Conformal Parameterization of a surface

Chi Po Choi
Department of Statistics
University of California, Davis
cpchoi@ucdavis.edu

Abstract

We computationally construct a conformal parameterization of a surface. We only consider the surface which is topologically equivalent to 2-dimensional disk. In other words, the surface is orientable, simply-connected and has only one boundary curve. We aim to find a conformal parameterization the surface onto a unit disk, with the boundary curve lying on the unit circle. This parameterization can be obtained by solving a Laplace equation with suitable boundary condition. The surface is discretized as a triangle mesh. Solving this Laplace equation is formulated as an optimization problem which has a quadratic objective with quadratic and linear constraints. We solve the optimization problem using augmented Lagrangian method.

1 Introduction

One picture worths more than thousands of words. Figure 1 illustrates the goal of this final project. See Figure 1. Figure 1(a) is a surface mesh of a hat. This surface is topologically equivalent to a disk. We want to find a parameterization of the surface on a unit disk, with the boundary curve of the surface lying on the unit circle. Figure 1(b) is a example of parameterization of this surface mesh. We can regard parameterization as a process of flattening the triangle mesh from 3D space to 2D plane. However, not any parameterization is useful. We want a special kind of surface parameterization: conformal parameterization, which preserves the angles of the triangle. Figure 1(c) is an example of conformal parameterization of this surface mesh. The angles of the triangles on the Figure 1(c) match those on the surface mesh, while those on Figure 1(b) does not.

2 Conformal parameterization as a solution to Laplace equation

One way to construct a conformal parameterization is variational method. Let S be a surface (smooth manifold topologically equivalent to a disk). Denote ∂S be the boundary curve of S .Let $f : S \to \mathbb{R}^2$ be a smooth function from the surface S to $\mathbb{R}^2$. Write $f = (u, v)$ where $u, v : S \to \mathbb{R}$. Let the Dirichlet's energy of f be

$$E(f) = \frac{1}{2}\int_S \|\nabla f\|^2 dg = \frac{1}{2}\int_S \|\nabla u\|^2 + \|\nabla v\|^2 dg$$

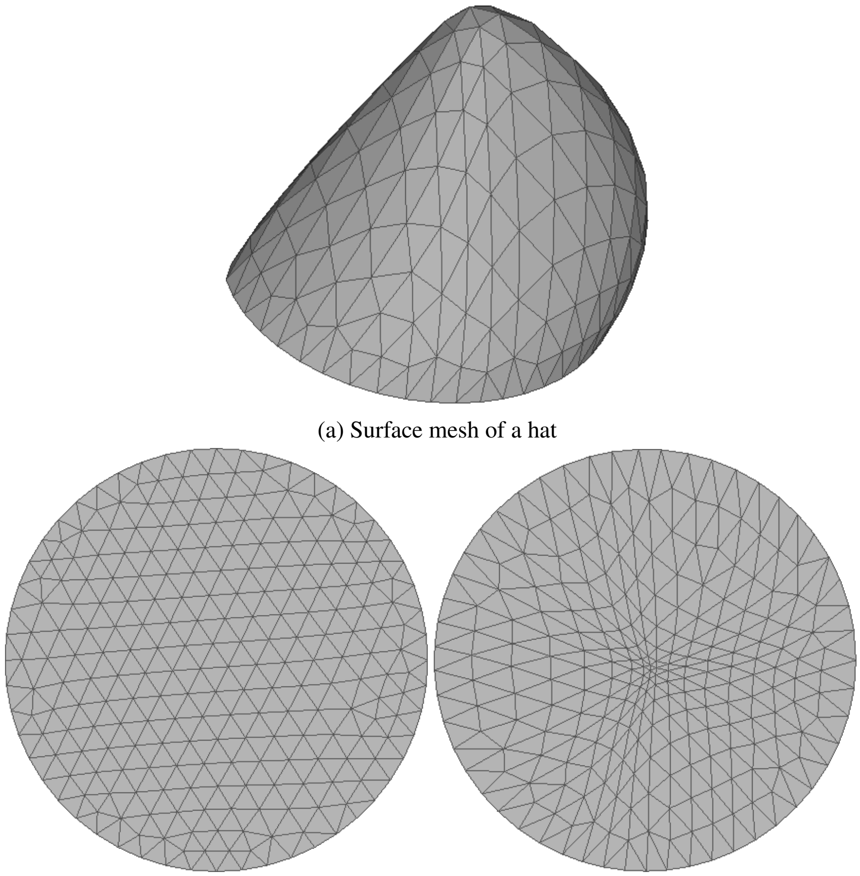
where g is the Riemannian metric on the surface S and $\|\cdot\|^2$ is the metric of the tangent space of S.

Let $\mathbb{D}$ be the unit disk on $\mathbb{R}^2$. Consider the following variational problem:

$$\min_f E(f) \quad \text{subject to } f(\partial S) = \partial\mathbb{D}.$$

According to [1], the minimizer of this variational problem is a conformal mapping, which preserves the angles between S and $f(S)$ under corresponding Riemannian metrics. Also, the minimizer of the above variational problem satisfies the Laplace equation

$$\Delta_S f = 0 \quad \text{with boundary condition } f(\partial S) = \partial\mathbb{D}$$



(a) Surface mesh of a hat

(b) *NOT* conformal parameterization (c) Conformal parameterization

Figure 1: Conformal Parameterization

where Δ_S is the Laplace-Beltrami operator on S under the Riemannian metric g.

Therefore, we will solve the Laplace equation in order to obtain a conformal parameterization.

3 Discretization

3.1 Triangle mesh

We aim to solve the above Laplace equation computationally. The surface is discretized as a triangle mesh. The triangle mesh defines a piece-wise continuous surface which approximate S. The triangle mesh consists of a set of vertices $V = \{V_i\} \subset S$ which are sample points on S, and also a set of 3-tuple $T = \{(V_{t_1}, V_{t_2}, V_{t_3})\}$ which define the triangles formed by the vertices.

$f : S \to \mathbb{R}^2$ is approximated by $f : \{V_i\} \to \mathbb{R}^2$. We write $(u_i, v_i) = f(V_i)$. Let n be the number of vertices in the triangle mesh. The discretization of f can be written as a vector of length $2n$: $\mathbf{x} = (u_1, \ldots, u_n, v_1, \ldots, v_n)'$.

3.2 Discrete Laplace-Beltrami operator

Let $N(V_i) \subset V$ be the neighborhood of V_i, i.e. the set of vertices which are connected to V_i. For each $V_j \in N(V_i)$, the two angles opposite to the edge (V_i, V_j) are denoted as α_{ij} and β_{ij}. See Figure 2 for the illustration.

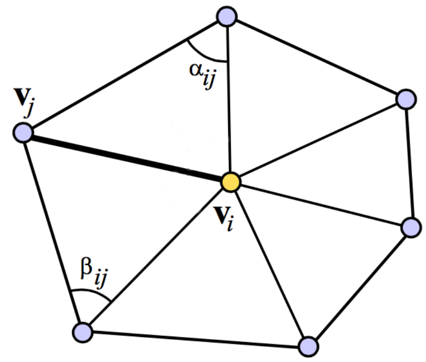


Figure 2: Cotangent formula (figure adopted from [3])

Let $f : V \to \mathbb{R}$ be a discrete function on the triangle mesh. The Laplacian $\Delta_S f$ of f at V_i is:

$$(\Delta_S f)(V_i) = \sum_{V_j \in N(V_i)} (\cot \alpha_{ij} + \cot_{ij})(f(V_j) - f(V_i)).$$

It is called "cotangent formula"[2] for the discrete Laplace-Beltrami operator.

The discrete Laplace-Beltrami operator of the triangle mesh can be written as an n-by-n symmetric matrix L defined as

$$L_{ij} = \begin{cases} 0 & \text{if } V_i \text{ is not connected to } V_j \\ \cot \alpha_{ij} + \cot \beta_{ij} & \text{if } V_i \text{ is connected to } V_j \end{cases}$$

$$L_{ii} = -\sum_{j \neq i} L_{ij}$$

Let $(u_i, v_i)' = f(V_i)$. Let $\mathbf{u} = (u_1, \ldots, u_n)'$ and $\mathbf{v} = (v_1, \ldots, v_n)'$. The Laplace equation $(\Delta_S f)(V_i) = 0$ for all $V_i \in V$ can be written as

$$\begin{cases} L\mathbf{u} = \mathbf{0} \\ L\mathbf{v} = \mathbf{0}. \end{cases}$$

Let $\mathbf{x} = (u_1, \ldots, u_n, v_1, \ldots, v_n)$. Then we need to solve

$$Q\mathbf{x} = \mathbf{0} \quad \text{where } Q = \begin{pmatrix} L & O_n \\ O_n & L \end{pmatrix} \text{ is a symmetric matrix.}$$

3.3 Boundary condition

The boundary condition $f(\partial S) = \partial \mathbb{D}$ can be written as $u_k^2 + v_k^2 = 1$ for all $V_k \in V \cap \partial S$. Note that those are quadratic constraints on $\mathbf{x}$. There exists $2n$-by-$2n$ symmetric matrices H_k such that $\mathbf{x}' H_k \mathbf{x} = u_k^2 + v_k^2$. Let n_b be the number of boundary vertices $V \cap \partial S$. We have n_b many quadratic constraints:

$$\mathbf{x}' H_k \mathbf{x} = 1 \quad \text{for } k = 1, \ldots, n_b.$$

Beside the boundary condition on the unit circle, we also need some another constraint for practical reason. It is intuitive that the solution of conformal parameterization is not unique, as any rigid motion on the mesh preserves all angle. Also, the constant mapping, i.e. $u_1 = u_2 = \cdots = u_n$ and $v_1 = v_2 = \cdots = v_n$, is a trivial solution. Therefore, in order to get rid of the above situation, we will need to fix at least three vertices to three different coordinates. Suppose we want to fix $(u_{l_1}, v_{l_1}) = (b_1, b_2)$, $(u_{l_2}, v_{l_2}) = (b_3, b_4)$ and $(u_{l_3}, v_{l_3}) = (b_5, b_6)$, where (b_1, b_2), (b_3, b_4) and $(b_5, b_6) \in \mathbb{D}$ are fixed coordinates. We have the following linear constraint:

$$A\mathbf{x} = \mathbf{b} \quad \text{where } A \text{ is such that } A\mathbf{x} = (u_{l_1}, v_{l_1}, u_{l_2}, v_{l_2}, u_{l_3}, v_{l_3})' \text{ and } \mathbf{b} = (b_1, b_2, b_3, b_4, b_5, b_6)'$$

In summary, we have $2n$ many quadratic constraints and also 4 linear constraints.

4 Optimization problem

The conformal parameterization can be written as a triangle mesh with a parameterization function $f : S \to \mathbb{R}^2$

$$
\begin{aligned}
V &= \{V_i \in S\}_{i=1}^n && \text{Vertex}\\
T &= \{(V_{t_1}, V_{t_2}, V_{t_3})\} && \text{Triangulation}\\
f(V) &= \{(u_i, v_i) \in \mathbb{R}^2\}_{i=1}^n && \text{Coordinates of vertices in } \mathbf{R}^2
\end{aligned}
$$

Given a triangle mesh (V, T) of a surface S, we want to find the parameterization function $f(V) = \{(u_i, v_i) \in \mathbb{R}^2\}_{i=1}^n$. Let $\mathbf{x} = (u_1, \ldots, u_n, v_1, \ldots, v_n)$. $\mathbf{x}$ is the solution of the following optimization problem:

$$
\min_{\mathbf{x} \in \mathbb{R}^{2n}} \mathbf{x}'Q\mathbf{x} \qquad \text{subject to } \begin{cases} \mathbf{x}'H_k\mathbf{x} = 1 & \text{for } k = 1, \ldots, n_b \\ A\mathbf{x} = \mathbf{b} \end{cases} \tag{$\star$}
$$

where Q, H_k, A and $\mathbf{b}$ are defined in the Section 3 **Discretization**. The optimization problem $(\star)$ has a quadratic objective with quadratic constraints and linear constraints.

We apply augmented Lagrangian method to solve $(\star)$.

The objective function, its gradient and its Hessian are:

$$
\begin{aligned}
\text{Obj}(\mathbf{x}) &= \mathbf{x}'Q\mathbf{x}\\
\text{Grad}(\mathbf{x}) &= Q\mathbf{x}\\
\text{Hess}(\mathbf{x}) &= Q
\end{aligned}
$$

The constraint function $c(\mathbf{x})$, gradient of constraint $\nabla c(\mathbf{x})$ and Hessian of constraint $\{\nabla^2 c_k(\mathbf{x})\}$ are:

$$
\begin{aligned}
c(\mathbf{x}) &= \begin{bmatrix} \frac{1}{2}\mathbf{x}'H_1\mathbf{x} \\ \vdots \\ \frac{1}{2}\mathbf{x}'H_{n_b}\mathbf{x} \\ A\mathbf{x} - \mathbf{b} \end{bmatrix}\\
\nabla c(\mathbf{x}) &= [H_1\mathbf{x} \quad \ldots \quad H_{n_b}\mathbf{x} \quad A']\\
\{\nabla^2 c_k(\mathbf{x})\} &= \{H_1, \ldots, H_{n_b}, O_n, \ldots, O_n\}
\end{aligned}
$$

The augmented Lagrangian, its gradient and its Hessian are:

$$
\begin{aligned}
L_\rho(\mathbf{x}, \mathbf{y}) &= \mathbf{x}'Q\mathbf{x} - \mathbf{y}'c(\mathbf{x}) + \frac{\rho}{2} c(\mathbf{x})'c(\mathbf{x})\\
\nabla_x L_\rho(\mathbf{x}, \mathbf{y}) &= Q\mathbf{x} - [\nabla c(\mathbf{x})]\mathbf{y} + \rho[\nabla c(\mathbf{x})]c(\mathbf{x})\\
\nabla^2_{xx} L_\rho(\mathbf{x}, \mathbf{y}) &= Q - \sum_{k=1}^{n_b+6} \nabla^2 c_k(\mathbf{x}) \cdot y_k + \rho \sum_{k=1}^{n_b+6} \cdot c_k(\mathbf{x}) + \rho[\nabla c(\mathbf{x})][\nabla c(\mathbf{x})]'
\end{aligned}
$$

With the above explicit formulas, we apply the augmented Lagrangian method with Newton's method to solve $(\star)$. See Algorithm 1.

$k = 0$, $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = \mathbf{0}$, $\rho_k = 10$;
while $\|c(\mathbf{x}_k)\| > \epsilon$ *or* $\|Grad(\mathbf{x}_k) - [\nabla c(\mathbf{x}_k)]\mathbf{y}_k\| > \epsilon$ **do**
    $\mathbf{x}_{k+1} = \min_x L_{\rho^k}(\mathbf{x}, \mathbf{y}_k)$ (by Newton's method) ;
    $k = k + 1$;
    if $\|c(\mathbf{x}_k)\| \leq 0.000001/2^k$ **then**
        $\rho_k = \rho_{k-1}$;
        $\mathbf{y}_k = \mathbf{y}_{k-1} - \rho c(\mathbf{x}_k)$;
    else
        $\rho_k = 2\rho_{k-1}$;
        $\mathbf{y}_k = \mathbf{y}_{k-1}$;
    end
end

Algorithm 1: Augmented Lagrangian method

5 Numerical experiment

The implementation is done in Julia. See Section 8 **Appendix** for the codes.

Below is the numerical experiment of the surface mesh shown in Figure 1. The surface mesh consists of 88 vertices, in which 31 vertices are at boundary. The augmented Lagrangian method takes 11 iterations to converge.

Iteration	ρ	$\|c(\mathbf{x}_k)\|$	$\|\text{Grad}(\mathbf{x}_k) - [\nabla c(\mathbf{x}_k)]\mathbf{y}_k\|$
1	10	0.219869662769	3.474162683887
2	20	0.110261610250	2.182805069440
3	40	0.055224136741	2.225978483471
4	80	0.027637315846	2.247778475219
5	160	0.013825249327	2.258770957372
6	320	0.006914310456	2.264297284670
7	640	0.003457581625	2.267069021520
8	1280	0.001728898043	2.268457178338
9	2560	0.000864475909	2.269151847922
10	2560	0.000000148032	0.000000000001
11	2560	0.000000000201	0.000000000002

6 Further work

In this project, we only consider the surface mesh which is topologically equivalent to unit disk. We may work on surface with different topological type. Spherical parameterization is another problem we can investigate. Given a surface which is topologically equivalent to sphere, we want to find a parameterization on the unit sphere. It is also sensible to talk about conformal parameterization in this case. In further work, the method in this project will extended for conformal parameterization on unit sphere.

7 References

[1] John E Hutchinson et al. Computing conformal maps and minimal surfaces. In *Theoretical and Numerical Aspects of Geometric Variational Problems*, pages 140–161. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University, 1991.

[2] Ulrich Pinkall and Konrad Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental mathematics*, 2(1):15–36, 1993.

[3] Olga Sorkine. Laplacian mesh processing. In *EUROGRAPHICS05 STAR–STATE OF THE ART REPORT*. Citeseer, 2005.

8 Appendix

The Julia notebook of the codes is on `https://github.com/pochoi/mat258A`.

8.1 Functions to construct objective and constraints

```
function getBoundary(t)
    nt = size(t)[2]
    edge_list = Array{Int64,1}[]
    for i in 1:nt
        push!(edge_list, t[1:2,i])
        push!(edge_list, t[2:3,i])
        push!(edge_list, t[[3,1],i])
    end
    boundary_list = Array{Int64,1}[]
    for i in 1:length(edge_list)
       myedge = edge_list[i][[2,1]]
        if !in(myedge, edge_list)
```

```
            push!(boundary_list, edge_list[i])
        end
    end
    boundary = zeros(Int64, length(boundary_list))
    boundary[1] = boundary_list[1][1]
    for i in 1:length(boundary)
       if i == length(boundary_list)
           break
       end
        for j in 1:length(boundary_list)
            if boundary_list[j][1] == boundary[i]
             boundary[i+1]  = boundary_list[j][2]
             break
            end
        end
    end
    return boundary
end

function getEdge(p, t)
    e1 = p[:, vec(t[3,:])] - p[:, vec(t[2,:])]
    e2 = p[:, vec(t[1,:])] - p[:, vec(t[3,:])]
    e3 = p[:, vec(t[2,:])] - p[:, vec(t[1,:])]
    e1norm = vec(sqrt(sum(e1.^2, 1)))
    e2norm = vec(sqrt(sum(e2.^2, 1)))
    e3norm = vec(sqrt(sum(e3.^2, 1)))
    return e1, e2, e3, e1norm, e2norm, e3norm
end

function mesh_cot(p, t)
    (e1, e2, e3, e1norm, e2norm, e3norm) = getEdge(p, t)
   e1cos = vec(sum(-e2 .* e3, 1)) ./ e2norm ./ e3norm
   e2cos = vec(sum(-e3 .* e1, 1)) ./ e3norm ./ e1norm
   e3cos = vec(sum(-e1 .* e2, 1)) ./ e1norm ./ e2norm
    r1 = zeros(Float64, size(t))
    r2 = zeros(Float64, size(t))
    r3 = zeros(Float64, size(t))
    for i in 1:size(t)[2]
      r1[:,i] = cross(e2[:,i], e3[:,i])
      r2[:,i] = cross(e3[:,i], e1[:,i])
      r3[:,i] = cross(e1[:,i], e2[:,i])
    end
   e1sin = vec(sqrt(sum(r1.^2,1))) ./ e2norm ./ e3norm
   e2sin = vec(sqrt(sum(r2.^2,1))) ./ e3norm ./ e1norm
   e3sin = vec(sqrt(sum(r3.^2,1))) ./ e1norm ./ e2norm
   e1cot = e1cos ./ e1sin;
   e2cot = e2cos ./ e2sin;
   e3cot = e3cos ./ e3sin;
   return vec(e1cot), vec(e2cot), vec(e3cot)
end

function WCot(p, t)
    n = size(p)[2]
    (e1cot, e2cot, e3cot) = mesh_cot(p, t);
    I = [vec(t[2,:]); vec(t[3,:]); vec(t[1,:])]
    J = [vec(t[3,:]); vec(t[1,:]); vec(t[2,:])]
    S = [e1cot; e2cot; e3cot]
    W = sparse(I,J,S, n, n)
    #Wfull = Float64[ W[i,j] for i in 1:n, j in 1:n]
    return W
end

function LCot(p, t)
    n = size(p)[2]
    W = WCot(p,t)
```

```
    W = W + transpose(W)
    L = sparse(1:n, 1:n, vec(sum(W,2))) - W
    #Lfull = Float64[ L[i,j] for i in 1:n, j in 1:n]
    return L
end

function areaMixMeyer(p, t)
    n = size(p)[2]
    (e1, e2, e3, e1norm, e2norm, e3norm) = getEdge(p, t)
    (e1cot, e2cot, e3cot) = mesh_cot(p, t);
    I = [vec(t[2,:]); vec(t[3,:]); vec(t[1,:])]
    J = [vec(t[3,:]); vec(t[1,:]); vec(t[2,:])]
    S = [e1cot .* vec(e1norm).^2; e2cot .* vec(e2norm).^2; e3cot .* vec(
        e3norm).^2]
    tri_angle = meshAngle(v,f)
    obtuse_flag = vec(any(tri_angle .> pi/2 , 1))
    #obtuse_flag = (e1angle > pi/2) | (e2angle > pi/2) | (e3angle > pi/2)
    tri_area, v_area = meshArea(p, t)
    tri_ring = vertexTriRing(p, t)
    mix_area = zeros(Float64, n)
    for i in 1:n
        a = 0
        for itri in tri_ring[i]
            v_index = find(t[:,itri] .== i)[]
            if !obtuse_flag[itri]
                a += v_area[v_index, itri]
            else
                a += tri_area[itri] / ( (tri_angle[v_index,itri] > pi/2)
                    ? 2 : 4)
            end
        end
        mix_area[i] = a
    end
    return mix_area
end

immutable ConfOpt
    Q::SparseMatrixCSC{Float64,Int64}
    H::Array{SparseMatrixCSC{Float64,Int64},1}
    A::Array{SparseMatrixCSC{Float64,Int64},1}
    b::Array{Float64,1}
    n::Int64
    m::Int64
    objFun::Function
    objGrad::Function
    objHess::Function
    constrFun::Function
    constrGrad::Function
    constrHess::Array{Function,1}
    augLagFun::Function
    augLagGrad::Function
    augLagHess::Function
    x0::Array{Float64,1}
    p::Array{Float64,2}
    t::Array{Int64,2}
    boundary::Array{Int64,1}
    boundary_3point::Array{Int64,1}
    boundary_3point_v::Array{Float64,2}
end

function Ltut(p, t)
        n = size(p)[2]
    I = [vec(t[2,:]); vec(t[3,:]); vec(t[1,:])]
    J = [vec(t[3,:]); vec(t[1,:]); vec(t[2,:])]
    S = ones(length(I))
```

```
    W = sparse(I,J,S, n, n)
    W = W + transpose(W)
    L = sparse(1:n, 1:n, vec(sum(W,2))) - W

    boundary = getBoundary(t)
    nb = length(boundary)
    theta = (2pi/nb) * (0:(nb-1))

        b1 = zeros(Float64, n)
        b2 = zeros(Float64, n)

        for k = 1:nb
                b1 = b1 - L[:, boundary[k]] * cos(theta[k])
                b2 = b2 - L[:, boundary[k]] * sin(theta[k])
        end
        b1[boundary] = cos(theta)
        b2[boundary] = sin(theta)

        Lw = copy(L)
        Lw[:, boundary] = 0
        Lw[boundary,:] = 0
        for k = 1:nb
                Lw[boundary[k],boundary[k]] = 1
        end

        uv = Lw \ [b1 b2]
        return uv
end


function ConfOpt(p::Array{Float64,2}, t::Array{Int64,2})
    n = size(p,2)
    boundary = getBoundary(t)
    nb = length(boundary)

    Qraw = LCot(p,t)
    Q = [Qraw zeros(n,n); zeros(n,n) Qraw]

        H = Array{Float64,2}[]
        for i in 1:nb
                G = zeros(Float64, 2, 2n)
                G[1,boundary[i]] = 1
                G[2,n + boundary[i]] = 1
                Hi = G.' * G
                push!(H, Hi)
        end

    objFun = function(x::Array{Float64,1}) 0.5 * getindex(x.' * Q * x, 1)
         end
    objGrad = function(x::Array{Float64,1}) Q * x end
    objHess = function(x::Array{Float64,1}) Q end

    theta = (2pi/3) * [0;1;2]
    boundary_3point = boundary[[1, 1 + div(nb,3), 1 + div(nb,3)*2]]
    boundary_3point_v = [cos(theta).'; sin(theta).']

    A = [zeros(Float64, 1, 2n) for i in 1:6]
    b = zeros(Float64, 6)
    for i in 1:3
        A[i][1, boundary_3point[i]] = 1
        A[3+i][1, boundary_3point[i] + n ] = 1
        b[i] = cos(theta[i])
        b[3+i] = sin(theta[i])
    end
```

```
    constrFun = function(x::Array{Float64,1})
        [Float64[ 0.5* ((x.' * H[i] * x) - 1)[] for i in 1:nb];
        Float64[ ((A[i] * x) - b[i])[] for i in 1:6]]
        end
    constrGrad = function(x::Array{Float64,1})
        J = zeros(Float64, 2n, nb+6)
        for i in 1:nb
                J[:,i] = H[i] * x
            end
            for i in 1:6
                J[:, nb + i] = A[i].'
            end
        return J
        end
    constrHess = Function[]
    for i in 1:nb
        fun = function(x::Array{Float64,1}) H[i] end
        push!(constrHess, fun)
    end
    for i in 1:6
        fun = function(x::Array{Float64,1}) zeros(Float64,2n,2n) end
        push!(constrHess, fun)
    end

    augLagFun = function(x::Array{Float64,1}, y::Array{Float64,1}, rho::
        Float64)
        objFun(x) - dot(y, constrFun(x)) + 0.5rho * dot(constrFun(x),
            constrFun(x))
        end
        augLagGrad = function(x::Array{Float64,1}, y::Array{Float64,1},
            rho::Float64)
        objGrad(x) - constrGrad(x) * y + rho * constrGrad(x) * constrFun(
            x)
        end
        augLagHess = function(x::Array{Float64,1}, y::Array{Float64,1},
            rho::Float64)
                Q - sum(H .* y[1:nb]) + rho * sum(H .* constrFun(x)[1:nb
                    ]) + rho * constrGrad(x) * transpose(constrGrad(x))
        end

        uv = Ltut(p, t)
        x0 = [uv[:,1];uv[:,2]]

    return ConfOpt(Q, H, A, b, 2n, nb+6,
                        objFun, objGrad, objHess,
                        constrFun, constrGrad, constrHess,
                        augLagFun, augLagGrad, augLagHess,
                        x0, p, t, boundary, boundary_3point,
                            boundary_3point_v)
end


function augLagObj(prob::ConfOpt, y::Array{Float64,1}, rho::Float64)
        augLagFun = function(x::Array{Float64,1})
        prob.objFun(x) - dot(y, prob.constrFun(x)) + 0.5rho * dot(prob.
            constrFun(x), prob.constrFun(x))
        end
        augLagGrad = function(x::Array{Float64,1})
        prob.objGrad(x) - prob.constrGrad(x) * y + rho * prob.constrGrad(
            x) * prob.constrFun(x)
        end
        augLagHess = function(x::Array{Float64,1})
                prob.Q - sum(prob.H .* y[1:(prob.m-6)]) + rho * sum(prob.
                    H .* prob.constrFun(x)[1:(prob.m-6)]) + rho * prob.
                    constrGrad(x) * transpose(prob.constrGrad(x))
```

```
        end

        obj = function(x::Array{Float64,1})
                return augLagFun(x), augLagGrad(x), augLagHess(x)
        end

        return obj
end
```

8.2 Newton's methods and augmented Lagrangian method

```
# Newton Method
function newtmin(obj, x0; maxIts=1000, optTol=1e-6, BkFlag = false,
    btFlag = true)
    # Minimize a function f using N e w t o n s method.
    # obj: a function that evaluates the objective value,
    # gradient, and Hessian at a point x, i.e.,
    # (f, g, H) = obj(x)
    # x0: starting point.
    # maxIts (optional): maximum number of iterations.
    # optTol (optional): optimality tolerance based on
    # ||grad(x)|| <= optTol*||grad(x0)||
    # BkFlag (optional): true for doing Modified Hessian
    # btFlag (optional): true for doing Back Tracking
    f0, g0, H0 = obj(x0)
    its = 0
    Opt = Float64[]
    xkp = copy(x0)
    for i in 1:maxIts
        xk = copy(xkp)
        fk, gk, Hk = obj(xk)
        opt = norm(gk, 2)
        push!(Opt, opt)
        #if opt < optTol*norm(g0)
        if opt < optTol
            break
        end
        if BkFlag == true
            Bkinv = BkFunInv(Hk, 0.01)
            dk = Bkinv * (- gk)
        else
            dk = Hk \ (- gk)
        end
        if btFlag == true
             k  = backTracking(obj, xk, dk, gk)
        else
             k  = 1
        end
        xkp = xk +   k  * dk
        its = its + 1
    end
    return xkp, its, Opt
end

function AugLag(prob::ConfOpt; maxItr=20, optTolck = 1e-8, optTolKKT = 1e
    -8, rho0 = 10.0)
        y0 = zeros(Float64, prob.m)

        rhok = rho0
        yk = copy(y0)
        xk = copy(prob.x0)
    #xk = zeros(Float64, prob.n)

    rho_counter = 0
```

```
        counter = 0

    @printf "%6s %6s %18s %18s" "Itr" "rho" "Norm Constr" "Norm AugLag
       Grad\n"
    storage_cnorm = Float64[]
    storage_lnorm = Float64[]
        for itr = 1:maxItr

                xkp = newtmin(augLagObj(prob, yk, rhok), xk, maxIts=20,
                    optTol = (1e-4 / 10^itr ) )[1]
        push!(storage_cnorm, norm(prob.constrFun(xkp)))
        push!(storage_lnorm, norm(prob.objGrad(xk) - prob.constrGrad(xk)
            * yk))
        @printf "%6d %6.0f %18.12f %18.12f\n" itr rhok storage_cnorm[end]
             storage_lnorm[end]
            if norm(prob.constrFun(xkp)) < 1e-3 * (1/2)^rho_counter
                        rhokp = rhok
                        ykp = yk - rhokp * prob.constrFun(xkp)
                        rho_counter = rho_counter + 1
                else
                        rhokp = 2.0 * rhok
                        ykp = copy(yk)
                end

                rhok = rhokp
                yk = copy(ykp)
                xk = copy(xkp)

                counter = itr
                if (norm(prob.constrFun(xkp)) < optTolck) & (norm( prob.
                    objGrad(xk) - prob.constrGrad(xk) * yk) < optTolKKT)
                        break
                end
        end

    return xk, yk, rhok, counter, storage_cnorm, storage_lnorm
end
```

8.3 How to run the codes

```
f = readdlm("test1_f.dat", ',', Int64)
v = readdlm("test1_v.dat", ',', Float64)

f = f.';
v = v.';

myprob = ConfOpt(v, f);
result_x = AugLag(myprob, maxItr=50);
```