An Introduction to Generating Functions

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What is a generating function?

A generating function $f(x)$ of a sequence $\{a_n\}_{n=0}^{\infty}$ is $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Example: Fibonacci Sequence

Fibonacci Sequence

Recall that F_n denotes the n-th Fibonacci number, where $F_0 = 0$ and $F_1 = 1$, with the recurrence

$$F_{n+2} = F_{n+1} + F_n$$

We want to find a function $f(x) = \sum_{n=0}^{\infty} F_n x^n$.

Example: Fibonacci Sequence, part 2

$$\sum_{n=0}^{\infty} F_{n+2}x^n = \sum_{n=0}^{\infty} F_{n+1}x^n + \sum_{n=0}^{\infty} F_n x^n$$

$$\implies \sum_{n=2}^{\infty} F_n x^{n-2} = \sum_{n=1}^{\infty} F_n x^{n-1} + \sum_{n=0}^{\infty} F_n x^n$$

$$\implies \frac{1}{x^2}\sum_{n=2}^{\infty} F_n x^n = \frac{1}{x}\sum_{n=1}^{\infty} F_n x^n + \sum_{n=0}^{\infty} F_n x^n$$

$$\implies \frac{1}{x^2}\sum_{n=0}^{\infty} F_n x^n - \frac{F_0}{x^2} - \frac{F_1}{x} = \frac{1}{x}\sum_{n=0}^{\infty} F_n x^n - \frac{F_0}{x} + \sum_{n=0}^{\infty} F_n x^n$$

$$\implies \frac{f(x)}{x^2} - \frac{1}{x^2} - \frac{1}{x} = \frac{f(x)}{x} - \frac{1}{x} + f(x)$$

$$\implies f(x) = \frac{1}{1-x-x^2}$$

Example: Fibonacci Sequence, part 2

$$\begin{aligned} f(x) &= \frac{1}{1-x-x^2} \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots \end{aligned}$$

Theorem

If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, then, $a_n = b_n$ for all $n \geq 0$. In particular, we can get bijections between two different sequences.

Why study generating functions?

1. Find an exact formula/recurrence relation
2. Proof of Identities
3. Applications of Analytic Tools (Asymptotic Formulas)
   Note: Justification for manipulations rely on the structure of $\mathbb{C}[[x]]$, formal power series in $\mathbb{C}$.

Example: Counting Partitions

Definition

A partition of a positive integer n is expressing it as the sum of positive integers, unique up to ordering.

Theorem (Euler's Method)

The number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts only.

Example: Partition of n=7

Odd Parts	Distinct Parts
$1+1+1+1+1+1+1$	$4+2+1$
$3+1+1+1+1$	$4+3$
$5+1+1$	$5+2$
$3+3+1$	$6+1$
7	7

Example: Counting Partitions, part 2

Counting Proof Sketch.

We exhibit an explicit bijection. Consider some partition into distinct parts $n = d_1 + d_2 + d_3 + \cdots + d_m$. Then, each d_i factors as $2^{k_i} o_i$, where o_i is odd. Thus, we have $n = 2^{k_1} o_1 + 2^{k_2} o_2 + \cdots + 2^{k_m} o_m$. We can regroup the odd numbers together to get

$$\begin{aligned} n &= (2^{\alpha_1} + 2^{\alpha_2} + \dots) \cdot 1 + (2^{\beta_1} + 2^{\beta_2} + \dots) \cdot 3 + \dots \\ &= m_1 \cdot 1 + m_2 \cdot 3 + m_3 \cdot 5 + \dots \end{aligned}$$

This last term describes our partition into odd parts. □

Example: Bijection

$$
\begin{aligned}
5 &= 3 + 2 \\
&= 2^0 \cdot 3 + 2^1 \cdot 1 \\
&= 2^1(1) + 2^0(3) \\
&= 3 + 1 + 1
\end{aligned}
$$

Example: Counting Partitions, part 3

Generating Function Proof.

Why is the generating function for the number of odd partitions for n is $f(x) = \prod_{i \text{ odd}}^{\infty} (\frac{1}{1-x^i})$?

$$f(x) = (\frac{1}{1-x})(\frac{1}{1-x^3})(\frac{1}{1-x^5})\dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + \dots$$

$$\frac{1}{1-x^5} = 1 + x^5 + x^{10} + \dots$$

Example: Counting Partitions, part 4

Generating Function Proof.

The generating function for the number of odd partitions for n is $f(x) = \prod_{i \text{ odd}}^{\infty} (\frac{1}{1-x^i})$. For distinct partitions, we have $g(x) = \prod_{i=1}^{\infty} (1 + x^i)$. Then,

$$\begin{aligned}
\prod_{i \text{ odd}}^{\infty} \frac{1}{1-x^i} &= (\frac{1}{1-x})(\frac{1}{1-x^3})(\frac{1}{1-x^5})\dots \\
&= (\frac{1-x^2}{1-x})(\frac{1-x^4}{1-x^2})(\frac{1-x^6}{1-x^3})\dots \\
&= (1+x)(1+x^2)(1+x^3)\dots \qquad = \prod_{i=1}^{\infty}(1+x^i)
\end{aligned}$$

□

Stirling Numbers

Definition

Stirling numbers of the first kind are the number of permutations on n elements with k cycles, denoted by $\left[{n \atop k}\right]$.

Definition

Stirling numbers of the second kind are the number of partitions of n elements into k non-empty subsets, denoted by $\left\{{n \atop k}\right\}$.

Example: $\left\{{4 \atop 3}\right\} = 6$

$$[\{1\},\{2\},\{3,4\}] \text{ and } [\{1\},\{2,3\},\{4\}] \text{ and } [\{1,2\},\{3\},\{4\}]$$
$$[\{1,4\},\{2\},\{3\}] \text{ and } [\{1,3\},\{2\},\{4\}] \text{ and } [\{1\},\{4,2\},\{3\}]$$

Stirling Numbers

Theorem (Identity for Stirling Numbers of the Second Kind)

$x^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} (x)_k$, *where* $(x)_k = x(x-1)\dots(x-k+1)$, *the falling factorial.*

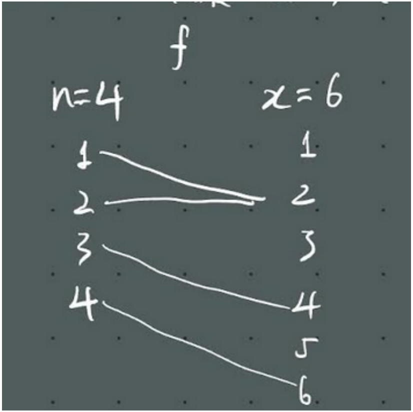
Proof.

Note that the LHS counts functions from $[n] \to [x]$. We can think of the RHS more clearly by rewriting as

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\} (x)_k = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} k! \binom{n}{k}$$

The latter term involves picking the "buckets" in the range, then assigning the elements in the domain. □

Proof pt.2



Some further results

While the examples we have worked with so far are relatively nice, it may be difficult to find an exact expression for the generating function of a sequence. However, this is where analytic tools can help us.

Some further results

We define some auxiliary functions.

$$a(r) = r\frac{f'(r)}{f(r)}$$

$$b(r) = ra'(r) = r\frac{f'(r)}{f(r)} + r^2\frac{f''(r)}{f(r)} - r^2(\frac{f'(r)}{f(r)})^2$$

Theorem (Hayman's Theorem)

Let $f(z) = \sum a_n z^n$ be admissible function. Let r_n be the positive real root of $a(r_n) = n$ for each $n \in \mathbb{Z}^+$. Then,

$$a_n \approx \frac{f(r_n)}{r_n^n\sqrt{2\pi b(r_n)}} \text{ as } n \to \infty$$

More details

Definition

Let $f(z)$ be analytic in a disk $|z| < R$. Define $M(r) = \max_{|z|=r}\{|f(z)|\}$. An admissible function means $M(r) = f(r)$ for sufficiently large r.

A consequence of Hayman's theorem is Stirling's formula.

$$\frac{1}{n!} \approx \frac{e^n}{n^n\sqrt{2n\pi}}$$

Conclusion

1. Generating functions provide a useful technique in our toolbox when standard approaches don't work.
2. They allow us to extract useful information from counting problems where an explicit formula is difficult to write down/work with.
3. They help simplify computations in proofs.

Credits

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