PDE Notes (Translated)

Week 1

What is PDE?

- PDE = Partial Differential Equation
- ODE = Ordinary Differential Equations

What is ODE?

An equation that contains unknown function(s) of one variable.

$$x + 2x + 3 = 5$$
$$f(x) + \sin f(x) = 5 \quad \text{(Not ODE)}$$

Ordinary Differential Equations (ODEs)

An ordinary differential equation (ODE) is an equation involving a function and its derivatives. For example, consider the equation:

$$f'(x) = \sin f(x) \tag{1}$$

This is an ODE.

Partial Differential Equations (PDEs)

A partial differential equation (PDE) is an equation that contains unknown functions and their partial derivatives. For example, a first-order PDE can be written as:

$$E(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0 \tag{2}$$

where $u(x, y)$ is a function of two variables x and y, and u_x and u_y are the partial derivatives of u with respect to x and y, respectively.

Another way to express a PDE is:

$$F(x, y, u, u_x, u_y) = 0 \tag{3}$$

Examples of Partial Differential Equations (PDEs)

$$\begin{aligned}
1.&\quad U_x + U_y = 0 \quad \text{(Transport Equation)} && (4)\\
2.&\quad U_x + yU_y = 0 \quad \text{(Transport Equation)} && (5)\\
3.&\quad U_x + UU_y = 0 \quad \text{(Shock Wave Equation)} && (6)\\
4.&\quad U_{xx} + U_{yy} = 0 \quad \text{(Laplace Equation)} && (7)\\
5.&\quad U_{xx} + 4 = 0 \quad \text{(Wave Equation / Interactor)} && (8)\\
6.&\quad U_t + UU_x + U_{xx} = 0 \quad \text{(Dispersive Equation)} && (9)\\
7.&\quad U_{tt} + U_{xx} = 0 \quad \text{(Vibration of a Bar)} && (10)\\
8.&\quad U - \nabla^2 U = 0 \quad \text{(Quantum Mechanics)} && (11)\\
\text{Parabolic Equation:}&\quad U_t = U_{xx} && (12)
\end{aligned}$$

Linear and Nonlinear Equations

In linear algebra, a transformation $T : V \to W$ is called a linear transformation if:

$$T(u+v) = T(u) + T(v), \tag{13}$$
$$T(au) = aT(u) \tag{14}$$

or equivalently,

$$T(au+bv) = aT(u) + bT(v) \tag{15}$$

Definition

An operator is called linear if:

$$L(u+v) = L(u) + L(v), \tag{16}$$
$$L(au) = aL(u) \tag{17}$$

A differential operator is linear if:

$$L(u+v) = L(u) + L(v), \tag{18}$$
$$L(au) = aL(u) \tag{19}$$

Example

Consider the differential equation:

$$u_{xx} + u_{yy} = 0 \tag{20}$$

This is an example of a linear partial differential equation (PDE).

The transport equation is given by:

$$u_x + u_y = 0 \tag{21}$$

A linear PDE can be written in the form:

$$Lu = f(x) \tag{22}$$

where L is a linear operator, and $f(x)$ is a given function.

The system of linear equations can be represented as:

$$Ax = b \tag{23}$$

Solutions to Linear Equations

Consider the linear equations:

$$Lu = f(x), \tag{24}$$
$$Ax = b. \tag{25}$$

For the homogeneous case, we have:

$$Qu = 0, \tag{26}$$
$$Ax = 0. \tag{27}$$

The general solution can be expressed as:

$$U = U_g + U_s, \tag{28}$$
$$X = X_g + X_s, \tag{29}$$

where:

- U_g is a general solution of $Lu = 0$,
- U_s is a special solution,
- X_g is a general solution of $Ax = 0$,
- X_s is a special solution of $Ax = b$.

Proposition

Let $U_1, U_2, \ldots, U_n$ be solutions of $Lu = 0$. Then the linear combination

$$u(t) = c_1U_1 + c_2U_2 + \cdots + c_nU_n \tag{30}$$

is also a solution of $Lu = 0$, where $c_1, c_2, \ldots, c_n$ are constants.

Example 1

$$x(t) = f(t) \tag{31}$$

$$x(t) = \int f(t)\,dt + C \tag{32}$$

$$u(x) = 0, \quad x = G \tag{33}$$

$$u''(x) = 0 \tag{34}$$

$$2(f) = C_1 + C_2 \tag{35}$$

$$u_{xx} = 0 \tag{36}$$

$$u(x, y) = 0 \tag{37}$$

$$u_x = 0 \tag{38}$$

$$u_x = F(y) \tag{39}$$

$$u(x, y) = XF(y) + G(x) \tag{40}$$

Example 2

$$U_x X + U = 0, \tag{41}$$

$$u'' + u = 0. \tag{42}$$

The solution for $u(x, y)$ is given by:

$$u(x, y) = C_1(y)\cos x + C_2(y)\sin x$$

Example 3

$$U_{xy} = 0, \tag{43}$$

$$u_{xx} = 0, \tag{44}$$

$$u_x = F(x). \tag{45}$$

The general solution is:

$$u = \int F(x)\,dx + H(x) + G(y)$$

List of Calculus Facts

1. Partial derivatives are local.
2. $U_{xy} = K_{yx}$
3. Chain rules:

$$\begin{aligned} f(g(x)) &= f'(g(x)) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x) \end{aligned}$$

   This is a special case of the chain rule. Assume $f = f(y_1, y_2, \ldots, y_n)$ where $y_i = g_i(x_1, \ldots, x_k)$ in the most general form of the chain rule.

4. Green's formula (later):

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

5.

$$\begin{aligned} I(t) &= \int_{a(t)}^{b(t)} f(x,t)\,dx \\ I'(t) &= f(b(t),t) \cdot b'(t) - f(a(t),t) \cdot a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,dx \end{aligned}$$

Jacobian

Consider the function $F : \mathbb{R}^n \to \mathbb{R}^m$, where

$$F = (F_1, F_2, \ldots, F_m)$$

Infinite Series

(to be covered later)

Directional Derivative (Geometry)

The directional derivative of a function $u(x,y)$ in the direction of a vector $\mathbf{u} = (u_1, u_2)$ is given by:

$$\frac{\partial u}{\partial x} u_1 + \frac{\partial u}{\partial y} u_2$$

This represents the rate of change of the function u in the direction of the vector $\mathbf{u}$.

Familiar with Math 3D

(to be covered later)

Example 1: Transport Equations

Consider the transport equation:

$$au_x + bu_y = 0 \tag{46}$$

where a and b are constants.

Method 1: Geometry / Gradient of u

The gradient of u is given by:

$$(a, b) \cdot (u_x, u_y) = -b \cdot a$$

The equation for the line is:

$$bx - ay = c \tag{47}$$

The solution is:

$$u(x, y) = F(bx - ay)$$

To check:

$$u_x = F' \cdot b, \quad u_y = F' \cdot (-a)$$

Substituting back into the transport equation:

$$au_x + bu_y = a \cdot F' \cdot b + b \cdot F' \cdot (-a) = 0$$

Method 2: Change of Variables

Let:

$$x' = ax + by, \quad y' = bx - ay$$

Example 2

$$u_x = Hy - a + Hyb = alx + bu_y, \tag{48}$$
$$My = u_y b + Hy(-a) = bu_y - any, \tag{49}$$
$$0 = alx + bu_y = (a+b)u_x = Hx = 0, \tag{50}$$
$$x = F(y) = F(bx - ay). \tag{51}$$

$$u_x + yu_y = 0, \tag{52}$$
$$2 = +y^2 u = 0. \tag{53}$$

$$(u_x, u_y) = 0, \tag{54}$$
$$(1, \%) - y = +y = ce^y. \tag{55}$$

Equations Along the Curve

Consider the following equations along the curve:

$$u(x, Ex + x) = 0 \tag{56}$$

$$u(x, x + c) = U_{10} \tag{57}$$

$$F(x) = F(y - Ex) = F(bx - ay) \tag{58}$$

Review

Example

$$ax + by = 0, \tag{59}$$
$$a + b = 70. \tag{60}$$

If we consider the equation:

$$y = Hx + My, \tag{61}$$
$$y = ux + My = 0. \tag{62}$$

We have:

$$u(x, x + c) = u(0, c), \tag{63}$$
$$u(x, y) = u(0, c) = u(0, y - tx) = F(y - Ex) = G(ay - bx) = H(bx - ay). \tag{64}$$

Example 2

Consider the differential equation:

$$u_x + yu_y = 0. \tag{65}$$

We have the following conditions:

$$y' = -y, \tag{66}$$
$$y = ce^{-x}, \tag{67}$$
$$c = ye^{x}, \tag{68}$$
$$u(x, ce^{-x}) = u(0, c) = u(0, ye^{x}) = F(ye^{x}). \tag{69}$$

Example ?

In the previous example, if in addition $u(0, y) = 3$, then what is the solution?

Solution

The solution is given by:

$$u(x, y) = F(ye^{x}) = u(0, y) = F(y) = 3. \tag{70}$$

Solution

The solution is given by

$$u(x,y) = F(ye) = (ye - x)^3$$

Integrating Factor

Consider the integrating factor:

$$e^y = e^{-y}(y' - y) = 0$$

Thus, we have:

$$e^y + y = y$$

$$xf_y(y) = x + C$$

$$f(y) = x + C$$

$$\frac{1}{y} = e^y$$

$$e^y = e^x + 6$$

The missing solution is:

$$Y = 0$$

$$y = ce^x$$

Example 4

Consider the partial differential equation:

$$U_x + 2xU_y = 0.$$

Solution

We start by considering the characteristic equations:

$$\frac{dy}{dx} = \frac{2x}{1} \quad \Rightarrow \quad y = x^2 + C.$$

Thus, the general solution can be expressed as:

$$u(x,y) = f(y - x^2),$$

where f is an arbitrary function.

Given the initial condition $u(0, -5) = 0$, we substitute into the general solution:

$$u(0, -5) = f(-5 - 0^2) = f(-5) = 0.$$

Therefore, the specific solution satisfying the initial condition is:

$$u(x, y) = f(y - x^2),$$

where $f(-5) = 0$.

Conclusion

The solution to the partial differential equation is determined by the function f which satisfies the initial condition. The characteristic curves are given by $y = x^2 + C$, and the solution is constant along these curves.

Implicit Differentiation

Consider the equation:

$$1 + xy = 0. \tag{71}$$

The function $u(x, y)$ is not defined on the curve $y = -y$.

In general, if $a(x, y)(x + b(x, y))y = 0$, the ordinary differential equation (ODE) is:

$$y = \dots \tag{72}$$

Assume $y = Y(x)$ is implicitly defined by the equation $U(x, y) = 0$. Then:

$$u(x, y(x)) = 0. \tag{73}$$

Differentiating implicitly, we have:

$$u_x + y' u_y = 0, \tag{74}$$

$$y' = -\frac{u_x}{u_y}. \tag{75}$$

This is equivalent to:

$$S \cdot Nx + y = 0. \tag{76}$$

$$y = \dots \tag{77}$$

Exercise 7 of Section 1.2

Solve the differential equation:

$$YU_x + Xy = 0 \tag{78}$$

with the initial condition:

$$x(0, y) = g \tag{79}$$

Determine in which region of the initial problem (IP) the solution is uniquely determined.

Solution

Consider the differential equation:

$$y' = y \tag{80}$$

which implies:

$$y = x + c \tag{81}$$

where c is a constant. Rewriting, we have:

$$y - x = c \tag{82}$$

If $c = 0$, then:

$$y = x \tag{83}$$

If $c \neq 0$, then:

$$y^2 - x = 1 \tag{84}$$

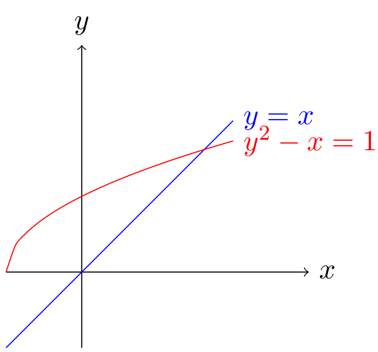
If $c = 0$, then:

$$y = x \tag{85}$$

If $c \neq 0$, then:

$$y = \ln U(x) \tag{86}$$

where $U(x) = u(x,t) = e$.



Mathematical Physical Equations

Wave Equations

$$
\begin{aligned}
u(x, y) &= e^{\cdots} \\
y &= x \ldots
\end{aligned}
$$

On the other hand, in the green region, we have

$$
\begin{gathered}
y - x = c < 0 \\
x = 1
\end{gathered}
$$

$$
u(1, y) = u(\ldots, 0) = u(\ldots, 0)
$$

In the blue region,

$$
u(x, y) = F(y)
$$

In the green region, sometimes

$$
F = G
$$

Heat Equations (Parabolic Equations)

Laplace Equations (Elliptic Equations)

$$
a, b, c, \ldots
$$

Chapter 2: Wave and Diffusion

2.1 The Wave Equation

Assume $U = U(x,t)$ and $U_{tt} = c^2 U_{xx}$ with $c > 0$.

Theorem: The general solutions of the wave equation are given by

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are arbitrary functions.

Proof:

Assume

$$U_{tt} - c^2 U_{xx} = 0.$$

Let $V = U_t + cU_x$ and $W = U_t - cU_x$. Then,

$$V_t - cV_x = 0 \quad \text{and} \quad W_t + cW_x = 0.$$

The solutions to these equations are

$$v(x,t) = h(x+ct) \quad \text{and} \quad w(x,t) = k(x-ct),$$

where h and k are arbitrary functions.

Thus, the general solution is

$$u(x,t) = f(x+ct) + g(x-ct).$$

Wave Equation Solutions

General Solution of the Homogeneous Equation

Find the general solution of the homogeneous equation, which is given by:

$$u_{tt} + c^2 u_{xx} = 0 \tag{87}$$

The solution can be expressed as:

$$u = g(x-ct) \tag{88}$$

Particular Solution

Find a particular solution by assuming:

$$u = f(x+ct) \tag{89}$$

Then, we have:

$$u_t = cf'(x+ct) \tag{90}$$
$$u_x = f'(x+ct) \tag{91}$$

Substituting into the wave equation:

$$u_{tt} + c^2 u_{xx} = c^2 f''(x+ct) = h(x+ct) \tag{92}$$

Thus, we find:

$$f(x) = \int h(x)\, dx \tag{93}$$

The general solution of the wave equation is:

$$u = g(x-ct) + f(x+ct) \tag{94}$$

Alternative Method: Change of Variables

Alternatively, we can solve the equation by changing variables. Let:

$$\xi = x + ct \tag{95}$$
$$\eta = x - ct \tag{96}$$

Then, the derivatives transform as follows:

$$u_t = cu_\xi - cu_\eta \tag{97}$$
$$u_x = u_\xi + u_\eta \tag{98}$$

This transformation simplifies the wave equation to:

$$u_{\xi\eta} = 0 \tag{99}$$

The solution in terms of the new variables is:

$$u(\xi, \eta) = F(\xi) + G(\eta) \tag{100}$$

where F and G are arbitrary functions determined by initial conditions.

Equations and Solutions

$$U_g = U_{se} + U_s = c(U_{ss} - U_{su}) \tag{101}$$
$$U_n = U_g + U_{ng} = c(U_{gy} - U_{nn}) \tag{102}$$
$$U_H = 2(U_{es} + U_{ng}) - 2cU_{gy} \tag{103}$$
$$U_x = 4 + Y_y \tag{104}$$
$$U_{xx} = U_{gg} + U_{ny} + 213 \tag{105}$$

$$0 = U_H - in_x = -45U_{gy} \tag{106}$$

$$\text{KeyO: } \Theta U_g() = \text{fixtc} + \text{gla} \tag{107}$$

$$U_H - cU_{xx} = 0 \tag{108}$$

$$u(x, 0) = f(x) \tag{109}$$

$$u_f(x, 0) = 4(X) \tag{110}$$

Solution

$$u(x, t) = f(x + t) + g(x - ct) \tag{111}$$

Let $t = 0$, then

$$u(x, 0) = f(x) + f(x) \tag{112}$$

$$U^+(x, t) = cf'(x + t) - cg(x - t) \tag{113}$$

Equations and Transformations

$$4(x) = (x + 0) = cf(x) - 38'(x) \tag{114}$$

$$\int (y(x))\,dx = f(x) - g(x) \tag{115}$$

$$f + y = 4 \tag{116}$$

$$f - g = \int 4(x)\,dx \tag{117}$$

$$f = k + \int t(y(x))\,dx \tag{118}$$

$$g = d - \int t(4(x))\,dx \tag{119}$$

Rewrite

$$f(x) = za(x) + So \tag{120}$$

$$g(x) = 1P(x) - E \tag{121}$$

Function Transformation

$$u(x+) = f(x + H) + g(x - t) \tag{122}$$

$$= (d(x + t) + q(x - Ct)) + y(s)\,ds \tag{123}$$

d'Alembert Formula

$$\int (d(x + t) + q(x - Ct)) + y(s)\,ds \tag{124}$$

Wave Equation and d'Alembert's Solution

Assume that $u = u(x,t)$ is a function of x and t. The wave equation is given by

$$u_{tt} - c^2 u_{xx} = 0. \tag{125}$$

General Solution

The general solution is given by

$$u(x,t) = f(x+ct) + g(x-ct), \tag{126}$$

where f and g are functions of one variable.

Initial Value Problem

For the initial value problem

$$\begin{align} u_{tt} - c^2 u_{xx} &= 0, \tag{127} \\ u(x,0) &= \phi(x), \tag{128} \\ u_t(x,0) &= \psi(x), \tag{129} \end{align}$$

we have the d'Alembert formula

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds. \tag{130}$$

Example 1

Assume $\phi(x) = 0$ and $\psi(x) = \cos x$. Then

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\, ds \\ &= \frac{1}{2}\int_{x-t}^{x+t} \cos s\, ds \\ &= \frac{1}{2}\left[\sin(x+t) - \sin(x-t)\right] \\ &= \sin x \cos t + \cos x \sin t. \end{aligned}$$

Example

The Plucked String

Assume $\phi(x) = 0$ and $\psi(x) = 1$. Then

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} 1\, ds \\ &= \frac{1}{2c}\left[(x+ct) - (x-ct)\right] \\ &= t. \end{aligned}$$

Example 3/Ex 11

Find the general solution of

$$3u'' + 10(x+3)u = \sin(x+\pi). \tag{131}$$

Solution

The linear operator is

$$\mathcal{L} = 3\frac{d^2}{dx^2} + 10(x+3). \tag{132}$$

So the equation can be written as

$$\mathcal{L}u = \sin(x+\pi). \tag{133}$$

To find a particular solution, we assume

$$u_p(x) = A\sin(x+\pi) + B\cos(x+\pi). \tag{134}$$

General Solutions of Differential Equations

We need to find general solutions of the differential equation $Lu = 0$.

$$UH - U_{xx} = 0, \tag{135}$$
$$3UH + 10U_x + 3U_{xx} = 0. \tag{136}$$

Let $v = (3 + 2)u$. Then we have:

$$30 + U_x = 0, \tag{137}$$
$$v = f(3x - t), \tag{138}$$
$$u = g(3t - x) + H(3x - t). \tag{139}$$

Causality and Energy

$$-TU_{xx} = 0 \tag{140}$$

where f, T are constants.

If we take $c = E$, then

$$-YU_x = 0 \tag{141}$$

Define Kinetic Energy E_{mr} as follows:

$$\mathrm{KE} = \int U\,dy \tag{142}$$
$$= \int UE\,dx \tag{143}$$
$$= \int (fU)\,dx \tag{144}$$
$$= p(u + u + dx) \tag{145}$$
$$= u + u_{xx}\,dx \tag{146}$$
$$= \int u + du_x \tag{147}$$
$$= TTu_x \tag{148}$$
$$= -T(U_xU_x + dX) \tag{149}$$

Potential Energy and Mechanical Energy Conservation

The potential energy is denoted by U. Then the total energy, which is the sum of kinetic energy (KE) and potential energy (PE), is given by:

$$\text{Total Energy} = \text{KE} + \text{PE} = 0$$

This defines the mechanical energy to be:

$$\text{Mechanical Energy} = \text{KE} + \text{PE}$$

Thus, we have the mechanical energy conservation equation:

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} = 0$$

with boundary conditions:

$$u(0) = q(x) = b$$

and

$$U(x, 0) = 0$$

The energy equation can be expressed as:

$$E = \int \left(\frac{1}{2} yup + \frac{\partial U}{\partial x} \right) dx$$

Mathematical Expressions

$$
\begin{aligned}
T(u) &= \int u(x)\,dx \\
a+b &= o \\
I + \int d(x)\,dx &= b \\
-b &< x < a \\
4(x) &= 2 \\
b+t &= -a \\
x_0 &= 12a \\
P(x) &= L \\
-a &< x < 0 \\
|x| &< a \\
2b^2 & \\
\int k\,dx &= 2 \\
u(x) &= d(x) + t \\
u(x) &= c + x \\
E(f) &= KE + PE \\
&= \int (i + u(x))\,dx
\end{aligned}
$$

$$
\begin{aligned}
U_t &= z(k'(x+c^+) - k'(x-t)), && (150) \\
J_u F &= \int (k'(x+c^+) - q'(x-c^+))\,dx, && (151) \\
J_u x &= \int (d'(x+c^+) + q'(x+c))\,dx, && (152) \\
E(t) &= \int (d'(x+c^+) + q'(x-c))\,dx && (153) \\
&= \int (k'(x+c^+) - c)(x-t)\,dx, && (154) \\
1'(x+c^+)\,dx &= H, && (155) \\
'(y)\,dy &= i, && (156) \\
E(CH) &= 2c\left(\int d\,dy - A\right). && (157)
\end{aligned}
$$

The Diffusion Equation

Equation

$$u_t = Ru_{xx} \tag{158}$$

where $u(x,t)$ is a function on the rectangular domain $[0,e] \times [0,T]$.

Maximum Principle

Theorem (Strong Maximum Principle)

The maximum value of $u(x,t)$ can only be reached on the lines:

$$\begin{aligned} t &= 0, \\ x &= 0, \\ x &= e, \\ t &= T. \end{aligned}$$

Theorem (Weak Maximum Principle)

$$\max u(x,t) = \max u(x,0) \tag{159}$$

Proof

What if at some point in the interior of $\mathbb{R}$ that reaches the maximum value of $U(x,y)$, then (x_0, y_0) satisfies $U_x(x_0, y_0) = 0$.

The Hessian matrix is given by:

$$\begin{bmatrix} U_{xx} & U_{xy} \\ U_{xy} & U_{yy} \end{bmatrix}$$

For a maximum, the Hessian must be negative definite, which is equivalent to:

$$U_{xx} < 0 \quad \text{and} \quad U_{xx}U_{yy} - U_{xy}^2 > 0$$

At (x_0, y_0), we have:

$$U_{xx}U_{yy} - U_{xy}^2 > 0$$

Let $v(x,t) = u(x,t) + 2x^2$. Then:

$$u_x = RU_x = RU_{xx} - 2kE$$

And:

$$U_{xx} + 2k$$

This leads to a contradiction if $U(x_0, y_0) \neq 0$.

Assumptions and Maximum Conditions

Assume (X_0, Y_0) is an interior point such that I reaches a maximum. Then

$$U_t(X_0, Y_0) = 0, \tag{160}$$
$$U_{xx}(X_0, Y_0) \neq 0. \tag{161}$$

This leads to a contradiction:

$$0 = V(X_0, Y_0) = RV_{xx} - 2k^2 - 2\delta s_0. \tag{162}$$

Maximum Value Analysis

Let (X_0, Y_0) be the maximum point of v. Then

$$\text{Max } v(x,t) = \text{Max } u(x,t) + 2. \tag{163}$$

Thus, we have

$$U(x,t) = v(x,t) - Ex, \tag{164}$$
$$\text{Max } u(x,t) < \text{Max } U(x,t). \tag{165}$$

Finally, we conclude

$$R = \text{Max } v(x,t) - \text{Max } U_t + 3. \tag{166}$$

Uniqueness of the Diffusion Equation

Theorem

Consider the diffusion equation:

$$
\begin{aligned}
u_t &= Ru_{xx}, \\
u(x,0) &= f(x), \\
u(0,t) &= g(t), \\
u(l,t) &= h(t).
\end{aligned}
$$

Then the solution is unique.

Proof

Let u_1 and u_2 be two solutions. Define

$$v(x,t) = u_1(x,t) - u_2(x,t).$$

Then $v(x,t)$ satisfies:

$$
\begin{aligned}
v_t &= Rv_{xx}, \\
v(x,0) &= 0, \\
v(0,t) &= 0, \\
v(l,t) &= 0.
\end{aligned}
$$

The maximum principle implies that

$$\max v(x,t) = \max v(x,0) = 0.$$

Thus, $v(x,t) = 0$ for all x and t, which implies $u_1(x,t) = u_2(x,t)$. Therefore, the solution is unique.

Uniqueness Theorem for Wave Equation

Consider the wave equation:

$$U_{tt} - c^2 U_{xx} = 0 \tag{167}$$

with initial conditions:

$$
\begin{aligned}
u(x,0) &= 0, \qquad (168) \\
u_t(x,0) &= 0. \qquad (169)
\end{aligned}
$$

We define the energy function $E(t)$ as:

$$E(t) = \int \left(U_t^2 + c^2 U_x^2 \right) \, dx \tag{170}$$

At $t = 0$, the energy is:

$$E(0) = 0 \tag{171}$$

Since the energy $E(t)$ is constant over time, we have:

$$E(t) = \text{const} = 0 \tag{172}$$

Thus, $U_t = 0$ and $U_x = 0$, implying that the solution is unique and zero everywhere.