732A91 Lab 2

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18 april 2018

1. Linear and polynomial regression

The data set TempLinkoping contains daily temperatures (in Celsius) at Malmslätt (Linköping) during 2016. The response variable is $temp$ and $time = \frac{\text{number of days since beginning of year}}{366}$ is the covariate. A Bayesian analysis of a quadratic regression $temp = \beta_0 + \beta_1 \cdot time + \beta_2 \cdot time^2 + \epsilon$, $\epsilon \overset{iid}{\sim} N(0, \sigma^2)$ is performed.

a)

Conjugate priors:

$$\beta|\sigma^2 \sim N(\mu_0, \sigma^2\Omega_0^{-1})\sigma^2 \sim Inv - \chi^2(\nu_0, \sigma_0^2)$$

The hyperparameters were chosen to be

$$\mu_0 = (-5, 90, -80)\nu_0 = 5\sigma_0^2 = 9\Omega_0 = I_3$$

The prior was based on data from NOAA, the weather of the current week, and results from the lm fit.

```
temps<-read.table("TempLinkoping.dat",header=TRUE)
#response: temp, covariate: time
temp <- temps$temp
time <- temps$time
temps_lm <-lm(temp ~ time + I(time^2))

#hyperparameters
mu0<-c(-5,90,-80) #based on lm
omega0<-diag(3)
inv_omega0 <- solve(omega0)
nu0<-5
sigma20<-9
```

b)

50 simulations from the prior were plotted and compared to the graph from above the prior seemed sensible.

```
library(mvtnorm)
n1 <- 50
sim_X <- rchisq(n1,nu0)
sigma2 <- nu0*sigma20/sim_X

# betas <- matrix(nrow=n1, ncol=3)
# for(i in 1:n1){
#   betas[i,] <- rmvnorm(n = 1, mean=mu0, sigma=sigma2[i]*inv_omega0)
# }
betas_mat1 <- t(apply(as.matrix(sigma2),1,function(x){rmvnorm(n=1,mean=mu0,sigma=x*inv_omega0)}))
betas <- as.data.frame(betas_mat1)
colnames(betas)<-c("beta0", "beta1", "beta2")
```

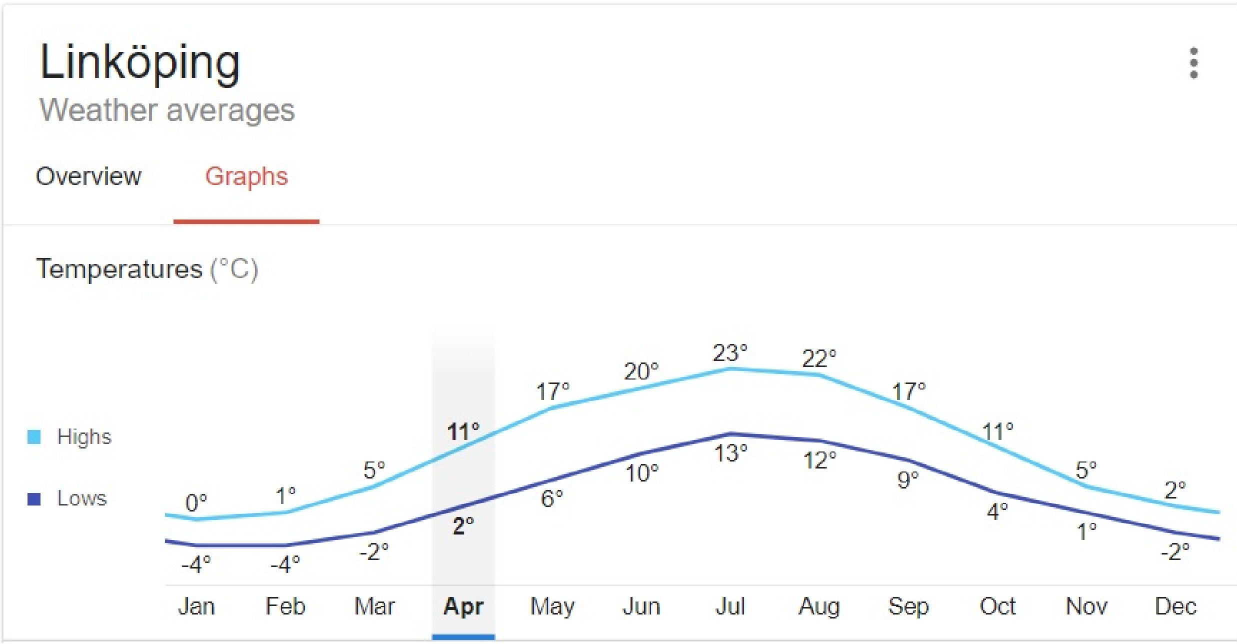
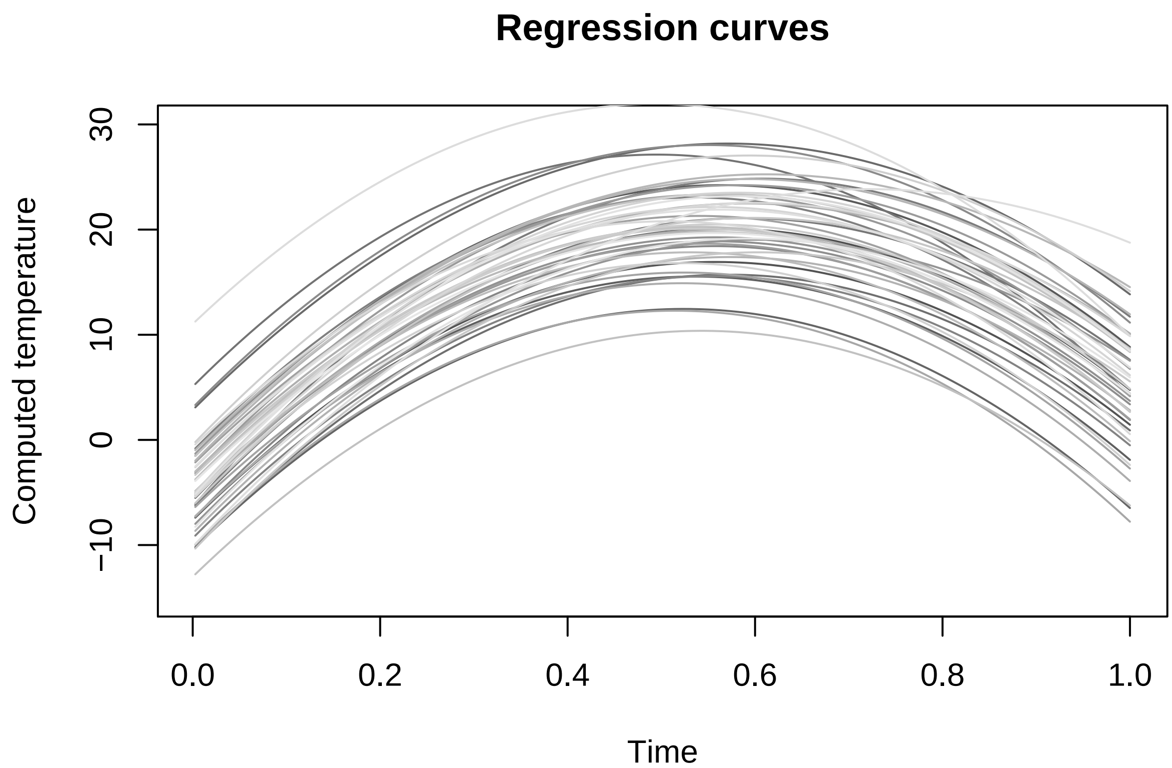


Figure 1: Weather averages in Linköping. Source: NOAA

```r
X <- as.matrix(data.frame(intercept=rep(1, length(time)), time, time^2))
y <- apply(betas_mat1,1,function(a){X%*%a})

cl <- grey.colors(n1)
plot(x=time, y=y[,1], col=cl[1], type="l", ylim=c(-15, 30),
     ylab="Computed temperature", xlab="Time", main="Regression curves")
for(i in 2:ncol(y)){
 lines(time, y[,i], col=cl[i])
}
```

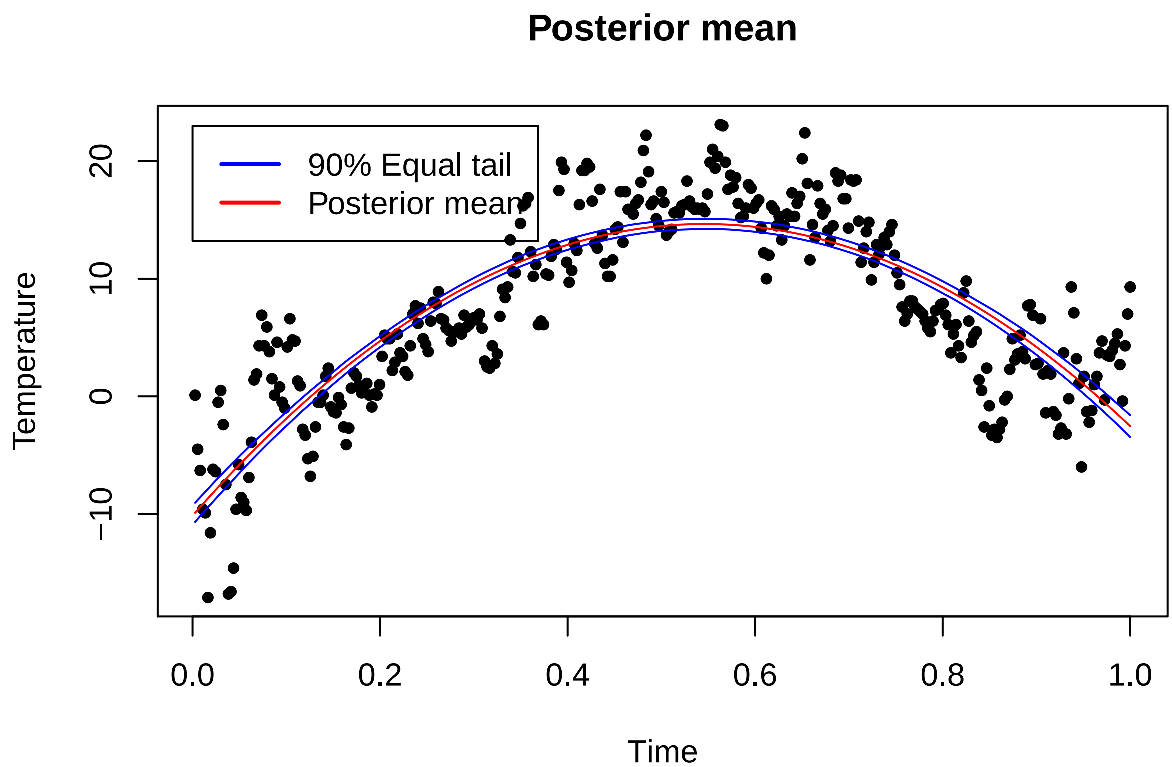


c)

The plot below shows the scatter plot of the temperatures and the posterior mean of the regression function $f(time)$ computed for every value of $time$. It also shows the 95% posterior credible interval for $f(time)$. The interval is narrow and includes only a few data points because the interval shows how certain we are about the posterior mean of $f(time)$ (the trend in the data).

```
n<-200
beta_hat<-solve(t(X)%*%X)%*%t(X)%*%temp #same as found with lm
mun <- solve(t(X)%*%X+omega0)%*%(t(X)%*%X%*%beta_hat+omega0%*%mu0)
omegan <-t(X)%*%X+omega0
inv_omegan <- solve(omegan)
nun <- nu0+nrow(temps)
sigma2n <- (nu0*sigma20+(t(temp)%*%temp+t(mu0)%*%omega0%*%mu0-t(mun)%*%omegan%*%mun))/nun

sim_X2 <- rchisq(n,nun)
sigma2_post <- nun*as.vector(sigma2n)/sim_X2

betas_mat<-t(apply(as.matrix(sigma2_post), 1, function(x){rmvnorm(n=1, mean=mun, sigma=x*inv_omegan)}))
betas_post <- as.data.frame(betas_mat)
colnames(betas_post)<-c("beta0", "beta1", "beta2")

y_post <- apply(betas_mat,1,function(a){X%*%a})
y_mean <- rowMeans(y_post)
```

```
perc = 0.05*n
low <- apply(y_post,1,function(x){x[order(x, decreasing = FALSE)[perc+1]]})
upp <- apply(y_post,1,function(x){x[order(x, decreasing = FALSE)[n-perc-1]]})

plot(x=time, y=temp, pch=20, ylab="Temperature", xlab="Time", main="Posterior mean")
lines(time, y_mean, col="red")
lines(time, low, col="blue")
lines(time, upp, col="blue")
legend(x = 0, y=23, c("90% Equal tail", "Posterior mean"), col=c("blue", "red"), lwd = 2)
```

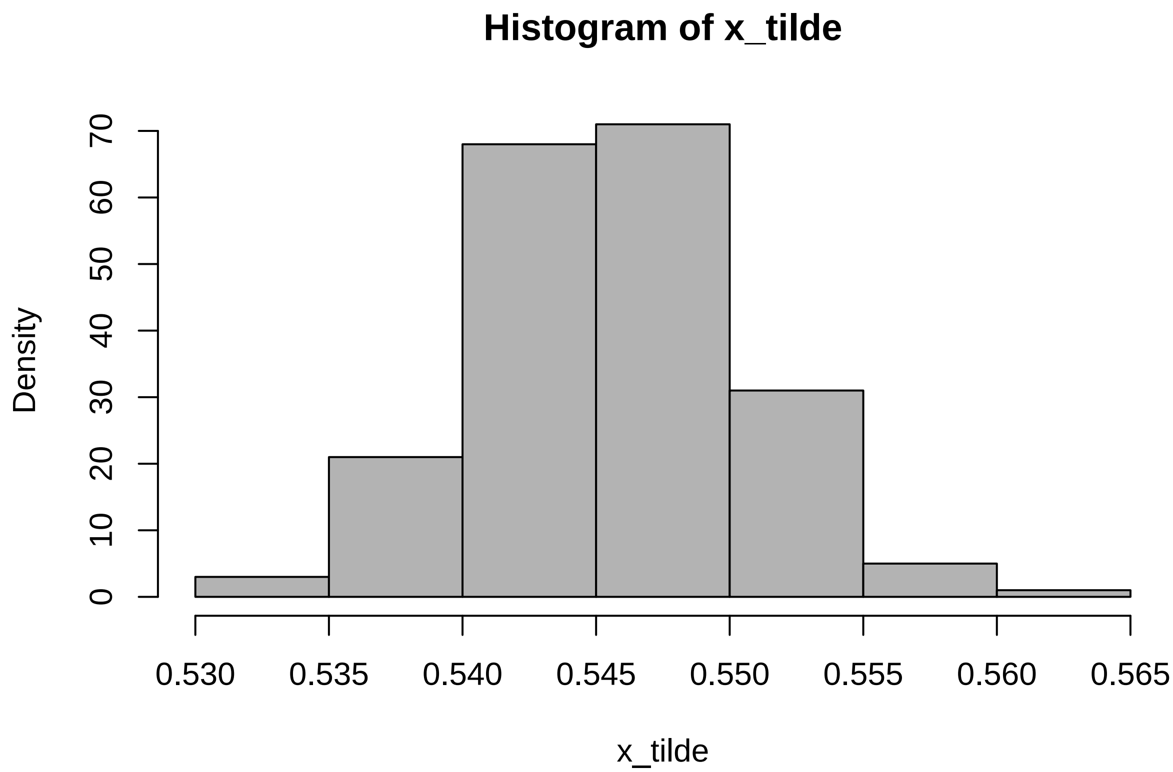


d)

$$f'(time) = \beta_1 + 2\beta_2 \cdot time$$

Setting the derivative to 0 and solving for $time$ we get that the $time$ that maximizes $f(time)$ is $\tilde{x} = -\frac{\beta_1}{2\beta_2}$. From the simulations in c) $\tilde{x}$ was found for each set of βs. The histogram below shows the distribution of $\tilde{x}$.

```
x_tilde <- apply(betas_mat,1,function(x){-x[2]/(2*x[3])})
hist(x_tilde, freq=FALSE, col="grey70")
```



e)

By results from Lecture 5, $\mu_0 = 0$ and $\Omega_o = \lambda I$, where λ would be found either using cross-validation (frequentist way) or using a prior on λ (Bayesian way). The prior would be

$$\beta|\sigma^2 \overset{iid}{\sim} N(0, \sigma^2(\lambda I)^{-1})$$

The larger the λ the more β values tend to 0 and in this way the risk of overfitting is decreased (but the risk of underfitting increases).

2. Posterior approximation for classification with logistic regression

a)

A logistic regression is used for classifying women as working ($y = 1$) and not working ($y = 0$):

$$Pr(y = 1|x) = \frac{exp(x^T\beta)}{1 + exp(x^T\beta)}$$

where x is a 8-dimensional vector containing features (including intercept).

A logistic regression was done using maximum likelihood estimation:

```r
work<-read.table("WomenWork.dat",header=TRUE)
glmModel <- glm(Work ~ 0 + ., data = work, family = binomial)
glmModel
```

```
##
## Call:  glm(formula = Work ~ 0 + ., family = binomial, data = work)
##
## Coefficients:
##    Constant   HusbandInc    EducYears     ExpYears    ExpYears2
##     0.64430     -0.01977      0.17988      0.16751     -0.14436
##         Age  NSmallChild    NBigChild
##    -0.08234     -1.36250     -0.02543
##
## Degrees of Freedom: 200 Total (i.e. Null);  192 Residual
## Null Deviance:       277.3
## Residual Deviance: 222.7     AIC: 238.7
```

b)

The posterior distribution of β is approximated by a multivariate normal distribution:

$$\beta|y, X \sim N(\tilde{\beta}, J_y^{-1}(\tilde{\beta}))$$

where $\tilde{\beta}$ is the posterior mode and $J(\tilde{\beta})$ is the observed Hessian evaluated at the posterior mode. The prior is $\beta \sim N(0, \tau^2 I)$, where $\tau = 10$.

```r
y_work <- work$Work
X_work <- work[,-work$Work]
nparam <- ncol(X_work)

#Prior
mu <- as.vector(rep(0, nparam))
tau2<-100
sigma2 <- tau2*diag(nparam)

logpost_logistic <- function(betas, y, X, mu, sigma2){
  npara <- length(betas)
  X<-as.matrix(X)
  lin_pred <- X%*%betas
  loglik <- sum(lin_pred*y-log(1+exp(lin_pred)))
  if(abs(loglik)==Inf){
    loglik <- -20000
  }
  log_prior <- dmvnorm(betas, mean=mu, sigma2, log=TRUE)
  log_post<-loglik+log_prior
  return(log_post)
}

betas_init <- as.vector(rep(0,nparam))
opt_results <- optim(betas_init, logpost_logistic, gr=NULL, y_work,
                     X_work, mu, sigma2, method=c("BFGS"),control=list(fnscale=-1), hessian=TRUE)

beta_tilde <- opt_results$par
beta_hessian <- -1*opt_results$hessian
```

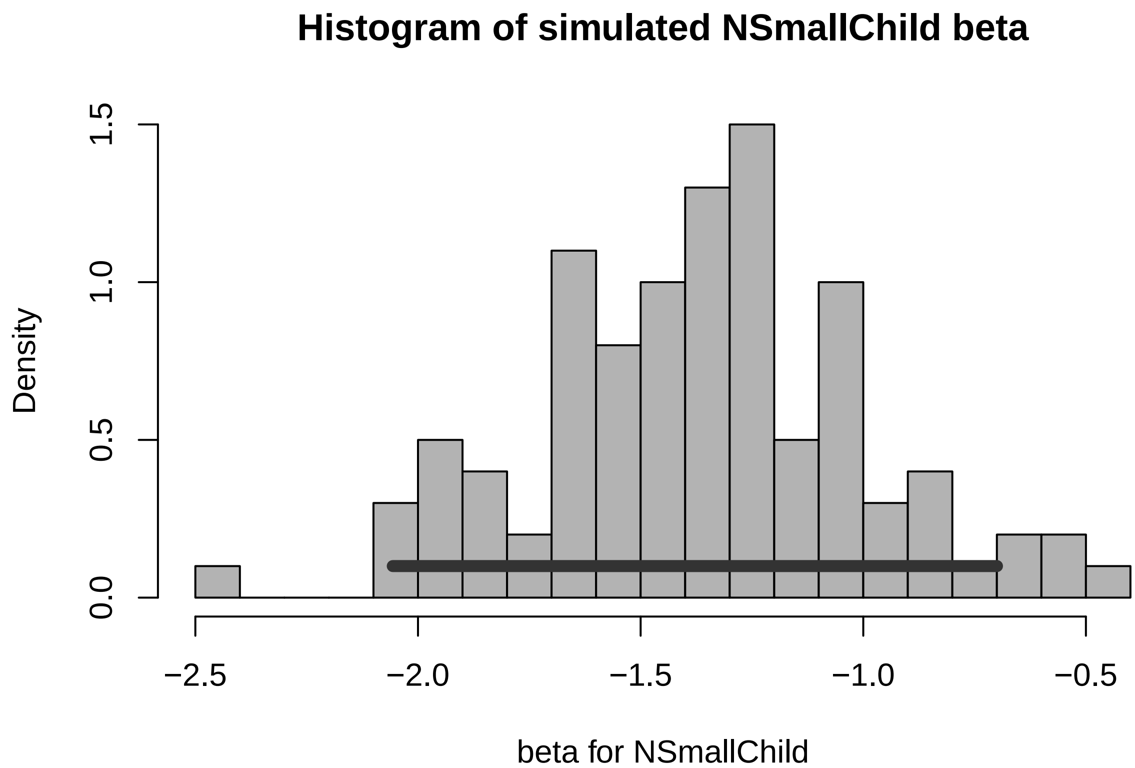
```r
inv_hessian <- solve(beta_hessian) # Posterior covariance matrix is -inv(Hessian)

print(beta_tilde)
```

```
## [1]  0.62672884 -0.01979113  0.18021897  0.16756670 -0.14459669 -0.08206561
## [7] -1.35913317 -0.02468351
```

```r
print(inv_hessian)
```

```
##              [,1]          [,2]          [,3]          [,4]          [,5]
## [1,]  2.266022568  3.338861e-03 -6.545121e-02 -1.179140e-02  0.0457807243
## [2,]  0.003338861  2.528045e-04 -5.610225e-04 -3.125413e-05  0.0001414915
## [3,] -0.065451206 -5.610225e-04  6.218199e-03 -3.558209e-04  0.0018962893
## [4,] -0.011791404 -3.125413e-05 -3.558209e-04  4.351716e-03 -0.0142490853
## [5,]  0.045780724  1.414915e-04  1.896289e-03 -1.424909e-02  0.0555786706
## [6,] -0.030293450 -3.588562e-05 -3.240448e-06 -1.340888e-04 -0.0003299398
## [7,] -0.188748354  5.066847e-04 -6.134564e-03 -1.468951e-03  0.0032082535
## [8,] -0.098023929 -1.444223e-04  1.752732e-03  5.437105e-04  0.0005120144
##               [,6]          [,7]          [,8]
## [1,] -3.029345e-02 -0.1887483542 -0.0980239285
## [2,] -3.588562e-05  0.0005066847 -0.0001444223
## [3,] -3.240448e-06 -0.0061345645  0.0017527317
## [4,] -1.340888e-04 -0.0014689508  0.0005437105
## [5,] -3.299398e-04  0.0032082535  0.0005120144
## [6,]  7.184611e-04  0.0051841611  0.0010952903
## [7,]  5.184161e-03  0.1512621814  0.0067688739
## [8,]  1.095290e-03  0.0067688739  0.0199722657
```

```r
sim_beta<-rmvnorm(n=100, mean=beta_tilde, sigma=inv_hessian)
colnames(sim_beta) <- colnames(X_work)
sim_NSmallChild <- sim_beta[,"NSmallChild"]
perc2 = 0.025*length(sim_NSmallChild)
lower <- sim_NSmallChild[order(sim_NSmallChild, decreasing = FALSE)[perc2+1]]
upper <- sim_NSmallChild[order(sim_NSmallChild, decreasing = FALSE)[length(sim_NSmallChild)-perc2-1]]

hist(sim_NSmallChild, freq = FALSE, breaks = 20, col = "grey70",
     xlab="beta for NSmallChild", main="Histogram of simulated NSmallChild beta")
lines(c(lower, upper), c(0.1,0.1), col="grey20", lwd=6)
```

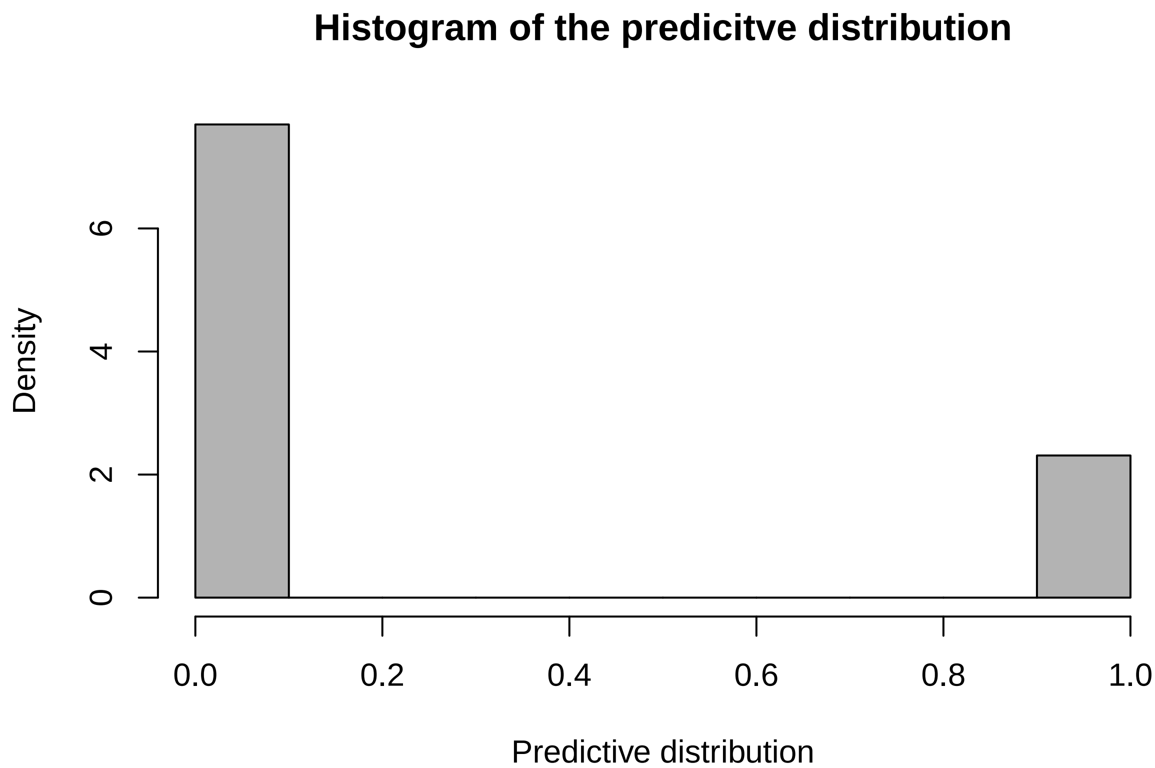


Since the 95% credible interval (black bar) for the simulated β values corresponding to the variable NSmallChild does not include 0, we conclude that the feature is an important determinant of the probability that a woman works.

c)

Using the normal approximation from b), the predictive distribution for the Work variable was simulated when a woman is 40 years old, has one small and one big child, 8 years of education, 10 years of experience and a husband with income of 10.

```
log_pred <- function(husband, edu, exper, age, smallchild, bigchild, n){
  sim_b<-rmvnorm(n=n, mean=beta_tilde, sigma=inv_hessian)
  x <- c(1, husband, edu, exper, (exper/10)^2, age, smallchild, bigchild)
  prob_pred <- exp(sim_b%*%x)/(1+exp(sim_b%*%x))
  u <- runif(n = n)
  y_pred <- ifelse(prob_pred>u, 1, 0)
  y_pred
}

woman <- log_pred(10, 8, 10, 40, 1, 1, 1000)
hist(woman, freq=FALSE, col = "grey70", xlab="Predictive distribution",
     main="Histogram of the predicitve distribution")
```



From the predictive distribution we conclude that it is more likely that the woman does not work.

Appendix

```
temps<-read.table("TempLinkoping.dat",header=TRUE)
#response: temp, covariate: time
temp <- temps$temp
time <- temps$time
temps_lm <-lm(temp ~ time + I(time^2))

#hyperparameters
mu0<-c(-5,90,-80) #based on lm
omega0<-diag(3)
inv_omega0 <- solve(omega0)
nu0<-5
sigma20<-9
library(mvtnorm)
n1 <- 50
sim_X <- rchisq(n1,nu0)
sigma2 <- nu0*sigma20/sim_X

# betas <- matrix(nrow=n1, ncol=3)
# for(i in 1:n1){
#   betas[i,] <- rmvnorm(n = 1, mean=mu0, sigma=sigma2[i]*inv_omega0)
```

```r
# }
betas_mat1 <- t(apply(as.matrix(sigma2),1,function(x){rmvnorm(n=1,mean=mu0,sigma=x*inv_omega0)}))
betas <- as.data.frame(betas_mat1)
colnames(betas)<-c("beta0", "beta1", "beta2")

X <- as.matrix(data.frame(intercept=rep(1, length(time)), time, time^2))
y <- apply(betas_mat1,1,function(a){X%*%a})

cl <- grey.colors(n1)
plot(x=time, y=y[,1], col=cl[1], type="l", ylim=c(-15, 30),
     ylab="Computed temperature", xlab="Time", main="Regression curves")
for(i in 2:ncol(y)){
 lines(time, y[,i], col=cl[i])
}
n<-200
beta_hat<-solve(t(X)%*%X)%*%t(X)%*%temp #same as found with lm
mun <- solve(t(X)%*%X+omega0)%*%(t(X)%*%X%*%beta_hat+omega0%*%mu0)
omegan <-t(X)%*%X+omega0
inv_omegan <- solve(omegan)
nun <- nu0+nrow(temps)
sigma2n <- (nu0*sigma20+(t(temp)%*%temp+t(mu0)%*%omega0%*%mu0-t(mun)%*%omegan%*%mun))/nun

sim_X2 <- rchisq(n,nun)
sigma2_post <- nun*as.vector(sigma2n)/sim_X2

betas_mat<-t(apply(as.matrix(sigma2_post), 1, function(x){rmvnorm(n=1, mean=mun, sigma=x*inv_omegan)}))
betas_post <- as.data.frame(betas_mat)
colnames(betas_post)<-c("beta0", "beta1", "beta2")

y_post <- apply(betas_mat,1,function(a){X%*%a})
y_mean <- rowMeans(y_post)

perc = 0.05*n
low <- apply(y_post,1,function(x){x[order(x, decreasing = FALSE)[perc+1]]})
upp <- apply(y_post,1,function(x){x[order(x, decreasing = FALSE)[n-perc-1]]})

plot(x=time, y=temp, pch=20, ylab="Temperature", xlab="Time", main="Posterior mean")
lines(time, y_mean, col="red")
lines(time, low, col="blue")
lines(time, upp, col="blue")
legend(x = 0, y=23, c("90% Equal tail", "Posterior mean"), col=c("blue", "red"), lwd = 2)
x_tilde <- apply(betas_mat,1,function(x){-x[2]/(2*x[3])})
hist(x_tilde, freq=FALSE, col="grey70")
work<-read.table("WomenWork.dat",header=TRUE)
glmModel <- glm(Work ~ 0 + ., data = work, family = binomial)
glmModel
y_work <- work$Work
X_work <- work[,-work$Work]
nparam <- ncol(X_work)

#Prior
mu <- as.vector(rep(0, nparam))
tau2<-100
```

```r
sigma2 <- tau2*diag(nparam)

logpost_logistic <- function(betas, y, X, mu, sigma2){
  npara <- length(betas)
  X<-as.matrix(X)
  lin_pred <- X%*%betas
  loglik <- sum(lin_pred*y-log(1+exp(lin_pred)))
  if(abs(loglik)==Inf){
    loglik <- -20000
  }
  log_prior <- dmvnorm(betas, mean=mu, sigma2, log=TRUE)
  log_post<-loglik+log_prior
  return(log_post)
}

betas_init <- as.vector(rep(0,nparam))
opt_results <- optim(betas_init, logpost_logistic, gr=NULL, y_work,
                     X_work, mu, sigma2, method=c("BFGS"),control=list(fnscale=-1), hessian=TRUE)

beta_tilde <- opt_results$par
beta_hessian <- -1*opt_results$hessian
inv_hessian <- solve(beta_hessian) # Posterior covariance matrix is -inv(Hessian)

print(beta_tilde)
print(inv_hessian)

sim_beta<-rmvnorm(n=100, mean=beta_tilde, sigma=inv_hessian)
colnames(sim_beta) <- colnames(X_work)
sim_NSmallChild <- sim_beta[,"NSmallChild"]
perc2 = 0.025*length(sim_NSmallChild)
lower <- sim_NSmallChild[order(sim_NSmallChild, decreasing = FALSE)[perc2+1]]
upper <- sim_NSmallChild[order(sim_NSmallChild, decreasing = FALSE)[length(sim_NSmallChild)-perc2-1]]

hist(sim_NSmallChild, freq = FALSE, breaks = 20, col = "grey70",
     xlab="beta for NSmallChild", main="Histogram of simulated NSmallChild beta")
lines(c(lower, upper), c(0.1,0.1), col="grey20", lwd=6)
log_pred <- function(husband, edu, exper, age, smallchild, bigchild, n){
  sim_b<-rmvnorm(n=n, mean=beta_tilde, sigma=inv_hessian)
  x <- c(1, husband, edu, exper, (exper/10)^2, age, smallchild, bigchild)
  prob_pred <- exp(sim_b%*%x)/(1+exp(sim_b%*%x))
  u <- runif(n = n)
  y_pred <- ifelse(prob_pred>u, 1, 0)
  y_pred
}

woman <- log_pred(10, 8, 10, 40, 1, 1, 1000)
hist(woman, freq=FALSE, col = "grey70", xlab="Predictive distribution",
     main="Histogram of the predicitve distribution")
```