My Humble Solution to

Volume 1 of Course of Theoretical Physics

Mechanics

Author
P.O. Bolduc

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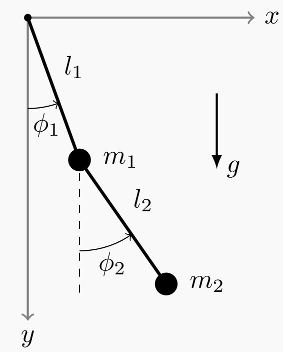
Chapter 1.

The Equations of Motion

PROBLEM 1.

Find the Lagrangian of a coplanar double pendulum when placed in a uniform gravitational field (acceleration g).



SOLUTION: The generalized co-ordinates of the system are the two angles ϕ_1 and ϕ_2. We need to express the Cartesian co-ordinates in terms of those two angles. First, the Cartesian position of the particle m_1 is

$$\begin{aligned} x_1 &= f_1(\phi_1) = l_1 \sin\phi_1 \\ y_1 &= g_1(\phi_1) = l_1 \cos\phi_1. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_1 &= \frac{\partial f_1}{\partial \phi_1}\dot{\phi}_1 = l_1 \cos\phi_1 \dot{\phi}_1 \\ \dot{y}_1 &= \frac{\partial g_1}{\partial \phi_1}\dot{\phi}_1 = -l_1 \sin\phi_1 \dot{\phi}_1. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_1 &= T_1 - U_1 \\ &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_1 g y_1 \\ &= \frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2 + m_1 g l_1 \cos\phi_1. \end{aligned} \tag{1.1}$$

Second, the Cartesian position of the particle m_2 is

$$\begin{aligned} x_2 &= f_2(\phi_1, \phi_2) = l_1 \sin\phi_1 + l_2 \sin\phi_2 \\ y_2 &= g_2(\phi_1, \phi_2) = l_1 \cos\phi_1 + l_2 \cos\phi_2. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x}_2 &= \sum_k \frac{\partial f_2}{\partial \phi_k}\dot{\phi}_k = l_1\dot{\phi}_1 \cos\phi_1 + l_2\dot{\phi}_2 \cos\phi_2 \\ \dot{y}_2 &= \sum_k \frac{\partial g_2}{\partial \phi_k}\dot{\phi}_k = -l_1\dot{\phi}_1 \sin\phi_1 - l_2\dot{\phi}_2 \sin\phi_2. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L_2 &= T_2 - U_2 \\ &= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + m_2 g y_2 \\ &= \frac{1}{2}m_2(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2l_1 l_2 \dot{\phi}_1\dot{\phi}_2 \cos(\phi_1 - \phi_2)) + m_2 g(l_1\cos\phi_1 + l_2\cos\phi_2), \end{aligned} \tag{1.2}$$

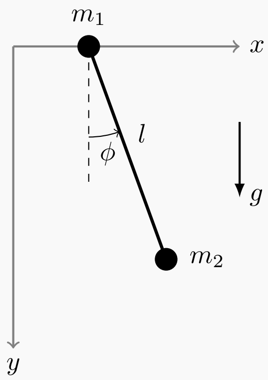
where the angle difference identity as been used. Finally, the Lagrangian of the complete system is simply the sum of Eq. (1.1) and Eq. (1.2), thus

Answer to Problem 1.

$$
\begin{aligned}
L =& \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \\
& + (m_1 + m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2
\end{aligned}
$$

Problem 2.

Find the Lagrangian of a simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves when placed in a uniform gravitational field (acceleration g).



Solution: The generalized co-ordinates q of the system are the position x and the angle ϕ. Therefore, the Lagrangian of the first particle is simply

$$L_1 = T_1 = \frac{1}{2} m_1 \dot{x}^2. \tag{1.3}$$

The Cartesian position of the particle m_2 can be express in terms of the generalized co-ordinates by

$$\begin{aligned} x_2 &= f_2(x, \phi) = x + l \sin\phi \\ y_2 &= g_2(x, \phi) = l \cos\phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x}_2 &= \sum_k \frac{\partial f_2}{\partial q_k} \dot{q}_k = \dot{x} + l\dot{\phi}\cos\phi \\ \dot{y}_2 &= \sum_k \frac{\partial g_2}{\partial q_k} \dot{q}_k = -l\dot{\phi}\sin\phi. \end{aligned}$$

$$\begin{aligned} L_2 &= T_2 - U_2 \\ &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + m_2 g y_2 \\ &= \frac{1}{2} m_2 (\dot{x}^2 + l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2 g l \cos\phi. \end{aligned} \tag{1.4}$$

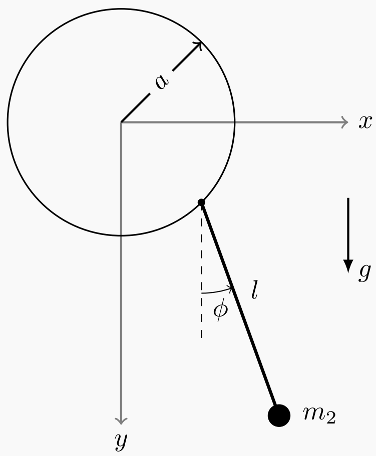
Finally, the Lagrangian of the complete system is the sum of the two Lagrangian Eq. (1.3) and Eq. (1.4), thus

Answer to Problem 2.

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 (l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2 g l \cos\phi$$

Problem 3.

Find the Lagrangian of a simple pendulum of mass m, when placed in a uniform gravitational field (acceleration g), whose point of support ...



(a)

moves uniformly on a vertical circle with constant frequency γ.

Solution: If we set that the rotation of the point of support is counterclockwise, the Cartesian position of the particle m is

$$\begin{aligned} x &= f(\phi_1) = a\cos\gamma t + l\sin\phi \\ y &= g(\phi_1) = -a\sin\gamma t + l\cos\phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi \\ \dot{y} &= -a\gamma\cos\gamma t - l\dot{\phi}\sin\phi. \end{aligned}$$

The potential energy of the system is

$$\begin{aligned} U &= -mgy \\ &= -mg(-a\sin\gamma t + l\cos\phi). \end{aligned}$$

The term $mga\sin\gamma t$ only depend on time and can therefore be ignored (does not contribute to the equations of motion). The kinetic energy of the particle, for its part, is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m\left(l^2\dot{\phi}^2 + a^2\gamma^2 + 2a\gamma l\dot{\phi}\sin(\phi - \gamma t)\right), \end{aligned}$$

using again the angle difference identity (I will stop mentioning it). We can observe that the term $\frac{1}{2}ma^2\gamma^2$ is a constant, thus can be ignored. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} ma\gamma l\dot{\phi}\sin(\phi - \gamma t) &= mal\gamma(\dot{\phi} - \gamma)\sin(\phi - \gamma t) + mal\gamma^2\sin(\gamma t - \phi) \\ &= \frac{d}{dt}\left(-mal\gamma\cos(\phi - \gamma t)\right) + mal\gamma^2\sin(\phi - \gamma t). \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

Answer to Problem 3 (a).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\sin(\phi - \gamma t) + mgl\cos\phi$$

(b)

oscillates horizontally in the plane of motion of the pendulum according to the law $x = a\cos\gamma t$.

Solution: The Cartesian position of the particle m is

$$\begin{aligned} x &= f(\phi_1) = a\cos\gamma t + l\sin\phi \\ y &= g(\phi_1) = l\cos\phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= -a\gamma\sin\gamma t + l\dot{\phi}\cos\phi \\ \dot{y} &= -l\dot{\phi}\sin\phi. \end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mgl\cos\phi,$$

and the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m\left(l^2\dot{\phi}^2 + a^2\gamma^2\sin^2\gamma t - 2a\gamma l\dot{\phi}\sin\gamma t\cos\phi\right). \end{aligned}$$

We first see that the term $\frac{1}{2}ma^2\gamma^2\sin^2\gamma t$ only depend on time. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} -ma\gamma l\dot{\phi}\sin\gamma t\cos\phi &= -mla\gamma(\gamma\cos\gamma t\sin\phi + \dot{\phi}\sin\gamma t\cos\phi) + mla\gamma^2\cos\gamma t\sin\phi \\ &= \frac{d}{dt}\left(-mla\gamma\sin\gamma t\sin\phi\right) + mla\gamma^2\cos\gamma t\sin\phi. \end{aligned}$$

After dropping the total time derivative, we can finally get the Lagrangian

Answer to Problem 3 (b).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mla\gamma^2\cos\gamma t\sin\phi + mgl\cos\phi$$

(c)

oscillates vertically according to the law $y = a\cos\gamma t$.

Solution: The Cartesian position of the particle m is

$$\begin{aligned} x &= f(\phi_1) = l\sin\phi \\ y &= g(\phi_1) = a\cos\gamma t + l\cos\phi. \end{aligned}$$

The time derivative of the position is then

$$\begin{aligned} \dot{x} &= l\dot{\phi}\cos\phi \\ \dot{y} &= -a\gamma\sin\gamma t - l\dot{\phi}\sin\phi. \end{aligned}$$

The potential energy of the system is

$$U = -mgy = -mg\left(a\cos\gamma t + l\cos\phi\right),$$

The term $mga\cos\gamma t$ only depend on time and can therefore be ignored. The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m\left(l^2\dot{\phi}^2 + a^2\gamma^2\sin^2\gamma t + 2a\gamma l\dot{\phi}\sin\gamma t\sin\phi\right). \end{aligned}$$

We first see that the term $\frac{1}{2}ma^2\gamma^2\sin^2\gamma t$ only depend on time. The last term of the kinetic energy can also be simplified. Indeed,

$$\begin{aligned} ma\gamma l\dot{\phi}\sin\gamma t\sin\phi &= mla\gamma(-\gamma\cos\gamma t\cos\phi + \dot{\phi}\sin\gamma t\sin\phi) + mla\gamma^2\cos\gamma t\cos\phi \\ &= \frac{d}{dt}\left(mla\gamma\cos\gamma t\sin\phi\right) + mla\gamma^2\cos\gamma t\cos\phi. \end{aligned}$$

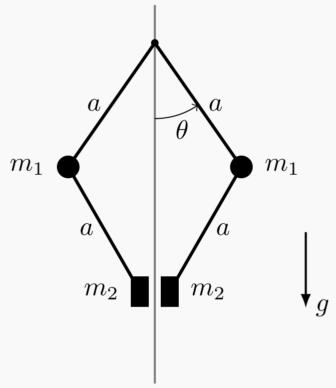
After dropping the total time derivative, we can finally get the Lagrangian

Answer to Problem 3 (c).

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mla\gamma^2\cos\gamma t\cos\phi + mgl\cos\phi$$

Problem 4.

Find the Lagrangian of a simple pendulum of the system below when placed in a uniform gravitational field (acceleration g). The particle m_2 moves on a vertical axis and the whole system rotates about this axis with a constant angular velocity Ω.



Solution: The position of each particle m_1 is best described in cylindrical coordinates, which is

$$\begin{aligned} r_1 &= a\sin\theta \\ \phi_1 &= \phi \\ z_1 &= a\cos\theta, \end{aligned}$$

where ϕ is the angle of rotation of the system about the axis; $\dot{\phi} = \Omega$. The kinetic energy of each particle m_1 is thus

$$\begin{aligned} T_1 &= \frac{1}{2}m_1(\dot{r}_1^2 + r_1^2\dot{\phi}_1^2 + \dot{z}_1^2) \\ &= \frac{1}{2}m_1(a^2\dot{\theta}^2\cos^2\theta + a^2\Omega^2\sin^2\theta + a^2\dot{\theta}^2\sin^2\theta) \\ &= \frac{1}{2}m_1a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta). \end{aligned} \tag{1.5}$$

The potential energy of this particle can be found by using the z component of his position, namely

$$\begin{aligned} V_1 &= -m_1gz_1 \\ &= -m_1ga\cos\theta. \end{aligned} \tag{1.6}$$

The particle m_2, in its case, can only move up and down, thus its position can be completely defined by

$$z_2 = 2a\cos\theta.$$

Then, the kinetic energy of each particle m_2 is

$$\begin{aligned} T_2 &= \frac{1}{2}m_2\dot{z}_2^2 \\ &= m_2a^2\dot{\theta}^2\sin^2\theta \end{aligned} \tag{1.7}$$

and the potential energy is

$$\begin{aligned} V_2 &= -m_2gz_2 \\ &= -2m_2ga\cos\theta. \end{aligned} \tag{1.8}$$

Using Eq. (1.5), Eq. (1.6), Eq. (1.7) and Eq. (1.8), the Lagrangian of the system is

$$L = 2(T_1 + T_2 - V_1 - V_2),$$

therefore

Answer to Problem 4.

$$L = m_1a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) + 2m_2a^2\dot{\theta}^2\sin^2\theta + 2(m_1 + 2m_2)ga\cos\theta$$

Chapter 2.

Conservation Laws

Problem 1.

A particle of mass m moving with velocity $\mathbf{v}_1$ leaves a half-space in which its potential energy is a constant U_1 and enters another in which its potential energy is a different constant U_2. Determine the change in the direction of motion of the particle.

Solution: The potential energy only depend on the co-ordinate perpendicular to the plane separating the half-space. Thus, the component of the momentum in that plane is conserved. Denoting by θ_1 and θ_2 the angles between the normal to the plane and the velocities v_1 and v_2 of the particle before and after passing the plane, we have

$$\begin{aligned} &P_1 \sin\theta_1 = P_2 \sin\theta_2 \\ \Rightarrow &v_1 \sin\theta_1 = v_2 \sin\theta_2 \\ \Rightarrow &\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1}. \end{aligned} \tag{2.1}$$

The potential energy of the system is also independent of time, therefore the energy of the particle is conserved. Posing $E_1 = T_1 + U_1$ and $E_2 = T_2 + U_2$ as the energy of the particle before and after passing the plane, the law of conservation of energy requires

$$\begin{aligned} &E_1 = E_2 \\ \Rightarrow &T_1 + U_1 = T_2 + U_2 \\ \Rightarrow &\frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2 \\ \Rightarrow &v_2^2 = v_1^2 + \frac{2}{m}(U_1 - U_2). \end{aligned} \tag{2.2}$$

By substituting Eq. (2.2) in the square of Eq. (2.1), we get

$$\left(\frac{\sin\theta_1}{\sin\theta_2}\right)^2 = \frac{v_1^2 + \frac{2}{m}(U_1 - U_2)}{v_1^2}.$$

After taking the square root, the result is

Answer to Problem 1.

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 + \frac{2}{mv_1^2}(U_1 - U_2)}$$

PROBLEM 2.

Find the law of transformation of the action S from one inertial frame to another.

SOLUTION: The Lagrangian L and L' of a mechanical system in two inertial frames of reference K and K' are respectively

$$L = T - U = \frac{1}{2}\sum_a m_a v_a^2 - U$$

and

$$L' = T' - U = \frac{1}{2}\sum_a m_a v_a'^2 - U.$$

If the frame K' moves with velocity $\mathbf{V}$ relative to the frame K, the velocities of the particles of the mechanical system relative to the two frames are related by $\mathbf{v}_a = \mathbf{v}_a' + \mathbf{V}$. We can now express the relation of the Lagrangian of the system in the two frames by

$$\begin{aligned} L &= \frac{1}{2}\sum_a m_a(\mathbf{v}_a' + \mathbf{V})^2 - U \\ &= \frac{1}{2}V^2\sum_a m_a + \mathbf{V}\cdot\sum_a m_a\mathbf{v}_a' + \frac{1}{2}\sum_a m_a v_a'^2 - U \\ &= L' + \mathbf{V}\cdot\mathbf{P}' + \frac{1}{2}\mu V^2. \end{aligned} \tag{2.3}$$

Integrating Eq. (2.3) with respect to time, we obtain

$$\begin{aligned} S &= \int L dt \\ &= S' + \int (\mathbf{V}\cdot\mathbf{P}')\, dt + \frac{1}{2}\mu V^2 t \\ &= S' + \mathbf{V}\cdot\int\sum_a m_a\mathbf{v}_a' dt + \frac{1}{2}\mu V^2 t \\ &= S' + \mathbf{V}\cdot\sum_a m_a\mathbf{r}_a' + \frac{1}{2}\mu V^2 t. \end{aligned}$$

The law of transformation of the action S is then

ANSWER TO PROBLEM 2.

$$S = S' + \mu\mathbf{V}\cdot\mathbf{R}' + \frac{1}{2}\mu V^2 t$$

Problem 3.

Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in cylindrical co-ordinates r, ϕ, z.

Solution: The Cartesian components of the angular momentum are simply

$$\mathbf{M} = \mathbf{r} \times \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times m \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix}. \tag{2.4}$$

In cylindrical co-ordinates, the Cartesian components are expressed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \\ z \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{z} \end{bmatrix}.$$

By substituting those in Eq. (2.4), we get

$$\begin{aligned} \mathbf{M} &= m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix} \\ &= m \begin{bmatrix} (r\dot{z} - z\dot{r})\sin\phi - rz\dot{\phi}\cos\phi \\ -(r\dot{z} - z\dot{r})\cos\phi - rz\dot{\phi}\sin\phi \\ r^2\dot{\phi} \end{bmatrix}. \end{aligned} \tag{2.5}$$

By taking the magnitude of Eq. (2.5), we finally get

Answer to Problem 3.

$$\begin{aligned} M_x &= m(r\dot{z} - z\dot{r})\sin\phi - mrz\dot{\phi}\cos\phi \\ M_y &= -m(r\dot{z} - z\dot{r})\cos\phi - mrz\dot{\phi}\sin\phi \\ M_z &= mr^2\dot{\phi} \\ M^2 &= m^2r^2\dot{\phi}^2(r^2 + z^2) + m^2(r\dot{z} - z\dot{r})^2 \end{aligned}$$

Problem 4.

Obtain the expressions for the Cartesian components and the magnitude of the angular momentum of a particle in spherical co-ordinates r, θ, ϕ.

Solution: In spherical co-ordinates, the Cartesian components are expressed by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{r}\sin\phi\cos\theta + r\dot{\phi}\cos\phi\cos\theta - r\dot{\theta}\sin\phi\sin\theta \\ \dot{r}\sin\phi\sin\theta + r\dot{\phi}\cos\phi\sin\theta + r\dot{\theta}\sin\phi\cos\theta \\ \dot{r}\cos\phi + r\sin\phi \end{bmatrix}.$$

By substituting those in Eq. (2.4), we get

$$\begin{aligned} \mathbf{M} &= m \begin{bmatrix} y\dot{z} - z\dot{y} \\ z\dot{x} - x\dot{z} \\ x\dot{y} - y\dot{x} \end{bmatrix} \\ &= m \begin{bmatrix} -r^2(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi) \\ r^2(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi) \\ r^2\dot{\phi}\sin^2\theta \end{bmatrix}. \end{aligned} \tag{2.6}$$

By taking the magnitude of Eq. (2.6), we finally get

Answer to Problem 4.

$$\begin{aligned} M_x &= -mr^2(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi) \\ M_y &= mr^2(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi) \\ M_z &= mr^2\dot{\phi}\sin^2\theta \\ M^2 &= m^2r^4(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \end{aligned}$$

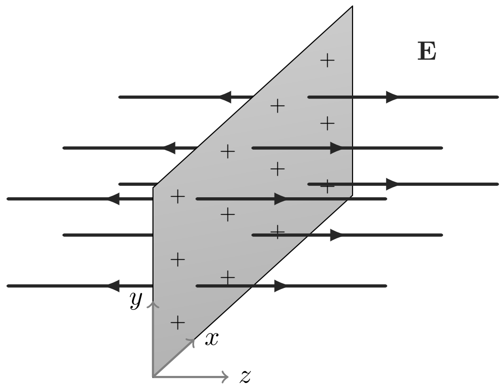
Problem 5.

Which components of momentum $\mathbf{P}$ and angular momentum $\mathbf{M}$ are conserved in motion in the following fields ?

(a)

The field of an infinite homogeneous plane.

Solution: solution



Answer to Problem 5 (a).

answer

(b)

The field of an infinite homogeneous cylinder.

Solution: solution

Answer to Problem 5 (b).

answer

(c)

The field of an infinite homogeneous prism.

Solution: solution

Answer to Problem 5 (c).

answer

(d)

The field of two points.

Solution: solution

Answer to Problem 5 (d).

answer

(e)

The field of an infinite homogeneous half-plane.

Solution: solution

Answer to Problem 5 (e).

answer

(f)

The field of a homogeneous cone.

Solution: solution

Answer to Problem 5 (f).

answer

(g)

The field of a homogeneous circular torus.

Solution: solution

Answer to Problem 5 (g).

answer

(h)

The field of an infinite homogeneous cylindrical helix.

Solution: solution

Answer to Problem 5 (h).

answer

Problem 6.

Find the ratio of the times in the same path for particles having different masses but the same potential energy.

Solution: If the two particles have the same path, then the ratio of linear dimension is

$$\frac{l'}{l} = \alpha = 1. \tag{2.7}$$

We can define the ratio of time and mass by

$$\frac{t'}{t} = \beta$$

and

$$\frac{m'}{m} = \gamma.$$

Then, the ratio of kinetic energy is

$$\frac{T'}{T} = \frac{m'\mathbf{v}'}{m\mathbf{v}} = \frac{m'}{m}\left(\frac{\mathbf{dr}'}{\mathbf{dr}}\frac{dt}{dt'}\right)^2 = \frac{\gamma\alpha^2}{\beta^2}.$$

To leave the equation of motion unaltered, the ratio of the kinetic energy and the potential energy must be the same, *i.e.* ,

$$\frac{U'}{U} = \frac{T'}{T}$$
$$\Rightarrow \alpha^k = \frac{\gamma\alpha^2}{\beta^2}.$$

Using Eq. (2.7), we get

$$1 = \frac{\gamma}{\beta^2}$$
$$\Rightarrow \beta = \sqrt{\gamma}.$$

The ratio of the times is then

Answer to Problem 6.

$$\frac{t'}{t} = \sqrt{\frac{m'}{m}}$$

Problem 7.

Find the ratio of the times in the same path for particles the same mass but potential energy differing by a constant factor.

Solution: If the two particles have the same path, then the ratio of linear dimension is

$$\frac{l'}{l} = \alpha = 1. \tag{2.8}$$

We can define the ratio of time by

$$\frac{t'}{t} = \beta.$$

Then, the ratio of kinetic energy is

$$\frac{T'}{T} = \frac{\alpha^2}{\beta^2}.$$

The potential energy of the two particles differ by a constant factor, which mean that

$$\frac{U'}{U} = \gamma.$$

To leave the equation of motion unaltered, the ratio of the kinetic energy and the potential energy must be the same, *i.e.* ,

$$\frac{U'}{U} = \frac{T'}{T}$$
$$\Rightarrow \gamma = \frac{\alpha^2}{\beta^2}.$$

Using Eq. (2.8), we get

$$\gamma = \frac{1}{\beta^2}$$
$$\Rightarrow \beta = \sqrt{\frac{1}{\gamma}}.$$

The ratio of the times is then

Answer to Problem 7.

$$\frac{t'}{t} = \sqrt{\frac{U}{U'}}$$

Chapter 3.

Integration of the Equations of Motion

Problem 1.

Determine the period of oscillations of a simple pendulum (a particle of mass m suspended by a string of length l in a gravitational field) as a function of the amplitude of the oscillations.

Solution: The potential energy of a simple pendulum is

$$U(\phi) = -mgl\cos\phi,$$

see Chap. 1 Prob. 1. If we define ϕ_0 as the maximum value of ϕ, the potential energy is equal to the total energy at this point, that is

$$U(\phi_0) = E = -mgl\cos\phi_0.$$

The energy of the pendulum could also be write using the kinetic energy,

$$E = \frac{1}{2}ml^2\dot{\phi}^2 + U(\phi).$$

This first order differential equation can be integrate to give the the period of oscillation

$$\begin{aligned}
T &= 2\sqrt{\frac{ml^2}{2}}\int_{-\phi_0}^{\phi_0}\frac{d\phi}{\sqrt{E-U(\phi)}} \\
&= 4\sqrt{\frac{ml^2}{2}}\int_{0}^{\phi_0}\frac{d\phi}{\sqrt{mgl\cos\phi - mgl\cos\phi_0}} \\
&= 4\sqrt{\frac{l}{2g}}\int_{0}^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi-\cos\phi_0}}.
\end{aligned}$$

To solve this integral, we need first to use a trigonometric identity (I'll let you find or derive this identity), giving us

$$T = 2\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2}-\sin^2\frac{\phi}{2}}}.$$

Next, we use the substitution

$$\sin\xi = \frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}}.$$

It's differential form is

$$\begin{aligned}
\frac{d\phi}{d\xi} &= \frac{d}{d\xi}\left(2\arcsin\left(\sin\frac{\phi_0}{2}\sin\xi\right)\right) \\
&= \frac{2\sin\frac{\phi_0}{2}\cos\xi}{\sqrt{1-\sin^2\frac{\phi_0}{2}\sin^2\xi}}
\end{aligned}$$

The substitution, finally, give

$$\begin{aligned}
T &= 2\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2}-\sin^2\xi\sin^2\frac{\phi_0}{2}}} \\
&= 2\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{d\phi}{\sin\frac{\phi_0}{2}\sqrt{1-\sin^2\xi}} \\
&= 2\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{d\phi}{\sin\frac{\phi_0}{2}\cos\xi} \\
&= 2\sqrt{\frac{l}{g}}\int_{\arcsin(0)}^{\arcsin(1)}\frac{2\sin\frac{\phi_0}{2}\cos\xi d\xi}{\sin\frac{\phi_0}{2}\cos\xi\sqrt{1-\sin^2\frac{\phi_0}{2}\sin^2\xi}} \\
&= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\xi}{\sqrt{1-\sin^2\frac{\phi_0}{2}\sin^2\xi}}.
\end{aligned}$$

We can also write

$$T = 4\sqrt{\frac{l}{g}} K\left(\sin\frac{\phi_0}{2}\right),$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1 - k\sin^2\xi}}. \tag{3.1}$$

The integral K is known as the complete elliptic integral of the first kind. To solve this integral, we first need to see that the integrand is of the form

$$f(x) = (1-x)^{-\frac{1}{2}}.$$

The n derivative of this function is

$$f^{(n)}(x) = \frac{(2n-1)!!}{2^n}(1-x)^{\frac{-2n+1}{2}},$$

thus the Maclaurin Series is

$$f(x) = \sum_{n\geq 0} \frac{(2n-1)!!}{2^n \cdot n!} x^n.$$

Using this result in Eq. (3.1), we get

$$\begin{aligned} K(k) &= \int_0^{\frac{\pi}{2}} \sum_{n\geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \sin^{2n}\xi \, d\xi \\ &= \sum_{n\geq 0} \frac{(2n-1)!!}{2^n \cdot n!} k^{2n} \int_0^{\frac{\pi}{2}} \sin^{2n}\xi \, d\xi. \end{aligned}$$

This last integral can be solve using the beta function

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1} t \cos^{2y-1} t \, dt,$$

with $x = \frac{2n+1}{2}$ and $y = \frac{1}{2}$. Thus, the beta function become

$$\begin{aligned} \frac{1}{2}\mathcal{B}(x,y) &= \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} \\ &= \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{2n+1}{2})}{n!} \\ &= \frac{\pi}{2} \frac{(2n+1)!!}{2^n \cdot n!} \end{aligned}$$

The elliptic integral is thereby

$$\begin{aligned} K(k) &= \frac{\pi}{2} \sum_{n\geq 0} \left[\frac{(2n-1)!!}{2^n \cdot n!} k^n\right]^2 \\ &= \frac{\pi}{2} \sum_{n\geq 0} \left[\frac{(2n-1)!!}{(2n)!!} k^n\right]^2. \end{aligned}$$

The period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}} \sum_{n\geq 0} \left[\frac{(2n-1)!!}{(2n)!!} \sin^n\frac{\phi_0}{2}\right]^2$$

By expanding the sum, we get

$$T = 2\pi\sqrt{\frac{l}{g}} \left(1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{\phi}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \sin^4\frac{\phi}{2} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \sin^6\frac{\phi}{2} + \dots\right).$$

Using the Maclaurin series

$$\sin\frac{\phi_0}{2} = \frac{1}{2}\phi_0 - \frac{1}{48}\phi_0^3 + \frac{1}{3840}\phi_0^5 - \frac{1}{645120}\phi_0^7 + \dots,$$

we finally get

Answer to Problem 1.

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \frac{11}{3072}\phi_0^4 + \dots\right)$$

Problem 2.

Determine the period of oscillation, as a function of the energy, when a particle of mass m moves in fields for which the potential energy is

(a)

$$U = A|x|^n$$

Solution: The total energy of the particle is

$$E = \frac{1}{2}m\dot{x}^2 + U(x).$$

Knowing that the maximum value of E is at x_1, this position is

$$\begin{aligned}
E &= U(x_1) = A|x_1|^n \\
\Rightarrow |x_1| &= \left(\frac{E}{A}\right)^{1/n}, \\
\Rightarrow x_1 &= \pm\left(\frac{E}{A}\right)^{1/n},
\end{aligned}$$

we get the period of oscillation,

$$\begin{aligned}
T &= 2\sqrt{\frac{m}{2}}\int_{-\left(\frac{E}{A}\right)^{1/n}}^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}} \\
&= 2\sqrt{2m}\int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - A|x|^n}}.
\end{aligned}$$

It is possible to remove the absolute value, because the integral is over positive x, thus

$$\begin{aligned}
T &= 2\sqrt{2m}\int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{E - Ax^n}} \\
&= 2\sqrt{\frac{2m}{E}}\int_0^{\left(\frac{E}{A}\right)^{1/n}} \frac{dx}{\sqrt{1 - \frac{A}{E}x^n}}.
\end{aligned}$$

Using the substitution $y = \left(\frac{A}{E}\right)^{\frac{1}{n}} x$, we get

$$\begin{aligned}
T &= 2\sqrt{\frac{2m}{E}}\int_0^1 \frac{\left(\frac{A}{E}\right)^{\frac{1}{n}} dy}{\sqrt{1 - \frac{A}{E}\left(\left(\frac{A}{E}\right)^{\frac{-1}{n}} y\right)^n}} \\
&= 2\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}}\int_0^1 \frac{dy}{\sqrt{1 - y^n}}.
\end{aligned}$$

Using another substitution $u = y^n$, we get

$$T = 2\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}}\int_0^1 \frac{u^{\frac{1}{n}} du}{nu\sqrt{1-u}}.$$

It is possible to express this integral in term of the beta function

$$\mathcal{B}(z_1, z_2) = \int_0^1 t^{z_1 - 1}(1-t)^{z_2 - 1}\, dt = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1 + z_2)},$$

with $z_1 = \frac{1}{n}$ and $z_2 = \frac{1}{2}$. This give us

$$T = \frac{2}{n}\sqrt{\frac{2m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}.$$

Knowing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we finally have

Answer to Problem 2 (a).

$$T = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{\frac{1}{n}}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

(b)

$$U = \frac{-U_0}{\cosh^2 \alpha x}$$

Solution: The two boundaries of the energy are at $E = 0 \Rightarrow x = x_1$ and $E = -U_0 \Rightarrow x = 0$. The period of oscillation is thus

$$\begin{aligned}
T &= 2\sqrt{2m}\int_0^{x_1}\frac{dx}{\sqrt{E-U}} \\
&= 2\sqrt{2m}\int_0^{x_1}\frac{dx}{\sqrt{E+\frac{U_0}{\cosh^2 \alpha x}}} \\
&= 2\sqrt{2m}\int_0^{x_1}\frac{\cosh^2 \alpha x \quad dx}{\sqrt{E\cosh^2 \alpha x + U_0}}.
\end{aligned}$$

Using the identity $\cosh^2 \alpha x = 1 + \sinh^2 \alpha x$, we get

$$T = 2\sqrt{2m}\int_0^{x_1}\frac{\cosh^2 \alpha x \quad dx}{\sqrt{E\left(1+\sinh^2 \alpha x\right)+U_0}}.$$

The substitution $y = \sinh \alpha x$, give us $dy = \alpha \cosh \alpha x dx$ and

$$\begin{aligned}
y(x_1) &= \sinh \alpha x \\
&= \sqrt{\sinh^2 \alpha x} \\
&= \sqrt{\cosh^2 \alpha x - 1} \\
&= \sqrt{\frac{-U_0}{E} - 1}.
\end{aligned}$$

Using this substitution, we get

$$\begin{aligned}
T &= \frac{2}{\alpha}\sqrt{2m}\int_0^{\sqrt{\frac{-U_0}{E}-1}}\frac{dy}{E\left(1+y^2\right)+U_0} \\
&= \frac{2}{\alpha}\sqrt{2m}\int_0^{\sqrt{\frac{-U_0}{E}-1}}\frac{dy}{E+Ey^2-E\cosh^2 \alpha x}
\end{aligned}$$

Factoring $|E|$ out of the square root (because $E < 0$), we have

$$\begin{aligned}
T &= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\int_0^{\sqrt{\frac{-U_0}{E}-1}}\frac{dy}{1-\cosh^2 \alpha x + y^2} \\
&= \frac{2}{\alpha}\sqrt{\frac{2m}{|E|}}\int_0^{\sqrt{\frac{-U_0}{E}-1}}\frac{dy}{\sqrt{\frac{-U_0}{E}-1+y^2}}
\end{aligned}$$

This integral is of the form

$$\int_0^a \frac{dz}{\sqrt{a+z^2}} = \sin^{-1}\left(\frac{z}{a}\right) = \frac{\pi}{2}.$$

Thus, the period of oscillation is

Answer to Problem 2 (b).

$$T = \frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}$$

(c)

$$U = U_0 \tan^2 \alpha x$$

Solution: The period of oscillation is

$$\begin{aligned} T &= 2\sqrt{2m}\int_0^{x_1} \frac{dx}{\sqrt{E-U}} \\ &= 2\sqrt{2m}\int_0^{x_1} \frac{dx}{\sqrt{E-U_0\tan^2\alpha x}} \\ &= 2\sqrt{2m}\int_0^{x_1} \frac{\cos\alpha x \quad dx}{\sqrt{E\cos^2\alpha x - U_0\sin^2\alpha x}} \\ &= 2\sqrt{2m}\int_0^{x_1} \frac{\cos\alpha x \quad dx}{\sqrt{E-(E+U_0)\sin^2\alpha x}}. \end{aligned}$$

Using the substitution $y = i\sin\alpha x$ $(dy = i\alpha\cos\alpha x)$, we get

$$\begin{aligned} T &= \frac{2\sqrt{2m}}{i\alpha}\int_0^{i\sin\alpha x_1} \frac{dy}{\sqrt{E+(E+U_0)\,y^2}} \\ &= \frac{2}{i\alpha}\sqrt{\frac{2m}{E+U_0}}\int_0^{i\sin\alpha x_1} \frac{dy}{\sqrt{\frac{E}{(E+U_0)}+y^2}} \\ &= \frac{2}{i\alpha}\sqrt{\frac{2m}{E+U_0}}\sin^{-1}\left(\frac{i\sin\alpha x_1}{\sqrt{\frac{E}{(E+U_0)}}}\right) \end{aligned}$$

Knowing

$$x_1 = \frac{1}{\alpha}\tan^{-1}\sqrt{\frac{E}{U_0}},$$

we finally get

$$\begin{aligned} T &= \frac{2}{i\alpha}\sqrt{\frac{2m}{E+U_0}}\sin^{-1}\left(\frac{i\sin\left(\tan^{-1}\sqrt{\frac{E}{U_0}}\right)}{\sqrt{\frac{E}{(E+U_0)}}}\right) \\ &= \frac{2}{i\alpha}\sqrt{\frac{2m}{E+U_0}}\sin^{-1}\left(\frac{i\sqrt{\frac{E}{U_0}}}{\sqrt{1+\frac{E}{U_0}}\sqrt{\frac{E}{(E+U_0)}}}\right) \\ &= \frac{2}{i\alpha}\sqrt{\frac{2m}{E+U_0}}\sin^{-1} i \\ &= \frac{2}{i\alpha}\sqrt{\frac{2m}{E+U_0}}\frac{\pi i}{2}. \end{aligned}$$

Thus, the period of oscillation is

Answer to Problem 2 (c).

$$T = \frac{\pi}{\alpha}\sqrt{\frac{2m}{E+U_0}}$$

Problem 3.

A system consists of one particle of mass M and n particles with equal masses m. Eliminate the motion of the centre of mass and so reduce the problem to one involving n particles.

Solution: Let $\mathbf{R}$ be the radius vector of the particle of mass M, and $\mathbf{R}_a$ $(a = 1, 2, \ldots, n)$ those of the particles of mass m. We put $\mathbf{r}_a \equiv \mathbf{R}_a - \mathbf{R}$ and take the origin to be at the centre of mass, namely

$$M\mathbf{R} + m\sum_a \mathbf{R}_a = 0.$$

Thus, we got

$$\mathbf{R} = -\frac{m}{M}\left(\sum_a \mathbf{r}_a + n\mathbf{R}\right) = -\frac{m}{\mu}\sum_a \mathbf{r}_a, \tag{3.2}$$

where $\mu \equiv M + nm$. The Lagrangian of the system is

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}m\sum_a \dot{\mathbf{R}}_a^2 - U.$$

If we substitute Eq. (3.2) and $\mathbf{R}_a = \mathbf{R} + \mathbf{r}_a$, we get

$$\begin{aligned}
L &= \frac{1}{2}\frac{Mm^2}{\mu^2}\left(\sum_a \dot{\mathbf{r}}_a\right)^2 + \frac{1}{2}nm\dot{\mathbf{R}}^2 + \frac{1}{2}m\sum_a \dot{\mathbf{r}}_a^2 + m\dot{\mathbf{R}}\sum_a \dot{\mathbf{r}}_a - U \\
&= \frac{1}{2}\frac{Mm^2}{\mu^2}\left(\sum_a \dot{\mathbf{r}}_a\right)^2 + \frac{1}{2}\frac{m^3}{\mu^2}\left(\sum_a \dot{\mathbf{r}}_a\right)^2 + \frac{1}{2}m\sum_a \dot{\mathbf{r}}_a^2 - \frac{m^2}{\mu}\left(\sum_a \dot{\mathbf{r}}_a\right)^2 - U \\
&= \frac{1}{2}m\sum_a \dot{\mathbf{r}}_a^2 - \frac{1}{2}\frac{m^2}{\mu}\left(\frac{2\mu - M - nm}{\mu}\right)\left(\sum_a \dot{\mathbf{r}}_a\right)^2.
\end{aligned}$$

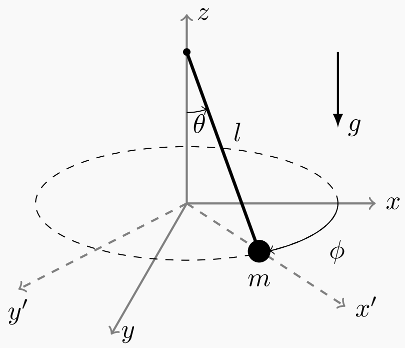
The Lagrangian is thus only a function of $\mathbf{r}_a$ and it's time derivative, which is

Answer to Problem 3.

$$L = \frac{1}{2}m\sum_a \dot{\mathbf{r}}_a^2 - \frac{1}{2}\frac{m^2}{\mu}\left(\sum_a \dot{\mathbf{r}}_a\right)^2$$

PROBLEM 4.

Integrate the equations of motion for a spherical pendulum (a particle of mass m moving on the surface of a sphere of radius l in a gravitational field).



SOLUTION: The Cartesian position of the particle m is

$$\begin{aligned} x &= l\sin\theta\cos\phi \\ y &= l\sin\theta\sin\phi \\ z &= -l\cos\theta. \end{aligned}$$

The time derivatives of these coordinates are

$$\begin{aligned} \dot{x} &= l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi \\ \dot{y} &= l\dot{\theta}\cos\theta\sin\phi + l\dot{\phi}\sin\theta\cos\phi \\ \dot{z} &= l\dot{\theta}\sin\theta. \end{aligned}$$

Substituting these expressions in the Lagrangian, we obtain the desired Lagrangian form, that is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - mgz \\ &= \frac{1}{2}ml^2\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right) + mgl\cos\theta. \end{aligned}$$

The Lagrangian does not involve ϕ explicitly, thus this co-ordinate is cyclic and the generalized momentum p_ϕ is conserved. This momentum is the same as the z-component of angular momentum M_z and is written

$$M_z = p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi}\sin^2\theta. \tag{3.3}$$

The energy of the pendulum is

$$E = \frac{1}{2}ml^2\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right) - mgl\cos\theta.$$

Substituting $\dot{\phi}$ with Eq. (3.3) in the energy we get

$$\begin{aligned} E &= \frac{1}{2}ml^2\dot{\theta}^2 + \frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + U_{\text{eff}}(\theta), \end{aligned}$$

where $U_{\text{eff}}(\theta)$ is the effective potential energy. Notice that the energy was re-written as a function of only one co-ordinate (*i.e.* θ). This is equivalent to the one particle problem we have already see, thus

$$\begin{aligned} \frac{d\theta}{dt} &= \sqrt{\frac{2\left(E-U_{\text{eff}}(\theta)\right)}{ml^2}} \\ \Rightarrow t &= \int\frac{d\theta}{\sqrt{2\left(E-U_{\text{eff}}(\theta)\right)}}. \end{aligned}$$

This integral lead to an elliptic integral of the first kind see Chap. 3 Prob. 1. We also need to find the solution to the angle ϕ. To do so, we can use the Eq. (3.3) with the chain rule from calculus, that is

$$\begin{aligned}\frac{M_z}{ml^2\sin^2\theta} &= \frac{d\phi}{dt} \\ &= \frac{d\phi}{d\theta}\frac{d\theta}{dt} \\ &= \frac{d\phi}{d\theta}\sqrt{\frac{2\left(E-U_{\text{eff}}(\theta)\right)}{ml^2}}.\end{aligned}$$

Finally, we get

$$\begin{aligned}\frac{d\phi}{d\theta} &= \frac{M_z}{l\sin^2\theta}\sqrt{\frac{1}{2m\left(E-U_{\text{eff}}(\theta)\right)}} \\ \Rightarrow \phi &= \frac{M_z}{l}\sqrt{\frac{1}{2m}}\int\frac{d\theta}{\sin^2\theta\left(E-U_{\text{eff}}(\theta)\right)}.\end{aligned}$$

This integral lead to an elliptic integral of the third kind (I really don't want to solve this). It is possible to find the range of angle θ. At the maximum and minimum value of θ, the pendulum as no kinetic energy, thus

$$\begin{aligned}E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2ml^2\sin^2\theta} - mgl\cos\theta \\ &= \frac{M_z^2}{2ml^2\left(1-\cos^2\theta\right)} - mgl\cos\theta\end{aligned}$$

We can find a cubic algebraic equation for $\cos\theta$, that is

$$\begin{aligned}0 &= \left(E + mgl\cos\theta\right)\left(2ml^2\left(1-\cos^2\theta\right)\right) - M_z^2 \\ &= mgl\left(\cos^3\theta - \cos\theta\right) + E\left(\cos^2\theta - 1\right) + \frac{M_z^2}{2ml^2}.\end{aligned}$$

The centrifugal part of the effective potential U_{eff}, namely

$$\frac{M_z^2}{2ml^2\sin^2\theta},$$

must be positive. As we can see, this term diverge at $\theta \to 0$ and $\theta \to \pi$, thus the angle θ is bound. This can be write as

$$\theta \in [\theta_p, \theta_a] \qquad\qquad 0 < \theta_p \leq \theta_a < \pi,$$

where the subscripts of the angles θ_p and θ_a are for the perigee and apogee. This mean that the two roots of the cubic algebraic equation for $\cos\theta$ is bound between -1 and $+1$. To summarize, the solution of the equation of motion of the spherical pendulum is

Answer to Problem 4.

$$t = \int_{\theta_p}^{\theta_a}\frac{d\theta}{\sqrt{2\left(E-U_{\text{eff}}(\theta)\right)}} \qquad\qquad \phi = \frac{M_z}{l}\sqrt{\frac{1}{2m}}\int_{\theta_p}^{\theta_a}\frac{d\theta}{\sin^2\theta\left(E-U_{\text{eff}}(\theta)\right)}$$

Problem 5.

Integrate the equations of motion for a particle moving on the surface of a cone (of vertical axis 2α) placed vertically and with vertex downwards in a gravitational field.

Solution: By using the spherical co-ordinate, we can write the position of the particle as

$$\begin{aligned} x &= r\cos\phi\sin\alpha \\ y &= r\sin\phi\sin\alpha \\ z &= r\cos\alpha. \end{aligned}$$

By taking the time derivative of those, we obtain

$$\begin{aligned} \dot{x} &= \dot{r}\cos\phi\sin\alpha - r\dot{\phi}\sin\phi\sin\alpha \\ \dot{y} &= \dot{r}\sin\phi\sin\alpha + r\dot{\phi}\cos\phi\sin\alpha \\ \dot{z} &= \dot{r}\cos\alpha. \end{aligned}$$

The Lagrangian is thus

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - mgz \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right) - mgr\cos\alpha. \end{aligned}$$

The co-ordinate ϕ is cyclic, thus

$$M_z = p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi}\sin^2\alpha \tag{3.4}$$

is conserved. The energy of the particle is

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\alpha\right) + mgr\cos\alpha. \tag{3.5}$$

We can rewrite Eq. (3.4) as

$$\dot{\phi}^2 = \frac{M_z^2}{m^2r^4\sin^4\alpha}$$

and substituting in Eq. (3.5), we get

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 + \frac{M_z^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha \\ &= \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r). \end{aligned}$$

Hence,

$$t = \sqrt{\frac{m}{2}}\int\frac{dr}{\sqrt{E-U_{\text{eff}}(r)}}.$$

From Eq. (3.4), it is possible to write

$$\begin{aligned} &\frac{d\phi}{dt} = \frac{M_z}{mr^2\sin^2\alpha} \\ \Rightarrow &\frac{d\phi}{dr}\frac{dr}{dt} = \frac{M_z}{mr^2\sin^2\alpha} \\ \Rightarrow &\frac{d\phi}{dr}\sqrt{\frac{2\left(E-U_{\text{eff}}(r)\right)}{m}} = \frac{M_z}{mr^2\sin^2\alpha} \\ \Rightarrow &\phi = \frac{M_z}{\sqrt{2m}\sin^2\alpha}\int\frac{dr}{r^2\sqrt{E-U_{\text{eff}}(r)}}. \end{aligned}$$

It is possible to find the range of r. At the maximum and minimum value of r, the particle as no kinetic energy, thus

$$\begin{aligned} E &= U_{\text{eff}} \\ &= \frac{M_z^2}{2mr^2 \sin^2 \alpha} + mgr \cos \alpha \end{aligned}$$

We can find a cubic algebraic equation for r, that is

$$\begin{aligned} 0 &= r^2 \left(E - mgr \cos \alpha \right) - \frac{M_z^2}{2m \sin^2 \alpha} \\ &= mgr^3 \cos \alpha - Er^2 + \frac{M_z^2}{2m \sin^2 \alpha}. \end{aligned}$$

This equation as two positive roots, r_p and r_a, which are the turning points of the motion. To summarize, the solution of the equation of motion for a particle moving on the surface of a cone is

Answer to Problem 5.

$$t = \sqrt{\frac{m}{2}} \int_{r_p}^{r_a} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} \qquad \phi = \frac{M_z}{\sqrt{2m} \sin^2 \alpha} \int_{r_p}^{r_a} \frac{dr}{r^2 \sqrt{E - U_{\text{eff}}(r)}}$$

Problem 6.

Integrate the equations of motion for a pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plane which m_2 moves (Chap. 1 Prob. 2).

Solution: From Chap. 1 Prob. 2, the Lagrangian of the system is

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) + m_2gl\cos\phi.$$

The co-ordinate x is cyclic, thus

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = \text{constant} \tag{3.6}$$

is conserved. It is always possible to find an inertial frame of reference where $p_x = 0$, using this frame we get

$$\begin{aligned}
&(m_1+m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = 0 \\
\Rightarrow &\int \Big((m_1+m_2)\dot{x} + m_2l\dot{\phi}\cos\phi\Big)\,dt = \text{constant} \\
\Rightarrow &(m_1+m_2)x + m_2l\sin\phi = (m_1+m_2)R_x = \text{constant}.
\end{aligned} \tag{3.7}$$

This express the fact that the centre of mass $\mathbf{R}$ of the system does not move horizontally. It is also possible to rewrite Eq. (3.6) as

$$\begin{aligned}
&(m_1+m_2)\dot{x} + m_2l\dot{\phi}\cos\phi = 0 \\
\Rightarrow &\dot{x} = \frac{-m_2l\dot{\phi}\cos\phi}{m_1+m_2}
\end{aligned}$$

Plugging this into the energy of the system we get

$$\begin{aligned}
E &= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) - m_2gl\cos\phi \\
&= \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{2(m_1+m_2)} + \frac{1}{2}m_2l^2\dot{\phi}^2 - \frac{m_2^2l^2\dot{\phi}^2\cos^2\phi}{m_1+m_2} - m_2gl\cos\phi \\
&= \frac{1}{2}m_2l^2\dot{\phi}^2\left(1 - \frac{m_2}{m_1+m_2}\cos^2\phi\right) - m_2gl\cos\phi.
\end{aligned}$$

Hence,

$$\begin{aligned}
t &= l\sqrt{\frac{m_2}{2}}\int\sqrt{\frac{1-\frac{m_2}{m_1+m_2}\cos^2\phi}{E+m_2gl\cos\phi}}\,d\phi \\
&= l\sqrt{\frac{m_2}{2(m_1+m_2)}}\int\sqrt{\frac{m_1+m_2\sin^2\phi}{E+m_2gl\cos\phi}}\,d\phi.
\end{aligned}$$

Using Eq. (3.7), we can express the position of the mass m_2 as

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x + l\sin\phi \\ l\cos\phi \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi\left(1-\frac{m_2}{m_1+m_2}\right) \\ l\cos\phi \end{bmatrix}.$$

The path of the particle of mass m_2 is thus an arc of an ellipse center at $(R_x, 0)$ with horizontal semi-axis $lm_1/(m_1+m_2)$ and vertical semi-axis l. It is possible to see that when $m_1 \to \infty$, the path return to the simple pendulum. The solution of the equations of motion is thus

Answer to Problem 6.

$$t = l\sqrt{\frac{m_2}{2}}\int\sqrt{\frac{1-\frac{m_2}{m_1+m_2}\cos^2\phi}{E+m_2gl\cos\phi}}\,d\phi \qquad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_x - l\sin\phi\left(1-\frac{m_2}{m_1+m_2}\right) \\ l\cos\phi \end{bmatrix}$$

Problem 7.

Find the time dependence of the co-ordinate of a particle with energy $E = 0$ moving in a parabola in a field $U = -\alpha/r$.

Solution: From the formulae (14.6) of the book, we have the integral

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2r^2}}} \\ &= \int \frac{dr}{\sqrt{\frac{2\alpha}{mr} - \frac{M^2}{m^2r^2}}} \\ &= \int \frac{rdr}{\sqrt{\frac{2\alpha}{m}r - \frac{M^2}{m^2}}}. \end{aligned}$$

We use the substitution

$$\begin{aligned} \frac{m}{M}\eta &= \sqrt{\frac{2\alpha}{m}r - \frac{M^2}{m^2}} \\ \Rightarrow r &= \frac{M^2}{2m\alpha}\left(1+\eta^2\right) = \frac{1}{2}p\left(1+\eta^2\right), \end{aligned} \tag{3.8}$$

with the differential form

$$dr = \frac{M^2}{m\alpha}\eta d\eta.$$

Hence, the integral become

$$\begin{aligned} t &= \frac{M^3}{2m\alpha^2}\int\left(1+\eta^2\right)d\eta \\ &= \frac{M^3}{2m\alpha^2}\left(\eta + \frac{1}{3}\eta^3\right) \\ &= \sqrt{\frac{mp^3}{\alpha}}\frac{1}{2}\eta\left(\eta + \frac{1}{3}\eta^3\right). \end{aligned}$$

It is important to specify that the parameter η varies from $-\infty$ to ∞. Using Eq. (3.8) and

$$\cos\phi = \frac{p}{r} - 1,$$

it is possible to find the Cartesian co-ordinates

$$x = r\cos\phi = \frac{1}{2}p\left(1-\eta^2\right)$$

and

$$y = \sqrt{r^2 - x^2} = p\eta.$$

The parametric form of the required dependence are thus

Answer to Problem 7.

$$r = \frac{1}{2}p\left(1+\eta^2\right) \qquad t = \sqrt{\frac{mp^3}{\alpha}}\frac{1}{2}\eta\left(\eta + \frac{1}{3}\eta^3\right)$$

$$x = \frac{1}{2}p\left(1-\eta^2\right) \qquad y = p\eta$$

Problem 8.

Integrate the equations of motion for a particle in a central field

$$U = -\frac{\alpha}{r^2} \qquad (\alpha > 0).$$

Solution: From the formulae (14.6) of the book, we have the integral

$$\begin{aligned} t &= \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2 r^2}}} \\ &= \int \frac{dr}{\sqrt{\frac{2E}{m} r^2 + \frac{2\alpha}{m} - \frac{M^2}{m^2}}} \\ &= \sqrt{\frac{m}{2E}} \int \frac{r dr}{\sqrt{r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)}}. \end{aligned}$$

We use the substitution

$$u = r^2 + \left(\frac{\alpha}{E} - \frac{M^2}{2mE}\right)$$

with the differential form

$$du = 2rdr.$$

Hence, the integral become

$$\begin{aligned} t &= \frac{1}{2}\sqrt{\frac{m}{2E}} \int \frac{du}{\sqrt{u}} \\ &= \sqrt{\frac{m}{2E}} \sqrt{u} \\ &= \sqrt{\frac{m}{2E}} \sqrt{r^2 + \frac{\alpha}{E} - \frac{M^2}{2mE}} \\ &= \frac{1}{E}\sqrt{\frac{m}{2}\left(Er^2 + \alpha - \frac{M^2}{2m}\right)}. \end{aligned}$$

The formulae (14.7) of the book give us the equation of the path

$$\begin{aligned} \phi &= \int \frac{M dr}{r^2 \sqrt{2m\left(E - U(r)\right) - \frac{M^2}{r^2}}} \\ &= \int \frac{M dr}{r^2 \sqrt{2mE + \frac{2m\alpha}{r^2} - \frac{M^2}{r^2}}}. \end{aligned}$$

Using the substitution

$$u = \frac{1}{r}$$

with the differential form

$$du = -\frac{1}{r^2} dr,$$

the integral become

$$\begin{aligned} \phi &= -\int \frac{du}{\sqrt{2mE + \left(2m\alpha - M^2\right) u^2}} \\ &= -\frac{1}{\sqrt{2mE}} \int \frac{du}{\sqrt{1 + (ku)^2}}, \end{aligned} \tag{3.9}$$

with

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}. \tag{3.10}$$

From there, the solution must be divided for the three possible cases : **(a)** $E > 0$, $M^2 > 2m\alpha$; **(b)** $E > 0$, $M^2 < 2m\alpha$; **(c)** $E < 0$ $M^2 < 2m\alpha$. It is also interesting to know the the path is a Cotes's spiral[1].

(c) : Eq. (3.10) is still

$$k = \sqrt{\frac{2m\alpha - M^2}{2mE}}.$$

By using the substitution

$$ku = \sinh\theta$$

with the differential form

$$kdu = \cosh\theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned}\phi &= -\frac{1}{\sqrt{2mE}}\int\frac{\cosh\theta d\theta}{k\sqrt{1+\sinh^2\theta}}\\ &= -\frac{1}{\sqrt{2m\alpha - M^2}}\theta\\ &= -\frac{1}{\sqrt{2m\alpha - M^2}}\sinh^{-1}\frac{k}{r}.\end{aligned}$$

The equation of the path for the case (c) is thus

$$\begin{aligned}\frac{1}{r} &= \frac{1}{k}\sinh\left(\phi\sqrt{2m\alpha - M^2}\right)\\ &= \sqrt{\frac{2mE}{2m\alpha - M^2}}\sinh\left(\phi\sqrt{\frac{2m\alpha}{M^2} - 1}\right).\end{aligned}$$

(b) : Eq. (3.10) become

$$k = \frac{\sqrt{2m\alpha - M^2}}{i\sqrt{2m|E|}}.$$

By using the substitution

$$iku = \cosh\theta$$

with the differential form

$$ikdu = \sinh\theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned}\phi &= -\frac{1}{i\sqrt{2m|E|}}\int\frac{\sinh\theta d\theta}{ik\sqrt{1-\cosh^2\theta}}\\ &= \frac{1}{\sqrt{2m\alpha - M^2}}\theta\\ &= \frac{1}{\sqrt{2m\alpha - M^2}}\cosh^{-1}\frac{ik}{r}.\end{aligned}$$

The equation of the path for the case (b) is thus

$$\begin{aligned}\frac{1}{r} &= \frac{1}{ik}\cosh\left(\phi\sqrt{2m\alpha - M^2}\right)\\ &= \sqrt{\frac{2m|E|}{2m\alpha - M^2}}\cosh\left(\phi\sqrt{\frac{2m\alpha}{M^2} - 1}\right).\end{aligned}$$

[1] Whittaker ET (1937). A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, with an Introduction to the Problem of Three Bodies (4th ed.). New York: Dover Publications. pp. 80–83. ISBN 978-0-521-35883-5.

(a) : Eq. (3.10) become

$$k = \frac{i\sqrt{2m\alpha - M^2}}{\sqrt{2mE}}.$$

By using the substitution

$$ku = i\cosh\theta$$

with the differential form

$$kdu = i\sinh\theta d\theta,$$

the integral Eq. (3.9) become

$$\begin{aligned}\phi &= -\frac{1}{\sqrt{2mE}}\int\frac{i\sinh\theta d\theta}{k\sqrt{1-\cosh^2\theta}}\\ &= -\frac{i}{\sqrt{2m\alpha - M^2}}\theta\\ &= -\frac{i}{\sqrt{2m\alpha - M^2}}\cosh^{-1}\frac{k}{ir}.\end{aligned}$$

The equation of the path for the case (b) is thus

$$\begin{aligned}\frac{1}{r} &= \frac{i}{k}\cosh\left(\frac{\phi}{i}\sqrt{2m\alpha - M^2}\right)\\ &= \sqrt{\frac{2mE}{M^2 - 2m\alpha}}\cos\left(\phi\sqrt{1-\frac{2m\alpha}{M^2}}\right).\end{aligned}$$

To summarize the equations of motion for a particle in a central inverse-cube law force is

Answer to Problem 8.

(a) for $E > 0$ and $\frac{M^2}{2m} > \alpha$, $\qquad \frac{1}{r} = \sqrt{\frac{2mE}{M^2 - 2m\alpha}}\cos\left(\phi\sqrt{1-\frac{2m\alpha}{M^2}}\right)$

(b) for $E > 0$ and $\frac{M^2}{2m} < \alpha$, $\qquad \frac{1}{r} = \sqrt{\frac{2mE}{2m\alpha - M^2}}\sinh\left(\phi\sqrt{\frac{2m\alpha}{M^2}-1}\right)$

(c) for $E < 0$ and $\frac{M^2}{2m} < \alpha$, $\qquad \frac{1}{r} = \sqrt{\frac{2m|E|}{2m\alpha - M^2}}\cosh\left(\phi\sqrt{\frac{2m\alpha}{M^2}-1}\right)$

In all three cases

$$t = \frac{1}{E}\sqrt{\frac{m}{2}\left(Er^2 + \alpha - \frac{M^2}{2m}\right)}$$

Problem 9.

When a small correction $\delta U(r)$ is added to the potential energy $U = -\alpha/r$, the paths of finite motion are no longer closed, and at each revolution the perihelion is displaced through a small angle $\delta\phi$. Find $\delta\phi$ when

(a)

$$\delta U = \frac{\beta}{r^2}$$

Solution: From the equation (14.10) of the book, we have

$$\begin{aligned}
\Delta\phi &= 2\int_{r_{\min}}^{r_{\max}} \frac{M dr}{r^2\sqrt{2m(E-U)-\frac{M^2}{r^2}}} \\
&= -2\frac{\partial}{\partial M}\int_{r_{\min}}^{r_{\max}} \sqrt{2m(E-U)-\frac{M^2}{r^2}}dr \\
&= -2\frac{\partial}{\partial M}\int_{r_{\min}}^{r_{\max}} \sqrt{2m\left(E+\frac{\alpha}{r}-\delta U\right)-\frac{M^2}{r^2}}dr.
\end{aligned}$$

Expanding the integrand in powers of δU involves using a Taylor series expansion. Let's denote the integrand as F:

$$F = \sqrt{2m\left(E+\frac{\alpha}{r}+\delta U\right)-\frac{M^2}{r^2}}$$

We want to expand F around $\delta U = 0$. The expansion will look like:

$$F = F_0 + F_1\delta U + F_2(\delta U)^2 + \ldots$$

Here, F_0 is the value of F at $\delta U = 0$, F_1 is the first derivative with respect to δU at $\delta U = 0$, and so on. Let's find the derivatives:

$$\begin{aligned}
F_0 &= \sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}} \\
F_1 &= \frac{\partial F}{\partial(\delta U)}\bigg|_{\delta U=0} = \frac{m}{\sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}}} \\
F_2 &= \frac{\partial^2 F}{\partial(\delta U)^2}\bigg|_{\delta U=0} = -\frac{m^2\delta U}{\left(2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}\right)^{3/2}}.
\end{aligned}$$

Therefore, the expanded expression in powers of δU is:

$$F = F_0 + \frac{m}{\sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}}}\delta U - \frac{m^2}{2\left(2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}\right)^{3/2}}\delta U^2 + \ldots$$

After plugging the expended expression of F in the integral, we see that the zero-order term gives 2π. The first-order term gives the required change $\delta\phi$:

$$\delta\phi = \frac{\partial}{\partial M}\int_{r_{\min}}^{r_{\max}} \frac{2m\delta U}{\sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}}}. \tag{3.11}$$

We can change the integration over r to one over ϕ, along the path of the unperturbed motion, using the substitution

$$\phi = \cos^{-1}\frac{(M/r)-(m\alpha/M)}{\sqrt{2mE+\frac{m^2\alpha^2}{M^2}}}$$

with the differential form

$$\frac{r^2}{M}d\phi = \frac{-1}{\sqrt{1-\left(\frac{(M/r)-(m\alpha/M)}{\sqrt{2mE+\frac{m^2\alpha^2}{M^2}}}\right)^2}}\frac{1}{\sqrt{2mE+\frac{m^2\alpha^2}{M^2}}}$$
$$= \sqrt{2m\left(E+\frac{\alpha}{r}\right)-\frac{M^2}{r^2}}.$$

Hence, the integral Eq. (3.11) become

$$\begin{aligned}\delta\phi &= \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^\pi r^2\delta U d\phi\right) \\ &= \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^\pi r^2\frac{\beta}{r^2}d\phi\right) \\ &= \frac{\partial}{\partial M}\left(\frac{2\pi m\beta}{M}\right) \\ &= -\frac{2\pi m\beta}{M^2}.\end{aligned} \tag{3.12}$$

Using the formulae (15.4) of the book we can finally write

Answer to Problem 9 (a).

$$\delta\phi = -\frac{2\pi\beta}{\alpha p}$$

(b)

$$\delta U = \frac{\gamma}{r^3}$$

SOLUTION: From Eq. (3.12), we have

$$\delta\phi = \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^\pi \frac{\gamma}{r}d\phi\right).$$

Using the formulae (15.5) from the book, the integral become

$$\begin{aligned}\delta\phi &= \frac{\partial}{\partial M}\left(\frac{2m}{M}\int_0^\pi \frac{\gamma}{p}\left(1+e\cos\phi\right)d\phi\right) \\ &= \frac{\partial}{\partial M}\left(\frac{2\pi\gamma m}{Mp}\right) \\ &= -\frac{6\pi\alpha\gamma m^2}{M^4}.\end{aligned}$$

The last expression can also be written as

Answer to Problem 9 (b).

$$\delta\phi = -\frac{6\pi\gamma}{\alpha p^2}$$

Chapter 4.

Collisions Between Particles

Problem 1.

Find the relation between the angles θ_1, θ_2 (in the L system) after disintegrating into two particles.

Solution: The angles of the particles, in the C system, are related by

$$\theta_0 = \theta_{10} = \pi - \theta_{20},$$

where θ_0 as been defined to simplify the notation. From the formulae (16.5) of the book, we have

$$\begin{aligned} V + v_{10}\cos\theta_0 &= v_{10}\sin\theta_0\cot\theta_1 \\ V - v_{20}\cos\theta_0 &= v_{20}\sin\theta_0\cot\theta_2. \end{aligned}$$

We must eliminate θ_0 from the two equations above. To do so, we can first and $\sin\theta_0$ which give us

$$\begin{aligned} \sin\theta_0 &= \frac{(V + v_{10}\cos\theta_0)}{v_{10}\cot\theta_1} = \frac{(V - v_{20}\cos\theta_0)}{v_{20}\cot\theta_2} \\ \Rightarrow & \end{aligned}$$

Answer to Problem 1.