcal{V}_p$. This is because A generates a flow in the G-direction, since $\underline{A}_p = \frac{d}{dt}\big|_{t=0}(pe^{tA})$. The integral curve of $\underline{A}_p$ should be contained in the fiber at $\pi(p)$.



 – In fact, for $j_p(g)=p\cdot g$, the differential $(j_p)_{*,e}:\mathfrak{g}\xrightarrow{\sim}\mathcal{V}_p\subset T_pP$ is an isomorphism, where note that $(j_p)_{*e}A=\underline{A}_p\in\mathcal{V}_p$. So, $\mathcal{V}_p\cong\mathfrak{g}$.

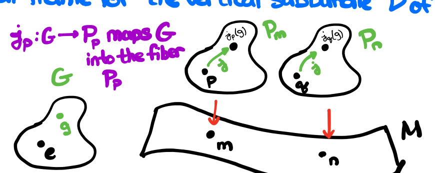
 – If $\{B_1,\dots,B_\ell\}\subset\mathfrak{g}$ is a basis, then this implies $\{\underline{B}_1,\dots,\underline{B}_\ell\}$ is a global frame for the vertical subbundle $\mathcal{V}$ of TP.

  – Note this implies $\mathcal{V}$ is the trivial bundle.

 – We also have a short exact sequence of bundles over P:

$$0\to\mathcal{V}\to TP\xrightarrow{\tilde{\pi}_*}\pi^*TM\to 0$$

where $\tilde{\pi}_*$ is the bundle map induced by π_*, $\tilde{\pi}_*(p,v)=(p,\pi_*(v))$.



• A subbundle $\mathcal{H}$ of P is a horizontal distribution if:

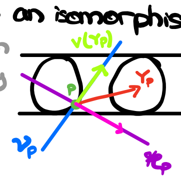
$$TP = \mathcal{H}\oplus\mathcal{V}$$

Although we have the SES above, this does not imply $TP=\pi^*TM\oplus\mathcal{V}$ unless the sequence is split, i.e. if there is a map $k:\pi^*TM\to TP$ such that $\tilde{\pi}_*\circ k = \mathrm{id}$.

– So, although $\mathcal{V}$ is always canonically defined, the dfns of $\mathcal{H}$ are in bijection w/ splittings of the above SES.

- Next we will define a connection on P in two equivalent ways: by a horizontal subbundle $\mathcal{H} \subset TP$, or by a $\mathfrak{g}$ valued 1-form on P which restricts to $id: \mathcal{V}_p \xrightarrow{\sim} \mathfrak{g}$.
- A distribution on a manifold P is a subbundle of TP. A distribution $\mathcal{H}$ on a principal bundle P is horizontal if it is a complement to $\mathcal{V}$, $TP = \mathcal{V} \oplus \mathcal{H}$.
  - Recall that for $j_p: G \to P$, $j_p(g) = p \cdot g$, the map $(j_p)_{*e}: \mathfrak{g} \to \mathcal{V}_p$, $A \mapsto \underline{A}_p = (j_p)_{*e} A$ is an isomorphism.
  - Let $v: T_pP = \mathcal{V}_p \oplus \mathcal{H}_p \to \mathcal{V}_p$ be the projection. Then $\ker(v) = \mathcal{H}_p$ and for $Y_p \in T_pP$, $v(Y_p) \in \mathcal{V}_p$ is the vertical component of Y_p.
    - Note $v(Y_p)$ depends on the choice of $\mathcal{H}_p$.

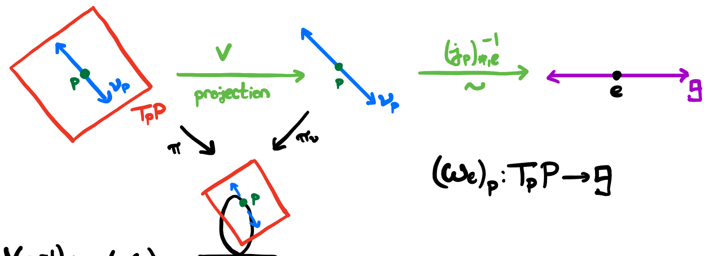
Note $\mathcal{H}_p$ can be gen. by any vector lin. ind. to the vertical subspace $\mathcal{V}_p$



- Given $\mathcal{H}_p$, one can define a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g}) = \mathfrak{g} \otimes \Omega^1(P)$:

$\omega_p: T_pP \to \mathfrak{g}$ (project onto vertical component of a vector)

$\omega_p := (j_p)_{*,e}^{-1} \circ v$



If $\mathcal{H}$ is right invariant, ω satisfies:

i) For each $A \in \mathfrak{g}$, $\omega_p(\underline{A}_p) = A$.

ii) For any $g \in G$, $r_g^* \omega = Ad(g^{-1}) \omega$

iii) ω is smooth.

- Note that ii) means $\omega_{pg}((r_g)_{*,p} Y_p) = Ad(g^{-1}) \omega_p(Y_p)$, since $(r_g)_{*,p}: T_pP \to T_{pg}P$ and $Ad(g^{-1}) = (c_{g^{-1}})_{*,e}: \mathfrak{g} \to \mathfrak{g}$, and $\omega_p(Y_p) \in \mathfrak{g}$. It also gives the commutative box:

$$\begin{tikzcd} T_pP \arrow[r, "(r_g)_{*,p}"] \arrow[d, "\omega_p"] & T_{pg}P \arrow[d, "\omega_{pg}"] \\ \mathfrak{g} \arrow[r, "Ad(g^{-1})"] & \mathfrak{g} \end{tikzcd}$$

- The kernel of ω_p is $\mathcal{H}_p$, since $V_p(X_p) = 0$ iff X_p is horizontal.

- Any connection $\omega \in \Omega^1(P, \mathfrak{g})$ on a principal bundle satisfying i) to iii) is called an Ehresmann connection.
- Note for $Ad(g): \mathfrak{g} \to \mathfrak{g}$, if we are working w/ matrix groups, for $X \in \mathfrak{g}$, $Ad(g)X = gXg^{-1}$. So condition ii) on a matrix Lie group can be rewritten:

$$\omega_{pg}((r_g)_{*,p} Y_p) = g^{-1} \omega_p(Y_p) g$$

This is also equivalent to ω being G-equivariant, where $\omega: TP \to \mathfrak{g}$ is a map from TP w/ a right action $(p, Y_p) \mapsto (p \cdot g, (r_g)_{*,p} Y_p)$ and $\mathfrak{g}$ w/ a left action (the adjoint), since $Ad(g^{-1}) \omega_p(Y_p) = g^{-1} * \omega_p(Y_p) = \omega_p(Y_p) * g$ and $(r_g^* \circ \omega)(Y_p) = \omega_{pg}((r_g)_* Y_p) = \omega_{pg}(Y_p * g)$, so an Ehresmann connection is a $\mathfrak{g}$-valued 1-form which acts as id on $\mathcal{V}_p$ and respects the group actions on TP and $\mathfrak{g}$.

- Given a horizontal distribution $\mathcal{H} \subset TP$, since $TP = \mathcal{V} \oplus \mathcal{H}$, for each vector field $Y \in \mathfrak{X}(P)$, we can decompose it at $p \in P$ as $Y_p = v(Y_p) + h(Y_p)$ w/ $v(Y_p) \in \mathcal{V}_p$, $h(Y_p) \in \mathcal{H}_p$. This gives us the decomposition:

$$Y = vY + hY, \quad vY \in \mathcal{V}, \; hY \in \mathcal{H}$$

for each $Y \in \mathfrak{X}(P)$. Note the map $v: \mathfrak{X}(P) \to \mathcal{V}$ is $v_p = (j_p)_{*,e} \circ \omega_p$. In terms of a basis $\{B_i\}_{i=1}^n$ for $\mathfrak{g}$, we can write $\omega \in \Omega^1(P, \mathfrak{g})$ as $\omega = \omega^i B_i$ with ω^i real valued, and here v can be expressed as:

$$v(Y) = \omega^i(Y) \underline{B_i}$$

- Right translation r_g works well w/ v and h. $\pi \circ r_g = \pi$ because r_g is a fiberwise action, so $\pi_* \circ r_{g*} = \pi_*$. Since $\mathcal{V}_p = \ker(\pi_{*,p})$, if $\underline{A}_p \in \mathcal{V}_p$ is a vertical vector, $\pi_*(r_{g*} \underline{A}_p) = 0$, so $r_{g*} \underline{A}_p \in \mathcal{V}_{pg}$, and similarly for a horizontal vector. Thus $\mathcal{V}$ and $\mathcal{H}$ are invariant under right translation. Furthermore, both v and h commute with r_{g*}:

$$v r_{g*} = r_{g*} v \qquad h r_{g*} = r_{g*} h$$

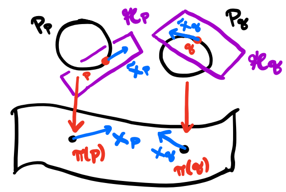
- An Ehresmann connection $\omega \in \Omega^1(P,\mathfrak{g})$ conversely determines a horizontal distribution $\mathcal{H}$ as its kernel:

  $\mathcal{H}_p := \ker(\omega_p)$

  – To prove this, one must verify the tangent space splits as $T_pP = \mathcal{V}_p \oplus \ker(\omega_p)$, that $\mathcal{H}$ is right invariant, $r_{g*}\mathcal{H}_p = \mathcal{H}_{pg}$, and that $\mathcal{H} \subset TP$ varies smoothly.

- The reason we care about $\mathcal{H}$ is that the definitions we've made allow us to *lift* a tangent vector on the base manifold M to a horizontal vector. Since $\mathcal{V}_p = \ker(\pi_*)$ and $\pi_*: T_pP \to T_{\pi(p)}M$, we have an isomorphism:

  $$\mathcal{H}_p \cong T_pP/_{\ker(\pi_*)} \xrightarrow{\sim} T_{\pi(p)}M$$

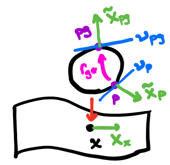
  

  Since $\pi_{*,p}$ is surjective. This means that given $X \in \mathfrak{X}(M)$, $\exists!$ lift $\tilde{X} \in \mathcal{H} \subset T_p$ such that $X_p = \pi_{*,p}(\tilde{X}_p)$. The vector field $\tilde{X} \in \mathcal{H}$ is called the horizontal lift of X.

  – If $\mathcal{H}$ is a right invariant distribution on P and $X \in \mathfrak{X}(M)$, the horizontal lift $\tilde{X} \in \mathcal{H}$ is a smooth right invariant distribution, i.e.:

  $$(r_g)_{*,p}\tilde{X}_p = \tilde{X}_{pg} \qquad \pi_*((r_g)_{*,p}\tilde{X}_p) = X_p$$

  The horizontal lift $\tilde{X}_p$ is right invariant and moves btwn each fiber.

- The Lie bracket respects notions of verticality. Let $A \in \mathfrak{g}$, and $\underline{A} \in \mathcal{V} \subset TP$. Then:

  i) For any horizontal vector field $H \in \mathcal{H}$,

  $$[\underline{A}, H] \in \mathcal{H}$$

  where $[X,Y]_p f = X_p(Yf) - Y_p(Xf)$ is the vector field Lie bracket.

  ii) For any right invariant vector field $Y \in \mathfrak{X}(P)$:

  $$[\underline{A}, Y] = 0$$

- Recall the Maurer-Cartan form $\Theta \in \Omega^1(G,\mathfrak{g})$ is defined as follows: Take a set of generators $\{T^a\}$ of $\mathfrak{g}$ and left translate them to $X^a_g := (\ell_g)_{*,e} T^a$, so $\{X^a\} \subset \mathfrak{X}(G)$ is a frame on TG. Then if $\Theta^a \in \Omega^1(G)$ is the dual coframe, $\Theta^a(X^b) = \delta^{ab}$, we have:

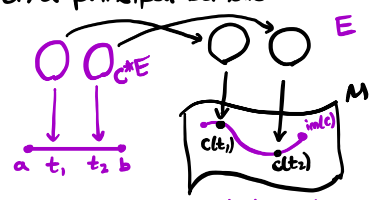
  $$\Theta = T^a \otimes \Theta^b$$

  is the Maurer-Cartan form, which is the unique form in $\Omega^1(G,\mathfrak{g})$ which restricts to id on $\mathfrak{g} = T_eG$. The Maurer Cartan form, when pulled back by $\rho: M \times G \to G$, is a connection on the trivial bundle $\pi: M \times G \to M$.

  – We have already shown the hard parts: when G acts on itself fiberwise by right mult. (as it does in a principal bundle) then $\underline{A}_p$ is the LI vector field generated by $A \in \mathcal{V}_{(\pi(p),e)}$, so $\rho_*\underline{A}_p = (X_A)_{\rho(p)} \in T_{\rho(p)}G$, where X_A is the LI invariant field. Then $\Theta(\rho_*\underline{A}_p) = \Theta(X_A) = A \in \mathfrak{g}$, so $\rho^*\Theta$ sends $\underline{A}$ to A. Likewise, we've already shown Θ is G-equivariant

- Next we will show how a connection on a frame bundle induces a connection on a principal bundle.
- A smooth section of a vector bundle $E \xrightarrow{\pi} M$ along a curve $c: [a,b] \to M$ is a section of the pullback bundle c^*E, i.e. an element of $\Gamma(c^*E)$
- The covariant derivative is the unique $\mathbb{R}$-linear map

  $$\frac{D}{dt}: \Gamma(c^*E) \to \Gamma(c^*E)$$

  $c^*E|_t = E|_{c(t)}$

  $\Gamma(c^*E)$ = vector fields along $c(t)$ valued in E.

  such that:

  i) $\frac{D}{dt}(fs) = \frac{df}{dt}s + f\frac{Ds}{dt}$

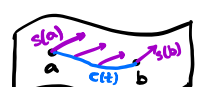
  ii) $\frac{Ds}{dt} = \nabla_{c'(t)}\tilde{s}$, where $s(t) = \tilde{s}(c(t))$ and $\tilde{s}: M \to E$ is a section.

  A section $s \in \Gamma(c^*E)$ is parallel along c if $Ds/dt \equiv 0$ on $[a,b]$.

  – If $im(c)$ is contained in a framed open set (U, e_i), we can expand $s(t) = s^i(t)e_i|_{c(t)}$ with $s^i \in C^\infty([a,b])$. Then $Ds/dt \equiv 0$ is equivalent to:

  $$\frac{ds^i}{dt}(t) + s^j(t)\omega^i_j(c'(t)) = 0$$

  for ω the connection form of ∇.

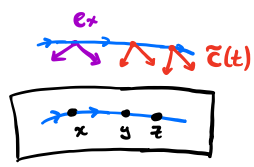
  – If $s \in \Gamma(c^*E)$ is parallel, we say s performs parallel transport from $s(a)$ to $s(b)$ along c, for $a,b \in im(c)$.

- For any vector bundle $E \to M$ w/ a connection and $c:[a,b]\to M$, there is a unique parallel transport $\varphi_{a,b}: E_{c(a)} \longrightarrow E_{c(b)}$. This map $\varphi_{a,b}$ is a linear isomorphism.

- A parallel frame $\{e_i\}$ on $U \subset M$ along c is a frame made of parallel sections. A curve $\tilde{c}:[a,b]\to Fr(E)$ is called a lift of $c:[a,b]\to M$ if $c = \pi\circ\tilde{c}$, and a horizontal lift if $\tilde{c}(t)$ is a parallel frame along $c(t)$.
  - A lift of c to the frame bundle is then just a frame $[e_1(t),\dots,e_r(t)]$ along $c(t)$, and it is a horizontal lift if the frame is parallel.
  - If $[s_1(t),\dots,s_r(t)]\in\Gamma(c^*E)$ are parallel and $[s_1(t_0),\dots,s_r(t_0)]$ is a basis for $E_{c(t_0)}$, then b/c parallel translation $\varphi_{t_0,t}$ is an isomorphism @ $\forall t\in[a,b]$, $[s_1(t),\dots,s_r(t)]$ is a frame on c^*E. Since φ is unique, any horizontal lift $\tilde{c}:[a,b]\to Fr(E)$ of $c:[a,b]\to M$ with an initial starting point $[s_{1,0},\dots,s_{r,0}]$ is uniquely determined.
    - So, given $c:[a,b]\to M$ and $[v_1,\dots,v_r]\subset E_{c(a)}$, there is a unique horizontal lift (parallel frame) $\tilde{c}:[a,b]\to Fr(E)$ of c with $\tilde{c}(a)=[v_1,\dots,v_r]$ and $c=\pi\circ\tilde{c}$.

- Let $x\in M$, $e_x\in Fr(E)_x$. A vector $v\in T_{e_x}Fr(E)$ is horizontal if $\exists$ a curve $c(t)$ through $x\in M$ such that $v=\tilde{c}'(0)$. Given such an $e_x\in Fr(E)_x$ and $c(t)$ through M, consider any section $s:U\to Fr(E_U)$ such that $im(c)\subset U$ and $s(x)=e_x$. (s not necessarily parallel). We can relate $\tilde{c}$ and $s\circ c$ by:

$\tilde{c}'(0)$ measures how much e_x changes as we parallel transport it along c.



$$s(c(t))=\tilde{c}(t)a(t)$$

with $a(t)\in GL(\mathbb{R}^r)$, since $\forall t$ $s(c(t))$ and $\tilde{c}(t)$ are ordered bases of $E|_{c(t)}$. Note $a(0)=I$, and $a'(0)\in\mathfrak{gl}(\mathbb{R}^r)$ as it is a derivative of a fn valued in GL. The relation btwn these quantities is:

$$S_*(c'(0))=\tilde{c}'(0)+\underline{a'(0)}_{e_x} \in T_{e_x}Fr(M)$$

where $S_*:T_xM\longrightarrow T_{e_x}Fr(M)$ is the push forward, and $\underline{a'(0)}_{e_x}$ is the fund. vector field gen. by $a'(0)\in\mathfrak{g}$ @ e_x.
Here we see $a(t)$ gives the difference between the horizontal lift $\tilde{c}$ of a curve and an arbitrary lift of the curve to $Fr(E)$.
If $s(c(t))=\tilde{c}(t)$, note $s_*c'(0)=\tilde{c}'(0)$ by definition, so $\underline{a'(0)}_{e_x}$ will vanish in that case.
- Let $\omega_s=\omega^i{}_j$ be the connection matrix for ∇ w.r.t. the frame $(s_1,\dots,s_r)\subset Fr(E)_U$. Then $a'(0)=\omega_s(c'(0))$, so we can write:

$$\underbrace{S_*c'(0)}_{\text{lift of } c'(0) \text{ by an arbitrary section}}=\underbrace{\tilde{c}'(0)}_{\text{horizontal piece}}+\underbrace{\underline{\omega_s(c'(0))}_{e_x}}_{\text{vertical piece, determined by connection in } c'(0) \text{ direction.}} \in T_{e_x}Fr(E)$$

- For any section $s:M\to Fr(E)$, the horizontal part of $S_*c'(0)$ will always be $\tilde{c}'(0)$, and the vertical piece is determined by the connection $\nabla_{c'(0)}$ acting on $\{s'(t),\dots,s^r(t)\}$. This gives a natural way to identify $\tilde{c}'(0)$ with $\mathcal{H}_p$, as $c'(0)$ ranges over $T_{\pi(p)}M$

- This formula shows the map:

$$\varphi: T_xM \xrightarrow{\sim} \mathcal{H}_{e_x}\subset T_{e_x}Fr(M)$$
$$c'(0)\mapsto\tilde{c}'(0)$$

is $\mathbb{R}$-linear and an isomorphism, and $\pi_*: T_{e_x}Fr(M)\xrightarrow{\sim}T_xM$ is the inverse of φ. Since the subset of horizontal vectors $\mathcal{H}_{e_x}\subset T_{e_x}Fr(M)$ in the frame bundle equals $im(\varphi)$, $\mathcal{H}_{e_x}$ is a subspace of $T_{e_x}Fr(M)$ of dimension $\dim(T_xM)$ - essentially the horizontal subspace at e_x is the same as T_xM.
- Note that at this point, this has all been determined w/ a connection on a vector bundle $E\to M$; the connection on $Fr(E)$ is determined by $\mathcal{H}$ if we show all the other properties discussed previously. Since π_* is φ^{-1}, this gives $\mathcal{V}_{e_x}\cap\mathcal{H}_{e_x}=\emptyset$ as $\pi_*(\mathcal{V}_p)=0$.

- As in the previous part, for $X\in\mathfrak{X}(M)$ we will denote its unique horizontal lift to $\mathcal{H}$ as $\tilde{X}$, so $\pi_*\tilde{X}_p=X_{\pi(p)}$. The above work gives us a horizontal lift formula for $\tilde{X}$ in terms of X. For a framed open set (U,s) w/ connection matrix ω_s relative to $s=[s_1,\dots,s_r]$, $x\in U$, and $p:=s(x)\in Fr(E)_x$, and $X_x\in T_xM$ w/ $\tilde{X}_p=\varphi(X_x)\in\mathcal{H}_p$, we get:

$$\tilde{X}_p=s_{*,x}(X_x)-\underline{\omega_s(X_x)}_p$$

So the connection determines how different $\tilde{X}$ will be from s_*X. Note ω depends on s, and for a parallel frame $\omega_s=0$, since $\nabla_X s_i=\omega_i^j(X)s_j=0$ for s_i parallel.

- Theorem: Let $E\to M$ be a vector bundle w/ connection ∇. Then the horizontal distribution $\mathcal{H}$ on $Fr(E)$ satisfies:
  i) $T_pFr(E)=\mathcal{V}_p\oplus\mathcal{H}_p$
  ii) $r_{g*}\mathcal{H}_p=\mathcal{H}_{pg}$

Thus $\mathcal{H}\subset TFr(E)$ is an Ehresmann connection on $Fr(E)$.

• Note that for ∇ on $E \to M$, $\omega_e(X) = [\underbrace{\omega^i{}_j(X)}_{\mathfrak{X}(M)}]$ is in $\mathfrak{gl}(\mathbb{R}^r)$, hence we can view a connection form for a vector bundle as a $\mathfrak{gl}(\mathbb{R}^r)$ valued form, $\omega_e \in \Omega^1(M, \mathfrak{g})$, just as how on $Fr(E)$, the connection $\omega \in \Omega^1(Fr(E), \mathfrak{g})$.
Thus, we can pull back ω to ω_e with a map $U \subset M \to Fr(E)_U$. Given a framed open set $(U, [e_1, \dots, e_r])$, and equivalently a section of the frame bundle $e = [e_1, \dots, e_r]$, $e : U \to Fr(E)_U$, and:

$$\omega_e = e^* \omega$$

where $\omega_e \in \Omega^1(U, \mathfrak{g})$ is the connection form w.r.t. the frame e.

• Summary of Ehresmann connection on $Fr(E)$: Let $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ be a connection on a vector bundle $E \to M$.

i) ∇ defines a covariant derivative $D/dt : \Gamma(c^*E) \to \Gamma(c^*E)$, which defines parallel transport, and the notion of a parallel frame along c. A horizontal lift of c is a curve $\tilde{c} : [a,b] \to Fr(E)$ s.t. $\pi \circ \tilde{c} = c$ and $\tilde{c}(t)$ is a parallel frame.

ii) Given $e_{c(a)} = [v_1, \dots, v_r] \in Fr(E)_{c(a)}$ (an initial point), there is a unique horizontal lift $\tilde{c}$ of c with $\tilde{c}(a) = e_{c(a)}$.
Although $\tilde{c}(t)$ is only defined modulo the initial condition, $\tilde{c}'(t) \in \mathfrak{gl}(\mathbb{R}^r)$ is defined unambiguously.

$$\begin{tikzcd} & Fr(E) \arrow[d, "\pi"] \\ {[a,b]} \arrow[ur, "\tilde{c}"] \arrow[r, "c"'] & M \end{tikzcd}$$

iii) This defines for us a map, for $e_x \in Fr(E)_x$ and $\tilde{c}$ through e_x:

$$\varphi : T_x M \to T_{e_x} Fr(E)$$
$$c'(0) \mapsto \tilde{c}'(0)$$

which is an embedding. This creates an isomorphism onto its image, which is the horizontal subspace $\mathcal{H} \subset TFr(E)$. The inverse of φ is π_*:

$$T_x M \underset{\pi_*}{\overset{\varphi}{\rightleftarrows}} T_{e_x} Fr(E)$$

iv) φ allows us to lift a vector field $X \in \mathfrak{X}(U)$ to its horizontal lift $\tilde{X} \in \mathcal{H}_U$ such that relative to a frame $s = [s_1, \dots, s_r]$ w/ connection form ω_s for ∇ relative to s, we have, w/ $x := \pi(p)$:

$$\tilde{X}_p = s_{*,x}(X_x) - \underline{\omega_s(X_x)}_p$$

v) The horizontal distribution $\mathcal{H}$ is C^∞, is a summand of $TFr(E) = \mathcal{V} \oplus \mathcal{H}$, and is right invariant $r_{g*}\mathcal{H}_p = \mathcal{H}_{pg}$, hence defines an Ehresmann connection ω_e as above, w/ $\mathcal{H}_p = \ker((\omega_e)_p)$. This connection pulls back to the connection form for ∇ relative to a frame $e : U \to Fr(E)_U$ as $\omega_e = e^* \omega$.

• All of this shows that ω is a globally defined object in $\Omega^1(Fr(E), \mathfrak{g})$, and the connection form w.r.t. a frame $[e_1, \dots, e_r]$ of a vector bundle w/ connection ∇ is a basis-dependent object which changes based on our coordinate choice.

• Now we turn to curvature. Given a connection $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, its curvature is the form:

$$\Omega := d\omega + \tfrac{1}{2}[\omega, \omega] \in \Omega^2(P) \otimes \mathfrak{g}$$

because the connection form relative to $e \in \Gamma(Fr(E))$ is $\omega_e = e^*\omega$, the curvature pulls back under e^* as well: If $\Omega \in \Omega^2(Fr(E)) \otimes \mathfrak{g}$ is the curvature form of $Fr(E)$ and e is a section, then:

$$\Omega_e = e^* \Omega$$

where $\Omega_e \in \Omega^2(M) \otimes \mathfrak{gl}(\mathbb{R}^r)$ is the curvature form of ∇ on E relative to the frame e.

• For a point $p \in P$, any vertical vector $A_p \in \mathcal{V}_p \subset T_p P$ can be extended to a fund. vector field $\underline{A} \in \Gamma(\mathcal{V})$ for some $A \in \mathfrak{g}$. Likewise for $B_p \in \mathcal{H}_p \subset T_p P$, B_p can be extended to a horizontal lift $\tilde{B} \in \Gamma(\mathcal{H})$ of a vector field $B \in \mathfrak{X}(M)$.
The curvature form satisfies the following:

i) Horizontality: Ω only acts on the horizontal components of vector fields:

$$\Omega_p(X_p, Y_p) = (d\omega)_p(hX_p, hY_p)$$

ii) $\mathfrak{g}$-equivariance: We have:

$$r_g^* \Omega = \mathrm{Ad}_{g^{-1}} \Omega$$

iii) Second Bianchi identity:

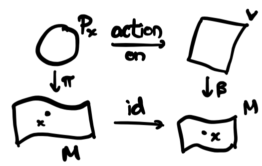
$$d\Omega = [\Omega, \omega]$$

- The associated bundle: Given a principal bundle $\pi: P \to M$ w/ structure group G and a representation $\rho: G \to GL(V)$, we can form a vector bundle $\beta: E \to M$ as follows. Let:

$$P\times_\rho V := P \times V/\sim \qquad\qquad (p,v)\sim(pg, g^{-1}\cdot v)$$

where $\rho(g^{-1})v = g^{-1}\cdot v$. Then $E := P\times_\rho V$ is the associated bundle, equipped w/ projection $\beta: E\to M$, $\beta([p,v]) := \pi(p) \in M$.
Locally, E looks like $M\times V$, i.e. there is an isomorphism:

$$f_p : V \xrightarrow{\sim} E_{\pi(p)}$$
$$f_p(v) = [(p,v)]$$



This map satisfies $f_{pg} = f_p \circ \rho(g)$.

- If $\pi: P\to M$ is a principal bundle w/ fibers $\cong G$ and $\rho: G\to GL(V)$ is a representation, then a V valued form $\varphi \in \Omega^k(P)\otimes V$ is right equivariant w.r.t. ρ if:

$$r_g^*\varphi = \rho(g^{-1})\cdot\varphi$$

where $(\rho(g^{-1})\cdot\varphi)(X_1,\dots,X_k) = \rho(g^{-1})\,\varphi(X_1,\dots,X_k)$. Such a form is also called pseudo-tensorial. Note that since $\rho(g^{-1})\cdot\varphi = \varphi\cdot\rho(g)$, such a k-form φ is just a right G-homomorphism, i.e.

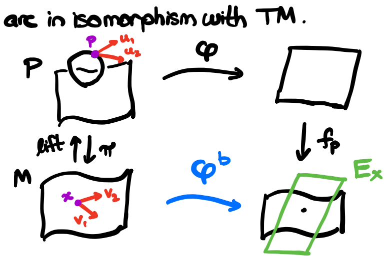
$$\varphi(r_{g*}X_1,\dots,r_{g*}X_k) = \varphi(X_1,\dots,X_k)\,\rho(g) = \rho(g^{-1})\,\varphi(X_1,\dots,X_k)$$

so the action of G factors out through φ.

- A form $\varphi\in\Omega^k(P)\otimes V$ is horizontal if $\varphi(X_1,\dots,X_k)=0$ whenever any $X_i \in \mathcal{V}$, so φ acts as 0 on any vertical vector.
  - Any 0-form is horizontal, since $f(x)$ can never take a vertical vector as an input.
- A form $\varphi\in\Omega^k(P)\otimes V$ is tensorial if it is right equivariant and horizontal. The set of ρ-tensorial V-valued k-forms is denoted $\Omega^k_\rho(P,V)$
  - Ex: The curvature $\Omega\in\Omega^2(P,\mathfrak{g})$ is tensorial of type $Ad: G\to GL(\mathfrak{g})$
  - We care about tensorial forms because $\varphi\in\Omega^k_\rho(P,V)$ can be dropped down to a form on M with values in the associated bundle, $E = P\times_\rho V$. This is intuitively because horizontal vectors in $\mathcal{H}$ are in isomorphism with TM.

  We define a map:

$$\Omega^k_\rho(P)\otimes V \xrightarrow{\sim} \Omega^k(M)\otimes E$$
$$\varphi \longmapsto \varphi^\flat$$



  where $\varphi^\flat$ ("φ-flat") is defined as follows. For $p\in P$, use the horizontal lift to lift a basis $\{v_1,\dots,v_n\}$ for $T_{\pi(p)}M$ to a basis $\{u_1,\dots,u_n\}$ of $\mathcal{H}_p\subset T_pP$, i.e. $\pi_{*,p}(u_i) = v_i$. Then $\varphi^\flat\in\Omega^k(M)\otimes E$ is defined as:

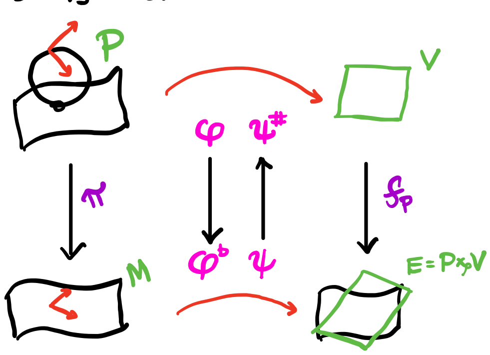
$$\varphi^\flat_x(v_1,\dots,v_k) := f_p(\varphi_p(u_1,\dots,u_k))$$

  where $f_p: V\xrightarrow{\sim} E_x$ is the isomorphism btwn V and E_x.
  - To show $\varphi^\flat$ is well defined:
    - The dfn of $\varphi^\flat$ is ind. of choice of horizontal lift, and this is b/c we assume φ is a vertical form. This is b/c for another lift $\{u_1',\dots,u_k'\}\subset T_pP$, since $\pi_* u_i' = v_i = \pi_* u_i$, we have $u_i - u_i' \in \mathcal{V}_p$ as it is killed by π_*. Then $\varphi_p(u_1',\dots,u_k') = \varphi_p(u_1+\text{vert},\dots,u_k+\text{vert}) = \varphi_p(u_1,\dots,u_k)$.
    - Also must be shown $\varphi_p(u_1,\dots,u_k) = \varphi_{pg}(\bar u_1,\dots,\bar u_k)$, if $\{\bar u_1,\dots,\bar u_k\}\subset T_{pg}P$ is a lift of $\{v_i\}$ to $T_{pg}P$. This is more involved, but right equivariance implies it.
  - The inverse of $\varphi\mapsto\varphi^\flat$ is:

$$\Omega^k(M)\otimes E \longrightarrow \Omega^k(P)\otimes V$$
$$\psi \longmapsto \psi^\#$$

  where $\psi^\#$ ("ψ-sharp") is defined for $\{u_1,\dots,u_k\}\subset T_pP$ as:

$$\psi^\#_p(u_1,\dots,u_k) := f_p^{-1}(\psi_x(\pi_* u_1,\dots,\pi_* u_k))$$



- For a principal bundle P w/ structure group G, recall $\mathrm{Ad}: G \to GL(\mathfrak{g})$ is a representation of $\mathfrak{g}$, by $\mathrm{Ad}(g)X = gXg^{-1}$ (for matrix Lie groups). The associated bundle is denoted $\mathrm{Ad}P := P \times_{\mathrm{Ad}} \mathfrak{g}$ is *locally* isomorphic to $M \times \mathfrak{g}$, i.e. a fiber bundle over M w/ fibers $\mathfrak{g}$.
- This machinery is often used in physics. With Yang Mills theory, we typically view curvature as a form on the base, $F = \Omega^b \in \Omega^2(M) \otimes \mathrm{Ad}(P)$ (a $\mathfrak{g}$-valued 2-form on M), since $F = \frac{1}{2} F^a_{\mu\nu} T^a dx^\mu dx^\nu$.
- For $k=0$, a map $f \in \Omega^0_\rho(P,V)$ is any map $f: P \to V$ with $f(pg) = \rho(g^{-1}) f(p)$. A map in $\Omega^0(M,E)$ is simply a section of the associated bundle $P \times_\rho V$. So, we get a bijection

$$\left\{\begin{matrix}\text{G-equivariant} \\ \text{maps } f: P \to V\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{Sections of} \\ P \times_\rho V \to M\end{matrix}\right\}$$

- A form $\varphi \in \Omega^k(P) \otimes V$ is basic if $\varphi = \pi^* \xi$ for a form $\xi \in \Omega^k(M) \otimes V$.
  - Note any basic form is horizontal, since if $X \in \mathcal{V} \subset TP$, $(\pi^*\xi)(X_1, \dots, X, \dots, X_n) = \xi(\pi_* X_1, \dots, \cancel{\pi_* X}^{\,0}, \dots, \pi_* X_n) = 0$.
- A form $\varphi \in \Omega^k(P) \otimes V$ is G-invariant if $r_g^* \varphi = \varphi$ for each $g \in G$.
  - If φ is G-equivariant, it plays nicely w/ G and is a homomorphism of the action of G on V, but if φ is G-invariant then it is fixed by the right action of G.
  - Similarly, a G-invariant form is exactly a G-equivariant form for the trivial representation $g \mapsto 1_V$.
- A form $\varphi \in \Omega^k(P) \otimes V$ is basic iff it is horizontal and G-invariant.
  - Already saw basic $\Rightarrow$ horizontal. Basic also implies G-equivariant, as $r_g^* \pi^* \omega = (\pi \circ r_g)^* \omega = \pi^* \omega$ since $\pi \circ r_g = \pi$. The iff is proved using the $\varphi \mapsto \varphi^b$ isomorphism when $\rho: G \to \mathrm{Aut}(V)$ is the trivial representation, and identifying $P \times_\rho V$ as the trivial bundle $M \times V$. For this rep, $\psi^\# \in \Omega_\rho(P,V)$ is just $\psi^\# = \pi^* \psi$.
- To the covariant derivative. If $\varphi \in \Omega^k(P) \otimes V$ is G-equivariant, then $d\varphi$ is as well.
  - Given $\varphi \in \Omega^k(P,V)$, its horizontal component $\varphi^h \in \Omega^k(P,V)$ is the form whose action is:

    $$\varphi^h(X_1, \dots, X_k) := \varphi(hX_1, \dots, hX_k)$$

    If φ is G-equivariant, then so is φ^h
    - Along w/ $d\varphi$ preserving horizontality, this implies for any right G-equivariant form, $(d\varphi)^h$ is tensorial, i.e. horizontal and G-equivariant.
- Let $\varphi \in \Omega^k(P) \otimes V$ be a k-form. The covariant derivative of φ is:

  $$D\varphi := (d\varphi)^h \in \Omega^{k+1}(P) \otimes V$$

  As stated previously, if φ is right G-equivariant, then $D\varphi$ is tensorial, $D\varphi \in \Omega^{k+1}_\rho(P) \otimes V$. In particular, D restricts to a map on tensorial forms:

  $$D: \Omega^k_\rho(P) \otimes V \to \Omega^{k+1}_\rho(P) \otimes V$$

  where D is an antiderivation w.r.t. $\wedge$:

  $$D(\omega \wedge \tau) = (D\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge (D\tau)$$

  However, generally $D^2 \neq 0$, so D is not a cochain map.
  - Since we saw above that $\Omega(X_1, \dots, X_k) = d\omega(hX_1, \dots, hX_k)$, we have:

    $$\Omega = D\omega \in \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$$

  - Typically we will look @ forms which are not always tensorial. Ex: $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ is not tensorial b/c it is not horizontal. However, $\Omega = D\omega \in \Omega^2_{\mathrm{Ad}}(P,V)$ is tensorial, and is D(non-tensorial form)
- We can consider a general wedge product: Given $\rho: G \to GL(V)$, the differential at e is a map $\rho_*: \mathfrak{g} \to \mathfrak{gl}(V)$. So, given $\tau \in \Omega^k(P, \mathfrak{g})$ and $\varphi \in \Omega^\ell(P,V)$, we can define $\omega \wedge \tau \in \Omega^{k+\ell}(P,V)$ as:

  $$\tau \wedge \omega(X_1, \dots, X_{k+\ell}) := \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma)\, \rho_*(\tau(X_{\sigma(1)}, \dots, X_{\sigma(k)}))\, \varphi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})$$

Note for $\rho = \mathrm{Ad}: \mathfrak{g} \to GL(\mathfrak{g})$, then $\tau \wedge \varphi$ agrees w/ the usual $\tau^a \wedge \varphi^b [t^a, t^b]$:

$$(\tau \wedge \varphi)(X_1, \dots, X_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma) [\tau(X_{\sigma(1)}, \dots, X_{\sigma(k)}), \varphi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})]$$

In this case we will also write $\tau \wedge \varphi$ as $[\tau, \varphi]$.

- This wedge product gives us a formula for $D\varphi$. For a general rep $\rho: G \to GL(V)$ and a tensorial form $\varphi \in \Omega^k_\rho(P, V)$, we have:

  $$D\varphi = d\varphi + \omega \wedge \varphi$$

  - If φ is not tensorial, this does not hold. For example, on a tensorial form $\xi \in \Omega^k_{\mathrm{Ad}}(P, \mathfrak{g})$, we see $D\xi = d\xi + [\omega, \xi]$. However, we already know $\omega \in \Omega^1(P, V)$ doesn't satisfy this:

    $$\Omega = D\omega = d\omega + \tfrac{1}{2}[\omega, \omega]$$

    The factor of $1/2$ is allowed b/c $\omega \notin \Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$, i.e. ω is not tensorial.
  - However, $\Omega \in \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$ is tensorial. Thus the second Bianchi identity implies:

    $$D\Omega = d\Omega + [\omega, \Omega] = 0$$

- Back to characteristic classes: Any degree k polynomial on V can be expressed relative to a basis $\{e_i\}_{i=1}^n$ in terms of its dual basis $\{\alpha^i\}_{i=1}^n \subset V^*$ as $f = a_I \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$, where $I = (i_1, \dots, i_k)$ is a multi-index.
  - Here we are interested in $\mathrm{Ad}(G)$ invariant polynomials $f: \mathfrak{g} \to \mathbb{R}$. Such an f satisfies:

    $$f(\mathrm{Ad}(g)X) = f(X) \qquad \forall g \in G, X \in \mathfrak{g}$$

    Note that for matrix groups, this is the statement that $f(gXg^{-1}) = f(X)$, so i.e. $\mathrm{tr}(X)$ and $\det(X)$ are $\mathrm{Ad}(GL(\mathbb{R}^n))$-invariant, which is the structure group of the frame bundle $Fr(E)$ of a rank n vector bundle.
  - Let $\{e_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$ w/ dual basis $\{\alpha^i\}_{i=1}^n$. Since $\Omega \in \Omega^2(P, \mathfrak{g})$, we can expand:

    $$\Omega = \Omega^i e_i \qquad \Omega^i \in \Omega^2(P)$$

    where Ω^i are real valued 2-forms on P. The polynomial $f(\Omega) \in \Omega^{2k}(P)$ is defined as:

    $$f(\Omega) := a_I \Omega^{i_1} \wedge \dots \wedge \Omega^{i_k}$$

    - Despite the basis-dependent dfn, $f(\Omega)$ is independent of basis.
  - The last thing we need: If φ is a basic form on P (so $\varphi = \pi^* \xi$), then $D\varphi = d\varphi$. This essentially comes b/c basic forms are horizontal.
- Chern-Weil homomorphism for principal bundles: $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$, f a degree k $\mathrm{Ad}(G)$ invariant polynomial:
  i) $f(\Omega)$ is a basic form, so $f(\Omega) = \pi^* \Lambda$ for $\Lambda \in \Omega^{2k}(M)$.
  ii) Λ is closed.
  iii) The cohomology class $[\Lambda]$ is independent of connection.
- The cohomology class $[\Lambda] \in H^{2k}(P)$ is a characteristic class. This induces a map, the Chern-Weil homomorphism, which maps $\mathrm{Ad}(G)$-invariant polynomials to the cohomology of M:

  $$w: \mathrm{Inv}(\mathfrak{g}) \longrightarrow H^*(M)$$
  $$f(X) \longmapsto [\Lambda], \quad f(\Omega) = \pi^* \Lambda$$

  This map is an algebra homomorphism.
  - Let $\det(X + \lambda I) = \sum f_k(X) \lambda^k$ be the characteristic polynomials. For a principal $GL(\mathbb{C}^r)$ bundle, $w(f_k)$ are the Chern classes of P. For a $GL(\mathbb{R}^r)$ bundle, $w(f_k)$ are the Pontryagin classes of P.

Applications to Physics

- Most applications to physics require a metric $g_{\mu\nu}\, dx^\mu \wedge dx^\nu$. Mathematically, this is only required to be a pseudo-metric. A pseudometric is a form $g \in \Omega^2(M)$ which is symmetric and non-degenerate; unlike a full metric, it need not be positive definite. A pair (M,g) for g a pseudo-metric is called a pseudo Riemannian manifold.
  - Lorentz spacetime $M=\mathbb{R}^4$ with $g = dt^2 - d\vec{x}^2$ is not a Riemannian manifold, but rather a pseudo-Riemannian manifold.
  - A pseudometric $g_{\mu\nu}$ is symmetric and nondegenerate $\Rightarrow$ real nonzero eigenvalues. If i and j are the # of positive and negative eigenvalues, then (i,j) is called the index of g.
  - A manifold is Riemannian if $j=0$, and Lorentzian if $j=1$.

- Recall: For a pseudo-metric, we can define the Hodge Star

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

$$*(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}) = \frac{\sqrt{|g|}}{(n-k)!} \epsilon^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_{n-k}}\, dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{n-k}}$$

The $*$ can be used to define a symmetric bilinear form on $\Omega^k(M)$ as:

$$\langle \omega, \tau \rangle = \int_M \omega \wedge * \tau$$

where $g = g_{ij}\, dx^i dx^j = g_i\, dx^i \otimes dx^i$ is taken to be diagonal. Note:

$$*1 = \mathrm{Vol} \qquad\qquad **\omega = \begin{cases} (-1)^{k(n-k)}\, \omega & g \text{ Riemannian} \\ (-1)^{1+k(n-k)}\, \omega & g \text{ Lorentzian} \end{cases}$$

- With this bilinear form $\langle \cdot , \cdot \rangle$, there is a natural chain map adjoint to d:

$$\delta : \Omega^k(M) \longrightarrow \Omega^{k-1}(M) \qquad\qquad \delta\omega = \begin{cases} (-1)^{n(k+1)+1} * d * \omega & g \text{ Riemannian} \\ (-1)^{n(k+1)} * d * \omega & g \text{ Lorentzian} \end{cases}$$

and $\delta = 0$ for $k=0$. Since $*^2 = (-1)^{\#}$, $\delta^2 = (-1)^{\#} * d^2 * = 0$, hence δ is a chain map. We have:

$$\langle d\omega, \tau \rangle = \langle \omega, \delta\tau \rangle$$

δ will often be denoted by $d^\dagger$ as well.

- Hodge duality: The Laplace deRham operator is defined as $\Delta : \Omega^k(M) \to \Omega^k(M)$ by:

$$\Delta = d\delta + \delta d$$

We have $\Delta\omega = 0$ iff $d\omega = \delta\omega = 0$, and we call such forms harmonic and denote the space of harmonic k-forms by $\mathcal{H}^k(M)$. For any $\omega \in \Omega^k(M)$, $\exists!$ decomposition:

$$\omega = d\alpha + \delta\beta + \gamma \qquad\qquad \Omega^k(M) = \mathrm{im}(d) \oplus \mathrm{im}(\delta) \oplus \mathcal{H}^k(M)$$

with $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^{k+1}(M)$, and $\gamma \in \mathcal{H}^k(M)$.

- On 0-forms $f \in \Omega^0(M)$, $\Delta f = -\frac{1}{\sqrt{|g|}} \partial_\nu (\sqrt{|g|}\, g^{\mu\nu} \partial_\nu f) = \partial^2 f$ if we use a flat metric. Similarly for $k=1$ in flat Euclidean space, $\Delta \omega = -(\partial_\mu \partial^\mu \omega_\nu)\, dx^\nu$
- The Hodge star explicitly gives an isomorphism between the δ-homology and the d-cohomology:

$$* : H_d^*(M) \longrightarrow H_{\delta, n-*}(M)$$

- For $\omega \in \Omega^k(M)$, we denote its harmonic projection as $P\omega \in \mathcal{H}^k$, where in the above decomposition we take $P\omega = \gamma$. Hodge's theorem asserts that for a cohomology class $[\omega] \in H^k(M)$, any representative $\tau \in [\omega]$ has the same harmonic representative $P\tau$.

• Hodge's Theorem: For a compact orientable Riemannian manifold M, there is an isomorphism:

$$H^k(M) \xrightarrow{\sim} \mathcal{H}^k$$
$$[\omega] \longmapsto P\omega$$

• Now we turn to the physics. In Minkowski space:

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = dt^2 - d\vec{x}^2$$

has index$(g) = 1$.

• In electromagnetism, quantities are naturally differential forms. We first work in 3 dimensions, then move to 4. In 3d:

$*1 = dx \wedge dy \wedge dz = \mathrm{Vol}$

$*dx = dy \wedge dz$	$*dy = dz \wedge dx$	$*dz = dx \wedge dy$
$*(dx \wedge dy) = dz$	$*(dz \wedge dx) = dy$	$*(dy \wedge dz) = dx$

$*(dx \wedge dy \wedge dz) = 1$

where index$(g) = 0$. Note that we have the correspondences to vector calculus:

Vector calc $\xleftrightarrow{\sim}$	Differential forms
functions f	0 forms f or 3 forms $f\, dx \wedge dy \wedge dz$
Vector fields $A^i \vec{e}_i$	1 forms $A_i dx^i$ or 2 forms $\epsilon^i{}_{jk} A_i\, dx^j \wedge dx^k = A_i (*dx^i)$
gradient ∇f	d of a 0 form $f \in \Omega^0(M)$
curl $\nabla \times \vec{F}$	d of a 1 form $F_i dx^i \in \Omega^1(M)$ (note $dF \in \Omega^2$, and $*dF \in \Omega^1$)
div $\nabla \cdot \vec{F}$	d of a 2 form $F_i dy \wedge dz + \ldots$, or $\delta = *d*$ of a 1 form $F_i dx^i$

- If we wish, the $*$ lets us take our identification to be with 0 and 1 forms.
- We see explicitly here that when $\Delta = d\delta + \delta d$ acts on 0 forms f, we get a Laplacian:

$$\Delta f = (d\overset{0}{\delta} + \delta d) f = \delta d f = *d*df = \nabla \cdot \nabla f = \nabla^2 f$$

- The electric field E is treated as a 1-form, while the magnetic field B is a 2-form:

$$E = E_1 dx + E_2 dy + E_3 dz$$
$$B = B_1 dy\wedge dz + B_2 dz\wedge dx + B_3 dx\wedge dy$$

  - This encodes how $E \mapsto -E$ and $B \mapsto B$ under parity. In general in $\mathbb{R}^3$, vectors are 1-forms in $\Omega^1(\mathbb{R}^3)$ and pseudovectors are 2 forms in $\Omega^2(\mathbb{R}^3)$, since $dx^i \overset{P}{\mapsto} -dx^i$ but $dx^i\wedge dy^j \overset{P}{\mapsto} dx^i\wedge dy^j$ under parity.
  - Since B is a closed 2form ($dB = \nabla\cdot\vec{B} = 0$), the Poincaré lemma for $\mathbb{R}^3$ implies it is exact:

$$B = dA \qquad A = A_i dx^i \in \Omega^1(\mathbb{R}^3) \qquad (\vec{B} = \nabla\times\vec{A})$$

  - The electric field is closed when $\partial_t B = 0$, and the Poincaré lemma applies. More generally:

$$dE + \partial_t B = 0 = d(E - \partial_t A) \xrightarrow[\text{lemma}]{\text{Poincaré}} E - \partial_t A = d\Phi, \quad \Phi \in \Omega^0(\mathbb{R}^3) \qquad (\nabla\times\vec{E} = -\partial\vec{B}/\partial t)$$

  - We combine these into a 2form $F \in \Omega^2(M)$, where $(M,g) = (\mathbb{R}^4, \text{Minkowski})$ is spacetime:

$$F = B + E\wedge dt$$

- In 4d, the $*$ acts differently, both b/c $n=4$ and b/c the index of g becomes $(3,1)$:

$$*dt = dx\wedge dy\wedge dz \qquad \dots \qquad *dz = dt\wedge dx\wedge dy$$

$$*(dt\wedge dx) = dy\wedge dz \qquad *(dt\wedge dy) = dz\wedge dx \qquad *(dt\wedge dz) = dx\wedge dy$$
$$*(dy\wedge dz) = -dt\wedge dx \qquad *(dz\wedge dx) = -dt\wedge dy \qquad *(dx\wedge dy) = -dt\wedge dz$$

  - For the 2 forms, if ω contains dt then there will be an extra $-$ sign (b/c g is a pseudometric)
  - Finally, we define the current $J \in \Omega^3(M)$:

$$J := g_{\mu\nu} j^\mu *(dx^\nu)$$
$$\rho\, dx\wedge dy\wedge dt - j_x dt\wedge dy\wedge dz - j_y dt\wedge dz\wedge dx - j_z dt\wedge dx\wedge dy$$

  - Maxwell's eqns can be succinctly combined into the assertion that F is a closed form, and $*F$ is closed except in the presence of an external field:

$$dF = 0 \qquad \left(\text{contains } \nabla\cdot B = 0,\ \nabla\times\vec{E} = -\tfrac{\partial\vec{B}}{\partial t}\right)$$
$$d*F = J$$

  - When in the presence of no background field, F and $*F$ pass to the cohomology.

- **Gauge theories**: Let $\pi : P \to M$ be a principal bundle w/ structure group G.
  - Given a covering $\{U_i\}$ of M and *local* sections $\{s_i : U_i \to P\}$, a **gauge potential** is defined by pulling ω back to the base:

    $$\mathcal{A}_i := s_i^* \omega \in \Omega^1(U_i) \otimes \mathfrak{g}$$

    Note gauge potentials are *locally defined*: if P is nontrivial, then it does not admit a global section, so the $\mathcal{A}_i$ must be *local*.