2. 2. 2023

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

1) a) $u(x,y) = 2 + 3x - y + x^2 - y^2 - 4xy$

$\frac{\partial u}{\partial x} = 3 + 2x - 4y = \frac{\partial v}{\partial y}$

$v(x,y) = \int \frac{\partial v}{\partial y}\, dy = 3y + 2xy - 2y^2 + C(x)$

$\frac{\partial u}{\partial y} = -1 - 2y - 4x = -\frac{\partial v}{\partial x}$

$\frac{d}{dx} v(x,y) = 2y + C'(x)$

$1 + 2y + 4x = 2y + C'(x)$

$C'(x) = 4x + 1$

$C(x) = 2x^2 + x + K, \quad K \in \mathbb{R}$

$v(x,y) = 3y + 2xy - 2y^2 + 2x^2 + x + K$

b) z bodu a

$f'(1+i) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3 + 2 - 4 + i(1 + 2 + 4) = 1 + 7i$

c) $g(z) = \alpha z\bar{z} + i\left(\beta \ln z + 2\ln(z^2)\right)$ dif. v $\mathrm{Re}\, z = 1$

$= \alpha(x^2+y^2) + i\left(\beta y + 2\ln(x+yi)^2\right) = \alpha(x^2+y^2) + \beta y i + 4xyi$ ($x^2 + 2xyi - y^2$)

$\mathrm{Re}\, g = \alpha(x^2+y^2)$

$\mathrm{Im}\, g = \beta y + 4xy$

$\frac{\partial u}{\partial x} = 2\alpha x$ $\quad x = 1$

$\frac{\partial v}{\partial y} = \beta + 4x$ $\quad 2\alpha = \beta + 4$

$\beta = 2\alpha - 4$

$\frac{\partial u}{\partial y} = 2\alpha y$

$-\frac{\partial v}{\partial x} = -4y$

$y \in \mathbb{R}$

$2\alpha y = -4y$

$\alpha = -2$

$\beta = -4 - 4 = -8$

$\alpha = -2 \quad \beta = -8$

3)

$$\int_{-\infty}^{+\infty} \frac{(x-3)\sin 2x}{(x^2-6x+10)^2}\,dx = \int_{-\infty}^{\infty} \frac{x-3}{(x^2-6x+10)^2} e^{2ix}\,dx =$$

$$x^2-6x+10 = 0$$

$$\frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i$$

$$= \int_{-\infty}^{\infty} \frac{z-3}{(z-3-i)^2(z-3+i)^2} e^{2iz}\,dz$$

$$\operatorname{res}_{z=3-i} = \lim_{z\to 3-i} \left(\frac{(z-3)e^{2iz}}{(z-3-i)^2}\right)' = \lim_{z\to 3-i} \frac{\left(e^{2iz}+(z-3)2ie^{2iz}\right)(z-3-i)^2 - 2(z-3-i)(z-3)e^{2iz}}{(z-3-i)^4} =$$

$$= \lim_{z\to 3-i} e^{2iz}\frac{\left(1+(z-3)2i\right)(z-3-i)^2 - 2(z-3-i)(z-3)}{(z-3-i)^4} = e^{2i(3-i)}\frac{\left(1+(-i)2i\right)(-2i)^2 - 2(-2i)(-i)}{(-2i)^4} =$$

$$= e^{6i+2}\frac{3\cdot 4+4}{16} = e^{6i+2}$$

$$\operatorname{res}_{z=3+i} = \lim_{z\to 3+i}\left(\frac{(z-3)e^{2iz}}{(z-3+1)^2}\right)' = \lim_{z\to 3+i} \frac{\left(e^{2iz}+(z-3)2ie^{2iz}\right)(z-3+i)^2 - 2(z-3+i)(z-3)e^{2iz}}{(z-3+i)^4} =$$

$$= \lim_{z\to 3+i} e^{2iz}\frac{\left(1+2i(z-3)\right)(2i)^2 - 2(z-3+i)(z-3)}{(z-3+i)^4} = e^{6i-2}\frac{(1-2)(-4) - 2(2i)(i)}{(2i)^4} =$$

$$= e^{6i-2}\frac{4+4}{16} = \frac{e^{6i-2}}{2}$$

$$\int_{-\infty}^{\infty} \frac{x-3}{(x^2-6x+10)^2} e^{2ix}\,dx = \frac{e^{6i-2}}{2}\cdot 2\pi i = e^{6i-2}\,\pi i$$

$$\operatorname{Im} I = \operatorname{Im}\left(\pi i\, e^{-2}(\cos 6 + i\sin 6)\right) = \frac{\pi\cos 6}{e^2}$$

3)

a)

$$F(z) = z \cdot \ln\left(1+\frac{3}{z^2}\right) = z \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{3}{z^2}\right)^n =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n\, z^{2n-1}} = \frac{3}{z} - \frac{9}{2z^3} - \frac{27}{3z^5}$$

$$a_2 = 0$$

$$a_3 = -\frac{9}{2}$$

$$a_4 = 0$$

$$a_5 = 9$$

b)

$$\left((ni^n) * \sin\left(\frac{3\pi}{2}(n+2)\right)\right)_{n=0}^{\infty}$$

$$\mathcal{Z}\left[(ni^n) * \sin\left(\frac{3\pi}{2}(n+2)\right)\right](z) = \mathcal{Z}\left[ni^n\right](z)\,\mathcal{Z}\left[\sin\left(\frac{3\pi}{2}(n+2)\right)\right](z) =$$

$$\mathcal{Z}\left[ni^n\right](z) = -z\frac{d}{dz}\mathcal{Z}\left[i^n\right](z) = -z\left(\frac{z}{z-i}\right)' = -z\,\frac{z-i-z}{(z-i)^2} = \frac{z}{(z-i)^2 i}$$

$$\mathcal{Z}\left[\sin\left(\frac{3\pi}{2}(n+2)\right)\right](z) =$$

$$\mathcal{Z}\left[\sin(\alpha n)\right](z) = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

c)

$$(c_n)_{n=0}^{\infty} = (3^n)_{n=0}^{\infty} * (n^2)_{n=0}^{\infty}$$

$$c_n = \sum_{k=0}^{n} 3^k \cdot (n-k)^2$$

$$c_2 = \sum_{k=0}^{2} 3^k \cdot (2-k)^2 = 4 + 3 + 0 = 7$$

$$c_3 = \sum_{k=0}^{3} 3^k (3-k)^2 = 9 + 12 + 9 + 0 = 30$$

4) $\mathcal{L}$ $\qquad y(0)=1$

$$y'(t)+9\int_0^t y(\tau)\,e^{-6(t-\tau)}\,d\tau = e^t$$

$$y'(t)+9\,y(t) * e^{-6t} = e^t$$

$$sY(s)+9Y(s)\frac{1}{s+6} = \frac{s}{s-1}$$

$$\left(\frac{s^2+6s+9}{s+6}\right)Y(s) = \frac{s}{s-1} \qquad Y(s) = \frac{s^2+6s}{(s-1)(s+3)^2}$$

$$s^2+6s+9 = (s+3)^2$$

$$y(t) = \operatorname{res}_{s=1}\frac{s^2+6s}{(s-1)(s+3)^2} + \operatorname{res}_{s=-3}\frac{s^2+6s}{(s-1)(s+3)^2}$$

$$y(t) = \lim_{s\to 1}\frac{s^2+6s}{(s+3)^2}e^t + \lim_{s\to -3}\left(\frac{(s^2+6s)e^{st}}{(s-1)}\right)' =$$

$$= \frac{7e^t}{16} + \lim_{s\to -3}\frac{\left[(2s+6)e^{st}+(s^2+6s)te^{st}\right](s-1)-(s^2+6s)e^{st}}{(s-1)^2} =$$

$$= \frac{7e^{st}}{16} + \lim_{s\to -3}\frac{-4\cdot(9-18)te^{-3t}-(9-18)e^{-3t}}{(s-1)^2} =$$

$$= \frac{7e^{st}}{16} + \frac{9(4t+1)e^{-3t}}{16}$$

9. 2. 2023

1) $f(z) = \frac{(z-i)^4}{(2-i+z)^2} = (z-i)^4 \frac{1}{(2-i+z)^2}$

$\frac{1}{2-i+z} = \frac{1}{2+(z-i)} = \frac{1}{2} \frac{1}{1-\left(\frac{-(z-i)}{2}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{2^n} \frac{1}{2}$ $\quad \left|\frac{-(z-i)}{2}\right| < 1$

$|z-i| < 2$

$\frac{1}{(2-i+z)^2} = -\left(\frac{1}{2-i+z}\right)' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} n (z-i)^{n-1}$

$= (z-i)^4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} n (z-i)^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} n (z-i)^{n+3}$ $\quad z \in U(i,2)$

b) $\sum_{n=0}^{\infty} a_n (z-6)^n$, $R=3$; kon v $z=4$? ANO

$|4-6| = 2 < 3$

2) $f(z) = \frac{e^{iz} - i - \cos z}{(1-\sin z)\left(z-\frac{\pi}{2}\right)}$

$\sin z = 1 \quad z = \frac{\pi}{2}$

$z_0 = \frac{\pi}{2} + 2k\pi$

$e^{iz} - i - \cos z\big|_{z_0} = i - i - 0 = 0$

$(1-\sin z)\left(z-\frac{\pi}{2}\right)\Big|_{z_0} = 0$

$(e^{iz} - i - \cos z)' = ie^{iz} + \sin z\big|_{z_0} = -1 + 1 = 0$

$(ie^{iz} + \sin z)' = -e^{iz} + \cos z\big|_{z_0} = -i + 0 \neq 0$

k. n. 2

$\left((1-\sin z)\left(z-\frac{\pi}{2}\right)\right)' = -\cos z\left(z-\frac{\pi}{2}\right) + 1 - \sin z\big|_{z_0} = 0$

$\left(-\cos z\left(z-\frac{\pi}{2}\right) + 1 - \sin z\right)' = \sin z\left(z-\frac{\pi}{2}\right) - \cos z - \cos z\big|_{z_0} = \begin{cases} 0 & k=0 \\ \neq 0 & k \neq 0 \end{cases}$ $\quad 2k\pi$

$\left(\sin z\left(z-\frac{\pi}{2}\right) - 2\cos z\right)' = \cos z\left(z-\frac{\pi}{2}\right) + \sin z - 2\sin z \neq 0$

k. n. 3

pro $k \neq 0$: $2-2=0 \rightarrow$ odst. sing.

pro $k=0$: $3-2=1 \rightarrow$ pól ř. 1

b) $\operatorname{res}_i\left((z-i) + \frac{3}{z-i} + \frac{4}{(z-4)^4} + \frac{a}{(z-i)^k}\right) = 5$ $\quad k=1$

$\frac{3}{z-i} + \frac{a}{z-i} = \frac{5}{z-i}$ $\quad a=2$

3) $$F(z)=\frac{z-2i}{z^4+8z^2+16}$$

$$z^4+8z^2+16=(z^2+4)^2=(z\pm 2i)^2$$

$$a_n=\operatorname{res}_{z=2i}\frac{z^{n-1}}{(z-2i)(z+2i)^2}+\operatorname{res}_{z=-2i}\frac{z^{n-1}}{(z-2i)(z+2i)^2}$$

$$\operatorname{res}_{z=-2i}\frac{1}{(z-2i)(z+2i)^2}=\lim_{z\to -2i}\left(\frac{z^{n-1}}{z-2i}\right)'=\lim_{z\to -2i}\frac{(n-1)z^{n-2}(z-2i)-z^{n-1}}{(z-2i)^2}=$$

$$=\frac{-4i(n-1)(-2i)^{n-2}-(-2i)^{n-1}}{-16}=\frac{\frac{-4i}{-2i}(n-1)(-2i)^{n-1}-(-2i)^{n-1}}{-16}=$$

$$=\frac{2(n-1)-1}{-16}(-2i)^{n-1}$$

$$\operatorname{res}_{z=2i}\frac{z^{n-1}}{(z-2i)(z+2i)^2}=\lim_{z\to 2i}\frac{z^{n-1}}{(z+2i)^2}=\frac{(2i)^{n-1}}{-16}$$

$$a_n=\frac{-(2i)^{n-1}}{16}-\frac{2(n-1)-1}{16}(-2i)^{n-1}$$

$$a_0=\lim_{z\to\infty}F(z)=\frac{1}{\infty}=0$$

6) $$\left(n\cos\left(\tfrac{\pi}{2}(n+2)\right)*(-i)^n\right)_{n=0}^{\infty}$$

$$\mathcal{Z}[(-i)^n](z)=\frac{z}{z+i}$$

$$\mathcal{Z}\left[n\cos\left(\tfrac{\pi}{2}(n+2)\right)*(-i)^n\right](z)=\mathcal{Z}\left[n\cos\left(\tfrac{\pi}{2}(n+2)\right)\right](z)\,\mathcal{Z}\left[(-i)^n\right](z)$$

$$\mathcal{Z}\left[n\cos\left(\tfrac{\pi}{2}(n+2)\right)\right](z)=-z\frac{d}{dz}\mathcal{Z}\left[\cos\left(\tfrac{\pi}{2}(n+2)\right)\right](z)=-z\left(\frac{-z^2}{z^2+1}\right)'=$$

$$=z\,\frac{2z(z^2+1)-z^2(2z)}{(z^2+1)^2}=\frac{2z^2}{(z^2+1)^2}$$

$$\mathcal{Z}\left[\cos\left(\tfrac{\pi}{2}(n+2)\right)\right](z)=z^2\frac{z^2}{z^2+1}-1\cdot z^2-0\cdot z=\frac{z^4-z^4-z^2}{z^2+1}=-\frac{z^2}{z^2+1}$$

$$\mathcal{Z}[\dots](z)=\frac{2z^2}{(z^2+1)^2}\,\frac{z}{z+i}=\frac{2z^3}{(z^2+1)^2(z+i)}$$

c) $(c_n)_{n=0}^{\infty} = (2^n)_{n=0}^{\infty} * (n^3)_{n=0}^{\infty}$

$$c_n = \sum_{k=0}^{n} 2^k \cdot (n-k)^3 \qquad c_2 = \sum_{k=0}^{2} 2^k (2-k)^3 = 8+2 = 10$$

$$c_3 = \sum_{k=0}^{3} 2^k (3-k)^3 = 27+16+4 = 47$$

4)

$$y''(t) - 4y'(t) + 4y(t) = e^{-t}\mathbb{1}(t)$$

$$(iw)^2 \hat{y}(s) - 4iw\hat{y}(s) + 4\hat{y}(s) = \frac{1}{1+iw}$$

$$-(w^2 + 4iw - 4)\hat{y}(s) = \frac{1}{1+iw} \qquad w^2 + 4iw - 4 = (w+2i)^2$$

$$\hat{y}(s) = \frac{-1}{(w+2i)^2(1+iw)}$$

$$y(t) = \mathcal{F}^{-1}\left[\frac{-1}{(w+2i)^2(1+iw)}\right](t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iwt}}{(w+2i)^2(1+iw)}\,dw$$

$t \geq 0$

$$\operatorname{res}_{w=i} \frac{e^{iwt}}{(w+2i)^2} = -\frac{e^{-t}}{9i} \qquad y(t) = -\frac{2\pi}{9} e^{-t}$$

$t < 0$

$$\operatorname{res}_{w=-2i} \left(\frac{e^{iwt}}{1+iw}\right)' = \lim_{z \to -2i} \frac{it e^{izt}(1+iz) - ie^{izt}}{(1+iz)^2} = \frac{3t-1}{9} i e^{2t}$$

$$y(t) = \frac{2\pi(3t-1)}{9} e^{2t}$$

$$y(t) = \begin{cases} \frac{e^{-t}}{9} & t \geq 0 \\ -\frac{(3t-1)}{9} e^{2t} & t < 0 \end{cases}$$

13. 2. 2023

1) $f(z)=\frac{1}{(z-1)^3(z^2+2z+1)}$ $\quad z^2+2z+1=(z+1)^2$

$f(z)=\frac{1}{(z-1)^3}\cdot\frac{1}{(z+1)^2}$

$\frac{1}{z+1}=\frac{1}{2+z-1}=\frac{1}{2}\cdot\frac{1}{1-\left(-\frac{z-1}{2}\right)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n \quad 0<|z-1|<2$

$\left(\frac{1}{z+1}\right)'=-\frac{1}{(z+1)^2}$ $\quad \left(\frac{1}{z+1}\right)^2=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}(z-1)^{n-1}$

$f(z)=\frac{1}{(z-1)^3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}(z-1)^{n-1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}(z-1)^{n-4}$

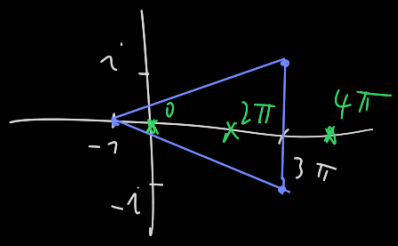
b) $g(z)=\frac{a}{(z-5)^k}-\cancel{\frac{1}{(z-5)^6}}+\sum_{n=0}^{\infty}\frac{(z-5)^{3n-6}}{(-4)^n}$

$+\cancel{\frac{1}{(z-5)^6}}-\frac{1}{4(z-5)^3}+\sum_{n=2}^{\infty}\frac{(z-5)^{3n-6}}{(-4)^n}$

$\frac{a}{(z-5)^k}=\frac{1}{4(z-5)^3}$ $\quad a=\frac{1}{4}$ $\quad k=3$

2) $\int_c \frac{1-\cos z}{z^4-2\pi z^3}+\cancel{\frac{\sin(z^3)}{(z-4\pi)^4}}\,dz$

Cauch. věta



$\int_c \frac{1-\cos z}{z^3(z-2\pi)}=2\pi i\left(\operatorname{res}_0\frac{1-\cos z}{z^3(z-2\pi)}+\operatorname{res}_{2\pi}\frac{1-\cos z}{z^3(z-2\pi)}\right)$

$1-\cos z\big|_{0,2\pi}=0$ $\quad (1-\cos z)'=-\sin t\big|_{0,2\pi}=0$ $\quad (-\sin t)'=-\cos z\big|_{0,2\pi}\neq 0$ kn. 2

$z^3(z-2\pi)\big|_{2\pi}=0$ $\quad z^3(z-2\pi)\big|_0=0$

$(\,)'=3z^2(z-2\pi)+z^3\big|_{2\pi}\neq 0$ kn. 1

$(\,)'=3z^2(z-2\pi)+z^3\big|_0=0$

$(\,)'=6z(z-2\pi)+6z^2\big|_0=0$

$(\,)'=6(z-2\pi)+6z+12z\big|_0\neq 0$ k. n. 3

0:

$$\operatorname{res}_0 \frac{1-\cos z}{z^3(z-2\pi)} = \lim_{z\to 0} \frac{1-\cos z}{z^2(z-2\pi)} \overset{l'H}{=} \lim_{z\to 0} \frac{\sin z}{2z(z-2\pi)+z^2} \overset{l'H}{=} \lim_{z\to 0} \frac{\cos z}{2(z-2\pi)+4z} = -\frac{1}{4\pi}$$

$$\operatorname{res}_{2\pi} \frac{1-\cos z}{z^3(z-2\pi)} = \lim_{z\to 2\pi} \frac{1-\cos z}{z^3} = 0$$

$$\int_C \frac{1-\cos z}{z^3(z-2\pi)} = 2\pi i \left(\operatorname{res}_0 \frac{1-\cos z}{z^3 z-2\pi)} + \operatorname{res}_{2\pi} \frac{1-\cos z}{z^3(z-2\pi)} \right| = 2\pi i \left(-\frac{1}{4\pi} + 0 \right) = -\frac{i}{2}$$

3) $\hat{g}(u) = \dfrac{2+iu+iu^3}{(u+i)^2(1+u^2)^2}$

$$y'(t) + \int_{-\infty}^{\infty} e^{-|\tau|} y(t-\tau)\, d\tau = g(t)$$

$$iu\,\hat{y}(u) + \frac{2}{1+u^2}\,\hat{y}(u) = \frac{2+iu+iu^3}{(u+2i)^2(1+u^2)^2}$$

$$\frac{iu+iu^3+2}{1+u^2}\,\hat{y}(u) = \frac{2+iu+iu^3}{(u+2i)^2(1+u^2)^2}$$

$$\hat{y}(u) = \frac{1}{(u+2i)^2(1+u^2)}$$

$$y(t) = \mathcal{F}\left[\frac{1}{(u+2i)^2(1+u^2)}\right](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iut}}{(u+2i)^2(1+u^2)}\, du$$

$$\operatorname{res}_{-2i} \frac{e^{iut}}{(u+2i)^2(1+u^2)} = \lim_{z\to -2i} \left(\frac{e^{izt}}{1+z^2} \right)' = \lim_{z\to -2i} \frac{it e^{izt}(1+z^2) - e^{izt} 2z}{(1+z^2)^2} =$$

$$= \frac{it e^{2t}(1-4) + e^{2t} 4i}{9} = \frac{-3t+4}{9} i e^{2t}$$

$$\operatorname{res}_i \frac{e^{iut}}{(u+2i)^2(1+u^2)} = \lim_{z\to i} \frac{e^{itz}}{(z+2i)^2 2z} = \frac{e^{-t}}{18} i$$

$$\operatorname{res}_{-i} \frac{e^{iut}}{(u+2i)^2(1+u^2)} = \lim_{z\to -i} \frac{e^{itz}}{(z+2i)^2 2z} = -\frac{e^{t}}{2} i$$

$t \geq 0$

$$y(t) = \frac{1}{2\pi} 2\pi i \frac{e^{-t}}{18} i = -\frac{e^{-t}}{18}$$

$t < 0$

$$y(t) = \frac{1}{2\pi} 2\pi i \left(\frac{4-3t}{9} i e^{2t} - \frac{e^{t}}{2} i \right) =$$

$$= \frac{4-3t}{9} e^{2t} - \frac{e^{t}}{2}$$

$$f(t)=\begin{cases} t^2 & t\in[0,1) \\ 0 & t\in[1,\pi) \\ \sin t & t\in[\pi,\infty) \end{cases}$$

$$f(t)= t^2[u(t)-u(t-1)]+0+\sin t[u(t-\pi)-u(t-\infty)]=$$

b) $\mathcal{L}$

$$\mathcal{L}[t^2u(t)](s)=\mathcal{L}[t^2](s)=\frac{2}{s^3} \qquad [t^n]=\frac{n}{s^{n+1}}$$

$$\mathcal{L}[t^2u(t-1)](s)=e^{-s}\mathcal{L}[(t+1)^2](s)=e^{-s}\mathcal{L}[t^2+2t+1](s)=$$

$$=e^{-s}[\mathcal{L}[t^2](s)+\mathcal{L}[2t](s)+\mathcal{L}[1](s)]=e^{-s}\left[\frac{2}{s^3}+2\frac{1}{s^2}+\frac{1}{s}\right]$$

$$\mathcal{L}[\sin t\,u(t-\pi)](s)=e^{-\pi s}\mathcal{L}[\sin(t+\pi)](s)=-e^{-\pi s}\mathcal{L}[\sin(t)](s)=$$

$$=-e^{\pi s}\frac{1}{s^2+1}$$

c) $T=5$ $\qquad u(t)-u(t-5)$

$$\mathcal{L}[h(t)](s)=\frac{\mathcal{L}[g(t)(u(t)-u(t-5))](s)}{1-e^{-5s}}=\frac{G(s)-\mathcal{L}[g(t)\cdot u(t-5)](s)}{1-e^{-5s}}=$$

$$=\frac{G(s)-e^{-5s}\mathcal{L}[g(t+5)](s)}{1-e^{-5s}}=\frac{G(s)-e^{-5s}\cdot(-G(s))}{1-e^{-5s}}=G(s)\frac{1+e^{-5s}}{1-e^{-5s}}$$

14.2.2023

1) a) $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (3n+1)}{4^{n+1}} z^{3n+2} = \frac{z^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (3n+1)}{4^n} z^{3n} =$

$= \frac{z^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (3n+1)}{4^n} \frac{z^{3n+1}}{3n+1} = \frac{z^2}{4} z \sum_{n=0}^{\infty} \left(-\frac{z^3}{4}\right)^n = \frac{z^2}{4} \frac{z}{1+\frac{z^3}{4}} =$

$|z^3| < 4$

$|z| < \sqrt[3]{4}$

$R = \sqrt[3]{4}$

$z \in U(0; \sqrt[3]{4})$

$= \frac{z^2}{4} \left(\frac{4z}{4+z^3}\right)' = \frac{z^2}{4} \frac{16+4z^3-12z^3}{(4+z^3)^2} = \frac{4z^2-2z^5}{(4+z^3)^2}$

b) $\sum_{n=0}^{\infty} a_n (z+2i)^n \qquad R = 3 \qquad z = -4i$

$|-4i+2i| < 3$

$|-2i| < 3 \qquad 2 < 3 \qquad ANO$

c) $g(z) = \frac{a}{(z-2)^k} + \sum_{n=-2}^{\infty} 3^n (z-2)^{2n} = \frac{a}{(z-2)^k} + \frac{1/4}{(z-2)^4} + \frac{1/3}{(z-2)^2} + \sum_{n=0}^{\infty} 3^n (z-2)^{2n} =$

$a = -1/4$

$k = 4$

3) $T = 4 \qquad f(t) = e^{2t} \mathbb{1}(t-3) \qquad t \in [0,4)$

$\mathcal{L}[f(t)](s) = \frac{\mathcal{L}[e^{2t}\mathbb{1}(t-3) - \mathbb{1}(t-4)](s)}{1-e^{-4s}}$

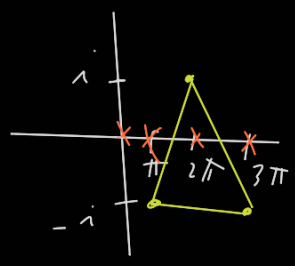
$\mathcal{L}[e^{2t}\mathbb{1}(t-3)](s) = e^{-3s} \mathcal{L}[e^{2(t+3)}](s) = e^{-3s} e^6 \mathcal{L}[e^{2t}](s) = e^{-3s+6} \frac{1}{s-2}$

$\mathcal{L}[e^{2t}\mathbb{1}(t-4)](s) = e^{-4s} \mathcal{L}[e^{2(t+4)}](s) = e^{-4s} e^8 \mathcal{L}[e^{2t}](s) = e^{-3s+8} \frac{1}{s-2}$

$F(s) = \frac{e^{6-3s} \frac{1}{s-2} + e^{8-3s} \frac{1}{s-2}}{1-e^{-4s}} = \frac{e^{6-3s} + e^{8-3s}}{(s-2)(1-e^{-4s})}$

2) $\int_c \frac{e^z}{z^2-\pi^2} + \frac{e^{\frac{z}{2}i}+1}{1-\cos z}\,dz$

$\pm\pi$ 0 $2k\pi$



$\int_c \frac{e^{\frac{z}{2}i}+1}{1-\cos z}\,dz$ $\qquad e^{\frac{z}{2}i}+1\Big|_{2\pi} = 0$

$(e^{\frac{z}{2}i}+1)' = \frac{i}{2}e^{\frac{z}{2}i}\Big|_{2\pi} \neq 0$ kn 1

$1-\cos z\Big|_{2\pi} = 0$

$(\quad)' = \sin z\Big|_{2\pi} = 0$

$(\quad)' = \cos z\Big|_{2\pi} \neq 0$ k.r. 2

pól 1. r.

$\operatorname{res}_{2\pi} \frac{e^{\frac{z}{2}i}+1}{1-\cos z} = \lim_{z\to 2\pi} \frac{(z-2\pi)(e^{\frac{z}{2}i}+1)}{1-\cos z} \overset{l'H}{=} \lim_{z\to 2\pi} \frac{e^{\frac{z}{2}i}+1+\frac{i}{2}e^{\frac{z}{2}i}(z-2\pi)}{\sin z} \overset{l'H}{=}$

$= \lim_{z\to 2\pi} \frac{\frac{i}{2}e^{\frac{z}{2}i} - \frac{i}{4}e^{\frac{z}{2}i}(z-2\pi) + \frac{i}{2}e^{\frac{z}{2}i}}{\cos z} = -\frac{i}{2} - \frac{i}{2} = -i$

$I = 2\pi i(-i) = 2\pi$

4) $y_{n+2} + y_{n+1} + 16\sum_{n=0}^{n} 7^k y_{n-k} = 1 \qquad y_0 = 0,\ y_1 = 1$

$z^2Y(z) - z^2y_0 - zy_1 + zY(z) - zy_0 + 16\mathcal{Z}[7^n](z) \qquad Y(z) = \frac{z}{z-1}$

$z^2Y(z) - z + zY(z) + 16\frac{z}{z-7}Y(z) = \frac{z}{z-1}$

$\left(z^2+z+\frac{16z}{z-7}\right)Y(z) = \frac{z^2}{z-1} \qquad \frac{z^3-7z^2+z^2-7z+16z}{z-7} = \frac{z^3-6z^2+9z}{z-7} = \frac{z(z-3)^2}{z-7}$

$Y(z) = \frac{z^2(z-7)}{(z-1)\,z(z-3)^2} = \frac{z(z-7)}{(z-1)(z-3)^2}$

$\operatorname{res}_1 \frac{z(z-7)}{(z-1)(z-3)^2}z^{n-1} = \operatorname{res}_1 \frac{(z-7)z^n}{(z-1)(z-3)^2} = \lim_{z\to 1}\frac{(z-7)z^n}{(z-3)^2} = \frac{-6\cdot 1^n}{4} = -\frac{3}{2}$

$\operatorname{res}_3 \frac{z(z-7)}{(z-1)(z-3)^2}z^{n-1} = \lim_{z\to 3}\left(\frac{z^{n+1}-7z^n}{z-1}\right)' = \lim_{z\to 3}\frac{((n+1)z^n - 7nz^{n-1})(z-1) - z^{n+1} + 7z^n}{(z-1)^2} =$

$= \frac{2(n+1)3^n - 14n3^{n-1} - 3^{n+1} + 7\cdot 3^n}{4} = \frac{1}{4}\left(2n+2-\frac{14}{3}n-3+7\right)3^n = \frac{1}{4}\left(\left(2-\frac{14}{3}\right)n+6\right)3^n = \left(\frac{3}{2}-\frac{7}{3}n\right)3^n$

$y_n = \left(\frac{3}{2}-\frac{7}{3}n\right)3^n - \frac{3}{2}$

$u(x,y) = e^{2y}\cos(\alpha x) + 2x^3 y + \beta x y^3,\ (x,y) \in \mathbb{R}^2,$

16.2.2023

1) a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} = -\alpha e^{2y}\sin\alpha x + 6x^2 y + \beta y^3$ $\qquad$ $\frac{\partial u}{\partial y} = 2e^{2y}\cos\alpha x + 2x^3 + 3\beta x y^2$

$\frac{\partial^2 u}{\partial x^2} = -\alpha^2 e^{2y}\cos\alpha x + 12xy$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = 4e^{2y}\cos\alpha x + 6\beta x y$

$$(4 - \alpha^2)e^{2y}\cos(\alpha x) + (12 + 6\beta)xy = 0$$

$4 - \alpha^2 = 0$ $\qquad$ $\alpha = \pm 2$

$12 + 6\beta = 0$ $\qquad$ $\beta = -2$

b) $\beta = 2$ $\qquad$ $\alpha = -2$ $\qquad$ $f(2\pi i) = e^4$

$u(x,y) = e^{2y}\cos(-2x) + 2x^3 y + 2xy^3$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2e^{2y}\sin(-2x) + 6x^2 y + 2y^3$

$v(x,y) = \int \frac{\partial v}{\partial y}\,dy = -e^{2y}\sin 2x + 3x^2 y^2 - \frac{y^4}{2} + C(x)$

$\frac{\partial v}{\partial x} = -2e^{2y}\cos 2x + 6xy^2 + C'(x)$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2e^{2y}\cos 2x + 2x^3 + 6xy^2$

$\Rightarrow C'(x) = -2x^3$, $C(x) = -\frac{x^4}{2} + k$

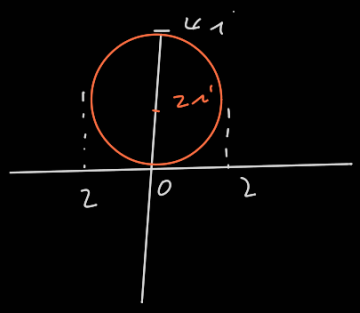
$$v(x,y) = -e^{2y}\sin 2x + 3x^2 y^2 - \frac{y^4}{2} - \frac{x^4}{2} + k$$

$f(2\pi i) = e^4 = u(0,2) + i\,v(0,2)$

$e^4 + i(-8 + k)$

$0 = -8 + k$ $\qquad$ $k = 8$

2)



$\oint_C \frac{\sin z}{z^2+2z} + \frac{e^{\pi z}+1}{(z^2+1)^2}$

$z^2 = 2i$ $\quad z = \pm\sqrt{2i}$ $\qquad z^2 = -1$ $\quad z = \pm i$

$I = 2\pi i \operatorname{res}_i \frac{e^{\pi z}+1}{(z^2+1)^2}$

$e^{\pi z}+1\big|_i = 0$

$(\ )' = \pi e^{\pi z}\big|_i \neq 0$ kn. 1

$(z^2+1)^2 = (z-i)^2(z+i)^2$ kn. 2

$\operatorname{res}_i \frac{e^{\pi z}+1}{(z^2+1)^2} = \lim_{z\to i} \frac{e^{\pi z}+1}{(z+i)^2(z-i)} \overset{l'H}{=} \lim_{z\to i} \frac{\pi e^{\pi z}}{2(z+i)(z-i)+(z+i)^2} = \frac{\pi}{4}$

$I = \frac{\pi^2}{2} i$

3)

$f(t) = \frac{t-2+i}{(t^2-4t+5)^2} = \frac{t-2+i}{(t-2-i)^2(t-2+i)^2} = \frac{1}{(t-2-i)^2(t-2+i)}$

$t^2-4t+5 = 2 \pm i$

$\hat{f}(w) = \int_{-\infty}^{\infty} \frac{e^{-iwt}}{(t-2-i)^2(t-2+i)}\,dt$

$w<0$

$\operatorname{res}_{2+i} \frac{e^{-iwz}}{(z-2-i)^2(z-2+i)} = \lim_{z\to 2+i} \left(\frac{e^{-iwz}}{z-2+i}\right)' = \lim_{z\to 2+i} \frac{-iwe^{-iwz}(z-2+i) - e^{-iwz}}{(z-2+i)^2} = \frac{-iw(2i)-1}{-4} e^{-iw(2+i)} =$

$= \frac{2w-1}{-4} e^{-2iw+w}$

$\hat{f}(w) = 2\pi i\left(-\frac{2w-1}{4}\right) e^{w-2iw} = \frac{\pi}{2} i\,(2w-1)\,e^{w-2iw}$

$w>0$

$\operatorname{res}_{2-i} \frac{e^{-iwz}}{(z-2-i)^2(z-2+i)} = \lim_{z\to 2-i} \frac{e^{-iwz}}{(z-2-i)^2} = \frac{e^{-iw(2-i)}}{-4} = -\frac{e^{-2iw-w}}{4}$

$\hat{f}(w) = -2\pi i\left(-\frac{e^{-2iw-w}}{4}\right) = \frac{\pi}{2} i\, e^{-2iw-w}$

6) $\mathcal{F}[(e^{-4(t+5)})'''] = (iw)^3 \mathcal{F}[e^{-4(t+5)}](w) = -iw^3 e^{-5iw} \mathcal{F}[e^{-4t}](w) =$

$= -iw^3 e^{-5iw} \sqrt{\frac{\pi}{4}}\, e^{-\frac{w^2}{16}}$

c) $\mathcal{F}\left[g(t) * \frac{1}{1+t^2}\right](w) = \pi e^{-2|w|}$

$\hat{g}(w) \cdot \pi e^{-|w|} = \pi e^{-2|w|}$

$\hat{g}(w) = e^{-|w|}$

4) $y''(t) + y'(t) - 2y(t) = e^t$ $\quad y(0) = 1$, $y'(0) = 0$

$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) = \frac{1}{s-1}$

$s^2Y(s) - s + sY(s) - 1 - 2Y(s) = \frac{1}{s-1}$

$(s^2+s-2)Y(s) = \frac{1}{s-1} + s + 1$ $\quad (s-1)(s-1)$

$(s+2)(s-1)Y(s) = \frac{s^2-s+s-1}{s-1} = \frac{s^2-1}{s-1}$ $\quad \frac{s^2}{s-1}$

$Y(s) = \frac{s^2}{(s-1)^2(s+2)}$

$y(s) = \operatorname{res}_1 \frac{s^2 e^{st}}{(s-1)^2(s+2)} + \operatorname{res}_{-2} \frac{s^2 e^{st}}{(s-1)^2(s+2)}$

$\operatorname{res}_1 \frac{s^2 e^{st}}{(s-1)^2(s+2)} = \lim_{z\to 1} \left(\frac{z^2 e^{zt}}{z+2}\right)' = \lim_{z\to 1} \frac{(2z e^{zt} + z^2 t e^{zt})(z+2) - z^2 e^{zt}}{(z+2)^2} =$

$= \frac{6e^t + 3te^t - e^t}{9} = \left(\frac{5}{9} + \frac{1}{3}t\right)e^t$

$\operatorname{res}_{-2} \frac{s^2 e^{st}}{(s-1)^2(s+2)} = \lim_{z\to -2} \frac{z^2 e^{zt}}{(z-1)^2} = \frac{4e^{-2t}}{9}$

$y(s) = \left(\frac{5}{9} + \frac{t}{3}\right)e^t + \frac{4}{9}e^{-2t}$

19.1.2023

1) a) $\sum_{n=0}^{\infty} \frac{z^{n+3}}{(n+1)\,3^{n+2}} = \frac{z^2}{9} \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)\,3^n} \overset{\frac{d}{dz}}{=} \frac{z^2}{9} \sum_{n=0}^{\infty} \frac{(n+1)z^n}{(n+1)\,3^n}$

$\frac{1}{1-\frac{z}{3}} = \int \frac{3}{3-z}\,dz = \left(-3\ln|3-z| + k\right)\Big|_{z_0}$ $\quad \Sigma \dots = 3[\ln(3) - \ln(3-z)]$

$|\frac{z}{3}| < 1$

$|z| < 3$

$0 = -3\ln|3| + k$

$k = 3\ln 3$

$f(z) = \frac{z^2}{9} \cdot 3[\ln(3) - \ln(3-z)] = \frac{z^2}{3}[\ln(3) - \ln(3-z)]$ $\quad |z| < 3$, $z \in U(0,1)$

b) $|z+5| < 6$ $\quad z = 1+i$

$|6+i| < 6$ $\quad$ NE $\quad$ diverguje

c) $f(z) = (z-i)^2 + \frac{3}{(z-i)^5} + \sum_{n=-2}^{\infty} \frac{n}{2}(z-i)^{3n},\ z \in P(i),$

$(z-i)^2 + \frac{3}{(z-i)^5} - \frac{1}{(z-i)^6} - \frac{1}{2(z-i)^3} + 0 + \dots$

hl. část

2) $\int_{-\infty}^{+\infty} \frac{\cos(2x)}{(x^2-2x+2)^2}\,dx$

$x^2 - 2x + 2 = 1 \pm i$

$I = Re\left(\int_{-\infty}^{\infty} \frac{e^{2ix}}{(x^2-2x+2)^2}\,dx\right)$

$\alpha = 2$

$\int_{-\infty}^{\infty} \frac{e^{2ix}}{(x^2-2x+2)^2}\,dx = 2\pi i \operatorname{res}_{1+i} \frac{e^{2iz}}{(z-1-i)^2(z-1+i)^2} = \lim_{z \to 1+i} \left(\frac{e^{2iz}}{(z-1+i)^2}\right)' =$

$= \lim_{z \to 1+i} \frac{2ie^{2iz}(z-1+i)^2 - e^{2iz} \cdot 2(z-1+i)}{(z-1+i)^4} = \frac{2ie^{2i(1+i)}(2i)^2 - e^{2i(1+i)} \cdot 2(2i)}{(2i)^4} =$

$= \frac{-8i-4i}{16} e^{2i(1+i)} = -\frac{3}{4}i\,e^{2i(1+i)} \cdot 2\pi i = \frac{3\pi}{2} e^{2i-2}$

$I = Re\left(\frac{3}{2}\pi e^{2i-2}\right) = \frac{3\pi}{2e^2}\cos 2$

3) a) $f(t) = \mathbb{1}(t+4) - \mathbb{1}(t-8)$

$$\hat{f}(t) = \int_{-4}^{8} e^{-iut}\,dt = \left[\frac{e^{-iut}}{-iu}\right]_{-4}^{8} = \frac{i}{u}\left[e^{-8iu} - e^{4iu}\right] \qquad u \neq 0$$

$u = 0 \quad \hat{f}(t) = \int_{-4}^{8} 1\,dt = 12$

b) ?

c)

$$\mathcal{F}\left[(t\,e^{-\frac{t^2}{4}}) * h''(2t+4)\right](u) = \mathcal{F}\left[t e^{-\frac{t^2}{4}}\right]\mathcal{F}\left[h''(2t+4)\right](u) =$$

$$= i\frac{d}{dt}\mathcal{F}\left[e^{-\frac{t^2}{4}}\right](u)\cdot(iu)^2\,\mathcal{F}\left[h(2t+4)\right](u) =$$

$$= i\frac{d}{du}\left(\sqrt{4\pi}\,e^{-u^2}\right)\cdot(-u^2)\,\frac{1}{2}\,\mathcal{F}\left[h(t+1)\right]\left(\frac{u}{2}\right) =$$

$$= -4ui\sqrt{\pi}\,e^{-u^2}\left(-\frac{u^2}{4}\right)\frac{1}{2}\,e^{2iu}\,\hat{h}\left(\frac{u}{2}\right) = u^3\sqrt{\pi}\,e^{-u^2}\,\frac{1}{2}\,e^{2iu}\,\hat{h}\left(\frac{u}{2}\right)$$

4) $y_{n+2} - 4y_n = (-2)^n \qquad y_0 = 1 \quad y_1 = 0$

$$z^2 Y(z) - z^2 y_0 - z y_1 - 4Y(z) = \frac{z}{z+2}$$

$$z^2 Y(z) - z^2 - 4Y(z) = \frac{z}{z+2}$$

$$(z^2-4)Y(z) = \frac{z^3+2z^2+z}{z+2}$$

$$Y(z) = \frac{z^3+2z^2+z}{(z+2)^2(z-2)}$$

24. 1. 2023

1) a) $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3) z^{2n+3}}{n! 4^n} = z \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3) z^{2n+2}}{n! 4^n}$

$\int \Sigma = \sum \frac{(-1)^n (2n+3) z^{2n+3}}{n! 4^n (2n+3)} = z^3 \sum \frac{\left(-\frac{z^2}{4}\right)^n}{n!} = \left(z^3 e^{\left(-\frac{z^2}{4}\right)} \right)' = 3z^2 e^{-\frac{z^2}{4}} - z^3 \frac{z}{2} e^{-\frac{z^2}{4}} =$

$= \left(3z^2 - \frac{z^4}{2}\right) e^{-\frac{z^2}{4}} \qquad z \in \mathbb{C} \qquad z \in U(0,\infty)$

b) $\frac{f(z)}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3) z^{2n+3}}{n! 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3) z^{2n-1}}{n! 4^n}$

$\text{res}_0 \frac{(-1)^n (2n+3) z^{2n-1}}{n! 4^n} = \frac{3 \cdot z^{-1}}{1 \cdot 1} = \frac{3}{z}$

c) $\sum_{n=-\infty}^{\infty} a_n (z+i)^n \qquad r = 4 \qquad R = 10 \qquad z = 2+i$

$4 < |z+1| < 10$

$4 < |2+2i| < 10$

$4 \not< \sqrt{8} < 10$ divergují

2) a) $f(z) = \frac{\sin z + z - \pi}{(1+e^{iz})^2}$

$z_{01} = \pi + 2k\pi$

$z_{02} = \pi$

pro $\pi + 2k\pi$ odst. sing.

pro π pól 2. řádu

$\sin z + z - \pi \big|_{\pi} = 0 \qquad |_{2k\pi} \neq 0$ kn. 0

$(\ )' = \cos z + 1 \big|_{\pi} = 0$

$(\ )' = -\sin z \big|_{\pi} = 0$ kn. 3

$(\ )' = -\cos z \big|_{\pi} \neq 0$

$(1+e^{iz})^2 \big|_{\pi} = 0$

$(\ )' = 2i(1+e^{iz}) \big|_{\pi} = 0$ kn. 2.

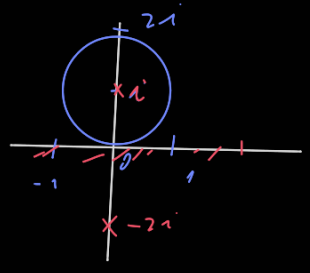
$(\ )' = -2e^{iz} \big|_{\pi} \neq 0$

b) $\int_C \frac{2}{z+2i} + \frac{3}{z-i} + \frac{4}{(z-i)^2} + z^{10} \cos(8z^3)\, dz,$

$\text{res}_i \frac{3}{z-i} = 3$

$\text{res}_i \frac{4}{(z-i)^2} = 0$

$I = 2\pi i \cdot 3 = 6\pi i$



3) a) $\mathcal{Z}\left[\left(n\sin\left(\frac{\pi}{2}(n+3)\right) * \frac{4^n}{n!}\right)_{n=0}^{\infty}\right](z) = \mathcal{Z}\left[n\sin\left(\frac{\pi}{2}(n+3)\right)\right](z)\,\mathcal{Z}\left[\frac{4^n}{n!}\right](z) =$

$\mathcal{Z}\left[n\sin\left(\frac{\pi}{2}(n+3)\right)\right](z) = -z\frac{d}{dz}\mathcal{Z}\left[\sin\left(\frac{\pi}{2}(n+3)\right)\right](z) =$

$\mathcal{Z}\left[\sin\left(\frac{\pi}{2}(n+3)\right)\right](z) = z^3\mathcal{Z}\left[\sin\left(\frac{\pi}{2}n\right)\right](z) - z^3 a_0 - z^2 a_1 - z a_2 = z^3\frac{z}{z^2+1} - z^2 = -\frac{z^2}{z^2+1}$

$= -z\left(-\frac{z^2}{z^2+1}\right)' = -z\left(-\frac{2z(z^2+1) - z^2\cdot 2z}{(z^2+1)^2}\right) = \frac{2z^2}{(z^2+1)^2}$

$\mathcal{Z}\left[\frac{4^n}{n!}\right](z) = \sum \frac{4^n}{n!\,z^n} = e^{\frac{4}{z}}$

$= \frac{2z^2}{(z^2+1)^2}e^{\frac{4}{z}}$

(b) Určete a_0, a_5 a a_{10}, kde $(a_n)_{n=0}^{\infty} \in Z_0$ je inverzní Z-transformace funkce

$$F(z) = \frac{3}{z^{12}} + \frac{4}{z^{10}} + \frac{10}{z^9} + \frac{7}{z^5} + 8 + \sum_{n=0}^{\infty}\frac{n^2+1}{z^{2n}},\ z \in U(\infty).$$

$a_0 = 8 + \frac{0^2+1}{z^{2\cdot 0}} = 9$

$a_5 = 7$

$a_{10} = 4 + 5^2 + 1 = 30$

4) $y''(t) - y(t) = t + \int_0^t \tau e^{t-\tau}\,d\tau$ (the integral is $t * e^t$) $\qquad y(0) = y'(0) = 0$

$s^2Y(s) - s\,y(0) - y'(0) - Y(s) = \frac{1}{s^2} + \frac{1}{s^2}\cdot\frac{1}{s-1}$

$(s^2-1)Y(s) = \frac{s}{s^2(s-1)} = \frac{1}{s(s-1)}$

$Y(s) = \frac{1}{s(s-1)^2(s+1)}$

$y(t) = \sum \text{res}$

$\text{res}_0\, Y(s) = \lim_{z\to 0}\frac{e^{st}}{(s-1)^2(s+1)} = 1$

$\text{res}_{-1}\, Y(s) = \lim_{z\to -1}\frac{e^{st}}{s(s-1)^2} = \frac{-e^{-t}}{4}$

$\text{res}_1\, Y(s) = \lim_{z\to 1}\left(\frac{e^{st}}{s(s+1)}\right)' = \lim_{z\to 1}\frac{te^{st}(s^2+s) - e^{st}(2s+1)}{(s^2+s)^2} = e^t\,\frac{2t-3}{4}$

$y(t) = \frac{2t-3}{4}e^t - \frac{e^{-t}}{4} + 1\ ;\ t \geq 0$

26.1.2023

1) $u(x,y) = e^{\alpha x}\cos y + xy^3 + \beta x^3 y,$

$\Delta\varphi \qquad \alpha, \beta$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} = \alpha e^{\alpha x}\cos y + y^3 + 3\beta x^2 y$

$\frac{\partial^2 u}{\partial x^2} = \alpha^2 e^{\alpha x}\cos y + 6\beta xy$

$\frac{\partial u}{\partial y} = -e^{\alpha x}\sin y + 3xy^2 + \beta x^3$

$\frac{\partial^2 u}{\partial y^2} = -e^{\alpha x}\cos y + 6xy$

$\alpha^2 e^{\alpha x}\cos y + 6\beta xy - e^{\alpha x}\cos y + 6xy = 0$

$(\alpha^2 - 1)\, e^{\alpha x}\cos y + (\beta + 1)\, 6xy = 0$

$\alpha = \pm 1$

$\beta = -1$

5) $\alpha = 1 \qquad \beta = -1$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y + y^3 - 3x^2 y$

$\int \frac{\partial v}{\partial y}\, dy = \left(e^x \sin y + \frac{y^4}{4} - \frac{3}{2}x^2 y^2 + C(x)\right)' = e^x \sin y - 3xy^2 + C'(x)$

$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = e^x \sin y - 3xy^2 + x^3$

$e^x \sin y - 3xy^2 + x^3 = e^x \sin y - 3xy^2 + C'(x)$

$C(x) = \frac{x^4}{4} + k$

$v(x,y) = e^x \sin y + \frac{y^4}{4} - \frac{3}{2}x^2 y^2 + \frac{x^4}{4} - \frac{\pi^4}{64}$

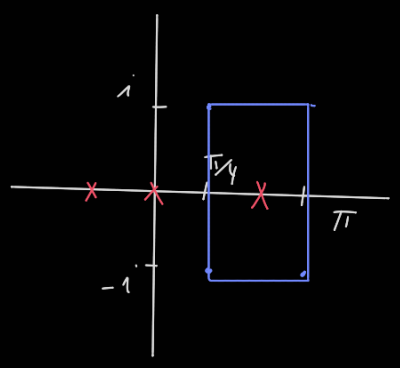
$f\left(\frac{\pi}{2} i\right) = u\left(0, \frac{\pi}{2}\right) + i\, v\left(0, \frac{\pi}{2}\right) = 0 + i\, 1 = i$

$v\left(0, \frac{\pi}{2}\right) = 1$

$1 + \frac{\pi^4}{64} + k = 1 \qquad k = -\frac{\pi^4}{64}$

2)

$$\int_C \frac{e^{iz} - i + z - \frac{\pi}{2}}{z(z-\frac{\pi}{2})^3} + \frac{\sin(z^2)}{e^z(z+1)^4}\,dz,$$



$$I = \int_c \frac{e^{iz} - i + z - \pi/2}{z(z-\pi/2)^3}$$

$z(z-\pi/2)^3$ kn. 3

pól r. 1

$e^{iz} - i + z - \pi/2 \,|_{\pi/2} = 0$

$(\ )' = ie^{iz} + 1 \,|_{\pi/2} = 0$

$(\ )' = -e^{iz}|_{\pi/2} \neq 0$

k.n. 2

$$\operatorname{res}_{\pi/2} \frac{e^{iz} - i + z - \pi/2}{z(z-\pi/2)^3} = \lim_{z\to\pi/2} \frac{e^{iz} - i + z - \pi/2}{z(z-\pi/2)^2} \overset{l'H}{=} \lim_{z\to\pi/2} \frac{ie^{iz} + 1}{(z-\pi/2)^2 + 2z(z-\pi/2)} \overset{l'H}{=} \lim_{z\to\pi/2} \frac{-e^{iz}}{2(z-\pi/2) + 2(z-\pi/2) + 2z} =$$

$$= \frac{-i}{\pi}$$

$$I = 2\pi i \left(\frac{-i}{\pi}\right) = 2$$

3) a) $F(s) = \dfrac{1}{(s+3)^2(1-e^{-2s})}$?

b) $G(s) = \dfrac{e^{-4s}}{s^2}$ $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) = t$ $g(t) = (t-4)1(t-4)$

c) $\mathcal{L}[h'''(t)](s) = s^3H(s) - s^3h(0) - s^2h'(0) - sh''(0) =$ ($h'(0)=1$, $h''(0)=2$, $h'''(0)=3$)

$= s^3H(s) - s^2 - 2s - 3$

4) $y_{n+2} - 2y_{n+1} + 5y_n = \sum_{k=0}^{n} k\, b_{n-k}$ $\quad n \mapsto b_n$ $\quad y_0 = y_1 = 0$

$$z^2Y(z) - z^2y_0 - zy_1 - 2zY(z) - zy_0 + 5Y(z) = \frac{z}{(z-1)^2} \cdot \frac{z-1-2i}{z}$$

$$z^2Y(z) - 2zY(z) + 5Y(z) = \frac{z-1-2i}{(z-1)^2}$$

$$(z^2 - 2z + 5)Y(z) = \frac{z-1-2i}{(z-1)^2} \qquad z^2 - 2z + 5 = (z-1+2i)(z-1-2i)$$

$$Y(z) = \frac{z-1-2i}{(z-1)^2(z-1+2i)(z-1-2i)}$$

$$\operatorname{res}_{1-2i} \frac{1}{(z-1)^2(z-1+2i)} = \lim_{z\to 1-2i} \frac{z^{n-1}}{(z-1)^2} = \frac{(1-2i)^{n-1}}{(-2i)^2} = -\frac{(1-2i)^{n-1}}{4}$$

$$\operatorname{res}_{1} \frac{1}{(z-1)^2(z-1+2i)} = \lim_{z\to 1}\left(\frac{z^{n-1}}{z-1+2i}\right)' = \lim_{z\to 1} \frac{(n-1)z^{n-2}(z-1+2i) - z^{n-1}}{(z-1+2i)^2} = \frac{(n-1)\cdot 2i - 1}{-4} = \frac{1-2i(n-1)}{4}$$

$$y_n = \frac{1-2i(n-1)}{4} - \frac{(1-2i)^{n-1}}{4}$$

31. 1. 2023

1) a) $f(z) = \frac{1}{z^{10} + 4z^{13}} = \frac{1}{z^{10}} \cdot \frac{1}{1+4z^3} = \frac{1}{4z^{13}} \cdot \frac{1}{1+\frac{1}{4z^3}}$

$\frac{1}{z^{10}} \sum_{n=0}^{\infty} (-1)^n 4^n z^{3n} = \sum_{n=0}^{\infty} (-1)^n 4^n z^{3n-10}$

$\frac{1}{4z^{13}} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n z^{3(n+1)}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1} z^{3n+13n}}$

$|z| < \sqrt[3]{\frac{1}{4}}$

b) $g(z) = z^5 \sin\left(\frac{1}{z^2}\right)$

$z^5 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n-3}(2n+1)!}$

2) a) $f(z) = \frac{1-e^{iz}}{(1-\cos z)(z-2\pi)}$

$1 = \cos z \qquad z = 2\pi$

$z = 2k\pi$

$1 - e^{iz}\big|_{2k\pi} = 0$

$(\ )' = -ie^{iz}\big|_{2k\pi} \neq 0$ k.n. 1

$(1-\cos z)(z-2\pi)\big|_{2k\pi} = 0$

$(\ )' = \sin z (z-2\pi) + 1 - \cos z\big|_{2k\pi} = 0$

$(\ )' = \cos z (z-2\pi) + 2\sin z\big|_{2k\pi} = 2\pi(k-1)$ k.n. 2, $k \neq 1$

$(\ )' = -\sin z (z-2\pi) + 3\cos z\big|_{2k\pi} \neq 0$ k.n. 3

if $k = 1 \Rightarrow$ pól r. 2

$k \neq 1 \Rightarrow$ pól r. 1

b) $g(z) = \frac{e^z + a}{z(z - \frac{\pi}{2}i)}$

$z(z - \frac{\pi}{2}i)\big|_{\frac{\pi}{2}i} = 0$

$(\ )' = z - \frac{\pi}{2}i + z\big|_{\pi/2 i} = \pi/2\, i$ kn. 1

$e^z + a\big|_{\pi/2 i} = i + a$

$a = -i$

3) $\hat{f} = \dfrac{1}{u^2+4}$ $\qquad \hat{g}(u) = \dfrac{1}{(u+i)^2}$

$\pm 2i$
$-i$

$$\mathcal{F}[(f*g)(t)](u) = \frac{1}{(u^2+4)(u+i)^2}$$

$$(f*g)(t) = \mathcal{F}^{-1}\left[\frac{1}{(u^2+4)(u+i)^2}\right](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iut}}{(u^2+4)(u+i)^2}\,du$$

$t > 0$

$$\int_{-\infty}^{\infty} \frac{e^{iut}}{(u^2+4)(u+i)^2}\,du = 2\pi i \left(Res_{2i} \frac{e^{iut}}{(u^2+4)(u+i)^2}\right) = 2\pi i \frac{e^{i\cdot 2i\, t}}{2\cdot 2i\,(2i+1)^2} = 2\pi i \frac{e^{-2t}}{-36\,i} =$$

$\dfrac{f(u)}{g'(u)}$

$$= -\frac{\pi}{18} e^{-2t}$$

$t < 0$

$$\int_{-\infty}^{\infty} \frac{e^{iut}}{(u^2+4)(u+i)^2}\,du = 2\pi i \left(Res_{-2i} \frac{e^{iut}}{(u^2+4)(u+i)^2}\right) = 2\pi i \frac{e^{-i\cdot 2i\, t}}{-2\cdot 2i(-2i+1)^2} = \frac{\pi}{2} e^{2t}$$

$\dfrac{f(u)}{g'(u)}$

$$\int_{-\infty}^{\infty} \frac{e^{iut}}{(u^2+4)(u+i)^2}\,du = -2\pi i \left(Res_{-i} \frac{e^{iut}}{(u^2+4)(u+i)^2}\right) = -2\pi i \left(\lim_{z\to -i} \left(\frac{e^{izt}}{(z^2+4)}\right)'\right) =$$

$$= -2\pi i \left(\lim_{z\to -i} \frac{it e^{izt}(z^2+4) - e^{izt}\cdot 2z}{(z^2+4)^2}\right) = -2\pi i \frac{it e^{t} 3 + e^{t} 2i}{9} = \frac{2\pi}{9} e^{t}(3t+2)$$

$$(f*g)(t) = \begin{cases} -\frac{1}{36} e^{-2t} & t \geq 0 \\ \frac{(3t+1)}{9} e^{t} - \frac{1}{4} e^{2t} & t < 0 \end{cases}$$

6) $\mathcal{F}[h(t)\sin(2t)](u)$ $\qquad \sin 2t = \dfrac{e^{2it} - e^{-2it}}{2i}$

$$= \frac{1}{2i}\left[\mathcal{F}[e^{2it}](u) - \mathcal{F}[e^{-2it}](u)\right] = \frac{1}{2i}\left[\hat{h}(u-2) - \hat{h}(u+2)\right]$$

4)

$$y''(t) - y'(t) - 2y(t) = 2e^{2t} \qquad y(0) = 0 \qquad y'(t) = 1$$

$$s^2 Y(s) - s\,y(0) - y'(0) - s Y(s) - y(0) - 2Y(s) = \frac{2}{s-2}$$

$$s^2 Y(s) - 1 - s Y(s) - 2 Y(s) = \frac{2}{s-2}$$

$$(s^2 - s - 2) Y(s) = \frac{s}{s-2}$$

$$s^2 - s - 2 \Rightarrow \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} < \begin{matrix} -1 \\ 2 \end{matrix}$$

$$(s+1)(s-2) Y(s) = \frac{s}{s-2}$$

$$Y(s) = \frac{s}{(s-2)^2 (s+1)}$$

$$y(t) = res_2 + res_{-1}$$

$$res_2 \frac{s e^{st}}{(s-2)^2 (s+1)} = \lim_{s \to 2} \left(\frac{s e^{st}}{s+1} \right)' = \lim_{s \to 2} \frac{(e^{st} + st e^{st})(s+1) - s e^{st}}{(s+1)^2} = \frac{3e^{2t} + 6te^{2t} - 2e^{2t}}{9} = \frac{1+6t}{9} e^{2t}$$

$$res_{-1} \frac{s e^{st}}{(s-2)^2 (s+1)} = \lim_{s \to -1} \frac{s e^{st}}{(s-2)^2} = \frac{-e^{-t}}{9}$$

$$y(t) = \frac{1+6t}{9} e^{2t} - \frac{e^{-t}}{9}$$