Differential Forms

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Differential forms generalize the notion of integration over Euclidean space to an arbitrary smooth manifold. Although they take a bit of manifold theory to even get to the definition, they are very useful for all areas of math and physics. In particular, they provide the ultimate generalization of multivariable calculus. We will see that there is a general analog of Stokes' theorem which is the origin of all the fundamental theorems of calculus you have studied (i.e. the one-dimensional version, divergence theorem, Green's theorem, etc.).

Once we work up to the definition of a form and how to integrate such an object, we will turn to cohomology theories which stem from the existence of forms. Namely, we will study the de Rham complex on a manifold, which is the cochain complex generated by the exterior derivative. We will then turn to Čech cohomology and its similarities with the de Rham cohomology.

1 Background

I will most likely write this up into a separate notes on differential manifolds later, but for now I will state the main definitions and theorems that we will need to study forms. I will assume the reader is familiar with the definition of a differentiable manifold M, but we will briefly recap the definitions for notations sake.

Note in these notes unless otherwise specified we will use summation convention, so any repeated index is summed over. We will also try to be accurate with upper and lower indices.

1.1 Smooth Manifolds

A **chart** on a space is a pair (U, ϕ) such that $U \subseteq M$ is open and $\phi : U \xrightarrow{\sim} \mathbb{R}^n$ is a homeomorphism onto an open subset $\phi(U) \subseteq \mathbb{R}^n$. We call n the **dimension** of M. A collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of charts which cover M is called an **atlas** if the overlap functions $\phi_\alpha \circ \phi_\beta^{-1}$ are smooth on $\mathbb{R}^n$ whenever their images overlap; this condition on the charts is known as being C^∞ **compatible**. A **smooth manifold** is a topological manifold with a **maximal atlas**, namely an atlas $\mathfrak{M}$ such that if $\mathfrak{A}$ is another atlas of M which contains $\mathfrak{M}$, then $\mathfrak{M} = \mathfrak{A}$.

These definitions are technical, and most likely will never be used in these notes. The important property that a smooth manifold M has is it can be covered by charts, which put a coordinate system around every point $p \in M$. We will often write a chart (U, ϕ) as $(U, x^1, ..., x^n)$, where x^i are the coordinates on the chart $U \subset M$. Formally, if we let $r^i : \mathbb{R}^n \to \mathbb{R}$ be the coordinate functions on $\mathbb{R}^n$, then the coordinates x^i are defined as $x^i := r^i \circ \phi$.

A function $f : M \to \mathbb{R}$ is **smooth**, or C^∞, at a point $p \in M$ if there is a chart (U, ϕ) about p such that $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth as a map between Euclidean spaces. f is a **smooth function** if it is smooth at each $p \in M$. We use the notation:

$$C^\infty(M) := \{f : M \to \mathbb{R} \mid f \text{ is smooth.}\} \tag{1}$$

A map $f : M \to N$ between two manifolds M and N (of dimensions m and n, respectively) is said to be **smooth** at $p \in M$ if there are charts (U, ϕ) about $p \in M$ and (V, ψ) about $f(p) \in N$ such that the composition $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth. f is smooth if it is smooth at each point in M. A **diffeomorphism** is an isomorphism in the category of smooth manifolds, and it is a bijective C^∞ map $M \to N$ which has a smooth inverse. If $F : M \to N$ is a map and $(V, y^1, ..., y^n)$ is a chart about $\phi(p) \in N$, then the components of F are functions $F^i := y^i \circ F : F^{-1}(V) \subset M \to \mathbb{R}$, and F is smooth iff its component functions are smooth on each chart (by the overlap condition, if F^i is smooth on one chart then it is smooth on every chart).

We can extend the notion of differentiation immediately to C^∞ functions given a chart. Let $p \in M$ be contained by a chart (U, ϕ). Let r^i be the standard coordinates on $\mathbb{R}^n$ and push these to coordinates x^i on M. For $f \in C^\infty(M)$ the **partial derivative** of f at $p \in M$ is:

$$\frac{\partial}{\partial x^i}\bigg|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} (f \circ \phi^{-1}) \tag{2}$$

where the last expression is defined as a derivate on $\mathbb{R}^n$. Note that $\partial x^i / \partial x^j = \delta^i_j$, as it should.

We conclude this section by stating the Inverse Function Theorem. A map $f : M \to N$ is called **locally invertible** at a point $p \in M$ if there is a neighborhood U about p such that $f|_U : U \to f(U)$ is a diffeomorphism.

Theorem 1.1 (Inverse Function Theorem)**.** Let $f : M \to N$ be a map between two manifolds of the same dimension, and $p \in M$. Let (U, x^i) be a chart about p, and suppose that there is a chart (V, y^i) about $f(p)$ such that $f(U) \subseteq V$. Then f is locally invertible iff the Jacobian determinant is nonzero, i.e. iff

$$det \left(\frac{\partial f^i}{\partial x^j}(p) \right) \neq 0 \tag{3}$$

where $f^i = y^i \circ f$.

1.2 The Tangent Bundle

To each point p on a manifold M, we can associate a space of tangent vectors, called the **tangent space** and denoted $T_p M$. Visualizing this in your head provides good intuition, but it is not perfect; namely, when you visualize tangent vectors to a point, you tend to think of the manifold being embedded into a higher dimensional Euclidean space. Although manifolds often come with such an embedding, we want a way to define tangent vectors independent of embedding in a larger space– a way to define a tangent space that is intrinsic to the manifold itself. To do this, we note that for each tangent vector $v \in T_p M$, we can define a curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. In other words, we can find a curve running through p whose derivative is v.

One way to make this definition is to define the tangent space as the tangent vectors of curves running through the point, but we will take a slightly different approach. For us, a tangent vector will be defined as an operator which acts like a directional derivative, since for each vector we can associate a such a directional derivative. For a function $f \in C^\infty(M)$, we define the **germ** of f at p to be the class of functions which equal f on some open neighborhood about p. This is an equivalence relation $\sim$ on $C^\infty(M)$, and we denote by $C^\infty_p(M)$ the set of germs of $C^\infty(M)$ functions at p. The notion of a derivative can be shown to be in one-to-one correspondence with any operator that satisfies the Leibniz rule, called a derivation:

Definition 1.1 (Derivation, tangent vector)**.** A **derivation** at a point $p \in M$ is a linear map $D : C^\infty_p(M) \to \mathbb{R}$ such that:

$$D(fg) = D(f)g(p) + f(p)D(g) \tag{4}$$

A **tangent vector** at a point $p \in M$ is a derivation at p, and we denote the space of tangent vectors by T_pM.

Given a chart (U, x^i), the set of partial derivatives evaluated at p are derivations at p, and in fact form a basis for the tangent space:

$$\text{span}\left\{ \left. \frac{\partial}{\partial x^i} \right|_p \right\}_{i=1}^n = T_pM \tag{5}$$

which is easy to prove. The tangent space of an n-dimensional manifold is thus an n-dimensional vector space. So, given a tangent vector $X_p \in T_pM$, we define X_p by its action X_pf on functions $f \in C^\infty(M)$ (since we can drop each function down to the germs at p, $C_p^\infty(M)$). We can similarly expand it about the derivatives:

$$X_p = \sum_{i=1}^n X^i \partial_i \tag{6}$$

where X^i are the coordinates of the vector X.

Given a map $f : M \to N$, we get a natural map $f_* : T_pM \to T_{f(p)}N$ for each $p \in M$ defined by:

$$(f_* X_p)(g) := X_p(g \circ f) \tag{7}$$

where here g is a function in $C^\infty(N)$, and so $g \circ f \in C^\infty(M)$. This map is generally called the **push-forward**, or the **differential**. When we dualize to study covectors, we will see its analog is a natural chain map between the de Rham cochain complexes. The push forward can be thought of as a sort of "derivative" since it approximates a smooth map as a linearized map between tangent spaces. It is often denoted as df instead of f_*, but we will reserve the use of df for discussing differential forms[1] later. Indeed, it even satisfies a functorial chain rule.

Theorem 1.2. Let $f : M \to N$ and $g : N \to P$ be smooth maps between manifolds. Then:

$$(g \circ f)_* = g_* \circ f_* : T_pM \to T_{(g \circ f)(p)}P \tag{8}$$

Let $U \subset M$ be open and $f : M \to N$ be a smooth map. If $f_* : T_pM \to T_{f(p)}N$ is an injective map as p ranges over U, then f is called an **immersion** on U, and if f_* is surjective on U, then f is called a **submersion**.

The set of tangent spaces to a manifold naturally has a fiber bundle structure. We define the **tangent bundle** TM to be the set:

$$TM := \coprod_{p \in M} T_pM \tag{9}$$

There is a natural projection map sending each vector to its base point:

$$\pi : TM \to M \tag{10}$$
$$X_p \mapsto p \tag{11}$$

We can define a topology on the total space TM. To define this topology, consider a chart $(U, \phi) = (U, x^i)$, and let $n = \dim(M)$. On TU, we can represent a general element by $(p, v^i \partial_i) = $

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[1]We will see that when the exterior derivative acts on a map $h : M \to \mathbb{R}$, the one-form dh is naturally the map $h_* : T_pM \to \mathbb{R} \in T_p^*M$, and so the notation df and f_* is really one and the same for real-valued maps in $C^\infty(M)$.

$(x^1, ..., x^n, v^1, ..., v^n)$, where $p = (x^1, ..., x^n)$ and the derivatives ∂_i are evaluated at p. So, we have a natural map:

$$\tilde{\phi} : TU \to \mathbb{R}^{2n} \tag{12}$$

$$(p, v^i \partial_i) \mapsto (x^1, ..., x^n, v^1, ..., v^n) \tag{13}$$

This is a bijection between TU and $U \times R^n$, and so we can use this to transfer the topology of R^{2n} to TU. This makes TU into a topological space. Furthermore, given a maximal atlas $\{(U_\alpha, \phi_\alpha)\}$ of M, the set $\{(TU_\alpha, \tilde{\phi}_{\alpha*})\}$ is a maximal atlas for TM. So, the bundle TM **is naturally a** $2n$ **dimensional manifold**. As a fiber bundle, the total space is TM and the base space is M.

Given a map $f : M \to N$, we can use the differential at each tangent space to define a push forward between the two bundles:

$$f_* : TM \to TN \tag{14}$$

$$(p, X_p) \mapsto (f(p), f_*(X_p)) \tag{15}$$

which is naturally a morphism of bundles.

Recall the definition of a section of a fiber bundle.

Definition 1.2 (Section). Let $\pi : E \to B$ be a fiber bundle. A **section** of (π, E, B) is a continuous map $s : B \to E$ such that $\pi \circ s = id_B$. A section is **smooth** if s is a smooth map. We denote the set of all C^∞ sections of of E by $\Gamma(E)$.

Definition 1.3 (Frame). A **frame** of a vector bundle $\pi : E \to B$ over an open set $U \subseteq B$ is a collection of sections $\{s_1, ..., s_r\}$ such that at each $p \in B$, $\{s_1(p), ..., s_r(p)\}$ is a basis for the fiber $\pi^{-1}(p)$.

A section[2] of the tangent bundle is called a **vector field**, which is perhaps the easiest way to visualize the definition of a section; it is simply a continuous assignment of each base point to an element in its fiber. We denote by $\mathfrak{X}(M)$ the set of all C^∞ vector fields on M, i.e:

$$\mathfrak{X}(M) := \{X \in \Gamma(TM) : X \text{ is } C^\infty\} \tag{16}$$

A frame on the bundle TM is then a set of pointwise linearly independent vector fields, and the requirement that the tangent bundle has a global frame means that there are n nowhere vanishing, linearly independent vector fields on the entire manifold. As an example of a frame, the vector fields $\{\partial_x, \partial_y, \partial_z\}$ form a global frame on $\mathbb{R}^3$.

1.3 Partitions of Unity

2 Differential Forms

At their most basic level, differential form generalize differential notions of length, area, volume, etc. in Euclidean space to arbitrary manifolds. These generalize the notion of integration to manifolds, and in the next section we will see how to integrate a form. If I give you a vector (a one-dimensional shape) and ask you its length, what do you say? You give me a real number! The length ℓ of a vector in a space V is simply a map $V \to \mathbb{R}$, i.e. it is a dual vector $\ell \in Hom(V, \mathbb{R})$.

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[2]In physics, whenever we think about a "field", we are really thinking about sections of a bundle (or we are just thinking about standard functions). For example, a QCD gauge field is a section of a fiber bundle with fibers $SU(3)$ over the base space $\mathbb{R}^4$. Sections are thus a concept that you've almost certainly seen before, albeit in a much less rigorous way.

This generalizes to higher dimensional objects, which we will take to be "oriented products" of vectors. For example, in two dimensions, suppose I represent a parallelogram with sides v and w by some object $v \wedge w$. Then to have an oriented area, I must have $v \wedge w = -w \wedge v$. There is a natural construction called the **exterior power** of a vector space which takes in a space V and spits an object with such an antisymmetric product of copies of V, and later we will use this to define a k-form, our generalized notion of volume.

2.1   1-forms

Before working with k-forms, we must define 1-forms.

Definition 2.1 (Cotangent space)**.** Let $p \in M$. The **cotangent space** at p is the dual of the tangent space at p, i.e. is:
$$T_p^* M := (T_p M)^* = Hom(T_p M, \mathbb{R}) \tag{17}$$

We call elements of the cotangent space **covectors**.

So, a covector takes in a vector and spits out a number, i.e. if $\omega_p \in T_p^* M$, then $\omega_p(X_p) \in \mathbb{R}$ for any vector $X_p \in T_p M$. Let (U, x^i) be a chart about $p \in M$. Then as $\{\partial_i\}_{i=1}^n$ is a basis for $T^p M$, we have a corresponding dual basis $\{dx^i\}_{i=1}^n$. The dual basis satisfies:
$$dx^i(\partial_j) = \delta_j^i \tag{18}$$

We will soon see that dx^i can be determined by acting the exterior derivative on the coordinate function x^i, and so this notation is quite suggestive. Note that we can use the cotangent spaces to define a fiber bundle:

Definition 2.2 (Cotangent bundle)**.** The **cotangent bundle** $T^* M$ of a manifold M is:
$$T^* M := \coprod_{p \in M} T_p^* M \tag{19}$$

We can define a bijection with $\mathbb{R}^{2n}$ by sending $(x^1, ..., x^n, \omega_i dx^i)$ to the corresponding element $(x^1, ..., x^n, \omega_1, ..., \omega_n)$, and use this to transfer the topology to $T^* M$, making it into a smooth manifold just like TM. We can now define a 1-form (for good measure we also define a 0-form):

Definition 2.3 (1-form)**.** A **differential 1-form** ω is a section of the cotangent bundle, i.e. a map $\omega : M \to T^* M$ with $\pi \circ \omega = id_M$. A 1-form is **smooth** if it is smooth as a map $M \to T^* M$. The set of all C^∞ 1-forms will be denoted by $\Omega^1(M)$, so we have:
$$\Omega^1(M) := \Gamma(T^* M) \tag{20}$$

Definition 2.4 (0-form)**.** We define a 0-form on M to be a smooth function:
$$\Omega^0(M) := C^\infty(M) \tag{21}$$

We already have seen some examples of 1-forms: consider a function $f \in C^\infty(M)$. The induced push forward map is $f_* : T_p M \to T_{f(p)} \mathbb{R}$. However, there is a canonical isomorphism $\iota : T_q R \to \mathbb{R}$, $a\frac{d}{dt}|_q \mapsto a \in \mathbb{R}$. Hence, we consider the **differential map** $df|_p := \iota \circ f_*$, which is a map $T_p M \to \mathbb{R}$. This has the exact same action as the push-forward, we just "forget" the basis vector and consider the corresponding scalar. Thus, $df|_p$ is a linear map from $T_p M$ into $\mathbb{R}$, hence is a covector, $df|_p \in$

T_p^*M. As we vary p across the entire manifold, we see that the map df is in fact a section of the cotangent bundle, and thus df is a 1-form, which we will soon see is a smooth 1-form.

The explicit action of $df \in \Omega^1(M)$ on a vector $X_p \in T_pM$ is:

$$df(X_p) = X_p(f) \tag{22}$$

which is valued in $\mathbb{R}$, as X_p maps $C^\infty(M)$ into $\mathbb{R}$. Note the difference with f_*; the push forward f_* maps X_p to an element of the tangent space of the codomain of f, while df maps X_p to a real number. For $f \in C^\infty(M)$ these are isomorphic, but it is an important distinction between the two. For example, f_*X_p acts on functions $\mathbb{R} \to \mathbb{R}$, and its action on $h : \mathbb{R} \to \mathbb{R}$ is $(f_*X_p)(h) = X_p(h \circ f)$, although when we consider $df(X_p)$ we need not act it on the a function h to get a real number; it just spits out a real number immediately.

Consider what happens when we apply this differential to the coordinate functions dx^i. Then acting on the basis elements, by definition of the induced map we have:

$$dx^i(\partial_j) = \partial_j x^i = \delta_j^i \tag{23}$$

and so the original notation of $\{dx^i\}$ as the dual basis was suggestive of this actually being the differential of the coordinate charts!

Now, suppose we have a function $f \in C^\infty(M)$. We can expand out $df = a_i dx^i$ in a chart (U, x^i), since at each point $\{dx^i|_p\}$ is a basis for T_p^*M. To determine the coefficients a_i, act each side on ∂_j:

$$df(\partial_j) = a_i dx^i(\partial_j) \implies \partial_j f = a_i \delta_j^i = a_j \tag{24}$$

This implies that for any function $f \in C^\infty(M)$, the differential of a function can be written as:

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{25}$$

Note that any 1-form $\omega = a_i dx^i$ is determined completely by the coordinate functions $a_i(p)$ for $p \in M$. It is a reasonably simple proof to show that ω is C^∞ iff each of its coordinate functions $a_i(p)$ are C^∞ as well. Thus we have the following corollary:

Corollary 2.0.1. If $f \in C^\infty(M)$, then $df \in \Omega^1(M)$, so df is a smooth 1-form.

Lemma 2.1. Let ω be a 1-form. If f is a function and X is a vector field on M, then $\omega(fX) = f\omega(X)$.

Finally, note that we can scalar multiply 1-forms by functions. For $g \in C^\infty(M)$ and $\omega \in \Omega^1(M)$, we can define a one form $g\omega \in \Omega^1(M)$ by scalar multiplication:

$$(g\omega)|_p := g(p)\omega_p \tag{26}$$

Because this is scalar multiplication, $g\omega = \omega g$ since $g(p)$ can simply pull out of 1-forms, since the action of a 1-form $g\omega$ on a vector field X at $p \in M$ is $(g\omega)X = (g\omega)|_p X_p = g(p)\omega_p X_p = \omega_p g(p) X_p$.

Since we have dualized the tangent bundle to define 1-forms, a map between manifolds induces a pullback map of forms, because the direction of the push forward map must reverse itself.

Definition 2.5 (Pullback). Let $f : M \to N$ be a smooth map. Then at each $p \in M$, we induce the **pullback map** $f^* : T_{f(p)}^*N \to T_p^*M$ defined pointwise by its action on vectors:

$$f^*(\omega)|_p X_p := \omega_{f(p)}(f_* X_p) \tag{27}$$

We will often write these as fields and instead write this relation between sections of the tangent and cotangent bundles:

$$f^*(\omega)X = \omega(f_*X) \tag{28}$$

Pictorially, the following diagram commutes:

$$
\begin{array}{ccc}
T_pM & \xrightarrow{f_*} & T_pN \\
& \searrow^{f^*\omega} & \downarrow^{\omega} \\
& & \mathbb{R}
\end{array}
\tag{29}
$$

Let $\omega \in \Omega^1(N)$ be a 1-form. Then f^* extends to a pullback on 1-forms:

$$f^* : \Omega^1(N) \to \Omega^1(M) \tag{30}$$

by pulling back each form pointwise, $f^*(\omega)|_p := f^*(\omega_p)$.

Any map can also pull back a function $g \in C^\infty(M)$ by precomposition, i.e. we get an induced map $f^* : C^\infty(N) \to C^\infty(M)$, $f^*(g) := g \circ f$, and so we have pullbacks $f^* : \Omega^k(N) \to \Omega^k(M)$ for $k \in \{0, 1\}$. There are a few particularly important properties of the pullback. These will later generalize and show us that f^* is a chain map on the de Rham complex, but for now we will show how this works with the differential $d : \Omega^0(M) \to \Omega^1(M)$.

Theorem 2.2. Let $f : M \to N$ be a C^∞ map between manifolds. Then the induced map $f^* : \Omega^k(N) \to \Omega^k(M)$ for $k = 0, 1$ satisfies:

1. $f^*d = df^*$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
C^\infty(N) & \xrightarrow{f^*} & C^\infty(M) \\
\downarrow^{d} & & \downarrow^{d} \\
\Omega^1(N) & \xrightarrow{f^*} & \Omega^1(M)
\end{array}
\tag{31}
$$

2. $f^*(\omega + \tau) = f^*\omega + f^*\tau$.

3. For $g \in C^\infty(N)$, $f^*(g\omega) = (f^*g)(f^*\omega)$.

Proof. We prove the first by picking $g \in C^\infty(N)$, and act $df^*(g)$ and $f^*(dg)$ (both in $\Omega^1(M)$ on $X_p \in T_pM$, as if these agree for arbitrary X_p then they are equal as linear functionals). We have:

$$df^*(g)X_p = d(g \circ f)X_p = (g \circ f)_*X_p = X_p(g \circ f) \tag{32}$$

$$f^*(dg)X_p = dg(f_*X_p) = (f_*X_p)(g) = X_p(g \circ f) \tag{33}$$

For 2, we $\omega, \tau \in \Omega^1(N)$ and must these on vectors $X \in TM$. This gives:

$$f^*(\omega + \tau)(X) = (\omega + \tau)(f_*X) = \omega f_*X + \tau f_*X = f^*(\omega)X + f^*(\tau)X \tag{34}$$

For 3, we act this on $X_p \in T_pM$ and have:

$$f^*(g\omega)X_p = (g\omega)_{f(p)}(f_*X_p) = g(f(p))\omega_{f(p)}(f_*X_p) = (f^*g)(f^*\omega)X_p \tag{35}$$

$\square$

This quick proof is a nice example to recall what the different types of maps do, and to work through it, and the best way to work through it is to draw out all the maps and induced maps. Note that although relations between 1-forms, vector fields, functions, and other objects are simplest to write out as sections of a bundle (i.e. vector / covector valued fields), when we do computations with them we typically pick a point $p \in M$ to do the computation over.

2.2 k-forms

At this point, we turn towards differential k-forms. If you are not familiar with the idea of the exterior algebra to a vector space, it would be a good idea to read my notes on module theory and familiarize yourself with it. Differential forms are exactly sections of the kth exterior power of the cotangent bundle; by taking the exterior power, we make forms implicitly antisymmetric, which makes them useful to define orientations.

Recall that for a vector space V, $\Lambda^k V$ is the quotient of $\bigotimes_k V$ by the ideal generated by $\{x_1 \otimes ... \otimes x_k : x_i = x_j, i \neq j\}$. The image of $\otimes$ in this algebra is denoted by the wedge product $\wedge$, and the exterior algebra of V is $\Lambda^* V := \bigoplus_{k=0}^{\infty} \Lambda^k V$. The exterior algebra $(\Lambda^* V, +, \times, \wedge)$ is the "best antisymmetric product" of V, as any alternating map from V^k to $\mathbb{R}$ factors to a linear map $\Lambda^* V \to \mathbb{R}$.

Definition 2.6 (Alternating map). A map $f : V^k \to K$ for a field K and a K-vector space V is called **alternating** if for each $\sigma \in S_k$, we have:

$$f(v_{\sigma(1)}, ..., v_{\sigma(k)}) = \text{sgn}(\sigma) f(v_1, ..., v_k) \tag{36}$$

where $\{v_1, ..., v_k\} \subseteq V$.

To study differential forms, let $p \in M$ and consider the kth exterior power of the cotangent space $\Lambda^k(T_p^* M)$. We can vary $p \in M$ and take a union to form a fiber bundle over M, called the kth exterior power of the cotangent bundle, and a section of this bundle is called a differential form.

Definition 2.7 (Differential form). The **kth exterior power of the cotangent bundle** is the fiber bundle:

$$\Lambda^k(T^* M) := \coprod_{p \in M} \Lambda^k(T_p^* M) \tag{37}$$

A k-**covector** is an element of this set. A smooth section of $\Lambda^k(T^* M)$ is called a **differential form**, and we call k its **rank**. We denote the set of all k-forms on a manifold M by $\Omega^k(M)$, so:

$$\Omega^k(M) := \Gamma\left(\Lambda^k(T^* M)\right) \tag{38}$$

Note $\Lambda^k(T_p^* M)$ is in bijection with the space of k-linear, alternating maps $(T_p^* M)^k \to \mathbb{R}$, so we may equivalently view a differential form as an alternating map on the tangent space at each point $p \in M$:

$$\omega \in \Omega^k(T^* M) \iff \forall p \in M, \ \omega_p \in \Lambda^k(T^* M) \tag{39}$$

$$\iff \forall p \in M, \ \omega_p : (T_p M)^k \to \mathbb{R} \text{ is linear and alternating} \tag{40}$$

We will typically view a differential form as an alternating map, as working with the abstract definition of a k-covector is very difficult to deal with and does not give us much.

A differential form is a specific example of a tensor field.

Definition 2.8 (Tensor). Let $p \in M$. A **tensor** at p is a multilinear map:

$$F : (T_p M)^k \times (T_p^* M)^\ell \to \mathbb{R} \tag{41}$$

where $k, \ell \in \mathbb{N}_{\geq 0}$. We say the tensor has **type** (k, ℓ).

A differential form of rank k may thus equivalently be categorized as an **alternating tensor of type $(\mathbf{0}, \mathbf{k})$** (where by an alternating tensor we mean it is an alternating map), and this definition often appears in literature.

Consider k vector fields $\{X_1, ..., X_k\}$ on M. Then we get a function $\omega(X_1, ..., X_k) : M \to \mathbb{R}$, $p \mapsto \omega_p((X_1)_p, ..., (X_k)_p)$. Because ω is an alternating function, note that for example $\omega(X_1, X_2, ..., X_k) = -\omega(X_2, X_1, ..., X_k)$ and so on. This function is linear with respect to functions $h : M \to \mathbb{R}$ in the sense that:

$$\omega(X_1, ..., hX_i, ..., X_k) = h\omega(X_1, ..., X_i, ..., X_k) \tag{42}$$

for each i.

We now consider what a general differential form looks like for a chart (U, x^i). At each $p \in M$, $\{dx^i|_p\}_{i=1}^n$ is a basis for the cotangent space. Because we are taking the kth exterior power, a basis is thus formed by taking all the combinations:

$$\beta = \{dx^{i_1} \wedge ... \wedge dx^{i_k} : 1 \leq i_1 < i_2 < ... < i_k \leq n\} \tag{43}$$

Note that this implies **the dimension of this exterior power is $dim(\Lambda^k(T_p^*M)) = \binom{n}{k}$, and $\Lambda^k(T_p^*M)$ vanishes for $k > n$**. A succinct way to write this is with a multi-index:

Definition 2.9 (Multi-index)**.** Let $k, n \in \mathbb{N}_{\geq 0}$. A **multi-index** between 1 and n of length k is an ordered k-tuple of the form:

$$I = (i_1, ..., i_k) \tag{44}$$

such that $1 \leq i_1 < ... < i_k \leq n$. We let $\mathfrak{I}_k^n$ denote the set of all multi-indices between 1 and n of length k.

With this notation, we may write a basis for $\Lambda^k(T_p^*M)$ as $\{dx^I : I \in \mathfrak{I}_k^n\}$, where by dx^I we mean $I = (i_1, ..., i_k)$ and $dx^I = dx^{i_1} \wedge ... \wedge dx^{i_k}$. So, we may expand each k-covector $\omega_p \in \Lambda^k(T_p^*M)$ as a linear combination:

$$\omega_p = a_{i_1, ..., i_k}(p)dx^{i_1}|_p \wedge ... \wedge dx^{i_k}|_p = \sum_{I \in \mathfrak{I}_k^n} a_I(p)dx^I|_p \tag{45}$$

This implies a k-form $\omega \in \Omega^k(M)$ can be expanded as:

$$\omega = a_{i_1, ..., i_k}dx^{i_1} \wedge ... \wedge dx^{i_k} = \sum_{I \in \mathfrak{I}_k^n} a_I dx^I \tag{46}$$

A k-form ω represented by Equation 46 is **smooth** (so $\omega \in \Omega^k(M)$) if and only if $a_I : M \to \mathbb{R}$ is smooth as a function on M. For each $p \in M$, the k-covectors $\Lambda^k(T_p^*M)$ are a vector space over $\mathbb{R}$. When we extend to sections of $\Lambda^k(T^*M)$ to form $\Omega^k(M)$, we thus see that $\Omega^k(M)$ has the structure of a module over $C^\infty(M)$, as we can multiply a differential form pointwise by a smooth function and still have a smooth section of $\Lambda^k(T^*M)$. However, we will mostly be concerned with the vector space structure of $\Omega^k(M)$, as this will drop to the cohomology. $\Omega^k(M)$ is a real valued vector space of uncountable dimensions (as different functions $f_I(x) \, dx^I$ will generate different linearly independent elements). We now switch gears a bit to induced maps.

Definition 2.10 (Pullback)**.** Let $f : M \to N$ be C^∞. Then there is an induced map:

$$f^* : \Lambda^k \left(T_{f(p)}^* N\right) \to \Lambda^k \left(T_p^* M\right) \tag{47}$$

which sends a k-covector $\omega_{f(p)}$ to $f^*(\omega_{f(p)})$ which acts on $\left(T_p^M\right)^k$ as follows:

$$f^*(\omega_{f(p)})(x_1, ..., x_k) := \omega_{f(p)}(f_* x_1, ..., f_* x_k) \tag{48}$$

for $x_i \in T_p M$. This extends to a **pullback map** of forms $f^* : \Omega^k(N) \to \Omega^k(M)$ which acts on vector fields:

$$f^*(\omega)(X_1, ..., X_k) := \omega(f_* X_1, ..., f_* X_k) \tag{49}$$

Lemma 2.3. Let $f : M \to N$ be C^∞. If $\omega, \tau \in \Omega^k(N)$ and $a \in \mathbb{R}$, then:

1. $f^*(\omega + \tau) = f^*\omega + f^*\tau$.

2. $f^*(a\omega) = af^*\omega$.

3. If ω is C^∞, then $f^*\omega$ is C^∞ as well.

2.3 The wedge product

We now turn to the wedge product, which will give the space of differential k-forms an algebra product. Abstractly, this is the same wedge product that is studied when the exterior power $\Lambda^k(T_p^*M)$ is considered, and extends to sections $\Omega^k(M)$ pointwise. We have already used this wedge product in the previous section. However, we will define this in a more concrete way than as "the image of the tensor product operation in the quotient $\Lambda^k(\cdot)$". We first define the wedge product generally on a vector space V.

Definition 2.11 (Wedge product)**.** Let V be a vector space, $\alpha \in \Lambda^k(V^*)$, and $\beta \in \Lambda^\ell(V^*)$ be two alternating tensors[3]. Then we define their **wedge product** to be an alternating tensor $\alpha \wedge \beta \in \Lambda^{k+\ell}(V^*)$ such that:

$$(\alpha \wedge \beta)(v_1, ..., v_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} sgn(\sigma)\alpha(v_{\sigma(1)}, ..., v_{\sigma(k)})\beta(v_{\sigma(k+1)},...,v_{\sigma(k+\ell)}) \tag{50}$$

where S_n is the symmetric group on n letters.

The wedge product may be rewritten as a slightly simpler sum without the extra factorial factors. We define a $(\mathbf{k}, \ell)$-**shuffle** to be a permutation $\tau \in S_{k+\ell}$ such that:

$$\tau(1) < ... < \tau(k) \& \tau(k+1) < ... < \tau(k+\ell) \tag{51}$$

Essentially, τ mixes the sets $\{1, ..., k\}$ and $\{k+1, ..., \ell\}$ in increasing order. Let $\mathfrak{S}_{k,\ell} \subseteq S_{k+\ell}$ be the set of all (k, ℓ) shuffles. Then Equation 50 may be rewritten as:

$$(\alpha \wedge \beta)(v_1, ..., v_{k+\ell}) = \sum_{\tau \in \mathfrak{S}_{k,\ell}} sgn(\tau)\alpha(v_{\tau(1)}, ..., v_{\tau(k)})\beta(v_{\tau(k+1)}, ..., v_{\tau(k+\ell)}) \tag{52}$$

The wedge product is totally anticommutative in the graded sense, in that if $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^\ell(V^*)$, then:

$$\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha \tag{53}$$

This implies that for any k-covector of odd degree $\alpha \in \Lambda^{2k}(V^*)$, $\alpha \wedge \alpha = 0$. The wedge product is also associative, and this associativity will give the spaces $\Omega^k(M)$ an algebra structure. One final nice property of the wedge product is that for $\alpha^1, ..., \alpha^k \in V^*$ and $v_1, ..., v_k \in V$, then:

$$(\alpha^1 \wedge ... \wedge \alpha^k)(v_1, ..., v_k) = \det\{a^i(v_j)\} \tag{54}$$

Now, we turn to using the wedge product as an operation on forms.

---

[3]Another common notation is to denote the space of alternating tensors $\Lambda^k(V^*)$ by $A^k(V)$.

Definition 2.12. Let $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$ be two differential forms. Then we define their **wedge product** to be $\omega \wedge \tau \in \Omega^{k+\ell}(M)$ defined pointwise as a wedge product on covectors:

$$(\omega \wedge \tau)_p := \omega_p \wedge \tau_p \tag{55}$$

It is easy to show that $\omega \wedge \tau$ is C^∞ if both ω and τ are. This structure makes $(\Omega^k(M), +, \cdot, \wedge)$ into algebra. We can further define the **algebra of C^∞ differential forms** $\Omega^*(M)$ as:

$$\Omega^*(M) := \bigoplus_{k=0}^\infty \Omega^k(M) \tag{56}$$

Then $(\Omega^*, +, \cdot, \wedge)$ is naturally a graded algebra over $\mathbb{R}$, and we will soon associate a differential with it to make it into a cochain complex. The pullback f^* is also an algebra homorphism.

Theorem 2.4. Let $f : M \to N$ be smooth. Then the pullback respects $\wedge$, i.e. if $\omega \in \Omega^k(N), \tau \in \Omega^\ell(N)$ are differential forms on N, we have:

$$f^*(\omega \wedge \tau) = f^*(\omega) \wedge f^*(\tau) \tag{57}$$

Proof. This is simply by working through the definitions. Act $f^*(\omega \wedge \tau)$ on vectors $X_1, ..., X_{k+\ell} \in T_pM$:

$$f^*(\omega \wedge \tau)_p(X_1, ..., X_{k+\ell}) = (\omega_p \wedge \tau_p)(f_*X_1, ..., f_*X_{k+\ell}) \tag{58}$$

$$= \sum_{\sigma \in \mathfrak{S}_{k,\ell}} sgn(\sigma)\omega_p(f_*X_{\sigma(1)}, ..., f_*X_{\sigma(k)})\tau_p(f_*X_{\sigma(k+1)}, ..., f_*X_{\sigma(k+\ell)}) \tag{59}$$

$$= \sum_{\sigma \in \mathfrak{S}_{k,\ell}} sgn(\sigma)f^*(\omega_p)(X_{\sigma(1)}, ..., X_{\sigma(k)})f^*(\tau_p)(X_{\sigma(k+1)}, ..., X_{\sigma(k+\ell)}) \tag{60}$$

$$= (f^*(\omega_p) \wedge f^*(\tau_p))(X_1, ..., X_{k+\ell}) \tag{61}$$

$\square$

2.4 The Exterior Derivative

We come to the last few major definitions for differential forms: we will first define a coboundary map, called the **exterior derivative**, that generalizes the standard derivative in $\mathbb{R}^n$ and gives $\Omega^*(M)$ the structure of a graded cochain complex. Later, we will generalize the notion of a derivative to a **Lie derivative**, and show that this is intimately connected with the exterior derivative.

Definition 2.13 (Antiderivation). Let $A = \bigoplus_{k=0}^\infty A^k$ be a graded algebra. A **antiderivation** on A is a $\mathbb{R}$-linear map $D : A \to A$ such that for $\omega \in A^k$ and $\tau \in A^\ell$:

$$D(\omega \cdot \tau) = D(\omega) \cdot \tau + (-1)^k \omega \cdot D(\tau) \tag{62}$$

We say the antiderivation has **degree** m if $\deg D(\omega) = \omega + m$ for each $\omega \in A^k$.

Definition 2.14 (Exterior derivative). An **exterior derivative** on a manifold M is a $\mathbb{R}$-linear map $d : \Omega^*(M) \to \Omega^*(M)$ such that:

1. d is an antiderivation of degree 1.

2. $d^2 = 0$.

3. If $f \in C^\infty(M)$ and X is a smooth vector field on M, then:

$$(df)(X) = X(f) \tag{63}$$

From condition 3 of the definition, it is clear that the differential $d : \Omega^0(M) \to \Omega^1(M)$ that we have defined in the previous section will be the action of the exterior derivative on 0-forms. This then means that if we take $\omega \in \Omega^k(M)$, when we expand $\omega = a_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, we have:

$$d\omega = (da_I) \wedge dx^I + a_I D(dx^I) = \frac{\partial a_{i_1,\ldots,i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \tag{64}$$

Thus on any chart (U, x^i), we can define an exterior derivative on U by:

$$d\omega := da_I \wedge dx^I = \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I \tag{65}$$

for $\omega = a_I dx^I$. It turns out that this definition is independent of the chart we have picked, and this proves the existence of such an exterior derivative. Furthermore, the exterior derivative defined in this way is unique, and we state this as a theorem.

Theorem 2.5. There is a unique exterior derivative $d : \Omega^*(M) \to \Omega^*(M)$ which agrees with all the all the properties in Definition 2.14.

The exterior derivative is a natural map which gives $\Omega^*(M)$ the structure of a cochain complex because it squares to 0:

$$\Omega^0(M) \xrightarrow{\ d\ } \Omega^1(M) \xrightarrow{\ d\ } \Omega^2(M) \xrightarrow{\ d\ } \Omega^3(M) \xrightarrow{\ d\ } \ldots \tag{66}$$

Furthermore, the pullback that was studied in the previous section is indeed a chain map with respect to $(\Omega^*(M), d)$.

Theorem 2.6. Let $f : M \to N$ be smooth. Then the pullback $f^* : \Omega^*(N) \to \Omega^*(M)$ is a chain map, i.e. the diagram commutes:

$$
\begin{array}{ccccccc}
\cdots \xrightarrow{\ d\ } & \Omega^{k-1}(N) & \xrightarrow{\ d\ } & \Omega^k(N) & \xrightarrow{\ d\ } & \Omega^{k+1}(N) & \xrightarrow{\ d\ } \cdots \\
& \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & \\
\cdots \xrightarrow{\ d\ } & \Omega^{k-1}(M) & \xrightarrow{\ d\ } & \Omega^k(M) & \xrightarrow{\ d\ } & \Omega^{k+1}(M) & \xrightarrow{\ d\ } \cdots
\end{array} \tag{67}
$$

Proof. Let $\omega \in \Omega^k(N)$. We must show that $f^* d\omega = df^*\omega \in \Omega^{k+1}(M)$. We expand $\omega = a_I dx^I$. Recall we have already shown $f^* dg = df^* g$ for $g \in C^\infty(M)$. Then:

$$df^*\omega = d\left[(f^* a_I) f^* dx^{i_1} \wedge \ldots \wedge f^* dx^{i_k}\right] = d\left[(a_I \circ f) df^{i_1} \wedge \ldots \wedge df^{i_k}\right] = d(a_I \circ f) \wedge df^I \tag{68}$$

$$f^* d\omega = f^*\left[da_I \wedge dx^I\right] = f^* da_I \wedge f^* dx^{i_1} \wedge \ldots \wedge f^* dx^{i_k} = d(a_I \circ f) \wedge df^I \tag{69}$$

$\square$

We will study the cochain complex $(\Omega^*(M), d)$ further later in these notes when we consider the cohomology obtained from this complex. It is called the **de Rham cohomology**, and is an essential tool in the study of smooth manifolds.

2.5 The Lie Derivative

We now seek to find a nice coordinate independent way to express the exterior derivative, which we will do by introducing the Lie derivative. We will denote by $\{X_t\}$ or $\{\omega_t\}$ a 1-parameter family of vector fields or manifolds, where t is varied over some subset of $\mathbb{R}$. For a coordinate chart (U, x^i), we may expand $X_t = a^i(t, p)\partial_i|_p$. We say $\lim_{t\to t_0} X_t$ exists if for each $p \in M$ and $i = 1, ..., n$, $\lim_{t\to t_0} a^i(t, p)$ exists, and if this limit exists we set it equal to the pointwise limit. We do the same thing with a 1-parameter family ω_t of k-forms, and expand $\omega_t = b_J(t, p)dx^J|_p$. Naturally, we can define a derivative of these objects:

$$\left(\frac{d}{dt}\bigg|_{t=t_0} X_t\right)_p := \frac{\partial a^i}{\partial t}(t_0, p)\partial_i|_p \qquad\qquad \left(\frac{d}{dt}\bigg|_{t=t_0} \omega_t\right)_p := \frac{\partial b_J}{\partial t}(t_0, p)dx^J \qquad (70)$$

Note that dX_t/dt and $d\omega_t/dt$ respectively remain 1-parameter families of vector fields and k-forms. This 1-parameter derivative has some nice properties which we state here.

Theorem 2.7. Let ω_t, τ_t be smooth 1-parameter families of k-forms. Then:

- d/dt satisfies the product rule:

$$\frac{d(\omega_t \wedge \tau_t)}{dt} = \frac{d\omega_t}{dt} \wedge \tau_t + \omega_t \wedge \frac{d\tau_t}{dt} \qquad (71)$$

- d/dt commutes with the differential:

$$\frac{d}{dt}\bigg|_{t=t_0} d\omega_t = d\left(\frac{d}{dt}\bigg|_{t=t_0} \omega_t\right) \qquad (72)$$

Recall the definition of an integral curve and a local flow. Given a vector field $X \in \mathfrak{X}(M)$, an integral curve of X is a curve γ for which X is a tangent vector field to γ at each point on the curve. A local flow generated by X is a neighborhood W and a family of curves $F_t(q)$ for each $q \in W$ such that $F_t(q)$ is an integral curve of X starting at each $q \in W$ in a "nice" manner. More formally:

Definition 2.15 (Integral curve). Let $X \in \mathfrak{X}(M)$. A curve $\gamma : (a, b) \to M$ is an **integral curve** of X if $\gamma'(t) = X_{\gamma(t)}$ at each $t \in (a, b)$. An integral curve is called **maximal** if its domain cannot be extended to a larger interval.

Definition 2.16 (Local Flow). Let $X \in \mathfrak{X}(M)$. A **local flow** generated by X about point p in an open set U is a C^∞ function $F : (-\epsilon, \epsilon) \times W \to U$ for $\epsilon > 0$ on neighborhood $p \in W \subseteq U$ such that for each $q \in W$, we have:

- $F_t(q)$ is an integral curve of X.

- $F_0(q) = q$.

- $F_t(F_s(q)) = F_{t+s}(q)$ whenever both sides are defined.

Given $X \in \mathfrak{X}(M)$ and $p \in M$, we will take for granted the existence of a neighborhood $p \in U$ for which there is a local flow $\phi : (-\epsilon, \epsilon) \times U \to M$ generated by X, so:

$$\frac{\partial \phi_t(q)}{\partial t} = X_{\phi_t(q)} \qquad (73)$$

and $\phi_0(q) = q$. Note that in particular, $\phi_{-t} \circ \phi_t = \phi_0 = 1$. We will use this to define the Lie derivative, which we will define for vector fields and k-forms.

Definition 2.17 (Lie derivative). Let $X \in \mathfrak{X}(M)$, $p \in M$, and $\phi : (-\epsilon, \epsilon) \times U \to M$ a local flow of X on a neighborhood $U \ni p$. Let $Y \in \mathfrak{X}(M)$, and $\omega \in \Omega^k(M)$. We define the **Lie derivative** $\mathcal{L}_X Y$ of Y with respect to X at p to be the vector:

$$(\mathcal{L}_X Y)_p := \lim_{t \to 0} \frac{\phi_{-t*}(Y_{\phi(t)}) - Y_p}{t} \tag{74}$$

and the **Lie derivative** $\mathcal{L}_X \omega$ to be the k-covector:

$$(\mathcal{L}_X \omega)_p := \lim_{t \to 0} \frac{\phi_{-t}^*(\omega_{\phi(t)}) - \omega_p}{t} \tag{75}$$

The idea here is that we transport either Y or ω along the path of X using the local flow generated by X, and then use this to define a derivative of the vector field T. On functions, the Lie derivative $\mathcal{L}_X f$ will be the exact same as the directional derivative. We can characterize the Lie derivatives of such simple objects as follows.

Theorem 2.8. Let $X \in \mathfrak{X}(M)$.

1. If $f \in C^\infty(M)$, then:

$$\mathcal{L}_X f = Xf \tag{76}$$

2. If $Y \in \mathfrak{X}(M)$, then:

$$\mathcal{L}_X Y = [X, Y] \tag{77}$$

Note that $X(f) \in C^\infty(M)$ by evaluation at $p \in M$, i.e. we view it as the map $p \mapsto X_p(f)$. The Lie bracket of vector fields is defined to be the vector field $[X, Y]$ given by the following action on $f \in C^\infty(M)$ at $p \in M$:

$$[X, Y]_p f := (X_p Y - Y_p X)f \tag{78}$$

where note that $X(f) \in C^\infty(M), p \mapsto X_p(f)$, so this is a well defined definition. The Lie bracket is intimately connected with the coordinate derivative because if $X = a^i \partial_i$ and $Y = b^j \partial_j$ are two vector fields, then:

$$[X, Y] = \left(a^k \partial_k b^i - b^k \partial_k a^i \right) \partial_i \tag{79}$$

The Lie bracket here gives $\mathfrak{X}(M)$ the structure of a **Lie algebra** $(\mathfrak{X}(M), [\cdot, \cdot])$. Intuitively, the Lie bracket is nonzero when the flows generated by X and Y do not commute via parallel transport. When $[X, Y]_p$ is nonzero, it ends up being proportional to the infinitesimal parallel transport of a point along the parallelpiped defined by X_p and Y_p. Finally, we relate some of the properties of the Lie derivative as it acts on differential forms.

Theorem 2.9. Let $X \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$, and $\tau \in \Omega^\ell(M)$.

1. $\mathcal{L}_X$ is a derivation:

$$\mathcal{L}_X(\omega \wedge \tau) = (\mathcal{L}_X \omega) \wedge \tau + \omega \wedge (\mathcal{L}_X \tau) \tag{80}$$

2. $\mathcal{L}_X$ commutes with the exterior derivative:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\ d\ } & \Omega^{k+1}(M) \\ \downarrow{\scriptstyle \mathcal{L}_X} & & \downarrow{\scriptstyle \mathcal{L}_X} \\ \Omega^k(M) & \xrightarrow{\ d\ } & \Omega^{k+1}(M) \end{array} \tag{81}$$

3. $\mathcal{L}_X$ satisfies a "product" rule. For $Y_1, ..., Y_k \in \mathfrak{X}(M)$, we have:

$$\mathcal{L}_X(\omega(Y_1, ..., Y_k)) = (\mathcal{L}_X\omega)(Y_1, ..., Y_k) + \sum_{i=1}^{k} \omega(Y_1, ..., \mathcal{L}_XY_i, ..., Y_k) \tag{82}$$

We next define interior multiplication, which is the final operation on forms that we will consider. Luckily for us, unlike the Lie derivative, this requires relatively little machinery.

Definition 2.18 (Interior multiplication). Let β be a k-covector on a vector space V and $v \in V$. Then for $k \geq 2$, the **interior multiplication** or **contraction** of β with v is the $(k-1)$ covector:

$$\iota_v(\beta)(v_2, ..., v_k) := \beta(v, v_2, ..., v_k) \tag{83}$$

For $k = 1$, we define $\iota_v(\beta) := \beta(v) \in \mathbb{R}$ and for $k = 0$, $\iota_v(\beta) := 0$.

We see that for $v \in T_pM$, $\iota_v : \Lambda^k(T_p^*M) \to \Lambda^{k-1}(T_p^*M)$. Similarly for a vector field $X \in \mathfrak{X}(M)$, we can define an interior derivative pointwise as $\iota_X(\omega)(X_2, ..., X_k) := \omega(X, X_2, ..., X_k)$, and so we see that $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$. Because each $\omega \in \Omega^k(M)$ is antisymmetric, note that:

$$\iota_X^2 = 0 \tag{84}$$

for each $X \in \mathfrak{X}(M)$ because $\iota_X^2(\omega)(X_3, ..., X_k) = \omega(X, X, X_2, ..., X_k) = 0$. Thus, ι_X gives $\Omega^*(M)$ the structure of a chain complex:

$$\dots \xrightarrow{\iota_X} \Omega^3(M) \xrightarrow{\iota_X} \Omega^2(M) \xrightarrow{\iota_X} \Omega^1(M) \xrightarrow{\iota_X} \Omega^0(M) \tag{85}$$

There is a natural duality of ι_X with the exterior derivative in this manner. Because of this, it is often called the **interior derivative** as well. As a derivative should, it is a graded derivation. For $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$:

$$\iota_X(\omega \wedge \tau) = \iota_X(\omega) \wedge \tau + (-1)^k \omega \wedge \iota_X(\tau) \tag{86}$$

The other appearance of a duality between the exterior and interior derivatives is **Cartan homotopy formula**, which states that ι_X is a cochain homotopy between $\mathcal{L}_X$ and 0, and likewise d is a chain homotopy between $\mathcal{L}_X$ and 0.

Theorem 2.10 (Cartan homotopy formula). For $X \in \mathfrak{X}(M)$:

$$\mathcal{L}_X = d\iota_X + \iota_X d \tag{87}$$

A corollary of this formula is that for any vector field, $\mathcal{L}_X$ is chain homotopic to the 0 map, and thus $\mathcal{L}_X$ induces the zero map on the de Rham cohomology.

For k-forms with small k, computing $\iota_X\omega$ by hand is not too difficult and can be done by brute force. However, it is often simpler to do this abstractly, and there is a nice formula to help with computations.

Lemma 2.11. Let $\alpha^1, ..., \alpha^k$ be 1-covectors on V. Then for $v \in V$:

$$\iota_v(\alpha^1 \wedge ... \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i+1}\alpha^i(v)\alpha^1 \wedge ... \wedge \hat{\alpha}^i \wedge ... \wedge \alpha^k \tag{88}$$

Example 2.1. Let $\omega = dx^1 \wedge ... \wedge dx^n$ and $X = x^i \partial_i$ on $\mathbb{R}^n$. To compute $\iota_X \omega$, we could note that $\iota_X \omega \in \Omega^{n-1}(M)$ and compute $(\iota_X \omega)(X_1, ..., X_{n-1})$, where $X_i = X_{ij} \partial_j$ are vector fields. This would get extraordinarily unwieldy, so it is simpler to use Equation 88. We can thus instead compute $dx^i(X) = dx^i(x^j \partial_j) = x^i$, thus:

$$\iota_X \omega = \sum_{i=1}^{k} (-1)^{i+1} x^i dx^1 \wedge ... \wedge \hat{dx}^i \wedge ... \wedge dx^n \tag{89}$$

$$= x^1 dx^2 \wedge dx^3 \wedge ... \wedge dx^n - x^2 dx^1 \wedge dx^3 \wedge ... \wedge dx^n + ... \tag{90}$$

Finally, this new machinery allows us to express the exterior derivative d globally without needing to refer to derivatives on a coordinate chart.

Theorem 2.12. Let $\omega \in \Omega^k(M)$ and $Y_0, Y_1, ..., Y_k \in \mathfrak{X}(M)$. Then:

$$(d\omega)(Y_0, ..., Y_k) = \sum_{i=1}^{k} (-1)^i \omega(Y_0, ..., \hat{Y}_i, ..., Y_k) Y_i \tag{91}$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, ..., \hat{Y}_i, ..., \hat{Y}_j, ..., Y_k) \tag{92}$$

Note $\hat{Y}$ means to omit Y from the argument of ω. The proof of this formula is done with induction and expressing $(d\omega)(Y_0, ..., Y_k) = (\iota_{Y_0} d\omega)(Y_1, ..., Y_k)$ and using the Cartan homotopy formula $\iota_{Y_0} d = (\mathcal{L}_{Y_0} \omega - d\iota_{Y_0})(Y_1, ..., Y_k)$ and using induction and Equation 82. As an example of this formula, we may write the action of the $d\omega$ for a 1-form ω explicitly for $X, Y \in \mathfrak{X}(M)$:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \tag{93}$$

Example 2.2. Consider $M = S^1$, and consider $\omega = -y\, dx + x\, dy \in \Omega^1(S^1)$ and $X = -y\, \partial_x + x\, \partial_y \in \mathfrak{X}(S^1)$. Note that since $dim(S^1) = 1$, we must have $d\omega = 2\, dx \wedge dy \in \Omega^2(S^1) \equiv 0$. Indeed on the circle we can see this because since the coordinates are defined by $x^2 + y^2 = R^2$, $x\, dx + y\, dy = 0$, hence wedging with dx we see $dx \wedge dy = 0$ when $y \neq 0$ (and when $x \neq 0$ by wedging with dy). Thus we of course have $d\omega = 0$, although it does not appear to vanish immediately from the computation. We can then compute $\mathcal{L}_X \omega = (d\iota_X + \iota_X d)\omega = d\iota_X \omega$:

$$\mathcal{L}_X \omega = d\iota_X \omega = d(\omega(X)) = d(y^2 + x^2) = d(R^2) = 0 \tag{94}$$

2.6   Summary

There were a lot of definitions introduced in this section, so I will summarize the major definitions and results here for a manifold M of dimension n.

- The **cotangent space** at $p \in M$ is the dual to the tangent space $T_p M$:

$$T_p^* M := Hom(T_p M, \mathbb{R}) \tag{95}$$

  A basis for this space is $\{dx^i\}_{i=1}^n$, and the dual vectors obey $dx^i(\partial_j) = \delta_j^i$, where d is the differential map $d : C^\infty(M) \to \Omega^1(M)$ defined by its action on a vector field $X \in \Gamma(TM)$:

$$(df)(X) := X(f) \tag{96}$$

  The differential can also be expressed in any chart (U, x^i) as:

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{97}$$

- A k-**covector** ω_p at a point $p \in M$ is an element of the kth exterior power of the cotangent space at p is:

$$\omega_p \in \Lambda^k \left(T_p^* M \right) = span \left\{ dx^{i_1} \wedge ... \wedge dx^{i_k} : 1 \le i_1 < ... < i_k \le n \right\} \tag{98}$$

  A basis for this space is $\{dx^{i_1} \wedge ... \wedge dx^{i_k}\}_{(i_j) \in \mathfrak{I}_k^n}$, and the dimension is $\binom{n}{k}$.

- The k**th exterior power of the cotangent bundle** is:

$$\Lambda^k \left(T^* M \right) := \coprod_{p \in M} \Lambda^k \left(T_p^* M \right) \tag{99}$$

- A **differential form** ω of rank k is a section of the kth exterior power of the cotangent bundle:

$$\omega \in \Omega^k(M) := \Gamma(\Lambda^k(T^* M)) \tag{100}$$

  and can be expressed as $\omega = a_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$.

- The **wedge product** of two differential forms $\omega \in \Omega^k(M), \tau \in \Omega^\ell(M)$ is defined pointwise as:

$$(\omega_p \wedge \tau_p)(v_1, ..., v_{k+\ell}) := \sum_{\sigma \in \mathfrak{S}_{k,\ell}} sgn(\sigma) \omega_p(v_{\sigma(1)}, ..., v_{\sigma(k)}) \tau_p(v_{\sigma(k+1)}, ..., v_{\sigma(k+\ell)}) \tag{101}$$

  This naturally gives $(\Omega^k(M), +, \cdot, \wedge)$ the structure of a $\mathbb{R}$-algebra.

- Given a smooth map $f : M \to N$, we induce a **pullback map** of differential forms $f^* : \Omega^k(N) \to \Omega^k(M)$ which acts on vector fields as:

$$f^*(\omega)(X_1, ..., X_k) = \omega(f_* X_1, ..., f_* X_k) \tag{102}$$

  Furthermore, f^* is a homomorphism of algebras and a chain map on the de Rham complex $(\Omega^*(M), d)$.

- The **exterior derivative** is a coboundary map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined as $df = (\partial f/\partial x^i)dx^i$ on $f \in \Omega^0(M)$ and:

$$d\omega := dg_{i_1,...,i_k} \wedge dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{103}$$

on $\omega = g_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k} \in \Omega^k(M)$.

- Given $X \in \mathfrak{X}(M)$, the **interior derivative** is a boundary map $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ defined as:

$$\iota_X(\omega)(X_2, ..., X_k) := \omega(X, X_2, ..., X_k) \tag{104}$$

- Given $X \in \mathfrak{X}(M)$, the **Lie derivative** measures the infinitesimal change in a tensor field as it flows along X. On a differential form $\omega \in \Omega^k(M)$:

$$(\mathcal{L}_X\omega)_p := \lim_{t \to 0} \frac{\phi_{-t}^*(\omega_{\phi(t)}) - \omega_p}{t} \tag{105}$$

where $\phi_t(p)$ is the flow generated by X at p, so $\phi_0(p) = p$ and $\phi_t(p)$ is an integral curve to X.

- The derivatives are all related via the **Cartan homotopy formula**:

$$\mathcal{L}_X = d\iota_X + \iota_X d \tag{106}$$

3   Integration

Differential forms exist to be integrated over. Since a k-form generalizes a notion of a k-dimensional volume, we can only integrate k-forms over a k-dimensional subset of a manifold. These forms are called **top forms**:

Definition 3.1 (Top form)**.** A **top form** on a k-dimensional manifold M is a k-form $\omega \in \Omega^k(M)$.

Before discussing how to integrate a differential form, we must discuss orientations on a manifold. Forms may only be integrated over an oriented manifold, and we will make that notion precise.

3.1   Orientation

Let V be a vector space of dimension n. We denote an ordered subset of V by $[v_1, ..., v_n]$, as opposed to the typical unordered subset $\{v_1, ..., v_n\}$. Let $u = [u_1, ..., u_n]$ and $v = [v_1, ..., v_n]$ be two ordered bases for V. We say that u is **equivalent** to v iff $u = vA$ with $det(A) > 0$, and we write $u \sim v$ in this case. This forms an equivalence relation on the space of ordered bases of V, and we define an **orientation** on V to be an equivalence class of ordered bases on V. There are two orientations on any vector space; a positive one, and a negative one, and one can get between ordered bases of the same orientation by a positive change of basis.

Orientations are nicely related to n-covectors on a space. Suppose that two ordered bases are related by $u_j = v_i A_{ij}$. Then for any n-covector β on V, we have:

$$\beta(u_1, ..., u_n) = det(A)\beta(v_1, ..., v_n) \tag{107}$$

This implies that $u \sim v$ iff $\beta(u_1, ..., u_n)$ and $\beta(v_1, ..., v_n)$ have the same sign for any covector β. In this case, we say that β **determines** the orientation $[v_1, ..., v_n]$ when $\beta(v_1, ..., v_n) > 0$. This is well defined because if β determines the orientation v and $u \sim v$, then β also determines the orientation u as well because $u = vA$ and $\beta(u) = det(A)\beta(v)$ with $det(A) > 0$. Furthermore, any multiple $a\beta$ with $a > 0$ **specifies the same orientation as** β because $a\beta(v)$ will have the same sign as $\beta(v)$. So, we say two n-covectors α and β are **equivalent**, $\alpha \sim \beta$, iff $\alpha = a\beta$ with $a > 0$. Note that any two n-covectors are multiples of one another because $dim(\Lambda^n(V)) = 1$. An orientation on V can thus be viewed as an equivalence class of ordered bases, or an equivalence class of n-covectors.

Now we work with a manifold M of dimension n. Recall a frame on an open set $U \subseteq M$ is a choice of n vector fields $X_1, ..., X_n \in \mathfrak{X}(U)$ such that $\{X_{1,p}, ..., X_{n,p}\}$ is linearly independent (forms a basis) at each $p \in U$. We can take this to an equivalence relation on frames $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ by decreeing:

$$(X_1, ..., X_n) \sim (Y_1, ..., Y_n) \iff \forall p \in U, \ (X_{1,p}, ..., X_{n,p}) \sim (Y_{1,p}, ..., Y_{n,p}) \tag{108}$$

A **pointwise orientation** μ assigns an orientation to each $p \in M$, i.e. $\mu = [X_1, ..., X_n]$ with $\mu_p = [X_{1,p}, ..., X_{n,p}]$. The vector fields contained in μ do not need to be continuous, but it is only interesting when they are continuous. Note that a pointwise orientation is global and assigns an orientation to each tangent space on M.

Definition 3.2. A continuous pointwise orientation on M is called an **orientation** on M. If M has an orientation, we call M **orientable**.

Theorem 3.1. A connected orientable manifold M has precisely two orientations.

Theorem 3.2. A pointwise orientation $[X_1, ..., X_n]$ on M is continuous iff at each $p \in M$, there is a chart (U, x^i) containing p such that the function:

$$(dx_1 \wedge ... \wedge dx_n)(X_1, ..., X_n) : U \to \mathbb{R} \tag{109}$$

is positive at each $q \in U$.

This leads to a classification theorem for a manifold being orientable. We can instead simply look at the top forms on M, and there is a simple condition on these top forms for us to see if M is orientable.

Theorem 3.3. An n-dimensional manifold M is orientable iff there exists a C^∞ nowhere vanishing n-form on M.

Proof. Given an orientable M, we can use the previous theorem and a partition of unity to "stitch" together the forms in the previous theorem into a single continuous top form which vanishes nowhere. In the other direction, suppose that ω vanishes nowhere. At each $p \in M$, we can find $[X_{1,p}, ..., X_{n,p}]$ such that $\omega_p(X_{1,p}, ..., X_{n,p}) > 0$. Then for any chart (U, x^i) containing p, we can write $\omega = f \, dx^1 \wedge ... \wedge dx^n$ where f is continuous and nowhere vanishing, hence $f > 0$ or $f < 0$ on U. If $f > 0$, then $\omega(X_1, ..., X_n) > 0$ everywhere on U, and if $f < 0$, then $-\omega(X_1, ..., X_n) > 0$ everywhere on U, and we can apply the previous theorem. $\square$

Thus on an orientable manifold M, there are two types of top forms– forms which are everywhere positive, and forms which are everywhere negative.

Definition 3.3. Let $\omega \in \Omega^n(M)$ be a top form. We call ω an **orientation (volume) form** if ω is nowhere vanishing, i.e. $\omega_p \neq 0$ at any $p \in M$. If we have an ordered frame $[X_1, ..., X_n]$ such that $\omega(X_1, ..., X_n) > 0$, we say that ω **specifies** the orientation $[X_1, ..., X_n]$.

We can partition the volume forms on M into two equivalence classes: let $\omega \sim \omega'$ iff $\omega = f\omega'$ with $f > 0$ everywhere. Then *each equivalence class of this relation exactly specifies an orientation of M.* Note in coordinates, one orientation form is given by $dx^1 \wedge ... \wedge dx^n$, and the other is $-dx^1 \wedge ... \wedge dx^n$. We will write an oriented manifold as a pair $(M, [\omega])$, where ω is a volume form on M specifying the positive orientation.

A diffeomorphism $f : (M, [\omega_M]) \to (N, [\omega_N])$ is **orientation preserving** if $[f^*\omega_N] = [\omega_M]$. On charts $U \subseteq M$ and $V \subseteq N$, this is equivalent to the Jacobian:

$$\det \left\{ \frac{\partial F^i}{\partial x^j} \right\} \tag{110}$$

being everywhere positive on U. Given an orientable manifold M, we can find an oriented atlas on M, i.e. an atlas such that for any two overlapping charts (U, x^i) and (V, y^i), the Jacobian $\det\{\partial y^i / \partial x^j\} > 0$ everywhere.

An orientation allows us to integrate a manifold. Before we dive into integration, we must define orientations on the boundaries of manifolds, as this will allow us to formulate Stokes's theorem rigorously. Recall on a manifold with boundary, the points in ∂M have coordinate neighborhoods diffeomorphic to a point on the boundary of $\mathcal{H}^n$, where $\mathcal{H}^n := \{(x_1, ..., x_n) : x_n \geq 0\}$ is the Euclidean half-plane.

Defining the tangent space to a manifold with boundary is slightly different than that on a manifold without boundary, because ∂M is a $(n-1)$ degree manifold, i.e. $dim(T_p(\partial M)) + 1 = dim(T_p M)$. However, since we defined the tangent space on a manifold with regards to derivations

on the entire manifold, if $p \in \partial M$, the tangent space to p in M is n-dimensional and *still includes derivative operators which point away from the boundary.* So, one must be careful to specify if we are considering $T_p M$ or $T_p \partial M$ when $p \in \partial M$; for our considerations we will want the full n-dimensional tangent space $T_p M$, but one may also examine the $(n-1)$ dimensional tangent space $T_p \partial M$.

Definition 3.4. Let M be an n-dimensional manifold with boundary and $p \in M$. We say that a tangent vector $X_p \in T_p M$ is **inward-pointing** if $X_p \notin T_p \partial M$ and if there is $\epsilon > 0$ and a curve $\gamma : [0, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. We say X_p is **outward-pointing** if $-X_p$ is inward pointing.

An inward pointing vector on the boundary is literally what it sounds like; for example, in $\mathcal{H}^3 = \mathbb{R}^3_{z \geq 0}$, an inward pointing vector at $\vec{0}$ is ∂_z, while an outward pointing vector is $-\partial_z$.

Theorem 3.4. On a manifold M with boundary ∂M, there is a smooth outward pointing vector field along ∂M.

An outward pointing vector field allows us to push an orientation for M onto an orientation for ∂M by means of the interior derivative.

Theorem 3.5. Let $(M, [\omega])$ be an oriented n dimensional manifold with boundary. If ω is a volume form on M representing the positive orientation and X is a smooth outward pointing vector field on ∂M, then $\iota_X \omega$ is a smooth nowhere vanishing $(n-1)$ form on ∂M, and hence ∂M is orientable.

Definition 3.5. In the notation of Theorem 3.5, we define the **boundary orientation** on ∂M to be $[\iota_X \omega]$, the orientation specified by $\iota_X \omega$.

Corollary 3.5.1. Let $p \in \partial M$ and $X_p \in T_p M$ be outward-pointed. Then an ordered basis $[v_1, ..., v_{n-1}]$ for $T_p \partial M$ represents the boundary orientation at p iff the ordered basis $[X_p, v_1, ..., v_{n-1}]$ for $T_p M$ represents the orientation on M at p.

In the manner of these theorems, this allows us to endow the boundary of an oriented manifold with an orientation which respects the orientation of M. Remember that it is conventional for the normal vector field to be put first in such an orientation: as Tu says, remember the mnemonic "outward vector first".

Example 3.1. Consider the boundary orientation on $\mathcal{H}^n$ induced by the positive orientation $\omega := dx^1 \wedge ... \wedge dx^n$. An outward pointed vector is $X := -\partial_n$, and we can contract this with the volume form to get the boundary orientation form:

$$\iota_X \omega = (-1)^{n+1} dx^n(-\partial_n) dx^1 \wedge ... \wedge dx^{n-1} = (-1)^n dx^1 \wedge ... \wedge dx^{n-1} \tag{111}$$

For $\mathcal{H}^3$, the boundary orientation form is thus $\iota_X \omega = (-1)^3 dx \wedge dy = dy \wedge dx$. Note the negative here is because we have put the $-\partial_z$ vector in first from the prescription for the boundary orientation, so our ordered basis is $[-\partial_z, \partial_y, \partial_x] \sim [\partial_x, \partial_y, \partial_z]$.

3.2   Integration

Integrating over forms follows in a natural way from integrating over $\mathbb{R}^n$, since a manifold is locally homeomorphic to $\mathbb{R}^n$. In Euclidean space, we integrate n-forms over $\mathbb{R}^n$. Suppose $\omega \in \Omega^n(\mathbb{R}^n)$ is

an n-form. Then we may express ω in the standard coordinates of $\mathbb{R}^n$ as $\omega = f(x)\, dx^1 \wedge ... \wedge dx^n$, and we define the **integral** of ω over $\mathbb{R}^n$ to be:

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x)\, dx^1 \wedge ... \wedge dx^n := \int_{\mathbb{R}^n} f(x)\, d^n x \tag{112}$$

where $d^n x$ is the standard Lebesgue measure on $\mathbb{R}^n$ and the last integral is the usual Lebesgue integral. So, integration of a form in $\mathbb{R}^n$ essentially amounts to forgetting the wedge products between the differentials. Because of the wedge product, integrating forms remembers the orientation of the form, i.e. $\int f(x)\, dx^2 \wedge dx^1 = - \int f(x)\, dx^1 \wedge dx^2$.

Now we are in position to define an integral over a n-dimensional manifold M. We may only integrate top forms with compact support so that we are guaranteed the integral converges. We denote the set of k-forms on M with compact support by $\Omega^k_C(M)$.

Let (U, ϕ) be a chart on M, and $\omega \in \Omega^n_C(U)$. Because $\phi : U \to \phi(U)$ is a diffeomorphism, the pullback $(\phi^{-1}) * \omega$ is a n-form on $\mathbb{R}^n$, so we can integrate it. We thus define:

$$\int_U \omega := \int_{\phi(U)} \left(\phi^{-1}\right)^* \omega \tag{113}$$

where $\phi(U) \subseteq \mathbb{R}^n$ and the integral on the right is the previously defined integral.

We can use a partition of unity on M to extend this definition to an integral over the total manifold. Let $\{U_\alpha, \phi_\alpha\}$ be an oriented atlas on M, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Recall that by definition for, any differential form ω satisfies $\omega = \sum_\alpha \rho_\alpha \omega$.

Definition 3.6 (Integral of a differential form)**.** Let $\omega \in \Omega^n_C(M)$. We define the **integral** of ω over M to be:

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega \tag{114}$$

This definition can be shown to be independent of choice of atlas and partition of unity, and so is well defined. Although the definition is intuitively nice, it is often difficult to use this to compute actual integrals. In practice, it is often easier to find a parameterization of a manifold M. A **parameterization** of M is a subset $A \subseteq M$ and a C^∞ map $F : D \to M$, where $D \subseteq \mathbb{R}^n$ is compact such that $A = im(F)$ and F restricts to an orientation preserving diffeomorphism on $int(D)$ to $F(int(D))$. The integral of $\omega \in \Omega^n(M)$ can thus be calculated over A as follows:

$$\int_A \omega = \int_D F^* \omega \tag{115}$$

Example 3.2. We can integrate the area form over the sphere S^2. This form is defined as:

$$\omega = \begin{cases} \frac{dy \wedge dz}{x} & x \neq 0 \\ \frac{dz \wedge dx}{y} & y \neq 0 \\ \frac{dx \wedge dy}{z} & z \neq 0 \end{cases} \tag{116}$$

We compute $\int_{S^2} \omega$ by parameterizing the sphere with the coordinates $F : [0, 2\pi] \times [0, \pi] \to S^2$ by:

$$F(\phi, \theta) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{117}$$

We must compute the pullback $F^*\omega$. Note that $F^*x = x \circ F : [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ is given by $(F^*x)(\phi, \theta) = \sin\theta\cos\phi$. Similarly, $F^*y = \sin\theta\sin\phi$, and $F^*z = \cos\theta$. We also have $F^*dy = dF^*y = d(\sin\theta\sin\phi) = \cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi$, and $F^*dz = dF^*z = d\cos\theta = -\sin\theta\, d\theta$. So:

$$F^*\omega = \frac{F^*(dy) \wedge F^*(dz)}{F^*x} = \frac{(\sin\theta\cos\phi\, d\phi) \wedge (-\sin\theta\, d\theta)}{\sin\theta\cos\phi} = \sin\theta\, d\theta \wedge d\phi \tag{118}$$

Therefore:

$$\int_{S^2} \omega = \int_{[0,2\pi] \times [0,\pi]} \sin\theta \, d\theta \wedge d\phi = \int_0^{2\pi} d\phi \int_0^{\phi} d\theta \, \sin\theta = 4\pi \tag{119}$$

Note integration over a form ω induces a premeasure μ_ω on the Borel sets (which are generated by the open sets on a topological space) by:

$$\mu_\omega(U) := \int_U \omega \tag{120}$$

The important thing to note is that integrating differential forms is different than integrating a function on a measure space because of the domain of integration. In measure theory, we integrate a function over a subset of the measure space. In particular, we have no notion of an "oriented volume", and instead just have an absolute notion of measure. When integrating a differential form, we have an implicit notion of orientation that comes built into our framework. Technically, the domain of integration is not a subset of a manifold; rather, it is an *oriented subset of M*, given formally by an equivalence class of orientation forms $[M] = (M, [\omega])$.

We end this section with Stokes' Theorem, which fully generalizes the fundamental theorem of calculus to manifolds. As a corollary to this theorem, it is easy to prove most of the nice multivariable integration theorems that you studied way back when in multivariable calculus.

Theorem 3.6 (Stokes' Theorem)**.** Let M be a smooth n dimensional manifold, and $\omega \in \Omega_C^{n-1}(M)$. Then:

$$\int_M d\omega = \int_{\partial M} \omega \tag{121}$$

where ∂M is realized with the boundary orientation.

In particular, if M has no boundary, then any exact form $\omega = d\tau$ integrates to 0 across M, $\int_M \omega = 0$.

Example 3.3 (Vector calculus in $\mathbb{R}^3$)**.** Let us study the exterior algebra in $\mathbb{R}^3$. We have four nonzero graded subsets of $\Omega^*(M)$. $\Omega^0(M)$ is 1-dimensional and is identified with functions $f(x)$. Similarly, $\Omega^3(M)$ is one-dimensional and can also be identified with functions $f(x) \leftrightarrow f(x) \, dx \wedge dy \wedge dz$. $\Omega^1(M)$ and $\Omega^2(M)$ can both be identified with vector fields $\vec{f}(x) := f_1(x)\hat{x} + f_2(x)\hat{y} + f_3(x)\hat{z}$. In $\Omega^1(M)$, $\vec{f} \leftrightarrow f_1 \, dx + f_2 \, dy + f_3 \, dz$, and in $\Omega^2(M)$ the identification is $\vec{f} \leftrightarrow f_1 \, dy \wedge dz + f_2 \, dz \wedge dx + f_3 \, dx \wedge dy$. We can compute the differential on these different functions. On $\Omega^0(M)$:

$$df = (\partial_i f) dx^i \leftrightarrow \partial_i f \, \hat{x}_i \tag{122}$$

and we see this is the standard gradient! On $\Omega^1(M)$:

$$d(f_1 \, dx + f_2 \, dy + f_3 \, dz) = (\partial_y f_3 - \partial_z f_2) \, dy \wedge dz + (\partial_z f_1 - \partial_x f_3) \, dz \wedge dx + (\partial_x f_2 - \partial_y f_1) \, dx \wedge dy$$
$$\leftrightarrow \nabla \times \vec{f} \tag{123}$$

which is the curl of the vector field. Similarly, on 2 forms:

$$d(f_1 \, dy \wedge dz + f_2 \, dz \wedge dx + f_3 \, dx \wedge dy) = (\partial_x f_1 + \partial_y f_2 + \partial_z f_3) \, dx \wedge dy \wedge dz \leftrightarrow \nabla \cdot \vec{f} \tag{124}$$

So, we see that every time we use multivariable calculus, we are secretly using the machinery of differential forms. The reason that this is not immediately obvious is because the exterior powers of $\mathbb{R}^3$ have similar dimensions, and we identify the 3-dimensional powers as vector fields and the 1-dimensional powers as standard functions.

4   de Rham Cohomology

The exterior derivative provides a powerful structure on differential forms; as mentioned earlier, the fact that $d^2 = 0$ gives $\Omega^*(M)$ the structure of a cochain complex. Accordingly, we can take the cohomology of this complex. We start with some terminology.

Definition 4.1. Let $\omega \in \Omega^k(M)$ be a k-form. If $d\omega = 0$, we say that ω is **closed**. If $\omega = d\tau$ for $\tau \in \Omega^{k-1}(M)$, we say that ω is **exact**. Let $Z^k(M)$ be the space of closed k-forms, and $B^k(M)$ be the space of exact k-forms. Then the **kth de Rham cohomology of** M is defined to be the vector space:

$$H^k(M) := Z^k(M)/B^k(M) \tag{125}$$

Because $\Omega^k(M)$ vanishes for $k > n$, $H^k(M)$ also vanishes for $k > n$. For $k = 0$, we can classify the cohomology group $H^0(M)$.

Theorem 4.1. Suppose M has r connected components. Then:

$$H^0(M) = \mathbb{R}^r \tag{126}$$

Proof. Since $B^0(M) = 0$, $H^0(M) = Z^0(M)$, and we must classify the closed 0-forms. Recall that 0-forms on a manifold are canonically isomorphic to functions on a manifold. So, let $f \in Z^0(M)$. Then:

$$df = \partial_i f \, dx^i = 0 \implies \forall i, \, \partial_i f = 0 \tag{127}$$

This implies f is locally constant, and therefore must be constant on each component of M. Because M has r components $M = \coprod_{i=1}^r M_i$ with $M_i \cap M_j = \emptyset$ for $i \neq j$, a basis for $Z^0(M)$ are the functions $\{f^i : M \to \mathbb{R}\}_{i=1}^r$ such that $f^i(x) = 1$ when $x \in M^i$ and $f^i = 0$ when $x \notin M^i$. Thus, $Z^0(M) = \mathbb{R}^r$. $\qquad \square$

The wedge product $\wedge$ gives the de Rham cohomology the structure of an algebra over $\mathbb{R}$, $(H^*(M), +, \cdot, \wedge)$.

4.1   Singular Cohomology

Recall the singular homology (integral coefficients) of a topological space X. We defined $Sin_n(X) := \{\sigma : \Delta^n \to X \mid \sigma \text{ is continuous.}\}$, then took the free abelian group $S_n(X) := \mathbb{Z}Sin_n(X)$ generated by the n-simplices in X. These are called n-chains in X. This space has a natural structure of a chain complex given by the boundary operator $\partial : \Delta^n \to \Delta^{n-1}$ sending each simplex to its boundary, and the **singular homology** is defined to be the homology of this chain complex.

One can dualize the n-chains in X to get a cochain complex:

$$S^n(X; \mathbb{Z}) := Hom(S_n(X), \mathbb{Z}) \tag{128}$$

This converts the chain complex $(S_*(X), \partial)$ into a cochain complex $(S^*(X), \delta)$, and the cohomology of this cochain complex is called the **singular cohomology** of X. Note that a cochain $f \in S^n(X)$ is a map $f : S_n(X) \to \mathbb{Z}$, so $f(\sigma)$ is an integer. Furthermore, the coboundary map δ is dual to the boundary map ∂ in the following manner:

$$(\delta f)(\sigma) = f(\partial \sigma) \tag{129}$$

where $f \in S^n(X; \mathbb{Z})$ and $\sigma \in S_{n+1}(X)$.

There are a variety of pairings on singular (co)homology that are of interest to us. Of primary interest will be the cup product, which will endow $S^*(X, \mathbb{Z})$ with a ring structure.

Singular cohomology is often the first introduction to cohomology that one sees in an algebraic topology course; we are discussing it now because there is a natural isomorphism between the de Rham cohomology and the singular cohomology of a manifold.

Theorem 4.2 (de Rham)**.** Let M be a n-manifold. Then the singular cohomology with real coefficients $H^*(M; \mathbb{R})$ and the de Rham cohomology $H^*_{DR}(M)$ are isomorphic as rings. The isomorphism is explicitly given by the map:

$$\gamma^* : H^*_{DR}(M) \to H^*(M; \mathbb{R}) \tag{130}$$

$$[\omega] \mapsto \left([\sigma] \mapsto \int_\sigma \omega \right) \tag{131}$$

To define this map, one works at the level of the cochain complexes. Define the map $\gamma : \Omega^k(M) \to S^k(M; \mathbb{Z})$ by $\gamma(\omega)(\sigma) := \int_\sigma \omega$ for $\sigma \in S_k(M)$. Then we must show that γ is a chain map, as then γ will drop to a homomorphism on the homology groups $\gamma^* : H^*_{DR}(M) \to H^*(M; \mathbb{R})$.

$$
\begin{array}{ccccccccc}
\dots & \xrightarrow{d} & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{d} & \dots \\
& & \downarrow{\gamma} & & \downarrow{\gamma} & & \downarrow{\gamma} & & \\
\dots & \xrightarrow{\delta} & S^{k-1}(M; \mathbb{R}) & \xrightarrow{\delta} & S^k(M; \mathbb{R}) & \xrightarrow{\delta} & S^{k+1}(M; \mathbb{R}) & \xrightarrow{\delta} & \dots
\end{array}
\tag{132}
$$

The fact that γ is a chain map is directly due to Stokes' theorem. For $\omega \in \Omega^k(M)$ and $\sigma \in S_{k+1}(M)$, we have:

$$(\gamma d)(\omega)(\sigma) = \gamma(d\omega)(\sigma) = \int_\sigma d\omega = \int_{\partial\sigma} \omega = \gamma(\omega)(\partial\sigma) = (\delta\gamma)(\omega)(\sigma) \tag{133}$$

This shows that γ is a chain map and therefore induces a homomorphism γ^* in cohomology.

5 Applications to physics

5.1 Electromagnetism

Problem 19.13 in Tu is a good start for this.

5.2 Relativity