Representation Theory

Patrick Oare

These notes will cover the basics of representation theory and its applications to physics. Most of the theory that we discuss will have a large range of applications, and you will almost certainly see it in a course on quantum field theory. This subject is integral to all of physics because representation theory provides a mathematical framework to study symmetries, both discrete and continuous. Indeed, whenever a physicist says the words “group theory” they really mean “representation theory”, and it is really impossible to study subjects like quantum field theory without at least a working knowledge of representation theory.

Although I'm writing these notes as a background for physics, I tend to write my notes more like a mathematician. As such, there will be some sections that read like definition, definition, theorem, corollary. I've found that bundling things up into theorems make them easier to remember and reference in the long run, and this is ultimately meant to be a reference for myself to use when doing physics. Generally, I will only include proofs of theorems if I find them to be informative or fun to do, so if a proof is there, it is generally pretty easy or will introduce methods that will be used later. We will assume background knowledge in algebra (specifically group theory and linear algebra) and some knowledge of topology, specifically smooth manifolds.

Now, on to some common notation. For a field $\mathbb{K}$, we let $M_{n\times m}(\mathbb{K})$ denote the set of all $m\times n$ matrices with values in $\mathbb{K}$ (usually here $\mathbb{K}$ will be either $\mathbb{R}$ or $\mathbb{C}$). Group representations will typically be denoted by Π or by D depending on whether the group is a Lie group or finite group, and Lie algebra representations will typically be denoted π. For algebraic objects M and N (by this I mean objects in a category), the set of morphisms between them will be denoted $Hom(M, N)$.

Let V be a n-dimensional $\mathbb{K}$-vector space. We denote its dual by V^*, and recall for finite dimensional vector spaces there is an isomorphism $V \to V^*$. An **endomorphism** of V is an homomorphism $\phi : V \to V$. If ϕ is continuous (an isomorphism $V \to V$) then we will call ϕ an **automorphism**. We will denote its automorphism group of V by $Aut(V)$ and its endomorphism ring by $End(V)$. The group $Aut(V)$ is naturally isomorphic to the group $GL(V) := GL(\mathbb{K}^n)$ of $n\times n$ invertible matrices with values in $\mathbb{K}$ (as once a basis is chosen for V, invertible linear maps $V \to V$ are in bijection with invertible matrices), and the group $End(V)$ is naturally isomorphic to $gl(V) := gl(\mathbb{K}^n)$, the set of all $n\times n$ matrices with values in $\mathbb{K}$.

1 Lie Groups, Algebras and Representations

Lie groups are used to describe sets of continuous symmetries. Although we will attempt to make definitions as general as possible, we will often restrict ourselves to the case of specific examples which are of interest for physics, namely the Lie groups $SU(N)$, $SO(N)$, and occasionally symplectic groups, which is where classical mechanics lives. We define Lie groups and algebras, and then will discuss the interplay between the two.

Definition 1.1 (Lie group)**.** A **Lie group** $(G, \cdot)$ is a group which is also a differentiable manifold, in which the group operation respects the structure of the manifold. Namely, we require that the maps $\cdot : G^2 \to G$ and $\cdot^{-1} : G \to G$ be smooth.

Definition 1.2 (Lie algebra)**.** A **Lie algebra** $\mathfrak{g}$ over a field k is a k-valued vector space equipped with a map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a **Lie bracket**, such that the following hold:

1. $[\cdot, \cdot]$ is bilinear.
2. $[\cdot, \cdot]$ is antisymmetric.
3. $[\cdot, \cdot]$ satisfies the **Jacobi identity**, i.e. for $A, B, C \in \mathfrak{g}$:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{1}$$

We will generally be considering matrix Lie groups and algebras, in which case the Lie bracket is simply the commutator, $[A, B] = AB - BA$. In the case of matrix groups, our Lie algebras will also be matrix-valued, so $\mathfrak{g}$ is a matrix-valued vector space. We call it an algebra because the map $[\cdot, \cdot]$ gives the vector space an algebra-like structure[1].

The essential idea behind Lie groups is this: Lie groups act on a vector space V as the symmetry operation (for example, the group $SO(3)$ of orthogonal real valued 3×3 matrices with determinant 1 act as rotations in $V = \mathbb{R}^3$). Lie algebras generate the Lie group via the exponential map in the following way: suppose $U \in G$ is an arbitrary element (assume G is path-connected, or at least that U is in the path-component of $1 \in G$). Then, there exists $X \in \mathfrak{g}$ such that:

$$U = \exp(iX) \tag{2}$$

The proof of this existence is one of the fundamental theorems of Lie theory, and is why we care about Lie algebras. In essence, the Lie algebra parameterizes the Lie group. Let $n = dim(\mathfrak{g})$. If we pick a basis $\{T^a\}_{a=1}^n$ for $\mathfrak{g}$, the elements of this basis are called **generators** of the Lie group G because an arbitrary element of G can be represented as:

$$\exp(iX^a T^a) \tag{3}$$

because T^a spans $\mathfrak{g}$. This means the coordinates X^a parameterize the Lie group G (really they parameterize the path-component of 1), and so we can specify an element of G by specifying a set of n coordinates.

Every Lie algebra is defined by its Lie bracket. Since $[\cdot, \cdot]$ maps $\mathfrak{g}^2$ into $\mathfrak{g}$, we must be able to expand:

$$[T^a, T^b] = if^{abc} T^c \tag{4}$$

[1] An algebra is simply a vector space with a ring structure, i.e. with a multiplication $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. However, the axioms that ring multiplication must satisfy are different than those that $[\cdot, \cdot]$ must satisfy.

where f^{abc} is an antisymmetric tensor of numbers known as the **structure constants** of the Lie algebra. Specifying the structure constants of an algebra exactly define the algebra and its Lie bracket.

The associated Lie algebra with a Lie group can be defined by taking its tangent space at the identity, and in this way we have an correspondence from Lie groups to Lie algebras, and back. Geometrically, I like to view a (connected) Lie group as a "smoothly deformed version" of a flat Lie algebra. The exponential map does the deformation, and to me this emphasizes the fact that a Lie group is first and foremost a manifold.

A common operation on Lie algebras we will use is the adjoint operation, of which there are two; the first is an action from the element of the Lie algebra, and the second is from the action of the Lie group. For $X \in \mathfrak{g}$, we define the **adjoint map** $ad_X : \mathfrak{g} \to \mathfrak{g}$ by:

$$ad_X(Y) := [X, Y] \tag{5}$$

For $A \in G$, there is a different type of adjoint map that we can consider. It is still an endomorphism of $\mathfrak{g}$, just defined slightly differently. Here we also require G to be a matrix Lie group so that we can multiply elements of the algebra with elements of G. The map is denoted $Ad_A : \mathfrak{g} \to \mathfrak{g}$:

$$Ad_A(Y) := AYA^{-1} \tag{6}$$

and we see that it is just the group adjoint. These two maps are connected via the exponential map:

$$e^{ad_X} = Ad_{e^X} \tag{7}$$

which is an expression of the familiar formula for Lie algebras:

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + ... \tag{8}$$

To act Lie groups on spaces, we must discuss representations. Note that the endomorphism ring $End(V)$ of a vector space is also a Lie algebra, by allowing the Lie bracket to equal the commutator of operators $A, B \in End(V)$.

Definition 1.3 (Representation)**.** A **representation** of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism[2] $\mathfrak{g} \to End(V)$. A representation of a Lie group G is a Lie group homomorphism $G \to Aut(V)$, where $Aut(V)$ denotes the space of vector space automorphisms on V. Every Lie algebra homomorphism can be extended in the obvious manner to a Lie group homomorphism.

Definition 1.4 (Dimension)**.** The **dimension** of a representation $\pi : G \to Aut(V)$ is the dimension of V. If $\pi : G \to GL(n, \mathbb{F})$ is a representation of a Lie group with $n \times n$ matrices, then n is the dimension of π.

Definition 1.5 (Equivalent representations)**.** Two representations $\Pi : G \to Aut(V)$ and $\Psi : G \to Aut(V)$ are said to be **equivalent** if there is $S \in Aut(V)$ such that:

$$\Pi(g) = S\Pi(g)S^{-1} \tag{9}$$

Definition 1.6 (Faithful)**.** A representation $\Pi : G \to Aut(V)$ is **faithful** if Π is injective. In other words, if Π is a faithful representation, then different group elements will map to different matrices in $Aut(V)$.

[2]Meaning it preserves $+$ and $[\cdot, \cdot]$

Equivalent representations are essentially the same representation, since a conjugation like in Equation **??** is the same as changing the basis that the two representations are expanded in.

A representation can likewise be defined as an action of G or $\mathfrak{g}$ on V, as given a representation $\pi : \mathfrak{g} \to V$, we have a $\mathfrak{g}$-action on V defined by $X \cdot v := \pi(X)(v)$. The easiest way to define a representation is to give an explicit embedding of the generators of $\mathfrak{g}$ into $End(V)$ as matrices, as every endomorphism can be realized as a matrix. Once the generators are embedded into matrices, then the rest of the representation can be embedded by linearity. Here are a few more definitions to keep in mind.

Representations are more than just a Lie group and a vector space: they also implicitly contain the map from $G \to Aut(V)$. Because of this, we have been labelling a representation $\Pi : G \to Aut(V)$ as Π WLOG. However, we also may label the representation by the triple (Π, G, V) (and similarly for an algebra representation we may label it as $(\pi, \mathfrak{g}, V)$. Since each G-representation induces a canonical $\mathfrak{g}$-representation (and vice versa), it is often useful to abuse notation and to let V denote the G-representation and the $\mathfrak{g}$-representation, since V is the common element of the triple denoting Π and π. This is often used in physics, where for example we may refer to the octet of $SU(3)$ as $\mathbf{8}$ to denote the representation $\pi : \mathfrak{su}(3) \to \mathbb{R}^8$ which has action $\pi(T^a)^{bc} = -if^{abc}$ (the adjoint representation).

We now consider what a morphism between representations looks like. This is called an intertwining map, and is essentially a vector space map which respects the action induced by the representation.

Definition 1.7 (Intertwining map)**.** Let $\Pi : G \to Aut(V)$ and $\Sigma : G \to Aut(W)$ be two representations of a Lie group G, and let $\phi : V \to W$ be a linear map. We call ϕ an **intertwining map** if for each $g \in G$ and $v \in V$ we have:

$$\phi(\Pi(g)v) = \Sigma(g)(\phi v) \tag{10}$$

In other words, the canonical diagram commutes for each element $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\Pi(g)} & V \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ W & \xrightarrow{\Sigma(g)} & W \end{array}$$

An intertwining map which is invertible is a **representation isomorphism**.

An intertwining map is simply a morphism in the category of representations: it preserves both the linear structure of the spaces it acts on and the action of the representation. A more obvious way to view such a map is as follows: denote by $\cdot$ the action of G on V or W, i.e. $g \cdot v := \Pi(g)v$. Then the definition of ϕ being an intertwining map can equivalently be written as:

$$\phi(g \cdot v) = g \cdot \phi(v) \tag{11}$$

which makes it completely obvious that such an intertwining map is a morphism in the category of linear G-representations.

1.1 Algebras vs. Groups

Let G be a Lie group with associated Lie algebra $\mathfrak{g}$. We will now discuss to what extent a representation of G induces a representation of $\mathfrak{g}$, and vice versa.

Definition 1.8. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A **Lie algebra homomorphism** between $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ which respects the Lie bracket, i.e. $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for each $X, Y \in \mathfrak{g}$.

On the other hand, a Lie group homomorphism is simply a group homomorphism which is smooth with respect to the manifold structure of the group. However, there is a subtle disparity between Lie groups and Lie algebras when it comes to maps between them. Lie algebras are simpler objects than Lie groups because they have a fundamentally linear structure. This means that if two Lie groups are the same, then their Lie algebras will be the same; unfortunately, the converse only holds in certain cases where the Lie group has a simple structure. Recall that we can move between $\mathfrak{g}$ and G by exponentiation, $X \mapsto \exp(iX)$, and consider the following theorem.

Theorem 1.1. Let G and H be Lie groups with algebras $\mathfrak{g}$ and $\mathfrak{h}$, and let $\Phi : G \to H$ be a Lie group homomorphism. Then, there is an unique homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{h}$ such that:

$$\Phi(e^{iX}) = e^{i\phi(X)} \tag{12}$$

for each $X \in \mathfrak{g}$. Furthermore, ϕ is given by:

$$\phi(X) := \left.\frac{d}{dt}\right|_{t=0} \Phi(e^{itX}) \tag{13}$$

In this theorem lies the disparity; a Lie group homomorphism Φ induces a Lie algebra homomorphism ϕ, but we have said nothing about the converse. Indeed, the converse is not always true. This associated map also acts functorially, in that if $\Phi : H \to K$ and $\Psi : G \to H$ are homomorphisms of Lie groups and $\Lambda = \Phi \circ \Psi$, then (denoting the map between algebras as lowercase):

$$\lambda = \phi \circ \psi \tag{14}$$

This immediately says that **if two Lie groups are isomorphic, then so are their Lie algebras**. In particular, consider the effect this has on representations of a group.

Corollary 1.1.1. Let $\Pi : G \to Aut(V)$ be a representation of a Lie group G. Then there is an associated representation of Lie algebras $\pi : \mathfrak{g} \to End(V)$. In other words, every representation of a Lie group induces a representation of its Lie algebra.

The converse of this is **only true when G is simply connected**, i.e. $\pi_1(G) = 0$, and can be proved with the Baker-Campbell-Hausdorff formula.

Theorem 1.2. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras. If G is simply connected, then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi(e^{iX}) = e^{i\phi(X)}$.

Corollary 1.2.1. Suppose G and H are simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$, then G is isomorphic to H.

Corollary 1.2.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\pi : \mathfrak{g} \to End(V)$ be a representation of $\mathfrak{g}$. Suppose that G is simply connected. Then there is a corresponding representation $\Pi : G \to Aut(V)$ such that $\Pi(e^{iX}) = e^{i\pi(X)}$ for $X \in \mathfrak{g}$.

As an example of all this, consider the Lie groups $SU(2)$ and $SO(3)$[3]. These have the same Lie algebras, but since $SU(2) \cong S^3$, it is simply connected, while $SO(3)$ has $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$. Let $\phi : \mathfrak{su}(2) \to \mathfrak{so}(3)$ be the corresponding Lie algebra isomorphism. In a later section we will show that every representation of $\mathfrak{su}(2)$ is of the form $\pi_m : \mathfrak{su}(2) \to End(V_m)$ with $dim(V_m) = m+1$. Then, we of course can form a representation of $\mathfrak{so}(3)$ by lifting π_m with the isomorphism, $\sigma_m := \pi_m \circ \phi^{-1}$, and these are precisely the isomorphisms of $\mathfrak{so}(3)$, since $\mathfrak{so}(2)$ and $\mathfrak{su}(3)$ are the same Lie algebra. Here's the catch: if m is even, then there is a representation Σ_m of $SO(3)$ such that $\Sigma_m(e^{iX}) = e^{i\sigma_m(X)}$ for each $X \in \mathfrak{so}(3)$. However, **if m is odd, then there is no representation of $SO(3)$ such that σ_m is the induced Lie algebra representation**.

You should know this fact inherently from physics: it is the statement that rotating systems with half integral spin behaves differently than 3D rotations, i.e. a full 2π rotation of a spin state is not the identity. The representations of $SU(2)$ are exactly the different angular momentum systems; a system with angular momentum j has Hilbert space V_m with $m = 2j$, as the dimension of the Hilbert space is $2j + 1 = m + 1$. The angular momentum operators on this Hilbert space are $\frac{1}{2}\pi_m(\sigma_i)$, where σ_i are the Pauli matrices which generate $\mathfrak{su}(2)$. So, a system with integer angular momentum j (m even) has is described representation of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ which is induced by a representation of the 3D rotation group $SO(3)$, and thus these rotations must behave like 3D rotations. In the half integer case, there is no corresponding representation Σ_m of $SO(3)$, and thus quantum mechanical rotations do not need to behave like rotations in $SO(3)$.

[3] $SU(2)$ is the "universal cover" of $SO(3)$, which means that $SU(2)$ is simply connected and has the same Lie algebra as $SO(3)$. I hope to include covering spaces in these notes at some point if I have time, because I believe that much of the correspondence between these two spaces generalizes to an arbitrary Lie group and its universal cover.

2 Irreducible Representations

Irreducible representations (irreps) are the simplest representations that one can create– they are akin to simple groups in standard group theory, or prime ideals in ring theory. They will give us a way to decompose complicated Lie groups into sums of irreps, which will allow us to study these groups based off of studying these simpler substructures. Any representation we are interested in will be able to be decomposed into irreps, which makes it quite important to study these irreps in the context of physics.

Definition 2.1 (Irreducible representation)**.** If $\Pi : G \to Aut(V)$ is a representation of G, we say that a subspace $W \subset V$ is an **invariant subspace** if $\pi(g)(W) \subseteq W$ for each $g \in G$, i.e. that the group *always* goes into itself under symmetry transformations. We say that W is **nontrivial** if W is a nonempty proper subspace. If the representation π has no nontrivial invariant subspaces, then we call π an **irreducible representation**.

Definition 2.2 (Irreducible subspace)**.** Let $\Pi : G \to Aut(V)$ be a representation of G. If $W \subseteq V$ is invariant and if it contains no proper invariant subspaces, then we call W an **irreducible subspace** of Π.

Note that saying V is an irreducible subspace of itself is the same as saying that the representation V is irreducible.

Because we are studying Lie theory in the context of physics, we are interested in representations of symmetry groups. In particular, these groups are unitary, and so are the representations of these groups.

Definition 2.3 (Unitary)**.** Let $\Pi : G \to Aut(V)$ be a representation. Then Π is an **unitary representation** if $\Pi(g)$ is a unitary operator on V for each $g \in G$. Note that this implies $\Pi(g^{-1}) = \Pi(g)^\dagger$.

Luckily for us, unitary representations are quite simple and can always be decomposed into the direct sum of irreps. We will make this precise with a few definitions.

Definition 2.4 (Completely reducible)**.** A finite dimensional representation of a Lie group or algebra is **completely reducible** if it is isomorphic to a direct sum of a finite number of irreps.

Prop 2.1. Let $\Pi : G \to Aut(V)$ be a completely reducible representation. Then:

1. For each invariant subspace $U \leq V$, V splits as a direct product $V = U \oplus W$ with W also invariant.

2. Every invariant subspace of V is completely reducible.

The last part of this proposition will allow us to prove that unitary representations are completely reducible, hence when studying representations in physics we can build up our representations from irreps without loss of generality.

Theorem 2.1. Any finite dimensional unitary representation of a Lie group or algebra is completely reducible.

Proof. Let $\Pi : G \to Aut(V)$ be unitary. If Π is an irrep, then we are done. So, suppose it is not, and pick a nontrivial invariant subspace $W \leq V$. Then V splits as $V = W \oplus W^\perp$. We show that $W^\perp$ is invariant as well. Let $v \in W^\perp$. Then for each $g \in G$ and $w \in W$, we have:

$$\langle \Pi(g)v|w\rangle = \langle v|\Pi(g^{-1})|w\rangle = 0 \tag{15}$$

because $\Pi(g^{-1})|w\rangle \in W$ as W is invariant. So, $\Pi(g)v \in W^\perp$ for each $g \in G$. We can inductively repeat this process on both W and $W^\perp$ a finite number of times until this terminates with each summand being an irrep, and the process must terminate because V is finite dimensional. $\square$

In addition, finite representations of compact Lie groups are also completely reducible. We will not prove this, but it is certainly a nice fact to know.

Theorem 2.2. If G is compact, then every finite dimensional representation of G is completely reducible.

An important theorem about irreps is called **Schur's lemma**. We will simply state the theorem.

Lemma 2.3 (Schur)**.**

1. Let V and W be irreducible real or complex representations of a group or Lie algebra and let $\phi : V \to W$ be an intertwining map (representation homomorphism). Then ϕ is either identically 0 or an isomorphism of representations.

2. Let V be a complex irrep of a group or Lie algebra and let $\phi : V \to V$ be an intertwining map. Then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.

3. Let V and W be complex irreps of a group or Lie algebra and $\phi_1, \phi_2 : V \to W$ be nonzero intertwining maps. Then $\phi_1 = \lambda\phi_2$ for some $\lambda \in \mathbb{C}$.

Schur's lemma essentially states that any two irreps which are homomorphic are also isomorphic, and so irreps are the simplest building blocks of representations. There is also a few nice corollaries which deal with commuting objects in the representation.

Corollary 2.3.1. Let $(\Pi_1, V), (\Pi_2, V)$ be two inequivalent irreps of G. If $\Pi_1(g)A = A\Pi_2(g)$ for each $g \in G$, then $A = 0$.

Corollary 2.3.2. Let (Π, V) be an irrep of G. If $\Pi(g)A = A\Pi(g)$ for each $g \in G$ and $A \in Aut(V)$, then $A = \lambda I$.

Corollary 2.3.3. Let $\mathfrak{g}$ be a Lie algebra and (π, V) is an irrep of $\mathfrak{g}$. If X is in the center of $\mathfrak{g}$ (so $[X, Y] = 0$ for each $Y \in \mathfrak{g}$), then $\pi(X) = \lambda I$.

These theorems are all very similar, and the main point is that **any element in an irrep of G which commutes with all other elements of G is proportional to the identity**. This can be a surprisingly powerful fact, and it often is applicable in places you might not expect it to be. It also proves the following fact, because if a group is commutative then each element must be mapped to a multiple of I.

Corollary 2.3.4. An irrep of a commutative group is one dimensional.

Finally, we must discuss some notation. We will often denote a representation by its dimension by simply writing a number for the representation. For example, the adjoint of $SU(3)$ is 8-dimensional, the fundamental and anti-fundamental representations are 3-dimensional, and the singlet (trivial) representation is 1-dimensional. We denote the adjoint by **8**, the fundamental by **3**, the anti-fundamental by **bar3**, and the singlet by **1**. One will often see equations written out in physics books that look like:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \tag{16}$$

All this means is that if we tensor together the fundamental with the anti-fundamental, it splits as a direct sum of the adjoint plus the singlet. We will discuss some of the common irreps of Lie groups in the following subsections.

2.1 The Fundamental Representation of $SU(N)$

$SU(N)$ plays a central role in the Standard model, so its representations are very relevant. Although we will describe its full representation theory later in Section **??**, there we will describe one of its specific representations here. Called the **fundamental representation** of $SU(N)$, this is the representation (π, V) of order N.

2.2 The Adjoint Representation

The adjoint representation is perhaps the most canonical of representations because every Lie algebra admits an adjoint representation on itself. For $X \in \mathfrak{g}$, recall we defined the adjoint map $ad_X : \mathfrak{g} \to \mathfrak{g}$ by:

$$ad_X(Y) := [X, Y] \tag{17}$$

It is easy to show that the map $\mathfrak{g} \to End(\mathfrak{g})$, $X \mapsto ad_X$ is a Lie algebra homomorphism. Thus this must define a representation, called the **adjoint representation** of $\mathfrak{g}$, in which the elements of $\mathfrak{g}$ can act on the algebra that they span. Suppose that our Lie algebra is d dimensional, so it has d generators T^a. We can explicitly embed the T^a into the adjoint representation by matrices T_A^a. To compute these matrices, suppose that we have an element $X = X^a T^a \in \mathfrak{g}$. Since the matrices T_A^a act on the column vector X^a, they must be $d \times d$ dimensional. Additionally, when T_A^a is multiplied against the vector X^a, it must give the components of $ad_{T^a}(X)$ (because T_A^a represents the operator ad_{T^a}), so:

$$(ad_{T^a}(X))^b = (T_A^a)^{bc} X^c \iff ([T^a, X^c T^c])^b = i X^c f^{acb} = (T_A^a)^{bc} X^c \tag{18}$$

And thus we see that the components of the generators in the adjoint representation must be given by:

$$(T_A^a)^{bc} = -i f^{abc} \tag{19}$$

Consider what the transformation law looks like in the adjoint representation. Of course, this is a matrix multiplication of X^a by the group element. Let $V = \exp(i\alpha^a T_A^a)$ be an element of G in the adjoint representation. Then:

$$X^a \mapsto (X^a)' = V^{ab} X^b \tag{20}$$

by definition. Infinitesimally, this is:

$$X^a \mapsto (1 + i\alpha^c T_A^c)^{ab} X^b = X^a + i\alpha^c(-i f^{cab}) X^b = X^a - f^{abc}\alpha^b X^c \tag{21}$$

Now for the interesting question: what does the transformation law for the element $X = X^a T^a$ look like? We claim that the matrix X actually transforms under conjugation by V:

$$X \mapsto X' = V X V^\dagger \tag{22}$$

We can check this by expanding out this transformation law for an infinitesimal α:

$$\begin{aligned} X^a T^a \mapsto (1 + i\alpha^a T^a)(X^b T^b)(1 - i\alpha^c T^c) &= X + i\alpha^a X^b [T^a, T^b] = X - \alpha^a X^b f^{abc} T^c \\ &= (X^a - f^{abc}\alpha^b X^c) T^a \end{aligned}$$

which is exactly the transformation law we were looking for. Thus, we can view transformation in the adjoint representation in two different ways. First, as how the vector of component fields transforms:

$$X^a \mapsto V^{ab} X^b \tag{23}$$

And secondly, how the entire matrix X transforms:

$$X \mapsto V X V^\dagger \tag{24}$$

3 Constructions

3.1 Complexification

It is often useful to study a real Lie algebra by complexifying it and studying the algebra as a complex vector space instead of a real vector space.

Definition 3.1 (Complex Lie group)**.** A Lie group G is **complex** if its Lie algebra $\mathfrak{g}$ is a vector space over $\mathbb{C}$, i.e. if $iX \in \mathfrak{g}$ for each $X \in \mathfrak{g}$.

We will next define how to complexify a real vector space and treat it as a $\mathbb{C}$-vector space. This will extend naturally to any Lie algebra (as a Lie algebra has a vector space structure in addition to its Lie bracket) in a functorial manner.

Definition 3.2 (Complexification)**.** Let V be a vector space over $\mathbb{R}$. Then the **complexification** of V, denoted $V_{\mathbb{C}}$, is the set of formal linear combinations:

$$V_{\mathbb{C}} = \{v_1 + iv_2 : v_1, v_2 \in V\} \tag{25}$$

We endow $V_{\mathbb{C}}$ with a vector space structure by defining for $v_1 + iv_2 \in V_{\mathbb{C}}$:

$$i(v_1 + iv_2) := -v_2 + iv_1 \tag{26}$$

This extends by linearity to all of $V_{\mathbb{C}}$, making it into a vector space over $\mathbb{C}$.

The complexification of a vector space is in itself also a real vector space of dimension $2dim(V)$, since as a set it is isomorphic to the direct sum $V \oplus V$. For a Lie algebra, we can use this construction to complexify the structure of the algebra as a vector space, and also extend the bracket to this new complex space. The complexification is also the "best" way we can embed a real Lie algebra into a complex Lie algebra, in the sense that it is a universal attractor in the category of maps $\mathfrak{g} \to h$, where h is a complex Lie algebra.

Theorem 3.1. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Let $\mathfrak{g}_{\mathbb{C}}$ be its complexification as a vector space. Then the bracket $[\cdot, \cdot]$ extends uniquely to $\mathfrak{g}_{\mathbb{C}}$ by:

$$[X_1 + iX_2, Y_1 + iY_2] := ([X_1, Y_1] - [X_2, Y_2]) + i([X_2, Y_1] + [X_1, Y_2]) \tag{27}$$

making $\mathfrak{g}_{\mathbb{C}}$ into a Lie algebra.

Prop 3.1. Let $\mathfrak{g} \subset M_n(\mathbb{C})$ be a real Lie algebra, and suppose that $iX \notin \mathfrak{g}$ for $X \in \mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to the set of matrices in $M_n(\mathbb{C})$ of the form $X + iY$ with $X, Y \in \mathfrak{g}$.

Theorem 3.2. Let $\mathfrak{g}$ be a real Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$, and let $\iota : \mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ be the canonical inclusion. Then $\mathfrak{g}$ is universal in the sense that if $\mathfrak{h}$ is any complex Lie algebra and $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then we can factor f through $\mathfrak{g}_{\mathbb{C}}$:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathfrak{g}_{\mathbb{C}} \arrow[d, "\overline{f}"] \\ & \mathfrak{h} \end{tikzcd}$$

Some common complexifications which are useful are:

- $\mathfrak{u}(N)_{\mathbb{C}} \cong gl(N;\mathbb{C})$
- $\mathfrak{s}u(N)_{\mathbb{C}} \cong \mathfrak{sl}(N;\mathbb{C})$

Complexification leaves irreps invariant, so studying the irreps of a complexified Lie algebra is equivalent to studying the irreps of the original algebra. This will be very useful as it will allow us to study the irreps of $\mathfrak{su}(N)$ through the irreps of $\mathfrak{sl}(n;\mathbb{C})$.

Theorem 3.3. Let $\mathfrak{g}$ be a real Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$. Then every finite dimensional complex representation π of $\mathfrak{g}$ has a unique extension to a complex representation of $\mathfrak{g}_{\mathbb{C}}$ by linearity, also denoted π. Furthermore, π is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$ iff it is irreducible as a representation of $\mathfrak{g}$.

3.2 Sums

Recall the direct sum $V_1 \oplus V_2$ of two vector spaces V_1 and V_2 is simply a fancy way of writing the Cartesian product $V_1 \times V_2$ after we give it a vector space structure. Let G be a Lie group with two representations $\pi_1 : G \to V_1$ and $\pi_2 : G \to V_2$. Then we can form a representation of G on $V_1 \oplus V_2$ in the obvious way:

$$\pi_1 \oplus \pi_2 : G \to Aut(V_1 \oplus V_2) \tag{28}$$

where $\pi_1 \oplus \pi_2(g)$ acts on elements $(v_1, v_2) \in V_1 \oplus V_2$ by:

$$[(\pi_1 \oplus \pi_2)g](v_1, v_2) := ((\pi_1 g)v_1, (\pi_2 g)v_2) \tag{29}$$

We define this in an analogous way for a representation of a Lie algebra.

When we sum together two representations, their dimensions are additive because the direct sum adds dimensions of vector spaces:

$$dim(\pi_1 \oplus \pi_2) = dim(\pi_1) + dim(\pi_2) \tag{30}$$

An important thing to note is that the direct sum of representations is not the same things as the direct sum of Lie algebras or Lie groups. While the operation is still defined in this case, it acts on different objects, and can have profound differences. Namely, note that (of course) the direct sum of irreps is not an irrep, because doing this gives an explicit decomposition of the representation. However, when we take a direct sum of Lie algebras, then in fact there is a way to get irreps from their individual irreps.

Theorem 3.4. Let $\mathfrak{g}, \mathfrak{h}$ be two Lie algebras. Then the irreps of $\mathfrak{g} \oplus \mathfrak{h}$ are precisely the tensor products of the irreps of $\mathfrak{g}$ and of $\mathfrak{h}$. In other words, if (π_a, V_a) are the irreps of $\mathfrak{g}$ and (ϕ_b, W_b) are the irreps of $\mathfrak{h}$, then the irreps of $\mathfrak{g} \oplus \mathfrak{h}$ are precisely the tensor products:

$$\{(\pi_a \otimes \phi_b, V_a \otimes W_b)\}_{a,b} \tag{31}$$

3.3 Products

Suppose that we have representations of groups G and H, namely $\Pi_1 : G \to Aut(V)$ and $\Pi_2 : H \to Aut(W)$. Then we can define a canonical representation of $G \times H$ by:

$$\Pi_1 \otimes \Pi_2 : G \times H \to Aut(V \otimes W) \qquad\qquad (g, h) \mapsto (\Pi_1 g) \otimes (\Pi_2 h) \tag{32}$$

where we see that $(\Pi_1 g) \otimes (\Pi_2 h)$ is a morphism on the space $V \otimes W$ by its definition.

Now, further suppose that $H = G$, and so we have two representations of G, $\Pi_1 : G \to V$ and $\Pi_2 : G \to W$. Then using the previous construction, we can embed G into $G \times G$ via the diagonal map:

$$G \hookrightarrow G \times G \xrightarrow{\Pi_1 \otimes \Pi_2} Aut(V \otimes W) \tag{33}$$

to form a new representation of G, one which acts on $V \otimes W$ as $(\Pi_1 \otimes \Pi_2)(g) = (\pi_1 g) \otimes (\pi_2 g)$. Note the dimensionality of the new representation is:

$$dim(\Pi_1 \otimes \Pi_2) = dim(\Pi_1)dim(\Pi_2) \tag{34}$$

In general, the product of irreducible representations is not reducible, and the factorization of the product of irreps into a sum of irreps is the basis for the Clebsch-Gordan theory often studied in the context of adding angular momentum in quantum mechanics. For example, consider the irreps of $SU(3)$. Then $\mathbf{3} \otimes \bar{\mathbf{3}}$ (the fundamental times the antifundamental) is reducible, and in fact $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$, i.e. it splits as the sum of the adjoint 8 and the singlet 1.

There is a corresponding induced representation of the Lie algebra on the tensor product. If we have two Lie algebra representations $\pi_1 : \mathfrak{g} \to End(V)$ and $\pi_2 : \mathfrak{g} \to End(W)$, then the tensor product representation of $\mathfrak{g}$ is the representation:

$$\pi_1 \otimes \pi_2 : \mathfrak{g} \to End(V \otimes W) \tag{35}$$

$$(\pi_1 \otimes \pi_2)(X) = \pi_1(X) \otimes I + I \otimes \pi_2(X) \tag{36}$$

Although this looks a bit clunky, this exponentiates to the correct representation of G since $[\pi_1(X) \otimes I, I \otimes \pi_2(X)] = 0$.

$$e^{i(\pi_1 \otimes \pi_2)(X)} = e^{i\pi_1(X) \otimes I + iI \otimes \pi_2(X)} = e^{i\pi_1(X)} \otimes e^{i\pi_2(X)} \tag{37}$$

3.4 Conjugate (dual) representations

Finally, given a complex representation (π, V) of a Lie algebra $\mathfrak{g}$, we can construct a conjugate representation. Then the **conjugate representation** to (π, V) is the representation $\pi^* : \mathfrak{g} \to End(V^*)$ (where V^* is the dual space of V) defined as:

$$\pi^*(X) := -\pi(X)^T \tag{38}$$

Note that for the case where the generators $\{\pi(X)\}$ are unitary (as in the case of the fundamental representation of $SU(N)$), then $-\pi(X)^T = -\pi(X)^*$ where $*$ is the conjugate, so one will often see this definition. For representations of Lie groups (Π, V), this induces the representation $\Pi^* : G \to Aut(V^*)$ defined as:

$$\Pi^*(g) = \Pi(g^{-1})^T \tag{39}$$

In physics, we will often see the generators of the fundamental representation written with less jargon. If $T^a_r = \pi(X^a)$ are the images of the generators $\{X^a\}$ of a Lie algebra $\mathfrak{g}$ in a representation r, then the generators of the conjugate representation $T^a_{\bar{r}}$ are:

$$(T^a_{\bar{r}})_{ij} = -(T^a_r)^*_{ij} = -(T^a_r)_{ji} \tag{40}$$

As a point of notation, if $\mathbf{r}$ is denoting a representation of G (i.e. for $SU(3)$ we denote the fundamental by $\mathbf{3}$) then we will denote its conjugate representation by $\bar{\mathbf{r}}$. The conjugate of the fundamental representation is called the **anti-fundamental** representation.

We define a representation to be **real** if it equals its conjugate representation. In the case of $SU(N)$, for $N > 2$ the fundamental representation is complex. For arbitrary $SU(N)$, the adjoint representation is always a real representation.

4 Finite Groups and Characters

Almost all of the theory we have discussed so far is in application to Lie groups and Lie algebras, which is the basis for how we study continuous symmetries in physics. However, it is also important to consider representations of finite groups, as these correspond to discrete symmetries. The representation theory of finite groups takes on a slightly different flavor than that of Lie groups; it is a bit more combinatorial, and we need to use different machinery than described above with Lie groups and algebras. To emphasize we are using a finite group, when we discuss a representation of a finite group G we will use the notation:

$$D : G \to Aut(V) \tag{41}$$

and refer to (D, V) as the representation of G. There are a few immediate theorems right off the bat pertaining to finite groups that we will not prove.

Theorem 4.1. Every representation of a finite group is equivalent to a unitary representation.

This fact means that for a finite group G, every representation (D, V) of G is completely reducible. So, if we can develop the machinery to classify all the irreps of G, then we can know exactly every representation of G.

4.1 The Regular Representation

4.2 Symmetric groups

5 Invariants

6 Roots and Weights

7 Representations of $SU(N)$

$SU(N)$ is the special unitary group of $N \times N$ matrices. Specifically, a matrix $A \in M_N(\mathbb{C})$ is in the **unitary group** of order N, denoted $U(N)$, if A is unitary, i.e. $A^\dagger A = AA^\dagger = 1$. $A \in U(N)$ is in the **special unitary group** if it has unit determinant, $det(A) = 1$. The group $U(1)$ is simply the unit circle in the complex plane, and for each N we have:

$$U(N) = U(1) \times SU(N) \tag{42}$$

The dimension of $U(N)$ as a real Lie group is N^2, and the dimension of $SU(N)$ is thus $N^2 - 1$.

When discussing the representation theory of $SU(n)$, one will often encounter the algebras $\mathfrak{sl}(N;\mathbb{C})$, and it behaves in quite a similar manner to $\mathfrak{su}(N)$. These $\mathfrak{sl}(N;\mathbb{C})$ algebras are simply the complexification of the real valued $\mathfrak{su}N$ algebras, i.e. $\mathfrak{sl}(N;\mathbb{C}) \cong \mathfrak{su}(N)_\mathbb{C}$, and as we have discussed earlier, this implies that they share the same irreps as $\mathfrak{su}(N)$. So, if you encounter an article about the irreps of $\mathfrak{sl}(N;\mathbb{C})$, fear not– for all intents and purposes, the article is about $\mathfrak{su}(N)$.

7.1 $SU(2)$ and Angular Momentum

This section will perform the classification of angular momentum done in quantum mechanics in a much more abstract sense. The Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ is simply the complexification of $\mathfrak{su}(2)$, and so has the same irreps and structure as $\mathfrak{su}(2)$. This is particularly important because of how the quantum mechanics of angular momentum works. **A quantum mechanical system with angular momentum j is described by the representation of $\mathfrak{su}(2)$ with dimension** $2j+1$, and thus studying the representations of $\mathfrak{sl}(2;\mathbb{C})$ is equivalent to studying quantum mechanical systems with different values of angular momentum. In particular, angular momentum operators are elements of the representation π_{2j} of $\mathfrak{su}(2)$, and rotation operators are elements of the representation Π_{2j} of $SU(2)$. We will describe and characterize these representations Π_m and π_m in this section.

Because of this, things like Clebsch-Gordan theory can be described purely mathematically in terms of representations of $SU(2)$. Addition of angular momentum is simply decomposing tensor product representations of $SU(2)$ into a direct sum of irreps of $SU(2)$. For example, the equation for adding angular momentum:

$$\mathbf{j_1} \otimes \mathbf{j_2} = \bigoplus_{\mathbf{j}=|\mathbf{j_1}-\mathbf{j_2}|}^{\mathbf{j_1}+\mathbf{j_2}} \mathbf{j} \tag{43}$$

allows one to fully unravel the addition of angular momentum in quantum mechanics. This may be viewed as a mathematical statement about adding representations; when the tensor product representation $\pi_{2j_1} \otimes \pi_{2j_2}$ of $\mathfrak{su}(2)$ is formed, it can be decomposed into the direct sum representation $\pi_{|j_1-j_2|} \oplus ... \oplus \pi_{j_1+j_2}$ of $\mathfrak{su}(2)$.

7.2 $SU(3)$ and the Quark Model

7.3 $SU(N)$

8 Representations of other groups

8.1 The Lorentz Group

Let $g_{\mu\nu} := diag(1,-1,-1,-1)$ be the standard Minkowski metric. The **Lorentz group**, denoted $SO(1,3)$, is the subgroup of $M_{4\times 4}(\mathbb{R})$ which leaves g invariant under conjugation, i.e. a 4 by 4 matrix Λ is in $SO(1,3)$ iff:

$$\Lambda^T g \Lambda = g \tag{44}$$

As you know from Special Relativity, flat spacetime has a Minkowski metric and elements of the Lorentz group boost states into different frames. One also may have studied the transformation properties of the Lorentz group in a quantum mechanics class; of primary interest to us is its Lie algebra. The algebra $\mathfrak{so}(1,3)$ has 6 generators, J_i and K_i for $i \in \{1,2,3\}$. They satisfy:

8.2 The Poincaré Groups

8.3 The Hypercubic Group