## Basic Definitions:

$\omega \in T^*M \Rightarrow \omega(\underset{\uparrow}{v}) : M \to \mathbb{R},\ p \mapsto \omega_p(v_p)$

$\Omega^1(M) := \Gamma(T^*M)$ $\qquad \mathfrak{X}(M) = \Gamma(TM) =$ vector fields on $M$.

$\Omega^k(M) := \Gamma(\Lambda^k(T^*M)) =$ sections of $k^{th}$ exterior power of $T^*M$

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ $\qquad \Omega^k(M) = \mathrm{span}\{dx_I\}$ w/ $I = (i_1, \dots, i_k)$ with $i_1 < \dots < i_k \le n = \dim(M)$ (multi-index)

$\omega = \omega_{\mu\dots\nu}\, dx^\mu \dots dx^\nu \mapsto (d\omega_{\mu\dots\nu}) \wedge dx^\mu \dots dx^\nu$

$df \in \Omega^1(M)$ for $f \in C^\infty(M) = \Omega^0(M)$, $df = (\partial_\mu f)\, dx^\mu$

- A form $\omega \in \Omega^k(M)$ is called:
  - Exact if $\omega = d\tau$ for $\tau \in \Omega^{k-1}(M)$ $\quad (\omega \in \mathrm{im}(d))$
  - Closed if $d\omega = 0$ $\quad (\omega \in \ker(d))$
- Since $d^2 = 0$, $\Omega^*(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M)$ forms a cochain complex:

$$\dots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \longrightarrow \dots$$

- The cohomology of $(\Omega^*(M), d)$ is called the de Rham cohomology:

$$H^k(M) := \ker(d : \Omega^k \to \Omega^{k+1}) / \mathrm{im}(d : \Omega^{k-1} \to \Omega^k)$$
$$= \text{closed forms} / \text{exact forms}$$

## The Wedge Product:

There is a natural map called the wedge product between the graded submodules of $\Lambda^*(V)$:

$$\wedge : \Lambda^k(V) \times \Lambda^\ell(V) \longrightarrow \Lambda^{k+\ell}(V), \quad (\omega, \tau) \mapsto \omega \wedge \tau$$

defined by passing $\omega \otimes \tau$ to the quotient, $[\omega \otimes \tau]$

- On alternating functions $f \in \Lambda^k(T_p^*M) = \Omega_p^k(M)$, $g = \Lambda^\ell(T_p^*M) = \Omega_p^k(M)$, this amounts to:

$$(f \wedge g)(v_1, \dots, v_{k+\ell}) = \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

$\omega \wedge \tau = (-1)^{\deg(\omega)\deg(\tau)}\, \tau \wedge \omega$ $\qquad \omega \wedge \tau = \omega_{\mu_I}(x)\, \tau_{\mu_J}(x)\, dx^{\mu_I} \wedge dx^{\mu_J}$

- The wedge product is an antiderivation:

$$d(\omega\wedge\tau) = (d\omega)\wedge d\tau + (-1)^{\deg(\omega)}\,\omega\wedge d\tau$$

thus drops to the deRham cohomology, $\wedge : H^k(M)\otimes H^l(M) \longrightarrow H^{k+l}(M)$

**Pullbacks:** Any map $f: M \longrightarrow N$ induces a cochain map $f^*: \Omega^*(N) \to \Omega^*(M)$. For $\omega \in \Omega^k(N)$, we define:

$$(f^*\omega)(v_1,\dots,v_k) := \omega(f_*v_1,\dots,f_*v_k)$$

$$\begin{tikzcd} v \arrow[r, mapsto] & f_*v \\ TM \arrow[r, "f_*"] \arrow[d, "f^*\omega"] & TN \arrow[d, "\omega"] \\ \mathbb{R} & \mathbb{R} \end{tikzcd}$$

- The pullback is a cochain map (commutes w/ d)

$\therefore$ it factors through $\Omega^*(M)$:

$$\begin{tikzcd} \cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r] & \Omega^k(M) \arrow[r] & \Omega^{k+1}(M) \arrow[r] & \cdots \\ \cdots \arrow[r] & \Omega^{k-1}(N) \arrow[r] \arrow[u, "f^*"] & \Omega^k(N) \arrow[r] \arrow[u, "f^*"] & \Omega^{k+1}(N) \arrow[r] \arrow[u, "f^*"] & \cdots \end{tikzcd}$$

$\Longrightarrow$ $f^*: H^k(N) \to H^k(M)$ is well defined

- Locally on a coordinate chart $(U, x^i)$ we can write any $\omega \in \Omega^k(M)$ as $\omega = \omega_{\mu_1\dots\mu_k}(x)\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_k}$ w/ $1 \le \mu_1 < \dots < \mu_k \le \underset{\dim N}{n}$, and on such an expression the pullback is:

$$f^*\omega = (\omega_{\mu_1\dots\mu_k}\circ f)\, df^{\mu_1}\wedge\dots\wedge df^{\mu_k}$$

where $df^\nu = d(x^\nu \circ f)$ is the $\nu^{th}$ coordinate function of $f$.

- The pullback is a ring homomorphism:

$$f^*(\omega\wedge\tau) = f^*(\omega)\wedge f^*(\tau)$$

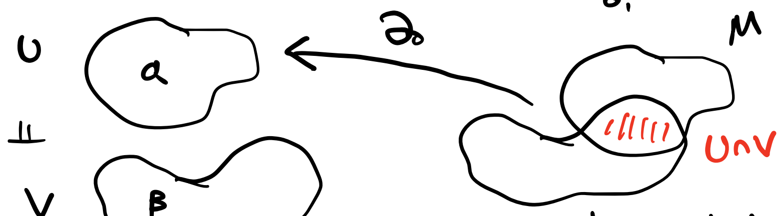
– This means on functions we can break it up, $f^*(g\omega) = (g\circ f)(f^*\omega)$

• **Mayer–Vietoris:** Suppose $M = U \cup V$ w/ $U, V$ open. Then applying $\Omega^*$ to:

$$M \leftarrow U \amalg V \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} U \cap V$$

we get the sequence:

$$\Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \overset{\partial_0^*}{\underset{\partial_1^*}{\rightrightarrows}} \Omega^*(U \cap V)$$




$\partial_0(x) = x \in U$

$$\begin{tikzcd} U \cap V \arrow[d, "\partial_0"'] \arrow[dr, dashed, "(\partial_0^*\omega)_{\mu_I}"] & \\ U \amalg V \arrow[r, "\omega_{\mu_I}"'] & \mathbb{R} \end{tikzcd}$$

$\omega = (\alpha, \beta) \in \Omega^*(U) \oplus \Omega^*(V)$ $\cong$

$$\partial_0^* : \Omega^*(U \amalg V) \longrightarrow \Omega^*(U \cap V)$$

$$\partial_0^* \omega_{\mu_I} dx^{\mu_I} = (\omega_{\mu_I} \circ \partial_0)(x)\, d(x^I \circ \partial_0)$$
$$= \alpha_{\mu_I}(x)|_{U\cap V}\, dx^I|_{U \cap V}$$

– Since $\omega = (\alpha, \beta)$ w/ $\alpha \in \Omega^*(U)$, $\beta \in \Omega^*(V)$, $\partial_0^*\omega = \alpha|_{U \cap V}$ as $\partial_0$ sends us from $U \cap V$ to $U$, and $\alpha$ is the component of $\omega$ on $U$.

The corollary of this is the Mayer Vietoris sequence.
The following sequence is exact: ← Note $\omega|_V - \omega|_U = 0$, so $g \circ f = 0$.

$$\omega \longmapsto (\omega|_U, \omega|_V)$$
$$0 \longrightarrow \Omega^*(M) \xrightarrow[f]{} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow[g]{} \Omega^*(U \cap V) \longrightarrow 0$$
$$(\omega, \tau) \longmapsto (\tau - \omega)|_{U \cap V}$$

which we get by subtracting the original two sequences.

– Exact @ middle: Clearly $(\omega|_U, \omega|_V) \mapsto (\omega - \omega)|_{U\cap V} = 0$. Conversely if $\tau|_{U \cap V} = \omega|_{U \cap V}$, then extending this to $\alpha$ w/ $\alpha|_{U \setminus V} = \omega|_{U \setminus V}$, $\alpha|_{V \setminus U} = \tau|_{V \setminus U}$, and $\alpha|_{U \cap V} = \tau|_{U \cap V} = \omega|_{U \cap V}$.

- The last joint is harder. It suffices to show this for $*=0$, i.e. $f\in\Omega^0(U\cap V)$. Take a partition of unity $\{\rho_U,\rho_V\}$ subordinate to $\{U,V\}$. Then the map $\bar{f}\in\Omega^0(U)\oplus\Omega^*(V)$ defined as:

$$\bar{f} := (\underbrace{-\rho_V f}_{\text{on } U},\ \underbrace{\rho_U f}_{\text{on } V})$$

Note $(\rho_V f)\big|_{U\setminus V}=0=(\rho_U f)\big|_{V\setminus U}$, which is good b/c $f$ is only defined on $U\cap V$.

maps to $f$, as $(\rho_U f-(-\rho_V f))\big|_{U\cap V}=(\rho_U+\rho_V)f\big|_{U\cap V}=f$ on $U\cap V$

- This sequence explicitly gives a snake map:

$(\alpha,\beta) \xmapsto{\text{lift by exactness}}$ $\qquad (d\omega, d\tau) \xmapsto{\text{commutativity}} 0$ $\qquad (\omega,\tau) \xmapsto{\text{lift by surjectivity}}$ $\qquad \xi\ (d\xi=0)$

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & \Omega^{k+1}(M) \arrow[r] \arrow[u] & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) \arrow[r] \arrow[u] & \Omega^{k+1}(U\cap V) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Omega^{k}(M) \arrow[r] \arrow[u, "d"] & \Omega^{k}(U)\oplus\Omega^{k}(V) \arrow[r] \arrow[u, "d"] & \Omega^{k}(U\cap V) \arrow[r] \arrow[u, "d"] & 0 \\
 & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd}$$

- Mayer Vietoris induces a long exact sequence in cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(M) \arrow[r] & H^q(U)\oplus H^q(V) \arrow[r] & H^q(U\cap V) \arrow[dll, "d^*"'] \\
 & H^{q+1}(M) \arrow[r] & H^{q+1}(U)\oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Where the snake map $d^*$ is:

$$d^*[\omega] = \begin{cases} [-d(\rho_V\omega)] & \text{on } U \\ [d(\rho_U\omega)] & \text{on } V \end{cases}$$

- Can use this to show:

$$H^*(S^1) = \begin{cases} \mathbb{R} & *=0,1 \\ 0 & \text{else} \end{cases}$$

- **Compact cohomology**: $\Omega_c^k(M) := \{\omega \in \Omega^k(M) : \text{supp}(\omega) \text{ compact}\}$
  - $d : \Omega^k(M) \to \Omega^{k+1}(M)$ restricts to $\Omega_c^*(M)$ since $d\omega|_{M \setminus \text{supp}(\omega)} = 0$, so $(\Omega_c^*, d)$ is a cochain complex. Its cohomology is the compact cohomology:

$$H_c^*(M) := \ker(d : \Omega_c^*(M) \to \Omega_c^{*+1}(M)) / \text{im}(d : \Omega_c^{*-1}(M) \to \Omega_c^*(M))$$

  - $\Omega_c^*$ is not a contravariant functor per say, b/c compactly supported forms can pull back to non-compactly supported forms, ex any $\omega$ under $\mathbb{R} \xrightarrow{f} S^1$.
  - Instead we take:
    i) $\Omega_c^*(\cdot)$ contravariant under proper (inverse image of every compact set is compact)
    ii) $\Omega_c^*(\cdot)$ is covariant under inclusions of open sets, i.e. if $U \subset V$ w/ $U$ open then $\Omega_c^*(U) \subset \Omega_c^*(V)$ $j_*$ = extension of a form by zero
      - For $j : U \hookrightarrow V$ the inclusion, $j_* : \Omega_c^*(U) \hookrightarrow \Omega_c^*(V)$ is defined by extending $\omega$ by $0$, i.e. $(j_*\omega)|_U = \omega$ and $(j_*\omega)|_{V \setminus U} = 0$.
  - Mayer Vietoris for $\Omega_c^*$ for open cover $\{U, V\}$ swaps directions:

$$\begin{array}{ccccccccc}
 & & & & (-j_*^U\omega, j_*^V\omega) & \longleftarrow & \omega & & \\
0 \leftarrow & \Omega_c^*(M) & \leftarrow & & \Omega_c^*(U) \oplus \Omega_c^*(V) & \leftarrow & \Omega_c^*(U \cap V) & \leftarrow 0 \\
 & \omega + \tau & \longleftarrow & & (\omega, \tau) & & & 
\end{array}$$

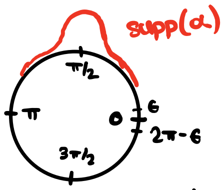
which induces the long exact sequence in compact cohomology:

$$\cdots \leftarrow H_c^{k+1}(M) \leftarrow H_c^{k+1}(U) \oplus H_c^{k+1}(V) \leftarrow H_c^{k+1}(U \cap V) \xleftarrow{d_*}$$

$$H_c^k(M) \leftarrow H_c^k(U) \oplus H_c^k(V) \leftarrow H_c^k(U \cap V) \leftarrow \cdots$$

$$d_*[\omega] = [d(\rho_V \omega)] = [-d(\rho_U \omega)]$$

- **Cohomology of $S^1$:** Every 1-form $\omega \in \Omega^1(S^1)$ is closed, since $\dim(S^1)=1$ and a form on a manifold must have degree $\leq n = \dim(M)$. Any non-exact 1-form $\omega \notin \mathrm{im}(d)$ will generate the cohomology $H^1(S^1)$. Consider:

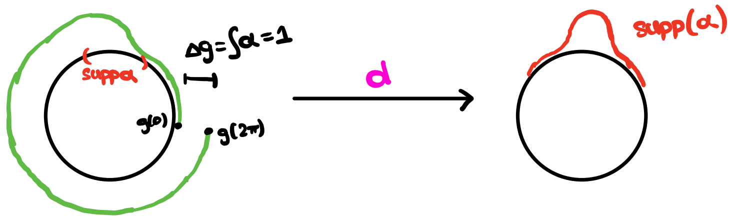



$\alpha \in \Omega^1(S^1)$, $\alpha = f(x)dx$ where $f(x)$ is a bump function.

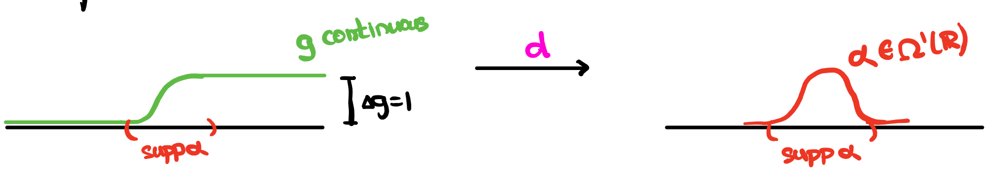
We claim $\alpha$ generates $H^1(S^1)$. To show this, suppose $\alpha$ is exact, so $\alpha = dg$ w/ $g \in C^\infty(S^1)$. Then:

$$g(2\pi-\epsilon) - g(\epsilon) = \int_\epsilon^{2\pi-\epsilon} \alpha = \int_{S^1} \alpha = 1 \qquad \text{as } \mathrm{supp}\,\alpha \cap (-\epsilon,\epsilon) = 0$$

So the function $g$ cannot be continuous: it looks like:



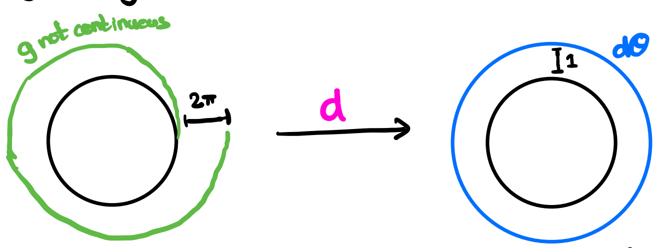

- On $\mathbb{R}$, this argument fails b/c $\mathbb{R}$ is not compact and functions do not need to be periodic:




$\alpha = dg$

However when we consider $H^1_c(S^1)$, forms w/ compact support, then $g$ cannot be used b/c $\mathrm{supp}(g)$ is <u>not</u> compact. Hence $\alpha$ will generate $H^1_c(\mathbb{R})$.

- Another generator of $H^1(S^1)$ is the constant 1-form $d\theta \in \Omega^1(S^1)$, by the same integral argument



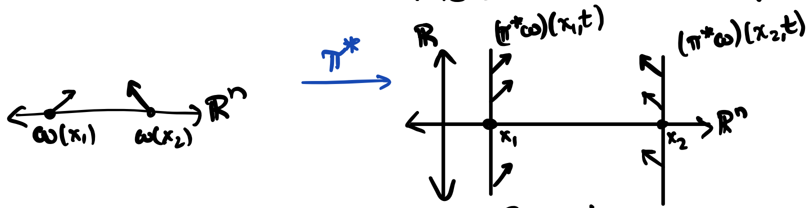

Note $d\theta$ is <u>locally</u> exact, since for $U \subseteq S^1$ open + connected $\theta|_U : U \to \mathbb{R}$ is a smooth function in $\Omega^0(U) = C^\infty(U)$. However, $d\theta$ is not globally exact since $\theta : S^1 \to \mathbb{R}$ is not smooth globally on $S^1$. Thus $[d\theta] \neq [0]$, although the notation $d\theta$ makes it <u>look</u> like $d\theta$ is exact

– Since $\alpha$ and $d\theta$ both generate $H^1(S^1)$, $\mathrm{span}\{[d\theta]\} = \mathrm{span}\{[\alpha]\}$, hence some multiple $k\,d\theta$ must be cohomologous to $\alpha$, i.e. we can write $k\,d\theta = d\tau - \alpha$, as follows:

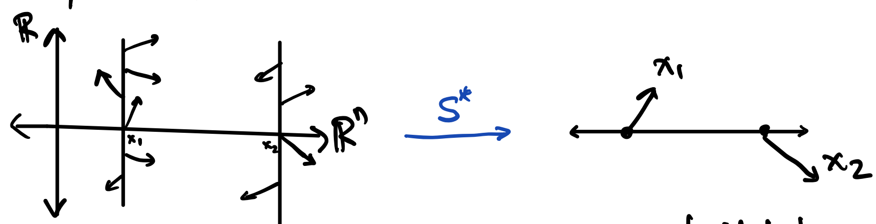

- We can see by requiring $\tau \in C^\infty(S^1)$ that $0 = \int_{S^1} d\tau = \int k\,d\theta - \int \alpha \Rightarrow k = \frac{1}{2\pi}$

- $\pi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $(x,t) \mapsto x \in \mathbb{R}^n$ | $\pi^*: \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^n \times \mathbb{R})$, $\omega_{\mu_I}(x)dx^{\mu_I} \mapsto (\omega_{\mu_I} \circ \pi)dx^{\mu_I}$
  $s: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$, $x \mapsto (x,0)$ | $s^*: \Omega^*(\mathbb{R}^n \times \mathbb{R}) \to \Omega^*(\mathbb{R}^n)$, $\omega_{\mu_I} dx^{\mu_I} \mapsto (\omega_{\mu_I} \circ s)dx^{\mu_I}$

– $\pi^*\omega$ is the $k$-form on $\mathbb{R}^{n+1}$ which is constant on each fiber




– $s^*$ picks out the zero section of each form.




- Note $\pi \circ s = id \Rightarrow s^* \circ \pi^* = id$. $\pi^* \circ s^*$ is not id, but we will show it induces id on $H^*$.
- Show $\pi^* \circ s^* = id$ by showing it is chain homotopic to id:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{n-1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[d, shift right, "\pi^* \circ s^*"'] \arrow[d, shift left, "id"] & \Omega^n(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[dl, "K^n"'] \arrow[d, shift right, "\pi^* \circ s^*"'] \arrow[d, shift left, "id"] & \Omega^{n+1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r] \arrow[dl, "K^{n+1}"'] \arrow[d, shift right, "\pi^* \circ s^*"'] \arrow[d, shift left, "id"] & \cdots \\
\cdots \arrow[r] & \Omega^{n-1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] & \Omega^n(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] & \Omega^{n+1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

So we want:

$$\pi^* \circ s^* - id = \pm(Kd \pm dK)$$

We will show $K$ is indefinite integration along the fiber.

- Note on this bundle-like structure $\mathbb{R}^n \times \mathbb{R}$, there are 2 types of forms:

$$\begin{array}{ll}
i) & (\pi^*\varphi) f(x,t) \\
ii) & (\pi^*\varphi) f(x,t) \wedge dt
\end{array}
\quad \overset{K}{\longmapsto} \quad
\begin{cases} 0 \\ (\pi^*\varphi)\int_0^t f(x,t')dt' \end{cases}$$

- Prp: $\pi^*$ and $s^*$ induce isomorphisms in cohomology:

$$H^*(\mathbb{R}^n \times \mathbb{R}) \underset{s^*}{\overset{\pi^*}{\rightleftarrows}} H^*(\mathbb{R}^n)$$

- Poincaré Lemma:

$$H^*(\mathbb{R}^n) = H^*(pt) = \begin{cases} \mathbb{R} & *=0 \\ 0 & \text{else} \end{cases}$$

- The same proof carries through to show for an arbitrary manifold $M$:

$$H^*(M \times \mathbb{R}) \cong H^*(M)$$

- Corollary (Homotopy invariance of $H^*$): If $f$ and $g$ are homotopic, they induce equal maps on de Rham cohomology.

$$f \simeq g : M \to N \Rightarrow g^* = f^* : H^*(N) \longrightarrow H^*(M)$$

– A homotopy is $h : M \times \mathbb{R} \to N$ w/ $h(x,0) = f(x)$, $h(x,1) = g(x)$.

- Corollary: If $M$ and $N$ are homotopy equivalent, then

$$H^*(M) \cong H^*(N).$$

- If $A \subset M$ and $M$ deformation retracts onto $A$, then $\Omega^*(M) = \Omega^*(A)$.
- Use this for $H^*(S^n)$:

$$H^*(S^n) = \begin{cases} \mathbb{R} & *=0,n \\ 0 & \text{else} \end{cases}$$

- Volume form on the sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$, i.e. $d^{n+1}x = dr \wedge \Omega$:

$$\Omega = \frac{1}{r} \sum_{i=1}^{n+1} (-1)^{i+1} x_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_{n+1}$$

- Compact Poincaré lemma: $\pi: M\times\mathbb{R}\to M$ the projection, the pullback $\pi^*: \Omega^*(M)\to\Omega^*(M\times\mathbb{R})$ does not restrict to compactly supported cohomology. Instead use a push-forward:

$$\pi_*: \Omega^*_c(M\times\mathbb{R})\longrightarrow\Omega^*_c(M)$$

called integration along the fiber defined on type i/ii forms:

$$\begin{aligned} &\text{i)}\ (\pi^*\varphi)\, f(x,t) \\ &\text{ii)}\ (\pi^*\varphi)\, f(x,t)\,dt \end{aligned} \quad \overset{\pi_*}{\longmapsto} \quad \begin{cases} 0 \\ \varphi\int_{\mathbb{R}} dt\, f(x,t) \end{cases}$$

here $\varphi\in\Omega^*(M)$ doesn't need compact support, but $f$ does.

- $\pi_*$ is a chain map:

$$\begin{tikzcd} \cdots \arrow[r] & \Omega^k_c(M\times\mathbb{R}) \arrow[r, "d"] \arrow[d, "\pi_*"] & \Omega^{k+1}_c(M\times\mathbb{R}) \arrow[r] \arrow[d, "\pi_*"] & \cdots \\ \cdots \arrow[r] & \Omega^{k-1}_c(M) \arrow[r, "d"] & \Omega^k_c(M) \arrow[r] & \cdots \end{tikzcd}$$

- $\pi_*: \Omega^*_c(M\times\mathbb{R})\longrightarrow\Omega^{*-1}_c(M)$ has an inverse. Let $e=e(t)dt$ be a compactly supported 1-form on $\mathbb{R}$ w/ $\int_{\mathbb{R}} e=1$. Define:

$$e_*: \Omega^{*-1}_c(M)\to\Omega^*_c(M\times\mathbb{R})$$
$$e_*(\varphi):=(\pi^*\varphi)\wedge e$$

- $e_*$ is a chain map $\therefore$ drops to $H^*_c$. $\pi_*\circ e_*: H^*_c(M)\to H^*_c(M)$ clearly equals id since $\int e=1$, so then WTS $e_*\circ\pi_*$ is chain homotopic to id.

- **Prp**: The maps $\pi_*$ and $e_*$ are inverse isomorphisms on $H_c^*$:

$$H_c^*(M\times\mathbb{R}) \underset{e_*}{\overset{\pi_*}{\rightleftarrows}} H_c^*(M)$$

- **Compact Poincaré Lemma**:

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & * = n \\ 0 & \text{else} \end{cases}$$

$$H_c^*(M\times\mathbb{R}^n) \cong H_c^{*-n}(M)$$

  - The generator of $H_c^n(\mathbb{R}^n)$ is a bump form $\omega = a(x)\,dx_1\wedge\dots\wedge dx_n$ w/ $\int\omega = 1$, i.e. $e_*^n(1)$.

- **Good covers**: A **good cover** of $M$ is an open cover $\{U_\alpha\}$ such that each finite nonempty intersection:

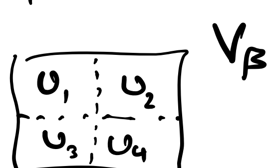
$$\bigcap_{k=1}^{n} U_{\alpha_k}$$

is diffeomorphic to $\mathbb{R}^n$. $M$ is called **finite type** if $M$ admits a finite good cover

- Prp: Every manifold admits a good cover. Every compact manifold is of finite type.

- We'll prove most of the following results for finite type manifolds.

- An open cover $\{U_\alpha\}$ is a **refinement** of $\{V_\beta\}$ if each $U_\alpha$ is contained in some $V_\beta$.
  - The set of open covers forms a directed set under $\prec$ = refinement




chop up $V_\beta$

- **Prp**: Any manifold of finite type has a finite dimensional cohomology $H^q(M)$ and compact cohomology $H^q_c(M)$.
- Proofs in this section are all very similar: Induct on the cardinality of a good cover. Assume the result holds for any manifold which has a good cover $\{U_1, \dots, U_p\}$, then prove for $M$ w/ good cover $\{U_0, U_1, \dots, U_p\}$
- Wedge products and integration factor through $H^*(M)$. This gives a pairing $(n = \dim M)$:

$$\int : H^q(M) \times H^{n-q}_c(M) \longrightarrow \mathbb{R} \qquad [\omega], [\tau] \longmapsto \int_M \omega \wedge \tau$$

- Poincaré duality: This pairing is nondegenerate if $M$ is orientable, hence defines an isomorphism:

$$H^q(M) \xrightarrow{\sim} \left[H^{n-q}_c(M)\right]^* \qquad \omega \mapsto \left(\tau \mapsto \int_M \omega \wedge \tau\right)$$

- Corollary: If $M$ is connected and orientable w/ $n = \dim(M)$, then:

$$H^n_c(M) \cong \mathbb{R}$$

If $M$ is also compact, then $H^n(M) \cong H^n_c(M) \cong \mathbb{R}$.
- The other way is <u>not</u> always true, i.e. $H^q_c(M) \not\cong H^{n-q}(M)^*$ in some situations.

## ***Fiber bundles***

- **Fiber bundle** with fiber $F$ and structure group $G$: A triple $(\pi, E, B)$ w/ $\pi: E \to B$ surjective and $G$ acting on $E$ from the left. **Local trivializations** $\{(U_\alpha, \varphi_\alpha)\}$ s.t. $\{U_\alpha\}$ is an open cover of $B$ and:

$$\varphi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times F$$

is a homeomorphism and the **transition functions**:

$$g_{\alpha\beta}(x) := \varphi_\alpha \varphi_\beta^{-1}\Big|_{\{x\}\times F} \longrightarrow G$$

are $G$-valued maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$.

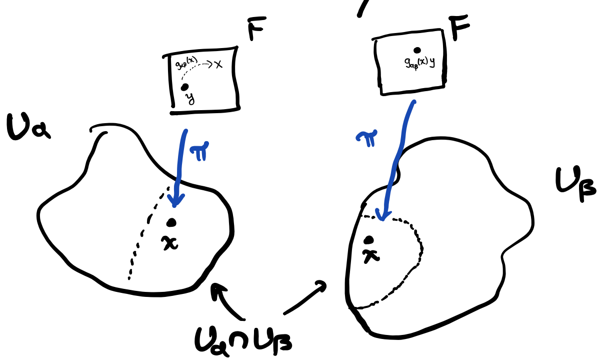
- So the transition fns are $G$-valued and satisfy the **cocycle condition**:

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$$

- Given an open cover $\{U_\alpha\}$ of $B$ w/ cocycles $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ acting on $F$ we can construct a fiber bundle $\pi : E \to B$ w/ transition maps $g_{\alpha\beta}$:

$$E := \coprod_\alpha (U_\alpha \times F)\Big/\sim \qquad (x,y) \sim (x, g_{\alpha\beta}(x)\, y)$$

for $x \in U_\alpha \cap U_\beta$, $y \in F$.



- Kunneth formula: The cohomology of a direct product $M\times F$ is a tensor product of cochain complexes:

$$H^*(M\times F)\cong H^*(M)\otimes H^*(F)$$

where $(C^*\otimes D^*)^n=\bigoplus_{p+q=n} C^p\otimes D^q$ and the codifferential is:

$$d:(C^*\otimes D^*)^n=\bigoplus_{p+q=n} C^p\otimes D^q\longrightarrow (C^*\otimes D^*)^{n+1}=\bigoplus_{p'+q'=n+1} C^{p'}\otimes D^{q'}$$

$$d(\omega\otimes\tau)=d\omega\otimes\tau+(-1)^{\deg(\omega)}\omega\otimes d\tau$$

- The isomorphism is explicitly given by:

$$\begin{array}{ccc} M\times F & \xrightarrow{\rho} & F \\ \downarrow{\scriptstyle\pi} & & \\ M & & \end{array}$$

$$\psi: H^*(M)\otimes H^*(F)\longrightarrow H^*(M\times F)$$

$$[\omega]\otimes[\varphi]\longmapsto[(\pi^*\omega)\wedge(\rho^*\varphi)]$$

- Leray-Hirsch: $\pi:E\to M$ a fiber bundle w/ fiber $F$, $M$ a finite good cover. Suppose there are global cohomology classes $\{e_1,\dots,e_n\}$ in $H^*(E)$ which generate $H^*(F)$ when restricted to each fiber. Then:

$$H^*(E)\cong H^*(M)\otimes\mathbb{R}\{e_1,\dots,e_n\}\cong H^*(M)\otimes H^*(F)$$

free $\mathbb{R}$ module generated by this basis

- The same formulas hold for compact cohomology.

- Poincaré dual: $M$ oriented dim $n$, $S \hookrightarrow M$ closed oriented submanifold of dim $k$, $\iota: S \hookrightarrow M$ the inclusion. The Poincaré dual of $S$ in $M$ is a cohomology class $[\eta_S] \in H^{n-k}(M)$ satisfying:

$$\int_S \iota^*\omega = \int_M \omega \wedge \eta_S \quad \text{for} \quad \omega \in H^k_c(M)$$

To construct $[\eta_S]$, note we have a functional $\int_S : H^k_c(M) \to \mathbb{R}$, $[\omega] \mapsto \int_S \iota^*\omega$ well defined on $H^k$ b/c $S$ is closed, $\int_S \in (H^k_c(M))^* \cong H^{n-k}(M)$, so we get a class $[\eta_S] \in H^{n-k}(M)$ such that the functional $\int_S(\omega) = \int_S \iota^*\omega$ equals $\omega \mapsto \int \omega \wedge \eta_S$.

- Compact Poincaré dual: If $S \subset M$ is compact, then there is a cohomology class $[\eta'_S] \in H^{n-k}_c(M)$ such that

$$\int_S \iota^*\omega = \int_M \omega \wedge \eta'_S \quad \text{for} \quad \omega \in H^k(M)$$

$\omega$ doesn't need compact support.

- As a form $\eta'_S$ will equal $\eta_S$, but as a form it can differ since $H^* \neq H^*_c$

• A vector bundle of rank $n$ is a fiber bundle $\pi: E \to M$ with fiber $F = \mathbb{R}^n$. Naturally, as: $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

are vector space automorphisms on each fiber and induce:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(\mathbb{R}^n) \qquad g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1}\Big|_{\{x\}\times\mathbb{R}^n}$$

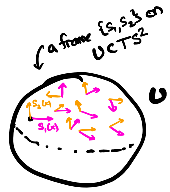
hence $\pi$ has structure group $G = \mathrm{Aut}(\mathbb{R}^n) = GL(\mathbb{R}, n)$

↑ invertible $n \times n$ matrices under $\times$, $\dim = n^2$

• $\Gamma(U, E) :=$ space of sections on $U \subset M$.

$$= \{ s : U \to E : \pi \circ s = id_U \}$$

• A set of $n$ sections $\{s_1, \dots, s_n\} \subset \Gamma(U, E)$ is a frame on $U$ if $\forall x \in U$, $\{s_1(x), \dots, s_n(x)\}$ is a basis for $\pi^{-1}(\{x\})$




• Two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are equivalent ($g_{\alpha\beta} \sim g'_{\alpha\beta}$) if there are maps $\lambda_\alpha : U_\alpha \to GL(\mathbb{R}^n)$ and $\lambda_\beta : U_\beta \to GL(\mathbb{R}^n)$ s.t.:

$$g_{\alpha\beta}(x) = \lambda_\alpha(x)\, g'_{\alpha\beta}(x)\, \lambda_\beta(x)^{-1} \qquad \forall x \in U_\alpha \cap U_\beta$$

- If $\{\varphi_\alpha\}$ and $\{\varphi'_\alpha\}$ are two local trivializations w/ cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, then $g_{\alpha\beta} \sim g'_{\alpha\beta}$

• A bundle map $f : E \to E'$ is a morphism of vector bundles, and it is a fiber-preserving smooth map which is linear on each fiber.

• If the cocycle $\{g_{\alpha\beta}\}$ for a vector bundle $\pi: E \to M$ is equivalent to a cocycle $\{h_{\alpha\beta}\}$ w/ values in $H \subset GL(\mathbb{R}^n)$, we say the structure group of $G$ may be reduced to $H$.

- $\pi: E \to M$ is orientable if its structure group can be reduced to $GL(\mathbb{R}^n)^+$

subgroup of $GL(\mathbb{R}^n)$ w/ positive det

• A trivialization $\{(\varphi_\alpha, U_\alpha)\}$ is oriented if the cocycle $g_{\alpha\beta}(x) \in GL(\mathbb{R}^n)^+$.

- **Orientations**: We say two trivializations $\{(\varphi_\alpha, U_\alpha)\}$ and $\{(\psi_\beta, V_\beta)\}$ are equivalent if $\forall x \in U_\alpha \cap V_\beta$, $\varphi_\alpha(x) \circ \psi_\beta(x)^{-1} \in GL(\mathbb{R}^n)$ has positive determinant. This is an equiv. relation and partitions the oriented trivializations into two equivalence classes. Each class is called an orientation on E.


- The structure group of any vector bundle may be reduced to $O(n)$ by endowing it w/ a Riemannian metric. For E orientable, it may be reduced to $SO(n)$.
- **Bundle operations**: Taking fiberwise operations for two bundles $\pi: E \to M$ and $\pi': E' \to M$ gives:

**Sum**: $(E \oplus E')_x = E_x \oplus E'_x$ ← fiber @ $x \in M$

$$g_{\alpha\beta}^{E\oplus E'}(x) = \begin{pmatrix} g_{\alpha\beta}^{E}(x) & 0 \\ 0 & g_{\alpha\beta}^{E'}(x) \end{pmatrix}$$

**Product**: $(E \otimes E')_x = E_x \otimes E'_x$

$$g_{\alpha\beta}^{E\otimes E'}(x) = g_{\alpha\beta}^{E}(x) \otimes g_{\alpha\beta}^{E'}(x)$$

**Dual**: $(E^*)_x = (E_x)^*$

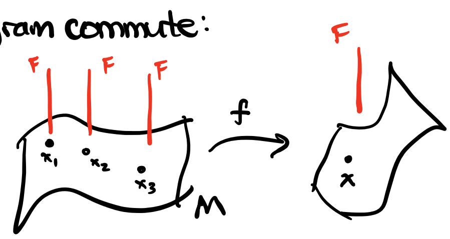
$$g_{\alpha\beta}^{E^*}(x) = \left[g_{\alpha\beta}^{E}(x)^T\right]^{-1}$$

- **Pullback bundle**: $\pi: E \to N$, $f: M \to N$ $C^\infty$. The pullback bundle $f^{-1}E$ is a bundle over M which is the maximal subset of $M \times E$ making the pullback diagram commute:

$$\begin{array}{ccc} f^{-1}E & \longrightarrow & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

$$f^{-1}E = \{(m,e) \in M \times E : f(m) = \pi(e)\}$$

$$(f^{-1}E)_{\underset{M}{x \in}} = E_{f(x)}$$




$f^{-1}(\{x\}) = \{x_1, x_2, x_3\}$

- **Theorem**: Let $Y$ be compact/paracompact. Homotopic maps induce isomorphic pullback bundles, i.e. if $f_0, f_1 : Y \to X$ are homotopic and $\pi : E \to X$ is a vector bundle, then:

$$f_0^{-1}E \cong f_1^{-1}E$$

- **Corollary**: Any vector bundle over a contractible manifold is trivial.
- **Notation**: $\mathrm{Vect}_k$ is a functor from DiffMan to PtSet, which associates w/ each $M$ a pointed set w/ basepoint the trivial $k$-bundle over $M$. The other elements of $\mathrm{Vect}_k(M)$ are the nontrivial $k$-bundles over $M$.
  - The corollary implies for $M$ contractible, $\mathrm{Vect}_k(M) \cong \{*\}$.
- **Theorem**: Let $E \to M$ be a vector bundle. Then:

$$H^*(E) \cong H^*(M)$$

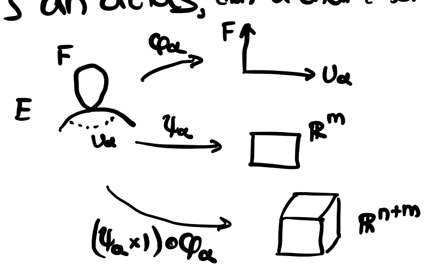
Furthermore if $E$ is orientable and $M$ is of finite type, then:

$$H_c^*(E) \cong H_c^{*-n}(M)$$

where $n$ is the rank of the bundle.

- **Local product orientation**: For $\pi : E \to M$ an oriented rank $n$ vector bundle, $\{(U_\alpha, \varphi_\alpha)\}$ a local triv., $\{(U_\alpha, \psi_\alpha)\}$ an atlas, then a chart for $E$

$$\{(\pi^{-1}(U_\alpha), (\psi_\alpha \times 1) \circ \varphi_\alpha)\}$$



**Vertical Cohomology:** $\pi: E \to M$ a rank $n$ vector bundle

- A $k$-form $\omega \in \Omega^k(E)$ has **compact support in the vertical direction** if for each compact set $K \subset M$, $\pi^{-1}(K) \cap \mathrm{supp}(\omega)$ is compact. We denote these forms on $E$ by:

$$\Omega^*_{cv}(E)$$

and this is a cochain complex w/ cohomology $H^*_{cv}(E)$

- A form on $E$ is a lin. combo of two types of forms:

| | |
|---|---|
| i) $(\pi^*\varphi) f(x, t_1, \dots, t_n)\, dt_{i_1} \wedge \dots \wedge dt_{i_r}$ | $1 \le i_1 < \dots < i_r = n$, $\varphi \in \Omega^{k-r}(M)$ |
| ii) $(\pi^*\varphi) f(x, t_1, \dots, t_n)\, dt_1 \wedge \dots \wedge dt_n$ | $\varphi \in \Omega^{k-n}(M)$ |

where $\varphi$ is a form on the base.

- **Integration along the fiber**: A map $\pi_*: \Omega^*_{cv}(E) \to \Omega^{*-n}_{cv}(M)$

$$\pi_*: \begin{array}{l} (\pi^*\varphi) f\, dt_{i_1} \wedge \dots \wedge dt_{i_r} \longmapsto 0 \\ (\pi^*\varphi) f\, dt_1 \wedge \dots \wedge dt_n \longmapsto \varphi \int_{\mathbb{R}^n} f(x, t_1, \dots, t_n)\, dt_1 \wedge \dots \wedge dt_n \end{array}$$

where the coordinates are taken on a local trivialization, and this is independent of coordinate chart

- <u>Prop</u>: Integration along the fiber $\pi_*$ is a cochain map.

$$\begin{array}{ccccccccc} \dots \longrightarrow & \Omega^{k-1}_{cv}(E) & \xrightarrow{d} & \Omega^{k}_{cv}(E) & \xrightarrow{d} & \Omega^{k+1}_{cv}(E) & \longrightarrow \dots \\ & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* & \\ \dots \longrightarrow & \Omega^{k-n-1}(M) & \xrightarrow{d} & \Omega^{k-n}(M) & \xrightarrow{d} & \Omega^{k-n+1}(M) & \longrightarrow \dots \end{array}$$

- For proofs, typically assume $E$ is a product bundle $M\times\mathbb{R}^n$ so we can define the form $\omega$ globally.
  - Prove this for $\rho_\alpha\omega = \omega_\alpha$, where $\{\rho_\alpha\}$ is a partition of unity subordinate to a trivialization $\{(U_\alpha,\varphi_\alpha)\}$. Then $\rho_\alpha\omega$ is $\cong$ a form on $M\times\mathbb{R}^n$
- **Projection formulas**: $\pi: E\to M$ oriented rank $n$ vector bundle, $\tau\in\Omega^*(M)$, $\omega\in\Omega^*_{cv}(E)$.

  i) $\pi_*((\pi^*\tau)\wedge\omega) = \tau\wedge(\pi_*\omega)$ ↙ $\omega\wedge\tau$ a top form on $E$

  ii) If $\dim(M)=m$ and $\omega\in\Omega^q_{cv}(E)$, $\tau\in\Omega^{m+n-q}(M)$, then:

$$\int_E (\pi^*\tau)\wedge\omega = \int_M \tau\wedge(\pi_*\omega)$$

($\pi_*$ is an adjoint to $\pi^*$ under the pairing $\langle\cdot,\cdot\rangle = \int\cdot\wedge\cdot$)

- **Theorem** (Thom isomorphism): $\pi: E\to M$ orientable rank $n$ vector bundle:

$$\pi_*: H^*_{cv}(E) \xrightarrow{\sim} H^{*-n}(M)$$

  defines an isomorphism.
  - For $M\times\mathbb{R}^n$, this gives $H^*_{cv}(M\times\mathbb{R}^n)\cong H^{*-n}(M)$.
  - We will denote the inverse of $\pi_*$ w/ $\tau$:

$$H^*_{cv}(E) \underset{\pi_*}{\overset{\tau}{\rightleftarrows}} H^{*-n}(M)$$

- The function $1\in C^\infty(M) = H^0(M)$ has image under $\tau$ defined to be the Thom class $\Phi\in H^n_{cv}(E)$:

$$\Phi := \tau(1) \qquad \Rightarrow \pi_*\Phi = 1$$

  ($\Phi$ is like a bump $n$ form on each fiber)

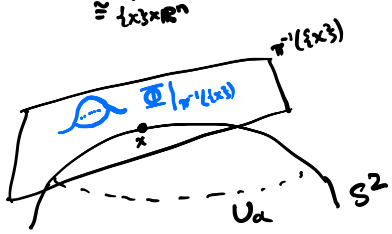
  - Using $\pi_*((\pi^*\omega)\wedge\Phi) = \omega\wedge\pi_*\Phi = \omega\wedge 1 = \omega$, we see:

$$\tau(\omega) = \pi^*(\omega)\wedge\Phi, \quad \forall\omega\in\Omega^*(M)$$

  - This means that when restricted to each fiber, $\underbrace{\Phi|_{\pi^{-1}(\{x\})}}_{\cong\{x\}\times\mathbb{R}^n}$ generates $H^n_c(\mathbb{R}^n)$

$$\Phi = a(x,t)\,dt_1\wedge\ldots\wedge dt_n$$

$$\int_{\mathbb{R}^n} a(x,t)\,d^nt = \int_{\mathbb{R}^n}\Phi = 1$$



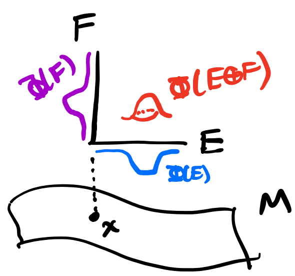

← Thom class of $TS^2$.

- The Thom isomorphism lifts a form from $M$ to being a form on $E$ by pulling it up to $E$ w/ $\pi^*$ and wedging w/ $\Phi$
- Under direct sums w/ we see:

$$\begin{tikzcd} & E\oplus F \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E & & F \end{tikzcd}$$

$$\Phi(E\oplus F) = \pi_1^*(\Phi(E)) \wedge \pi_2^*(\Phi(F))$$

  - Ex: For two $\mathbb{R}^1$ bundles, $\Phi(E)$ and $\Phi(F)$ are each bump 1-forms, and $\pi_1^*(\cdot)\wedge\pi_2^*(\cdot)$ is a bump two form




- Normal bundles: For $S \subset M$, $k=\dim(S)$ and $n=\dim(M)$, the normal bundle $N_{S/M}$ is defined by making:

$$0 \longrightarrow TS \longrightarrow TM|_S \longrightarrow N_{S/M} \longrightarrow 0$$

exact (i.e. $N_{S/M}$ is the quotient @ each point $T_pM/T_pS$). A tubular neighborhood $T$ of $S$ in $M$ is an open neighborhood of $S$ in $M$ which is diffeomorphic to a rank $n-k$ vector bundle over $S$ s.t. $S$ embeds as the zero section.

  - Every submanifold $S \subset M$ has a tubular neighborhood $T$, and in fact $T$ is diffeomorphic to $N_{S/M}$.

- For $S \subset M$ a closed submanifold and $T$ a tubular n'borhood of $S$, we get:

$$H^*(S) \xrightarrow{\tau} H_{cv}^{*+n-k}(T) \xrightarrow{j_*} H^{*+n-k}(M)$$

where $\tau(\omega) = (\pi^*\omega)\wedge\Phi$ is the Thom map and $j_*$ is extension by $0$. The Poincaré dual of $S$ in $M$ is the image of $1 \in H^0(S)$:

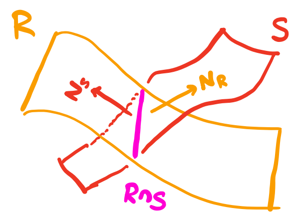
$$[\eta_S] = j_*(\Phi) \in H^{n-k}(M)$$

where recall this class satisfies:

$$\int_S \imath^*\omega = \int_M \omega\wedge\eta_S \qquad \forall \omega \in H_c^k(M).$$

- Prp: The Poincaré dual of a closed oriented submanifold $S \subset M$ and the Thom class of the normal bundle $N_{S/M}$ can be represented by the same forms.
  - Intuitively this is saying that $\int_M \omega \wedge \Phi$ is equal to $\int_S \omega$ and wedging w/ $\Phi$ lets you "factor" the integral w/ $1 = \int_F \Phi = \pi_* \Phi$
- Prp: The support of the Poincaré dual of $S$ may be shrunk into any open neighborhood of $S$.
- Two submanifolds $R, S \subset M$ intersect transversally if:

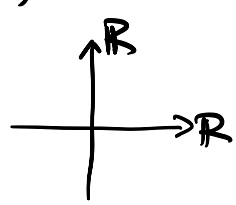
$$T_x R + T_x S = T_x M \qquad \forall x \in R \cap S$$




In this case the normal bundle splits as a sum $N_{R\cap S} = N_R \oplus N_S$, so the Thom class on $R \cap S$ factors as $\Phi(N_{R\cap S}) = \pi_1^* \Phi(N_R) \wedge \pi_2^* \Phi(N_S)$. Hence the Poincaré dual satisfies:

$$\eta_{R\cap S} = \eta_R \wedge \eta_S$$

  - We can evidently see this for the 2 dim case, as $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ w/ each copy of $\mathbb{R}$ intersecting transversally.



- Orientation of forms: The top forms on $M$, $\Omega^n(M)$ split into two classes which are related by multiplication by a positive function $f: M \to \mathbb{R}^+$. A top form in the orientation class of $M$ is called positive.

| Negative orientation | Positive orientation |
|---|---|
| $-dx_1 \wedge \ldots \wedge dx_n$ | $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$ |
| $-f(x) dx_1 \wedge \ldots \wedge dx_n$ | $\underset{\uparrow \atop >0}{f(x)}\, dx_1 \wedge \ldots \wedge dx_n$ |

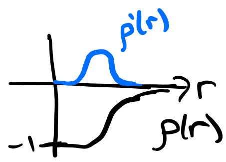
  - Orientation on $S^{n-1}$: A form $\sigma \in H^{n-1}(S^{n-1})$ is positive (and defines the orientation on $S^{n-1}$) if $dr \wedge \pi^*\sigma \in H^n(\mathbb{R}^n \setminus \{0\})$ is positive w.r.t. the standard orientation $dx_1 \wedge \ldots \wedge dx_n$ on $\mathbb{R}^n \setminus \{0\} \subset \mathbb{R}^n$.

- Angular form: On $\mathbb{R}^n \setminus \{0\}$, the angular form is:

$$\psi = \pi^*\sigma \in H^{n-1}(\mathbb{R}^n \setminus \{0\})$$

where $\sigma \in H^{n-1}(S^{n-1})$ is a positive generator of $H^{n-1}(S^{n-1})$

  - Note this is essentially of the form $\psi(\theta^i) d\theta^i$, ← sphere coords
    and $d\psi = 0$ since $d\sigma = 0$.
  - Let $\rho(r)$ be a fn on $\mathbb{R}^n$ s.t. $\mathrm{supp}(\rho'(r)) \not\ni 0$ and $\int \rho'(r) dr = \int d\rho = 1$. (so $d\rho = \rho'(r) dr$ is a bump 1-form). Then:




$$d\rho \wedge \psi = d(\rho \wedge \psi) \in H^n_c(\mathbb{R}^n)$$

generates the compact cohomology of $\mathbb{R}^n$.

- Global angular form: Given a rank $n$ vector bundle, endow it w/ a metric to give a radial fn $r$ on each fiber. Let $E^0$ be the complement of the zero section of $\pi: E \to M$ ($E^0 \cong E \setminus M$). The global angular form $\psi$ on $E^0$ is a form which restricts to the angular form $\pi^*\sigma$

on each fiber of $E^\circ \cong \mathbb{R}^n \setminus \{0\}$.

- Rank 2 case: Trivialize $E$ w/ an open cover $\{U_\alpha\}$ of $M$. On each $E^\circ|_{U_\alpha}$, we can put polar coords s.t. coords on $E^\circ|_{U_\alpha}$ are $(\pi^*x_1, \dots, \pi^*x_n, r_\alpha, \theta_\alpha)$ where $(x_1, \dots, x_n)$ are coordinates on $U_\alpha$.
  - On each $U_\alpha \cap U_\beta$, $r_\alpha$ will be the same as $r_\beta$ (as the two charts inherit $r$ from $\mathbb{R}^n$) but $\theta_\alpha$ and $\theta_\beta$ will be different (how $\mathbb{R}^2$ is stitched onto $M$). So we define $\varphi_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{R}$ as the difference:

$$\pi^*\varphi_{\alpha\beta} = \theta_\beta - \theta_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

$\rho_\gamma \varphi_{\gamma\alpha}$ dies @ boundary of $U_\gamma$.

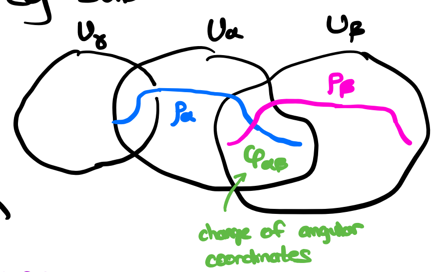
We define one forms $\xi_\alpha$ on each $U_\alpha$ as:

$$\xi_\alpha := \frac{1}{2\pi} \sum_\gamma \rho_\gamma \, d\varphi_{\gamma\alpha}$$

(clearly a well defined form on $U_\alpha$)

where $\{\rho_\gamma\}$ is a partition of unity subordinate to $\{U_\gamma\}$.

$$d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha$$




on $U_\alpha \cap U_\beta$. This gives $d\xi_\alpha = d\xi_\beta$ on $U_\alpha \cap U_\beta$, so we get a global 2 form $e$ on $M$ whose cohomology class is called the Euler class $e(E) \in H^2(M)$:

$$e|_{U_\alpha} = d\xi_\alpha$$

- The Euler class $e(E) \in H^2(M)$ will be studied for rank $n$ vector bundles later. It measures the "twisting" of a vector bundle by quantifying how hard it is to put angular coordinates on each fiber $\mathbb{R}^n \setminus \{0\} \cong E^\circ|_{\pi^{-1}(\{p\})}$

- On a rank 2 vector bundle, the global angular form can be patched together on each chart:

$$\psi\big|_{U_\alpha} := \frac{d\theta_\alpha}{2\pi} - \pi^*\xi_\alpha$$

  - On each fiber, $\psi_{\pi^{-1}(\{x\})} = \frac{d\theta}{2\pi}$ is just the angular form on $\mathbb{R}^2 \setminus \{0\}$. The important part is that $\psi$ is no longer closed:

$$d\psi = -\pi^* e$$

hence $e$ measures the failure to patch together a set of global angular coordinates on $E^0$.

- From $\psi$, we again can show the Thom class is:

$$\Phi = d(p(r)\cdot\psi)$$

  - If $s: M \to E$ is the zero section:

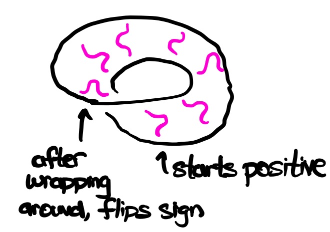
$$s^*\Phi = e$$

$p(r)\cdot\psi$ is not a global cohomology class on $E$ b/c it is not defined @ $r=0$, but $d(p(r)\psi)$ is, so $\Phi$ is not necessarily trivial in cohomology.

- The Euler class is functorial in the sense that:

$$e(f^{-1}E) = f^* e(E)$$

- Non-orientable case: For the Möbiüs strip there is no Thom class b/c a bump form on the fiber @ $\theta = 0$ with $\int_F \alpha = 1$ will have its integral at $\theta = 2\pi$ become $-1$.
  - Are there any good examples of a vector bundle which is oriented and has a nontrivial Euler class? If so, what does its Thom/Euler class look like?



## Applications to Physics

- Most applications to physics require a metric $g_{\mu\nu}\, dx^\mu \wedge dx^\nu$. Mathematically, this is only required to be a pseudo-metric. A pseudometric is a form $g \in \Omega^2(M)$ which is symmetric and non-degenerate; unlike a full metric, it need not be positive definite. A pair $(M,g)$ for $g$ a pseudo-metric is called a pseudo Riemannian manifold.
  - Lorentz spacetime $M=\mathbb{R}^4$ with $g = dt^2 - d\vec{x}^2$ is not a Riemannian manifold, but rather a pseudo-Riemannian manifold.
  - A pseudometric $g_{\mu\nu}$ is symmetric and nondegenerate $\Rightarrow$ real nonzero eigenvalues. If $i$ and $j$ are the # of positive and negative eigenvalues, then $(i,j)$ is called the index of $g$.
  - A manifold is Riemannian if $j=0$, and Lorentzian if $j=1$.

- For a pseudo-metric, we can define the Hodge Star

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$$

$$*(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}) = \frac{\sqrt{|g|}}{(n-k)!} \epsilon^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_{n-k}}\, dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{n-k}}$$

The $*$ can be used to define a symmetric bilinear form on $\Omega^k(M)$ as:

$$\langle \omega, \tau \rangle = \int_M \omega \wedge * \tau$$

where $g = g_{ij}\, dx^i dx^j = g_i\, dx^i \otimes dx^i$ is taken to be diagonal. Note:

$$*1 = \mathrm{Vol} \qquad **\omega = \begin{cases} (-1)^{k(n-k)}\, \omega & g \text{ Riemannian} \\ (-1)^{1+k(n-k)}\, \omega & g \text{ Lorentzian} \end{cases}$$

- With this bilinear form $\langle \cdot, \cdot \rangle$, there is a natural chain map adjoint to $d$:

$$\delta : \Omega^k(M) \longrightarrow \Omega^{k-1}(M) \qquad \delta\omega = \begin{cases} (-1)^{n(k+1)+1} *d*\omega & g \text{ Riemannian} \\ (-1)^{n(k+1)} *d*\omega & g \text{ Lorentzian} \end{cases}$$

and $\delta = 0$ for $k=0$. Since $*^2 = (-1)^\#$, $\delta^2 = (-1)^\# * d^2 * = 0$, hence $\delta$ is a chain map. We have:

$$\langle d\omega, \tau \rangle = \langle \omega, \delta\tau \rangle$$

$\delta$ will often be denoted by $d^\dagger$ as well.

- Hodge duality: The Laplace deRham operator is defined as $\Delta : \Omega^k(M) \rightarrow \Omega^k(M)$ by:

$$\Delta = d\delta + \delta d$$

We have $\Delta\omega = 0$ iff $d\omega = \delta\omega = 0$, and we call such forms harmonic and denote the space of harmonic $k$-forms by $\mathcal{H}^k(M)$. For any $\omega \in \Omega^k(M)$, $\exists!$ decomposition:

$$\omega = d\alpha + \delta\beta + \gamma$$

with $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^{k+1}(M)$, and $\gamma \in \mathcal{H}^k(M)$.

- On 0-forms $f\in\Omega^0(M)$, $\Delta f = -\frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu f) = \partial^2 f$ if we use a flat metric. Similarly for $k=1$ in flat Euclidean space, $\Delta\omega = -(\partial_\mu\partial^\mu\omega_\nu)dx^\nu$
- The Hodge star explicitly gives an isomorphism between the $\delta$-homology and the d-cohomology:

$$*: H^*_d(M) \longrightarrow H_{\delta, n-*}(M)$$

- Now we turn to the physics. In Minkowski space:

$$g = g_{\mu\nu}\,dx^\mu \otimes dx^\nu = dt^2 - d\vec{x}^2 \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

has $\mathrm{index}(g) = 1$.

- In electromagnetism, quantities are naturally differential forms. We first work in 3 dimensions, then move to 4. In 3d:

$*1 = dx\wedge dy\wedge dz = \mathrm{Vol}$

| | | |
|---|---|---|
| $*dx = dy\wedge dz$ | $*dy = dz\wedge dx$ | $*dz = dx\wedge dy$ |
| $*(dx\wedge dy) = dz$ | $*(dz\wedge dx) = dy$ | $*(dy\wedge dz) = dx$ |

$*(dx\wedge dy\wedge dz) = 1$

where $\mathrm{index}(g) = 0$. Note that we have the correspondences to vector calculus:

| Vector calc $\overset{\sim}{\longleftrightarrow}$ | Differential forms |
|---|---|
| functions $f$ | 0 forms $f$ or 3-forms $f\,dx\wedge dy\wedge dz$ |
| Vector fields $A^i\vec{e}_i$ | 1 forms $A_i dx^i$ or 2 forms $\epsilon^i{}_{jk}A_i\,dx^j\wedge dx^k = A_i(*dx^i)$ |
| gradient $\nabla f$ | d of a 0 form $f\in\Omega^0(M)$ |
| curl $\nabla\times\vec{F}$ | d of a 1 form $F_i dx^i \in\Omega^1(M)$ (note $dF\in\Omega^2$, and $*dF\in\Omega^1$) |
| div $\nabla\cdot\vec{F}$ | d of a 2 form $F_i dy\wedge dz + \ldots$, or $\delta = *d*$ of a 1 form $F_i dx^i$ |

- If we wish, the $*$ lets us take our identification to be with 0 and 1 forms.
- We see explicitly here that when $\Delta = d\delta + \delta d$ acts on 0 forms $f$, we get a Laplacian:

$$\Delta f = (d\overset{0}{\delta} + \delta d)f = \delta df = *d*df = \nabla\cdot\nabla f = \nabla^2 f$$

- The electric field $E$ is treated as a 1-form, while the magnetic field $B$ is a 2-form:

$$E = E_1 dx + E_2 dy + E_3 dz$$
$$B = B_1 dy\wedge dz + B_2 dz\wedge dx + B_3 dx\wedge dy$$

- This encodes how $E\mapsto -E$ and $B\mapsto B$ under parity. In general in $\mathbb{R}^3$, vectors are 1-forms in $\Omega^1(\mathbb{R}^3)$ and pseudovectors are 2 forms in $\Omega^2(\mathbb{R}^3)$, since $dx^i \overset{P}{\mapsto} -dx^i$ but $dx^i\wedge dy^j \overset{P}{\mapsto} dx^i\wedge dy^j$ under parity.
- Since $B$ is a closed 2 form ($dB = \nabla\cdot\vec{B} = 0$), the Poincaré lemma for $\mathbb{R}^3$ implies it is exact:

$$B = dA \qquad A = A_i dx^i \in\Omega^1(\mathbb{R}^3) \qquad (\vec{B} = \nabla\times\vec{A})$$

- The electric field is closed when $\partial_t B = 0$, and the Poincaré lemma applies. More generally:

$$dE + \partial_t B = 0 = d(E - \partial_t A) \xRightarrow[\text{lemma}]{\text{Poincaré}} E - \partial_t A = d\Phi,\quad \Phi\in\Omega^0(\mathbb{R}^3) \qquad (\nabla\times\vec{E} = -\partial\vec{B}/\partial t)$$

- We combine these into a 2-form $F \in \Omega^2(M)$, where $(M,g) = (\mathbb{R}^4, \text{Minkowski})$ is spacetime:

$$F = B + E \wedge dt$$

- In 4d, the $*$ acts differently, both b/c $n=4$ and b/c the index of $g$ becomes $(3,1)$:

$*dt = dx \wedge dy \wedge dz$ ... $*dz = dt \wedge dx \wedge dy$

$*(dt \wedge dx) = dy \wedge dz$ $\quad$ $*(dt \wedge dy) = dz \wedge dx$ $\quad$ $*(dt \wedge dz) = dx \wedge dy$

$*(dy \wedge dz) = -dt \wedge dx$ $\quad$ $*(dz \wedge dx) = -dt \wedge dy$ $\quad$ $*(dx \wedge dy) = -dt \wedge dz$

- For the 2 forms, if $\omega$ contains $dt$ then there will be an extra $-$ sign (b/c $g$ is a pseudometric)
- Finally, we define the current $J \in \Omega^3(M)$:

$$J := g_{\mu\nu} j^\mu * (dx^\nu)$$

$$\rho\, dx \wedge dy \wedge dt - j_x dt \wedge dy \wedge dz - j_y dt \wedge dz \wedge dx - j_z dt \wedge dx \wedge dy$$

- Maxwell's eqns can be succinctly combined into the assertion that $F$ is a closed form, and $*F$ is closed except in the presence of an external field:

$$dF = 0 \qquad \left(\text{contains } \nabla \cdot B = 0,\ \nabla \times \vec{E} = -\tfrac{\partial \vec{B}}{\partial t}\right)$$

$$d*F = J$$

- When in the presence of no background field, $F$ and $*F$ pass to the cohomology.

# Čech cohomology

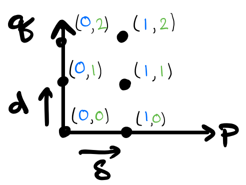
- Formulate Mayer-Vietoris w/ a double complex $K^{**} = C^*(\mathcal{U}, \Omega^*)$, where $\mathcal{U} = (U, V)$ is an open cover of $M$.

$$K^{0,q} := C^0(\mathcal{U}, \Omega^q) := \Omega^q(U) \oplus \Omega^q(V)$$
$$K^{1,q} := C^1(\mathcal{U}, \Omega^q) := \Omega^q(U \cap V)$$
$$K^{p,q} := 0 \quad \text{for } p > 1$$

– We get two distinct cochain maps:

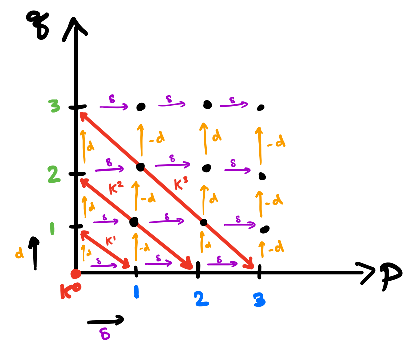
$$d : K^{p,q} \longrightarrow K^{p,q+1} \qquad\qquad \delta : K^{p,q} \longrightarrow K^{p+1,q}$$
$$\omega \longmapsto d\omega \qquad\qquad \omega \longmapsto \begin{cases} \omega|_V - \omega|_U & p = 0 \\ 0 & \text{else} \end{cases}$$

so we can represent these on a commutative grid:




- Given a double complex $(K^{**}, d, \delta)$ w/ commuting differentials, we can form a single complex $(K^*, D)$ with:

$$K^n := \bigoplus_{p+q=n} K^{p,q} \qquad\qquad D := \delta + (-1)^p d$$




- <u>Theorem</u>: The $D$-cohomology of $C^*(\mathcal{U}, \Omega^*)$ is isomorphic to the deRham cohomology of $M$.

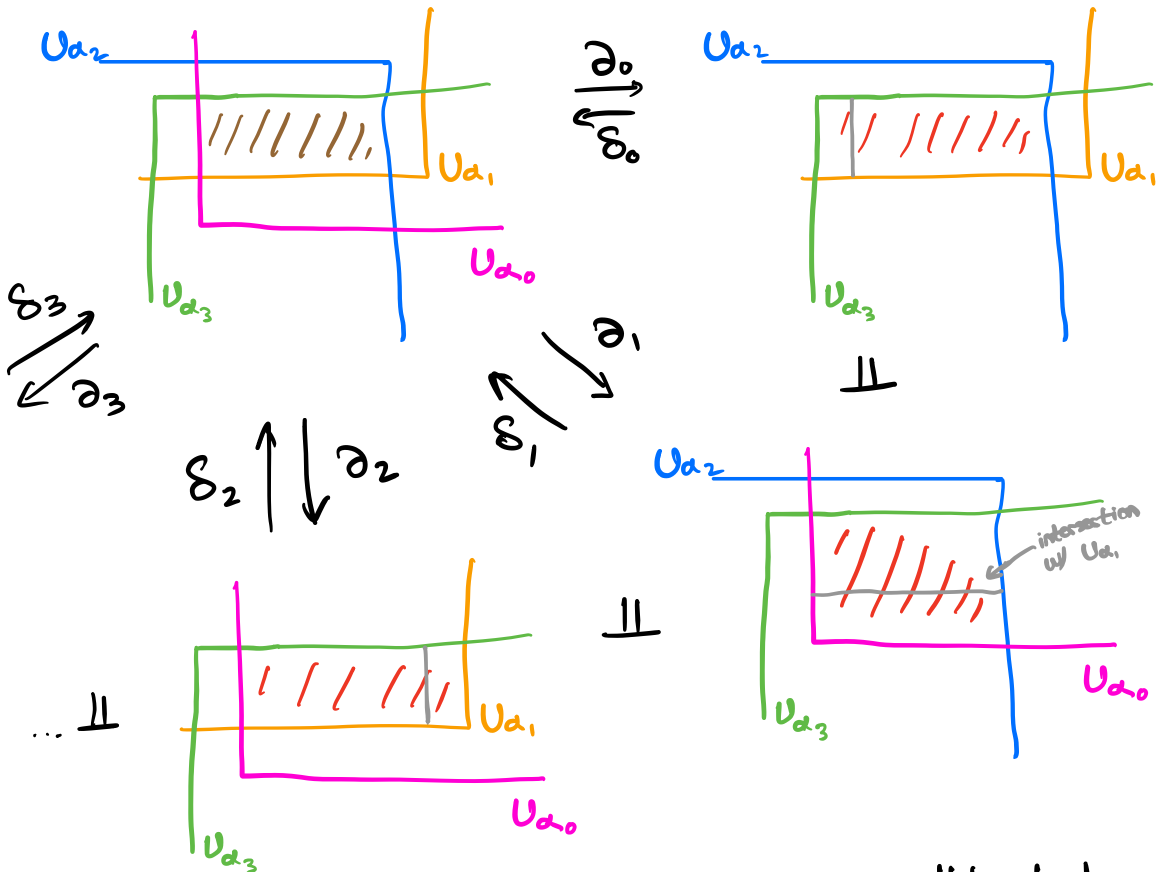
$$H_D(C^*(\mathcal{U}, \Omega^*)) \cong H_{DR}(M)$$

To show this, show $r : \Omega^q(M) \longrightarrow \Omega^q(U) \oplus \Omega^q(V) = C^0(\mathcal{U}, \Omega^q) \subset [C^*(\mathcal{U}, \Omega^*)]^q$, $\omega \mapsto (\omega|_U, \omega|_V)$ is a chain map and drops to a cohomology isomorphism.

- Generalization: $\mathcal{U} := \{U_\alpha\}_{\alpha \in J}$ a countable open cover of $M$, denote $U_\alpha \cap U_\beta \cap \ldots \cap U_\gamma = U_{\alpha\beta\ldots\gamma}$. We get inclusions

$$M \xleftarrow{\partial} \coprod_{\alpha} U_\alpha \;\underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}}\; \coprod_{\alpha<\beta} U_{\alpha\beta} \;\underset{\partial_2}{\overset{\partial_0}{\Lleftarrow}}\; \coprod_{\alpha<\beta<\gamma} U_{\alpha\beta\gamma} \;\Lleftarrow\; \cdots$$

where $\partial_i : \coprod U_{\alpha\ldots\alpha_i\ldots\beta} \longrightarrow \coprod U_{\alpha\ldots\beta}$ are inclusions




These induce restrictions on forms which pull back to:

$$\Omega^*(M) \xrightarrow{\delta} \prod_{\alpha} \Omega^*(U_\alpha) \;\underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}}\; \prod_{\alpha<\beta} \Omega^*(U_{\alpha\beta}) \;\underset{\delta_2}{\overset{\delta_0}{\Rrightarrow}}\; \prod_{\alpha<\beta<\gamma} \Omega^*(U_{\alpha\beta\gamma}) \;\Rrightarrow\; \cdots$$

If $\omega \in \prod_{\alpha_0<\ldots<\alpha_p} \Omega^*(U_{\alpha_0\ldots\alpha_p})$, $\omega$ has components $\omega_{\alpha_0\ldots\alpha_p} \in \Omega^*(U_{\alpha_0\ldots\alpha_p})$,

and we can define a cochain map $\delta : \coprod_{\alpha_0<\ldots<\alpha_p} \Omega^*(U_{\alpha_0\ldots\alpha_p}) \to \coprod_{\alpha_0<\ldots<\alpha_{p+1}} \Omega^*(U_{\alpha_0\ldots\alpha_{p+1}})$

$$(\delta\omega)_{\alpha_0\ldots\alpha_{p+1}} := \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}} \quad \longleftarrow$$

(This is a generalization of Mayer-Vietoris w/ a 2 piece cover $\{U,V\}$)

- This defines a cochain complex:

$$\delta^2 = 0$$

much like what we get in simplicial cohomology. The alternating nature of the sum is what forces $\delta^2$ to $0$.

- We define $\omega_{\dots\beta\dots\alpha\dots} = -\omega_{\dots\alpha\dots\beta\dots}$ when $\alpha<\beta$
- Generalized Mayer-Vietoris: This is exact:

$$0 \longrightarrow \Omega^*(M) \xrightarrow{r} \prod_{\alpha} \Omega^*(U_\alpha) \xrightarrow{\delta} \prod_{\alpha_1<\alpha_2} \Omega^*(U_{\alpha_1\alpha_2}) \xrightarrow{\delta} \dots$$

i.e. the $\delta$-cohomology of this cochain complex vanishes, so each $\delta$-cocycle is a $\delta$-coboundary.

$C^0(\mathcal{U},\Omega^q) = \coprod_{\alpha_0} \Omega^q(U_{\alpha_0})$
$C^1(\mathcal{U},\Omega^q) = \coprod_{\alpha_0<\alpha_1} \Omega^q(U_{\alpha_0}\cap U_{\alpha_1})$

- Terminology: Let:

$$C^p(\mathcal{U},\Omega^q) := K^{p,q} := \prod_{\alpha_0<\dots<\alpha_p} \Omega^q(U_{\alpha_1\dots\alpha_p})$$

(Note $C^0 = \prod_{\alpha_0} \Omega^q(U_{\alpha_0})$)

This is an extension of the previous double graded complex.

$r$ = restriction, $\omega \mapsto (\omega|_{U_\alpha})$

$$\begin{tikzcd}
 & \uparrow d & q & & & & \\
0 \arrow[r] & \Omega^2(M) \arrow[r, "r"] & C^0(\mathcal{U},\Omega^2) & C^1(\mathcal{U},\Omega^2) & C^2(\mathcal{U},\Omega^2) & C^3(\mathcal{U},\Omega^2) & \text{2-forms} \\
0 \arrow[r] & \Omega^1(M) \arrow[r, "r"] \arrow[u, "d"] & C^0(\mathcal{U},\Omega^1) & C^1(\mathcal{U},\Omega^1) & C^2(\mathcal{U},\Omega^1) & C^3(\mathcal{U},\Omega^1) & \text{1-forms} \\
 & & & & & \uparrow d & \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "r"] \arrow[u, "d"] & C^0(\mathcal{U},\Omega^0) & C^1(\mathcal{U},\Omega^0) & C^2(\mathcal{U},\Omega^0) & C^3(\mathcal{U},\Omega^0) \quad p & \text{fns} \\
 & & \xrightarrow{\delta} & & & &
\end{tikzcd}$$

Let $\omega\in C^p(\mathcal{U},\Omega^q)$. We call $\omega$:

i) A cocycle if $\delta\omega=0$
ii) A coboundary if $\omega=\delta\tau$

$\Big\}$ $(C^*(\mathcal{U},\Omega^q),\delta)$ cohomology

iii) A closed form if $d\omega=0$
iv) An exact form if $\omega=d\tau$

$\Big\}$ $(C^p(\mathcal{U},\Omega^*),d)$ cohomology

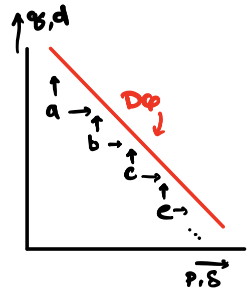
$$C^*(\mathcal{U},\Omega^*) := \bigoplus_{p,q\geq 0} C^p(\mathcal{U},\Omega^q)$$ is the Čech-deRham complex.

This is formed into a single cochain complex as before:

$$[C^*(\mathcal{U},\Omega^*)]^n = \bigoplus_{p+q=n} C^p(\mathcal{U},\Omega^q) \qquad D = \delta + (-1)^p d$$

- Note if $\varphi$ is a cocycle, $D\varphi = 0 \Rightarrow 0 = da + (\delta a - db) + (\delta b + dc) + (\delta c - de) + \dots$. Since each of these terms live in their own gradings, they must be $0$, i.e. $da = 0$, $\delta a = db$, $\delta b = -dc$, $\delta c = de, \dots$

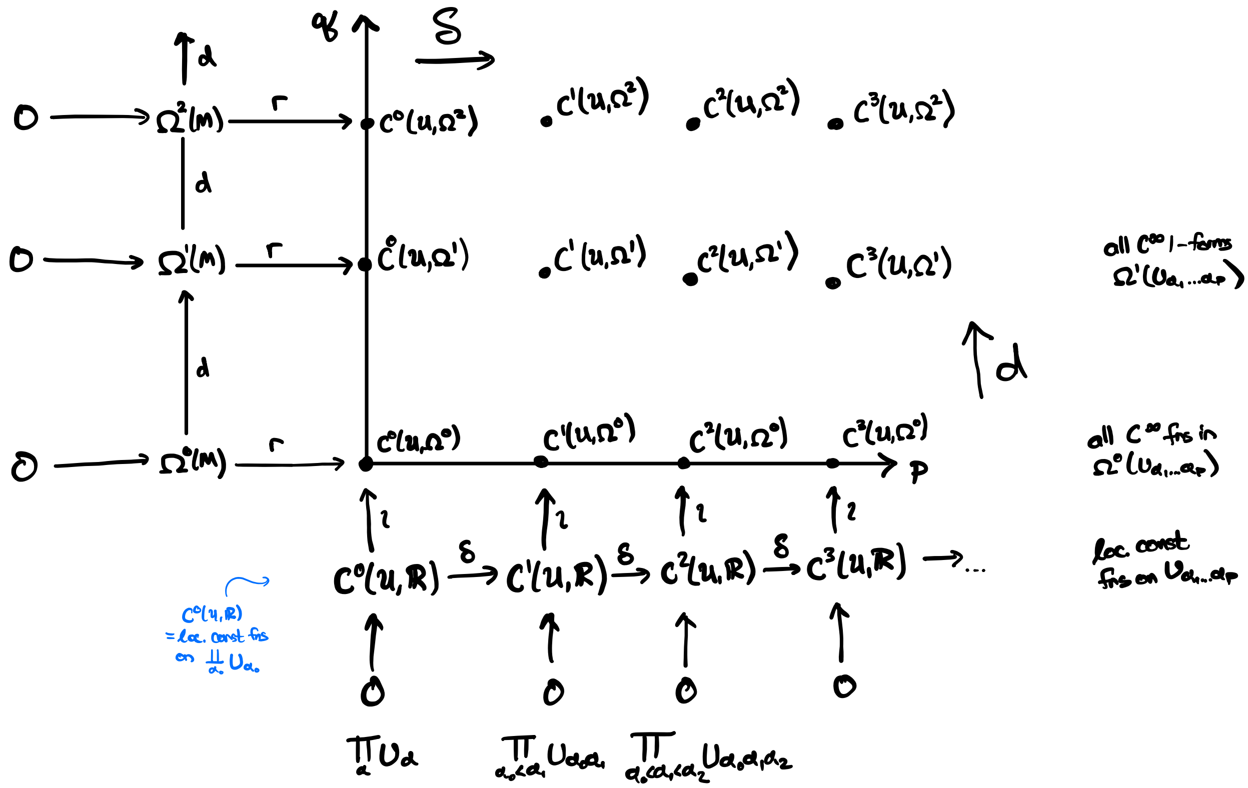
$\varphi \in [C^*(\mathcal{U},\Omega^*)]^n$ has $\varphi = a+b+c+d+\dots$ w/ each element in a grading $(p,q)$ w/ $p+q=n$.




- Rows of a double complex exact $\Rightarrow$ vertical d-cohomology $\cong$ D-cohomology
  - Vice versa if the columns are exact, we get $H^*_D \cong H^*_\delta$.
  - In this case one can show each cocycle is cohomologous to a cocycle w/ only a top component, i.e. $D\varphi = 0 \Rightarrow \varphi = \varphi' + D\tau$ w/ $\varphi' \in C^0(\mathcal{U},\Omega^q)$.
- Generalized Mayer-Vietoris then implies:

$$H^*_{dR}(M) \cong H^*_D(C^*(\mathcal{U},\Omega^*))$$

- Augment the complex $C^*(\mathcal{U},\Omega^*)$ with spaces $C^p(\mathcal{U},\mathbb{R})$ of the locally constant functions on $\prod_{\alpha_0<\dots<\alpha_p} U_{\alpha_0\dots\alpha_p}$. Then we can extend the double complex:



- Note although the bottom joint is exact, the exactness of the column $(C^p(\mathcal{U},\Omega^*), d)$ is measured by:

$$\prod_{\substack{q\geq 1 \\ \alpha_0<\dots<\alpha_p}} H^q(U_{\alpha_1\dots\alpha_p})$$

p-cochain = element of $C^p(\mathcal{U},\mathbb{R})$

- The $\delta$ cohomology of $(C^*(\mathcal{U},\mathbb{R}),\delta)$ is called the Čech cohomology of the cover $\mathcal{U}$, $\check{H}^*(\mathcal{U})$

- If the columns of the double complex are exact (equivalently $\prod_{\substack{q\geq 1 \\ \alpha_0<\dots<\alpha_p}} H^q(U_{\alpha_1\dots\alpha_p})$ vanishes) then $\check{H}^*(\mathcal{U},\mathbb{R}) \cong H_D^*(C^*(\mathcal{U},\Omega^*))$, and thus:

$$\check{H}^*(\mathcal{U},\mathbb{R}) \cong H^*_{dR}(M)$$

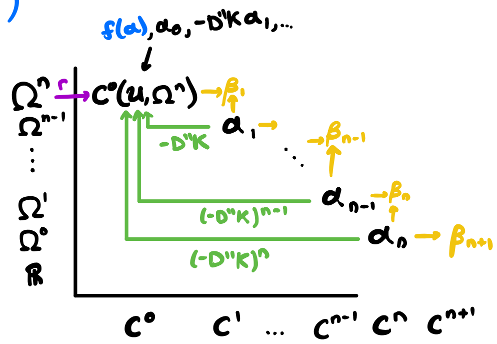
- When does this happen? For good covers $\mathcal{U}$ of $M$.
- Theorem: If $M$ has a good cover $\mathcal{U}$, then $\check{H}^*(\mathcal{U},\mathbb{R}) \cong H_{dR}(M)$
  - Recall every manifold has a good cover. If $M$ is of finite type, this is easier to work with.
  - Corollary: The Čech cohomology is the same for all good covers $\mathcal{U}$ of $M$.
  - Corollary: If $M$ is compact or of finite type, the deRham cohomology of $M$ is finite dimensional.
- The Čech complex $C^*(\mathcal{U},\mathbb{R})$ quantifies the intersections of the cover $\mathcal{U}$. If $\mathcal{U}$ is a good cover, all its intersections $\cong \mathbb{R}^n$ and are connected, so in the Čech complex each intersection will contribute $\simeq \mathbb{R}$ = locally const fns on $\mathbb{R}^n$. Then each $C^q(\mathcal{U},\mathbb{R}) \cong \mathbb{R}^{n(q)}$, where $n(q)$ = # of nonzero $q$-intersections of the good cover $\mathcal{U}$.

- The nerve of a cover $\mathcal{U}$, $N(\mathcal{U})$, is a simplicial complex constructed by:
  - To each $U_\alpha \in \mathcal{U}$, associate a vertex $v_\alpha$.
  - For each nonzero $U_\alpha \cap U_\beta$, attach the 0-cells $v_\alpha$ and $v_\beta$ w/ a 1-cell (line) $\ell_{\alpha\beta}$
  - For each nonzero $U_\alpha \cap U_\beta \cap U_\gamma$, attach $\ell_{\alpha\beta}, \ell_{\beta\gamma}$, and $\ell_{\gamma\alpha}$ w/ a 2 cell $c_{\alpha\beta\gamma}$ (fill in the face)
  - Continue for higher dimensionality

• The Collating Formula: The explicit isomorphism between $H^*_{dR}$ and $H^*_D$ is induced by $r: \Omega^*(M) \to C^*(\mathcal{U}, \Omega^*)$, $\omega \mapsto (\omega|_{U_\alpha})_\alpha$. The inverse isomorphism:

$$f: C^*(\mathcal{U}, \Omega^*) \to \Omega^*(M)$$
$$f(\alpha) := \sum_{i=0}^{n} (-D''K)^i \alpha_i + \sum_{i=1}^{n+1} K(-D''K)^{i-1} \beta_i \quad \in C^0(\mathcal{U}, \Omega^n)$$

where $D\alpha = \beta = \sum_{i=0}^{n+1} \beta_i$ w/ $\beta_i \in C^i(\mathcal{U}, \Omega^{n+1-i})$.




- $f$ essentially takes all the pieces of $\alpha$ and $D\alpha$ and moves them to the top component of the cochain, which stitches together to a global form (that is not obvious).
- $f$ satisfies:
  i) $f \circ r = id$
  ii) $r \circ f$ is chain homotopic to id.

hence $f, r$ induce inverse isomorphisms in cohomology $H^*_{dR}(M) \underset{f}{\overset{r}{\rightleftarrows}} H^*_D(C^*(\mathcal{U}, \Omega^*))$.

- If we have a 0-cochain $\omega_\alpha \in C^0(\mathcal{U}, \Omega^*)$, we can extend $[\omega_\alpha]$ to a global cohomology class $[r f(\omega)]$ if we can extend $\omega_\alpha$ to a D-cochain in $C^*(\mathcal{U}, \Omega^*)$.

• Given a cover $\mathcal{U}$, there is a natural homotopy operator:

$$K: C^p(\mathcal{U}, \Omega^q) \to C^{p-1}(\mathcal{U}, \Omega^q)$$
$$(K\omega)_{\alpha_0 \ldots \alpha_{p-1}} := \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0 \ldots \alpha_{p-1}}$$

$\{\rho_\alpha\}$ partition of unity subordinate to $\{U_\alpha\} = \mathcal{U}$

- This map is chain homotopic to $\delta$, $K\delta + \delta K = id$, and can be used in the case of $\mathcal{U}$ being a good cover to explicitly give the isomorphism between Čech and de Rham cohomology for a good cover $\mathcal{U}$ of $M$:

$$f: \check{H}^n(\mathcal{U}, \mathbb{R}) \xrightarrow{\sim} H^n_{dR}(M)$$
$$[\eta] \mapsto [(-1)^n (D''K)^n \eta]$$

$D'' = (-1)^p d$ = vertical differential

- A **presheaf** $\mathcal{F}$ on a topological space $X$ is a map assigning to each open $U \subset X$ an abelian group $\mathcal{F}(U)$ s.t. for any inclusion $i_U^V : V \to U$ there is a **restriction map** $\mathcal{F}(i_U^V) : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfying (let $\rho_V^U := \mathcal{F}(i_U^V)$)
  i) Identity, $\rho_U^U = id_{\mathcal{F}(U)}$
  ii) Transitivity, $\rho_W^V \circ \rho_V^U = \rho_W^U \; \forall W \subset V \subset U$.
- A **presheaf homomorphism** $f : \mathcal{F} \to \mathcal{G}$ preserves the restriction making this commute:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "f_U"'] & \mathcal{F}(V) \arrow[d, "f_V"] \\ \mathcal{G}(U) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$

  where $f$ is formally a collection of maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$.
- Categorically a presheaf $\mathcal{F}$ is a contravariant functor from **$Open(X)$ to $Ab$**, where $Open(X)$ has objects being the open sets of $X$ and whose morphisms are inclusions $i_U^V$
  - Presheaf homomorphisms are natural transformations $f : \mathcal{F} \to \mathcal{G}$
- The **constant presheaf w/ group $G$** is the presheaf which assigns to each open $U$ the space of constant functions $U \to G$, i.e. $\mathcal{F}_{const} = \{f : U \to G \mid f \text{ is constant}\}$
  - For Čech cohomology, since we are looking @ $C^p(\mathcal{U}, \mathbb{R}) =$ locally const. functions on $\coprod_{\alpha_0 < \ldots < \alpha_p} U_{\alpha_0 \ldots \alpha_p}$, we see there is a relation btwn $C^p(\mathcal{U}, \mathbb{R})$ and the constant presheaf w/ group $\mathbb{R}$.
- Let $\mathcal{F}$ be a presheaf on $X$, w/ $\mathcal{U} = \{U_\alpha\}$ an open cover of $X$. We define the **p-cochains on $\mathcal{U}$ with values in $\mathcal{F}$** to be:

$\mathcal{F}(U) \sim$ functions on $U$, $\{f : U \to G\}$

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0 < \ldots < \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \ldots \cap U_{\alpha_p})$$

  - Our previous dfn of $C^p(\mathcal{U}, \mathbb{R})$ matches this, where $\mathcal{F}$ is the locally const. presheaf valued in $\mathbb{R}$, since $x \in C^p(\mathcal{U}, \mathcal{F})$ means $x$ is a product of constant fns $U_{\alpha_0 \ldots \alpha_p} \to \mathbb{R}$, which matches our dfn of $C^p(\mathcal{U}, \mathbb{R})$

think of $\mathcal{F}(U)$ as a set of fns $\mathcal{F}(U) = \{f : U \to G\}$

  - Given an inclusion $U_{\alpha_0 \ldots \alpha_p} \xleftarrow{\partial_i} U_{\alpha_0 \ldots \alpha_{p+1}}$, we get a restriction $\mathcal{F}(\partial_i) : \mathcal{F}(U_{\alpha_0 \ldots \alpha_p}) \to \mathcal{F}(U_{\alpha_0 \ldots \alpha_{p+1}})$, so applying $\mathcal{F}$ to the sequence $U_\alpha \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} U_{\alpha\beta} \Lleftarrow U_{\alpha\beta\gamma} \Lleftarrow \ldots$ we can define a differential:

$$\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} := \sum_{i=0}^{p+1} (-1)^i \mathcal{F}(\partial_i) \omega_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}} = \sum (-1)^i \rho^{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}}_{\alpha_0 \ldots \alpha_i \ldots \alpha_{p+1}} \omega_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}}$$

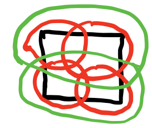
$$\mathcal{F}(\partial_i) = \rho^{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p}_{\alpha_0 \ldots \alpha_i \ldots \alpha_p}$$

The **Čech cohomology of the cover $\mathcal{U}$ w/ values in $\mathcal{F}$, $\check{H}^*(\mathcal{U}, \mathcal{F})$** is the cohomology of $(C^*(\mathcal{U}, \mathcal{F}), \delta)$.

- A cover $\mathcal{B} := \{V_\beta\}_{\beta \in J}$ is a refinement of $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ if there is a map $\varphi : J \to I$ such that $V_\beta \subset U_{\varphi(\beta)}$ for each $\beta \in J$. We write $\mathcal{U} < \mathcal{B}$. Given a refinement $\varphi$, there is a canonical map:

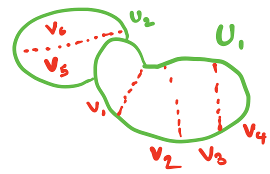
$$\varphi^\# : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{B}, \mathcal{F})$$
$$(\varphi^\# \omega)_{\beta_0 \dots \beta_p} := \omega_{\varphi(\beta_0) \dots \varphi(\beta_p)}$$

(we can drop a cochain from the coarser cover $\mathcal{U}$ to a refinement $\mathcal{B}$ naturally since each $V_\beta \in \mathcal{B}$ is a subset of a $U_\alpha \in \mathcal{U}$.)



$\mathcal{U} < \mathcal{B}$

  - In the picture, a cochain $\omega \in C^p(\mathcal{U}, \mathcal{F})$ induces a cochain $\varphi^\# \omega$ on $C^p(V, \mathcal{F})$ by restriction: if $M = U_1 \cup U_2$ and $\omega = (\omega_1, \omega_2) \in C^0(\mathcal{U}, \mathcal{F})$ w/ $\omega_i$ forms on $\mathcal{U}_i$, then $\varphi^\# \omega = (\bar\omega_1, \bar\omega_2, \bar\omega_3, \bar\omega_4, \bar\omega_5, \bar\omega_6)$ w/ $\bar\omega_1 = \omega_1|_{V_1}, \bar\omega_2 = \omega_1|_{V_2}, \dots, \bar\omega_6 = \omega_2|_{V_6}$.




$\varphi : 1,2,3,4 \longmapsto 1$
$\quad 5,6 \longmapsto 2$

  - $\varphi^\#$ is a cochain map and drops to cohomology
  - If $\varphi, \psi : J \to I$ are two refinement maps, then:

$$K : C^q(\mathcal{U}, \mathcal{F}) \longrightarrow C^{q-1}(\mathcal{B}, \mathcal{F})$$
$$(K\omega)|_{\beta_0 \dots \beta_{q-1}} := \sum_i (-1)^i \omega_{\varphi(\beta_0) \dots \varphi(\beta_i) \psi(\beta_i) \dots \psi(\beta_{q-1})}$$

is a chain homotopy between $\varphi^\#$ and $\psi^\#$, hence a refinement $\mathcal{U} < \mathcal{B}$ induces a unique map $\check{H}^q(\mathcal{U}, \mathcal{F}) \to \check{H}^q(\mathcal{B}, \mathcal{F})$

- Let $X$ be a topological space. The collection $\{\mathcal{U}\}$ of open covers of $X$ is a directed set under refinement b/c given $\mathcal{U}_1, \mathcal{U}_2 \in \{\mathcal{U}\}$, we can form $\mathcal{B} := \{U_1 \cap U_2 : U_i \in \mathcal{U}_i\}$ which is a refinement of both. Hence we can take the direct limit:

$$\begin{array}{ccccc} \check{H}^*(\{X\}, \mathcal{F}) \to \dots \to & \check{H}^*(\mathcal{U}, \mathcal{F}) & \to \dots \to & \check{H}^*(\mathcal{B}, \mathcal{F}) \to \dots \\ \searrow & \downarrow & \swarrow & \\ & \varinjlim_{\mathcal{U}} \check{H}^*(\mathcal{U}, \mathcal{F}) & & \end{array}$$

and we define this limit to be the Čech cohomology of $X$:

$$\check{H}^*(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^*(\mathcal{U}, \mathcal{F})$$

- If we let $\mathbb{R}$ denote the constant presheaf w/ values in $\mathbb{R}$, then for any manifold $M$:

$$\check{H}^*(M, \mathbb{R}) \cong H_{dR}(M)$$

  - In $\varinjlim_{\mathcal{U}}$, since good covers are cofinal in the directed system, one is allowed to compute $\varinjlim_{\mathcal{U}}$ using only the case when $\mathcal{U}$ is a good cover.
    - By cofinal, we mean for any cover $\mathcal{U}$, $\exists$ a good cover $\mathcal{B}$ s.t. $\mathcal{U} < \mathcal{B}$.

• We previously defined the global angular form $\psi \in H^1(E^0)$ for a rank 2 vector bundle $E$ (w/ $E^0 = E \setminus$ zero section). Since $\psi|_{\text{fiber}}$ generates $H^1(S^1)$, if $\psi$ is closed we can apply Leray Hirsch and $H^*(E^0) = H^*(S^1) \otimes H^*(M)$. We also have $d\psi = -\pi^* e$ w/ $[e] \in H^2(M)$ the Euler class, so $[e] = 0 \in H^2(M)$ implies $H^*(E^0) = H^*(S^1) \otimes H^*(M)$.

• Orientation of a sphere bundle $E \to M$: $E$ is orientable if @ each fiber $F_x \cong S^n$ w/ $x \in M$, we can choose a generator $[\sigma_x] \in H^n(F_x)$ satisfying *local compatibility*: At each $x \in M$, $\exists x \in U \subset M$ open and $[\sigma_U] \in H^n(E|_U)$ generating the cohomology s.t. @ each $p \in U$, $[\sigma_U|_{F_p}] = [\sigma_p]$.

  - Equivalently we can find an open cover $\{U_\alpha\}$ and generators $[\sigma_\alpha] \in H^n(E|_{U_\alpha})$ s.t. $[\sigma_\alpha] = [\sigma_\beta]$ on $U_\alpha \cap U_\beta$.
  - Each fiber has $H^n(F_x) \cong \mathbb{R}$ and 2 possible generators, $\pm[\alpha]$, where $\alpha$ is a bump $n$-form on $S^n$.

• Let $E \to M$ be a rank $n+1$ vector bundle w/ a Riemannian metric + structure group $O(n+1)$. The sphere bundle $S(E)$ is the subbundle with fibers $\tilde{F}_x := \{v \in F_x : \|v\| = 1\}$ with structure group $O(n+1)$.

  - A vector bundle $E$ is orientable iff $S(E)$ is orientable.
    - Recall the dfn of orientability for $E$ is if its structure group may be reduced to $SO(n+1)$.

• Some extra facts:

  i) A line bundle is orientable iff it is trivial

  ii) Every vector bundle over a simply connected base space is orientable

  iii) Every simply connected manifold is orientable
    - This follows from ii b/c $M$ is orientable iff $TM$ is orientable as a bundle. Note $TM$ is always orientable as a manifold.

• Construction of the Euler class for an $S^1$ bundle $E \to M$: Pick a good cover $\mathcal{U} = \{U_\alpha\}$ of $M$, $[\sigma_\alpha^{0,1}] \in H^1(E|_{U_\alpha})$ a generator. Assume $E$ is orientable and each $[\sigma_\alpha^{0,1}]$ is in the same orientation class.

  Goal: Because $H^*_{dR}(E) \cong H^*_D(C^*(\pi^{-1}\mathcal{U}, \Omega^*))$, extending $[\sigma_\alpha^{0,1}]$ to a $D$-cocycle will yield a global cohomology class which restricts to $[\sigma_\alpha^{0,1}]$ on each $U_\alpha$ (collating formula).

$$
\begin{tikzcd}
\Omega^2 & 0 & & \\
\Omega^1 \arrow[u] & \sigma^{0,1} \arrow[u, "d"] \arrow[r, "\delta"] & {} & \\
\Omega^0 \arrow[u, "d"] & & \sigma^{1,0} \arrow[u, "d"] \arrow[r, "\delta"] & {} \\
\mathbb{R} \arrow[u, "\iota"] & & & \varepsilon \arrow[u, "\iota"] \\
& C^0 \arrow[r, "\delta"] & C^1 \arrow[r, "\delta"] & C^2
\end{tikzcd}
$$

exact columns

  i) Find $\sigma^{1,0}$ s.t. $\delta\sigma^{0,1} = d\sigma^{1,0}$. We have $[\delta\sigma_{\alpha\beta}^{0,1}] = [\sigma_\beta^{0,1}] - [\sigma_\alpha^{0,1}] = 0$ b/c $\{[\sigma_\alpha^{0,1}]\}$ define the orientation of $E$

  $$\Longrightarrow \delta\sigma^{0,1} = d\sigma^{1,0} = -D''\sigma^{1,0} \quad \text{for } \sigma^{1,0} \in C^1(\pi^{-1}\mathcal{U}, \Omega^0)$$

  ii) $\sigma^{0,1} + \sigma^{1,0}$ is almost a $D$-cocycle:

  $$D(\sigma^{0,1} + \sigma^{1,0}) = \underbrace{d\sigma^{0,1}}_{0} + (\underbrace{\delta\sigma^{0,1} - d\sigma^{1,0}}_{0}) + \delta\sigma^{1,0} = \delta\sigma^{1,0}$$

  iii) Does $\delta\sigma^{1,0} = 0$? We see $d\delta\sigma^{1,0} = \delta d\sigma^{1,0} = \delta^2\sigma^{0,1} = 0$, so $\delta\sigma^{1,0} = \iota\varepsilon$ for $\varepsilon \in C^2(\pi^{-1}\mathcal{U}, \mathbb{R})$ by exactness of $\xrightarrow{\iota} \xrightarrow{d}$. We can consider $\varepsilon \in C^2(\mathcal{U}, \mathbb{R})$ since $C^2(\pi^{-1}\mathcal{U}, \mathbb{R}) \cong C^2(\mathcal{U}, \mathbb{R})$ b/c const. fns on $\mathcal{U}$ and $\pi^{-1}\mathcal{U}$ are equivalent, and $\delta\varepsilon = \delta(\delta\sigma^{0,1}) = 0 \Rightarrow \varepsilon$ defines a cohomology class:

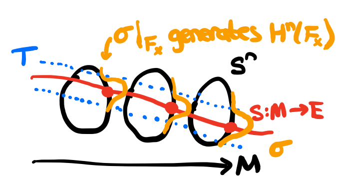
$$\begin{array}{ccc} \check{H}^2(\mathcal{U},\mathbb{R}) & \xrightarrow{\sim} & H^2_{dR}(M) \\ \varepsilon & \longmapsto & e(E) \end{array} \qquad e(E) = \text{Euler class}$$

iv) $\delta\sigma^{0,1} = 0$ iff $[\varepsilon] = 0$ in $\check{H}^2(\mathcal{U},\mathbb{R})$, so we can extend $[\sigma^{0,1}]$ to a D-cocycle iff $e(E)$ vanishes.

- This same construction defines the Euler class $e(E)$ for any sphere bundle. We have no trouble in intermediate steps b/c @any middle step, $d\delta\sigma^{k,n-k} = \delta(\delta\sigma^{k-1,n-k+1}) = 0$, and for $k \neq 0, n$ $H^{n-k}(E|_{U_{\alpha_0\ldots\alpha_k}}) \cong H^{n-k}(S^n) = 0$, so $\delta\sigma^{k,n-k} = d\sigma^{k+1,n-k-1}$

  $\sigma^{0,n} \to$, $\uparrow$, $\sigma^{1,n-1} \to$, $\uparrow$, $\ldots$, $\sigma^{n,0}$

  - $e(E)$ will live in $H^n(M)$, and $e(E) = (-D''K)^{n+1}\varepsilon$
- $e(E)$ is well defined, independent of choice of generator $[\sigma^{0,n}_\alpha] \in H^n(E|_{U_\alpha})$ and of choice of good cover.
- Summary: On a sphere bundle $E$, there is a global form $\sigma \in H^n(E)$ which restricts to a generator of each fiber iff:
  i) $E$ is orientable
  ii) $[e(E)] \in H^n(M)$ vanishes
- $e(E)$ for any trivial bundle vanishes b/c $\delta\sigma^{0,n} = 0$
  - Since $E = M \times S^n$, on a good cover $\{U_\alpha\}$ of $M$, we can take $[\sigma^{0,n}_\alpha] \in H^n(U_\alpha \times S^n)$ to restrict to the same generator on each fiber, $[\sigma^{0,n}_\alpha|_x] = [\sigma^{0,n}_\beta|_y] \in H^n(S^n)$ for $\forall \alpha,\beta$, $x \in U_\alpha$, $y \in U_\beta$. This means $(\delta\sigma^{0,n})_{\alpha\beta} = \sigma^{0,n}_\beta - \sigma^{0,n}_\alpha = 0$ everywhere, so the construction stops @ the first level.
- Theorem: If an oriented $S^n$ bundle $E \to M$ has a global section, then $e(E) = 0$
  - However, the converse is not true: If $e(E)$ vanishes $E$ may not have a section.
  - The intuition for this is similar to that for a trivial bundle, because intuitively a global section $M \to E$ yields a well defined form $\sigma \in H^n(E)$ s.t. $\sigma|_{F_x}$ generates $H^n(S^n)$. If $s$ is a section, create a cylindrical neighborhood $T$ of $s(M)$ in $E$. Then on each fiber, define $\sigma|_{F_x}$ to be a bump form w/ support in $T \cap F_x$. Taking $\sigma|_{F_x}$ to be smooth as we vary $x$, we see there is a "correspondence" btwn:




$$\left\{ \begin{array}{c} \text{global sections} \\ s: M \to E \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{Forms } \sigma \in H^n(E) \text{ which restrict} \\ \text{to a generator of } H^n(F_x) \end{array} \right\}$$

- Given a vector bundle $E \to M$, its Euler class is the Euler class of $E^0 = E \setminus s(M)$ where $s: M \to E$ is the zero section.
- Relation to the previous $e(E)$ construction for $n=1$:
  - Recall previously we had a good cover $\{U_\alpha\}$ on $M$ w/ polar coordinates $(r_\alpha, \theta_\alpha)$, and the map $\varphi_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{R}$, $\varphi_{\alpha\beta} := \theta_\beta - \theta_\alpha$. Note that $\frac{\pi^*\varphi_{\alpha\beta}}{2\pi} \in \coprod_{\alpha<\beta} \Omega^0(U_{\alpha\beta}) = C^1(\mathcal{U}, \Omega^0)$ and $\frac{d\theta_\alpha}{2\pi} \in \coprod_\alpha \Omega^1(U_\alpha) = C^0(\mathcal{U}, \Omega^1)$ generates $H^1(E|_{U_\alpha})$. So, taking:

$$\begin{cases} \sigma_\alpha^{0,1} = \dfrac{d\theta_\alpha}{2\pi} \\ \sigma_{\alpha\beta}^{1,0} = \dfrac{\pi^*\varphi_{\alpha\beta}}{2\pi} \end{cases}$$

$$\begin{array}{c|ccc}
\Omega^1 & \frac{d\theta_\alpha}{2\pi} \xrightarrow{\delta} & \frac{\pi^* d\varphi_{\alpha\beta}}{2\pi} & \\
 & & \uparrow d & \\
\Omega^0 & & \frac{\pi^*\varphi_{\alpha\beta}}{2\pi} \xrightarrow{\delta} & -\pi^*\varepsilon \\
 & & & \uparrow i \\
\mathbb{R} & & & -\pi^*\varepsilon \\
\hline
 & C^0 & C^1 & C^2 \\
 & \coprod_\alpha U_\alpha & \coprod_{\alpha\beta} U_{\alpha\beta} &
\end{array}$$

  We see everything we did previously works, as:

$$(\delta\sigma^{0,1})_{\alpha\beta} = \frac{d\theta_\beta}{2\pi} - \frac{d\theta_\alpha}{2\pi} = d\left(\frac{\pi^*\varphi_{\alpha\beta}}{2\pi}\right) = d\sigma_{\alpha\beta}^{1,0}$$

  - Note that $-\pi^*\varepsilon$ satisfies:

$$(-\pi^*\varepsilon)_{\alpha\beta\gamma} = \delta\left(\frac{\pi^*\varphi}{2\pi}\right)_{\alpha\beta\gamma}$$

$$\Rightarrow \varepsilon_{\alpha\beta\gamma} = \left(\frac{\delta\varphi}{2\pi}\right)_{\alpha\beta\gamma} = \frac{1}{2\pi}(\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta})$$

- Using the isomorphism $\check{H}^n \xrightarrow{\sim} H^n_{dR}$, we see the Euler class is:

$$[e(E)] = [(-D''K)^2\varepsilon] = [-dKdK\varepsilon]$$

  - Since $\delta K \pm K\delta = \mathrm{id}$, $K$ is almost an inverse of $\delta$, namely that $\varepsilon = -\frac{\delta\varphi_{\alpha\beta}}{2\pi}$ implies $(K\varepsilon)_{\alpha\beta} = -\varphi_{\alpha\beta}/2\pi + \delta\tau$. Then defining:

$$\xi_\alpha := -(KdK\varepsilon)_\alpha = K\frac{d\varphi_{\alpha\beta}}{2\pi} = \frac{1}{2\pi}\sum_\gamma \rho_\gamma \, d\varphi_{\gamma\alpha}$$

  We see the Euler class is:

$$[e(E)] = [d\xi]$$

  because $dKd\delta\tau = \delta d\alpha$ with $\alpha = Kd\tau \in \Omega^1(M)$ a global form, so $[dKd\delta\tau] = [d\delta\alpha] = 0$.
- The Euler class on a rank $n$ sphere bundle is thus defined by this diagram:

$$\begin{array}{c|cccccc}
\Omega^n & \alpha_0 & & & & & \\
\Omega^{n-1} & & \alpha_1 & \ddots & & & \\
\vdots & & & & & & \\
\Omega^1 & & & & \alpha_{n-1} & & \\
\Omega^0 & & & & & \alpha_n \to & \delta\alpha_n \\
 & & & & & & \uparrow i \\
\mathbb{R} & & & & & & -\pi^*\varepsilon \\
\hline
 & C^0 & C^1 & \ldots & C^{n-1} & C^n & C^{n+1}
\end{array}$$

$\alpha^0 = \sigma^{0,n}$ = orientation form in the orientation class of $E$.

$$[e(M)] = [(-D''K)^{n+1}\varepsilon] \in H^{n+1}(M)$$

for an $S^n$-bundle / rank $n+1$ vector bundle

• **Global angular form**: If $\alpha = \sum_{i=0}^{n} \alpha_i \in [C^*(\pi^{-1}\mathcal{U}, \Omega^*)]^n$, then:

$$D\alpha = -\delta\pi^*\varepsilon \qquad\qquad D = \delta + (-1)^p d$$

since at every level $\delta\alpha_i = -D''\alpha_{i+1}$. Recall the Euler class is:

$$e(M) = (-D''K)^n\varepsilon$$

$K$ is a chain map and commutes w/ $\pi^*$. We can use the collating formula to construct a global form on $E$ from $\alpha$, our $D$-cocycle:

$$\psi := \sum_{i=0}^{n}(-D''K)^i\alpha_i + (-1)^{n+1}K(D''K)^n(-\pi^*\varepsilon)$$

This form $\psi \in H^n(E)$ is a global form which satisfies:

$$d\psi = -\pi^* e$$

and $\psi$ is called the global angular form. When restricted to each fiber, $D''K(-\pi^*\varepsilon)$ vanishes since $K\pi^*\varepsilon$ is const on each fiber, and b/c $(D''K)^i\alpha_i$ is $d$-exact on each fiber:

$$[\psi|_{F_x}] \text{ generates } H^n(F_x) = H^n(S^n)$$

Ex: $S^3$ as an $S^1$ bundle over $S^2$.

– Again, we see if $e(E)$ vanishes, $\psi$ is closed and Leray-Hirsch applies.

• Let $E \to M$ be an oriented $(k-1)$ sphere bundle over a compact $k$-manifold $M$. Note $H^k(M) \cong H^k_c(M) \cong H^0(M)^* \cong \mathbb{R}$. We define the Euler number of $E$ to be:

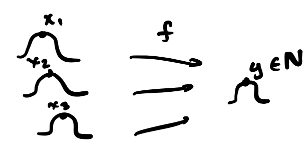
$$\int_M e(E)$$

For a general manifold $M$ its Euler number is defined as the Euler number for $E = S(TM)$.

– If the structure group of such a sphere bundle is $O(k)$, we can take $E$ to be the unit sphere bundle $S(E')$ for a vector bundle $E' \to M$.

• Let $E \to M$ be a sphere bundle as above. Then $E$ has a section over $M \setminus \{x_1, \dots, x_q\}$ for a finite set $\{x_1, \dots, x_q\}$, called a partial section.

• Recall the degree of a proper map $f: N \to M$ is defined as:




$$\deg(f) := \int_M f^*\alpha$$

where $\alpha \in H^n_c(M)$ generates the top cohomology. The degree essentially tells you how many copies of a point the pullback makes.

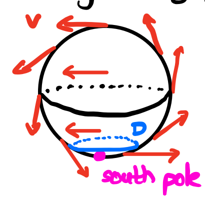
• For the sphere bundle above and for any partial section over $M \setminus \{x_1, \dots, x_q\}$, the Euler number is the sum of local degrees of $s$ at $\{x_1, \dots, x_q\}$.

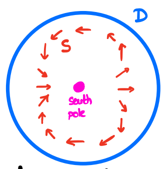
– Given a finite set $\{x_1, \dots, x_q\} \subset M$, we can always take the Euler class to have support away from each $x_i$.

• For a section $s: M \to E$ of a vector bundle, a zero $x$ of $s$ has multiplicity defined as the local degree of the partial section $s/\|s\|: M \to S(E)$ at $x$.

• For such a section w/ a finite number of zeros $\{x_1, \dots, x_q\}$ the Euler class is the Poincaré dual of the $\{x_1, \dots, x_q\}$, counted w/ their multiplicities.

- We are typically concerned w/ finding sections of sphere bundles. Let $E \to M$ be a vector bundle w/ unit sphere bundle $S(E)$. Then a global section of $S(E)$ is exactly a nowhere zero section of $E$.
  - This lets us translate info about the Euler class to vector bundles: if there is a nowhere zero section of a vector bundle, it must have vanishing $e(E)$.
- The Euler class of $TS^2$: A nonzero partial section of $S^2 \setminus \{\text{south pole}\}$ is given by parallel transporting a vector $v \in T_N S^2 \setminus \{\vec{0}\}$. Locally at a disc $D$ around the south pole, the partial section looks like:




This partial section winds twice hence has degree 2 (the zero of the section $S^2 \to TS^2$ has multiplicity 2) so the Euler number is:

$$\int_{S^2} e(TS^2) = 2$$

- For an even dimensional orientable sphere bundle $E \to M$ with fibers $F_x \cong S^{2n}$:

$$e(E) = 0$$

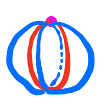
hence:

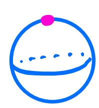
$$H^*(E) \cong H^*(M) \otimes H^*(S^{2n})$$

- Euler characteristic: For a CW complex $M$ with $k_n$ $n$-cells, the Euler characteristic is:

$$\chi(M) = \sum_{n=0}^{\infty} (-1)^n k_n$$

  - This is independent of CW structure. Ex: $M = S^2$, can take:

 $k_0 = 1$, $k_1 = 1$, $k_2 = 2$ $\Rightarrow \chi(S^2) = 2$

 $k_0 = 1$, $k_1 = 0$, $k_2 = 1$ $\Rightarrow \chi(S^2) = 2$

  - Even without a CW structure, the Euler characteristic is defined as the alternating sum of Betti numbers $b_q := \dim H^q(M)$:

$$\chi(M) = \sum_{q=0}^{\infty} (-1)^q \dim H^q(M)$$

- Let $M$ be compact, w/ basis $\{\omega_i\}$ for $H^*(M)$ and dual basis $\{\tau_j\}$ for $H^*(M)$, $\int_M \omega_i \wedge \tau_j = \delta_{ij}$. We may write the Poincaré dual of the diagonal $\Delta = \operatorname{im}(x \mapsto (x,x)) \subset M \times M$ as $\eta_\Delta = \sum_i (-1)^{\deg(\omega_i)} \pi^* \omega_i \wedge \rho^* \tau_i$, and by regarding $\eta_\Delta$ and $\Phi(N_\Delta)$ (the Thom class of the normal bundle to $\Delta$) as equal, we get:

$$\int_M e(TM) = \chi(M)$$

for any compact oriented manifold.

- Theorem (Hopf index): For a vector field $V \in \mathfrak{X}(M)$ with isolated zeros, the index of $V$ at a zero $u \in M$ is the local degree of $V/\|V\|$ as a section of $S(TM)$. The sum of the indices of a vector field $V$ on $M$ equals $\chi(M)$.
  - Note the degree is equivalently an amount of "wrapping" of a map.
- Thom isomorphism: We have a Thom isomorphism for vector bundles which are not orientable. Let $\mathcal{H}^q_{cv}$ be the presheaf on $M$:

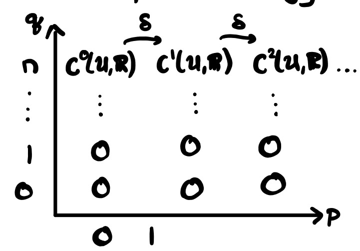
$$\mathcal{H}^q_{cv}(U) := H^q_{cv}(\pi^{-1}U)$$

  - If $U$ is contractible (if we look @ $U \in \mathcal{U}$, a good cover) then $\mathcal{H}^q_{cv}(U) = \begin{cases} \mathbb{R} & q=n \\ 0 & \text{else} \end{cases}$
  - The $d$-cohomology of $C^*(\pi^{-1}\mathcal{U}, \Omega^*_{cv})$ can be written:

$$H^{p,q}_d(C^*(\pi^{-1}\mathcal{U}, \Omega^*_{cv})) = C^p(\mathcal{U}, \mathcal{H}^q_{cv})$$

  - For a fixed $q$, $H^{*,q}_d(C^*(\pi^{-1}\mathcal{U}, \Omega^*_{cv})) = C^*(\mathcal{U}, \mathcal{H}^q_{cv})$ is a cochain complex in $p$, and the $p$-cohomology of this is written as:

$$H^{p,q}_\delta H_d := H^p(H^{*,q}_d(C^*(\pi^{-1}\mathcal{U}, \Omega^*_{cv}))) = H^p(C^*(\mathcal{U}, \mathcal{H}^q_{cv}))$$




    - Since $\mathcal{H}^q_{cv}$ vanishes unless $q=n$, $H^{p,q}_\delta H_d$ is only nonzero in the row w/ $q=n$.
  - Lemma: For any double complex, if $H_\delta H_d$ vanishes along all but one row or column, then $H^*_D \cong \bigoplus_{p+q=*} H^{p,q}_\delta H_d$
  - This implies the general Thom isomorphism:

$$H^*_{cv}(E) \cong H^{*-n}(\mathcal{U}, \mathcal{H}^n_{cv})$$

    where $\mathcal{U}$ is a good cover of $M$.
- We can use this construction to prove generally that for an oriented vector bundle, the Thom class is:

$$\Phi = d(\rho(r)\psi) \in \Omega^n_{cv}(E)$$

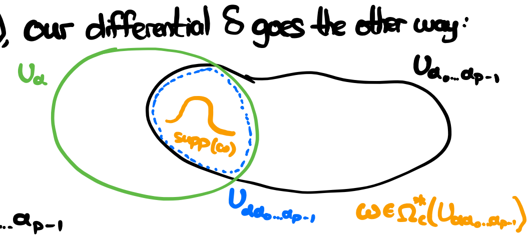
  where $\psi$ is the pullback of the global angular form $\psi_S$ on the bundle $S(E)$ to $E^\circ$.
  - The Euler class is the pullback of $\Phi$ under the zero section $s: M \to E$.
  - This means the Euler class respects direct sums:

$$e(E \oplus F) = e(E) \wedge e(F)$$

- Čech cohomology for compactly supported forms: For $\Omega^*_c(U_{\alpha_0 \ldots \alpha_p})$, our differential $\delta$ goes the other way:

$$\delta : \bigoplus_{\alpha_0 < \ldots < \alpha_p} \Omega^*_c(U_{\alpha_0 \ldots \alpha_p}) \to \bigoplus_{\alpha_0 < \ldots < \alpha_{p-1}} \Omega^*_c(U_{\alpha_0 \ldots \alpha_{p-1}})$$

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p-1}} := \sum_\alpha \omega_{\alpha\alpha_0 \ldots \alpha_{p-1}}$$




  where $\omega_{\alpha\alpha_0 \ldots \alpha_{p-1}}$ is the extension by zero of itself to $U_{\alpha_0 \ldots \alpha_{p-1}}$
  - This forms a Mayer-Vietoris generalization, which is exact:

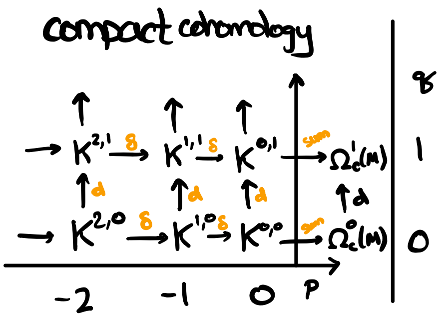
$$0 \leftarrow \Omega^*_c(M) \xleftarrow{\text{sum}} \bigoplus_{\alpha_0} \Omega^*_c(U_{\alpha_0}) \xleftarrow{\delta} \ldots \xleftarrow{\delta} \bigoplus_{\alpha_0 < \ldots < \alpha_p} \Omega^*_c(U_{\alpha_0 \ldots \alpha_p}) \leftarrow \ldots$$

for $\mathcal{U}=\{U_\alpha\}$ any locally finite open cover.

- This allows us to form a double complex w/ $p$ negative:

  $$K^{-p,q} := C^p(\mathcal{U}, \Omega^q_*)$$

  - By studying this complex, we get a more general version of Poincaré duality.




• Poincaré duality: $M$ n-dimensional, $\mathcal{U}$ a locally finite ($\forall\alpha, \exists$ finite $\beta$ w/ $U_\alpha \cap U_\beta \neq \emptyset$) good cover of $M$. Then:

$$H^*_c(M) \cong H_{*-n}(\mathcal{U}, \mathcal{H}^n_c)$$

where $\check{H}_*$ is the Čech homology and $\mathcal{H}^n_c$ is the functor assigning to each $U$ $\mathcal{H}^n_c(U) = H^n_c(U)$ and to $\iota: U \hookrightarrow V$ the extension by $0$ $\iota_*$.

# *Spectral Sequences*

- An exact couple is an exact sequence of Abelian groups:

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd} \qquad d: B \to B, \quad d := j \circ k$$

w/ this dfn, $d^2 = 0$, so we can define its homology $H(B) = \ker(d)/\mathrm{im}(d)$. From an exact couple we can define a new exact couple, called the derived couple, as follows:

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

i) $A' := i(A)$
ii) $B' := H(B)$
iii) $i' := i|_{A'}$
iv) For $a' := ia \in A'$, $j'a' := [ja]$
v) For $[b] \in B' = H(B)$: $k'[b] := kb$

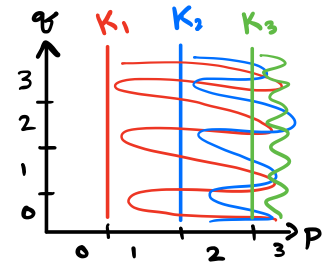
- A subcomplex of a cochain complex $(K, D)$ is a subgroup $K' \subset K$ s.t. $DK' \subset K'$. A sequence of subcomplexes $\{K_p\}_{p \in \mathbb{Z}}$ s.t. $K_{-q} = K$ for $q \geq 0$ and:

(abelian group)

$K = K_0 \supset K_1 \supset K_2 \supset \dots$

is a filtration on $K$. The associated graded complex is:

$GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$

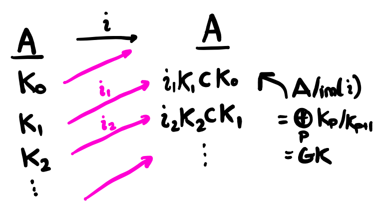



– Ex: For double complex $K^{**}$ w/ $D = \delta + (-1)^p d$, a filtration of $(K, D)$ is:

$K_p := \bigoplus_{i \geq p} \bigoplus_q K^{i,q}$

- Given a filtration $K = K_0 \supset K_1 \supset K_2 \supset \dots$, define:

$A := \bigoplus_p K_p$




$A$ comes equipped w/ a map $i: A \to A$ by summing each inclusion $i_{p+1}: K_{p+1} \to K_p$.
We define $B$ to be the quotient $B := A/\mathrm{im}(i)$, so:

$0 \to A \xrightarrow{i} A \xrightarrow{j} B \to 0$

is exact, w/ $j = \pi: A \to B$. Note that $B = GK$, since @ each step in the filtration $(A/\mathrm{im}(i))_p = K_p/K_{p+1}$. Then this induces a LES in cohomology (in the graded case, $A$ is $A^*$ and $H(A)$ is $H^*(A)$):

$\dots \to H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \to \dots$

where here $i_1: H(A) \to H(A)$, $j_1: H(A) \to H(B)$, and $k: H(B) \to H(A)$ are chain maps, and we have an exact couple:

$$\begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd} := \begin{tikzcd} H(A) \arrow[rr, "i_1"] & & H(A) \arrow[dl, "j_1"] \\ & H(B) \arrow[ul, "k_1"] & \end{tikzcd}$$

– We can then consider the sequence of derived couples to this exact couple:

$$\begin{tikzcd} A_r \arrow[rr, "i_r"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd}$$

where $(A_{r+1}, B_{r+1})$ is derived from $(A_r, B_r)$

– Note $H(A)$ is the sum $\bigoplus_p H(K_p)$ and the induced $i_1$ maps $H(K_{p+1}) \hookrightarrow H(K_p)$ for $p \geq 0$ and is id for $p < 0$.

We also have:

$A_{r+1} = i(A_r)$ $\qquad$ $B_{r+1} = H(B_r) =: E_{r+1}$, $E_1 = H(B)$

for each $r$ by the dfn of the derived couple.

- S'pose the filtration stops at some finite value $q$ (i.e. $K_q \neq \emptyset = K_{q+1} = K_{q+2} = \dots$), $\{A_r\}$ will become stationary for $r > q$:

$$A_1 = \bigoplus (H(K) \leftarrow H(K_1) \leftarrow H(K_2) \leftarrow \dots \leftarrow H(K_q) \leftarrow 0)$$
$$A_2 = \bigoplus (H(K) \supset iH(K_1) \leftarrow iH(K_2) \leftarrow \dots \leftarrow iH(K_q) \leftarrow 0) = i(A_1)$$
$$A_3 = \bigoplus (H(K) \supset iH(K_1) \supset iiH(K_2) \leftarrow \dots \leftarrow iiH(K_q) \leftarrow 0) = i(A_2)$$

$iH(K_1) \subset H(K)$ and $i: H(K) \to H(K)$ is id, moving it up by $\Delta p = 1$

$\vdots$

$$A_{q+1} = \bigoplus (H(K) \supset iH(K_1) \supset iiH(K_2) \supset \dots \supset i^q H(K_q) \leftarrow 0) = i(A_{q-1})$$
$$A_{q+2} = A_{q+1}$$

Filtration of $H(K)$

$iiH(K_1)$ is the inclusion of $H(K_1)$ into $H(K)$, as is $iH(K_1)$. Thus $iH(K_1) = ii\,H(K_1) \subset H(K)$, and likewise for $r \geq p$, $i^r H(K_p) = \underbrace{i^p H(K_p)}_{\text{filtration } F_p \text{ of } H(K)}$.

- We define this stationary value to be $A_\infty := A_{q+1} = A_{q+2} = \dots$
- Looking @ the derived couple, for $r > q$, $i_r : A_r \to A_r$ is the identity cycle, so by exactness $k_r$ must be the $0$ map, $\therefore$ $d : B_r \to B_r$ is $0$, thus $B_{r+1} = H(B) = B_r =: B_\infty$ is stationary.

$\operatorname{im}(k_r) = \ker(i) = 0$ $\Rightarrow k_r = 0$.

$$\begin{tikzcd} A_r \arrow[rr, "i"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd}$$

$\Rightarrow$ for $r > q$, $A_r = A_\infty$ and $B_r = B_\infty$

- We define the image of each $H(K_p)$ in $K$ to be $F_p := i^p H(K_p)$, so $A_\infty = \bigoplus_p F_p$. Then $B_r = A_\infty / \operatorname{im}(i)$, so $B_\infty = \bigoplus_p F_p / F_{p+1}$.
- The filtration:

$$H(K) := F_0 \supset F_1 \supset F_2 \supset \dots$$

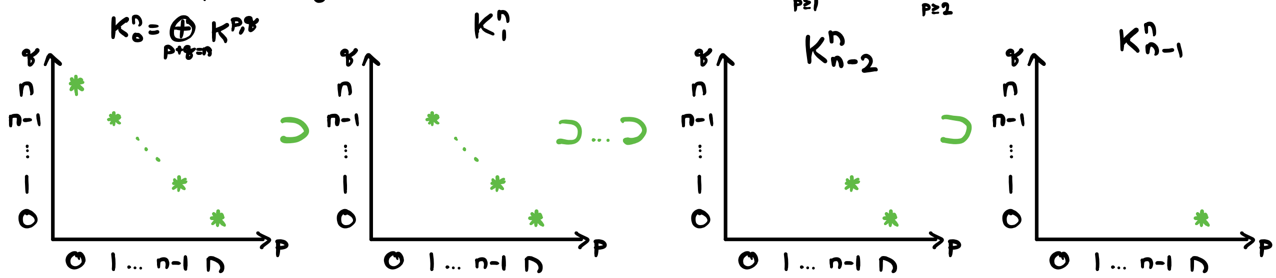
is called the induced filtration on $H(K)$, and we see $B_\infty = GH(K)$ w/ this filtration.

- We will usually write $E_r$ for $B_r$.

- A spectral sequence is a sequence of cochain complexes $\{(E_r, d_r)\}$ s.t. each $E_r$ is the cohomology of its predecessor, $E_{r+1} = H(E_r)$.
  - If $\{(E_r, d_r)\}$ becomes stationary w/ value $(E_\infty, d_\infty)$ and if $E_\infty$ is the graded complex of some group $H$, $E_\infty = GH$, we say $\{(E_r, d_r)\}$ converges to $H$.
  - The idea of spectral sequence is to approximate the cohomology of $E_\infty$ w/ the other parts of the sequence

- Suppose $K$ is graded w/ grading $K = \bigoplus_{n \in \mathbb{Z}} K^n$ ($K^n$ has dimension $n$). A filtration $K_p$ induces a filtration $K^n = K_0^n \supset K_1^n \supset K_2^n \supset \dots$ of $K^n$, where $K_p^n := K^n \cap K_p$.
  - For the double complex, we get a filtration of $K^n := \bigoplus_{p+q=n} K^{p,q} \supset \bigoplus_{\substack{p+q=n \\ p \geq 1}} K^{p,q} \supset \bigoplus_{\substack{p+q=n \\ p \geq 2}} K^{p,q} \supset \dots$

$K^{p,q} = C^p(\mathfrak{U}, \Omega^q) = \prod_{\alpha_0 < \dots < \alpha_p} \Omega^q(U_{\alpha_0 \dots \alpha_p})$



• **Theorem**: $\{K_p^n\}$ a graded filtered complex w/ differential $D$ s.t. for each dimension $n$, the filtration $K_0^n \supset K_1^n \supset K_2^n \supset \ldots$ has finite length. Then the SES:

$$0 \to \bigoplus K_{p+1} \xrightarrow{i} \bigoplus K_p \to GK \to 0$$

induces a spectral sequence which converges to $H_D^*(K)$, i.e. $H_D(K) = E_\infty$.

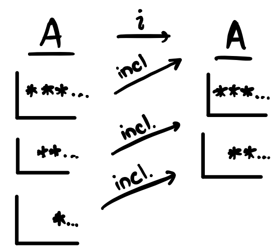
– This induces the spectral sequence from the exact couple (LES in cohomology):

$$\ldots \to \bigoplus_p H^k(K_{p+1}) \xrightarrow{i_1} \bigoplus_p H^k(K_p) \to \bigoplus_p H^k(K_{p+1}/K_p) \to \bigoplus_p H^{k+1}(K_{p+1}) \to \ldots$$

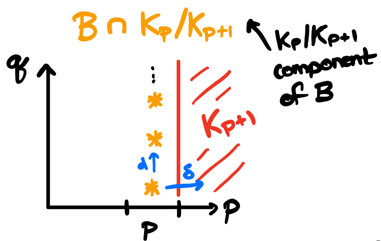
$$\Longrightarrow \begin{tikzcd} \bigoplus H(K_{p+1}) \arrow[rr, "i_1"] & & \bigoplus H(K_p) \arrow[dl, "j_1"] \\ & \bigoplus H(K_{p+1}/K_p) \arrow[ul, "k_1"] & \end{tikzcd} =: \begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd} \qquad A_r = \bigoplus_p i^{r-1} H(K_p), \quad E_r = H(E^{r-1}),\ E_1 = H(GK)$$

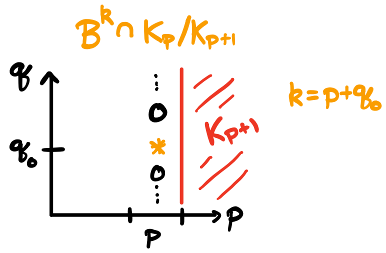
– Note $k$ increases the dimension by 1.

• **The double complex $K^{p,q}$**: Take the filtration $K = K_0 \supset K_1 \supset K_2 \supset \ldots$ as before, w/ $K_p = \bigoplus_{q,\, i \geq p} K^{i,q}$ and the grading $K^n = \bigoplus_{p+q=n} K^{p,q}$. Then $A^n = \bigoplus_p K_p^n$ and $B = A/\mathrm{im}(i) = \bigoplus_p K_p/K_{p+1} = GK$.




Each $K_p/K_{p+1}$ is a vertical strip and $D$ acts on $K_p/K_{p+1}$ as $(-1)^p d$.

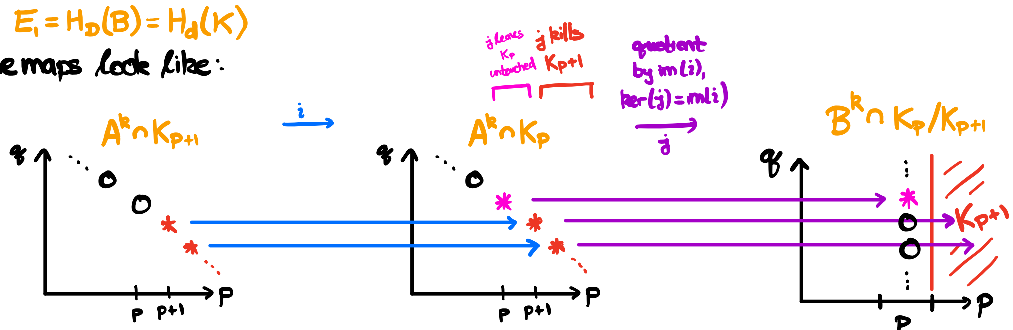



– Thus $(B^*, D) = (B^*, D'')$, where $D'' = (-1)^p d$, so:

$E_1 = H_D(B) = H_d(K)$

– The maps look like:




• To determine the associated spectral sequence, we must determine $k_1: H_D(B) \to H_D(A)$. This is the associated snake map induced from $0 \to A^* \xrightarrow{i} A^* \xrightarrow{j} B^* \to 0$. We can look at what $k$ does to each degree $p$ of the filtration: $0 \to A^* \cap K_{p+1} \xrightarrow{i} A^* \cap K_p \xrightarrow{j} B^* \cap (K_p/K_{p+1}) \to 0$ induces the map $k_1 : H^n(B) \to H^{n+1}(A)$:

$$\begin{tikzcd} & \cdots \ k_1[b] & \cdots \ Db & \cdots & \\ 0 \arrow[r] & A^{n+1} \cap K_{p+1} \arrow[r, "i"] \arrow[u] & A^{n+1} \cap K_p \arrow[r, "j"] \arrow[u] & B^{n+1} \cap (K_p/K_{p+1}) \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & A^n \cap K_{p+1} \arrow[r, "i"] \arrow[u, "D"] & A^n \cap K_p \ b \arrow[r, "j"] \arrow[u, "D"] & B^n \cap (K_p/K_{p+1}) \ [b] \arrow[r] \arrow[u, "D"] \arrow[ull, "k_1"] & 0 \\ & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] & \end{tikzcd}$$

since determining the map on cohomology

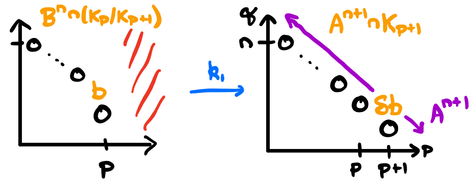
– Via the usual diagram chase, let $[b] \in H^n(B \cap (K_p/K_{p+1}))$, so $b \in B^n \cap K_p/K_{p+1}$ is a cocycle, $Db = 0 = (-1)^p db$. Then $Db$ (in $A^{n+1} \cap K_p$) equals $\delta b + (-1)^p \cancel{db}^{\,0} = \delta b$, so lifting $Db = \delta b$ via $i$ to $A^{n+1} \cap K_{p+1}$ gives

$k_1 : H^n(B \cap (K_p/K_{p+1})) \to H^{n+1}(A \cap K_{p+1})$

↑ note $i$ doesn't do anything to the form $\delta b$ but change where it lives: $\delta b \in A^{k+1} \cap K_{p+1}$, or $\delta b \in A^{k+1} \cap K_p$ $\xrightarrow{i(\cdot)}$

$k_1[b] = [\delta b]$

- Here the $\delta$ augments the dimension by 1, and pictorially is on the right. Since $E_1 = B_1 = H(B)$ and $d_1 = j_1 \circ k_1$, we have:




$$E_1 = H_d(K) \qquad d_1 = \delta$$

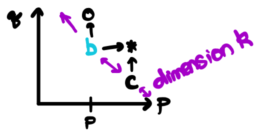
- Then $E_2$ is the cohomology of $(H_d(K), \delta)$, so:

$$E_2 = H_\delta H_d(K)$$

To compute $d_2$, we need $k_2 : E_2 = H(E_1) \to A_2 = i(A_1)$. An element of $H(E_1) = H_\delta H_d(K)$ is represented by a cocycle b w/:

$db = 0$

$\delta b = -D''c$




$$\begin{tikzcd} i(A_1) \arrow[rr, "i_2"] & & i(A_1) \arrow[dl, "j_2"] \\ & H_\delta H_d(K) \arrow[ul, "k_2"] & \end{tikzcd}$$

- We need to compute $d_2 = j_2 k_2$:

$$d_2[b_2] = j_2 k_2 [b]_2 = j_2(k_1[b]_1) \qquad \longleftarrow \quad (k_{r+1}[b]_{r+1} = k_r[b]_r \in i(A_r) \text{ by dfn})$$

where $[\cdot]_r$ denotes the class of $\cdot$ in $E_r$. Since for $a' = ia \in i(A_r) = A_{r+1}$, $j_{r+1}a' = j_r a$, we need to find $a \in A^{k+1} \cap K_{p+2}$ s.t $k_1[b]_1 = i[a]_1$. This is done by lifting $[b]_1$ via $j_1$ to $b+c$, since $c \in A^k \cap K_{p+1}$ is killed by $j_1$.

↑ increases grading by 1, and sends $[b]_1$ to $H(K_{p+1})$

↑ increases p by 1

- Upon taking $D(b+c)$ and pulling back via $i$, we get $D(b+c) = \underset{A^{k+1} \cap K_{p+1}}{\delta c} = \underset{A^{k+1} \cap K_{p+2}}{i(\delta c)}$, so evidently $a = \delta c$.

- Why do we lift the cocycles differently for $E_1$ and $E_2$, i.e. first lift $[b]$ to just $b$ and next lift $[b]$ to $b+c$?
  - First case: determine $d_1, k_1$ on $E_1$. $[b] \in E_1 = H_d(K)$, so the form $b \in A^k \cap K_p$ has $db = 0$, as $b$ represents a cocycle in $E_1$.
  - For the next case, b represents a cocycle in $E_2 = H(E_1) = H_\delta H_d(K)$. Thus in addition to $db = 0$, b is also extendable to the zig-zag:

$$\begin{tikzcd} 0 & \\ b \arrow[u] \arrow[r] & * \\ & c \arrow[u] \end{tikzcd}$$

  - $k_2[b]_2 = k_1[b]_1$ is determined by $k_1[b]_1 = i[a]_1$. In determining $k_1[b]_1$ before for $E_1$, we did not have this additional constraint on it living in $i(A_1) = iH(A)$, only that it was in $H(A)$. I would assume this extra condition is b/c we are looking in $E_2$ now, not $E_1$ — only certain elements of $E_1$ will remain $E_2$-cocycles.
  - If we lift $[b]$ to $b \in A^k \cap K_p$, the resulting $k_1[b_1]$ will not be in $i(A_1)$, hence will not help us to define $k_2$, and we see this b/c $\delta b \in A^{k+1} \cap (K_{p+1} \setminus K_{p+2})$, so we can't write $\delta b = ia$ with $a \in K_{p+2}$. Thus we need more: we lift $[b]$ to $b+c$ instead, b/c although it must generate the same image in $E_1$, i.e. $[\delta b]_1 = [\delta c]_1$ independent of lift, only $\delta c$ will give us a valid dfn for $k_2$.

• An important point: For different $(E_r, d_r)$, the forms we study are the same — they live in $K^{p,q}$. However for different r, the forms which are in $E_r = H(E_{r-1})$ change: a form b represents an element $[b]_r$ of $E_r$ iff it is in $H(E_{r-1})$, i.e. iff $d_{r-1}b = 0$.

- A form $[b]_r \in E_r$ also represents an element of $E_k$ for $k < r$, as in the example for $k=2$; we can look at the same form b in each different $E_k$ as a different cohomology class $[b]_k \in E_k$ for each k.
  - This is exactly the statement that if $d_r b = 0$, $d_k b = 0$ for $k < r$. This holds b/c $E_r = H(E_{r-1}) = \ldots = HH\ldots H(E_1)$, so at each step $d_k b = 0$ for b to represent $[b]_k \in \underbrace{H\ldots H}_{k\text{-times}}(E_1)$.

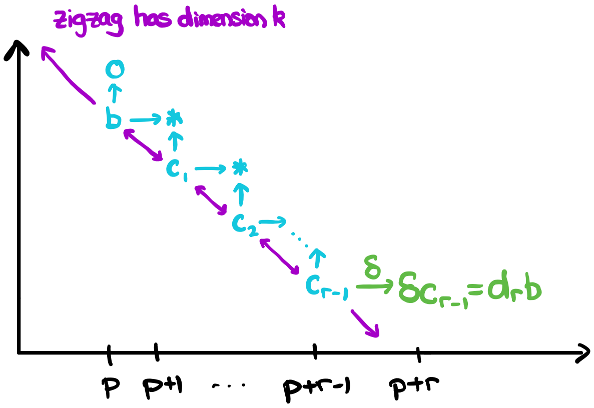
• In general for the double complex:

– An element $b \in K$ lives in $E_r$ if it represents a cohomology class in $E_r = H(E_{r-1})$. $b$ lives in $E_r$ iff $b$ is a cocycle in $E_{r-1}, E_{r-2}, \dots, E_1$, equivalently:

$$d_{r-1} b = 0$$

(this implies $d_k b = 0$ for $r < k-1$)

– For $r=1$ and $2$, we saw $b$ lives in $E_r$ iff it can be extended to a zigzag, and $d_r b$ is $\delta$(lowest part of the zigzag). This holds generally: $b$ lives in $E_r$ iff it can be extended to a zigzag of length $r$:




$b \in A^k \cap K_p$

$c_i \in A^k \cap K_{p+i}$

$d_r b = \delta c_{r-1} \in A^{k+1} \cap K_{p+r}$

The differential on an element $[b]_r \in E_r$ is the $\delta$ of the tail of its zigzag:

$$d_r [b]_r = [\delta c_{r-1}]_r$$

• Ex: Let us determine what a form $b \in K^{p,q}$ must look like if it represents a cohomology class $[b]_3$ in $E_3$.

– Lemma: Iff $[a]_2 = 0$ in $H_\delta H_d(K)$, then $\exists a'$ s.t. $a = da'$.

For $b$ to live in $E_3$, it must live in $E_2$, so we can write it as a 2-zigzag:

$$\begin{array}{ll} 0 & \\ \uparrow & \\ b \longrightarrow & * \\ & \uparrow \\ & c_1 \end{array}$$

Since $E_3 = H(E_2)$, $b$ lives in $E_3 = H(E_2)$ iff $d_2[b_2] = 0$. Since $d_2[b_2] = [\delta c_1]_2 = 0$ and $E_2 = H_\delta H_d(K)$, the lemma implies we can lift $\delta c_1$ to $c_2$ with D", i.e. we can extend $b$ to a 3-zigzag:

$$\begin{array}{lll} 0 & & \\ \uparrow & & \\ b \longrightarrow & * & \\ & \uparrow & \\ & c_1 \longrightarrow & * \\ & & \uparrow \\ & & c_2 \end{array}$$

By lifting $b$ w/ the snake map to $b + c_1 + c_2$, we get $D(b + c_1 + c_2) = \delta c_1 \in i(A_2) = ii\, H(K_{p+1}) = H(K_{p+3})$, so:

$$d_3[b]_3 = [c_2]_3$$

– For $b$ living in $E_3$, we claimed it lives in $E_1$ and $E_2$ as well. We have:

$[d_1 b]_1 = [\delta b]_1 = [dc_1]_1 = [dc_1]_d = 0$ $\quad$ ($E_1 = H_d(K)$, so $dc_1$ equals $0$ in $E_1$)

$[d_2 b]_2 = [\delta c_1]_2 = [dc_2]_2 = [dc_2]_{\delta d} = 0$ $\quad$ (by lemma)

• We let $E_r = \bigoplus_{p,q} E_r^{p,q}$ have the same bidegree as $K^{p,q}$ (so if $b$ lives in $E_r^{p,q}$, as a form $b \in K^{p,q}$) and $d_r$ is a map:

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r+1,q-r}$$

that maps the bidegree $(p,q) \longmapsto (p+r+1, q-r)$ and increases the grading dimension by 1.

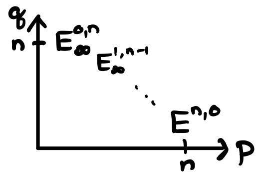
• For grading dimension $K^n_{;}$ the filtration induced by $K_{p,q}$ has length $n$ as seen in the picture above. This means:

$$A_\infty^n = \bigoplus \{H^n(K_0) \supset iH^n(K_1) \supset iiH^n(K_2) \supset \dots \supset i^{n-1}H^n(K_{n-1})\} \qquad F_p^n = i^p H^n(K_p)$$

Since $H^n(K) = (H^n \cap F_0) \supset (H^n \cap F_1) \supset \dots$, we define $E_\infty^{p,q} := H^{p+q} \cap (F_p / F_{p+1})$ such that:

$$E_\infty^n = \bigoplus_{p+q=n} E_\infty^{p,q} = A_\infty^n / \mathrm{im}(i)$$

– Note $E_\infty^n = GH_D^n(K)$, so $\{(E_r^n, d_r)\}$ converges to $H_D^n(K)$.




• Summary: For a double complex $K^{p,q}_{;}$ there is a spectral sequence $\{(E_r, d_r)\}$ with bigrading $E_r^{p,q}$ such that:

i) $d_r : E_r^{p,q} \longrightarrow E_r^{p+r+1,q-r}$

ii) $E_1^{p,q} = H_d^{p,q}(K)$

iii) $E_2^{p,q} = H_\delta^{p,q} H_d(K)$

• A spectral sequence is degenerate at $E_r$ if $d_r = d_{r+1} = \dots = 0$, and for a degenerate spectral sequence $E_r = E_{r+1} = \dots = E_\infty$.

• If $K$ is a vector space, then:

$$H_D^n(K) \cong GH_D^n(K) \cong E_\infty^n = \bigoplus_{p+q=n} E_\infty^{p,q}$$

Since $V \cong W$ iff $\dim V = \dim W$.

– This is not true for an arbitrary abelian group $K$. This is the extension problem: for the SES $0 \to A \to B \to C \to 0$, knowledge of $A$ and $C$ does not determine $B$.

• Leray's theorem: For a fiber bundle $\pi : E \to M$ and a good cover $\mathcal{U}$ of $M$, let $\mathcal{H}^q$ be the locally constant presheaf $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$ for each $U \in \mathcal{U}$. Then there is a spectral sequence $\{(E_r, d_r)\}$ converging to $H^*(E)$, so:

$$H^*(E) \cong \bigoplus_{p+q=*} E_\infty^{p,q}$$

such that:

i) $E_1^{p,q} = C^p(\mathcal{U}, \mathcal{H}^q)$

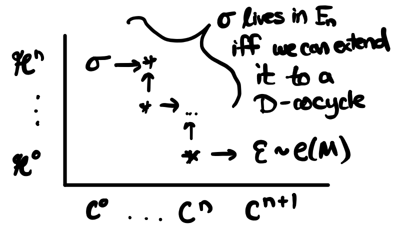
ii) $E_2^{p,q} = H_\delta(E_1) = \check{H}^p(\mathcal{U}, \mathcal{H}^q)$

Furthermore, if $M$ is simply connected and $H^q(F)$ is finite dimensional, then:

$$E_\infty^{p,q} = E_2^{p,q} = H^p(M) \otimes H^q(F)$$

• For a $S^n$-bundle $\pi : E \to M$, let $\mathcal{U}$ be a good cover of $M$ and $\{(E_r, d_r)\}$ the spectral sequence converging to $H^*(E)$ in Leray's thm, let $\sigma \in E_1^{0,n} = C^0(\mathcal{U}, \mathcal{H}^n(S^n))$ be a local angular form. Then $E$ is orientable iff there is a $\sigma$ which can be extended to a D-cocycle $\Longleftrightarrow$ global form, so such a form has $d_1[\sigma]_1 = [\delta\sigma]_1 = 0 \Rightarrow$ lives in $E_2$, ..., lives in $E_n$, and then the Euler class is:

$$e(M) = f(d_n[\sigma]_n) \in H^{n+1}(M)$$




where $f$ is the isomorphism $\check{H}^{n+1}(\mathcal{U},\mathcal{K}^0) \to H^{n+1}(M)$

- The **cup product**: The cup product is a pairing on the Čech complex:

$$\cup : C^p(\mathcal{U},\Omega^q) \otimes C^r(\mathcal{U},\Omega^s) \to C^{p+r}(\mathcal{U},\Omega^{q+s})$$

$$(\omega \cup \tau)\big|_{U_{\alpha_0 \dots \alpha_{q+s}}} := (-1)^{qr}\, \omega\big|_{U_{\alpha_0 \dots \alpha_q}} \wedge \tau\big|_{U_{\alpha_{q+1} \dots \alpha_{q+r}}}$$

This is an antiderivation on the double complex $(C^*(\mathcal{U},\Omega^*), D)$, i.e. $D(\omega \cup \tau) = (D\omega) \cup \tau + (-1)^{\deg(\omega)}\omega \cup D\tau$, where $\deg(\omega)$ is the bidegree $p+q$.

- Restricted to $\check{C}^p(\mathcal{U},\mathbb{R})$, $\cup$ induces an antiderivation w.r.t. the Čech complex $(\check{C}^p(\mathcal{U},\mathbb{R}),\delta)$, so this is a graded algebra.

- The isomorphism $f : \check{H}^p(\mathcal{U},\mathbb{R}) \xrightarrow{\sim} H^p_{dR}(M)$ is an isomorphism of algebras

$$(\check{H}^*(\mathcal{U},\mathbb{R}), \delta, \cup) \xleftrightarrow{\sim} (H^*_{dR}(M), d, \wedge)$$

- **Simplicial homology**: For a topological space $X$, we define (if $G = \mathbb{Z}$, we omit $G$):

$$\mathrm{Sing}_q(X) := \{\text{continuous maps } \Delta^q \to X\} \qquad (q\text{-simplices})$$

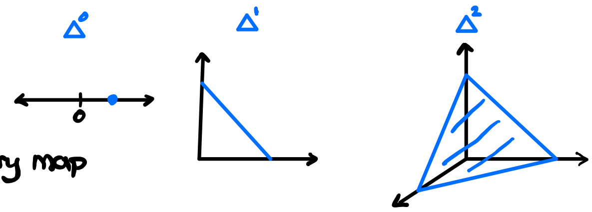
$$S_q(X;G) := G\,\mathrm{Sing}_q(X) \qquad (q\text{-chains})$$

where $\Delta^q \subset \mathbb{R}^{q+1}$ is the $q$-dimensional simplex $\Delta^q = \{\sum_{i=0}^{q} t_i e_i \in \mathbb{R}^{q+1} : \sum_{i=0}^{q} t_i = 1\}$ and $G$ is an abelian group.

- Define face maps which embed $\Delta^{q-1}$ into $\Delta^q$:

$$\partial_q^i : \Delta^{q-1} \to \Delta^q$$

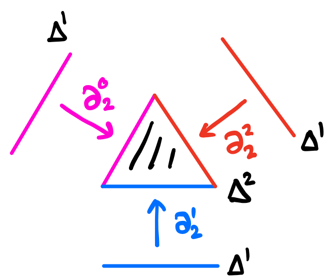
$$\partial_q^i\Big(\sum_{i=0}^{q-1} t_j e_j\Big) := \sum_{j=0}^{i-1} t_j e_j + \sum_{j=i+1}^{q} t_j e_j$$




for $i \in \{0,\dots,q\}$. The alternating sum defines a boundary map when extended by linearity to the singular chains:

$$\partial : S_q(X;G) \to S_{q-1}(X;G)$$

$$\partial\sigma := \sum_{i=0}^{q} (-1)^i \sigma \circ \partial_q^i$$




where $\sigma \in \mathrm{Sin}_q(X;G)$ are $q$-simplices $\sigma_i : \Delta^q \to X$ and $\partial$ naturally extends by linearity. The homology of the chain complex:

$$\dots \xrightarrow{\partial} S_{q+1}(X;G) \xrightarrow{\partial} S_q(X;G) \xrightarrow{\partial} S_{q-1}(X;G) \to \dots$$

is called the simplicial homology of $X$ with values in $G$ and denoted:

$$H_*(X;G) = \ker(\partial : S_q \to S_{q-1}) / \mathrm{im}(S_{q+1} \to S_q)$$

- The rank (recall for an abelian group $A = \mathbb{Z}^r \oplus (\text{torsion})$ the rank is $r$) of $H_0(X;\mathbb{Z})$ is the number of path components of $X$:

$$\mathrm{rank}(H_0(X;\mathbb{Z})) = \#\text{ path components of } X$$

- The simplicial homology of $\mathbb{R}^n$ is:

$$H_q(\mathbb{R}^n) = \begin{cases} 0 & q > 0 \\ \mathbb{Z} & q = 0 \end{cases}$$

• **Mayer–Vietoris for singular chains**: Define a Čech boundary operator:

$$\delta : \bigoplus_{\alpha_0<\dots<\alpha_p} S_q(U_{\alpha_0\dots\alpha_p}) \longrightarrow \bigoplus_{\alpha_0<\dots<\alpha_{p-1}} S_q(U_{\alpha_0\dots\alpha_{p-1}})$$

$$(\delta\sigma)_{\alpha_0\dots\alpha_{p-1}} := \sum_{\alpha} \sigma_{\alpha\alpha_0\dots\alpha_{p-1}}$$

where $\sigma_{\dots\alpha_i\dots\alpha_j\dots} = -\sigma_{\dots\alpha_j\dots\alpha_i\dots}$. Let $\mathcal{U}$ be a cover of $X$, and let $S_q^{\mathcal{U}}(X)$ be the set of $\mathcal{U}$-small $q$-chains on $X$, which is the subset of $S_q(X)$ generated by $q$-simplices $\Delta^q \xrightarrow{\sigma} X$ s.t. $\sigma(\Delta^q) \subset U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Note by subdivision, $H_*(S_*^{\mathcal{U}}(X)) \cong H_*(S_*(X))$. Let $\varepsilon : \bigoplus_{\alpha} S_q(U_\alpha) \to S_q^{\mathcal{U}}(X)$ be the sum $\varepsilon\sigma = \sum_\alpha \sigma_\alpha \in S_q^{\mathcal{U}}(X)$. Then this sequence is exact:

$$0 \leftarrow S_q^{\mathcal{U}}(X) \xleftarrow{\varepsilon} \bigoplus_{\alpha_0} S_q(U_{\alpha_0}) \xleftarrow{\delta} \dots \xleftarrow{\delta} \bigoplus_{\alpha_0<\dots<\alpha_p} S_q(U_{\alpha_0\dots\alpha_p}) \xleftarrow{\delta} \dots$$

– Corollary: For an open cover $\{U,V\}$ of $X$, there is a long exact sequence in homology:

$$\dots \to H_q(U\cap V) \xrightarrow{f} H_q(U)\oplus H_q(V) \xrightarrow{g} H_q(U\cup V) \xrightarrow{\text{snake}} H_{q-1}(U\cap V) \to \dots$$

induced by the SES $0 \leftarrow S_*^{\mathcal{U}}(X) \leftarrow S_*(U)\oplus S_*(V) \leftarrow S_*(U\cap V) \leftarrow 0$. Here $f$ is induced by $a \mapsto (a,-a) \in S_*(U)\oplus S_*(V)$ and $g$ is induced by $(a,b) \xmapsto{\varepsilon} a+b$.

• A **singular cochain** on $X$ is a functional $S_q(X;G) \longrightarrow G$, denoted by:

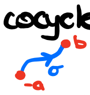
$$S^q(X;G) := \mathrm{Hom}_G(S_q(X;G), G)$$

The boundary operator $\partial : S_{q+1} \to S_q$ induces a coboundary operator on $S^q$:

$$d : S^q(X;G) \to S^{q+1}(X;G)$$

$$d(\omega)(\sigma) := \omega(\partial\sigma)$$

for $\sigma \in S_{q+1}(X;G)$. This turns $(S^*(X;G), d)$ **into a cochain complex**.

– Suppose $\omega \in S^0(X)$ is a $0$-cocycle. Then $d\omega = 0$, so $\omega(\partial\sigma) = 0$ for each $\sigma \in S_1(X)$, i.e. $\omega$ sends the boundary of a $1$-chain  to $0 \Rightarrow \omega$ is constant on each path component of $X$, hence rank $H_0$ = rank $H^0$ = # path components of $X$.

– We can dualize the Mayer–Vietoris sequence in homology to get one in cohomology:

$$0 \to S_{\mathcal{U}}^*(X) \xrightarrow{\varepsilon^*} \bigoplus_{\alpha_0} S^*(U_{\alpha_0}) \xrightarrow{\delta^*} \dots \xrightarrow{\delta^*} \bigoplus_{\alpha_0<\dots<\alpha_p} S^*(U_{\alpha_0\dots\alpha_p}) \xrightarrow{\delta^*} \dots$$

which is exact. Here $\delta^*$ is the alternating difference:

$$(\delta^*\omega)_{\alpha_0\dots\alpha_p} = \sum_{i=0}^{p} (-1)^i \omega_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}$$

– Together with $d$, $\delta^*$ turns:

$$C^p(\mathcal{U}, S^q) := \bigoplus_{\alpha_0<\dots<\alpha_p} S^q(U_{\alpha_0\dots\alpha_p})$$

into a double graded complex, like the Čech–de Rham complex

$$\begin{array}{c|ccc}
q & \uparrow & & \uparrow \\
 & C^0(\mathcal{U},S^2) & \longrightarrow & C^1(\mathcal{U},S^2) \xrightarrow{\delta^*} \dots \\
 & d\uparrow & & d\uparrow \\
 & C^0(\mathcal{U},S^1) & \xrightarrow{\delta^*} & C^1(\mathcal{U},S^1) \xrightarrow{\delta^*} \dots \\
 & d\uparrow & & \uparrow d \\
S^0 & C^0(\mathcal{U},S^0) & \xrightarrow{\delta^*} & C^1(\mathcal{U},S^0) \xrightarrow{\delta^*} \dots \\
\hline
 & C^0 & & \to p
\end{array}$$

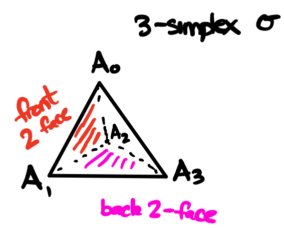
– Because Mayer–Vietoris is exact (note at the first slot $S_{\mathcal{U}}^*(X)$, the map $\varepsilon^*$ is injective; however, $\delta^* : S_{\mathcal{U}}^*(X) \to C^0(\mathcal{U},S^*)$ is not and has $\ker(\delta^*) = S_{\mathcal{U}}^*(X)$), $H_{\delta^*}(C^*(\mathcal{U},S^*))$ is only nonzero in column $0$, so:

$$H^*(C^*(\mathcal{U},S^*)) \cong H_d^*(H_\delta^*(C^*(\mathcal{U},S^*))) \cong H_{\text{sing}}^*(X)$$

- If $X$ is triangularizable, then the singular cohomology of $X$ is isomorphic to its Čech cohomology with $\mathbb{Z}$-coefficients. For a good cover $\mathcal{U}$:

$$H^*_{sing}(X) \cong \check{H}^*(\mathcal{U},\mathbb{Z}) \cong \check{H}^*(X,\mathbb{Z})$$

- On singular cohomology, there is a corresponding cup product: For a simplex $\sigma=(A_0 \dots A_q)$, its front $r$-face is $(A_0 \dots A_r)$ and its back $r$-face is $(A_{q-r} \dots A_q)$. For $\omega \in S^p(X)$ and $\eta \in S^r(X)$, we define $\omega \cup \eta \in S^{p+r}(X)$ by extending the following dfn through linearity:




$$(\omega \cup \eta)(A_0 \dots A_{p+r}) := \omega(A_0 \dots A_p)\,\eta(A_p \dots A_{p+r})$$

  - This is an antiderivation and makes $(S^*(X), d, \delta^*, \cup)$ a double complex with a graded algebra structure.
  - Note we can use $\cup$ if $G$ is a ring. Most of the theorems we proved previously for dR cohomology hold in singular cohomology if the cohomology is freely generated as a $G$-module, otherwise we must also include torsion.

- Leray's thm (Singular cohomology): $\pi: E \to X$ a fiber bundle, $A$ a CRng. Then there is a spectral sequence converging to $H^*(E;A)$ with $E_2$ term:

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \mathcal{H}^q(F;A))$$

where $\mathcal{H}^q(F,A)$ is the loc. const presheaf assigning $U \mapsto H^q(F;A)$. For $X$ simply connected:

$$E_2^{p,q} = \check{H}^p(X, H^q(F;A))$$

and if $H^*(F;A)$ is a free finitely generated $A$-module:

$$E_2^{p,q} = H^*(X;A) \otimes H^*(F;A)$$

- Ext and Tor: Let $A$ be abelian with $\{a_i\} \subset A$ a set of generators. Let $F := \langle a_i \rangle$ be the free group on these letters. Then for the natural map $p: F \to A$ with kernel $R$,

$$0 \to R \xrightarrow{i} F \xrightarrow{p} A \to 0$$

is exact. Note $R$ is also free, so this is a free resolution of $A$. For any $G$, the sequences:

$$0 \longrightarrow \mathrm{Hom}(A,G) \xrightarrow{p^*} \mathrm{Hom}(F,G) \xrightarrow{i^*} \mathrm{Hom}(R,G)$$

$$R \otimes G \xrightarrow{i \otimes 1} F \otimes G \xrightarrow{p \otimes 1} A \otimes G \longrightarrow 0$$

are exact b/c $\mathrm{Hom}(\cdot,G)$ and $\cdot \otimes G$ are right exact. We define:

$$\mathrm{Ext}(A,G) := \mathrm{coker}(i^*)$$
$$\mathrm{Tor}(A,G) := \ker(i \otimes 1)$$

- Ext measures the failure of $\mathrm{Hom}(\cdot,G)$ to be exact, and Tor does the same for $\cdot \otimes G$. If Ext vanishes, $\mathrm{Hom}(\cdot,G)$ is exact when applied to a free resolution of $A$, and likewise for Tor. For $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$:

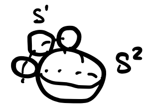
| $\mathrm{Ext}(\cdot,\cdot)$ | $\mathbb{Z}$ | $\mathbb{Z}/n\mathbb{Z}$ |
|---|---|---|
| $\mathbb{Z}$ | $0$ | $0$ |
| $\mathbb{Z}/m\mathbb{Z}$ | $\mathbb{Z}/m\mathbb{Z}$ | $\mathbb{Z}/\gcd(n,m)\mathbb{Z}$ |

| $\mathrm{Tor}(\cdot,\cdot)$ | $\mathbb{Z}$ | $\mathbb{Z}/n\mathbb{Z}$ |
|---|---|---|
| $\mathbb{Z}$ | $0$ | $0$ |
| $\mathbb{Z}/m\mathbb{Z}$ | $0$ | $\mathbb{Z}/\gcd(n,m)\mathbb{Z}$ |

- **Universal coefficient theorem**: For any space $X$ and group $G$:
  i) The homology of $X$ with $G$ coefficients splits as:
  $$H_q(X;G) = (H_q(X)\otimes G)\oplus \mathrm{Tor}(H_{q-1}(X),G)$$
  ii) The cohomology splits as:
  $$H^q(X;G) = \mathrm{Hom}(H_q(X),G)\oplus \mathrm{Ext}(H_{q-1}(G),G)$$
  – For $G=\mathbb{Z}$, this implies $H^q(X) = F_q\oplus T_{q-1}$, where $F_q$ is free and $T_{q-1}$ is torsion. When we study dR cohomology, we lose all info about the torsion piece.
- Ex: Cohomology of $S(TS^2)$. Since $\pi_1(S^2)=0$, Leray's thm implies:

$$E_2^{p,q} = H^p(S^2)\otimes H^q(S^1)$$

$$E_2:\quad \begin{array}{c|ccc} 1 & \mathbb{Z} & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \mathbb{Z} \\ \hline q\,/\,p & 0 & 1 & 2 \end{array}\qquad \mathbb{Z}^{0,1} \xrightarrow{d_2} \mathbb{Z}^{2,0}$$




and b/c $d_3=d_4=\ldots=0$ as we cannot extend zigzag, $E_\infty = E_3$.
  – For $\pi: E\to M$ an oriented $S^n$ bundle, let $\sigma$ be an angular form. Then b/c $\pi$ is orientable, $d_1\sigma=0$, so $\sigma$ can be chosen to be in $C^0(\mathcal{U},\mathcal{H}^n(S^n))$ and $[d_k\sigma]_k = 0$ for $k<n$, so:
  $$d_n\sigma_n = \pm e(M)$$
  by the construction of the Euler class in a previous section.
  – On $S(TS^2)$, this means $d_2[\alpha]=2[\alpha]$ since $[e(S^2)]=2[\sigma]$ for $[\sigma]$ a generator of $H^2(S^2)$.
  We have $E_3 = \ker(d_2)/\mathrm{im}(d_2)$, so:
  $$H^*(SO(3))\cong H^*(S(TS^2)) = \begin{cases}\mathbb{Z} & *=0,3\\ \mathbb{Z}/2\mathbb{Z} & *=2\\ 0 & \text{else}\end{cases}$$
  because $S(TS^2)\cong SO(3)\cong \mathbb{RP}^3$.
- **Cohomology of $\mathbb{CP}^n$**: Let $S^{2n+1} := \{(z_0,\ldots,z_n)\in\mathbb{R}^{n+1} : \sum_i |z_i|^2 = 1\}$ and act $S^1$ on $S^{2n+1}$ by multiplication, $\lambda\cdot z := (\lambda z_0,\ldots,\lambda z_n)$. $\mathbb{CP}^n$ is defined as the orbit space:
  $$\mathbb{CP}^n := S^{2n+1}/\sim$$
  where $\sim$ are the equivalence classes of the action. Note this is the extension of identifying antipodal points on the real n-sphere. $\mathbb{CP}^n$ can be shown to be simply connected and gives $S^{2n+1}$ the fibering:
  $$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{CP}^n \end{array}$$
  Since $\mathbb{CP}^n$ is the base and is simply connected, there is a spectral sequence conv. to $H^*(E)$ w/ $E := S^{2n+1}$ s.t.:
  $$E_2^{p,q} = H^p(S^1)\otimes H^q(\mathbb{CP}^n)$$
  We don't know $H^q(CP^n)$, but it is 4 dim over $\mathbb{R}$ as a manifold, and since $H^p(S^1)=0$ for $p>1$, we have:
  $$E_2 = \begin{array}{c|cccccc} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbb{R} & A & B & C & D & 0 \\ 0 & \mathbb{R} & A & B & C & D & 0 \\ \hline q\,/\,p & 0 & 1 & 2 & 3 & 4 & 5 \end{array}\qquad \xrightarrow{d_2}$$
  The two rows are equal b/c $H^0(S^1)=\mathbb{R}=H^1(S^1)$, and $\mathbb{R}\otimes_{\mathbb{R}} H^q(\mathbb{CP}^n) = H^q(\mathbb{CP}^n)$. Now specialize to $n=2$.
  Since $d_3, d_4,\ldots$ move an element 3 rows down, $d_3=d_4=\ldots=0$, hence $\{(E_r,d_r)\}$ degenerates at $E_3$, and so:
  $$E_3^* = E_\infty^* = H^*(S^5) = \begin{array}{c|cccccc} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \mathbb{R} & 0 \\ 0 & \mathbb{R} & 0 & 0 & 0 & 0 & 0 \\ \hline q\,/\,p & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$
  (since $H^*_D(S^5) = H^*_{dR}(S^5)$ is only nonzero in degree 5, $p+q=5$ for nonzero $E_3^{p,q}$, and the only spot allowed to be nonzero by dimensions is $p=4, q=1$)

- Since $E_3^{p,q} = \ker(d_2)/\operatorname{im}(d_2)$, $0 \to \mathbb{R} \to B \to 0$, $0 \to A \to C \to 0$, and $0 \to C \to 0$ are exact, establishing isomorphisms $\mathbb{R} \cong B$, $A \cong C \cong 0$. For $D$, $\mathbb{R} = \ker(d_2)/\operatorname{im}(d_2)$, and $\operatorname{im}(d_2 : D \to 0) = 0$, so $\ker(d_2 : 0 \to D) = \mathbb{R}$, hence $D = \mathbb{R}$. Thus:

$$H^*(\mathbb{CP}^2) = \begin{cases} \mathbb{R} & * = 0,2,4 \\ 0 & \text{else} \end{cases}$$

This extends to $\mathbb{CP}^n$:

$$H^*_{dR}(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & * = 0,2,\dots,2n \\ 0 & \text{else} \end{cases}$$

- Looking at the algebra structure of $\mathbb{CP}^2$, we let $a \in H^1(S^1) = H^0(\mathbb{CP}^2) \otimes H^1(S^1) = E_2$ be a generator. Then $x := d_2 a \in H^2(\mathbb{CP}^2)$ generates $H^2(\mathbb{CP}^2)$. $x \wedge a \in E_2^{2,1}$ is a generator, so $d_2(x \wedge a)$ must generate $H^4(\mathbb{CP}^2)$. But $d_2(x \wedge a) = x^2$, so $x^2$ generates $H^4(\mathbb{CP}^2)$, where $x = [\omega]$ is a generator of $H^2(\mathbb{CP}^2)$. As a ring, we see:

$$H^*(\mathbb{CP}^2) \cong \mathbb{R}[x]/(x^3)$$

where the isomorphism sends a generator $[\omega]$ of $H^2(\mathbb{CP}^2)$ to $x$. In general:

$$H^*(\mathbb{CP}^n) \cong \mathbb{R}[x]/(x^{n+1})$$

$$[\omega] \in H^2(\mathbb{CP}^n) \overset{\sim}{\longmapsto} x \in \mathbb{R}[x]/(x^{n+1})$$

# *Characteristic Classes*

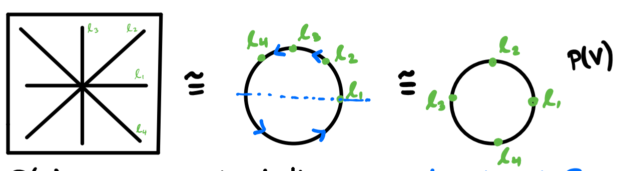
- A complex line bundle is a complex vector bundle of rank 1. By dropping the algebra structure on $\mathbb{C}$, we identify any complex line bundle w/ a real vector bundle of rank 2. For a complex line bundle $L$, we define its first Chern class to be the Euler class of this plane bundle $L_{\mathbb{R}}$:

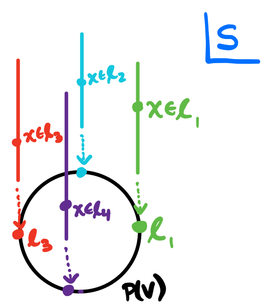
  $$c_1(L) := e(L_{\mathbb{R}})$$

  - For a line bundle by properties of $e(M)$:
    i) $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$
    ii) $c_1(L^*) = -c_1(L)$, where $L^*_p = (L^*_p) = \mathrm{Hom}(L_p, \mathbb{C})$.
- The projectivization $P(V)$ of $V$: For a complex vector space $V$, define:

  $$P(V) := \{1\text{ dim subspaces } \ell \subset V\}$$

- For example, for $\mathbb{R}^2$ the result is $\mathbb{RP}^1 = S^1$:

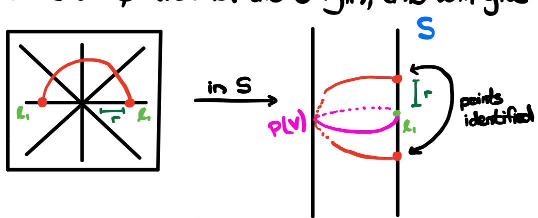



- Given $P(V)$, we can construct the universal subbundle $S$:

  $$S := \{(\ell, v) \in P(V) \times V : v \in L\}$$

  The fiber above a point $L$ is $L$ itself, but viewed as a subspace of $V$.
  - As we wrap around the origin, this will give us some twisting:




$S$ comes equipped w/ a projection called the quadratic transformation $\sigma$:

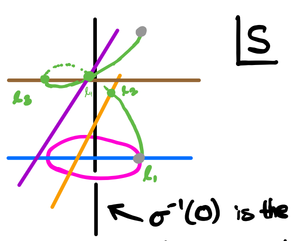
$$\sigma : S \hookrightarrow P(V) \times V \longrightarrow V \qquad \sigma((\ell, v)) = v$$

$\sigma$ is a natural map sending a point in the fiber above $\ell$ to its point in $V$. $\sigma^{-1}$ is almost injective:

$$\sigma^{-1}(v) = \{(\ell, v) : v \in \ell\}$$

i.e. $\sigma^{-1}(v)$ sends $v$ to the single point $(\ell, v) \in S$ for nonzero $v$. For $v = 0$, $\sigma^{-1}(0) = P(V) \subset S$, since $\forall \ell \in P(V)$, $0 \in \ell$.

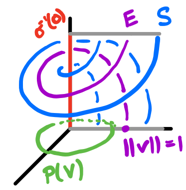
- We can visualize $S$ as separating out all the lines in $V$, since we attach each line to a point in $P(V)$ representing the line:




$\sigma^{-1}(0)$ is the vertical line

- The gray points are identified, so the curve in $\mathbb{R}^2$ lifts to a curve in $S$.

- **Cohomology of $P(V)$:** Note $\sigma^{-1}(0)=\{(\ell,0):\ell\in P(V)\}\subset S$ is the zero section of $S$. Let $E$ be the unit sphere bundle of $S$, $E=\{(\ell,v):v\in\ell,\|v\|=1\}$. B/c of the identification after wrapping ⊙ (the gray line), $S\setminus\sigma^{-1}(0)\cong V\setminus\{0\}$, and so $E\cong S^{2n-1}$!




This gives us a fibering:

$$\begin{array}{c} S^1\longrightarrow S^{2n-1}\cong E=\text{unit sphere bundle of } S\\ \downarrow\\ P(V)\end{array}$$

This looks much like $\mathbb{C}P^n$, and indeed the cohomologies are the same:

$$H^*(P(V))\cong\mathbb{R}[x]/(x^n)\qquad n=\dim_{\mathbb{C}}V$$

where the isomorphism sends the Euler class of the circle bundle $E$ to $x$:

$$[c_1(S)]=[e(E)]\in H^2(P(V))\longmapsto -x\in\mathbb{R}[x]/(x^n)$$

The generator $x$ is chosen like this so $x\longleftrightarrow c_1(S^*)$

- Dimensions: For $n=\dim_{\mathbb{C}}V$, note $\dim_{\mathbb{C}}P(V)=n-1$. $S$ is a vector bundle over $P(V)$ of rank (over $\mathbb{C}$) of 1, and $e(E)=c_1(S)\in H^2(P(V))$. Since $E=\{(\ell,v):v\in\ell,\|v\|=1\}$, each fiber is $\cong$ unit sphere of $\mathbb{C}$ w/ base space $P(V)$. Dimensions of $H^*(P(V))$? Note dR cohomology is over $\mathbb{R}$, so $H^*(P(V))$ can be nonzero for $*=0,1,\dots,\dim_{\mathbb{R}}P(V)=2(n-1)$. Sending $e(E)$ to $x$ satisfies this, as $x^{n-1}=[e(M)]^{n-1}\in H^{2(n-1)}(P(V))$ and $x^n=0$ as we quotient by $(x^n)$.

- For a manifold $M$, we define its Poincaré Series $P_t(M)$ to be:

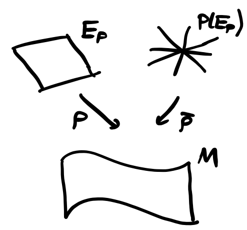
$$P_t(M):=\sum_{i=0}^{\infty}(-1)^i\dim(H^i(M))t^i$$

  - For $P(V)$, we get:

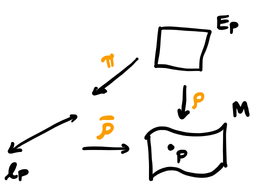
$$P_t(P(V))=1+t^2+t^4+\dots+t^{2(n-1)}=\frac{1-t^{2n}}{1-t^2}$$

- Given a vector bundle $p:E\to M$ w/ structure fns $g_{\alpha\beta}:U_\alpha\cap U_\beta\to GL(\mathbb{C}^n)$, the projectivization $P(E)$ of $E$ is $\pi:P(E)\to M$, w/ fibers $P(E_p)$, and structure fns $\overline{g}_{\alpha\beta}:U_\alpha\cap U_\beta\to PGL(\mathbb{C}^n)$ induced by $g_{\alpha\beta}$, where $PGL(V)=GL(V)/\{\text{scalar matrices}\}$

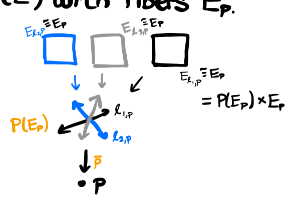
  - There are multiple natural vector bundles one can consider having $P(E)$ as the base space.




  - The pullback bundle $\pi^{-1}E$ is the vector bundle over $P(E)$ with fibers $E_p$.




$$(\pi^{-1}E)_{\ell_p}=E_p$$

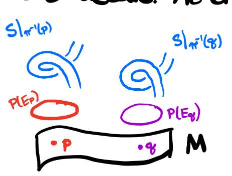



  - Restricted to $\pi^{-1}(\{p\})$, $(\pi^{-1}E)|_{P(E_p)}$ is trivial, i.e. $\pi^{-1}E|_{P(E_p)}=P(E_p)\times E_p$

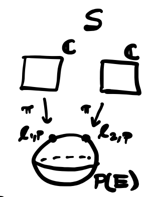
- The universal subbundle $S$ over $P(E)$ is the subbundle of $\pi^{-1}E$ defined as:

$$S:=\{(\ell_p,v)\in\pi^{-1}E:v\in\ell_p\subset E_p\}$$

This is essentially the same as in the vector space case, but now the $P(E_p)$ have an additional fiber bundle structure. At one $p\in M$, $S|_{\pi^{-1}(\{p\})}$ is just like $S$ constructed for the vector space $E_p$.



• Chern classes: Let $\rho: E \to M$ be a vector bundle, $P(E)$ its projectivization, and $S$ the universal subbundle. Let $x := c_1(S^*) \in H^2(P(E))$. If we restrict $S$ to $P(E_p)$ for $p \in M$, we get the universal subbundle $\tilde{S} = S|_{P(E_p)}$ of the vector space $P(E_p)$, and $c_1(\tilde{S})$ is the restriction of $c_1(S) = -x$ to $P(E_p)$ by the functoriality of the Euler class. As such, $\{1, x, x^2, \dots, x^{n-1}\}$ are global cohomology classes on $P(E)$ which generate the cohomology of each fiber $P(E_p)$ when restricted. Thus Leray-Hirsch implies $H^*(P(E))$ is freely generated by $\{1, x, x^2, \dots, x^{n-1}\}$ as a module over $H^*(M)$. We can thus define the Chern classes of $E$ to be the coefficients of $x^n$ w.r.t. this basis:




S is a $\mathbb{C}$-line bundle over P(E)

$$x^n + c_1(E)x^{n-1} + c_2(E)x^{n-2} + \dots + c_n(E) = 0$$

$$c_i(E) \in H^{2i}(M)$$

where the product here is $\wedge$ and $c_i(E)$ is really $\pi^* c_i(E)$. The total Chern class is:

$$c(E) := 1 + c_1(E) + \dots + c_n(E) \in H^*(M)$$

Using the dfn of Chern classes, we see:

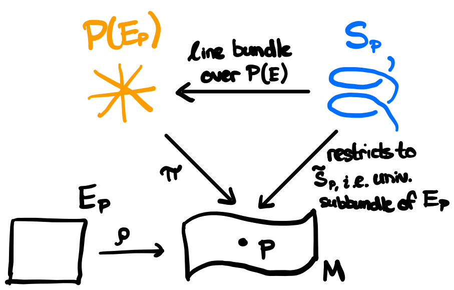
$$H^*(P(E)) = H^*(M) / (x^n + c_1(E)x^{n-1} + \dots + c_n(E))$$

w/ $n = \text{rank}(E)$, and the cohomology splits as a product:

$$H^*(P(E)) = H^*(M) \otimes H^*(P^{n-1}) \qquad P^{n-1} := P(\mathbb{C}^n),\ H^*(P^{n-1}) \cong \mathbb{R}[x]/(x^n)$$

– $S$ has a line bundle structure over $P(E)$, and a fiber bundle structure over $M$, where each fiber over $p \in M$ is $\tilde{S}$, the univ. subbundle of the vector space $E_p$, and looks like the space $E_p$ where we separated out the 1D subspaces $\cong \mathbb{C}$.

– To summarize, we have many fiber bundle structures here:




• $x := c_1(S^*) \to$ as a line bundle over $P(E)$, so $x \in H^2(P(E))$, and $x^i \in H^{2i}(P(E))$

$c_1(S^*) = -e(S^*)$

• Problem to solve: global cohomology of $P(E)$, when we already know $H^*$ of each fiber $P(E_p)$

– Prior to this, we only know the cohomology of each fiber $H^*(P(E_p)) \cong \mathbb{R}[x]/(x^n)$. When we stitch the fibers together, the classes $\{1, x, \dots, x^{n-1}\}$ generate each fiber's cohomology, and $x^n \neq 0$ as a global cohomology class. But, because of Leray-Hirsch, we can write $x^n = -\sum_{i=1}^{n} c_i(E) \wedge x^{n-i}$, so the relation $x^n + \sum c_i x^{n-i} = 0$ defines the ring structure on $H^*(P(E))$.

– We care about the bundle $S \to P(E)$ b/c its first Chern class (the Euler class of its realization) gives us the form $x \in H^2(P(E))$ which is used to generate the cohomology of the whole vector bundle.

• The Poincaré series of $P(E)$ is:

$$P_t(P(E)) = P_t(M) \frac{1 - t^{2n}}{1 - t^2}$$

- Let $L \to M$ be a complex line bundle. Then $P(L) = M$, so $S = L$, and $x = e(S^*_{\mathbb{R}}) = e(L^*_{\mathbb{R}})$ $= -e(L_{\mathbb{R}})$, so $x^2 = -e(L_{\mathbb{R}}) \wedge x \Rightarrow x^2 + e(L_{\mathbb{R}})x = 0$. This means the identification of $c_1(L)$ as $e(L_{\mathbb{R}})$ is well defined - in the construction above we could have used $e(S^*_{\mathbb{R}})$ instead of $c_1(S^*)$ and everything would be identical.
- Trivial bundle: For $E = M \times V$, $P(E) = M \times P(V)$, so $x^n = 0$ b/c it is the same on all fibers. Thus:

  $$c_i(M \times V) = 0$$
- Naturality: If $f: Y \to X$ and $E \to X$ is a complex vector bundle, then:

  $$c_i(f^{-1}E) = f^* c_i(E)$$

  This implies isomorphic vector bundles have the same Chern classes.
- Whitney Product formula: For two vector bundles $E \to M \leftarrow E'$:

  $$c_i(E \oplus E') = c_i(E) c_i(E')$$
- If $n = \text{rank}_{\mathbb{C}}(E)$, we define $c_i(E) := 0$ for $i > n$.
- If $E$ has a nowhere vanishing section, then the top Chern class vanishes, $c_n(E) = 0$.
  - This can be seen immediately from the next statement.
- The top Chern class $c_n(E) \in H^{2n}(M)$ is the Euler class of the realization of $E$:

  $$c_n(E) = e(E_{\mathbb{R}}) \in H^{2n}(M) \qquad n = \text{rank}\, E$$