Besicovitch Sets & Hausdorff Dimension

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Abstract

Let S be a set that contains a line in every direction. How small can S be? What do we mean by small sets in mathematics? This write-up seeks to construct these special sets and demonstrate some surprising properties.

Contents

1 Problem

Definition 1. A set $S \subseteq \mathbb{R}^n$ is called a **Besicovitch** (or **Kakeya**) if we can fit a unit length line segment of every direction inside S.

In the plane, a circle with diameter 1 will be a Besicovitch set, as well as an equilateral triangle with height 1. For higher dimensions, the problem becomes even more complicated. Notice that these shapes will not all

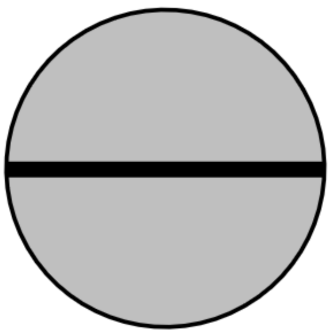


Figure 1: Circle with radius $\frac{1}{2}$

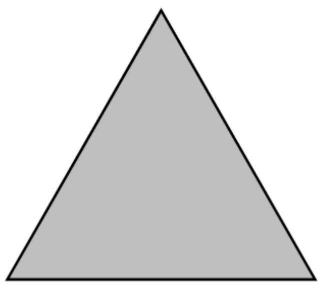


Figure 2: Triangle with height 1

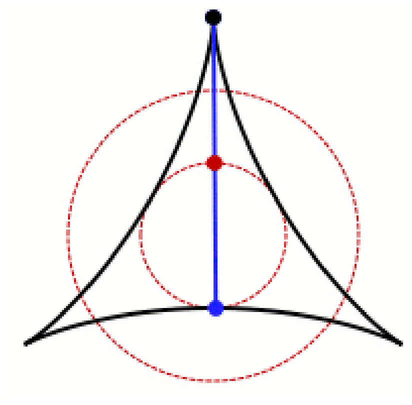


Figure 3: Deltoid (Wikipedia)

have necessarily the same area, nor does a Kakeya needle set need to be convex, as given by the curved triangle shape above. The classical Besicovitch set problem asks what is the smallest amongst all Besicovitch sets.

2 Background

First, we need to ask what we even mean by "the smallest." While we'd like to define size of a set S by area, this first approach does not seem to clearly extend to all sets. For example, consider the line from a point a to b, denoted as $E = \{a(1-t) + bt \mid t \in [0, 1]\}$. Should the size of this set be 0, corresponding to our intuitive notion that lines take up no area, or $\epsilon > 0$, since the set is not empty? What is the size of a point set, or subset of the line containing only points such that at least one coordinate is irrational?

To begin measuring more exotic sets in any space, mathematicians usually use something called a measure.

Definition 2. Let X be a set and Σ a σ-algebra over X. We call a function μ from $\Sigma \to [0, \infty)$ a measure if it satisfies two properties

1. $\mu(\emptyset) = 0$.
2. For all $E \in \Sigma$, $\mu(E) \geq 0$.
3. $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$, where E_i are all disjoint

Roughly speaking, a measure assigns a notion of size to subsets of a set. Readers may be familiar with the Lebesgue measure $\mathcal{L}$, which generalizes the traditional Riemann integral. The Lebesgue measure is defined on Borel subsets of $\mathbb{R}$, all sets that are the countable union/intersection of open and closed subsets of $\mathbb{R}$. This Lebesgue measure extends naturally to $\mathbb{R}^n$ using product measure constructions; we will denote $\mathcal{L}^n$. For more specific details, the reader should look at [2].

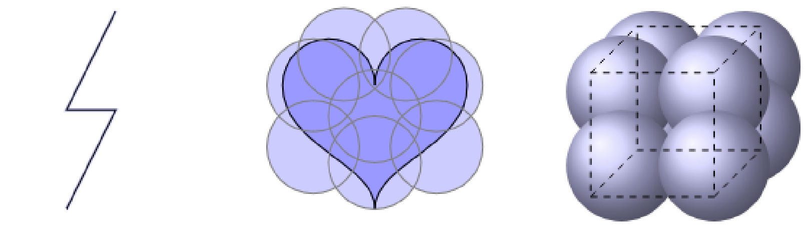
However, measures on a space are hardly unique; there may be many ways to assign size to sets. For many problems in geometric measure theory, Hausdorff measure is another convenient notion.

Definition 3. Let $S \subseteq \mathbb{R}^n$. For all $s \geq 0$ and $\delta > 0$, let

$$\mathcal{H}_\delta^s(S) = \inf\{\sum_j (\text{diam} A_j)^s \mid S \subseteq \bigcup_j A_j, \text{diam}(A_j) \leq \delta \text{ for all } j\} \text{ where } \text{diam}(S) := \sup_{x,y \in S} d(x,y)$$

Define the **s-Hausdorff measure** $\mathcal{H}^s(S) = \lim_{\delta \to 0} \mathcal{H}_\delta^s S$ if the limit exists.

Intuitively, the s-Hausdorff measure seeks to minimize the size of a particular covering A_j. We take all coverings of our set S using a family of sets A_j, whose sizes are less than δ (we use diameter to essentially measure the size of each part of the cover). Then, the size of the cover is $\sum_j (\text{diam} A_j)^s$, where the exponent of s corresponds to the scaling of volume in proportion to the diameter roughly in some dimension. For example, if $s = 2$, then we can consider our covering to be by balls B_j of diameter d_j. The area of each ball is $\frac{\pi d_j^2}{4}$, corresponding to a quadratic relationship between diameter and dimension. Notice that s could very well be non-integer; this is particularly useful in the study of fractals.



$\propto \Sigma\, \text{diam}(B)$ $\qquad$ $\propto \Sigma\, \text{diam}(B)^2$ $\qquad$ $\propto \Sigma\, \text{diam}(B)^3$

Figure 4: Examples of Coverings ([4])

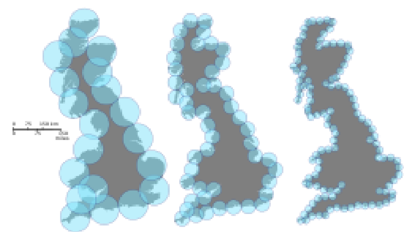


Figure 5: Famous example of use on Great Britain Coast

Example 2.1. Consider the line segment $L(t) = \{a(1-t) + bt \mid t \in [0,1]\}$ in $\mathbb{R}^2$ with length l. We take a cover using balls B_j of radius $\frac{l}{n}$ centered at n points along the line segment. Then, this is a cover for all $n \geq 1$ and the s-Hausdorff measure will be upper bounded by $\sum_{j=1}^{n-1} \text{diam}(B_j)^s = l^s \cdot \sum_{j=1}^{n-1} \frac{1}{n^s}$. If $s = 1$, then $0 \leq \mathcal{H}^1(L) \leq l$.

Example 2.2. Consider an n-dimensional unit cube C in $\mathbb{R}^n$. Then, C is the union of t^n cubes. Each cube has side length $\frac{1}{t}$, so the diameter is the length of the diagonal, which is $\frac{\sqrt{n}}{t}$. Let t be large enough, so that this diagonal is less than $\delta > 0$. Then, $\mathcal{H}_\delta^s \leq \sum_{j=1}^{t^n} (\frac{n}{t})^s = t^n (\frac{\sqrt{n}}{t})^s$. In particular, note that for $s = n$, this becomes $\mathcal{H}_\delta^s \leq n^{\frac{n}{2}}$, which is finite.

Frequently, it is hard to compute precisely the s-Hausdorff measure. It is easy to establish upper bounds by smart choices of coverings, but lower bounds are far more difficult. However, Hausdorff measures allow us to define something much more useful.

Definition 4. Let $S \subseteq \mathbb{R}^n$. The **Hausdorff dimension** of S is $d := \inf s \mid \mathcal{H}^s(S) = 0$, or equivalently, $\sup s \mid \mathcal{H}^s(S) = \infty$. In other words, the Hausdorff dimension is the least value of $s \geq 0$ such that $H^s(S)$ is non-zero.

First, it is not immediately clear that this dimension is well-defined. The following theorem tells us that we can find a unique value for the dimension if it exists (thus, we can speak of the Hausdorff dimension).

Theorem 2.1. *For any $S \subseteq \mathbb{R}^n$, $\mathcal{H}^s(S)$ is finite, non-zero for at most one value of s. In fact, if $\mathcal{H}^s(S) = C < \infty$, we get that the following is true:*

(a) For all $t > s$, $\mathcal{H}^t(S) = 0$.

(b) For all $t < s$, $\mathcal{H}^t(S) = \infty$.

Proof. Fix s such that $\mathcal{H}^s(S) = C < \infty$. Suppose that $t > s$. Then, take a cover of S by A_j, where $\operatorname{diam}(A_j) \leq \delta$. Then, we can write

$$\sum_j \operatorname{diam}(A_j)^t = \sum_j \operatorname{diam}(A_j)^s \operatorname{diam}(A_j)^{t-s} \leq \delta^{t-s} \sum_j \operatorname{diam}(A_j)^s$$

Since this held for any initial choice of cover A_j, we can take the infimum of both sides and note that this means $\mathcal{H}^t_\delta(S) \leq \delta^{t-s}\mathcal{H}^s_\delta(S)$. Taking $\delta \to 0$, we find

$$\mathcal{H}^t(S) = \lim_{\delta\to\infty} \mathcal{H}^t_\delta(S) \leq \lim_{\delta\to\infty} \delta^{t-s}\mathcal{H}^s_\delta(S) \leq \lim_{\delta\to\infty} \delta^{t-s} \cdot \mathcal{H}^s(S) = \lim_{\delta\to\infty} \delta^{t-s} \cdot C = 0$$

For the second part, suppose instead that $t < s$. Then, similar to before, we have

$$0 \neq C = \mathcal{H}^s(S) \leq \lim_{\delta\to 0} \delta^{s-t}\mathcal{H}^t(S)$$

If $\mathcal{H}^t$ were finite, then the limit distributes over the product and we arrive at a contradiction. □

Hausdorff dimension represents the critical value in which our Hausdorff measures change from ∞ to 0. Using theorem 2.1 and our previous examples, we can see that the Hausdorff dimension of a line segment is 1 and the Hausdorff dimension of a n-cube is n. The Hausdorff dimension of $\mathbb{R}^n$ is n.

As an aside, it is possible that the s-Hausdorff measures never take a finite and non-zero value on some set S. As an example, one can construct cantor sets with arbitrary Hausdorff measure $C > 0$ in a particular dimension s. Taking a countable union of these with $\frac{1}{n}$, the s-Hausdorff measure would be 0. This doesn't pose an issue for our concept of dimension, since we define it using infimums/supremums.

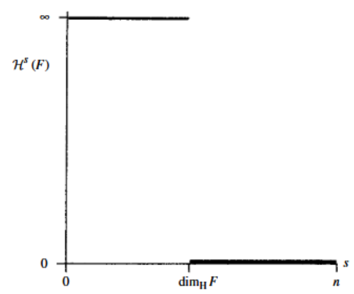


Figure 2.3 Graph of $\mathcal{H}^s(F)$ against s for a set F. The Hausdorff dimension is the value of s at which the 'jump' from ∞ to 0 occurs

Figure 6: Image from Pg 31, [3]

3 Construction of Besicovitch Sets

Now, with the appropriate definitions in hand, let's return to Besicovitch sets. One method of constructing Besicovitch sets with arbitrary Lebesgue measure $\epsilon > 0$ uses two main tricks - a sliding action and Pál joins.

Heuristically, consider starting with a unit length line segment in the plane. It does not add area to slide it along the extended line that contains that line segment. Next, we might rotate the line a bit by some angle θ. Then, we can slide it further out along the new line that contains it, rotate by another angle θ, and repeat until the line segment has been in every possible direction. The area of the Besicovitch set is just the area traced by the line segment during the rotation step. This area can be made sufficiently small by sending the line segment far enough away, such that the area of the rotation by θ is negligible. Surprisingly, one can do better than constructing a Besicovitch set with arbitrarily small Lebesgue measure. We follow mostly the steps from Chapter 11 of [1].

Theorem 3.1. *There exists a compact Besicovitch set $B \subseteq \mathbb{R}^n$ with Lebesgue measure $\mathcal{L}^n(B) = 0$.*

Proof. It suffices to find the desired set B in $\mathbb{R}^2$; then $S = B \times \mathbb{R}^{n-2}$ would still be Besicovitch and have Lebesgue measure 0. Let C be a compact set in the plane that is bounded between $x = 0$ and $x = 1$. Define $\pi(x, y) : \mathbb{R}^2 \to \mathbb{R}$ to be the projection to the first component, and $p_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ to be the orthogonal projection onto the unique line through the origin with angle θ from the x-axis.

Let's define $C(\frac{1}{4})$ as the four corners Cantor set, formed by iteratively removing everything but the four corners of each square at the n-step, then taking the intersection of these. See the image below. Let's fix C to be a copy of $C(\frac{1}{4})$ after being rotated by $\frac{\pi}{4}$ and scale down by a factor of $\sqrt{2}$. Notice that $\pi(C)$ covers $[0,1]$.

Consider the lines $L(a,b) = \{(x, ax+b) \mid x \in \mathbb{R}\}$ for each $(a,b) \in C$ and let the union of these lines be B. By construction, B is the union of countably many closed, compact spaces, so it is Borel, so measurable. If we restrict B so to that $x \in [0,1]$, then B is bounded, so compact. Furthermore, since $\pi(C) = [0,1]$ for all $a \in [0,1]$, B contains a line $l(a,b)$ for some b. This means that B contains one line segment with an angle $\theta \in [0, \frac{\pi}{4}]$ with the x-axis. By taking 4 copies of B rotated, we get every possible angle in $[0, \pi]$, so our set B is a compact Besicovitch set.

We now need to show that the Lebesgue measure $\mathcal{L}^2(C) = 0$. Consider the vertical line $x = t$, denoted as V_t. Then,

$$B \bigcap V_t = \{(t, at+b) \mid (a,b) \in C\} = \{t\} \times \{at+b \mid (a,b) \in C\}$$

Note that the second component is basically the image of some projection of C onto a line with angle θ. We know that the Lebesgue measure $\mathcal{L}^1(p_\theta(C)) = 0$ (See a more rigorous explanation in Chapter 4 of [1]). Since the two-dimensional Lebesgue measure is a product measure, $\mathcal{L}^2(B) = \mathcal{L}^1(\{t\})\mathcal{L}^1(\{at+b \mid (a,b) \in C\}) = 0$ as desired. $\square$

In the construction, above, notice that the fact our resulting Besicovitch set is compact implies by Heine-Borel that it is actually bounded, meaning it is contained in some ball. This is a notable difference between this construction and other common ones, such as those using Perron Trees, or Pál joins. These two constructions can construct Besicovitch sets, but may not be bounded.

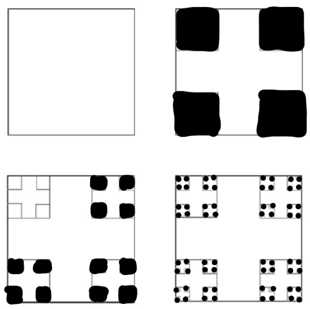


Figure 7: Image of Cantor's 4-corners set

Despite being Lebesgue measure zero, roughly speaking, we expect Besicovitch sets to be at least Hausdorff dimension 1, since they contain line segments. On the other hand, they cannot have Hausdorff dimension greater than 2, since they embed into the plane.

Theorem 3.2. *For every Besicovitch set $B \subseteq \mathbb{R}^n$, the Hausdorff dimension of B, denoted* $\dim B$*, is at least* 2.

Proof. Let B be a Besicovitch set in $\mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathbb{R}^2$ be the projection map onto the xy-plane. It suffices to show that $\dim \pi(B) \geq 2$, since projections can only decrease the Hausdorff dimension.

Assume now that $B \subseteq \mathbb{R}^2$. Define $I(a,b,q)$ to be the line segment $\{(q+t, at+b) \mid 0 \leq t \leq \frac{1}{2}\}$, where $a \in (0,1)$, $b \in \mathbb{R}$, and $q \in \mathbb{Q}$. Notice that these lines segments join the points (q,b) and $(q + \frac{1}{2}, b + \frac{a}{2})$.

Let C_q be the set of (a,b) such that the line $I(a,b,q)$ are contained in B. For every $a \in (0,1)$, we get a line of this form in B. Thus, the projection $\pi(\bigcup_{q \in \mathbb{Q}} C_q)$ is $(0,1)$. Since the 1-Hausdorff measure of the interval is non-zero, we must have that $\mathcal{H}^1(\bigcup_{q \in \mathbb{Q}} C_q) > 0$. Using countable additivity of the measure, this means some $\mathcal{H}^1(C_q) > 0$ for at least one $q \in \mathbb{Q}$.

We have that $\dim\{at+b \mid (a,b) \in C_q\} = 1$ (Hausdorff dimension of lines) for all $t \in \mathbb{R}$. Now, we can write this as an intersection with vertical lines similar to the proof of Theorem 3.1. For $t \in [0, \frac{1}{2}]$

$$\{q+t\} \times \{at+b \mid (a,b) \in C_q\} \subset B \cap \{(x,y) \mid x = q+t\}$$

These vertical sections of B each have Hausdorff dimension 1. Since we have this for all $t \in [0, \frac{1}{2}]$, the original set B must have dimension 2. $\square$

In particular, this means that Besicovitch sets in $\mathbb{R}^2$ have a Hausdorff dimension of exactly 2.

4 Extensions

The topics discussed in this write-up are only the beginnings of a rich class of problems, many of which are open.

(a) What are the size of Besicovitch sets under other notions of dimension? Two other common notions are the Minkowski and packing dimensions.

(b) Define a Besicovitch (or Kakeya) needle set $S \subseteq \mathbb{R}^2$ to be a set in which you can rotate a unit length line segment completely within S. Notice that this is a strictly stronger property than just containing a line segment in every direction. Surprisingly, despite there existing a Kakeya needle set with Lebesgue measure ϵ for all $\epsilon > 0$, there does not exist one with measure 0.

(c) While we could determine the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^2$, our results only show that for Besicovitch sets $B \subseteq \mathbb{R}^n$, $2 \leq \dim B \leq n$. It is conjectured that $\dim B = n$.

(d) What happens if instead of unit length lines segments, we require lines, or other general shapes, such as circles? Does their exist natural lower bounds on the size of sets containing these objects?

(e) Let S be a compact set in $\mathbb{R}^n$ with Hausdorff dimension $d > \frac{n}{2}$. Does the distance set $D(S) := \{d(x, y) \mid x, y \in S\}$ have strictly positive Lebesgue measure? This is called Falconer's conjecture and remains unsolved at the time of writing.

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