V.10

1] a) $2y' = \frac{-1}{x^2 y}$ $\quad y(1)=2$

$x, y \neq 0$

$2\frac{dy}{dx} = \frac{-1}{x^2 y}$

$2\int y\,dy = -\int \frac{1}{x^2}\,dx$

$y^2 = \frac{1}{x} + C$

$y = \pm\sqrt{\frac{1}{x}+C}$

$2 = \sqrt{\frac{1}{1}+C}$

$2 = \sqrt{1+C}$

$C = 3$

b) $y' = \frac{1-y}{y^2}$ $\quad$ eq: $y=0$, $y=1$

c) $y' = \ln(x)\, y^2$

2] a) $y'' - 2y' = 6e^{3x} + 4x$

hmg: $\lambda^2 - 2\lambda = 0$

$\lambda(\lambda-2)=0$

$\{1, e^{2x}\}$

part.:

$y_p = Ae^{3x} + Bx + C$

$y_p' = 3Ae^{3x} + B$

$y_p'' = 9Ae^{3x}$

Lineární rovnice

1) $y'' - 4y' + 3y = x^2 + 1$

$y_h = \lambda^2 - 4\lambda + 3 = 0$

$y_h = (\lambda - 3)(\lambda - 1) = 0$

$f.s. = \{e^x; e^{3x}\}$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$= (x+1)e^{3x}$ $\quad y_p = (Ax+B)xe^{3x}$

$= 12\sin(3x)$

$y_p = A\cos(3x) + B\sin(3x)$

$y'' - 2y' + 5y = 2e^x \sin(2x) + (x-1)e^x \cos(2x)$

$\lambda^2 - 2\lambda + 5 = 0$

$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$y_p = xe^x[(Ax+B)\sin 2x + (Cx+D)\cos(2x)]$

$f.s. = \{e^x \cos 2x, e^x \sin 2x\}$

$y''' - 4y'' + 13y'$

$\lambda^3 - 4\lambda^2 + 13\lambda = 0$

$\lambda(\lambda^2 - 4\lambda + 13) = 0$

$\frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$

$f.s. = \{e^{2x}\cos 3x; e^{2x}\sin 3x\}$

$y^{(4)} + 9y''$

$\lambda^4 + 9\lambda^2 = 0$

$\lambda^2(\lambda^2 + 9) = 0$

$\lambda = 0\,(2x), \pm 3i$

$f.s. = \{1, x, \cos 3x, \sin 3x\}$

$y''' - 2y'' = 2e^x - 1$ $\quad y_p = Ae^x + x^2C$

$\lambda^2(\lambda - 2) = 0$

$f.s. \{1, x, e^{2x}\}$ $\quad y(x) = \alpha + \beta x + \gamma e^{2x} + Ae^x + x^2C$

$y_h = \alpha + \beta x + \gamma e^{2x}, x \in \mathbb{R}$

$y'' + 4y = 1 + 2\sin(2x)$

$y_p(x) = A + xB\sin 2x + xC\cos 2x$

$\lambda^2 + 4 = 0$

$f.s. = \{\cos(2x), \sin 2x\}$

$y(x) = \alpha\cos 2x + \beta\sin 2x + A + xB\sin 2x + xC\cos 2x$

$y_h(x) = \alpha\cos 2x + \beta\sin 2x \quad x \in \mathbb{R}$

$y'' - 2y' = 2x - 1 + xe^x$

$y_p = Ax^2 + Bx + (Cx + D)e^x$

$\lambda^2 - 2\lambda = 0$

$\lambda = 0, 2$

$f.s. = \{1, e^{2x}\}$

$y_h = \alpha + \beta e^{2x}$

$x'' - 3x' + 2x = 2e^t + 2t^2 - 1$

$y_p = Ate^t + Bt^2 - C$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$f.s. = \{e^t, e^{2t}\}$

$y_h(t) = \alpha e^t + \beta e^{2t}, \quad t \in \mathbb{R}$

$y'' - 2y' = 2e^{2x} - 5\cos x + 6 \qquad y(0) = 2 \quad y'(0) = 2$

hmg

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

f. s. $= \{1, e^{2x}\}$

$y_h(x) = \alpha + \beta e^{2x}$

$y'_h(x) = 2\beta e^{2x}$

$\alpha + \beta = 2 \quad \alpha = 1$

$2\beta = 2 \quad \beta = 1$

part

$y_p(x) = Ax e^{2x} + B\cos x + C\sin x + Dx$

$y'_p(x) = Ae^{2x} + 2Axe^{2x} - B\sin x + C\cos x + D$

$y''_p(x) = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} - B\cos x - C\sin x$

$4Ae^{2x} + 4Axe^{2x} - B\cos x - C\sin x - 2Ae^{2x} - 4Axe^{2x} + 2B\sin x - 2C\cos x - 2D = 2e^{2x} - 5\cos x + 6$

$2A = 2 \quad A = 1$

$0 = 0$

$-B - 2C = -5 \quad B = 1$

$2B - C = 0 \quad C = 2$

$-2D = 6 \quad D = -3$

$y(x) = 1 + e^{2x} + 2xe^{2x} + \cos x + 2\sin x - 3x$

$y'' + 4y = 9x\sin(x) - 5e^x \qquad y(0) = -3 \quad y'(0) = 1$

hmg

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$\{\cos 2x, \sin 2x\}$

$y_h(x) = \alpha\cos 2x + \beta\sin 2x$

$y'_h(x) = -2\alpha\sin 2x + 2\beta\cos 2x$

part

$y_p = (Ax + B)\sin x + (Cx + D)\cos x - Ee^x$

$y'_p = A\sin x + (Ax + B)\cos x + C\cos x - (Cx + D)\sin x + Ee^x$

$y''_p = 2A\cos x - (Ax + B)\sin x - 2C\sin x - (Cx + D)\cos x - Ee^x$

$2A\cos x - (Ax + B)\sin x - 2C\sin x - (Cx + D)\cos x - Ee^x + 4(Ax + B)\sin x + 4(Cx + D)\cos x - 4Ee^x = 9x\sin x - 5e^x$

$\alpha = -3$

$\beta = 1/2$

$2A - D + 4D = 0 \quad D = -1$

$-C + 4C = 0 \quad C = 0$

$-A + 4A = 9 \quad A = 3$

$-B - 2C + 4B = 0 \quad B = 0$

$-E + 4E = -5 \quad E = -1$

2) Separace

$\dot{x} = \frac{x^2 - x}{t}$ $\quad x^2 - x \neq 0$, $x \neq 0,1$, $t \neq 0$

$\frac{dx}{dt} = \frac{x^2 - x}{t}$

$\int \frac{1}{x(x-1)} dx = \int \frac{1}{t} dt$

$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$1 = A(x-1) + Bx$

$1 = (A+B)x - A$

$A + B = 0$

$A = -1$

$B = 1$

$\int -\frac{1}{x} + \frac{1}{x-1} dx = \int \frac{1}{t} dt$

$-\ln|x| + \ln|x-1| = \ln|t| + C$

$\ln\left|\frac{x-1}{x}\right| = \ln|t| + C$

$\frac{x-1}{x} = t \cdot k$

$1 - \frac{1}{x} = t \cdot k$

$x(t) = \frac{1}{1 - k \cdot t}$ $\quad k \in \mathbb{R}$, $t \neq \frac{1}{k}$, $k \neq 0$

$x(1) = 2$

$2 = \frac{1}{1-k}$

$2 - 2k = 1$

$k = \frac{1}{2}$

$x(t) = \frac{1}{1 - \frac{t}{2}}$

$x(4) = \frac{1}{2}$

$\frac{1}{2} = \frac{1}{1 - 4k}$

$\frac{1}{2} - 2k = 1$

$k = -\frac{1}{4}$

$x(t) = \frac{1}{1 + \frac{t}{4}}$

$\frac{y'}{y+1} = -4x^3$

$\int \frac{dy}{y+1} = \int -4x^3 dx$

$\ln|y+1| = -x^4 + C$ $\quad C \in \mathbb{R}$

$y = k \cdot e^{-x^4} + 1$ $\quad k \in \mathbb{R}$

$y' = \cos x \cdot y^2$

$\int \frac{1}{y^2} dy = \int \cos x \, dx$

$-\frac{1}{y} = \sin x + C$

$y = \frac{1}{\sin x - C}$

$2y' = \frac{1}{y^2}$

$y^3 = x + C$

$y = \sqrt[3]{x + C}$

$y' - y^2 = 1$

$y' = 1 + y^2$

$\int \frac{1}{1+y^2} dy = \int 1 \, dx$

$\operatorname{arctg} y = x + C$

$y = \operatorname{tg}(x + C)$

$\frac{y'}{y} = -\frac{1}{x}$

$\ln|y| = -\ln|x| + C$

$y = -k \cdot x$

$$y' = 2y + \frac{1}{\sqrt{x}} e^{2x}$$

$$y' - 2y = \frac{e^{2x}}{\sqrt{x}}$$

$$y' = 2y$$

$$\int \frac{1}{2y}\, dy = \int 1\, dx$$

$$\frac{1}{2}\ln|y| = x + C$$

$$y = e^{2(x+C)}$$

$$y = k(x)\, e^{2x}$$

$$k'(x)e^{2x} + k(x)\cdot 2e^{2x} - 2k(x)e^{2x} = \frac{1}{\sqrt{x}} e^{2x}$$

$$k'(x) = \frac{1}{\sqrt{x}}$$

$$k(x) = \int \frac{1}{\sqrt{x}}\, dx$$

$$k(x) = -\sqrt{x}$$

$$y(x) = -\sqrt{x}\, e^{2x}$$

$$y' + y = 13x$$

$$y' = -y$$

$$\int -\frac{1}{y}\, dy = \int 1\, dx$$

$$\ln|y| = -x + C$$

$$y = k_y e^{-x}$$

$$k'(x)\cdot e^{-x} - k(x)\cdot e^{-x} + k(x)e^{-x} = 13x$$

$$k'(x) = 13x\, e^{x}$$

$$k(x) = \int 13x\, e^{x}\, dx$$

$$y(x) = \int 13x\, e^{x}\, dx \cdot e^{-x}$$

$$y_1' = -2y_1 + 4y_2$$

$$y_2' = y_1 + y_2$$

$$\begin{pmatrix} -2-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} = (-2-\lambda)(1-\lambda) - 4 = -2 + 2\lambda - \lambda + \lambda^2 - 4 = \lambda^2 + \lambda - 6$$

$$(\lambda - 2)(\lambda + 3) = 0$$

$\lambda = 2$ $\quad \begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

$$y_\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2x}$$

$\lambda = -3$ $\quad \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} = \left\langle \begin{pmatrix} 4 \\ -1 \end{pmatrix} \right\rangle$

$$y_\beta = \begin{pmatrix} 4 \\ -1 \end{pmatrix} e^{-3x}$$

$$y_1 = \alpha e^{2x} + 4\beta e^{-3x}$$

$$y_2 = \beta e^{2x} - \beta e^{-3x} \; ; \; x \in \mathbb{R}$$

$$y_1' = y_1 + 4y_2$$
$$y_2' = 3y_1 + 2y_2$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = 0 \qquad (\lambda+2)(\lambda-5) = 0$$

$\lambda = -2$

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 \\ 3 \end{pmatrix} e^{-2x}$$

$\lambda = 5$

$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5x}$$

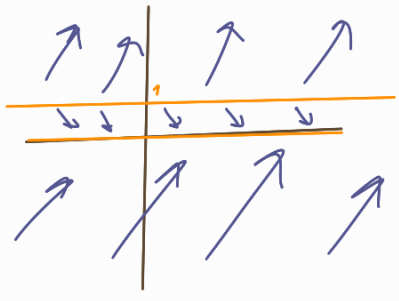
$$y_1 = -4\alpha e^{-2x} + 3\beta e^{5x}$$
$$y_2 = 3\alpha e^{-2x} - 4\beta e^{5x}$$

Vektorové pole

$y' = y^2 - y$

$y = 0$

$y = 1$

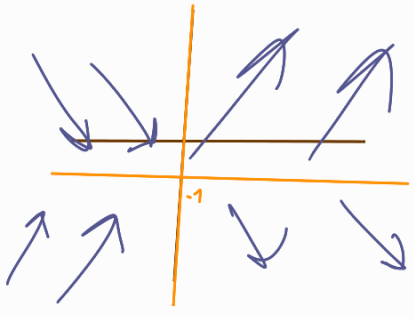


$y' = xy + x$

$y' = x(y+1)$

$y = -1$

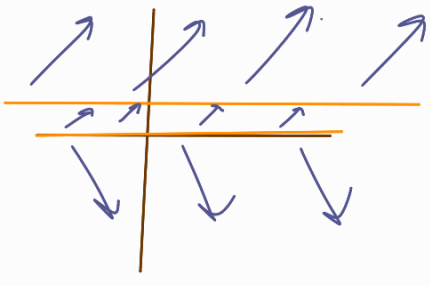
$x = 0$



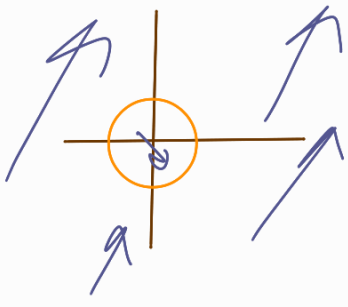
$y' = \frac{(y-1)^2}{e^y - 1}$

$y = 0$

$y = 1$



$y' = x^2 + y^2 - 1$



$$y'' - 4y' + 3y = 2e^{3x} + 9x$$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0$

$f.s. = \{e^x, e^{3x}\}$

$y_h(x) = \alpha e^x + \beta e^{3x}, \quad x \in \mathbb{R}$

$y_h'(x) = \alpha e^x + 3\beta e^{3x}$

$\alpha + \beta = 3 \quad \alpha = 2$

$\alpha + 3\beta = 3 \quad \beta = 0$

$y_p(x) = Axe^{3x} + Bx + C$

$y_p'(x) = 3Axe^{3x} + Ae^{3x} + B$

$y_p''(x) = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x}$

$9Axe^{3x} + 6Ae^{3x} - 12Axe^{3x} - 4Ae^{3x} - 4B + 3Axe^{3x} + 3Bx + 3C = 2e^{3x} + 9x$

$2A = 2 \quad A = 1$

$-4B + 3C = 0 \quad C = 4$

$3B = 9 \quad B = 3$

$y(x) = 2e^x + \beta e^{3x} + xe^{3x} + 3x + 4, \quad x \in \mathbb{R}$

$y'' - 2y' + 2y = 0$

$y(x) = \alpha e^x \cos(x) + \beta e^x \sin(x)$

$\lambda^2 - 2\lambda + 2 = 0$

$1 \pm i$

$y' = \frac{2x-5}{x^2-5x} y \qquad y \neq 0 \qquad y(4) = 8$

$\int \frac{1}{y} dy = \int \frac{2x-5}{x^2-5x} dx$

$\ln|y| = \ln|x^2-5x| + C, \quad C \in \mathbb{R}$

$y = k \cdot (x^2-5x), \quad k \in \mathbb{R}$

$8 = k(4^2 - 5 \cdot 4)$

$k = -2$

$y = -2(x^2-5x)$

$y_1' = y_1 - 2y_2$

$y_2' = y_1 + 4y_2$

$\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$

$\lambda = 3$

$\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \Rightarrow \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3x}$

$\lambda = 2$

$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \Rightarrow \beta \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2x}$

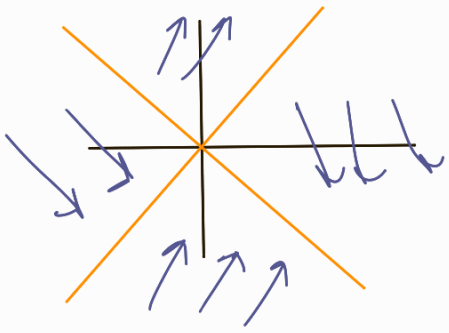
$y_1 = \alpha e^{3x} + 2\beta e^{2x}$

$y_2 = -\alpha e^{3x} - \beta e^{2x}$

$y' = \frac{y-x}{x+y}$

$y = -x$

$y = x$



$f(h) = \sqrt{1+2h}$ aproximace

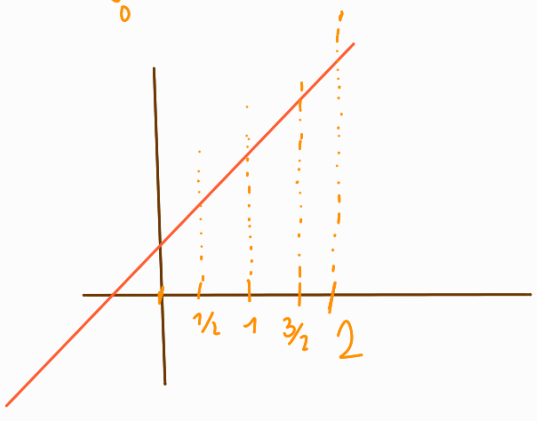
$f'(h) = (1+2h)^{-1/2}$

$f''(h) = -(1+2h)^{-3/2}$

$T = f(0) + f'(0)\cdot h + \frac{1}{2}f''(0)\,h^2 + O(h^3)$

$T = 1 + h - \frac{1}{2}h^2 + O(h^3)$

$\int_0^2 x+1 \, dx$



$h = 4$ $\quad h = \frac{2}{4} = \frac{1}{2}$

LO

$\frac{1}{2}\left[0+1 + \frac{1}{2}+1+ 2+1+\frac{3}{2}+1\right] = 4$

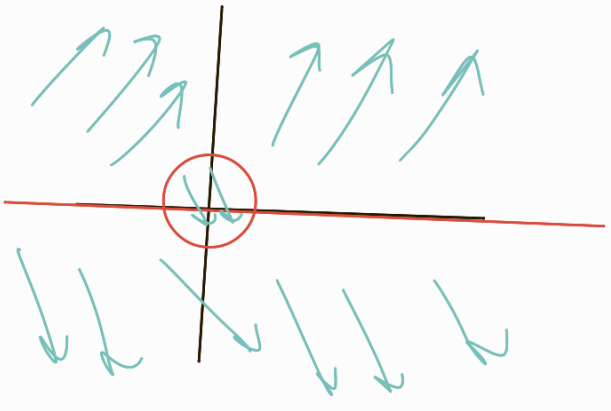
$y' = \frac{\cos x \, y}{\sin x}$

$\int \frac{1}{y}\,dy = \int \frac{\cos x}{\sin x}\,dx$ sto

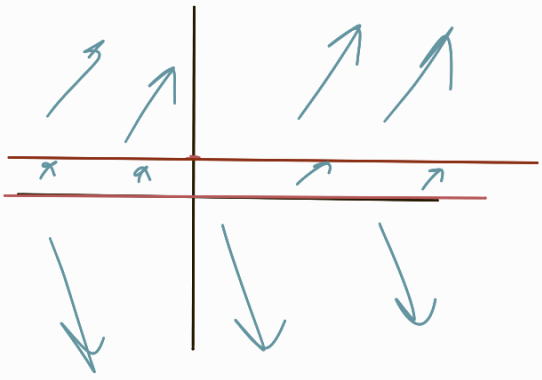
$\ln|y| = \ln|\sin x| + C$, $C \in \mathbb{R}$

$y = k\cdot(\sin x)$; $k \in \mathbb{R}$

$y' = y(x^2+y^2-1)$



$y' = (y-1)^2 y$



$y'' - 2y' + py = 0$

$\lambda^2 - 2\lambda + p = 0$

$\frac{2 \pm \sqrt{4-4p}}{2} = 1 \pm \sqrt{2-2p}$ $\begin{cases} 3 \dots p = -1 \\ x = -1 \end{cases}$

e^{3x}, e^{-x}