# Numerical Analysis Homework11

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## 1 Introduction

编写一个用有限差分法求解线性两点边值问题的通用的计算机程序,允许用户提供$\alpha,\beta,a,b,n$,以及函数$u,v,w$.

对下面的例子测试上题中编写的程序:

$$a.\begin{cases} x^{''}=-x \\ x(0)=0 \quad x(\pi/2)=7 \end{cases}$$

$$b.\begin{cases} x^{''}=2e^t-x \\ x(0)=2 \quad x(1)=e+cos1 \end{cases}$$

然后计算这两种测试情况中数值解的误差,对应的解分别是:$a.x(t)=7sint+3cost.\quad b.x(t)=e^t+cost.$

## 2 Method

假定我们求解的问题是:

$$\begin{cases} x^{''}=f(t,x,x') \\ x(a)=\alpha \quad x(b)=\beta \end{cases}$$

设区间$[a,b]$用点$a=t_0,t_1,t_2,...,t_{n+1}=b$分割,为简单起见,我们假设

$$t_i=a+ih \quad 0\leq i\leq n+1 \quad h=\frac{b-a}{n+1}$$

若$f$以非线性方式包含$y_i$,则这些方程是非线性的,一般来说求解很困难.因而我们总假定$f$关于$x$和$x'$是线性的,则它具有下列形式

$$f(t,x,x')=u(t)+v(t)x+w(t)x'$$

以及下面的公式

$$x'(t)=(2h)^{-1}[x(t+h)-x(t-h)]-\frac{1}{6}h^2x^{'''}(\xi)$$

$$x^{''}(t)=h^{-2}[x(t+h)-2x(t)+x(t-h)]-\frac{1}{12}h^2x^{(4)}(\tau)$$

那么我们可以将方程组写成下面的离散形式:

$$\begin{cases} y_0 = \alpha \\ (-1-\frac{1}{2}hw_i)y_{i-1} + (2+h^2v_i)y_i + (-1+\frac{1}{2}hw_i)y_{i+1} = -h^2u_i(1 \le i \le n) \\ y_{n+1} = \beta \end{cases}$$

我们记$u_i = u(t_i), v_i = v(t_i)$.
下面我们引进缩写:

$$a_i = -1 - \frac{1}{2}hw_{i+1}$$
$$d_i = 2 + h^2v_i$$
$$c_i = -1 + \frac{1}{2}hw_i$$
$$b_i = -h^2u_i$$

解线性方程组

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & c_3 & \\ & \cdots & \cdots & \cdots & \\ & & a_{n-2} & d_{n-1} & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 - a_0\alpha \\ b_2 \\ b_3 \\ \cdots \\ b_{n-1} \\ b_n - c_n\beta \end{bmatrix} \tag{1}$$

解得$y_i$,作为近似的数值解.

## 3 Results

| n | 例1误差 | order | 例2误差 | order |
|---|---|---|---|---|
| 10 | 0.0047 | —— | 2.4401e-04 | —— |
| 20 | 0.0013 | 1.8650 | 6.7120e-05 | 1.8621 |
| 40 | 3.4022e-04 | 1.9317 | 1.7607e-05 | 1.9306 |
| 80 | 8.7174e-04 | 1.9645 | 4.5114e-06 | 1.9645 |
| 160 | 2.2064e-04 | 1.9822 | 1.1419e-06 | 1.9822 |

这里的误差是所有节点与解析解误差的最大值.

## 4 Discussion

我们可以看到有限差分法求解边值问题的误差随着$N$的增大,大致是$\mathcal{O}(h^2)$量级的.该方法的误差阶大致也是2阶的,并且比较稳定.这说明该方法的性质比较稳定,误差较小,性能优良.

# 5 Computer Code

```
u1=@(t)0;
u2=@(t)(2*exp(t));
v=@(t)-1;
w=@(t)0;

f=@(t)(7*sin(t)+3*cos(t));
g=@(t)(exp(t)+cos(t));



[y1]=differ(0,pi/2,10,3,7,u1,v,w,f);
[y2]=differ(0,pi/2,20,3,7,u1,v,w,f);
[y3]=differ(0,pi/2,40,3,7,u1,v,w,f);
[y4]=differ(0,pi/2,80,3,7,u1,v,w,f);
[y5]=differ(0,pi/2,160,3,7,u1,v,w,f);

[z1]=differ(0,1,10,2,(exp(1)+cos(1)),u2,v,w,g);
[z2]=differ(0,1,20,2,(exp(1)+cos(1)),u2,v,w,g);
[z3]=differ(0,1,40,2,(exp(1)+cos(1)),u2,v,w,g);
[z4]=differ(0,1,80,2,(exp(1)+cos(1)),u2,v,w,g);
[z5]=differ(0,1,160,2,(exp(1)+cos(1)),u2,v,w,g);

error11=max(y1);
error12=max(y2);
error13=max(y3);
error14=max(y4);
error15=max(y5);

error21=max(z1);
error22=max(z2);
error23=max(z3);
error24=max(z4);
error25=max(z5);



order12=log(error11/error12)/log(2);
order13=log(error12/error13)/log(2);
order14=log(error13/error14)/log(2);
order15=log(error14/error15)/log(2);
```

```
order22=log(error21/error22)/log(2);
order23=log(error22/error23)/log(2);
order24=log(error23/error24)/log(2);
order25=log(error24/error25)/log(2);


function [y]=differ(a,b,n,alpha,beta,u,v,w,f)

h=(b-a)/(n+1);
y(1)=alpha;
y(n+2)=beta;

for i=1:n
    al(i)=-1-0.5*h*w(a+(i+1)*h);
    dl(i)=2+h*h*v(a+i*h);
    cl(i)=-1+0.5*h*w(a+i*h);
    bl(i)=-h*h*u(a+i*h);
end

A=zeros(n,n);
A(1,1)=dl(1);
A(1,2)=cl(1);
A(n,n-1)=al(n-1);
A(n,n)=dl(n);
for k=2:n-1
    A(k,k)=dl(k);
    A(k,k+1)=cl(k);
    A(k,k-1)=al(k-1);
end

l=zeros(n,1);

l(1)=bl(1)-alpha*(-1-0.5*h*w(a+h));
l(n)=bl(n)-beta*cl(n);

for k=2:n-1
l(k)=bl(k);
end

M = Chase_method(A,l);
```

```
for k=2:n+1

y(k)=M(k-1);

end

for k=1:n+2
y(k)=y(k)-f(a+(k-1)*h);
end

function [ x ] = Chase_method( A, b )


T = A;
for i = 2 : size(T,1)
    T(i,i-1) = T(i,i-1)/T(i-1,i-1);
    T(i,i) = T(i,i) - T(i-1,i) * T(i,i-1);
end
L = zeros(size(T));
L(logical(eye(size(T)))) = 1;
for i = 2:size(T,1)
    for j = i-1:size(T,1)
        L(i,j) = T(i,j);
        break;
    end
end
U = zeros(size(T));
U(logical(eye(size(T)))) = T(logical(eye(size(T))));
for i = 1:size(T,1)
    for j = i+1:size(T,1)
        U(i,j) = T(i,j);
        break;
    end
end

y = L\b;
x = U\y;
end
```