pog’s Problems

pog

March 2, 2022

Contents

1 Introduction

This handout contains every problem which I, pog, have proposed which has appeared in a test format, mock or official. Note that this also means that I will not be putting problems here which I posted outside of a contest or are currently not public.

These problems will be listed in chronological order and in the order in which the problems are listed in their respective tests. I will be updating this document as I continue to propose problems.

I have written problems for the following:

- The Bingo Forum Mock AMC 8
- Karate Masters Mathematics Competitions (KMMC)
- Karate Masters Mathematics Competitions 2 (KMMC 2)
- De Mathematics Competitions (DMC)
- Cyclic National Competitive Math Group (CNCM) Problem of the Day
- AMC Learning Program (ALP) Problem of the Day

Without further ado, please sit back and enjoy the problems! (and yes, this introduction was a reference to the AoPS user **DeToasty3**)

2 Contest Problems (2020–2021)

1. **(The Bingo Forum Mock AMC 8 P1)** Compute $2^{1^{2^1}} - 2^{0^{2^0}}$.

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Answer (B):

   Note that $2^{1^{2^1}} = 2^{1^2} = 2^1 = 2$ and $2^{0^{2^0}} = 2^{0^1} = 2^0 = 1$, for an answer of $2 - 1 = \boxed{\textbf{(B) } 2}$. ■

2. **(The Bingo Forum Mock AMC 8 P2)** Isabella goes to the grocery store. She has \$20 and spends \$5. If one can of Sprrite sells for \$1.50, how many 6-packs of Sprrite can she buy?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 10

   Answer (A):

   Isabella now has \$20 − \$5 = \$15. Hence, she can buy \$15 ÷ \$1.50 = 10 cans of Sprrite, so she can buy at most $\boxed{\textbf{(A) } 1}$ 6-pack of sprite. ■

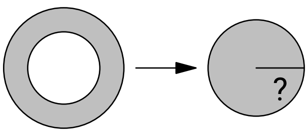
3. **(The Bingo Forum Mock AMC 8 P4)** If $a \bigstar b = \sqrt{a^2+b^2}$ and $a \star b = \sqrt{a^2-b^2}$, what is $7 \bigstar 24 - 29 \star 20$?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Answer (E):

   We get that $7 \bigstar 24 = \sqrt{7^2+24^2} = \sqrt{49+576} = \sqrt{625} = 25$ and $29 \star 20 = \sqrt{29^2-20^2} = \sqrt{(29+20)(29-20)} = \sqrt{49 \cdot 9} = 7 \cdot 3 = 21$. Hence, our answer is $25 - 21 = \boxed{\textbf{(E) } 4}$. ■

4. **(The Bingo Forum Mock AMC 8 P6)** A cookie cutter of radius 3 cuts out the center of a cookie with radius 5. The remainder is then formed into a different cookie. What is the radius of the new cookie?

   (A) $\frac{5}{3}$ **(B)** 2 **(C)** π **(D)** 4 **(E)** $\frac{5}{3}\pi$

Answer (D):

The remaining area is equal to $5^2 \cdot \pi - 3^2 \cdot \pi = 16\pi$. Let the radius of the new cookie be r. Then $r^2 \cdot \pi = 16\pi$, so $r = \sqrt{16} = \boxed{\textbf{(D) } 4}$. ■

5. **(The Bingo Forum Mock AMC 8 P17)** If the area of a right triangle is 175 and its hypotenuse is 30, find the perimeter of the triangle.

(A) 40 **(B)** 50 **(C)** 60 **(D)** 70 **(E)** 80

Answer (D):

Let the legs of the triangle be a and b. Then $\frac{a \cdot b}{2} = 175$, so $a \cdot b = 350$. As well, by the Pythagorean theorem, $\sqrt{a^2 + b^2} = 30$, so $a^2 + b^2 = 900$.

Note that $a^2 + b^2 = (a + b)^2 - 2ab$, so $900 = (a + b)^2 - 2 \cdot 350$. Thus, $1600 = (a + b)^2$, so $a + b = 40$. Consequently, the perimeter of the triangle is

$$a + b + 30 = 40 + 30 = \boxed{\textbf{(D) } 70}.$$

■

6. **(2020 KMMC 10 P1)** What is the value of

$$\frac{2021^2 - 2021}{2020} + \frac{2021^2 - 2020^2}{2021 - 2020}?$$

(A) 1010 **(B)** 2020 **(C)** 2021 **(D)** 4041 **(E)** 6062

Answer (E):

We have that the former fraction is equal to

$$\frac{2021(2021) - 2021}{2020} = \frac{(2021 - 1)(2021)}{2020} = 2021$$

and the latter fraction, by differences of squares, is equal to

$$\frac{2021^2 - 2020^2}{2021 - 2020} = \frac{(2021 + 2020)(2021 - 2020)}{2021 - 2020} = \frac{4041 \cdot 1}{1} = 4041,$$

so our answer is equal to $2021 + 4041 = \boxed{\textbf{(E) } 6062}$. ■

7. **(2020 KMMC 10 P8)** Every day in an eight-day interval, the Cents Lord puts a number of cents into her piggy bank. If the number of cents she puts into her piggy bank from one day to the next forms an increasing arithmetic progression and she puts 2008 cents into her piggy bank in total over the eight days, what is the least possible number of cents she could have put into her piggy bank on any one of the days?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Answer (A):

Let the first term of the arithmetic progression be equal to a (this is also the smallest term, since the arithmetic progression is increasing), and let the common difference between its terms be equal to b. Then the sum of its terms is equal to

$$a+(a+b)+(a+2b)+\cdots+(a+7b) = 8a+(1+2+\cdots+7)b = 8a+\frac{7(8)}{2}b = 8a+\frac{56}{2}b = 8a+28b.$$

Since we wish to minimize a, we should make b as large as possible. However, $2008-28b = 8a$ has to be divisible by 8 for a to be an integer, so since 2008 is already divisible by 8, we must have $28b$ also be divisible by 8. Thus, since 28 is only divisible by 4, we find that b has to be even (so their product will be divisible by $4 \cdot 2 = 8$), so b cannot be equal to $\left\lfloor \frac{2008}{28} \right\rfloor$, but can be equal to $\left\lfloor \frac{2008}{28} \right\rfloor - 1 = 70$. Thus $8a = 2008 - 28 \cdot 70 = 48$ and $a = \frac{48}{8} = \boxed{\text{(A) } 6}$. ■

8. **(2020 KMMC 10 P10)** Let n be a positive integer between 1458 and 2021, inclusive. What is the largest possible value of the sum of the digits of n when n is expressed in base-9? (Express your answer in base-10.)

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

Answer (C):

Note that $1458_{10} = 2000_9$ and $2021_{10} = 2685_9$. Clearly 2688_9 has a larger digit sum than any of the numbers from 2000_9 to 2685_9, so the upper bound of the sum of the digits when n is expressed in base-9 is $(2+6+8+8)-1 = 23$. We can achieve a digit sum of 23 in several ways, such as 2588_9 and 2678_9, so the maximum possible value of the sum of the digits of n when n is expressed in base-9 is thus $\boxed{\text{(C) } 23}$. ■

9. **(2020 KMMC 10 P11)** For a real number x, the median of the list of numbers

$$10, 14, 18, 22, x+3, x+6, x+9, x+12$$

is equal to 13.5. What is the sum of the unique mode and range of the list?

(A) 21 **(B)** 23 **(C)** 25 **(D)** 27 **(E)** 29

Answer (C):

Note that, in a list of 8 numbers, 4 of them will be greater than the median, so since 14, 18, and 22 are already larger than the median of the list, $x+12$ is the only other number larger than the median.

Case 1: The fourth largest number is equal to 10.

Subcase 1.1: The fifth largest number is equal to $x+12$.

We have that $\frac{(x+12)+10}{2} = 13.5$, so $\frac{x+22}{2} = 13.5$. Multiplying both sides of this equation by 2 gives $x+22 = 27$, so $x = 5$ and $x+12 = 17$. However, then the new set would be

$$8, 10, 11, 14, 14, 17, 18, 22,$$

so $x+12$ cannot be the fifth largest number of the list, because then the solution for x would make it so that $x+12$ is not the fifth largest number of the list.

Subcase 1.2: The fifth largest number is equal to 14.

Then the median is equal to $\frac{10+14}{2} = 12$, which is not equal to 13.5.

Case 2: The fourth largest number is equal to $x+9$.

Subcase 2.1: The fifth largest number is equal to $x+12$.

Then $\frac{(x+9)+(x+12)}{2} = \frac{2x+21}{2} = 13.5$, so $x+10.5 = 13.5$ and $x = 3$. However, then the new set would be

$$6, 9, 10, 12, 14, 15, 18, 22,$$

so $x+12$ cannot be the fifth largest number of the list, because then the solution for x would make it so that $x+12$ is not the fifth largest number of the list.

Subcase 2.2: The fifth largest number is equal to 14.

Then $\frac{(x+9)+14}{2} = \frac{x+23}{2} = 13.5$, so multiplying both sides of the equation by 2 gives $x+23 = 27$, so $x = 4$. Then the new set would be

$$7, 10, 10, 13, 14, 16, 18, 22,$$

which does have a median of 13.5. Thus, since the range of the list is equal to $22 - 7 = 15$ and the unique mode of the list is equal to 10, the sum of the unique mode and range of the list is equal to $10 + 15 = \boxed{\textbf{(C) } 25}$. ■

10. **(2020 KMMC 10 P17)** If there exist three distinct positive primes p that satisfy the equation

$$p^4 + ap^3 + bp^2 + cp + 2020 = 0$$

for integers a, b, and c, what is the absolute value of $a + b + c$?

(A) 871 **(B)** 1621 **(C)** 2019 **(D)** 2020 **(E)** 2419

Answer (B):

Note that, by Vieta's formulas, the product of the four roots of the equation is equal to 2020. If three of the roots are distinct primes, then since $2020 = 2^2 \cdot 5 \cdot 101$, the three roots can only be 2, 5, and 101. Thus, since the product of the roots is equal to 2020, the fourth root is equal to 2, so since the leading coefficient of the given polynomial is 1, the factored form of the equation is equal to $(p-2)^2(p-5)(p-101)$. Let $p = 1$. Then

$$(p-2)^2(p-5)(p-101) = 1^4 + a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + 2020 = 1 + a + b + c + 2020.$$

Thus, expanding the left side, we get that $(p-2)^2(p-5)(p-101)$ is equal to

$$(1-2)^2(1-5)(1-101) = (-1)^2(-4)(-100) = 1^2 \cdot 4 \cdot 100 = 400,$$

so $1 + a + b + c + 2020 = 400$ and $a + b + c = 400 - 2020 - 1 = 400 - 2021 = -1621$. Hence, the absolute value of $a + b + c$ is equal to $|-1621| = \boxed{\textbf{(B) } 1621}$. ■

11. **(2020 KMMC 10 P18)** An infinite number of jars are lined up in a row, where the nth jar from the left has n red marbles and 2 blue marbles. For a positive integer k, Karate randomly selects a marble from each of the first k jars from the left. If the probability that he draws exactly k red marbles is strictly less than $\frac{1}{2020}$, what is the least possible value of k?

(A) 44 **(B)** 45 **(C)** 51 **(D)** 57 **(E)** 63

Answer (E):

Note that the probability of getting a red marble from the nth jar is equal to $\dfrac{n}{n+2}$. Thus, the probability of getting k red marbles from the first k jars is equal to

$$\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdots \frac{k-2}{k} \cdot \frac{k-1}{k+1} \cdot \frac{k}{k+2}.$$

Note that all of the numerators cancel except for 1 and 2, and all of the denominators cancel except for $k + 1$ and $k + 2$.

Thus, the probability of getting k red marbles from the first k jars is equal to $\dfrac{1 \cdot 2}{(k+1)(k+2)}$, which must be less than $\frac{1}{2020}$. We take the reciprocal of both sides of

$$\frac{2}{(k+1)(k+2)} < \frac{1}{2020}$$

(we have to flip the inequality sign), giving

$$\frac{(k+1)(k+2)}{2} > 2020.$$

Multiplying both sides of the inequality by 2 gives $(k+1)(k+2) > 4040$. Thus, seeing as $(63)(64) = 4032$ and $(64)(65) = 4160$, so the least possible value of k is $\boxed{\textbf{(E) } 63}$, which gives the probability of drawing 63 red marbles from the first 63 jars as $\frac{2}{(63+1)(63+2)} = \frac{2}{64 \cdot 65} = \frac{1}{32 \cdot 65} = \frac{1}{2080}$. ■

12. **(2021 DIME P1)** Find the remainder when the number of positive divisors of the value

$$(3^{2020} + 3^{2021})(3^{2021} + 3^{2022})(3^{2022} + 3^{2023})(3^{2023} + 3^{2024})$$

is divided by 1000.

Answer (783):

The given value can be written as

$$(1 \cdot 3^{2020} + 3 \cdot 3^{2020})(1 \cdot 3^{2021} + 3 \cdot 3^{2021})(1 \cdot 3^{2022} + 3 \cdot 3^{2022})(1 \cdot 3^{2023} + 3 \cdot 3^{2023}).$$

This is equal to $(4 \cdot 3^{2020})(4 \cdot 3^{2021})(4 \cdot 3^{2022})(4 \cdot 3^{2023})$. Rearranging gives $4^4 \cdot 3^{2020} \cdot 3^{2021} \cdot 3^{2022} \cdot 3^{2023} = 4^4 \cdot 3^{2020+2021+2022+2023} = (2^2)^4 \cdot 3^{8086} = 2^8 \cdot 3^{8086}$.

Each positive divisor of $2^8 \cdot 3^{8086}$ can be written as $2^a \cdot 3^b$, where $a \in [0, 8]$ and $b \in [0, 8086]$. Hence, there are 9 possible choices of a and 8087 possible choices of b, so by the Fundamental Counting Principle, the given value has $9 \cdot 8087 = 72783$ positive divisors, which has a remainder of $\boxed{783}$ when divided by 1000. ■

13. **(2021 DIME P2)** If x is a real number satisfying the equation

$$9 \log_3 x - 10 \log_9 x = 18 \log_{27} 45,$$

then the value of x is equal to $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

Answer (140):

Note that $3^x = (3^2)^{\left(\frac{1}{2}x\right)} = 9^{\left(\frac{1}{2}x\right)}$ and $3^x = (3^3)^{\left(\frac{1}{3}x\right)} = 27^{\left(\frac{1}{3}x\right)}$, so $\log_9 x = \frac{1}{2}x \log_3 x$ and

$\log_{27} x = \frac{1}{3}\log_3 x$ and the given equation is equal to

$$9\log_3 x - 5\log_3 x = 6\log_3 45.$$

Simplifying, we get $4\log_3 x = 6\log_3 45$, so thus dividing both sides of this equation by 4 gives $\log_3 x = \frac{3}{2}\log_3 45$.

Since $\frac{3}{2}\log_3 45 = 45^{\frac{3}{2}} = 45^1 \cdot 45^{\frac{1}{2}} = 45 \cdot \sqrt{45}$, we get that $\log_3 x = \log_3(45 \cdot \sqrt{45})$. Hence, $x = 45\sqrt{45}$.

Consequently, since $45 = 3^2 \cdot 5$, we get that

$$x = 45\sqrt{45} = 45\sqrt{3^2 \cdot 5} = 45 \cdot \sqrt{3^2} \cdot \sqrt{5} = 45 \cdot 3 \cdot \sqrt{5} = 135\sqrt{5},$$

so our answer is $135 + 5 = \boxed{140}$. ■

3 PoTD Problems (2020–2021)

1. **(CNCM Problem of the Day — March 31, 2021)** If θ is a real number such that

$$\sin(\theta)+\cos(\theta)=\frac{199}{187},$$

then $\sin(\theta)\cos(\theta)=\frac{m}{n}$, where m and n are relatively prime positive integers. What are the last two digits of n?

Answer (69):

Note that $\sin^2(\theta)+\cos^2(\theta)=(\sin(\theta)+\cos(\theta))^2-2(\sin(\theta)\cos(\theta))=1$. Hence,

$$\begin{aligned}\left(\frac{199}{187}\right)^2-2(\sin(\theta)\cos(\theta)&=1\\ \frac{199^2}{187^2}-2(\sin(\theta)\cos(\theta)&=\frac{187^2}{187^2}\\ \frac{199^2}{187^2}-\frac{187^2}{187^2}&=2(\sin(\theta)\cos(\theta)\\ \frac{1}{2}\cdot\frac{199^2-187^2}{187^2}&=\sin(\theta)\cos(\theta).\end{aligned}$$

By differences of squares, we get that

$$\frac{\frac{1}{2}(199^2-187^2)}{187^2}=\frac{\frac{1}{2}(386)(12)}{187^2}.$$

Simplifying, we get that $\sin(\theta)\cos(\theta)=\frac{2^2\cdot 3\cdot 193}{11^2\cdot 17^2}$. These are relatively prime, so $n=11^2\cdot 17^2=34969$, for an answer of $\boxed{69}$. ■

4 Contest Problems (2021–2022)

1. **(2021 DMC 10A P3)** Let n be a positive integer less than 2021. It is given that if a regular hexagon is rotated n degrees clockwise about its center, the resulting hexagon coincides with the original hexagon. How many possible values of n are there?

   (A) 8 **(B)** 16 **(C)** 17 **(D)** 32 **(E)** 33

   Answer (E):

   Note that the 6 vertices of the hexagon are equidistant, so rotating the hexagon a multiple of $\frac{360^\circ}{6} = 60^\circ$ will give a new hexagon that coincides with the original.

   Thus, we wish to find how many positive multiples of 60 are less than 2021; since $2021 = 60 \cdot 33 + 41$, we have that the $\boxed{\textbf{(E) } 33}$ possible values of n are

   $$60 \cdot 1, 60 \cdot 2, 60 \cdot 3, \cdots, 60 \cdot 33.$$

   ■

2. **(2021 DMC 10A P5)** If the product of three distinct positive real numbers forming a geometric progression is equal to 2197, what is the median of the three numbers?

   (A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

   Answer (C):

   Let the first term of the geometric progression be equal to a, and let the common ratio between two consecutive terms of the geometric progression be equal to r. Then, the first term of the geometric progression is equal to a, the second term (or the median, since r is clearly positive) of the geometric progression is equal to ar, and the third term of the geometric progression is equal to ar^2. Thus, the product of the terms of the geometric progression is equal to

   $$a \cdot ar \cdot ar^2 = a^3r^3.$$

   Since the product of the terms in the geometric progression is equal to 2197, we have that $a^3r^3 = 2197$, so since we wish to find ar (the median), we can take the cube root of both sides, giving the requested median as $\sqrt[3]{2197} = \boxed{\textbf{(C) } 13}$. ■

3. **(2021 DMC 10A P9)** In the coordinate plane, let $\mathcal{P}$ be the figure formed by the set of points with coordinates satisfying $0.5x + y = 1$, and let $\mathcal{Q}$ be the figure formed by the set of points with coordinates satisfying $0.25x^2 + y^2 = 1$. How many points lie on both $\mathcal{P}$ and $\mathcal{Q}$?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

Answer (C):

Squaring both sides of the first equation gives

$$0.25x^2 + y^2 + xy = 1,$$

so since we are given that $0.25x^2 + y^2 = 1$, we have that xy must be equal to 0.

However, squaring both sides of an equation may produce extraneous solutions, so we must check the solutions for $xy = 0$ against the solutions for $0.5x + y = 1$, giving either $x = 0$, $y = 1$ or $y = 0, x = 2$; there are thus $\boxed{\textbf{(C) } 2}$ requested points that lie on both $\mathcal{P}$ and $\mathcal{Q}$, namely $(0, 1)$ and $(2, 0)$.

OR

Note that the equation for $\mathcal{Q}$ can be rewritten as

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1,$$

which, when graphing, will give an ellipse with center $(0, 0)$, horizontal major axis 4, and vertical minor axis 2, so an approximate graph of the line $\mathcal{P}$ formed by the equation $0.5x+y = 1$ will intersect the ellipse twice, for a total of $\boxed{\textbf{(C) } 2}$ requested points that lie on both $\mathcal{P}$ and $\mathcal{Q}$.

Remark: Briefly checking the infinitely many solutions to $xy = 0$ by the graph in the second solution will make it easier to spot that there are extraneous solutions. ■

4. **(2021 DMC 10A P10)** How many real numbers x satisfy the equation

$$9^x + 3^{3x} = 3^{x+1} + 3?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (B):

Let $3^x = n$. Then

$$\begin{aligned} 9^x + 3^{3x} &= 3^{x+1} + 3 \\ (3^2)^x + (3^x)^3 &= 3 \cdot 3^x + 3 \\ (3^x)^2 + n^3 &= 3n + 3 \\ n^2 + n^3 &= 3n + 3. \end{aligned}$$

We can factor this as $n^2(n + 1) = 3(n + 1)$, so subtracting $3(n + 1)$ from both sides gives

$(n^2-3)(n+1)=0$. Thus, either $n^2-3=0$ or $n+1=0$, so the solutions for n are

$$n=-\sqrt{3},\sqrt{3},-1.$$

Since x is real, $3^x=n$ cannot be negative, so our only solution is $n=\sqrt{3}$ (where $x=\frac{1}{2}$), so $\boxed{\textbf{(B)}\ 1}$ real number x satisfies the equation $9^x+3^{3x}=3^{x+1}+3$, as requested. ■

5. **(2021 DMC 10A P16)** If a and b are the distinct roots of the polynomial $x^2+2021x+2019$, then

$$\frac{1}{a^2+2019a+2019}+\frac{1}{b^2+2019b+2019}=\frac{m}{n},$$

where m and n are relatively prime positive integers. What is $m+n$?

(A) 2020 **(B)** 2021 **(C)** 4040 **(D)** 6059 **(E)** 6061

Answer (D):

Note that since a is a root of $x^2+2021x+2019$, we have that $a^2+2021a+2019=0$ and

$$a^2+2019a+2019=(a^2+2021a+2019)-2a=0-2a=-2a.$$

Similarly, note that since b is a root of $x^2+2021x+2019$, we have that $b^2+2021b+2019=0$ and

$$b^2+2019b+2019=(b^2+2021b+2019)-2b=0-2b=-2b.$$

Thus, $\dfrac{1}{a^2+2019a+2019}+\dfrac{1}{b^2+2019b+2019}=\dfrac{1}{-2a}+\dfrac{1}{-2b}=\dfrac{-2a-2b}{4ab}$.

Since a and b are the roots of $x^2+2021x+2019$, we have that $(x-a)(x-b)=x^2-ax-bx+2019=x^2-(a+b)x+2019=x^2+2021x+2019$, so $a+b=-2021$ and $ab=2019$.

Finally, we get that $\dfrac{-2a-2b}{4ab}=-\dfrac{a+b}{2ab}$, so since $a+b=-2021$ and $ab=2019$,

$$\frac{1}{a^2+2019a+2019}+\frac{1}{b^2+2019b+2019}=-\frac{-2021}{2\cdot 2019}=\frac{2021}{4038}$$

and the requested sum is equal to $2021+4038=\boxed{\textbf{(D)}\ 6059}$. ■

6. **(2021 DMC 10B P3)** If n is a positive integer such that $n\times 3^5=3^7-3^5$, what is n?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

Answer (D):

We can rewrite the right-hand side of the equation as

$$(3^2 - 1) \cdot 3^5 = 8 \cdot 3^5,$$

so the requested answer is $\boxed{\textbf{(D) } 8}$. ■

7. **(2021 DMC 10B P8, with DeToasty3)** At Test Academy, there are four classes, one on each of the four floors of the building. For each class, the class which is one floor above it has twice as many students and half the average grade of that class. If the average grade of all four classes combined is 20, what is the average grade of the class on the bottom floor?

(A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

Answer (A):

Let n be the number of students in the class on the bottom floor, and let g be the average grade of the class on the bottom floor. Our requested answer is

$$\frac{gn + \frac{g}{2} \cdot 2n + \frac{g}{4} \cdot 4n + \frac{g}{8} \cdot 8n}{n + 2n + 4n + 8n} = 20 \implies \frac{4gn}{15n} = 20 \implies g = \boxed{\textbf{(A) } 75}.$$

■

8. **(2021 DMC 10B P10, with DeToasty3)** How many ordered triples of integers (a, b, c) are there such that the product

$$(a - 2020)(2b - 2021)(3c - 2022)$$

is positive and has exactly three positive divisors?

(A) 3 **(B)** 9 **(C)** 12 **(D)** 24 **(E)** infinitely many

Answer (C):

The key observation is that 2022 is divisible by 3, namely $2022 = 3 \cdot 674$. This means that our product becomes

$$3(a - 2020)(2b - 2021)(c - 674).$$

This means that our product is divisible by 3. Now, in order for our product to have three positive divisors, we must have that the product is equal to $3^2 = 9$, which has $2 + 1 = 3$ factors. Note that this is the only possibility since 3 is prime.

This means that for each of the three factors $a - 2020$, $2b - 2021$, and $c - 674$, we must have that they multiply to 3. We can either choose two of the three to be negative or all three to be positive, for 4 possibilities, and there are three ways to choose which one has the factor of 3. This gives us a total of $4 \cdot 3 = \boxed{\textbf{(C) } 12}$ ordered triples, as requested. Note that since -3, -1, 1, and 3 are all odd, and $2b - 2021$ is always odd, we do not over count. ■

9. **(2021 DMC 10C P5)** A group of 200 people were invited to see a movie, where each person either had first row seats, second row seats, or third row seats. It is given that four-fifths of the people invited chose to watch the movie, one-ninth of the viewers were not invited, one-fifth of the viewers had first row seats, and 60 of the viewers had third row seats. What is the probability that a randomly selected viewer had second row seats?

(A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{5}$

Answer (C):

From the condition that four-fifths of the people invited chose to watch the movie, we get that

$$\frac{4}{5} \cdot 200 = 160$$

of the people invited chose to watch the movie. Next, from the condition that one-ninth of the viewers were not invited, so eight-ninths of the viewers invited. Letting n be the total number of viewers, we get

$$\frac{8n}{9} = 160 \implies n = 180.$$

Finally, from the condition that one-fifth of the viewers had first row seats, we get that

$$\frac{1}{5} \cdot 180 = 36$$

viewers had first row seats. Since 60 of the viewers had third row seats, the total number of viewers with second row seats is $180 - 36 - 60 = 84$, so the probability is

$$\frac{84}{180} = \boxed{\textbf{(C)}\ \frac{7}{15}},$$

as requested. ■

10. **(2021 DMC 10C P15)** For certain real numbers x and y, the first 3 terms of a geometric progression are $x - 2$, $2y$, and $x + 2$ in that order, and the sum of these terms is 4. What is the fifth term?

(A) $\frac{64}{3}$ **(B)** $\frac{196}{9}$ **(C)** $\frac{512}{23}$ **(D)** $\frac{256}{111}$ **(E)** $\frac{128}{5}$

Answer (A):

We have that $(x - 2) + 2y + (x + 2) = 4$, so since

$$(x - 2) + 2y + (x + 2) = 2x + 2y,$$

we get that $2x + 2y = 4$. Thus, $x + y = 2$, so $y = 2 - x$. Since $x - 2$, $2y$, and $x + 2$ are the three consecutive terms of a geometric sequence, we have that $\frac{2(2-x)}{x-2} = \frac{x+2}{2(2-x)}$. Hence

$$\frac{2(\cancel{2-x}^{\,-1})}{\cancel{x-2}} = \frac{x+2}{2(2-x)} \implies -2 = \frac{x+2}{2(2-x)},$$

so by cross-multiplying, we get that

$$\begin{aligned} -2(2)(2-x) &= x+2 \\ -8+4x &= x+2 \\ 3x &= 10 \\ x &= \tfrac{10}{3}. \end{aligned}$$

If $x = \frac{10}{3}$, then the resulting three terms will be equal to $\frac{4}{3}$, $-\frac{8}{3}$, and $\frac{16}{3}$ in that order, which works. We see that the common ratio of the sequence is equal to -2, so the fifth term of the sequence is equal to

$$\frac{4}{3}\cdot(-2)^4 = \boxed{\textbf{(A)}\ \frac{64}{3}},$$

as requested. ■

11. **(2021 DMC 10C P17)** For positive integers n, let the nth triangular number be the sum of the first n positive integers. For how many integers n between 1 and 100, inclusive, does the nth triangular number have the same last digit as the product of the first n triangular numbers?

(A) 11 **(B)** 12 **(C)** 20 **(D)** 21 **(E)** 22

Answer (E):

Note that the nth triangular number is equal to $\frac{n(n+1)}{2}$. As well, note that for all $n \geq 4$, the first n triangular numbers will be divisible by the fourth triangular number, 10, so they will have a last digit of 0.

Thus, for $n \geq 4$, we want $\frac{n(n+1)}{2}$ to have a last digit of 0; (e.g., $\frac{n(n+1)}{2}$ is divisible by 10).

Consequently, we can set $n(n+1) = 2k$, where k is a multiple of 10. This is equivalent to $n = 20\left(\frac{1}{10}k\right)$, and since k is a multiple of 10, we have that $\frac{1}{10}k$ is an integer.

Hence, $n(n+1)$ must be divisible by 20. Since exactly one of n and $n+1$ will be even, one of them must contain all of the powers of 2 in 20, and the other one cannot contain any powers of 2.

Case 1: n is divisible by both 4 and 5

This happens when n has a remainder of 0 when divided by 20.

Case 2: $n + 1$ is divisible by both 4 and 5.

This happens when n has a remainder of 19 when divided by 20.

Case 3: n is divisible by 4 and $n + 1$ is divisible by 5.

This happens when n has a remainder of 4 when divided by 20.

Case 4: n is divisible by 5 and $n + 1$ is divisible by 4.

This happens when n has a remainder of 5 when divided by 20.

Thus, if $n \geq 4$, we get that n must have a remainder of 0, 4, 15, or 19 when divided by 20.

Note that, for each of the intervals $[21, 40]$, $[41, 60]$, $[61, 80]$, $[81, 100]$, there are 4 possible values of n. If $4 \leq n \leq 20$, then the possible values of n are 4, 15, 19, and 20. Finally, testing $n < 4$, we also see that $n = 1$ and $n = 2$ satisfy our condition. Hence, the requested answer is $4 \cdot 5 + 2 = \boxed{\textbf{(E) } 22}$. ■

12. **(2021 KMMC 8A P4, with DeToasty3)** Karate has a recipe for hot chocolate which requires 2 grams of cocoa powder and 5 grams of milk. After adding 5 grams of milk, Karate accidentally adds 3 grams of cocoa powder, so he adds additional milk in the same proportion as the recipe to balance the cocoa powder out. How many grams of additional milk does he add?

(A) 1 **(B)** 2.5 **(C)** 5 **(D)** 7.5 **(E)** 10

Answer (B):

Since Karate adds 50% too much cocoa powder, he needs to add 50% more grams of milk, or $50\% \cdot 5 = \boxed{\textbf{(B) } 2.5}$. ■

13. **(2021 KMMC 8A P5)** Let $P = 2^2 + 3^2 + 4^2 + \cdots + 10^2$ and $Q = 1^2 + 2^2 + 3^2 + \cdots + 9^2$. What is $P - Q$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 99 **(E)** 100

Answer (D):

We can pair the terms from 2^2 to 9^2, giving

$$10^2 + (9^2 - 9^2) + (8^2 - 8^2) + \cdots + (2^2 - 2^2) - 1^2 = 10^2 - 1^2,$$

so $P - Q = \boxed{\textbf{(D) } 99}$. ■

14. **(2021 KMMC 8A P6, with DeToasty3)** Karate has a bag of sweets consisting of 30% pieces of chocolate, 45% pieces of toffee, and the rest pieces of caramel. After giving half of his caramel to his wife, he has 15 pieces of caramel left. How many pieces of toffee does he have?

(A) 27 **(B)** 36 **(C)** 54 **(D)** 60 **(E)** 120

Answer (C):

First, we find that Karate has $100\% - (30\% + 45\%) = 25\%$ pieces of caramel in his bag. Next, we have that half of the total pieces of caramel is equal to 15 pieces of caramel, so letting p be the total number of pieces of sweets in Karate's bag, we have that $12.5\% \cdot p = 15 \implies p = 120$. Finally, we find that the number of pieces of toffee in Karate's bag is equal to $45\% \cdot 120 = \boxed{\textbf{(C) } 54}$. ■

15. **(2021 KMMC 8A P17)** Suppose that $a \clubsuit b$ means $a^2 - ab + b^2$. What is the value of x^2 if

$$(x+4) \clubsuit (x-4) = 75?$$

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 27

Answer (E):

Note that $(a-b)^2 = a^2 - 2ab + b^2$, so $a \clubsuit b = (a-b)^2 + ab$. Thus, we get that

$$\begin{aligned}(x+4) \clubsuit (x-4) &= ((x+4)-(x-4))^2 + (x+4)(x-4) \\ &= 8^2 + (x+4)(x-4) \\ &= 64 + (x+4)(x-4) \\ &= 75.\end{aligned}$$

By difference of squares, $(a+b)(a-b) = a^2 - b^2$, so $(x+4)(x-4) = x^2 - 4^2 = x^2 - 16$. From earlier, we have that $64 + (x+4)(x-4) = 75$, so by substitution, $64 + (x^2 - 16) = 75$. Subtracting 64 from both sides of $64 + (x^2 - 16) = 75$ gives $x^2 - 16 = 11$, so $x^2 = \boxed{\textbf{(E) } 27}$.
■

16. **(2021 KMMC 10 P7)** Let a be a positive integer. A geometric sequence b, c, d in that order satisfies

$$ab = 15, \quad bd = 16, \quad \text{and} \quad ac = 120.$$

What is the value of $a + d$?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

Answer (D):

By the first and third equations, $8ab = ac$. Dividing both sides by a, we get that $8b = c$, so the common ratio of the sequence is 8.

Thus, $c = 8b$ and $d = 8c = 8(8b) = 64b$. By the second equation, we have that $b \cdot 64b = 16$, so $64b^2 = 16$. Thus, $b^2 = \frac{1}{4}$. Since a is a positive integer and $ab = 15$, b must be positive, so thus $b = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

Finally, we have that $a \cdot \dfrac{1}{2} = 15$, so $a = 30$, and thus

$$a + d = a + 64b = 30 + 32 = \boxed{\textbf{(D) } 62},$$

as requested. ■

17. **(2021 KMMC 10 P13)** Karate writes the number 2021 on a blackboard. He then repeatedly erases and writes a new number on the blackboard, where if the current number on the blackboard is odd, he will erase the number and write $8n$ on the blackboard, and if the current number on the blackboard is even, he will erase the number and write $n + 2019$ on the blackboard. Eventually, Karate will have written 2020 numbers on the board (including the initial 2021). What is the remainder when his 2020th number is divided by 9?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

Answer (A):

Note that if a number is odd, then the succeeding number will be even, as it will be a multiple of 8. As well, if a number is even, then the succeeding number will be odd, as it will be added to 2019.

Consider the terms on the board when taken $\bmod 9$. Note that $8n \equiv -n \mod 9$ and $n + 2019 \equiv 3 \mod 9$. We start with an odd number, and each succeeding number on the board will switch between even and odd. Computing, we get

$$\begin{aligned}
2021 &\equiv 5 \pmod 9, && (1)\\
-5 &\equiv 4 \pmod 9, && (2)\\
4+3 &\equiv 7 \pmod 9, && (3)\\
-7 &\equiv 2 \pmod 9, && (4)\\
2+3 &\equiv 5 \pmod 9, && (5)\\
-5 &\equiv 4 \pmod 9, && (6)\\
4+3 &\equiv 7 \pmod 9, && (7)\\
-7 &\equiv 2 \pmod 9, && (8)\\
2+3 &\equiv 5 \pmod 9, && (9)\\
&\cdots. && (10)
\end{aligned}$$

Since every other number on the board is odd, the fifth term is odd and has a remainder of 5 when divided by 9. Note that, as Karate will apply the same process of addition and multiplication on the next 4 terms, this process of remainders will repeat every 4 terms.

Thus, the 4th term, 8th term, 16th term, and so forth, all the way to the 2020th term, will all have a remainder of $\boxed{\textbf{(A) } 2}$ when divided by 9. ■

18. **(2021 KMMC 8B P2/9 P2)** What is the value of the expression $\dfrac{4 \cdot 5 \cdot 6}{2(0+2+1)}$?

(A) 20 **(B)** 28 **(C)** 30 **(D)** 36 **(E)** 40

Answer (A):

We have that the denominator of the given expression is $2(0+2+1) = 2(3) = 6$, so the requested answer is equal to

$$\frac{4 \cdot 5 \cdot 6}{2(0+2+1)} = \frac{4 \cdot 5 \cdot 6}{6} = 4 \cdot 5 = \boxed{\textbf{(A) } 20}.$$

■

19. **(2021 KMMC 8B P3/9 P3)** Naruto is running forwards. At some point, he turns around and runs backwards. If Naruto ran a total of 60 equal steps and ended up 24 steps behind where he started running, for how many steps was Naruto running backwards?

(A) 18 **(B)** 36 **(C)** 38 **(D)** 40 **(E)** 42

Answer (E):

For every step Naruto runs backwards, he will end up 2 steps behind if he had ran forwards.

Let b be how many steps Naruto was running backwards. Then $60 - 2b = -24$, so $-2b = -84$ and solving gives $b = \boxed{\textbf{(E) } 42}$. ■

20. **(2021 KMMC 8B P5/9 P4)** Karate and Judo are playing a game with 100 rounds. Each round, either Karate wins, Judo wins, or they tie. If Karate wins 3 rounds, and the number of rounds Judo wins is a perfect square, what is the least possible number of rounds where they tie?

(A) 16 **(B)** 19 **(C)** 22 **(D)** 25 **(E)** 28

Answer (A):

Since Karate won 3 rounds, Judo can win at most 97 rounds. To minimize the number of rounds where nobody wins, we want Judo to win as many rounds as possible. The largest perfect square that is less than 97 is 81, so Judo won 81 rounds and there are $97 - 81 = \boxed{\textbf{(A) } 16}$ rounds where they tie. ■

21. **(2021 KMMC 8B P12/9 P10, with treemath)** What is the probability that a randomly chosen arrangement of the letters of the word *KARATE* will have an *A* as the first letter or the last letter (or both)?

(A) $\frac{7}{15}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{9}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{2}{3}$

Answer (D):

Note that

$$P(A \text{ is first letter or last letter}) + P(A \text{ is neither first letter nor last letter}) = 1,$$

so the desired probability is equal to $1 - P(A \text{ is neither first letter nor last letter})$. Hence, we can start by finding the probability that A is neither the first nor the last letter of a randomly chosen arrangement of the letters of the word *KARATE* and subtract it from 1.

There is a $\frac{4}{6}$ probability that the first A is neither first nor last. Then, after this A is placed, there is a $\frac{3}{5}$ probability that the second A is neither first nor last. Thus, the complementary probability is $\frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5}$. Hence, the requested probability is $1 - \frac{2}{5} = \boxed{\textbf{(D) } \frac{3}{5}}$. ■

22. **(2021 KMMC 9 P19)** If m and n are positive integers and $m^2 - n^2$ is equal to a prime number p, which of the following statements must always be true?

(A) $m + n$ is divisible by 3 **(B)** $p = m - n$ **(C)** $p - m - n$ is odd
(D) $p^2 + m^2 + n^2$ is not prime **(E)** $p - n^2$ is even

Answer (D):

First, note that $m > n$. By differences of squares, $m^2 - n^2 = p$ is equal to

$$(m+n)(m-n) = p,$$

and since $m > n$, we have that $m+n$ and $m-n$ are both positive integers. Seeing as p is prime, it will have no divisors other than 1 and itself. Since b is positive, $m+n$ must be the larger divisor, so thus $m+n = p$ and $m-n = 1$. Now, we look at the answer choices.

Since $m+n = (n+1)+n = 2b+1$, it can be any odd prime depending on the value of b. In this case, $m+n$ is only divisible by 3 when $(m,n) = (2,1)$, and thus **(A)** is not always true.

Since $m - n = 1$, which is not prime, it cannot be equal to p and thus **(B)** is not always true.

Note that $p - m - n = p - (m+n) = p - p = 0$, so $p - m - n$ is never odd and thus **(C)** is not always true.

Since $p - n^2$ is only even when n is odd and n can be odd or even, **(E)** is not always true.

Finally, seeing as $m = n+1$, we get that

$$(m,n) = (\text{even}, \text{odd}) \quad \text{or} \quad (\text{odd}, \text{even}),$$

so $m^2 + n^2$ is always (even) + (odd) in some order. Consequently, $m^2 + n^2$ is always odd. Thus, $p^2 + m^2 + n^2$ is always even and greater than 2, so $\boxed{\textbf{(D) } p^2 + m^2 + n^2 \text{ is not prime}}$ is always true.

Remark: Alternatively, we can provide a counterexample for every other option, such as $(m,n,p) = (3,2,5)$. ■

23. **(2021 KMMC 9 P20)** How many different real numbers x satisfy the equation

$$\left|5|x| - x^2\right| = 10?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

Answer (C):

For convenience, substitute $y = |x|$.

Hence, $\left|5y - x^2\right| = 10$. Note that $x^2 = |x|^2 = y^2$, so $\left|5y - x^2\right| = \left|5y - y^2\right| = 10$.

Consequently, either

$$5y - y^2 = 10 \quad \text{or} \quad 5y - y^2 = -10.$$

Case 1: $5y - y^2 = 10$

If $5y - y^2 = 10$, then $y^2 - 5y - 10 = 0$. This equation has a discriminant of

$$-5 - 4(1)(-10) = 35,$$

so in this case there are two real solutions for y. However, since the product of these two real solutions is -10, exactly one of the solutions for y here is negative. Since $y = |x|$, only the positive solution for y works, so thus there is only one solution for y in this case.

Case 2: $5y - y^2 = -10$

If $5y - y^2 = -10$, then $y^2 - 5y + 10 = 0$. This equation has a discriminant of

$$-5 - 4(1)(10) = -45,$$

so in this case there are no real solutions for y.

Thus, $|x|$ must be the positive root of $y^2 - 5y - 10$, so hence $x \in \{|x|, -|x|\}$ and the requested answer is $\boxed{\textbf{(C) } 2}$. ■

24. **(2021 KMMC 8B P23/9 P22)** Karate thinks of a two-digit number. Judo then asks the following questions in order:

- "Is the number a multiple of 2?"
- "Is the number a multiple of 3?"
- "Is the number a multiple of 4?"

Karate answers "yes," "yes," and "no" to each of the questions, but Judo forgot which order Karate said his responses in. How many possible values of the number are there?

(A) 21 **(B)** 24 **(C)** 29 **(D)** 33 **(E)** 37

Answer (A):

Let Karate's number be n. We can split the possible values of n into cases, as follows.

Case 1: n is a multiple of 4

If n is a multiple of 4, then it is also a multiple of 2. Seeing as Karate said no to one of the questions, it cannot also be a multiple of 3. Thus, n can be any two-digit multiple of 4 that is not also a multiple of 3. The smallest two-digit multiple of 4 is $3 \cdot 4$ and the largest two-digit multiple of 4 is $24 \cdot 4$, so there are $24 - 3 + 1 = 22$ two-digit multiples of 4. Of these, $\{12, 24, 36, 48, 60, 72, 84, 96\}$ are also multiples of 3, which don't work, so there are $22 - 8 = 14$ possibilities for this case.

Case 2: n is not a multiple of 4

Seeing as Karate said yes to two of the questions, n must be a multiple of both 2 and 3, so n can be any two-digit multiple of 6 that is not also a multiple of 4. The smallest two-digit multiple of 6 is $2 \cdot 6$ and the largest two-digit multiple of 6 is $16 \cdot 6$, so there are $16 - 2 + 1 = 15$ two-digit multiples of 6. Of these, $\{12, 24, 36, 48, 60, 72, 84, 96\}$ are also multiples of 4, which don't work, so there are $15 - 8 = 7$ possibilities for this case.

There are no overlaps between the two groups as the first group cannot contain multiples of 3 and the second group can only contain multiples of 3, so our answer is $14+7 = \boxed{\textbf{(A) } 21}$.
■

25. **(2021 KMMC 8B P24/9 P23, with treemath)** Define the operation $a \ominus b = a + b - \sqrt{4ab}$. If N is a two-digit number such that

$$(4 \ominus N) \ominus (81 \ominus 196) = 4,$$

what is the sum of the possible values of N?

(A) 73 **(B)** 106 **(C)** 113 **(D)** 130 **(E)** 145

Answer (B):

Note that $a + b - \sqrt{4ab} = a + b - 2\sqrt{ab}$, which we notice is equal to $\left(\sqrt{a} - \sqrt{b}\right)^2$.

By the given equation, we have that

$$\begin{aligned}
(4 \ominus N) \ominus (81 \ominus 196) &= 4 \\
\left(2 - \sqrt{N}\right)^2 \ominus \left(\sqrt{81} - \sqrt{196}\right)^2 &= 4 \\
\left(2 - \sqrt{N}\right)^2 \ominus 25 &= 4 \\
\left(\sqrt{\left(2 - \sqrt{N}\right)^2} - \sqrt{25}\right)^2 &= 4 \\
\sqrt{\left(2 - \sqrt{N}\right)^2} - 5 &= \pm 2.
\end{aligned}$$

Note that $\sqrt{\left(2 - \sqrt{N}\right)^2} = \left|2 - \sqrt{N}\right|$. Since N is a two-digit number, $\sqrt{N} > 2$, so thus $2 - \sqrt{N}$ is negative.

Hence, $\left|2-\sqrt{N}\right| = -(2-\sqrt{N}) = \sqrt{N}-2$. Thus, $\sqrt{N}-2-5 = \pm 2$, so either $\sqrt{N} = 9$ or $\sqrt{N} = 5$. Consequently, the possible values of N are 81 and 25, for an answer of $25+81 = \boxed{\textbf{(B) } 106}$.

OR

Let $4 \ominus N = \gamma$. Then $\gamma \ominus (81 \ominus 196) = 4$. By computation, we get $81 \ominus 196 = 81+196-\sqrt{4 \cdot 81 \cdot 196} = 277 - 2 \cdot 9 \cdot 14 = 25$, so $\gamma \ominus 25 = 4$.

Thus, $\gamma + 25 - \sqrt{4 \cdot 25 \cdot \gamma} = 4$. Rearranging, we get $\gamma - 10\sqrt{\gamma} + 21 = 0$, which factors as $(\sqrt{\gamma}-3)(\sqrt{\gamma}-7) = 0$.

Hence, $\sqrt{\gamma} = 3$ or $\sqrt{\gamma} = 7$, so either $\gamma = 9$ or $\gamma = 49$.

Case 1: $\gamma = 9$

We get that $4 + N - \sqrt{4 \cdot 4 \cdot N} = 9$. Rearranging, we get $N - 4\sqrt{N} - 5 = 0$, which factors as $\left(\sqrt{N}+1\right)\left(\sqrt{N}-5\right) = 0$. Since $\sqrt{N}$ must be positive, we get that $\sqrt{N} = 5$, so if $\gamma = 9$, then $N = 25$.

Case 2: $\gamma = 49$

We get that $4+N-\sqrt{4 \cdot 4 \cdot N} = 49$. Rearranging, we get $N-4\sqrt{N}-45 = 0$, which factors as $\left(\sqrt{N}+5\right)\left(\sqrt{N}-9\right) = 0$. Since $\sqrt{N}$ must be positive, we get that $\sqrt{N} = 9$, so if $\gamma = 49$, then $N = 81$.

Consequently, the possible values of N are 25 and 81, for an answer of $81+25 = \boxed{\textbf{(B) } 106}$.

■

26. **(2021 KMMC 8B P25/9 P24)** Karate's favorite positive integer A has B positive integer factors. If the product of A and B is equal to 13,500, what is the sum of the digits of the sum of A and B?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

Answer (D):

Note that $13500 = 2^2 \cdot 3^3 \cdot 5^3$, and thus $A \cdot B$ contains 2 powers of 2, 3 powers of 3, and 3 powers of 5.

Let $A = 2^x \cdot 3^y \cdot 5^z$, where $x \in \{0,1,2\}$, $y \in \{0,1,2,3\}$, and $z \in \{0,1,2,3\}$. Then $B = (x+1)(y+1)(z+1)$.

If A is a multiple of 5, then since 5 is prime, $x+1$, $y+1$, or $z+1$ would have to contain a multiple of 5, which is impossible as x, y, and z cannot be greater than 3, so B cannot be a multiple of 5.

However, $A \cdot B$ has exactly three powers of 5, so A must contain all three powers of 5, and thus $z = 3$.

Consequently, $B = (x+1)(y+1)(3+1)$, so B must contain two powers of 2. Since $A \cdot B$ has exactly two powers of 2, we get that A contains no powers of 2, so $x = 0$.

Hence, $A = 2^0 \cdot 3^y \cdot 5^3$, where $y \in \{0, 1, 2, 3\}$.

If y is odd, then $B = (0+1)(\text{odd}+1)(3+1)$. However, then B would contain at least three powers of 2, and since $A \cdot B$ has exactly two powers of 2, we get that y cannot be odd.

If $y = 0$, then A has no powers of 3 and $B = (0+1)(0+1)(3+1) = 4$ has no powers of 3 either. However, $A \cdot B$ must have exactly three powers of 3, so y cannot be 0.

If $y = 2$, then A has two powers of 3 and $B = (0+1)(2+1)(3+1) = 12$ has one power of 3, which works.

Therefore, $(A, B) = (1125, 12)$ and $A+B = 1125+12 = 1137$. Hence, our answer is $1+1+3+7 = \boxed{\textbf{(D) } 12}$. ■

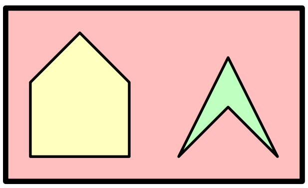
27. **(2022 KMMC 2A P1)** If $5 + \square = 8$, what is the value of $\square$?

(A) 2 **(B)** 3 **(C)** 6 **(D)** 8 **(E)** 13

Answer (B):

Since $5 + \square = 8$, we get that $\square = 8 - 5 = \boxed{\textbf{(B) } 3}$. ■

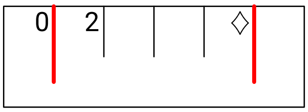
28. **(2022 KMMC 2A P5, with pandabearcat)** What is the number of sides of each of the yellow and green shapes added together?



(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

> **Answer (C):**
>
> The yellow shape has 5 sides, and the green shape has 4 sides. Thus, our answer is $5 + 4 = \boxed{\text{(C) } 9}$. ■

29. **(2022 KMMC 2A P6)** If the spaces between every two markings next to each other on the ruler are equal in length, what is the value of $\diamondsuit$?



(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

> **Answer (C):**
>
> Since the space between every two markings on the ruler is equal, the number of units between every two markings is equal. Hence, the third marking is $2 + 2 = 4$, the fourth marking is $4 + 2 = 6$, and $\diamondsuit$ is $6 + 2 = \boxed{\text{(C) } 8}$. ■

30. **(2022 KMMC 2A P8, with DeToasty3)** In a race between 10 people, Karate finished first in the race, and Judo finished last in the race. If there were no ties in the race, how many people finished behind Karate but ahead of Judo?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

> **Answer (B):**
>
> Note that since Karate finished first and Judo finished last, the number of people that finished behind Karate but ahead of Judo is equal to the number of people in the race other than Karate and Judo. Thus, our answer is $10 - 2 = \boxed{\text{(B) } 8}$. ■

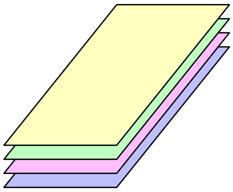
31. **(2022 KMMC 2A P11, with pandabearcat)** Karate is thinking of a number whose ones digit is 2. How many different numbers from 10 to 100 could Karate be thinking of?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

> **Answer (A):**
>
> The numbers from 10 to 100 are the two-digit numbers. Thus, the tens digit of the number

can be any number from 1 to 9, while the ones digit must be 2, for an answer of $\boxed{\text{(A) } 9}$. ■

32. (2022 KMMC 2A P15, with pandabearcat) Karate has 4 worksheets he needs to do: a math worksheet, a logic worksheet, a reading worksheet, and a science worksheet. He wants to do the math worksheet last, the logic worksheet before the science worksheet, and the reading worksheet after the science worksheet. In what order does Karate have to do the worksheets?

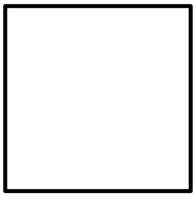


(A) logic, reading, science, math
(B) math, science, reading, logic
(C) science, reading, logic, math
(D) logic, science, reading, math
(E) reading, science, math, logic

Answer (D):

The math worksheet is last. Since the logic worksheet is before the science worksheet and the reading worksheet is after the science worksheet, the science worksheet is between the logic worksheet and the reading worksheet. Thus, our answer is $\boxed{\text{(D) logic, science, reading, math}}$ ■

33. (2022 KMMC 2A P16, with DeToasty3) Which of the following is **not** true about a square?



(A) Squares have four sides.
(B) All sides of a square are equal.
(C) Squares are rectangles.
(D) Squares have two lines of symmetry.
(E) Squares have more sides than triangles.

Answer (D):

Squares have four sides, so **(A)** is true.

All sides of a square are equal, so **(B)** is true.

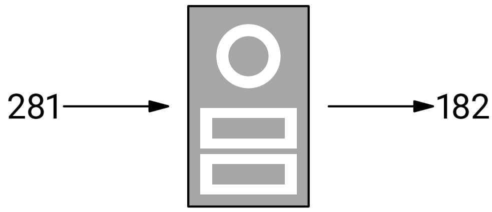
Squares are rectangles as they have four right angles and four sides, so **(C)** is true.

Squares have four lines of symmetry, so **(D)** is not true.

Squares have four sides, while triangles have three sides, so **(E)** is true.

Hence, our answer is $\boxed{\text{(D) Squares have two lines of symmetry.}}$ ■

34. **(2022 KMMC 2A P17)** A machine reads the digits of a number from right to left and outputs what it read. (For example, putting the number 281 into the machine would output the number 182). When put into the machine, which of these numbers would **not** cause the machine to output a valid number?

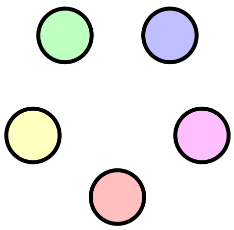


(A) 0 **(B)** 37 **(C)** 64 **(D)** 121 **(E)** 140

Answer (E):

For each answer choice, the machine will output 0, 73, 46, 121, and 041, respectively. Of these, 041 is not a valid number, so our answer is $\boxed{\text{(E) 140}}$. ■

35. **(2022 KMMC 2A P19, with pandabearcat)** Karate has a red, a yellow, a green, a blue, and a pink marble in a bag. He picks a random marble from the bag. What is the chance that he picks the red marble?

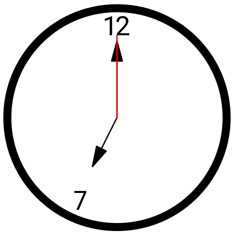


(A) 1 in 7 **(B)** 1 in 6 **(C)** 1 in 5 **(D)** 1 in 4 **(E)** 1 in 3

Answer (C):

If Karate picks a random marble of the 5 marbles and 1 of the marbles is the red marble, the chance that he picks the red marble is $\boxed{\text{(C) 1 in 5}}$. ■

36. (2022 KMMC 2A P20, with DeToasty3) At 7:00, all of the gears of an analog clock stopped working. Karate wants to display 7:30 on the clock. Which of the three hands of the analog clock does Karate have to move?



(A) hour hand only **(B)** minute hand only **(C)** seconds hand only
(D) both hour hand and minute hand **(E)** all three hands

Answer (D):

We have to move the hour hand to halfway between the 7 and the 8, and we have to move the minute hand from the 12 to the 6. Since we don't have to move the seconds hand, our answer is $\boxed{\text{(D) both hour hand and minute hand}}$. ■

37. (2022 KMMC 2B P3, with DeToasty3) How many letters of the word *YUKI* also appear in the word *KARATE*?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (B):

Inspecting each of the letters Y, U, K, and I, we get that only the letter K appears in the word *KARATE*. Thus, the answer is $\boxed{\text{(B) 1}}$. ■

38. (2022 KMMC 2B P14) What is the value of $99 + 89 + 1 + 1$?

(A) 150 **(B)** 160 **(C)** 170 **(D)** 180 **(E)** 190

Answer (E):

Rearrange the numbers in the sum as $99 + 1 + 89 + 1$. This is equal to $100 + 90$, which is equal to $\boxed{\text{(E) 190}}$. ■

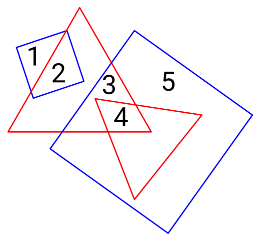
39. **(2022 KMMC 2B P15, with DeToasty3 & pandabearcat)** The composers Johann Sebastian Bach and Franz Joseph Haydn share the same birthday, where Bach was born in the year 1685, and Haydn was born in the year 1732. How many years before Haydn was born was Bach born?

(A) 41 **(B)** 43 **(C)** 45 **(D)** 47 **(E)** 49

Answer (D):

Since Bach and Haydn share the same birthday, Bach was born exactly $1732 - 1685 = \boxed{\text{(D) } 47}$ years before Haydn was born. ■

40. **(2022 KMMC 2B P20, with DeToasty3 & pandabearcat)** How many of the five numbers are **not** inside of any red triangles?

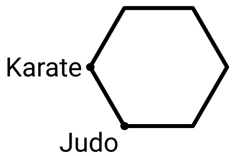


(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (B):

Inspecting the picture, we see that the numbers 1 and 5 are not inside of any red triangles, but the numbers 2, 3, and 4 are. Thus, the answer is $\boxed{\text{(B) } 2}$. ■

41. **(2022 KMMC 2B P24, with pandabearcat)** Karate and Judo each stand at one of the six different corners of a hexagon, but not the same corner. In how many ways can Karate and Judo stand on corners which are next to each other? (One way is shown in the picture.)



(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 18

Answer (D):

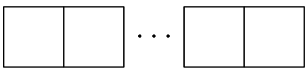
We will look at Karate and Judo's corners separately. For each of the six corners that Karate can be on, we see that Judo can either be one corner clockwise of Karate or one corner counterclockwise of Karate. Since there are six corners and two ways for Karate and Judo to stand on corners which are next to each other, the answer is $2+2+2+2+2+2 = \boxed{\text{(D) } 12}$.
■

42. **(2022 KMJJIME P5)** Aki is hiking in a straight line. He passes the first trail marker after hiking for 3 total miles and passes the second trail marker after hiking for 11 total miles. If the spaces from one trail marker to the next are equal in length, find how many total miles Aki will have hiked once he passes the 11th trail marker.

Answer (83):

Note that, each subsequent trail marker, Aki will have hiked for 8 more miles than the last one. Hence, on the third marker, Aki will hike $11 + 8$ miles, on the fourth marker, Aki will hike $11 + 8 + 8$ miles. We see that this pattern will continue, so Aki will have walked $11 + 8 \times 9 = \boxed{83}$ miles by the time he passes his 11th trail marker. ■

43. **(2022 KMJJIME P7)** Karate glues together ten identical squares by their sides so that they line up exactly, and they form a rectangle with a perimeter of 132 inches, as shown in the picture. Find the area of one of the squares, in square inches.



Answer (36):

Let the side length of one square be $\square$. The rectangle has a width of $10 \times \square$ and a height of $\square$, so its perimeter is $2(10 \times \square + \square) = 22 \times \square$. Hence,

$$22 \times \square = 132,$$

so $\square = \frac{132}{22} = 6$. Thus, one of the squares has an area of $6 \times 6 = \boxed{36}$. ■

44. **(2022 KMJJIME P8, with pandabearcat)** Karate currently has 84 pennies. Karate can trade five of his pennies for a nickel. After making some number of trades, Karate ends up with 64 coins (pennies and nickels). Find how many of these 64 coins are pennies.

Answer (59):

Note that, for each trade Karate makes, he gains 1 coin and loses 5 coins, for a net loss of 4 coins. Thus, if $\ominus$ is the number of trades that Karate made, $84 - 4 \times \ominus = 64$, so $4 \times \ominus =$

20. Hence, Karate made $\ominus = \frac{20}{4} = 5$ trades, so 5 of his coins are nickels and the other $64 - 5 = \boxed{59}$ coins are pennies. ■

45. **(2022 KMJJIME P11, with DeToasty3)** Aki and Hanami each have a whole number from 1 to 20 (including 1 and 20). They only know that at least one of their numbers is odd. Aki says, "My number is less than 7, but it is greater than 4." Hanami says, "Then, I know that my number is at least three times as large as yours." Aki says, "Then, I know your number." Find the sum of Aki and Hanami's numbers.

Answer (25):

Since Aki's number is less than 7 but greater than 4, it is either 5 or 6. Thus, if Hanami knows her number must be at least three times Aki's number, it must be at least $3 \times 6 = 18$, so Hanami's number can be 18, 19, or 20.

First, if Aki's number is 5, then he cannot find out Hanami's number, as it can be any of 18, 19, or 20. Thus, Aki's number cannot be 5. Next, if Aki's number is 6, then he can find out Hanami's number, as he knows that at least one of their numbers is odd, so since his number is not odd, he knows that Hanami's number must be odd, so it must be 19.

Consequently, Aki's number is 6 and Hanami's number is 19, for an answer of $6+19 = \boxed{25}$. ■

46. **(2022 KMJJIME P13, with DeToasty3)** Karate chooses two different whole numbers and writes their sum, their positive difference, and their product on a board. If there is exactly one even number on the board, and that number is equal to 72, find the sum of all possible values of the smaller of Karate's two numbers.

Answer (12):

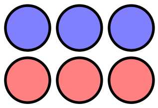
Let $\square$ and $\triangle$ be the two numbers, where $\triangle$ is the smaller number. Note that if $\square + \triangle$ is even, then so is $\square - \triangle$, and if $\square - \triangle$ is even, then so is $\square + \triangle$. However, we are given that exactly one even number is on the board, so $\square + \triangle$ and $\square - \triangle$ are both odd, and the even number on the board is $\square \times \triangle$, which we are given is equal to 72.

Note that, because 72 is even, at least one of $\square$ and $\triangle$ will also be even. However, $\square$ and $\triangle$ cannot both be even, so we wish to find values of $\square$ and $\triangle$ such that one of them is even and the other one is odd. Looking at the numbers that multiply to 72, we get that the possible values of $\square$ and $\triangle$ are

$$72 \times 1, 24 \times 3, \quad \text{and} \quad 9 \times 8,$$

so the sum of all possible values of $\triangle$ is $1 + 3 + 8 = \boxed{12}$. ■

47. **(2022 KMJJIME P14, with DeToasty3)** Karate, Judo, Naruto, Haruka, Ayaka, and Saya each stand on one of the six circles in the picture so that each circle has exactly one person standing on it. Find how many ways they can stand so that Karate and Judo both stand on blue circles, and Naruto and Saya stand on circles with different colors.



Answer (72):

First, note that if Naruto and Saya have to stand on circles with different colors, then one of the two must be standing on a blue circle. Also, since Karate and Judo have to stand on blue circles, the three people who are standing on blue circles must be either Karate, Judo, and Naruto or Karate, Judo, and Saya. As a result, the three people who are standing on red circles must be either Haruka, Ayaka, and Saya or Haruka, Ayaka, and Naruto.

Next, we take a look at how many ways the six people can be ordered from left to right. Note that for each row of circles, there are three choices for who goes in the leftmost circle, then there are two choices left for who goes in the middle circle, and finally, there is one choice left for who goes in the rightmost circle. For each way to put someone in the leftmost circle, there are two ways to put someone in the middle circle and one way to put someone in the rightmost circle, so there are $3 \times 2 \times 1 = 6$ ways to order the three people in each row of circles. Since there are two rows, we realize that for each of the 6 ways to order the people in the blue circles row, there are 6 ways to order the people in the red circles row. Thus, there are $6 \times 6 = 36$ ways to order the six people from left to right.

Finally, for each of the 36 ways to order the six people from left to right, there are two choices for which people stand on the blue circles, which leaves one choice for which people stand on the red circles. Thus, there are $36 \times 2 = \boxed{72}$ ways for the six people to stand on the six circles. ■

48. **(2022 KMJJIME P15, with DeToasty3)** Karate has two whole numbers $\square$ and $\triangle$ such that

$$\square + \triangle = 2022 \quad \text{and} \quad 4 < \frac{\square}{\triangle} < 5.$$

Find the number of possible values of $\triangle$.

Answer (67):

Note that the equation $\square + \triangle = 2022$ tells us that $\square = 2022 - \triangle$. From here, we may plug this value into the chain of inequalities to get

$$4 < \frac{2022 - \triangle}{\triangle} < 5.$$

Now, since $\triangle$ is a positive number, we can multiply all three sides of the inequality chain by $\triangle$ to get

$$4 \times \triangle < 2022 - \triangle < 5 \times \triangle.$$

Finally, adding $\triangle$ to all three sides of the inequality chain and combining like terms, we get

$$5 \times \triangle < 2022 < 6 \times \triangle.$$

Next, if we divide all three sides of the inequality chain by 5, we get

$$\triangle < \frac{2022}{5} < \frac{6}{5} \times \triangle.$$

This means that $\triangle$ is less than $\frac{2022}{5} = 404\frac{2}{5}$. If we instead divide all three sides of the inequality chain by 6, we get

$$\frac{5}{6} \times \triangle < \frac{2022}{6} < \triangle.$$

This means that $\triangle$ is greater than $\frac{2022}{6} = 337$. This means that $\triangle$ can be any whole number from 338 to 404 (including 338 and 404). Thus, our answer is $404 - 338 + 1 = \boxed{67}$.

OR

Note that we can add $\frac{\triangle}{\triangle} = 1$ to the inequality, as then

$$\frac{\square}{\triangle} + \frac{\triangle}{\triangle} = \frac{\square + \triangle}{\triangle} = \frac{2022}{\triangle},$$

which removes one of the unknown numbers.

Thus, we get that

$$5 < \frac{2022}{\triangle} < 6.$$

As $\triangle$ increases, $\frac{2022}{\triangle}$ decreases. Since $\triangle = \frac{2022}{404} > 5$ and $\frac{2022}{403} < 5$, the smallest possible value of $\triangle$ is 403. From here, all values of $\triangle$ will work until $\frac{2022}{337} = 6$, while $\frac{2022}{338} < 6$ works.

Hence, $\triangle$ can be any whole number from 338 to 404 (including 338 and 404). This is equivalent to finding the number of whole numbers from $338 - 337 = 1$ to $404 - 337 = 67$, for an answer of $\boxed{67}$. ■

5 PoTD Problems (2021–2022)

1. **(CNCM Problem of the Day — July 28, 2021)** Define a sequence a recursively by $a_1 = 0$ and

$$a_n = n \cdot a_{n-1} + n^2 - 2n$$

for all integers $n \geq 2$. If k is a positive integer such that exactly 2021 of the k smallest terms of a are divisible by 10, what is the sum of all possible values of k?

Answer (201845):

Factoring, we get that $a_n = n(a_{n-1} + n - 2)$. Note that $n! - n = n((n-1)! - (n-1) + n - 2)$, so $a_n = n! - n$. This sequence is increasing as n increases, so hence the k smallest terms of (a_n) is the same as the first k terms of (a_n).

Thus, we wish to find when $n! - n$ will be divisible by 10. Note that if $n \geq 5$, then $n!$ is divisible by 10, so if $n \geq 5$, we get that n must also be a multiple of 10.

Checking $n < 5$, we get that $n = 1$, $n = 2$, and $n = 4$ work as well. Hence, a_n is divisible by 10 when

$$n = 1, 2, 4, 10, 20, 30, \cdots.$$

Thus, a_{20180} is the 2021th smallest term of (a_n) that is divisible by 10, while a_{20190} is the 2022th smallest term of (a_n) that is divisible by 10. Hence, the sum of the possible values of k is

$$20180 + 20181 + 20182 + \cdots + 20187 + 20188 + 20189 = \boxed{201845}.$$

2. **(ALP Problem of the Day — January 4, 2022)** If

$$23! = 25{,}852{,}016{,}738{,}884{,}976{,}640{,}000$$

has P distinct positive divisors and

$$24! = 620{,}448{,}401{,}733{,}239{,}439{,}360{,}000$$

has Q distinct positive divisors, then $\frac{Q}{P} = \frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Answer (41):

Since $\nu_2(23!) = 19$ and $\nu_3(23!) = 9$, we get that

$$\nu_2(24!) = \nu_2(23!) + \nu_2(24) = 19 + 3 = 22$$

and

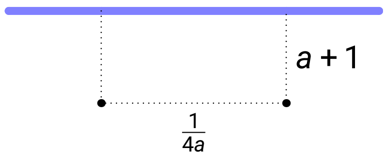
$$\nu_3(24!) = \nu_3(23!) + \nu_3(24) = 9 + 1 = 10.$$

Otherwise, the prime factorizations of a and b are the same, so hence

$$\frac{Q}{P} = \frac{22+1}{19+1} \cdot \frac{10+1}{9+1}.$$

Hence, $\frac{m}{n} = \frac{253}{200}$, and thus our answer is $m + n = \boxed{453}$. ■

3. **(ALP Problem of the Day — January 5, 2022)** In a flat plane, Karate lives in a rectangular, fenced-off compound next to an infinitely long river, as shown in the diagram below. The fence has a height of $a + 1$ ft and a width of $\frac{1}{4a}$ ft, where a is positive. If the least possible amount of fencing surrounding Karate's compound is k feet long, what is $\lfloor 100k \rfloor$?



Answer (341):

The amount of fencing used for Karate's compound is equal to $2(a+1) + \frac{1}{4a} = 2a + 2 + \frac{1}{4a}$, or $2 + \left(2a + \frac{1}{4a}\right)$.

Since a is positive, $2a$ and $\frac{1}{4a}$ are both positive. Hence, by AM-GM, we get

$$\frac{2a + \frac{1}{4a}}{2} \geq \sqrt{2a \cdot \frac{1}{4a}}.$$

Multiplying both sides of this inequality by 2, we get that $2a + \frac{1}{4a} \geq 2\left(\sqrt{\frac{1}{2}}\right)$, so

$$2a + \frac{1}{4a} \geq \sqrt{2}.$$

Note that $2a + \frac{1}{4a} = \sqrt{2}$ is achievable for $a = \frac{1}{2\sqrt{2}}$, so thus the least possible amount of fencing that was used for Karate's compound is $2 + \left(2a + \frac{1}{4a}\right) = 2 + \sqrt{2}$, so the answer is $\left\lfloor 100(2+\sqrt{2}) \right\rfloor = \boxed{341}$. ■

4. **(ALP Problem of the Day — January 10, 2022)** Given that

$$0.5\overline{83} - 0.58\overline{3} = \frac{1}{n},$$

what is the value of n?

Answer (1980):

Note that $0.5\overline{83} - 0.58\overline{3} = 0.00\overline{05}$. Since $0.\overline{ab} = \frac{\overline{ab}}{99}$, we get that $0.\overline{05} = \frac{5}{99}$. Thus, $0.00\overline{05} = \frac{1}{100} \cdot \frac{5}{99} = \frac{1}{20 \cdot 99}$ and $n = \boxed{1980}$. ■

5. **(ALP Problem of the Day — January 11, 2022)** There exists a 12-digit number n with 221 positive divisors. How many positive divisors does n^2 have?

Answer (825):

Note that $221 = 13 \cdot 17$, so n is either in the form p^{220} or $(p_1)^{12} \cdot (p_2)^{16}$, where p, p_1, and p_2 are arbitrary prime numbers.

Case 1: $n = p^{220}$

Note that n is at least 2^{220}. Seeing as $2^{10} > 10^3$, we get that $2^{10} > (2^{10})^{22} > (10^3)^{22}$, which has at least 67 digits. Hence, in this case, no 12-digit n exist.

Case 2: $n = (p_1)^{12} \cdot (p_2)^{16}$

Thus, n must be in the form $(p_1)^{12} \cdot (p_2)^{16}$, which means $n^2 = (p_1)^{24} \cdot (p_2)^{32}$. So n^2 has $(24+1)(32+1) = \boxed{825}$ divisors. ■

6. **(ALP Problem of the Day — January 12, 2022)** Let

$$f(x) = x^3 + ax^2 + bx + c$$

be a polynomial such that $f(1) = 43$ and $f(-1) = 11$. What is b?

Answer (15):

Note that $f(1) - f(-1) = (1 + a + b + c) - (-1 + a - b + c) = 2b + 2 = 43 - 11 = 32$, so $b = \boxed{15}$.
■

7. **(ALP Problem of the Day — January 17, 2022)** If k is a real number such that the value of the product

$$\left(1-\tfrac{2}{2^2-1^2}\right)\left(1-\tfrac{2}{2.5^2-1.5^2}\right)\left(1-\tfrac{2}{3^2-2^2}\right)\cdots\left(1-\tfrac{2}{(k+1)^2-k^2}\right)$$

is $\frac{1}{2016}$, then what is $2k$?

Answer (63):

Note that, by differences of squares, $1-\dfrac{2}{(k+1)^2-k^2}=1-\dfrac{2}{2k+1}=\dfrac{(2k+1)-2}{2k+1}=\dfrac{2k-1}{2k+1}$. Thus

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdot\ldots\cdot\frac{2k-3}{2k-1}\cdot\frac{2k-2}{2k}\cdot\frac{2k-1}{2k+1}=\frac{1}{2016}.$$

Note that everything cancels except for the first two numerators and the last two denominators, so $\dfrac{2}{2k(2k+1)}=\dfrac{1}{2016}$, which we can simplify to give $\dfrac{1}{k(2k+1)}=\dfrac{1}{2016}$. Thus $k(2k+1)=2016$, so $2k^2+k=2016$ and $2k^2+k-2016=0$. This factors into $(k+32)(2k-63)=0$. Clearly the negative solution is nonsensical, so $2k-63=0$ and $2k=\boxed{63}$. ■

8. **(ALP Problem of the Day — January 18, 2022)** For any positive integer x, let $\clubsuit(x)$ be the number of powers of x in the result when the prime factorization of 2022! is written out. (For example, $\clubsuit(5)=503$.) What is $\clubsuit(21)$?

Answer (0):

Note that 21 is not prime, for an answer of $\boxed{0}$. ■

9. **(ALP Problem of the Day — January 19, 2022)** Call a five-digit number *cumbersome* if at least two consecutive digits of the number are either all odd or all even. Karate randomly chooses a five-digit number whose digits are all distinct. The probability that it is cumbersome is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

Answer (121):

Note that all five-digit positive integers whose digits are all distinct are either cumbersome or not cumbersome, so the requested probability is equal to

$$1-\frac{\text{number of five-digit positive integers that are not cumbersome with distinct digits}}{\text{number of five-digit positive integers with distinct digits}}.$$

Finding the denominator, there are $9\cdot9\cdot8\cdot7\cdot6$ five-digit positive integers with distinct digits. (The first digit can be any digit but 0, the second digit can be any digit but the first digit, the third digit can be any digit but the first digit and the second digit, and so forth.)

Next, to find the number of five-digit positive integers that are not cumbersome with distinct digits, we get that five-digit positive integers with distinct digits that are not cumbersome will either be in the form $\overline{O_1}\,\overline{E_1}\,\overline{O_2}\,\overline{E_2}\,\overline{O_3}$ or $\overline{E_1}\,\overline{O_1}\,\overline{E_2}\,\overline{O_2}\,\overline{E_3}$, where O_1, O_2, O_3 are distinct odd digits and E_1, E_2, E_3 are distinct even digits.

For the first case, there are $5 \cdot 5 \cdot 4 \cdot 4 \cdot 3$ numbers, and for the second case, there are $4 \cdot 5 \cdot 4 \cdot 4 \cdot 3$ numbers (note that the first digit cannot be 0).

Thus, there are $5 \cdot (5 \cdot 4 \cdot 4 \cdot 3) + 4 \cdot (5 \cdot 4 \cdot 4 \cdot 3) = 9 \cdot (5 \cdot 4 \cdot 4 \cdot 3)$ five-digit positive integers that are not cumbersome with distinct digits.

Consequently, the probability is equal to $1 - \frac{9 \cdot 5 \cdot 4 \cdot 4 \cdot 3}{9 \cdot 9 \cdot 8 \cdot 7 \cdot 6}$, and canceling gives

$$1 - \frac{\cancel{9} \cdot 5 \cdot \cancel{4 \cdot 4 \cdot 3}}{\cancel{9} \cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6}} = 1 - \frac{5}{9 \cdot 7} = \frac{58}{63}.$$

Hence, our answer is $m + n = 58 + 63 = \boxed{121}$. ■

10. **(ALP Problem of the Day — January 25, 2022)** Let x and y be real numbers that satisfy

$$404x + 404y = 2019$$
$$673x - 673y = 2020$$

What is $x^2 - y^2$?

Answer (15):

We convert the equations to

$$x + y = \tfrac{2019}{404}$$
$$x - y = \tfrac{2020}{673}$$

Note that $x^2 - y^2 = (x + y)(x - y)$, so we multiply both equations to get

$$x^2 - y^2 = \frac{2019 \cdot 2020}{404 \cdot 673} = \frac{3 \cdot \cancel{673} \cdot 5 \cdot \cancel{404}}{\cancel{404} \cdot \cancel{673}} = \boxed{15}.$$

■

11. **(ALP Problem of the Day — January 26, 2022)** Karate randomly picks, with replacement, two numbers from a set of 2020 integers. If the probability that their product is positive is $\frac{37}{400}$ and the probability that their product is negative is $\frac{3}{100}$, how many integers in the set are equal to 0?

Answer (1313):

Let the probability that a number from the set is positive be a and let the probability that a number from the set is negative is b, so the probability that a number from the set is equal to 0 is $1 - a - b$.

Note that the probability that the product of two randomly chosen numbers is positive or negative is $(a+b)^2$, and we are given that this is also equal to $\frac{37}{400} + \frac{3}{100} = \frac{49}{400}$. Thus, $a+b = \sqrt{\frac{49}{400}} = \frac{7}{20}$.

Hence, the probability that a number from the set is equal to 0 is $1 - a - b = \frac{13}{20}$, so there are $\frac{13}{20} \cdot 2020 = \boxed{1313}$ integers in the set that are equal to 0.

Remark: Note that a and b have to be rational. In this case, they are $\frac{1}{20}$ and $\frac{3}{10}$ (in some order). ■

12. **(ALP Problem of the Day — February 1, 2022)** If a and b are positive integers such that

$$\left(\sqrt{5}\right)^{a-b} \cdot \left(\sqrt{10}\right)^{a+b} = 20^{21},$$

what is b?

Answer (63):

Note that the given expression is equal to

$$\left(\sqrt{5}\right)^{(a-b)+(a+b)} \cdot \left(\sqrt{2}\right)^{a+b}.$$

Hence $2a = 42$ and $a + b = 84$, which gives $a = 21$ and $b = \boxed{63}$. ■

13. **(ALP Problem of the Day — February 2, 2022)** Let x be a positive real number such that $\log_2(\log_3(x)) = 571$. Find $\log_{32}(\log_{81}(x^{64}))$.

Answer (115):

Note that $x = 3^{\left(2^{571}\right)}$, so $x^{64} = 3^{\left(2^{571}\right)^{64}} = 3^{2^6 \cdot 2^{571}} = 3^{2^{571+6}} = 3^{\left(2^{577}\right)}$.

Since we want $\log_{32}(\log_{81}(x^{64}))$, we start by computing $\log_{81}(x^{64})$. We can express $3^{\left(2^{577}\right)}$ as a power of 81, since

$$3^{\left(2^{577}\right)} = 3^{\left(4 \cdot \frac{2^{577}}{4}\right)} = \left(3^4\right)^{\frac{2^{577}}{4}} = 81^{\frac{2^{577}}{4}},$$

so $\log_{81}(x^{64}) = \frac{2^{577}}{4} = \frac{2^{577}}{2^2} = 2^{577-2} = 2^{575}$.

Thus, $\log_{32}(\log_{81}(x^{64})) = \log_{32}(2^{575})$. Similarly, we can express 2^{575} as a power of 32, since

$$2^{575} = 2^{\left(5 \cdot \frac{575}{5}\right)} = (2^5)^{\frac{575}{5}} = 32^{\frac{575}{5}} = 32^{115}.$$

Hence, our final answer is $\log_{32}(32^{115}) = \boxed{115}$. ■

14. **(ALP Problem of the Day — February 7, 2022)** Let a, b, and c are positive real numbers such that

$$\begin{aligned} ab + a + b &= 2019 \\ ac + a + c &= 2073 \\ a^2 + 2a &= 2115. \end{aligned}$$

What is $a + b + c$?

Answer (132):

Factoring, we get that $(a+1)(b+1) - 1 = 2019$, $(a+1)(c+1) - 1 = 2073$, and $(a+1)(a+1) - 1 = 2115$. Since $(a+1)(a+1) = 2116$ and a is positive, we get that $a = \sqrt{2116} - 1 = 45$. Hence,

$$\frac{(a+1)(b+1) + (a+1)(c+1) + (a+1)(a+1)}{a+1} = \frac{2020 + 2073 + 2116}{46} = 135,$$

so $(a+1) + (b+1) + (c+1) = 135$ and $a + b + c = 135 - 3 = \boxed{132}$. ■

15. **(ALP Problem of the Day — February 21, 2022)** If a and b are numbers such that $ab = 8$ and $a + b = 12$, what is

$$\frac{a^3b + ab + b^3a}{8}?$$

Answer (129):

Since $ab = 8$, the requested expression is equal to

$$\frac{a^3b}{ab} + \frac{ab}{ab} + \frac{b^3a}{ab} = a^2 + 1 + b^2 = (a+b)^2 - 2ab + 1 = 144 - 16 + 1 = \boxed{129}.$$

■