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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(6 Marks)
Consider the following Python code:

```
def mystery(L):
    '''
    :param L: List of size n
    :return: A mystery number
    '''
    sum1 = 0
    sum2 = 0
    bound = 1
    while bound <= len(L):
        i = 0
        while i < bound:
            j = 0
            while j < len(L):
                if L[j] > L[i]:
                    sum1 = sum1 + L[j]
                j = j + 2
            j = 1
            while j < len(L):
                sum2 = sum2 + L[j]
                j = j*2
            i = i + 1
        bound = bound * 2
    return sum1 + sum2
```

1. (3 Marks) Denote the time complexity of the given code $T(n)$ as a function of n where n is the size of the list L. Compute $T(n)$. Justify all steps.
2. (3 Marks) Prove that $T(n) \in O(n^{\frac{5}{2}})$.
   Hint: You can use without proof the following: $\forall \alpha \in \mathbb{R}^+ : \log_2 n \in O(n^\alpha)$.

Solution

1. Let $len(L) = n$

    1. Loop while j < n
       it iterates $\frac{n}{2}$ times with 3 lines inside
    2. Loop while j < n
       it iterates $\sqrt{n} - 1$ times with 2 lines inside
    3. Loop while i < bound
       iterates $log_2 n$ times with 5 lines and 2 loops inside
    4. Loop while bound $\leq n$
       iterates $\sqrt{n}$ times with 3 lines and one loop inside
    5. There are 5 lines outside the loop
       So:
       $$5 + \sqrt{n}(3 + \log_2 n(5 + \frac{3}{2}n + 2\sqrt{n} - 2)) =$$
       $$5 + \sqrt{n}(3 + 5 \cdot \log_2 n + \frac{3}{2}n \cdot \log_2 n + 2\sqrt{n} \cdot \log_2 n - 2\log_2 n) =$$
       $$5 + 3\sqrt{n} + 5 \cdot \log_2 n\sqrt{n} + \frac{3}{2}n \cdot \log_2 n\sqrt{n} + 2n \cdot \log_2 n - \sqrt{n} \cdot 2\log_2 n$$

2. # Need to prove if $f \in O(h)$ and $g \in O(h)$ $then (f + g) \in O(h)$
   if $f \in O(h)$ then $\exists C_1 \in \mathbb{R}^+ : [\exists B_1 \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B_1 \implies f \leq C_1 h]]$
   if $g \in O(h)$ then $\exists C_2 \in \mathbb{R}^+ : [\exists B_2 \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B_2 \implies g \leq C_2 h]]$
   Let $C_3 = \max(C_1, C_2)$
   Let $B_3 = \max(B_1, B_2)$
   Then $f + g \leq C_3 \cdot h$, for $n \geq B_3$
   Therefore, $f \in O(h) \wedge g \in O(h) \implies (f + g) \in O(h)$ # Let this definition be *

prove that equation from 1a is in $O(n^{\frac{5}{2}})$

$5 + 3\sqrt{n} + 5 \cdot \log_2 n\sqrt{n} + \dfrac{3}{2}n \cdot \log_2 n\sqrt{n} + 2n \cdot \log_2 n - \sqrt{n} \cdot 2\log_2 n \geq$

$\geq 5n + 3n + 5 \cdot \log_2 n \cdot n + 2n \cdot \sqrt{n} \cdot \sqrt{n} \geq$

$\geq 8n + 5 \cdot n \cdot \log_2 n + 2n^2$

prove 8n $\in O(n^{\frac{5}{2}})$

Let $c_0 = 8$

Then $c_0 \in \mathbb{R}^+$

Let $B_0 = 0$

Then $B_0 \in \mathbb{N}$

Then $8n \leq 8n^2 \leq c_0 \cdot n^2 \leq c_0 \cdot n^{\frac{5}{2}}$

Therefore, $\exists c \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies 8n \leq c(n^{\frac{5}{2}})]]$

Therefore, $8n \in O(n^{\frac{5}{2}})$

prove $2n^2 \in O(n^{\frac{5}{2}})$

Let $c_1 = 2$

Then $c_1 \in \mathbb{R}^+$

Let $B_1 = 0$

Then $B_1 \in \mathbb{N}$

Then $2n^2 \leq c_1 \cdot n^2 \leq c_1 \cdot n^{\frac{5}{2}}$

Therefore, $\exists c \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies 2n^2 \leq c(n^{\frac{5}{2}})]]$

Therefore, $2n^2 \in O(n^{\frac{5}{2}})$

prove $5n \cdot \log_2 n \in O(n^{\frac{5}{2}})$

Let $c_2 = 5$

Then $c_2 \in \mathbb{R}^+$

Let $B_2 = 1$

Then $B_2 \in \mathbb{N}$

Then $5n \cdot \log_2 n \leq 5n \cdot n \leq 5n^2 \leq c_2 \cdot n^2 \leq c_2 \cdot n^{\frac{5}{2}}$

Therefore, $\exists c \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies 5n \cdot \log_2 n \leq c(n^{\frac{5}{2}})]]$

Therefore, $5n \cdot \log_2 n \in O(n^{\frac{5}{2}})$

Let $c_3 = max(c_0, c_1, c_2)$

Then $c_3 = 8$

Let $B_3 = max(B_0, B_1, B_2)$

Then $B_3 = 1$

Then, by *

$\exists c \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies 8n + 5 \cdot n \cdot \log_2 n + 2n^2 \leq c(n^{\frac{5}{2}})]]$

Therefore, $\exists c \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies$

$(5 + 3\sqrt{n} + 5 \cdot \log_2 n\sqrt{n} + \frac{3}{2}n \cdot \log_2 n\sqrt{n} + 2n \cdot \log_2 n - \sqrt{n} \cdot 2\log_2 n) \leq c(n^{\frac{5}{2}})]]$

Therefore, $5+3\sqrt{n}+5\cdot\log_2 n\sqrt{n}+\frac{3}{2}n\cdot\log_2 n\sqrt{n}+2n\cdot\log_2 n-\sqrt{n}\cdot 2\log_2 n \in O(n^{\frac{5}{2}})$

Problem 2.

(6 Marks) Using the appropriate definitions, prove the following:

1. (3 Marks)
$$7n^2 + 77n + 1 \in \Theta(n^2 + n + 165).$$

2. (3 Marks)
$$n \log(n^7) + n^{\frac{7}{2}} \in O(n^{\frac{7}{2}}).$$

Solution

1. # Need to prove
$$7n^2 + 77n + 1 \in O(n^2 + n + 165).$$
and
$$7n^2 + 77n + 1 \in \Omega(n^2 + n + 165).$$

prove for big O

$\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies 7n^2 + 77n + 1 \leq C(n^2 + n + 165)]]$

Let $C_0 = 85$

Then $C_0 \in \mathbb{R}^+$

Let $B_0 = 0$

Then $B_0 \in \mathbb{N}$

Assume $n \geq B_0, n \geq 0$

Then $7n^2 + 77n + 1 \leq 7n^2 + 77n^2 + n^2 = 85n^2$
$= C_0 n^2 \leq C_0(n^2 + n) \leq C_0(n^2 + n + 165)$

Then $7n^2 + 77n + 1 \leq C_0(n^2 + n + 165)$

Then $\forall n \in \mathbb{N} : n \geq B \implies 7n^2 + 77n + 1 \leq C(n^2 + n + 165)$

Therefore, $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies$
$7n^2 + 77n + 1 \leq C(n^2 + n + 165)]]$

Therefore, $7n^2 + 77n + 1 \in O(n^2 + n + 165)$.

Continued...

prove for big Ω

$\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies 7n^2 + 77n + 1 \geq C(n^2 + n + 165)]]$

Let $C_0 = \dfrac{7}{167}$

Then $C_0 \in \mathbb{R}^+$

Let $B_0 = 0$

Then $B_0 \in \mathbb{N}$

Assume $n \geq B_0, n \geq 0$

Then $7n^2 + 77n + 1 \geq 7n^2 + 77n \geq 7n^2 \geq \dfrac{7}{167} \cdot 167n^2$

Then $167n^2 \geq n^2 + n^2 + 165n^2 \geq n^2 + n + 165n \geq n^2 + n + 165$

Then $C_0 167n^2 \geq C_0(n^2 + n + 165)$

Then $7n^2 + 77n + 1 \geq C_0(n^2 + n + 165)$

Then $\forall n \in \mathbb{N} : n \geq B \implies 7n^2 + 77n + 1 \geq C_0(n^2 + n + 165)$

Therefore, $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies 7n^2 + 77n + 1 \geq C(n^2 + n + 165)]]$

Therefore, $7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$

Therefore, $7n^2 + 77n + 1 \in \Theta(n^2 + n + 165)$.

2. Continued...

Need to prove $n\ \log(n^7) \in O(n^{\frac{7}{2}})$ and $n^{\frac{7}{2}} \in O(n^{\frac{7}{2}})$

Proof for $n\ \log(n^7)$

$\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies n\ \log(n^7) \leq C \cdot n^{\frac{7}{2}}]]$

Let $C_0 = 7$

Then $C_0 \in \mathbb{R}^+$

Let $B_0 = 0$

Then $B_0 \in \mathbb{N}$

Assume $n \geq B_0, n \geq 0$

Then $n\ \log(n^{\frac{7}{2}}) \leq 7n\ \log n \leq 7n \cdot n \leq 7n^2 \leq C_0 n^2 \leq C_0 n^{\frac{7}{3,5}} \leq C_0 n^{\frac{7}{2}}$

Then $n\ \log(n^7) \leq C_0 n^{\frac{7}{2}}$

Then $\forall n \in \mathbb{N} : n \geq B \implies n\ \log(n^7) \leq C_0 n^{\frac{7}{2}}$

Therefore, $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies n\ \log(n^7) \leq C \cdot n^{\frac{7}{2}}]]$

Therefore, $n\ \log(n^7) \in O(n^{\frac{7}{2}})$

Proof for $n^{\frac{7}{2}}$

$\exists C \in \mathbb{R} : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies n^{\frac{7}{2}} \leq C \cdot n^{\frac{7}{2}}]]$

Let $C_0 = 1$

Then $C_0 \in \mathbb{R}^+$

Let $B_0 = 1$

Then $B_0 \in \mathbb{N}$

Assume $n \geq B_0, n \geq 0$

Then $n^{\frac{7}{2}} \leq C_0 n^{\frac{7}{2}}$

Then $\forall n \in \mathbb{N} : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies n^{\frac{7}{2}} \leq C \cdot n^{\frac{7}{2}}]]$

Therefore, $\exists C \in \mathbb{R} : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies n^{\frac{7}{2}} \leq C \cdot n^{\frac{7}{2}}]]$

Therefore, $n^{\frac{7}{2}} \in O(n^{\frac{7}{2}})$

Therefore, $n\ \log(n^7) + n^{\frac{7}{2}} \in O(n^{\frac{7}{2}})$ # by *

Problem 3.

(6 MARKS) Let $\mathcal{F} = \{f | f : \mathbb{N} \to \mathbb{R}^+\}$. Using the appropriate definitions, prove or disprove the following:

1. (3 MARKS)

$$\forall f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \implies f(n) \in O(3^{g(n)}).$$

2. (3 MARKS)

$$\forall f \in \mathcal{F} : \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \in O(\sqrt{f(n)}).$$

Solution

1. # Want to disprove by proving its negation

   $\exists f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \wedge f(n) \notin O(3^{g(n)})$

   Let $f(n) = n^3, g(n) = \log n$

   Then $f, g \in \mathcal{F}$

   Let $C_0 = 4$

   Then $C_0 \in \mathbb{R}^+$

   Let $B_0 = 0$

   Then $B_0 \in \mathbb{N}$

   Assume $n \geq B_0, n \geq 0$

   Then $\log f(n) = \log n^3 = 3logn \leq C_0 \log n \leq C_0 g(n)$

   Therefore, $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies \log f(n) \leq Cg(n)]]$

   Therefore, $\log f(n) \in O(g(n))$

Continued...

Let $C \in \mathbb{R}^+$

Let $B \in \mathbb{N}$

Let $n = B + \lceil C \rceil$

Then $n \in \mathbb{N}$

Then $n \geq B$

Then $n > C$

Then $f(n) = n^3 = n \cdot n^2$

Then $n \cdot n^2 > C \cdot n^2$ # since $n > c$

Then $C \cdot 3^{\log_3 n} = Cn$ # property of loagarithm

Then $n^3 > C \cdot n^2 > Cn = C \cdot 3^{\log_3 n}$

Then $\forall C \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists \mathbb{N}, n \geq B \wedge f(n) > C3^{g(n)}$

Then $f(n) \notin O(3^{g(n)})$

Therefore, statement is false

2. Continued...

$\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies \left\lfloor \sqrt{\lfloor f(n) \rfloor} \right\rfloor \leq C \cdot \sqrt{f(n)}$

Let $C_0 = 1$

Then $C_0 \in \mathbb{R}^+$

Let $B_0 = 0$

Then $B_0 \in \mathbb{N}$

Assume $n \geq B_0, n \geq 0$

Then $f(n) \geq 0$

Then $f(n) - 1 < \lfloor f(n) \rfloor \leq f(n)$ # from class

Then $\lfloor f(n) \rfloor \leq f(n)$

Then $\sqrt{\lfloor f(n) \rfloor} \leq \sqrt{f(n)}$

Then $\sqrt{\lfloor f(n) \rfloor} - 1 < \left\lfloor \sqrt{\lfloor f(n) \rfloor} \right\rfloor \leq \sqrt{\lfloor f(n) \rfloor}$ # from class

Then $\left\lfloor \sqrt{\lfloor f(n) \rfloor} \right\rfloor \leq \sqrt{\lfloor f(n) \rfloor} \leq \sqrt{f(n)}$

Then $\left\lfloor \sqrt{\lfloor f(n) \rfloor} \right\rfloor \leq C_0 \sqrt{f(n)}$

Then $\forall n \in \mathbb{N} : n \geq B \implies \left\lfloor \sqrt{\lfloor f(n) \rfloor} \right\rfloor \leq C \cdot \sqrt{f(n)}$

Therefore, $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies$
$\left\lfloor \sqrt{\lfloor f(n) \rfloor} \right\rfloor \leq C \cdot \sqrt{f(n)}$

Therefore, $\forall f \in \mathcal{F} : \left\lfloor \sqrt{\lfloor f(n) \rfloor} \right\rfloor \in O(\sqrt{f(n)})$

Problem 4.

(6 Marks) Recall that $n! = 1 \cdot 2 \ldots n$. Also, by convention, $0! = 1$. Using the method of mathematical induction, prove the following:

1. (3 Marks)
$$\forall n \in \mathbb{N} : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1.$$

2. (3 Marks)
$$\forall n \in \mathbb{N} : n \geq 1 \to 2^n \leq 2^{n+1} - 2^{n-1} - 1.$$

Solution

1. Continued...

Proof by induction

Let $P(n)$ denote: $\forall n \in \mathbb{N} : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$

<u>Base Case</u>

$i = 0$

$0 \cdot 0! = 0 \cdot 1 = 0$ # LHS

$(0+1)! = 1! - 1 = 1 - 1 = 0$ # RHS

Therefore, $P(0)$ # LHS = RHS

<u>Inductive Step</u>

$\forall k \in \mathbb{N} : k \geq 0 : P(k) \implies P(k+1)$

Assume $P(k)$

Then $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$

Then $i \cdot i! + (i+1)(i+1)! ... k \cdot k! = (k+1)! - 1$

Prove $P(k+1)$

Then $i \cdot i! + (i+1)(i+1)! ... k \cdot k! + (k+1)(k+1)! = (k+2)! - 1$

Then $(k+2)! - 1 = (k+2)(k+1)k! - 1$ # RHS

Then $i \cdot i! + (i+1)(i+1)! ... k \cdot k! + (k+1)(k+1)! =$

$= (k+1)! - 1 + (k+1)(k+1)!$ # substitution P(k)

$$
\begin{aligned}
\text{Then } (k+1)! + (k+1)(k+1)! - 1 &= (k+1)! + (k+1)(k+1)k! - 1 \\
&= (k+1)! + (k+1)^2 k! - 1 \\
&= (k+1)k! + (k+1)^2 k! - 1 \\
&= (k+1)k! + (1 + k + 1) - 1 \\
&= (k+1)k! + (k+2) - 1 \quad \text{\# LHS}
\end{aligned}
$$

Then LHS = RHS

Therfore, $P(k+1)! - 1$

Then $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$

Therefore, $\forall n \in \mathbb{N} : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$

2. # Proof by induction

Let $P(n)$ denote: $\forall n \in \mathbb{N} : n \geq 1 \rightarrow 2^n \leq 2^{n+1} - 2^{n-1} - 1$

Base Case

$P(1) : 2^1 \leq 2^2 - 2^0 - 1$

$P(1) : 2 \leq 4 - 1 - 1$

$P(1) : 2 \leq 2$

Then $(P1)$

Inductive Step

$\forall k \in \mathbb{N} : k \geq 1 : P(k) \implies P(k+1)$

Let $k \geq 1$

Assume $P(k)$

Then $2^k \leq 2^{k+1} - 2^{k-1} - 1$

Prove for $P(k+1)$

$2^{k+1} = 2 \cdot 2^k$

$2 \cdot 2^k \leq 2 \cdot (2^{k+1} - 2^{k-1} - 1)$ # substitution P(k)

$2^{k+1} \leq 2^{k+2} - 2^k - 2$# P(k+1)

$2^{k+1} \leq 2^{k+2} - 2^k - 1$

Then $2^{k+1} \leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1$

Then $P(k)$

Therefore, $\forall k \in \mathbb{N} : k \geq 1 : P(k) \implies P(k+1)$

Therefore, $\forall n \in \mathbb{N} : n \geq 1 \rightarrow 2^n \leq 2^{n+1} - 2^{n-1} - 1$