Variational Formula for f-Divergence

Theorem 1 (Variational Representation of f-Divergence)**.** *Let P and Q be probability measures on a measurable space $(\mathcal{S} \times \mathcal{A}, \mathcal{F})$ such that $P \ll Q$. Then*

$$D_f(P\|Q) = \mathbb{E}_Q\big[f(\tfrac{dP}{dQ})\big] = \sup_{g:\mathcal{S}\times\mathcal{A}\to\mathrm{dom}\, f^*} \Big\{\mathbb{E}_P[g] - \mathbb{E}_Q[f^*(g)]\Big\}.$$

Proof. **1. Definitions and Setup.** Since $P \ll Q$, define the Radon-Nikodym derivative

$$r(s,a) := \frac{dP}{dQ}(s,a).$$

By definition,

$$D_f(P\|Q) = \int f\big(r(s,a)\big)\, Q(ds\, da) = \mathbb{E}_Q[f(r)].$$

The convex conjugate of f is defined by

$$f^*(u) = \sup_{t\in\mathrm{dom}\, f} \{u\, t - f(t)\}.$$

2. Pointwise Fenchel–Young. For each (s,a), by FenchelYoung,

$$f\big(r(s,a)\big) = \sup_{u\in\mathrm{dom}\, f^*} \{u\, r(s,a) - f^*(u)\}.$$

Hence for any measurable $g : \mathcal{S} \times \mathcal{A} \to \mathrm{dom}\, f^*$,

$$f(r(s,a)) \geq g(s,a)\, r(s,a) - f^*\big(g(s,a)\big),$$

with equality if $g(s,a)$ attains the supremum.

3. Integrate and Swap Supremum and Expectation. Integrating both sides w.r.t. Q:

$$\mathbb{E}_Q[f(r)] \geq \mathbb{E}_Q[g\,r] - \mathbb{E}_Q[f^*(g)] = \mathbb{E}_P[g] - \mathbb{E}_Q[f^*(g)].$$

Since this holds for all such g, we obtain

$$D_f(P\|Q) \geq \sup_g \{\mathbb{E}_P[g] - \mathbb{E}_Q[f^*(g)]\}.$$

Choosing $g(s, a)$ to realize the pointwise supremum yields equality, completing the proof. $\square$