HADAMARD MATRICES IN CENTRALISER ALGEBRAS OF MONOMIAL REPRESENTATIONS

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ABSTRACT. Centraliser algebras of monomial representations of finite groups may be constructed and studied using methods similar to those employed in the study of permutation groups. Guided by results of D. G. Higman and others, we give an explicit construction for a basis of the centraliser algebra of a monomial representation. The character table of this algebra is then constructed via character sums over double cosets. We locate the theory of group-developed and cocyclic-developed Hadamard matrices within this framework. We apply Gröbner bases to produce a new classification of highly symmetric complex Hadamard matrices.
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1. INTRODUCTION

An $n \times n$ complex Hadamard matrix has complex entries of norm 1 and satisfies $MM^* = nI_n$ where M^* denotes the conjugate-transpose of M and I_n is the $n \times n$ identity matrix. Complex Hadamard matrices find applications in combinatorics, signal processing, and quantum information theory; see e.g. Horadam's book [31]. In this paper, we combine combinatorics, representation theory, and computational algebra to construct new complex Hadamard matrices in the centraliser algebra of a suitable group representation.

Let Ω be a finite set, $k > 0$ an integer, and $\mathcal{B}$ a collection of k-subsets of Ω. A permutation $g \in \mathrm{Sym}(\Omega)$ acts on the set of all k-sets in the natural way: $B \subseteq \Omega$ is mapped to $B^g = \{b^g \mid b \in B\}$. The automorphism group of $\mathcal{B}$ is the largest subgroup $G \leqslant \mathrm{Sym}(\Omega)$ that satisfies $B^g \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $g \in G$. Taking $k = 2$, one recovers the automorphism group of a graph, while the automorphism groups of block designs and finite geometries arise by imposing suitable conditions on $\mathcal{B}$. The interplay between algebraic properties of the group G and structural properties of the underlying combinatorial structure is one of the classical topics in algebraic combinatorics. Graphs, designs, and geometries are described by $\{0, 1\}$-matrices, and so it is natural to consider groups of automorphisms that are permutation groups. Our work here can be seen as a generalisation of this theory to combinatorial objects defined over larger alphabets.

A matrix is monomial if it contains a unique non-zero entry in each row and column. The automorphism group of a Hadamard matrix M is defined to be the group G of all pairs of monomial matrices (P, Q) such that $PMQ^* = M$. Since M is invertible, we see that $P = MQM^{-1}$, so the projections $\pi_1(P, Q) = P$ and $\pi_2(P, Q) = Q$ are equivalent representation that satisfy $\pi_1(g)M = M\pi_2(g)$ for all $g \in G$, and so M belongs to the *intertwiner* of π_1 and π_2. In the special case that M satisfies $\pi(g)M = M\pi(g)$ for all $g \in G$ and a representation π, the matrix M belongs to the *centraliser* of π. While an intertwiner carries the structure of a $\mathbb{C}$-vector space, the centraliser is naturally a $\mathbb{C}$-algebra. We show how classical results on associative algebras, induced representations, and character theory of finite groups can be combined to give detailed information about a centraliser algebra, see Section 2.2 for further details. In particular, the eigenvalues of a commutative centraliser algebra are expressed as character sums of G. Since Hadamard matrices are characterised by norm conditions on their entries and eigenvalues, the property of acting by monomial automorphisms on a Hadamard matrix can be reduced to the computation of character sums over certain double cosets of G and the solution of a system of norm equations, see Sections 5 and 6 for details.

Our results do in fact generalise to other matrix structures that are determined by entry and eigenvalue conditions, e.g. weighing matrices, equiangular lines, mutually unbiased bases. The restriction to complex Hadamard matrices was chosen to impose coherence on the narrative, and since this choice illustrates the complexities of the generalisation from $\{0, 1\}$-matrices to general complex entries. Historically, the natural measures of complexity for Hadamard matrices were the size of the matrix and the size of the set of entries: thus matrices with k^{th} roots of unity as entries were studied extensively. In this paper we propose an alternative measure of complexity: the dimension of a centraliser algebra containing the Hadamard matrix M. Our constructions have locally determined entries: by Proposition 3.2, each entry of M is an explicit scalar multiple of (at least) linearly many other entries. Furthermore the eigenvalues of M are expressed as linear combinations of character sums, see Proposition 5.1 and the subsequent discussion. While the computations required are too extensive to carry out by hand except for the smallest cases, advances in computational algebra make these methods practical for matrices with several hundred rows.

1.1. **Main Results.** We outline the structure of this work and, at the same time, highlight some main results. In Section 2, we introduce notation and discuss preliminary results. In Section 3, we provide a description of the centraliser algebra of a monomial representation that allows for effective computations.

Our results in Section 4.1 concern Hadamard matrices invariant under a group of permutations that acts regularly. More precisely, let G be a group of order n and let $f\colon G \to \mathbb{C}$ be a function. Relative to a fixed ordering of the elements of G, define the matrix $M_f = [f(gh^{-1})]_{g,h\in G}$. Such a matrix is often called *group-developed* or *group-invariant* in the literature. In Theorem 4.2 we show that a matrix is group-developed over a finite group G if and only if there exist monomial matrices P, Q such that PM_fQ^* is in the centraliser algebra of the right regular representation of G. Results of this type are well-known, and real Hadamard matrices that are group-developed are equivalent to Menon difference sets, [5, 41]. Our main result in Section 4.2 is to locate the so-called *cocyclic Hadamard matrices* within a similar framework. If $\psi \in Z^2(G, \mathbb{C}^\times)$ is a 2-cocycle then $M_\psi = [\psi(g,h)]_{g,h\in G}$ is a (strictly) cocyclic-developed matrix, [15, 31]. In Theorem 4.4 we show that a matrix is cocyclic-developed over G if and only if the matrix belongs to the centraliser algebra of a monomial representation associated with the central extension of G determined by ψ. One sees that group-developed matrices are cocyclic-developed, and correspond to splitting extensions. Group development involves permutation representations and linear combinations of $\{0, 1\}$-matrices. Cocyclic development involves monomial matrices, though it can be related to the group-developed case via group cohomology.

One goal of this paper is to explain how complex Hadamard matrices with suitable symmetry assumptions may be constructed in the centraliser algebra of a monomial representation. Section 4 deals with regular representations, while Section 5 deals with general representations. The eigenvalues of a matrix that is group-developed over an abelian group are expressible in terms of its characters: continuing the notation of the above paragraph, the matrix M_f can be decomposed as $M_f = \sum_{g\in G} f(g)L_g$ where each $L_g = [\delta_g^{xy^{-1}}]_{x,y\in G}$; here δ is the usual Kronecker delta. With this definition, $\rho\colon g \to L_g$ is the left regular representation of G. Since G is finite, each L_g is diagonalisable over $\mathbb{C}$; since G is abelian, all the matrices L_g commute, hence they are simultaneously diagonalisable. It follows that M_f lies in the centraliser algebra of ρ and that the eigenvalues of M_f are the Fourier coefficients of f, that is, $\sum_{g\in G} f(g)\chi(g)$ where χ is a (linear) character χ of G. It is known that a complex $n \times n$ matrix is complex Hadamard if and only if all entries have unit norm and all eigenvalues have complex norm $\sqrt{n}$, see Lemma 2.2. Thus, the matrix M_f is complex Hadamard if and only if the function $f\colon G \to \mathbb{C}$ has all values of norm 1 and all of its Fourier coefficients are of norm $\sqrt{n}$. As mentioned earlier, this case is considered further in Section 4.1. If the group G is nonabelian, then in general one no-longer obtains the eigenvalues of M_f as explicit functions of the characters of G. However, if the monomial representation is multiplicity-free, there exist explicit formulae for the eigenvalues of M_f in terms of character sums over certain double cosets, see Proposition 5.1.

The construction of complex Hadamard matrices in the centraliser algebra of a monomial group requires several computational steps. First, given a primitive permutation group, we must construct all (perfect) monomial groups supported on this group: under suitable restrictions, this is solved by computing the Schur multiplier. Once we have obtained a monomial group, we construct the *character table* of the centraliser algebra using routines that will be described in Section 5. Finally, to construct complex Hadamard matrices, we must solve a linear system $T\underline{\alpha} = \underline{\lambda}$, where T is the character table of the centraliser algebra, $\underline{\alpha}$ is a vector of unknowns (essentially the entries of the resulting matrix) which must have norm 1, and $\underline{\lambda}$ is a vector of unknowns (the eigenvalues of the matrix) which must have norm $\sqrt{n}$. In essence, this is a system of quadratic equations. We use Gröbner basis routines to construct all solutions. Pursued systematically, this allows us to classify all Hadamard matrices under appropriate symmetry assumptions; we discuss the details in Section 6. To illustrate this approach, the paper concludes in Section 7 with computer constructions of complex Hadamard matrices that are invariant under a monomial cover of a primitive group of degree at most 15 and rank 3.

Related work. Centraliser algebras of induced representations are a well-studied topic in group theory, and for background information we refer particularly to work of Higman [27–30], Müller [38], and the textbook of Curtis and Reiner [13]. The theory of cocyclic development has been extensively surveyed; see the monographs of Horadam and de Launey and Flannery, [15, 31]. A cohomological approach to some of the results in this paper has been developed independently by Goldberger and collaborators, see [4, 22, 23].

2. Preliminaries

Unless mentioned otherwise, all groups are finite.

2.1. **Group actions.** We refer to [16] for background reading on this section and recall notation here. Let Ω be a finite set and denote by $\mathrm{Sym}(\Omega)$ the symmetric group on Ω. A group G acts on Ω if there is a homomorphism $\pi\colon G \to \mathrm{Sym}(\Omega)$; in this case π is called a *permutation representation* of G, and its image is a *permutation group on* Ω. We denote the image of $\omega \in \Omega$ under $\pi(g)$ by $\omega.g = \omega.\pi(g)$. The stabiliser of $\omega \in \Omega$ is $G_\omega = \{g \in G : \omega.g = \omega\}$, and the G-orbit of $\omega \in \Omega$ is $\omega.G = \{\omega.g : g \in G\}$.

The action is *faithful* if $\ker \pi$ is trivial. The G-orbits form a partition of Ω. If Ω forms a single G-orbit then the action is *transitive*; it is n-transitive if the induced action on n-tuples over Ω with pairwise distinct entries is transitive. The action is *semiregular* (or *free*) if every stabiliser is trivial; and *regular* if it is semiregular and transitive. Permutation actions of G on sets Ω_1 and Ω_2 are *equivalent* if there exists a bijection $\phi\colon \Omega_1 \to \Omega_2$ such that $\phi(\omega).g = \phi(\omega.g)$ for all $\omega \in \Omega_1$ and $g \in G$. There is a bijection between equivalence classes of transitive G-actions and conjugacy classes of subgroups of G. Specifically, a transitive G-action on Ω is equivalent to the G-action on right-cosets of a point stabiliser G_ω via right multiplication.

Let G be transitive on Ω, and let $g \in G$ act on $\Omega \times \Omega$ by $(\alpha, \beta).g = (\alpha.g, \beta.g)$. An *orbital* of G is a G-orbit on $\Omega \times \Omega$, and the *rank* of G is the number of orbitals. An orbital $\mathcal{O}$ is *self-paired* if $(\alpha, \beta) \in \mathcal{O}$ whenever $(\beta, \alpha) \in \mathcal{O}$. Since G is transitive on Ω, we can fix $\omega \in \Omega$ such that every $(\alpha, \beta) \in \Omega \times \Omega$ can be written as $(\alpha, \beta) = (\omega.g_1, \omega.g_2)$ for suitable $g_1, g_2 \in G$. As a result, there is a bijection between the orbitals of G and the orbits of G_ω on Ω given by the map $(\omega, \omega.g).G \mapsto (\omega.g).G_\omega$. The G_ω-orbits on Ω are *suborbits*, and their cardinalities are the *subdegrees* of G; the latter are independent of ω. In addition, there is a bijection between the orbitals of G and the G_ω-double cosets of G given by the maps $(\omega.g_1, \omega.g_2).G \mapsto G_\omega g_2 g_1^{-1} G_\omega$ and $G_\omega g G_\omega \mapsto (\omega, \omega.g).G$; for details and proofs see [46, Chapter 3]. Let $H = G_\omega$; the orbital of G corresponding to HgH is denoted $\mathcal{O}_{HgH}$ and has elements $(\omega.k, \omega.gk)$ for $k \in G$. For a set S, we define $\delta_S(x)$ to be 1 if $x \in S$, and 0 otherwise; the Kronecker delta δ_x^y is 1 if $x = y$, and 0 otherwise.

2.2. **$\mathbb{C}$-algebras and representation theory.** Let $\mathrm{M}_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over $\mathbb{C}$, and let A be a $\mathbb{C}$–algebra. An n-dimensional *representation* of A is an algebra homomorphism $\rho\colon A \to M_n(\mathbb{C})$. The induced A-module structure on the n–dimensional row space $\mathbb{C}^n$ is defined by $v.a = v\rho(a)$ for $v \in \mathbb{C}^n$ and $a \in A$. A representation is *reducible* if there exists a nontrivial submodule, and *irreducible* otherwise. If the associated A–module decomposes as a direct sum of irreducible submodules, then the representation is *completely reducible*. The *character* of ρ is the trace map $\chi_\rho\colon A \to \mathbb{C}$, $a \mapsto \mathrm{Tr}(\rho(a))$; it is called irreducible if and only if ρ is irreducible.

By Maschke's Theorem [33, Theorem 1.9], the complex *group algebra* $\mathbb{C}[G]$ is completely reducible. If $\rho = \rho_1 + \ldots + \rho_r$ is the sum of irreducible representations $\rho_1, \ldots, \rho_r$, then each ρ_i is an *irreducible constituent* of ρ, and r is the *rank* of ρ. A group algebra representation $\rho\colon \mathbb{C}[G] \to M_n(\mathbb{C})$ restricts to a group homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$ into the group of invertible complex $n \times n$ matrices; this restriction is an n-dimensional (complex) representation of G. The irreducible submodules of a representation of G coincide with those of $\mathbb{C}[G]$. The number of nonisomorphic irreducible representations of G is equal to the number of conjugacy classes in G. The centraliser algebra $\mathrm{C}(\rho)$ of ρ is the subalgebra of $M_n(\mathbb{C})$ consisting of all matrices that commute with every element of $\rho(G)$. Schur's Lemma [33, Lemma 1.5] states that the centraliser of an irreducible representation consists of scalar matrices. A corollary of this is that $\mathrm{C}(\rho)$ is commutative if and only if each irreducible constituent of ρ occurs with multiplicity 1, see [42, Theorem 1.7.8].

2.3. **Monomial matrices.** A matrix $M \in \mathrm{M}_n(\mathbb{C})$ is *monomial* if it has exactly one nonzero element per row and column. A monomial matrix with entries in $\{0, 1\}$ is a *permutation matrix*. A matrix is monomial if and only if $M = PD$ for a permutation matrix P and diagonal matrix D. A permutation representation $\pi\colon G \to \mathrm{Sym}(\Omega)$, over a finite set Ω, yields a representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ of permutation matrices. We also refer to ρ as a permutation representation; it can be extended to a group algebra representation $\rho\colon \mathbb{C}[G] \to M_n(\mathbb{C})$. A representation ρ is *monomial* if each $\rho(g)$ factorises as $\rho(g) = P_g D_g$ for a permutation matrix P_g and diagonal matrix D_g. The *associated permutation representation* is defined by $\pi_\rho(g) = P_g$. By abuse of notation, we say a monomial representation has a permutation group property $\mathcal{P}$ (such as transitive, primitive, etc) if the associated permutation representation has it.

The set of $n \times n$ monomial matrices forms a group under matrix multiplication, and the direct product of this group with itself acts on $\mathrm{M}_n(\mathbb{C})$ via $(P, Q) \cdot R = PRQ^*$. Two matrices are *equivalent* if they lie in the same orbit, and the *automorphism group* $\mathrm{Aut}(R)$ of $R \in \mathrm{M}_n(\mathbb{C})$ is the stabiliser of R under this action. A subgroup $U \leqslant \mathrm{Aut}(R)$ acts *regularly* (transitively) if the induced actions on rows and columns are regular (transitive). The *strong automorphism group* $\mathrm{SAut}(R)$ is the subgroup of all $(P, P) \in \mathrm{Aut}(R)$.

2.4. **Gröbner bases.** By Hilbert's Nullstellensatz [12, Section 4.1], there is a one-to-one correspondence between ideals in a polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ and algebraic varieties in $\mathbb{C}^n$. This correspondence is fundamental in algebraic geometry, as it allows the translation of geometric questions into questions about sets of polynomials, for which algorithmic methods are often available. Any serious discussion of these topics would take us far afield from the subject of this paper, we refer the reader to standard references such as Shafarevich [43] or Cox, Little & O'Shea [12]. A Gröbner basis for an ideal is defined with respect to an ordering on the monomials of the polynomial ring, and facilitates computation with the ideal. In particular, the irreducible components of the ideal can generally be read from the Gröbner basis without difficulty. Methods to compute Gröbner bases are provided by standard computational algebra systems, such as GAP [20] and Magma [6]. We will be exclusively interested in ideals whose elements are polynomials of degree 2 such that every indeterminate appears with degree at most 1, corresponding to solutions of linear systems on which norm conditions are imposed. The next example illustrates our use of Gröbner bases.

Example 2.1. Let

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_1 & \alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

and suppose we want to determine all complex Hadamard matrices of the form M where $\alpha_1, \ldots, \alpha_4$ are complex units. Note that the matrix M is group-invariant under the Klein four-group C_2^2. Here we note without proof that T is the character table of the associated centraliser algebra and the eigenvalues $\lambda_1, \ldots, \lambda_4$ of M are given by the linear system $T\underline{\alpha} = \underline{\lambda}$, where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_4)^\intercal$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_4)^\intercal$; the latter is proved in Section 4.1. Up to Hadamard equivalence, α_1 can be set to 1. Lemma 2.2 shows that the set of tuples $(1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{C}^4$ for which M is complex Hadamard is defined by the norm equations $\alpha_i \alpha_i^* = 1$ for $i = 2, 3, 4$ and $\lambda_j \lambda_j^* = 2$ for $j = 1, \ldots, 4$. Since complex conjugation is not $\mathbb{C}$-linear, we introduce variables α_{ic} and λ_{jc} denoting the complex conjugates of α_i and λ_j respectively. Thus, the polynomials in $\mathcal{R} = \mathbb{Q}[\alpha_2, \alpha_{2c}, \alpha_3, \alpha_{3c}, \alpha_4, \alpha_{4c}]$ describing these conditions are

$$P_2 = \alpha_2\alpha_{2c} - 1 \qquad P_3 = \alpha_3\alpha_{3c} - 1 \qquad P_4 = \alpha_4\alpha_{4c} - 1$$
$$Q_1 = (1 + \alpha_2 + \alpha_3 + \alpha_4)(1 + \alpha_{2c} + \alpha_{3c} + \alpha_{4c}) - 2$$
$$Q_2 = (1 - \alpha_2 + \alpha_3 - \alpha_4)(1 - \alpha_{2c} + \alpha_{3c} - \alpha_{4c}) - 2$$
$$Q_3 = (1 + \alpha_2 - \alpha_3 - \alpha_4)(1 + \alpha_{2c} - \alpha_{3c} - \alpha_{4c}) - 2$$
$$Q_4 = (1 - \alpha_2 - \alpha_3 + \alpha_4)(1 - \alpha_{2c} - \alpha_{3c} + \alpha_{4c}) - 2$$

These polynomials generate an ideal $\mathcal{I}$ of $\mathcal{R}$. A Gröbner basis for this ideal consists of a collection of ideals, each describing one irreducible component of the variety of $\mathcal{I}$. In this case, there are 6 irreducible components, one of them being the ideal $\mathcal{J}$ generated by $\{\alpha_2 - 1, \alpha_{2c} - 1, \alpha_3 + \alpha_4, \alpha_{3c} + \alpha_{4c}, \alpha_4\alpha_{4c} - 1\}$. Geometrically, $\mathcal{J}$ is a circle, in which α_4 can be any complex unit, $\alpha_3 = -\alpha_4$, and $\alpha_1 = \alpha_2 = 1$. Every point on this circle corresponds to a complex Hadamard matrix when substituted into M. The remainder of the Gröbner basis consists of five similar ideals obtained by freely permuting $\alpha_2, \alpha_3, \alpha_4$.

We conclude the preliminaries with a relating the entries and eigenvalues of a complex Hadamard matrix, which is an immediate consequence of the Hadamard Inequality.

Lemma 2.2. *A complex $n \times n$ matrix M is a complex Hadamard matrix if and only if every entry of M has complex norm 1 and every eigenvalue of M has complex norm $\sqrt{n}$.*

Proof. If M is a complex Hadamard matrix, then its entries have norm 1 by definition, and it follows from $MM^* = nI_n$ that every eigenvalue of M has norm $\sqrt{n}$. Conversely, suppose M is an $n \times n$ matrix with the stated properties. Since M has n complex eigenvalues of norm $\sqrt{n}$, its determinant meets the Hadamard bound, that is, $|\det(M)| = n^{n/2}$. It follows that MM^* is positive definite with diagonal entries of norm n and determinant n^n. A fundamental inequality for positive definite matrices $D = [d_{ij}]_{i,j=1}^n$ is that $|\det(D)| \leqslant \prod_{i=1}^n d_{ii}$, with equality if and only if D is diagonal, see [8, Theorem 1] and the discussion afterwards. Thus, MM^* is diagonal. Since every entry of M has unit norm, M is a Hadamard matrix. □

3. Centralisers of monomial representations

In this section, we develop ideas of Higman [28, 30] to give an explicit construction for a basis of the centraliser algebra of a monomial representation. This is closely related to the transfer homomorphism of finite group theory, see [26, Chapter 14]. Most of our results here follow from the existing literature.

In the following, let G be a finite group with subgroup $H \leqslant G$ of index n. Let $T = \{t_1, \ldots, t_n\}$ with $t_1 = 1$ be a right transversal to H in G, that is, every $g \in G$ admits a factorisation as

$$g = h_g t_g$$

for uniquely determined $h_g \in H$ and $t_g \in T$. We define the maps $\mathbf{H}\colon G \to H$ and $\mathbf{T}\colon G \to T$ by $\mathbf{H}(g) = h_g$ and $\mathbf{T}(g) = t_g$. We let G act on T by setting $t_i \cdot g = \mathbf{T}(t_i g)$ for $t_i \in T$; this defines a group action.

Let $\chi\colon H \to \mathbb{C}^\times$ be a 1-dimensional representation of H (commonly referred to in the literature as a linear character), and extend χ from H to G by

$$\chi^+(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

We write $\chi_{\mathbf{H}}(g) = \chi(\mathbf{H}(g))$ for the χ-value of the H-part of g; note that $\chi_{\mathbf{H}}$ is usually not a homomorphism.

The next proposition gives the construction of the representation of G induced from χ. It is known that every n-dimensional transitive monomial representation of G is induced from some 1-dimensional representation of a subgroup H of index n, see [14, Section 43, Exercise 1].

Proposition 3.1. *With the previous notation, the monomial representation induced from χ is the n-dimensional representation $\rho_\chi = \chi\uparrow_H^G$ that maps $g \in G$ to the matrix*

$$\rho_\chi(g) = \left[\chi^+(t_i g t_k^{-1})\right]_{i,k}. \tag{1}$$

Proof. Let $g_1, g_2 \in G$. The entry in row i and column k in $\rho_\chi(g_1)\rho_\chi(g_2)$ is $\sum_{j=1}^n \chi^+(t_i g_1 t_j^{-1})\chi^+(t_j g_2 t_k^{-1})$. It is nonzero if and only if $\mathbf{T}(t_i g_1) = t_j$ and $\mathbf{T}(t_j g_2) = t_k$. Thus, there is a unique nonzero entry in row i, namely, $\chi^+(t_i g_1 t_j^{-1})\chi^+(t_j g_2 t_k^{-1}) = \chi(t_i g_1 g_2 t_k^{-1})$ in column k where $t_k = \mathbf{T}(\mathbf{T}(t_i g_1) g_2)$. This coincides with the entry in $\rho_\chi(g_1g_2)$ in row i and column k, which shows that ρ_χ is a homomorphism defining a monomial representation. $\square$

Recall that $g \in G$ acts on $T \times T$ via $(t,s)\cdot g = (\mathbf{T}(tg), \mathbf{T}(sg))$. If M is a matrix whose rows and columns are labelled by the ordered set $T = \{t_1,\dots,t_n\}$, then we denote by $m(t_i,t_j)$ the entry in M in row t_i and column t_j. The next proposition shows that a matrix M lies in the centraliser algebra $\mathrm{C}(\rho)$ if and only if for every G-orbital $\mathcal{O}$ the elements in M labelled by $\mathcal{O}$ satisfy certain relations.

Proposition 3.2. *With the previous notation, a matrix M with rows and columns labelled by the transversal T, is in the centraliser algebra $C(\rho)$ if and only if for all $g \in G$ and $t \in T$ we have*

$$m(\mathbf{T}(g), \mathbf{T}(tg)) = m(1,t)\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg). \tag{2}$$

Proof. Since $t_igt_j^{-1} \in H$ if and only if $t_j = \mathbf{T}(t_ig)$ and $t_igt_j^{-1} = \mathbf{H}(t_ig)$, we observe

$$\rho(g)M = \left[\sum\nolimits_{j=1}^{n} \chi^+(t_igt_j^{-1})m(t_j,t_k)\right]_{i,k} = \left[\chi_{\mathbf{H}}(t_ig)m(\mathbf{T}(t_ig),t_k)\right]_{i,k}.$$

Similarly, $t_jgt_k^{-1} = h \in H$ if and only if $h^{-1}t_j = t_kg^{-1}$, if and only if $t_j = \mathbf{T}(t_kg^{-1})$ and $h = \mathbf{H}(t_kg^{-1})^{-1}$, and so

$$M\rho(g) = \left[\sum\nolimits_{j=1}^{n} m(t_i,t_j)\chi^+(t_jgt_k^{-1})\right]_{i,k} = \left[m(t_i,\mathbf{T}(t_kg^{-1}))\chi_{\mathbf{H}}(t_kg^{-1})^{-1}\right]_{i,k}.$$

Now $\rho(g)M = M\rho(g)$ holds for all $g \in G$ if and only if for all $g \in G$ and $i,k \in \{1,\dots,n\}$ we have

$$m(\mathbf{T}(t_ig),t_k) = m(t_i,\mathbf{T}(t_kg^{-1}))\chi_{\mathbf{H}}(t_ig)^{-1}\chi_{\mathbf{H}}(t_kg^{-1})^{-1}.$$

Let $t \in T$ such that $Ht_k = Htg$, say $t_k = htg$. This means that $\mathbf{H}(t_kg^{-1}) = h = \mathbf{H}(tg)^{-1}$, $\mathbf{T}(tg) = t_k$, and $\mathbf{T}(t_kg^{-1}) = t$, which shows that $M \in \mathrm{C}(\rho)$ if and only if for all $g \in G$ and $i \in \{1,\dots,n\}$ we have

$$m(\mathbf{T}(t_ig), \mathbf{T}(tg)) = m(t_i,t)\chi_{\mathbf{H}}(t_ig)^{-1}\chi_{\mathbf{H}}(tg) \tag{3}$$

This defines $m(\mathbf{T}(t_ig), \mathbf{T}(tg))$ in terms of $m(t_i,t)$; for $i = 1$, we obtain (2). $\square$

Note that (2) is well-defined if and only if $\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg) = \chi_{\mathbf{H}}(k)^{-1}\chi_{\mathbf{H}}(tk)$ whenever $g,k \in G$ satisfy $(\mathbf{T}(g),\mathbf{T}(tg)) = (u,v) = (\mathbf{T}(k),\mathbf{T}(tk))$, and the latter equation holds if and only if $g,k \in Hu \cap t^{-1}Hv$. This is captured by the following definition.

Definition 3.3. Let G be a group with subgroup H. Let $\rho = \rho_\chi$ be the monomial representation of G induced from a linear character χ of H. Let T be a set of right coset representatives of H. The orbital $\mathcal{O}$ associated with $(1,t)$ is *orientable* if and only if for all $(u,v) \in \mathcal{O}$ and $g,k \in Hu \cap t^{-1}Hv$ we have

$$\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg) = \chi_{\mathbf{H}}(k)^{-1}\chi_{\mathbf{H}}(tk). \tag{4}$$

Otherwise, $\mathcal{O}$ is *non-orientable*.

It follows that a matrix M in the centraliser algebra must have zero entries at all positions of non-orientable orbitals or, in other words, M must be supported on orientable orbitals. In particular, two matrices supported on the same orbital must be linearly dependent, whereas two orbital matrices corresponding to distinct orbitals are linearly independent. In conclusion, the next result follows.

Theorem 3.4. *With the previous notation, the centraliser algebra $C(\rho)$ has a $\mathbb{C}$-basis spanned by the orientable orbital matrices.*

Example 3.5. Let $G = \mathrm{Sym}_n$ with $n \geqslant 4$ be the symmetric group of degree n, acting on the set of unordered pairs of elements in $\{1,2,\dots,n\}$. The point stabiliser H of $\{1,2\}$ is isomorphic to $\mathrm{Sym}_{n-2} \times \mathrm{Sym}_2$, with transversal $T = \{1_G, (1,2,i), (2,1,i), (1,j)(2,k) \mid 3 \leqslant i \leqslant n, 3 \leqslant j < k \leqslant n\}$. The action has rank 3 with orbitals $\mathcal{O}_1 = \{\{1,2\}\}$, $\mathcal{O}_2 = \{\{1,x\},\{2,x\} \mid x \neq 1,2\}$, $\mathcal{O}_3 = \{\{x,y\} \mid x,y \neq 1,2\}$. Since the commutator subgroup of H has index 4 in H, it follows that H has three nontrivial linear characters, [33, (2.23)]. We choose χ to be the character which has kernel Sym_{n-2}, so $\chi(x)$ is nontrivial if and only if the projection onto the direct factor Sym_2 is nontrivial. We claim that $\mathcal{O}_3$ is non-orientable. To see this, pick $u = 1_G$ and $t = v = (1,3)(2,4)$, so that $Hu \cap t^{-1}Hv$ is a subgroup of Sym_n isomorphic to $\mathrm{Sym}_2 \times \mathrm{Sym}_2 \times \mathrm{Sym}_{n-4}$, fixing $\{1,2\}$ and $\{3,4\}$ setwise. Since $(1,2) \in Hu \cap t^{-1}Hv$, non-orientability is witnessed by

$$\chi_{\mathbf{H}}(u)^{-1}\chi_{\mathbf{H}}(tu) = 1 \neq -1 = \chi_{\mathbf{H}}((1,2))^{-1}\chi_{\mathbf{H}}(t(1,2));$$

indeed, we have $\chi_{\mathbf{H}}((1,2))^{-1}\chi_{\mathbf{H}}(t(1,2)) = \chi((1,2))^{-1}\chi((3,4)) = -1$ since $t(1,2) = (1,3,2,4) = (3,4)(1,3)(2,4)$, and $\chi_{\mathbf{H}}(u)^{-1}\chi_{\mathbf{H}}(tu) = 1$ holds by definition.

We conclude this section with a convenient test for orientability of orbitals. Given an orbital $\mathcal{O} = (1,t)\cdot G$, the next proposition shows that instead of verifying (4) for all $(u,v) \in \mathcal{O}$ and $g,h \in Hu \cap t^{-1}Hv$, it suffices to only consider elements in $H \cap t^{-1}Ht$ as explained below.

Proposition 3.6. *Let $H \leqslant G$, let T be a right transversal of H in G, and let χ be a linear character of H. The orbital $\mathcal{O}$ containing $(1,t)$ is orientable if and only if $\chi(tht^{-1}h^{-1}) = 1$ for all $h \in H \cap t^{-1}Ht$.*

Proof. Note that $H \cap t^{-1}Ht$ is the stabiliser of $(1,t)$ in G. Let $g,k \in G$ with $(1,t)\cdot g = (1,t)\cdot k = (u,v)$. Then $g = h_1u$, $tg = h_2v$, $k = h_3u$ and $tk = h_4v$ for some $h_1,\ldots,h_4 \in H$. It follows that $gk^{-1} = h_1h_3^{-1}$ and $tgk^{-1}t^{-1} = h_2h_4^{-1}$. Now suppose that $\mathcal{O}$ is orientable. Since $g,k \in Hu \cap t^{-1}Hv$, the orientability assumption implies that $\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg) = \chi_{\mathbf{H}}(k)^{-1}\chi_{\mathbf{H}}(tk)$. Using that χ is a homomorphism on H, this can be rephrased as follows:

$$\begin{array}{rrcl} & \chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg) & = & \chi_{\mathbf{H}}(k)^{-1}\chi_{\mathbf{H}}(tk) \\ \Longleftrightarrow & \chi_{\mathbf{H}}(h_1u)^{-1}\chi_{\mathbf{H}}(h_2v) & = & \chi_{\mathbf{H}}(h_3u)^{-1}\chi_{\mathbf{H}}(h_4v) \\ \Longleftrightarrow & \chi(h_2h_4^{-1}) & = & \chi(h_1h_3^{-1}) \\ \Longleftrightarrow & \chi(tgk^{-1}t^{-1}) & = & \chi(gk^{-1}). \end{array}$$

The stabiliser of $(1,t)$ in G is generated by all gk^{-1} where $g,k \in G$ which satisfy $(1,t)\cdot g = (1,t)\cdot k$, see [16, Theorem 3.6A]. Thus, the last equation in the above equivalences implies that $\chi(tht^{-1}) = \chi(h)$ for all $h \in H \cap tHt^{-1}$.

In the other direction, assume that $\chi(h) = \chi(tht^{-1})$ for all $h \in H \cap t^{-1}Ht$. Suppose that $(1,t)\cdot g = (1,t)\cdot k$ and proceed as before: we can write $g = h_1u$, $tg = h_2v$, $k = h_3u$ and $tk = h_4v$ for suitable $h_i \in H$, and now the argument follows from the implications above. □

4. Locating group-developed and cocyclic matrices in centraliser algebras

The main results of this section are characterisations of group-developed and cocyclic matrices; we recall the definitions below. In Theorem 4.2 we see that a matrix is group-developed over a finite group G if and only if there exists an equivalent matrix in the centraliser algebra of the right regular representation of G. While this result is well-known, the reader is encouraged to compare this with Theorem 4.4, which shows that a matrix is cocyclic over G if and only if there exists an equivalent matrix in the centraliser algebra of a certain monomial cover of G.

Throughout, let G be a finite group and let $\mathcal{A}$ be a finite (hence cyclic) subgroup of $\mathbb{C}^\times$.

4.1. **Group-development and permutation representations.** All matrices in this section will have rows and columns labelled by the elements of G with respect to some fixed ordering. A matrix M with entries in $\mathcal{A}$ is called *strictly group-developed over* G if there exists a map $f\colon G \to \mathcal{A}$ such that $M = [f(gh)]_{g,h\in G}$, and *strictly group-invariant* if $M = [f(gh^{-1})]_{g,h\in G}$, see [31, Definition 2.17] and [15, Definition 10.2.1]. These definitions differ by a permutation of columns, hence M is strictly group-invariant if it is $\mathcal{A}$-equivalent to a strictly group-developed matrix, and vice versa. It is convenient to define M to be *group-developed* if it is $\mathcal{A}$-equivalent to a strictly group-developed matrix.

The *right regular representation* of G is defined by $R(g) = [\delta_y^{xg}]_{x,y\in G}$ for $g \in G$, where δ_a^b is the usual Kronecker delta. Similarly, the left regular representation is defined by $L(g) = [\delta_y^{g^{-1}x}]_{x,y\in G}$. A short direct calculation confirms that $N = [\delta_{y^{-1}}^x]_{x,y\in G}$ satisfies

$$N^2 = I_n \quad \text{and} \quad NR(g)N = L(g) = L(g^{-1})^\intercal \quad (g \in G), \tag{5}$$

which shows that $L(g) = NR(g)N^* = NR(g)N$ for all $g \in G$, so R and L are conjugate representations.

Lemma 4.1. *With the previous notation, a complex $n \times n$ matrix M is strictly group-developed over G if and only if $R(g)ML(g)^\intercal = M$ for all $g \in G$.*

Proof. Recall that for $x, y \in G$ we denote by $m(x, y)$ the entry in M in row x and column y. If M is strictly group-developed over G with map $f\colon G \to \mathbb{C}$, then $m(x, y) = f(xy)$; if $g \in G$, then $L(g)^\intercal = L(g^{-1})$ implies

$$R(g)ML(g^{-1}) = \left[\sum_{x,y} \delta_x^{wg} f(xy) \delta_z^{gy}\right]_{w,z\in G} = \left[f\left((wg)(g^{-1}z)\right)\right]_{w,z\in G} = M.$$

Conversely, if $M = [m(x, y)]_{x,y\in G}$ satisfies $M = R(g)ML(g)^\intercal = R(g)ML(g^{-1})$ for all $g \in G$, then a calculation similar to the one before shows that $[m(xg, g^{-1}y)]_{x,y\in G} = [m(x, y)]_{x,y\in G}$ for all $g \in G$. By choosing $g = x^{-1}$ we find that $m(x, y) = m(1, xy)$ for all y, and so $M = [f(xy)]_{x,y\in G}$ where $f\colon G \to \mathbb{C}$ is defined by $f(g) = m(1, g)$. □

The next theorem characterises the existence of group-developed matrices.

Theorem 4.2. *A matrix with entries in a finite group $\mathcal{A} \leqslant \mathbb{C}^\times$ is group-developed over G if and only if there exists an $\mathcal{A}$-equivalent matrix in $\mathrm{C}(R)$, where R is the right regular representation of G.*

Proof. Any group-developed matrix is $\mathcal{A}$-equivalent to a strictly group-developed matrix, so we can assume that M is a strictly group-developed matrix over G. Lemma 4.1 shows that $R(g)ML(g)^\intercal = M$ for all $g \in G$. Thus, $R(g)M = ML(g^{-1})^\intercal$, and (5) implies that $R(g)MN = MNR(g)$, hence $MN \in \mathrm{C}(R)$.

Conversely, let $M \in \mathrm{C}(R)$. Then $R(g)M = MR(g)$ for all $g \in G$, and using $R(g) = NL(g)N$, we obtain $R(g)M = MNL(g)N$. Since $L(g)^{-1} = L(g)^\intercal$, see (5), this is equivalent to $R(g)MNL(g)^\intercal = MN$. Now Lemma 4.1 shows that M is group-developed over G, as claimed. □

4.2. **Cocyclic development and monomial representations.** Having discussed the relation between group-developed matrices and permutation representations in the previous section, we now locate cocyclic development in the theory of monomial representations. As described by Goldberger and collaborators [22], these ideas lead to a more general theory of cohomology development of matrices.

As before, let G be a finite group and let $\mathcal{A} \leqslant \mathbb{C}^\times$ be a finite subgroup; note that $\mathcal{A}$ is a cyclic group, which we consider a G-module with trivial action. Let Γ be a central extension of G by $\mathcal{A}$. By standard theory of group extensions, Γ is isomorphic to a group with underlying set $\mathcal{A} \times G$ and multiplication

$$(a, g)(b, h) = (ab\psi(g, h), gh) \tag{6}$$

where $\psi\colon G \times G \to \mathcal{A}$ is a (normalised) 2-cocycle, that is, a function satisfying $\psi(g, 1) = \psi(1, g) = 1$ and $\psi(g, h)\psi(gh, k) = \psi(g, hk)\psi(h, k)$ for all $g, h, k \in G$; we refer to [26, Chapter 15] or [15, Chapter 12] for further details. In the following we write $\Gamma = (G, \mathcal{A}, \psi)$ to record these data about the group extension.

A matrix M with entries in $\mathcal{A}$ is called *strictly cocyclic over G* if there exists a cocycle $\psi : G \times G \to \mathcal{A}$ and a function $\phi : G \to \mathcal{A}$ such that

$$M = [\psi(x, y)\phi(xy)]_{x,y\in G}\,,$$

where the rows and columns are indexed by the elements of G in a fixed ordering. As for the group-developed case, it is often convenient to consider $\mathcal{A}$-equivalence: the matrix M is *cocyclic over* G if it is $\mathcal{A}$-equivalent to a strictly cocyclic matrix.

Motivated by the right and left regular representations of G, we define the following monomial representations R and L of Γ: if (a, g) is an element of the extension Γ, then

$$R(a,g) = a\left[\psi(x,g)\delta_y^{xg}\right]_{x,y\in G} \quad \text{and} \quad L(a,g) = a\left[\psi(g,g^{-1}x)\delta_{gy}^{x}\right]_{x,y\in G}. \tag{7}$$

That R is a homomorphism follows from the cocycle identity and a short calculation:

$$\begin{aligned}
R(a,g)R(b,h) &= ab\left[\sum\nolimits_{y\in G}\psi(x,g)\delta_y^{xg}\psi(y,h)\delta_z^{yh}\right]_{x,z\in G}\\
&= ab\left[\psi(x,g)\psi(xg,h)\delta_z^{xgh}\right]_{x,z\in G}\\
&= ab\left[\psi(x,gh)\psi(g,h)\delta_z^{xgh}\right]_{x,z\in G}\\
&= ab\psi(g,h)\left[\psi(x,gh)\delta_z^{xgh}\right]_{x,z\in G}\\
&= R(ab\psi(g,h), gh)\,.
\end{aligned}$$

In applications it will be necessary to consider L^* instead of L. The following property holds for L^*, which also shows that L is a homomorphism; recall that $\mathcal{A}$ is a finite subgroup of $\mathbb{C}^\times$, so $a^* = a^{-1}$ for $a \in \mathcal{A}$:

$$\begin{aligned}
L(a,g)^*L(b,h)^* &= a^{-1}b^{-1}\left[\sum\nolimits_{y\in G}\psi(g,g^{-1}y)^{-1}\delta_y^{gx}\psi(h,h^{-1}z)^{-1}\delta_z^{hy}\right]_{x,z\in G}\\
&= a^{-1}b^{-1}\left[\psi(g,x)^{-1}\psi(h,gx)^{-1}\delta_z^{hgx}\right]_{x,z\in G}\\
&= a^{-1}b^{-1}\left[\psi(h,g)^{-1}\psi(hg,x)^{-1}\delta_z^{hgx}\right]_{x,z\in G}\\
&= a^{-1}b^{-1}\psi(h,g)^{-1}\left[\psi(hg,g^{-1}h^{-1}z)^{-1}\delta_z^{hgx}\right]_{x,z\in G}\\
&= L(ba\psi(h,g), hg)^*\,.
\end{aligned}$$

As in the group-developed case, the representations R and L are conjugate. To see this, we define $N = [\psi(x,x^{-1})\delta_{y^{-1}}^{x}]_{x,y\in G}$ and use the cocycle identity to deduce that

$$\begin{aligned}
R(a,g)NL(a,g)^* &= aa^{-1}\left[\sum\nolimits_{x,y}\psi(w,g)\delta_x^{wg}\psi(x,x^{-1})\delta_{y^{-1}}^{x}\psi(g,g^{-1}z)^{-1}\delta_z^{gy}\right]_{w,z}\\
&= \left[\sum\nolimits_{y}\psi(w,g)\psi(wg,g^{-1}w^{-1})\delta_{y^{-1}}^{wg}\psi(g,g^{-1}z)^{-1}\delta_z^{gy}\right]_{w,z}\\
&= \left[\psi(w,g)\psi(wg,g^{-1}w^{-1})\psi(g,g^{-1}w^{-1})^{-1}\delta_{z^{-1}}^{w}\right]_{w,z}\\
&= [\psi(w,w^{-1})\delta_{z^{-1}}^{w}]_{w,z\in G} = N.
\end{aligned}$$

Thus, if $(a,g) \in \Gamma$, then

$$L(a,g) = NR(a,g)N^*. \tag{8}$$

Lemma 4.3. *With the previous notation, let* $\Gamma = (G, \mathcal{A}, \psi)$ *be a group extension and* $\phi\colon G \to \mathcal{A}$ *a map. If* $M = [\psi(x,y)\phi(xy)]_{x,y\in G}$ *then* $R(a,g)ML(a,g)^* = M$ *for all* $(a,g) \in \Gamma$.

Proof. This follows from a direct computation using the cocycle identity:

$$\begin{aligned}
R(a,g)ML(a,g)^* &= aa^{-1}\left[\sum\nolimits_{x,y}\psi(w,g)\delta_x^{wg}\psi(x,y)\phi(xy)\psi(g,g^{-1}z)^{-1}\delta_z^{gy}\right]_{w,z}\\
&= \left[\psi(w,g)\psi(wg,g^{-1}z)\psi(g,g^{-1}z)^{-1}\phi(wz)\right]_{w,z}\\
&= [\psi(w,z)\phi(wz)]_{w,z} = M.
\end{aligned}$$

□

The main result of this section is the following.

Theorem 4.4. *Let M be a square matrix with entries in the finite subgroup $\mathcal{A} \leqslant \mathbb{C}^\times$ and with rows and columns labelled by a finite group G. Then M is cocyclic over G if and only if there exists an $\mathcal{A}$-equivalent matrix in $\mathrm{C}(R)$, where R is the monomial representation* (7) *of an extension $\Gamma = \Gamma(G, \mathcal{A}, \psi)$ of G.*

Proof. Any cocyclic matrix is $\mathcal{A}$-equivalent to a strictly cocyclic matrix, so suppose that M is strictly cocyclic over G, with extension group $\Gamma = (G, \mathcal{A}, \psi)$. Lemma 4.3 shows that $R(a,g)ML(a,g)^* = M$ for all $(a,g) \in \Gamma$. Together with (8), it follows that

$$R(a,g)MN^* = MN^*R(a,g),$$

and therefore $MN^* \in \mathrm{C}(R)$. Conversely, let $M \in \mathrm{C}(R)$ for an extension Γ and representation R as in the statement. By definition, $R(a,g)M = MR(a,g)$ for all $(a,g) \in \Gamma$, and using $R(a,g) = NL(a,g)N^*$, we obtain $R(a,g)M = MNL(a,g)N^*$, hence $R(a,g)MNL(a,g)^* = MN$. □

Group-developed matrices have constant row and column sums, since each row and each column is a permutation of the first. If H is a group-developed $n \times n$ Hadamard matrix with row and column sum s, then $HJ_n = sJ_n$ and $H^\intercal J_n = sJ_n$ for the $n \times n$ all-1s matrix J_n. Thus, multiplying $nI_n = HH^\intercal$ from the right by J_n yields

$$nJ_n = HH^\intercal J_n = sHJ_n = s^2 J_n$$

which forces that $n = s^2$ is a perfect square. This well-known observation restricts the orders at which group-developed Hadamard matrices exist. (Recall that the order of an $n \times n$ Hadamard matrix refers to the dimension n.) There are no known restrictions on the orders of cocyclic Hadamard matrices: indeed, it has been conjectured by Horadam [31, Research Problem 38] that there exists a cocyclic Hadamard matrix of order $4n$ for all n. This conjecture has been verified for all $n < 188$. Many constructions of Hadamard matrices are known to be cocyclic, including Sylvester and Paley matrices, [18,40]. Some families of cocyclic real and complex Hadamard matrices have also been classified computationally, [3, 17].

5. Character tables of centraliser algebras

Recall from Lemma 2.2 that a complex Hadamard matrix is characterised by norm conditions on its entries and eigenvalues. Theorem 4.4 explains that the existence of a complex Hadamard matrix that is cocyclic with respect to some indexing group can be verified by studying a suitable centraliser algebra of a monomial representation. Theorem 3.4 and Proposition 3.6 allow us to determine a basis of a centraliser algebra. Thus, to locate complex cocyclic Hadamard matrices, it remains to consider linear combinations of the basis elements of the centraliser algebra and verify the norm conditions. For this the so-called character table of the centraliser algebra will be useful. We discuss this character table here, and focus on the construction of complex Hadamard matrices in Sections 6 and 7.

The representation theory of finite groups is closely related to the representation theory of associative algebras applied to the group algebra $\mathbb{C}[G]$. Several accessible expositions of this theory are available, including the books [1, 33, 34]. We recall that a finite dimensional associative algebra over $\mathbb{C}$ is *semisimple* if its Jacobson radical is trivial, in which case the algebra is a direct sum of simple algebras.

Let A be a finite dimensional semisimple $\mathbb{C}$-algebra. It is well-known that A is a direct sum of matrix algebras, such that the number r of matrix algebras occurring in this direct sum is equal to the number of isomorphism types of simple A-modules; see [1, Lemma 13.14 and Theorem 13.16] and the details given in the proofs thereof. Since the centre of each matrix algebra consists of the scalar matrices, it also follows that r is the dimension of the centre of A. We denote by $\{M_1, \ldots, M_r\}$ a basis of the centre of A and let $\chi_1, \ldots, \chi_r$ be the irreducible characters of A. The *character table* of A is defined to be the $r \times r$ matrix

$$\mathrm{CT}(A) = [\chi_i(M_j)]_{i,j}\,.$$

For a representation ρ of a finite group G induced from a linear character χ of a subgroup H, the character table of the centraliser algebra $\mathrm{C}(\rho)$ may be constructed from the character table of G, together with some additional data about double cosets of H in G. Let $\{t_1, \ldots, t_r\}$ be a set of representatives of the H-double cosets in G. For $i = 1, \ldots r$, let $H_i = H \cap t_i^{-1} H t_i \leqslant H$ and $k_i = |H : H_i|$. Let $M_{G,H}$ be the matrix that contains the rows of the character table of G corresponding to the irreducible constituents of ρ. Let L be a matrix whose rows are indexed by $t_1, \ldots, t_r$, whose columns are indexed by conjugacy classes of G, and whose entry $\ell(t_i, C)$ is defined as

$$\ell(t_i, C) = \sum_{h \in H} \delta_C(ht_i)\chi(h^{-1})$$

where δ_C is the Kronecker delta for the conjugacy class C. While not especially well-known, the following result has appeared in the literature multiple times. Seemingly first obtained by Tamaschke [45], a version for general induced representations is given by Curtis and Reiner [13], while the special case of monomial representations was considered by Higman [28] and by Müller [38].

Proposition 5.1 ((3.20) in [38])**.** *Let ρ be the monomial representation of G induced from a linear character χ of a subgroup $H \leqslant G$. With the previous notation, the character table of the centraliser algebra $C(\rho)$ is*

$$\mathrm{CT}(C(\rho)) = \frac{1}{|H|} \cdot M_{G,H} \cdot L^{\mathsf{T}} \cdot \mathrm{diag}(k_1, \ldots, k_r).$$

The computations required by Proposition 5.1 are practical for reasonably sized groups. In the remainder of this paper we restrict to the case of monomial representations that have commutative centraliser algebras. It is well known that $\mathrm{C}(\rho)$ is commutative if and only if ρ is multiplicity-free, see e.g. [42, Theorem 1.7.8]. The semisimplicity of $\mathrm{C}(\rho)$ implies that $M_1, \ldots, M_r$ are simultaneously diagonalisable, with r common eigenspaces, denoted $V_1, \ldots, V_r$. Write $\lambda_{i,j}$ for the eigenvalue of M_j on the eigenspace V_i. In this case the character table may be described as

$$\mathrm{CT}(\mathrm{C}(\rho)) = [\lambda_{i,j}]_{i,j} \,. \tag{9}$$

Every $M \in \mathrm{C}(\rho)$ can be written as $M = \sum_{i=1}^{r} \alpha_i M_i$ for complex coefficients $\alpha_1, \ldots, \alpha_r$. The eigenvalue of M on the eigenspace V_i is given given by the i^{th} entry of the vector $T\underline{\alpha}$ where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)^{\mathsf{T}}$. It follows from Lemma 2.2 that M is a complex Hadamard matrix if each entry in $\underline{\alpha}$ has norm 1 and each entry in $T\underline{\alpha}$ has norm $\sqrt{n}$. In other words, there exists a complex Hadamard matrix, say M, in $\mathrm{C}(\rho)$ if and only if there is a solution of the system $T\underline{\alpha} = \underline{\lambda}$ where the entries of $\underline{\alpha}$ (the entries of M) have norm 1 and the entries of $\underline{\lambda}$ (the eigenvalues of M) have norm $\sqrt{n}$. As discussed in Section 2.4, this yields a system of linear equations and norm equations over a subfield of the complex numbers. While norm equations are not polynomial over $\mathbb{C}$, the system of equations can be rewritten as a real system of quadratic equations in twice as many variables. The next two examples illustrate the construction of complex Hadamard matrices in the centraliser algebra of a permutation group.

Example 5.2. Let $G \leqslant \mathrm{Sym}_{16}$ be the group $G = \langle \sigma, \tau \rangle$ where

$$\begin{aligned} \sigma &= (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16), \\ \tau &= (2,3,5,9,16)(4,7,13,8,15)(6,11,12,10,14). \end{aligned}$$

The group G is a Frobenius group of order 80, with an elementary abelian subgroup of order 16 and a point stabiliser $H = \langle \tau \rangle$ of order 5. Let ρ be the permutation representation induced by the trivial character χ of H. As a list of H-double coset representatives in G, we choose the identity, σ, $\sigma^{-1}\tau^{-1}\sigma\tau$, and $\sigma^{-1}\tau^{-2}\sigma\tau^2$. We denote by $M_1, \ldots, M_4$ the corresponding orbital matrices; these form a basis for the (commutative) centraliser algebra. We note that M_1 is the identity matrix and $M_2, \ldots, M_4$ have constant row sum 5. Using

Proposition 5.1, the character table of the centraliser algebra is

$$\begin{array}{cc} & \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ T = & \begin{pmatrix} 1 & 5 & 5 & 5 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix} \end{array}$$

The eigenvalues of the matrix $M = M_1 + M_2 - M_3 - M_4$ with entries $\{\pm 1\}$ are -4 and 4, so M is a 16×16 Hadamard matrix by Lemma 2.2. More generally, all solutions to $T\underline{\alpha} = \underline{\lambda}$ with $\alpha_i \alpha_{ic} = 1$ and $\lambda_j \lambda_{jc} = 4$ are obtained via computing the Gröbner basis as described in Section 2.4. All solutions are of the form $\underline{\alpha} = (1, z, -z, -1)$, or a cyclic permutation of the last three coordinates, where z is of norm 1.

Example 5.3. Let $p \equiv 1 \bmod 4$ be a prime and denote by $\mathbb{F}_p$ the field with p elements. Let $\mathrm{AGL}_1(p)$ be the permutation group consisting of affine transformations of $\mathbb{F}_p$ of the form $x \mapsto ax + b$, where $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$; this group is 2-transitive of degree p. The index-2 subgroup G of $\mathrm{AGL}_1(p)$ consisting of transformations $x \mapsto a^2x + b$ has rank 3. The stabiliser of a point is cyclic of order $(p-1)/2$, and we consider the permutation representation of degree p induced from the trivial character of the point stabiliser. The centraliser algebra C of this representation has dimension 3, and it is well known that one of the basis elements (constructed as in Theorem 3.4) is an adjacency matrix for the so-called *Paley graph*, which has vertices labelled by the elements of $\mathbb{F}_p$, and vertices x and y are connected by an edge if and only if $x - y$ is a quadratic residue in $\mathbb{F}_p$, see [7, Section 7.4.4]. The Paley graph is regular of degree $k = (p-1)/2$, so k occurs as an eigenvalue of the adjacency matrix A with multiplicity 1. By standard results in algebraic graph theory, the other eigenvalues are $\mu = (-1 + \sqrt{p})/2$ and $\nu = (-1 - \sqrt{p})/2$, each with multiplicity k.

Let A be the adjacency matrix of the Paley graph. It follows that a basis for the centraliser algebra of G can be constructed as $\{M_1, M_2, M_3\}$, where $M_1 = I_p$, $M_2 = A$, and $M_3 = J_p - I_p - A$. The all-1s matrix J_p has a 1-dimensional p-eigenspace and a $(p-1)$-dimensional 0-eigenspace. The matrices J_p, I_p, A are simultaneously diagonalisable, so they share common eigenspaces. This means that the nullspace of J_p is the direct sum of the μ- and ν-eigenspaces of A. Since M_3 is a linear combination, its value on the p-eigenspace of J_p is $p - k - 1 = k$, on the μ-eigenspace it is $-1 - \mu = \nu$, and on the ν-eigenspace it is $-1 - \nu = \mu$. Thus, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix $M = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3$ are computed as follows:

$$\begin{pmatrix} 1 & k & k \\ 1 & \mu & \nu \\ 1 & \nu & \mu \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}.$$

Lemma 2.2 shows that M is Hadamard if and only if each $\alpha_i \alpha_i^* = 1$ and each $\lambda_i \lambda_i^* = p$. For this consider the rational polynomial ring $\mathcal{R} = \mathbb{Q}[p, \mu, \alpha_2, \alpha_{2c}, \alpha_3, \alpha_{3c}]$. We construct the polynomials P_2, P_3, Q_1, Q_2, Q_3 to encode the norm conditions on α_i and λ_j as in Section 2.4. In addition, we introduce a polynomial $R = (2\mu + 1)^2 - p$ to encode the relation between μ and p. In this case, the computation yields a number of isolated points, all of which require $p \leqslant 4$, and one nontrivial component. This component may be parametrised in terms of $\mu = (-1 + \sqrt{p})/2$ as

$$\alpha_1 = 1, \quad \alpha_2 = \tfrac{-1+\sqrt{1-4\mu^2}}{2\mu}, \quad \text{and} \quad \alpha_3 = \alpha_{2c} = \alpha_2^*;$$

recall that $p \equiv 1 \bmod 4$ is a prime. This solution is unique up to permuting α_2 and α_3, and replacing both by complex conjugates. For example, if $p = 5$, then $\alpha_1 = 1$, $\alpha_2 = -1 + \imath\sqrt{(5 - \sqrt{5})/8}$, and $\alpha_3 = \alpha_2^*$.

6. Constructing complex Hadamard matrices: Schur multipliers

Let M be an $n \times n$ complex Hadamard matrix and recall that $\mathrm{SAut}(M)$ is the subgroup of $\mathrm{Aut}(M)$ consisting of pairs (P, P) with $PMP^* = M$. Denote by π_1 and π_2 the projections of $\mathrm{Aut}(M)$ onto the first and second components, respectively, and set $\Gamma = \pi_1(\mathrm{SAut}(M))$. Let $\pi\colon \Gamma \to \mathrm{Sym}_n$ be the homomorphism that maps a monomial matrix to the induced permutation matrix (identified with a permutation in Sym_n). For the rest of this section, we assume that $G = \pi(\Gamma)$ is transitive. We describe relations between the groups G, Γ, and $\mathrm{SAut}(M)$, and the matrix M. This is required for Section 7 where we start with a permutation group G and construct complex Hadamard matrices such that $\pi \circ \pi_1(\mathrm{SAut}(M)) \leqslant G$.

The next proposition shows that $\mathrm{SAut}(M)$ contains a finite subgroup, specified entirely by G, which determines the centraliser algebra $\mathrm{C}(\Gamma)$ completely. Here we use the convention that for a matrix group K we denote by $\mathrm{C}(K)$ the centraliser algebra of the identity representation $K \to K$. For an integer m we denote by ζ_m a primitive complex root of unity; if K is a group, then $K' = [K, K]$ is the commutator subgroup.

Proposition 6.1. *Let M be an $n \times n$ complex Hadamard matrix, and let $\Gamma = \pi_1(\mathrm{SAut}(M))$ and $G = \pi(\Gamma)$ as defined above. Let $\Gamma_{\mathrm{f}} = \{L \in \Gamma \mid \det(L) = 1\}$. The projection $\Gamma_{\mathrm{f}} \to G$ induced by π is surjective with cyclic and central kernel $\langle \zeta_n I_n \rangle$, so $|\Gamma_{\mathrm{f}}| = n|G|$ is finite. Moreover, $M \in C(\Gamma)$ and $C(\Gamma) = C(\Gamma_{\mathrm{f}})$.*

Proof. By definition, $M \in \mathrm{C}(\Gamma)$. The kernel of π consists of diagonal matrices $D = \mathrm{diag}(a_1, \ldots, a_n)$ with $DMD^* = M$. Since the entries of M are all nonzero, this forces $a_i a_j^* = 1$ for all i, j, which shows that $\ker \pi$ consists exactly of all scalar matrices whose entries are complex units.

Now consider $L \in \Gamma$. Let d be the (finite) exponent of G/G'. Since π maps Γ' to G', the d^{th} power of L satisfies $\pi(L^d) \in G'$. In particular, $L^d = SA$ for some scalar matrix $S = \zeta I_n \in \ker \pi$ and a matrix $A \in \Gamma'$. Note that $\det(A) = 1$, so $\det(L^d) = \zeta^n$. Set $\sigma = (\det(L)^{1/n})^*$. Then $\sigma I_n \in \ker \pi$ and hence $\sigma L \in \Gamma$. By construction, $\det(\sigma L) = 1$ and $\pi(\sigma L) = \pi(L)$. Since $\pi\colon \Gamma \to G$ is surjective, this implies that also the projection $\Gamma_{\mathrm{f}} \to G$ is surjective. The elements in the kernel of that projection are scalar matrices of determinant 1, that is, $\langle \zeta_n I_n \rangle$.

Since $\Gamma_{\mathrm{f}} \leqslant \Gamma$, we clearly have $\mathrm{C}(\Gamma) \leqslant \mathrm{C}(\Gamma_{\mathrm{f}})$. For the converse, consider $B \in \mathrm{C}(\Gamma_{\mathrm{f}})$. If $L \in \Gamma$, then the previous paragraph shows that $L = \sigma^{-1} L'$ where $\sigma^{-1} I_n \in \ker \pi$ and $L' \in \Gamma_{\mathrm{f}}$. Since $B \in \mathrm{C}(\Gamma_{\mathrm{f}})$, we have $BL = B\sigma^{-1}L' = \sigma^{-1}BL' = \sigma^{-1}L'B = LB$, that is, $B \in \mathrm{C}(\Gamma)$. Thus, $\mathrm{C}(\Gamma) = \mathrm{C}(\Gamma_{\mathrm{f}})$, as claimed. □

In early work on group representations, Schur studied the following covering problem: given a projective representation $\rho\colon G \to \mathrm{PGL}_n(\mathbb{C})$, construct a group $\hat{G}$ with representation $\hat{\rho}\colon \hat{G} \to \mathrm{GL}_n(\mathbb{C})$ such that the natural projection of $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$ induces a surjective homomorphism $\pi\colon \hat{G} \to G$. Recall that a *stem extension* of a finite group G is a group S containing a central subgroup $L \leqslant Z(S) \cap S'$ such that $S/L \cong G$. To solve this problem, Schur introduced what is now known as the *Schur multiplier* of G, which is a group isomorphic to $H^2(G, \mathbb{C}^*)$. A *Schur cover* of G is a stem extension of G by its Schur multiplier. A Schur cover is not generally unique up to isomorphism, but this is so if G is perfect (i.e. if $G = G'$), see Aschbacher [2, Section 33]. The next result shows that, under suitable hypotheses, Γ is determined by a representation of a Schur cover of G.

Proposition 6.2. *With the notation of Proposition 6.1, suppose that G and Γ_{f} are perfect. Let $\hat{G}$ be a Schur cover of G, let $H \leqslant G$ be a point stabiliser, and let $\hat{H} \leqslant \hat{G}$ be the full preimage of H under the projection $\hat{G} \to G$. Then $\Gamma_{\mathrm{f}} = \rho(\hat{G})$ for some representation ρ induced from a linear character of $\hat{H}$.*

Proof. Since G is perfect, the Schur cover $\hat{G}$ is finite, perfect, unique up to isomorphism, and *universal*, in the sense that the natural projection $\hat{G} \to G$ factors through any other central extension of G, [2, (33.1)–(33.4),(33.10)]. We saw in Proposition 6.1 that Γ_{f} is a central extension of G by a cyclic scalar subgroup $\langle \zeta_n I_n \rangle$. By assumption, Γ_{f} is perfect, which implies that Γ_{f} is an epimorphic image of $\hat{G}$, see [2, (33.8)], say with epimorphism $\psi\colon \hat{G} \to \Gamma_{\mathrm{f}}$. Since G is by hypothesis a transitive permutation group, its permutation representation is induced from the trivial character of a point stabiliser H. Let $L \leqslant \hat{G}$ be the central subgroup such that $\hat{G}/L \cong G$. By construction, $L \leqslant \hat{H}$ and $\hat{H}/L \cong H$. Since L is central, it follows that the permutation action of G on left costs of H coincides with the permutation action of $\hat{G}$ on left cosets of $\hat{H}$. Together with [14, Section 43, Exercise 1], this implies that an n-dimensional monomial representation ρ of $\hat{G}$ satisfies $\pi \circ \rho(\hat{G}) = \pi(\Gamma_{\mathrm{f}})$ if and only if ρ is induced from $\hat{H}$. This holds in particular for the epimorphism $\psi\colon \hat{G} \to \Gamma_{\mathrm{f}}$, as claimed. □

The conditions of Proposition 6.2 can be relaxed somewhat. If G is not perfect, a universal central extension does not exist, and a Schur cover is no longer unique up to isomorphism. Without assuming a perfect extension, the possibilities for Γ_{f} are classified by the group of 2-cocycles $Z^2(G, \langle \zeta_n I_n \rangle)$. Computational techniques are known for working with matrices developed from cocycles [15], but in the remainder of this paper we introduce a technique using Gröbner bases to build complex Hadamard matrices.

We now provide an example that illustrates how the Schur multiplier arises naturally in the construction of the Paley Hadamard matrices. Recall that the centraliser algebra of any 2-transitive permutation matrix group of degree n is 2-dimensional, so it is spanned by I_n and $J_n - I_n$, where J_n is the all-1s matrix. If $M = \alpha I_n + \beta J_n$ is a Hadamard matrix for complex α, β of norm 1, then $\alpha\alpha^* + (n-1)\beta\beta^* = n$ and $\alpha\beta^* + \beta\alpha^* + (n-2)\beta\beta^* = 0$, which implies $n = \alpha\alpha^* - \alpha\beta^* - \beta\alpha^* + \beta\beta^*$, and therefore $n \leqslant 4$. The action of $\mathrm{PSL}_2(q)$ on $q+1$ points is 2-transitive, which implies that for $q > 3$ there is no complex Hadamard matrix in the centraliser algebra of the $(q+1) \times (q+1)$ permutation matrix group $\mathrm{PSL}_2(q)$. In contrast to this, the next example shows that $\mathrm{SL}_2(q)$, considered as a suitable monomial cover of $\mathrm{PSL}_2(q)$, admits a Hadamard matrix in its centraliser algebra when $q \equiv 3 \bmod 4$.

Example 6.3 (Paley I Hadamard matrices)**.** Let $q \equiv 3 \bmod 4$ be a prime power and consider $G = \mathrm{PSL}_2(q)$ as a 2-transitive permutation matrix group of degree $q+1$. The Schur cover of G is isomorphic to $\mathrm{SL}_2(q)$ for all $q > 3$, see [35, Theorem 7.1.1]. Write $\mathbb{F}_q$ for the finite field with q elements, and let

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{F}_q, a \neq 0 \right\}$$

be the stabiliser in $\mathrm{SL}_2(q)$ of a 1-dimensional subspace. Let χ be the quadratic character of $\mathbb{F}_q$, that is, $\chi(a) = 1$ if $a \in \mathbb{F}_q$ is a nonzero quadratic residue, $\chi(a) = -1$ if $a \in \mathbb{F}_q$ is a nonzero quadratic non-residue, and $\chi(0) = 0$. By abuse of notation define

$$\chi \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \chi(a);$$

this is easily seen to be a character of H. The induced representation ρ of $\mathrm{SL}_2(q)$ has a centraliser algebra of rank 2. We now show that this centraliser algebra contains the $(q+1) \times (q+1)$ Paley I Hadamard matrix which is defined as

$$P = \begin{pmatrix} 1 & 1 \\ -1 & Q + I \end{pmatrix} \quad \text{where} \quad Q = \big(\chi(i-j)\big)_{i,j \in \mathbb{F}_q}.$$

We start by showing that

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ t_x = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \mid x \in \mathbb{F}_q \right\},$$

is a transversal to H in $\mathrm{SL}_2(q)$; this follows because every $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$ can be written uniquely as $m = ht$ where

$$h = \begin{pmatrix} ac^{-1}d - b & a \\ 0 & c \end{pmatrix} \in H \quad \text{and} \quad t = t_{c^{-1}d} \in T.$$

We now want to construct a matrix M with entries $m(s,t)$, $s,t \in T$, that lies in the centraliser algebra of ρ and equals P. Proposition 3.2 shows that we require $m(\mathbf{T}(g),\mathbf{T}(tg)) = m(1,t)\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(tg)$ for every $g \in G$ and $t \in T$; here we write $1 \in T$ for the identity matrix. First we consider $g = h \in H$. If $t \in T$, then

$$m(1,\mathbf{T}(th)) = m(\mathbf{T}(h),\mathbf{T}(th)) = m(1,t)\chi_{\mathbf{H}}(h)^{-1}\chi_{\mathbf{H}}(th)\,.$$

If $x \in \mathbb{F}_q$ and $h = \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in H$, then

$$t_x h = \begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & a^{-1}(b+xa^{-1})\end{pmatrix},$$

which implies that $\chi_{\mathbf{H}}(t_x h) = \chi(a^{-1})$, and therefore $\chi_{\mathbf{H}}(h)^{-1}\chi_{\mathbf{H}}(t_x h) = \chi(a)^{-1}\chi(a^{-1}) = \chi(a^{-1})^2 = 1$. If we define $m(1,t_x) = 1$ for all $x \in \mathbb{F}_q$, then it follows that the first row of M only has entries 1. To see that $m(t_i,1) = -1$ for $i \in \mathbb{F}_q$, choose $g = t_i$ and $t = t_0$; then $tg \in H$, and since $p \equiv 3 \bmod 4$, we obtain

$$m(t_i,1) = m(\mathbf{T}(g),\mathbf{T}(tg)) = m(1,t_0)\chi_{\mathbf{H}}(t_i)^{-1}\chi_{\mathbf{H}}(tg) = \chi(tg) = \chi(-1) = -1.$$

It remains to show that the bottom right $q \times q$ block matrix of P is $Q = (\chi(i-j))_{i,j\in\mathbb{F}_q}$. For this let $i,j \in \mathbb{F}_q$ be distinct and consider

$$g = \begin{pmatrix} (i-j)^{-1} & j(i-j)^{-1}) \\ 1 & i\end{pmatrix} \in G.$$

The factorisations

$$g = \begin{pmatrix} 1 & (i-j)^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & i\end{pmatrix} \quad \text{and} \quad t_0 g = \begin{pmatrix} i-j & -1 \\ 0 & (i-j)^{-1}\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & j\end{pmatrix}$$

show that $m(t_i,t_j) = m(\mathbf{T}(g),\mathbf{T}(t_0 g)) = m(1,t_0)\chi_{\mathbf{H}}(g)^{-1}\chi_{\mathbf{H}}(t_0 g) = \chi(1)\chi(i-j)$, as claimed.

In the previous example, we explicitly computed a basis $\{I_{q+1}, M\}$ for the centraliser algebra of $\mathrm{SL}_2(q)$ acting as a monomial group on $q+1$ points. It is also possible to apply the method of Section 5. One finds that the character table is equal to

$$T = \begin{pmatrix} 1 & g \\ 1 & -g\end{pmatrix} \quad \text{where} \quad g = \sum_{j\in\mathbb{F}_q}\left(\frac{j}{q}\right)e^{2\pi\imath j/q}.$$

is a Gauss sum (see [32, Section 6.3]) and (j/q) denotes a Legendre symbol. For a prime $q \equiv 3 \bmod 4$ the sum is evaluated as $g = \imath\sqrt{q}$, while $g = \sqrt{q}$ is real for $q \equiv 1 \bmod 4$, see [32, Proposition 6.3.2]. The extension of this result to prime powers q is routine. Hence, when $q \equiv 3 \bmod 4$, then $(g+1)(g^*+1) = q+1$, and the existence of the Paley I matrices is immediate from Lemma 2.2. Similarly, for $q \equiv 1 \bmod 4$, a computation shows that $M_q = I_{q+1} + \imath M$ is complex Hadamard.

When $q \equiv 1 \bmod 4$, the group $\mathrm{SAut}(M_q)$ contains both a monomial cover of $\mathrm{SL}_2(q)$ and the scalar subgroup $\langle(\imath I_{q+1}, \imath I_{q+1})\rangle$, see [15, Section 17.2]. Since $q+1 \equiv 2 \bmod 4$, the determinant of $\imath I_{q+1}$ is -1. Since $\mathrm{SL}_2(q)$ is perfect, it follows from the definition of the Schur multiplier that the monomial preimage of $\mathrm{SL}_2(q)$ in Γ is contained in the commutator subgroup Γ', and hence cannot contain $\pm\imath I_{q+1}$. Thus the automorphism group is isomorphic to a central product of C_4 and $\mathrm{SL}_2(q)$, intersecting in a cyclic subgroup of order 2.

We have shown that if M is a complex Hadamard matrix with $\pi_1(\mathrm{SAut}(M)) = \Gamma$, then $M \in \mathrm{C}(\Gamma)$. The next example shows that this conclusion no longer holds when the strong automorphism group is replaced with the (ordinary) automorphism group. It may happen that $\Gamma_1 = \pi_1(\mathrm{Aut}(M))$ and $\Gamma_2 = \pi_2(\mathrm{Aut}(M))$ are induced from non-conjugate subgroups, in which case M belongs to the *intertwiner* of distinct representations rather than a centraliser algebra. This is illustrated in the next example.

Example 6.4. The Sylvester Hadamard matrix of order 2^n can be defined as $S_n = [(-1)^{x^\intercal y}]_{x,y\in\mathbb{F}_2^n}$ where $\mathbb{F}_2^n$ denotes the space of n-dimensional column vectors over $\{0,1\}$. Moreover, for $n \geqslant 2$ it is known that

$$\mathrm{Aut}(S_n) \cong Z(\mathrm{Aut}(S_n)) \times (C_2^n \rtimes \mathrm{AGL}_n(2)), \tag{10}$$

where C_2^n is the n-fold direct product of the cyclic group of size 2 and $Z(\mathrm{Aut}(S_n)) = \langle(-I_{2^n}, -I_{2^n})\rangle$, see [15, Theorem 9.2.4]. For $x, y \in \mathbb{F}_2^n$ denote by r_x and c_y the row and column of S_n labelled by x and y, respectively. The action of $\mathrm{AGL}_n(2)$ on rows and columns is described in detail in [19]: if $(v, A) \in \mathrm{AGL}_n(2)$ is the transformation $x \mapsto Ax + v$, then

$$r_x \cdot (v, A) = r_{Ax+v} \quad \text{and} \quad c_y \cdot (v, A) = (-1)^{v^\intercal A^{-1} y} c_{(A^{-1})^\intercal y}.$$

Observe that the action on rows is a 2-transitive permutation action, whereas the column c_0 is stabilised. Let $V \leqslant \mathrm{AGL}_n(2)$ be the subgroup of translations, and observe that $\pi_1(V)$ is a regular permutation group while $\pi_2(V)$ is trivial. Hence, S_n does not belong to the centraliser of $\mathrm{AGL}_n(2)$, although it *does* admit an action of a 2-transitive permutation group. In fact, the stabiliser of a row in $\mathrm{Aut}(S_n)$ is not conjugate to the stabiliser of a column (the projections onto $\mathrm{GL}_n(2)$ are the stabiliser of a point and of a hyperplane, respectively). Hence the actions on rows and columns are linearly equivalent, but not monomially equivalent, and S_n belongs to the intertwiner of these representations, but not to the centraliser algebra of either of these representations. Specifically, $\pi_1(\mathrm{Aut}(S_n))$ and $\pi_2(\mathrm{Aut}(S_n))$ are equivalent as monomial representations, but the permutation representations $\pi(\pi_1(\mathrm{Aut}(S_n)))$ and $\pi(\pi_2(\mathrm{Aut}(S_n)))$ are inequivalent as the former is the the regular representation of V while the latter is trivial.

A similar phenomenon where inequivalent permutation representations lift to equivalent monomial representations can be used to construct complex Hadamard matrices of orders 6 and 12 related to the outer automorphisms of Sym_6 and the Mathieu group M_{12} respectively, [21, 24].

7. COMPLEX HADAMARD MATRICES ADMITTING A LOW RANK AUTOMORPHISM GROUP

In Example 6.3, we saw that for $q \equiv 3 \bmod 4$ the group $\mathrm{SL}_2(q)$ is isomorphic to a subgroup of the strong automorphism group of the $(q+1) \times (q+1)$ Paley type I matrix. It is natural to ask which other low-rank permutation groups act as the group of strong automorphisms of a complex Hadamard matrix: we recall results in this direction by Moorhouse and Chan, and then describe the results of a computer classification. Detailed information about the 2-transitive permutation groups suffices to carry out this programme.

Theorem 7.1 (Moorhouse, [37]). *Suppose that M is a complex Hadamard matrix, and that G is a 2-transitive permutation group contained in $\pi \circ \pi_1(\mathrm{Aut}(M))$. Then one of the following occurs.*

(1) *$G \cong \mathrm{AGL}_n(p)$ in its natural action on p^n points, and M is a generalised Sylvester matrix (the character table of an elementary abelian p-group),*

(2) *$G \cong \mathrm{PSL}_2(q)$ acting on $q+1$ points, and M is a Paley matrix of order $q+1$ that is real for $q \equiv 3 \bmod 4$ and over 4^{th} roots of unity for $q \equiv 1 \bmod 4$,*

(3) *$G \cong \mathrm{Sp}_{2d}(q)$ where q is a power of 2 and $q^{2d} \geqslant 16$, and M is of order q^{2d},*

(4) *G is isomorphic to one of $\mathrm{Alt}_6, M_{12}, \mathrm{P\Sigma L}_2(8)$ or $\mathrm{Sp}_6(2)$; and M is of order 6, 12, 28 or 36 respectively.*

We emphasise that Moorhouse studies the full automorphism group rather than the group of strong automorphisms. Throughout our classification, we require that $\pi_1(\mathrm{Aut}(M))$ and $\pi_2(\mathrm{Aut}(M))$ are conjugate not only as linear representations, but as monomial representations: that is, the representations are induced from the same subgroup, or equivalently, M belongs to the centraliser algebra of $\pi_1(\mathrm{Aut}(M))$. Moorhouse does not make this assumption: he allows representations induced from non-conjugate subgroups, equivalently M may belong to the intertwiner of representations which are not monomially equivalent. The Sylvester matrices are an example of this phenomenon, see Example 6.4.

Complex Hadamard matrices in the centraliser algebra of a strongly regular graph have been considered by Chan and Godsil [10, 11]. Recall that a (v, k, λ, μ)-strongly regular graph is a k-regular graphs on v vertices in which any two adjacent vertices share λ common neighbours, while any pair of non-adjacent vertices share μ neighbours. The centraliser algebra of a rank 3 permutation group of even order is spanned by the adjacency matrix of a strongly regular graph, see [7, Section 1.1]. Not every strongly regular graph admits a rank 3 group action, thus we only state the following special case of relevant to our purposes.

Theorem 7.2 (Chan, Godsil, [10, 11]). *Suppose that M is a complex Hadamard matrix, and that G is a rank 3 permutation group contained in $\pi \circ \pi_1(\mathrm{SAut}(M))$. Then one of the following holds:*

(1) *$n = 4t^2$ for an integer t, and G is a group of automorphisms of a $(4t^2, 2t^2 - t, t^2 - t, t^2 - t)$-strongly regular graph (equivalently a Menon Hadamard design).*
(2) *$n = 4t^2 - 1$ for an integer t, and G is a group of automorphisms of a $(4t^2 - 1, 2t^2, t^2, t^2)$-strongly regular graph.*
(3) *$n = 4t^2 + 4t + 1$ and G is a group of automorphisms of a $(4t^2 + 4t + 1, 2t^2 + 2t, t^2 + t - 1, t^2 + t)$-strongly regular graph; here either t or $t^2 + t$ is an integer.*
(4) *$n = 4t^2 + 4t + 2$ for an integer t, and G is a group of automorphisms of a $(4t^2 + 4t + 2, 2t^2 + t, t^2 - 1, t^2)$-strongly regular graph.*

Recall that the subdegrees of a transitive permutation group are the lengths of the orbits of a point stabiliser. These subdegrees are given in the row of the character table of the centraliser algebra corresponding to the trivial irreducible character. In fact, for the groups related to strongly regular graphs considered above, it follows from [7, Section 1.1.4] that the subdegrees determine all remaining entries of the character table. Thus, Theorem 7.2 gives a condition on the subdegrees of a rank 3 permutation matrix group G which is necessary and sufficient for $\mathrm{C}(G)$ to contain a complex Hadamard matrix. A classification of rank 3 permutation groups, including their subdegrees is available in the literature, see for example [36]. Thus, while we do not give the classification explicitly here, the classification is in principle known. Chan's classification applies only to rank 3 permutation matrix groups in which all orbitals are self-paired (that is, the centraliser algebra has a basis of symmetric matrices). This omits, for example, the Frobenius groups of order $\binom{p}{2}$ where $p \equiv 3 \bmod 4$; such groups have been considered previously in the PhD thesis of Nuñez Ponasso [39].

Apart from these results, the literature on classifying complex Hadamard matrices by their automorphism groups is rather sparse. It appears that there is only a single real Hadamard matrix in the literature which admits a primitive-but-not-2-transitive automorphism group. This matrix has order 144 and was described by Marshall Hall in [25]. To our knowledge, monomial covers of rank 3 permutation groups have not been investigated, nor have groups of higher rank.

7.1. **Computational classification results.** We conclude this paper with some computational results, building on the theory developed thus far. Given a transitive permutation group $G \leqslant \mathrm{Sym}(\Omega)$ and point stabiliser $H = G_\omega$ (with $\omega \in \Omega$), our algorithm proceeds as follows.

(1) Construct a Schur cover $\hat{G}$ for G, and compute the full preimage $\hat{H} \leqslant \hat{G}$ of $H \leqslant G$.
(2) For each linear character χ of $\hat{H}$, compute the character of the induced representation $\rho = \chi \uparrow_{\hat{H}}^{\hat{G}}$, and compute the character table $T = \mathrm{CT}(\rho)$ of its centraliser algebra via Proposition 5.1. Denote by $\mathbb{K}$ the field of definition of T, and by r the number of rows in T.
(3) Proceed as in Example 2.1: Define $K = \mathbb{K}[\alpha_1, \alpha_{1c}, \alpha_2, \alpha_{2c}, \ldots, \alpha_r, \alpha_{rc}]$ and construct a Gröbner basis of the ideal $\mathcal{I}$ generated by the polynomials that encode the norm conditions for $\alpha_1, \ldots, \alpha_r$ to define a complex Hadamard matrix. The result is an ideal (defined over $\mathbb{K}$) in which variables are eliminated according to a monomial ordering; due to the structure of our original polynomial

equations, there exists a polynomial in the Gröbner basis that expresses one of the variables in terms of a univariate polynomial.

(4) Solve for the roots of a univariate polynomial in the Gröbner basis; for each solution, substitute the values in the remaining polynomials and then iterate this process. This way it is possible to find all points in the variety.

(5) If $(\alpha_1, \alpha_{1c}, \ldots, \alpha_r, \alpha_{rc})$ is one of the points in the variety, then this defines a complex Hadamard matrix provided that α_{ic} is the complex conjugate of α_i for all i; the latter test is still necessary because it does not follow from the imposed condition $\alpha_i\alpha_{ci} = 1$. Once this is verified, the resulting complex Hadamard matrix may be constructed explicitly via Proposition 3.2.

If G and Γ_f are perfect, then, by Proposition 6.2, the above algorithm produces all complex Hadamard matrices such that $\pi \circ \pi_1(\mathrm{SAut}(M) \cong G$. When these hypotheses do not hold, the algorithm produces matrices, but without a guarantee of completeness. This procedure requires some heavy machinery: computation of Schur multipliers is a notorious problem in group theory, naive implementations of Proposition 5.1 require an iteration over all elements of G, and the complexity of computing a Gröbner basis is well-known to be doubly exponential in the number of variables. Nevertheless, the algorithm seems practical for permutation groups of order $\leqslant 10^8$ and of rank $\leqslant 5$. We illustrate this approach with an explicit example.

Example 7.3. There is, up to conjugacy, a unique group $G = C_7 \rtimes C_3$ that acts transitively on 7 points, with point stabiliser $H \cong C_3$. Let χ be a non-trivial character of H, with induced monomial representation ρ. Let $\omega = \zeta_3$ be a primitive 3^{rd} root of unity. The centraliser $\mathrm{C}(\rho)$ is spanned by $\{I_7, M_1, M_2\}$ where

$$M_1 = \begin{pmatrix} 0 & 1 & \omega^2 & \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & \omega & \omega^2 \\ 0 & 0 & 0 & \omega^2 & \omega & \omega^2 & 0 \\ 0 & \omega^2 & 0 & 0 & \omega^2 & 0 & \omega \\ 1 & 1 & 0 & 0 & 0 & \omega & 0 \\ \omega^2 & 0 & 0 & 1 & 0 & 0 & \omega \\ \omega & 0 & 1 & 0 & \omega & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 \\ 1 & 0 & 0 & \omega & 1 & 0 & 0 \\ \omega & \omega & 0 & 0 & 0 & 0 & 1 \\ \omega^2 & 0 & \omega & 0 & 0 & 1 & 0 \\ 0 & 0 & \omega^2 & \omega & 0 & 0 & \omega^2 \\ 0 & \omega^2 & \omega & 0 & \omega^2 & 0 & 0 \\ 0 & \omega & 0 & \omega^2 & 0 & \omega^2 & 0 \end{pmatrix}.$$

These matrices are unique up to conjugation by permutation matrices and multiplication by scalars. It is not immediately obvious whether G acts on a complex Hadamard matrix M, that is, whether $\mathrm{C}(\rho)$ contains a complex Hadamard matrix. The latter holds if and only if there exist complex numbers α_1, α_2 of norm 1 such that $M = I_7 + \alpha_1 M_1 + \alpha_2 M_2$ with $MM^* = 7I_7$. The character table for $\mathrm{C}(\rho)$ is given below, along with the linear equation corresponding to a complex Hadamard matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & \frac{-1-\imath\sqrt{7}}{2} & \frac{-1+\imath\sqrt{7}}{2} \\ 3 & \frac{-1+\imath\sqrt{7}}{2} & \frac{-1-\imath\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

One solution is given by $\alpha_1 = (-3 + \imath\sqrt{7})/4$ and $\alpha_2 = 1$; there are three more solutions, obtained by swapping α_1 with α_2 and taking complex conjugates. It follows that G does act on complex Hadamard matrices defined over the field $\mathbb{Q}(\zeta_{21})$; the latter is the smallest cyclotomic field containing $\imath\sqrt{7}$ and ω.

In Table 1 we report on our findings for primitive permutations groups of degree $n \leqslant 15$ and of rank 3; a database of such groups is available in MAGMA [6]. The notation used in the table is as follows. We denote by $\hat{G}$ a Schur cover of G, and $\hat{H}$ is the preimage in $\hat{G}$ of a point stabiliser in G. We denote by d the largest value for which a primitive d^{th} root of unity appears in the character (thus, a "1" indicates a permutation representation). The next column gives a minimal polynomial for the smallest field extension containing the entries of the complex Hadamard matrix. As before, C_n denotes a cylic group of order n, and ζ_n is a primitive n^{th} root of unity.

n	G	$\lvert\hat{G}\rvert/\lvert G\rvert$	$\hat{H}/\hat{H}'$	d	Minimal polynomial over $\mathbb{Q}(\zeta_d)$	subdegrees
7	$C_7 \rtimes C_3$	1	C_3	$d \mid 3$	$x^2 + \frac{3}{2}x + 1$	$[1, 3, 3]$
9	$C_3^2 \rtimes C_4$	3	C_{12}	$d \mid 12$	$x^2 - \frac{1}{2}x + 1$	$[1, 4, 4]$
10	Alt_5	2	C_4	1	$x^2 + \frac{1}{2}x + 1; \quad x^2 + 1$	$[1, 3, 6]$
10	Alt_5	2	C_4	2	$x^4 - 8x^2 + 36$	$[1, 3, 6]$
11	$C_{11} \rtimes C_5$	1	C_5	$d \mid 5$	$x^2 + \frac{5}{3}x + 1$	$[1, 5, 5]$
13	$C_{13} \rtimes C_6$	1	C_6	1	$x^4 + \frac{1}{3}x^3 + \frac{5}{3}x^2 + \frac{1}{3}x + 1$	$[1, 6, 6]$
13	$C_{13} \rtimes C_6$	1	C_6	2	$x^4 - \frac{1}{3}x^3 + \frac{5}{3}x^2 - \frac{1}{3}x + 1$	$[1, 6, 6]$
13	$C_{13} \rtimes C_6$	1	C_6	3	$x^8 - \frac{1}{3}x^7 - \frac{14}{9}x^6 - \frac{1}{9}x^5 + \frac{5}{3}x^4 - \frac{1}{9}x^3 - \frac{14}{9}x^2 - \frac{1}{3}x + 1$	$[1, 6, 6]$
13	$C_{13} \rtimes C_6$	1	C_6	6	$x^8 + \frac{1}{3}x^7 - \frac{14}{9}x^6 - \frac{1}{9}x^5 + \frac{5}{3}x^4 - \frac{1}{9}x^3 - \frac{14}{9}x^2 + \frac{1}{3}x + 1$	$[1, 6, 6]$
15	Alt_6	6	C_6	1	$x^2 + \frac{5}{3}x + 1; \quad x^2 - \frac{7}{4}x + 1$	$[1, 6, 8]$

TABLE 1. Primitive groups degree $n \leqslant 15$, rank 3 acting on complex Hadamard matrices.

Apart from the matrices in the centraliser algebra of a permutation group of rank 3 (described by Godsil, Chan and Nuñez Ponasso), we believe that none of the matrices described below have appeared in the literature; they do not appear in the database of complex Hadamard matrices [9, 44]. We performed extensive computations with primitive groups of rank $\leqslant 5$ and degree $\leqslant 200$; the resulting complex Hadamard matrices will be made available at `https://github.com/pocathain/CHM`.

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