Renormalization

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1 Overview

2 Regulators

There are two steps to a renormalization calculation: regulation and renormalization. Regulation is the process of deforming your QFT to give something formally finite and to give you a way to determine "how infinite" a divergence is. After you have regulated your theory, you must renormalize it, in which you swap out the parameters in your Lagrangian for renormalized parameters.

3 Counterterms and Renormalization

4 Renormalization of QED

5 Renormalization Schemes

6 The Renormalization Group

7 Local Operator Renormalization

Renormalization of a theory is often seen as a program that replaces the bare couplings and fields in a Lagrangian by renormalized quantities to make the observable quantities we can compute from a theory finite in the UV or the IR. However, renormalization can be viewed from a much more general perspective in terms of local composite operators. A **local composite operator** is an operator $\mathcal{O}(x)$ which is built up from the fields and derivatives in a theory at a single spacetime point x. Schematically for a theory with fermion fields $\{\Psi_i(x)\}$ and a gauge field A_μ, such an operator looks like:

$$\mathcal{O}_{\mu...\nu\rho...\sigma}(x) = \mathcal{Z}_{\mathcal{O}} D_\mu ... D_\nu A_\sigma ... A_\rho \Gamma_{\ell_1} ... \Gamma_{\ell_k} \Psi_{i_1} ... \Psi_{i_k} \overline{\Psi}_{j_1} ... \overline{\Psi}_{j_k} \tag{1}$$

where $\mathcal{Z}_{\mathcal{O}}$ is a normalization constant which we will use to renormalize the operator $\mathcal{O}$, and Γ is any matrix with a Dirac structure. The operator can have any number of spin indices that one wants, but often we will consider a relatively small value for its spin because the operators which are dominant in the OPE are those with the smallest twist[1].

Even after renormalizing all the fields and couplings in the Lagrangian, matrix elements of composite operators are often divergent, which is why we need to renormalize them separately. Take for example the renormalization of the operator $\mathcal{O} := \phi^2$ in the ϕ^4 scalar field theory. When

[1] The twist of an operator is its spin s (number of Lorentz indices) minus the mass dimension d. Operators must have twist ≥ 2, so we will often consider twist 2 operators.

we renormalize the bare field $\phi_0 = \mathcal{Z}_\phi(\mu)\phi(\mu)$, this does not immediately renormalize $\mathcal{O}$. This is because as an operator, ϕ^2 is implicitly the normal ordering $:\phi^2:$ of the actual ϕ^2 operator. Because of this normal ordering, renormalization constants are not multiplicative[2], i.e. we cannot just take $\mathcal{O} = \mathcal{Z}(\mu)^2$.

To renormalize a composite operator, it suffices to compute a (divergent) Green's function and impose renormalization conditions on that. Typically the renormalization condition on this Green's function will be setting it equal to its tree level value.

When we study composite operator renormalization, we do not need to even insert the operator into the Lagrangian. Instead, we will view the operator $\mathcal{O}$ we wish to renormalize as an **external operator** which is separate from the theory's Lagrangian. The goal of renormalizing $\mathcal{O}$ is to make any Green's function of $\mathcal{O}$ with another operator finite, and we can do this without introducing a coupling. Later we will see what happens when we renormalize an external operator then add it to a Lagrangian; based on the operator renormalization, we will immediately be able to read off how its Wilson coefficient flows.

For any of these computations, we will need to understand how to diagrammatically compute Green's functions. We can use what Schwartz calls **off-shell Feynman rules** to diagrammatically compute this Green's function. Off-shell Feynman rules are the same as the typical Feynman rules for a theory, with some extra caveats:

- We replace external polarizations with propagators. Because of this, we will often want to consider amputated Green's functions without the propagators as the objects to renormalize.
- Current insertions have their own set of rules, and are taken at non-zero momenta. This means currents inject momentum into our diagrams, and this must be considered when labeling all the momenta.
- We do not always "work backwards" from the tip of the arrow to its bottom when computing diagrams with fermion lines (as we will see in the example).

The main tool we will use to expand these diagrams in perturbation theory is the following relation between operators in the interacting theory and operators in the free theory:

$$\langle 0|T\left\{\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\right\}|0\rangle = \frac{1}{\mathcal{A}}\langle 0|T\left\{\exp\left[i\int d^4z\,\mathcal{L}^0_{int}(z)\right]\mathcal{O}^0_1(x_1)...\mathcal{O}^0_n(x_n))\right\}|0\rangle \tag{2}$$

where $\mathcal{O}^0$ denotes an operator in the free theory, and $\mathcal{L}^0_{int}$ is the interaction Lagrangian in the free theory. $\mathcal{A}$ is a normalization given by:

$$\mathcal{A} = \langle 0|T\left\{\exp\left(i\int d^4z\,\mathcal{L}^0_{int}(z)\right)\right\}|0\rangle \tag{3}$$

With this expansion, we can draw Feynman diagrams. Intuitively it helps to compare this to a derivation of Feynman diagrams in Schwartz, where he shows how to compute n-point functions diagrammatically using Equation 7.64 to relate them to their free field values:

$$\langle 0|T\left\{\phi(x_1)...\phi(x_n)\right\}|0\rangle = \frac{1}{\mathcal{A}}\langle 0|T\left\{\exp\left[i\int d^4z\,\mathcal{L}^0_{int}(z)\right]\phi^0(x_1)...\phi^0(x_n)\right\}|0\rangle \tag{4}$$

[2]More on that here: https://www.physicsoverflow.org/27963/renormalization-determined-renormalization-elementary

The key point in this equation (as in Equation 2) is that we can Taylor expand the exponential and contract the fields on the right hand side of the equation to give ourselves propagators and derive Feynman rules for the theory.

Once we renormalize an operator, an important quantity to compute is the operator's **anomalous dimension** $\gamma_{\mathcal{O}}$, which describes how the operator flows under renormalization:

$$\gamma_{\mathcal{O}} := -\frac{\mu}{Z_{\mathcal{O}}}\frac{d\mathcal{Z}_{\mathcal{O}}}{d\mu} = -\mu\frac{d}{d\mu}\log\mathcal{Z}_{\mathcal{O}} \tag{5}$$

Once we have determined the counterterm for the operator $\mathcal{Z}_{\mathcal{O}}$, we can immediately get the anomalous dimension. This quantity acts similarly to the β function for a coupling, and also appears in the Callan-Symanzik equation on equal terms.

TODO discuss Callan-Symanzik.

7.1 Example: QED current

We will do an example in QED, and compute the renormalization of $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$. The simplest non-vanishing Green's function involving $j^\mu(x)$ is the three point function, which we will expand in momentum space:

$$\langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle = \int \text{đ}^4p\,\text{đ}^4q_1\,\text{đ}^4q_2\,e^{-ipx}e^{-ip_1x_1}e^{iq_2x_2}i\mathcal{M}^\mu(p,q_1,q_2)(2\pi)^4\delta^4(p+q_1-q_2) \tag{6}$$

We have chosen the signs on the momenta to replicate a current insertion of momentum p; namely, if a particle is propagating initially with momentum q_1, the current insertion knocks the momentum of the particle so that:

$$q_2 = p + q_1 \tag{7}$$

We will compute this in perturbation theory to one loop and associate a diagram with each term. Using the expansion in Equation 2, we can expand the exponential to one loop:

$$\exp\left(i\int d^4z\,\mathcal{L}^0_{int}(z)\right) \sim 1 + i\int d^4z\,\mathcal{L}^0_{int} + \frac{i^2}{2}\int d^4z\,d^4z'\,\mathcal{L}^0_{int}(z)\mathcal{L}^0_{int}(z') + \ldots \tag{8}$$

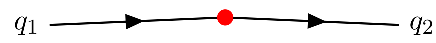
When inserted into the equation, the first piece gives the tree level vertex and the piece with two integrals gives the first loop correction. Plugging in at first order, we can use Wick's theorem to evaluate the free field correlators (here α,β,ρ,σ are Dirac indices):

$$\begin{aligned}
\langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle_{\text{tree}} &= \langle j_0^\mu(x)\psi_0(x_1)\overline{\psi}_0(x_2)\rangle &(9)\\
&= \langle \overline{\psi}_0^\alpha(x)\gamma^\mu_{\alpha\beta}\psi_0^\beta(x)\psi_0^\rho(x_1)\overline{\psi}_0^\sigma(x_2)\rangle &(10)\\
&= S(x_1,x)\gamma^\mu S(x,x_2) &(11)
\end{aligned}$$

as $\langle\psi(x)\overline{\psi}(y)\rangle = S(x,y) = i\int \text{đ}^4k\,\frac{e^{-ik(x-y)}}{\not{k}-m}$. We can take this tree level result to momentum space, where we see:

$$\begin{aligned}
\mathcal{M}^\mu(p,q_1,q_2)_{\text{tree}} &= \int d^4x\,d^4x_1\,d^4x_2\,e^{ipx}e^{iq_1x_1}e^{-iq_2x_2}\langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle_{tree}\\
&= \int d^4x\,d^4x_1\,d^4x_2\,e^{ipx}e^{iq_1x_1}e^{-iq_2x_2}\int \text{đ}^4k\int \text{đ}^4k'\,\frac{ie^{-ik(x_1-x)}}{\not{k}-m}\gamma^\mu\frac{ie^{-ik'(x-x_2)}}{\not{k}'-m}\\
&= \int \text{đ}^4k\,\text{đ}^4k'\,d^4x\,d^4x_1\,d^4x_2 e^{i(p+k-k')x}e^{i(q_1-k)x_1}e^{i(k'-q_2)x_2}\frac{i}{\not{k}-m}\gamma^\mu\frac{i}{\not{k}'-m}\\
&= \frac{i}{\not{q}_1-m}\gamma^\mu\frac{i}{\not{q}_2-m}\delta^4(p+q_1-q_2) &(12)
\end{aligned}$$

where we leave the delta function in to emphasize that we **must enforce momentum conservation from current insertion**. Diagrammatically, this corresponds to the diagram



(13)

where the red dot denotes a current insertion of momentum p.

Now we move towards the actual process of renormalization: we can expand Equation 2 to one loop and compute the one loop 1PI correction, then renormalize the operator. This is going to get very tiring very soon, but I think it is instructive to get familiar with where the Feynman rules come from for operator insertions. At one loop, we pick up the $(\mathcal{L}^0_{int})^2$ piece in our free theory Green's function (recall $\mathcal{L}^0_{int}(z) = -e\overline{\psi}_0(z)\not{A}^0(z)\psi_0(z)$). This gives us (also switching the Dirac indices to be Latin indices because there are a lot of them and I don't know the entire Greek alphabet):

$$\begin{aligned}\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} = -\frac{e^2}{2}\int d^4z\, d^4z'\, \langle \overline{\psi}^c_0(z)\gamma^\nu_{cd}A^0_\nu(z)\psi^d_0(z)\times \\ \overline{\psi}^e_0(z')\gamma^\lambda_{ef}A^0_\lambda(z')\psi^f_0(z')\overline{\psi}^g_0(x)\gamma^\mu_{gh}\psi^h_0(x)\psi^a_0(x_1)\overline{\psi}^b_0(x_2)\rangle \qquad (14)\end{aligned}$$

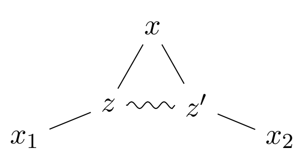
We now have to sum on all contractions of each of the fields. However, we can simplify this by remembering that we are only dealing with connected, 1PI diagrams and throw away many of the contractions. Namely, the general structure of the correlator will look like this:

$$\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} \sim \frac{(-ie)^2}{2}\int \langle A(z)A(z')\rangle\langle\overline{\psi}(z)\psi(z)\overline{\psi}(z')\psi(z')\overline{\psi}(x)\psi(x)\overline{\psi}(x_1)\psi(x_2)\rangle$$

where we sum over contractions for the fermionic correlator. Because the photon propagator $S^{(\gamma)}_{\nu\lambda}(z,z') = \langle A^0_\nu(z)A^0_\lambda(z')\rangle$ already connects the z and z' points, this limits the structure of this to two specific contractions:

$$\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} \sim \frac{(-ie)^2}{2}\int S^{(\gamma)}_{z,z'}\left[\langle\overline{\psi}_z\psi_z\overline{\psi}_{z'}\psi_{z'}\overline{\psi}_x\psi_x\psi_{x_1}\overline{\psi}_{x_2}\rangle + \langle\overline{\psi}_z\psi_z\overline{\psi}_{z'}\psi_{z'}\overline{\psi}_x\psi_x\psi_{x_1}\overline{\psi}_{x_2}\rangle\right]$$

Diagrammatically, two diagrams which contribute look like this:



(15)

where the second term has the z and z' vertices flipped. These will end up being the same and canceling the factor of 1/2.

We only get these two contractions for a few reasons. First, because of the photon propagator connecting z and z', we cannot contract the fermion fields at z and z' together. If we did, we would get a graph which is not 1P1, because we could cut the bubble out of the diagram. Second, we cannot contract the x_1 and x_2 points together, or else we would have a disconnected diagram, hence we must contract x_1 and x_2 to one of z or z' (therefore we have two diagrams). This is easiest to see pictorially– in the manner of the previous diagram.

So, we have reduced this computation to calculating (now restoring the Dirac structure):

$$\begin{aligned}\langle j^\mu(x)\psi^a(x_1)\overline{\psi}^b(x_2)\rangle_{\text{1-loop}} &= (-ie)^2\int d^4z\, d^4z'\, S^{(\gamma)}_{\nu\lambda}(z,z')\langle \overline{\psi}^c_z\gamma^\nu_{cd}\psi^d_z\overline{\psi}^e_{z'}\gamma^\lambda_{ef}\psi^f_{z'}\overline{\psi}^g_x\gamma^\mu_{gh}\psi^h_x\psi^a_{x_1}\overline{\psi}^b_{x_2}\rangle\\ &= (-ie)^2\int d^4z\, d^4z'\, S^{(\gamma)}_{\nu\lambda}(z,z')S^{ac}(x_1,z)\gamma^\nu_{cd}S^{dg}(z,x)\gamma^\lambda_{ef}S^{he}(x,z')\gamma^\mu_{gh}S_{fb}(z',x_2)\\ &= (-ie)^2\int d^4z\int d^4z'\, S^{(\gamma)}_{\nu\lambda}(z,z')S(x_1,z)\gamma^\nu S(z,x)\gamma^\mu S(x,z')\gamma^\lambda S(z',x_2)\end{aligned} \tag{16}$$

where:

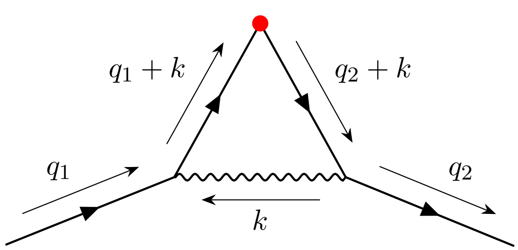
$$S^{(\gamma)}_{\mu\nu}(x,y) = \int d^4k\frac{-ie^{ik(x-y)}g_{\mu\nu}}{k^2+i\epsilon} \tag{17}$$

is the photon propagator. We can now plug this into our expression for $\mathcal{M}^\mu(p,q_1,q_2)$ and associate a one-loop diagram to this computation (where the measure $d^{20}x = d^4x\, d^4x_1\, d^4x_2\, d^4z\, d^4z'$):

$$\begin{aligned}\mathcal{M}^\mu(p,q_1,q_2)_{\text{1-loop}} &= (-ie)^2\int d^{20}x\, e^{i(px+q_1x_1-q_2x_2)}S^{(\gamma)}_{\nu\lambda}(z,z')S(x_1,z)\gamma^\nu S(z,x)\gamma^\mu S(x,z')\gamma^\lambda S(z',x_2)\\ &= (-ie)^2\int d^{20}x đ^{20}k\, e^{i(px+q_1x_1-q_2x_2)}\frac{-ie^{ik(z-z')}}{k^2}\frac{ie^{-ik_1(x_1-z)}}{\not{k}_1-m}\gamma^\nu\frac{ie^{-ik_2(z-x)}}{\not{k}_2-m}\gamma^\mu\frac{ie^{-ik_3(x-z')}}{\not{k}_3-m}\gamma_\nu\frac{ie^{-ik_4(z'-x_2)}}{\not{k}_4-m}\\ &= (-ie)^2\int d^{20}x đ^{20}k\, e^{i(p+k_2-k_3)x}e^{i(k+k_1-k_2)z}e^{i(k_3-k_4-k)z'}e^{i(q_1-k_1)x_1}e^{-i(q_2-k_4)x_2}\times\\ &\qquad\left(\frac{-i}{k^2}\frac{i}{\not{k}_1-m}\gamma^\nu\frac{i}{\not{k}_2-m}\gamma^\mu\frac{i}{\not{k}_3-m}\gamma_\nu\frac{i}{\not{k}_4-m}\right)\\ &= (-ie)^2\frac{i}{\not{q}_1-m}\left[\int đ^4k\frac{i}{k^2}\gamma^\nu\frac{i}{\not{q}_1+\not{k}-m}\gamma^\mu\frac{i}{\not{q}_2+\not{k}-m}\gamma_\nu\right]\frac{i}{\not{q}_2-m}\end{aligned} \tag{18}$$

The delta functions enforce momentum conservation at each vertex in the following diagram, and we can thus encode this amplitude using the off-shell Feynman rules on the following diagram, where the current insertion injects momentum p into the diagram, as well as a γ^μ factor. Our result is thus:

$$i\mathcal{M}^\mu(p,q_1,q_2)_{\text{1-loop}} = \tag{19}$$



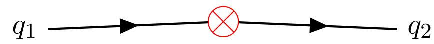
$$= (-ie)^2\frac{i}{\not{q}_1-m}\left[\int đ^4k\frac{i}{k^2}\gamma^\nu\frac{i}{\not{q}_1+\not{k}-m}\gamma^\mu\frac{i}{\not{q}_2+\not{k}-m}\gamma_\nu\right]\frac{i}{\not{q}_2-m} \tag{20}$$

where $q_2 = p+q_1$ is the result of current insertion at the red dot in the diagram. At this point, we can evaluate this loop integral in dimensional regularization. Stripping off the mass dimensions of the electron coupling with $e\mapsto\mu^\epsilon e$ in $d=4-\epsilon$ dimensions, we have:

$$i\mathcal{M}^\mu(p,q_1,q_2) = i\mathcal{M}^\mu_{\text{tree}}(p,q_1,q_2)\frac{e^2}{16\pi^2}\frac{2}{\epsilon} \tag{21}$$

To finish our renormalization process, we must include the counterterm diagram as well by expanding $\mathcal{Z}_j = 1 + \delta_j$ in the Green's function. We have:

$$\delta_j \mathcal{M}^\mu(p, q_1, q_2) = \tag{22}$$



$$= i\mathcal{M}^\mu_{\text{tree}}(p, q_1, q_2)\delta_j \tag{23}$$

where the red $\otimes$ denotes a counterterm insertion. We can thus read off the counterterm:

$$\delta_j = -\frac{e^2}{16\pi^2}\frac{2}{\epsilon} \tag{24}$$

Now that we have read off the counterterm, the bulk of the work is done! Notice that this calculation is much easier to do diagrammatically with Feynman rules than from first principles; however, I wanted to illustrate how to do build up and motivate Feynman rules for operator insertions, i.e. the off-shell Feynman rules that Schwartz discusses. Note that in this case, we read the fermion arrow from backwards to forwards because of the position of the propagators. In general, I would suggest doing Wick contractions at tree level first to determine the order of the propagators in the diagram, and then using this to move to the loop diagrams.

To compute the anomalous dimension, the bare current can be expressed as:

$$J^\mu_{bare} = \overline{\psi}_0 \gamma^\mu \psi_0 = \mathcal{Z}_J^{-1}\mathcal{Z}_\psi J^\mu_R \tag{25}$$

where ψ_0 is the bare field. Since this is purely a function of the bare parameters, it cannot run with μ. Using $\mathcal{Z}_J = \mathcal{Z}_\psi$, we can differentiate this to find:

$$\mu \frac{dJ^\mu}{d\mu} = 0 \tag{26}$$

where J^μ is the renormalized current.

8 Renormalons