pog's Problems

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Contents

1 Introduction

This handout contains every problem which I, pog, have proposed which has appeared in a test format, mock or official. Note that this also means that I will not be putting problems here which I posted outside of a contest or are currently not public.

These problems will be listed in chronological order and in the order in which the problems are listed in their respective tests. This document will not contain any solutions to my problems; I will be updating this document as I continue to propose problems.

I have written problems for the following:

- The Bingo Forum Mock AMC 8
- Karate Masters Mathematics Competitions (KMMC)
- De Mathematics Competitions (DMC)
- Cyclic National Competitive Math Group (CNCM) Problem of the Day
- AMC Learning Program (ALP) Problem of the Day

Without further ado, please sit back and enjoy the problems! (and yes, this introduction was a reference to the AoPS user **DeToasty3**)

2 Problems (2020-2021)

1. **(The Bingo Forum Mock AMC 8 – Problem 1)** Compute $2^{1^{2^1}} - 2^{0^{2^0}}$.

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

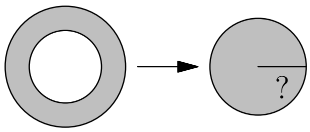
2. **(The Bingo Forum Mock AMC 8 – Problem 2)** Isabella goes to the grocery store. She has \$20 and spends \$5. If one can of Sprrite sells for \$1.50, how many 6-packs of Sprrite can she buy?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 10

3. **(The Bingo Forum Mock AMC 8 – Problem 4)** If $a\bigstar b = \sqrt{a^2+b^2}$ and $a \star b = \sqrt{a^2-b^2}$, what is $7\bigstar 24 - 29\star 20$?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

4. **(The Bingo Forum Mock AMC 8 – Problem 6)** A cookie cutter of radius 3 cuts out the center of a cookie with radius 5. The remainder is then formed into a different cookie. What is the radius of the new cookie?



   (A) $\frac{5}{3}$ **(B)** 2 **(C)** π **(D)** 4 **(E)** $\frac{5}{3}\pi$

5. **(The Bingo Forum Mock AMC 8 – Problem 17)** If the area of a right triangle is 175 and its hypotenuse is 30, find the perimeter of the triangle.

   (A) 40 **(B)** 50 **(C)** 60 **(D)** 70 **(E)** 80

6. **(2020 KMMC 10 – Problem 1)** What is the value of

$$\frac{2021^2 - 2021}{2020} + \frac{2021^2 - 2020^2}{2021 - 2020}?$$

   (A) 1010 **(B)** 2020 **(C)** 2021 **(D)** 4041 **(E)** 6062

7. **(2020 KMMC 10 – Problem 8)** Every day in an eight-day interval, the Cents Lord puts a number of cents into her piggy bank. If the number of cents she puts into her piggy bank from one day to the next forms an increasing arithmetic progression and she puts 2008 cents into her piggy bank in total over the eight days, what is the least possible number of cents she could have put into her piggy bank on any one of the days?

   (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

8. **(2020 KMMC 10 – Problem 10)** Let n be a positive integer between 1458 and 2021, inclusive. What is the largest possible value of the sum of the digits of n when n is expressed in base-9? (Express your answer in base-10.)

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

9. **(2020 KMMC 10 – Problem 11)** For a real number x, the median of the list of numbers

$$10, 14, 18, 22, x+3, x+6, x+9, x+12$$

is equal to 13.5. What is the sum of the unique mode and range of the list?

(A) 21 **(B)** 23 **(C)** 25 **(D)** 27 **(E)** 29

10. **(2020 KMMC 10 – Problem 17)** If there exist three distinct positive primes p that satisfy the equation

$$p^4 + ap^3 + bp^2 + cp + 2020 = 0$$

for integers a, b, and c, what is the absolute value of $a+b+c$?

(A) 871 **(B)** 1621 **(C)** 2019 **(D)** 2020 **(E)** 2419

11. **(2020 KMMC 10 – Problem 18)** An infinite number of jars are lined up in a row, where the nth jar from the left has n red marbles and 2 blue marbles. For a positive integer k, Karate randomly selects a marble from each of the first k jars from the left. If the probability that he draws exactly k red marbles is strictly less than $\frac{1}{2020}$, what is the least possible value of k?

(A) 44 **(B)** 45 **(C)** 51 **(D)** 57 **(E)** 63

12. **(2021 DIME – Problem 1)** Find the remainder when the number of positive divisors of the value

$$(3^{2020}+3^{2021})(3^{2021}+3^{2022})(3^{2022}+3^{2023})(3^{2023}+3^{2024})$$

is divided by 1000.

13. **(2021 DIME – Problem 2)** If x is a real number satisfying the equation

$$9\log_3 x - 10\log_9 x = 18\log_{27} 45,$$

then the value of x is equal to $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

14. **(CNCM Problem of the Day – March 31, 2021)** If θ is a real number such that

$$\sin(\theta) + \cos(\theta) = \frac{199}{187},$$

then $\sin(\theta)\cos(\theta) = \dfrac{m}{n}$, where m and n are relatively prime positive integers. What are the last two digits of n?

3 Problems (2021-2022)

15. (2021 DMC 10A – Problem 3) Let n be a positive integer less than 2021. It is given that if a regular hexagon is rotated n degrees clockwise about its center, the resulting hexagon coincides with the original hexagon. How many possible values of n are there?

(A) 8 **(B)** 16 **(C)** 17 **(D)** 32 **(E)** 33

16. (2021 DMC 10A – Problem 5) If the product of three distinct positive real numbers forming a geometric progression is equal to 2197, what is the median of the three numbers?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

17. (2021 DMC 10A – Problem 9) In the coordinate plane, let $\mathcal{P}$ be the figure formed by the set of points with coordinates satisfying $0.5x + y = 1$, and let $\mathcal{Q}$ be the figure formed by the set of points with coordinates satisfying $0.25x^2 + y^2 = 1$. How many points lie on both $\mathcal{P}$ and $\mathcal{Q}$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

18. (2021 DMC 10A – Problem 10) How many real numbers x satisfy the equation

$$9^x + 3^{3x} = 3^{x+1} + 3?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

19. (2021 DMC 10A – Problem 16) If a and b are the distinct roots of the polynomial $x^2 + 2021x + 2019$, then

$$\frac{1}{a^2 + 2019a + 2019} + \frac{1}{b^2 + 2019b + 2019} = \frac{m}{n},$$

where m and n are relatively prime positive integers. What is $m + n$?

(A) 2020 **(B)** 2021 **(C)** 4040 **(D)** 6059 **(E)** 6061

20. (CNCM Problem of the Day – July 28, 2021) Define a sequence a recursively by $a_1 = 0$ and

$$a_n = n \cdot a_{n-1} + n^2 - 2n$$

for all integers $n \geq 2$. If k is a positive integer such that exactly 2021 of the k smallest terms of a are divisible by 10, what is the sum of all possible values of k?

21. (2021 DMC 10B – Problem 3) If n is a positive integer such that $n \times 3^5 = 3^7 - 3^5$, what is n?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

22. **(2021 DMC 10B – Problem 8, with DeToasty3)** At Test Academy, there are four classes, one on each of the four floors of the building. For each class, the class which is one floor above it has twice as many students and half the average grade of that class. If the average grade of all four classes combined is 20, what is the average grade of the class on the bottom floor?

(A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

23. **(2021 DMC 10B – Problem 10, with DeToasty3)** How many ordered triples of integers (a, b, c) are there such that the product

$$(a - 2020)(2b - 2021)(3c - 2022)$$

is positive and has exactly three positive divisors?

(A) 3 **(B)** 9 **(C)** 12 **(D)** 24 **(E)** infinitely many

24. **(2021 DMC 10C – Problem 5)** A group of 200 people were invited to see a movie, where each person either had first row seats, second row seats, or third row seats. It is given that four-fifths of the people invited chose to watch the movie, one-ninth of the viewers were not invited, one-fifth of the viewers had first row seats, and 60 of the viewers had third row seats. What is the probability that a randomly selected viewer had second row seats?

(A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{5}$

25. **(2021 DMC 10C – Problem 15)** For certain real numbers x and y, the first 3 terms of a geometric progression are $x - 2$, $2y$, and $x + 2$ in that order, and the sum of these terms is 4. What is the fifth term?

(A) $\frac{64}{3}$ **(B)** $\frac{196}{9}$ **(C)** $\frac{512}{23}$ **(D)** $\frac{256}{111}$ **(E)** $\frac{128}{5}$

26. **(2021 DMC 10C – Problem 17)** For positive integers n, let the nth triangular number be the sum of the first n positive integers. For how many integers n between 1 and 100, inclusive, does the nth triangular number have the same last digit as the product of the first n triangular numbers?

(A) 11 **(B)** 12 **(C)** 20 **(D)** 21 **(E)** 22

27. **(2021 KMMC 8A – Problem 4, with DeToasty3)** Karate has a recipe for hot chocolate which requires 2 grams of cocoa powder and 5 grams of milk. After adding 5 grams of milk, Karate accidentally adds 3 grams of cocoa powder, so he adds additional milk in the same proportion as the recipe to balance the cocoa powder out. How many grams of additional milk does he add?

(A) 1 **(B)** 2.5 **(C)** 5 **(D)** 7.5 **(E)** 10

28. **(2021 KMMC 8A – Problem 5)** Let $P = 2^2+3^2+4^2+\cdots+10^2$ and $Q = 1^2+2^2+3^2+\cdots+9^2$. What is $P-Q$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 99 **(E)** 100

29. **(2021 KMMC 8A – Problem 6, with DeToasty3)** Karate has a bag of sweets consisting of 30% pieces of chocolate, 45% pieces of toffee, and the rest pieces of caramel. After giving half of his caramel to his wife, he has 15 pieces of caramel left. How many pieces of toffee does he have?

(A) 27 **(B)** 36 **(C)** 54 **(D)** 60 **(E)** 120

30. **(2021 KMMC 8A – Problem 17)** Suppose that $a \clubsuit b$ means $a^2 - ab + b^2$. What is the value of x^2 if

$$(x+4) \clubsuit (x-4) = 75?$$

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 27

31. **(2021 KMMC 10 – Problem 7)** Let a be a positive integer. A geometric sequence b, c, d in that order satisfies

$$ab = 15, \quad bd = 16, \quad \text{and} \quad ac = 120.$$

What is the value of $a+d$?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

32. **(2021 KMMC 10 – Problem 13)** Karate writes the number 2021 on a blackboard. He then repeatedly erases and writes a new number on the blackboard, where if the current number on the blackboard is odd, he will erase the number and write $8n$ on the blackboard, and if the current number on the blackboard is even, he will erase the number and write $n+2019$ on the blackboard. Eventually, Karate will have written 2020 numbers on the board (including the initial 2021). What is the remainder when his 2020th number is divided by 9?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

33. **(ALP Problem of the Day – January 4, 2022)** If

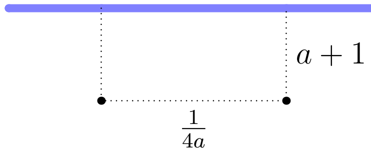
$$23! = 25{,}852{,}016{,}738{,}884{,}976{,}640{,}000$$

has P distinct positive divisors and

$$24! = 620{,}448{,}401{,}733{,}239{,}439{,}360{,}000$$

has Q distinct positive divisors, then $\frac{Q}{P} = \frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

34. **(ALP Problem of the Day – January 5, 2022)** In a flat plane, Karate lives in a rectangular, fenced-off compound next to an infinitely long river, as shown in the diagram below. The fence has a height of $a+1$ ft and a width of $\frac{1}{4a}$ ft, where a is positive. If the least possible amount of fencing surrounding Karate's compound is k feet long, what is $\lfloor 100k \rfloor$?



35. **(ALP Problem of the Day – January 10, 2022)** Given that

$$0.5\overline{83} - 0.58\overline{3} = \frac{1}{n},$$

what is the value of n?

36. **(ALP Problem of the Day – January 11, 2022)** There exists a 12-digit number n with 221 positive divisors. How many positive divisors does n^2 have?

37. **(ALP Problem of the Day – January 12, 2022)** Let

$$f(x) = x^3 + ax^2 + bx + c$$

be a polynomial such that $f(1) = 43$ and $f(-1) = 11$. What is b?