Introduction to Fundamental Groups

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July 14, 2024

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- Compactness ($[0, 1]$ v.s. $(0, 1)$)
- Connectedness ($\mathbb{R}$ v.s. $\mathbb{Q}$)
- "Deleting a point" ($\mathbb{R}$ v.s. $\mathbb{R}^2$)
- Separation Axioms/Countable/Lindelöf

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Is this good enough?

No. In fact, we run into trouble fast. Does there exist a homeomorphism f between the following?

- $\mathbb{R}^2 \to \mathbb{R}^3$
- $S^2 \to T$
- $S^2 \to T\#T$

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- $S^2 \to T \# T$

The answer is unclear. None of our previous invariants are helpful.

Our solution? The fundamental group. First, some definitions.

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Definition

Let f, g be two continuous mapping between topological spaces $X \to Y$. We say f is homotopic to g if there is a continuous map $H : X \times I \to Y$, such that

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

H above is called a homotopy between f and g.

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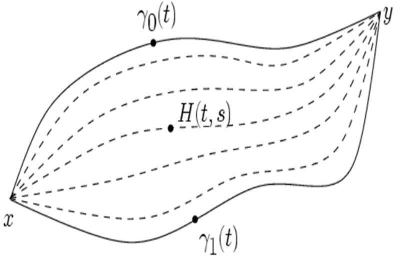
We can think of this like frames of a movie.


x_0
x_1


$\gamma_0(t)$
y
$H(t,s)$
x
$\gamma_1(t)$

Definition

A path in Y from x_0 to x_1 is a continuous map $f : I \to Y$ such that $f(0) = x_0$ and $f(1) = x_1$.

We can consider the special case where $X = I$, the closed unit interval.

Definition

Two paths $f, g : I \to Y$ are path homotopic if

(1) $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$

(2) there is a continuous map $H : I \times I \to Y$, such that

$$H(x, 0) = f(x) \text{ and } \mathrm{H(s,\ 1) = g(x)}$$

$$H(0, t) = x_0 \text{ and } H(1, t) = x_1$$

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$$H(x, 0) = f(x) \text{ and H(s, 1) = g(x)}$$

$$H(0, t) = x_0 \text{ and } H(1, t) = x_1$$

Is path homotopic equivalent to homotopic paths? No. The endpoints of the path may "wiggle" during a normal homotopy.

Lemma

Being homotopic and path homotopic are both equivalence relations.

Proof.

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1. (Reflexive) $H(x,t) = f(x)$ is a homotopy from f to itself
2. (Symmetric) Given H, we can reverse it via $G(x,t) = F(x, 1-t)$
3. (Transitive) Given homotopies F from f to g and G from g to h, we can construct

$$H(x,t) = \begin{cases} F(x,2t) \text{ for } t \in [0, \frac{1}{2}] \\ G(x, 2t-1) \text{ for } t \in [\frac{1}{2}, 1] \end{cases}$$

□

Now, we can create path homotopy equivalence classes $[f]$.

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What paths in $\mathbb{R}^2$ are homotopic to one another? Are there two maps that are not homotopic? Any maps f, g from $X \to \mathbb{R}^2$ are homotopic by the homotopy

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$$H(x, t) = (1 - t)f(x) + tg(x)$$

This is called the straight-line homotopy. This means there is only a single equivalence class $[e]$.

But intuitively, some of the spaces should have more than one equivalence class. For example,

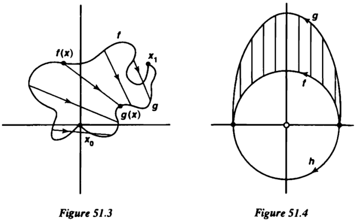

Figure: Straight-line homotopy in $\mathbb{R}^2$ on right, $[f]$ and $[h]$ are not the same on left

$$f(t) = (\cos \pi t, \sin \pi t)$$
$$g(t) = (\cos \pi t, 2 \sin \pi t)$$
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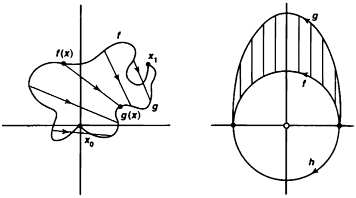

Figure 51.3

Figure 51.4

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How many equivalence classes are there?

Let's return to studying these path-homotopy classes. Can we get a better algebraic structure on it?

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Definition

If $f : I \to X$ from x_0 to x_1 and $g : I \to X$ from x_1 to x_2, then we define the product

$$f \star g(t) = \begin{cases} f(2t) \text{ for } t \in [0, \frac{1}{2}] \\ g(2t-1) \text{ for } t \in [\frac{1}{2}, 1] \end{cases}$$

Think of traveling f, then g, both at double speed.

- What properties does this operation satisfy on the set of paths in X?

- What properties does this operation satisfy on the set of paths in X? Not much as it requires $f(1) = g(0)$.
- However, let's restrict our attention to loops (paths that satisfy $f(0) = f(1)$). If we consider the set of all loops that start at a fixed point x_0, do we have a group?

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- However, let's restrict our attention to loops (paths that satisfy $f(0) = f(1)$). If we consider the set of all loops that start at a fixed point x_0, do we have a group? Technically no; We don't have associativity, identities, or inverses.
- But if we consider path-homotopy classes of loops, then we are there!

Definition

We call (X, x_0), where $x_0 \in X$, a based space at x_0. Given such a space, we define the fundamental group, or first homotopy group, of (X, x_0) as the set of path-homotopy classes of loops based at x_0. It is denoted by $\pi_1(X, x_0)$,

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How much does this base-point matter? For path-connected spaces, we can ignore the choice of basepoint.

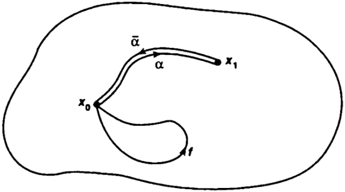

Figure: Basepoint x_0 is shifted by a conjugation by $[\alpha]$ to x_1. Thus, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. We will usually consider a path connected component of topological spaces.

What is the fundamental group of the following?

- $(\mathbb{R}^2, (0,0))$
- $(\mathbb{R}, 0)$
- $(S^2, \text{ north pole})$

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Is there any non-trivial fundamental group?

What do we think $\pi_1(S^1, (1, 0))$ is?

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Theorem

The map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ *given by* $n \to [w_n]$, *where* $w_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ *for* $s \in [0,1]$ *is a loop at* $(1,0)$, *is an isomorphism. In other words,* $\pi_1(S^1) \cong \mathbb{Z}$

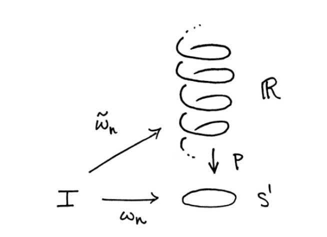
Proof.

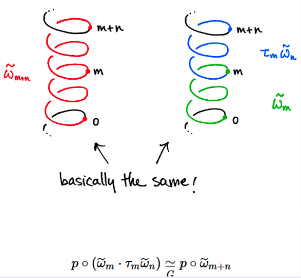
The sketch of the proof is a few parts.

- Φ is a group homomorphism (and well-defined)
- Φ is surjective
- Φ is injective

Verifying homomorphism.

Consider the following diagrams.





Verifying homomorphism.

$$
\begin{aligned}
\Phi(m)\cdot\Phi(n) = [\omega_m]\cdot[\omega_n] &= [p\circ\widetilde{\omega}_m]\cdot[p\circ\widetilde{\omega}_n] \\
&= [p\circ\widetilde{\omega}_m]\cdot[p\circ(\tau_m\widetilde{\omega}_n)] \\
&= [(p\circ\widetilde{\omega}_m)\cdot(p\circ\tau_m\widetilde{\omega}_n)] \\
&= [p\circ(\widetilde{\omega}_m\cdot\tau_m\widetilde{\omega}_n)] \\
&= [p\circ\widetilde{\omega}_{m+n}] \\
&= [\omega_{m+n}] \\
&= \Phi(m+n)\,.
\end{aligned}
$$

because shifting $\widetilde{\omega}_n$ up/down by m doesn't change the fact that the projection of $\widetilde{\omega}_n$ (or $\tau_m\widetilde{\omega}_n$) is still a path that winds n times around S^1!

because winding around S^1 m times and then winding around n times (the 3rd line) is the same as winding $m+n$ times (the 4th line)

because $\widetilde{\omega}_m\cdot\tau_m\widetilde{\omega}_n \simeq \widetilde{\omega}_{m+n}$

Lemma (Unique Lifting of Paths)

For each path $f : I \to S^1$ s.t. $f(0) = x_0$ and each $\tilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \mathbb{R}$, such that $\tilde{f}(0) = \tilde{x_0}$.

Surjective.

Let $[f] \in \pi_1(S^1)$. We WTS there is some n s.t. $\Phi(n) = [f]$ aka $f \simeq w_n$. Note that $0 \in p^{-1}((1,0)) = \mathbb{Z}$ by construction. By unique lifting of paths, we get a lift $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$. Now,

$$(p \circ \tilde{f})(1) = f(1) = (1,0) \in S^1$$

Thus, $\tilde{f}(1)$ must be an integer n; We find $\tilde{f} \simeq \tilde{w}_n$.

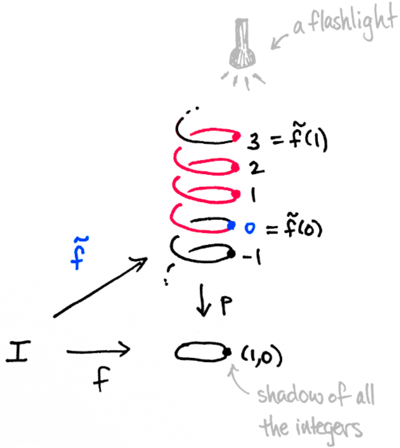


Figure: $f = p \circ \tilde{f} \simeq p \circ \tilde{w}_n = w_n$. Thus, we find $[f] = [w_n] = \Phi(n)$

Lemma (Unique Lifting of Homotopies)

For each homotopy $H(x,t) : I \times I \to S^1$ of paths with $H(0,t) = x_0$ and for each $\tilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\tilde{H}(x,t) : I \times I \to \mathbb{R}$, such that $\tilde{H}(0,t) = \tilde{x_0}$

Injective.

Suppose that $\Phi(m) = \Phi(n)$, which means $w_m \simeq w_n$. We WTS $m = n$. Let $H(x,t)$ be the homotopy from w_m to w_n. Then, note $H(0,t) = (1,0)$ is the starting point of both as $w_m(0) = w_n(1) = (1,0)$, and $0 \in p^{-1}(1,0) = \mathbb{Z}$. Then, we can apply unique lifting of paths to get a unique lift $\tilde{H}(x,t)$ such that $\tilde{H}(0,t) = 0 \in \mathbb{R}$. In particular, the lift is unique, so

$$\tilde{H}(x,0) = \tilde{w_m}(x) \text{ and } \tilde{H}(x,1) = \tilde{w_n}(x)$$

Injective.

By uniqueness of lifts of paths, we find out that the shared endpoints of the loops of $\tilde{H}(1, t)$ must be the same for all t. Since this is a map into integers, we know that the endpoints, which are in $\mathbb{Z}$, are the same. $\square$

Conclusion

- Homeomorphic spaces have the same fundamental group, but the converse is not true $(\pi_1(S^2) = \pi_1(*) = 0)$
- The fundamental group provides a way to use algebraic tools to study topological spaces
- Higher homotopy groups, homology, and more!

Bibliography & Questions

- Munkres' Topology
- Math3MA Blog
- Notes by Aaron Landesman
- Hatcher's Algebraic Topology
- Google Images