Lab3 SSM

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Question 1

A Hidden Markov model is implemented with the following features:

Transition model:

$$p(z_t|z_{t-1}) = \frac{1}{3}[N(z_t|z_{t-1},1) + N(z_t|z_{t-1}+1,1) + N(z_t|z_{t-1}+2,1)]$$

Emission model:

$$p(x_t|z_t) = \frac{1}{3}[N(x_t|z_t,1) + N(x_t|z_t+1,1) + N(x_t|z_t-1,1)]$$

Initial model:

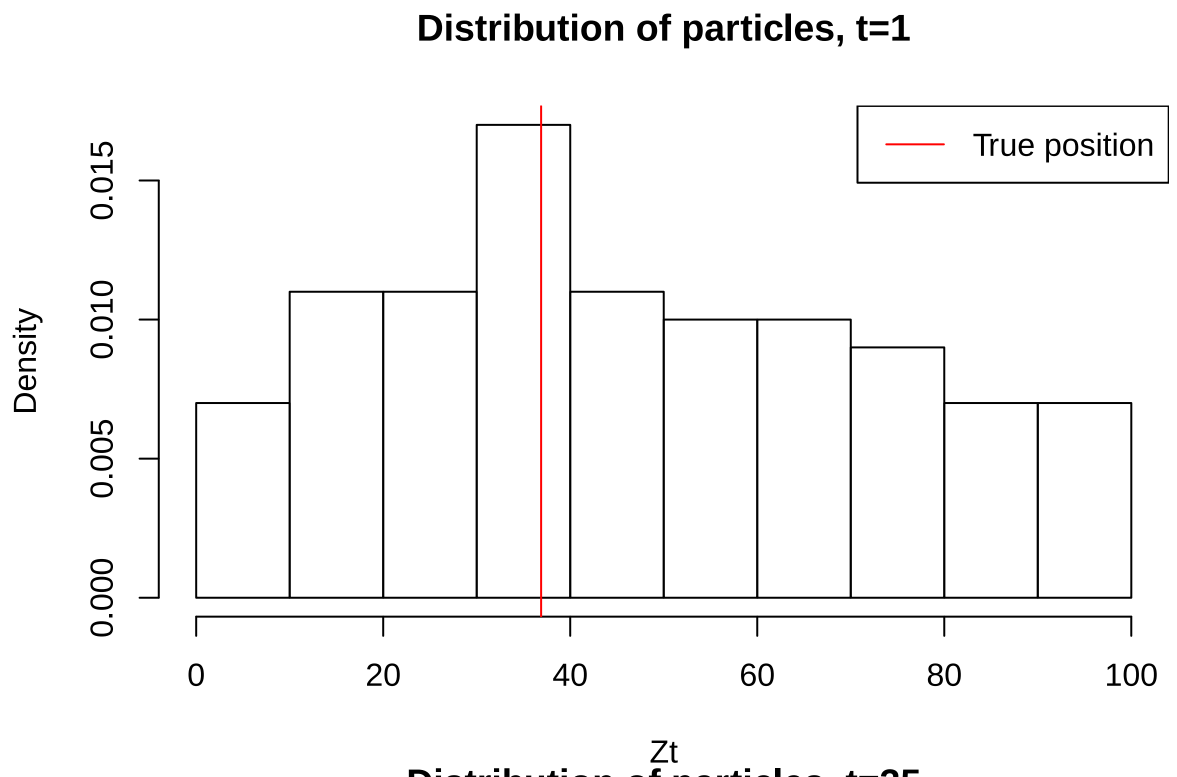
$$p(z_1) = Uniform(0,100)$$

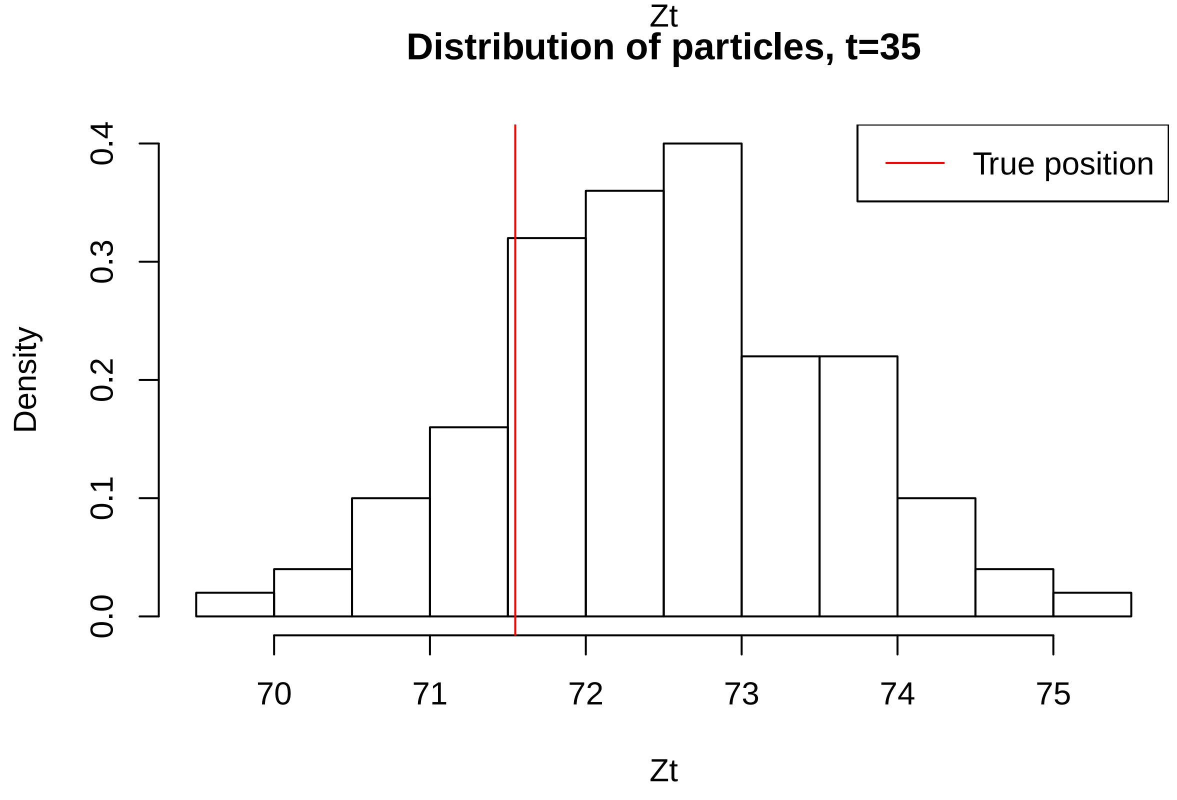
The functions are as follows:

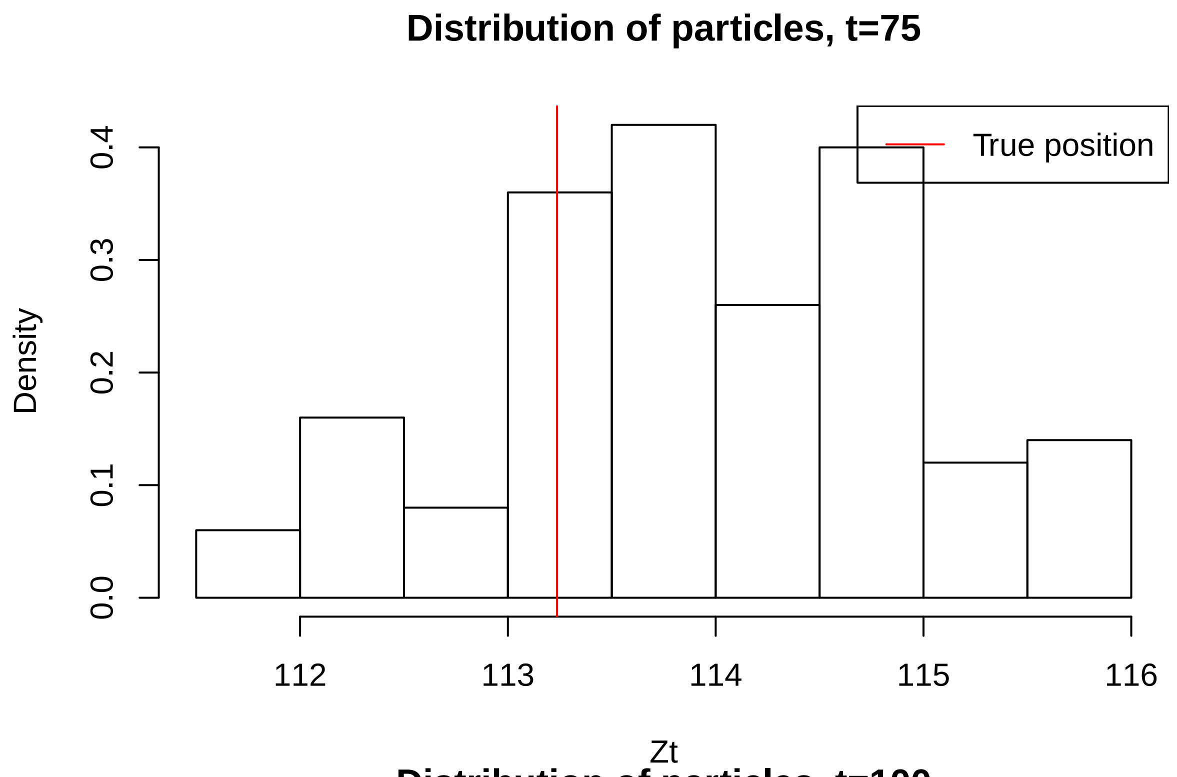
```r
# M - number of particals Iter - number of time steps T z is
# latent variable x is the observation

# Function for sampling from a mixture model
sampleMix <- function(mean1, mean2, mean3, sdX) {
    u <- sample(1:3, 1)
    means <- c(mean1, mean2, mean3)
    return(rnorm(1, mean = means[u], sd = sdX))
}

## Function that creates the SSM model
create_SSM <- function(z0, TT, sdX = 1) {
    Zt <- c(z0)

    Xt <- sampleMix(mean1 = z0, mean2 = (z0 - 1), mean3 = (z0 +
        1), sd = 1)

    for (i in 2:TT) {
        Zt[i] <- sampleMix(mean1 = Zt[i - 1], mean2 = (Zt[i -
            1] + 1), mean3 = (Zt[i - 1] + 2), sdX = 1)
        Xt[i] <- sampleMix(mean1 = Zt[i], mean2 = (Zt[i] - 1),
            mean3 = (Zt[i] + 1), sdX = sdX)
    }
    return(list(sates = Zt, observations = Xt))

}

generateX <- function(Zt, TT, sdX = 1) {
    Xt <- c()
    for (i in 1:TT) {
        Xt[i] <- sampleMix(mean1 = Zt[i], mean2 = (Zt[i] - 1),
            mean3 = (Zt[i] + 1), sdX = sdX)
    }
    return(Xt)
}

## Particle filter function
particles <- function(Obs, M, Iter, sdX = 1, correction = TRUE) {
    Xt <- Obs

    # Initialise particles for the latent variable
    Zt <- matrix(ncol = Iter, nrow = M)
    wt <- matrix(ncol = Iter, nrow = M)
    Zt_bar <- matrix(ncol = Iter, nrow = M)
    # fill the initial time step
    Zt[, 1] <- runif(M, 0, 100)
    # Add arbitrary values in order for the indexing to work
    # later
    Zt_bar[, 1] <- rep.int(1, M)
    wt[, 1] <- rep.int(1, M)

    for (t in 2:Iter) {
        # Prediction
        for (m in 1:M) {
            # sample from the transition model (current Z_t depends on
            # the Z_{t-1}) Zt_bar[m,t] <-
            # (rnorm(1,mean=Zt[m,t-1],sd=1)+rnorm(1,mean=(Zt[m,t-1]+1),sd=1)+rnorm(1,mean=(Zt[m,t-1]+2)
            Zt_bar[m, t] <- sampleMix(mean1 = Zt[m, t - 1], mean2 = (Zt[m,
                t - 1] + 1), mean3 = (Zt[m, t - 1] + 2), sdX = 1)
            # Calculate weights
            wt[m, t] <- (dnorm(Xt[t], mean = Zt_bar[m, t], sd = sdX) +
                dnorm(Xt[t], mean = (Zt_bar[m, t] - 1), sd = sdX) +
                dnorm(Xt[t], mean = (Zt_bar[m, t] + sdX), sd = 1))/3
        }
        # Correction
        if (correction) {
            Zt[, t] <- sample(Zt_bar[, t], M, replace = TRUE,
                prob = wt[, t])
        } else {
            Zt[, t] <- Zt_bar[, t]
        }

    }
    return(list(Zt = Zt, Zt_bar = Zt_bar, wt = wt))

}
```

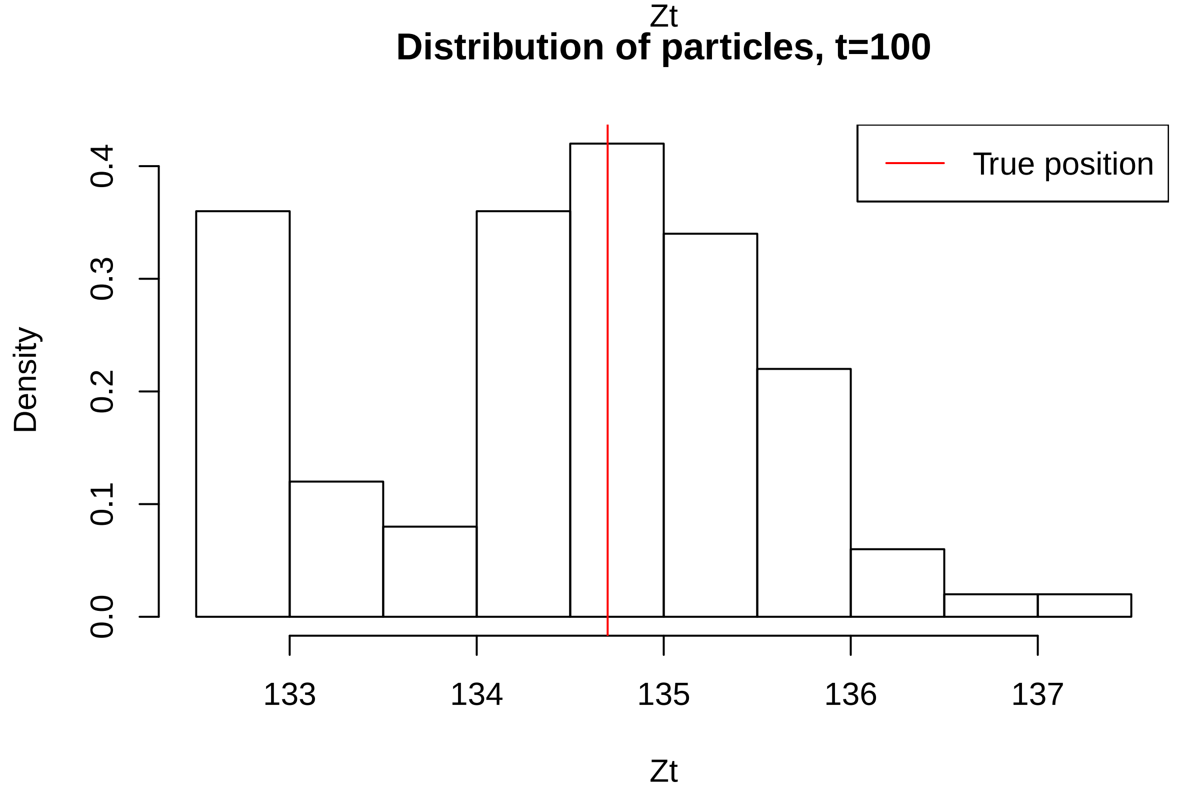
Afterward the states $z_{1:T}$ (using the initial and transition models) and observations $x_{1:T}$ of sensor readings (using the emission model) are simulated for $T = 100$ time steps. The observations are used in the particle filter algorithm with 100 particles in order to predict the possible states (location of the robot).

I chose time points 1, 35, 75, and 100 for which I plot the distribution of the particles (a histogram), and I indicate on them what is the true value of the z_t at that particular time step.


Distribution of particles, t=1
True position
Density
0.000
0.005
0.010
0.015
0
20
40
60
80
100
Zt


Distribution of particles, t=35
True position
Density
0.0
0.1
0.2
0.3
0.4
70
71
72
73
74
75
Zt

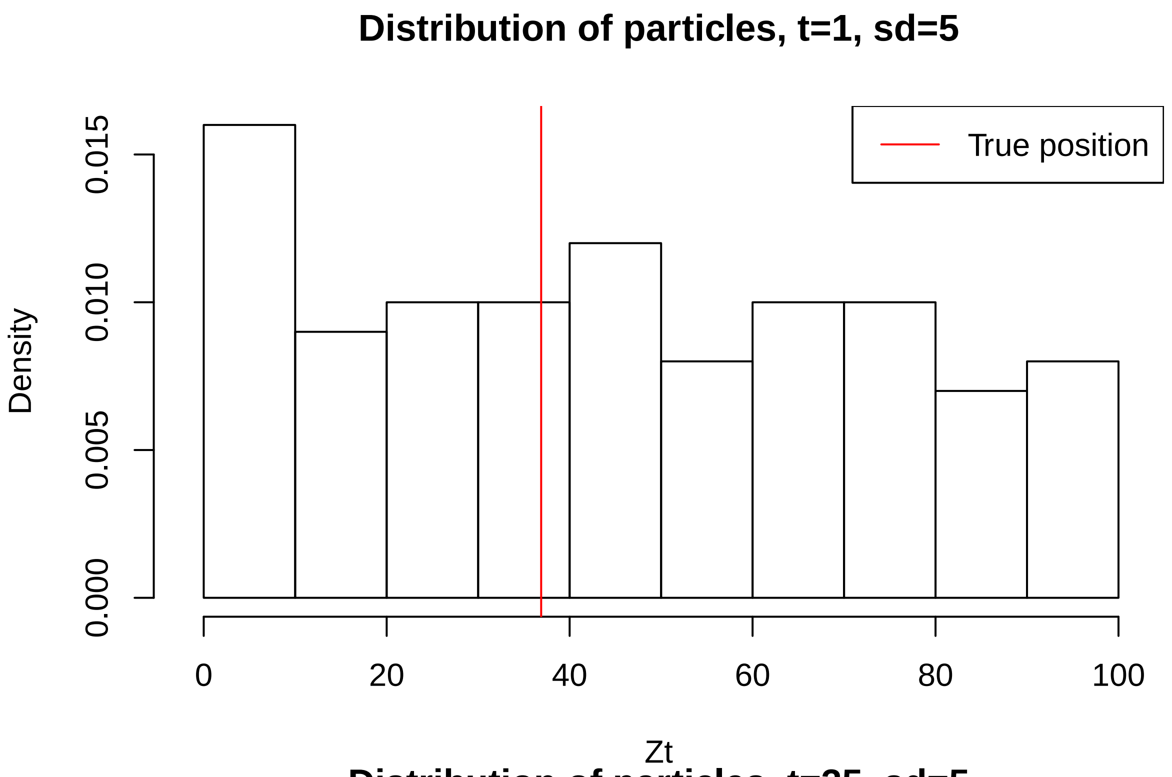


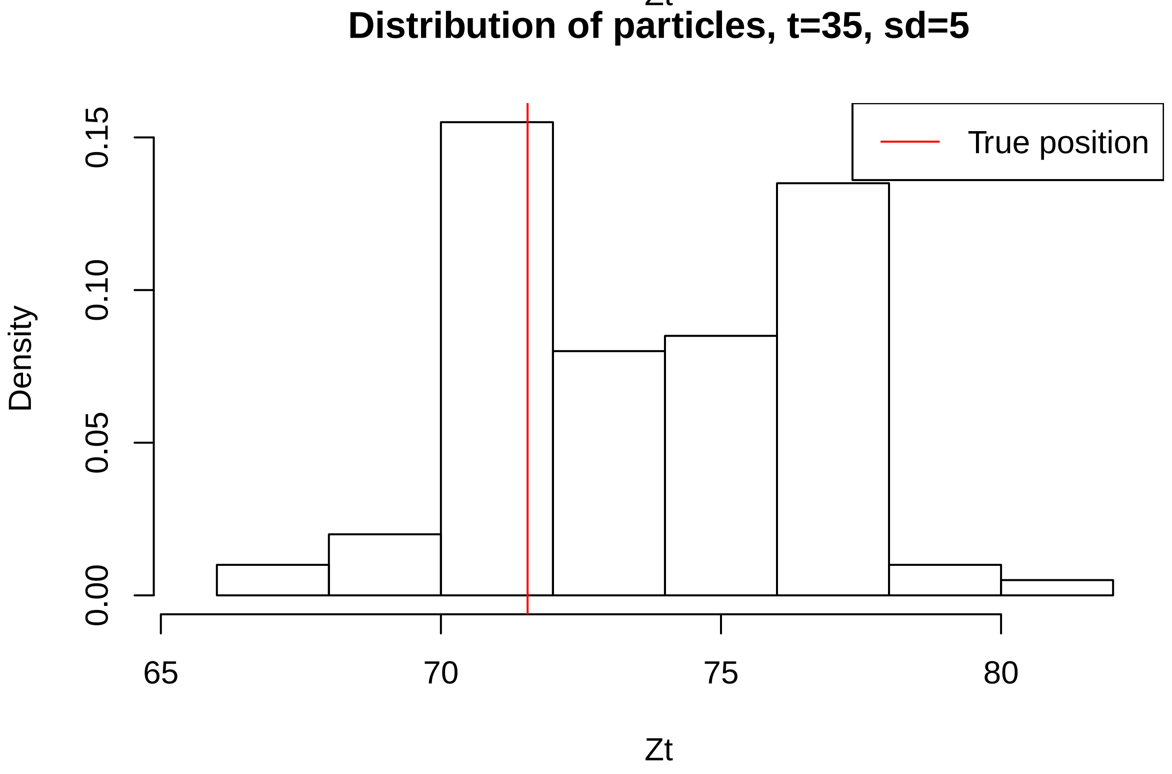


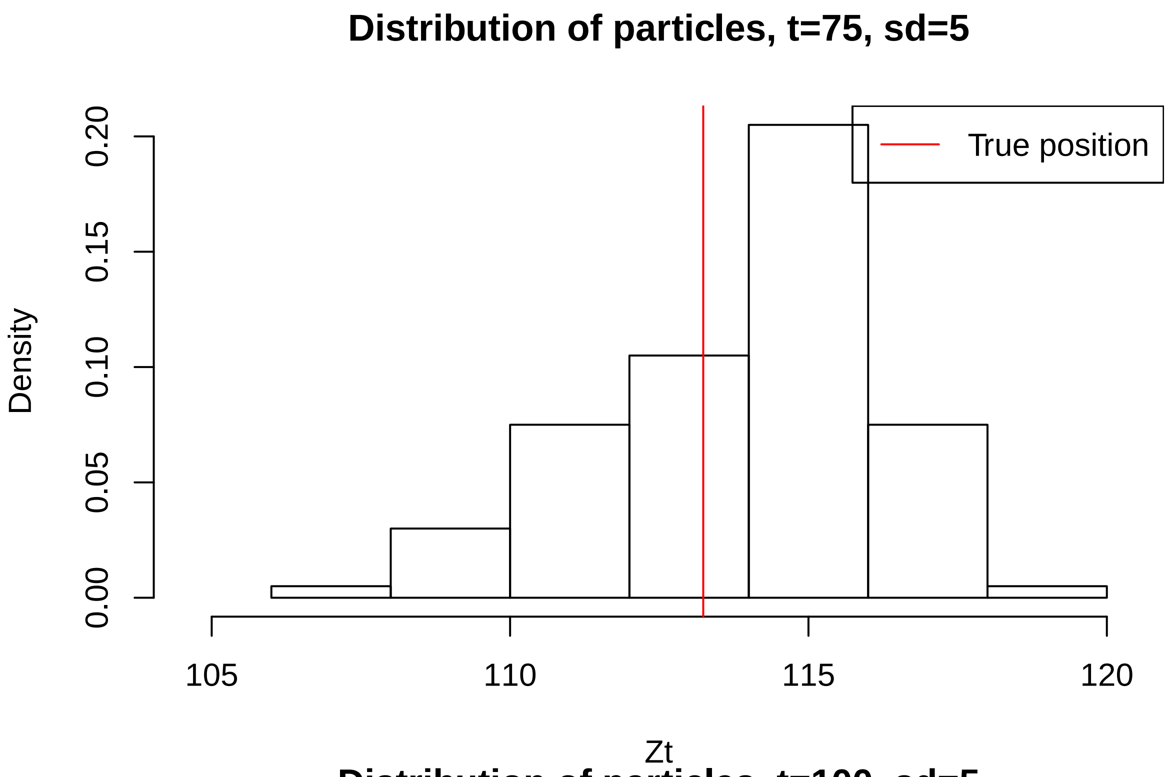
From the histograms, I conclude that the particle algorithm worked pretty well when the standard deviation was 1. In all four plots, the red line falls on one of the highest bars: this indicates that the algorithm quite accurately predicted the robots location (with a reasonable error).

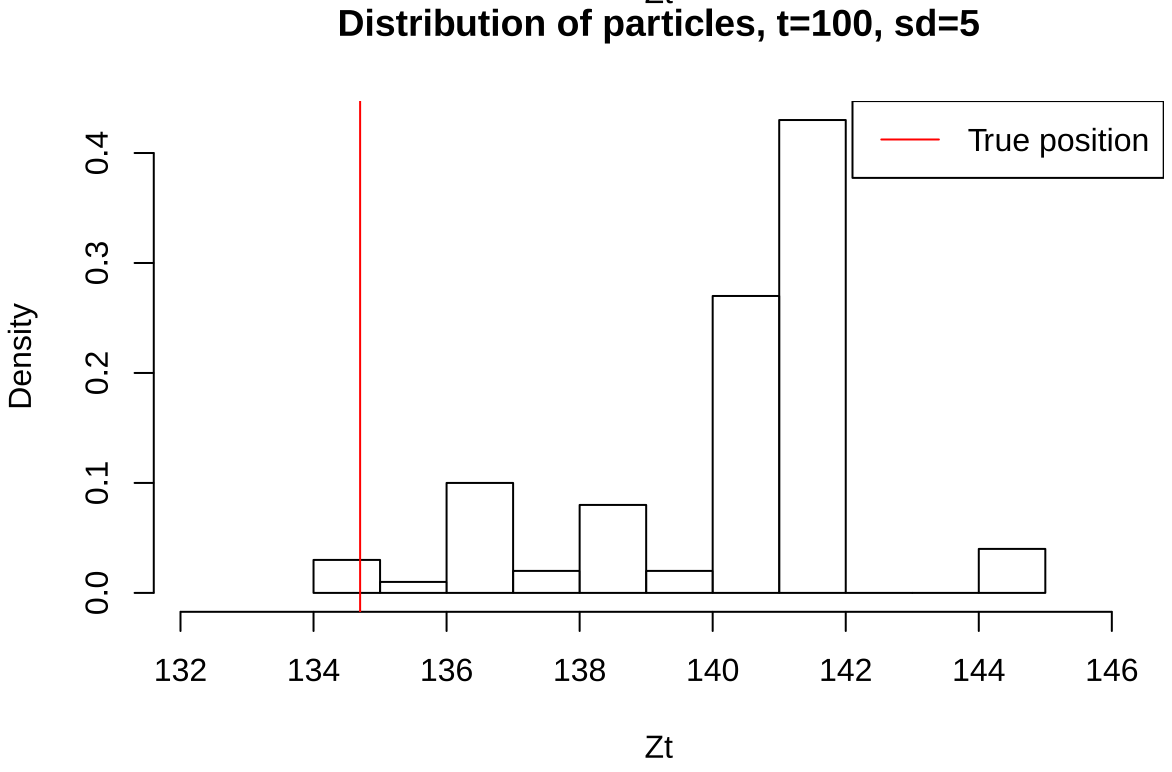
Question 2

In this question the HMM is modified by changing the emission models. Firstly, the standard deviation is set to 5, and later it is increased to 50. The states $z_{1:T}$ are kept the same, but new observations $x_{1:T}$ are generated. The particle filter algorithm is applied again on both cases.

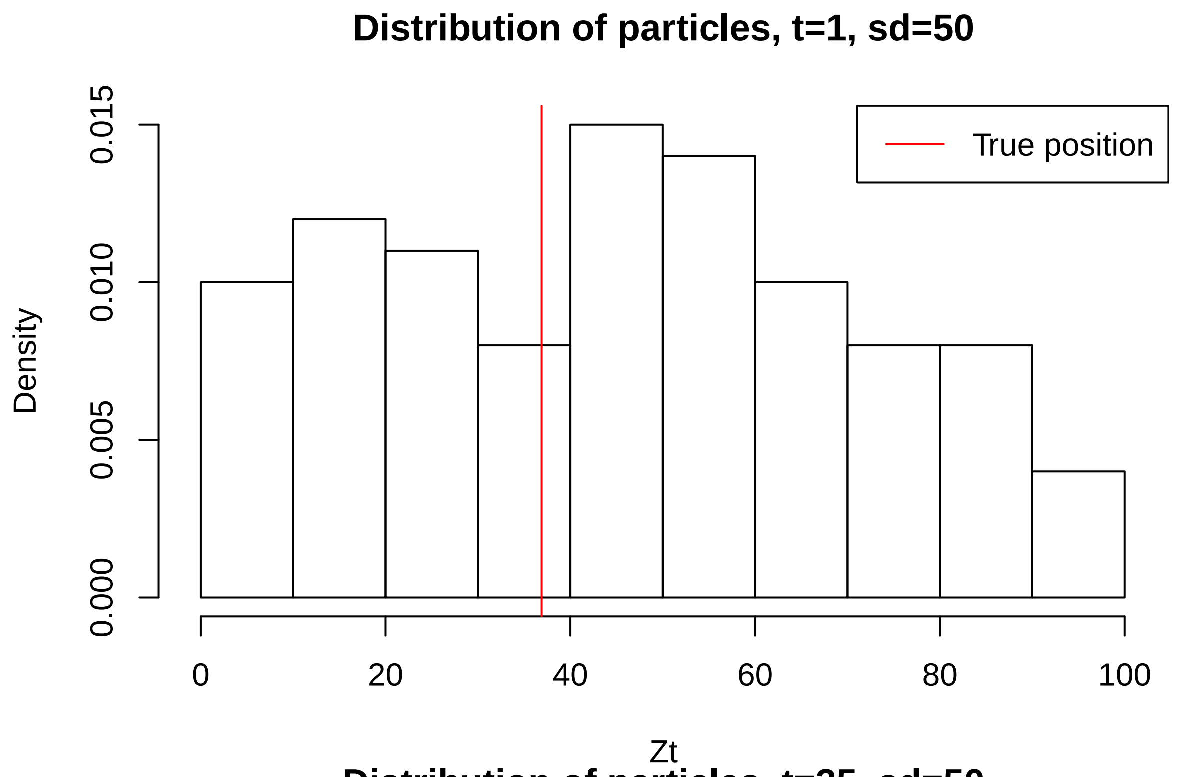


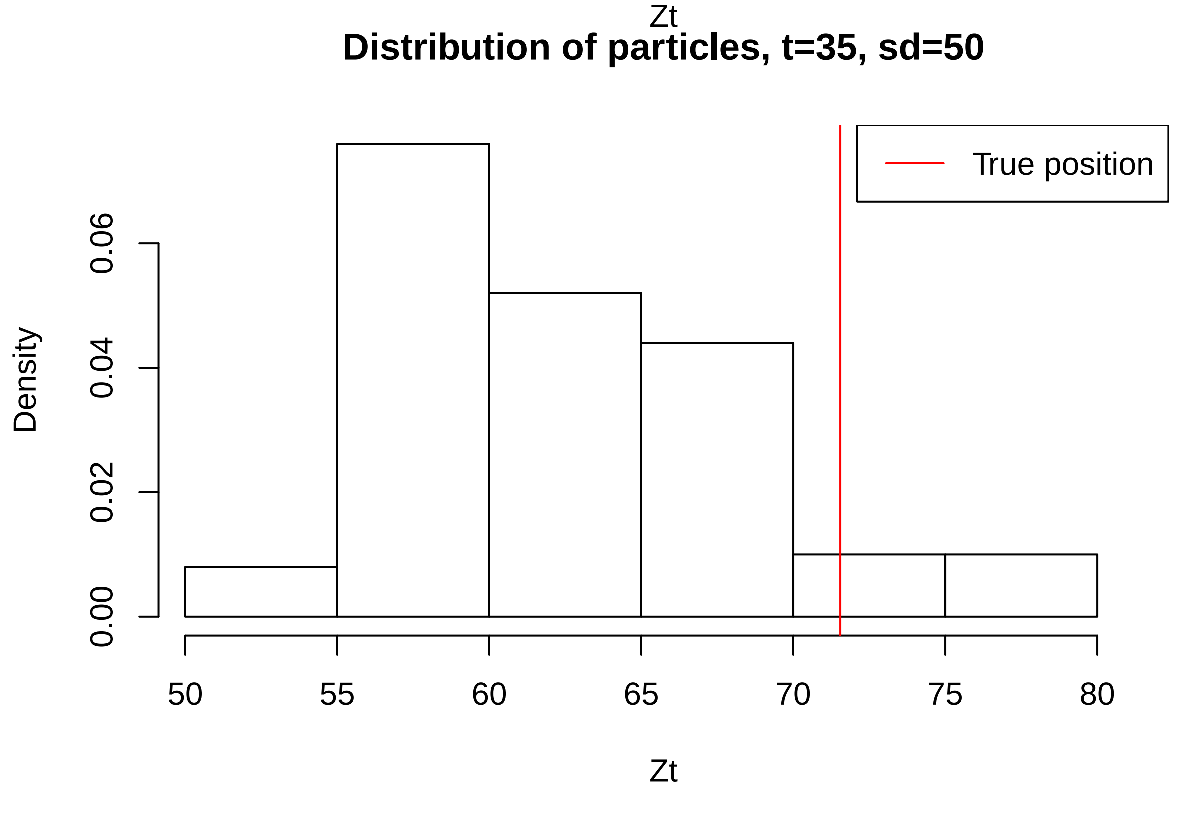


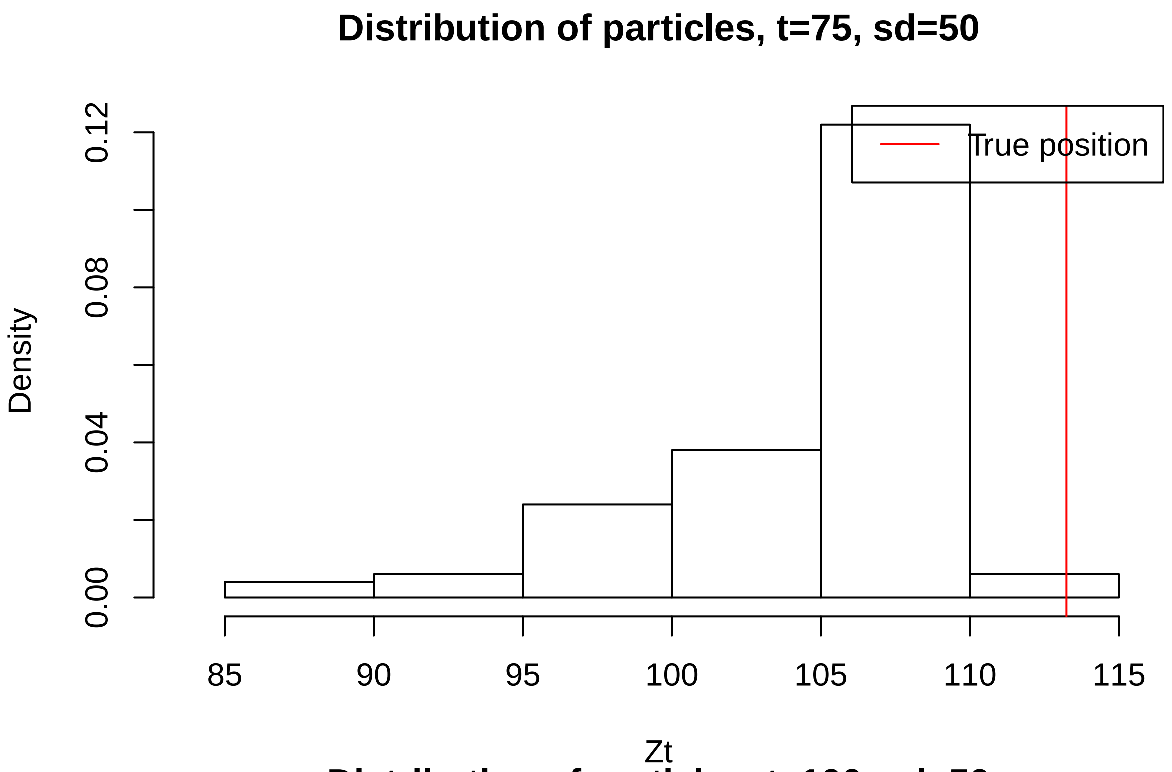


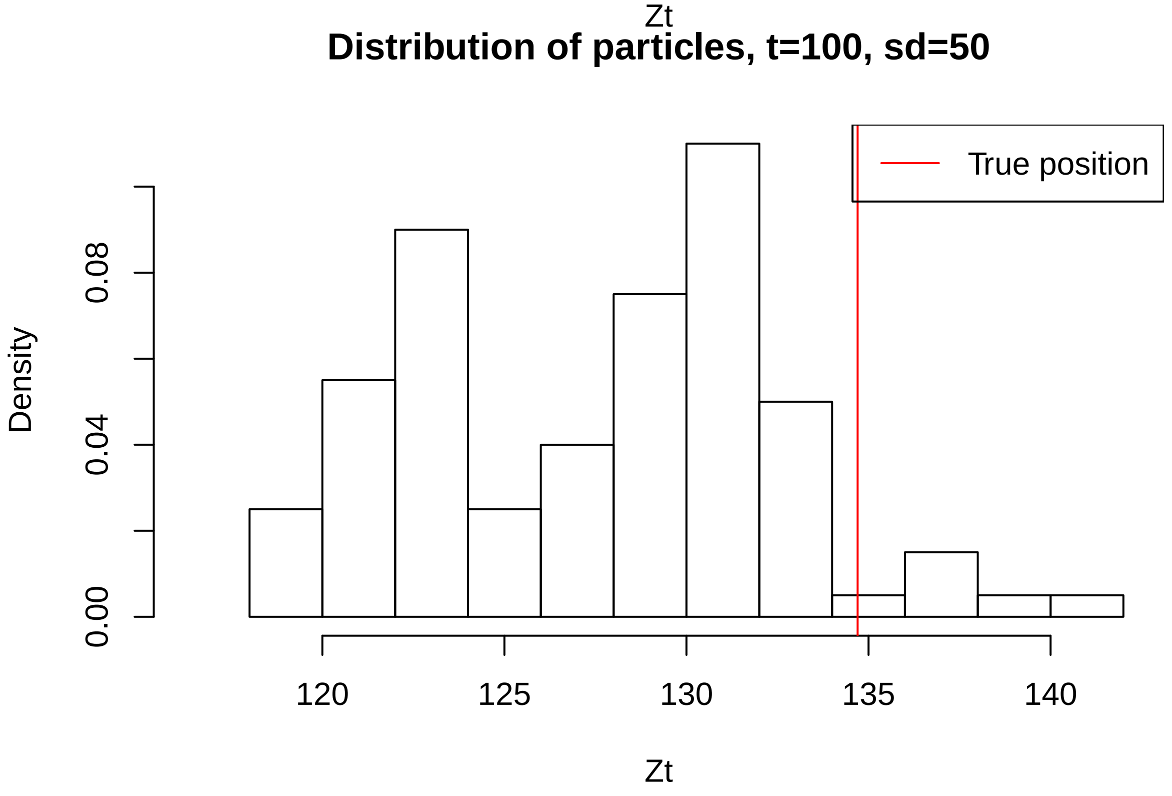


The histogram plots above show the distribution of the particles when standard deviation is 5 (for time steps 1, 35, 75, and 100). This time the algorithm possibly performed worse: at time step 100 the true location of the robot falls into one of the shortest bars of the distribution. This indicates that the algorithm predicted quite a low probability that $z_1 00$ is the true value. However, the overall performance cannot be deduced by looking at only 4 time points: each new step introduces some new uncertainty, and the results at it might be worse then the results from the previous time step.


Distribution of particles, t=1, sd=50
Density
Zt
True position
0.000
0.005
0.010
0.015
0
20
40
60
80
100

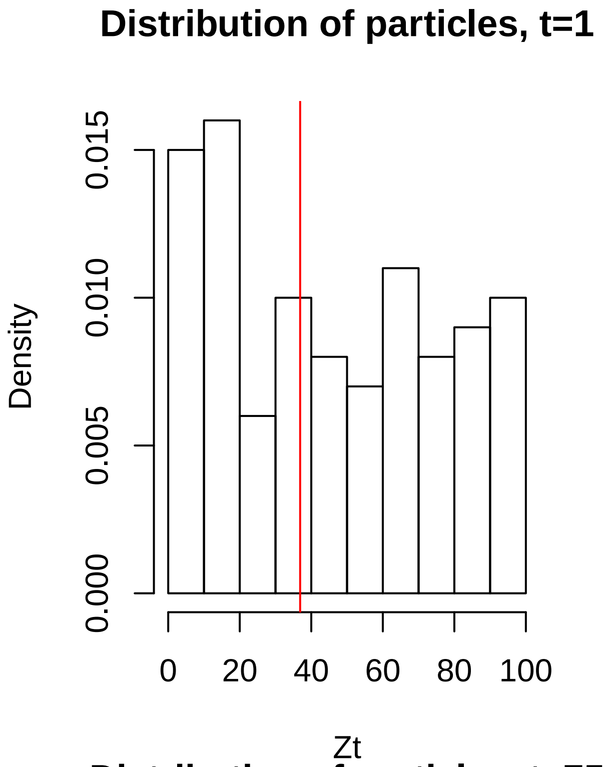

Distribution of particles, t=35, sd=50
Density
Zt
True position
0.00
0.02
0.04
0.06
50
55
60
65
70
75
80

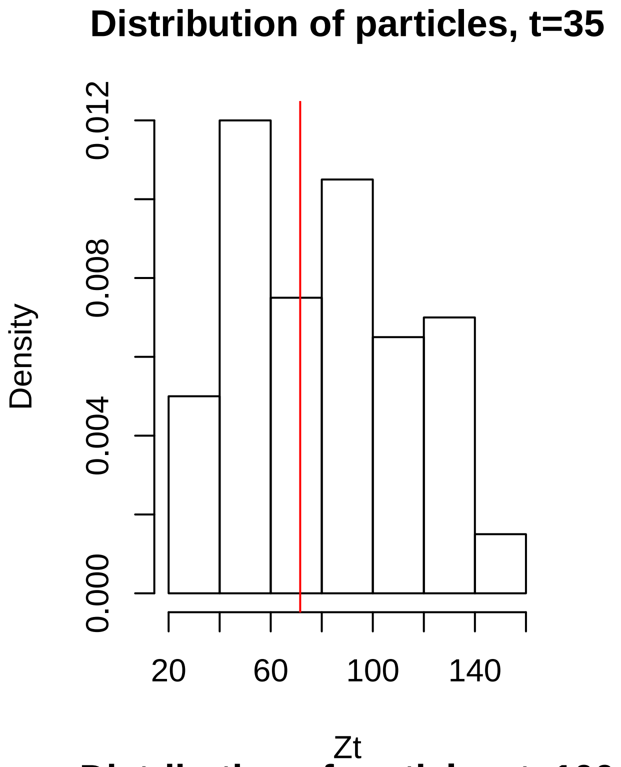


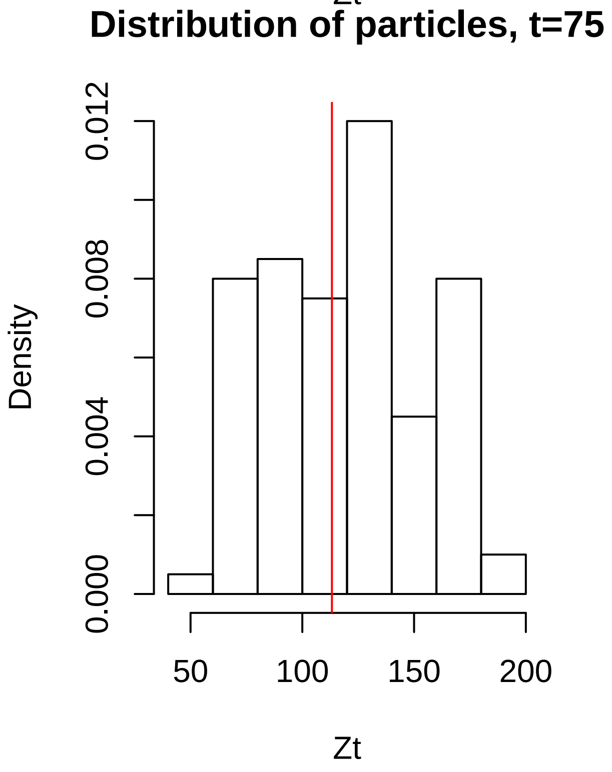


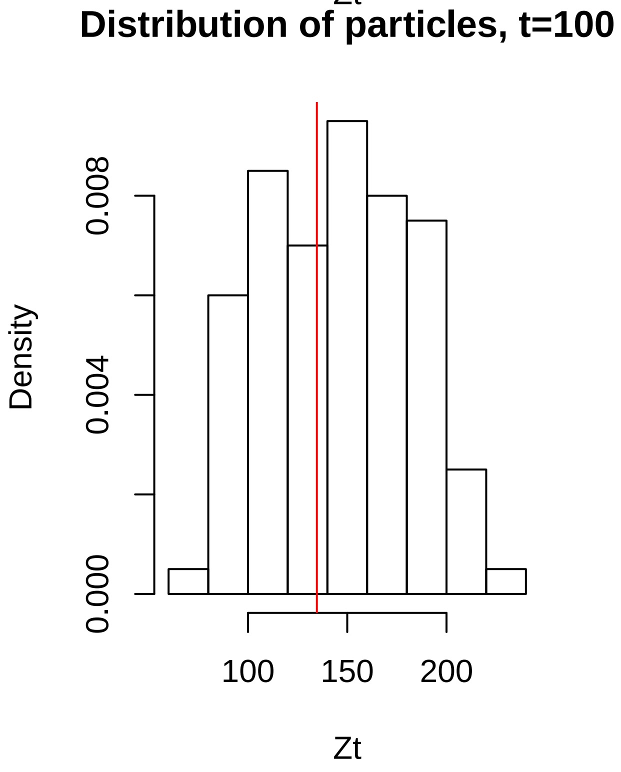
The histogram plots above show the distribution of the particles when standard deviation is 50 (for time steps 1, 35, 75, and 100). This time the true location of the robot for time steps 35, 75, and 100 falls into one of the shortest bars of the distribution of that time step. This indicates that the algorithm predicted quite a low probability that z_t is the true value. It is reasonable to assume that the performance of the particle filter worsens as the standard deviation of the observations from the true states is increased: there is more uncertainty involved.

Question 3





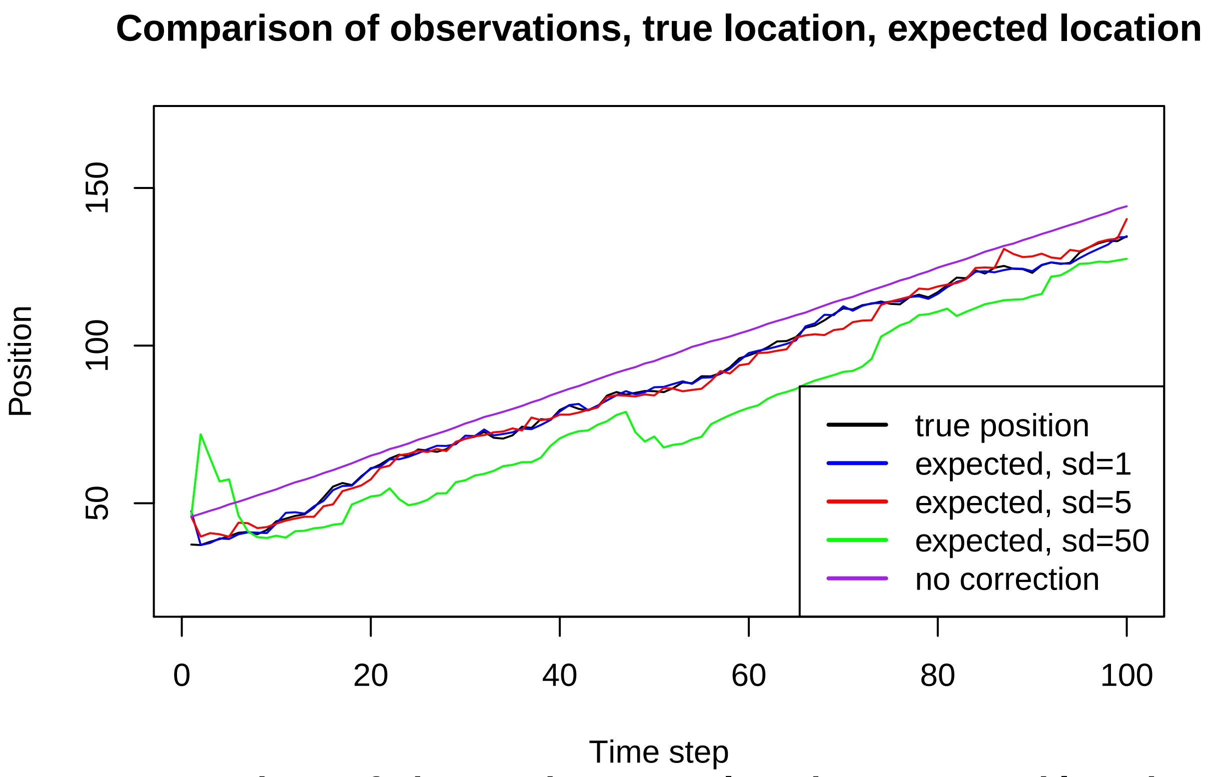


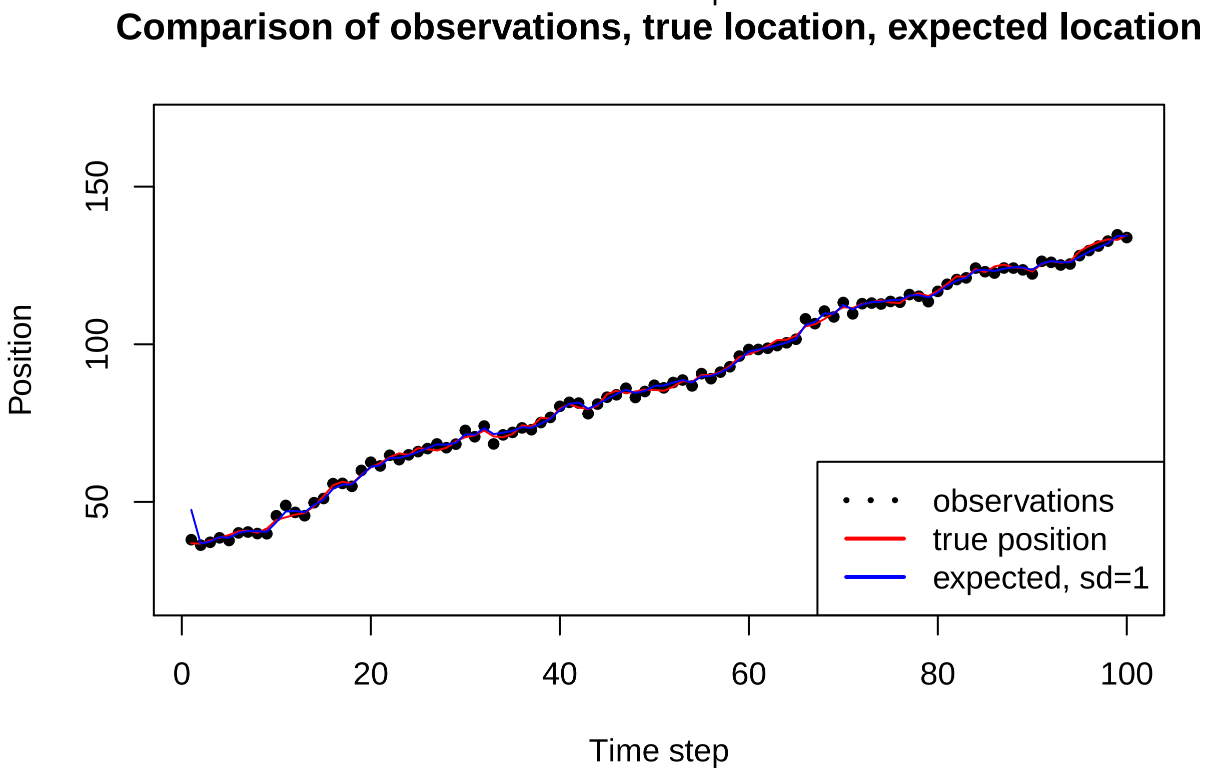


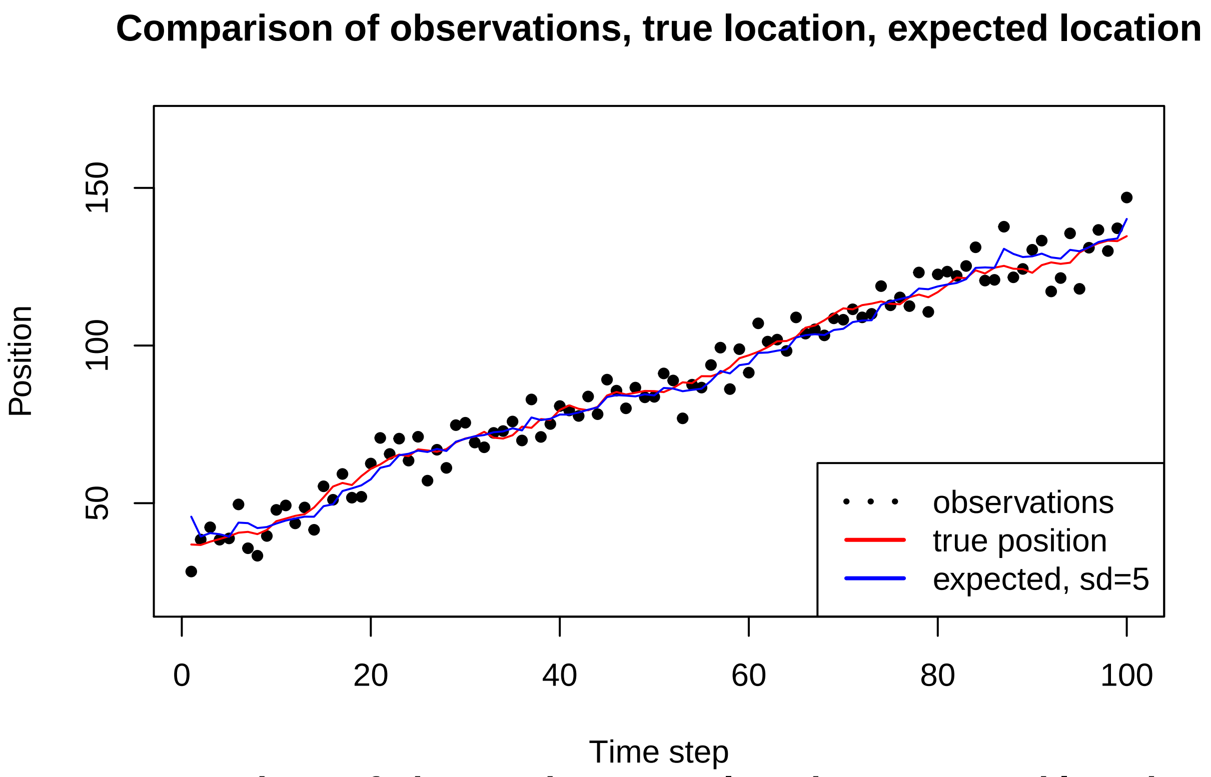
From the distribution plots, it seems that the particle filtering algorithm without correction step did not performed much worse than for the same model with correction. However, as mentioned above, some more plots are needed in order to see the full picture (the full comparison is done below).

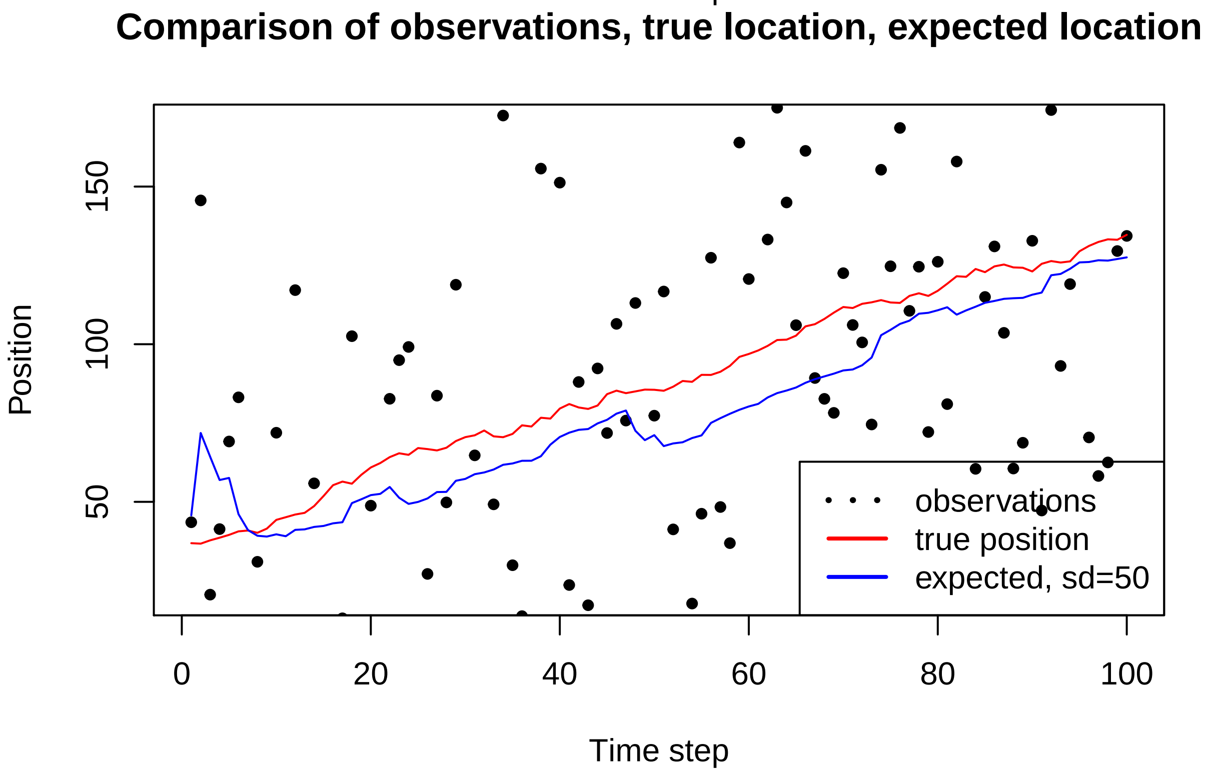
Comparison of all methods

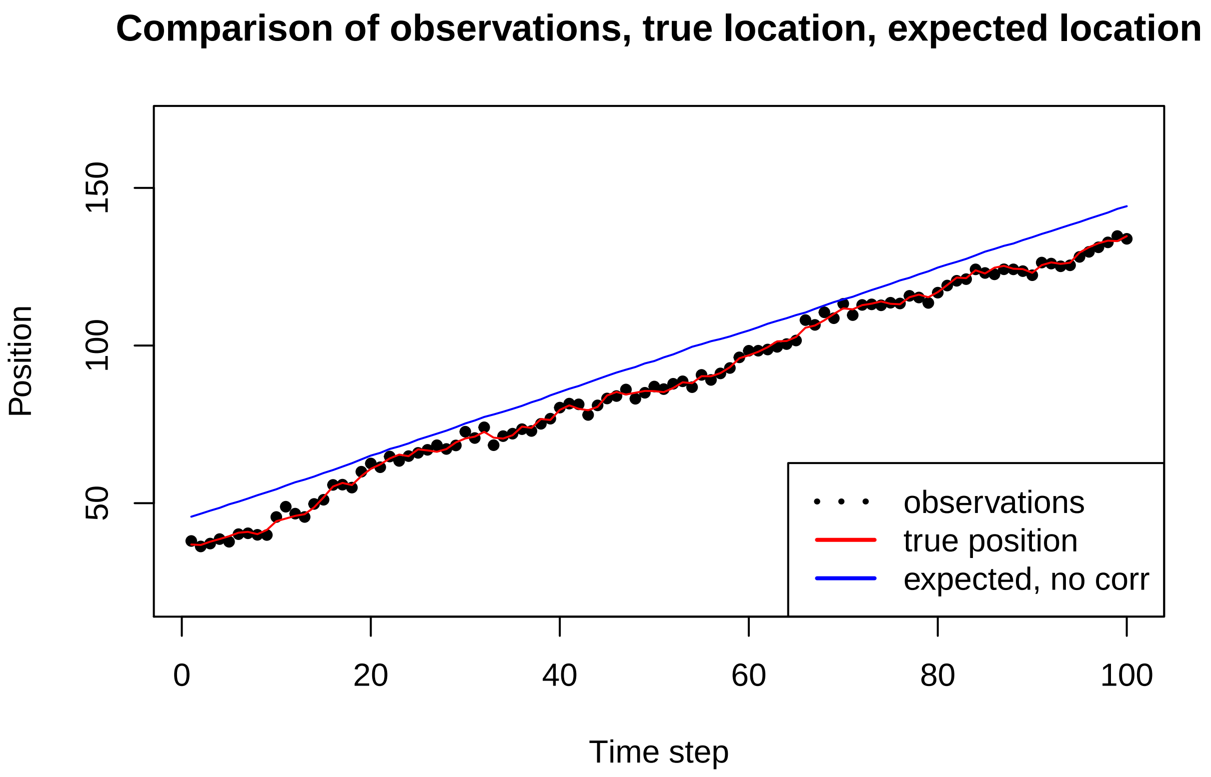
I have taken the mean value of particles for each time step and plotted the true value of z_t with likely predictions from the different models. Also the plots of observations, the true value of z_t and the predictions from each model is plotted separately.






Comparison of observations, true location, expected location
Position
Time step
0
20
40
60
80
100
50
100
150
observations
true position
expected, sd=5


Comparison of observations, true location, expected location
Position
Time step
0
20
40
60
80
100
50
100
150
observations
true position
expected, sd=50



It seems that the model worked best when emission had standard deviation of 1 and the correction step was done. The performance of the particle filter decreased as the standard deviation in the emission model was increased. This is logical, as the more observations deviate from the true location, the more difficult it is to predict correctly what was the true location.

When the correction step is skipped in the particle filter, it seems that the prediction is highly influenced by the random choice of the first state value, and then the consequent time steps is just a gradual increase from that (as this is how the transition model is done). It does not update the predicted z_t values by how likely they are for each observation x_t. Hence it does capture the overall tendency (which is due to the transition model), but it misses the fluctuations in the location change.

Appendix

```r
knitr::opts_chunk$set(echo = TRUE, tidy.opts = list(width.cutoff = 60),
    tidy = TRUE)
set.seed(987654321)
# M - number of particals Iter - number of time steps T z is
# latent variable x is the observation

# Function for sampling from a mixture model
sampleMix <- function(mean1, mean2, mean3, sdX) {
    u <- sample(1:3, 1)
    means <- c(mean1, mean2, mean3)
    return(rnorm(1, mean = means[u], sd = sdX))
}

## Function that creates the SSM model
create_SSM <- function(z0, TT, sdX = 1) {
    Zt <- c(z0)

    Xt <- sampleMix(mean1 = z0, mean2 = (z0 - 1), mean3 = (z0 +
        1), sd = 1)

    for (i in 2:TT) {
        Zt[i] <- sampleMix(mean1 = Zt[i - 1], mean2 = (Zt[i -
            1] + 1), mean3 = (Zt[i - 1] + 2), sdX = 1)
        Xt[i] <- sampleMix(mean1 = Zt[i], mean2 = (Zt[i] - 1),
            mean3 = (Zt[i] + 1), sdX = sdX)
    }
    return(list(sates = Zt, observations = Xt))

}

generateX <- function(Zt, TT, sdX = 1) {
    Xt <- c()
    for (i in 1:TT) {
        Xt[i] <- sampleMix(mean1 = Zt[i], mean2 = (Zt[i] - 1),
            mean3 = (Zt[i] + 1), sdX = sdX)
    }
    return(Xt)
}

## Particle filter function
particles <- function(Obs, M, Iter, sdX = 1, correction = TRUE) {
    Xt <- Obs

    # Initialise particles for the latent variable
    Zt <- matrix(ncol = Iter, nrow = M)
    wt <- matrix(ncol = Iter, nrow = M)
    Zt_bar <- matrix(ncol = Iter, nrow = M)
    # fill the initial time step
    Zt[, 1] <- runif(M, 0, 100)
    # Add arbitrary values in order for the indexing to work
    # later
    Zt_bar[, 1] <- rep.int(1, M)
    wt[, 1] <- rep.int(1, M)

    for (t in 2:Iter) {
        # Prediction
        for (m in 1:M) {
            # sample from the transition model (current Z_t depends on
            # the Z_{t-1}) Zt_bar[m,t] <-
            # (rnorm(1,mean=Zt[m,t-1],sd=1)+rnorm(1,mean=(Zt[m,t-1]+1),sd=1)+rnorm(1,mean=(Zt[m,t-1]+2)
            Zt_bar[m, t] <- sampleMix(mean1 = Zt[m, t - 1], mean2 = (Zt[m,
                t - 1] + 1), mean3 = (Zt[m, t - 1] + 2), sdX = 1)
            # Calculate weights
            wt[m, t] <- (dnorm(Xt[t], mean = Zt_bar[m, t], sd = sdX) +
                dnorm(Xt[t], mean = (Zt_bar[m, t] - 1), sd = sdX) +
                dnorm(Xt[t], mean = (Zt_bar[m, t] + sdX), sd = 1))/3
        }
        # Correction
```

```r
        if (correction) {
            Zt[, t] <- sample(Zt_bar[, t], M, replace = TRUE,
                prob = wt[, t])
        } else {
            Zt[, t] <- Zt_bar[, t]
        }

    }
    return(list(Zt = Zt, Zt_bar = Zt_bar, wt = wt))

}

# Create the model
my_SSM <- create_SSM(runif(1, 0, 100), 100)

# Filter particles
results_q1 <- particles(my_SSM$observations, 100, 100)
hist(results_q1$Zt[, 1], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=1")
abline(v = my_SSM$sates[1], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))

hist(results_q1$Zt[, 35], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=35")
abline(v = my_SSM$sates[35], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))

hist(results_q1$Zt[, 75], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=75")
abline(v = my_SSM$sates[75], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))

hist(results_q1$Zt[, 100], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=100")
abline(v = my_SSM$sates[100], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))
# Create observations for the model with sd=5
my_SSM_2a <- list()
my_SSM_2a$sates <- my_SSM$sates
my_SSM_2a$observations <- generateX(my_SSM$sates, 100, sdX = 5)
# Filter particles, sd=5
results_q2a <- particles(my_SSM_2a$observations, 100, 100, sdX = 5)

# Create the model, sd=50
my_SSM_2b <- list()
my_SSM_2b$sates <- my_SSM$sates
my_SSM_2b$observations <- generateX(my_SSM$sates, 100, sdX = 50)
# Filter particles, sd=50
results_q2b <- particles(my_SSM_2b$observations, 100, 100, sdX = 50)

# Plot results from 2a, sd =5
hist(results_q2a$Zt[, 1], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=1, sd=5",
    xlim = c(min(min(results_q2a$Zt[, 1], my_SSM_2a$sates[1])) -
        2, max(max(results_q2a$Zt[, 1]), my_SSM_2a$sates[1]) +
        2))
abline(v = my_SSM_2a$sates[1], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))
```

```r
hist(results_q2a$Zt[, 35], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=35, sd=5",
    xlim = c(min(min(results_q2a$Zt[, 35], my_SSM_2a$sates[35])) -
        2, max(max(results_q2a$Zt[, 35]), my_SSM_2a$sates[35]) +
        2))
abline(v = my_SSM_2a$sates[35], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))

hist(results_q2a$Zt[, 75], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=75, sd=5",
    xlim = c(min(min(results_q2a$Zt[, 75], my_SSM_2a$sates[75])) -
        2, max(max(results_q2a$Zt[, 75]), my_SSM_2a$sates[75]) +
        2))
abline(v = my_SSM_2a$sates[75], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))

hist(results_q2a$Zt[, 100], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=100, sd=5",
    xlim = c(min(min(results_q2a$Zt[, 100], my_SSM_2a$sates[100])) -
        2, max(max(results_q2a$Zt[, 100]), my_SSM_2a$sates[100]) +
        2))
abline(v = my_SSM_2a$sates[100], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))


# Plot results from 2b, sd =50
hist(results_q2b$Zt[, 1], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=1, sd=50",
    xlim = c(min(min(results_q2b$Zt[, 1], my_SSM_2b$sates[1])) -
        2, max(max(results_q2b$Zt[, 1]), my_SSM_2b$sates[1]) +
        2))
abline(v = my_SSM_2b$sates[1], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))


hist(results_q2b$Zt[, 35], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=35, sd=50",
    xlim = c(min(min(results_q2b$Zt[, 35], my_SSM_2b$sates[35])) -
        2, max(max(results_q2b$Zt[, 35]), my_SSM_2b$sates[35]) +
        2))
abline(v = my_SSM_2b$sates[35], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))


hist(results_q2b$Zt[, 75], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=75, sd=50",
    xlim = c(min(min(results_q2b$Zt[, 75], my_SSM_2b$sates[75])) -
        2, max(max(results_q2b$Zt[, 75]), my_SSM_2b$sates[75]) +
        2))
abline(v = my_SSM_2b$sates[75], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))


hist(results_q2b$Zt[, 100], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=100, sd=50"
    xlim = c(min(min(results_q2b$Zt[, 100], my_SSM_2b$sates[100])) -
        2, max(max(results_q2b$Zt[, 100]), my_SSM_2b$sates[100]) +
        2))
abline(v = my_SSM_2b$sates[100], col = "red")
legend("topright", c("True position"), col = c("red"), lty = c(1))
```

```r
# Filter particles
results_q3 <- particles(my_SSM$observations, 100, 100, correction = FALSE)
par(mfrow = c(1, 2))

hist(results_q3$Zt[, 1], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=1")
abline(v = my_SSM$sates[1], col = "red")


hist(results_q3$Zt[, 35], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=35")
abline(v = my_SSM$sates[35], col = "red")


hist(results_q3$Zt[, 75], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=75")
abline(v = my_SSM$sates[75], col = "red")


hist(results_q3$Zt[, 100], freq = FALSE, xlab = "Zt", main = "Distribution of particles, t=100")
abline(v = my_SSM$sates[100], col = "red")

exp_loc_q1 <- colMeans(results_q1$Zt)
exp_loc_q2a <- colMeans(results_q2a$Zt)
exp_loc_q2b <- colMeans(results_q2b$Zt)
exp_loc_q3 <- colMeans(results_q3$Zt)

plot(my_SSM$sates, col = "black", type = "l", ylab = "Position",
    xlab = "Time step", main = "Comparison of observations, true location, expected location",
    ylim = c(20, 170))
lines(exp_loc_q1, col = "blue")
lines(exp_loc_q2a, col = "red")
lines(exp_loc_q2b, col = "green")
lines(exp_loc_q3, col = "purple")
legend("bottomright", c("true position", "expected, sd=1", "expected, sd=5",
    "expected, sd=50", "no correction"), col = c("black", "blue",
    "red", "green", "purple"), lty = c(1, 1, 1, 1, 1), lwd = c(rep(2,
    5)))


plot(my_SSM$observations, pch = 20, ylab = "Position", xlab = "Time step",
    main = "Comparison of observations, true location, expected location",
    ylim = c(20, 170))
lines(my_SSM$sates, col = "red")
lines(exp_loc_q1, col = "blue")
legend("bottomright", c("observations", "true position", "expected, sd=1"),
    col = c("black", "red", "blue"), lty = c(3, 1, 1), lwd = c(3,
        rep(2, 2)))


plot(my_SSM_2a$observations, pch = 20, ylab = "Position", xlab = "Time step",
    main = "Comparison of observations, true location, expected location",
    ylim = c(20, 170))
lines(my_SSM$sates, col = "red")
lines(exp_loc_q2a, col = "blue")
legend("bottomright", c("observations", "true position", "expected, sd=5"),
    col = c("black", "red", "blue"), lty = c(3, 1, 1), lwd = c(3,
        rep(2, 2)))

plot(my_SSM_2b$observations, pch = 20, ylab = "Position", xlab = "Time step",
    main = "Comparison of observations, true location, expected location",
    ylim = c(20, 170))
lines(my_SSM$sates, col = "red")
lines(exp_loc_q2b, col = "blue")
legend("bottomright", c("observations", "true position", "expected, sd=50"),
    col = c("black", "red", "blue"), lty = c(3, 1, 1), lwd = c(3,
        rep(2, 2)))

plot(my_SSM$observations, pch = 20, ylab = "Position", xlab = "Time step",
    main = "Comparison of observations, true location, expected location",
    ylim = c(20, 170))
lines(my_SSM$sates, col = "red")
lines(exp_loc_q3, col = "blue")
legend("bottomright", c("observations", "true position", "expected, no corr"),
    col = c("black", "red", "blue"), lty = c(3, 1, 1), lwd = c(3,
        rep(2, 2)))
```