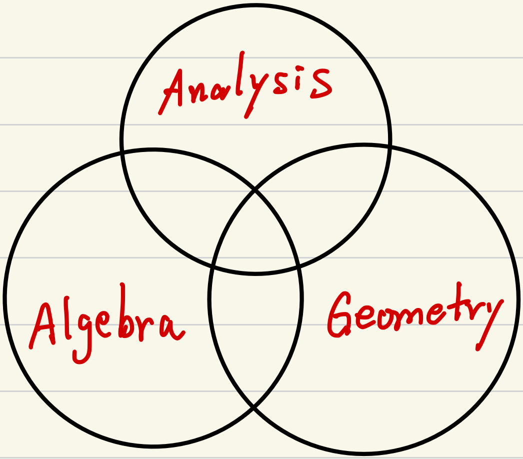
Week 1 What is PDE?



Calculus (2ABDE, 140ABC, 3D)

$\boxed{90\%}$ Analysis = PDE

PDE = Partial Differential Equation

ODE = Ordinary Differential Equations

What is ODE?

and its derivatives

An equation contains unknown function(s) of 1-variable.

$$x^2 + 2x + 3 = 5e^x$$

$$f^2(x) + \sin f(x) = 5 \quad (\text{Not an ODE})$$

$f'(x) = \sin f(x)$ (Yes. it is an ODE)

$f'(x) = \sin x$ Yes! ODE

What is PDE?

A equation contains unknown functions and their partial derivatives is called a PDE.

Suppose we have a function $u(x,y)$.

$$\frac{\partial u}{\partial x}(x,y), \quad u_x(x,y), \quad u_x$$

Example: A first order PDE can be written as

$$F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0.$$

$$F(x, y, u, u_x, u_y) = 0$$

Examples of PDE: u is a function of (x,y)

1. $u_x + u_y = 0$ (Transport Equation)
2. $u_x + y u_y = 0$ (Transport Equation)
3. $u_x + u u_y = 0$ (Shock Wave Equation)
4. ✓ $u_{xx} + u_{yy} = 0$ (Laplace Equation)

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

5. ✓ $u_{tt} - u_{xx} + u^2 = 0$ (Wave equation w/ interaction)
6. $u_t + u u_x + u_{xxx} = 0$ (Dispersive equation)
7. $u_{tt} + u_{xxxx} = 0$ (Vibration bar)
8. $u_t - \sqrt{-1}\, u_{xx} = 0$ (Quantum Mechanics)

parabolic equation $u_t = u_{xx}$

Linear and Nonlinear Equations

In linear algebra, $T: V \to W$ is called a linear transformation, if

① $T(u+v) = T(u) + T(v)$
② $T(au) = a\,T(u)$

Or

$$T(au+bv) = aT(u) + bT(v)$$

Definition An operator is called linear,
differential

if ① $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$
② $\mathcal{L}(au) = a\,\mathcal{L}(u)$

Example: ① $\mathcal{L} = \frac{\partial}{\partial x}$, $\mathcal{L}(u) = \frac{\partial u}{\partial x}$

② $\mathcal{L} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. $\mathcal{L}(u) = 0 \Leftrightarrow$

$u_x + y\,u_y = 0$: Transport equation

A linear PDE is written as

$$\mathcal{L}u = f(x) \qquad Ax = b$$

$$\mathcal{L}u = f(x)$$

homo $\mathcal{L}u = 0$

$$u = u_g + u_s$$

where: u_g is a general solution of $\mathcal{L}u = 0$

u_s is a special solution of

$$\mathcal{L}u = f$$

$$Ax = b.$$

$$Ax = 0.$$

$$x = x_g + x_s$$

x_g is a general solution of $Ax = 0$

x_s is a special solution of

$$Ax = b$$

proposition Let $u_1, u_2, \cdots, u_n$ be solutions of

$$\mathcal{L}u = 0$$

Then

$$u(x) = C_1 u_1 + C_2 u_2 + \cdots + C_n u_n = \sum_{j=1}^{n} C_j u_j$$

is also a solution of $\mathcal{L}u = 0$.

$$\frac{\partial u}{\partial x}(x,y) \qquad \frac{\partial u}{\partial x} \qquad u_x \qquad \frac{u_x(\sin y, \cos z)}{\frac{\partial u}{\partial x_1}(\sin y, \cos z)}$$

$$A \qquad (a_1, \cdots a_n) \qquad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1}, & a_{m2}, & \cdots & a_{mn} \end{pmatrix} \qquad (a_{ij})$$

Simplest ODE:

$$u'(t) = f(t)$$

$$u(t) = \int f(t)\,dt + C$$

$$u'(t) = 0, \qquad u(t) = C_1$$

$$u''(t) = 0, \qquad u(t) = C_1 t + C_2$$

Example 1. $u_{xx} = 0$ $\qquad \frac{\partial^2 u}{\partial x^2}(x,y) = 0$

$$\frac{\partial}{\partial x}(u_x) = 0.$$

$$u_x = F(y) \qquad \frac{\partial u}{\partial x}(x,y) = F(y)$$

$$u(x,y) = x F(y) + G(y)$$

Example 2. $u_{xx} + u = 0$ $u'' + u = 0.$

$$u(x,y) = C_1(y)\cos x + C_2(y)\sin x$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) + u(x,y) = 0.$$

Example 3. $u_{xy} = 0$

$$\boxed{u_{xy} = u_{yx}}$$

$$\frac{\partial}{\partial y}(u_x) = 0$$

$$u_x = F(x)$$

$$u = \int F(x)\,dx + \underline{G(y)}$$

$$= H(x) + G(y).$$

List of Calculus Facts that are useful.

1. partial derivatives are local.
2. $u_{xy} = u_{yx}$
3. (Chain rules)

$$\frac{\partial}{\partial x}(f(g(x,t))) = f'(g(x,t)) \cdot \frac{\partial g}{\partial x}(x,t)$$

$$\frac{\partial f}{\partial x} = f' \cdot g_x$$

special case of chain rule.

Assume

$$f = f(y_1, y_2, \cdots, y_n)$$

$$y_i = g_i(x_1, \cdots, x_k) \qquad 1 \le i \le n$$

$$\frac{\partial f}{\partial x_j} = \sum_{e=1}^{n} \frac{\partial f}{\partial y_e} \cdot \frac{\partial g_e}{\partial x_j}$$

most general form of chain rule.

4. Green's formula (later).

$$\boxed{\begin{aligned} &\left(\int_0^x f(t)\,dt\right)' = f(x) \\ &\left(\int_0^{b(x)} f(t)\,dt\right)' = f(b(x)) \cdot b'(x) \end{aligned}}$$

5. $$I(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

$$I'(t) = f(b(t),t) \cdot b'(t) - f(a(t),t)\,a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,dx$$

6. Jacobian $F: \mathbb{R}^n \to \mathbb{R}^m$ (later)

$$\left(\frac{\partial F_i}{\partial x_j}\right)$$

where $F = (F_1, \dots, F_m)$

7. Infinite series (later)

8. Directional derivative (geometry)

$u(x,y)$ (u_x, u_y)

$a u_x + b u_y$ as the directional derivative in (a,b) direction.

9. Familiar with Math 3D.

Example. 1 (Transport equation)

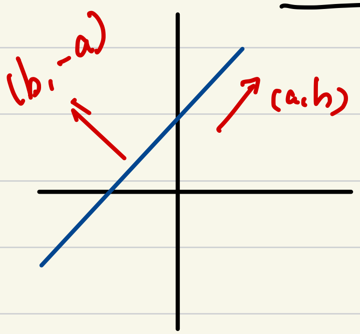
$$a u_x + b u_y = 0.$$

where a, b are constants. $a^2 + b^2 \neq 0$

Method 1 (Geometry)

gradient of u.

$$\underline{(a, b) \cdot (u_x, u_y) = 0}$$



Equation for the line is

$$bx - ay = C$$

Solution is $F(c)$

$$u(x, y) = F(bx - ay).$$

Check:

$$u_x = F' \cdot b$$
$$u_y = F' \cdot (-a)$$

$$a u_x + b u_y = a F' \cdot b + b \cdot F'(-a) = 0.$$

Method 2 (Change of variable)

Let

$$x' = ax + by$$
$$y' = bx - ay$$

$$u_x = u_{x'} \cdot a + u_{y'} b = a u_{x'} + b u_{y'}$$

$$u_y = u_{x'} b + u_{y'} (-a) = b u_{x'} - a u_{y'}$$

$$0 = a u_x + b u_y = (a^2 + b^2) u_{x'}$$

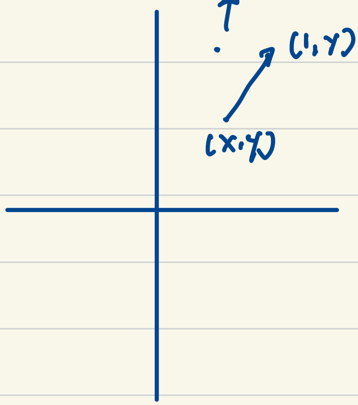
$$\Rightarrow \quad u_{x'} = 0$$

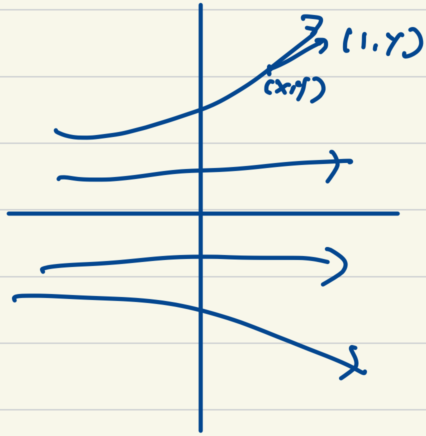
$$u = F(y') = F(bx - ay)$$

Example 2. $u_x + y u_y = 0.$

It is linear $\mathcal{L} = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ $\quad \mathcal{L} u = 0.$

$$(1, y) \cdot (u_x, u_y) = 0$$





$$y' = \frac{y}{1} \qquad y = ce^x$$

$u(x, y)$ along the curve

$$\frac{d}{dx}u(x, ce^x)$$

$$= u_x + u_y(ce^x)$$

$$= u_x + yu_y = 0.$$

$$u(x, ce^x) = u(0, c) = F(c)$$

$$u(x, y) = F(ye^{-x})$$

$$y' = \frac{b}{a}, y = \frac{b}{a}x + c.$$

$$aux + buy$$

$$\frac{\partial}{\partial x}\left(u\left(x, \frac{b}{a}x + c\right)\right) = 0.$$

$$u\left(x, \frac{b}{a}x + c\right) = u(0, c)$$

$$= F(c) = F\left(y - \frac{b}{a}x\right)$$

$$= \tilde{F}(bx - ay)$$

Review.

Example : $a u_x + b u_y = 0.$ $a^2+b^2 \neq 0.$

if we consider the equation

$$y' = \frac{b}{a} \quad (*)$$

$$y = \frac{b}{a} x + C \quad *$$

$$\frac{d}{dx} u\left(x, \frac{b}{a} x + c\right)$$

$$= u_x + u_y \cdot y'$$

$$= u_x + \frac{b}{a} u_y = 0$$

$$u\left(x, \frac{b}{a} x + c\right) = u(0, c)$$

$$u(x, y) = u(0, C)$$

$$= u\left(0, y - \frac{b}{a} x\right)$$

$$= F\left(y - \frac{b}{a} x\right)$$

$$= G(ay - bx)$$

$$= H(bx - ay)$$

Example 2.

$$u_x + y\, u_y = 0.$$

Consider $y' = \frac{y}{1} = y$

$$y = Ce^{x} \Rightarrow C = ye^{-x}$$

$$\frac{d}{dx} u(x, Ce^{x})$$

$$= u_x + u_y\, Ce^{x} = u_x + y\, u_y = 0$$

$$u(x, Ce^{x}) = u(0, C)$$

$$= u(0, ye^{-x})$$

$$= F(ye^{-x})$$

Example 3 In the previous example, if in addition, $u(0, y) = y^3$. Then what is the solution?

Solution. $u(x, y) = F(ye^{-x})$

$$y^3 = u(0, y) = F(y)$$

Then the solution is

$$u(x,y) = F(ye^{-x}) = (ye^{-x})^3$$
$$= y^3 e^{-3x}$$

Remark: How to solve $y' = y$.

① Integrating factor.

$$(e^{-x}y)' = e^{-x}(y'-y) = 0.$$

$$e^{-x}y = c \Rightarrow \underline{y = ce^{x}}$$

② $\dfrac{dy}{dx} = y \Rightarrow \dfrac{dy}{y} = dx$

$\boxed{\log = \ln}$

$$\log|y| = x + C$$
$$|y| = e^{x+c}$$
$$y = \pm e^{x+c} = \pm e^{c} e^{x}$$

Missing solution $y = 0$.

$$\Rightarrow y = c_1 e^{x}$$

Example 4. $u_x + 2xy^2 u_y = 0.$

Solution: $y' = \frac{2xy^2}{1} = 2xy^2$

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{y^2} = 2x\,dx \qquad \text{(Missing } y=0\text{)}$$

$$-\frac{1}{y} = x^2 + C.$$

$$y = -\frac{1}{x^2+C} \qquad C \in \mathbb{R}$$

One more $y = 0$

$$u\left(x, -\frac{1}{x^2+C}\right) = \text{const}$$

$$= u\left(0, -\frac{1}{C}\right)$$

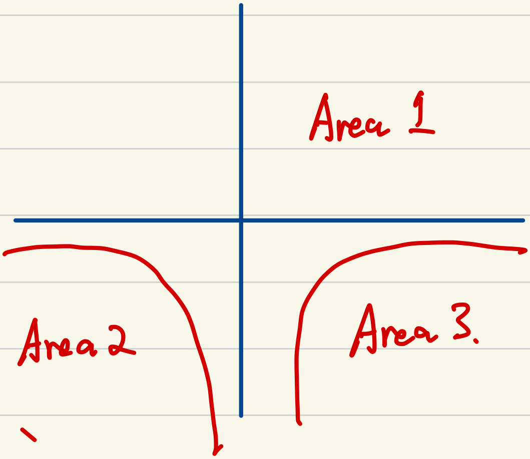
$$y = -\frac{1}{x^2+C} \Rightarrow x^2+C = -\frac{1}{y}$$

$$\Rightarrow C = -\frac{1}{y} - x^2$$

$$u(x,y) = u\left(0, -\frac{1}{C}\right) = u\left(0, \frac{1}{\frac{1}{y}+x^2}\right)$$

$$= u\left(0, \frac{y}{1+x^2y}\right) = F\left(\frac{y}{1+x^2y}\right)$$

$1+x^2y=0$. $\quad y=-\frac{1}{x^2}$ The function $u(x,y)$ is NOT defined on the curve $y=-\frac{1}{x^2}$



In general, if

$$a(x,y)\,u_x + b(x,y)\,u_y = 0. \qquad (A)$$

The ODE: $$y' = \frac{b(x,y)}{a(x,y)}$$

Implicit Differentiation: Assume $y=y(x)$ is implicitly defined by the equation

$$u(x,y)=0 \Leftrightarrow u(x,y(x))=0$$

$$y' = -\frac{u_x}{u_y}. \Rightarrow \begin{cases} u_x + y'u_y = 0. \\ u_x + \frac{b(x,y)}{a(x,y)}u_y = 0. \end{cases}$$

(A) is equivalent to

$$y' = \frac{b(x,y)}{a(x,y)}$$

Ex 7 of Section 1.2

ⓐ. Solve $y u_x + x u_y = 0$

$u(0,y) = e^{-y^2}$.

ⓑ In which region of $\mathbb{R}^2$ is the solution uniquely determined

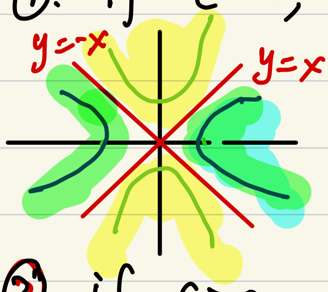
Solution. ⓐ $y' = \frac{x}{y} \Rightarrow yy' = x$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$. $y^2 - x^2 = 2c$

Rewrite

$$y^2 - x^2 = C$$

①. if $c=0$, then $y^2 - x^2 = 0 \Rightarrow y = \pm x$



$$\boxed{u(x,y) = e^{y^2-x^2} \text{ on the yellow regions.}}$$

② if $c>0$, then $y^2 - x^2 = 1$

③ if $c<0$, then $y^2 - x^2 = -1$

ⓑ.

① if $c>0$, $y = \pm\sqrt{x^2 + C}$

$u(x, \pm\sqrt{x^2+c}) = u(0, \pm\sqrt{c}) = e^{-c}$

$$u(x,y) = e^{-c} = e^{y^2-x^2} \qquad y^2-x^2>0. \quad |y| \geq |x|$$

On the other hand, On the green region, we have $y^2-x^2 = c < 0$

$$x = \pm\sqrt{y^2-c}$$

$$u(\pm\sqrt{y^2-c}, y) = u(\pm\sqrt{-c}, 0)$$

$$= u(\pm\sqrt{x^2-y^2}, 0)$$

On the blue region,

$$u(x,y) = F(\sqrt{x^2-y^2})$$

On the green region

$$u(x,y) = F(-\sqrt{x^2-y^2}) = G(\sqrt{x^2-y^2})$$

Sometimes we call the course

$$\boxed{\text{Mathematical Physical Equations}}$$

1. Wave equations — hyperbolic Eqs
2. Heat equations — parabolic Eqs
3. Laplace equations — elliptic Eqs.

Chapter 2 Wave and Diffusion

§2.1 The wave equation.

Assume $u = u(t, x)$

$$u_{tt} = c^2 u_{xx} \qquad \begin{matrix} -\infty < x < \infty \\ t > 0. \end{matrix}$$

Theorem The general solutions of $u(x,t)$ are

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f, g are arbitrary functions.

proof:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u}_{u_t + c u_x = v} = 0.$$

$$\frac{\partial}{\partial t}(u_t + c u_x) = u_{tt} + c u_{xt}$$

$$\frac{\partial}{\partial x}(u_t + c u_x) = u_{tx} + c u_{xx}$$

if $\quad v = u_t + c u_x \iff$

$$v_t - c v_x = 0.$$

$$\Rightarrow \quad v(x,t) = h(x+ct)$$

$$u_t + c u_x = h(x+ct) \qquad (*)$$

① Find general solution of the homogeneous equation, which is

$$u_t + c u_x = 0$$

$$u = g(x-ct)$$

② Find a particular solution of (*)

Assume $u = f(x+ct)$.

$$u_t = c f'(x+ct)$$

$$u_x = f'(x)$$

$$u_t + c u_x = 2c\, f'(x+ct) = h(x+ct)$$

$$f(x) = \frac{1}{2c}\int h(x)\,dx$$

$$u = g(x-ct) + f(x+ct)$$ ⟵ general solution of the wave eq.

Alternatively, we can solve the equation by changing of variables. We let

$$\xi = x+ct, \quad \eta = x-ct$$

$$u_t = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = u_\xi \cdot c + u_\eta(-c)$$

$$= c(u_\xi - u_\eta)$$

$$\frac{\partial}{\partial t} u_\xi = u_{\xi\xi}\frac{\partial \xi}{\partial t} + u_{\xi\eta}\cdot\frac{\partial \eta}{\partial t} = c(u_{\xi\xi} - u_{\xi\eta})$$

$$\frac{\partial}{\partial t} u_\eta = u_{\eta\xi}\frac{\partial \xi}{\partial t} + u_{\eta\eta}\cdot\frac{\partial \eta}{\partial t} = c(u_{\xi\eta} - u_{\eta\eta})$$

$$u_{tt} = c^2(u_{\xi\xi} + u_{\eta\eta}) - 2c^2 u_{\xi\eta}$$

$$u_x = u_\xi + u_\eta$$

$$u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$$

$$0 = u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}$$

$$\Rightarrow u_{\xi\eta} = 0.$$

$$u = f(\xi) + g(\eta) = f(x+ct) + g(x-ct)$$

Initial value problem.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \qquad -\infty < x < \infty$$

Solution :

$$u(x,t) = f(x+ct) + g(x-ct)$$

Let $t=0$

$$\phi(x) = u(x,0) = f(x) + g(x)$$

$$u_t(x,t) = c f'(x+ct) - c g'(x-ct)$$

$$\psi(x) = u_t(x,0) = c f'(x) - c g'(x)$$

$$\frac{1}{c}\int \psi(x)\,dx = f(x) - g(x)$$

$$f + g = \phi$$
$$f - g = \frac{1}{c}\int \psi(x)\,dx$$

$$f = \frac{1}{2}\left(\phi + \frac{1}{c}\int \psi(x)\,dx\right)$$
$$g = \frac{1}{2}\left(\phi - \frac{1}{c}\int \psi(x)\,dx\right)$$

Rewrite

$$f(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds$$
$$g(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds$$

$$u(x,t) = f(x+ct) + g(x-ct)$$
$$= \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right)$$
$$+ \frac{1}{2c}\left(\int_0^{x+ct} \psi(s)\,ds - \int_0^{x-ct} \psi(s)\,ds\right)$$
$$= \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right)$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

d'Alembert formula.

Assume that $u = u(x,t)$ be a function of $x \in \mathbb{R}$, $t > 0$. The wave equation is given by

$$u_{tt} - c^2 u_{xx} = 0.$$

1. The general solution is given by

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f, g are functions of 1-variable.

2. For the initial value problem

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

Then we have the d'Alembert formula

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Example 1. Assume $\phi(x) = 0$

$\psi(x) = \cos x.$

Then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \cos s\,ds$$

$$= \frac{1}{2c}\left(\sin(x+ct) - \sin(x-ct)\right)$$

$$= \frac{1}{2c}\Big(\cancel{\sin x\cos ct} + \cos x \sin ct$$

$$- \cancel{\sin x \cos ct} + \cos x \sin ct\Big)$$

$$= \frac{1}{c}\cos x \sin ct$$

Example. The Plucked string

Assume. $\psi(x) = 0$

$$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & |x| < a \\ 0 & |x| > a \end{cases}$$

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right)$$

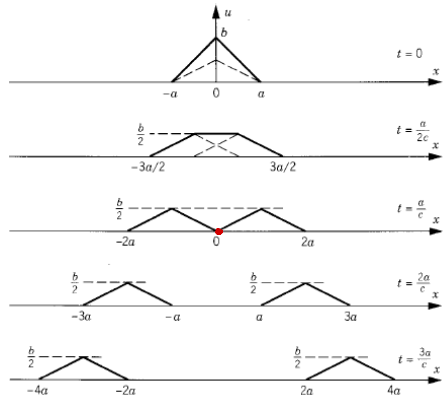


Figure 2

picture $\phi(x)$

Example 3 (Ex 11)

Find the general solution of

$$3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$$

Solution: The linear operator

$$\mathcal{L} = 3\frac{\partial^2}{\partial t^2} + 10\frac{\partial^2}{\partial x\partial t} + 3\frac{\partial^2}{\partial x^2}$$

So the equation can be written as

$$\mathcal{L}u = \sin(x+t).$$

1. To find a special solution, we assume

$$u(x,t) = C_1 \sin(x+t) + C_2 \cancel{\cos(x+t)}$$

$$u_{tt} = -C_1 \sin(x+t) = u_{xt} = u_{xx}$$

$$\mathcal{L}(u) = -16 C_1 \sin(x+t) = \sin(x+t)$$

$$u(x,t) = -\frac{1}{16}\sin(x+t)$$

2. We need find general solutions of

$$\mathcal{L}u = 0.$$

$$3u_{tt} + 10u_{xt} + 3u_{xx} = 0$$

$$\mathcal{L} = 3\frac{\partial^2}{\partial t^2} + 10\frac{\partial^2}{\partial x \partial t} + 3\frac{\partial^2}{\partial x^2}$$

$$= \left(3\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right)$$

$$u_{tt} - u_{xx} = 0$$

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

$$= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) \cdot \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$

Let $v = \left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right)u.$

$$\left(3\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}\right)v = 0$$

$$3v_t + v_x = 0.$$

$$v = f(3x - t)$$

$$\underline{u_t + 3u_x} = v = f(3x - t)$$

$$u = g(3t - x) + H(3x - t)$$

§ 2.2 Causality and energy

$$\rho u_{tt} - T u_{xx} = 0$$

where $\rho, T > 0$ constants. If we take

$$c = \sqrt{\frac{T}{\rho}} \qquad u_{tt} - c^2 u_{xx} = 0.$$

Define kinetic energy $\frac{1}{2} m v^2$

$$KE = \frac{1}{2} \int \rho u_t^2 \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \rho u_t^2 \, dx$$

$$\frac{dKE}{dt} = \frac{\partial}{\partial t} \int \frac{1}{2} \rho u_t^2 \, dx$$

$$= \int \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u_t^2 \right) dx$$

$$= \rho \int u_t u_{tt} \, dx$$

$$= T \int_{-\infty}^{\infty} u_t u_{xx} \, dx$$

$$= T \int_{-\infty}^{\infty} u_t \, d u_x$$

$$= \xcancel{T u_t u_x \Big|_{-\infty}^{\infty}}^{\,0} - T \int_{-\infty}^{\infty} u_x \, d u_t$$

$$= - T \int_{-\infty}^{\infty} u_x u_{xt} \, dx$$

$$= -\frac{1}{2} T \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (u_x^2)\, dx$$

$$= - \frac{\partial}{\partial t} \underline{\int (\frac{1}{2} T) u_x^2 \, dx}$$

potential energy

$$P \qquad \int \frac{1}{2} T u_x^2 \, dx$$

Then

$$\frac{d}{dt} (KE + PE) = 0$$

Defines the mechanical energy to be

$$KE + PE$$

Then we have the

Mechanical energy conservation Law.

Example plucked String.

$$\begin{cases} \rho u_{tt} - T u_{xx} = 0 \\ u(x,0) = \phi(x) = \begin{cases} b - \frac{b|x|}{a} & |x| < a \\ 0 & |x| > a \end{cases} \\ u_t(x,0) = 0 \end{cases}$$

$$E = \frac{1}{2} \int (\rho u_t^2 + T u_x^2)\, dx$$

$$= \frac{1}{2} T \int u_x^2 \, dx \qquad \text{at } t=0$$

$$= \frac{1}{2} T \int \phi'(x)^2 \, dx$$

$$\phi(x) = \begin{cases} b - \frac{bx}{a} & 0<x<a \\ b + \frac{bx}{a} & -a<x<0 \\ 0 & |x|>a. \end{cases}$$

$$\phi'(x) = \begin{cases} -\frac{b}{a} & 0<x<a \\ \frac{b}{a} & -a<x<0 \\ 0 & |x|>a \end{cases}$$

$$\int \phi'(x)^2 dx = \frac{b^2}{a^2} \cdot 2a = \frac{2b^2}{a}.$$

$$E = \frac{1}{2} T \cdot \frac{2b^2}{a} = \frac{Tb^2}{a}.$$

Assume $\psi(x)=0$. $\qquad u_{tt} - c^2 u_{xx} = 0$

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right).$$

$$E(t) = KE + PE$$

$$= \frac{1}{2} \int \left(u_t^2 + c^2 u_x^2\right) dx$$

$$u_t = \frac{c}{2}\left(\phi'(x+ct) - \phi'(x-ct)\right).$$

$$\frac{1}{2}\int u_t^2 = \frac{c^2}{2}\int \left(\phi'(x+ct) - \phi'(x-ct)\right)^2 dx$$

$$\frac{c^2}{2}\int u_x^2 = \frac{c^2}{2}\int \left(\phi'(x+ct) + \phi'(x+ct)\right)^2 dx$$

$$E(t) = \frac{c^2}{2}\int 2\left(\left(\phi'(x+ct)\right)^2 + \left(\phi'(x-ct)\right)^2\right) dx$$

$$= c^2\int \left(\phi'(x+ct)\right)^2 dx$$

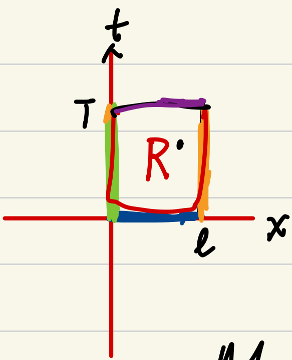
$$+ c^2\int \left(\phi'(x-ct)\right)^2 dx$$

$$\int_{-\infty}^{+\infty} \phi'(x+ct)^2 dx \overset{y=x+ct}{=\!=} \int_{\mathbb{R}} \phi'(y)^2 dy$$

$$= \int_{\mathbb{R}} \phi'(x)^2 dx$$

$$E(t) = 2c^2\int_{-\infty}^{\infty} \phi'(x)^2 dx$$

§2.3 The Diffusion Equation. (heat equation)



$$u_t = k u_{xx}$$

$u(x,t)$ is a function on the rectangular $[0,\ell] \times [0,T)$

Maximum Principle.

Theorem The maximum value of $u(x,t)$ can only be reached on lines

$$\left.\begin{array}{l} \{(x,t) \mid t=0,\ 0 \le x \le \ell\} \\ \{(x,t) \mid 0 \le t \le T,\ x=0\} \\ \{(x,t) \mid 0 \le t \le T,\ x=\ell\} \end{array}\right\} S.$$

(Strong maximum principle).

Theorem (Weak maximum principle).

$$\max_R u(x,t) = \max_S u(x,t).$$

proof: What if at (x_0, y_0) in the interior of $\mathbb{R}$ that reaches the maximum value of $u(x,t)$

Then

① $u_t(x_0, y_0) = u_x(x_0, y_0) = 0.$

②. Hessian

$$\begin{pmatrix} u_{xx} & u_{xt} \\ u_{xt} & u_{tt} \end{pmatrix}$$

inonpositive definite. which is equivalent to

ⓐ $u_{xx} \leq 0$, $u_{tt} \leq 0$ at (x_0, y_0)

ⓑ $u_{xx} u_{tt} - u_{xt}^2 \geq 0.$

$$u_t = k u_{xx}$$

$$0 = u_t(x_0, y_0) = k\, u_{xx}(x_0, y_0) \leq 0$$

if $u_{xx}(x_0, y_0) < 0$. we get a contradiction

Let

$$v(x,t) = u(x,t) + \varepsilon x^2$$

$$\boxed{v_t = u_t = k u_{xx} = k v_{xx} - 2k\varepsilon}$$

$$v_{xx} = u_{xx} + 2\varepsilon$$

Assume $(x_0, y_0) \in R$ is an interior point such that v reaches maxiumum. Then

$$v_t(x_0, y_0) = 0$$

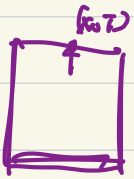
$$v_{xx}(x_0, y_0) \le 0.$$

Contradiction

$$0 = v_t(x_0, y_0) = k v_{xx} - 2k\varepsilon \le -2k\varepsilon < 0.$$

$$\max_R v(x,t) = \max_S v(x,t)$$

Let (x_0, T) be the maxium point of $v(x,t)$



$$v_t = \lim_{t \to 0^+} \left(-\frac{1}{t}\right)\left(u(x_0, T) - u(x_0, T-t)\right)$$

$$\ge 0.$$

$$0 \le v_t = k v_{xx} - 2k\varepsilon \le 2k\varepsilon < 0.$$

$$v(x,t) = u(x,t) + \underline{\varepsilon x^2}$$

$$u(x,t) = v(x,t) - \varepsilon x^2$$

$$\Rightarrow \quad \max_R u(x,t) \le \max_R v(x,t)$$

$$= \max_S v(x,t)$$

$$\max_S v(x,t) \le \max_S u(x,t) + \varepsilon l^2$$

$$\underset{R}{\text{Max}}\, u(x,t) \leq \underset{S}{\text{Max}}\, u(x,t) + \varepsilon l^2$$

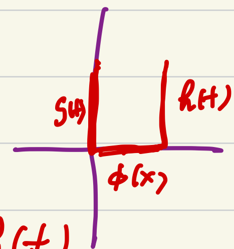
Let $\varepsilon \to 0$. $\Rightarrow$

$$\underset{R}{\text{Max}}\, u(x,t) \leq \underset{S}{\text{Max}}\, u(x,t)$$

$$=$$

Uniqueness of the Diffusion Equation.

Theorem Consider



$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = \phi(x) \\ u(0,t) = g(t), \quad u(l,t) = h(t) \end{cases}$$

Then the solution is unique.

proof: Let u_1 u_2 be the two solutions.

Let $u(x,t) = u_1(x,t) - u_2(x,t)$

$\Rightarrow$

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = 0 \\ u(0,t) = 0, \quad u(l,t) = 0 \end{cases}$$

$$\underset{R}{\text{Max}}\, u = \underset{S}{\text{Max}}\, u = 0$$

Moreover, $-u$ also solves the above equation

$$\underset{R}{\text{Max}}\, (-u) \leq 0. \quad \Rightarrow \quad u \equiv 0$$

Uniqueness theorem for wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

$$E(t) = \frac{1}{2}\int (u_t^2 + c^2 u_x^2)\,dx \equiv 0$$

$E(0) = 0$

$u_t \equiv 0$, $u_x \equiv 0$, $\Rightarrow$ $u(x,t) = \text{const}$
$= 0$