Lecture 2

It turns out that the spatial fluctuations of $T(\vec{x})$ are very well described by Gaussian Random Fields.

GRF intro

Let R be some region in $\mathbb{R}^d$.

e.g. $R = [0,1]^2 \subset \mathbb{R}^2$

e.g. $R = S^2 := \{ x \in \mathbb{R}^3 : \|x\| = 1 \}$

e.g. $R = \mathbb{R}$.

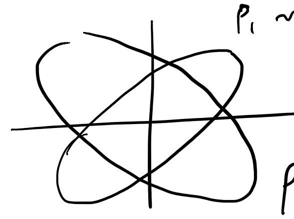
$f(x) : R \to \mathbb{R}$ denotes a function mapping R into $\mathbb{R}$.

Informally a GRF on R is a random function $f(x) : R \to \mathbb{R}$ with the property that all finite dimensional distributions (fdd) are multivariate Gaussian, i.e. that $\forall n \geq 1$, $\forall x_1, \dots, x_n \in R$ the random vector $(f(x_1), \dots, f(x_n))^T$ is jointly Gaussian.

Note: A random vector $(X_1, \dots, X_n)^T$ can have all marginals X_i Gaussian but not be jointly Gaussian.

Here is an example

$p_1 \sim N\left(0, \begin{pmatrix} 1 & \frac{1}{2} \\ & 1 \end{pmatrix}\right)$



$p_2 \sim N\left(0, \begin{pmatrix} 1 & -\frac{1}{2} \\ & 1 \end{pmatrix}\right)$

The density $\frac{1}{2}(p_1 + p_2)$ has $N(0,1)$ marginals but is not jointly Gaussian.

Thm: $X \in \mathbb{R}^n$ is jointly Gaussian iff $\alpha^T X$ is univariate Gaussian $\forall \alpha \in \mathbb{R}^n$.

Proof: Use characteristic functions.

It is sometimes more natural to think of a GRF on R as a collection of univariate R.V.s indexed by R.

$$\{ f(x) : x \in R \}$$

For a fixed x, $f(x)$ is just a R.V.

Need that all these R.V.s are defined on the same probability space $(\Omega, \mathcal{F}, P)$.

i.e. the R.V.s $f(x)$ are actually a function of x & ω

$$f(x) = f(x, \omega).$$

2 ways to Model GRF's

First way: Direct map level construction.

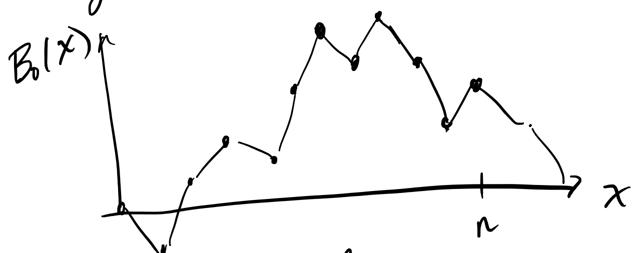
e.g. Let $\xi_1, \xi_2, \ldots \xi_n$ be iid $N(0,1)$ & $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^d$ be non-random..

random coefs.

Then $f(x) := \sum_{i=1}^{n} \xi_i \underbrace{\cos\left(\langle \lambda_i, x_i \rangle\right)}_{\text{basis fns}}$

is a GRF on $\mathbb{R}^d$

e.g. Construct Brownian Motion by limiting scaled Random walks.



where $B_0(n) = \sum_{i=1}^{n} Z_i$ &

$Z_1, \ldots$ iid $N(0,1)$.

$\therefore$ $B(x)$ is defined as the $L_2(\Omega, \mathcal{F}, P)$ limit $\lim_{k \to \infty} \frac{1}{\sqrt{k}} B_0(kx)$.

Second way: Implicity by specifying

The mean function $\mu(x)$, the covariance function $C(x,y)$ and postulating the existance of a GRF $\{f(x) : x \in R\}$ which satisfies

$$E(f(x)) = \mu(x)$$

$$\operatorname{cov}(f(x), f(y)) = C(x,y).$$

Does such an $f(x)$ always exist?

Yes if $C(x,y) : R \times R \to R$ is positive definite.

Def: $C(x,y)$ is positive definite on $R \times R$ if $\forall n \geq 1$, $\forall x_1, \ldots, x_n \in R$, $\forall b_1, \ldots, b_n \in \mathbb{R}$ one has:

i) $\sum_{i,j=1}^{n} b_i b_j C(x_i, x_j) \geq 0$

ii) $C(x,y) = C(y,x)$, $\forall x, y \in R$.

Note that i) is required since

$$LHS = \operatorname{var}\left(\sum_{i=1}^{n} b_i f(x_i)\right) \geq 0.$$

e.g. $C(x,y) := \min(x,y)$ is p.d. on $[0, \infty)$

$\therefore$ $\exists$ a GRF $B(x)$ on $x \in [0,\infty)$ s.t.

$$E(B(x)) = 0$$

$$\operatorname{cov}(B(x), B(y)) = C(x,y)$$

This is the exact same Brownian Motion we constructed before.

Under either construction a (separable) GRF is <u>completely characterized</u> by $\mu(x)$ & $C(x,y)$.

$\therefore$ the fdd satisfy

$$\begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix} \sim N\left(\begin{pmatrix} \mu(x_1) \\ \vdots \\ \mu(x_n) \end{pmatrix}, \begin{array}{cc} & j \\ i & \begin{pmatrix} C(x_i, x_j) \end{pmatrix} \end{array} \right)$$

This gives you a way to simulate predict & do likelihood inference when working on a finite number of observation points $x_1, \dots, x_n \in \mathbb{R}$ without worrying about the rest of $\mathbb{R}$.

e.g. Simulating a GRF f with a given $\mu(x)$ & $C(x,y)$ at points $x_1, \dots, x_n \in \mathbb{R}$.

Let $\Sigma = \left(C(x_i, x_j) \right)_{i,j=1}^{n}$

$\vec{\mu} = \left(\mu(x_i) \right)_{i=1}^{n}$

$\vec{f} = \left(f(x_i) \right)_{i=1}^{n}$

Cholesky $\Sigma = L L^T$, L is lower triangular

SVD $\Sigma = U \Lambda U^T$, U is orthonormal, Λ is diag with <u>pos. entries</u>

follows from p.d. of Σ

Simulate $\vec{f}$ by

$$\vec{f} := L \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \vec{\mu}$$

$\downarrow$ iid $N(0,1)$

or by $$\vec{f} := U \Lambda^{1/2} U^T \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \vec{\mu}$$

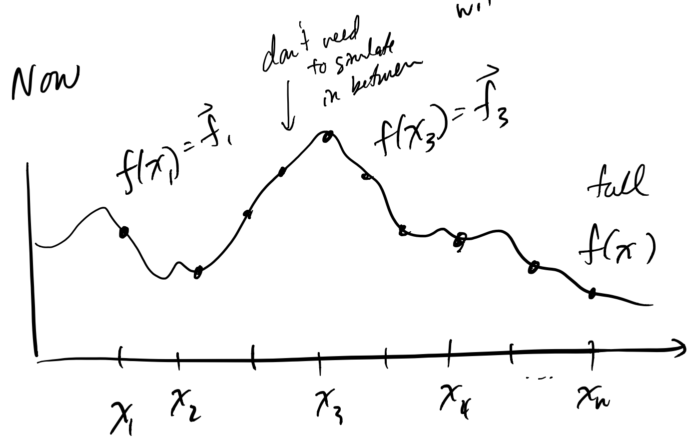
$$\overset{\mathcal{D}}{=} U \Lambda^{1/2} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \vec{\mu}$$

$$= \sum_{i=1}^{n} u_i \underbrace{\lambda_i^{1/2} z_i}_{} + \vec{\mu}$$

eigen basis of Σ

indep random coeffs with var $= \lambda_i$.

Now



<u>Physics perspective</u>:

$$\left\{ T(x) : x \in \mathbb{R}^3 \right\}$$

is predicted, under standard models, to be a GRF... Note the Randomness & Gaussianity is predicted via <u>Quantum mechanics</u>

Stationary & Isotropic GRFs

Def: For two RFs f, g write

$$\{f(x): x\in R\} \overset{\mathcal{D}}{=} \{g(x): x \in R\}$$

if the fdds are the same, i.e.

$$(f(x_1),\dots,f(x_n))^T \overset{\mathcal{D}}{=} (g(x_1),\dots,g(x_n))^T$$

$\forall n \geq 1, \forall x_1,\dots,x_n \in R$.

Def: A RF f on $\mathbb{R}^d$ is stationary if

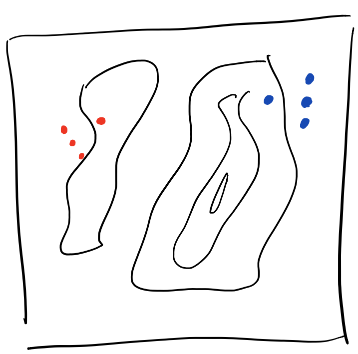
$$\{f(x): x\in\mathbb{R}^d\} \overset{\mathcal{D}}{=} \{f(x+v): x\in\mathbb{R}^d\}$$

$\forall v \in \mathbb{R}^d$ i.e. distributionally translation invariant.

Def: A RF f on $\mathbb{R}^d$ is isotropic if

$$\{f(x): x\in\mathbb{R}^d\} \overset{\mathcal{D}}{=} \{f(\mathcal{U}x+v): x\in\mathbb{R}^d\}$$

$\forall$ orthogonal matrix $\mathcal{U}$ & $v \in \mathbb{R}^d$.



Contours of isotropic $f: \mathbb{R}^d \to \mathbb{R}$

- f evaluated at • dots has the same dist as f evaluated at • dots.

Thm: Let $\mu(x)$ & $C(x,y)$ be the mean & cov fun for a GRF $f(x)$ on $\mathbb{R}^d$.

f is stationary $\implies C(x,y) = C(x-y)$, $\mu(x) = c \in \mathbb{R}$

f is isotropic $\implies C(x,y) = C(|x-y|)$, $\mu(x) = c \in \mathbb{R}$

$C(x-y)$ or $C(|x-y|)$ are called auto covariance functions in this case.

Spectral Representation for stationary GRF

Thm (Bochner): If f is a stationary GRF on $\mathbb{R}^d$ with $\mu(x)=0$ iff $\exists\, C_k^{ff}$ mapping $k\in\mathbb{R}^d$ to $\mathbb{R}^+ := \{x : x \geq 0\}$ s.t.

$$\{f(x): x\in\mathbb{R}^d\} \overset{\mathcal{D}}{=} \left\{\int_{\mathbb{R}^d} e^{ix\cdot k}\sqrt{C_k^{ff}}\, W_k \frac{dk}{(2\pi)^{d/2}}\right\}$$

where W_k is complex white noise with unit variance.

Remark 1

usually we will just write

$$f(x) = \int e^{ix\cdot k} \underbrace{\sqrt{C_k^{ff}}\, W_k}\, \frac{dk}{(2\pi)^{d/2}}$$

Think of this as the Fourier transform of $f(x)$.

Remark 2

For an ordinary (non random) function $g(x)$

let $$g_k := \int e^{-ix\cdot k} g(x) \frac{dx}{(2\pi)^{d/2}} = \text{Fourier transform}$$

$$g(x) = \int e^{ix\cdot k} g_k \frac{dk}{(2\pi)^{d/2}} = \text{inverse Fourier transform}$$

we'll not worry about when these transforms exist & when $\mathcal{F}^{-1} = IF$ (generally need $L_2(\mathbb{R}^d)$ or $L_1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$)

Remark 3.

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Think of f_k as the random coeffs in a basis expansion of $f(x)$ where k indexes the basis elements:

$$f(x) \text{ "="} \sum_k f_k \, u_k(x)$$

random coeffs $\sqrt{C_k^{ff}}\, W_k$

basis funs $e^{ix\cdot k} \frac{dk}{(2\pi)^{d/2}}$

Recall the SVD simulation of $\vec{f} = (f(x_1), \dots, f(x_n))^T$

$$= \underbrace{U \Lambda U^T}_{(c(x_i - x_j))_{ij}} \vec{Z}$$

$$= \sum_i \underbrace{\lambda_i^{1/2} Z_i}_{\text{random coeffs} \overset{\text{indep}}{\sim} N(0, \lambda_i)} \vec{u}_i$$

$\lambda_i \geq 0$ by pos. def cov mat.

basis eigenvectors of $(c(x_i - x_j))_{ij}$

$\therefore$ think of $e^{ix\cdot k}\frac{dk}{(2\pi)^{d/2}}$ as basis "eigenvectors" of $C(x-y)$ & $\sqrt{C_k^{ff}}\, W_k$ as indep coeffs $\sim N(0, C_k^{ff})$.

$C_k^{ff} \geq 0$ by pos. def. cov fun (Bochners' Thm)

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The Physics way of characterizing a mean zero stationary GRF is

$$E(f_k \overline{f_{k'}}) = \delta_{k-k'} C_k^{ff}$$

dirac delta

Sometimes written $\langle f_k f_{k'}^* \rangle = \delta_{k-k'} C_k^{ff}$

physics notation for expected value.

Heuristically:

$$E(f_k \overline{f_{k'}}) = E\left(W_k \sqrt{C_k^{ff}}\, \overline{W_{k'} \sqrt{C_{k'}^{ff}}}\right)$$

$$= \sqrt{C_k^{ff}} \sqrt{C_{k'}^{ff}}\, \underbrace{E(W_k \overline{W_{k'}})}_{\delta_{k-k'}}$$

only non-zero when $k = k'$

$$= C_k^{ff} \, \delta_{k-k'}$$

Also

$$E(f_k \overline{f_{k'}}) = E\left(\int e^{-ix\cdot k} f(x) \frac{dx}{(2\pi)^{d/2}} \overline{\int e^{-iy\cdot k'} f(y) \frac{dy}{(2\pi)^{d/2}}} \right)$$

$$= \iint e^{-ix\cdot k + iyk'} \underbrace{E(f(x)\overline{f(y)})}_{C(x-y)} \frac{dx}{(2\pi)^{d/2}} \frac{dy}{(2\pi)^{d/2}}$$

$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ y \end{pmatrix}$, $dz\,dy = dx\,dy$

$$= \iint e^{-i(z+y)k + iyk'} C(z) \frac{dz}{(2\pi)^{d/2}} \frac{dy}{(2\pi)^{d/2}}$$

$$= \underbrace{\int e^{-iz\cdot k} C(z) \frac{dz}{(2\pi)^{d/2}}}_{= C_k \text{ i.e. the fourier transform of the a.c.f } C(x)} \underbrace{\int e^{-iy(k-k')} \frac{dy}{(2\pi)^{d/2}}}_{= 2\pi^{d/2} \delta_{k-k'}} \quad (*)$$

where (*) holds since

$$\int g_k \left[\int e^{iy(k-k')} \frac{dy}{(2\pi)^d} \right] dk = \int \underbrace{\left(\int e^{iyk} g_k \frac{dk}{(2\pi)^{d/2}} \right)}_{g(y)} e^{-iy\cdot k'} \frac{dy}{(2\pi)^{d/2}}$$

$$= \int g(y) e^{-iy\cdot k'} \frac{dy}{(2\pi)^{d/2}}$$

$$= g_{k'}$$

$$\therefore \quad E(f_k \overline{f}_{k'}) = c_k^{ff} \, \delta_{k-k'}$$

$$= (2\pi)^{d/2} C_k \, \delta_{k-k'}$$

Def: For a mean zero stationary GRF f, c_k^{ff} is called the spectral density & equals $(2\pi)^{d/2} C_k$ where

$$\mathrm{cov}(f(x), f(y)) = C(x-y).$$

Remark 4.

If $f(x)$ is also isotropic then $c_k^{ff} = c_{|k|}^{ff}$

i.e. c_k^{ff} is rotationally symmetric about the origin.

White Noise & the dirac delta

Think of two different dirac delta functions:

1) δ_k defined in Fourier space $k \in \mathbb{R}^d$.

$$\int_{\mathbb{R}^d} \delta_k g_k \, dk = g_0$$

2) $\delta(x)$ defined in "pixel space" $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \delta(x) g(x) dx = g(0)$$

Heuristically:

$$\delta_k = I_{\{k=0\}} \frac{1}{dk} = \begin{cases} 0 & \text{if } k \neq 0 \\ \frac{1}{dk} & \text{if } k = 0 \end{cases}$$

$$\delta(x) = I_{\{x=0\}} \frac{1}{dx} = \begin{cases} 0 & \text{if } x \neq 0 \\ \frac{1}{dx} & \text{if } x = 0 \end{cases}$$

where dx & dk are infinitesimal area elements & $I_{\{\cdots\}}$ is an indicator.

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Real Gaussian white Noise in pixel space $W(x)$, $x \in \mathbb{R}^d$ with unit variance satisfies

$$E\big(W(x)W(y)\big) = \delta(x-y).$$

$$E\big(W_k \overline{W_{k'}}\big) = \delta_{k-k'}.$$

This is very "non rigourous" but can be made rigourous.

Since $\delta(x-y) = I_{x=y} \dfrac{1}{dx}$

You can think of $W(x)$ as $W(x) = Z(x) \cdot \dfrac{1}{\sqrt{dx}}$

when at each x, $Z(x)$ is an independent $N(0,1)$ R.V.

If you observe $W(x)$ on a grid then $\dfrac{1}{\sqrt{dx}} = \dfrac{1}{\sqrt{\text{grid pixel area}}}$.

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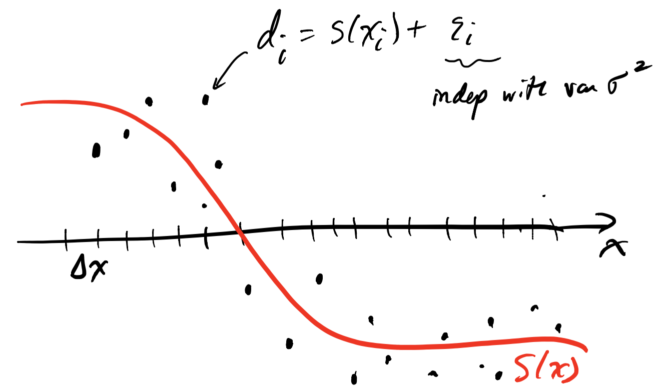
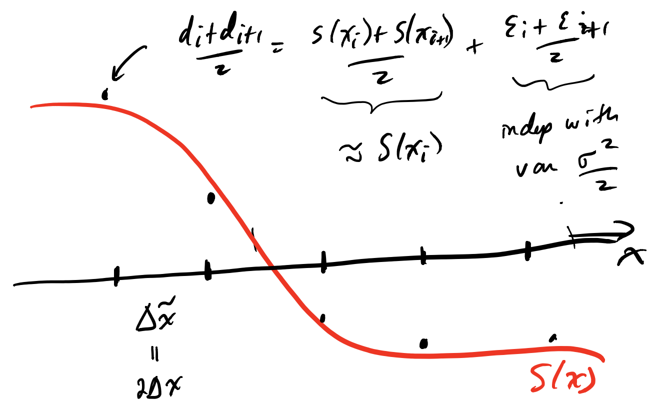
The brilliant thing about $W(x)$ is that it gives "grid invariant" quantifation of information or signal-to-noise ratio.

e.g. suppose you observe a stationary GRF $s(x)$ (i.e. the signal) with iid $N(0,\sigma^2)$ Noise on a grid with area Δx



Now consider constructing a new data set $\tilde{d}$ obtained by averaging 2 pixels.



If $s(x)$ is sufficiently smooth Both of the above inferences (one based on d, the other on $\tilde{d}$) should be about the same.

i.e. the information content is the same (i.e. SNR) but the Noise level changes.

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- Instead of writing $d_i = s(x_i) + \varepsilon_i$ when x_i's are on a grid, you can write

$$(*) \quad d(x) = s(x) + \varepsilon(x)$$

"observed" on $x \in \mathbb{R}^d$; where

$$\varepsilon(x) = \sigma \underbrace{W(x)}_{\text{white noise with unit var}} \text{"="} \sigma \frac{Z(x)}{\sqrt{dx}}$$

iid $N(0,1)$ at each $x \in \mathbb{R}^d$

- $\therefore \operatorname{cov}(\varepsilon(x), \varepsilon(y)) = \sigma^2 \frac{I_{\{x=y\}}}{dx} = \sigma^2 \delta(x-y)$

- For a particular grid x_i ... with grid spacing Δx, one has

$$d_i = d(x_i) = s(x_i) + \varepsilon(x_i)$$

where $\varepsilon(x_i) = \sigma \frac{Z(x_i)}{\sqrt{\Delta x}}$ ← replace dx with the grid spacing.

$$\overset{iid}{\sim} N\left(0, \frac{\sigma^2}{\Delta x}\right)$$

- Developing estimates based on $(*)$ allows you to work with integrals vrs sums over grid elements.

- when $\varepsilon(x) = \sigma W(x)$ you can think of σ^2 as variance of Noise per unit area pixel.

i.e. $\Delta x = 1 \Rightarrow \varepsilon(x_i) \sim N(0, \sigma^2)$

$\Delta x = 5 \Rightarrow \varepsilon(x_i) \sim N\left(0, \frac{\sigma^2}{5}\right)$

$\Delta x = 0.2 \Rightarrow \varepsilon(x_i) \sim N\left(0, \frac{\sigma^2}{0.2}\right)$

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- In the statistic literature often written:

$$dY(x) = s(x)\,dx + \sigma\, dB(x)$$

↖ Brownian Motion.

where $\frac{dY(x)}{dx} \equiv d(x)$

$$\sigma \frac{dB(x)}{dx} \equiv \varepsilon(x)$$

follows since

$$\operatorname{cov}\left(\sigma \frac{dB(x)}{dx}, \sigma \frac{dB(y)}{dy}\right)$$

$$\simeq \operatorname{cov}\left(\sigma \frac{B(x+\Delta x) - B(x)}{\Delta x}, \sigma \frac{B(y+\Delta x) - B(y)}{\Delta x}\right)$$

$$= \frac{\sigma^2}{(\Delta x)^2} \underbrace{\operatorname{cov}\big(B(x+\Delta x) - B(x), B(y+\Delta x) - B(y)\big)}_{= I_{x=y}\ \Delta x}$$

↑ By indep increments; ↑ $\operatorname{var}(B(x+\Delta x) - B(x))$

$$= \frac{\sigma^2}{\Delta x} I_{x=y}$$

$$= \sigma^2 \delta(x-y)$$

- to work with $W(x)$ Rigourously you need to always have $W(x)$ integrated against test fns.

$\varphi, \psi \in \underbrace{C_b^\infty(\mathbb{R}^d)}_{\text{compact support, infinity diff}}$

$\therefore \left(\int \varphi(x) W(x) dx, \int \psi(x) W(x) dx\right)^T$ is a Bivariate Gaussian $N(0, \Sigma)$

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where

$$\mathrm{cov}\left(\int \varphi(x) w(x) dx, \int \psi(y) w(y) dy \right)$$

$$= \iint \varphi(x) \psi(y) \underbrace{E(w(x) w(y))}_{= I_{x=y} \frac{1}{dx} = \delta(x-y)} dx dy$$

$$= \int \varphi(x) \psi(x) dx.$$