Patrick Oare | Physics H7A Midterm 1

Formulas:

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 \ldots a_nb_n = |\vec{a}||\vec{b}|\cos\theta$

$\vec{V}_{ab} = \vec{V}_{ac} + \vec{V}_{cb} = \vec{V}_{ac} - \vec{V}_{bc}$ $\vec{p} = m\vec{v}$

$\vec{F}_g = -G\frac{m_1 m_2}{r^2}\hat{r}$ $\vec{J} = \Delta\vec{p} = \int \vec{F}\,dt = \langle F\rangle\Delta t$

$\vec{F}_{app} = \vec{F}_{true} - M\ddot{\vec{R}}$

$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ $\omega = \dot{\theta}$ $s = r\theta$ $\frac{d}{dt}(|\vec{a}|^2) = 2\vec{a}\cdot\dot{\vec{a}}$ $V_{sphere} = \frac{4}{3}\pi r^3$

$\vec{a} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$\hat{r} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$

$\hat{\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$ $\dot{\hat{\theta}} = -\dot{\theta}\hat{r}$

Circle: $\vec{r}(t) = r_0(\cos\omega t\,\hat{\imath} + \sin\omega t\,\hat{\jmath})$

$F_D \sim -bv - cv^2$, for sphere $b = 2\beta r$, $c = (2r)^2\gamma$ $T = \frac{2\pi}{\omega}$

$\vec{F}_{net} = d\vec{p}/dt = m\vec{a}$ (unless mass Δs) $F_f = \mu N$ $F_s = -kx$

$V = V_0 + at$ $\Delta x = \frac{1}{2}(V + V_0)t$

$\Delta x = V_0 t + \frac{1}{2}at^2$ $V^2 = V_0^2 + 2a\Delta x$

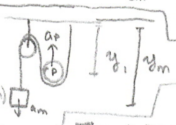
$\vec{R}_{cm} = \frac{\sum m\vec{r}}{\sum m} = \frac{1}{M}\int \vec{r}\,dm = \frac{1}{M}\iiint_E \rho\vec{r}\,dV$

$\Delta v = u\ln\left(\frac{M_0}{M_f}\right) - gt$ (only w/ gravity)

SHM: $\ddot{x} + \omega^2 x = 0$

• **Constraints:**

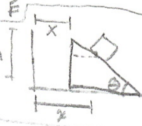
— Length of rope:



$\ell = y_1 + \pi R + (y_1 - h) + \pi R + (y_1 - y_m)$

$\ddot{\ell} = 0 = 2\ddot{y}_1 + \ddot{y}_m \Rightarrow a_m = -2a_p$

— Block on wedge:



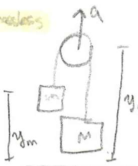
$\tan\theta = \frac{h - y}{x - X}$

$\ddot{y} = \tan\theta(\ddot{x} - \ddot{X})$

$ma = F - F'$

$Ma = F'$

— Accelerating massless Pulley



$2a_p = a_M + a_m$

$\ell = \pi R + (y_p - y_M) + (y_p - y_m)$

• **Tension:**

M, L $\lambda = M/L$

$T(r) \leftarrow \rightarrow T(r + \Delta r)$

$T(r) - T(r + \Delta r) = \Delta m a$

$a = r\dot{\theta}^2 = r\omega^2$ $\Delta m = \frac{M}{L}\Delta r$

$T(r + \Delta r) - T(r) = -\frac{M}{L}\Delta r(r\omega^2)$

$\frac{dT}{dr} \leftarrow \frac{T(r + \Delta r) - T(r)}{\Delta r} = -\frac{M\omega^2}{L}r$

For m:

$-T = ma_r = m(\ddot{r} - r\dot{\theta}^2)$ (2)

$N - mg = 0$

$0 = ma_T = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$

Constraint: $z < 0$

$\ell = r - z$

$\ddot{\ell} = 0 \Rightarrow \ddot{z} = \ddot{r}$ (3)

Moves w/ ω_0 starting at r_0

For M: $T - Mg = M\ddot{z}$ (1)

Solve the system w/ (1), (2), (3) knowing $\dot{\theta}(0) = \omega$ and $r(0) = r_0$

• **Polar:** Analyze radial/tangential separately

$\theta = \omega t \rightarrow \dot{\theta} = \omega$ Find $\vec{v}(t)$ and $\vec{a}(t)$

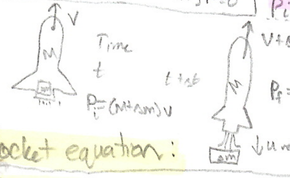
$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\vec{v} = u\hat{r} + \omega u t\hat{\theta}$

$V_R = u \Rightarrow r = ut,\ \ddot{r} = 0$

$\vec{a}(t) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$\vec{a}(t) = (-ut\omega^2)\hat{r} + (2u\omega)\hat{\theta}$



Time t $P_i = (M + \Delta m)v$

$t + \Delta t$ $V + \Delta V$ $\downarrow u$ rel to v

$P_i = P_f \therefore (M + \Delta m)v = M(v + \Delta v) + \Delta m(v - u)$

$P_f = M(v + \Delta v) + \Delta m(v - u)$

$0 = M\Delta v - \Delta m u$

$M\frac{dv}{dt} = \frac{dm}{dt}u$

$\frac{dm}{dt} = -\frac{dM}{dt}$

$M\frac{dv}{dt} = -u\frac{dM}{dt}$

$\int_{M_0}^{M_f}\frac{dM}{M} = -\int_{v_0}^{v_f}\frac{dv}{u} \rightarrow V = u\ln\frac{M_0}{M_f}$

Rocket equation:

Continuous fluid: Stream at v, mass/length λ

— $\vec{P}$ per length: $\frac{mv}{\ell} = \lambda v$ $= dp/d\ell$

— Rate of transport of $\vec{p}$: $\frac{d\vec{p}}{dt} = \frac{dp}{d\ell}\cdot\frac{d\ell}{dt} = (\lambda v)(v) = \lambda v^2$

— $\vec{F} = d\vec{p}/dt = \lambda v^2$

Fluid w/ cross section A:

n particles/unit volume w/ mass m_0, total volume V

$M_{total} = m_0(nV)$ $\lambda = \frac{M_{total}}{L} = \frac{m_0 n V}{L} = m_0 n A$ $V = AL$

$F = \lambda v^2 = m_0 n A v^2$ $P = F/A = m_0 n v^2$

Drag Force: Say $F_D = -cv^2$

$ma = mg - cv^2$ $m\frac{dv}{dt} = mg - cv^2$

$\int_{v_0}^{v_f}\frac{dv}{mg - cv^2} = \int_0^t\frac{dt}{m}$

• COM always conserved w/ no F_{ext}

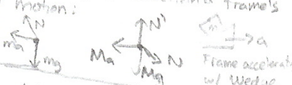
Spherical: (r, ϕ, θ)

$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

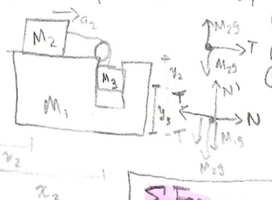
$z = r\cos\phi$

• Fictitious forces: Tack on an ma in opposite direction of frame's motion:



Frame accelerating w/ wedge

• Dynamics problem: Find a_1



(1) $M_2 a_2 = T$

(2) $M_1 a_1 = N - T$

(3) $M_3 a_{3x} = -N$

(4) $M_3 a_{3y} = T - M_3 g$

(5) $a_1 = a_{3x}$

— $\ell = (x_3 - x_2) + \frac{\pi R}{2} + (y_2 - y_3)$

$\ddot{\ell} = 0 = \ddot{x}_3 - \ddot{x}_2 - \ddot{y}_3$

(6) $a_{3x} = a_2 + a_{3y}$

6 vars $(a_1, a_2, a_{3x}, a_{3y}, T, N)$, 6 eqns

$\sum$ Forces in a massless pulley is 0

Spinning loop

— Find tension

— Analyze a small $\Delta\theta$

Mass M, circ. ℓ

$\phi = \frac{\Delta\theta}{2}$

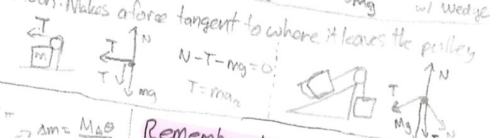
$(a_r = \ddot{r} - r\dot{\theta}^2) = -r\omega^2$ $r = \ell/2\pi$

$-2T\sin\phi = -\Delta m r\omega^2$

$\Delta m = \lambda\Delta s = \frac{M}{\ell}\left(\frac{\ell}{2\pi}\right)\Delta\theta \rightarrow \Delta m = \frac{M\Delta\theta}{2\pi}$

$2T\sin\phi \sim 2T\phi = T\Delta\theta = \left(\frac{M\Delta\theta}{2\pi}\right)\left(\frac{\ell}{2\pi}\right)\omega^2 \therefore T = \frac{M\ell}{4\pi^2}\omega^2$

• Tension: Makes a force tangent to where it leaves the pulley

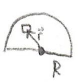


$N - T - mg = 0$

$T = ma_x$

Remember to multiply $\vec{a}$ in polar by the mass to get Force!

• COM of a semicircular disk:



$\vec{R}_{cm} = \frac{1}{M}\iint_D \sigma\vec{r}\,dA = \frac{\sigma}{M}\iint_D \vec{r}\,dA = \frac{2}{\pi R^2}\iint_D \vec{r}\,dA$

$\sigma = \frac{2M}{\pi R^2} \therefore \frac{\sigma}{M} = \frac{2}{\pi R^2}$

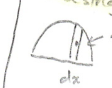
$x \in [-R, R]$

$y \in [0, \sqrt{R^2 - x^2}]$

• By sym, $x_{cm} = 0 \rightarrow Y_{cm} = \frac{2}{\pi R^2}\iint_D y\,dA$

$= \frac{2}{\pi R^2}\int_{-R}^{R}\int_0^{\sqrt{R^2 - x^2}} y\,dy\,dx$

w/ a single integral:

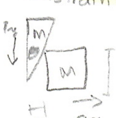


$y_{cm} = \frac{y}{2}$ $dA = y\,dx = \sqrt{R^2 - x^2}\,dx$

$Y_{cm} = \frac{2}{\pi R^2}\int\left(\frac{y}{2}\right)dA$

$Y_{cm} = \frac{1}{\pi R^2}\int y\sqrt{R^2 - x^2}\,dx = \frac{1}{\pi R^2}\int_{-R}^{R} R^2 - x^2\,dx$

• Constraint: Wedge on block:



$\tan\theta = y/x$

$y = x\tan\theta$

$\ddot{y} = \ddot{x}\tan\theta$

$\vec{r}_1 = (R, \pi/4, 0) = R\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

$\vec{r}_2 = (R, \pi/4, \pi/2) = R\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Spaceship moving through cloud w/ cross section A:

Mass of dust it hits Δm is

$\Delta m = \rho\Delta V = \rho A\Delta\ell = \rho A\frac{\Delta\ell}{\Delta t}\Delta t$

$\Delta m = \rho A v\Delta t$

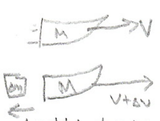
$\vec{r}(t) = r_0(\cos\omega t\,\hat{\imath} + \sin\omega t\,\hat{\jmath})$

$x(t)$ and $y(t)$ if you know $\theta(t)$:

$x(t) = r_0\cos(\theta(t))$

$y(t) = r_0\sin(\theta(t))$

Jetski: Pumps water from sea behind it at u m/s relative to velocity, jetski + rider mass m, pumps at $b = dm/dt$



Time t $P_i = Mv$

Time $t + \Delta t$ $V + \Delta V$ u rel to boat $\rightarrow V - u$

$P_f = M(v + \Delta v) + \Delta m(v - u)$

$\Delta P = 0 = M\Delta v + \Delta m(v - u)$

$\lim_{\Delta t \to 0}\left(\frac{M\Delta v}{\Delta t} = -\Delta m(v - u) = \frac{\Delta m(u - v)}{\Delta t}\right)$

$M\frac{dv}{dt} = \frac{dm}{dt}(u - v) = b(u - v)$

$\int_{v_0}^{v}\frac{dv}{u - v} = \int_0^t\frac{b}{M}dt$

w/ drag $\vec{F}_D = -\lambda\vec{v}$

$\vec{F}_D = \vec{F}_{ext} = \Delta\vec{p} \therefore -\lambda v = M\Delta v + \Delta m(v - u)$

• Gravity inside Mass: $r < R$

$\vec{F}_g = -G\frac{Mm}{R^3}r\hat{r}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Patrick Oare $W_{ba} = \int_a^b \vec{F} \cdot d\vec{r} = T_b - T_a = U_a - U_b$ $\vec{F}_g = -\frac{Gm_1m_2}{r^2}\hat{r}$ $U_g = -\frac{Gm_1m_2}{r}$ $\vec{F} = -\frac{dU}{dx} = -\nabla U$ Physics H7A Midterm 2 Cheat Sheet

Energy: $T = \frac{1}{2}mv^2 = \frac{|\vec{p}|^2}{2m}$ $U_g = mgh$ • **Energy diagram:** Equilibrium $\rightarrow \frac{dU}{dx} = 0$ • Turning points when $T = 0 \therefore U = E_{sys}$

Same format as $U = \frac{1}{2}kx^2$: $k = d^2U/dx^2|_{x=x_0}$, $x = (x - x_0)$

$T_a + U_a + W_{ba} = T_b + U_b$ – Stable when $\frac{d^2U}{dx^2} > 0$, unstable if $\frac{d^2U}{dx^2} < 0$ choice of potential at $U(x_0) = 0$; $U'(x_0)$ by def $= 0$

$\omega^2 = \frac{1}{m}\frac{d^2U}{dx^2}\Big|_{x=x_0}$ $\ddot{x} + \omega^2 x = 0$

• Bound systems: $U = U(x) \rightarrow U$ has min at x_0 so $U'(x_0) = 0$; Taylor expand around x_0: $U(x) \approx U(x_0) + U'(x_0)(x - x_0) + \frac{U''(x_0)(x-x_0)^2}{2} \rightarrow U(x) \approx \frac{1}{2}(x - x_0)^2 \frac{d^2U}{dx^2}\Big|_{x=x_0}$

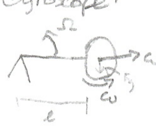
• Power $= \frac{dW}{dt} = \vec{F} \cdot \vec{v} = P$ • Elastic collision $\rightarrow E, p$ conserved

$\frac{d\vec{B}}{dt} = \vec{\Omega} \times \vec{B}$ $\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$ $\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}$ $I = \iiint \rho \rho^2 dV$ $T = \frac{1}{2}I\omega^2$ $\vec{v} = \vec{\omega} \times \vec{r}$ $I = I_{cm} + mh^2$ $\vec{\tau} = \dot{\vec{L}}$ $|\vec{a}| = r\alpha$

Rotation: • Angular momentum conserved w/ no net external torque • Any motion/ $\vec{L}/\vec{\tau}$ can be broken into rotation about the CoM and translation $W = \int \tau \, d\theta$

• Torque from gravity acts on CoM • Make sure to check signs on N2 equations • Rolling w/o slipping $\rightarrow a = r\alpha$ • **Perpendicular axis theorem:** For any planar object, the I value about a point is the sum of the I values of two perpendicular axes $\rightarrow I_z = I_x + I_y$

• Gyroscope:

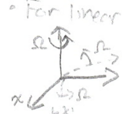


$\vec{L}_0 = I_0\vec{\omega}$ $\Omega = \frac{\tau}{L_{spin}}$

$\vec{\tau} = mg\ell = \frac{d\vec{L}}{dt} = \vec{\Omega} \times \vec{L} = \Omega L \rightarrow \Omega = \frac{mg\ell}{I_0\omega}$

Non-Inertial Frames:

• For linearly accelerating, tack on a $F_{fict} = ma$ opposite direction of acceleration



• Rotating frame:

$\left(\frac{d\vec{C}}{dt}\right)_{IP} = \left(\frac{d\vec{C}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{C}$

$F_{cor} = 2m(\vec{v}_{rot} \times \vec{\Omega})$

$F_{CF} = m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} = m\Omega^2 \rho\hat{\rho}$

– CF much stronger than F_{cor} unless after long time or slow rotation speed

• F_{fict} can also produce a torque at CoM

• Centrifugal potential: $U(\vec{r}) = -\frac{1}{2}m\omega^2 r^2$

• NIF: $E_2 - E_1 = W_{21}^{NC} + W_{21}^{Fict}$ ← Fict forces can do work

Misc. Formulas: $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$, $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$

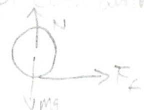
$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$ $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ $\dot{\hat{\theta}} = -\dot{\theta}\hat{r}$

$a_c = r\omega^2 = v^2/r$ $\Delta\vec{r} = \vec{v}_0\Delta t + \frac{1}{2}\vec{a}\Delta t^2$ $\vec{F} = \dot{\vec{p}}$ $\vec{v} = \vec{v}_0 + \vec{a}\Delta t$

$\vec{R}_{cm} = \frac{1}{m}\iiint \rho\vec{r}\,dV$, $\vec{r} = x\hat{i} + y\hat{j} = r\hat{r}$ $\vec{J} = \Delta\vec{p} = \int \vec{F}\,dt$

For oscillations, try a force diagram and look for $\frac{d^2\theta}{dt^2} = -\omega^2\theta$ or $\frac{d^2x}{dt^2} = -\omega^2 x$

Disk w/ ω_0 placed on surface, radius R, what is final v_f of CoM when it rolls w/o slipping?



$v = ?$ $v_0 = 0$ $a = F_f/m$ $\Delta t = t$

$ma = F_f$ $a = \frac{F_f}{m}$ For rolling w/o slipping, $v = r\omega$ – Same time t this happens

$I\alpha = \frac{1}{2}mR^2\alpha = F_fR \rightarrow \alpha = \frac{2F_f}{mR}$ $\omega = ?$ $\omega_0 = \omega_0$ $\alpha = -\frac{2F_f}{mR}$ $\Delta t = t$

$v = 0 + \frac{F_f}{m}t$ $\omega = \omega_0 - \frac{2F_f}{mR}t$ $v = R\omega \rightarrow \frac{F_f}{m}t = \omega_0 R - \frac{2F_f}{m}t$ $t = \frac{m\omega_0 R}{3F_f}$

$v = \frac{\cancel{F_f}}{\cancel{m}}\left(\frac{\cancel{m}\omega_0 R}{3\cancel{F_f}}\right) = \boxed{\frac{\omega_0 R}{3}}$

I values:

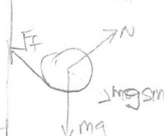
$1/3\,mL^2$

$1/12\,mL^2$

$\frac{1}{2}mR^2$

Cylinder: $\frac{1}{2}mr^2$ Hollow sphere: $\frac{2}{3}mr^2$ Solid sphere: $\frac{2}{5}mr^2$

Find period of oscillations of a marble w/ radius b in a well of radius R:



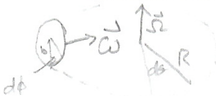
$ma = mg\sin\theta - F_f$

$\tau = \alpha \frac{2}{5}mb^2\left(\frac{a}{b}\right) = F_f b$

$ma = mg\sin\theta - \frac{2}{5}ma \rightarrow \frac{7}{5}ma = mg\sin\theta$

$a = R\frac{d^2\theta}{dt^2} = \frac{5g}{7}\sin\theta \approx \frac{5g}{7}\theta$

$\frac{d^2\theta}{dt^2} = \frac{5g}{7R}\theta \rightarrow \omega = \sqrt{\frac{5g}{7R}} \therefore T = 2\pi\sqrt{\frac{7R}{5g}}$



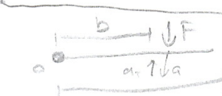
S is same for big R/little b:

$S = R\,d\theta = b\,d\phi \rightarrow R\Omega = b\omega$

$\omega = \frac{d\phi}{dt}$, $\Omega = \frac{d\theta}{dt}$ $\omega = \frac{R}{b}\Omega$

$U_{fict} = \frac{L^2}{2mr}$ (?)

Baseball bat (might be wrong):



Find b such that $\tau_0 = 0$

$F = ma$ $I = \frac{1}{3}m\ell^2$

$\tau = bF = \frac{1}{3}m\ell^2\alpha$ $\alpha = \frac{a_T}{b} \rightarrow bF = \frac{1}{3}m\ell^2\left(\frac{a_T}{b}\right) \rightarrow a_T = \frac{3b^2F}{m\ell^2}$

Want $|a_T| = |a| \rightarrow \frac{3b^2F}{m\ell^2} = \frac{F}{m} \rightarrow b^2 = \frac{\ell^2}{3} \rightarrow \boxed{b = \frac{1}{\sqrt{3}}\ell}$

Add effective potential to final sheet

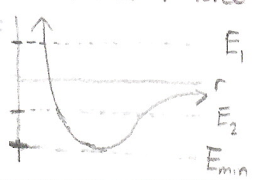
Patrick Oare | $\vec F = f(r)\hat r$ $\mu = \frac{m_1 m_2}{m_1+m_2}$ $L=\mu r^2\dot\theta$ $E=\frac{1}{2}\mu\dot r^2+U_{eff}(r)$ $U_{eff}(r)=\frac{L^2}{2\mu r^2}+U(r)$ | Physics H7A Final Cheat Sheet

Central Forces: Conserve $|\vec L|$ and E

• Plot energy diagrams w/ $U_{eff}(r)$

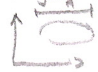
$\Delta\theta = L\int \frac{dr}{r^2\sqrt{2\mu(E-U_{eff})}}$ $r=\frac{r_0}{1-\varepsilon\cos\theta}$ $r_0=\frac{L^2}{\mu C}$

• Examine a $F\propto 1/r^2$ force: $\vec F=-\frac{C}{r^2}\hat r$



- $E=E_1 \to$ Open orbit $\to$ Hyperbola
- $E=0 \to$ Open $\to$ Parabola
- $E=E_2 \to$ Closed $\to$ Ellipse
- $E=E_{min} \to$ Closed $\to$ Circle

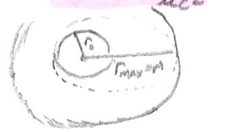
• Ellipse energy w/ semimajor axis A is $E=-\frac{C}{A}$



• Can only say $L=mrv$ at perigee and apogee

• Orbit transfer:
- Ellipse w/ r_{min} at r_0 and r_{max} at r'
- Solve for r_0, ε and use equations to find E, L

$\varepsilon=\sqrt{1+\frac{2EL^2}{\mu C^2}}$



Oscillations: $e^{i\theta}=\cos\theta+i\sin\theta$

• Solve for A, ϕ w/ I.C.'s

Solution to: $\ddot x+\omega^2 x=0$ is $x(t)=A\cos(\omega t+\phi)$

• Solve equations by guessing $x(t)=Ae^{\gamma t}$

Damped: $m\vec a=-b\vec v-k\vec x \to \ddot x+2\beta\dot x+\omega_0^2 x=0$

$2\beta=b/m=\gamma$ $\omega_0^2=k/m$

• General soln $x(t)=e^{-\beta t}\left(c_1e^{\sqrt{\beta^2-\omega_0^2}t}+c_2e^{-\sqrt{\beta^2-\omega_0^2}t}\right)$

• 3 cases: ① Underdamped, ② Overdamped, ③ Critical

① $\beta<\omega_0$: $x(t)=e^{-\beta t}\left(c_1e^{i\omega_1 t}+c_2e^{-i\omega_1 t}\right)$ $\omega_1^2=\omega_0^2-\beta^2$

② $\beta>\omega_0$: $x(t)=e^{-\beta t}\left(c_1e^{\omega_1 t}+c_2e^{-\omega_1 t}\right)$ $\omega_1^2=\beta^2-\omega_0^2$

③ $\beta=\omega_0$: $x(t)=e^{-\beta t}(c_1+c_2t)$ - Fastest return to eq - No oscillation

• Time to decay to $1/e$: $\tau=1/\beta$ • $T=2\pi/\omega_1$ for ①

$a(t)=a_0e^{-\beta t}$

• Quality Factor:
- $\lambda=\ln\left(\frac{a(t)}{a(t+T)}\right)\to\frac{a(t)}{a(t+T)}=e^{\beta T}\to\lambda=\beta T$
- $Q=\frac{\pi}{\lambda}=\pi N_e=\pi\frac{\tau}{T}=\frac{\pi}{\beta T}=\frac{\omega_1}{2\beta}$
- Energy stored in oscillator w/ k_{eff}:

$E(t)=\frac{1}{2}ka_0^2e^{-2\beta t}\to\frac{dE}{dt}=-2\beta E(t)$

- ΔE is E lost/period, $\Delta E'$ is E lost/rad:

$Q=2\pi E/\Delta E = E/\Delta E'$

Driven Oscillations: Driving force $F=F_0 m\cos(\omega t)$

• EoM: $\ddot x+\omega_0^2 x=F_0\cos(\omega t)$
- Solution is $x_p(t)+x_h(t)$; x_p is soln to above eqn $\ddot x_p+\omega_0^2x_p=F_0\cos\omega t$, x_h is homogeneous soln: $\ddot x_h+\omega_0^2x_h=0\to x_h(t)=a\cos(\omega_0t+\delta)$

$x(t)=x_p(t)+a\cos(\omega_0t+\delta)$

Driven, Damped Oscillations:

• EoM: $\ddot x+2\beta\dot x+\omega_0^2x=F_0\cos(\omega t)$
- $x_h(t)$ is now soln to damped oscillator:

$x_h(t)=a_0e^{-\beta t}(\cos(\omega_1t+\phi))$

- Find $x_p(t)$: Let $x(t)=\Re[z]$ where z satisfies $\ddot z+2\beta\dot z+\omega_0^2z=F_0e^{i\omega t}$
- Let $z(t)=z_0e^{i\omega t}=Ae^{-i\delta}$
- Solving, get: $z_0=Ae^{-i\delta}=\frac{F_0}{\omega_0^2-\omega^2+2\beta i\omega}$
- Simplify and get $\frac{F_0}{A}e^{i\delta}=\omega_0^2-\omega^2+2i\beta\omega$
- Set Re and Im parts = to get:

$\tan\delta=\frac{2\beta\omega}{\omega_0^2-\omega^2}$ $A=\frac{F_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$

- Soln is: $x(t)=A\cos(\omega t-\delta)$

• Resonance: Max A when denom of A at min:

$\frac{d}{d\omega}\left((\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right)\Big|_{\omega=\omega_R}=0$

- End up w/ $\omega_R^2=\omega_0^2-2\beta^2$

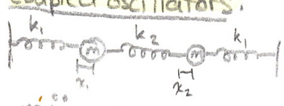
Many Coupled Oscillators:

• EoM: $\ddot x_j+2\omega_0^2x_j-\omega_0^2(x_{j+1}+x_{j-1})=0$
- Solution: $x(t)=A_{jn}\cos(\omega_nt)$

$A_{jn}=c_n\sin\left(\frac{n\pi}{N+1}j\right)$ $\omega_n=2\omega_0\sin\left(\frac{n\pi}{2(N+1)}\right)$

- Continuous limit as $N\to\infty$ is a wave

Coupled oscillators:



$m\ddot x_1=-k_1x_1-k_2(x_1-x_2)$

$m\ddot x_2=-k_1x_2-k_2(x_2-x_1)$

$$\begin{cases}\ddot x_1=-\frac{k_1}{m}x_1-\frac{k_2}{m}(x_1-x_2)\\ \ddot x_2=-\frac{k_1}{m}x_2-\frac{k_2}{m}(x_2-x_1)\end{cases}$$

Solve by adding x_1+x_2 and subtracting to get eqns for x_1+x_2 and x_1-x_2

Waves: $y(x,t)=A\sin(k'x\pm\omega t)$

- Positive x traveling wave is $k'x-\omega t$, $-x$ moving is $kx+\omega t$

• $\omega_n=\frac{n\pi v}{L}=\frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$ $\lambda=\frac{2L}{n}$
- For n^{th} normal mode

• Group Velocity: Speed of modulated packet $V_g=\frac{df}{dk}=\frac{d\omega}{dk'}$

• All harmonics multiples of fundamental frequency

• $\omega=2\pi f=\frac{2\pi}{T}$ $k'=\frac{2\pi}{\lambda}$ $k=1/\lambda$

• Phase velocity: Speed of an individual wave $V_{ph}=\sqrt{\frac{T}{\mu}}=\lambda f=\lambda/k=\omega/k'$

• Wave equation: $\frac{\partial^2y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2y}{\partial t^2}$

• Energy: $\frac{dT}{dx}=\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$ $\frac{dU}{dx}=\frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2$

• Traveling waves always of form $f(x\mp vt)$

• Doppler Effect: $f=\frac{(V+u_R)}{(V-u_s)}f_0$ - u_R is speed of reciever, u_s is speed of source

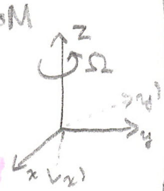
$\frac{d\vec B}{dt}=\vec\Omega\times\vec B$ $I=\iiint\rho p^2dV$ $I=I_{cm}+mh^2$

Non-Inertial Frames:

• Linearly accelerating - Add on $F_{fict}=-m\vec a$ opposite direction of acceleration

• F_{fict} produces a torque about CoM

• Rotating Frames:



$\vec F_{cor}=2m(\vec v_{rot}\times\vec\Omega)$

$\vec F_{CF}=m(\vec\Omega\times\vec r)\times\vec\Omega=m\Omega^2p\hat p$

$\left(\frac{d\vec C}{dt}\right)_{IF}=\left(\frac{d\vec C}{dt}\right)_{rot}+\vec\Omega\times\vec C$ p is $\perp$ distance from axis

Rotation: $\vec L=\vec r\times\vec p=\tilde I\vec\omega$ $\vec\tau=\vec r\times\vec F=\tilde I\vec\alpha=\frac{d\vec L}{dt}$ $T_{rot}=\frac{1}{2}I\omega^2$ $W=\int\tau d\theta$

• Torque from gravity acts on CoM

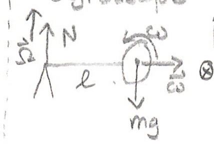
• L conserved w/ no τ_{ext}

• Make sure to check signs on N2

• Rolling - Point of contact not moving w/ respect to surface ($a=R\alpha$ and $v=R\omega$)

• Any motion can be broken into translation + rotation

• Gyroscope



$\vec L_s=I\vec\omega$ $\tau=mgl=\Omega L_s$

$\Omega=\frac{\tau}{L_{spin}}=\frac{mgl}{I\omega_s}$

I values: $I=kmr^2$
• Rod about end: $k=\frac{1}{3}$
• Rod about center: $k=\frac{1}{12}$
• Disk/Cylinder: $k=\frac{1}{2}$
• Hollow Sphere: $k=\frac{2}{3}$
• Solid Sphere: $k=\frac{2}{5}$

Energy:
$W_{ba} = \int_a^b \vec{F} \cdot d\vec{r} = T_b - T_a = U_a - U_b$ $\quad T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

$U_g = mgh \quad \vec{F}_g = -G\frac{mM}{r^2}\hat{r} \quad U_g = -G\frac{mM}{r} \quad \vec{F} = -\nabla U$

$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$ • Elastic collision $\rightarrow \vec{p}$ and E conserved

- Energy diagram: Equilibrium $\Rightarrow F = -\frac{dU}{dx} = 0$
  - Stable eq. when $\frac{d^2U}{dx^2} > 0$; unstable when $\frac{d^2U}{dx^2} < 0$
  - Turning point when $T = 0 \therefore U = E$ $\quad \swarrow x_0$ is min so $U'(x_0) = 0$
  - Bound systems: $U(x) \sim U(x_0) + U'(x_0)(x - x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2$
    - Let $U(x_0) = 0$ so: $U(x) = \frac{1}{2}(x - x_0)^2 \left.\frac{d^2U}{dx^2}\right|_{x = x_0}$
    - Spring w/ $k_{eff} = \left.\frac{d^2U}{dx^2}\right|_{x = x_0}$ so $\omega^2 = \frac{1}{m}\left.\frac{d^2U}{dx^2}\right|_{x = x_0}$

Trig Identities:
$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$

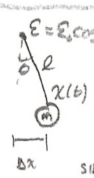
$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$

$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$

$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$

$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$



$T \sim mg$ b/c small θ

$T_x = T\sin\theta = \frac{mg}{\ell}(x - \varepsilon)$

$\sin\theta = \frac{\Delta x}{\ell} = \frac{x - \varepsilon}{\ell}$

$m\ddot{x} = -T_x = -\frac{mg}{\ell}(x - \varepsilon)$

$\boxed{\ddot{x} + \frac{g}{\ell}x = \frac{g}{\ell}\varepsilon}$

Random Formulas:
$\vec{J} = \Delta\vec{p} = \int \vec{F}dt = \langle \vec{F} \rangle \Delta t \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = r\hat{r} + z\hat{z}$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \quad \hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \quad \hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$

$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\theta} \quad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \quad \vec{p} = m\vec{v} \quad \vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = m\vec{a}$

$\vec{v} = \vec{v}_0 + \vec{a}t \quad \Delta\vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \quad v^2 = v_0^2 + 2a\Delta r \quad V_{sphere} = \frac{4}{3}\pi r^3 \quad \Delta v = u\ln\left(\frac{M_0}{M_f}\right) - gt$

$\vec{R}_{cm} = \frac{\sum m\vec{r}}{\sum m} = \frac{1}{M}\iiint_E \rho\vec{r}dV$ $\quad$ Circle: $\vec{r}(t) = r_0(\cos(\omega t)\hat{i} + \sin(\omega t)\hat{j})$

$a_c = v^2/r = r\omega^2 \quad v = r\omega \quad \vec{F}_s = -k\vec{x} \quad U_s = \frac{1}{2}kx^2$

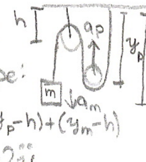
$\vec{g} = -9.8\hat{j}\ m/s^2 \quad G = 6.67 \times 10^{-11}\ \frac{N \cdot m^2}{kg^2}$

Dynamics:
n unknowns, n variables • $\sum\vec{F}$ on massless pulley $= 0$

- Always set a coordinate system before solving the problem
- General method: (1) FBD (2) Draw physical forces (3) Introduce coordinate system (4) Break forces into components along $\perp$ axes (5) Apply N_2 to each axis (6) Apply N_3 to relate forces (7) Include constraints (8) Solve the system

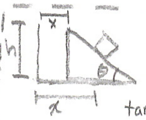
Constraints:

- Length of rope:



$\ell = y_p + 2\pi R + (y_p - h) + (y_m - h)$

$\ddot{\ell} = 0 \rightarrow \ddot{y}_m = -2\ddot{y}_p$



$h - y$, $x - X$, θ

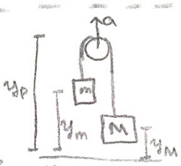
$\tan\theta = \frac{h - y}{x - X}$

$h - y = (x - X)\tan\theta$

$\ddot{y} = (\ddot{X} - \ddot{x})\tan\theta$

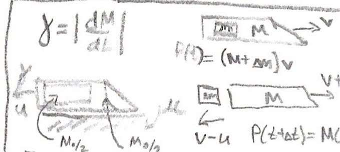
$\ell = \pi R + (y_p - y_m) + (y_p - y_M)$

$\ddot{\ell} = 0 \downarrow$



$2\ddot{y}_p = \ddot{y}_m + \ddot{y}_M$

$\gamma = \left|\frac{dM}{dt}\right|$



$\Delta P = P(t + \Delta t) - P(t)$

$= Mv + M\Delta v + \Delta m v - \Delta m u - (Mv + \Delta m v)$

$= M\Delta v - \Delta m u \quad \Delta m = -\Delta M$

$P(t + \Delta t) = M(v + \Delta v) + \Delta m(v - u)$

$\Delta P = M\Delta v + \Delta M u$

$\frac{\Delta P}{\Delta t} = M\frac{\Delta v}{\Delta t} + u\frac{\Delta M}{\Delta t}$

$\frac{dP}{dt} = F_{ext} = M\frac{dv}{dt} + u\frac{dM}{dt}$

$-\mu Mg = M\frac{dv}{dt} + u\frac{dM}{dt}$

$F_{ext} = -\mu N = -\mu Mg \rightarrow \frac{dv}{dt} = -\frac{u}{M}\frac{dM}{dt} - \mu g \rightarrow \int_0^{v(t)} dv = -u\int_{M_0}^{M(t)}\frac{dM}{M} - \int_0^t \mu g\,dt \rightarrow \boxed{v(t) = u\ln\left(\frac{M_0}{M_f}\right) - \mu g t}$

$\gamma = \frac{dM}{dt} \rightarrow \int_{M_0}^{M(t)} dM = \int_0^t \gamma\,dt \rightarrow M(t) = M_0 + \gamma t \rightarrow v(t) = u\ln\left(\frac{M_0}{M_0 - \gamma t}\right) - \mu g t$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

- If any problem has a central force, think to use U_{eff}
- On fictitious force problems, try g_{eff} instead