Varianta A

1) a) Re a Im

$$z = \frac{2-3i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{2-7i-6}{1+4} = \frac{-4-7i}{5}$$

$\operatorname{Re} z = -\frac{4}{5}$ $\operatorname{Im} z = -\frac{7}{5}$

b) $1-6i = re^{i\varphi}$

$|1-6i| = \sqrt{37} = r$

$\varphi = \operatorname{arctg} -6$

c)

$$g(z) = \frac{a}{(z-4)^k} + \sum_{n=-2}^{\infty} 3^n (z-4)^{3n}$$

$$+ \frac{1/9}{(z-4)^6} + \frac{1/3}{(z-4)^3}$$

$a = -1/9$ $k = 6$

2) a)

$$f(z) = \frac{1}{(z-4)^5(z^2+9z+9)} = \frac{1}{(z-4)^5} \frac{1}{(z+3)^2} \qquad z_0 = 4$$

$$\frac{1}{z+3} = \frac{1}{7} \frac{1}{1+\frac{(z-4)}{7}}$$

$$\frac{1}{1-\left(-\frac{z-4}{7}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n (z-4)^n}{7^n} \overset{\frac{d}{dz}}{\Rightarrow} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n (z-4)^{n-1}}{7^n}$$

$$-\frac{1}{(\ )^2} = \left(\frac{1}{(\ )}\right)' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n (z-4)^{n-6}}{7^{n+1}}$$

$0 < \left|-\frac{z-4}{7}\right| < 1$

$0 < |z-4| < 7$

b) $\sum_{n=0}^{\infty} a_n (z+5)^n$ $R = 2$ $z = -2+i$

$|z+5| < 2$

$|-2+i+5| < 2$

$|3+i| < 2$ divergují

$\sqrt{10} < 2$

3)

$$f(z) = \frac{(z-\pi)(\cos z + 1)}{(1+e^{iz})^3}$$

$e^{iz} = -1 \quad z_0 = \pi + 2k\pi$

$$1+e^{iz}\Big|_{z_0} = 0$$

$$\frac{d}{dz} = ie^{iz}\Big|_{z_0} \neq 0 \quad k.n.1$$

$$3 \cdot 1 = 3 \Rightarrow k.n.\ 3$$

$$(z-\pi)(\cos z + 1)\Big|_{z_0} = 0$$

$$\frac{d}{dz} = \cos z + 1 - (z-\pi)\sin z \Big|_{z_0} = 0$$

$$\frac{d^2}{dz^2} = -\sin z - \sin z - (z-\pi)\cos z = -2k\pi$$

$k \neq 0 \quad k.n.2$

$$\frac{d^3}{dz^3} = -2\cos z - \cos z + (z-\pi)\sin z\Big|_{z_0} \neq 0$$

$k = 0 \quad k.n.3$

pro $k=0$ ods. sing

pro $k \neq 0$ pól ř. 1

4) a) $(a_n)_{n=0}^{\infty}$

$$F(z) = \frac{1}{z^6 + 3z^2} = \frac{1}{z^6} \frac{1}{1+\frac{3}{z^4}}$$

$$\frac{1}{z^6} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{z^{4n}} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{z^{4n+6}}$$

$a_3 = 0 \qquad \frac{1}{z^3} = 0$

$a_{14} = 9 \qquad \frac{1}{z^{14}} = \frac{(-3)^2}{z^{14}}$

b) $$\left(\left[\cos\left(\frac{\pi}{2}(n+2)\right)\right] * \left(n(2i)^n\right) \right)_{n=0}^{\infty} \overset{\mathcal{Z}}{=}$$

$$\mathcal{Z}\left[\cos\left(\frac{\pi}{2}(n+2)\right)\right](z) = z^2 \mathcal{Z}\left[\cos\left(\frac{\pi}{2}n\right)\right](z) - z^2 \overset{a_0}{\cos 0} - z \overset{a_1}{\cos\frac{\pi}{2}} =$$

$$= z^2 \mathcal{Z}\left[\cos\left(\frac{\pi}{2}n\right)\right](z) - z^2 = z^2 \frac{z^2}{z^2+1} - z^2 = \frac{z^4 - z^4 - z^2}{z^2+1}$$

$$\mathcal{Z}\left[n(2i)^n\right](z) = -z\frac{d}{dz}\mathcal{Z}\left[(2i)^n\right](z) = z\left(\frac{z}{z-2i}\right)' = -z\frac{z-2i-z}{(z-2i)^2} = \frac{2iz}{(z-2i)^2}$$

5) a) $\mathcal{F}$

$$y'''(t) + \int_{-\infty}^{\infty} e^{-|\tau|} y(t-\tau)\, d\tau = e^{-\frac{t^2}{4}}$$

$$(iu)^3 \hat{y}(t) + \mathcal{F}\left[e^{-|t|} * y(t)\right](u) = 2\sqrt{\pi}\, e^{-u^2}$$

$$-iu^3 \hat{y}(t) + \frac{2}{1+u^2}\, \hat{y}(u) = 2\sqrt{\pi}\, e^{-u^2}$$

$$\left(-iu^3 + \frac{2}{1+u^2}\right) \hat{y}(u) = 2\sqrt{\pi}\, e^{-u^2}$$

$$\frac{-iu^3 - iu^5 + 2}{1+u^2}\, \hat{y}(u) = 2\sqrt{\pi}\, e^{-u^2}$$

$$\hat{y}(u) = \frac{2\sqrt{\pi}\, e^{-u^2} (1+u^2)}{-iu^5 - iu^3 + 2}$$

b)

$$\hat{y}(u) = \frac{u}{(u+2i)^2 (u-i)}$$

$$y(t) = \mathcal{F}^{-1}\left[\frac{u}{(u+2i)^2(u-i)}\right](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u\, e^{iut}}{(u+2i)^2(u-i)}\, du$$

$t \geq 0$

$$y(t) = \frac{2\pi i}{2\pi}\, \operatorname{res}_i \frac{z\, e^{izt}}{(z+2i)^2(z-i)} = i \lim_{z\to i} \frac{z\, e^{izt}}{(z+2i)^2} = \frac{e^{-t}}{9}$$

$t < 0$

$$y(t) = \frac{-2\pi i}{2\pi}\, \operatorname{res}_{-2i} \frac{u\, e^{iut}}{(u+2i)^2(u-i)} = -i \lim_{z\to -2i} \left(\frac{z \cdot e^{izt}}{z-i}\right)' =$$

$$= -i \lim_{z\to -2i} \frac{\left(e^{izt} + zit\, e^{izt}\right)(z-i) - z e^{izt}}{(z-i)^2} = -i\, \frac{-3i\left(e^{2t} + 2t e^{2t}\right) + 2i e^{2t}}{-9} =$$

$$= -i\, \frac{(6t+1)\, i}{9}\, e^{2t} = \frac{6t+1}{9}\, e^{2t}$$

Varianta B

1) a)

$$z = \frac{i^{17}}{(2-i)^2} = \frac{i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{-4+3i}{25}$$

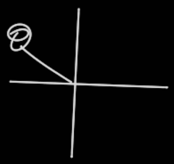
$$\operatorname{Re} z = -\frac{4}{25} \quad \operatorname{Im} z = \frac{3}{25}$$

b)

$$z = (-2-2i)\, e^{1+\frac{3}{2}\pi i} = \sqrt{8}\, e^{\frac{\pi}{4} i}\, e \cdot e^{\frac{3}{2}\pi i}$$

$$|z| = \sqrt{8}\, e$$

$$\varphi = -\frac{3\pi}{4} + \frac{3}{2}\pi = \frac{3}{4}\pi$$



c)

$$\operatorname{res}_1 \left((z-1)^2 + \frac{4}{z-1} + \frac{2}{(z-1)^8} + \frac{a}{(z-1)^k} \right) = -2$$

$$k = 1 \qquad a = -6$$

2) a)

$$\sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+1)\, 2^{n+4}} = z \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)\, 2^{n+4}}$$

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)\, 2^{n+4}} \overset{\frac{d}{dz}}{=} \sum_{n=0}^{\infty} \frac{z^n}{2^{n+4}} = \frac{1}{16} \frac{1}{1-\frac{z}{2}} = \int \frac{1}{16-8z}\, dz = -\frac{1}{8} \ln|16-8z| + C$$

$$\left|\frac{z}{2}\right| < 1$$
$$|z| < 2$$

$$-\frac{1}{8} \ln(16) + c = 0$$
$$c = \frac{\ln(16)}{8}$$

$$f(z) = z\left(-\frac{1}{8} \ln(16-8z) + \frac{\ln 16}{8}\right)$$

3)

$$\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+4x+5)^2}\, dx \qquad \int_{-\infty}^{\infty} \frac{e^{3iz}}{(z+2+i)^2 (z+2-i)^2}\, dz \qquad \alpha = 3$$

$$-2 \pm \sqrt{4-5} = -2 \pm i$$

$$f(z) = 2\pi i \operatorname*{res}_{z=-2+i} \frac{e^{3iz}}{(z+2+i)^2 (z+2-i)^2} = 2\pi i \lim_{z \to -2+i} \left(\frac{e^{3iz}}{(z+2+i)^2} \right)' =$$

$$= 2\pi i \lim_{z \to -2+i} \frac{3i e^{3iz} (z+2+i)^2 - e^{3iz}\, 2\,(z+2+i)}{(z+2+i)^4} = 2\pi i\, e^{3i(i-2)} \frac{3(2i)^2 i - 2(2i)}{(2i)^4} =$$

$$= -2\pi i e^{3i(i-2)} \frac{-12i - 4i}{-16} = 2\pi e^{-3-6i}$$

4) a)

$$f(t) = \begin{cases} e^{it} & t \in [0,2) \\ 0 & t \in [2;4) \\ (t-3)^2 & t \in [4;\infty) \end{cases}$$

$$f(t) = e^{it}\left[\mathbb{1}(t) - \mathbb{1}(t-2)\right] + (t-3)^2 \cdot \mathbb{1}(t-4)$$

$$F(s) = \mathcal{L}\left[e^{it}[\mathbb{1}(t) - \mathbb{1}(t-2)]\right](s) + \mathcal{L}[(t-3)^2\, \mathbb{1}(t-4)](s) =$$

$$= \mathcal{L}[e^{it}](s) - \mathcal{L}[e^{it}\, \mathbb{1}(t-2)](s) + \mathcal{L}[(t-3)^2\, \mathbb{1}(t-4)](s) =$$

$$= \frac{1}{s-i} - e^{-2s}\mathcal{L}[e^{i(t+2)}](s) + e^{-4s}\mathcal{L}[(t+1)^2](s) =$$

$$= \frac{1}{s-i} - e^{-2s+2i}\frac{1}{s-i} + e^{-4s}\left(\mathcal{L}[t^2](s) + \mathcal{L}[2t](s) + \mathcal{L}[1](s)\right) =$$

$$= \frac{1}{s-i} - e^{-2s+2i}\frac{1}{s-i} + e^{-4s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

b)

$$G(s) = \frac{e^{-2s}}{(s+2i)^2} \qquad H(s) = \frac{1}{(s+2i)^2}$$

$$h(t) = \operatorname{res}_{-2i} \frac{e^{st}}{(s+2i)^2} = \lim_{s \to -2i} (e^{st})' = t e^{-2it}$$

$$g(t) = h(t-2)\, \mathbb{1}(t-2) = (t-2)\, e^{-2i(t-2)}\, \mathbb{1}(t-2)$$

5) a) $y''(t) + 2y'(t) = \sin(3t) \qquad y(0) = 1 \qquad y'(0) = -2$

$$s^2Y(s) - s\,y(0) - y'(0) + 2sY(s) - y(0) = \frac{3}{s^2+9}$$

$$s^2Y(s) - s + 2 + 2sY(s) - 2 = \frac{3}{s^2+9}$$

$$(s^2 + 2s)Y(s) = \frac{3}{s^2-9} + s$$

$$Y(s) = \frac{1 + s^3 + 9s}{(s^2+9)(s^2+2s)}$$

b)

$$Y(s) = \frac{1}{(s-3)^2(s+1)}$$

$$y(t) = \operatorname{res}_3 + \operatorname{res}_{-1}$$

$$\operatorname{res}_3 = \lim_{s \to 3} \left(\frac{e^{st}}{s+1}\right)' = \frac{te^{st}(s+1) - e^{st}}{(s+1)^2} =$$

$$= \frac{4t-1}{16} e^{3t}$$

$$\operatorname{res}_{-1} = \lim_{s \to -1} \frac{e^{st}}{(s-3)^2} = \frac{e^{-t}}{16}$$

Varianta C

1) a) $z^2+6z+15=0$

$$\frac{-6\pm\sqrt{36-60}}{2} = -3\pm\sqrt{6}\,i$$

b) $z=(3-3i)^5$

$$3-3i = \sqrt{18}\,e^{-\frac{\pi}{4}i}$$

$$|z|=\sqrt{18}^5$$

$$(3-3i)^5 = (\sqrt{18})^5\,e^{5\cdot(-\frac{\pi}{4}i)}$$

c) $g(z)=\dfrac{e^{zi}+a}{z^5(z+\pi)^2}$

pól ř. 1. $z=-\pi)$

$a=1$

$$z^5(z+\pi)^2\Big|_{z_0}=0$$

$$\frac{d}{dz}=5z^4(z+\pi)^2+2z^5(z+\pi)\Big|_{z_0}=0$$

$$\frac{d^2}{dz^2}=20z^3(z+\pi)^2+2z^5\neq 0$$

k.n.2

$$e^{zi}+a\Big|_{z_0}=-1+1=0$$

$$\frac{d}{dz}=ie^{zi}\neq 0$$

k.n.1

2) a) $u(x,y)=2+3x-y+x^2-y^2-4xy$

$v(x,y)$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=3+2x-4y$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-1-2y-4x$$

$$\int\frac{\partial v}{\partial y}\,dy=3y+2xy-2y^2+c(x)\overset{\frac{\partial}{\partial x}}{=}2y+c'(x)$$

$$2y+c'(x)=1+2y+4x$$

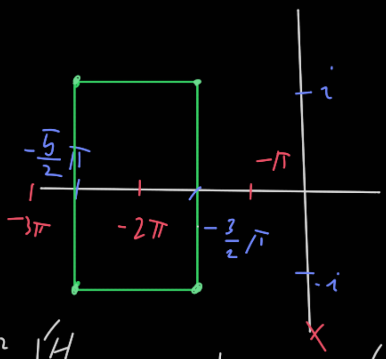
$$c'(x)=1+4x$$

$$c(x)=x+2x^2+k$$

$$v(x,y)=3y+2xy-2y^2+2x^2+x+k$$

$$f'(1+i)=\frac{\partial u}{\partial x}(1,1)+i\frac{\partial v}{\partial x}(1,1)=3+2-4+i(2+4+1)=1+7i$$

3) $\int_c \cancel{\frac{\cos z}{(z+10i)^2}} + \frac{z+2\pi}{(\sin z)^2}\, dz$



$\int \frac{z+2\pi}{(\sin z)^2}\, dz \qquad I = 2\pi i \cdot (res_{-2\pi})$

$res_{-2\pi} \frac{z+2\pi}{(\sin z)^2} = \lim_{z\to-2\pi} (z+2\pi)\frac{z+2\pi}{(\sin z)^2} = \lim_{z\to-2\pi} \frac{(z+2\pi)^2}{(\sin z)^2} \overset{l'H}{=} \lim_{z\to-2\pi} \frac{2(z+2\pi)}{\underbrace{2\sin z\cos z}_{\sin 2z}} \overset{l'H}{=}$

$= \lim_{z\to-2\pi} \frac{2}{2\cos 2z} = \frac{2}{2} = 1$

$I = 2\pi i$

4) a) $f(t) = \mathbb{1}(t+1) - \mathbb{1}(t-5)$

$u \neq 0$

$\hat{f}(u) = \int_{-\infty}^{\infty} [\mathbb{1}(t+1) - \mathbb{1}(t-5)]\, e^{-iut}\, dt = \int_{-1}^{5} e^{-iut}\, dt = \left[\frac{e^{-iut}}{-iu}\right]_{-1}^{5} = \frac{i}{u}\left[e^{-5ui} - e^{-ui}\right]$

$u = 0$

$\int_1^5 1\, dt = 6$

b) $\mathcal{F}[(te^{-\frac{t^2}{2}}) * (e^{2it}\, h'''(t)](u) = \mathcal{F}[te^{-\frac{t^2}{2}}](u)\ \mathcal{F}[e^{2it} h'''(t)](u)$

$\mathcal{F}[te^{-\frac{t^2}{2}}](u) = i\frac{d}{du}\ \mathcal{F}[e^{-\frac{t^2}{2}}](u) = i\frac{d}{du}\left(\sqrt{2\pi}\, e^{-\frac{u^2}{2}}\right) = -iu\ \sqrt{2\pi}\, e^{-\frac{u^2}{2}}$

5) a) $y_{n+2} - y_{n+1} + 2y_n = \sum_{k=0}^{n} k\ b_{n-k} \qquad y_0 = 2 \quad y_1 = 1$

$y_{n+2} - y_{n+1} + 2y_n = n * b_n$

$z^2Y(z) - z^2y_0 - zy_1 - zY(z) - zy_0 + 2Y(z) = \frac{z}{(z-1)^2}\ \frac{(z-1)^2}{z^4}$

$(z^2 - z + 2)Y(z) - 2z^2 - 3z = \frac{1}{z^3}$

$Y(z) = \frac{2z^5 + 3z^4 + 1}{z^3(z^2 - z + 2)}$

b) $Y(z) = \frac{1}{(z-3+i)(z-3)^2}$

$y_n = res_{3-i} + res_3$

$res_{3-i} \frac{z^{n-1}}{(z-3+i)(z-3)^2} = \lim_{z\to 3-i} \frac{z^{n-1}}{(z-3)^2} = -(3-i)^{n-1}$

$res_3 \frac{z^{n-1}}{(z-3+i)(z-3)^2} = \lim_{z\to 3} \left(\frac{z^{n-1}}{z-3+i}\right)' =$

$= \lim_{z\to 3} \frac{(n-1)z^{n-2}(z-3+i) - z^{n-1}}{(z-3+i)^2} = -(n-1)i\,3^{n-2} + 3^{n-1}$

Varianta D

1)

a) $z = \frac{(3-4i)^2}{i^{83}} = \frac{9-24i-16}{-i} \cdot \frac{i}{i} = 24-7i$

b)

$-2-3i = r(\cos\varphi + i\sin\varphi)$

$r = \sqrt{13}$ $\quad \varphi = -\pi + \operatorname{arctg}\frac{3}{2}$

c) $g(z) = \frac{a}{(z-2)^k} + \frac{3}{(z-2)^3} + \sum_{n=-1}^{\infty} n(z-2)^{3n}$ $\quad$ odst. sing.

$-\frac{1}{(z-2)^3}$ $\quad k = 3$

$a = -2$

2)

a) $f(z) = \frac{(z-i)^4}{(2+i-z)^2}$ $\quad z_0 = i$

$(z-i)^4 \quad \frac{1}{(2+i-z)^2}$

$\frac{1}{2+i-z} = \frac{1}{2}\frac{1}{1-\frac{z-i}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-i)^n}{2^n}$

$\frac{1}{(2+i-z)^2} = -\left(\frac{1}{2+i-z}\right)'$

$\left|\frac{z-i}{2}\right| < 1$ $\quad R = 2$

$|z-i| < 2$ $\quad z \in U(0,2)$

$\frac{d}{dz} = \sum_{n=0}^{\infty}\frac{n(z-i)^{n-1}}{2^{n+1}}$

$f(z) = \sum_{n=0}^{\infty}\frac{n(z-i)^{n+3}}{2^{n+1}}$

b) $\sum_{n=0}^{\infty} a_n (z-6)^n$ $\quad R=3$ $\quad z=4$

$|z-6| < 3$

$|4-6| < 3$ $\quad$ ANO konverguje

3)

$f(z) = \frac{\sin z}{z(1-\cos z)}$ $\quad z = 0$

$\cos z = 1$ $\quad z = 2k\pi$

$\sin z|_{2k\pi} = 0$

$\frac{d}{dz} = \cos z|_{2k\pi} \neq 0$ $\quad$ kn 1

$z(1-\cos z)|_{2k\pi} = 0$

$\frac{d}{dz} = 1-\cos z + z\sin z|_{2k\pi} = 0$

$\frac{d^2}{dz^2} = \sin z + \sin z + z\cos z|_{2k\pi} = 2k\pi$ $\quad$ kn. 2

$k=0$ $\quad \frac{d^3}{dz^3} = 2\cos z + \cos z - z\sin z|_{2k\pi} \neq 0$

kn. 3

$k=0$ $\quad$ pól r. 2

$k \neq 0$ $\quad$ pól r. 1

4) a) $F(z) = z^3 \sin\left(\frac{3}{z^5}\right) = z^3 \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3}{z^5}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} \frac{1}{z^{10n+2}}$

$a_2 = 3$

$a_{30} = 0$

b) $\left((b_{n+3}) * (n^2)\right)_{n=0}^{\infty}$ $\quad b_0 = 0 \quad b_1 = 2 \quad b_2 = 4$

$$\mathcal{Z}[b_{n+3}](z) = z^3 B(z) - z^3 b_0 - z^2 b_1 - z b_2 = z^3 B(z) - 2z^2 - 4z$$

$$\mathcal{Z}[n^2](z) = \mathcal{Z}[n \cdot n](z) = -z \frac{d}{dz} \mathcal{Z}[n](z) = -z\left(\frac{z}{(z-1)^2}\right)' = -z \frac{(z-1)^2 - 2z(z-1)}{(z-1)^4} =$$
$$= \frac{z(z+1)}{(z-1)^3}$$

5) a) $y'''(t) + 2y''(t) + y(t) = \frac{1}{1+t^2}$

$$(i\omega)^3 \hat{y}(\omega) + 2(i\omega)^2 \hat{y}(\omega) + \hat{y}(\omega) = \pi e^{-|\omega|}$$

$$(-i\omega^3 - 2\omega^2 + 1)\hat{y}(\omega) = \pi e^{-|\omega|}$$
$$\hat{y}(\omega) = \frac{\pi e^{-|\omega|}}{(-i\omega^3 - 2\omega^2 + 1)}$$

b) $\hat{y}(\omega) = i \frac{\omega + 2i}{(\omega^2+4)^2}$

$$y(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\omega + 2i}{(\omega^2+4)^2} e^{i\omega t} dt \qquad \omega^2 + 4 = (\omega+2i)(\omega-2i)$$

$$\int_{-\infty}^{\infty} \frac{\omega + 2i}{(\omega^2+4)^2} e^{i\omega t} dt = 2\pi i \operatorname{res}_{\pm 2i}$$

$t > 0$

$$2\pi i \operatorname{res}_{2i} \frac{e^{i\omega t}}{(\omega+2i)(\omega-2i)^2} = 2\pi i \lim_{\omega \to 2i} \left(\frac{e^{i\omega t}}{\omega + 2i}\right)' =$$
$$= 2\pi i \lim_{\omega \to 2i} \frac{it e^{i\omega t}(\omega+2i) - e^{i\omega t}}{(\omega+2i)^2} = 2\pi i \frac{4t+1}{16} e^{-2t} = \frac{(4t+1)\pi i}{8} e^{-2t}$$

$t < 0$

$$-2\pi i \operatorname{res}_{-2i} \frac{e^{i\omega t}}{(\omega+2i)(\omega-2i)^2} = -2\pi i \lim_{\omega \to -2i} \frac{e^{i\omega t}}{(\omega-2i)^2} = \frac{\pi i e^{2t}}{8}$$

$$y(t) = \begin{cases} -\frac{(4t+1)}{16} e^{-2t} & t > 0 \\ -\frac{e^{2t}}{16} & t < 0 \end{cases}$$

Varianta E

1) a) $z^2 - 8z + 18 = 0$

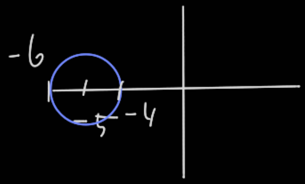
$(z-4)^2 + 2 = 0$

$z = 4 \pm i\sqrt{2}$

b) $-3+5i = re^{i\varphi}$

$r = \sqrt{34}$ $\quad \varphi = \pi - \operatorname{arctg} \frac{5}{3}$

c) $\int_c \frac{3}{(z+5)^{10}} + \frac{1}{(z-5)^2} + \frac{4}{z+5}\, dz$



$\operatorname{res}_{-5}\left(\frac{3}{(z+5)^{10}} + \frac{4}{z+5}\right) = 4$

$I = 2\pi i \cdot 4 = 8\pi i$

2) a) $u(x,y) = e^{\alpha x}\cos y + xy^3 + \beta x^3 y$ $\qquad$ harm.

$\frac{\partial u}{\partial x} = \alpha e^{\alpha x}\cos y + y^3 + 3\beta x^2 y$ $\qquad \frac{\partial u}{\partial y} = -e^{\alpha x}\sin y + 3xy^2 + \beta x^3$

$\frac{\partial^2 u}{\partial x^2} = \alpha^2 e^{\alpha x}\cos y + 6\beta x y$ $\qquad \frac{\partial^2 u}{\partial y^2} = -e^{\alpha x}\cos y + 6xy$

$(\alpha^2 - 1)e^{\alpha x}\cos y + 6(\beta+1)xy = 0$ $\qquad \begin{array}{l}\alpha = \pm 1 \\ \beta = -1\end{array}$

C-R $\quad$ b) $f(z) = \operatorname{Re}(z^2) + i(z+\bar{z})^2 + 2i \operatorname{Im} z$ $\qquad z_0 = 1+4i$

$= x^2 - y^2 + i(4x+2y)$

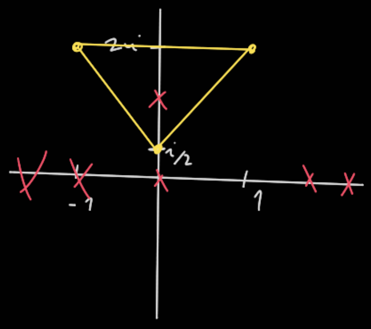
$\frac{\partial u}{\partial x} = 2x\big|_{z_0} = 2 \quad = \quad \frac{\partial v}{\partial y} = 2$

$\frac{\partial u}{\partial y} = -2y\big|_{z_0} = -8 \quad = \quad -\frac{\partial v}{\partial x} = 8x\big|_{z_0} = 8$

$f'(1+4i) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i\,8x = 2 + 8i$

3)

$$\int_C \frac{e^{z\pi}+z\pi-\pi i+1}{z(z-i)^3} + \cancel{\frac{\sin(z^2)}{\cos z}}\, dz$$



$$I = 2\pi i \operatorname{res}_i \frac{e^{z\pi}+z\pi-\pi i+1}{z(z-i)^3}$$

$$e^{z\pi}+z\pi-\pi i+1 \Big|_i = 0 \qquad z(z-i)^3\Big|_i = 0$$

$$\frac{d}{dz} = \pi e^{z\pi}+\pi \Big|_i = 0 \qquad \frac{d}{dz} = (z-i)^3 + 3z(z-i)^2\Big|_i = 0$$

$$\frac{d^2}{dz^2} = \pi^2 e^{z\pi}\Big|_i \neq 0 \quad k.n.2 \qquad \frac{d^2}{dz^2} = 3(z-i)^2+3(z-i)^2+6z(z-i)\Big|_i = 0$$

$$\frac{d^3}{dz^3} = 12(z-i)+6(z-i)+6z\Big|_i \neq 0 \quad k.n.3$$

pól ř.1

$$\operatorname{res}_i \frac{e^{z\pi}+z\pi-\pi i+1}{z(z-i)^3} = \lim_{z\to i} \cancel{(z-i)}\frac{e^{z\pi}+z\pi-\pi i+1}{z(z-i)^{\cancel{3}}} = \lim_{z\to i}\frac{e^{z\pi}+z\pi-\pi i+1}{z(z-i)^2} \overset{l'H}{=}$$

$$= \lim_{z\to i}\frac{\pi e^{z\pi}+\pi}{(z-i)^2+2z(z-i)} \overset{l'H}{=} \lim_{z\to i}\frac{\pi^2 e^{z\pi}}{2(z-i)+2z} = \frac{\pi^2\cdot(-1)}{0+2i} = \frac{\pi^2 i}{2}$$

$$I = -\pi^3$$

4) a)

$$\hat{g}(w) = \frac{1}{w^2+9} \qquad \widehat{f*g}(w) = \frac{1}{(w+3i)^2(w-3i)(w-i)}$$

$$\hat{f}\cdot\frac{1}{w^2+9} = \frac{1}{(w+3i)^2(w-3i)(w-i)} \qquad w^2+9 = (w-3i)(w+3i)$$

$$\hat{f} = \frac{\cancel{(w-3i)}\cancel{(w+3i)}}{(w+3i)^{\cancel{2}}\cancel{(w-3i)}(w-i)} = \frac{1}{(w+3i)(w-i)}$$

$$f = \frac{1}{2\pi}\int \frac{e^{iwt}}{(w+3i)(w-i)}\, dz$$

$$\int \frac{e^{iwt}}{(w+3i)(w-i)}\, dz$$

$t<0$

$$-2\pi i \operatorname{res}_{-3i}\frac{e^{iwt}}{(w+3i)(w-i)} = \lim_{z\to -3i}\frac{e^{iwt}}{w-i} = \frac{e^{3t}}{4}i$$

$t\geq 0$

$$2\pi i \operatorname{res}_i \frac{e^{iwt}}{(w+3i)(w-i)} = \lim_{z\to i}\frac{e^{iwt}}{w+3i} = -\frac{e^{-t}}{4}i$$

$$f(t) = \begin{cases} \dfrac{e^{3t}}{4} & t<0 \\ \dfrac{e^{-t}}{4} & t\geq 0 \end{cases}$$

b) $\mathcal{F}[h(t-1)\cos(3t)](\omega) = \mathcal{F}[h(t-1)\ \frac{1}{2}(e^{3t} - e^{-3t})](\omega) =$

$= \frac{1}{2}\mathcal{F}[h(t-1)\ e^{3ti}](\omega) - \mathcal{F}[h(\tau-1)\ e^{-3ti}](\omega) =$

$= \frac{1}{2}\ \mathcal{F}[h(t-1)](\omega-3)\ \mathcal{F}[h(t-1)](\omega+3) = \frac{1}{2}\ e^{-(\omega-3)}\ \mathcal{F}[h(t)](\omega-3)\ e^{-(\omega+3)}\ \mathcal{F}[h(t)](\omega+3)$

5)

a) $\mathcal{L}$: $y''(t) + 2y'(t) + y(t) = \int_0^t (t-\tau)^3 \sin(\tau)\, d\tau$ $\quad y(0) = -1$

$y(0) = 2$

$$s^2 Y(s) - s\,y(0) - y'(0) + 2(s\,Y(s) - y(0)) + Y(s) = \mathcal{L}[t^3 * \sin t]$$

$$s^2 Y(s) + s - 2 + 2s\,Y(s) + 2 + Y(s) = \frac{6}{s^4(s^2+1)}$$

$$(s^2 + 2s + 1)\ Y(s) = \frac{6}{s^4(s^2+1)} - s$$

$$Y(s) = \frac{6 - s^5(s^2+1)}{(s^6+s^4)(s^2+2s+1)} = \frac{6 - s^7 - s^5}{(s^6+s^4)(s^2+2s+1)}$$

b)

$Y(s) = \frac{s+2}{(s^2+s-2)^2}$ $\qquad y(t) = ?$

$$s^2+s-2 = \frac{-1 \pm 3}{2} = \begin{matrix}-2\\1\end{matrix} \qquad (s+2)(s-1)$$

$$Y(s) = \frac{s+2}{(s+2)^2(s-1)^2} = \frac{1}{(s+2)(s-1)^2}$$

$$y(t) = \operatorname{res}_{-2} \frac{e^{st}}{(s+2)(s-1)^2} + \operatorname{res}_{1} \frac{e^{st}}{(s+2)(s-1)^2}$$

$$\operatorname{res}_{-2} \frac{e^{st}}{(s+2)(s-1)^2} = \lim_{z\to-2} \frac{e^{st}}{(s-1)^2} = \frac{e^{-2t}}{9}$$

$$\operatorname{res}_{1} \frac{e^{st}}{(s+2)(s-1)^2} = \lim_{z\to1} \left(\frac{e^{st}}{s+2}\right)' = \lim_{z\to1} \frac{t\,e^{st}(s+2) - e^{st}}{(s+2)^2} = \frac{3t\,e^{t} - e^{t}}{9} = \frac{3t-1}{9} e^{t}$$

$$y(t) = \frac{3t-1}{9} e^{t} + \frac{e^{-2t}}{9}$$

Varianta F

1) a)

$$z=(2-3i)(3+i)+\frac{2}{i^{13}}$$

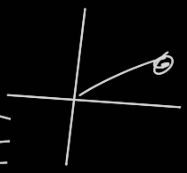
$$z=9-7i\quad -2i=9-9i$$

b)

$$z=5\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)^{16}$$

$$z=5\left(e^{\frac{\pi}{7}i}\right)^{16}=5\,e^{\frac{16}{7}\pi i}$$

$$|z|=5\qquad \arg z=\frac{16}{7}\pi-2\pi=\frac{2}{7}\pi<\frac{\pi}{2}$$



c)

$$\operatorname{res}_i\left(\frac{4}{(z-i)^7}+\frac{a}{(z-i)^k}+\sum_{n=0}^{\infty}n^3(z-i)^{3n-7}\right)=1+i$$

$$\frac{4}{(z-i)^3}+\frac{a}{(z-i)^k}+\frac{1}{(z-i)^7}+\frac{8}{(z-i)}+\sum_{n=3}^{\infty}n^3(z-i)^{3n-7}$$

$$k=1\qquad a=-7+i$$

2)

a)

$$f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(3n+1)\,z^{3n+2}}{n!\,2^n}$$

$$=z^2\int\sum_{n=0}^{\infty}\frac{(-1)^n(3n+1)\,z^{3n}}{n!\,2^n}\,dz=z^2\sum_{n=0}^{\infty}\frac{(-1)^n\cancel{(3n+1)}\,z^{3n+1}}{n!\,2^n\,\cancel{(3n+1)}}=$$

$$=z^2\cdot z\sum_{n=0}^{\infty}\frac{\left(\frac{-z^3}{2}\right)^n}{n!}=z^2\left(z\,e^{-\frac{z^3}{2}}\right)'=z^2\left(e^{-\frac{z^3}{2}}-\frac{3}{2}z^3e^{-\frac{z^3}{2}}\right)=z^2e^{-\frac{z^3}{2}}\left(1-\frac{3}{2}z^3\right)$$

$$R=\infty$$

$$z\in\mathbb{C}$$

b)

$$\sum_{n=-\infty}^{\infty}a_n(z+i)^n\qquad r=\sqrt{6}\qquad R=\infty\qquad z=2+i$$

$$\sqrt{6}<|z+i|<\infty$$

$$\sqrt{6}<|2+2i|<\infty$$

$$\sqrt{6}<8<\infty$$

ANO konverguje

3)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2-2x+5)^2}\,dx$$

$$x^2-2x+5=0$$
$$(x-1)^2+4=0$$
$$x=1\pm 2i$$

$$\int_{-\infty}^{\infty} \frac{z^2}{(z-1-i)^2(z-1+i)^2}\,dz$$

$\alpha = 0$

$$I = 2\pi i \operatorname{res}_{1+2i} \frac{z^2}{(z-1-2i)^2(z-1+2i)^2} = 2\pi i \lim_{z\to 1+2i}\left(\frac{z^2}{(z-1+2i)^2}\right)' =$$

$$= 2\pi i \lim_{z\to 1+2i} \frac{2z(z-1+2i)^2-2z^2(z-1+2i)}{(z-1+2i)^4} = 2\pi i \lim \frac{2z(z-1+2i)-2z^2}{(z-1+2i)^3} =$$

$$= 2\pi i \frac{2(1+2i)(4i)-2(1+2i)^2}{(4i)^3} = 2\pi i \frac{8i-16-2-8i+8}{-4^3 i} = 2\pi i \frac{-10}{-64i} = \frac{5}{16}\pi$$

4) a) $\mathcal{L}$ $f(t)=e^{4t}(\mathbb{1}(t-1)-\mathbb{1}(t-3))$ $\quad t\in[0,5]$

$$\mathcal{L}[e^{4t}(\mathbb{1}(t-1)-\mathbb{1}(t-3))](s) = \frac{1}{1-e^{-5s}}\mathcal{L}[e^{4t}(\mathbb{1}(t-1)-\mathbb{1}(t-3)(\mathbb{1}(t)-\mathbb{1}(t-5))](s) =$$

$$= \frac{1}{1-e^{-5s}}\mathcal{L}[e^{4t}(\mathbb{1}(t-1)-\mathbb{1}(t-3))](s) = \frac{1}{1-e^{-5s}}\left[\mathcal{L}[e^{4t}\mathbb{1}(t-1)](s) - \mathcal{L}[e^{4t}(t-3)](s)\right] =$$

$$= \frac{1}{1-e^{-5s}}\left[e^{-s}\mathcal{L}[e^{4(t+1)}](s) - e^{-3s}\mathcal{L}[e^{4(t+3)}](s)\right] = \frac{1}{1-e^{-5s}}\left[e^{-s}\frac{e^4}{s-4} - e^{-3s}\frac{e^{12}}{s-4}\right]$$

b) $\mathcal{L}[((t\sin(it)) * g''(t)](s)$ $\quad g(0)=2 \quad g'(0)=1$

$$\mathcal{L}[t\sin(it)](s)\mathcal{L}[g''(t)](s) = -\frac{d}{ds}\mathcal{L}[\sin(it)](s)\cdot\left[s^2G(s)-s\,g(0)-g'(0)\right] =$$

$$= -\frac{d}{ds}\left(\frac{i}{s^2-1}\right)\cdot\left[s^2G(s)-2s-1\right] = \frac{2si}{(s^2-1)^2}\left[s^2G(s)-2s-1\right]$$

5) a) $y_{n+2} + 2y_{n+1} - 3y_n = \cos\left(\frac{\pi}{2} n\right) \qquad y_0 = 0 \qquad y_1 = 1$

$$z^2 Y(z) - z^2 y_0 - z y_1 + 2z Y(z) - 2z y_0 - 3Y(z) = \frac{z^2}{z^2+1}$$

$$(z^2+2z-3)Y(z) = \frac{z^2}{z^2+1} + z$$

$$(z-1)(z+3)\ Y(z) = \frac{z^3+z^2-z}{z^2+1}$$

$$Y(z) = \frac{z^3+z^2+z}{(z^2+1)(z-1)(z+3)}$$

b) $Y(z) = \frac{z(z-2i)}{(z^2+4)^2}$ $\qquad z^2 = -4$

$z = \pm 2i$

$$Y(z) = \frac{z(z-2i)}{(z-2i)^2(z+2i)^2}$$

$$y(t) = res_{2i} + res_{-2i}$$

$$res_{2i} \frac{z \quad z^{n-1}}{(z-2i)(z+2i)^2} = \lim_{z \to 2i} \frac{z^n}{(z+2i)^2} = \frac{(2i)^n}{(4i)^2} = -\frac{(2i)^n}{16}$$

$$res_{-2i} \frac{z \cdot z^{n-1}}{(z-2i)(z+2i)^2} = \lim_{z \to -2i} \left(\frac{z^n}{z-2i}\right)' = \lim_{z \to -2i} \frac{nz^{n-1}(z-2i) - z^n}{(z-2i)^2} = \frac{-4in(-2i)^{n-1} + (-2i)^n}{-16} =$$

$$= -\frac{(-2i)^{n-1}}{16}(-4in + 2i)$$