8.513 Term Paper: Symmetry and Topology in Lattice QCD

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1 Introduction

Quantum Chromodynamics (QCD) is the theory of the strong nuclear force. It is a $SU(3)$ gauge theory coupled to 6 flavors of fermions, which are known as the quarks. At low energies, the coupling of QCD is reasonably large and perturbation theory cannot be applied to the theory, which means physics must be extracted by non-perturbative means. One such way to do this is to formulate QCD as a **lattice gauge theory** by discretizing spacetime–the advantage of this is that the path integral becomes a finite (albeit large) dimensional integral which can be evaluated numerically using computers. Although lattice QCD is used primarily as a calculational tool for full QCD observables, the theory by itself has many interesting properties. In this paper, I will discuss the details of such a theory and consider the role of topology in studying lattice gauge theories.

We will begin by making some initial definitions: denote our spacetime lattice with spacing a by Λ, and Wick rotate to imaginary time so that Λ is a Euclidean lattice[1]. In the full theory of QCD, the dynamical fields in the path integral are the quark fields ψ_f and gluon fields A_μ. When we discretize QCD, we will still work with the quark fields, but instead of directly working with the gauge fields A_μ we will work **link fields** $U_\mu(n)$ which transform in the following way under a gauge transformation $\Omega : \Lambda \to SU(3)$:

$$U_\mu(n) \xrightarrow{\Omega(n)} \Omega(n)U_\mu(n)\Omega(n+\hat{\mu})^\dagger \tag{1}$$

Here $\hat{\mu}$ is the unit vector in the μ direction, and $n \in \Lambda$ denotes a site in the lattice. The link fields can be taken to be $U_\mu(n) = \exp(iaA_\mu(n))$, so are intimately related to the gauge field, and take values (as they must) in $SU(3)$. With this transformation law, the link fields act as a connection between the fibers at different points in Λ:

$$U_\mu(n)\psi(n+\hat{\mu}) \xrightarrow{\Omega} \Omega(n)U_\mu(n)\psi(n+\hat{\mu}) \tag{2}$$

This transformation means that $U_\mu(n)\psi(n+\hat{\mu})$ is valued in the fiber at point n, and so can directly be compared with $\psi(n)$. This allows us to add and subtract fermion fields at different points in a gauge invariant way, and so define a covariant derivative.

We may now write down a first pass at a fermion action, which will be equivalent to $\not{D} + m$ upon taking the continuum limit. The direct discretization of this action is thus:

$$\begin{aligned} S_f^0[\psi_f, \bar{\psi}_f, U] &= a^4 \sum_{n\in\Lambda} \sum_f \bar{\psi}_f(n) \left(\gamma^\mu \frac{U_\mu(n)\psi_f(n+\hat{\mu}) - U_{-\mu}(n)\psi_f(n-\hat{\mu})}{2a} - m_f\psi_f(n) \right) &(3) \\ &= a^4 \sum_{n,m\in\Lambda} \sum_f \bar{\psi}_f(n)^a_\alpha D_f^0(n|m)^{ab}_{\alpha\beta}\psi_f(m)^b_\beta &(4) \end{aligned}$$

where we have defined the **Dirac operator** $D^{ab}_{\alpha\beta}(n|m)$ to be:

$$D_f^0(n|m)^{ab}_{\alpha\beta} := (\gamma^\mu)_{\alpha\beta} \left(\frac{U_\mu(n)^{ab}\delta_{n+\hat{\mu},m} - U_{-\mu}(n)^{ab}\delta_{n-\hat{\mu},m}}{2a} \right) + m_f\delta_{\alpha\beta}\delta^{ab}\delta_{nm} \tag{5}$$

Note the Greek indices α, β are in Dirac space, and the Latin indices a, b are in color space. The Dirac operator is one of the fundamental objects we will study later when considering the role of topology in lattice QCD, and is the discretized version of $i\not{D} - m$ in Euclidean space.

[1]We must Wick rotate so that the Boltzmann factor of e^{iS} in the path integral becomes a valid probability density e^{-S}, in order to perform any computations at all.

However, there is a slight problem with the action in Equation 4. Because we are working on a lattice, the Fourier transform $\tilde{D}^0(p)$ of the Dirac operator $D^0(n|m)$ has extra unphysical poles at the edges of the Brillioun zone. These extra poles are known in the literature as **doublers**, and must be eliminated by adjusting the action. We do this by adding a corresponding Wilson term to the Dirac operator, so that the full Dirac operator and action now become:

$$D_f^W(n|m)^{ab}_{\alpha\beta} := \left(m_f + \frac{4}{a}\right)\delta_{\alpha\beta}\delta^{ab}\delta_{nm} - \frac{1}{2a}\sum_{\mu=\pm 1}^{\pm 4}(1-\gamma^\mu)_{\alpha\beta}U_\mu(n)^{ab}\delta_{n+\hat{\mu},m} \tag{6}$$

$$S_f[\psi,\bar{\psi},U] = a^4\sum_{n,m\in\Lambda}\sum_f \bar{\psi}_f(n)^a_\alpha D_f^W(n|m)^{ab}_{\alpha\beta}\psi_f(m)^b_\beta \tag{7}$$

We will soon see that the Wilson term makes it much more difficult to deal with chiral symmetry on the lattice than in the continuum, and is responsible for many of the interesting topological properties of lattice gauge theories as compared to their continuum counterparts.

For completeness we will record the glue action as well:

$$S_g[U] = \frac{2}{g^2}\sum_{n\in\Lambda}\sum_{\mu<\nu}\mathbb{R}e\ tr\{1 - U_{\mu\nu}(n)\} \tag{8}$$

where $U_{\mu\nu}(n) := U_\mu(n)U_\nu(n+\hat{\mu})U_{-\mu}(n+\hat{\mu}+\hat{\nu})U_{-\nu}(n+\hat{\nu})$ is known as a **plaquette**, and performs parallel transport around a closed loop. The partition function for the full theory of lattice QCD is thus:

$$Z = \int D\psi D\bar{\psi} DU e^{-S_f - S_g} \tag{9}$$

where the measures $D\psi, D\bar{\psi}, DU$ now contain only a finite amount of of sites to be integrated over.

2 Chiral Symmetry and the Ginsparg-Wilson Relation

Chiral symmetry in standard QCD is realized by taking the approximations that the light quarks are massless. Let $D_{cont} := \gamma^\mu D_\mu + m$ be the continuum Dirac operator in Euclidean space. Then for a massless fermion described by $\mathcal{L} = \bar{\psi}D_{cont}\psi$, the Lagrangian is invariant under:

$$\psi \mapsto \exp(i\theta\gamma_5)\psi \tag{10}$$

because of the anticommutation relation $\{D_{cont},\gamma_5\}$. This is the core of chiral symmetry in the continuum because it allows us to split our Lagrangian into two pieces with definite chirality:

$$\mathcal{L} \supset \bar{\psi}_L D_{cont}\psi_L + \bar{\psi}_R D_{cont}\psi_R \tag{11}$$

and rotate ψ_L and ψ_R separately, giving the chiral symmetry $SU(N_f)_L \times SU(N_f)_R$, where N_f is the number of (approximately) massless quarks.

Upon discretization, the relation $\{D,\gamma_5\} = 0$ falls apart because of the Wilson term. Even if we assume our lattice quarks are massless, the piece proportional to $1_{\alpha\beta}$ does not anticommute with γ_5 (just as a mass term does not because it is proportional to $1_{\alpha\beta}$). More generally, it was shown by Nielson and Ninomiya [1] that *any attempt* to remove the doublers from a lattice regularized theory would result in such a breaking of this anticommutation relation, and thus the essence of chiral symmetry in lattice theories must be reformulated.

In 1998, Luscher [2] proposed an alternative symmetry on the lattice that acts as chiral symmetry, and indeed goes into chiral symmetry in the continuum limit $a \to 0$. The modified anticommutation relation is known as the **Ginsparg-Wilson equation**:

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \tag{12}$$

Although this is not satisfied by the Wilson-Dirac operator in Equation 6, it is satisfied by a different class of Dirac operators on the lattice. One in particular was defined by Neuberger [3] and is known as the **overlap operator**:

$$D_{over} = \tag{13}$$

3 The Index Theorem and Topological Charge

References

[1] A No-Go Theorem For Regularizing Chiral Fermions TODO

[2] Luscher

[3] Neuberger