Conformal Parameterization of a surface

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Abstract

We computationally construct a conformal parameterization of a surface. We only consider the surface which is topologically equivalent to 2-dimensional disk. In other words, the surface is orientable, simply-connected and has only one boundary curve. We aim to find a conformal parameterization the surface onto a unit disk, with the boundary curve lying on the unit circle. This parameterization can be obtained by solving a Laplace equation with suitable boundary condition. The surface is discretized as a triangle mesh. Solving this Laplace equation is formulated as an optimization problem which has a quadratic objective with quadratic and linear constraints. We solve the optimization problem using augmented Lagrangian method.

1 Introduction

One picture worths more than thousands of words. Figure 1 illustrates the goal of this final project. See Figure 1. Figure 1(a) is a surface mesh of a hat. This surface is topologically equivalent to a disk. We want to find a parameterization of the surface on a unit disk, with the boundary curve of the surface lying on the unit circle. Figure 1(b) is a example of parameterization of this surface mesh. We can regard parameterization as a process of flattening the triangle mesh from 3D space to 2D plane. However, not any parameterization is useful. We want a special kind of surface parameterization: conformal parameterization, which preserves the angles of the triangle. Figure 1(c) is an example of conformal parameterization of this surface mesh. The angles of the triangles on the Figure 1(c) match those on the surface mesh, while those on Figure 1(b) does not.

2 Conformal parameterization as a solution to Laplace equation

One way to construct a conformal parameterization is variational method. Let S be a surface (smooth manifold topologically equivalent to a disk). Denote ∂S be the boundary curve of S .Let $f : S \to \mathbb{R}^2$ be a smooth function from the surface S to $\mathbb{R}^2$. Write $f = (u, v)$ where $u, v : S \to \mathbb{R}$. Let the Dirichlet's energy of f be

$$E(f) = \frac{1}{2}\int_S \|\nabla f\|^2 dg = \frac{1}{2}\int_S \|\nabla u\|^2 + \|\nabla v\|^2 dg$$

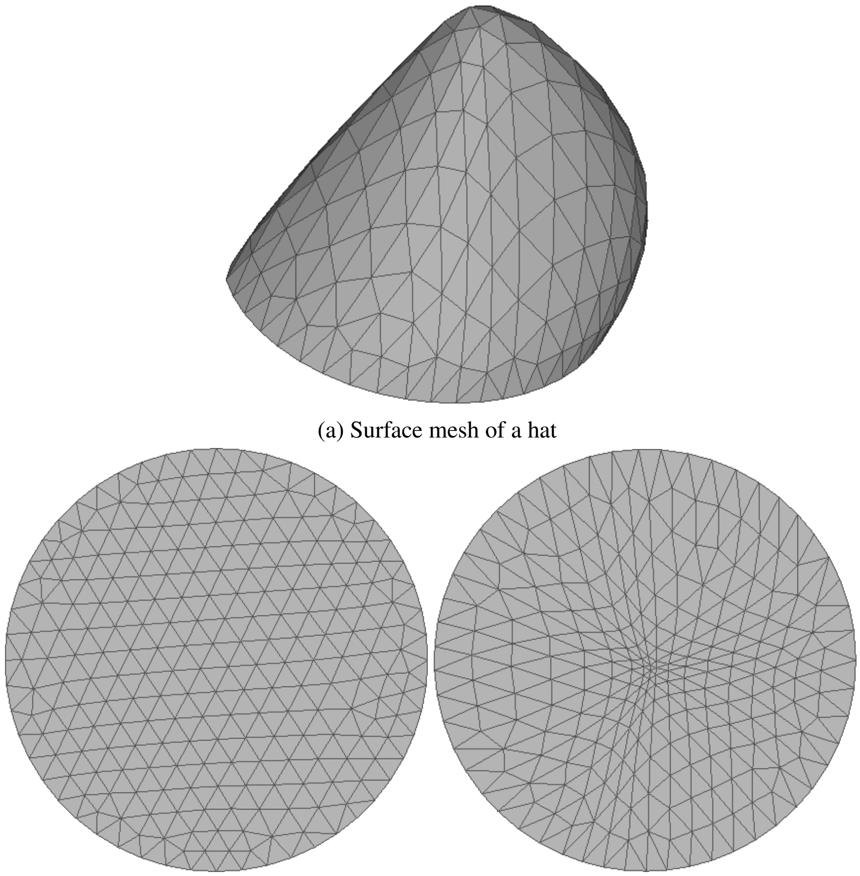
where g is the Riemannian metric on the surface S and $\|\cdot\|^2$ is the metric of the tangent space of S.

Let $\mathbb{D}$ be the unit disk on $\mathbb{R}^2$. Consider the following variational problem:

$$\min_f E(f) \quad \text{subject to } f(\partial S) = \partial\mathbb{D}.$$

According to [1], the minimizer of this variational problem is a conformal mapping, which preserves the angles between S and $f(S)$ under corresponding Riemannian metrics. Also, the minimizer of the above variational problem satisfies the Laplace equation

$$\Delta_S f = 0 \quad \text{with boundary condition } f(\partial S) = \partial\mathbb{D}$$



(a) Surface mesh of a hat

(b) *NOT* conformal parameterization

(c) Conformal parameterization

Figure 1: Conformal Parameterization

where Δ_S is the Laplace-Beltrami operator on S under the Riemannian metric g.

Therefore, we will solve the Laplace equation in order to obtain a conformal parameterization.

3 Discretization

3.1 Triangle mesh

We aim to solve the above Laplace equation computationally. The surface is discretized as a triangle mesh. The triangle mesh defines a piece-wise continuous surface which approximate S. The triangle mesh consists of a set of vertices $V = \{V_i\} \subset S$ which are sample points on S, and also a set of 3-tuple $T = \{(V_{t_1}, V_{t_2}, V_{t_3})\}$ which define the triangles formed by the vertices.

$f : S \to \mathbb{R}^2$ is approximated by $f : \{V_i\} \to \mathbb{R}^2$. We write $(u_i, v_i) = f(V_i)$. Let n be the number of vertices in the triangle mesh. The discretization of f can be written as a vector of length $2n$: $\mathbf{x} = (u_1, \ldots, u_n, v_1, \ldots, v_n)'$.

3.2 Discrete Laplace-Beltrami operator

Let $N(V_i) \subset V$ be the neighborhood of V_i, i.e. the set of vertices which are connected to V_i. For each $V_j \in N(V_i)$, the two angles opposite to the edge (V_i, V_j) are denoted as α_{ij} and β_{ij}. See Figure 2 for the illustration.

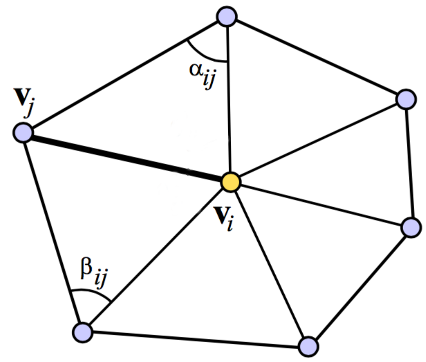


Figure 2: Cotangent formula (figure adopted from [3])

Let $f : V \to \mathbb{R}$ be a discrete function on the triangle mesh. The Laplacian $\Delta_S f$ of f at V_i is:

$$(\Delta_S f)(V_i) = \sum_{V_j \in N(V_i)} (\cot \alpha_{ij} + \cot_{ij})(f(V_j) - f(V_i)).$$

It is called “cotangent formula”[2] for the discrete Laplace-Beltrami operator.

The discrete Laplace-Beltrami operator of the triangle mesh can be written as an n-by-n symmetric matrix L defined as

$$L_{ij} = \begin{cases} 0 & \text{if } V_i \text{ is not connected to } V_j \\ \cot \alpha_{ij} + \cot \beta_{ij} & \text{if } V_i \text{ is connected to } V_j \end{cases}$$
$$L_{ii} = -\sum_{j \neq i} L_{ij}$$

Let $(u_i, v_i)' = f(V_i)$. Let $\mathbf{u} = (u_1, \ldots, u_n)'$ and $\mathbf{v} = (v_1, \ldots, v_n)'$. The Laplace equation $(\Delta_S f)(V_i) = 0$ for all $V_i \in V$ can be written as

$$\begin{cases} L\mathbf{u} = \mathbf{0} \\ L\mathbf{v} = \mathbf{0}. \end{cases}$$

Let $\mathbf{x} = (u_1, \ldots, u_n, v_1, \ldots, v_n)$. Then we need to solve

$$Q\mathbf{x} = \mathbf{0} \quad \text{where } Q = \begin{pmatrix} L & O_n \\ O_n & L \end{pmatrix} \text{ is a symmetric matrix.}$$

3.3 Boundary condition

The boundary condition $f(\partial S) = \partial \mathbb{D}$ can be written as $u_k^2 + v_k^2 = 1$ for all $V_k \in V \cap \partial S$. Note that those are quadratic constraints on $\mathbf{x}$. There exists $2n$-by-$2n$ symmetric matrices H_k such that $\mathbf{x}' H_k \mathbf{x} = u_k^2 + v_k^2$. Let n_b be the number of boundary vertices $V \cap \partial S$. We have n_b many quadratic constraints:

$$\mathbf{x}' H_k \mathbf{x} = 1 \quad \text{for } k = 1, \ldots, n_b.$$

Beside the boundary condition on the unit circle, we also need some another constraint for practical reason. It is intuitive that the solution of conformal parameterization is not unique, as any rigid motion on the mesh preserves all angle. Also, the constant mapping, i.e. $u_1 = u_2 = \cdots = u_n$ and $v_1 = v_2 = \cdots = v_n$, is a trivial solution. Therefore, in order to get rid of the above situation, we will need to fix at least three vertices to three different coordinates. Suppose we want to fix $(u_{l_1}, v_{l_1}) = (b_1, b_2)$, $(u_{l_2}, v_{l_2}) = (b_3, b_4)$ and $(u_{l_3}, v_{l_3}) = (b_5, b_6)$, where (b_1, b_2), (b_3, b_4) and $(b_5, b_6) \in \mathbb{D}$ are fixed coordinates. We have the following linear constraint:

$$A\mathbf{x} = \mathbf{b} \quad \text{where } A \text{ is such that } A\mathbf{x} = (u_{l_1}, v_{l_1}, u_{l_2}, v_{l_2}, u_{l_3}, v_{l_3})' \text{ and } \mathbf{b} = (b_1, b_2, b_3, b_4, b_5, b_6)'$$

In summary, we have $2n$ many quadratic constraints and also 4 linear constraints.

4 Optimization problem

The conformal parameterization can be written as a triangle mesh with a parameterization function $f : S \to \mathbb{R}^2$

$$\begin{aligned} V &= \{V_i \in S\}_{i=1}^n && \text{Vertex} \\ T &= \{(V_{t_1}, V_{t_2}, V_{t_3})\} && \text{Triangulation} \\ f(V) &= \{(u_i, v_i) \in \mathbb{R}^2\}_{i=1}^n && \text{Coordinates of vertices in } \mathbf{R}^2 \end{aligned}$$

Given a triangle mesh (V, T) of a surface S, we want to find the parameterization function $f(V) = \{(u_i, v_i) \in \mathbb{R}^2\}_{i=1}^n$. Let $\mathbf{x} = (u_1, \dots, u_n, v_1, \dots, v_n)$. $\mathbf{x}$ is the solution of the following optimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^{2n}} \mathbf{x}' Q \mathbf{x} \qquad \text{subject to } \begin{cases} \mathbf{x}' H_k \mathbf{x} = 1 & \text{for } k = 1, \dots, n_b \\ A\mathbf{x} = \mathbf{b} \end{cases} \tag{$\star$}$$

where Q, H_k, A and $\mathbf{b}$ are defined in the Section 3 **Discretization**. The optimization problem $(\star)$ has a quadratic objective with quadratic constraints and linear constraints.

We apply augmented Lagrangian method to solve $(\star)$.

The objective function, its gradient and its Hessian are:

$$\begin{aligned} \text{Obj}(\mathbf{x}) &= \mathbf{x}' Q \mathbf{x} \\ \text{Grad}(\mathbf{x}) &= Q\mathbf{x} \\ \text{Hess}(\mathbf{x}) &= Q \end{aligned}$$

The constraint function $c(\mathbf{x})$, gradient of constraint $\nabla c(\mathbf{x})$ and Hessian of constraint $\{\nabla^2 c_k(\mathbf{x})\}$ are:

$$\begin{aligned} c(\mathbf{x}) &= \begin{bmatrix} \frac{1}{2}\mathbf{x}' H_1 \mathbf{x} \\ \vdots \\ \frac{1}{2}\mathbf{x}' H_{n_b} \mathbf{x} \\ A\mathbf{x} - \mathbf{b} \end{bmatrix} \\ \nabla c(\mathbf{x}) &= [H_1 \mathbf{x} \quad \dots \quad H_{n_b}\mathbf{x} \quad A'] \\ \{\nabla^2 c_k(\mathbf{x})\} &= \{H_1, \dots, H_{n_b}, O_n, \dots, O_n\} \end{aligned}$$

The augmented Lagrangian, its gradient and its Hessian are:

$$\begin{aligned} L_\rho(\mathbf{x}, \mathbf{y}) &= \mathbf{x}' Q \mathbf{x} - \mathbf{y}' c(\mathbf{x}) + \frac{\rho}{2} c(\mathbf{x})' c(\mathbf{x}) \\ \nabla_x L_\rho(\mathbf{x}, \mathbf{y}) &= Q\mathbf{x} - [\nabla c(\mathbf{x})]\mathbf{y} + \rho[\nabla c(\mathbf{x})] c(\mathbf{x}) \\ \nabla^2_{xx} L_\rho(\mathbf{x}, \mathbf{y}) &= Q - \sum_{k=1}^{n_b+6} \nabla^2 c_k(\mathbf{x}) \cdot y_k + \rho \sum_{k=1}^{n_b+6} \cdot c_k(\mathbf{x}) + \rho[\nabla c(\mathbf{x})][\nabla c(\mathbf{x})]' \end{aligned}$$

With the above explicit formulas, we apply the augmented Lagrangian method with Newton's method to solve $(\star)$. See Algorithm 1.

$k = 0$, $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = \mathbf{0}$, $\rho_k = 10$;
while $\|c(\mathbf{x}_k)\| > \epsilon$ *or* $\|Grad(\mathbf{x}_k) - [\nabla c(\mathbf{x}_k)]\mathbf{y}_k\| > \epsilon$ **do**
    $\mathbf{x}_{k+1} = \min_x L_{\rho^k}(\mathbf{x}, \mathbf{y}_k)$ (by Newton's method) ;
    $k = k + 1$;
    if $\|c(\mathbf{x}_k)\| \le 0.000001/2^k$ **then**
        $\rho_k = \rho_{k-1}$;
        $\mathbf{y}_k = \mathbf{y}_{k-1} - \rho c(\mathbf{x}_k)$;
    else
        $\rho_k = 2\rho_{k-1}$;
        $\mathbf{y}_k = \mathbf{y}_{k-1}$;
    end
end

Algorithm 1: Augmented Lagrangian method

5 Numerical experiment

The implementation is done in Julia. See Section 8 **Appendix** for the codes.

Below is the numerical experiment of the surface mesh shown in Figure 1. The surface mesh consists of 88 vertices, in which 31 vertices are at boundary. The augmented Lagrangian method takes 12 iterations to converge.

Iteration	ρ	$\|c(\mathbf{x}_k)\|$	$\|\text{Grad}(\mathbf{x}_k) - [\nabla c(\mathbf{x}_k)]\mathbf{y}_k\|$
1	10	0.322200556833	4.296581324074
2	20	0.162975113062	3.112602891839
3	40	0.082049621748	3.275987170477
4	80	0.041174790104	3.363717465205
5	160	0.020626047938	3.409291123305
6	320	0.010322832301	3.432540894415
7	640	0.005163886443	3.444286516261
8	1280	0.002582563077	3.450190167892
9	2560	0.001291436788	3.453149787933
10	5120	0.000645757242	3.454631557140
11	5120	0.000000117213	0.000000000001
12	5120	0.000000000072	0.000000000001

6 Further work

In this project, we only consider the surface mesh which is topologically equivalent to unit disk. We may work on surface with different topological type. Spherical parameterization is another problem we can investigate. Given a surface which is topologically equivalent to sphere, we want to find a parameterization on the unit sphere. It is also sensible to talk about conformal parameterization in this case. In further work, the method in this project will extended for conformal parameterization on unit sphere.

7 References

[1] John E Hutchinson et al. Computing conformal maps and minimal surfaces. In *Theoretical and Numerical Aspects of Geometric Variational Problems*, pages 140–161. Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University, 1991.

[2] Ulrich Pinkall and Konrad Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental mathematics*, 2(1):15–36, 1993.

[3] Olga Sorkine. Laplacian mesh processing. In *EUROGRAPHICS05 STAR–STATE OF THE ART REPORT*. Citeseer, 2005.

8 Appendix

The Julia notebook of the codes is on `https://github.com/pochoi/mat258A`.

8.1 Functions to construct objective and constraints

```
# Get the indices of the boundary vertices
function getBoundary(t)
    nt = size(t)[2]
    edge_list = Array{Int64,1}[]
    for i in 1:nt
        push!(edge_list, t[1:2,i])
        push!(edge_list, t[2:3,i])
        push!(edge_list, t[[3,1],i])
    end
    boundary_list = Array{Int64,1}[]
    for i in 1:length(edge_list)
```

```
        myedge = edge_list[i][[2,1]]
         if !in(myedge, edge_list)
             push!(boundary_list, edge_list[i])
         end
    end
    boundary = zeros(Int64, length(boundary_list))
    boundary[1] = boundary_list[1][1]
    for i in 1:length(boundary)
       if i == length(boundary_list)
           break
       end
        for j in 1:length(boundary_list)
            if boundary_list[j][1] == boundary[i]
             boundary[i+1]  = boundary_list[j][2]
             break
            end
        end
    end
    return boundary
end

# Function to help compute cotangent formula
function getEdge(p, t)
    e1 = p[:, vec(t[3,:])] - p[:, vec(t[2,:])]
    e2 = p[:, vec(t[1,:])] - p[:, vec(t[3,:])]
    e3 = p[:, vec(t[2,:])] - p[:, vec(t[1,:])]
    e1norm = vec(sqrt(sum(e1.^2, 1)))
    e2norm = vec(sqrt(sum(e2.^2, 1)))
    e3norm = vec(sqrt(sum(e3.^2, 1)))
    return e1, e2, e3, e1norm, e2norm, e3norm
end

# Function to compute cotangent formula
function mesh_cot(p, t)
    (e1, e2, e3, e1norm, e2norm, e3norm) = getEdge(p, t)
   e1cos = vec(sum(-e2 .* e3, 1)) ./ e2norm ./ e3norm
   e2cos = vec(sum(-e3 .* e1, 1)) ./ e3norm ./ e1norm
   e3cos = vec(sum(-e1 .* e2, 1)) ./ e1norm ./ e2norm
    r1 = zeros(Float64, size(t))
    r2 = zeros(Float64, size(t))
    r3 = zeros(Float64, size(t))
    for i in 1:size(t)[2]
      r1[:,i] = cross(e2[:,i], e3[:,i])
      r2[:,i] = cross(e3[:,i], e1[:,i])
      r3[:,i] = cross(e1[:,i], e2[:,i])
    end
   e1sin = vec(sqrt(sum(r1.^2,1))) ./ e2norm ./ e3norm
   e2sin = vec(sqrt(sum(r2.^2,1))) ./ e3norm ./ e1norm
   e3sin = vec(sqrt(sum(r3.^2,1))) ./ e1norm ./ e2norm
   e1cot = e1cos ./ e1sin;
   e2cot = e2cos ./ e2sin;
   e3cot = e3cos ./ e3sin;
   return vec(e1cot), vec(e2cot), vec(e3cot)
end

# Function to make Laplacian matrix
function WCot(p, t)
    n = size(p)[2]
    (e1cot, e2cot, e3cot) = mesh_cot(p, t);
    I = [vec(t[2,:]); vec(t[3,:]); vec(t[1,:])]
    J = [vec(t[3,:]); vec(t[1,:]); vec(t[2,:])]
    S = [e1cot; e2cot; e3cot]
    W = sparse(I,J,S, n, n)
    #Wfull = Float64[ W[i,j] for i in 1:n, j in 1:n]
    return W
```

```
end

# Function to make Laplacian matrix
function LCot(p, t)
    n = size(p)[2]
    W = WCot(p,t)
    W = W + transpose(W)
    L = sparse(1:n, 1:n, vec(sum(W,2))) - W
    #Lfull = Float64[ L[i,j] for i in 1:n, j in 1:n]
    return L
end

# An ad hoc way to get an initial x0 for the augrumented Lagrangian
   method
# It is called Tutte embedding.
function Ltut(p, t)
  n = size(p)[2]
    I = [vec(t[2,:]); vec(t[3,:]); vec(t[1,:])]
    J = [vec(t[3,:]); vec(t[1,:]); vec(t[2,:])]
    S = ones(length(I))
    W = sparse(I,J,S, n, n)
    W = W + transpose(W)
    L = sparse(1:n, 1:n, vec(sum(W,2))) - W

    boundary = getBoundary(t)
    nb = length(boundary)
    theta = (2pi/nb) * (0:(nb-1))

  b1 = zeros(Float64, n)
  b2 = zeros(Float64, n)

  for k = 1:nb
    b1 = b1 - L[:, boundary[k]] * cos(theta[k])
    b2 = b2 - L[:, boundary[k]] * sin(theta[k])
  end
  b1[boundary] = cos(theta)
  b2[boundary] = sin(theta)

  Lw = copy(L)
  Lw[:, boundary] = 0
  Lw[boundary,:] = 0
  for k = 1:nb
    Lw[boundary[k],boundary[k]] = 1
  end

  uv = Lw \ [b1 b2]
  return uv
end

# A type which stores all the things we need for the optimization problem
immutable ConfOpt
    Q::SparseMatrixCSC{Float64,Int64}
    H::Array{SparseMatrixCSC{Float64,Int64},1}
    A::Array{SparseMatrixCSC{Float64,Int64},1}
    b::Array{Float64,1}
    n::Int64
    m::Int64
    objFun::Function
    objGrad::Function
    objHess::Function
    constrFun::Function
    constrGrad::Function
    constrHess::Array{Function,1}
    augLagFun::Function
    augLagGrad::Function
```

```
    augLagHess::Function
    x0::Array{Float64,1}
    p::Array{Float64,2}
    t::Array{Int64,2}
    boundary::Array{Int64,1}
    boundary_3point::Array{Int64,1}
    boundary_3point_v::Array{Float64,2}
end

# Type constructor for the type ConfOpt
function ConfOpt(p::Array{Float64,2}, t::Array{Int64,2})
    n = size(p,2)
    boundary = getBoundary(t)
    nb = length(boundary)

    Qraw = LCot(p,t)
    Q = [Qraw zeros(n,n); zeros(n,n) Qraw]

  H = Array{Float64,2}[]
  for i in 1:nb
    G = zeros(Float64, 2, 2n)
    G[1,boundary[i]] = 1
    G[2,n + boundary[i]] = 1
    Hi = G.' * G
    push!(H, Hi)
  end

    objFun = function(x::Array{Float64,1}) 0.5 * getindex(x.' * Q * x, 1)
         end
    objGrad = function(x::Array{Float64,1}) Q * x end
    objHess = function(x::Array{Float64,1}) Q end

    theta = (2pi/3) * [0;1;2]
    boundary_3point = boundary[[1, 1 + div(nb,3), 1 + div(nb,3)*2]]
    boundary_3point_v = [cos(theta).'; sin(theta).']

    A = [zeros(Float64, 1, 2n) for i in 1:6]
    b = zeros(Float64, 6)
    for i in 1:3
      A[i][1, boundary_3point[i]] = 1
      A[3+i][1, boundary_3point[i] + n ] = 1
      b[i] = cos(theta[i])
      b[3+i] = sin(theta[i])
    end

    constrFun = function(x::Array{Float64,1})
      [Float64[ 0.5* ((x.' * H[i] * x) - 1)[] for i in 1:nb];
      Float64[ ((A[i] * x) - b[i])[] for i in 1:6]]
  end
    constrGrad = function(x::Array{Float64,1})
      J = zeros(Float64, 2n, nb+6)
      for i in 1:nb
        J[:,i] = H[i] * x
      end
      for i in 1:6
        J[:, nb + i] = A[i].'
      end
      return J
  end
    constrHess = Function[]
    for i in 1:nb
      fun = function(x::Array{Float64,1}) H[i] end
      push!(constrHess, fun)
    end
    for i in 1:6
```

```
      fun = function(x::Array{Float64,1}) zeros(Float64,2n,2n) end
      push!(constrHess, fun)
    end

    augLagFun = function(x::Array{Float64,1}, y::Array{Float64,1}, rho::
        Float64)
      objFun(x) - dot(y, constrFun(x)) + 0.5rho * dot(constrFun(x),
          constrFun(x))
  end
  augLagGrad = function(x::Array{Float64,1}, y::Array{Float64,1}, rho::
      Float64)
      objGrad(x) - constrGrad(x) * y + rho * constrGrad(x) * constrFun(x)
  end
  augLagHess = function(x::Array{Float64,1}, y::Array{Float64,1}, rho::
      Float64)
    Q - sum(H .* y[1:nb]) + rho * sum(H .* constrFun(x)[1:nb]) + rho *
        constrGrad(x) * transpose(constrGrad(x))
  end

  uv = Ltut(p, t)
  x0 = [uv[:,1];uv[:,2]]

    return ConfOpt(Q, H, A, b, 2n, nb+6,
          objFun, objGrad, objHess,
          constrFun, constrGrad, constrHess,
          augLagFun, augLagGrad, augLagHess,
          x0, p, t, boundary, boundary_3point, boundary_3point_v)
end

# Compute the augrumented Lagrangian
function augLagObj(prob::ConfOpt, y::Array{Float64,1}, rho::Float64)
  augLagFun = function(x::Array{Float64,1})
      prob.objFun(x) - dot(y, prob.constrFun(x)) + 0.5rho * dot(prob.
          constrFun(x), prob.constrFun(x))
  end
  augLagGrad = function(x::Array{Float64,1})
      prob.objGrad(x) - prob.constrGrad(x) * y + rho * prob.constrGrad(x)
           * prob.constrFun(x)
  end
  augLagHess = function(x::Array{Float64,1})
    prob.Q - sum(prob.H .* y[1:(prob.m-6)]) + rho * sum(prob.H .* prob.
        constrFun(x)[1:(prob.m-6)]) + rho * prob.constrGrad(x) *
        transpose(prob.constrGrad(x))
  end

  obj = function(x::Array{Float64,1})
    return augLagFun(x), augLagGrad(x), augLagHess(x)
  end

  return obj
end
```

8.2 Newton's methods and augmented Lagrangian method

```
# Modified Newton method
# Modify the eigenvalues to be positive
function BkFunInv(H,   )
    D, V = eig(Symmetric(H))
    Dp = ifelse(D .>   , D,   )
    Dpinv = 1./Dp
    return V * Diagonal(Dpinv) * V'
end

# Back tracking method
```

```
function backTracking(obj, x, d, g)
       = 1
    while (obj(x +   *d)[1] > obj(x)[1] +    * 1e-4 * dot(g, d))
          =    * 0.5
       if    < 1e-6
           break
       end
    end
    return
end

# Newton Method
function newtmin(obj, x0; maxIts=1000, optTol=1e-6, BkFlag = false,
    btFlag = true)
    # BkFlag (optional): true for doing Modified Hessian
    # btFlag (optional): true for doing Back Tracking
    f0, g0, H0 = obj(x0)
    its = 0
    Opt = Float64[]
    xkp = copy(x0)
    for i in 1:maxIts
        xk = copy(xkp)
        fk, gk, Hk = obj(xk)
        opt = norm(gk, 2)
        push!(Opt, opt)
        #if opt < optTol*norm(g0)
        if opt < optTol
            break
        end
        if BkFlag == true
            Bkinv = BkFunInv(Hk, 0.01)
            dk = Bkinv * (- gk)
        else
            dk = Hk \ (- gk)
        end
        if btFlag == true
             k  = backTracking(obj, xk, dk, gk)
        else
             k  = 1
        end
        xkp = xk +   k  * dk
        its = its + 1
    end
    return xkp, its, Opt
end

function AugLag(prob::ConfOpt; maxItr=20, optTolck = 1e-8, optTolKKT = 1e
    -8, rho0 = 10.0)
  y0 = zeros(Float64, prob.m)

  rhok = rho0
  yk = copy(y0)
  xk = copy(prob.x0)
    #xk = zeros(Float64, prob.n)

    rho_counter = 0
  counter = 0

    @printf "%6s %6s %18s %18s" "Itr" "rho" "Norm Constr" "Norm AugLag
        Grad\n"
    storage_cnorm = Float64[]
    storage_lnorm = Float64[]
  for itr = 1:maxItr
```

```
    xkp = newtmin(augLagObj(prob, yk, rhok), xk, maxIts=20, optTol = (1e
        -4 / 10^itr ) )[1]
        push!(storage_cnorm, norm(prob.constrFun(xkp)))
        push!(storage_lnorm, norm(prob.objGrad(xk) - prob.constrGrad(xk)
            * yk))
        @printf "%6d %6.0f %18.12f %18.12f\n" itr rhok storage_cnorm[end]
             storage_lnorm[end]
            if norm(prob.constrFun(xkp)) < 1e-3 * (1/2)^rho_counter
      rhokp = rhok
      ykp = yk - rhokp * prob.constrFun(xkp)
      rho_counter = rho_counter + 1
    else
      rhokp = 2.0 * rhok
      ykp = copy(yk)
    end

    rhok = rhokp
    yk = copy(ykp)
    xk = copy(xkp)

    counter = itr
    if (norm(prob.constrFun(xkp)) < optTolck) & (norm( prob.objGrad(xk) -
         prob.constrGrad(xk) * yk) < optTolKKT)
      break
    end
  end

    return xk, yk, rhok, counter, storage_cnorm, storage_lnorm
end
```

8.3 How to run the codes

```
f = readdlm("testr_f.dat", ',', Int64)
v = readdlm("testr_v.dat", ',', Float64)

f = f.';
v = v.';

myprob = ConfOpt(v, f);
result_x = AugLag(myprob, maxItr=50);
```