Intrinsic Morse Functions on Grassmannians

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Overview

Motivation from Multivariable Calculus

- Morse theory focuses on extracting topological properties on (smooth) manifolds by looking at behavior of critical points of (Morse) functions
- Example: A metric space (X, d) is compact iff all continuous functions $f : X \to \mathbb{R}$ are bounded

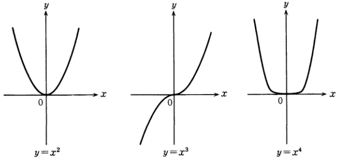


Figure: Graphs of x^n for $n = 2, 3, 4$

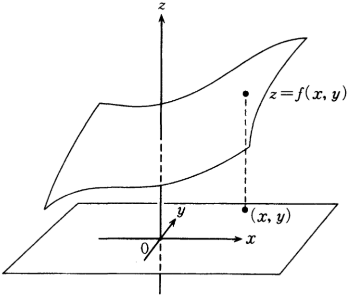


Figure: Some surface

Definitions

Let M be an m-manifold (second-countable, Hausdorff) and $f : M \to \mathbb{R}$ be smooth.

Definition (Critical points)

A point p of M is a critical point of f if we have

$$\frac{\partial f}{\partial x_1}(p) = \frac{\partial f}{\partial x_2}(p) = \cdots = \frac{\partial f}{\partial x_m} = 0$$

with respect to a local coordinate system $(x_1, \ldots, x_m)$ about p.

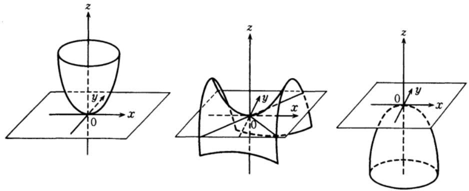
Remark: Choice of local coordinates does not change the critical points

Definition (Hessian)

We define the Hessian of f at a critical point p to be the matrix of second derivatives, shown below.

$$H_f(p_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(p) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(p) & \cdots & \frac{\partial^2 f}{\partial x_m^2}(p) \end{bmatrix}$$

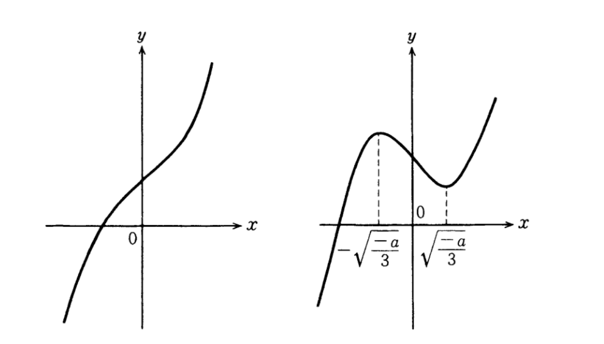
Remark: Notice that the Hessian is symmetric and its determinant remains invariant under change of coordinates.



Definition (Degeneracy)

A critical point p_0 is non-degenerate if $\det H_f(p_0) \neq 0$ and degenerate if $\det H_f(p_0) = 0$.

Example



Consider the above function $y = x^3$. If we introduce a small perturbation, the unstable critical point might vanish, or split!

Definition (Morse function)

Suppose that every critical point of a function $f : M \to \mathbb{R}$ is non-degenerate. Then, we say f is a Morse function on M.

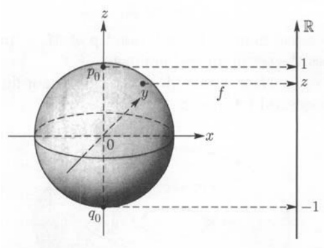


Figure: Height function $f : S^2 \to \mathbb{R}$

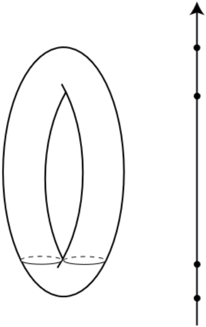


Figure: Height function on sideways torus

Generic Existence

Theorem (Genericity of Morse Functions)

Suppose that $M \subset \mathbb{R}^n$ is a submanifold. Then, for almost every point $p \in \mathbb{R}^n$, the function $f_p : M \to \mathbb{R}$ sending

$$x \to ||x - p||^2$$

is a Morse function.

- In fact, any C^∞ function on M can be uniformly approximated by some Morse function on each compact subset (including its derivatives).
- There are many possible Morse functions, yet may be difficult to show that a particular function is explicitly.

Two Nice Results

Theorem (Reeb's Theorem)

Let M be a compact manifold. If there exists a Morse function f with exactly two critical points, then M is homeomorphic to the n-sphere S^n.

Theorem (Morse Lemma)

Let p be a non-degenerate critical point of f. Then, there exists a local coordinate system $(X_1, \ldots, X_m)$ around p, such that

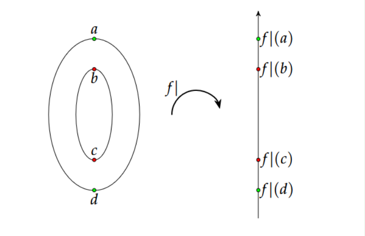
$$f = f(p_0) - X_1^2 - X_2^2 - \cdots - X_i^2 + X_{i+1}^2 + \cdots + X_m^2$$

We say that p is the origin $(0, \ldots, 0)$. Furthermore, we remark that i above is called the index of a critical point p.

Theorem

Given an manifold M and Morse function f, define $M_a = f^{-1}((-\infty, a])$. If f has no critical values in the real interval $[a, b]$, then M_a and M_b are diffeomorphic.

Example



$$\mathbb{T}^2_a \begin{cases} = \varnothing & \text{if } a < f|(d), \\ \cong \mathbb{D}^2 & \text{if } f|(d) < a < f|(c), \\ \cong \mathbb{S}^1 \times I \text{ (a cylinder)} & \text{if } f|(c) < a < f|(b), \\ \cong \mathbb{T}^2 \setminus \mathring{\mathbb{D}}^2 & \text{if } f|(b) < a < f|(a), \\ = \mathbb{T}^2 & \text{if } a > f|(a). \end{cases}$$

The topological data only changes at the critical points!

Morse Homology

Definition (Index of a critical point)

The index of a critical point p is the number of negative eigenvalues of the Hessian at that point. Equivalently, we can define it using the statement in the Morse lemma.

Remark: Sylvester's theorem helps us show that the index is well-defined.

Theorem (Handle-body Decomposition)

The structure of a handle-body on M is determined by the choice of Morse function f. Each k-handle ($D^k \times D^{m-k}$) corresponds to a critical point of index k.

Remark: Handle-body representations give us a great way of constructing manifolds.

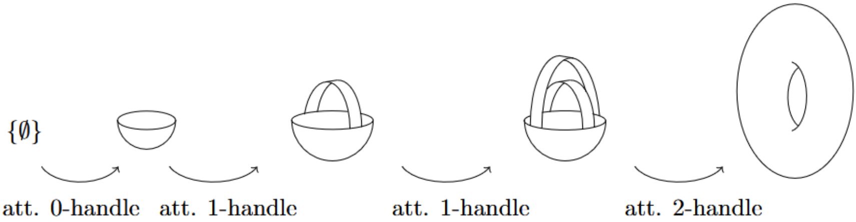


Figure: Construction of a torus

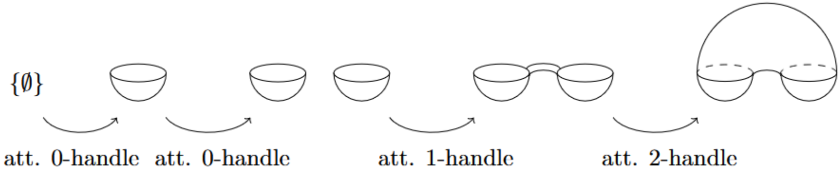


Figure: Construction of S^2

Definition (Morse Homology)

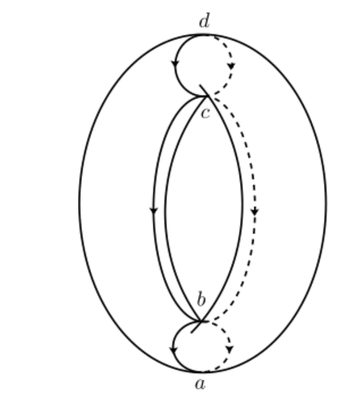
Define a vector space $C_k(f) = \{\sum_{c \in \mathrm{Crit}_k(f)} a_c c : a_c \in \mathbb{Z}\}$. Let's define boundary map ∂ send the critical point p of index k

$$\partial(p) = \sum_{b \in \mathrm{Crit}_{k-1}} n(p, b) b,$$

where $n(p, b)$ is the number of trajectories between p to b. Then, our Morse homology groups are $H_k(f, X) = \mathrm{Ker}\ \partial / \mathrm{Im}\ \partial$

- Equivalent to cellular homology and uses flows
- Independent of the choice of Morse function and gradient-like vector field assuming it satisfies the Smale condition.
- Provides the right perspective for homology theories of infinite-dimensional manifolds
- Remark: See Audin-Damian for more details

Example



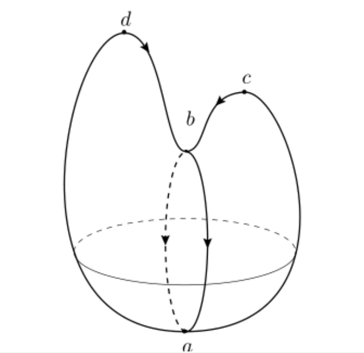


Figure: Torus and "Other" Sphere with flow lines

Grassmannians as a Manifold

Definition

Let $G_k(V^n)$ be the set of k-dimensional subspaces of an n-dimensional vector space V. This is called the Grassmannian.

- For our purposes, we will mainly focus on $V = \mathbb{R}$.
- $G_k(\mathbb{R}^n)$ is a smooth manifold of dimension $k(n-k)$
- Can be seen as homogeneous space

$$G_k(\mathbb{R}^n) \cong O(n)/O(r) \times O(n-r)$$

- Is compact by embedding into $\prod_{i=1}^{k} \mathbb{R}^n$ & using Heine-Borel
- $G_k(\mathbb{R}^n) \cong G_{n-k}(\mathbb{R}^n)$, where the isomorphism is given by the orthogonal complement map sending $X \to X^{\perp}$

Our Problem

- We know abstractly that Morse functions exist on any Grassmannian $G_k(\mathbb{R}^n)$. In the literature, people have done this by embedding in some sufficiently large $\mathbb{R}^n$, then taking an appropriate ”height-like” function
- Goal: Do this intrinsically in coordinates to get something concrete
- Inspiration: Consider $G_2(\mathbb{R}^4) \cong S^2 \times S^2$.

Set-Up

- Let V, W be 2-planes with O.N. bases $\{v_1, v_2\}$, $\{w_1, w_2\}$ respectively. Define

$$f(V, W) = \det \begin{bmatrix} \langle v_1, w_1 \rangle & \langle v_1, w_2 \rangle \\ \langle v_2, w_1 \rangle & \langle v_2, w_2 \rangle \end{bmatrix}$$

  Remark: f is independent of choice of oriented basis as rotation matrices (elements of $SO(2)$) have determinant 1. Thus, f is really a function of the planes V, W

- Let $\{e_i\}_{i=1}^n$ be an O.N. basis of $\mathbb{R}^n$ and $V_j = \text{span}\{e_{2j-1}, e_{2j}\}$, where j ranges from 1 to $k = \lfloor \frac{n}{2} \rfloor$

  Remark: One can think of the V_j as fixed planes we are measuring a ”distance” from

Morse Functions for $G_k(\mathbb{R}^n)$

Theorem (Morse function for $G_2(\mathbb{R}^n)$)

Suppose we have the set-up from the last slide. For any 2-plane $X \in G_k(\mathbb{R}^n)$, define

$$g(X) = a_1 f(V_1, X) + a_2 f(V_2, X) + \cdots + a_k f(V_k, X)$$

Let $0 \leq a_1 < a_2 < \cdots < a_k < 1$. Then, $g(X)$ is a Morse function. Furthermore,

- *The critical points are precisely $X = \pm V_j$*
- *The critical point V_j has index $2(n-2) - 2(j-1) = 2n - 2j - 2$ and $-V_j$ has index $2(j-1)$ for $1 \leq j \leq k$*

Outline of Proof

The proof involves three major steps.

1. Show $\{\pm V_j\}$ are actually non-degenerate critical points
2. Calculate the index of these critical points
3. Prove no other critical points exist

We will walk through part of the illustrative case of $G_2(\mathbb{R}^4)$. Full details are given in my write-up.

V_1 is a non-degenerate critical point.

1. Looking at $f(V_1, X)$, we can put local coordinates about V_1 corresponding to $\{v_1, v_2\} = \{[1, 0, a, b], [0, 1, c, d]\}$, where both $[0, 0, a, b]$ and $[0, 0, c, d]$ are perpendicular to V_1.
2. We use Gram-Schmidt to turn this into an O.N. basis. Importantly, during the calculation, we only care about the terms up to second order.

$$x_1 = \frac{v_1}{||v_1||} = [1 - \frac{1}{2}(a^2 + b^2), 0, a, b] + O(r^3)$$

$$x_2 = \frac{v_2 - \langle x_1, v_2 \rangle v_1}{||v_2 - \langle x_1, v_2 \rangle v_1||} = [-ac - bd, 1 - \frac{1}{2}(c^2 + d^2), c, d] + O(r^3)$$

$$\implies f(V_1, X) = 1 - \frac{1}{2}(a^2 + b^2) - \frac{1}{2}(c^2 + d^2) + O(r^3)$$

3. Thus, V_1 corresponds to a critical point.

□

V_1 is a critical point of index 4.

The Hessian of f at V_1 is given by $H_f(V_1) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
Thus, V_1 is a critical point of index 4, meaning maxima. Notice that $-V_1$ only swaps some columns, changing only the sign of f. Thus, $-V_1$ is a minima, a critical point of index 0. $\square$

$$G_2(\mathbb{R}^6) : \begin{bmatrix} -a_1 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & -a_1 & -a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_2 & -a_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_1 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & -a_1 & -a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_3 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & 0 & -a_1 \end{bmatrix}$$

No other critical points exist.

Let X be a 2-plane with O.N. basis $\{x_1, x_2\}$. We can compute the directional derivative of g at X by choosing rotations R satisfying $R(0) = I$ and $R'(0) = S$, where S is skew-symmetric matrix. If we can find S, such that $\frac{d}{dt}|_{t=0} g(R(t)X) \neq 0$, then X cannot be a critical points (locally, we have a direction increasing g!) $\square$

- To extend the proof for $n \geq 4$, you need to do a similar analysis for $f(V_j, X)$. The role of α_i are in the first and second step (particularly so that the eigenvalues are non-zero).
- By using the homeomorphism given by orthogonal complement, we get some data for $G_{n-2}(\mathbb{R}^n)$

Homology of Grassmannians

With a Morse function whose critical points and indices are known, we can use our knowledge of Morse homology to get the following corollary. Recall that $\dim G_2(\mathbb{R}^n) = 2(n-2) = 2n-4$.

Corollary

If n is odd, the homology (in $\mathbb{R}$ coefficients) of $G_2(\mathbb{R}^n)$ is

$$H_k(G_2(\mathbb{R}^n); \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0, 2, \ldots, 2n-4 \\ 0 & else \end{cases}$$

If n is even, the homology (in $\mathbb{R}$ coefficients) of $G_2(\mathbb{R}^n)$ is

$$H_k(G_2(\mathbb{R}^n); \mathbb{R}) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & k = n-2 \\ \mathbb{R} & k = 0, 2, \ldots n-4, n, \ldots, 2n-4 \\ 0 & else \end{cases}$$

Example

In the case $G_2(\mathbb{R}^4)$, we have $H_0 = \mathbb{R}$, $H_2 = \mathbb{R} \oplus \mathbb{R}$ and $H_4 = \mathbb{R}$. This coincides with the homology of $S^2 \times S^2 \cong G_2(\mathbb{R}^4)$.

Proof.

Critical points represent the generator in the homology in degree corresponding to its index. We can count $2k$ critical points of the form $\{\pm V_j\}_{j=1}^k$. Recall the index of V_j is $2n - 2j - 2$, while $-V_j$ is $2j - 2$.

- For n odd, all the indices have distinct values. The index is always even and ranges from 0 to $2(n-2)$ for $j \in [1, \frac{n-1}{2}]$.
- For n even, similar reasoning holds. Note however, that when $j = \frac{1}{2}n$, V_j and $-V_j$ have the same index $n-2$, resulting in $H_{n-2} = \mathbb{R} \oplus \mathbb{R}$.

$\square$

Main References

- ”Morse Theory and Floer Homology” by Audin and Damian
- ”Morse Theory” by Milnor
- ”An Introduction to Morse Theory” by Matsumoto
- ”The wild world of 4-manifolds” by Scorpan
- ”Riemmanian Geometry” by do Carmo
- ”Morse Theory Indomitable” by Bott

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Questions