Privacy-Preserving Computation with Fully Homomorphic Encryption

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Our Work on FHE

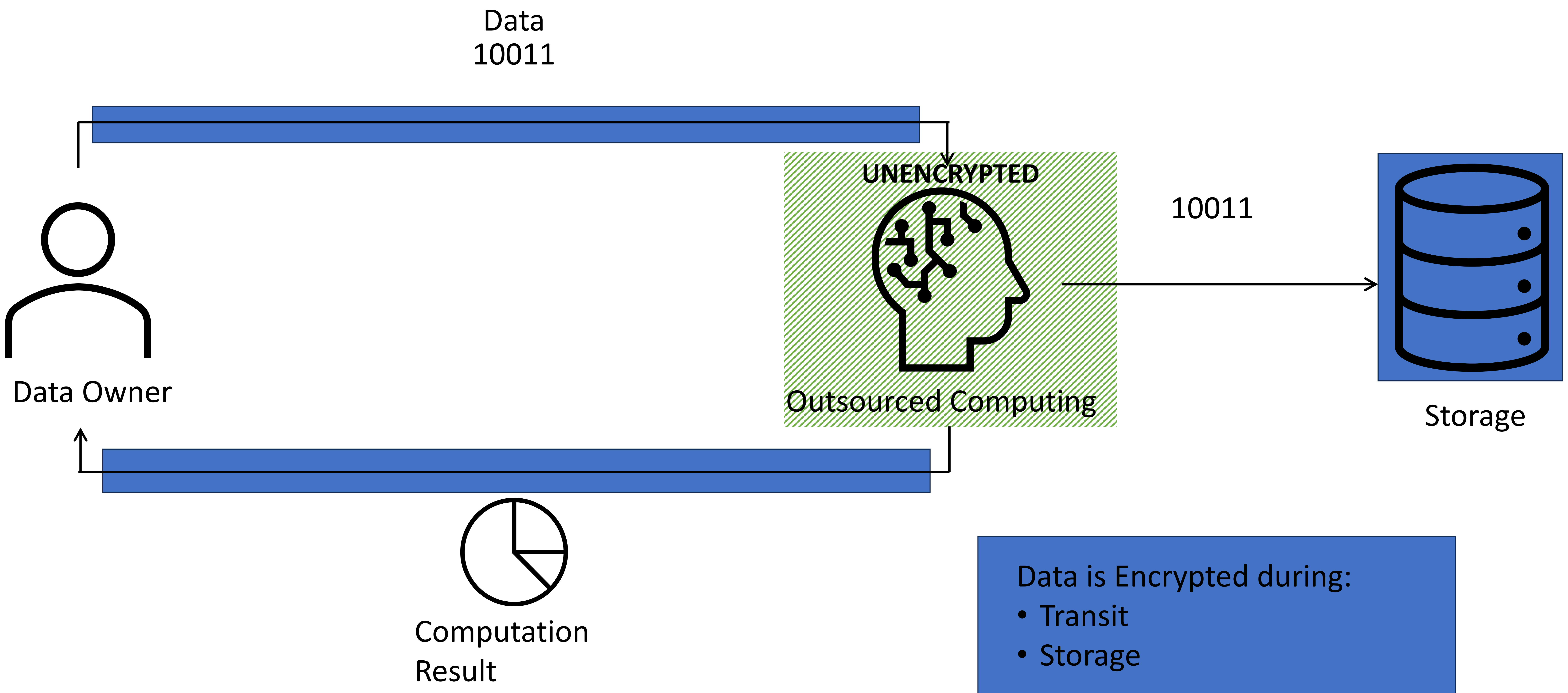
- Since 2015
- Primarily focused on privacy-preserving machine learning (PPML)
  - Convolutional Neural Networks (CNN)
  - Recurrent Neural Networks (RNN)
  - GPU Acceleration
  - Image Transformers
  - Memory Efficiency
  - ML pruning/ compression
  - Ongoing Work: compiler tooling, applications & use cases
- Published 20+ papers

Outline

1. Introduction to Fully Homomorphic Encryption (FHE)
2. Mathematical Background
3. FHE schemes and their Properties
4. CKKS: The Details
5. Computation with FHE
6. Hands-on

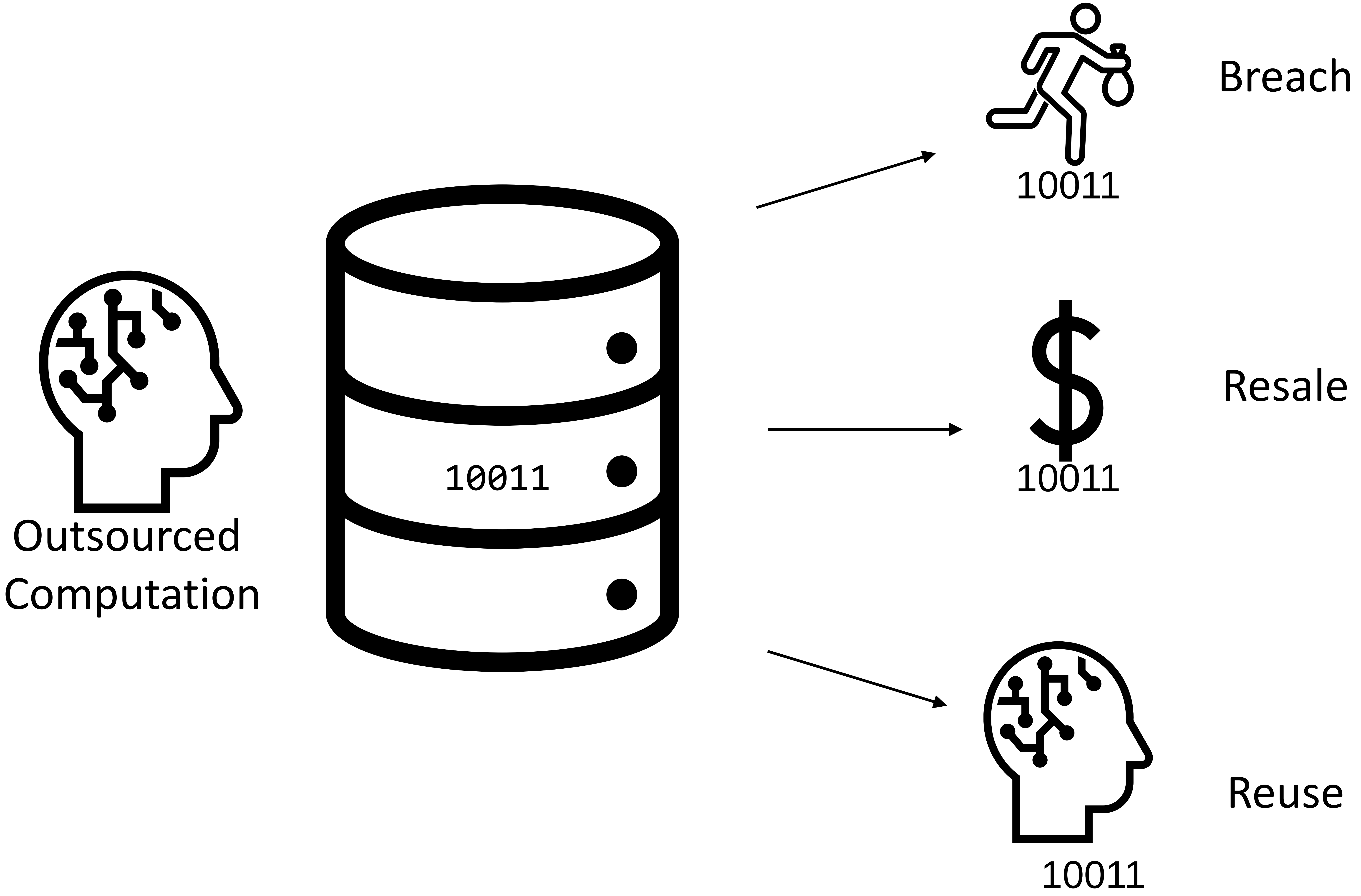
Introduction to Fully Homomorphic Encryption (FHE)

The Importance of Privacy Preserving Computation

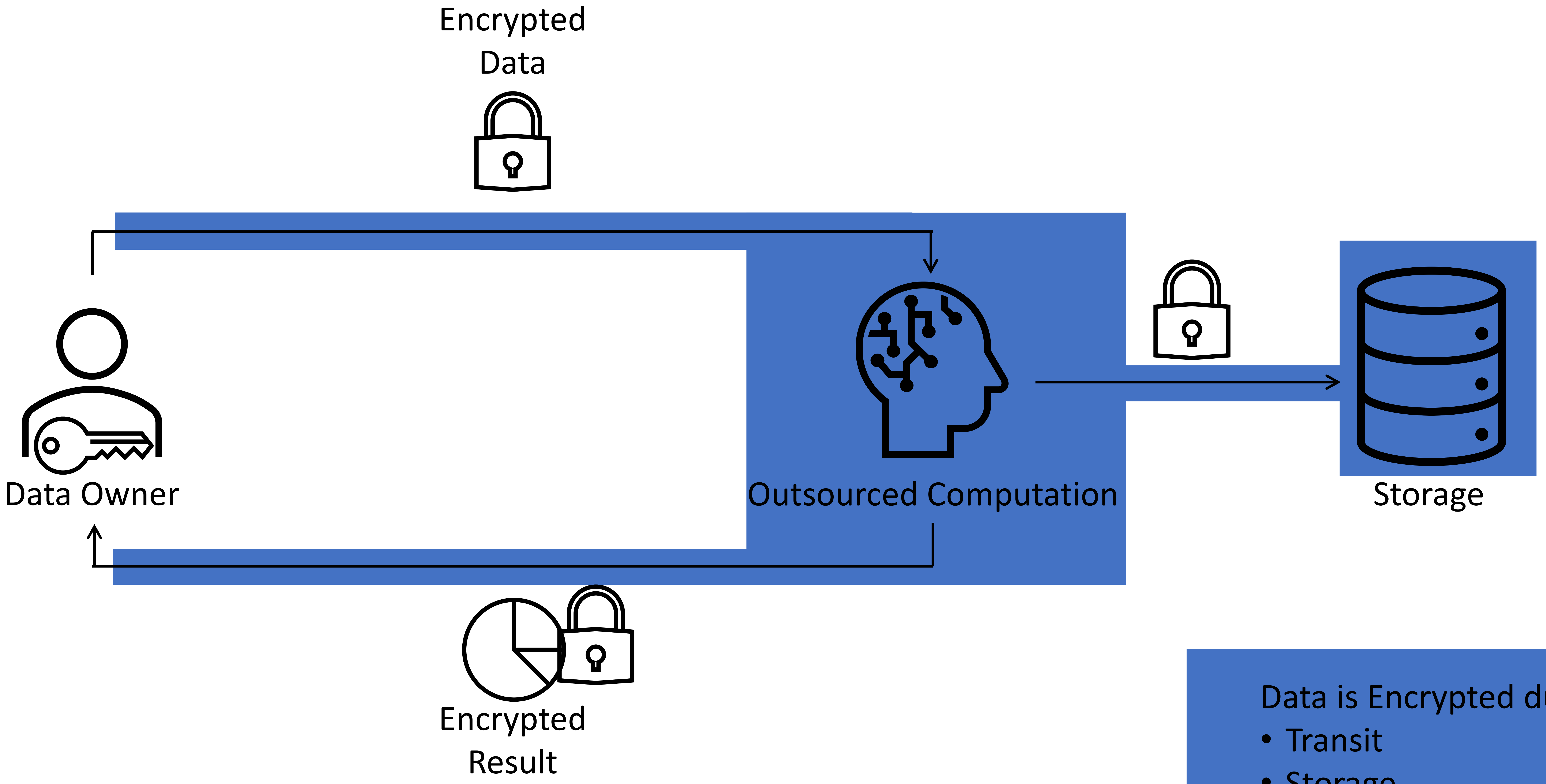


Data is Encrypted during:
- Transit
- Storage

Data Leakage



Encrypted Computation



Data is Encrypted during:
- Transit
- Storage
- **Computation**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Homomorphic Encryption

- HE allows for operation on the encrypted data without the need for decryption or access to the secret key
- Perfect for outsourced computation…..
- … but there is a catch (multiple catches, actually)
  - Only supports addition and multiplication
  - Noise growth
  - Execution time
  - Size
- Pro: No interactions that are not in the plaintext version
- Con: High computational cost
- Basically trade-off network traffic for CPU load

The High-level View

- Encrypt data by adding noise
- Decryption removes the noise
- Multiplication increases the noise in the Ciphertext
- Too much noise prevents correct decryption
- With leveled FHE we can configure the number of multiplications we can perform
- Once all levels are used we can do no further computation
- With bootstrapping we can refresh the number of levels…
- … unlimited computation!

Homomorphic Encryption - Types

- Partially homomorphic encryption
  - Only support one type of operation like addition or multiplication
- Somewhat homomorphic encryption
  - Support addition and multiplication but not all types of circuits
- Leveled fully homomorphic encryption
  - Support addition and multiplication and circuits with a predefined depth
- Fully homomorphic encryption
  - Support addition and multiplication and circuits of arbitrary depth

Fully Homomorphic Encryption – History 1/2

- First proposed in 1978 after the publication of RSA
  - 30 years of partial results
- First FHE
  - Done by Craig Gentry in 2010
  - Based on ideal lattices
  - Encryption adds noise to the data
  - Operations increase the noise
  - If the noise is too large decryption is impossible
  - Introduction of the bootstrapping trick
    - Use scheme that can evaluate its own decryption function homomorphically
    - This refreshes the noise and allows for further computation

Fully Homomorphic Encryption – History 1/2

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- Improved FHE schemes
  - Mostly based on Ring Learning with Errors Problem (RLWE)
  - BGV https://eprint.iacr.org/2011/277
  - BFV https://eprint.iacr.org/2012/144
  - CKKS https://link.springer.com/chapter/10.1007%2F978-3-319-70694-8_15
  - Support for bootstrapping or leveled FHE
  - Faster bootstrapping
  - TFHE https://tfhe.github.io/tfhe/
    - Supports binary gates
    - High level operations need to pieced together

Open-Source Libraries

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- HELib (https://github.com/HomEnc/HElib)
  - One of the earliest libraries
  - Supports BGV and CKKS
- HEAAN (https://github.com/snucrypto/HEAAN)
  - Original implementation of the CKKS scheme
- TFHE (https://github.com/tfhe/tfhe)
  - Original implementation of the TFHE scheme
- OpenFHE (https://github.com/openfheorg/openfhe-development)
  - Implementation of the most common schemes with bootstrapping and multiparty support
- SEAL (https://github.com/microsoft/SEAL)
  - BFV and CKKS support
  - Used to be popular among researchers
- Lattigo (https://github.com/ldsec/lattigo)
  - Go implementation of various schemes

Mathematical Background

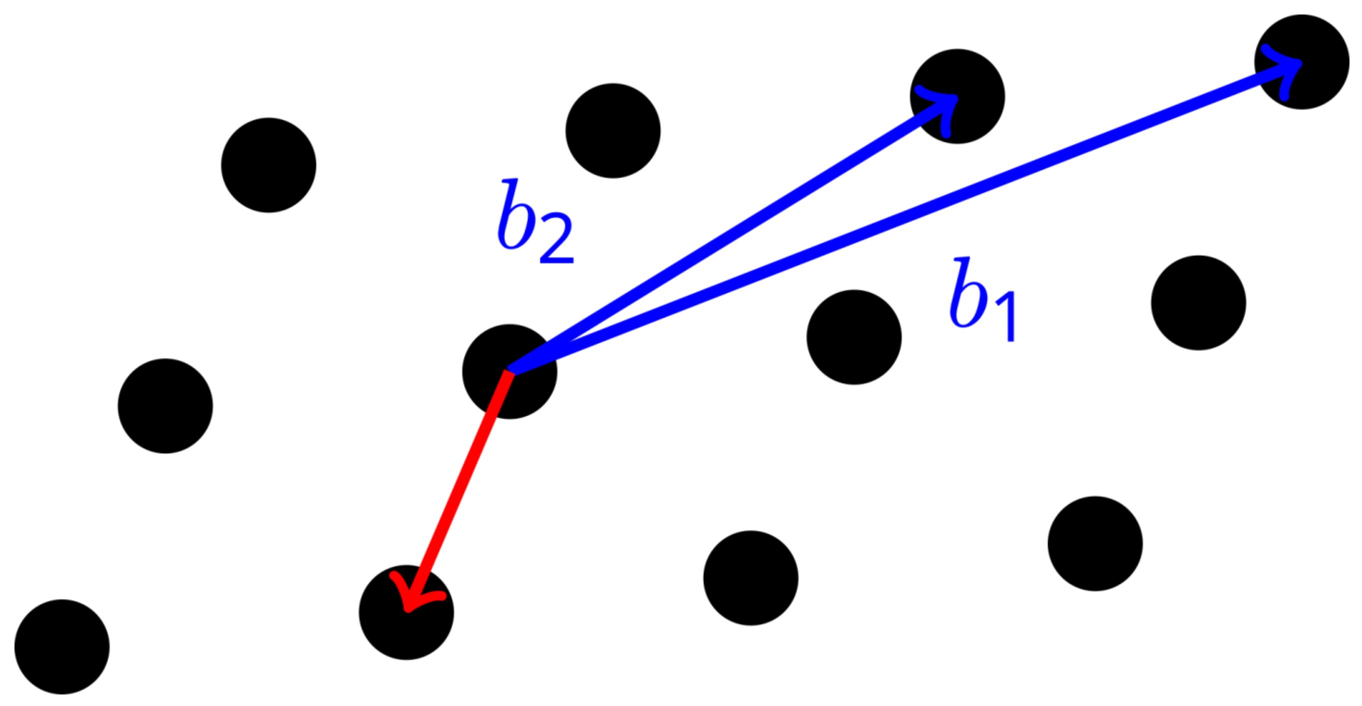
Learning with Errors

- LWE: adding small, random noise to linear equations makes solving them difficult.
- q: a large prime modulus.
- n: the dimension (typically chosen large for security).
- $s \in Z_q^n$: a secret vector.
- $A \in Z_q^{m \times n}$: a randomly chosen matrix.
- $e \in Z_q^m$ a small error vector, where each entry is drawn from a specific error distribution (usually a discrete Gaussian distribution).

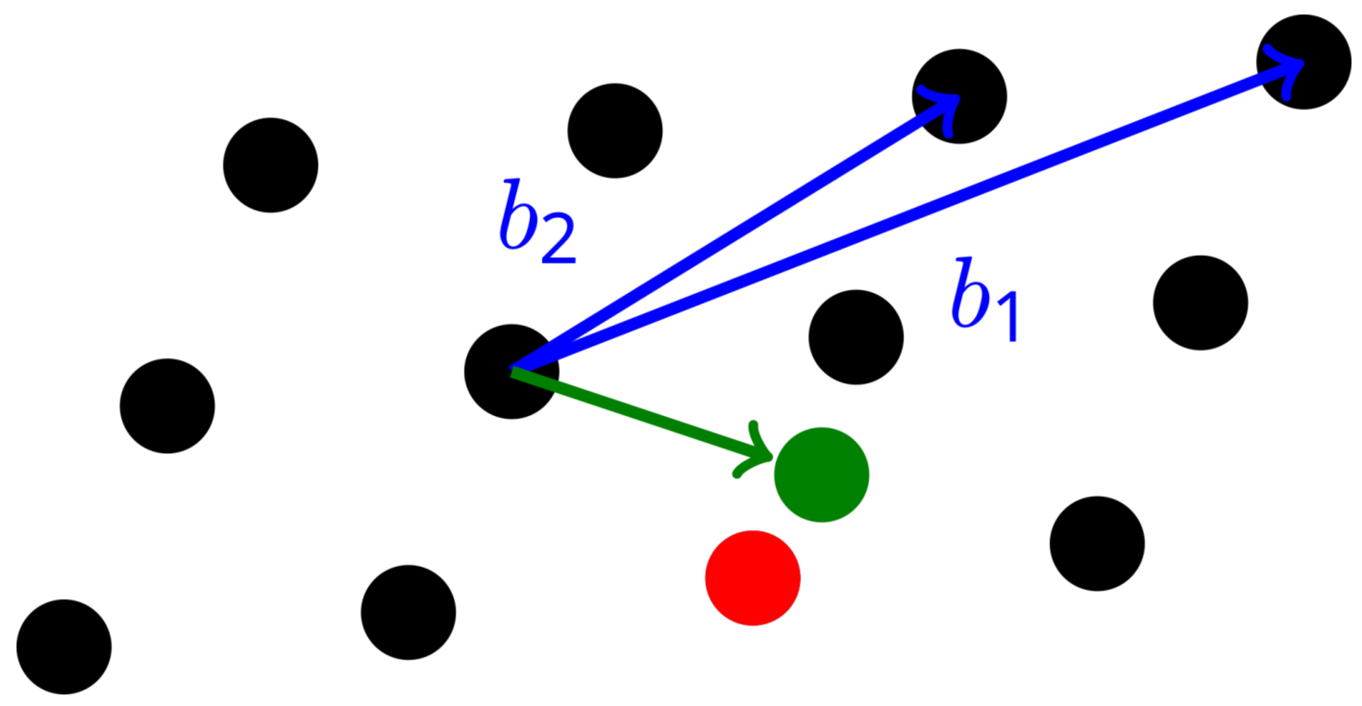
Hardness of Learning with Errors

- Search LWE
  - Given $b = A \cdot s + e \ (mod\ q)$
  - the hard problem is to find s
- Decision LWE
  - Given A and b, decide if b was generated using noise or if it is just a random vector
- Solving these problems can be linked to lattice problems
  - Problems are believed to quantum hard

Shortest Vector Problem



Closest Vector Problem



LWE Example

- Modulus $q = 7$
- Secret $s = [3, 5]$
- Randomly choose $A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$
- Random small vector $e = [1, 0]$
- Compute $b = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ mod 7
  - $b = \begin{bmatrix} 1 \cdot 3 + 4 \cdot 5 \\ 2 \cdot 3 + 6 \cdot 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ mod 7
  - $b = \begin{bmatrix} 24 \\ 36 \end{bmatrix}$ mod 7
  - $b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Ring Learning With Errors 1/2

- Extension of LWE to polynomial rings
- A polynomial ring is a mathematical structure
  - With addition and multiplication
  - Elements are polynomials
- We work with quotient rings
  - All polynomials are modulo q
  - Cyclotomic polynomial $\mathrm{f}(x)$

$$R_q = \mathbb{Z}_q[X]/(f(x))$$

Is the ring with of polynomials with integer coefficients mod q and polynomials are reduced by $\mathrm{f}(x)$

Ring Learning With Errors 2/2

- The components are similar to the LWE
  - $a(x)$ a random polynomial
  - $s(x)$ a secret polynomial
  - $e(x)$ a small error polynomial
- RLWE is more efficient than LWE

Example:

- $q = 7, R_7 = \mathbb{Z}_7[X]/(x^2 + 1)$
- $a(x) = 4 + x$
- $s(x) = 3 + 2x$
- $e(x) = 1$

$$b(x) = a(x) \cdot s(x) + e(x) \; mod \; 7$$

$$b(x) = (4 + x) \cdot (3 + 2x) + 1 \; mod \; 7$$

Polynomial multiplication:

$$(4 + x) \cdot (3 + 2x) = 12 + 8x + 3x + 2x\text{^}2$$

Reduce modulo $\mathrm{x}^2 + 1$:

$$= 12 + 8x + 3x + 2(-1) = 10 + 11x$$

Reduce coefficients modulo 7 and error $e(x) = 1$

$$= 10 + 11x = 3 + 4x$$

Add error $e(x) = 1$

$$\mathbf{4 + 4x}$$

FHE Schemes

Single Instruction Multiple Data (SIMD)

- Most schemes allow encoding multiple messages into a plaintext/ciphertext
- All operations on the plaintext/ciphertext are performed on all messages encoded at no extra cost
- Max. number of messages is called *slots*
- Number of slots typically >1000

BFV/BGV Schemes

- BFV/BGV Schemes (Brakerski-Fan-Vercauteren/Brakerski-Gentry-Vaikuntanathan):
  - Structure:
    - The BFV/BGV scheme supports operations on integers or fixed-point numbers, making it useful for exact computations. BFV focuses more on optimizations.
  - Use Cases:
    - Best for scenarios where exact arithmetic is needed, such as financial computations (e.g., balance calculations) or voting systems.
  - Properties:
    - Precise results and supports both addition and multiplication on ciphertexts.
    - Supports batching (processing multiple encrypted data points at once)

CKKS Scheme

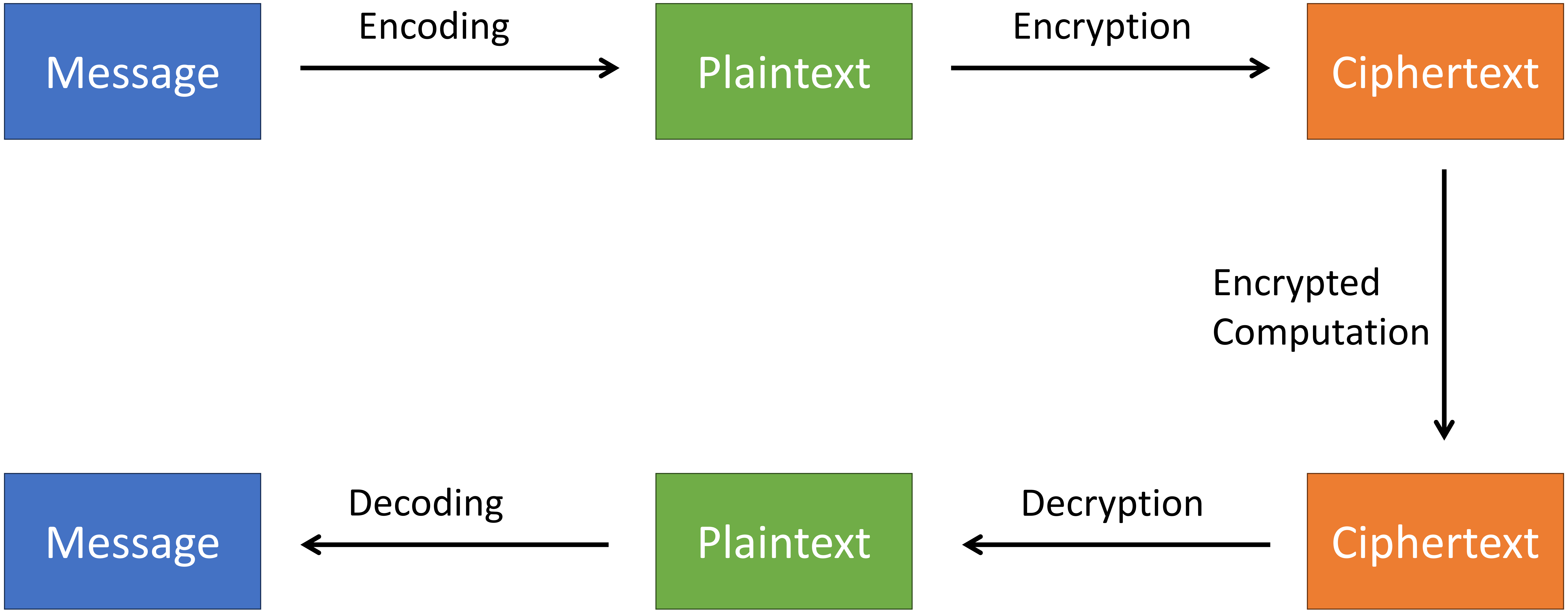
- CKKS (Cheon-Kim-Kim-Song) Scheme
  - Structure:
    - Unlike BFV and BGV, CKKS is designed for computations on real numbers. It enables approximate arithmetic, meaning computations will introduce a small amount of error.
  - Use Cases:
    - Applications requiring computations on real or floating-point numbers, such as machine learning algorithms, image processing, or signal processing. Commonly used in AI/ML, where a high degree of precision is often not needed (approximations are acceptable).
  - Properties:
    - Supports addition and multiplication but introduces a level of approximation (rounding). Provides excellent performance for large-scale computations due to reduced noise accumulation compared to other schemes.
    - Supports batching (processing multiple encrypted data points at once)

TFHE

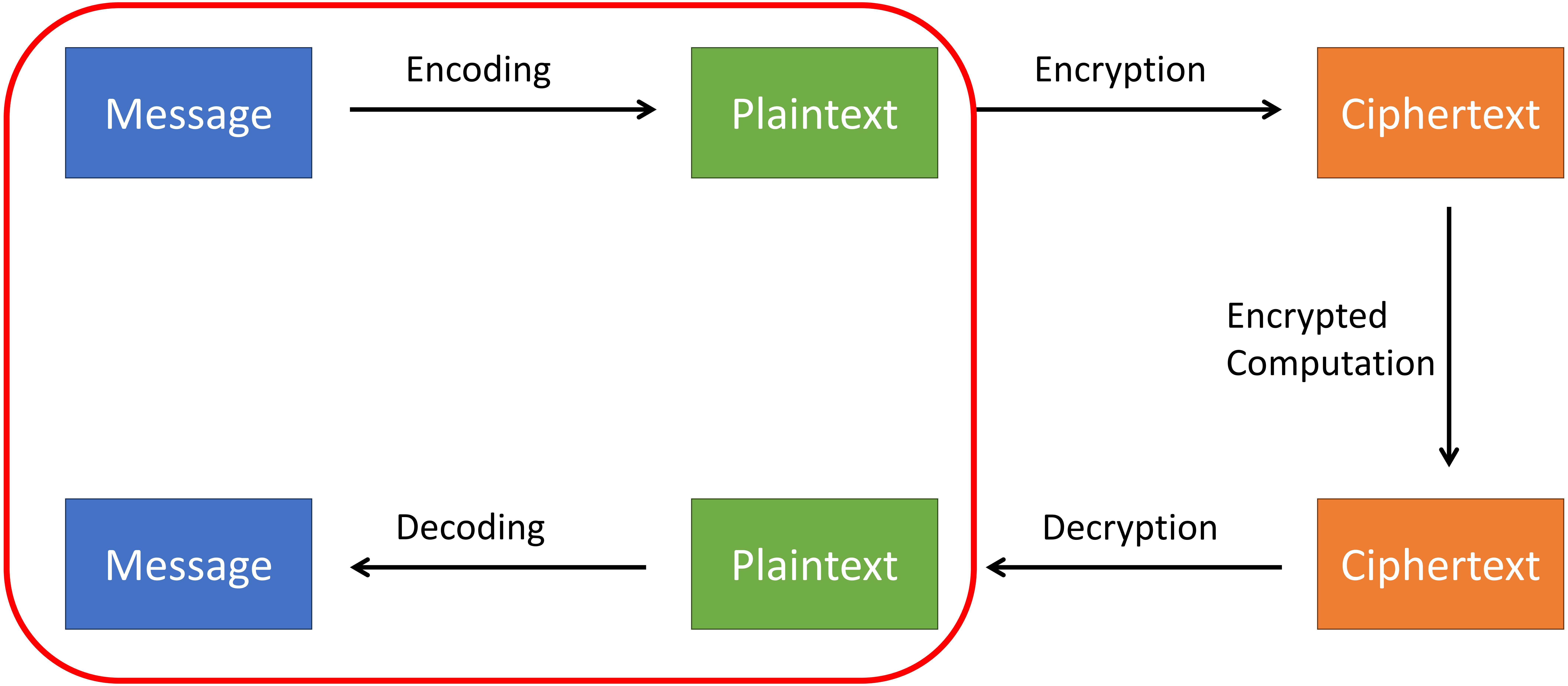
- TFHE (Torus Fully Homomorphic Encryption)
  - AKA. CGGI (Chillotti-Gama-Georgieva-Izabachène)
  - Structure:
    - TFHE is optimized for fast and efficient boolean circuit evaluation. It uses operations over the torus, which makes binary computations faster.
  - Use Cases:
    - Ideal for applications that require boolean logic, such as circuits performing logical AND/OR/NOT operations. Examples include secure voting systems, logic gate-based algorithms, or encrypted control systems.
  - Properties:
    - Low Latency: TFHE allows for fast, low-latency operations, making it one of the most efficient schemes for binary data.
    - Binary Gates: Supports the evaluation of encrypted logic gates, making it particularly useful for cryptographic protocols and arbitrary computation.

CKKS: The Details

CKKS Overview



CKKS Overview



CKKS Encoding

Encoding: complex vector $\mathbb{C}^{\frac{N}{2}} \rightarrow m(x) \in \mathbb{Z}[X]/(X^N + 1)$

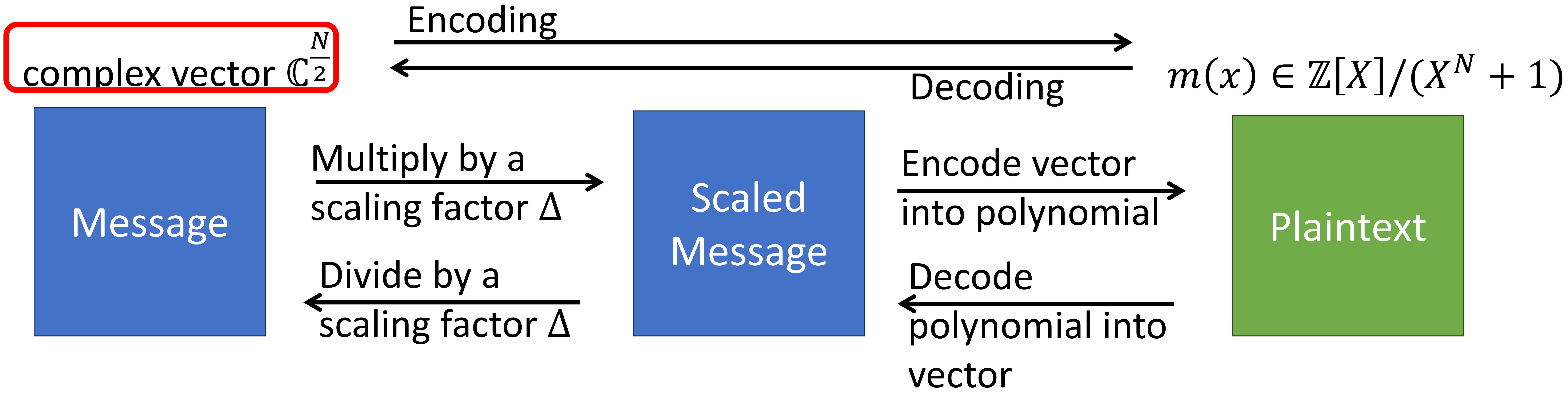
Decoding: $m(x) \in \mathbb{Z}[X]/(X^N + 1) \rightarrow$ complex vector $\mathbb{C}^{\frac{N}{2}}$

Message $\xrightarrow{\text{Multiply by a scaling factor } \Delta}$ Scaled Message $\xrightarrow{\text{Encode vector into polynomial}}$ Plaintext

Plaintext $\xrightarrow{\text{Decode polynomial into vector}}$ Scaled Message $\xrightarrow{\text{Divide by a scaling factor } \Delta}$ Message

- Message is the vector $z = [z_1, z_2, \ldots, z_{N/2}]$
- In the encoded plaintext $m(x)$ the slot i contains the value Δz_i (approximately)
- $m(\zeta_i) \approx \Delta \cdot z_i$ for root ζ_i of $X^N + 1 = 0$

CKKS Encoding



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CKKS Encoding

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Encoding

complex vector $\mathbb{C}^{\frac{N}{2}}$

Decoding

$m(x) \in \mathbb{Z}[X]/(X^N + 1)$

Message

Multiply by a scaling factor Δ

Divide by a scaling factor Δ

Scaled Message

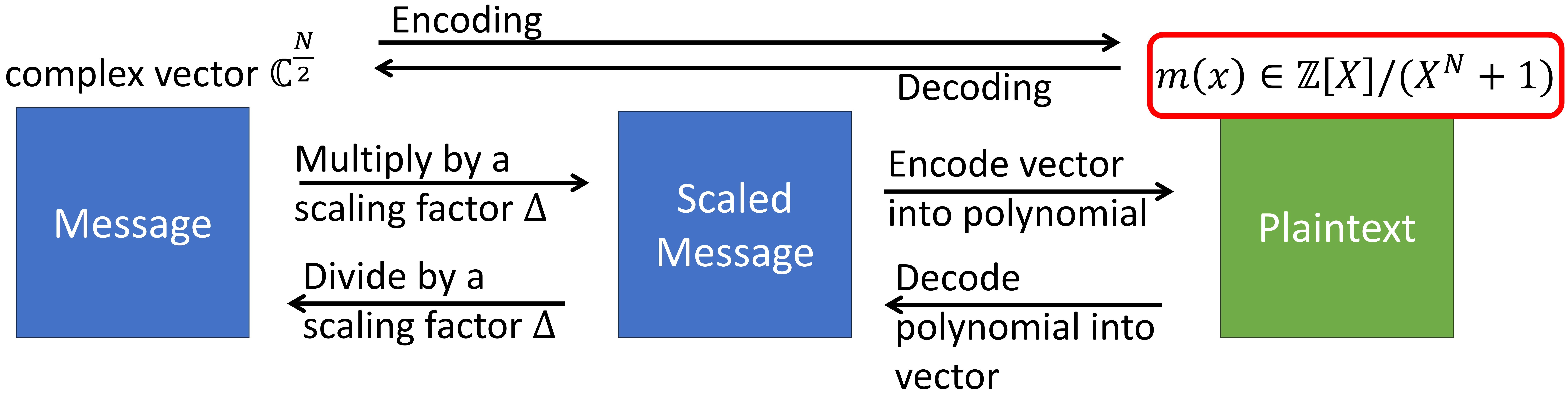
Encode vector into polynomial

Decode polynomial into vector

Plaintext

- Message is the vector $z = [z_1, z_2, \dots, z_{N/2}]$
- In the encoded plaintext $m(x)$ the slot i contains the value $\boxed{\Delta z_i}$ (approximately)
- $m(\zeta_i) \approx \Delta \cdot z_i$ for root ζ_i of $X^N + 1 = 0$

CKKS Encoding



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- In the encoded plaintext $m(x)$ the slot i contains the value Δz_i (approximately)
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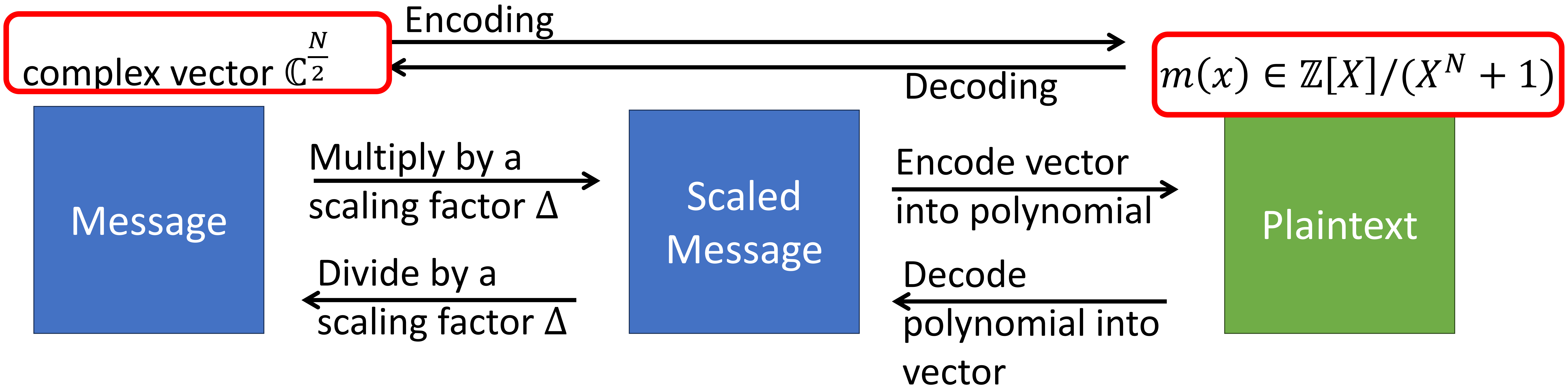
CKKS Encoding



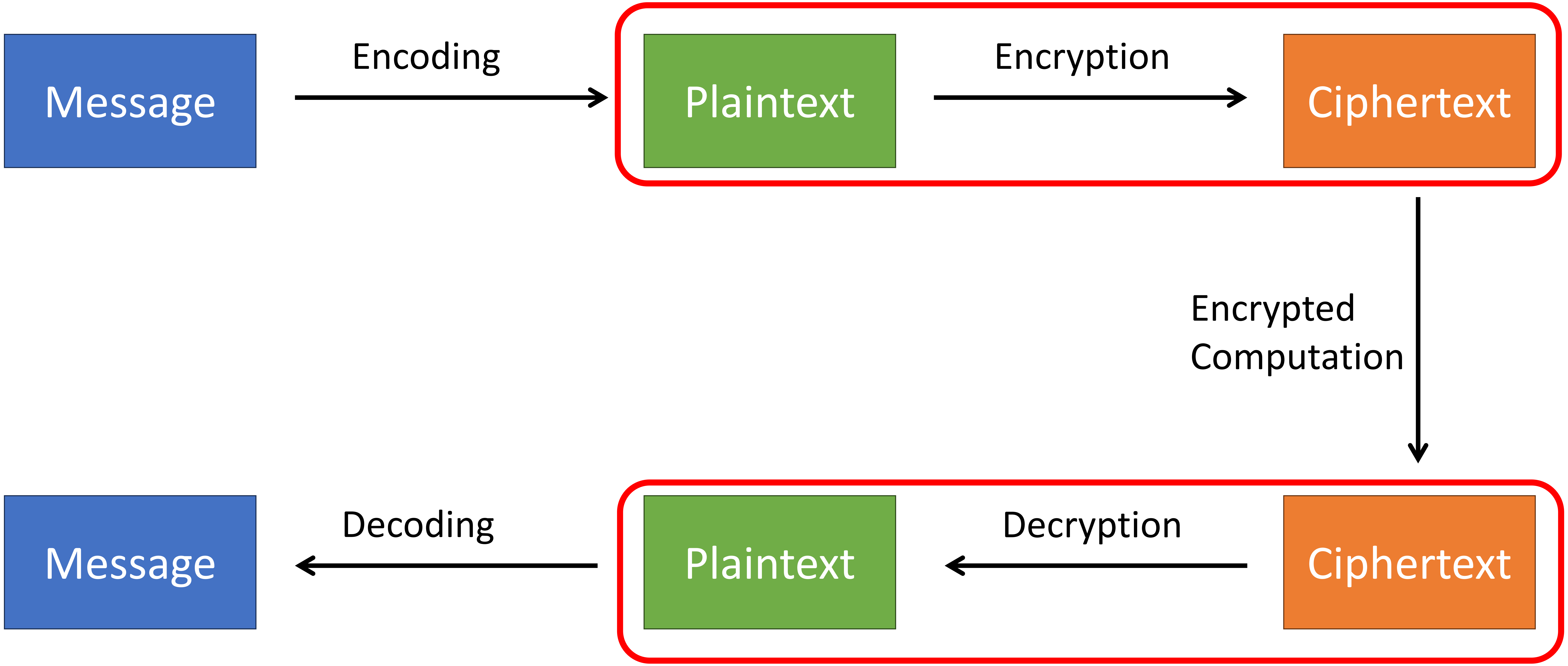
The message vector z is encoded in the plaintext polynomial $m(x)$

CKKS Encoding Example

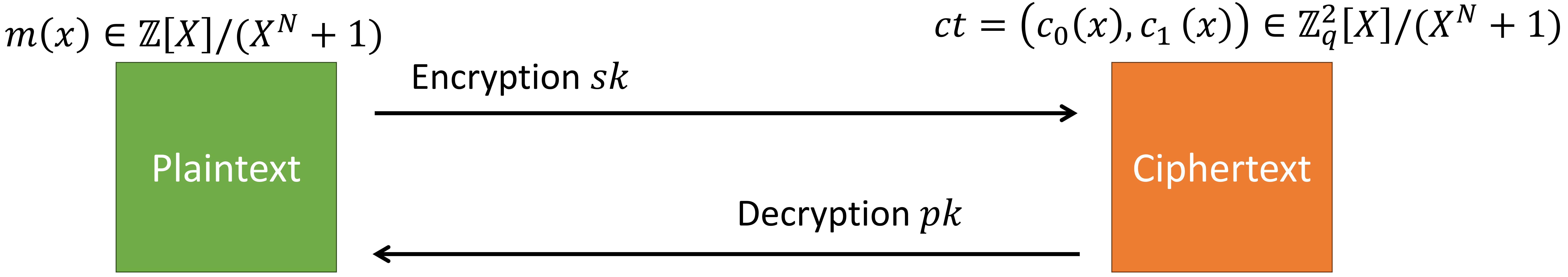
- $\mathrm{N} = 4, \Delta = 2^7$
- Message: $(z_1, z_2) = (1.2 - 3.4i, 5.6 + 7.8i)$
- Encoded message: $m(x) = 435 - 706x + 282x^2 - 308x^3$
- Complex roots of $X^4 + 1 = 0$
  - $\zeta_1 = (1 + i)/\sqrt{2}$
  - $\zeta_2 = -(1 + i)/\sqrt{2}$
  - …
- $m(\zeta_1) \approx 153.5 - i \cdot 435.0$
- $m(\zeta_1)/\Delta \approx 1.998 - i \cdot 3.398$

- $m(\zeta_2) \approx 716.4 + i \cdot 999.0$
- $m(\zeta_2)/\Delta \approx 5.597 + i \cdot 7.805$

CKKS Overview



CKKS Secret and Public Key



- Sample three polynomials from $\mathbb{Z}_q[X]/(X^N + 1)$: a, sk, e
  - Secret key: sk
  - Public key: $pk = (-a \cdot sk + e, a)$
  - A small error polynomial e

CKKS Encryption

$m(x) \in \mathbb{Z}[X]/(X^N + 1)$

$c = \left(c_0(x), c_1(x)\right) \in R_q^2$

- Encryption:
  - Encrypt the message polynomial m into two polynomials c_0, c_1

$$Enc(m) =$$

$$(m, 0) + pk =$$

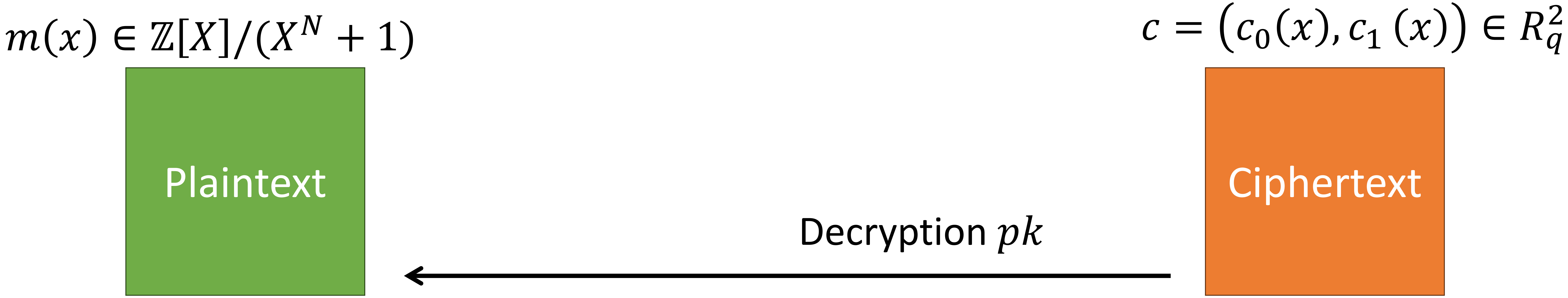
$$(m - a \cdot sk + e, a) =$$

$$(c_0, c_1) = c$$

(we use m and c_i instead of $m(x), c_i(x)$

for simplicity)

CKKS Decryption



- Decryption:
  - Decrypt the ciphertext polynomials c_0, c_1 into the message polynomial m

Recall:
$c_0, c_1 = (m - a \cdot sk + e, a)$

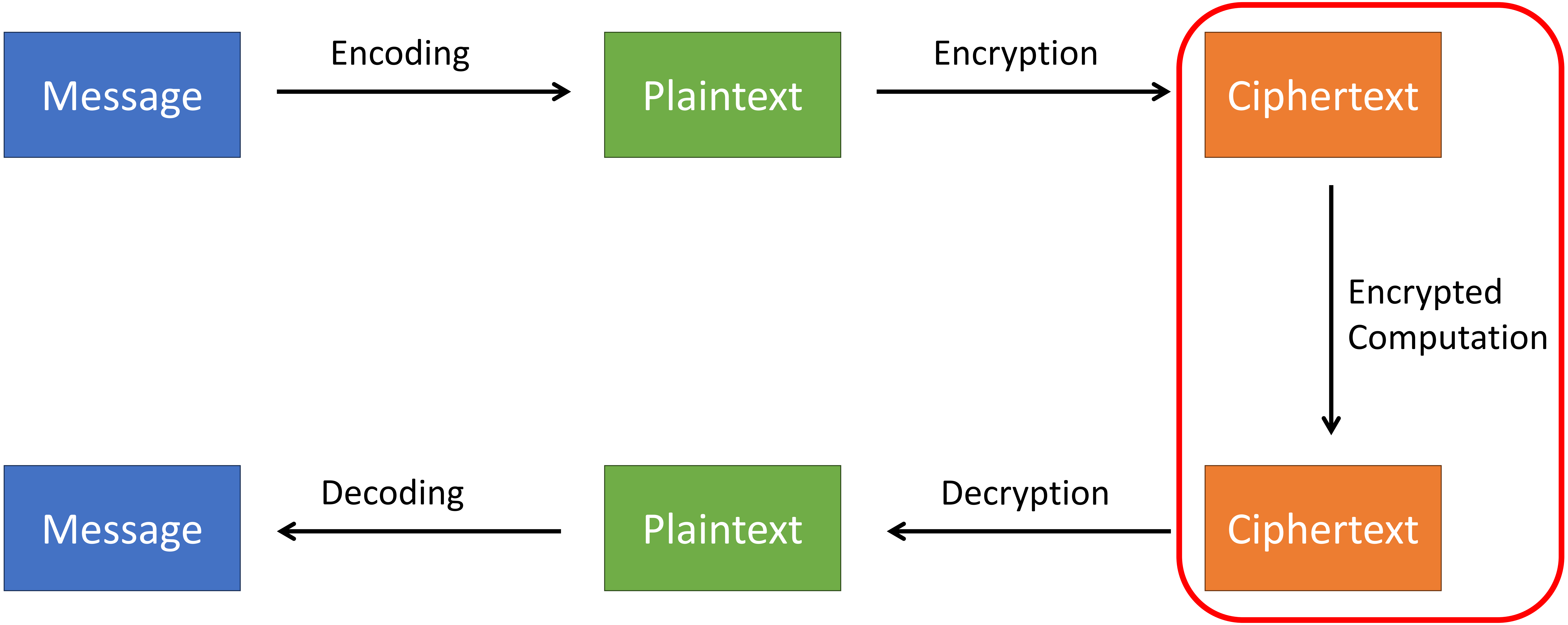
$$Dec(c) =$$

$$c_0 + c_1 sk =$$

$$m - \cancel{a \cdot sk} + e + \cancel{a \cdot sk} =$$

$$m + e \approx m$$

CKKS Overview



CKKS Addition

- Given two ciphertexts $c = (c_0, c_1), d = (d_0, d_1)$
- Addition:
  - Straight forward. Add polynomials of both chiphertexts

$$Add(c, d) = (c_0 + d_0, c_1 + d_1)$$

CKKS Multiplication

- Given two ciphertexts $c = (c_0, c_1), d = (d_0, d_1)$
- Multiplication
  - What we want is: $\mathrm{Dec(cd)} = \ Dec(c) \cdot Dec(d)$

$$(c_0 + c_1 \cdot sk) \cdot (d_0 + d_1 \cdot sk) =$$

$$c_0 d_0 + (c_0 d_1 + d_0 c_1) \cdot sk + c_1 d_1 \cdot sk^2$$

  - We can compute the product as

$$cd = (c_0 d_0, c_0 d_1 + d_0 c_1, c_1 d_1)$$

  - BUT the ciphertext consist of three polynomials now

CKKS Relinearization

- After multiplication the ciphertext consists of three parts
- To bring it back down to two we use relinearization
  - Create a relinearization key:
    - e_0 small random polynomial
    - a_0 random polynomial
    - v a large integer
  - relinearization key $rk = (-a_o sk + e_0 + sk^2, a_0)\ mod\ vq$
  - $(-a_o sk + e_0 + sk^2, a_0)$ decrypts to $e_0 + sk^2$
  - p is used to control the noise introduced
  - $Relin(Mult(c, d), rk) = (c_0 d_0, c_0 d_1 + d_0 c_1) + \lfloor \frac{c_1 d_1 \cdot rk}{v} \rceil$

CKKS Rescaling Motivation

- Recall:
  - The ciphertext c encrypts some message z scaled by Δ
  - $c \cdot c$ encrypts $z^2\Delta^2$
  - Multiplication causes the scale Δ to grow quadratically
- We want to keep the scale Δ the same size after multiplication to prevent overflow
- Rescaling allows us to reduce the size of Δ after a multiplication

Residue Number System 1/2

- Time for a small detour
- Residue Number System (RNS), related to the Chinese Remainder Theorem
- Given a set of “small” coprime numbers we can represent a large integer as a set of smaller integers
- Given n coprimes $c_1, \dots, c_n$ we can represent numbers between 0 and $-1 + \prod_{i=1}^{n} c_i$
- We represent a number x as an n tuple where each element is the remainder of $\frac{x}{c_i}$

Residue Number System 2/2

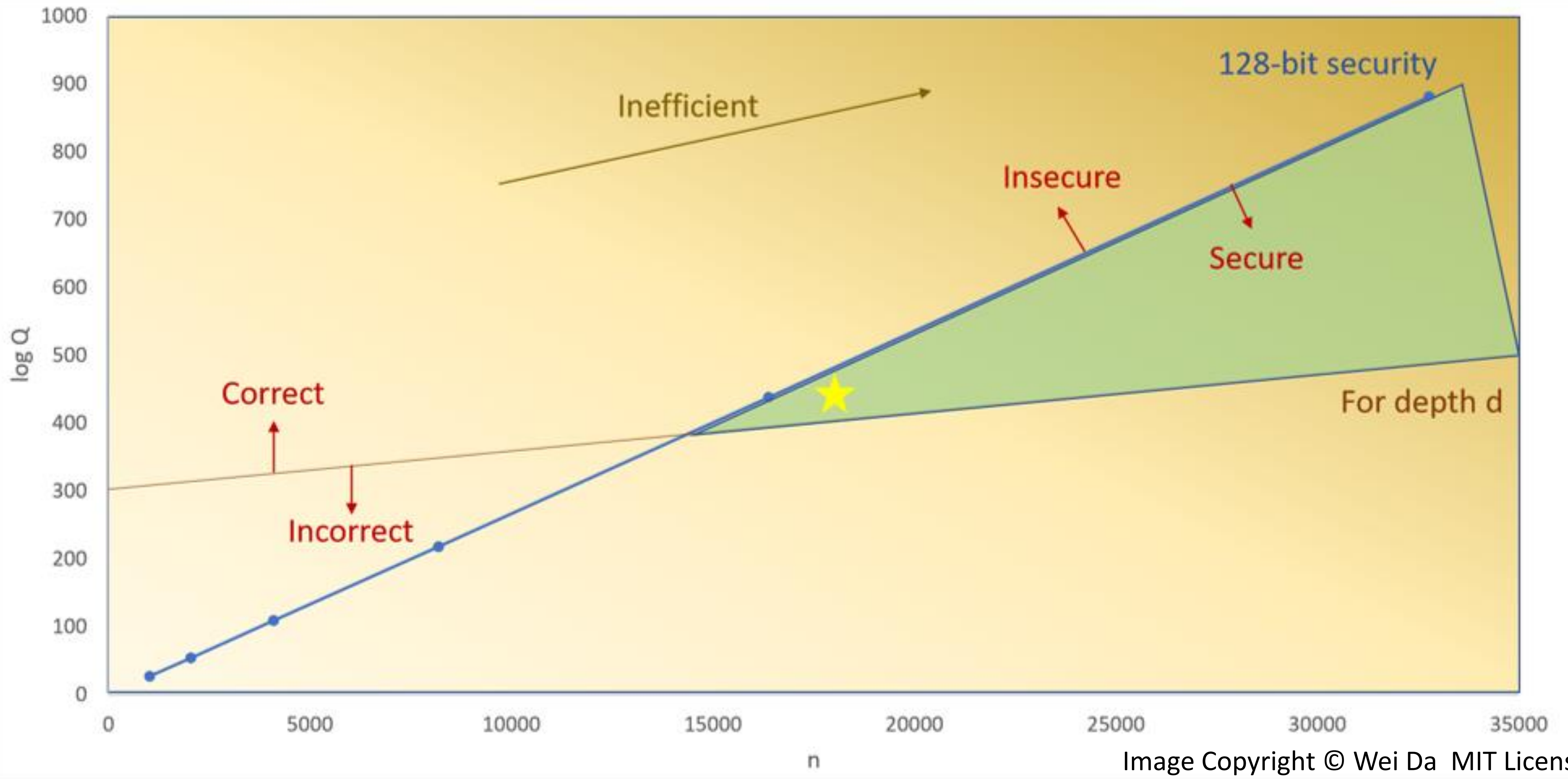
- Almost back on track
- Addition and multiplication is element-wise
- Example:
  - Co-primes: 3, 5, 11
  - Can represent numbers between 0 and 164
  - 16 -> (1,1,5), 9 -> (0, 4, 9)
  - 16+9 = 25 -> (1,0,3) = (1,1,5) + (0,4,9) = (1+0, 1+4, 5+9) = (1,0,3)
  - 16*9 = 144 -> (0,4,1) = (1,1,5) * (0,4,9) = (1*0, 4*1, 5*9) = (0,4,1)
- Why do we need this?
  - Numbers can get 100s of bits large
  - Working with numbers larger than a word (64bit) is slow

CKKS Rescaling

- We can select the ciphertext modulus q as the product of multiple smaller (less than word-size) primes p_l and a prime q_0
- L is the number of the smaller primes p_l
- Select L primes $p_1, \ldots p_L$, each $p_l \approx \Delta$, and a prime $q_0 > \Delta$
- After each multiplication, we can “discard” one of the primes
  - Ciphertext c is now $c' \in R_{q'}^2$, with $q' = \frac{q}{\Delta}$
  - Scaling factor Δ^2 is reduced to Δ
  - Doesn’t change the encrypted message only the representation
- We can only preform L multiplications -> leveled HE

Security Parameters

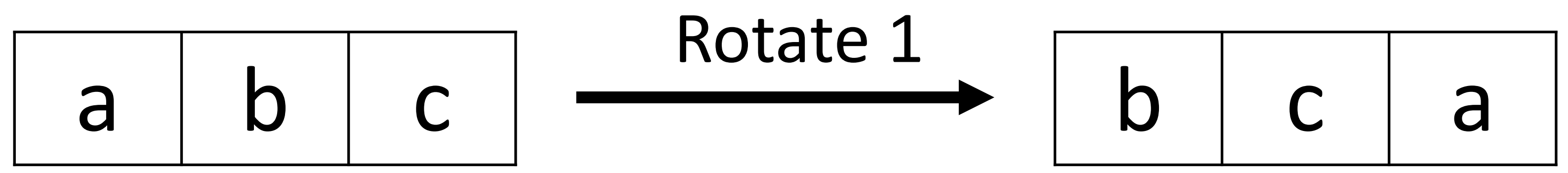
- For security increase n
- For more levels increase q
- Security of the scheme relies on $\frac{n}{q}$
  - as q increases so must n
- Larger values increase the computational cost
- The HE standard provides values for n and q that provide 128bit security



N	log q
1024	29
2048	56
4096	111
8192	220
16384	440
32768	880

Additional CKKS operations

- Rotation
  - Ciphertexts are encryptions of vectors
  - We can rotate the elements in the vector with wrap around
    - Requires rotation (galois) keys



- Bootstrapping
  - Resets the level of the Ciphertext to allow additional computation
  - Expensive operation

Computation with FHE

Constraints

- No “random” access to slots in the encrypted vector
  - Can’t do $c[i]$
- No inter slot operations
  - Can’t do $c[i] + c[j]$
- With CKKS we can only evaluate Polynomial functions
- Given a ciphertext c we can’t (easily) compute, e.g.:
  - $\max(y, c)$
  - Sigmoid: $\frac{1}{1+e^{-c}}$
  - $\sqrt{c}$
  - y^c
  - $\frac{y}{c}$
  - ….

Vector Computation

- Element-wise operations are simple

a	b	c	+	d	e	f	=	a+d	b+e	c+f

- But what if we want to compute the inner product?
  - The first part is simple

a	b	c	x	d	e	f	=	ad	be	cf

  - But how do we sum up the rest?

$$\underbrace{\boxed{ad}\boxed{be}\boxed{cf}}_{\Sigma}$$

- Simple Solution:
  - More ciphertexts:

a	a	a	x	d	d	d	=	ad	ad	ad
b	b	b	x	e	e	e	=	be	be	be
c	c	c	x	f	f	f	=	cf	cf	cf

+

ad+be+cf	ad+be+cf	ad+be+cf

Better Vector Computation

- Using multiple ciphertexts is not very efficient
- Better way:
  - Use rotations

a	b	c

x

d	e	f

=

ad	be	cf

ad	be	cf

Rotate by 1 →

be	cf	ad

ad	be	cf

+

be	cf	ad

=

ad+be	be+cf	ad+cf

ad	be	cf

Rotate by 2 →

cf	ad	be

ad+be	be+cf	ad+cf

+

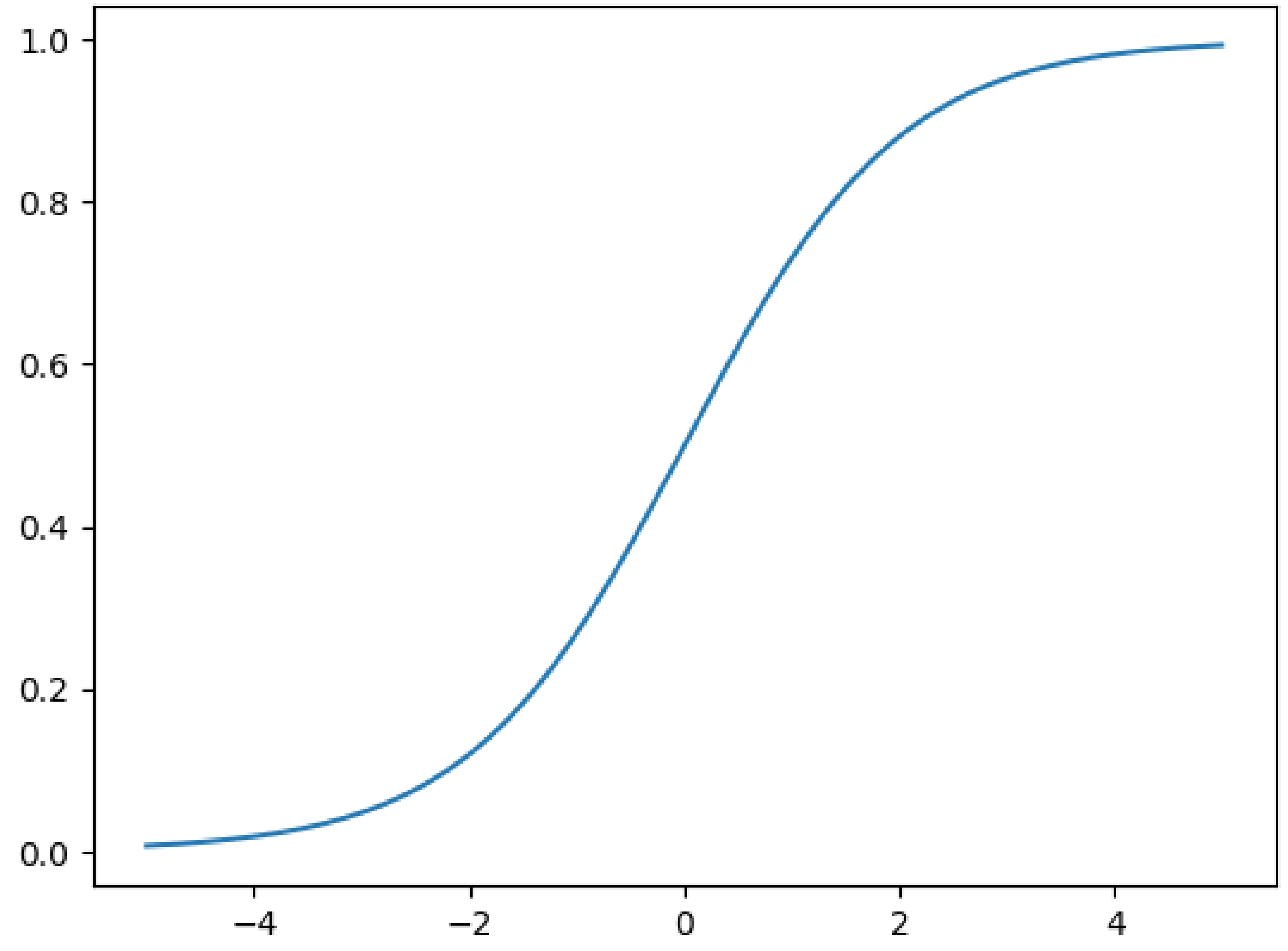
cf	ad	be

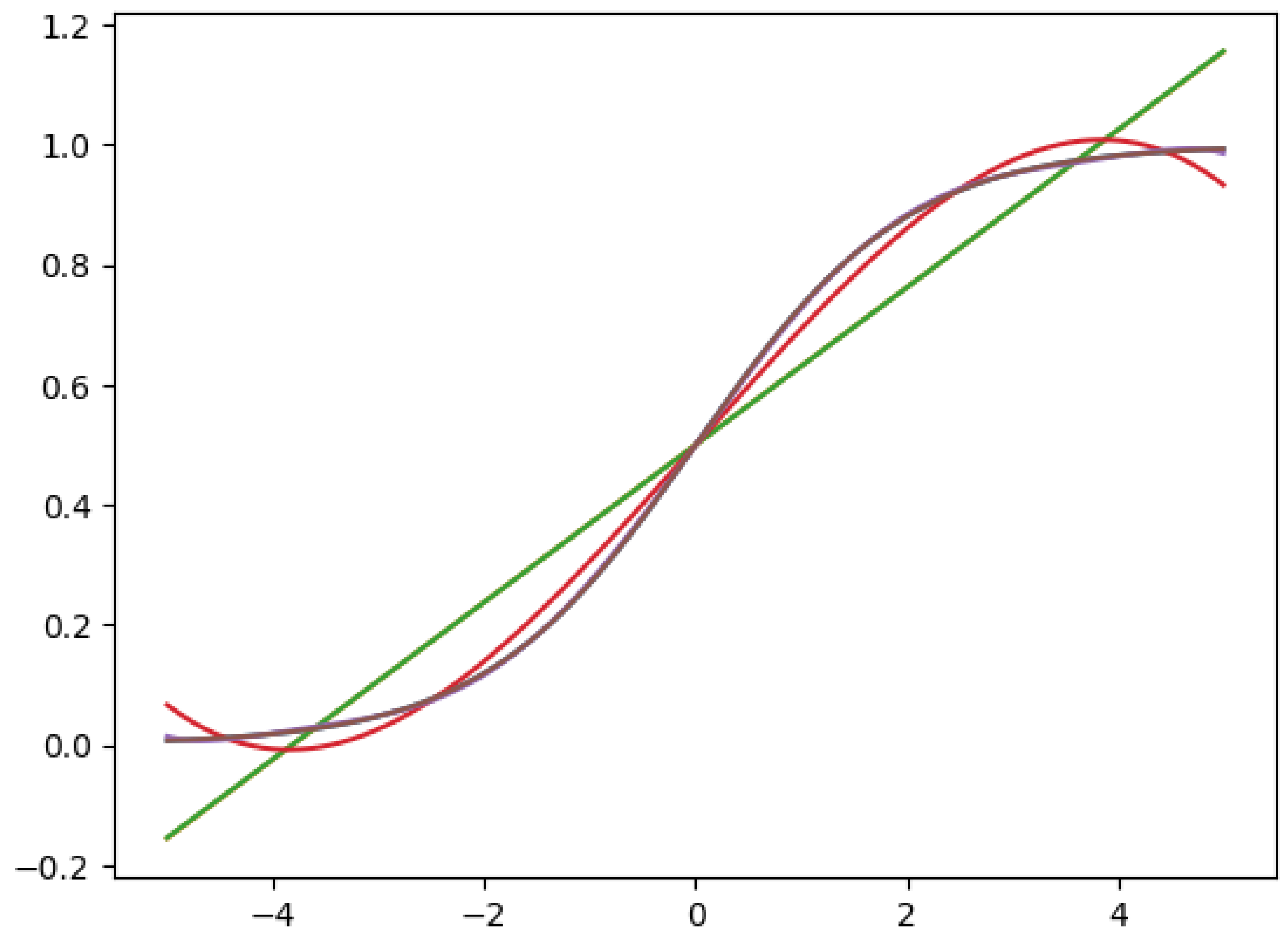
=

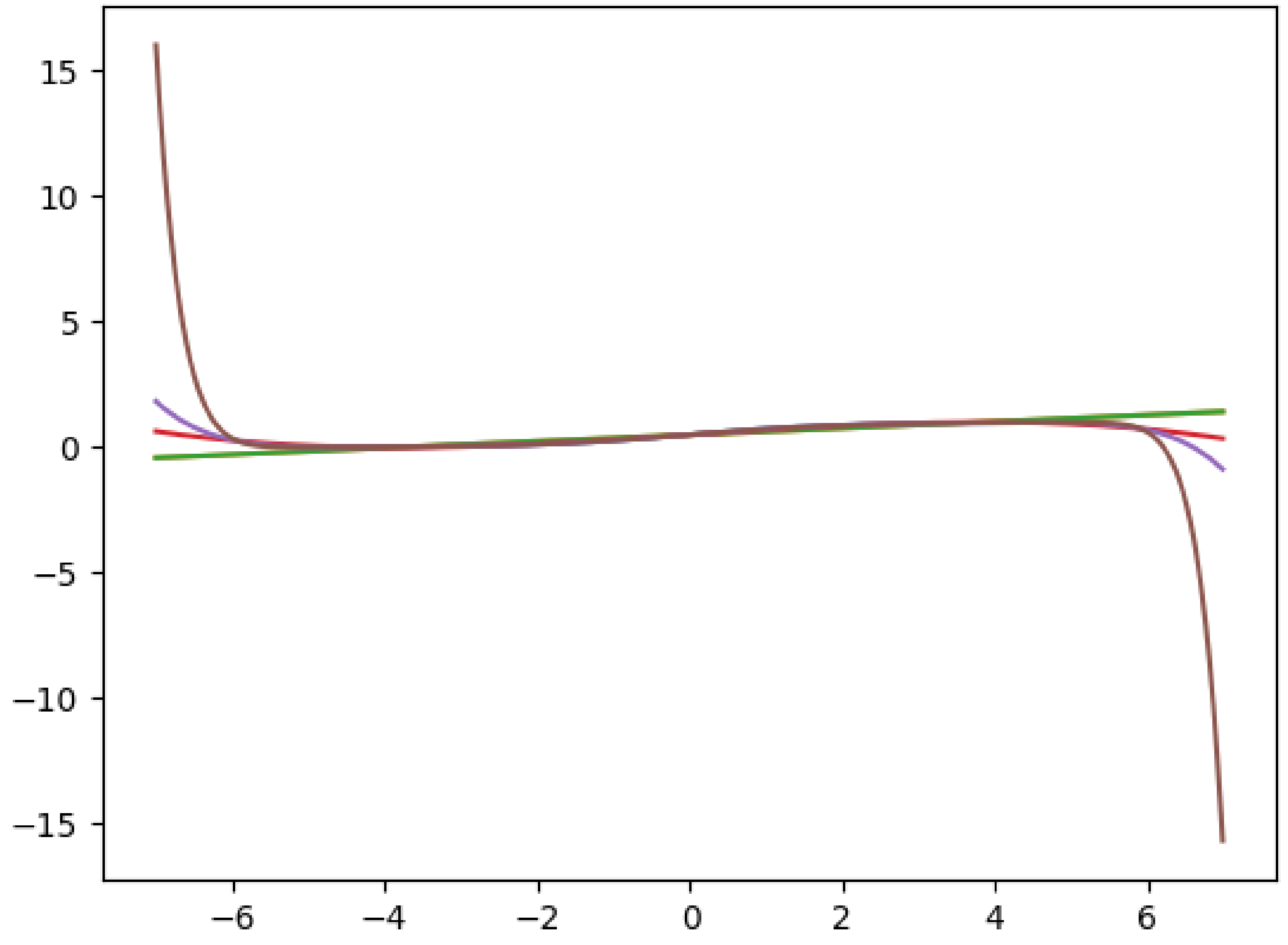
ad+be+cf	ad+be+cf	ad+be+cf

Evaluating Non-Polynomial Functions

- What if we want to evaluate functions that are not easily expressed as polynomials?
- Example:
- $S(x) = \frac{1}{1+e^{-x}}$
- We can approximate the function using polynomials
- Trade-off between accuracy and complexity
- We need to carefully consider the interval. Polynomials can get out of hand quickly

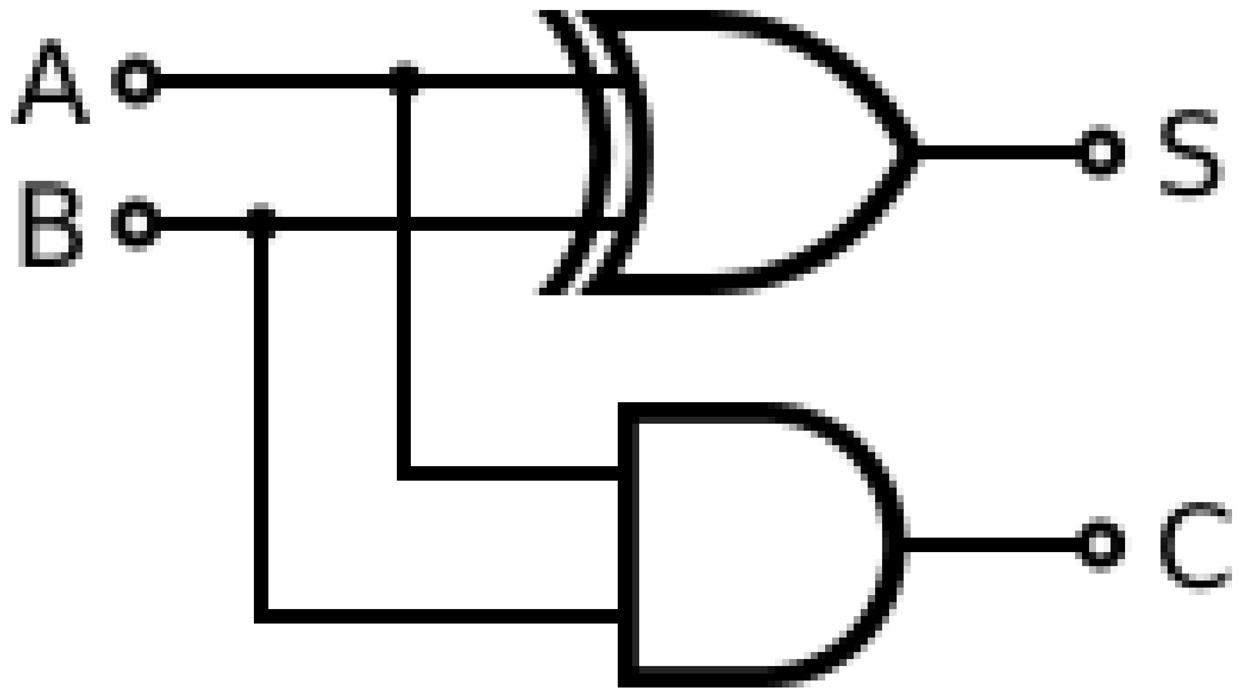


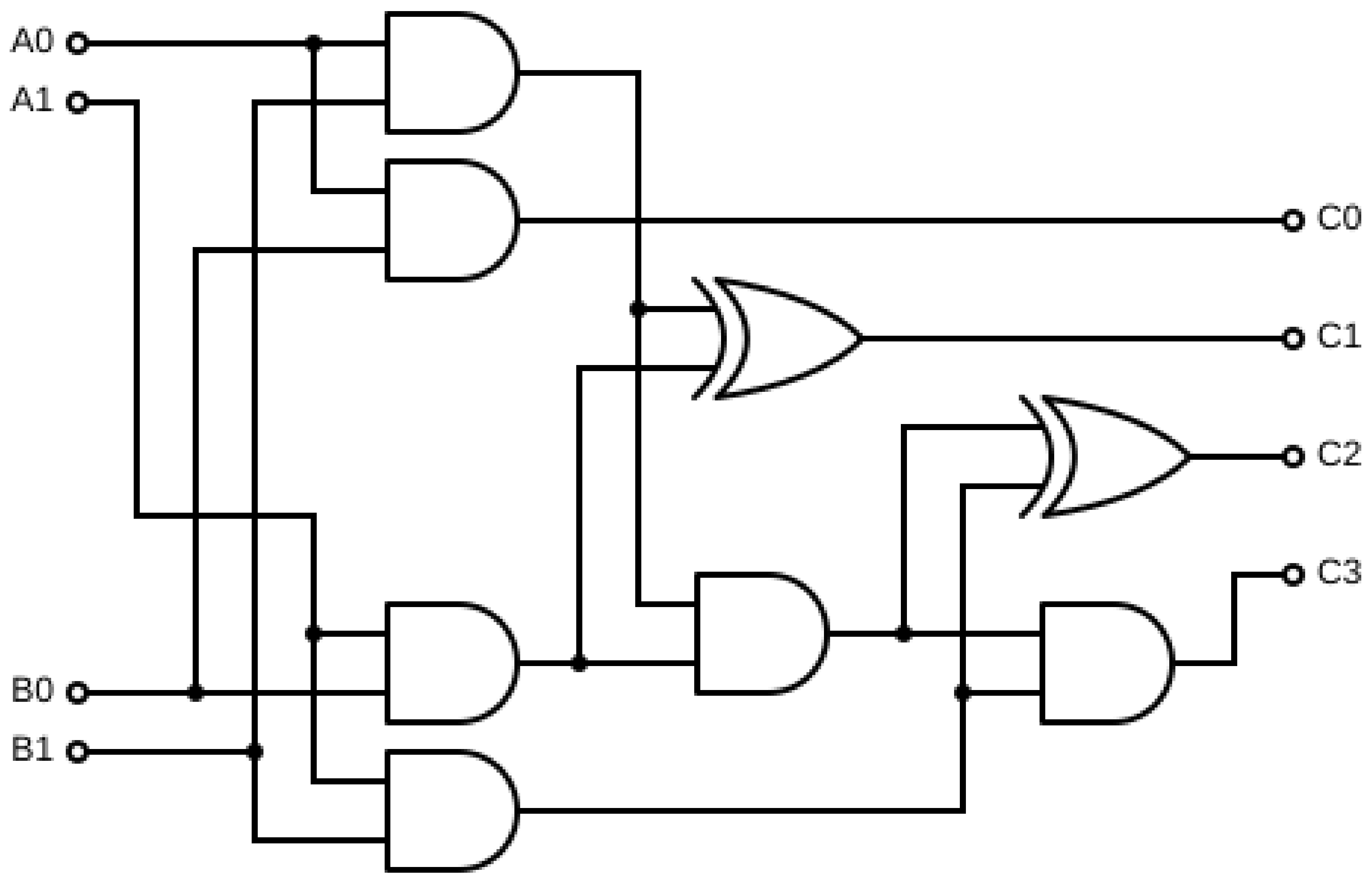




Binary Gate Computation

- FHEW/TFHE support binary gate evaluation
  - AND, OR, NAND, NOR, XOR, XNOR
- We can use the binary gates to (theoretically) express any function or program





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Lookup Tables

- Another way to compute functions is lookup tables
- Lookup up tables store a mapping from input to output
- Example: Sigmoid Function

x	f(x)
-1	0.26894142,
-0.5	0.37754067
0	0.
0.5	0.62245933
1	0.7310585786300049

- Lookup tables can be efficiently computed using binary schemes

Scheme Switching

- If the only tool you have is a hammer every problem looks like nail
- Schemes are great a different things
  - BFV,BGV, CKKS are great for arithmetic
  - TFHE/FHEW are great for binary computation
- We can use the best scheme for the operation
- From CKKS -> FHEW
  - Perform the decoding homomorphically
  - One CKKS ciphertext into multiple FHEW ciphertexts
- From FHEW -> CKKS
  - Homomorphically evaluate the decryption function
  - Multiple FHEW into one CKKS ciphertexts

Tools For An Easier Life

- LWE/Latice Estimator
  - https://github.com/malb/lattice-estimator
  - Use to estimate security of parameters
- TenSEAL
  - https://github.com/OpenMined/TenSEAL
  - Tensor library build on top of SEAL
- HEIR
  - https://heir.dev/
  - Compiler Toolchain for FHE
- Concrete and Concrete-ML
  - https://github.com/zama-ai/concrete and https://github.com/zama-ai/concrete-ml
  - Concrete is a TFHE compiler
  - Concrete-ML is a machine learning built on top of Concrete

Hands On

Getting Ready

- Here is what you need:
  - A browser with internet access
  - A Google account

github.com/podschwadt/fhe_tutorial