

MY HUMBLE SOLUTION TO

VOLUME 1 OF COURSE OF THEORETICAL PHYSICS

MECHANICS

Author
P.O. Bolduc

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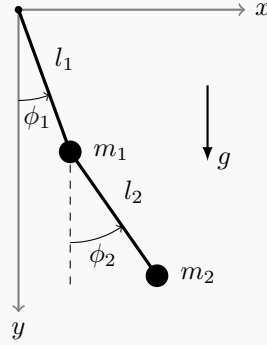
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CHAPTER 1.

THE EQUATIONS OF MOTION

PROBLEM 1.

Find the Lagrangian of a coplanar double pendulum when placed in a uniform gravitational field (acceleration g).



SOLUTION: The generalized co-ordinates of the system are the two angles ϕ_1 and ϕ_2 . We need to express the Cartesian co-ordinates in terms of those two angles. First, the Cartesian position of the particle m_1 is

$$\begin{aligned} x_1 &= f_1(\phi_1) = l_1 \sin \phi_1 \\ y_1 &= g_1(\phi_1) = l_1 \cos \phi_1. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_1 &= \frac{\partial f_1}{\partial \phi_1} \dot{\phi}_1 = l_1 \cos \phi_1 \dot{\phi}_1 \\ \dot{y}_1 &= \frac{\partial g_1}{\partial \phi_1} \dot{\phi}_1 = -l_1 \sin \phi_1 \dot{\phi}_1. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_1 &= T_1 - U_1 \\ &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + m_1 g y_1 \\ &= \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + m_1 g l_1 \cos \phi_1. \end{aligned} \tag{1.1}$$

Second, the Cartesian position of the particle m_2 is

$$\begin{aligned} x_2 &= f_2(\phi_1, \phi_2) = l_1 \sin \phi_1 + l_2 \sin \phi_2 \\ y_2 &= g_2(\phi_1, \phi_2) = l_1 \cos \phi_1 + l_2 \cos \phi_2. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_2 &= \sum_k \frac{\partial f_2}{\partial \phi_k} \dot{\phi}_k = l_1 \dot{\phi}_1 \cos \phi_1 + l_2 \dot{\phi}_2 \cos \phi_2 \\ \dot{y}_2 &= \sum_k \frac{\partial g_2}{\partial \phi_k} \dot{\phi}_k = -l_1 \dot{\phi}_1 \sin \phi_1 - l_2 \dot{\phi}_2 \sin \phi_2. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_2 &= T_2 - U_2 \\ &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + m_2 g y_2 \\ &= \frac{1}{2} m_2 (l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)) + m_2 g (l_1 \cos \phi_1 + l_2 \cos \phi_2), \end{aligned} \tag{1.2}$$

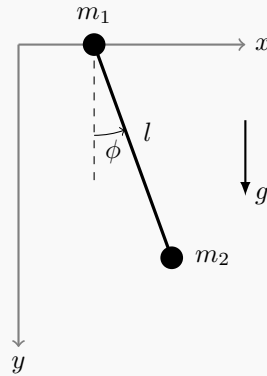
where the angle difference identity as been used. Finally, the Lagrangian of the complete system is simply the sum of Eq. (1.1) and Eq. (1.2), thus

ANSWER TO PROBLEM 1.

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) \\ + (m_1 + m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2$$

PROBLEM 2.

Find the Lagrangian of a simple pendulum of mass m_2 , with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves when placed in a uniform gravitational field (acceleration g).



SOLUTION: The generalized co-ordinates q of the system are the position x and the angle ϕ . Therefore, the Lagrangian of the first particle is simply

$$L_1 = T_1 = \frac{1}{2}m_1\dot{x}^2. \quad (1.3)$$

The Cartesian position of the particle m_2 can be express in terms of the generalized co-ordinates by

$$\begin{aligned} x_2 &= f_2(x, \phi) = x + l \sin \phi \\ y_2 &= g_2(x, \phi) = l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x}_2 &= \sum_k \frac{\partial f_2}{\partial q_k} \dot{q}_k = \dot{x} + l\dot{\phi} \cos \phi \\ \dot{y}_2 &= \sum_k \frac{\partial g_2}{\partial q_k} \dot{q}_k = -l\dot{\phi} \sin \phi. \end{aligned}$$

$$\begin{aligned} L_2 &= T_2 - U_2 \\ &= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + m_2gy_2 \\ &= \frac{1}{2}m_2(\dot{x}^2 + l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi} \cos \phi) + m_2gl \cos \phi. \end{aligned} \quad (1.4)$$

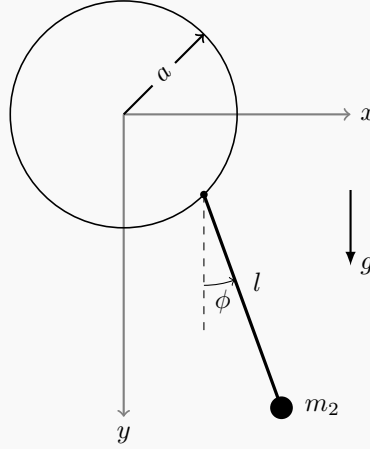
Finally, the Lagrangian of the complete system is the sum of the two Lagrangian Eq. (1.3) and Eq. (1.4), thus

ANSWER TO PROBLEM 2.

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi} \cos \phi) + m_2gl \cos \phi$$

PROBLEM 3.

Find the Lagrangian of a simple pendulum of mass m , when placed in a uniform gravitational field (acceleration g), whose point of support ...



(a)

moves uniformly on a vertical circle with constant frequency γ .

SOLUTION: If we set that the rotation of the point of support is counterclockwise, the Cartesian position of the particle m is

$$\begin{aligned} x &= f(\phi_1) = a \cos \gamma t + l \sin \phi \\ y &= g(\phi_1) = -a \sin \gamma t + l \cos \phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= -a\gamma \sin \gamma t + l\dot{\phi} \cos \phi \\ \dot{y} &= -a\gamma \cos \gamma t - l\dot{\phi} \sin \phi. \end{aligned}$$

The potential energy of the system is

$$\begin{aligned} U &= -mgy \\ &= -mg(-a \sin \gamma t + l \cos \phi). \end{aligned}$$

The term $mg a \sin \gamma t$ only depend on time and can therefore be ignored (does not contribute to the equations of motion). The kinetic energy of the particle, for its part, is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left(l^2 \dot{\phi}^2 + a^2 \gamma^2 + 2a\gamma l \dot{\phi} \sin(\phi - \gamma t) \right), \end{aligned}$$

using again the angle difference identity (I will stop mentioning it). We can observe that the term $\frac{1}{2}ma^2\gamma^2$ is a constant, thus can be ignored. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} ma\gamma l \dot{\phi} \sin(\phi - \gamma t) &= mal\gamma(\dot{\phi} - \gamma) \sin(\phi - \gamma t) + mal\gamma^2 \sin(\gamma t - \phi) \\ &= \frac{d}{dt} (-mal\gamma \cos(\phi - \gamma t)) + mal\gamma^2 \sin(\phi - \gamma t). \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

Answer to Problem 3 (a).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2 \sin(\phi - \gamma t) + mgl \cos \phi$$

(b)

oscillates horizontally in the plane of motion of the pendulum according to the law $x = a \cos \gamma t$.

SOLUTION: The Cartesian position of the particle m is

$$\begin{aligned}x &= f(\phi_1) = a \cos \gamma t + l \sin \phi \\y &= g(\phi_1) = l \cos \phi.\end{aligned}$$

The time derivative of the position is then

$$\begin{aligned}\dot{x} &= -a\gamma \sin \gamma t + l\dot{\phi} \cos \phi \\ \dot{y} &= -l\dot{\phi} \sin \phi.\end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mgl \cos \phi,$$

and the kinetic energy

$$\begin{aligned}T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left(l^2 \dot{\phi}^2 + a^2 \gamma^2 \sin^2 \gamma t - 2a\gamma l \dot{\phi} \sin \gamma t \cos \phi \right).\end{aligned}$$

We first see that the term $\frac{1}{2}ma^2\gamma^2 \sin^2 \gamma t$ only depend on time. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned}-ma\gamma l \dot{\phi} \sin \gamma t \cos \phi &= -mla\gamma(\gamma \cos \gamma t \sin \phi + \dot{\phi} \sin \gamma t \cos \phi) + mla\gamma^2 \cos \gamma t \sin \phi \\ &= \frac{d}{dt}(-mla\gamma \sin \gamma t \sin \phi) + mla\gamma^2 \cos \gamma t \sin \phi.\end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

Answer to Problem 3 (b).

$$L = \frac{1}{2}ml^2 \dot{\phi}^2 + mla\gamma^2 \cos \gamma t \sin \phi + mgl \cos \phi$$

(c)

oscillates vertically according to the law $y = a \cos \gamma t$.

SOLUTION: The Cartesian position of the particle m is

$$\begin{aligned}x &= f(\phi_1) = l \sin \phi \\ y &= g(\phi_1) = a \cos \gamma t + l \cos \phi.\end{aligned}$$

The time derivative of the position is then

$$\begin{aligned}\dot{x} &= l\dot{\phi} \cos \phi \\ \dot{y} &= -a\gamma \sin \gamma t - l\dot{\phi} \sin \phi.\end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mg(a \cos \gamma t + l \cos \phi),$$

The term $mga \cos \gamma t$ only depend on time and can therefore be ignored. The kinetic energy is

$$\begin{aligned}T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m \left(l^2 \dot{\phi}^2 + a^2 \gamma^2 \sin^2 \gamma t + 2a\gamma l \dot{\phi} \sin \gamma t \sin \phi \right).\end{aligned}$$

We first see that the term $\frac{1}{2}ma^2\gamma^2\sin^2\gamma t$ only depend on time. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} m\gamma l\dot{\phi}\sin\gamma t\sin\phi &= mla\gamma(-\gamma\cos\gamma t\cos\phi + \dot{\phi}\sin\gamma t\sin\phi) + mla\gamma^2\cos\gamma t\cos\phi \\ &= \frac{d}{dt}(mla\gamma\cos\gamma t\sin\phi) + mla\gamma^2\cos\gamma t\cos\phi. \end{aligned}$$

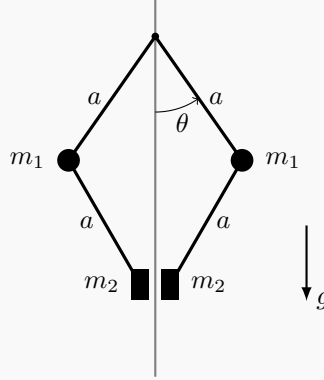
After dropping the total time derivative, we can finally get the Lagrangian

Answer to Problem 3 (c).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mla\gamma^2\cos\gamma t\cos\phi + mgl\cos\phi$$

PROBLEM 4.

Find the Lagrangian of a simple pendulum of the system below when placed in a uniform gravitational field (acceleration g). The particle m_2 moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity Ω .



SOLUTION: The position of each particle m_1 is best described in cylindrical coordinates, which is

$$\begin{aligned} r_1 &= a \sin \theta \\ \phi_1 &= \phi \\ z_1 &= a \cos \theta, \end{aligned}$$

where ϕ is the angle of rotation of the system about the axis; $\dot{\phi} = \Omega$. The kinetic energy of each particle m_1 is thus

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 (\dot{r}_1^2 + r_1^2 \dot{\phi}_1^2 + \dot{z}_1^2) \\ &= \frac{1}{2} m_1 (a^2 \dot{\theta}^2 \cos^2 \theta + a^2 \Omega^2 \sin^2 \theta + a^2 \dot{\theta}^2 \sin^2 \theta) \\ &= \frac{1}{2} m_1 a^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta). \end{aligned} \quad (1.5)$$

The potential energy of this particle can be found by using the z component of his position, namely

$$\begin{aligned} V_1 &= -m_1 g z_1 \\ &= -m_1 g a \cos \theta. \end{aligned} \quad (1.6)$$

The particle m_2 , in its case, can only move up and down, thus its position can be completely defined by

$$z_2 = 2a \cos \theta.$$

Then, the kinetic energy of each particle m_2 is

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \dot{z}_2^2 \\ &= m_2 a^2 \dot{\theta}^2 \sin^2 \theta \end{aligned} \quad (1.7)$$

and the potential energy is

$$\begin{aligned} V_2 &= -m_2 g z_2 \\ &= -2m_2 g a \cos \theta. \end{aligned} \quad (1.8)$$

Using Eq. (1.5), Eq. (1.6), Eq. (1.7) and Eq. (1.8), the Lagrangian of the system is

$$L = 2(T_1 + T_2 - V_1 - V_2),$$

therefore

ANSWER TO PROBLEM 4.

$$L = m_1 a^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) + 2m_2 a^2 \dot{\theta}^2 \sin^2 \theta + 2(m_1 + 2m_2) g a \cos \theta$$

CHAPTER 2.

CONSERVATION LAWS

PROBLEM 1.

A particle of mass m moving with velocity \mathbf{v}_1 leaves a half-space in which its potential energy is a constant U_1 and enters another in which its potential energy is a different constant U_2 . Determine the change in the direction of motion of the particle.

SOLUTION: The potential energy only depend on the co-ordinate perpendicular to the plane separating the half-space. Thus, the component of the momentum in that plane is conserved. Denoting by θ_1 and θ_2 the angles between the normal to the plane and the velocities v_1 and v_2 of the particle before and after passing the plane, we have

$$\begin{aligned} P_1 \sin \theta_1 &= P_2 \sin \theta_2 \\ \Rightarrow v_1 \sin \theta_1 &= v_2 \sin \theta_2 \\ \Rightarrow \frac{\sin \theta_1}{\sin \theta_2} &= \frac{v_2}{v_1}. \end{aligned} \quad (2.1)$$

The potential energy of the system is also independent of time, therefore the energy of the particle is conserved. Posing $E_1 = T_1 + U_1$ and $E_2 = T_2 + U_2$ as the energy of the particle before and after passing the plane, the law of conservation of energy requires

$$\begin{aligned} E_1 &= E_2 \\ \Rightarrow T_1 + U_1 &= T_2 + U_2 \\ \Rightarrow \frac{1}{2}mv_1^2 + U_1 &= \frac{1}{2}mv_2^2 + U_2 \\ \Rightarrow v_2^2 &= v_1^2 + \frac{2}{m}(U_1 - U_2). \end{aligned} \quad (2.2)$$

By substituting Eq. (2.2) in the square of Eq. (2.1), we get

$$\left(\frac{\sin \theta_1}{\sin \theta_2} \right)^2 = \frac{v_1^2 + \frac{2}{m}(U_1 - U_2)}{v_1^2}.$$

After taking the square root, the result is

ANSWER TO PROBLEM 1.

$$\frac{\sin \theta_1}{\sin \theta_2} = \sqrt{1 + \frac{2}{mv_1^2}(U_1 - U_2)}$$

PROBLEM 2.

Find the law of transformation of the action S from one inertial frame to another.

SOLUTION: The Lagrangian L and L' of a mechanical system in two inertial frames of reference K and K' are respectively

$$L = T - U = \frac{1}{2} \sum_a m_a v_a^2 - U$$

and

$$L' = T' - U = \frac{1}{2} \sum_a m_a v_a'^2 - U.$$

If the frame K' moves with velocity \mathbf{V} relative to the frame K , the velocities of the particles of the mechanical system relative to the two frames are related by $\mathbf{v}_a = \mathbf{v}_a' + \mathbf{V}$. We can now express the relation of the Lagrangian of the system in the two frames by

$$\begin{aligned} L &= \frac{1}{2} \sum_a m_a (\mathbf{v}_a' + \mathbf{V})^2 - U \\ &= \frac{1}{2} V^2 \sum_a m_a + \mathbf{V} \cdot \sum_a m_a \mathbf{v}_a' + \frac{1}{2} \sum_a m_a v_a'^2 - U \\ &= L' + \mathbf{V} \cdot \mathbf{P}' + \frac{1}{2} \mu V^2. \end{aligned} \quad (2.3)$$

Integrating Eq. (2.3) with respect to time, we obtain

$$\begin{aligned} S &= \int L dt \\ &= S' + \int (\mathbf{V} \cdot \mathbf{P}') dt + \frac{1}{2} \mu V^2 t \\ &= S' + \mathbf{V} \cdot \int \sum_a m_a \mathbf{v}_a' dt + \frac{1}{2} \mu V^2 t \\ &= S' + \mathbf{V} \cdot \sum_a m_a \mathbf{r}_a' + \frac{1}{2} \mu V^2 t. \end{aligned}$$

The law of transformation of the action S is then

ANSWER TO PROBLEM 2.

$$S = S' + \mu \mathbf{V} \cdot \mathbf{R}' + \frac{1}{2} \mu V^2 t$$

PROBLEM 3.

Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in cylindrical co-ordinates r, ϕ, z .

SOLUTION: The Cartesian components of the angular momentum are simply

$$\mathbf{M} = \mathbf{r} \times \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times m \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix}. \quad (2.4)$$

In cylindrical co-ordinates, the Cartesian components are expressed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \cos \phi \\ r \sin \phi \\ z \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{r} \cos \phi - r\dot{\phi} \sin \phi \\ \dot{r} \sin \phi + r\dot{\phi} \cos \phi \\ \dot{z} \end{bmatrix}.$$

By substituting those in Eq. (2.4), we get

$$\begin{aligned} \mathbf{M} &= m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix} \\ &= m \begin{bmatrix} (r\dot{z} - z\dot{r}) \sin \phi - rz\dot{\phi} \cos \phi \\ -(r\dot{z} - z\dot{r}) \cos \phi - rz\dot{\phi} \sin \phi \\ r^2\dot{\phi} \end{bmatrix}. \end{aligned} \quad (2.5)$$

By taking the magnitude of Eq. (2.5), we finally get

ANSWER TO PROBLEM 3.

$$\begin{aligned} M_x &= m(r\dot{z} - z\dot{r}) \sin \phi - mrz\dot{\phi} \cos \phi \\ M_y &= -m(r\dot{z} - z\dot{r}) \cos \phi - mrz\dot{\phi} \sin \phi \\ M_z &= mr^2\dot{\phi} \\ M^2 &= m^2 r^2 \dot{\phi}^2 (r^2 + z^2) + m^2 (r\dot{z} - z\dot{r})^2 \end{aligned}$$

PROBLEM 4.

Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in spherical co-ordinates r, θ, ϕ .

SOLUTION: In spherical co-ordinates, the Cartesian components are expressed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{r} \sin \phi \cos \theta + r \dot{\phi} \cos \phi \cos \theta - r \dot{\theta} \sin \phi \sin \theta \\ \dot{r} \sin \phi \sin \theta + r \dot{\phi} \cos \phi \sin \theta + r \dot{\theta} \sin \phi \cos \theta \\ \dot{r} \cos \phi - r \sin \phi \end{bmatrix}.$$

By substituting those in Eq. (2.4), we get

$$\begin{aligned} \mathbf{M} &= m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix} \\ &= m \begin{bmatrix} -r^2(\dot{\theta} \sin \phi + \dot{\phi} \sin \theta \cos \theta \cos \phi) \\ r^2(\dot{\theta} \cos \phi - \dot{\phi} \sin \theta \cos \theta \sin \phi) \\ r^2 \dot{\phi} \sin^2 \theta \end{bmatrix}. \end{aligned} \quad (2.6)$$

By taking the magnitude of Eq. (2.6), we finally get

ANSWER TO PROBLEM 4.

$$\begin{aligned} M_x &= -mr^2(\dot{\theta} \sin \phi + \dot{\phi} \sin \theta \cos \theta \cos \phi) \\ M_y &= mr^2(\dot{\theta} \cos \phi - \dot{\phi} \sin \theta \cos \theta \sin \phi) \\ M_z &= mr^2 \dot{\phi} \sin^2 \theta \\ M^2 &= m^2 r^4 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) \end{aligned}$$

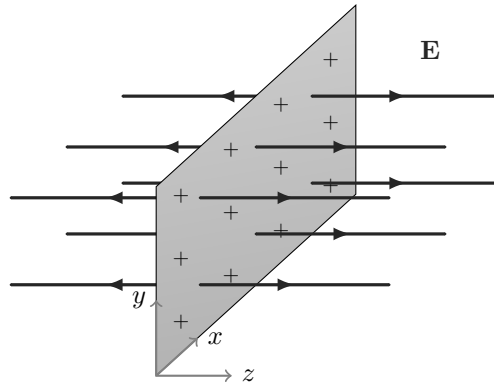
PROBLEM 5.

Which components of momentum \mathbf{P} and angular momentum \mathbf{M} are conserved in motion in the following fields ?

(a)

The field of an infinite homogeneous plane.

SOLUTION: solution



Answer to Problem 5 (a).

answer

(b)

The field of an infinite homogeneous cylinder.

SOLUTION: solution

Answer to Problem 5 (b).

answer

(c)

The field of an infinite homogeneous prism.

SOLUTION: solution

Answer to Problem 5 (c).

answer

(d)

The field of two points.

SOLUTION: solution

Answer to Problem 5 (d).

answer

(e)

The field of an infinite homogeneous half-plane.

SOLUTION: solution

Answer to Problem 5 (e).

answer

(f)

The field of a homogeneous cone.

SOLUTION: solution

Answer to Problem 5 (f).

answer

(g)

The field of a homogeneous circular torus.

SOLUTION: solution

Answer to Problem 5 (g).

answer

(h)

The field of an infinite homogeneous cylindrical helix.

SOLUTION: solution

Answer to Problem 5 (h).

answer

PROBLEM 6.

Find the ratio of the times in the same path for particles having different masses but the same potential energy.

SOLUTION: If the two particles have the same path, then the ratio of linear dimension is

$$\frac{l'}{l} = \alpha = 1. \quad (2.7)$$

We can define the ratio of time and mass by

$$\frac{t'}{t} = \beta$$

and

$$\frac{m'}{m} = \gamma.$$

Then, the ratio of kinetic energy is

$$\frac{T'}{T} = \frac{m'\mathbf{v}'}{m\mathbf{v}} = \frac{m'}{m} \left(\frac{d\mathbf{r}'}{d\mathbf{r}} \frac{dt}{dt'} \right)^2 = \frac{\gamma\alpha^2}{\beta^2}.$$

To leave the equation of motion unaltered, the ratio of the kinetic energy and the potential energy must be the same, *i.e.* ,

$$\begin{aligned} \frac{U'}{U} &= \frac{T'}{T} \\ \Rightarrow \alpha^k &= \frac{\gamma\alpha^2}{\beta^2}. \end{aligned}$$

Using Eq. (2.7), we get

$$\begin{aligned} 1 &= \frac{\gamma}{\beta^2} \\ \Rightarrow \beta &= \sqrt{\gamma}. \end{aligned}$$

The ratio of the times is then

ANSWER TO PROBLEM 6.

$$\frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

PROBLEM 7.

Find the ratio of the times in the same path for particles the same mass but potential energy differing by a constant factor.

SOLUTION: If the two particles have the same path, then the ratio of linear dimension is

$$\frac{l'}{l} = \alpha = 1. \quad (2.8)$$

We can define the ratio of time by

$$\frac{t'}{t} = \beta.$$

Then, the ratio of kinetic energy is

$$\frac{T'}{T} = \frac{\alpha^2}{\beta^2}.$$

The potential energy of the two particles differ by a constant factor, which mean that

$$\frac{U'}{U} = \gamma.$$

To leave the equation of motion unaltered, the ratio of the kinetic energy and the potential energy must be the same, *i.e.* ,

$$\begin{aligned} \frac{U'}{U} &= \frac{T'}{T} \\ \Rightarrow \gamma &= \frac{\alpha^2}{\beta^2}. \end{aligned}$$

Using Eq. (2.8), we get

$$\begin{aligned} \gamma &= \frac{1}{\beta^2} \\ \Rightarrow \beta &= \sqrt{\frac{1}{\gamma}}. \end{aligned}$$

The ratio of the times is then

ANSWER TO PROBLEM 7.

$$\frac{t'}{t} = \sqrt{\frac{U}{U'}}$$

CHAPTER 3.

INTEGRATION OF THE EQUATIONS OF MOTION

PROBLEM 1.

Determine the period of oscillations of a simple pendulum (a particle of mass m suspended by a string of length l in a gravitational field) as a function of the amplitude of the oscillations.

SOLUTION: The potential energy of a simple pendulum is

$$U(\phi) = -mgl \cos \phi,$$

see Chap. 1 Prob. 1. If we define ϕ_0 as the maximum value of ϕ , the potential energy is equal to the total energy at this point, that is

$$U(\phi_0) = E = -mgl \cos \phi_0.$$

The energy of the pendulum could also be write using the kinetic energy,

$$E = \frac{1}{2}ml^2\dot{\phi}^2 + U(\phi).$$

This first order differential equation can be integrate to give the the period of oscillation

$$\begin{aligned} T &= 2\sqrt{\frac{ml^2}{2}} \int_{-\phi_0}^{\phi_0} \frac{d\phi}{\sqrt{E - U(\phi)}} \\ &= 4\sqrt{\frac{ml^2}{2}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{mgl \cos \phi - mgl \cos \phi_0}} \\ &= 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos \phi - \cos \phi_0}}. \end{aligned}$$

To solve this integral, we need first to use a trigonometric identity (I'll let you find or derive this identity), giving us

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \frac{\phi}{2}}}.$$

Next, we use the substitution

$$\sin \xi = \frac{\sin \frac{\phi}{2}}{\sin \frac{\phi_0}{2}}.$$

It's differential form is

$$\begin{aligned} \frac{d\phi}{d\xi} &= \frac{d}{d\xi} \left(2 \arcsin \left(\sin \frac{\phi_0}{2} \sin \xi \right) \right) \\ &= \frac{2 \sin \frac{\phi_0}{2} \cos \xi}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}} \end{aligned}$$

The substitution, finally, give

$$\begin{aligned} T &= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\phi_0}{2} - \sin^2 \xi \sin^2 \frac{\phi_0}{2}}} \\ &= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sin \frac{\phi_0}{2} \sqrt{1 - \sin^2 \xi}} \\ &= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sin \frac{\phi_0}{2} \cos \xi} \\ &= 2\sqrt{\frac{l}{g}} \int_{\arcsin(0)}^{\arcsin(1)} \frac{2 \sin \frac{\phi_0}{2} \cos \xi d\xi}{\sin \frac{\phi_0}{2} \cos \xi \sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}} \\ &= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - \sin^2 \frac{\phi_0}{2} \sin^2 \xi}}. \end{aligned}$$

We can also write

$$T = 4\sqrt{\frac{l}{g}} K\left(\sin \frac{\phi_0}{2}\right),$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - k \sin^2 \xi}}. \quad (3.1)$$

The integral K is known as the complete elliptic integral of the first kind. To solve this integral, we first need to see that the integrand is of the form

$$f(x) = (1 - x)^{-\frac{1}{2}}.$$

The n derivative of this function is

$$f^{(n)}(x) = \frac{(2n-1)!!}{2^n} (1-x)^{-\frac{2n+1}{2}},$$

thus the Maclaurin Series is

$$f(x) = \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} x^n.$$

Using this result in Eq. (3.1), we get

$$\begin{aligned} K(k) &= \int_0^{\frac{\pi}{2}} \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \sin^{2n} \xi \, d\xi \\ &= \sum_{n \geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \int_0^{\frac{\pi}{2}} \sin^{2n} \xi \, d\xi. \end{aligned}$$

This last integral can be solve using the beta function

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} t \cos^{2y-1} t \, dt,$$

with $x = \frac{2n+1}{2}$ and $y = \frac{1}{2}$. Thus, the beta function become

$$\begin{aligned} \frac{1}{2} \mathcal{B}(x, y) &= \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} \\ &= \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{2n+1}{2})}{n!} \\ &= \frac{\pi}{2} \frac{(2n+1)!!}{2^n \cdot n!} \end{aligned}$$

The elliptic integral is thereby

$$\begin{aligned} K(k) &= \frac{\pi}{2} \sum_{n \geq 0} \left[\frac{(2n-1)!!}{2^n \cdot n!} k^n \right]^2 \\ &= \frac{\pi}{2} \sum_{n \geq 0} \left[\frac{(2n-1)!!}{(2n)!!} k^n \right]^2. \end{aligned}$$

The period of a simple pendulum is

$$T = 2\pi \sqrt{\frac{l}{g}} \sum_{n \geq 0} \left[\frac{(2n-1)!!}{(2n)!!} \sin^n \frac{\phi_0}{2} \right]^2$$

By expanding the sum, we get

$$T = 2\pi \sqrt{\frac{l}{g}} \left(1 + \left(\frac{1}{2}\right)^2 \sin^2 \frac{\phi}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \sin^4 \frac{\phi}{2} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \sin^6 \frac{\phi}{2} + \dots \right).$$

Using the Maclaurin series

$$\sin \frac{\phi_0}{2} = \frac{1}{2}\phi_0 - \frac{1}{48}\phi_0^3 + \frac{1}{3840}\phi_0^5 - \frac{1}{645120}\phi_0^7 + \dots,$$

we finally get

ANSWER TO PROBLEM 1.

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \frac{11}{3072}\phi_0^4 + \dots\right)$$

PROBLEM 2.

Determine the period of oscillation, as a function of the energy, when a particle of mass m moves in fields for which the potential energy is

(a)

$$U = A|x|^n$$

SOLUTION: The total energy of the particle is

$$E = \frac{1}{2}m\dot{x}^2 + U(x).$$

Knowing that the maximum value of E is at x_1 , this position is

$$\begin{aligned} E &= U(x_1) = A|x_1|^n \\ \Rightarrow |x_1| &= \left(\frac{E}{A}\right)^{1/n}, \\ \Rightarrow x_1 &= \pm \left(\frac{E}{A}\right)^{1/n}, \end{aligned}$$

we get the period of oscillation,

$$\begin{aligned} T &= 2\sqrt{\frac{m}{2}} \int_{-\left(\frac{E}{A}\right)^{1/n}}^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}} \\ &= 2\sqrt{2m} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}}. \end{aligned}$$

It is possible to remove the absolute value, because the integral is over positive x , thus

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - Ax^n}} \\ &= 2\sqrt{\frac{2m}{E}} \int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{1 - \frac{A}{E}x^n}}. \end{aligned}$$

Using the substitution $y = \left(\frac{A}{E}\right)^{\frac{1}{n}} x$, we get

$$\begin{aligned} T &= 2\sqrt{\frac{2m}{E}} \int_0^1 \frac{\left(\frac{A}{E}\right)^{\frac{1}{n}} dy}{\sqrt{1 - \frac{A}{E} \left(\left(\frac{A}{E}\right)^{\frac{1}{n}} y\right)^n}} \\ &= 2\sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{dy}{\sqrt{1 - y^n}}. \end{aligned}$$

Using another substitution $u = y^n$, we get

$$T = 2\sqrt{\frac{2m}{E}} \left(\frac{E}{A}\right)^{\frac{1}{n}} \int_0^1 \frac{u^{\frac{1}{n}} du}{nu\sqrt{1-u}}.$$

It is possible to express this integral in term of the beta function

$$\mathcal{B}(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)},$$

with $z_1 = \frac{1}{n}$ and $z_2 = \frac{1}{2}$. This give us

$$T = \frac{2}{n} \sqrt{\frac{2m}{E}} \left(\frac{E}{A} \right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}.$$

Knowing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we finally have

Answer to Problem 2 (a).

$$T = \frac{2}{n} \sqrt{\frac{2\pi m}{E}} \left(\frac{E}{A} \right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}$$

(b)

$$U = \frac{-U_0}{\cosh^2 \alpha x}$$

SOLUTION: The two boundaries of the energy are at $E = 0 \Rightarrow x = x_1$ and $E = -U_0 \Rightarrow x = 0$. The period of oscillation is thus

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E - U}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E + \frac{U_0}{\cosh^2 \alpha x}}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cosh^2 \alpha x \, dx}{\sqrt{E \cosh^2 \alpha x + U_0}}. \end{aligned}$$

Using the identity $\cosh^2 \alpha x = 1 + \sinh^2 \alpha x$, we get

$$T = 2\sqrt{2m} \int_0^{x_1} \frac{\cosh^2 \alpha x \, dx}{\sqrt{E(1 + \sinh^2 \alpha x) + U_0}}.$$

The substitution $y = \sinh \alpha x$, give us $dy = \alpha \cosh \alpha x dx$ and

$$\begin{aligned} y(x_1) &= \sinh \alpha x \\ &= \sqrt{\sinh^2 \alpha x} \\ &= \sqrt{\cosh^2 \alpha x - 1} \\ &= \sqrt{\frac{-U_0}{E} - 1}. \end{aligned}$$

Using this substitution, we get

$$\begin{aligned} T &= \frac{2}{\alpha} \sqrt{2m} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{E(1 + y^2) + U_0} \\ &= \frac{2}{\alpha} \sqrt{2m} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{E + Ey^2 - E \cosh^2 \alpha x} \end{aligned}$$

Factoring $|E|$ out of the square root (because $E < 0$), we have

$$\begin{aligned} T &= \frac{2}{\alpha} \sqrt{\frac{2m}{|E|}} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{1 - \cosh^2 \alpha x + y^2} \\ &= \frac{2}{\alpha} \sqrt{\frac{2m}{|E|}} \int_0^{\sqrt{\frac{-U_0}{E} - 1}} \frac{dy}{\sqrt{\frac{-U_0}{E} - 1} + y^2} \end{aligned}$$

This integral is of the form

$$\int_0^a \frac{dz}{\sqrt{a+z^2}} = \sin^{-1} \left(\frac{z}{a} \right) = \frac{\pi}{2}.$$

Thus, the period of oscillation is

Answer to Problem 2 (b).

$$T = \frac{\pi}{\alpha} \sqrt{\frac{2m}{|E|}}$$

(c)

$$U = U_0 \tan^2 \alpha x$$

SOLUTION: The period of oscillation is

$$\begin{aligned} T &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E-U}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{dx}{\sqrt{E-U_0 \tan^2 \alpha x}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cos \alpha x \, dx}{\sqrt{E \cos^2 \alpha x - U_0 \sin^2 \alpha x}} \\ &= 2\sqrt{2m} \int_0^{x_1} \frac{\cos \alpha x \, dx}{\sqrt{E - (E+U_0) \sin^2 \alpha x}}. \end{aligned}$$

Using the substitution $y = i \sin \alpha x$ ($dy = i\alpha \cos \alpha x$), we get

$$\begin{aligned} T &= \frac{2\sqrt{2m}}{i\alpha} \int_0^{i \sin \alpha x_1} \frac{dy}{\sqrt{E + (E+U_0)y^2}} \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \int_0^{i \sin \alpha x_1} \frac{dy}{\sqrt{\frac{E}{(E+U_0)} + y^2}} \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left(\frac{i \sin \alpha x_1}{\sqrt{\frac{E}{(E+U_0)}}} \right) \end{aligned}$$

Knowing

$$x_1 = \frac{1}{\alpha} \tan^{-1} \sqrt{\frac{E}{U_0}},$$

we finally get

$$\begin{aligned} T &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left(\frac{i \sin \left(\tan^{-1} \sqrt{\frac{E}{U_0}} \right)}{\sqrt{\frac{E}{(E+U_0)}}} \right) \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} \left(\frac{i \sqrt{\frac{E}{U_0}}}{\sqrt{1 + \frac{E}{U_0} \sqrt{\frac{E}{(E+U_0)}}}} \right) \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \sin^{-1} i \\ &= \frac{2}{i\alpha} \sqrt{\frac{2m}{E+U_0}} \frac{\pi i}{2}. \end{aligned}$$

Thus, the period of oscillation is

Answer to Problem 2 (c).

$$T = \frac{\pi}{\alpha} \sqrt{\frac{2m}{E + U_0}}$$

PROBLEM 3.

A system consists of one particle of mass M and n particles with equal masses m . Eliminate the motion of the centre of mass and so reduce the problem to one involving n particles.

SOLUTION: Let \mathbf{R} be the radius vector of the particle of mass M , and \mathbf{R}_a ($a = 1, 2, \dots, n$) those of the particles of mass m . We put $\mathbf{r}_a \equiv \mathbf{R}_a - \mathbf{R}$ and take the origin to be at the centre of mass, namely

$$M\mathbf{R} + m \sum_a \mathbf{R}_a = 0.$$

Thus, we got

$$\mathbf{R} = -\frac{m}{M} \left(\sum_a \mathbf{r}_a + n\mathbf{R} \right) = -\frac{m}{\mu} \sum_a \mathbf{r}_a, \quad (3.2)$$

where $\mu \equiv M + nm$. The Lagrangian of the system is

$$L = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} m \sum_a \dot{\mathbf{R}}_a^2 - U.$$

If we substitute Eq. (3.2) and $\mathbf{R}_a = \mathbf{R} + \mathbf{r}_a$, we get

$$\begin{aligned} L &= \frac{1}{2} \frac{Mm^2}{\mu^2} \left(\sum_a \dot{\mathbf{r}}_a \right)^2 + \frac{1}{2} nm \dot{\mathbf{R}}^2 + \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 + m \dot{\mathbf{R}} \sum_a \dot{\mathbf{r}}_a - U \\ &= \frac{1}{2} \frac{Mm^2}{\mu^2} \left(\sum_a \dot{\mathbf{r}}_a \right)^2 + \frac{1}{2} \frac{m^3}{\mu^2} \left(\sum_a \dot{\mathbf{r}}_a \right)^2 + \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 - \frac{m^2}{\mu} \left(\sum_a \dot{\mathbf{r}}_a \right)^2 - U \\ &= \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 - \frac{1}{2} \frac{m^2}{\mu} \left(\frac{2\mu - M - nm}{\mu} \right) \left(\sum_a \dot{\mathbf{r}}_a \right)^2. \end{aligned}$$

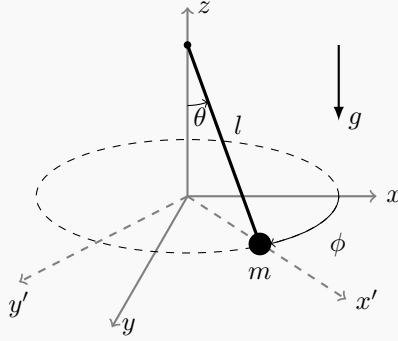
The Lagrangian is thus only a function of \mathbf{r}_a and it's time derivative, which is

ANSWER TO PROBLEM 3.

$$L = \frac{1}{2} m \sum_a \dot{\mathbf{r}}_a^2 - \frac{1}{2} \frac{m^2}{\mu} \left(\sum_a \dot{\mathbf{r}}_a \right)^2$$

PROBLEM 4.

Integrate the equations of motion for a spherical pendulum (a particle of mass m moving on the surface of a sphere of radius l in a gravitational field).



SOLUTION: The Cartesian position of the particle m is

$$\begin{aligned} x &= l \sin \theta \cos \phi \\ y &= l \sin \theta \sin \phi \\ z &= -l \cos \theta. \end{aligned}$$

The time derivatives of these coordinates are

$$\begin{aligned} \dot{x} &= l\dot{\theta} \cos \theta \cos \phi - l\dot{\phi} \sin \theta \sin \phi \\ \dot{y} &= l\dot{\theta} \cos \theta \sin \phi + l\dot{\phi} \sin \theta \cos \phi \\ \dot{z} &= l\dot{\theta} \sin \theta. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgl \cos \theta. \end{aligned}$$

The Lagrangian does not involve ϕ explicitly, thus this co-ordinate is cyclic and the generalized momentum p_ϕ is conserved. This momentum is the same as the z -component of angular momentum M_z and is written

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2 \dot{\phi} \sin^2 \theta. \quad (3.3)$$

The energy of the pendulum is

$$E = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - mgl \cos \theta.$$

Substituting $\dot{\phi}$ with Eq. (3.3) in the energy we get

$$\begin{aligned} E &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{M_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + U_{\text{eff}}(\theta), \end{aligned}$$

where $U_{\text{eff}}(\theta)$ is the effective potential energy. Notice that the energy was re-written as a function of only one co-ordinate (*i.e.* θ). This is equivalent to the one particle problem we have already see, thus

$$\begin{aligned} \frac{d\theta}{dt} &= \sqrt{\frac{2(E - U_{\text{eff}}(\theta))}{ml^2}} \\ \Rightarrow t &= \int \frac{d\theta}{\sqrt{2(E - U_{\text{eff}}(\theta))}}. \end{aligned}$$

This integral lead to an elliptic integral of the first kind see **Chap. 3 Prob. 1**. We also need to find the solution to the angle ϕ . To do so, we can use the Eq. (3.3) with the chain rule from calculus, that is

$$\begin{aligned}\frac{M_z}{ml^2 \sin^2 \theta} &= \frac{d\phi}{dt} \\ &= \frac{d\phi}{d\theta} \frac{d\theta}{dt} \\ &= \frac{d\phi}{d\theta} \sqrt{\frac{2(E - U_{\text{eff}}(\theta))}{ml^2}}.\end{aligned}$$

Finally, we get

$$\begin{aligned}\frac{d\phi}{d\theta} &= \frac{M_z}{l \sin^2 \theta} \sqrt{\frac{1}{2m(E - U_{\text{eff}}(\theta))}} \\ \Rightarrow \phi &= \frac{M_z}{l} \sqrt{\frac{1}{2m}} \int \frac{d\theta}{\sin^2 \theta (E - U_{\text{eff}}(\theta))}.\end{aligned}$$

This integral lead to an elliptic integral of the third kind (I really don't want to solve this). It is possible to find the range of angle θ . At the maximum and minimum value of θ , the pendulum as no kinetic energy, thus

$$\begin{aligned}E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta \\ &= \frac{M_z^2}{2ml^2 (1 - \cos^2 \theta)} - mgl \cos \theta\end{aligned}$$

We can find a cubic algebraic equation for $\cos \theta$, that is

$$\begin{aligned}0 &= (E + mgl \cos \theta) (2ml^2 (1 - \cos^2 \theta)) - M_z^2 \\ &= mgl (\cos^3 \theta - \cos \theta) + E (\cos^2 \theta - 1) + \frac{M_z^2}{2ml^2}.\end{aligned}$$

The centrifugal part of the effective potential U_{eff} , namely

$$\frac{M_z^2}{2ml^2 \sin^2 \theta},$$

must be positive. As we can see, this term diverge at $\theta \rightarrow 0$ and $\theta \rightarrow \pi$, thus the angle θ is bound. This can be write as

$$\theta \in [\theta_p, \theta_a] \qquad 0 < \theta_p \leq \theta_a < \pi,$$

where the subscripts of the angles θ_p and θ_a are for the perigee and apogee. This mean that the two roots of the cubic algebraic equation for $\cos \theta$ is bound between -1 and $+1$. To summarize, the solution of the equation of motion of the spherical pendulum is

ANSWER TO PROBLEM 4.

$$t = \int_{\theta_p}^{\theta_a} \frac{d\theta}{\sqrt{2(E - U_{\text{eff}}(\theta))}} \qquad \phi = \frac{M_z}{l} \sqrt{\frac{1}{2m}} \int_{\theta_p}^{\theta_a} \frac{d\theta}{\sin^2 \theta (E - U_{\text{eff}}(\theta))}$$

PROBLEM 5.

Integrate the equations of motion for a particle moving on the surface of a cone (of vertical axis 2α) placed vertically and with vertex downwards in a gravitational field.

SOLUTION: By using the spherical co-ordinate, we can write the position of the particle as

$$\begin{aligned}x &= r \cos \phi \sin \alpha \\y &= r \sin \phi \sin \alpha \\z &= r \cos \alpha.\end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned}\dot{x} &= \dot{r} \cos \phi \sin \alpha - r \dot{\phi} \sin \phi \sin \alpha \\ \dot{y} &= \dot{r} \sin \phi \sin \alpha + r \dot{\phi} \cos \phi \sin \alpha \\ \dot{z} &= \dot{r} \cos \alpha.\end{aligned}$$

The Lagrangian is thus

$$\begin{aligned}L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2 \alpha) - mgr \cos \alpha.\end{aligned}$$

The co-ordinate ϕ is cyclic, thus

$$M_z = p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \dot{\phi} \sin^2 \alpha \quad (3.4)$$

is conserved. The energy of the particle is

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2 \alpha) + mgr \cos \alpha. \quad (3.5)$$

We can rewrite Eq. (3.4) as

$$\dot{\phi}^2 = \frac{M_z^2}{m^2 r^4 \sin^4 \alpha}$$

and substituting in Eq. (3.5), we get

$$\begin{aligned}E &= \frac{1}{2}m\dot{r}^2 + \frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha \\ &= \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r).\end{aligned}$$

Hence,

$$t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}}.$$

From Eq. (3.4), it is possible to write

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \frac{d\phi}{dr} \frac{dr}{dt} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \frac{d\phi}{dr} \sqrt{\frac{2(E - U_{\text{eff}}(r))}{m}} &= \frac{M_z}{mr^2 \sin^2 \alpha} \\ \Rightarrow \phi &= \frac{M_z}{\sqrt{2m \sin^2 \alpha}} \int \frac{dr}{r^2 \sqrt{E - U_{\text{eff}}(r)}}.\end{aligned}$$

It is possible to find the range of r . At the maximum and minimum value of r , the particle has no kinetic energy, thus

$$\begin{aligned} E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha \end{aligned}$$

We can find a cubic algebraic equation for r , that is

$$\begin{aligned} 0 &= r^2 (E - mgr \cos \alpha) - \frac{M_z^2}{2m \sin^2 \alpha} \\ &= mgr^3 \cos \alpha - Er^2 + \frac{M_z^2}{2m \sin^2 \alpha}. \end{aligned}$$

This equation has two positive roots, r_p and r_a , which are the turning points of the motion. To summarize, the solution of the equation of motion for a particle moving on the surface of a cone is

ANSWER TO PROBLEM 5.

$$t = \sqrt{\frac{m}{2}} \int_{r_p}^{r_a} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} \quad \phi = \frac{M_z}{\sqrt{2m} \sin^2 \alpha} \int_{r_p}^{r_a} \frac{dr}{r^2 \sqrt{E - U_{\text{eff}}(r)}}$$

PROBLEM 6.

Integrate the equations of motion for a pendulum of mass m_2 , with a mass m_1 at the point of support which can move on a horizontal line lying in the plane which m_2 moves (Chap. 1 Prob. 2).

SOLUTION: From Chap. 1 Prob. 2, the Lagrangian of the system is

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2gl\cos\phi.$$

The co-ordinate x is cyclic, thus

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = \text{constant} \quad (3.6)$$

is conserved. It is always possible to find an inertial frame of reference where $p_x = 0$, using this frame we get

$$\begin{aligned} (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi &= 0 \\ \Rightarrow \int \left((m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi \right) dt &= \text{constant} \\ \Rightarrow (m_1 + m_2)x + m_2l\sin\phi &= (m_1 + m_2)R_x = \text{constant}. \end{aligned} \quad (3.7)$$

This express the fact that the centre of mass **R** of the system does not move horizontally. It is also possible to rewrite Eq. (3.6) as

$$\begin{aligned} (m_1 + m_2)\dot{x} + m_2l\dot{\phi}\cos\phi &= 0 \\ \Rightarrow \dot{x} &= \frac{-m_2l\dot{\phi}\cos\phi}{m_1 + m_2} \end{aligned}$$

Plugging this into the energy of the system we get

$$\begin{aligned} E &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2gl\cos\phi \\ &= \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{2(m_1 + m_2)} + \frac{1}{2}m_2l^2\dot{\phi}^2 - \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{m_1 + m_2} - m_2gl\cos\phi \\ &= \frac{1}{2}m_2l^2\dot{\phi}^2 \left(1 - \frac{m_2}{m_1 + m_2}\cos^2\phi \right) - m_2gl\cos\phi. \end{aligned}$$

Hence,

$$\begin{aligned} t &= l\sqrt{\frac{m_2}{2}} \int \sqrt{\frac{1 - \frac{m_2}{m_1 + m_2}\cos^2\phi}{E + m_2gl\cos\phi}} d\phi \\ &= l\sqrt{\frac{m_2}{2(m_1 + m_2)}} \int \sqrt{\frac{m_1 + m_2\sin^2\phi}{E + m_2gl\cos\phi}} d\phi. \end{aligned}$$

Using Eq. (3.7), we can express the position of the mass m_2 as

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x + l\sin\phi \\ l\cos\phi \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi \left(1 - \frac{m_2}{m_1 + m_2} \right) \\ l\cos\phi \end{bmatrix}.$$

The path of the particle of mass m_2 is thus an arc of an ellipse center at $(R_x, 0)$ with horizontal semi-axis $lm_1/(m_1 + m_2)$ and vertical semi-axis l . It is possible to see that when $m_1 \rightarrow \infty$, the path return to the simple pendulum. The solution of the equations of motion is thus

ANSWER TO PROBLEM 6.

$$t = l\sqrt{\frac{m_2}{2}} \int \sqrt{\frac{1 - \frac{m_2}{m_1 + m_2}\cos^2\phi}{E + m_2gl\cos\phi}} d\phi \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi \left(1 - \frac{m_2}{m_1 + m_2} \right) \\ l\cos\phi \end{bmatrix}$$

PROBLEM 7.

Find the time dependence of the co-ordinate of a particle with energy $E = 0$ moving in a parabola in a field $U = -\alpha/r$.

SOLUTION: From the formulae (14.6) of the book, we have the integral

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{r dr}{\sqrt{\frac{2\alpha}{m}r - \frac{M^2}{m^2}}}. \end{aligned}$$

We use the substitution

$$\begin{aligned} \frac{m}{M}\eta &= \sqrt{\frac{2\alpha}{m}r - \frac{M^2}{m^2}} \\ \Rightarrow r &= \frac{M^2}{2m\alpha} (1 + \eta^2) = \frac{1}{2}p (1 + \eta^2), \end{aligned} \quad (3.8)$$

with the differential form

$$dr = \frac{M^2}{m\alpha} \eta d\eta.$$

Hence, the integral become

$$\begin{aligned} t &= \frac{M^3}{2m\alpha^2} \int (1 + \eta^2) d\eta \\ &= \frac{M^3}{2m\alpha^2} \left(\eta + \frac{1}{3}\eta^3 \right) \\ &= \sqrt{\frac{mp^3}{\alpha}} \frac{1}{2} \eta \left(\eta + \frac{1}{3}\eta^3 \right). \end{aligned}$$

It is important to specify that the parameter η varies from $-\infty$ to ∞ . Using Eq. (3.8) and

$$\cos \phi = \frac{p}{r} - 1,$$

it is possible to find the Cartesian co-ordinates

$$x = r \cos \phi = \frac{1}{2}p (1 - \eta^2)$$

and

$$y = \sqrt{r^2 - x^2} = p\eta.$$

The parametric form of the required dependence are thus

ANSWER TO PROBLEM 7.

$$\begin{aligned} r &= \frac{1}{2}p (1 + \eta^2) & t &= \sqrt{\frac{mp^3}{\alpha}} \frac{1}{2} \eta \left(\eta + \frac{1}{3}\eta^3 \right) \\ x &= \frac{1}{2}p (1 - \eta^2) & y &= p\eta \end{aligned}$$

PROBLEM 8.

Integrate the equations of motion for a particle in a central field

$$U = -\frac{\alpha}{r^2} \quad (\alpha > 0).$$

SOLUTION: From the formulae (14.6) of the book, we have the integral

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{dr}{\sqrt{\frac{2E}{m} r^2 + \frac{2\alpha}{m} - \frac{M^2}{m^2}}} \\ &= \sqrt{\frac{m}{2E}} \int \frac{r dr}{\sqrt{r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)}}. \end{aligned}$$

We use the substitution

$$u = r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)$$

with the differential form

$$du = 2r dr.$$

Hence, the integral become

$$\begin{aligned} t &= \frac{1}{2} \sqrt{\frac{m}{2E}} \int \frac{du}{\sqrt{u}} \\ &= \sqrt{\frac{m}{2E}} \sqrt{u} \\ &= \sqrt{\frac{m}{2E}} \sqrt{r^2 + \frac{\alpha}{E} - \frac{M^2}{2mE}} \\ &= \frac{1}{E} \sqrt{\frac{m}{2}} \left(Er^2 + \alpha - \frac{M^2}{2m} \right). \end{aligned}$$

The formulae (14.7) of the book give us the equation of the path

$$\begin{aligned} \phi &= \int \frac{M dr}{r^2 \sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}} \\ &= \int \frac{M dr}{r^2 \sqrt{2mE + \frac{2m\alpha}{r^2} - \frac{M^2}{r^2}}}. \end{aligned}$$

Using the substitution

$$u = \frac{1}{r}$$

with the differential form

$$du = -\frac{1}{r^2} dr,$$

the integral become

$$\begin{aligned} \phi &= - \int \frac{du}{\sqrt{2mE + (2m\alpha - M^2) u^2}} \\ &= - \frac{1}{\sqrt{2mE}} \int \frac{du}{\sqrt{1 + (ku)^2}}, \end{aligned} \tag{3.9}$$

with

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}. \tag{3.10}$$

From there, the solution must be divided for the three possible cases : **(a)** $E > 0$, $M^2 > 2m\alpha$; **(b)** $E > 0$, $M^2 < 2m\alpha$; **(c)** $E < 0$, $M^2 < 2m\alpha$. It is also interesting to know the the path is a Cotes's spiral¹.

(c) : Eq. (3.10) is still

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}.$$

By using the substitution

$$ku = \sinh \theta$$

with the differential form

$$kdu = \cosh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{\sqrt{2mE}} \int \frac{\cosh \theta d\theta}{k \sqrt{1 + \sinh^2 \theta}} \\ &= -\frac{1}{\sqrt{2m\alpha - M^2}} \theta \\ &= -\frac{1}{\sqrt{2m\alpha - M^2}} \sinh^{-1} \frac{k}{r}. \end{aligned}$$

The equation of the path for the case (c) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{1}{k} \sinh \left(\phi \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2mE}{2m\alpha - M^2}} \sinh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right). \end{aligned}$$

(b) : Eq. (3.10) become

$$k = \frac{\sqrt{2m\alpha - M^2}}{i\sqrt{2m|E|}}.$$

By using the substitution

$$iku = \cosh \theta$$

with the differential form

$$ikdu = \sinh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{i\sqrt{2m|E|}} \int \frac{\sinh \theta d\theta}{ik \sqrt{1 - \cosh^2 \theta}} \\ &= \frac{1}{\sqrt{2m\alpha - M^2}} \theta \\ &= \frac{1}{\sqrt{2m\alpha - M^2}} \cosh^{-1} \frac{ik}{r}. \end{aligned}$$

The equation of the path for the case (b) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{1}{ik} \cosh \left(\phi \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2m|E|}{2m\alpha - M^2}} \cosh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right). \end{aligned}$$

¹Whittaker ET (1937). A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, with an Introduction to the Problem of Three Bodies (4th ed.). New York: Dover Publications. pp. 80–83. ISBN 978-0-521-35883-5.

(a) : Eq. (3.10) become

$$k = \frac{i\sqrt{2m\alpha - M^2}}{\sqrt{2mE}}.$$

By using the substitution

$$ku = i \cosh \theta$$

with the differential form

$$kdu = i \sinh \theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned} \phi &= -\frac{1}{\sqrt{2mE}} \int \frac{i \sinh \theta d\theta}{k \sqrt{1 - \cosh^2 \theta}} \\ &= -\frac{i}{\sqrt{2m\alpha - M^2}} \theta \\ &= -\frac{i}{\sqrt{2m\alpha - M^2}} \cosh^{-1} \frac{k}{ir}. \end{aligned}$$

The equation of the path for the case (b) is thus

$$\begin{aligned} \frac{1}{r} &= \frac{i}{k} \cosh \left(\frac{\phi}{i} \sqrt{2m\alpha - M^2} \right) \\ &= \sqrt{\frac{2mE}{M^2 - 2m\alpha}} \cos \left(\phi \sqrt{1 - \frac{2m\alpha}{M^2}} \right). \end{aligned}$$

To summarize the equations of motion for a particle in a central inverse-cube law force is

ANSWER TO PROBLEM 8.

(a) for $E > 0$ and $\frac{M^2}{2m} > \alpha$,

$$\frac{1}{r} = \sqrt{\frac{2mE}{M^2 - 2m\alpha}} \cos \left(\phi \sqrt{1 - \frac{2m\alpha}{M^2}} \right)$$

(b) for $E > 0$ and $\frac{M^2}{2m} < \alpha$,

$$\frac{1}{r} = \sqrt{\frac{2mE}{2m\alpha - M^2}} \sinh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right)$$

(c) for $E < 0$ and $\frac{M^2}{2m} < \alpha$,

$$\frac{1}{r} = \sqrt{\frac{2m|E|}{2m\alpha - M^2}} \cosh \left(\phi \sqrt{\frac{2m\alpha}{M^2} - 1} \right)$$

In all three cases

$$t = \frac{1}{E} \sqrt{\frac{m}{2} \left(Er^2 + \alpha - \frac{M^2}{2m} \right)}$$

PROBLEM 9.

When a small correction $\delta U(r)$ is added to the potential energy $U = -\alpha/r$, the paths of finite motion are no longer closed, and at each revolution the perihelion is displaced through a small angle $\delta\phi$. Find $\delta\phi$ when

(a)

$$\delta U = \frac{\beta}{r^2}$$

SOLUTION: From the equation (14.10) of the book, we have

$$\begin{aligned}\Delta\phi &= 2 \int_{r_{\min}}^{r_{\max}} \frac{M dr}{r^2 \sqrt{2m(E - U) - \frac{M^2}{r^2}}} \\ &= -2 \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \sqrt{2m(E - U) - \frac{M^2}{r^2}} dr \\ &= -2 \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \sqrt{2m \left(E + \frac{\alpha}{r} - \delta U \right) - \frac{M^2}{r^2}} dr.\end{aligned}$$

Expanding the integrand in powers of δU involves using a Taylor series expansion. Let's denote the integrand as F :

$$F = \sqrt{2m \left(E + \frac{\alpha}{r} + \delta U \right) - \frac{M^2}{r^2}}$$

We want to expand F around $\delta U = 0$. The expansion will look like:

$$F = F_0 + F_1 \delta U + F_2 (\delta U)^2 + \dots$$

Here, F_0 is the value of F at $\delta U = 0$, F_1 is the first derivative with respect to δU at $\delta U = 0$, and so on. Let's find the derivatives:

$$\begin{aligned}F_0 &= \sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}} \\ F_1 &= \left. \frac{\partial F}{\partial (\delta U)} \right|_{\delta U=0} = \frac{m}{\sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}} \\ F_2 &= \left. \frac{\partial^2 F}{\partial (\delta U)^2} \right|_{\delta U=0} = -\frac{m^2 \delta U}{\left(2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2} \right)^{3/2}}.\end{aligned}$$

Therefore, the expanded expression in powers of δU is:

$$F = F_0 + \frac{m}{\sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}} \delta U - \frac{m^2}{2 \left(2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2} \right)^{3/2}} \delta U^2 + \dots$$

After plugging the expanded expression of F in the integral, we see that the zero-order term gives 2π . The first-order term gives the required change $\delta\phi$:

$$\delta\phi = \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \frac{2m\delta U}{\sqrt{2m \left(E + \frac{\alpha}{r} \right) - \frac{M^2}{r^2}}} dr. \quad (3.11)$$

We can change the integration over r to one over ϕ , along the path of the unperturbed motion, using the substitution

$$\phi = \cos^{-1} \frac{(M/r) - (m\alpha/M)}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}}$$

with the differential form

$$\begin{aligned}\frac{r^2}{M}d\phi &= \frac{-1}{\sqrt{1 - \left(\frac{(M/r) - (m\alpha/M)}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}}\right)^2}} \frac{1}{\sqrt{2mE + \frac{m^2\alpha^2}{M^2}}} \\ &= \sqrt{2m \left(E + \frac{\alpha}{r}\right) - \frac{M^2}{r^2}}.\end{aligned}$$

Hence, the integral Eq. (3.11) become

$$\begin{aligned}\delta\phi &= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi r^2 \delta U d\phi \right) \\ &= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi r^2 \frac{\beta}{r^2} d\phi \right) \\ &= \frac{\partial}{\partial M} \left(\frac{2\pi m\beta}{M} \right) \\ &= -\frac{2\pi m\beta}{M^2}.\end{aligned}\tag{3.12}$$

Using the formulae (15.4) of the book we can finally write

Answer to Problem 9 (a).

$$\delta\phi = -\frac{2\pi\beta}{\alpha p}$$

(b)

$$\delta U = \frac{\gamma}{r^3}$$

SOLUTION: From Eq. (3.12), we have

$$\delta\phi = \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi \frac{\gamma}{r} d\phi \right).$$

Using the formulae (15.5) from the book, the integral become

$$\begin{aligned}\delta\phi &= \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi \frac{\gamma}{p} (1 + e \cos \phi) d\phi \right) \\ &= \frac{\partial}{\partial M} \left(\frac{2\pi\gamma m}{Mp} \right) \\ &= -\frac{6\pi\alpha\gamma m^2}{M^4}.\end{aligned}$$

The last expression can also be written as

Answer to Problem 9 (b).

$$\delta\phi = -\frac{6\pi\gamma}{\alpha p^2}$$

CHAPTER 4.

COLLISIONS BETWEEN PARTICLES

PROBLEM 1.

Find the relation between the angles θ_1, θ_2 (in the L system) after disintegrating into two particles.

SOLUTION: The angles of the particles, in the C system, are related by

$$\theta_0 = \theta_{10} = \pi - \theta_{20},$$

where θ_0 as been defined to simplify the notation. From the formulae (16.5) of the book, we have

$$\begin{aligned} V + v_{10} \cos \theta_0 &= v_{10} \sin \theta_0 \cot \theta_1 \\ V - v_{20} \cos \theta_0 &= v_{20} \sin \theta_0 \cot \theta_2. \end{aligned}$$

We must eliminate θ_0 from the two equations above. To do so, we can first solve for $\cos \theta_0$ and $\sin \theta_0$ which give us

$$\sin \theta_0 = \frac{V + v_{10} \cos \theta_0}{v_{10} \cot \theta_1} = \frac{V - v_{20} \cos \theta_0}{v_{20} \cot \theta_2} \quad (4.1)$$

$$\begin{aligned} \Rightarrow V v_{20} \cot \theta_2 + v_{10} v_{20} \cos \theta_0 \cot \theta_2 &= V v_{10} \cot \theta_1 - v_{10} v_{20} \cos \theta_0 \cot \theta_1 \\ \Rightarrow \cos \theta_0 &= \frac{V (v_{10} \cot \theta_1 - v_{20} \cot \theta_2)}{v_{10} v_{20} (\cot \theta_1 + \cot \theta_2)}. \end{aligned} \quad (4.2)$$

Using the sum of square of Eq. (4.1) and Eq. (4.2), we get

$$\begin{aligned} \left(\frac{V + v_{10} \cos \theta_0}{v_{10} \cot \theta_1} \right)^2 + \left(\frac{V (v_{10} \cot \theta_1 - v_{20} \cot \theta_2)}{v_{10} v_{20} (\cot \theta_1 + \cot \theta_2)} \right)^2 &= 1 \\ \Rightarrow (v_{10} + v_{20})^2 + (v_{10} \cot \theta_1 - v_{20} \cot \theta_2)^2 &= \frac{v_{10}^2 v_{20}^2}{V^2} (\cot \theta_1 + \cot \theta_2)^2 \\ \Rightarrow v_{10}^2 \csc^2 \theta_1 + v_{20}^2 \csc^2 \theta_2 - 2 v_{10} v_{20} \cot \theta_1 \cot \theta_2 &= \frac{v_{10}^2 v_{20}^2 \sin^2 (\theta_1 + \theta_2)}{V^2 \sin^2 \theta_1 \sin^2 \theta_2} \\ \Rightarrow v_{20}^2 \sin^2 \theta_1 + v_{10}^2 \sin^2 \theta_2 - 2 v_{10} v_{20} \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 &= \frac{v_{10}^2 v_{20}^2}{V^2} \sin^2 (\theta_1 + \theta_2). \end{aligned}$$

Using the equation (16.2) from the book and the relation $v_{10}/v_{20} = m_2/m_1$, we obtain

ANSWER TO PROBLEM 1.

$$\frac{m_2}{m_1} \sin^2 \theta_2 + \frac{m_1}{m_2} \sin^2 \theta_1 - 2 \sin \theta_1 \sin \theta_2 \cos \theta_1 \cos \theta_2 = \frac{2\epsilon}{(m_1 + m_2)V^2} \sin^2 (\theta_1 + \theta_2).$$

PROBLEM 2.

Find the angular distribution of the resulting particles in the L system.

SOLUTION: The distribution of the resulting particles with respect to the angle θ_0 is given by the equation (16.7) of the book, *i.e.*

$$\frac{1}{2} \sin \theta_0 d\theta_0 = \frac{1}{2} d(\cos \theta_0).$$

Using equation (16.6) of the book yield

$$\frac{1}{2} d(\cos \theta_0) = -\frac{V}{v_0} \sin \theta \cos \theta d\theta \pm \left(\sin \theta \sqrt{1 - (V^2/v_0^2) \sin^2 \theta} + \cos \theta \frac{(V^2/v_0^2) \sin \theta \cos \theta}{\sqrt{1 - (V^2/v_0^2) \sin^2 \theta}} \right) d\theta.$$

After simplification, the angular distribution become

ANSWER TO PROBLEM 2.

$$\frac{1}{2} \sin \theta d\theta \left(2 \frac{V}{v_0} \cos \theta \pm \frac{1 + (V^2/v_0^2) \cos 2\theta}{\sqrt{1 - (V^2/v_0^2) \sin^2 \theta}} \right)$$

PROBLEM 3.

Determine the range of possible values of the angle θ between the directions of motion of the two resulting particles in the L system.

SOLUTION: The tangent of the angle $\theta = \theta_1 + \theta_2$ is

$$\tan \theta = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}.$$

Using equation (16.5) of the book to get $\tan \theta_1$ and $\tan \theta_2$, and using $\theta_0 = \theta_{10} = \pi - \theta_{20}$, the last equation become

$$\tan \theta = \frac{(v_{10} + v_{20})V \sin \theta_0}{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}$$

or equivalently

$$\cot \theta = \frac{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}{(v_{10} + v_{20})V \sin \theta_0}.$$

In order to find the extrema, we take the derivative with respect to θ_0 ,

$$\begin{aligned} \frac{\partial \cot \theta}{\partial \theta_0} = 0 &= \frac{(-v_{10} + v_{20})V^2 - V \cos \theta_0(V^2 - v_{10}v_{20})}{(v_{10} + v_{20})V^2 \sin^2 \theta_0} \\ \Rightarrow \cos \theta_0 &= \frac{(v_{20} - v_{10})V}{V^2 - v_{10}v_{20}}. \end{aligned}$$

We know that $-1 \leq \cos \theta_0 \leq 1$, thus $v_{10} < V < v_{20}$. Moreover, we could calculate the derivative of the cotangent with respect to V and find the expression:

$$\frac{\partial \cot \theta}{\partial V} = \frac{V^2 + v_{10}v_{20}}{(v_{10} + v_{20})V^2 \sin \theta_0}$$

which is positive for all values of V .

This is why $\cot \theta$ may assume all the values between 0 and π when $v_{10} < V < v_{20}$, because $\frac{\partial \cot \theta}{\partial \theta_0} \neq 0$ and $\frac{\partial \cot \theta}{\partial V} \neq 0$ for all V, θ_0 in that interval. It is easy to see that if V is kept constant and θ_0 is varied between 0 and π , $\cot \theta$ spans the interval $(-\infty, +\infty)$; similarly, if θ_0 is kept constant and V is varied between 0 and $+\infty$, $\cot \theta$ spans the interval $(-\infty, +\infty)$.

When $V \leq v_{10}$, it is easy to see that $\cot \theta$ has a maximum value. In fact,

$$\lim_{\theta_0 \rightarrow 0^+} \cot(x) = -\infty, \quad \lim_{\theta_0 \rightarrow \pi^-} \cot(x) = -\infty$$

and the maximum value of the cotangent is reached when:

$$\cos \theta_0 = \frac{(v_{20} - v_{10})V}{V^2 - v_{10}v_{20}}$$

The calculation of the maximum value leads to:

$$\cot \theta = \frac{V^2 + (v_{10} - v_{20})V \cos \theta_0 - v_{10}v_{20}}{(v_{10} + v_{20})V \sin \theta_0} = -\frac{\sqrt{(V^2 - v_{10}v_{20})^2 - (v_{20} - v_{10})^2 V^2}}{(v_{10} + v_{20})V}$$

Then, after transforming the cotangent into a sine, one gets:

$$\sin \theta = \frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}}$$

If we put:

$$\alpha = \sin^{-1} \left(\frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}} \right)$$

then, in the interval where $V \leq v_{10}$, we have the following ranges for θ : $\pi - \alpha \leq \theta \leq \pi$. Similarly, when $V \geq v_{20}$, it is easy to see that $\cot \theta$ has a minimum value. In fact,

$$\lim_{\theta_0 \rightarrow 0^+} \cot(x) = +\infty, \quad \lim_{\theta_0 \rightarrow \pi^-} \cot(x) = +\infty$$

and the minimum value of the cotangent is reached when $\theta = \alpha$. In this case, the ranges for θ are $0 \leq \theta \leq \alpha$ (the equality $\theta = 0$ is satisfied only when $V \rightarrow +\infty$). The final solution is thus

ANSWER TO PROBLEM 3.

$$\text{if } v_{10} < V < v_{20} \quad \Rightarrow \quad 0 < \theta < \pi$$

$$\text{if } V \leq v_{10} \quad \Rightarrow \quad \pi - \alpha \leq \theta \leq \pi$$

$$\text{if } V \geq v_{20} \quad \Rightarrow \quad 0 \leq \theta \leq \alpha$$

where

$$\alpha = \sin^{-1} \left(\frac{(v_{20} + v_{10})V}{V^2 + v_{10}v_{20}} \right).$$