INTRINSIC MORSE FUNCTIONS ON GRASSMANNIANS

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Abstract

The purpose of this thesis is to be a gentle, but rigorous introduction into Morse theory. A relatively recent field developed in the past century, the theory has gained attention for its power, despite it relatively simple origins in abstracting ideas from multivariable calculus. This thesis will focus on introducing the standard tools in the finite-dimensional setting, both the traditional handlebody perspective and the modern flow perspective. Afterwards, we will discuss the Morse Homology briefly. Finally, we end this thesis with an chapter constructing an explicit intrinsic functions on certain Grassmannian manifolds, our main result.

Contents

1 Introduction

We assume the reader to have some familiarity with multivariable calculus, linear algebra, topology, and differential geometry. It will be useful to have background in algebraic topology as well for the later section on Morse homology.

Formally, a part of differential topology, Morse theory's origins are attributed to mathematician H.C. Marston Morse. In the decades after his initial work in the 1920s, John Milnor, Raoul Bott, and Stephen Smale would popularize these techniques in proving some important results, cementing Morse theory as a worthy tool in a topologists toolbox. Modern Morse theory has found applications in many other subfields, such as gauge theory, Floer homology, and symplectic topology.

Broadly speaking, Morse theory studies how we can extract topological data of a space (smooth manifold) by studying certain nice functions on that space (a so-called "Morse function"). For example, consider the 1-dimensional manifolds of a line (identified as the x-axis $y = 0$) versus circle (identified as the unit circle $x^2 + y^2 = 1$). The continuous function $f(x) = x^2$ would take on arbitrarily high values on the line, while being bounded on the circle, a consequence of the extreme value theorem. Thus, we would be able to tell lines and circles apart topologically by determining whether any unbounded, continuous functions exist on that manifold. This observation is not really new, but can be recast in the language of Morse theory; readers may be familiar with the following result from point-set topology.

Lemma. *Let (X, d) be a metric space. Then, X is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.*

With a bit of motivation in hand, let's begin our journey into Morse theory.

2 Morse Theory

Going forward, if unspecified, we assume all manifolds and functions are smooth, meaning they are in C^∞.

2.1 Morse Functions

Let M be a smooth m-manifold without boundary and $f : M \to \mathbb{R}$ a smooth map. Much of the introduction that follows is adapted from Yukio Matsumoto's wonderful introductory text.

Definition 1 (Critical Points & Values)**.** *A point p_0 of M is a critical point of f if we have*

$$\frac{\partial f}{\partial x_1}(p_0) = \frac{\partial f}{\partial x_2}(p_0) = \cdots = \frac{\partial f}{\partial x_m}(p_0) = 0,$$

where $(x_1, \ldots, x_m)$ is a local coordinate system around p_0. Furthermore, we call $f(p_0)$ the critical value.

Since we will generally be working in multiple different coordinate systems on a manifold M, it would be trouble if being a critical point depended on the choice of coordinate system. Luckily, this is not the case.

Proposition 2. *If p_0 is a critical point in some local coordinates about p_0, then it is a critical point for all local coordinate systems.*

Proof. Suppose we knew that p_0 was a critical point in the local $(x_1, \ldots, x_m)$ coordinate system and let $(y_1, \ldots, y_m)$ be another local coordinate system. Then, using the chain rule,

$$\frac{\partial f}{\partial y_i}(p_0) = \sum_{j=1}^{m} \frac{\partial x_j}{y_i} \frac{\partial f}{\partial x_j}(p_0)$$

Then, p_0 being a critical points means that $\frac{\partial f}{\partial x_j}(p_0) = 0$ for all $1 \leq j \leq m$, so the left-hand side must be 0 as well for all i. Swapping the roles of these coordinates, we can see that p_0 does not dependent on the choice of coordinates. $\square$

Once we have a critical point, it will be useful to try to find out whether it represents an extrema (a minimum or maximum of our function f). Thinking about the second partial derivative test from multivariable calculus motivates our next definition.

Definition 3 (Hessian)**.** *The Hessian of f at a critical point p_0 is the matrix of second derivatives, shown below.*

$$H_f(p_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(p_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(p_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(p_0) & \cdots & \frac{\partial^2 f}{\partial x_m^2}(p_0) \end{bmatrix}$$

We remark that the Hessian matrix is symmetric; partial derivatives commute when they are all continuous by our C^∞ assumption. This Hessian will play an important role, similar to the how one uses the determinant of the Hessian to classify the type of critical point (saddle, max, or min).

Definition 4 (Degeneracy)**.** *A critical point p_0 of f is called a non-degenerate critical point if $det(H_f(p_0)) \neq 0$. Otherwise, if $det(H_f(p_0)) = 0$, then p_0 is a degenerate critical point.*

Again, since we may be working in different local coordinate systems, we need to study how the Hessian changes under coordinate transformations in order to show that degeneracy of critical points are independent of the choice of coordinates.

Definition 5 (Jacobian)**.** *The Jacobian of $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point p is the* *matrix of derivatives below.*

$$J(p) = \begin{bmatrix} \frac{\partial x_1}{\partial y_1}(p) & \dots & \frac{\partial x_1}{\partial y_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial y_1}(p) & \dots & \frac{\partial x_m}{\partial y_m}(p) \end{bmatrix}$$

Lemma 6. *Suppose we have two coordinate systems $(y_1, \dots, y_m)$ and $(x_1, \dots, x_m)$ around p_0 with respective Hessians $H_f^x(p_0)$ and $H_f^y(p_0)$. Then, we have the following relationship*

$$H_f^y(p_0) = J^T(p_0)H_f^x(p_0)J(p_0),$$

where $J(p_0)$ is the Jacobian of the coordinate transformation from $(y_1, \dots, y_m) \to (x_1, \dots, x_m)$ at p_0.

Proof. Suppose that $(x_1, \dots, x_m)$ is the original coordinate system. We want to compute the second order partial derivatives in the new system $(y_1, \dots, y_m)$. Using the chain rule twice,

$$\frac{\partial f}{\partial y_h} = \sum_{i=1}^{m} \frac{\partial x_i}{\partial y_h}\frac{\partial f}{\partial x_i}$$

$$\frac{\partial^2 f}{\partial y_h \partial y_k}(p) = \sum_{i,j=1}^{m} \frac{\partial x_i}{\partial y_h}(p)\frac{\partial x_j}{\partial y_k}(p)\frac{\partial^2 f}{\partial x_i \partial x_j}(p) + \sum_{i=1}^{m} \frac{\partial x_i}{\partial y_h \partial y_k}(p)\frac{\partial f}{\partial x_i}(p)$$

If we plug in a critical point $p = p_0$ above, then this second summation vanishes as $\frac{\partial f}{\partial x_i}(p_0) = 0$ for all $1 \leq i \leq m$. It immediate follows y comparing coordinates that

$$\frac{\partial^2 f}{\partial y_h \partial y_k}(p_0) = \sum_{i,j=1}^{m} \frac{\partial x_i}{\partial y_h}(p_0)\frac{\partial x_j}{\partial y_k}(p_0)\frac{\partial^2 f}{\partial x_i \partial x_j}(p_0)$$

$$\implies H_f^y(p_0) = J^T(p_0)H_f^x(p_0)J(p_0)$$

□

As an immediate corollary, the degeneracy of a critical point does not depend on coordinates.

Corollary 7. *Degeneracy of critical points is independent of the choice of coordinates.*

Proof. Let p_0 be a degenerate critical point in $(x_1, \ldots, x_m)$. From the previous lemma, we have in a new coordinate system $(y_1, \ldots, y_m)$

$$H_f^y(p_0) = J^T(p_0) H_f^x(p_0) J(p_0)$$

since the determinant is multiplicative, we have

$$\det\left(H_f^y(p_0)\right) = \det\left(J^T(p_0) H_f^x(p_0) J(p_0)\right) = \det\left(J^T(p_0)\right) \det\left(H_f^x(p_0)\right) \det\left(J(p_0)\right)$$

Since our coordinate systems have the same number of coordinates, the determinant of the Jacobian of this transformation must be non-zero, so $\det\left(J^T(p_0)\right) = \det\left(J(p_0)\right) \neq 0$. Thus, $\det\left(H_f^y(p_0)\right) = 0$ if and only if $\det\left(H_f^x(p_0)\right)$. $\square$

Finally, we can define our Morse functions!

Definition 8. *We say that a function $f : M \to \mathbb{R}$ is Morse if every critical point of f is non-degenerate.*

2.2 Morse Lemma

With our definitions in hand, we can finally prove our first main result.

Theorem 9 (Morse Lemma)**.** *Let p_0 be a non-degenerate critical point of f. Then, there exists a local coordinate system $(X_1, \ldots, X_m)$ around p_0, such that*

$$f = f(p_0) - X_1^2 - X_2^2 - \cdots - X_i^2 + X_{i+1}^2 + \cdots + X_m^2$$

We say that p_0 is the origin $(0, \ldots, 0)$.

We remark here that there is an alternative proof in Audin-Damian using the implicit function theorem. We will follow the treatment given in Matsumoto's text. First, we need to establish an important fact.

Lemma 10 (Hadamard's lemma)**.** *Let f be a smooth, real-valued function on an open, convex neighborhood U about a point a. Then, we may express f in local coordinates $(x_1, \ldots, x_m)$ in U as*

$$f(x_1, \ldots, x_m) = f(a) + \sum_{i=1}^{n} (x_i - a_i) g_i(x_1, \ldots, x_m)$$

, where each g_i is a smooth function on U.

Hadamard's lemma doesn't quite give us the quadratic form we are looking for. The smooth functions g_i need not be quadratic in coordinates.

Proof. We follow the proof as outlined in Matsumoto's text. Choose a local coordinate system $(x_1, \ldots, x_m)$ at the critical point p_0, so that p_0 is the origin. We can assume without loss of generality that $f(p_0) = 0$; subtracting a Morse function by a constant is still Morse.

Using Hadamard's lemma, we can obtain m smooth functions

$$g_1(x_1, \ldots, x_m), g_2(x_1, \ldots, x_m), \ldots, g_m(x_1, \ldots, x_m)$$

such that

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i g_i(x_1, \ldots, x_m)$$

Taking the partial derivatives and using that $p_0 = (0, \ldots, 0)$ in our coordinates also tells us that

$$\frac{\partial f}{\partial x_i}(0, \ldots, 0) = g_i(0, \ldots, 0) \implies \frac{\partial f}{\partial x_i}(p_0) = g_i(p_0) = 0$$

Thus, we may use Hadamard's lemma again on each $g_i(x_1, \ldots, x_m)$ to find m smooth functions of the form

$$h_{i1}(x_1, \ldots, x_m), h_{i2}(x_1, \ldots, x_m), \ldots, h_{im}(x_1, \ldots, x_m)$$

such that locally

$$g_i(x_1, \ldots, x_m) = \sum_{j=1}^{m} x_j h_{ij}(x_1, \ldots, x_m)$$

We can plug this back to get a expression for f in terms of these h_{ij}'s.

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j h_{ij}(x_1, \ldots, x_m)$$

We can pair up terms h_{ij} and h_{ji} as $H_{ij} = \frac{h_{ij}+h_{ji}}{2}$. Substituting,

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j H_{ij}(x_1, \ldots, x_m)$$

Recognize that $H_{ij} = H_{ji}$. Computing the second partial derivatives of f by linearity, we find for each (i, j) pair

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0, \ldots, 0) = 2H_{ij}(0, \ldots, 0)$$

Since p_0 is a non-degenerate critical point, the left-hand side is non-zero implying $H_{ij}(0, \ldots, 0) \neq 0$. By a linear transformation of the local coordinate system $(x_1, \ldots, x_m)$, we get that

$$\frac{\partial^2 f}{\partial x_1^2}(0, \ldots, 0) \neq 0 \implies H_{11}(0, \ldots, 0) \neq 0$$

Since the function $H_1$is continuous, it is non-zero in a neighborhood of the origin. We can define a new coordinate system $(X_1, x_2, \ldots, x_m)$, where we define X_1 by

$$X_1 = \sqrt{|H_{11}|}(x_1 + \sum_{i=2}^{m} x_i \frac{H_{1i}}{H_{11}})$$

The Jacobian of the coordinate transform from $(X_1, x_2, \ldots, x_m)$ to $(x_1, x_2, \ldots, x_m)$ is non-zero at the origin, so it is a valid local coordinate system. Computing the square,

$$\begin{aligned} X_1^2 &= |H_{11}|\,(x_1 + \sum_{i=2}^{m} x_i \frac{H_{1i}}{H_{11}})^2 \\ &= \begin{cases} H_{11}x_1^2 + 2\sum_{i=2}^{m} x_1 x_i H_{1i} + (\sum_{i=2}^{m} x_i H_{1i})^2/H_{11} & (H_{11} > 0) \\ -H_{11}x_1^2 - 2\sum_{i=2}^{m} x_1 x_i H_{1i} - (\sum_{i=2}^{m} x_i H_{1i})^2/H_{11} & (H_{11} < 0) \end{cases} \end{aligned}$$

A substitution back into the original form of f reduces our quadratic expression to one with one less variable. We can repeat this above argument again for x_2, so by induction, the proof is complete. $\square$

The Morse lemma tells us that in a neighborhood near non-degenerate critical points, our function is well-behaved for a specific choice of coordinates. The chart, where every open set's coordinates are given by the Morse lemma, is called a Morse chart.

We also want to pay special attention to the value of i in the statement of the Morse lemma. The number i above is called the index of a non-degenerate critical point p_0. We make two remarks. First, the index must belong to $[0, m]$, where m is the dimension of the manifold. Second, a critical point of index i of f is a critical point of index $n - i$ of $-f$.

If we have $-X_j^2$ in the local coordinates, the value of the function f decreases as we move from p_0 in the direction of X_j. Intuitively, the index tells us locally, the number of directions we can move to decrease f. Thus, equivalently, we can define it more directly using the Hessian. We will use both definitions interchangeably.

Definition 11 (Index of a critical point)**.** *The index of a critical point p_0 to be the number of negative eigenvalues of the Hessian $H_f(p_0)$.*

While we know the Hessian changes under coordinate transformation is captured by the Jacobian, it isn't sufficient to tell us that the index remains invariant under coordinate transformations. Luckily, a version of Sylvester's law of inertia from linear algebra fills in the gap.

Theorem 12. *The index of a critical point is invariant under coordinate transformation.*

Proof. Sylvester's law of inertia states that if A is a symmetric matrix, then for any invertible S, the number of positive, negative and zero eigenvalues of $D = S^T AS$ is constant. In our case, the Hessian is always symmetric and Jacobian has non-zero determinant, as it is full rank. □

Two immediate corollaries of the Morse lemma are the following.

Corollary 13. *The non-degenerate critical points of a function are isolated.*

Proof. From the Morse lemma, we get local coordinates such that $f = f(p_0) - X_1^2 - X_2^2 - \cdots - X_i^2 + X_{i+1}^2 + \cdots + X_m^2$. If we compute the partial derivatives in this local neighborhood, we get for all i,

$$\frac{\partial f}{\partial X_i} = \pm 2x_i = 0 \iff x_i = 0$$

Thus, we only have the critical point corresponding to the origin p_0. □

Corollary 14. *A Morse function defined on a compact manifold admit only finitely many critical points*

Proof. Suppose that a Morse function f on compact M had infinitely many critical points. Denote the set of critical points as C. Since a set of isolated critical points is closed and closed subsets of compact spaces are compact, C is compact. By Bolzano-Weierstrass, we could find a subsequence $\{c_i\}$ of C that has a limit point $p \in M$. If $p \in C$, then p is an isolated point by the previous corollary, yet every neighborhood contains a critical point in $\{c_i\}$, a contradiction.

If $p \notin C$, then the derivative $\frac{\partial f}{\partial x_i}(p) \neq 0$ for at least one i. Since partial derivatives are continuous by our C^∞ assumptions, there is a neighborhood U containing p, such that $\frac{\partial f}{\partial u}(p) \neq 0$ for all $u \in U$. This means that U contains no critical points. However, this contradicts the fact that p is a limit point, which means U must contain some critical point in $\{c_i\}$. Thus, our proof is complete. □

2.3 Existence of Morse Functions

As we have seen, Morse functions on manifolds have some nice properties. Even just the Morse lemma gives us some conveniently nice charts for our manifold. We pause now to ask whether every manifold even has a Morse function.

Theorem 15 (Existence of Morse Functions)**.** *Let $V \subset \mathbb{R}^n$ be a submanifold. For almost every point $p \in \mathbb{R}^n$, the function $f_p : V \to \mathbb{R}$ given by sending $x \to ||x - p||^2$ is a Morse function.*

Proof. We sketch the proof here from Audin-Damian. Consider the differential f_p, where T_x represents the differential induced by the tangent space. By smooth manifold theory, we can choose a local parameterization of V as

$$(y_1, \ldots, y_d) \to x(y_1, \ldots, y_d)$$

With these new coordinates,

$$\frac{\partial f_p}{\partial y_i} = 2(x-p) \cdot \frac{\partial x}{\partial u_i}$$
$$\frac{\partial^2 f_p}{\partial y_i \partial y_j} = 2(\frac{\partial x}{\partial y_j} \cdot \frac{\partial x}{\partial y_i} + (x-p) \cdot \frac{\partial^2 x}{\partial y_i \partial y_j})$$

Looking at these equations, x is a non-degenerate critical point if and only if $x-p$ is orthogonal to T_xV and the Hessian matrix has rank d. Thus, to finish the proof, it suffices to show that the points not satisfying the latter condition are critical values of a C^∞ function. From there, Sard's theorem (the set of critical values of a smooth map form a set of measure zero.) The rest of the details can be found in the textbook. □

The previous result tells us that there exists many Morse functions on any manifold that can be embedded in $\mathbb{R}^n$. In fact, these Morse functions can be used to approximate smooth functions very well.

Theorem 16 (Genericity of Morse Functions)**.** *Let V be a manifold that embeds into $\mathbb{R}^n$, $f : V \to \mathbb{R}$ be smooth, and fix an integer k. Then, f and its derivatives of order less than k can be uniformly approximated by Morse functions on every compact subset.*

One can find an alternative approach to existence and genericity in section 2.2 of Matsumoto.

Proof. We can choose an embedding h of V into $\mathbb{R}^n$ of the following form.

$$h(x) = (f(x), h_2(x), \ldots, h_n(x))$$

By our existence theorem, for almost every $p \in \mathbb{R}^n$, we can write near $(-c, 0, \ldots, 0)$,

$$p = (-c + \epsilon_1, \epsilon_2, \ldots, \epsilon_n)$$

, where c is a sufficiently large fixed real number. Consequently, f_p is a Morse function. Since Morse functions remain Morse under adding constants and scaling,

$$g(x) = \frac{f_p(x) - c^2}{2c}$$

is also a Morse function. Furthermore, a quick calculation in coordinates shows

$$\begin{aligned} g(x) &= \frac{1}{2c}[(f(x) + c - \epsilon_1)^2 + (h_2(x) + \epsilon_2)^2 + \cdots + (h_n(x) - \epsilon_n)^2 - c^2] \\ &= f(x) + \frac{[f(x)]^2 + \sum_{i=1}^n [h_i(x)]^2}{2c} - \frac{\epsilon_1 f(x) + \sum_{i=1}^n \epsilon_i h_i(x)}{c} + \sum_{i=1}^n \epsilon_i^2 - \epsilon_1 \end{aligned}$$

If we take a sufficiently large c and ϵ_i small, then $g(x)$ is the desired uniform approximation and $g(x)$ is Morse as desired. □

2.4 An Application

We end this introductory chapter with a neat application of the techniques so far developed. The goal is to showcase some of the power in the theory.

Theorem 17 (Reeb's theorem)**.** *Let M be a compact manifold. If there exists a Morse function $f : M \to \mathbb{R}$ with exactly two critical points, then M is homeomorphic to a sphere S^m.*

Proof. By the extreme value theorem, the critical points must be a minimum and maximum. We can assume that $f(M) = [0, 1]$ by rescaling. For $\epsilon > 0$ sufficiently small, the Morse lemma tells us that $f^{-1}([0, \epsilon])$ and $f^{-1}([1 - \epsilon, 1])$ are n-dimensional disks. Thus, the preimages $f^{-1}([0, \epsilon])$ and $f^{-1}([0, 1 - \epsilon])$ are diffeomorphic, since the topology of a manifold only changes at these critical points. Therefore, M is the gluing of two disks along their boundary, which is the desired sphere. □

3 Morse Homology

This section is a collection of useful definitions and results from Chapter 2 and 3 of Audin-Damian. For brevity, we will mainly state some results and comment on them.

3.1 The Flow Perspective

Historically, Morse theory was used to prove certain inequalities about the nature of critical points. In the modern day, topologists have used flows between critical points to study the manifold.

Definition 18 (Pseudo-Gradient)**.** *Let $f : M \to \mathbb{R}$ be a Morse function. Then, a pseudo-gradient, or gradient-like, field on f is a vector field X on M, such that*

- *We have that $(df)_x(X_x) \leq 0$, where equality holds if and only if x is a critical point*
- *In a Morse chart about any critical point of f, X coincides with the negative gradient for the metric on $\mathbb{R}^n$.*

As the name suggests, these special vector fields capture the most important nice properties of traditional gradient vector fields.

An important fact one can prove with gradient-like vector fields is the following.

Theorem 19. *Let a and b be real numbers, such that f does not have a critical value in $[a, b]$. Then, if $f^{-1}([a, b])$ is compact, $f^{[-\infty,b]}$ is diffeomorphic to $f^{-1}([-\infty, a])$.*

These preimages above are called (sub)level sets. This theorem captures that the topological data of level sets with respect to a Morse function f on a manifold doesn't change unless we hit a critical point. We already saw this phenomenon appear in Reed's theorem.

Proof. We can use the flow of our pseudo-gradient vector field X to perform a retraction from $f^{[-\infty,b]}$ to $f^{[-\infty,a]}$. Consider,

$$p(x) = \begin{cases} \frac{-1}{(df)_x(X)} & \text{on } f^{-1}([a,b]) \\ 0 & \text{outside of compact neighborhood of } f^{-1}([a,b]) \end{cases}$$

Then, $Y = pX$ defines a flow ϕ^s for all $s \in \mathbb{R}$. For fixed $x \in V$ and $\phi^s(x) \in f^{-1}([a,b])$, we have

$$\begin{aligned} \frac{d}{ds}(f \circ \phi^s(x)) &= (df)_{\psi^s(x)}(\frac{d}{ds}\psi^s(x)) \\ &= (df)_{\psi^s(x)}(Y_{\phi^s(x)}) = -1 \end{aligned}$$

Thus, we have $f \circ \phi^s(x) = -s + f(x)$ by rearrangement. When $s = b - a$, ϕ^s is the desired diffeomorphism. $\square$

When we eventually compute the Morse homology, we cannot just take any pseudo-gradient field. We want it to satisfy the so-called Smale conditional. This insures that the intersections of certain manifolds about our critical points are well-behaved.

Definition 20. *Let p be a critical point of f. For a flow ϕ^s induced by a pseudo-gradient, we let the stable manifold $W^s(p)$ be*

$$W^s(p) = \{x \in V : \lim_{s \to \infty \phi^s(x) = p}\}$$

Analogously, we let the unstable manifold $W^u(p)$ be

$$W^u(p) = \{x \in V : \lim_{s \to -\infty \phi^s(x) = p}\}$$

The unstable and stable manifolds will help us count trajectories later.

Definition 21 (Smale condition)**.** *We say that a pseudo gradient field satisfies the Smale condition if all stable and unstable manifolds at critical points meet transversely.*

3.2 The Morse Complex

Let $\mathrm{Crit}_k(f)$ denote the set of critical points of f that have index k. Then, we can consider the vector space

$$C_k(f) = \{\sum_{c \in \mathrm{Crit}_k(f)} a_c c | a_c \in \mathbb{Z}/2\}$$

To develop a valid homology theory, we want to take a chain complex of these vector spaces and define a differential map.

$$\ldots \xrightarrow{\partial_k} C_{k+1}(f) \xrightarrow{\partial_{k-1}} C_k(f) \to C_{k-1}(f) \to \ldots$$

We occasionally may omit the indices when the domain is clear.

Definition 22. *The differential ∂_X acts on a critical point p of index k as a generator for $C_k(f)$ by*

$$\partial_X(p) = \sum_{q \in Crit_{k-1}} n_X(p,q)q$$

, where $n_X(p,q)$ is the number of trajectories of X mod 2. We can extend the differential to act on all elements of $C_k(f)$ by linearity.

Firstly, one can show that the about $n_X(p, q)$ is always finite, so the above is well-defined. The number of trajectories depends on the particular pseudo-gradient field we select and perhaps the Morse function f.

With a chain complex of vector spaces and the differential defined, we can go ahead and compute the homology as in any theory by defining

$$H_k = \mathrm{Ker}\partial^k / \mathrm{Im}\partial^{k+1}$$

However, we might worry that the choice of vector field and Morse function can change the resulting homology groups we get on the same manifold.

Theorem 23. *Let M be a compact manifold. Let f, g be two Morse functions and their respective pseudo-gradients be X, Y with the Smale condition. Then, the mapping on chain complexes*

$$\Phi_\star : (C_\star(f), \partial_X) \to (C_\star(g), \partial_Y)$$

induces an isomorphism of homology groups.

We won't delve into the proof, but remark that this is similar to how one constructs simplicial/celluluar and singular homology. We can show that simplicial/cellular homology does not depend on the choice of simplicial/cellular structure by comparing with a less concrete-to-compute, but nicer abstractly homology theory.

4 Our Result

4.1 Grassmannians as a Manifold

With our new tools from Morse theory, let's try applying them to concrete manifolds. By our previously established existence and uniqueness theorems, Morse functions are known to be abundant on Grassmannians, since these manifolds are abstractly metrizable. Authors generally embed Grassmannians into a sufficiently high-dimensional Euclidean space $\mathbb{R}^n$ and take the appropriate height function to derive a Morse function, or exploit some abstract embedding that arises from the Grassmannian viewed as an affine algebraic variety. Our goal in this section is to derive the real homology of the Grassmannian manifold by constructing an intrinsic Morse function that does not rely on such embeddings. In doing so, we have a nice concrete function we can play with. Before that, we will rattle off some quick fast facts about Grassmannians.

Definition 24 (Grassmannian)**.** *Let $G_k(V^k)$ be the set of k-dimensional subspaces of an n-dimensional vector space V. $G_k(V^k)$ is called the Grassmannian.*

For our purposes, we will focus on $V = \mathbb{R}$.

Proposition 25. *$G_k(\mathbb{R}^n)$ is a smooth manifold of dimension $k(n-k)$.*

Proof. We will briefly describe the coordinates construction for later reference. For full details, check Lee's book on smooth manifolds. Begin by choosing a basis for $\mathbb{R}^n$. Then, for any k-dimensional subspace of $\mathbb{R}^n$, we can choose a basis of size k. The coordinates (in fact, homogeneous coordinates) of $s \in G_k(\mathbb{R}^n)$ correspond to to full rank matrices E of dimension $n \times k$, up to some equivalence. We say E and D are equivalent if there is some invertible $k \times k$ matrix K, such that $E = KD$. One can check that the elementary column operations give an appropriate atlas. In fact, this sketch also shows the next proposition. □

Proposition 26. *$G_k(\mathbb{R}^n)$ is the homogeneous space $\frac{O(n)}{O(r)\times O(n-r)}$*

Proposition 27. *$G_k(\mathbb{R}^n)$ is a compact manifold*

Proof. An element of the Grassmannian is an $n \times k$ matrix with orthonormal columns, representing a k-dimensional subspace of $\mathbb{R}^n$, up to equivalence by the orthogonal group action. Ignoring the group action, it suffices to show that the set of these matrices is closed and bounded. Note that the entries of the matrix can be normalized to be between -1 and 1. Thus, these are bounded.

Furthermore, it is closed, since orthogonality of the columns is equivalent to satisfying some set of polynomial equations (variables being the entries of the $n \times k$ matrix). In particular, taking the standard matrix embedding, $G_k(\mathbb{R}^n)$ is a closed and bounded subset of $\mathbb{R}^{nk}$. By Heine-Borel, closed and bounded subsets are compact, so we are done. ☐

Proposition 28. $G_k(\mathbb{R}^n) \cong G_{n-k}(\mathbb{R}^n)$

Proof. Take the map $\phi : G_k(\mathbb{R}^n) \to G_{n-k}(\mathbb{R}^n$ which maps $\phi(X) = X^{\perp}$. In other words, ϕ is the orthogonal complement map. One can easily verify that this map is its own inverse and is continuous. ☐

4.2 Constructing a Morse Function

Now that we know our manifold better, we can construct a Morse function on $G_2(\mathbb{R}^n)$.

Let $V, W \in G_2(\mathbb{R}^n)$ be oriented 2-planes with orthonormal bases $\{v_1, v_2\}$ and $\{w_1, w_2\}$ respectively. As a shorthand, we will write $M_V = M(v_1, v_2)$ to the $n \times 2$ matrix with columns formed by $\{v_1, v_2\}$. Then, we define

$$f(V, W) = \det\left(\begin{bmatrix} \langle v_1, w_1 \rangle & \langle v_1, w_2 \rangle \\ \langle v_2, w_1 \rangle & \langle v_2, w_2 \rangle \end{bmatrix}\right) = \det\left(M_V^T M_W\right)$$

Observe the following.

Lemma 29. *As defined above, $f(V, W)$ is independent of the choice of orthonormal bases of V and W, up to orientation.*

Proof. Since V is a 2-plane, any other orthonormal basis will be a rotated form of $\{v_1.v_2\}$. For some $R \in \mathrm{SO}(2)$We can express this new basis as

$$M(v_1, v_2)R = M(v_1, v_2) \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = M(\cos\theta v_1 - \sin\theta v_2, \sin\theta v_1 + \cos\theta v_2)$$

We can repeat the same argument for W with a rotation matrix $S \in \mathrm{SO}(2)$. Then, since the determinant of rotation matrices are 1, we have

$$f(M_V R, M_W S) = \det\left((M_V R)^T (M_W S)\right) = \det\left(R^T M_V M_W S\right) = \det\left(M_V M_W\right)$$

□

As a consequence of the above lemma, f is truly a function of the planes, not the choice of bases on them. Furthermore, it is easy to verify we have $f(V, W) = f(W, V)$, so f is a symmetric function. As a remark, f turns out to be a Morse function on $G_2(\mathbb{R}^4)$. The reader may find it useful to walk through this calculation as an exercise before tackling the general case.

For a general Morse function on $G_2(\mathbb{R}^n)$, let $k = \lfloor \frac{n}{2} \rfloor$. We take k orthogonal 2-planes $V_1, \ldots, V_k$ in $\mathbb{R}^n$ with corresponding orthonormal bases $V_j = \mathrm{span}\{e_{2j-1}, e_{2j}\}$, where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. Consider constants $\alpha_1, \ldots, \alpha_k$. For a variable 2-plane $X \in G_2(\mathbb{R}^n)$, define a function $g : G_2(\mathbb{R}^n) \to \mathbb{R}$ as

$$g(X) = \sum_{i=1}^{k} \alpha_i f(V_i, X) = \alpha_1 f(V_1, X) + \alpha_2 f(V_2, X) + \cdots + \alpha_k f(V_k, X)$$

Recall that f was defined in the previous section. We claim that g is a Morse function if the constants $\alpha_1, \ldots, \alpha_k$ are distinct.

Theorem 30. *Let $0 < a_1 < a_2 < \cdots < a_k < 1$. Then, $g(X)$ is a Morse* *function. Furthermore,*

- *The critical points are precisely* $X = \pm V_j$
- *The critical point* $+V_j$ *has index* $2n - 2j - 2$ *and* $-V_j$ *has index* $2j - 2$ *for* $1 \leq j \leq k$.

Proof. Our proof consists of three major steps (which we will break into propositions throughout the course of this chapter).

1. Show that $\pm V_j$ are actually non-degenerate critical points
2. Calculate the index of these critical points
3. Prove that no other critical points exist

□

Proposition 31. $X = V_l$ *is a critical point.*

Proof. For the first step towards showing our result, we put local coordinates on a neighborhood of V_l, where $1 \leq l \leq k$, via the unique "row-reduced" $\{v_1 = e_{2l-1} + q_1, v_2 = e_{2l-1} + q_2\}$, where $\{q_1, q_2\}$ are perpendicular vectors to V_l. While we may be tempted to jump right into computing the partial derivatives, this new basis $\{v_1, v_2\}$ is no longer orthonormal. Thus, we need to apply the Gram-Schmidt process.

Since our end goal is to show that V_1 is a non-degenerate critical point, we only need to compute the Taylor series of g about V_1 up to the terms of second order. For simplicity, we collect the extraneous terms into $O(r^3)$, where $r = \max\{||q_1||\,, ||q_2||\}$. Then, using the binomial theorem for exponent $-\frac{1}{2}$,

$$\begin{aligned}\frac{1}{||v_1||} &= \frac{1}{\sqrt{1 + \langle q_1, q_1\rangle}} = (1 + \langle q_1, q_1\rangle)^{-\frac{1}{2}} \\ &= 1 - \frac{1}{2}\langle q_1, q_1\rangle + O(r^3)\end{aligned}$$

We can normalize v_1, defining

$$\begin{aligned} x_1 = \frac{v_1}{||v_1||} &= (1 - \frac{1}{2}\langle q_1, q_1\rangle)(e_{2l-1} + q_1) + O(r^3) \\ &= (1 - \frac{1}{2}\langle q_1, q_1\rangle)e_{2l-1} + q_1 + O(r^3) \end{aligned}$$

Since $\langle x_1, v_1\rangle = \langle q_1, q_2\rangle + O(r^3)$, we can finish the Gram-Schmidt process.

$$\begin{aligned} w_2 &= v_2 - \langle x_1, v_2\rangle x_1 \\ &= e_{2l} + q_2 - \langle q_1, q_2\rangle e_{2l-1} + O(r^3) \\ \frac{1}{||w_2||} &= (1 + \langle w_2, w_2\rangle)^{-\frac{1}{2}} \\ &= 1 - \frac{1}{2}\langle w_2, w_2\rangle + O(r^3) = 1 - \frac{1}{2}\langle v_2 - \langle x_1, v_2\rangle x_1, v_2 - \langle x_1, v_2\rangle x_1\rangle + O(r^3) \\ &= 1 - \frac{1}{2}[\langle v_2, v_2\rangle + \langle x_1, x_1\rangle - 2\langle x_1, v_2\rangle \langle v_2, x_1\rangle] = 1 - \frac{1}{2}\langle q_2, q_2\rangle + O(r^3) \\ \implies x_2 = \frac{w_2}{||w_2||} &= (1 - \frac{1}{2}\langle q_2, q_2\rangle)e_{2l} + q_2 - \langle q_1, q_2\rangle e_{2l-1} + O(r^3) \end{aligned}$$

With our new orthonormal basis $\{x_1, x_2\}$ above, we can look at the parts of

$g(X)$. For $f(V_l, X)$, we have two forms.

$$
\begin{aligned}
f(V_l, X) &= \det\left(\begin{bmatrix} \langle e_{2l-1}, x_1\rangle & \langle e_{2l}, x_1\rangle \\ \langle e_{2l-1}, x_2\rangle & \langle e_{2l}, x_1\rangle \end{bmatrix}\right) \\
&= \langle e_{2l-1}, x_1\rangle \langle e_{2l}, x_1\rangle - \langle e_{2l}, x_1\rangle \langle e_{2l-1}, x_2\rangle \\
&= (1 - \frac{1}{2}\langle q_1, q_1\rangle)(1 - \frac{1}{2}\langle q_2, q_2\rangle) + O(r^3) \\
&= 1 - \frac{1}{2}\langle q_1, q_1\rangle - \frac{1}{2}\langle q_2, q_2\rangle + O(r^3) = 1 - \frac{1}{2}[\langle q_1, q_1\rangle + \langle q_2, q_2\rangle] + O(r^3) \\
\text{For } j \neq l,\ f(V_j, X) &= \det\left(\begin{bmatrix} \langle e_{2j-1}, x_1\rangle & \langle e_{2j}, x_1\rangle \\ \langle e_{2j-1}, x_2\rangle & \langle e_{2j}, x_2\rangle \end{bmatrix}\right) \\
&= \langle e_{2j-1}, x_1\rangle \langle e_{2j}, x_2\rangle - \langle e_{2j}, x_1\rangle \langle e_{2j-1}, x_2\rangle \\
&= \langle e_{2j-1}, q_1\rangle \langle e_{2j}, q_2\rangle - \langle e_{2j}, q_1\rangle \langle e_{2j-1}, q_2\rangle + O(r^3)
\end{aligned}
$$

We can recognize that this is the determinant of a particular 2×2 minor of the matrix $M(x_1, x_2)$. Looking at the $f(V_j, X)$ for $j \neq l$, we can see that none of the terms are linear in q_1 or q_2. On the other hand, if $X = V_l$, then $q_1 = q_2 = 0 \implies \langle q_1, q_1\rangle = \langle q_2, q_2\rangle = 0$. This maximizes the expression for $f(V_l, X)$, meaning V_l is a critical point of g.

Notice that we've also shown that $-V_l$ is a critical point. We just take the opposite orientation, which swaps rows of the determinant expression. This only flips the sign of the final expressions for $f(V_j, X)$ or $f(V_l, X)$ by a negative sign. Thus, $-V_l$ is a critical point as well. □

Proposition 32. *$X = V_l$ is a non-degenerate critical point of index $2(n-2) - 2(l-1) = 2n - 2l - 2$.*

Proof. Our next goal is to show that these critical points are non-degenerate and determine their indices. Now, we want to compute the Hessians of $f(V_j, X)$ at $X = V_l$. Let $a_j = \langle x_1, e_j\rangle$ and $b_j = \langle x_2, e_j\rangle$ for $j \notin \{2l-1, 2l\}$. These

a_j, b_j are all 0 for $X = V_l$. Thus, these variables represent local coordinates in a neighborhood of $X = V_l$!

Computing the Hessian of g and using linearity, we only have non-zero terms on the main diagonal. Plugging in $X = V_l$, our Hessian (which has dimension $(2n-4) \times (2n-4)$) is the negative identity matrix.

On the other hand, for $j \neq l$, the only second order mixed partials of $f(V_j, X)$ that don't disappear fall into two categories

- $\frac{\partial^2 f(V_j, X)}{\partial a_{2j-1} \partial b_{2j}}$, which evaluates to $+1$ at $X = V_l$.
- $\frac{\partial^2 f(V_j, X)}{\partial a_{2j} \partial b_{2j-1}}$, which evaluates to -1 at $X = V_l$

There are two in each of these classes, since the $(2n-4) \times (2n-4)$ Hessian is symmetric. Thus, when accounting for our coefficients, we get entries corresponding to $\pm\alpha_l$ and $\pm\alpha_j$.

As a concrete example to keep in mind, in the setting of $G_2(\mathbb{R}^6)$, we get the following 8×8 Hessian of g evaluated at $X = V_1$.

$$\begin{bmatrix} -\alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1 & -\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2 & -\alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & -\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_1 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_1 & -\alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 & -\alpha_1 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & -\alpha_1 \end{bmatrix}$$

If n is even, then, the Hessian can be decomposed into 2×2 blocks of either the form

$$\begin{bmatrix} -\alpha_l & \alpha_j \\ \alpha_j & -\alpha_l \end{bmatrix} \text{ or } \begin{bmatrix} -\alpha_l & -\alpha_j \\ -\alpha_j & -\alpha_l \end{bmatrix}$$

From this, we get that the eigenvectors of the Hessian correspond to vectors with exactly 2 non-zero entries. The eigenvalues are of the form $-\alpha_l \pm \alpha_j$ for $j \neq l$. Each eigenvalue has multiplicity 2. All the eigenvalues are of this form, so the summation of the Hessians will have cancellation. Only the $H_{f(V_l,X)}(V_l)$ will survive, which is the negative identity. Thus, the determinant is non-zero, so V_l is a non-degenerate critical point. Since we assumed that the coefficients were strictly increasing (distinct), we can also count the number of positive eigenvalues.

We know that there are $2(l-1)$ positive eigenvalues of the form $-\alpha_l + \alpha_j$ for $j > l$. Therefore, the index of $X = V_l$ is $2(n-2) - 2(l-1)$.

The case where n is odd is easily tackled by the same argument as above. Observe that the vector e_n does not belong to any of the V_j's by construction. The only difference in our Hessian not is that there will be a 2×2 "block" (two rows and two columns selected), such that two $-\alpha_l$'s appear on the diagonal. These contribute two more negative eigenvalues, so we still count the same $2(l-1)$ positive eigenvalues. Thus, we again find the index of $X = V_l$ is $2(n-2) - 2(l-1)$.

For the negatively oriented critical points $X = -V_l$, we get that the index is $2(l-1)$ immediately, since the calculation remains the same, except for a change of sign. $\square$

We've established two of the three steps at this point in the proof. Our final goal, which is the most difficult, is to show the following.

Proposition 33. *There are no other critical points of g beyond $X = \pm V_j$.*

Proof. Let X be a 2-plane with orthonormal basis $\{x, y\}$. Suppose that X is a critical point, not equal to any of the $\pm V_l$. We want to calculate the directional derivative of g at X by picking a particular family of rotations $R(t) \in SO(n)$. We require that $R(0) = I$ and $R'(0) = S$, where S is some skew-symmetric matrix. We should imagine $R(t)$ representing a perturbation of X; we can compute this effect on g as a directional derivative. If X were a critical point, then $\frac{d}{dt}g(R(t)X)|_{t=0}=0$. Thus, taking the contrapositive, if there is a choice of S, such that $\frac{d}{dt}g(R(t)X)|_{t=0}\neq 0$, then we can conclude X is not a critical point of g.

We write

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j \quad \text{and} \quad \mathbf{y} = \sum_{j=1}^{n} y_j \mathbf{e}_j$$

Then $R(t)\mathbf{x} = \sum (R(t)\mathbf{x})_j \mathbf{e}_j$, where

$$(R(t)\mathbf{x})_j = x_j + t\left(-\sum_{\ell=1}^{j-1} s_{\ell j} x_\ell + \sum_{\ell=j+1}^{n} s_{j\ell} x_\ell\right) + O\left(t^2\right)$$

As a convention, we will write $s_{j\ell}$ only for $j < \ell$ and $-s_{j\ell}$ if $\ell < j$. If the lower index of summation is greater than the upper one, we declare that empty sum to be zero.

Similarly, $R(t)\mathbf{y} = \sum (R(t)\mathbf{y})_j \mathbf{e}_j$, where

$$(R(t)\mathbf{y})_j = y_j + t\left(-\sum_{\ell=1}^{j-1} s_{\ell} y_\ell + \sum_{\ell=j+1}^{n} s_{j\ell} y_\ell\right) + \boldsymbol{O}\left(t^2\right)$$

Let $X(t)$ be the 2-plane with orthonormal basis $\{R(t)\mathbf{x}, R(t)\mathbf{y}\}$ and consider.

$$\begin{aligned}
f\left(V_j, X(t)\right) =& \begin{vmatrix} \langle \mathbf{e}_{2j-1}, R(t)\mathbf{x}\rangle & \langle \mathbf{e}_{2j-1}, R(t)\mathbf{y}\rangle \\ \langle \mathbf{e}_{2j}, R(t)\mathbf{x}\rangle & \langle \mathbf{e}_{2j}, R(t)\mathbf{y}\rangle \end{vmatrix} = \begin{vmatrix} (R(t)\mathbf{x})_{2j-1} & (R(t)\mathbf{y})_{2j-1} \\ (R(t)\mathbf{x})_{2j} & (R(t)\mathbf{y})_{2j} \end{vmatrix} \\
=& x_{2j-1}y_{2j} - x_{2j}y_{2j-1} \\
&+ t\left(-\sum_{\ell=1}^{2j-1} s_{\ell(2j)}x_{2j-1}y_\ell + \sum_{\ell=2j+1}^{n} s_{(2j)\ell}x_{2j-1}y_\ell - \sum_{\ell=1}^{2j-2} s_{\ell(2j-1)}x_\ell y_{2j} + \sum_{\ell=2j}^{n} s_{(2j-1)\ell}x_\ell y_{2j}\right. \\
&\left. + \sum_{\ell=1}^{2j-1} s_{\ell(2j)}x_\ell y_{2j-1} - \sum_{\ell=2j+1}^{n} s_{(2j)\ell}x_\ell y_{2j-1} + \sum_{\ell=1}^{2j-2} s_{\ell(2j-1)}x_{2j}y_\ell - \sum_{\ell=2j}^{n} s_{(2j-1)\ell}x_{2j}y_\ell\right) \\
&+ O\left(t^2\right)
\end{aligned}$$

We combine each sum in the first row of sums with the corresponding one directly below it to obtain

$$\begin{aligned}
f\left(V_j, X(t)\right) =& x_{2j-1}y_{2j} - x_{2j}y_{2j-1} \\
&+ t\left(\sum_{\ell=1}^{2j-1} s_{\ell(2j)} \begin{vmatrix} x_\ell & y_{2j-1} \\ y_\ell & x_{2j-1} \end{vmatrix} - \sum_{\ell=1}^{2j-2} s_{\ell(2j-1)} \begin{vmatrix} x_\ell & x_{2j} \\ y_\ell & y_{2j} \end{vmatrix}\right. \\
&\left. + \sum_{\ell=2j+1}^{n} s_{(2j)\ell} \begin{vmatrix} x_{2j-1} & x_\ell \\ y_{2j-1} & y\ell \end{vmatrix} - \sum_{\ell=2j}^{n} s_{(2j-1)\ell} \begin{vmatrix} x_{2j} & x_\ell \\ y_{2j} & y_\ell \end{vmatrix}\right) + O\left(t^2\right)
\end{aligned}$$

Next, observe that the term for $\ell = 2j-1$ in the first sum is zero, as is the term for $\ell = 2j$ in the last sum. Writing $M(p,q)$ for $\begin{vmatrix} x_p & x_q \\ y_p & y_q \end{vmatrix}$ for $p < q$ we have

$$
\begin{aligned}
f\left(V_j, X(t)\right) = & M(2j-1, 2j) \\
& + t\left(\sum_{\ell=1}^{2j-2} s_{\ell(2j)} M(\ell, 2j-1) - s_{\ell(2j-1)} M(\ell, 2j)\right. \\
& \left. + \sum_{\ell=2j+1}^{n} s_{(2j)\ell} M(2j-1, \ell) - s_{(2j-1)\ell} M(2j, \ell)\right) + O\left(t^2\right)
\end{aligned}
$$

This last expression shows us that each s_{pq} and each $M(p,q)$ appears in $f\left(V_j, X\right)$ for at most two values of j (and each of p and q will play the role of $2j-1$ or $2j$, depending on their parity). Also observe that neither $M(2j-1, 2j)$ nor $s_{(2j-1)(2j)}$ appears in the linear terms of $f\left(V_j, X(t)\right)$. This is expected since linear terms would correspond to rotations of the plane $X = X(0)$, or V_j, within themselves, respectively.

Based on these observations, we can express the function $g(X(t))$ as follows. If n is even (so $k = \frac{1}{2}n$),

$$
\begin{aligned}
g(X(t)) &= \sum_{j=1}^{d} \alpha_j f\left(V_j, X(t)\right) \\
&= \sum_{j=1}^{d} \alpha_j M(2j-1, 2j) \\
&+ t\left(\sum_{j=1}^{d-1} \sum_{\ell=j+1}^{d} \left(\alpha_j s_{(2j)(2\ell)} - \alpha_\ell s_{(2j-1)(2\ell-1)}\right) M(2j-1, 2\ell)\right. \\
&+ \left(\alpha_\ell s_{(2j)(2\ell)} - \alpha_j s_{(2j-1)(2\ell-1)}\right) M(2j, 2\ell-1) \\
&+ \left(\alpha_j s_{(2j)(2\ell-1)} + \alpha_\ell s_{(2j-1)(2\ell)}\right) M(2j-1, 2\ell-1) \\
&\left.- \left(\alpha_\ell s_{(2j)(2\ell-1)} + \alpha_j s_{(2j-1)(2\ell)}\right) M(2j, 2\ell)\right) + O\left(t^2\right)
\end{aligned}
$$

If the α_j's are all distinct, we can reorganize this expression as follows:

$$
\begin{aligned}
g(X(t)) &= \sum_{j=1}^{d} \alpha_j (M(2j-1,2j)) \\
&+ t[\sum_{j=1}^{d-1} \sum_{l=j+1}^{d} (\alpha_j M(2j-1,2l) + \alpha_l M(2j,2l-1)) s_{(2j)(2l)} \\
&- (\alpha_l M(2j-1,2l) + \alpha_j M(2j,2l-1)) s_{(2j-1)(2l-1)} \\
&+ (\alpha_j M(2j-1,2l-1) + \alpha_j M(2j,2l)) s_{(2j)(2l-1)} \\
&+ ((\alpha_l M(2j-1,2l-1) - \alpha_j M(2j,2l)) s_{(2j-1)(2l-1)})] + O(t^2)
\end{aligned}
$$

If X is a critical point of g, then the coefficients of all the s_{pq} in this expression must be zero. In other words, for $1 \leqslant j \leqslant d-1$ and $j+1 \leqslant \ell \leqslant d$, we must have

$$
\begin{aligned}
\alpha_j M(2j-1,2\ell) + \alpha_\ell M(2j,2\ell-1) &= 0 \\
\alpha_\ell M(2j-1,2\ell) + \alpha_j M(2j,2\ell-1) &= 0 \\
\alpha_j M(2j-1,2\ell-1) - \alpha_\ell M(2j,2\ell) &= 0 \\
\alpha_\ell M(2j-1,2\ell-1) - \alpha_j M(2j,2\ell) &= 0
\end{aligned}
$$

This the coefficient matrix of this system of four equations in four unknowns $M(2j-1,2\ell), M(2j,2\ell-1)$, $M(2j-1,2\ell-1), M(2j,2\ell)$ has determinant $-\left(\alpha_j^2 - \alpha_\ell^2\right)^2$ and is non-singular if all the α_j's are positive and distinct.

If n is odd, so $k = \frac{1}{2}(n-1)$, there are $n-1$ additional terms of the form:

$$
t \sum_{j=1}^{k} \alpha_j s_{(2j)n} M(2j-1,n) - \alpha_j s_{(2j-1)n} M(2j,n)
$$

which shows that all of the $M(j,n)$ must be zero at a critical point of g in this case. Altogether, this shows that if X is a critical point of g then all of the determinants $M(p,q)$ for $p < q$ must be odd except for those of the form $M(2j-1,2j)$ for $1 \leqslant j \leqslant k$. From the lemma shown after this proof, X is one of the planes V_j, and $\{\mathbf{x}, \mathbf{y}\}$ is an orthonormal basis of this plane. $\square$

Notice that we've also found the maximum and minimum of g as a result. $X = V_1$ has index $2n-4$, matching the dimension of $G_2(\mathbb{R}^n)$. On the other hand, $X = -V_1$ has index 0, representing a minimum.

Now, we address the lemma cited in the proof above.

Lemma 34. *Let X be a 2-plane in $\mathbb{R}^n$ with orthonormal basis*

$$\left\{\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i, \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{e}_i\right\}$$

Let $d = \frac{1}{2}n$, or $\frac{1}{2}(n-1)$ depending on parity of n, $V_j = \operatorname{span}\{\mathbf{e}_{2j-1}, \mathbf{e}_{2j}\}$ for $j = 1, \ldots, d$, and $M(p,q) = \begin{vmatrix} x_p & x_q \\ y_p & y_q \end{vmatrix}$. Then,

1. *For all choices of a, b, c and d,*

$$M(a,b)M(c,d) - M(a,c)M(b,d) + M(a,d)M(b,c) = 0$$

2. *If $M(p,q) = 0$ except for those of the form $M(2j-1, 2j)$, then at most one of the $M(2j-1, 2j) \neq 0$.*

3. *If $M(p,q) = 0$ except for those of the form $M(2j-1, 2j)$, then $x_{2j-1} = x_{2j} = y_{2j-1} = y_{2j} = 0$ except for exactly one value of j.*

Proof. 1. We have

$$
\begin{aligned}
&M(a,b)M(c,d) - M(a,c)M(b,d) + M(a,d)M(b,c) \\
&= (x_a y_b - x_b y_a)(x_c y_d - x_d y_c) - (x_a y_c - x_c y_a)(x_b y_d - x_d y_b) + (x_a y_d - x_d y_a)(x_b y_c - x_c y_b) \\
&= (x_a y_b x_c y_d - x_a y_b y_c x_d - y_a x_b x_c y_d + y_a x_b y_c x_d) \\
&+ (-x_a x_b y_c y_d + x_a y_b y_c x_d + y_a x_b x_c y_d - y_a y_b x_c x_d) \\
&\qquad + (x_a x_b y_c y_d - x_a y_b x_c y_d - y_a x_b y_c x_d + y_a y_b x_c x_d) \\
&\quad = 0
\end{aligned}
$$

2. Suppose $M(2j-1, 2j) \neq 0$ and $M(2\ell-1, 2\ell) \neq 0$ for $j \neq \ell$. Then,

$$
\begin{aligned}
M(2j-1,2j)M(2\ell-1,2\ell) - M(2j-1,2\ell-1)(M(2j,2\ell) + M(2j-1,2\ell)M(2j,2\ell-1) \\
= M(2j-1,2j)M(2\ell-1,2\ell) + 0 + 0 \\
\neq 0
\end{aligned}
$$

3. Since $\{\mathbf{x}, \mathbf{y}\}$ is an orthonormal basis of X, there must be at least one nonzero $M(p,q)$, and we're assuming that only the $M(2j-1,2j)$ are candidates. So there is at least one value of j for which not all of $x_{2j-1}, x_{2j}, y_{2j-1}, y_{2j}$ are zero. If any of $x_{2\ell-1}, x_{2\ell}, y_{2\ell-1}, y_{2\ell}$ are non-zero for some $\ell \neq j$, then at least one of $M(2j-1, 2\ell-1), M(2j-1, 2\ell), M(2j, 2\ell-1)$ or $M(2j, 2\ell)$ will be non-zero, contradicting the hypothesis.

□

Before our conclusion, I want to note that the techniques used above do not generalize nicely for the $k \neq 2$ case (in terms of a manageable calculation). This suggests that completely different techniques are necessary to tackle higher dimensions. We do get a Morse function on $G_{n-2}(\mathbb{R}^n)$ by the orthogonal complement map, but writing this explicitly is difficult due to the complexity of the

Gram-Schmidt algorithm here.

With a Morse function found, we can compute the Morse homology to get an immediate corollary. Recall that the dimension of $G_2(\mathbb{R}^n)$ is $2(n-2) = 2n-4$.

Corollary 35. *If n is odd, the homology (in $\mathbb{R}$ coefficients) of $G_2(\mathbb{R}^n)$ is*

$$H_k(G_2(\mathbb{R}^n);\mathbb{R}) = \begin{cases} \mathbb{R} & k = 0, 2, \ldots, 2n-4 \\ 0 & \textit{else} \end{cases}$$

If n is even, the homology (in $\mathbb{R}$ coefficients) of $G_2(\mathbb{R}^n)$ is

$$H_k(G_2(\mathbb{R}^n);\mathbb{R}) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & k = n-2 \\ \mathbb{R} & k = 0, 2, \ldots n-4, n, \ldots, 2n-4 \\ 0 & \textit{else} \end{cases}$$

Proof. Critical points represent generators in the homology in degree corresponding to its index. We can count $2k$ critical points of the form $\{\pm V_j\}_{j=1}^k$. Recall the index of V_j is $2n-2j-2$, while $-V_j$ is $2j-2$.

- For n odd, all the indices have distinct values. The index is always even and ranges from 0 to $2(n-2)$ for $j \in [1, \frac{n-1}{2}]$.
- For n even, similar reasoning holds. Note however, that when $j = \frac{1}{2}n$, V_j and $-V_j$ have the same index $n-2$, resulting in $H_{n-2} = \mathbb{R} \oplus \mathbb{R}$.

$\square$

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