

STATISTICS OF SMOOTHED COSMIC FIELDS IN PERTURBATION THEORY. I. FORMULATION AND USEFUL FORMULAE IN SECOND-ORDER PERTURBATION THEORY

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ABSTRACT

We formulate a general method for perturbative evaluations of statistics of smoothed cosmic fields and provide useful formulae for application of the perturbation theory to various statistics. This formalism is an extensive generalization of the method used by Matsubara, who derived a weakly nonlinear formula of the genus statistic in a three-dimensional density field. After describing the general method, we apply the formalism to a series of statistics, including genus statistics, level-crossing statistics, Minkowski functionals, and a density extrema statistic, regardless of the dimensions in which each statistic is defined. The relation between the Minkowski functionals and other geometrical statistics is clarified. These statistics can be applied to several cosmic fields, including three-dimensional density field, three-dimensional velocity field, two-dimensional projected density field, and so forth. The results are detailed for second-order theory of the formalism. The effect of the bias is discussed. The statistics of smoothed cosmic fields as functions of rescaled threshold by volume fraction are discussed in the framework of second-order perturbation theory. In CDM-like models, their functional deviations from linear predictions plotted against the rescaled threshold are generally much smaller than that plotted against the direct threshold. There is still a slight meatball shift against rescaled threshold, which is characterized by asymmetry in depths of troughs in the genus curve. A theory-motivated asymmetry factor in the genus curve is proposed.

Subject headings: cosmology: theory — large-scale structure of universe — methods: statistical

1. INTRODUCTION

Any conceivable theory of structure formation in the universe predicts statistical properties of observable quantities. Therefore, the analysis of the present inhomogeneity of the universe is inevitably statistical. However, what kind of statistics is useful is not obvious, since we can invent an infinite number of statistics to be analyzed. Whether we can adopt better statistical descriptions is of great importance in this sense. Each statistic has both advantages and disadvantages. The power spectrum, for example, can fully characterize the random Gaussian fields, while it does not contain any information on the non-Gaussianity, which contains significant information in the gravitational instability theory. Among various statistical quantities, there is a promising class of statistics that utilizes smoothed cosmic fields. The smoothed field has less noisy property than the actual (unsmoothed, or raw) cosmic fields, such as galaxy distributions, temperature fluctuations of cosmic microwave background (CMB), shear fields of the gravitational lensing, and so forth. Perhaps the simplest example of such statistics is the variance $\langle \delta_R^2 \rangle$ of smoothed density contrast δ_R , which is a function of smoothing length R . Similarly, higher order cumulants $\langle \delta_R^N \rangle_c$ ($N = 3, 4, \dots$) are also simple statistics. The density probability distribution function (PDF) $P(\delta_R)$, which in principle can be constructed from the hierarchy of cumulants (e.g., Balian & Schaeffer 1989), is another example of popular statistics of smoothed cosmic fields.

Rather recently, more complex statistics of smoothed cosmic fields have become popular in cosmology, such as the genus statistic (Gott, Melott, & Dickinson 1986), density peak statistics (Bardeen et al. 1986), area, length, and level-crossing statistics (Ryden 1988a), Minkowski functionals (Schmalzing & Buchert 1997), etc. These statistics provide assuring characterizations of the clustering pattern that cannot be perceived only by the hierarchy of cumulants or by the PDF. The genus statistic is a powerful measure of the morphology in the three-dimensional (Gott et al. 1989; Moore et al. 1992; Park, Gott, & da Costa 1992; Beaky, Scherrer, & Villumsen 1992; Rhoads, Gott, & Postman 1994; Vogeley et al. 1994; Protogeros & Weinberg 1997; Sahni, Sathyaprakash, & Shandarin 1997; Canavezes et al. 1998; Springel et al. 1998; Colley et al. 2000) and two-dimensional (Melott et al. 1989; Gott et al. 1992; Coles & Plionis 1991; Plionis, Valdarnini, & Coles 1992; Davies & Coles 1993; Coles et al. 1993; Colley 1997) clustering of galaxies and clusters and also of the pattern of temperature fluctuations of CMB radiation (Bond & Efstathiou 1987; Torres 1994; Smoot et al. 1994; Torres et al. 1995; Park et al. 1998). The peak statistics of the three-dimensional density field are frequently used in connection with the statistics of the collapsing object (Kaiser 1984; Mann, Heavens, & Peacock 1993; Croft & Efstathiou 1994; Watanabe, Matsubara, & Suto 1994; Cen 1998; Gabrielli, Labini, & Durrer 2000), while the peak statistics of the CMB fluctuations (Sazhin 1985; Bond & Efstathiou 1987; Coles & Barrow 1987; Vittorio & Juskiewicz 1987; Kogut et al. 1995; Cayon & Smoot 1995; Fabbri & Torres 1996; Heavens & Sheth 1999) and of the weak lensing fields (van Waerbeke 2000; Jain & Van Waerbeke 2000) are suitable for constraining cosmological models. The area, length, and level-crossing statistics directly quantify the amount of contour surfaces (Ryden 1988b; Ryden et al. 1989; Torres 1994). The Minkowski functionals (Mecke, Buchert, & Wagner 1994; Sahni, Sathyaprakash, & Shandarin 1998; Sathyaprakash, Sahni, & Shandarin 1998; Kerscher et al. 1998; Bharadwaj et al. 2000), which are closely related to the above statistics, are also applied to smoothed cosmic fields (Winitzki & Kosowsky 1997; Naselsky & Novikov 1998; Schmalzing &

Gorski 1998; Schmalzing et al. 1999; Schmalzing & Diaferio 2000). These statistics of smoothed density fields are considered as powerful descriptors of the statistical information of the universe. It is therefore essential to establish theoretical predictions of the behavior of such statistics so that we can effectively and ideally analyze the data of our universe.

Recent developments of the perturbation theory (for review see Bernardeau et al. 2002) in calculating the variance, the cumulants, and the PDF are remarkable. The perturbation theory becomes more and more useful because the recent developments of observations enable us to have a widely covered sample volume of the universe, which can minimize undesirable strongly nonlinear effects, which we do not understand well. The direct comparison of the theoretical predictions of the perturbation theory with the actual data is a promising field of research in this sense. In the case of the top-hat smoothing function, Juszkiewicz, Bouchet, & Colombi (1993) and Bernardeau (1994b) used the tree-level (i.e., lowest order) perturbation theory to obtain the third- and fourth-order cumulants, i.e., the skewness and the kurtosis. The same quantities with Gaussian smoothing are calculated by Goroff et al. (1986), Matsubara (1994), and Łokas et al. (1995). Quite cleverly, Bernardeau (1994a) took advantage of special properties of the top-hat smoothing function and succeeded in obtaining full hierarchy of higher order moments in the tree-level perturbation theory. He also obtained the PDF from this hierarchy of cumulants, which remarkably describes the nonlinear behavior of the gravitational instability in numerical simulations. Beyond the tree-level calculation, the perturbation theory with loop corrections has also been developed (Juszkiewicz 1981; Vishniac 1983; Juszkiewicz, Sonoda, & Barrow 1984; Coles 1990; Suto & Sasaki 1991; Makino, Sasaki, & Suto 1992; Jain & Bertschinger 1994; Baugh & Efstathiou 1994; Scoccimarro & Frieman 1996a, 1996b). Because of the simplicity of the statistics, calculating the variance, the cumulants, and the PDF was primarily the playground of perturbation theorists. The evaluation of other statistics by the perturbation theory is less trivial. A quite useful technique is the Edgeworth expansion, which was first applied to cosmology in calculating the PDF from seed perturbations by Scherrer & Bertschinger (1991) and from nonlinear perturbation theory by Juszkiewicz et al. (1995) and Bernardeau & Kofman (1995). The analytic expression of the genus statistics in the perturbation theory is derived by Matsubara (1994), whose technique corresponds to a multivariate version of the Edgeworth expansion. In some literatures the Edgeworth expansion is used in connecting the statistics and dynamics of the universe (Chodorowski & Łokas 1997; Łokas 1998; Taruya & Soda 1999). The purpose of this paper is to give a comprehensive description of the formalism by which the perturbative evaluation is possible for a wide range of nontrivial statistics of smoothed cosmic fields in general.

Since the number of spatial dimensions of our universe is three, the statistics of large-scale cosmic fields are defined in either one-, two-, or three-dimensional space. For example, the space of the density field in a redshift map of galaxies or quasars is three-dimensional. A projected galaxy map on the sky, a shear field of gravitational lensing, and temperature fluctuations of the CMB on the sky are fields in two-dimensional space. The absorption lines of quasar spectra and pencil beam surveys of galaxies define fields in one-dimensional space. Thus, it is useful to develop our statistical method in d dimensions for generality. As illustrative examples of applications of our method, we calculate the level-crossing statistic (or equivalently length and area statistics), genus statistics, density extrema statistics, and Minkowski functionals. As cosmic fields, we consider density and velocity fields in three dimensions and the projected density field of galaxies in two dimensions. The basic formalism and results of the second-order theory are presented in this paper. We will give results of the third-order theory in a future paper that are technically more involved.

The primary purpose of the present paper is to fully describe the basic formalism for the statistics of smoothed cosmic fields in perturbation theory. Applications of the second-order theory to popular statistics and cosmic fields are systematically presented. Thus, this paper is in some way a mixture of the new results and a comprehensive review of the old results. The major new results in this paper are as follows:

1. Explicit formulae of the lowest non-Gaussian correction in various smoothed cosmic fields are given in d dimensions, including all the Minkowski functionals. They are presented as functions of both density threshold and rescaled threshold by volume fraction.
2. Relations among several statistics of the smoothed field are clarified.
3. Derivatives of skewness parameter of the velocity field and projected two-dimensional field are derived in second-order perturbation theory of gravitational instability theory. Especially, skewness parameters with Gaussian smoothing are detailed. These quantities are particularly important to perturbatively evaluate the statistics of the smoothed field.
4. The genus curve against the scaled threshold in perturbation theory is discussed, and a theory-motivated new asymmetry parameter in the genus curve is introduced.

This paper is structured in the following way. Sections 2 and 3 carry out a mathematical exercise of expressing high-order terms in terms of skewness parameters; the physics of gravity only enters once one actually calculates the skewness parameters in § 4. Thus, §§ 2 and 3 are a derivation of an extension of the Edgeworth expansion, and the basic results are equations (22) and (85). In § 3, popular statistics of smoothed fields are examined. The second-order expressions in terms of the skewness parameters are given for the PDF, level-crossing statistics, two- and three-dimensional genus statistics, two-dimensional weighted extrema, and Minkowski functionals in §§ 3.1–3.6. These quantities are reexpressed as functions of the volume fraction threshold in § 3.7. The skewness parameters for several cosmic fields are calculated in § 4, applying the second-order perturbation theory. Detailed calculations of the simple hierarchical model, the three-dimensional density field, the velocity field, the two-dimensional projected density field, and the weak lensing field are given in §§ 4.1–4.5. The effects of biasing on the skewness parameters are discussed in § 4.6. We discuss implications of our second-order results in § 5. We introduce a theory-motivated asymmetry factor to characterize the shift of the genus curve in this section. The conclusions are given in § 6. Useful Gaussian integrals including Hermite polynomials are given in Appendix A. The bispectrum in a two-dimensional projected field is reviewed in Appendix B. The symbols in this paper are summarized in Appendix C.

2. PERTURBATIVE EXPANSION OF STATISTICS

2.1. Smoothed Fields

We consider a cosmic random field $f(\mathbf{x})$ that represents any field constructed from observable quantities of the universe, such as three-dimensional density field, velocity field, two-dimensional projected density field, shear or convergence field of weak lensing, temperature fluctuations of the CMB, etc. The coordinates \mathbf{x} can be either one, two, or three dimensions.

We assume that the field f is already smoothed by a smoothing function W_R with smoothing length R that cuts the high-frequency fluctuations that suffer strongly nonlinear effects:

$$f(\mathbf{x}) = \int d^d x' W_R(|\mathbf{x} - \mathbf{x}'|) f_{\text{raw}}(\mathbf{x}') , \quad (1)$$

where $d = 1, 2, 3$ is the dimension of the space \mathbf{x} and f_{raw} is a raw, unsmoothed field. Two of the most popular three-dimensional smoothing functions are the top-hat smoothing function,

$$W_R(x) = \frac{3}{4\pi R^3} \Theta(R - x) , \quad (2)$$

and the Gaussian smoothing function,

$$W_R(x) = \frac{1}{(2\pi)^{3/2} R^3} \exp\left(-\frac{x^2}{2R^2}\right) . \quad (3)$$

We also assume that the mean value of the field f is zero and that the variance σ_0^2 exists:

$$\langle f \rangle = 0 , \quad \langle f^2 \rangle = \sigma_0^2 . \quad (4)$$

It is convenient to introduce a normalized field α that has a unit variance as follows:

$$\alpha \equiv \frac{f}{\sigma_0} , \quad \langle \alpha^2 \rangle = 1 . \quad (5)$$

2.2. Expressing the Non-Gaussian Statistics by Gaussian Integration

The statistics of a smoothed cosmic field we are interested in are the functions of the field α and its spatial derivatives, as we will see in the following sections. We denote the series of spatial derivatives by a set of variables (A_μ) that is defined by, for example, in the three-dimensional case,

$$(A_\mu) = (\alpha, \partial_1 \alpha, \partial_2 \alpha, \partial_3 \alpha, \partial_1^2 \alpha, \partial_2^2 \alpha, \partial_3^2 \alpha, \partial_1 \partial_2 \alpha, \partial_1 \partial_3 \alpha, \partial_2 \partial_3 \alpha, \dots) , \quad (6)$$

where $\partial_i = \partial/\partial x_i$. For convenience, the index is denoted as $\mu = 0, 1, 2, 3, (11), (22), (33), (12), (13), (23), \dots$ in such cases. In this example of equation (6), only the value of the field and the value of the derivatives on a single point are considered, but, in general, more than two points can be considered. The set A_μ forms multivariate random fields, which is denoted as an N -dimensional vector \mathbf{A} in the following. The dimension N is the total number of derivatives that appear in the definition of the statistics we are interested in. The statistical information is described by the multivariate PDF, $P(\mathbf{A})$. The Fourier transform of the PDF is the partition function:

$$Z(\mathbf{J}) = \int_{-\infty}^{\infty} d^N A P(\mathbf{A}) \exp(i\mathbf{J} \cdot \mathbf{A}) . \quad (7)$$

At this point, the cumulant expansion theorem (e.g., Ma 1985) is very useful. This theorem states that $\ln Z$ is the generating function of the cumulants, $M_{\mu_1 \dots \mu_n}^{(n)} \equiv \langle A_{\mu_1} \dots A_{\mu_n} \rangle_c$:

$$\ln Z(\mathbf{J}) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \sum_{\mu_1=1}^N \dots \sum_{\mu_n=1}^N M_{\mu_1 \dots \mu_n}^{(n)} J_{\mu_1} \dots J_{\mu_n} . \quad (8)$$

It follows from $\langle f \rangle = 0$ that $\langle A_\mu \rangle = 0$, and the first several cumulants are given by

$$M_\mu^{(1)} = 0 , \quad (9)$$

$$M_{\mu_1 \mu_2}^{(2)} = \langle A_{\mu_1} A_{\mu_2} \rangle , \quad (10)$$

$$M_{\mu_1 \mu_2 \mu_3}^{(3)} = \langle A_{\mu_1} A_{\mu_2} A_{\mu_3} \rangle , \quad (11)$$

$$M_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4)} = \langle A_{\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} \rangle - \langle A_{\mu_1} A_{\mu_2} \rangle \langle A_{\mu_3} A_{\mu_4} \rangle - \langle A_{\mu_1} A_{\mu_3} \rangle \langle A_{\mu_2} A_{\mu_4} \rangle - \langle A_{\mu_1} A_{\mu_4} \rangle \langle A_{\mu_2} A_{\mu_3} \rangle , \quad (12)$$

and so forth. From equations (8) and (9), the partition function is given by

$$Z(\mathbf{J}) = \exp\left(-\frac{1}{2}\mathbf{J}^T \mathbf{M} \mathbf{J}\right) \exp\left(\sum_{n=3}^{\infty} \frac{i^n}{n!} \sum_{\mu_1, \dots, \mu_n} M_{\mu_1 \dots \mu_n}^{(n)} J_{\mu_1} \cdots J_{\mu_n}\right), \quad (13)$$

where \mathbf{M} is an $N \times N$ matrix whose components are given by $M_{\mu\nu}^{(2)}$. On the other hand, equation (7) is inverted as

$$P(\mathbf{A}) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} d^N J Z(\mathbf{J}) \exp(-i\mathbf{J} \cdot \mathbf{A}). \quad (14)$$

Substituting $J_{\mu} \rightarrow i\partial/\partial A_{\mu}$ in the last term of equation (13), the distribution function of equation (14) can be transformed into

$$P(\mathbf{A}) = \exp\left[\sum_{n=3}^{\infty} \frac{(-)^n}{n!} \sum_{\mu_1, \dots, \mu_n} M_{\mu_1 \dots \mu_n}^{(n)} \frac{\partial^n}{\partial A_{\mu_1} \cdots \partial A_{\mu_n}}\right] P_G(\mathbf{A}), \quad (15)$$

where

$$P_G(\mathbf{A}) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} d^N J \exp\left(-i\mathbf{J} \cdot \mathbf{A} - \frac{1}{2}\mathbf{J}^T \mathbf{M} \mathbf{J}\right) \quad (16)$$

$$= \frac{1}{(2\pi)^{N/2} \sqrt{\det \mathbf{M}}} \exp\left(-\frac{1}{2}\mathbf{A}^T \mathbf{M}^{-1} \mathbf{A}\right) \quad (17)$$

is the multivariate Gaussian distribution function characterized by the correlation matrix \mathbf{M} .

Any statistical quantity of a smoothed cosmic field is expressed by an average $\langle F \rangle$ of a certain function $F(\mathbf{A})$, as we will see in the following sections. Thus, from equation (15), we obtain

$$\langle F \rangle = \int_{-\infty}^{\infty} d^N A P(\mathbf{A}) F(\mathbf{A}) \quad (18)$$

$$= \left\langle \exp\left(\sum_{n=3}^{\infty} \frac{1}{n!} \sum_{\mu_1, \dots, \mu_n} M_{\mu_1 \dots \mu_n}^{(n)} \frac{\partial^n}{\partial A_{\mu_1} \cdots \partial A_{\mu_n}}\right) F(\mathbf{A}) \right\rangle_G, \quad (19)$$

where

$$\langle \cdots \rangle_G \equiv \int_{-\infty}^{\infty} d^N A P_G(\mathbf{A}) F(\mathbf{A}) \quad (20)$$

denotes the averaging by the Gaussian distribution function of equation (17).

This form, equation (19), is useful when the deviation from the Gaussian distribution is not large. In principle, this equation reduces the general statistical averaging procedure to Gaussian integrations. However, it contains the infinite series; thus, we have to truncate this expression by some criteria. In most cases of interest, the weakly nonlinear evolution of cosmic fields satisfies $M^{(n)} \sim \mathcal{O}(\sigma_0^{n-2})$, as we will see in § 4. When this relation holds, we can expand the distribution function to arbitrary order in σ_0 . In the following, we assume this relation and introduce the normalized cumulants,

$$\hat{M}_{\mu_1 \dots \mu_n}^{(n)} = \frac{M_{\mu_1 \dots \mu_n}^{(n)}}{\sigma_0^{n-2}}, \quad (21)$$

which are assumed to be of order 1 in terms of σ_0 . In this case, equation (19) is expanded as, up to $\mathcal{O}(\sigma_0^2)$,

$$\begin{aligned} \langle F \rangle &= \langle F \rangle_G + \frac{1}{3!} \sum \hat{M}_{\mu_1 \mu_2 \mu_3}^{(3)} \langle F_{,\mu_1 \mu_2 \mu_3} \rangle_G \sigma_0 \\ &\quad + \left[\frac{1}{4!} \sum \hat{M}_{\mu_1 \mu_2 \mu_3 \mu_4}^{(4)} \langle F_{,\mu_1 \mu_2 \mu_3 \mu_4} \rangle_G + \frac{1}{2(3!)^2} \sum \hat{M}_{\mu_1 \mu_2 \mu_3}^{(3)} \hat{M}_{\nu_1 \nu_2 \nu_3}^{(3)} \langle F_{,\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3} \rangle_G \right] \sigma_0^2 + \mathcal{O}(\sigma_0^3), \end{aligned} \quad (22)$$

where we introduce the notation $F_{,\mu_1 \mu_2 \mu_3} \equiv \partial^3 F / \partial A_{\mu_1} \partial A_{\mu_2} \partial A_{\mu_3}$, etc. The calculation of the factors $\langle F_{,\mu \dots} \rangle_G$ is performed for an individual statistic that gives the explicit form of the function F .

2.3. Two-Point Correlations

In the expansion of equation (22), we need to calculate the Gaussian average of derivatives of the function F :

$$\langle F_{,\mu_1 \mu_2 \dots} \rangle_G = \int d^N A P_G(\mathbf{A}) F_{,\mu_1 \mu_2 \dots}(\mathbf{A}) = \frac{1}{(2\pi)^{N/2} \sqrt{\det \mathbf{M}}} \int d^N A \exp\left(-\frac{1}{2}\mathbf{A}^T \mathbf{M}^{-1} \mathbf{A}\right) F_{,\mu_1 \mu_2 \dots}(\mathbf{A}). \quad (23)$$

In most cases, the evaluation is analytically feasible because only Gaussian integration is needed. For the evaluation of such

integration, we need the correlation matrix \mathbf{M} . Throughout this paper we consider statistically homogeneous and isotropic fields, in which case the correlation matrix \mathbf{M} is simplified because of the symmetry. In fact, the correlation matrix takes the following forms:

$$\langle \alpha \alpha \rangle = 1, \quad (24)$$

$$\langle \alpha \alpha_{,i} \rangle = 0, \quad (25)$$

$$\langle \alpha \alpha_{,ij} \rangle = -\frac{1}{d} \frac{\sigma_1^2}{\sigma_0^2} \delta_{ij}, \quad (26)$$

$$\langle \alpha_{,i} \alpha_{,j} \rangle = \frac{1}{d} \frac{\sigma_1^2}{\sigma_0^2} \delta_{ij}, \quad (27)$$

$$\langle \alpha_{,i} \alpha_{,ijk} \rangle = 0, \quad (28)$$

$$\langle \alpha_{,ij} \alpha_{,kl} \rangle = \frac{1}{d(d+2)} \frac{\sigma_2^2}{\sigma_0^2} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (29)$$

where $\alpha_{,ij} = \partial \alpha / \partial x_i \partial x_j$, etc., and we define the following quantities:

$$\sigma_1^2 = -\langle f \nabla^2 f \rangle = -\langle \alpha \nabla^2 \alpha \rangle \sigma_0^2. \quad (30)$$

$$\sigma_2^2 = \langle \nabla^2 f \nabla^2 f \rangle = \langle \nabla^2 \alpha \nabla^2 \alpha \rangle \sigma_0^2. \quad (31)$$

In most cases of interest, the derivatives of order higher than 2 do not appear in the definition of the statistics, so we do not give the explicit form of those correlations.

In the following, we use the notation $\eta_i = \alpha_{,i}$ and $\zeta_{ij} = \alpha_{,ij}$ following Bardeen et al. (1986). As is often the case, when the second-order derivatives ζ_{ij} appear in F as simple polynomials, the following transform is particularly useful:

$$\tilde{\zeta}_{ij} = \zeta_{ij} + \frac{1}{d} \frac{\sigma_1^2}{\sigma_0^2} \delta_{ij} \alpha. \quad (32)$$

This transform erases the correlation between α and $\tilde{\zeta}_{ij}$, and the nonzero correlations are only

$$\langle \alpha^2 \rangle = 1, \quad (33)$$

$$\langle \eta_1^2 \rangle = \langle \eta_2^2 \rangle = \langle \eta_3^2 \rangle = \frac{1}{d} \frac{\sigma_1^2}{\sigma_0^2}, \quad (34)$$

$$\langle \tilde{\zeta}_{11}^2 \rangle = \langle \tilde{\zeta}_{22}^2 \rangle = \langle \tilde{\zeta}_{33}^2 \rangle = \frac{3}{d(d+2)} \frac{\sigma_2^2}{\sigma_0^2} \left(1 - \frac{d+2}{3d} \gamma^2 \right), \quad (35)$$

$$\langle \tilde{\zeta}_{11} \tilde{\zeta}_{22} \rangle = \langle \tilde{\zeta}_{11} \tilde{\zeta}_{33} \rangle = \langle \tilde{\zeta}_{22} \tilde{\zeta}_{33} \rangle = \frac{1}{d(d+2)} \frac{\sigma_2^2}{\sigma_0^2} \left(1 - \frac{d+2}{d} \gamma^2 \right), \quad (36)$$

$$\langle \tilde{\zeta}_{12}^2 \rangle = \langle \tilde{\zeta}_{13}^2 \rangle = \langle \tilde{\zeta}_{23}^2 \rangle = \frac{1}{d(d+2)} \frac{\sigma_2^2}{\sigma_0^2}, \quad (37)$$

where

$$\gamma = \frac{\sigma_1^2}{\sigma_0 \sigma_2}. \quad (38)$$

The Gaussian integration is straightforward if the function F is expressed by polynomials of $\tilde{\zeta}_{ij}$.

If the function F is more complicated in which ζ_{ij} are not simply given by polynomials, it is useful to completely diagonalize the correlation matrix \mathbf{M} of equations (24)–(29). We introduce the following transform, which is quite similar to the one in Bardeen et al. (1986):

$$x = -\frac{\sigma_0}{\sigma_2} \left(\sum_i \zeta_{ii} + \frac{\sigma_1^2}{\sigma_0^2} \alpha \right), \quad (39)$$

$$y = -\frac{\sigma_0}{\sigma_2} \frac{\zeta_{11} - \zeta_{22}}{2}, \quad (40)$$

$$z = -\frac{\sigma_0}{\sigma_2} \frac{\zeta_{11} + \zeta_{22} - 2\zeta_{33}}{2}. \quad (41)$$

If $d = 1$, we ignore the variables y, z , and similarly if $d = 2$, we ignore the variable z in the above equations and in the following. The above transform is similar to Bardeen et al. (1986), but notice that it is not identical. This transform completely

diagonalizes the correlation matrix:

$$\langle \alpha^2 \rangle = 1, \quad (42)$$

$$\langle \eta_1^2 \rangle = \langle \eta_2^2 \rangle = \langle \eta_3^2 \rangle = \frac{1}{d} \frac{\sigma_1^2}{\sigma_0^2}, \quad (43)$$

$$\langle x^2 \rangle = 1 - \gamma^2, \quad (44)$$

$$\langle y^2 \rangle = \frac{1}{d(d+2)}, \quad (45)$$

$$\langle z^2 \rangle = \frac{3}{d(d+2)}, \quad (46)$$

$$\langle \zeta_{12}^2 \rangle = \langle \zeta_{13}^2 \rangle = \langle \zeta_{23}^2 \rangle = \frac{1}{d(d+2)} \frac{\sigma_2^2}{\sigma_0^2}, \quad (47)$$

and all nondiagonal correlations are zero. For later convenience, we write the inverse relation of the transform of x , y , and z :

$$\zeta_{11} = -\frac{\sigma_2}{4\sigma_0} (x + 4y + 2z + \gamma\alpha), \quad (48)$$

$$\zeta_{22} = -\frac{\sigma_2}{4\sigma_0} (x - 4y + 2z + \gamma\alpha), \quad (49)$$

$$\zeta_{33} = -\frac{\sigma_2}{2\sigma_0} (x - 2z + \gamma\alpha). \quad (50)$$

If ζ_{33} does not appear in the function F , the variable z should be omitted in the above equations. If both ζ_{22} and ζ_{33} do not appear in the function F , the variables y and z should be omitted.

After expressing the function F in terms of the diagonalized variables α , η_i , x , y , z , and ζ_{ij} ($i < j$), the calculation of the Gaussian integration of equation (23) is performed.

2.4. Three-Point Correlations

In this paper only the first two terms in equation (22) are considered. Thus, we evaluate $M^{(3)}$ here. When the spatial symmetry is taken into account, the quantity $M^{(3)}$ reduces to the following expressions:

$$\langle \alpha^2 \alpha_{,i} \rangle = 0, \quad (51)$$

$$\langle \alpha^2 \alpha_{,ij} \rangle = \frac{\langle \alpha^2 \nabla^2 \alpha \rangle}{d} \delta_{ij}, \quad (52)$$

$$\langle \alpha \alpha_{,i} \alpha_{,j} \rangle = -\frac{\langle \alpha^2 \nabla^2 \alpha \rangle}{2d} \delta_{ij}, \quad (53)$$

$$\langle \alpha \alpha_{,i} \alpha_{,jk} \rangle = 0, \quad (54)$$

$$\langle \alpha \alpha_{,ij} \alpha_{,kl} \rangle = \frac{\langle \alpha \nabla^2 \alpha \nabla^2 \alpha \rangle}{d(d+2)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{3 \langle (\nabla \alpha \cdot \nabla \alpha) \nabla^2 \alpha \rangle}{d(d-1)(d+2)} \left[\delta_{ij} \delta_{kl} - \frac{d}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right], \quad (55)$$

$$\langle \alpha_{,i} \alpha_{,j} \alpha_{,k} \rangle = 0, \quad (56)$$

$$\langle \alpha_{,i} \alpha_{,j} \alpha_{,kl} \rangle = \frac{\langle (\nabla \alpha \cdot \nabla \alpha) \nabla^2 \alpha \rangle}{d(d-1)} \left[\delta_{ij} \delta_{kl} - \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right], \quad (57)$$

$$\langle \alpha_{,i} \alpha_{,jk} \alpha_{,lm} \rangle = 0, \quad (58)$$

etc. To prove the above equations, the following identities for isotropic fields are useful:

$$\langle \alpha_{,i} \alpha_{,j} \alpha_{,ij} \rangle = -\frac{1}{2} \langle (\nabla \alpha \cdot \nabla \alpha) \nabla^2 \alpha \rangle, \quad (59)$$

$$\langle \alpha \alpha_{,ij} \alpha_{,ij} \rangle = \frac{3}{2} \langle (\nabla \alpha \cdot \nabla \alpha) \nabla^2 \alpha \rangle + \langle \alpha (\nabla^2 \alpha) (\nabla^2 \alpha) \rangle. \quad (60)$$

Although the above equations are valid for $d \neq 0, 1$, the case $d = 1$ is obtained by just ignoring the terms with $(d-1)^{-1}$. Generally, more complicated quantities can appear for $M^{(3)}$, but the above relations are sufficient for our applications in this paper.

According to the spatial symmetry of the above equations, the third-order correlations $\hat{M}_{\mu\nu\lambda}^{(3)}$ are explicitly given. At this point, it is useful to define the following quantities:

$$S^{(0)} = \frac{\langle f^3 \rangle}{\sigma_0^4} = \frac{\langle \alpha^3 \rangle}{\sigma_0}, \quad (61)$$

$$S^{(1)} = -\frac{3}{4} \frac{\langle f^2 (\nabla^2 f) \rangle}{\sigma_0^2 \sigma_1^2} = -\frac{3}{4} \frac{\langle \alpha^2 (\nabla^2 \alpha) \rangle \sigma_0}{\sigma_1^2}, \quad (62)$$

$$S^{(2)} = -\frac{3d}{2(d-1)} \frac{\langle (\nabla f \cdot \nabla f) (\nabla^2 f) \rangle}{\sigma_1^4} = -\frac{3d}{2(d-1)} \frac{\langle (\nabla \alpha \cdot \nabla \alpha) (\nabla^2 \alpha) \rangle \sigma_0^3}{\sigma_1^4}, \quad (63)$$

$$S_2^{(2)} = \frac{\langle f (\nabla^2 f) (\nabla^2 f) \rangle}{\sigma_1^4} = \frac{\langle \alpha (\nabla^2 \alpha) (\nabla^2 \alpha) \rangle \sigma_0^3}{\sigma_1^4}. \quad (64)$$

We call quantities $S^{(a)}$ skewness parameters. The first one $S^{(0)}$ is usually denoted as skewness, and the others are its derivatives. They are to be calculated from usual perturbation theories in § 4. Using these quantities, the third-order correlations are given by

$$\hat{M}_{000}^{(3)} = S^{(0)}, \quad (65)$$

$$\hat{M}_{00i}^{(3)} = 0, \quad (66)$$

$$\hat{M}_{00(ii)}^{(3)} = -\frac{4}{3d} \frac{\sigma_1^2}{\sigma_0^2} S^{(1)}, \quad (67)$$

$$\hat{M}_{00(ij)}^{(3)} = 0 \quad (i < j), \quad (68)$$

$$\hat{M}_{0ii}^{(3)} = \frac{2}{3d} \frac{\sigma_1^2}{\sigma_0^2} S^{(1)}, \quad (69)$$

$$\hat{M}_{0ij}^{(3)} = 0 \quad (i \neq j), \quad (70)$$

$$\hat{M}_{0i(jk)}^{(3)} = 0, \quad (71)$$

$$\hat{M}_{0(ii)(ii)}^{(3)} = -\frac{1}{d^2(d+2)} \frac{\sigma_1^4}{\sigma_0^4} [2(d-1)S^{(2)} - 3dS_2^{(2)}], \quad (72)$$

$$\hat{M}_{0(ii)(jj)}^{(3)} = \frac{1}{d^2(d+2)} \frac{\sigma_1^4}{\sigma_0^4} [2S^{(2)} + dS_2^{(2)}] \quad (i \neq j), \quad (73)$$

$$\hat{M}_{0(ii)(jk)}^{(3)} = 0 \quad (j < k), \quad (74)$$

$$\hat{M}_{0(ij)(ij)}^{(3)} = -\frac{1}{d(d+2)} \frac{\sigma_1^4}{\sigma_0^4} [S^{(2)} - S_2^{(2)}] \quad (i < j), \quad (75)$$

$$\hat{M}_{0(ij)(kl)}^{(3)} = 0 \quad (i < j, k < l, i \neq k), \quad (76)$$

$$\hat{M}_{ijk}^{(3)} = 0, \quad (77)$$

$$\hat{M}_{ii(ii)}^{(3)} = 0, \quad (78)$$

$$\hat{M}_{ii(jj)}^{(3)} = -\frac{2}{3d^2} \frac{\sigma_1^4}{\sigma_0^4} S^{(2)} \quad (i \neq j), \quad (79)$$

$$\hat{M}_{ii(jk)}^{(3)} = 0 \quad (j < k), \quad (80)$$

$$\hat{M}_{ij(kk)}^{(3)} = 0 \quad (i \neq j), \quad (81)$$

$$\hat{M}_{ij(ij)}^{(3)} = \frac{1}{3d^2} \frac{\sigma_1^4}{\sigma_0^4} S^{(2)} \quad (i < j), \quad (82)$$

$$\hat{M}_{ij(kl)}^{(3)} = 0 \quad (i < j, k < l, i \neq k), \quad (83)$$

$$\hat{M}_{i(jk)(lm)}^{(3)} = 0, \quad (84)$$

and so forth, where repeated indices are *not* summed over in the above equations and $\hat{M}^{(3)}$ is symmetric under

permutation of its indices. Thus, denoting $F_{\mu\nu\lambda} \equiv \langle F_{,\mu\nu\lambda} \rangle_G$ for simplicity,

$$\begin{aligned}
\sum_{\mu,\nu,\lambda} \hat{M}_{\mu\nu\lambda}^{(3)} F_{\mu\nu\lambda} &= \hat{M}_{000}^{(3)} F_{000} + 3 \sum_i \hat{M}_{00(ii)}^{(3)} F_{00(ii)} + 3 \sum_i \hat{M}_{0ii}^{(3)} F_{0ii} + 3 \sum_i \hat{M}_{0(ii)(ii)}^{(3)} F_{0(ii)(ii)} \\
&\quad + 6 \sum_{i<j} \hat{M}_{0(ii)(jj)}^{(3)} F_{0(ii)(jj)} + 3 \sum_{i<j} \hat{M}_{0(ij)(ij)}^{(3)} F_{0(ij)(ij)} + 3 \sum_{i \neq j} \hat{M}_{ii(jj)}^{(3)} F_{ii(jj)} + 6 \sum_{i<j} \hat{M}_{ij(ij)}^{(3)} F_{ij(ij)} + \dots \\
&= S^{(0)} F_{000} - \frac{2S^{(1)}}{d} \frac{\sigma_1^2}{\sigma_0^2} \left(2 \sum_i F_{00(ii)} - \sum_i F_{0ii} \right) - \frac{2S^{(2)}}{d^2(d+2)} \frac{\sigma_1^4}{\sigma_0^4} \\
&\quad \times \left[3(d-1) \sum_i F_{0(ii)(ii)} - 6 \sum_{i<j} F_{0(ii)(jj)} + \frac{3d}{2} \sum_{i<j} F_{0(ij)(ij)} + (d+2) \sum_{i \neq j} F_{ii(jj)} - (d+2) \sum_{i<j} F_{ij(ij)} \right] \\
&\quad + \frac{3S_2^{(2)}}{d(d+2)} \frac{\sigma_1^4}{\sigma_0^4} \left(3 \sum_i F_{0(ii)(ii)} + 2 \sum_{i<j} F_{0(ii)(jj)} + \sum_{i<j} F_{0(ij)(ij)} \right) + \dots
\end{aligned} \tag{85}$$

This equation gives the second-order correction term of the statistical quantity $\langle F \rangle$ through equation (22). Once the field f is specified, the skewness parameters $S^{(a)}$ are calculated by dynamical perturbation theory of the field f . The remaining factors in the above equation are the Gaussian integrations of the derivatives of the function F , i.e., $\langle F_{,\mu\nu\lambda} \rangle_G$. These factors can be calculated once the function F is given. In the next section we calculate the latter factors for individual statistics.

3. STATISTICS OF SMOOTHED COSMIC FIELD

In this section we calculate the factor $\langle F_{,\mu_1\mu_2\dots} \rangle_G$ for each statistic. Some of the results in this section have been previously presented. The Edgeworth expansion of the PDF in § 3.1 is a familiar result. The result of the three-dimensional genus statistic in § 3.4 was already given by Matsubara (1994). We include these old results for completeness. Other subsections present new results.

3.1. Probability Distribution Function

Perhaps the simplest yet nontrivial statistic is the PDF, $P(f)$. The perturbative expansion of the PDF is known as the Edgeworth expansion (Scherrer & Bertschinger 1991; Juszkiewicz et al. 1995; Bernardeau & Kofman 1995). As the simplest example, we rederive the known Edgeworth expansion from the point of view of our general formalism above (see also Matsubara 1995a).

Since the PDF is simply given by $P(f) = \langle \delta(f' - f) \rangle_{f'}$, where δ is the Dirac δ -function, the function F in the previous section for PDF $P(f)$ is given by

$$F = \frac{1}{\sigma_0} \delta \left(\alpha - \frac{f}{\sigma_0} \right). \tag{86}$$

Since this form of F does not depend on derivatives of α , only F_{000} survives in equation (85). From equation (A5) with $n = 0$ and $k = 3$, $F_{000} = (2\pi)^{1/2} e^{-\nu^2/2} H_3(\nu)$, where $\nu = f/\sigma_0$. Thus, the PDF is derived from equation (22):

$$P(f) = \frac{e^{-\nu^2/2}}{\sqrt{2\pi}\sigma_0} \left[1 + \sigma_0 \frac{S^{(0)}}{6} H_3(\nu) + \mathcal{O}(\sigma_0^2) \right], \tag{87}$$

which reproduces the well-known result.

There is less advantage of applying our formalism to this simple statistic, which can be treated by standard methods. Our formalism has advantages when more nontrivial statistics are considered as shown below.

3.2. Level-crossing, Length, and Area Statistics

The next three statistics we consider here are the level-crossing statistic N_1 , the length statistic N_2 , and the area statistic N_3 . The level-crossing statistic is defined by the mean number of intersections of a straight line and threshold contours of the field. The length statistic is defined by the mean length of intersection of a two-dimensional surface and the threshold contours of the field. The area statistic is defined by the mean area of the contour surface in a three-dimensional space (Ryden 1988a; Ryden et al. 1989; Matsubara 1996). The level-crossing statistic is defined for one-, two-, and three-dimensional cosmic fields, the length statistic is defined for two- and three-dimensional cosmic fields, while the area statistic is defined only for three-dimensional cosmic fields. For statistically isotropic fields, those three statistics are proportional to each other.

In general, a statistic of the smoothed field f is a function of the threshold f_t , or of the normalized threshold $\nu = f_t/\sigma_0$. The explicit expressions of statistics N_1 , N_2 , and N_3 are given by (Ryden 1988a)

$$N_1(\nu) = \langle \delta(\alpha - \nu) |\eta_1| \rangle, \quad (88)$$

$$N_2(\nu) = \langle \delta(\alpha - \nu) [(\eta_1)^2 + (\eta_2)^2]^{1/2} \rangle, \quad (89)$$

$$N_3(\nu) = \langle \delta(\alpha - \nu) [(\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2]^{1/2} \rangle. \quad (90)$$

For isotropic fields, these statistics are actually equivalent (Ryden 1988a). In fact, the distribution function of $\eta_i \equiv \alpha_i$ for fixed $\alpha = \nu$ is the function of only the magnitude $|\boldsymbol{\eta}|$. Thus, using spherical coordinates for $\boldsymbol{\eta}$, one can see that

$$N_1(\nu) = N_2(\nu) \int \frac{d\phi}{2\pi} |\cos \phi| = N_3(\nu) \int \frac{d\Omega}{4\pi} |\sin \theta \cos \phi|, \quad (91)$$

i.e.,

$$N_1(\nu) = \frac{2}{\pi} N_2(\nu) = \frac{1}{2} N_3(\nu). \quad (92)$$

Thus, we only need to consider N_1 , which has the simplest expression, and the rest of the statistics are automatically given by equation (92). However, if the field is anisotropic, such as the density field in redshift space (Matsubara 1996), equation (92) no longer holds, and equations (88)–(90) should be used for each statistic. Equation (85) only holds for statistically isotropic fields, but equation (22) is applicable even for anisotropic fields.

Now we calculate the factor $\langle F_{,\mu_1\mu_2\cdots} \rangle_G$ for the particular statistic N_1 . The indices μ_1, μ_2, \dots only take 0 and 1 for the N_1 statistic. Let the number of 0 be k and the number of 1 be l . Then the factor is given by

$$\langle F_{,\mu_1\mu_2\cdots} \rangle_G = R(k, l) \equiv \left\langle \left(\frac{\partial}{\partial \alpha} \right)^k \left(\frac{\partial}{\partial \eta_1} \right)^l \delta(\alpha - \nu) |\eta_1| \right\rangle_G. \quad (93)$$

Since the variables α and η_i are uncorrelated in the Gaussian averaging, we can use equations (A5) and (A7) in Appendix A. Thus, the above Gaussian integration results in

$$R(k, l) = \frac{h_{l-2}}{\pi} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^{1-l} e^{-\nu^2/2} H_k(\nu), \quad (94)$$

where h_l is given by equation (A4). Now calculation of equation (85) is straightforward:

$$\sum_{\mu,\nu,\lambda} \hat{M}_{\mu\nu\lambda}^{(3)} F_{\mu\nu\lambda} = S^{(0)} R(3, 0) + 2S^{(1)} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^2 R(1, 2) = \frac{1}{\pi} \frac{\sigma_1}{\sqrt{d}\sigma_0} e^{-\nu^2/2} [S^{(0)} H_3(\nu) + 2S^{(1)} H_1(\nu)]. \quad (95)$$

On the other hand, the Gaussian contribution is simply given by

$$\langle F \rangle_G = R(0, 0) = \frac{1}{\pi} \frac{\sigma_1}{\sqrt{d}\sigma_0} e^{-\nu^2/2}. \quad (96)$$

Thus, the perturbative expansion of equation (22) up to second order is finally given by

$$N_1(\nu) = \frac{1}{\pi} \frac{\sigma_1}{\sqrt{d}\sigma_0} e^{-\nu^2/2} \left\{ 1 + \left[\frac{S^{(0)}}{6} H_3(\nu) + \frac{S^{(1)}}{3} H_1(\nu) \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}. \quad (97)$$

The second-order formulae for area and length statistics are given by equation (92) with the above equation. To evaluate the above formula, factors $S^{(0)}$ and $S^{(1)}$ should be known. Those factors are evaluated by usual perturbation theory in the following section.

3.3. Two-dimensional Genus Statistic

The next statistics we consider are the genus statistics. The genus statistics have been attractive because they have a geometrical meaning of the clustering as well as the cosmological significance. The two-dimensional genus statistic G_2 is defined in a two-dimensional plane S in a d -dimensional space, so that $d \geq 2$ is required. In this plane S , there are contours corresponding to each threshold ν . The two-dimensional genus statistic is defined by the number of contours surrounding regions higher than the threshold value minus the number of contours surrounding regions lower than the threshold value (Adler 1981; Coles 1988; Melott et al. 1989; Gott et al. 1990). This definition is intuitive, but an alternative, equivalent definition is more useful. First we set an arbitrary, fixed direction on the plane S . Then there are maxima and minima of contours according to that direction. These points are classified into upcrossing points and downcrossing points with respect to that chosen direction.

The numbers of those points are used to define the two-dimensional genus statistic in the following way:

$$G_2 = \frac{1}{2}[(\text{number of upcrossing minima}) - (\text{number of upcrossing maxima}) - (\text{number of downcrossing minima}) + (\text{number of downcrossing maxima})], \quad (98)$$

per unit area of the surface. According to this definition, the explicit expression of the two-dimensional genus statistic is given by

$$G_2(\nu) = -\frac{1}{2}\langle\delta(\alpha - \nu)\delta(\eta_1)|\eta_2|\zeta_{11}\rangle. \quad (99)$$

For this statistic, the indices μ_1, μ_2, \dots only take 0, 1, 2, and (11). Let the number of 0 be k , 1 be l_1 , 2 be l_2 , and (11) be m . Then we need to calculate the following quantity:

$$\langle F_{\mu_1\mu_2\dots} \rangle_G = R(k; l_1, l_2; m) \equiv -\frac{1}{2} \left\langle \left(\frac{\partial}{\partial \alpha} \right)^k \left(\frac{\partial}{\partial \eta_1} \right)^{l_1} \left(\frac{\partial}{\partial \eta_2} \right)^{l_2} \left(\frac{\partial}{\partial \zeta_{11}} \right)^m \delta(\alpha - \nu) \delta(\eta_1) |\eta_2| \zeta_{11} \right\rangle_G. \quad (100)$$

Since the second derivative ζ_{11} appears as a polynomial, we just use the transform of equation (32). Then, from equations (A5), (A7), and (A8), the above equation reduces to

$$R(k; l_1, l_2; m) = \frac{h_{l_1} h_{l_2-2}}{(2\pi)^{3/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^{2-l_1-l_2-2m} e^{-\nu^2/2} [H_{k+1}(\nu)\delta_{m0} - H_k(\nu)\delta_{m1}]. \quad (101)$$

Thus, from equation (85),

$$\begin{aligned} \sum_{\mu, \nu, \lambda} \hat{M}_{\mu\nu\lambda}^{(3)} F_{\mu\nu\lambda} &= -2S^{(1)} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^2 [2R(2; 0, 0; 1) - R(1; 2, 0; 0) - R(1; 0, 2; 0)] - 2S^{(2)} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^4 R(0; 0; 2, 1) \\ &= \frac{1}{(2\pi)^{3/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^2 e^{-\nu^2/2} [S^{(0)} H_4(\nu) + 4S^{(1)} H_2(\nu) + 2S^{(2)}]. \end{aligned} \quad (102)$$

On the other hand, the Gaussian contribution is simply given by

$$\langle F \rangle_G = R(0; 0, 0; 0) = \frac{1}{(2\pi)^{3/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^2 e^{-\nu^2/2} H_1(\nu). \quad (103)$$

Thus, the perturbative expansion of equation (22) up to second order is finally given by

$$G_2(\nu) = \frac{1}{(2\pi)^{3/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^2 e^{-\nu^2/2} \left\{ H_1(\nu) + \left[\frac{S^{(0)}}{6} H_4(\nu) + \frac{2S^{(1)}}{3} H_2(\nu) + \frac{S^{(2)}}{3} \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}. \quad (104)$$

3.4. Three-dimensional Genus Statistic

The second-order formula for the three-dimensional genus statistic was already derived by Matsubara (1994), but the detailed derivation was omitted. For completeness, we revisit the same quantity from our general point of view here. While the two-dimensional genus statistic is defined by the number of contour lines in two-dimensional surface, the three-dimensional genus statistic (Gott et al. 1986) is defined by the number of contour surfaces and the number of handles in three-dimensional space. Thus, the three-dimensional genus is defined only for cosmic fields of $d = 3$. The three-dimensional genus statistic G_3 is defined by

$$G_3 = [(\text{number of handles of contours}) - (\text{number of isolated contours})], \quad (105)$$

per unit volume of the three-dimensional space. This quantity is mathematically equivalent to $-\frac{1}{2}$ times the Euler characteristic of the contour surfaces and thus is proportional to the total surface integral of local curvature of contours from the Gauss-Bonnet theorem. Although this definition is intuitive, an alternative, equivalent definition is more useful as in the two-dimensional genus case. We set an arbitrary direction in the three-dimensional space. Then there are maxima, minima, and saddle points according to that direction. From the number of these points, the three-dimensional genus is defined by

$$G_3 = -\frac{1}{2}[(\text{number of maxima}) + (\text{number of minima}) - (\text{number of saddle points})], \quad (106)$$

TABLE 1
DEFINITION OF $J_m^{(2)}$

m_{11}	m_{22}	m_{12}	$J_0^{(2)}$	$J_1^{(2)}$	$J_2^{(2)}$
0	0	0	1	0	0
1	0	0	0	1	0
0	1	0	0	1	0
1	1	0	0	0	1
0	0	2	0	0	-2

NOTE.—For other sets of $\{m_{ij}\}$ not listed in this table, $J_m^{(2)} = 0$.

per unit volume. According to this definition, the explicit expression of the three-dimensional genus statistic is given by (Doroshkevich 1970; Adler 1981; Bardeen et al. 1986)

$$G_3(\nu) = -\frac{1}{2} \langle \delta(\alpha - \nu) \delta(\eta_1) \delta(\eta_2) |\eta_3| (\zeta_{11} \zeta_{22} - \zeta_{12}^2) \rangle. \quad (107)$$

We need to calculate the following quantity:

$$\begin{aligned} \langle F_{,\mu_1 \mu_2 \dots} \rangle_G &= R(k; l_1, l_2, l_3; m_{11}, m_{22}, m_{12}) \\ &\equiv -\frac{1}{2} \left\langle \left(\frac{\partial}{\partial \alpha} \right)^k \prod_{i=1}^3 \left(\frac{\partial}{\partial \eta_i} \right)^{l_i} \prod_{i \leq j}^2 \left(\frac{\partial}{\partial \zeta_{ij}} \right)^{m_{ij}} \delta(\alpha - \nu) \delta(\eta_1) \delta(\eta_2) |\eta_3| (\zeta_{11} \zeta_{22} - \zeta_{12}^2) \right\rangle_G. \end{aligned} \quad (108)$$

After applying the transform of equation (32), the Gaussian integration of $\tilde{\zeta}_{ij}$ fixing α is given by

$$\left\langle \prod_{i \leq j}^2 \left(\frac{\partial}{\partial \zeta_{ij}} \right)^{m_{ij}} (\zeta_{11} \zeta_{22} - \zeta_{12}^2) \middle| \alpha \right\rangle_G = \left(\frac{\sigma_1^2}{d \sigma_0^2} \right)^{2 - \sum_{i \leq j} m_{ij}} [H_2(\alpha) J_0^{(2)}(\{m_{ij}\}) - H_1(\alpha) J_1^{(2)}(\{m_{ij}\}) + J_2^{(2)}(\{m_{ij}\})], \quad (109)$$

where $J_m^{(2)}$ is defined in Table 1.

Then, from equations (A5), (A7), and (A8), equation (108) reduces to

$$\begin{aligned} R(k; l_1, l_2, l_3; m_{11}, m_{22}, m_{12}) &= -\frac{1}{(2\pi)^2} \left(\frac{\sigma_1}{\sqrt{d} \sigma_0} \right)^{3 - \sum_{i=1}^3 l_i - 2 \sum_{i \leq j} m_{ij}} h_{l_1} h_{l_2} h_{l_3-2} e^{-\nu^2/2} [J_0^{(2)}(\{m_{ij}\}) H_{k+2}(\nu) \\ &\quad - J_1^{(2)}(\{m_{ij}\}) H_{k+1}(\nu) + J_2^{(2)}(\{m_{ij}\}) H_k(\nu)]. \end{aligned} \quad (110)$$

Since the three-dimensional genus is only defined for $d \geq 3$ and our universe has the spatial dimension 3, only $d = 3$ is meaningful for actual cosmic fields. Nevertheless, we preserve the general dimension d for some flavor of generality. Thus, from equation (85),

$$\sum_{\mu, \nu, \lambda} \hat{M}_{\mu\nu\lambda}^{(3)} F_{\mu\nu\lambda} = -\frac{1}{(2\pi)^2} \left(\frac{\sigma_1}{\sqrt{d} \sigma_0} \right)^3 e^{-\nu^2/2} [S^{(0)} H_5(\nu) + 6S^{(1)} H_3(\nu) + 6S^{(2)} H_1(\nu)]. \quad (111)$$

The Gaussian contribution is given by (Doroshkevich 1970; Hamilton, Gott, & Weinberg 1986)

$$\langle F \rangle_G = R(0; 0, 0, 0; 0, 0, 0) = -\frac{1}{(2\pi)^2} \left(\frac{\sigma_1}{\sqrt{d} \sigma_0} \right)^3 e^{-\nu^2/2} H_2(\nu). \quad (112)$$

Thus, the perturbative expansion of equation (22) up to second order is finally given by

$$G_3(\nu) = -\frac{1}{(2\pi)^2} \left(\frac{\sigma_1}{\sqrt{d} \sigma_0} \right)^3 e^{-\nu^2/2} \left\{ H_2(\nu) + \left[\frac{S^{(0)}}{6} H_5(\nu) + S^{(1)} H_3(\nu) + S^{(2)} H_1(\nu) \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}. \quad (113)$$

The above equation with $d = 3$ agrees with Matsubara (1994).¹

3.5. Two-dimensional Weighted Extrema Density

Next, we consider the weighted extrema density above a threshold ν . The field extrema are defined to be points where all the first-order spatial derivatives of the field f vanish: $\partial f / \partial x_i = 0$. The weight ± 1 is associated to each extremum according to the

¹ The notations S , T , and U in Matsubara (1994) are related to the notations here by $S = S^{(0)}$, $T = 2S^{(1)}/3$, and $U = S^{(2)}/3$.

number of negative eigenvalues of spatial second-order derivatives of the field. When the threshold is high enough, the weighted extrema are approximately identified with the field peaks.

Mathematically, this statistic is equivalent to the two-dimensional genus statistics, according to Morse's theorem (Morse & Cairns 1969; Adler 1981). Therefore, we do not need to calculate separately to obtain the result on this statistic. However, we present the derivation of the extrema density as an alternative calculation to the two-dimensional genus.

The weighted extrema in the two-dimensional field are given by

$$\rho_{e2}(\nu) = \langle \theta(\alpha - \nu) \delta(\eta_1) \delta(\eta_2) (\zeta_{11} \zeta_{22} - \zeta_{12}^2) \rangle. \quad (114)$$

Following the similar calculation of previous examples and using equations in Appendix A, we obtain

$$\begin{aligned} \langle F_{,\mu_1 \mu_2 \dots} \rangle_G &= R(k; l_1, l_2; m_{11}, m_{22}, m_{12}) \\ &\equiv -\frac{1}{2} \left\langle \left(\frac{\partial}{\partial \alpha} \right)^k \prod_{i=1}^2 \left(\frac{\partial}{\partial \eta_i} \right)^{l_i} \prod_{i \leq j}^2 \left(\frac{\partial}{\partial \zeta_{ij}} \right)^{m_{ij}} \delta(\alpha - \nu) \delta(\eta_1) \delta(\eta_2) (\zeta_{11} \zeta_{22} - \zeta_{12}^2) \right\rangle_G \\ &= \frac{1}{(2\pi)^{3/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^{2 - \sum_{i=1}^2 l_i - 2 \sum_{i \leq j}^2 m_{ij}} h_{l_1} h_{l_2} e^{-\nu^2/2} [J_0^{(2)}(\{m_{ij}\}) H_{k+1}(\nu) \\ &\quad - J_1^{(2)}(\{m_{ij}\}) H_k(\nu) + J_2^{(2)}(\{m_{ij}\}) H_{k-1}(\nu)] , \end{aligned} \quad (115)$$

and the perturbative expansion of equation (22) up to second order is finally given by

$$\rho_{e2}(\nu) = \frac{1}{(2\pi)^{3/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^2 e^{-\nu^2/2} \left\{ H_1(\nu) + \left[\frac{S^{(0)}}{6} H_4(\nu) + \frac{2S^{(1)}}{3} H_2(\nu) + \frac{S^{(2)}}{3} \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}. \quad (116)$$

This result is in fact equivalent to the two-dimensional genus statistics given by equation (104).

3.6. Minkowski Functionals

Most of the Minkowski functionals are closely related to statistics discussed above. In this subsection we comprehensively describe the exact relation between the Minkowski functionals $V_k^{(d)}$ of a smoothed field and the statistical quantities considered above.

The Minkowski functional of $k = 0$ is simply the volume fraction of the excursion set K , which is defined by high-density regions above a given threshold ν :

$$V_0^{(d)}(\nu) = \frac{1}{V} \int_K dV. \quad (117)$$

The other functionals with $k = 1, 2, \dots, d$ are defined by the integral of the curvatures on isodensity surfaces of the threshold ν (Schmalzing & Buchert 1997; Schmalzing & Gorski 1998). In three dimensions, $d = 3$, they are evaluated by a surface integration averaged over the whole system of volume V (Schmalzing & Buchert 1997), i.e.,

$$V_k^{(3)}(\nu) = \frac{1}{V} \int_{\partial K} d^2 A(\mathbf{x}) v_k^{(3)}(\nu, \mathbf{x}), \quad (118)$$

where the local Minkowski functionals,

$$v_1^{(3)}(\nu, \mathbf{x}) = \frac{1}{6}, \quad (119)$$

$$v_2^{(3)}(\nu, \mathbf{x}) = \frac{1}{6\pi} \left(\frac{1}{R_1} + \frac{1}{R_2} \right), \quad (120)$$

$$v_3^{(3)}(\nu, \mathbf{x}) = \frac{1}{4\pi} \frac{1}{R_1 R_2}, \quad (121)$$

are defined by the principal curvatures $1/R_1$ and $1/R_2$ of the surface oriented toward lower density values. In two dimensions, $d = 2$, the Minkowski functionals of $k = 1, 2$ are evaluated by a line integration averaged over the whole system of two-dimensional volume (surface) V (Schmalzing & Gorski 1998), i.e.,

$$V_k^{(2)}(\nu) = \frac{1}{V} \int_{\partial K} dL(\mathbf{x}) v_k^{(2)}(\nu, \mathbf{x}), \quad (122)$$

where the local Minkowski functionals,

$$v_1^{(2)}(\nu, \mathbf{x}) = \frac{1}{4}, \quad (123)$$

$$v_2^{(2)}(\nu, \mathbf{x}) = \frac{1}{2\pi} \frac{1}{R_1}, \quad (124)$$

are defined by the principal curvature $1/R_1$ of the line oriented toward lower density values.

All the Minkowski functionals for a Gaussian random field are analytically derived by Tomita (1986):

$$V_k^{(d)}(\nu) = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_d}{\omega_{d-k}\omega_k} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^k e^{-\nu^2/2} H_{k-1}(\nu), \quad (125)$$

where the factor $\omega_k = \pi^{k/2}/\Gamma(k/2 + 1)$ is the volume of the unit ball in k dimensions, so that $\omega_0 = 1$, $\omega_1 = 2$, $\omega_2 = \pi$, and $\omega_3 = 4\pi/3$ (Schmalzing & Buchert 1997).

It turns out that the Minkowski functionals in two and three dimensions are identical to the statistics N_1 , G_2 , and G_3 in each dimension, except for normalization factors. In fact, from Crofton's formula (Crofton 1868), the k th Minkowski functional is given by

$$V_k^{(d)} = \frac{\omega_d}{\omega_{d-k}\omega_k} \int_{\mathcal{E}_k^{(d)}} d\mu_k(E) \chi^{(k)}(K \cap E). \quad (126)$$

In this formula for body K in d dimensions, we consider an arbitrary k -dimensional hypersurface E and calculate the Euler characteristic $\chi^{(k)}$ of the intersection $K \cap E$ in k dimensions. This quantity is integrated over the space $\mathcal{E}_k^{(d)}$ of all conceivable hypersurfaces. The integration measure $d\mu_k(E)$ is normalized to give $\int_{\mathcal{E}_k^{(d)}} d\mu_k(E) = 1$. From this formula, we can see that the statistics G_3 , G_2 , and N_1 are proportional to the Minkowski functionals of $V_3^{(d)}$, $V_2^{(d)}$, and $V_1^{(d)}$. In fact, $\chi^{(3)}$ is given by -1 times the three-dimensional genus (or, equivalently, $\frac{1}{2}$ times the Euler number of boundaries, $\chi[\partial(K \cap E)]$), $\chi^{(2)}$ is identical to the two-dimensional genus, and $\chi^{(1)}$ is just $\frac{1}{2}$ times the number of level-crossing points (Adler 1981). Thus, Minkowski functionals are given by

$$V_3^{(d)}(\nu) = -\frac{\omega_d}{\omega_{d-3}\omega_3} G_3(\nu), \quad (127)$$

$$V_2^{(d)}(\nu) = \frac{\omega_d}{\omega_{d-2}\omega_2} G_2(\nu), \quad (128)$$

$$V_1^{(d)}(\nu) = \frac{\omega_d}{2\omega_{d-1}\omega_1} N_1(\nu), \quad (129)$$

where the boundary of the body K is identified with the isodensity contours of threshold ν and the statistics on the right-hand sides are defined in d dimensions. Therefore, we have already obtained the weakly non-Gaussian expressions for Minkowski functionals; i.e., the Minkowski functionals of $k = 1, 2, 3$ are given by the above equations and equations (97), (104), and (113). The Gaussian parts of the above equations exactly reproduce Tomita's formula (eq. [125]).

The remaining Minkowski functional is the volume functional

$$V_0^{(d)}(\nu) = \langle \theta(\nu - \alpha) \rangle. \quad (130)$$

In this case, from equation (A5) in Appendix A,

$$R(k) \equiv \left\langle \left(\frac{\partial}{\partial \alpha} \right)^k \theta(\alpha - \nu) \right\rangle_G = \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} H_{k-1}(\nu), \quad (131)$$

so that equation (85) reduces to

$$V_0^{(d)}(\nu) = \frac{1}{2} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}}\right) + \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} \frac{S^{(0)}}{6} H_2(\nu) \sigma_0 + \mathcal{O}(\sigma_0^2). \quad (132)$$

The equivalent form can be obtained by integrating the Edgeworth expansion of the PDF, equation (87).

All the formulae of Minkowski functionals derived above for $0 \leq k \leq 3$ fit into a single expression:

$$V_k^{(d)}(\nu) = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_d}{\omega_{d-k}\omega_k} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^k \times e^{-\nu^2/2} \left\{ H_{k-1}(\nu) + \left[\frac{1}{6} S^{(0)} H_{k+2}(\nu) + \frac{k}{3} S^{(1)} H_k(\nu) + \frac{k(k-1)}{6} S^{(2)} H_{k-2}(\nu) \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}. \quad (133)$$

3.7. Rescaling the Threshold Density by Volume Fractions

The density threshold ν so far is simply defined so that the isodensity surface is identified by $f = \nu\sigma_0$. However, the horizontal shift of the nonlinear genus curve, etc., is considerably attributed to the nonlinear shift of the probability distribution of the density field (e.g., Gott, Weinberg, & Melott 1987; Matsubara & Yokoyama 1996). In order to cancel the latter shift, the threshold $\tilde{\nu}$ is defined so that the volume fraction f_V on the high-density side of the isodensity surface equals

$$f_V = \frac{1}{\sqrt{2\pi}} \int_{\tilde{\nu}}^{\infty} dt e^{-t^2/2}. \quad (134)$$

In fact, most of the work on genus analysis uses the genus curve plotted against the volume fraction threshold $\tilde{\nu}$. Recently, Seto (2000) reexpressed the weakly non-Gaussian formula of the genus curve (Matsubara 1994) in terms of $\tilde{\nu}$, using perturbative expansion of the PDF of the density field (Juszkiewicz et al. 1995). We follow this method to reexpress the weakly non-Gaussian formulae of various statistical quantities derived above in terms of the volume fraction threshold $\tilde{\nu}$.

The relation of ν and $\tilde{\nu}$ of weakly non-Gaussian field is simply given by equating the two equations (132) and (134). Up to first order in σ_0 , the relation reduces to

$$\nu = \tilde{\nu} + \frac{S^{(0)}}{6} H_2(\tilde{\nu}) \sigma_0 + \mathcal{O}(\sigma_0^2). \quad (135)$$

It is straightforward to rewrite the various analytical formulae we derived above. The results for the level-crossing statistic, the two-dimensional genus, and the three-dimensional genus are, respectively, given by

$$N_1(\tilde{\nu}) = \frac{1}{\pi} \frac{\sigma_1}{\sqrt{d}\sigma_0} e^{-\tilde{\nu}^2/2} \left\{ 1 + \left[\frac{1}{3} (S^{(1)} - S^{(0)}) H_1(\tilde{\nu}) \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}, \quad (136)$$

$$G_2(\tilde{\nu}) = \rho_{e2}(\tilde{\nu}) = \frac{1}{(2\pi)^{3/2}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^2 e^{-\tilde{\nu}^2/2} \left\{ H_1(\tilde{\nu}) + \left[\frac{2}{3} (S^{(1)} - S^{(0)}) H_2(\tilde{\nu}) + \frac{1}{3} (S^{(2)} - S^{(0)}) \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}, \quad (137)$$

$$G_3(\tilde{\nu}) = -\frac{1}{(2\pi)^2} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^3 e^{-\tilde{\nu}^2/2} \left\{ H_2(\tilde{\nu}) + \left[(S^{(1)} - S^{(0)}) H_3(\tilde{\nu}) + (S^{(2)} - S^{(0)}) H_1(\tilde{\nu}) \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}. \quad (138)$$

The results for Minkowski functionals of $k = 1, 2, 3$ are again given by these equations and equations (127)–(129). All the Minkowski functionals for $0 \leq k \leq 3$ fit into a single expression:

$$V_k^{(d)}(\tilde{\nu}) = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_d}{\omega_{d-k}\omega_k} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^k \times e^{-\tilde{\nu}^2/2} \left\{ H_{k-1}(\tilde{\nu}) + \left[\frac{k}{3} (S^{(1)} - S^{(0)}) H_k(\tilde{\nu}) + \frac{k(k-1)}{6} (S^{(2)} - S^{(0)}) H_{k-2}(\tilde{\nu}) \right] \sigma_0 + \mathcal{O}(\sigma_0^2) \right\}. \quad (139)$$

Remarkably, the highest order Hermite polynomial in the non-Gaussian correction terms vanishes in each case. In addition, the skewness parameters only appear as combinations of the form $S^{(a)} - S^{(0)}$, which makes the result simpler compared with the original form with the direct threshold ν . As we will see in the following sections, the numerical values of $S^{(a)}$ ($a = 0, 1, 2$) are quite close, or even identical in some special models. This means that the non-Gaussian corrections of the above statistics are smaller with the rescaled threshold $\tilde{\nu}$ than with the original threshold ν . This is one of the central results in this paper. This tendency is in agreement with the analyses of numerical simulations.

4. SKEWNESS PARAMETERS FOR SMOOTHED FIELDS

We need to know the skewness parameters $S^{(a)}$ for the evaluation of the second-order perturbative terms of equation (85). These quantities can be calculated by usual perturbation theory, once we specify the cosmic field f . In each example of the previous section, the quantity $S_2^{(2)}$ does not appear so that we evaluate $S^{(0)}$, $S^{(1)}$, and $S^{(2)}$ in this section. The other kinds of skewness parameters like $S_2^{(2)}$ can be similarly evaluated without difficulty.

4.1. Hierarchical Model

Before we explore actual cosmic fields, we consider a simple, phenomenological statistical model, i.e., the hierarchical model of higher order correlation functions (e.g., Peebles 1980). In this model, the N -point correlation function is a sum of $N - 1$ products of the two-point correlation function. Specifically, the three-point correlation function is given by

$$\langle f(\mathbf{x}_1)f(\mathbf{x}_2)f(\mathbf{x}_3) \rangle = \mathcal{Q}[\langle f(\mathbf{x}_1)f(\mathbf{x}_2) \rangle \langle f(\mathbf{x}_2)f(\mathbf{x}_3) \rangle + \langle f(\mathbf{x}_2)f(\mathbf{x}_3) \rangle \langle f(\mathbf{x}_3)f(\mathbf{x}_1) \rangle + \langle f(\mathbf{x}_3)f(\mathbf{x}_1) \rangle \langle f(\mathbf{x}_1)f(\mathbf{x}_2) \rangle]. \quad (140)$$

We assume that the field f in the above equation is already smoothed. In this case, skewness parameters (eqs. [61]–[64]) are given by straightforward calculations (Matsubara 1994):

$$S^{(0)} = S^{(1)} = S^{(2)} = 3\mathcal{Q}. \quad (141)$$

These values depend on a hierarchical amplitude, Q , which is a free parameter of this model. The relative amplitudes among $S^{(a)}$ are not freely adjusted in the above equation. In the case of volume fraction threshold, the first non-Gaussian correction of the various statistics considered in the previous section is absent, since they depend only on $S^{(a)} - S^{(0)}$.

4.2. Three-dimensional Density Field

Next, we consider the three-dimensional density field. The skewness parameters of this field in perturbation theory are already calculated by Matsubara (1994) and Matsubara & Suto (1996), using Fourier transforms of the field. We comprehensively review this calculation here for completeness. There is an alternative way to calculate the skewness not depending on Fourier transforms (Buchalter & Kamionkowski 1999).

The cosmic field f is identified with the three-dimensional density contrast, $\rho/\bar{\rho} - 1$, where ρ is the density field. The dimension in which this field is defined is $d = 3$. The Fourier transform of the field is useful in the following:

$$\tilde{f}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) . \quad (142)$$

In this notation, two- and three-point correlations in Fourier space have the forms

$$\langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \rangle = (2\pi)^2 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) P(k_1) , \quad (143)$$

$$\langle \tilde{f}(\mathbf{k}_1) \tilde{f}(\mathbf{k}_2) \tilde{f}(\mathbf{k}_3) \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(k_1, k_2, k_3) . \quad (144)$$

The above forms are the consequence of the statistical homogeneity of the space, where the functions P and B are the power spectrum and the bispectrum, respectively. Thus, the variance and its variants (eqs. [4], [30], and [31]), in their Fourier representation, are given by

$$\sigma_j^2 = \int \frac{k^2 dk}{2\pi^2} k^{2j} P(k) , \quad (145)$$

and the skewness parameters of equations (61)–(64) are given by

$$S^{(0)} = \frac{1}{\sigma_0^4} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} B(k_1, k_2, |\mathbf{k}_1 + \mathbf{k}_2|) , \quad (146)$$

$$S^{(1)} = \frac{3}{4\sigma_0^2\sigma_1^2} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} |\mathbf{k}_1 + \mathbf{k}_2|^2 B(k_1, k_2, |\mathbf{k}_1 + \mathbf{k}_2|) , \quad (147)$$

$$S^{(2)} = \frac{9}{4\sigma_1^4} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} (\mathbf{k}_1 \cdot \mathbf{k}_2) |\mathbf{k}_1 + \mathbf{k}_2|^2 B(k_1, k_2, |\mathbf{k}_1 + \mathbf{k}_2|) . \quad (148)$$

For initial random Gaussian density field, the second-order perturbation theory predicts the power spectrum and the bispectrum as follows (e.g., Peebles 1980; Fry 1984; Bouchet et al. 1992; Bernardeau 1994b):

$$P(k) = P_{\text{LIN}}(k) W^2(kR) + \mathcal{O}(\sigma_0^4) , \quad (149)$$

$$B(k_1, k_2, k_3) = \left[1 + E + \left(\frac{k_2}{k_1} + \frac{k_1}{k_2} \right) \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} + (1 - E) \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 \right] P_{\text{LIN}}(k_1) P_{\text{LIN}}(k_2) W(k_1 R) W(k_2 R) W(k_3 R) + \text{cyc.}(1, 2, 3) + \mathcal{O}(\sigma_0^6) , \quad (150)$$

where $P_{\text{LIN}}(k)$ is the linear power spectrum and E is a weak function of cosmology (Bernardeau 1994b; Bernardeau et al. 1995). The field smoothing corresponds to the multiplication of the window function $W(kR)$, which is the three-dimensional Fourier transform of the smoothing function W_R . It is a good approximation to use the value of E for an Einstein–de Sitter universe, $E = \frac{3}{7}$, in most cases. The explicit form of the function E in terms of cosmological parameters Ω_0 and λ_0 is given by Matsubara (1995b), which is accurately fitted by

$$E \approx \frac{3}{7} \Omega_0^{-1/30} - \frac{\lambda_0}{80} \left(1 - \frac{3}{2} \lambda_0 \log_{10} \Omega_0 \right) . \quad (151)$$

The perturbation theory is considered to be an expansion by a parameter σ_0 . In this respect, the power spectrum P and the bispectrum B are of order σ_0^2 and σ_0^4 , respectively.

Substituting equation (149) into equation (145), we obtain

$$\sigma_j^2(R) = \int \frac{k^2 dk}{2\pi^2} k^{2j} P_{\text{LIN}}(k) W^2(kR) , \quad (152)$$

up to the lowest order. Similarly, substituting equation (150) into equations (146)–(148) and introducing new integration

variables, $l_1 \equiv |\mathbf{k}_1|R$, $l_2 \equiv |\mathbf{k}_2|R$, and $\mu = \mathbf{k}_1 \cdot \mathbf{k}_2 / (k_1 k_2)$, we obtain

$$S^{(a)}(R) = \frac{1}{\sigma_0^4} \left(\frac{\sigma_0}{\sigma_1 R} \right)^{2a} \int \frac{l_1^2 dl_1}{2\pi^2 R^3} \frac{l_2^2 dl_2}{2\pi^2 R^3} P_{\text{LIN}} \left(\frac{l_1}{R} \right) P_{\text{LIN}} \left(\frac{l_2}{R} \right) W^2(l_1) W^2(l_2) \tilde{S}^{(a)}(l_1, l_2), \quad (153)$$

where $a = 0, 1, 2$ and

$$\tilde{S}^{(a)}(l_1, l_2) = \frac{3}{4} \int_{-1}^1 d\mu \left[1 + E + \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) \mu + (1 - E) \mu^2 \right] \frac{W \left(\sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \mu} \right)}{W(l_1) W(l_2)} \times \begin{cases} 2, & a = 0, \\ l_1^2 + l_2^2 + l_1 l_2 \mu, & a = 1, \\ 3l_1^2 l_2^2 (1 - \mu^2), & a = 2. \end{cases} \quad (154)$$

So far the smoothing function is arbitrary. For a general smoothing function, the above equations can be numerically integrated to obtain the skewness parameters for each model of the power spectrum. For some smoothing functions, further analytical reductions of the above equations are possible. As a popular example, we consider the Gaussian smoothing, $W(l) = \exp(-l^2/2)$, which is frequently adopted for practical purposes. We follow a technique similar to that of Łokas et al. (1995), in which they derived the skewness of the density field with Gaussian smoothing.

For the Gaussian smoothing,

$$\frac{W \left(\sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \mu} \right)}{W(l_1) W(l_2)} = e^{-l_1 l_2 \mu}. \quad (155)$$

In this case, the following formula of the modified Bessel function $I_\nu(z)$ is useful:

$$\int_{-1}^1 d\mu P_l(\mu) e^{-\mu z} = (-1)^l \sqrt{\frac{2\pi}{z}} I_{l+1/2}(z), \quad (156)$$

where P_l is the l th Legendre polynomial. From this formula, the angular integration of μ in equation (154) can be analytically performed, and the result is

$$\tilde{S}^{(0)} = \sqrt{\frac{2\pi}{l_1 l_2}} \left[(2 + E) I_{1/2}(l_1 l_2) - \frac{3}{2} \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) I_{3/2}(l_1 l_2) + (1 - E) I_{5/2}(l_1 l_2) \right], \quad (157)$$

$$\begin{aligned} \tilde{S}^{(1)} = \sqrt{2\pi l_1 l_2} \left\{ \frac{5 + 2E}{4} \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) I_{1/2}(l_1 l_2) - \left[\frac{3(9 + E)}{10} + \frac{l_2^2}{l_1^2} + \frac{l_1^2}{l_2^2} \right] I_{3/2}(l_1 l_2) \right. \\ \left. + \frac{2 - E}{2} \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) I_{5/2}(l_1 l_2) - \frac{3(1 - E)}{10} I_{7/2}(l_1 l_2) \right\}, \quad (158) \end{aligned}$$

$$\begin{aligned} \tilde{S}^{(2)} = \sqrt{2\pi} (l_1 l_2)^{3/2} \left[\frac{3(3 + 2E)}{5} I_{1/2}(l_1 l_2) - \frac{9}{10} \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) I_{3/2}(l_1 l_2) - \frac{3(3 + 4E)}{7} I_{5/2}(l_1 l_2) \right. \\ \left. + \frac{9}{10} \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) I_{7/2}(l_1 l_2) - \frac{18(1 - E)}{35} I_{9/2}(l_1 l_2) \right]. \quad (159) \end{aligned}$$

At this point, it is useful to define the following quantity:

$$S_m^{\alpha\beta}(R) \equiv \frac{\sqrt{2\pi}}{\sigma_0^4} \left(\frac{\sigma_0}{\sigma_1 R} \right)^{\alpha+\beta-2} \int \frac{l_1^2 dl_1}{2\pi^2 R^3} \frac{l_2^2 dl_2}{2\pi^2 R^3} P_{\text{LIN}} \left(\frac{l_1}{R} \right) P_{\text{LIN}} \left(\frac{l_2}{R} \right) e^{-l_1^2 - l_2^2} l_1^{\alpha-3/2} l_2^{\beta-3/2} I_{m+1/2}(l_1 l_2). \quad (160)$$

In the above equation, variance parameters σ_0 and σ_1 are given by equation (152). The nonlinear correction for σ_j is not needed because our estimate of $S^{(a)}$ is only lowest order in σ_0 . When the higher order corrections, e.g., third-order perturbation corrections, are estimated, one should be sure that all the necessary nonlinear corrections are properly taken into account. With this quantity, the skewness parameters are given by

$$S^{(0)}(R) = (2 + E) S_0^{11} - 3 S_1^{02} + (1 - E) S_2^{11}, \quad (161)$$

$$S^{(1)}(R) = \frac{3}{2} \left[\frac{5 + 2E}{3} S_0^{13} - \frac{9 + E}{5} S_1^{22} - S_1^{04} + \frac{2(2 - E)}{3} S_2^{13} - \frac{1 - E}{5} S_3^{22} \right], \quad (162)$$

$$S^{(2)}(R) = 9 \left[\frac{3 + 2E}{15} S_0^{33} - \frac{1}{5} S_1^{24} - \frac{3 + 4E}{21} S_2^{33} + \frac{1}{5} S_3^{24} - \frac{2(1 - E)}{35} S_4^{33} \right]. \quad (163)$$

Thus, the lowest order estimates of skewness parameters are given by equations (160)–(163). For each given power spectrum, the integration of equation (160) is straightforward.

The resulting skewness parameters are independent of the amplitude of the power spectrum. When the power spectrum is given by a CDM-like model, $P_{\text{LIN}}(k) \propto k T_{\text{CDM}}^2(k/\Gamma)$, where

$$T_{\text{CDM}}(p) = \frac{\ln(1 + 2.34p)}{2.34p} [1 + 3.89p + (16.1p)^2 + (5.46p)^3 + (6.71p)^4]^{-1/4} \quad (164)$$

is the CDM-like transfer function fitted by Bardeen et al. (1986) and Γ is the shape parameter of this model, then the skewness parameters are functions of ΓR . In Table 2 we give the values of skewness parameters $S^{(a)}$ for CDM-like models for several values of ΓR .

In this table the value of E is approximated by $\frac{3}{7}$. Since the skewness parameters are weak functions of ΓR as seen from the table, one can interpolate the values in this table to obtain the values of arbitrary scales for practical purposes.

When the power spectrum is given by a power-law form,

$$P_{\text{LIN}}(k) = A k^{n_s}, \quad (165)$$

the integration of equation (160) can be analytically performed. First, the simple Gaussian integration gives

$$\sigma_j^2 = \frac{A}{4\pi^2 R^{n_s+2j+3}} \Gamma\left(\frac{n_s + 2j + 3}{2}\right). \quad (166)$$

Second, we expand the modified Bessel function as

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty} \frac{(z/2)^{2r}}{r! \Gamma(\nu + r + 1)}. \quad (167)$$

Then equation (160) reduces to

$$S_m^{\alpha\beta} = \frac{2}{(2m+1)!!} \left(\frac{n_s+3}{2}\right)^{1-(\alpha+\beta)/2} \left(\frac{n_s+3}{2}\right)_{(\alpha+m-1)/2} \left(\frac{n_s+3}{2}\right)_{(\beta+m-1)/2} \\ \times F\left(\frac{n_s+\alpha+m+2}{2}, \frac{n_s+\beta+m+2}{2}, m+\frac{3}{2}; \frac{1}{4}\right), \quad (168)$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1)$ and F is the Gauss hypergeometric function:

$$F(\alpha, \beta, \gamma; z) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{z^r}{r!}. \quad (169)$$

Equations (161)–(163) and equation (168) give the skewness parameters $S^{(a)}$. The recursion relations for the hypergeometric function,

$$\alpha F(\alpha+1, \beta, \gamma+1; z) = \gamma F(\alpha, \beta, \gamma; z) + (\alpha - \gamma) F(\alpha, \beta, \gamma+1; z), \quad (170)$$

$$\alpha\beta F(\alpha+1, \beta+1, \gamma+1; z) = \frac{\gamma(\gamma-1)}{z} [F(\alpha, \beta, \gamma-1; z) - F(\alpha, \beta, \gamma; z)], \quad (171)$$

$$(\gamma - \alpha)(\gamma - \beta)zF(\alpha, \beta, \gamma+1; z) + \gamma[(\alpha + \beta - 2\gamma + 1)z + \gamma - 1]F(\alpha, \beta, \gamma; z) + \gamma(\gamma-1)(z-1)F(\alpha, \beta, \gamma-1; z) = 0, \quad (172)$$

simplify the result

$$S^{(0)}(n_s) = S^{(1)}(n_s) = 3F\left(\frac{n_s+3}{2}, \frac{n_s+3}{2}, \frac{3}{2}; \frac{1}{4}\right) - (n_s+2-2E)F\left(\frac{n_s+3}{2}, \frac{n_s+3}{2}, \frac{5}{2}; \frac{1}{4}\right), \quad (173)$$

TABLE 2
THREE-DIMENSIONAL SKEWNESS PARAMETERS OF GAUSSIAN-SMOOTHED DENSITY FIELD
FOR CDM-LIKE MODELS

PARAMETER	ΓR					
	1.0	2.0	4.0	8.0	16.0	32.0
$S^{(0)}$	3.678	3.500	3.332	3.201	3.115	3.065
$S^{(1)}$	3.757	3.566	3.377	3.228	3.129	3.072
$S^{(2)}$	3.657	3.662	3.701	3.783	3.903	4.046

NOTE.—The models are functions of the product of shape parameter Γ and smoothing length R . The parameter E is set as $E = \frac{3}{7}$.

TABLE 3
THREE-DIMENSIONAL SKEWNESS PARAMETERS OF GAUSSIAN-SMOOTHED DENSITY FIELD FOR POWER-LAW MODELS

PARAMETER	n								
	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0
$S^{(0,1)}$	4.857	4.400	4.022	3.714	3.468	3.280	3.144	3.061	3.029
$S^{(2)}$	3.815	3.720	3.665	3.652	3.680	3.754	3.877	4.054	4.294

NOTE.—The models are functions of the spectral index n .

$$S^{(2)}(n_s) = 3F\left(\frac{n_s+5}{2}, \frac{n_s+5}{2}, \frac{5}{2}; \frac{1}{4}\right) - \frac{3}{5}(n_s+4-4E)F\left(\frac{n_s+5}{2}, \frac{n_s+5}{2}, \frac{7}{2}; \frac{1}{4}\right). \quad (174)$$

In this power-law case, skewness parameters do not depend on scales R but only on power-law index n_s . Incidentally, $S^{(0)}(n_s)$ and $S^{(1)}(n_s)$ are identical. This is just a coincidence and is not generally the case when the power spectrum is not given by a power law. Several numerical values are shown in Table 3, where $E = \frac{3}{7}$ is assumed.

4.3. Three-dimensional Velocity Field

Next, we consider the three-dimensional velocity field as the cosmic field f . Since the rotational components of the velocity field are decaying modes of gravitational evolution in perturbation theory, we only consider the rotation-free component. The cosmic field f is identified with the dimensionless scalar field,

$$f(\mathbf{x}) = \frac{1}{H} \nabla \cdot \mathbf{v}(\mathbf{x}), \quad (175)$$

where H is the Hubble parameter. One can also consider other quantities like the radial component of the velocity field, $V = \mathbf{n} \cdot \mathbf{v}$, where \mathbf{n} is the line-of-sight normal vector. Those quantities are more complicated than a simple divergence. We illustrate only the simplest case in this paper. The second-order perturbation theory predicts the power spectrum and bispectrum of the velocity field as follows (e.g., Bernardeau 1994b):

$$P(k) = g_f^2 P_{\text{LIN}}(k) W^2(kR) + \mathcal{O}(\sigma_0^4), \quad (176)$$

$$B(k_1, k_2, k_3) = -g_f^3 \left[1 + E_v + \left(\frac{k_2}{k_1} + \frac{k_1}{k_2} \right) \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} + (1 - E_v) \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 \right] P_{\text{LIN}}(k_1) P_{\text{LIN}}(k_2) W(k_1 R) W(k_2 R) W(k_3 R) + \text{cyc.}(1, 2, 3) + \mathcal{O}(\sigma_0^6). \quad (177)$$

In the above equation, the factor g_f is a logarithmic derivative of the growth factor, given by

$$g_f(\Omega_0, \lambda_0) = \frac{d \ln D}{d \ln a} \Big|_0 \approx \Omega_0^{4/7} + \frac{\lambda_0}{70} \left(1 + \frac{\Omega_0}{2} \right) \quad (178)$$

(Lightman & Schechter 1990; Lahav et al. 1991), D is the linear growth factor, and a is the expansion factor. The logarithmic derivative is evaluated at present. The factor E_v is a weak function of cosmology. It is a good approximation to use the value for an Einstein–de Sitter universe, $E_v = -\frac{1}{7}$. The explicit form of the function E_v in terms of Ω and λ is given by Matsubara (1995b), which is accurately fitted by

$$\frac{E_v + 1}{2} \approx \frac{3}{7} \Omega_0^{-11/200} - \frac{\lambda_0}{70} \left(1 - \frac{7}{3} \lambda_0 \log_{10} \Omega_0 \right). \quad (179)$$

The similarity of equations (176) and (177) for the velocity field to equations (149) and (150) for the density field is obvious. We can easily see that the skewness parameters for the velocity field are obtained by formulae similar to equations (161)–(163):

$$S^{(0)}(R) = -\frac{1}{g_f} [(2 + E_v) S_0^{11} - 3 S_1^{02} + (1 - E_v) S_2^{11}], \quad (180)$$

$$S^{(1)}(R) = -\frac{3}{2g_f} \left[\frac{5 + 2E_v}{3} S_0^{13} - \frac{9 + E_v}{5} S_1^{22} - S_1^{04} + \frac{2(2 - E_v)}{3} S_2^{13} - \frac{1 - E_v}{5} S_3^{22} \right], \quad (181)$$

$$S^{(2)}(R) = -\frac{9}{g_f} \left[\frac{3 + 2E_v}{15} S_0^{33} - \frac{1}{5} S_1^{24} - \frac{3 + 4E_v}{21} S_2^{33} + \frac{1}{5} S_3^{24} - \frac{2(1 - E_v)}{35} S_4^{33} \right], \quad (182)$$

where $S_m^{\alpha\beta}$ is given by equation (160) without any modification. In Table 4 we show the values of skewness parameters of the velocity field for the CDM-like models for several values of ΓR . In this table the value of E_v is set as $-\frac{1}{7}$.

TABLE 4
THREE-DIMENSIONAL SKEWNESS PARAMETERS OF GAUSSIAN-SMOOTHED VELOCITY
FIELD FOR CDM-LIKE MODELS

PARAMETER	ΓR					
	1.0	2.0	4.0	8.0	16.0	32.0
$-g_f S^{(0)}$	2.456	2.232	1.995	1.773	1.581	1.420
$-g_f S^{(1)}$	2.551	2.319	2.065	1.826	1.620	1.448
$-g_f S^{(2)}$	1.963	1.877	1.779	1.683	1.601	1.530

NOTE.—The factor $-g_f$ is multiplied and the values are almost independent of cosmological parameters.

For the power spectrum of the power-law form,

$$S^{(0)}(n_s) = S^{(1)}(n_s) = -\frac{1}{g_f} \left[3F\left(\frac{n_s+3}{2}, \frac{n_s+3}{2}, \frac{3}{2}; \frac{1}{4}\right) - (n_s+2-2E_v)F\left(\frac{n_s+3}{2}, \frac{n_s+3}{2}, \frac{5}{2}; \frac{1}{4}\right) \right], \quad (183)$$

$$S^{(2)}(n_s) = -\frac{3}{g_f} \left[F\left(\frac{n_s+5}{2}, \frac{n_s+5}{2}, \frac{5}{2}; \frac{1}{4}\right) - \frac{1}{5}(n_s+4-4E_v)F\left(\frac{n_s+5}{2}, \frac{n_s+5}{2}, \frac{7}{2}; \frac{1}{4}\right) \right]. \quad (184)$$

Several numerical values are shown in Table 5, where $E_v = -\frac{1}{7}$ is assumed.

4.4. Two-dimensional Projected Density Field

The projection of the density field ρ_p defines the two-dimensional cosmic fields on the sky ($d = 2$). Here we derive the skewness parameters for this field. The two-dimensional projected density field with a top-hat kernel is investigated by Bernardeau (1995). Here we are interested in the Gaussian kernel for our purpose.

In a Friedman-Lemaître universe, the comoving angular diameter distance at a comoving distance χ is given by

$$S_K(\chi) = \begin{cases} \frac{\sinh(\chi\sqrt{-K})}{\sqrt{-K}}, & K < 0, \\ \chi, & K = 0, \\ \frac{\sin(\chi\sqrt{K})}{\sqrt{K}}, & K > 0, \end{cases} \quad (185)$$

depending on the sign of the spatial curvature $K = H_0^2(\Omega_0 + \lambda_0 - 1)$. Thus, in spherical coordinates, the projected density field in two dimensions is given by

$$\rho_p(\theta, \phi) = \int d\chi S_K^2(\chi) n(\chi) \rho(\chi, \theta, \phi; \tau_0 - \chi), \quad (186)$$

where $n(\chi)$ is the selection function without volume factor, normalized as $\int d\chi S_K^2(\chi) n(\chi) = 1$, and $\rho(\chi, \theta, \phi; \tau)$ is the three-dimensional comoving density field.² The present value of the conformal time $\tau = \int dt/a$ is τ_0 .

The projected density contrast is defined by $\rho_p/\bar{\rho}_p - 1$, where one can see $\bar{\rho}_p = \bar{\rho}$. We identify the projected density contrast with the two-dimensional ($d = 2$) field f . Since the smoothing angle θ_f is much smaller than π in most of the interesting cases,

² The comoving density field is defined by the density per unit comoving volume and thus satisfies $\bar{\rho} = \text{const.}$

TABLE 5
THREE-DIMENSIONAL SKEWNESS PARAMETERS OF GAUSSIAN-SMOOTHED VELOCITY FIELD FOR
POWER-LAW MODELS

PARAMETER	n								
	-3.0	-2.5	-2.0	-1.5	-1.0	-0.5	0.0	0.5	1.0
$-g_f S^{(0,1)}$	3.714	3.250	2.848	2.498	2.191	1.918	1.670	1.441	1.222
$-g_f S^{(2)}$	2.333	2.170	2.026	1.900	1.788	1.687	1.595	1.509	1.425

NOTE.—The models are functions of the spectral index n .

we consider the small patch of the sky in the vicinity of the polar axis, $\theta \ll 1$. With this approximation, we introduce the variables $\theta_1 = \theta \cos \phi$ and $\theta_2 = \theta \sin \phi$, which are considered as two-dimensional Euclidean coordinates, $\boldsymbol{\theta}$. Therefore, the projection equation is given by

$$f(\boldsymbol{\theta}) = \int d\chi S_K^2(\chi) n(\chi) \delta_{3D}[\chi, S_K(\chi) \boldsymbol{\theta}; \tau_0 - \chi], \quad (187)$$

where $\delta_{3D}(\mathbf{x}, \tau) = \rho/\bar{\rho} - 1$ is the density contrast at comoving coordinates \mathbf{x} and conformal time τ .

The power spectrum and the bispectrum for the above projected field are given by Limber's equations (B2) and (B10) in Appendix B, with $q(\chi) = n(\chi)$:

$$P_{2D}(\omega) = \int d\chi S_K^2(\chi) n^2(\chi) P_{3D}\left[\frac{\omega}{S_K(\chi)}; \tau_0 - \chi\right], \quad (188)$$

$$B_{2D}(\omega_1, \omega_2, \omega_3) = \int d\chi S_K^2(\chi) n^3(\chi) B_{3D}\left[\frac{\omega_1}{S_K(\chi)}, \frac{\omega_2}{S_K(\chi)}, \frac{\omega_3}{S_K(\chi)}; \tau_0 - \chi\right], \quad (189)$$

where P and B are the two-dimensional projected power spectrum and bispectrum, respectively, of the field f and P_{3D} and B_{3D} are the three-dimensional power spectrum and bispectrum, respectively. The three-dimensional power spectrum and the three-dimensional bispectrum are evaluated by the second-order perturbation theory. They are similar to equations (149) and (150), but we have to take into account the time dependence here. They are given by

$$P_{3D}(k; \tau_0 - \chi) = D^2(\chi) P_{\text{LIN}}(k), \quad (190)$$

$$B_{3D}(k_1, k_2, k_3; \tau_0 - \chi)$$

$$= D^4(\chi) \left\{ 1 + E(\chi) + \left(\frac{k_2}{k_1} + \frac{k_1}{k_2} \right) \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} + [1 - E(\chi)] \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 \right\} P_{\text{LIN}}(k_1) P_{\text{LIN}}(k_2) + \text{cyc.}(1, 2, 3), \quad (191)$$

where $D(\chi)$ is the linear growth factor at conformal look-back time χ (i.e., at conformal time $\tau = \tau_0 - \chi$), which is normalized as $D(0) = 1$. The following fitting formula (Carroll, Press, & Turner 1992) is useful:

$$D \approx \frac{a\Omega}{\Omega_0} \frac{\Omega_0^{4/7} - \lambda_0 + (1 + \Omega_0/2)(1 + \lambda_0/70)}{\Omega_0^{4/7} - \lambda + (1 + \Omega/2)(1 + \lambda/70)}, \quad (192)$$

where Ω and λ are time-dependent cosmological parameters at conformal look-back time χ . The variable $E(\chi)$ is a weak function of time and cosmology, and for the Einstein–de Sitter universe, $E = \frac{3}{7}$. This quantity E is the same we used in the three-dimensional density field, but here we also take into account the time dependence. It is accurately approximated by

$$E \approx \frac{3}{7} \Omega^{-1/30} - \frac{\lambda}{80} \left(1 - \frac{3}{2} \lambda \log_{10} \Omega \right). \quad (193)$$

The variance parameters of the smoothed projected field are given by

$$\sigma_f^2(\theta_f) = \int \frac{\omega d\omega}{2\pi} \omega^{2j} P(\omega) W^2(\omega \theta_f) = \frac{1}{\theta_f^{2j+2}} \int d\chi S_K^2(\chi) n^2(\chi) D^2(\chi) \Sigma_j^2[S_K(\chi) \theta_f], \quad (194)$$

where

$$\Sigma_j^2(R) = R^{2j+2} \int \frac{k dk}{2\pi} k^{2j} P_{\text{LIN}}(k) W^2(kR) = \int \frac{l dl}{2\pi} l^{2j} P_{\text{LIN}}\left(\frac{l}{R}\right) W^2(l). \quad (195)$$

The skewness parameters of the smoothed projected field are given by

$$S^{(a)}(\theta_f) = \frac{1}{\sigma_0^4 \theta_f^4} \left(\frac{\sigma_0}{\sigma_1 \theta_f} \right)^{2a} \int d\chi S_K^2(\chi) n^3(\chi) D^4(\chi) \Sigma_0^{4-2a}[S_K(\chi) \theta_f] \Sigma_1^{2a}[S_K(\chi) \theta_f] C^{(a)}[S_K(\chi) \theta_f], \quad (196)$$

where

$$C^{(a)}(R) = \frac{1}{\Sigma_0^4} \left(\frac{\Sigma_0}{\Sigma_1} \right)^{2a} \int \frac{l_1 dl_1}{2\pi} \frac{l_2 dl_2}{2\pi} P_{\text{LIN}}\left(\frac{l_1}{R}\right) P_{\text{LIN}}\left(\frac{l_2}{R}\right) W^2(l_1) W^2(l_2) \tilde{C}^{(a)}(l_1, l_2) \quad (197)$$

and

$$\begin{aligned} \tilde{C}^{(a)}(l_1, l_2) = & \frac{3}{2\pi} \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} \\ & \times \left[1 + E + \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) \mu + (1-E)\mu^2 \right] \frac{W\left(\sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \mu}\right)}{W(l_1)W(l_2)} \times \begin{cases} 2, & a=0, \\ l_1^2 + l_2^2 + l_1 l_2 \mu, & a=1, \\ 4l_1^2 l_2^2 (1-\mu^2), & a=2. \end{cases} \end{aligned} \quad (198)$$

So far the smoothing function is arbitrary. For a general smoothing function, the above equations can be numerically integrated to obtain the skewness parameters for each model of the power spectrum. In the following, we adopt the Gaussian smoothing function, $W(l) = \exp(-l^2/2)$. For this smoothing function, equation (155) holds even for this two-dimensional case. In this case, the following integral representation of the modified Bessel function $I_\nu(z)$ for $\nu = 0$ is useful:

$$I_0(z) = \frac{1}{\pi} \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} e^{-z\mu}. \quad (199)$$

Actually, the derivatives of the above equation,

$$\frac{1}{\pi} \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} \mu^m e^{-z\mu} = \left(-\frac{d}{dz} \right)^m I_0(z), \quad (200)$$

are sufficient to perform the angular integration in equation (198). Moreover, one can use the property of the Bessel function, $I'_0 = I_1$, and $I'_m = (I_{m-1} + I_{m+1})/2$ to obtain

$$\frac{1}{\pi} \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} \mu e^{-z\mu} = -I_1(z), \quad (201)$$

$$\frac{1}{\pi} \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} \mu^2 e^{-z\mu} = \frac{1}{2} I_0(z) + \frac{1}{2} I_2(z), \quad (202)$$

$$\frac{1}{\pi} \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} \mu^3 e^{-z\mu} = -\frac{3}{4} I_1(z) - \frac{1}{4} I_3(z), \quad (203)$$

$$\frac{1}{\pi} \int_{-1}^1 \frac{d\mu}{\sqrt{1-\mu^2}} \mu^4 e^{-z\mu} = \frac{3}{8} I_0(z) + \frac{1}{2} I_2(z) + \frac{1}{8} I_4(z). \quad (204)$$

From these formulae, equation (198) reduces to

$$\tilde{C}^{(0)} = \frac{3}{2} \left[(3+E)I_0(l_1 l_2) - 2 \left(\frac{l_2}{l_1} + \frac{l_1}{l_2} \right) I_1(l_1 l_2) + (1-E)I_2(l_1 l_2) \right], \quad (205)$$

$$\begin{aligned} \tilde{C}^{(1)} = & \frac{3}{4} \left\{ (4+E)(l_1^2 + l_2^2)I_0(l_1 l_2) - \left[\frac{15+E}{2} l_1 l_2 + 2 \left(\frac{l_1^3}{l_2} + \frac{l_2^3}{l_1} \right) \right] I_1(l_1 l_2) \right. \\ & \left. + (2-E)(l_1^2 + l_2^2)I_2(l_1 l_2) - \frac{1-E}{2} l_1 l_2 I_3(l_1 l_2) \right\}, \end{aligned} \quad (206)$$

$$\begin{aligned} \tilde{C}^{(2)} = & \frac{3}{4} \left[(5+3E)l_1^2 l_2^2 I_0(l_1 l_2) - 2l_1 l_2 (l_1^2 + l_2^2) I_1(l_1 l_2) - 4(1+E)l_1^2 l_2^2 I_2(l_1 l_2) \right. \\ & \left. + 2l_1 l_2 (l_1^2 + l_2^2) I_3(l_1 l_2) - (1-E)l_1^2 l_2^2 I_4(l_1 l_2) \right]. \end{aligned} \quad (207)$$

At this point, defining

$$C_m^{\alpha\beta}(R) = \frac{1}{\Sigma_0^4} \left(\frac{\Sigma_0}{\Sigma_1} \right)^{\alpha+\beta-2} \int \frac{l_1 dl_1}{2\pi} \frac{l_2 dl_2}{2\pi} P_{\text{LIN}} \left(\frac{l_1}{R} \right) P_{\text{LIN}} \left(\frac{l_2}{R} \right) e^{-l_1^2 - l_2^2} l_1^{\alpha-1} l_2^{\beta-1} I_m(l_1 l_2), \quad (208)$$

equation (197) reduces to

$$C^{(0)} = \frac{3}{2} \left[(3+E)C_0^{11} - 4C_1^{02} + (1-E)C_2^{11} \right], \quad (209)$$

$$C^{(1)} = \frac{3}{4} \left[2(4+E)C_0^{13} - \frac{15+E}{2} C_1^{22} - 4C_1^{04} + 2(2-E)C_2^{13} - \frac{1-E}{2} C_3^{22} \right], \quad (210)$$

$$C^{(2)} = \frac{3}{4} \left[(5+3E)C_0^{33} - 4C_1^{24} - 4(1+E)C_2^{33} + 4C_3^{24} - (1-E)C_4^{33} \right]. \quad (211)$$

The two-dimensional integration of equation (208) is performed only once as a function of R . The result is stored as a table

TABLE 6
FUNCTIONS $C^{(a)}(R)$ FOR CDM-LIKE MODELS

PARAMETER	ΓR								
	0.25	0.5	1.0	2.0	4.0	8.0	16.0	32.0	64.0
$C^{(0)}$	4.544	4.311	4.020	3.685	3.345	3.044	2.797	2.613	2.504
$C^{(1)}$	4.885	4.614	4.279	3.890	3.492	3.143	2.863	2.653	2.523
$C^{(2)}$	3.723	3.660	3.589	3.518	3.467	3.453	3.479	3.532	3.584

NOTE.—The models are functions of the product of shape parameter Γ and smoothing length R . The parameter E is set as $E = \frac{3}{7}$.

and is used in the one-dimensional numerical integration of equation (196) for finally obtaining the skewness parameters in two-dimensional projected density fields. Functions $C^{(a)}$ for CDM-like models are given in Table 6 that are functions of ΓR .

When the three-dimensional power spectrum is given by a power-law form of equation (165), the integration by wavelength l_1, l_2 can be analytically performed as in the three-dimensional case. In fact, the parameter Σ_j of equation (195) with Gaussian smoothing $W(l) = e^{-l^2/2}$ is given by

$$\Sigma_j^2(R) = \frac{A}{4\pi R^{n_s}} \Gamma\left(\frac{n_s + 2j + 2}{2}\right), \quad (212)$$

and the variance parameters are given by

$$\sigma_j^2(\theta_f) = \frac{A}{4\pi\theta_f^{n_s+2j+2}} \Gamma\left(\frac{n_s + 2j + 2}{2}\right) \int d\chi S_K^{2-n_s}(\chi) n^2(\chi) D^2(\chi). \quad (213)$$

With a similar technique used in the three-dimensional density field,

$$C_m^{\alpha\beta} = \frac{1}{2^m m!} \left(\frac{n+2}{2}\right)^{1-(\alpha+\beta)/2} \left(\frac{n+2}{2}\right)_{(\alpha+m-1)/2} \left(\frac{n+2}{2}\right)_{(\beta+m-1)/2} \times F\left(\frac{n+\alpha+m+1}{2}, \frac{n+\beta+m+1}{2}, m+1; \frac{1}{4}\right). \quad (214)$$

Equations (209)–(211) and the above equation finally give the values of $C^{(a)}$. The recursion relations of equations (170) and (171) simplify the result:

$$C^{(0)}(n_s) = C^{(1)}(n_s) = 3F\left(\frac{n_s+2}{2}, \frac{n_s+2}{2}, 1; \frac{1}{4}\right) - \frac{3}{2}(n_s+1-E)F\left(\frac{n_s+2}{2}, \frac{n_s+2}{2}, 2; \frac{1}{4}\right), \quad (215)$$

$$C^{(2)}(n_s) = 3F\left(\frac{n_s+4}{2}, \frac{n_s+4}{2}, 2; \frac{1}{4}\right) - \frac{3(n_s+3-3E)}{4}F\left(\frac{n_s+4}{2}, \frac{n_s+4}{2}, 3; \frac{1}{4}\right). \quad (216)$$

These functions are independent of scale R in the power-law case but dependent on power-law index n_s . Again, $C^{(0)}(n_s)$ and $C^{(1)}(n_s)$ are identical only in the power-law case. Numerical values are given in Table 7, where $E = \frac{3}{7}$ is assumed.

For the power-law case, the skewness parameters of equation (196) reduce to the following simple form:

$$S^{(a)}(n_s) = \frac{\int d\chi [S_K(\chi)]^{2-2n_s} n^3(\chi) D^4(\chi)}{\left\{ \int d\chi [S_K(\chi)]^{2-n_s} n^2(\chi) D^2(\chi) \right\}^2} C^{(a)}(n_s). \quad (217)$$

It is interesting to compare the results with those of top-hat smoothing. According to Bernardeau (1995), the top-hat smoothing gives $C^{(0)}(n_s) = 36/7 + 3(n_s+2)/2$ for the power-law case. Comparing this expression with our Table 7, they

TABLE 7
FUNCTIONS $C^{(a)}$ FOR POWER-LAW MODELS

PARAMETER	n_s								
	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0
$C^{(0,1)}$	6.949	5.965	5.143	4.457	3.885	3.409	3.014	2.688	2.421
$C^{(2)}$	4.090	3.863	3.687	3.560	3.478	3.443	3.445	3.518	3.635

NOTE.—The models are functions of the spectral index n . The parameter E is set as $E = \frac{3}{7}$.

roughly agree with each other and have similar behavior with spectral index. In detail, the Gaussian smoothing gives more or less larger values. For example, top-hat smoothing gives $C^{(0)} = 5.14, 3.64, 2.14$ for $n_s = -2, -1, 0$, respectively, while Gaussian smoothing gives $C^{(0)} = 5.14, 3.89, 3.01$, respectively. It is reasonable that the $C^{(0)}$ of top-hat smoothing differs from that of Gaussian smoothing for large spectral index because the top-hat smoothing gathers more power on small scales than Gaussian smoothing.

4.5. Weak Lensing Field

The local convergence field of the weak lensing is commonly used for studying the large-scale structure of the universe (e.g., Kaiser 1998; Bartelmann & Schneider 1999). Assuming the situation in which Limber's equation (see Appendix B) and also the Born approximation (Kaiser 1998) hold, the following correspondence between the convergence field κ and the three-dimensional density contrast δ_{3D} is useful (e.g., Mellier 1999):

$$\kappa(\theta) = \frac{3}{2} H_0^2 \Omega_0 \int_0^\infty d\chi' S_K^2(\chi') n(\chi') \int_0^{\chi'} d\chi \frac{S_K(\chi) S_K(\chi' - \chi)}{a(\chi) S_K(\chi')} \delta_{3D}[\chi', \theta S_K(\chi'); \tau_0 - \chi'] , \quad (218)$$

where $a(\chi)$ is the scale factor at conformal look-back time $\chi = \tau_0 - \tau$. The above equation is reduced to exactly the same form as the projected field of equation (187), but with the substitution

$$n(\chi) \rightarrow n_{wl}(\chi) = \frac{3}{2} \frac{H_0^2 \Omega_0}{a(\chi)} \int_\chi^\infty d\chi' S_K^2(\chi') \frac{S_K(\chi' - \chi)}{S_K(\chi') S_K(\chi)} n(\chi') . \quad (219)$$

However, one should note that this substitution is valid only under the assumption of the Born approximation. Although the effect of the Born approximation on the skewness is known to be weak (Bernardeau, van Waerbeke, & Mellier 1997), the validity of the Born approximation in general has not been tested in detail (Mellier 1999). There is some subtlety that could arise when various combinations of skewness are considered.

4.6. The Biases

The above expressions of the skewness parameters are for unbiased fields. The skewness parameters for biased fields are nontrivial. They depend on the details of the biasing scheme in the real universe, which is poorly known so far. However, the skewness parameters are well-defined quantities, so that they are calculated from the first principle once the biasing scheme is given.

Perhaps one of the simplest yet nontrivial cases is the local, deterministic biasing. In this case, we can follow Fry & Gaztanaga (1993) to obtain the perturbative expansion of the biasing:

$$\delta_g = b\delta + \frac{b_2}{2} (\delta^2 - \langle \delta^2 \rangle) + \dots , \quad (220)$$

where δ_g and δ are the galaxy and mass density contrast, respectively. The spatial dimension is arbitrary, so that δ_g and δ can be functions in either one-, two-, or three-dimensional space. The biased variance in lowest order is given by

$$\sigma_{0,g} = b\sigma_0 . \quad (221)$$

It is also straightforward to calculate skewness parameters. After some algebra, all skewness parameters are shown to transform in the same way:

$$S_g^{(a)} = \frac{S^{(a)}}{b} + \frac{3b_2}{b^2} , \quad (222)$$

irrespective of spatial dimensions, d , and of kinds of skewness parameters, $a = 0, 1, 2$. In this framework, the parameters b and b_2 are needed. A possible way to determine these parameters is to measure the variance σ_g^2 and the skewness S_g from the observation. Theoretical models predict the variance σ_m^2 and the skewness S_m of the mass distribution. We obtain biasing parameters by $b = \sigma_g/\sigma_m$ and $b_2 = b(bS_g - S_m)/3$. The derivatives of the skewness, $S_g^{(1)}$ and $S_g^{(2)}$, are then obtained from equation (222).

In the case of level crossing and genus statistics as functions of scaled threshold \tilde{v} , the second-order corrections in equations (136)–(138) only depend on the difference of the skewness parameters times the variance. Quite remarkably, this type of second-order correction term does not depend on the bias parameter at all:

$$(S_g^{(a)} - S_g^{(0)})\sigma_{0,g} = (S^{(a)} - S^{(0)})\sigma_0 . \quad (223)$$

Thus, the second-order nonlinear corrections for level crossing and genus statistics of the locally biased field are exactly the same as that of the unbiased mass density field. This can be considered as an advantage of the scaled threshold \tilde{v} in these statistics because the biasing is one of the most embarrassing uncertainties in the analysis of galaxy distribution. However, one should note that this result is derived under the local biasing scheme. So far the locality of the bias is not guaranteed in general.

Stochastic argument (Dekel & Lahav 1999) of the biasing is more complicated for derivative skewness $S^{(1)}$ and $S^{(2)}$ than for usual skewness $S^{(0)}$ because they involve the correlation between field derivatives. The phenomenological nature of the stochastic biasing requires many parameters, which is not calculable from first principles. Therefore, the stochastic biasing scheme does not effectively work for our problem. More physical treatment of the biasing schemes of galaxy formation is needed, in which case the nonlocality of the bias could also be important (Matsubara 1999).

5. IMPLICATIONS OF SECOND-ORDER RESULTS

The perturbative calculations offer valuable aspects of weakly nonlinear evolution of the various statistics without a laborious parameter survey by numerical simulations. In this paper we obtain the lowest nonlinear corrections to relatively popular statistics of smoothed cosmic fields. Using these results, it is interesting to see how the weakly nonlinear effect tends to distort the Gaussian prediction for those exemplified statistics.

In Figure 1, various statistics for the three-dimensional density field are shown. The amplitude of each statistic is appropriately normalized as we are interested in the deviation from the Gaussian prediction. If we neglect the normalization, Minkowski functionals of $k = 1, 2, 3$ are equivalent to the statistics N_1 , G_2 , and G_3 , respectively, so that they degenerate in this figure (we should note that the sign of $V_3^{(3)}$ is inverted).

The rms σ_0 , which is considered as a weakly nonlinear parameter, is set as $\sigma_0 = 0.3$. A limit $\sigma_0 \rightarrow 0$ corresponds to the prediction of the linear theory, which is given by thin solid lines in the figure. This linear prediction is equivalent to the Gaussian fluctuations because we assume that the initial density field is random Gaussian.

In general, the curves of statistics plotted against the direct density threshold, ν (*dotted lines*), exhibit considerable deviations from Gaussian predictions. The overall tendency does not depend much on the shape of the spectrum we consider here, i.e., a power-law spectrum with index -2 , -1 , 0 , and a CDM model with smoothing length $R = 4/\Gamma$, where Γ is the shape

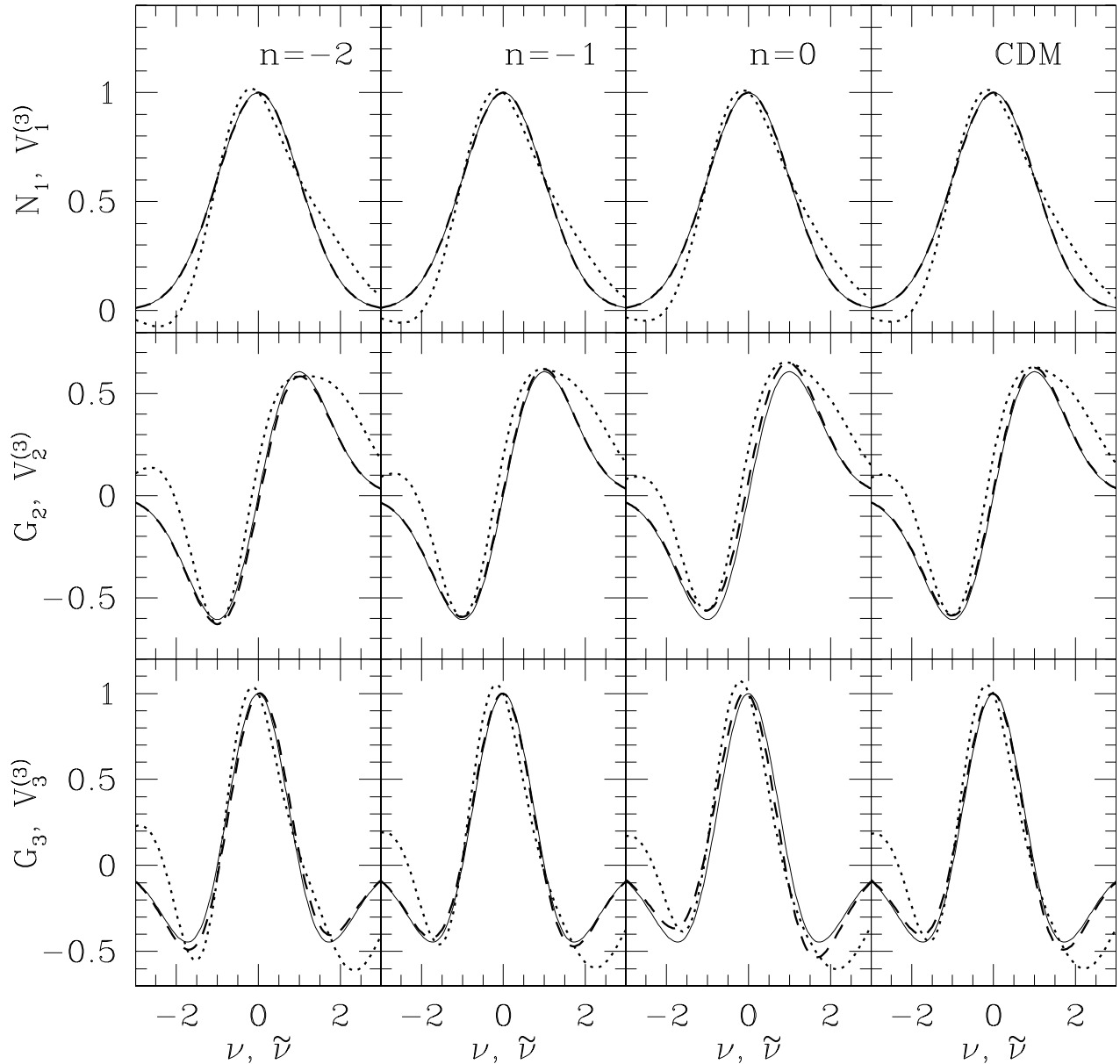


FIG. 1.—Three-dimensional genus G_3 , two-dimensional genus G_2 , level-crossing statistic N_1 , and Minkowski functionals $V_k^{(3)}$ of the three-dimensional density field. All curves are appropriately normalized. The variance is set as $\sigma_0 = 0.3$. *Solid lines*: Gaussian predictions; *dotted lines*: second-order predictions in terms of density threshold ν ; *dashed lines*: second-order predictions in terms of volume fraction threshold $\tilde{\nu}$. The initial density fluctuation spectrum is given by the power law with $n = -2, -1, 0$ and also by the CDM-like model with smoothing length $R = 4/\Gamma$ (see text).

parameter of the CDM spectrum. They are consistent with the so-called meatball shift, which means that there are more isolated regions in a nonlinear field than in a Gaussian field for a fixed threshold. In fact, N_1 , G_2 , and $-G_3$ of a high value of threshold, e.g., $\nu \sim 2$, virtually correspond to the number of isolated regions, and each figure shows that the number is indeed increased by weakly nonlinear evolution.

The weakly nonlinear formula for the genus curve against the density threshold, $G_3(\nu)$, which was first derived by Matsubara (1994), has been compared with numerical simulations in the literature. Matsubara & Suto (1996) show the good agreement of the analytic prediction with the simulations for various spectra. Colley et al. (2000) compared the prediction of the genus curve against direct ν with the simulated SDSS data. Unfortunately, in their published paper they have transcribing errors, which incorrectly made the perturbation theory considerably disagree with their data. There are also ambiguities in their comparison on biasing that alter the values of skewness parameters. One can guess the biasing effect on skewness parameters by equation (220). They chose the peak particles as galaxies, and the linear biasing parameter is inferred as $b \sim 1.3$, but the nonlinear parameter b_2 is not obvious in their work. Some literatures indicate that the skewness $S^{(0)}$ of peaks is roughly given by 1–2 (Watanabe et al. 1994; Plionis & Valdarnini 1995), but for highly biased peaks $b \sim 2$. If we adopt $b_2 = -0.5$, the skewness is given by $S^{(0)} = 1.8$, which is not unreasonable. If it is the case for their simulation, the perturbative prediction and their data completely agree with each other. The χ^2 value per degrees of freedom reduces to only 1.03 (W. N. Colley & D. H. Weinberg 2000, private communication). Obviously, we have to further investigate the biasing effects in numerical simulations to obtain a conclusive result.

Most of the topological analyses of the previous work use the scaled threshold $\tilde{\nu}$. The dashed lines in Figure 1 show the corresponding curves. The deviations from the linear theory are dramatically reduced. This fact is empirically known by the analyses of numerical simulations (Gott et al. 1986, 1987). The reason for this reduction is mathematically due to the closeness of the values of skewness parameters $S^{(a)}$ ($a = 0, 1, 2$), since all the terms of the nonlinear corrections in equations (136)–(138) depend only on $S^{(a)} - S^{(0)}$ ($a = 1, 2$). For the hierarchical model of equation (140), they are exactly zero, which means that there is not any (second-order) nonlinear correction for the hierarchical model. Since the hierarchical model is known to roughly approximate the nonlinear evolution, it is not surprising that more realistic fields have only small corrections of nonlinearity if they are plotted against the volume fraction threshold $\tilde{\nu}$.

For power-law models, $S^{(0)}$ and $S^{(1)}$ are exactly the same. That makes the nonlinear correction for N_1 or $V_1^{(3)}$ exactly vanish. Thus, the nonlinear corrections for other statistics arise from the difference between $S^{(0)}$ and $S^{(2)}$. For the CDM model, there still is a difference between $S^{(0)}$ and $S^{(1)}$, but it is relatively small as seen from Table 2.

As for the topological statistics, G_2 , G_3 , $V_2^{(3)}$, and $V_3^{(3)}$, deviations of curves against $\tilde{\nu}$ from the linear theory prediction depend on the underlying spectrum through differences of skewness parameters. The weakly nonlinear effect for the redder spectrum of $n_s = -2$ induces a spongelike shift, which means that the number of holes in isolated regions increases. On the other hand, the bluer spectrum of $n_s = 0$ indicates a meatball shift. These tendencies are qualitatively in agreement with the numerical results (e.g., Ryden et al. 1989; Melott et al. 1989; Park & Gott 1991).

It is not trivial to estimate errors expected in observationally estimating statistics of smoothed cosmic fields. The observational errors mainly consist of the shot noise and the cosmic variance. The systematic comparison with simulations should be used for the error estimates. Unfortunately, previous earlier simulations do not have enough resolution to be quantitatively compared in the weakly nonlinear regime. Canavezes et al. (1998) use much larger simulations than those in earlier works and give the genus curve in the weakly nonlinear regime in their Figure 9. Relative depths of left and right troughs in the genus curve show a slight meatball shift, which is consistent with our prediction. Although their variation of smoothing length is rather limited, their plots ensure that the slight meatball shift predicted by the perturbation theory can definitely be observed. Colley et al. (2000) plot the two-dimensional genus curve, but they use the thin slice that suffers strongly nonlinear effects so that the quantitative comparison is not appropriate with weakly nonlinear results. They report the meatball shift in the two-dimensional genus, which is the same direction of the weakly nonlinear correction besides amplitude. We find that there is still necessity of proper, systematic comparison between the perturbation theory and numerical simulations in the weakly nonlinear regime.

The decisive factor between spongelike shift and meatball shift is the sign and amplitude of $S^{(2)} - S^{(0)}$, since $S^{(1)} - S^{(0)}$ is exactly or approximately zero for a wide range of models like power-law or CDM-like models. If the factor $S^{(2)} - S^{(0)}$ is positive, the meatball shift takes place. If this factor is negative, the spongelike shift occurs. For the power-law case, that factor is positive for $n_s > -1.4$ and negative for $n_s < -1.4$. The amplitude of this factor times the amplitude of the nonlinearity parameter, $(S^{(2)} - S^{(0)})\sigma_0$, determines the amplitude of the meatball shift in the genus curve.

Prominent shifts are seen around the two troughs of the genus curve at $\tilde{\nu} = \pm\sqrt{3}$. At these troughs, the factor $S^{(1)} - S^{(0)}$ does not contribute to the genus curve since the accompanying factor $H_3(\tilde{\nu}) = \tilde{\nu}^3 - \tilde{\nu}$ vanishes. Therefore, one can almost completely characterize the shifts by a factor $(S^{(2)} - S^{(0)})\sigma_0$, which we call the “genus asymmetry parameter,” around the troughs. We propose this genus asymmetry parameter as a theory-motivated parameter of the genus asymmetry. This factor is observationally determined by fitting the shape of the genus curve by a form

$$G(\tilde{\nu}) \propto H_2(\tilde{\nu}) + AH_3(\tilde{\nu}) + BH_1(\tilde{\nu}), \quad (224)$$

where B is the genus asymmetry parameter. In this fitting, A is much smaller if the non-Gaussian features are from purely gravitational evolution of the initial Gaussian field and if the power spectrum is smooth enough as CDM models. The genus asymmetry parameter for CDM models with shape parameter Γ , normalized by σ_8 , is listed for various smoothing lengths in Table 8.

Since the factor $S^{(2)} - S^{(0)}$ increases with the spectral index, as seen from Table 3, this factor also increases with the smoothing length for the CDM model on scales of interest. On the other hand, the factor σ_0 is a decreasing function of the smoothing

TABLE 8
GENUS ASYMMETRY PARAMETER $(S^{(2)} - S^{(0)})\sigma_0$ FOR A CDM MODEL WITH SHAPE PARAMETER Γ

Γ	SMOOTHING LENGTH R (h^{-1} Mpc)							
	6.0	8.0	10.0	12.0	14.0	16.0	18.0	20.0
0.1.....	-0.101	-0.045	-0.011	0.011	0.025	0.034	0.041	0.045
0.2.....	0.017	0.054	0.071	0.078	0.081	0.080	0.079	0.076
0.3.....	0.088	0.107	0.111	0.108	0.102	0.096	0.090	0.084
0.4.....	0.138	0.141	0.134	0.123	0.113	0.102	0.093	0.085
0.5.....	0.175	0.165	0.149	0.132	0.118	0.105	0.094	0.085

NOTE.—The power spectrum is normalized by $\sigma_8 = 1$. The parameter E is set as $E = \frac{3}{7}$.

length. As a result, the genus asymmetry parameter is not a monotonic function of the smoothing length. For example, the genus asymmetry parameter in the $\Gamma = 0.2$, $\sigma_8 = 1$ CDM model is approximately 0.08 on a wide range of smoothing lengths of $10 h^{-1}$ Mpc $\lesssim R \lesssim 20 h^{-1}$ Mpc.

From the second-order formula for the genus of equation (138), the genus at the troughs is proportional to $-2 \pm \sqrt{3}(S^{(2)} - S^{(0)})\sigma_0$. Therefore, the fraction of the deviation from the Gaussian prediction at $\tilde{\nu} = \pm\sqrt{3}$ is $\pm\sqrt{3}(S^{(2)} - S^{(0)})\sigma_0/2$. When the genus asymmetry parameter is 0.08, this fraction is about $\pm 7\%$. Thus, the perturbation theory predicts that the values of genus at positive and negative troughs differ by 14% for the $\Gamma = 0.3$, $\sigma_8 = 1$ CDM model, and so on. This difference increases with the shape parameter Γ . The degree of deviation is qualitatively consistent with the numerical result in Figure 9 of Canavezes et al. (1998), although quantitative comparison is still difficult because of the noise in the simulation.

The statistics for the velocity field is plotted in Figure 2. Since the sign of the skewness parameters is negative in this case, the weakly nonlinear evolution of statistics against the density threshold ν indicates the spongelike shift. In terms of the volume fraction threshold $\tilde{\nu}$, on the other hand, meatball shifts are observed even for relatively bluer spectrum with $n_s \leq 0$.

The two-dimensional projected galaxy statistics are dependent on the selection function of galaxies and cosmological models. As an example, we assume the Automated Plate Measurement (APM) luminosity function (Loveday et al. 1992) for galaxies with B -band magnitude limit $m_{\text{lim}} = 19$. The differential number count $dN/dz(z)$ for this sample is plotted in Figure 3.

The resulting mean redshift is $\langle z \rangle = 0.12$. The relation between the differential number count and the normalized mean number density $n(\chi)$ in comoving coordinates is given by

$$n[\chi(z)] = \frac{H(z)(dN/dz)(z)}{\{S_K[\chi(z)]\}^2 \int_0^\infty (dN/dz) dz}, \quad (225)$$

where

$$H(z) = H_0 \sqrt{(1+z)^3 \Omega_0 + (1+z)^2 (1 - \Omega_0 - \lambda_0) + \lambda_0}, \quad (226)$$

$$\chi(z) = \int_0^z \frac{dz'}{H(z')} \quad (227)$$

are the Hubble parameter and the comoving distance at redshift z , respectively. Once the selection function $n(\chi)$ is given by equation (225), the integration of equations (194)–(196) and the interpolation of tabulated values of $C^{(a)}$ in Table 6 give the skewness parameters. For power-law power spectra, the skewness parameters are given by the simpler integration of equation (217) and the values of $C^{(a)}$ in Table 7.

In Figure 4 two-dimensional statistics, N_1 , G_2 , $V_1^{(2)}$, and $V_2^{(2)}$, are plotted with assumed cosmological parameters, $\Omega_0 = 0.3$, $\lambda_0 = 0.7$, and APM luminosity function with limiting magnitude 19. The nonlinear parameter is assumed as $\sigma_0 = 0.2$. For the CDM model, the shape parameter is assumed as $\Gamma = 0.25$, and we take the smoothing angle as $\theta_f = 1^\circ$. With this smoothing angle the value $\sigma_0 = 0.2$ corresponds to the normalization $\sigma_8 = 1.22$. Basic features for these two-dimensional statistics are the same as for the three-dimensional density field, except that the smoothing scale in the CDM model we adopt corresponds to smaller scale than in the example of the three-dimensional density field in Figure 1.

6. CONCLUSIONS

In this paper we comprehensively presented a basic formalism to treat the statistics of smoothed cosmic fields in perturbation theory. This formalism provides a methodology on evaluating how various statistics deviate from the prediction of simple random Gaussian fields. As long as the non-Gaussianity is weak, the behavior of the statistics caused by non-Gaussianity is predicted by our formalism, which enables us to quantitatively compare the statistical quantities and the source of the non-Gaussianity. This method is considered as an extension of the Edgeworth expansion, which has been proven to be useful in various fields of research, as long as the non-Gaussianity is weak. In this paper we derive useful formulae and relations focusing on application of the second-order perturbation theory to various cosmic fields.

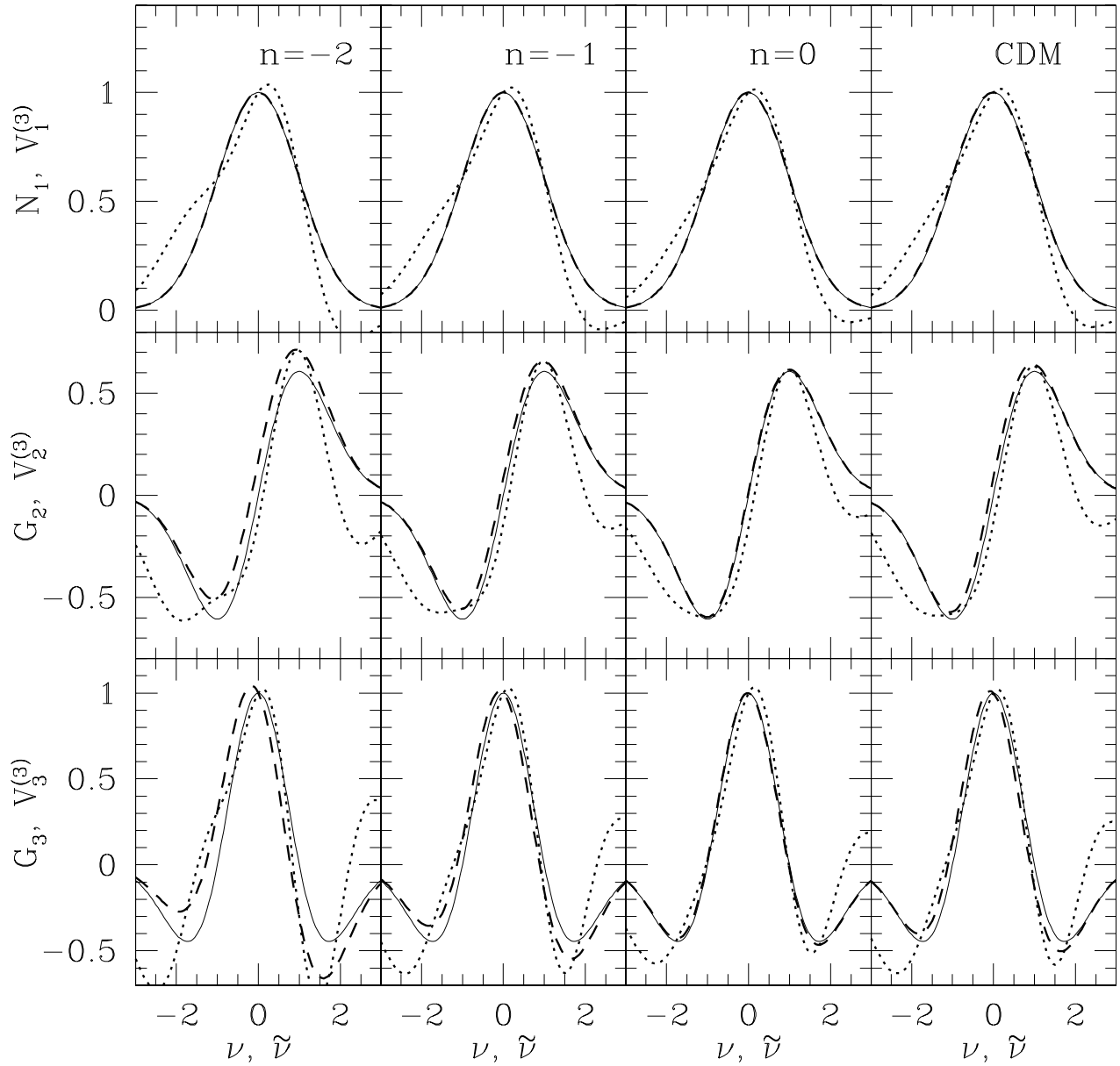


FIG. 2.—Same as Fig. 1, but for the three-dimensional velocity field

Several examples of statistics of cosmic fields in second-order perturbation theory are investigated in detail, including level-crossing statistics, two-dimensional and three-dimensional genus statistics, two-dimensional extrema statistics, and the Minkowski functionals, which are extensively used in cosmology. More complicated statistics, such as two-dimensional and three-dimensional density peaks, can also be calculated, although they are more tedious.

A particular interest in cosmology of our method is in the application to the cosmic fields. Even if the cosmic field was random Gaussian at the initial stage, the gravitational evolution induces the non-Gaussianity. The gravitational instability is a well-defined process, so that we can evaluate the non-Gaussianity without any ambiguity when the evolution remains in the quasi-linear regime provided that the biasing from the nongravitational process is simple enough on large scales. Therefore, we performed the perturbative analysis to obtain the necessary skewness parameters based on the gravitational instability theory. We considered the three-dimensional density field, the three-dimensional velocity field, and the two-dimensional projected density field. In the application of second-order theory to various statistics of smoothed cosmic fields, three types of skewness parameters are commonly useful, i.e., $S^{(0)}$, $S^{(1)}$, and $S^{(2)}$. Extensive calculations of these parameters for various cosmic fields are one of the new results of this paper. It would be true that other skewness parameters are needed when other complex statistics are considered. Such other parameters, if needed, are similarly calculated by the method we outlined in this paper.

We find that the lowest order deviations from the Gaussian predictions of various statistics of smoothed cosmic fields depend only on the differences of the skewness parameters when we use a threshold $\tilde{\nu}$, which is rescaled by a volume

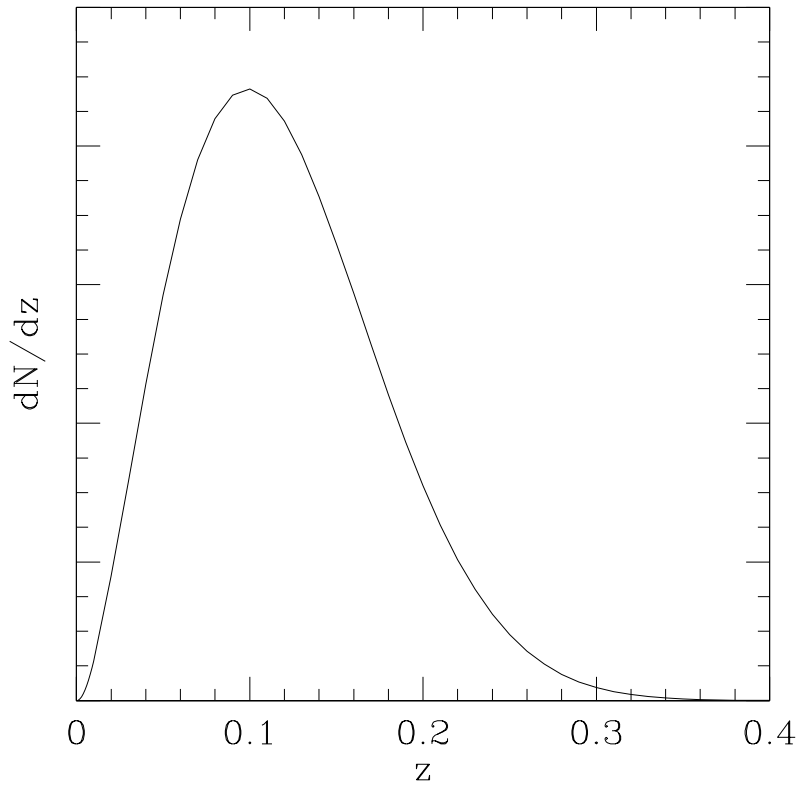


FIG. 3.—Differential number count for the APM luminosity function with B -band limiting magnitude 19. The normalization is arbitrary.

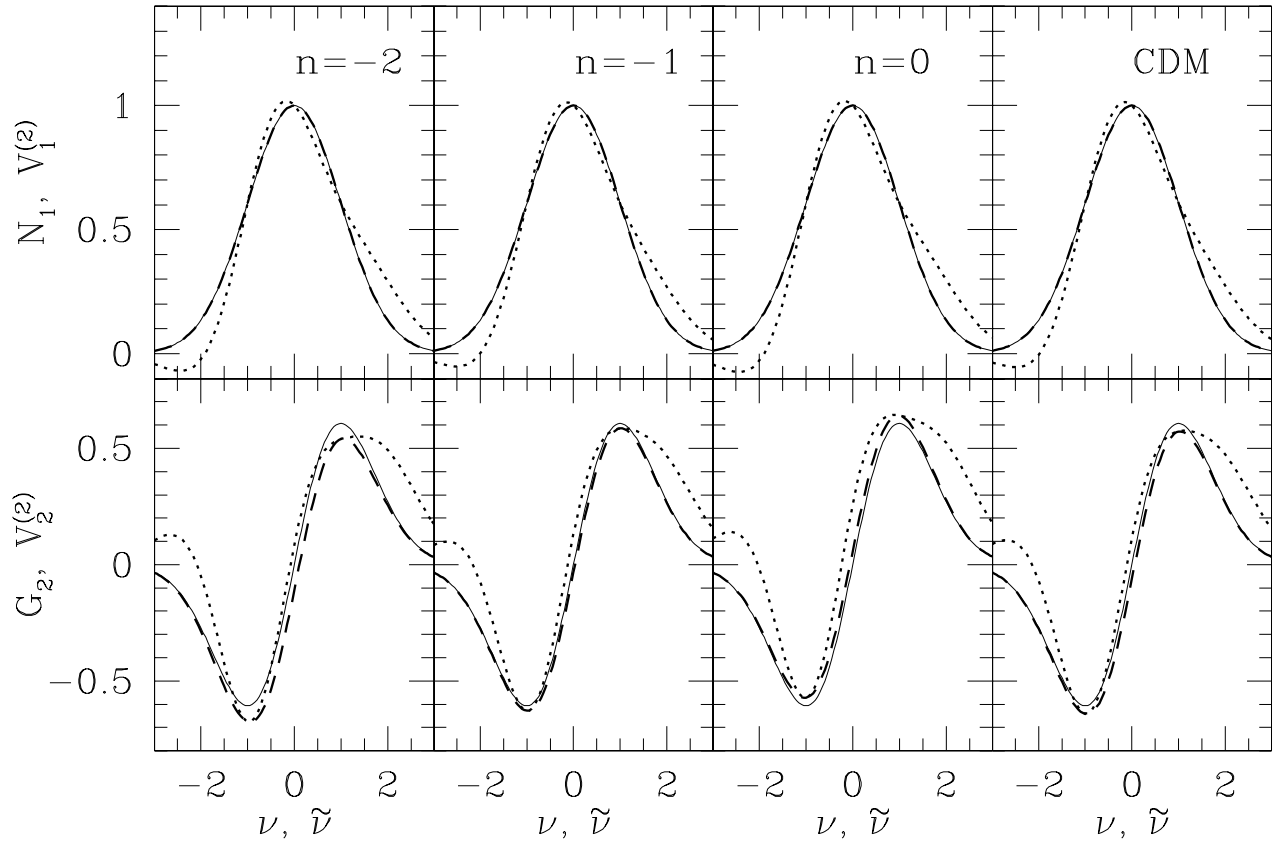


FIG. 4.—Two-dimensional genus G_2 , level-crossing statistic N_1 , and the Minkowski functionals $V_k^{(3)}$ of the two-dimensional projected density field. All curves are appropriately normalized. The meanings of lines are the same as in Fig. 1. The selection function from the APM luminosity function is assumed.

fraction of the smoothed field. This rescaling makes the lowest deviations much smaller than in the case of direct threshold ν . This is because three skewness parameters $S^{(a)}$ take similar values if it arises from the gravitational evolution. For the phenomenological hierarchical model, these parameters are identical. In this case, the weakly nonlinear correction of the statistical quantities in terms of the volume fraction threshold vanishes. When evaluated by the second-order perturbation theory of density fluctuations, those three types of skewness parameter are still close to each other. This fact explains the smallness of the deviations from Gaussian predictions of statistical quantities like genus, level crossing, or Minkowski functionals when the rescaled threshold by volume fraction is used.

We discussed small but detectable deviations from Gaussian prediction of the three-dimensional genus curve against rescaled threshold in detail. In the framework of the second-order perturbation theory, a prominent deviation of the genus curve occurs at the two troughs of the curve. Relative depths of left and right troughs in the genus curve show a slight meatball shift for CDM-like models. We found that the degree of this asymmetry is proportional to the combination $(S^{(2)} - S^{(0)})\sigma_0$. We call this factor a genus asymmetry parameter, which we propose as a theory-motivated parameter that characterizes the asymmetry of the genus curve. The genus asymmetry parameter can observationally be obtained by fitting the genus curve by Hermite polynomials as equation (224). Qualitative comparison with numerical simulations in the literature suggests that such asymmetry can actually be observable. We have not estimated over what dynamic range in nonlinearity parameters like σ_0 , $\nu\sigma_0$, or $\tilde{\nu}\sigma_0$ is each perturbative expression valid. This estimation requires a systematic comparison with large N -body simulations, which is beyond the scope of this paper and will be given in a subsequent paper of the series.

In principle, any order in the perturbation theory can be calculated as far as one would like. Although the computation of the higher order theory becomes more and more tedious, the necessity of the comparison with large-scale cosmological observations is a good reason to perform such computation as far as we can. One of the spectacular examples of the detailed comparison between perturbation theory and observations is the fine-structure constant in quantum electrodynamics (e.g., Kinoshita 1996). Our analysis in this paper will be extended to the third-order perturbation theory in a subsequent paper of the series. The present time is in a unique decade when the observations of cosmic fields are in unforeseen progress, like large-scale redshift surveys, detailed mapping of CMB fluctuations, gravitational lensing surveys, and so forth. Statistics of smoothed density field with higher order perturbation theory will provide a unique method to analyze those high-precision data. The precision cosmology is undoubtedly providing clues to unlock the door to the origin of the universe.

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APPENDIX A

USEFUL GAUSSIAN INTEGRALS

In this appendix we give Gaussian integrals that are useful in this paper. In the following, H_n are the Hermite polynomials,

$$H_n(\nu) = e^{\nu^2/2} \left(-\frac{\partial}{\partial \nu} \right)^n e^{-\nu^2/2}, \quad (\text{A1})$$

and we further employ the notation

$$H_{-1}(\nu) \equiv e^{\nu^2/2} \int_{\nu}^{\infty} d\nu' e^{-\nu'^2/2} = \sqrt{\frac{\pi}{2}} e^{\nu^2/2} \text{erfc}\left(\frac{\nu}{\sqrt{2}}\right). \quad (\text{A2})$$

Several Hermite polynomials are

$$\begin{aligned} H_0(\nu) &= 1, & H_1(\nu) &= \nu, & H_2(\nu) &= \nu^2 - 1, & H_3(\nu) &= \nu^3 - 3\nu, & H_4(\nu) &= \nu^4 - 6\nu^2 + 3, \\ H_5(\nu) &= \nu^5 - 10\nu^3 + 15\nu. \end{aligned} \quad (\text{A3})$$

The Hermite polynomial at zero is given by

$$\begin{aligned} H_n(0) &= \begin{cases} 0 & (n : \text{odd}) \\ (-1)^{n/2} (n-1)!! & (l : \text{even}) \end{cases} \\ &\equiv h_n. \end{aligned} \quad (\text{A4})$$

We generalize the above definition of h_n to the case $n < 0$ by interpreting $(n-1)!!$ as the appropriate gamma function so that $(-1)!! = 1$, $(-3)!! = -1$, etc. For example, $h_{-2} = 1$, $h_0 = 1$, $h_2 = -1$, $h_4 = 3$, and so forth. In this appendix, we give useful Gaussian averages $\langle \cdots \rangle_G$ of equation (20) for normalized cosmic field α defined by equation (5) and its spatial derivatives

$\eta_i = \alpha_i$. In the following, η represents any one of the components η_i . First, concerning α ,

$$\left\langle \frac{\partial^k \delta(\alpha - \nu)}{\partial \alpha^k} H_n(\alpha) \right\rangle_G = \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} H_{k+n}(\nu), \quad (\text{A5})$$

$$\left\langle \frac{\partial^k \theta(\alpha - \nu)}{\partial \alpha^k} H_n(\alpha) \right\rangle_G = \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} H_{k+n-1}(\nu). \quad (\text{A6})$$

Second, concerning η , in the notation of equation (A4),

$$\left\langle \frac{\partial^l |\eta|}{\partial \eta^l} \right\rangle_G = \sqrt{\frac{2}{\pi}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^{1-l} h_{l-2}, \quad (\text{A7})$$

$$\left\langle \frac{\partial^l \delta(\eta)}{\partial \eta^l} \right\rangle_G = \sqrt{\frac{1}{2\pi}} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0} \right)^{-l-1} h_l. \quad (\text{A8})$$

APPENDIX B

LIMBER'S EQUATION FOR BISPECTRUM

The correlation functions on projected sky are expressible by the three-dimensional correlation functions. The explicit relation for the two-point correlation function is given by Limber's equation (Limber 1954). The Fourier space version of Limber's equation is given by Kaiser (1998) and is somewhat simpler. His argument was generalized to higher order correlation functions and their Fourier transforms (Scoccimarro, Zaldarriaga, & Hui 1999; Buchalter, Kamionkowski, & Jaffe 2000). Since the higher order Limber equation in Fourier space was discussed in the context of weak lensing field in the literature, here we review the derivation in a way more useful in this paper. Let two-dimensional projected field f be the projection of a time-dependent three-dimensional field $F(\mathbf{x}; \tau)$ along a light cone:

$$f(\boldsymbol{\theta}) = \int d\chi S_K^2(\chi) q(\chi) F[\chi, \boldsymbol{\theta} S_K(\chi); \tau_0 - \chi], \quad (\text{B1})$$

where $q(\chi)$ is some radial weighting function, χ is the radial comoving distance, and τ_0 is the conformal time at the observer. The past light cone of the observer is specified by the equation $\chi = \tau_0 - \tau$. The comoving angular distance $S_K(\chi)$ is defined by equation (185).

In the following, we explicitly derive the relation for the three-point correlation function and bispectrum. From Limber's equation, the power spectrum is already derived by Kaiser (1998):

$$P_f(\omega) = \int d\chi S_K^2(\chi) q^2(\chi) P_F \left[\frac{\omega}{S_K(\chi)}; \tau_0 - \chi \right], \quad (\text{B2})$$

where P_f and P_F are the power spectrum of fields f and F , respectively. We generalize this equation to the one for three-point statistics below. The generalization of the following derivation to higher order statistics is straightforward. The angular three-point correlation function $w_f^{(3)}$ of f is

$$\begin{aligned} w_f^{(3)}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) &= \int d\chi_1 S_K^2(\chi_1) q(\chi_1) \int d\chi_2 S_K^2(\chi_2) q(\chi_2) \int d\chi_3 S_K^2(\chi_3) q(\chi_3) \\ &\quad \times \langle F(\chi_1, \boldsymbol{\theta}_1 S_K(\chi_1); \tau_0 - \chi) F(\chi_2, \boldsymbol{\theta}_2 S_K(\chi_2); \tau_0 - \chi) F(\chi_3, \boldsymbol{\theta}_3 S_K(\chi_3); \tau_0 - \chi) \rangle \\ &\simeq \int d\chi S_K^6(\chi) q^3(\chi) \int d\chi_1 d\chi_2 \zeta_F[\chi_1, \boldsymbol{\theta}_1 S_K(\chi_1); \chi_2, \boldsymbol{\theta}_2 S_K(\chi_2); \chi, \boldsymbol{\theta}_3 S_K(\chi); \tau_0 - \chi], \end{aligned} \quad (\text{B3})$$

where $\zeta_F(\mathbf{x}_1; \dots; \mathbf{x}_3; \tau)$ is the spatial three-point correlation function of the field F with the three-point configuration $(\mathbf{x}_1; \mathbf{x}_2; \mathbf{x}_3)$ at conformal time τ . According to the spirit of Limber's equation, we assume that $S_K^2(\chi) q(\chi)$ is slowly varying compared to the scale of the fluctuations of interest and also that these fluctuations occur on a scale much smaller than the curvature scale. Equation (B3) is the generalization of Limber's equation to higher order correlation functions.

Now we transform equation (103) to obtain the two-dimensional bispectrum. We use the following convention of the Fourier transforms:

$$\tilde{f}(\boldsymbol{\omega}) = \int d^2\theta f(\boldsymbol{\theta}) e^{-i\boldsymbol{\omega} \cdot \boldsymbol{\theta}}, \quad (\text{B4})$$

$$\tilde{F}(\mathbf{k}; \tau) = \int d^3x F(\mathbf{x}; \tau) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (\text{B5})$$

The bispectrum B_f of the two-dimensional field f and B_F of the three-dimensional field F are defined by

$$\langle \tilde{f}(\omega_1) \tilde{f}(\omega_2) \tilde{f}(\omega_3) \rangle = (2\pi)^2 \delta^2(\omega_1 + \omega_2 + \omega_3) B_f(\omega_1, \omega_2, \omega_3), \quad (\text{B6})$$

$$\langle \tilde{F}(\mathbf{k}_1; \tau) \tilde{F}(\mathbf{k}_2; \tau) \tilde{F}(\mathbf{k}_3; \tau) \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_F(k_1, k_2, k_3; \tau), \quad (\text{B7})$$

where $\omega_i = |\omega_i|$ and $k_i = |\mathbf{k}_i|$. The Dirac δ -function comes from the translational invariance of statistics. From the relations

$$B_f(\omega_1, \omega_2, \omega_3) = \int d^2\theta_1 d^2\theta_2 v_f(\theta_1, \theta_2, 0) e^{-i\omega_1 \cdot \theta_1 - i\omega_2 \cdot \theta_2} \quad (\omega_3 = |\omega_1 + \omega_2|), \quad (\text{B8})$$

$$\zeta_F(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \tau) = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} B_F(k_1, k_2, k_3; \tau) e^{ik_1 \cdot (\mathbf{x}_1 - \mathbf{x}_3) + ik_2 \cdot (\mathbf{x}_2 - \mathbf{x}_3)}, \quad (\text{B9})$$

the Fourier transform of equation (B3) reduces to a simple equation,

$$B_f(\omega_1, \omega_2, \omega_3) = \int d\chi S_K^2(\chi) q^3(\chi) B_F \left[\frac{\omega_1}{S_K(\chi)}, \frac{\omega_2}{S_K(\chi)}, \frac{\omega_3}{S_K(\chi)}; \tau_0 - \chi \right]. \quad (\text{B10})$$

This is a bispectrum version of Limber's equation and the generalization of Kaiser's equation for the power spectrum (eq. [B2]).

APPENDIX C

SYMBOL INDEX

The quantities used in this paper are listed in Table 9.

TABLE 9
SYMBOL INDEX

Symbol	Definition	Equation
$\langle \cdots \rangle$	Ensemble average	(18)
$\langle \cdots \rangle_G$	Ensemble average by Gaussian field	(20)
Σ_f	Two-dimensional spectral parameter	(194)
Ω	Time-dependent density parameter	(192)
Ω_0	Density parameter	(192)
α	Normalized field of f	(5)
α_{ij} , etc.	$\partial^2 \alpha / \partial x_i \partial x_j = \partial_i \partial_j \alpha$, etc.	(6)
γ	Spectral parameter	(38)
δ	Mass density contrast	(220)
δ_{3D}	Three-dimensional density contrast	(187)
δ_g	Galaxy number density contrast	(220)
ζ_{ij}	$= \alpha_{ij}$, tensor notation	
$\tilde{\zeta}_{ij}$	Linear combination of α and ζ_{ij}	(32)
η_i	$= \alpha_{,i}$, vector notation	
θ	Angular coordinates in a small patch of sky	(187)
κ	Local convergence field	(218)
λ	Time-dependent scaled cosmological constant	(192)
λ_0	Scaled cosmological constant	(192)
ν	Threshold by variance	
$\bar{\nu}$	Threshold by volume fraction	(134)
ρ	Three-dimensional comoving density field	(186)
ρ_{e2}	Two-dimensional weighted extrema	(114)
ρ_p	Two-dimensional projected density field	(186)
σ_0	Variance of f	(4)
σ_1, σ_2	Spectral parameters	(30), (31)
τ_0	Conformal time at present	(186)
χ	Comoving distance	(185)
ω_k	Volume of unit ball in k dimensions	(125)
A_μ, \mathbf{A}	Vector of spatial derivatives of α	(6)
$B(k_1, k_2, k_3)$	Bispectrum at present time	(144)
$B_{2D}(\omega_1, \omega_2, \omega_3)$	Two-dimensional bispectrum	(189)
$B_{3D}(k_1, k_2, k_3; \tau)$	Three-dimensional bispectrum at conformal time τ	(189)
$C^{(a)}$	Integrand of two-dimensional skewness parameters	(197)
$\tilde{C}^{(a)}$	Integrand of two-dimensional skewness parameters	(198)
$C_m^{\alpha\beta}$	Contributions to two-dimensional skewness parameters	(208)

TABLE 9—*Continued*

Symbol	Definition	Equation
D	Linear growth rate	(190)
E	Parameter of perturbation theory	(151)
E_v	Parameter of perturbation theory	(177)
F	Cosmic field, in general	(18)
$F_{\mu_1\mu_2\mu_3}$, etc.	$\partial^3 F / \partial A_{\mu_1} \partial A_{\mu_2} \partial A_{\mu_3}$, etc.	(22)
$F(\alpha, \beta, \gamma; z)$	Gauss hypergeometric function ${}_2F_1$	(169)
G_2	Two-dimensional genus statistic	(98), (99)
G_3	Three-dimensional genus statistic	(105), (107)
H	Time-dependent Hubble parameter	(175)
H_0	Hubble constant	(226)
I_ν	Modified Bessel function	(156)
J, \mathbf{J}	Fourier counterpart of A_μ	(7)
$J_m^{(2)}(\{m_{ij}\})$	Integers defined by Table 1	(109)
K	Spatial curvature	(185)
$H_n(\nu)$	Hermite polynomials	(A1)
$H_{-1}(\nu)$	Extended Hermite polynomial of order -1	(A2)
$M_{\mu_1 \dots \mu_n}^{(n)}$	n th-order cumulants of A_μ	(8)–(12)
$\bar{M}_{\mu_1 \dots \mu_n}^{(n)}$	Normalized cumulants of A_μ	(21)
M	Second-order cumulant of A_μ	(13)
N_1	Level-crossing statistic	(88)
N_2	Length statistic	(89)
N_3	Area statistic	(90)
P_G	Multivariate Gaussian distribution function	(7)
P	Multivariate probability distribution function	(7)
$P(k)$	Power spectrum at present	(143)
$P_{2D}(\omega)$	Two-dimensional power spectrum	(188)
$P_{3D}(k; \tau)$	Three-dimensional power spectrum at conformal time τ	(188)
$P_{LIN}(k)$	Linear power spectrum	(149)
Q	Hierarchical amplitude of three-point function	(140)
$S^{(a)}, S_2^{(2)}$	Skewness parameters	(61)–(64)
$\tilde{S}^{(a)}$	Integrand of skewness parameters	(154)
$S_m^{\alpha\beta}(R)$	Contributions to skewness parameters	(160)
$S_g^{(a)}$	Galaxy skewness parameters	(185)
S_K	Comoving angular diameter distance	(185)
T_{CDM}	CDM transfer function	(164)
$V_k^{(d)}$	Minkowski functionals	(117), (118), (122), (125)
W	Smoothing function in real space	(1)
W_R	Smoothing function in real space	(1)
Z	Generating function	(7)
b	Bias parameter	(220)
b_2	Nonlinear bias parameter	(220)
d	Dimension of sample space	(1)
dN/dz	Differential number count	(225)
f	Smoothed cosmic field, in general	(1)
f_V	Volume fraction on high-density side	(134)
g_f	Logarithmic derivative of growth factor	(178)
h_l	Hermite polynomial at zero, $H_l(0)$, extended to $l < 0$	(A4)
n	Selection function per unit comoving volume	(186)
n_s	Index for power-law power spectrum	(165)
x, y, z	Linear combinations of α and ζ_{ij} values	(39)–(41)

REFERENCES

- Adler, R. J. 1981, *The Geometry of Random Fields* (Chichester: Wiley)
- Balian, R., & Schaeffer, R. 1989, *A&A*, 220, 1
- Bardeen, J. M., Bond, J. R., Kaiser, N., & Szalay, A. S. 1986, *ApJ*, 304, 15
- Bartelmann, M., & Schneider, P. 2001, *Phys. Rep.*, 340, 291
- Baugh, C. M., & Efstathiou, G. 1994, *MNRAS*, 270, 183
- Beaky, M. M., Scherrer, R. J., & Villumsen, J. V. 1992, *ApJ*, 387, 443
- . 1994a, *A&A*, 291, 697
- . 1994b, *ApJ*, 433, 1
- . 1995, *A&A*, 301, 309
- Bernardeau, F., Colombi, S., Gaztanaga, E., & Scoccimarro, R. 2002, *Phys. Rep.*, 367, 1
- Bernardeau, F., Juszkiewicz, R., Dekel, A., & Bouchet, F. R. 1995, *MNRAS*, 274, 20
- Bernardeau, F., & Kofman, L. 1995, *ApJ*, 443, 479
- Bernardeau, F., van Waerbeke, L., & Mellier, Y. 1997, *A&A*, 322, 1
- Bharadwaj, S., Sahni, V., Sathyaprakash, B. S., Shandarin, S. F., & Yess, C. 2000, *ApJ*, 528, 21
- Bond, J. R., & Efstathiou, G. 1987, *MNRAS*, 226, 655
- Bouchet, F. R., Juszkiewicz, R., Colombi, S., & Pellat, R. 1992, *ApJ*, 394, L5
- Buchalter, A., & Kamionkowski, M. 1999, *ApJ*, 521, 1
- Buchalter, A., Kamionkowski, M., & Jaffe, A. H. 2000, *ApJ*, 530, 36
- Canavezes, A., et al. 1998, *MNRAS*, 297, 777
- Carroll, S. M., Press, W. H., & Turner, E. L. 1992, *ARA&A*, 30, 499
- Cayon, L., & Smoot, G. 1995, *ApJ*, 452, 487
- Cen, R. 1998, *ApJ*, 509, 494
- Chodorowski, M. J., & Łokas, E. L. 1997, *MNRAS*, 287, 591
- Coles, P. 1988, *MNRAS*, 234, 509
- . 1990, *MNRAS*, 243, 171
- Coles, P., & Barrow, J. D. 1987, *MNRAS*, 228, 407

- Coles, P., Moscardini, L., Plionis, M., Lucchin, F., Matarrese, S., & Messina, A. 1993, *MNRAS*, 260, 572
- Coles, P., & Plionis, M. 1991, *MNRAS*, 250, 75
- Colley, W. N. 1997, *ApJ*, 489, 471
- Colley, W. N., Gott, J. R., III, Weinberg, D. H., Park, C., & Berlind, A. A. 2000, *ApJ*, 529, 795
- Croft, R. A. C., & Efstathiou, G. 1994, *MNRAS*, 267, 390
- Crofton, M. W. 1868, *Philos. Trans. R. Soc. London*, A158, 181
- Davies, A., & Coles, P. 1993, *MNRAS*, 260, 553
- Dekel, A., & Lahav, O. 1999, *ApJ*, 520, 24
- Doroshkevich, A. G. 1970, *Astrophysica*, 6, 320
- Fabbri, R., & Torres, S. 1996, *A&A*, 307, 703
- Fry, J. N. 1984, *ApJ*, 279, 499
- Fry, J. N., & Gaztanaga, E. 1993, *ApJ*, 413, 447
- Gabrieli, A., Labini, F. S., & Durrer, R. 2000, *ApJ*, 531, L1
- Goroff, M. H., Grinstein, B., Rey, S.-J., & Wise, M. B. 1986, *ApJ*, 311, 6
- Gott, J. R., III, Mao, S., Park, C., & Lahav, O. 1992, *ApJ*, 385, 26
- Gott, J. R., III, Melott, A. L., & Dickinson, M. 1986, *ApJ*, 306, 341
- Gott, J. R., III, Park, C., Juszkiewicz, R., Bies, W. E., Bennett, D. P., Bouchet, F. R., & Stebbins, A. 1990, *ApJ*, 352, 1
- Gott, J. R., III, Weinberg, D. H., & Melott, A. L. 1987, *ApJ*, 319, 1
- Gott, J. R., III, et al. 1989, *ApJ*, 340, 625
- Hamilton, A. J. S., Gott, J. R., III, & Weinberg, D. 1986, *ApJ*, 309, 1
- Heavens, A. F., & Sheth, R. K. 1999, *MNRAS*, 310, 1062
- Jain, B., & Bertschinger, E. 1994, *ApJ*, 431, 495
- Jain, B., & Van Waerbeke, L. 2000, *ApJ*, 530, L1
- Juszkiewicz, R. 1981, *MNRAS*, 197, 931
- Juszkiewicz, R., Bouchet, F. R., & Colombi, S. 1993, *ApJ*, 412, L9
- Juszkiewicz, R., Sonoda, D. H., & Barrow, J. D. 1984, *MNRAS*, 209, 139
- Juszkiewicz, R., Weinberg, D. H., Amsterdamski, P., Chodorowski, M., & Bouchet, F. R. 1995, *ApJ*, 442, 39
- Kaiser, N. 1984, *ApJ*, 284, L9
- . 1998, *ApJ*, 498, 26
- Kerscher, M., Schmalzing, J., Buchert, T., & Wagner, H. 1998, *A&A*, 333, 1
- Kinoshita, T. 1996, *Rep. Prog. Phys.*, 59, 1459
- Kogut, A., Banday, A. J., Bennett, C. L., Hinshaw, G., Lubin, P. M., & Smoot, G. F. 1995, *ApJ*, 439, L29
- Lahav, O., Lilje, P. B., Primack, J. R., & Rees, M. J. 1991, *MNRAS*, 251, 128
- Lightman, A. P., & Schechter, P. L. 1990, *ApJS*, 74, 831
- Limber, D. N. 1954, *ApJ*, 119, 655
- Łokas, E. L. 1998, *MNRAS*, 296, 491
- Łokas, E. L., Juszkiewicz, R., Weinberg, D. H., & Bouchet, F. R. 1995, *MNRAS*, 274, 730
- Loveday, J., Peterson, B. A., Efstathiou, G., & Maddox, S. J. 1992, *ApJ*, 390, 338
- Ma, S.-K. 1985, *Statistical Mechanics* (Philadelphia: World Scientific)
- Makino, N., Sasaki, M., & Suto, Y. 1992, *Phys. Rev. D*, 46, 585
- Mann, R. G., Heavens, A. F., & Peacock, J. A. 1993, *MNRAS*, 263, 798
- Matsubara, T. 1994, *ApJ*, 434, L43
- . 1995a, *ApJS*, 101, 1
- . 1995b, *Prog. Theor. Phys.*, 94, 1151
- . 1996, *ApJ*, 457, 13
- . 1999, *ApJ*, 525, 543
- Matsubara, T., & Suto, Y. 1996, *ApJ*, 460, 51
- Matsubara, T., & Yokoyama, J. 1996, *ApJ*, 463, 409
- Mecke, K. R., Buchert, T., & Wagner, H. 1994, *A&A*, 288, 697
- Mellier, Y. 1999, *ARA&A*, 37, 127
- Melott, A. L., Cohen, A. P., Hamilton, A. J. S., Gott, J. R., III, & Weinberg, D. H. 1989, *ApJ*, 345, 618
- Moore, B., et al. 1992, *MNRAS*, 256, 477
- Morse, M., & Cairns, S. 1969, *Critical Point Theory in Global Analysis and Differential Topology* (New York: Academic Press)
- Naselsky, P. D., & Novikov, D. I. 1998, *ApJ*, 507, 31
- Park, C., Colley, W. N., Gott, J. R., III, Ratra, B., Spergel, D. N., & Sugiyama, N. 1998, *ApJ*, 506, 473
- Park, C., & Gott, J. R., III 1991, *ApJ*, 378, 457
- Park, C., Gott, J. R., III, & da Costa, L. N. 1992, *ApJ*, 392, L51
- Peebles, P. J. E. 1980, *The Large-Scale Structure of the Universe* (Princeton: Princeton Univ. Press)
- Plionis, M., & Valdarnini, R. 1995, *MNRAS*, 272, 869
- Plionis, M., Valdarnini, R., & Coles, P. 1992, *MNRAS*, 258, 114
- Protogeros, Z. A. M., & Weinberg, D. H. 1997, *ApJ*, 489, 457
- Rhoads, J. E., Gott, J. R., III, & Postman, M. 1994, *ApJ*, 421, 1
- Ryden, B. S. 1988a, *ApJ*, 333, L41
- . 1988b, *PASP*, 100, 1360
- Ryden, B. S., Melott, A. L., Craig, D. A., Gott, J. R., III, Wenberg, D. H., Scherrer, R. J., Bhavsar, S. P., & Miller, J. M. 1989, *ApJ*, 340, 647
- Sahni, V., Sathyaprakash, B. S., & Shandarin, S. F. 1997, *ApJ*, 476, L1
- . 1998, *ApJ*, 495, L5
- Sathyaprakash, B. S., Sahni, V., & Shandarin, S. 1998, *ApJ*, 508, 551
- Sazhin, M. V. 1985, *MNRAS*, 216, 25P
- Scherrer, R. J., & Bertschinger, E. 1991, *ApJ*, 381, 349
- Schmalzing, J., & Buchert, T. 1997, *ApJ*, 482, L1
- Schmalzing, J., Buchert, T., Melott, A. L., Sahni, V., Sathyaprakash, B. S., & Shandarin, S. F. 1999, *ApJ*, 526, 568
- Schmalzing, J., & Diaferio, A. 2000, *MNRAS*, 312, 638
- Schmalzing, J., & Gorski, K. M. 1998, *MNRAS*, 297, 355
- Scoccimarro, R., & Frieman, J. A. 1996a, *ApJ*, 473, 620
- . 1996b, *ApJS*, 105, 37
- Scoccimarro, R., Zaldarriaga, M., & Hui, L. 1999, *ApJ*, 527, 1
- Seto, N. 2000, *ApJ*, 537, 21
- Smoot, G. F., Tenorio, L., Banday, A. J., Kogut, A., Wright, E. L., Hinshaw, G., & Bennett, C. L. 1994, *ApJ*, 437, 1
- Springel, V., et al. 1998, *MNRAS*, 298, 1169
- Suto, Y., & Sasaki, M. 1991, *Phys. Rev. Lett.*, 66, 264
- Taruya, A., & Soda, J. 1999, *ApJ*, 522, 46
- Tomita, H. 1986, *Prog. Theor. Phys.*, 76, 952
- Torres, S. 1994, *ApJ*, 423, L9
- Torres, S., Cayon, L., Martinez-Gonzalez, E., & Sanz, J. L. 1995, *MNRAS*, 274, 853
- van Waerbeke, L. 2000, *MNRAS*, 313, 524
- Vishniac, E. T. 1983, *MNRAS*, 203, 345
- Vittorio, N., & Juszkiewicz, R. 1987, *ApJ*, 314, L29
- Vogeley, M. S., Park, C., Geller, M. J., Huchra, J. P., & Gott, J. R., III 1994, *ApJ*, 420, 525
- Watanabe, T., Matsubara, T., & Suto, Y. 1994, *ApJ*, 432, 17
- Winitzki, S., & Kosowsky, A. 1997, *NewA*, 3, 75