DeMathCounts

Official Solutions

DeMathCounts

1st Annual

DMC Chapter Competition

Monday, March 7, 2022

This official solutions booklet gives at least one solution for each problem on this year's competition. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be
sent by private message to

pog.

Sprint Round Answer Key:

1. 5.50	2. 204222	3. 12	4. 81	5. -15
6. 30	7. 66	8. 3	9. 8	10. 28
11. 2 or 2.00	12. 167	13. 3	14. -225	15. 3/2
16. 4/45	17. 10/3	18. 8:40	19. 4 - pi	20. 2821
21. 91/216	22. 151	23. 48518	24. 7	25. 39
26. 20	27. -64	28. 153	29. 26	30. 600

Target Round Answer Key:

1. 312	2. 3
3. 66	4. 20
5. 36877.36	6. 63.52
7. 1/9	8. 1372

Problem 1:

(pog) Dwayne spent \$22.00 at the movies on soda and popcorn. If soda costs three times as much as popcorn, how much did Dwayne spend on popcorn?

Answer (5.50):

He spent $\frac{1}{4}$ of his money on popcorn, so our answer is $\$22.00 \cdot \frac{1}{4} = \boxed{5.50}$. ■

Problem 2:

(pog) What is the value of $2022 \cdot 100 + 2022$?

Answer (204222):

We get $202200 + 2022 = \boxed{204222}$. ■

Problem 3:

(pog) Katherine multiplies the number $\Diamond$ by 7 and then subtracts 2 from the result, while Taiki subtracts 2 from $\Diamond$ and then multiplies the resulting number by 7. What is the positive difference between Katherine and Taiki's final numbers?

Answer (12):

Katherine's final number is $7 \cdot \diamond - 2$, while Taiki's final number is $7 \cdot (\diamond - 2) = 7 \cdot \diamond - 14$. Hence, the positive difference between Katherine and Taiki's final numbers is $14 - 2 = \boxed{12}$. ■

Problem 4:

(pog) Vikram wrote an expression on the board, but he accidentally smudged one of the digits with his hand.

$$19\blacksquare \cdot 9$$

What is the result when the minimum possible value of Vikram's expression is subtracted from the maximum possible value of Vikram's expression?

Answer (81):

The minimum possible value of Vikram's expression is $190 \cdot 9$, and the maximum possible value of Vikram's expression is $199 \cdot 9$. Hence, the requested answer is $199 \cdot 9 - 190 \cdot 9 = 9 \cdot 9 = \boxed{81}$. ■

Problem 5:

(pog) There are two numbers with a square of 15. What is their product?

Answer (-15):

Let the possible numbers be x. We have that $x^2 = 15$, so $x = \pm\sqrt{15}$. Hence, the product of the two numbers is $(\sqrt{15})(-\sqrt{15}) = \boxed{-15}$. ■

Problem 6:

(pog) The first five terms of an arithmetic sequence are 3, a, b, c, and 17. What is the value of $a + b + c$?

Answer (30):

Note that $(3, b, 17)$ is an arithmetic sequence, so $b = \dfrac{3 + 17}{2} = 10$. Hence, since (a, b, c) is an arithmetic sequence, we get that $a + b + c = 3b = \boxed{30}$. ■

Problem 7:

(pog) Two of the sides of an isosceles triangle have lengths 10 units and 12 units. What is the sum of the possible perimeters of the triangle?

Answer (66):

Since the triangle is isosceles, there are either two sides with length 10 units or two sides with length 12 units. Consequently, the possible perimeters are $10 + 10 + 12 = 32$ and $10 + 12 + 12 = 34$, so our answer is $32 + 34 = \boxed{66}$. ■

Problem 8:

(pog) In the equations below, what is the value of e?

$$\begin{aligned} a+b+c+d+e &= 11 \\ a+b+c+d+f &= 12 \\ a+b+c+d+e+f &= 15 \end{aligned}$$

Answer (3):

We get that $(a+b+c+d+e+f) - (a+b+c+d+f) = 15 - 12 = 3$, so $e = \boxed{3}$. ■

Problem 9:

(pog) Alice and Barbara both draw a circle. If Alice draws a circle with an area of 2 in^2, and Barbara draws twice as much as Alice, what is the area of Barbara's circle?

Answer (8):

Let the radius of Alice's circle be r. Then the radius of Barbara's circle is $2r$. We have that $r^2\pi = 2$, so hence the area of Barbara's circle is $(2r)^2\pi = 4(r^2\pi) = \boxed{8}$. ■

Problem 10:

(pog) If $\sqrt{x} + \sqrt{y} = 7$ and $\sqrt{x} - \sqrt{y} = 4$, what is the value of $x - y$?

Answer (28):

Let $a = \sqrt{x}$ and $b = \sqrt{y}$. Then $a + b = 7$ and $a - b = 4$, so $(a + b)(a - b) = a^2 - b^2 = x - y = \boxed{28}$ by differences of squares. ■

Problem 11:

(pog) Evan has twice the amount of nickels as he has quarters. If the value of his nickels and quarters combined is \$7.00, how much money are his nickels worth?

Answer (2 or 2.00):

Let the value of the nickels be s. Then the value of the quarters is $\frac{0.25}{0.05} \cdot \frac{s}{2} = \frac{5}{2}s$, so $s + \frac{5}{2}s = \frac{7}{2}s = \$7.00$ and his nickels are worth $\boxed{\$2.00}$. ■

Problem 12:

(pog) Katie has a bag containing blue, pink, and red marbles. The ratio of blue to pink marbles is $5 : 7$, and the ratio of pink to red marbles is $18 : 11$. If the ratio of blue to red marbles is $a : b$, what is the value of $a + b$?

Answer (167):

Let Katie have b blue marbles, p pink marbles, and r red marbles, so $\frac{b}{p} = \frac{5}{7}$ and $\frac{p}{r} = \frac{18}{11}$, so $\frac{b}{r} = \frac{b}{p} \cdot \frac{p}{r} = \frac{90}{77}$ and our answer is $90 + 77 = \boxed{167}$. ■

Problem 13:

(pog) Every day, if Katherine remembers to water her bamboo plant, it will get 10 inches taller, but if she forgets to water it, it will get 5 inches shorter. Over a seven-day period, Katherine's bamboo plant grew from 25 inches tall to 50 inches tall. On how many days during the seven-day period did Katherine forget to water her bamboo plant?

Answer (3):

If Katherine never forgot to water her bamboo plant, it would grow from 25 inches tall to 95 inches tall. Note that, for each day Katherine forgets to water her bamboo plant, it will be 15 inches shorter than it would be if she remembered to water it.
So, let x be the number of days that Katherine forgot to water her bamboo plant. Then $95 - 15x = 50$, so $15x = 45$. Thus, $x = 3$, and Katherine forgot to water her bamboo plant for $\boxed{3}$ days. ■

Problem 14:

(pog) Hanami has 3 consecutive integers. If the product of two of these integers is 224, what is the product of the possible values of the third number?

Answer (-225):

Let the average of the two numbers be x.

If the two numbers have a positive difference of 1, then $(x - 0.5)(x + 0.5) = 224$, so $x^2 - 0.25 = 224$ by differences of squares. However, then $x \approx 14.9$, so it cannot be the average of two consecutive integers.

If the two numbers have a positive difference of 2, then the third number is x. We get that $(x - 1)(x + 1) = 224$, so $x^2 - 1 = 224$ by differences of squares. Hence, $x = \pm 15$, so the product of the possible values of the third number is $(15)(-15) = \boxed{-225}$. ■

Problem 15:

(pog) If x and y are numbers such that $2^{12} = x^{2022}$ and $x^{1011} = 16^y$, what is the value of y? Express your answer as a common fraction.

Answer (3/2):

Note that

$$x^{1011} = \sqrt{x^{2022}} = \sqrt{2^{12}} = 2^6,$$

so thus $2^6 = 16^y = (2^4)^y = 2^{4y}$. Hence, $6 = 4y$, so $y = \frac{6}{4} = \boxed{\frac{3}{2}}$. ■

Problem 16:

(pog) Two numbers are randomly chosen, without replacement, from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. What is the probability that their product is prime? Express your answer as a common fraction.

Answer (4/45):

There are $10 \cdot 9 = 90$ possible ways to choose the two numbers. Note that if both numbers are greater than 1, then their product will be composite, as it will have at least three positive divisors. Hence, one of the numbers must be equal to 1 and the other number must be prime, so we get

$$(1, 2), (1, 3), (1, 5), (1, 7), (2, 1), (3, 1), (5, 1), \text{ and } (7, 1)$$

for an answer of $\dfrac{8}{90} = \boxed{\dfrac{4}{45}}$. ■

Problem 17:

(pog) Let $ABCD$ be a square, and let P be a point on side $\overline{AB}$. If the area of $\triangle BCP$ is 25 in^2 and the area of quadrilateral $APCD$ is 200 in^2, what is the length of $\overline{BP}$? Express your answer as a common fraction.

Answer (10/3):

Note that the sum of the two areas is equal to the area of square $ABCD$. Let the side length of the square be s. Then $s^2 = \sqrt{200 + 25}$, so $s = 15$. Hence, $AB = BC = 15$. Since the area of $\triangle BCP$ is 25, we get that $BP \cdot BC = 50$, so hence $BP \cdot 15 = 50$ and $BP = \boxed{\dfrac{10}{3}}$. ■

Problem 18:

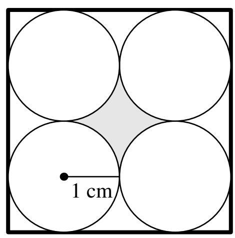
(pog) At 6:00 AM, Manuel started running a marathon at 6 miles per hour at a trail in Honolulu, Hawaii. Then, Parkes started running a marathon at 8 miles per hour at the same trail. If Parkes met Manuel 2 hours after Parkes started running, at what time did the two meet?

Answer (8:40):

Every hour after Parkes started running, he travelled 2 miles more than Manuel did in the same time. If they met 2 hours after Parkes started running, that means that Manuel travelled 4 miles before Parkes started running. Manuel travelled 4 miles by 6:40, so Parkes started running at 6:40 and the two met at $\boxed{8{:}40}$. ■

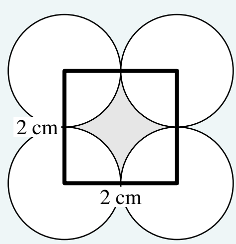
Problem 19:

(pog) There are four congruent circles, each with a radius of 1 centimeter, that are internally tangent to a square and externally tangent to each other, as shown. What is the area of the shaded region? Express your answer in terms of π.



Answer (4 - pi):

Note that the area of the shaded region is equal to $4 - 4 \cdot \frac{1}{4}\pi = \boxed{4 - \pi}$.



■

Problem 20:

(pog) If k is a real number such that

$$k\left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}\right)^{(5 \cdot 6)} = 1,$$

how many positive divisors does k have?

Answer (2821):

We get that $k = (\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4})^{(-30)} = 2^{30} \cdot 3^{30} \cdot 4^{30} = 2^{90} \cdot 3^{30}$. For each divisor of k, we can choose a power of 2 from 0 to 90 and a power of 3 from 0 to 30 to uniquely determine the divisor, so hence k has $91 \cdot 31 = \boxed{2821}$ positive divisors. ■

Problem 21:

(pog) Bill rolls a standard six-sided die 4 times. What is the probability that he gets any number twice in a row? Express your answer as a common fraction.

Answer (91/216):

There are $6^4 = 1296$ ways for Bill to roll the die. The number of ways for Bill to **not** roll the same number twice is $6 \cdot 5 \cdot 5 \cdot 5 = 750$, since he can roll any number the first time and cannot roll the same number as the roll before each subsequent time. Hence, the requested probability is $\frac{1296-750}{1296} = \boxed{\frac{91}{216}}$. ■

Problem 22:

(pog) The grid below has been filled with positive integers in such a way that the sum of the numbers in each row is the same. However, each number in the grid has been expressed in a different base, from base-two to base-ten. What is the result when the number covered by ★ is expressed in base-ten?

★	27	30
97	86	15
201	63	442

Answer (151):

Note that 97 must be 97_{10}, as no other possible bases use the digit 9. Similarly, we deduce

$$86_9,\ 27_8,\ 63_7,\ 15_6,\ 442_5,\ 30_4,\ 201_3.$$

Hence, $\bigstar + 27_8 + 30_4 = 97_{10} + 86_9 + 15_6$, so $\bigstar + 23 + 12 = 97 + 78 + 11$ and $\bigstar = \boxed{151}$. ■

Problem 23:

(pog) If

$$\sqrt{4038 \cdot 2021 + 4040 \cdot 2020 + k} = 4046,$$

what is the value of k?

Answer (48518):

Divide both sides of the given equation by $\sqrt{2}$. Then

$$\sqrt{2019 \cdot 2021 + 2020 \cdot 2020 + \tfrac{k}{2}} = 2023\sqrt{2}.$$

Let $x = 2020$. Then

$$2x^2 - 1 + \tfrac{k}{2} = 2(x+3)^2,$$

so hence $2x^2 - 1 + \frac{k}{2} = 2x^2 + 12x + 18$. Consequently, $k = 2(12 \cdot 2020 + 18 + 1) = \boxed{48518}$. ■

Problem 24:

(pog) For how many integer values of n is a triangle with side lengths 7, 9, and n obtuse?

Answer (7):

Let the longest side of an arbitrary triangle be c, and let its legs be a and b. Then, note that if c^2 is larger than $a^2 + b^2$, the angle opposite side c will be obtuse (the larger c is, the more the angle opposite side c has to stretch out). Hence, we can split the possible triangles into several cases.

Case 1: n is the longest side

Then n^2 must be larger than $7^2 + 9^2$, so since $7^2 + 9^2 = 130$ and $7 + 9 > n$ by the triangle inequality theorem, n can be equal to 12, 13, 14, or 15, giving 4 triangles.

Case 2: 9 is the longest side

Then 9^2 must be larger than $7^2 + n^2$, so since $9^2 - 7^2 = 32$ and $7 + n > 9$ by the triangle inequality theorem, n can be equal to 3, 4, or 5, giving 3 triangles.

Thus, there are a total of $3 + 4 = \boxed{7}$ obtuse triangles with side lengths 7, 9, and n. ■

Problem 25:

(pog) The sequence a_n is defined for all integers n. If

$$(n-1) \cdot a_{n-2} = a_n - 3$$

for all odd $n \geq 1$, what is the value of a_5?

Answer (39):

Note that if $n = 1$, then $0 \cdot a_{(-1)} = a_1 - 3$. Since $a_{(-1)}$ is defined, we get that $0 = a_1 - 3$, so hence $a_1 = 3$. If $n = 3$, then $2 \cdot a_1 = a_3 - 3$, so $2 \cdot 3 = a_3 - 3$ and hence $a_3 = 9$. If $n = 5$, then $4 \cdot a_3 = a_5 - 3$, so $4 \cdot 9 = a_5 - 3$ and hence $a_5 = \boxed{39}$. ■

Problem 26:

(pog) The midpoints of the sides of a triangle are $(1, 0)$, $(3, 0)$, and $(2, 5)$. What is the area of this triangle?

Answer (20):

The midpoints of a triangle split the triangle into four triangles. Note that these four are congruent, since they all share their side lengths. Hence, the area of the four triangles is the same.

The given triangle has a base of 2 and a height of 5, so its area is $\frac{2 \cdot 5}{2} = 5$. Consequently, the area of the whole triangle is $5 \cdot 4 = \boxed{20}$. ■

Problem 27:

(pog) If p, q, and r are the roots of the polynomial

$$x^3 + 20x^2 + 21x + 22,$$

what is the value of $(p-1)(q-1)(r-1)$?

Answer (-64):

Since the roots of $x^3 + 20x^2 + 21x + 22$ are p, q, and r, the factored form of the polynomial is $(x-p)(x-q)(x-r)$. If we plug in $x = 1$, then the factored equation and the polynomial output must be equal, so

$$(1-p)(1-q)(1-r) = 1 + 20 + 21 + 22 = 64.$$

Note that $1 - p = -(p-1)$, $1 - q = -(q-1)$, and $1 - r = -(r-1)$. Hence, the requested expression is equal to

$$(p-1)(q-1)(r-1) = (-1)^3(1-p)(1-q)(1-r) = (-1)^3(64) = \boxed{-64}.$$

■

Problem 28:

(pog) If m and n are positive integers such that $\sqrt[m]{5}\sqrt[n]{25} = \sqrt[5]{5}$, what is the sum of the possible values of n?

Answer (153):

Note that the given expression is equivalent to

$$\frac{1}{m} + \frac{2}{n} = \frac{1}{5}.$$

Hence, $\frac{2m+n}{mn} = \frac{1}{5}$, so $10m + 5n = mn$ and $mn - 10m - 5n = 0$. Note that $mn - 10m - 5n = (m-5)(n-10) - 50$. Hence, $(m-5)(n-10) = 50$, so $m - 5$ and $n - 10$ must be divisors of 50. Looking for possible values, we get that

$$(m, n) = (6, 60), (7, 35), (10, 20), (15, 15), (30, 12), (55, 11),$$

so our answer is $60 + 35 + 20 + 15 + 12 + 11 = \boxed{153}$. ■

Problem 29:

(pog) Ashley chooses a nonempty subset of $\{1, 2, 3, 4, 5, 6\}$ and finds the product of its elements. How many distinct products can she get from this process?

Answer (26):

Every product that can be obtained from this process must be a divisor of $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$. As well, note that, for a subset that obtains product n, Ashley may exclude the members of the subset to obtain $\frac{720}{n}$. Thus, we only have to consider the 15 divisors of 720 that are less than $\sqrt{720} \approx 26.8$, as $\frac{720}{n}$ is achievable if and only if n is achievable. We get that 9 and 16 are not possible, so there are $13 \cdot 2 = \boxed{26}$ possible products. ■

Problem 30:

(pog) How many ordered triples of positive integers (a, b, c) are there such that $a \cdot b \cdot c = 13500$?

Answer (600):

Note that $13500 = 2^2 \cdot 3^3 \cdot 5^3$. As well, note that a, b, and c are uniquely determined by their powers of 2, 3, and 5. The possible powers of 2 of a, b, and c are $(0, 0, 2)$, $(0, 1, 1)$, so hence there are $3 + 3 = 6$ possibilities to distribute the powers of 2. The possible powers of 3 of a, b, and c are $(0, 0, 3)$, $(0, 1, 2)$, $(1, 1, 1)$, so hence there are $3 + 6 + 1 = 10$ possibilities to distribute the powers of 3. The possible powers of 5 are the same as the possible powers of 3, so hence there are 10 possibilities to distribute the powers of 5. Consequently, our answer is $6 \cdot 10 \cdot 10 = \boxed{600}$. ■

Problem 1:

(pog) Ayaka, Brian, and Taiki's ages form an arithmetic sequence, in that order. Six years ago, Brian's age was a quarter of what it is now. Two years ago, Taiki's age was a third of what it is now. What is the product of Ayaka, Brian, and Taiki's ages now?

Answer (312):

We get that $\frac{3}{4}$ of Brian's age now is 6, so Bryan is $6 \cdot \frac{4}{3} = 8$ years old now. Similarly, $\frac{2}{3}$ of Taiki's age now is 2, so Taiki is $2 \cdot \frac{3}{2} = 3$ years old now. Hence, Ayaka's age is $8 + (8 - 3) = 13$, so the requested product is equal to $13 \cdot 8 \cdot 3 = \boxed{312}$. ■

Problem 2:

(pog) How many squares have vertices at $(0,0)$ and $(4,0)$?

Answer (3):

If $(0,0)$ and $(4,0)$ are adjacent vertices, then the possible squares are $(0,0),(4,0),(4,4),(0,4)$ and $(0,0),(4,0),(4,-4),(0,-4)$. If $(0,0)$ and $(4,0)$ are opposite vertices, then the possible square is $(0,0)$, $(2,2)$, $(4,0)$, $(2,-2)$. Hence, there are $2+1=\boxed{3}$ possible squares. ■

Problem 3:

(pog) The average height of Harry, Jerry, Kerry, Mary, and Terry is 60 inches. If the average height of Harry, Jerry, and Kerry is 58 inches, while the average height of Kerry, Mary, and Terry is 54 inches, what is the average height of Harry, Jerry, Mary, and Terry?

Answer (66):

Let their heights be h, j, k, m, and t, respectively. Then

$$\begin{aligned} h+j+k+m+t &= 300 \\ h+j+k &= 174 \\ k+m+t &= 162. \end{aligned}$$

Hence, $(h+j+k)+(k+m+t) = 174+162 = 336$, so thus

$$(h+j+2k+m+t)-(h+j+k+m+t) = k = 336-300 = 36.$$

Consequently, $\frac{h+j+m+t}{4} = \frac{(h+j+k+m+t)-k}{4} = \frac{300-36}{4} = \boxed{66}$. ■

Problem 4:

(pog) How many numbers $\underline{a}\ \underline{b}\ \underline{c}$, where a, b, and c are digits and a is nonzero, are there such that the number $\underline{a}\ \underline{b}\ \underline{c}_7$ is divisible by 2 and the number $\underline{a}\ \underline{b}\ \underline{c}_6$ is divisible by 5?

Answer (20):

Note that $\underline{a}\ \underline{b}\ \underline{c}_7 = 49a + 7b + c$ has a remainder of $a + b + c$ when divided by 2 and $\underline{a}\ \underline{b}\ \underline{c}_6 = 36a + 6b + c$ has a remainder of $a + b + c$ when divided by 5. Hence, $a + b + c$ is divisible by both 2 and 5, and since a, b, and c are less than 6, we get that $a + b + c = 10$.

The possible values of (a, b, c) are $(0, 5, 5)$, $(1, 4, 5)$, $(2, 3, 5)$, $(2, 4, 4)$, and $(3, 3, 4)$, so there are $2 + 6 + 6 + 3 + 3 = \boxed{20}$ possible numbers. ■

Problem 5:

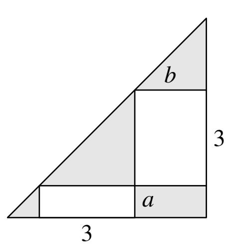
(pog) Analise drives a red car which costs \$48,312 and travels 20 miles per gallon of gasoline, while Henry drives a white car which costs \$41,797 and travels 12 miles per gallon of gasoline. If gasoline costs \$5.30 per gallon, what is the smallest possible value of n such that the total cost (car and fuel) for Analise when travelling n miles is less than the total cost for Henry when travelling n miles? Express your answer to the nearest hundredth.

Answer (36877.36):

For it to cost less for Analise than for Henry, the cost for gasoline needs to be $\$48312 - \$41797 = \$6515$ less. Each mile costs $\frac{1}{20} \cdot \$5.30$ for Analise and $\frac{1}{12} \cdot \$5.30$ for Henry, so hence each mile costs $(\frac{1}{12} - \frac{1}{20}) \cdot \$5.30$ less for Analise than for Henry. Hence, our answer is $\frac{\$6515}{\left(\frac{1}{12}-\frac{1}{20}\right)\cdot\$5.30} \approx \boxed{36877.36}$. ■

Problem 6:

(pog) An isosceles right triangle passes through the vertices of two rectangles that have heights of 3 inches and widths of a inches and b inches, as shown. If the area of the shaded region is 2022 in^2, what is $a + b$? Express your answer to the nearest hundredth.



Answer (63.52):

Note that the three triangles are all similar to the triangle with legs 3 and 3, so hence they are all isosceles right triangles. Hence, the area of the triangle is $\frac{(3+a+b)^2}{2}$. If we subtract the areas of the rectangles out, we will have the area of the shaded region, so the area of the shaded region is

$$\frac{(3+a+b)^2}{2} - 3a - 3b = 2022.$$

Let $x = a + b$. Then the given expression is equal to $\frac{(3+x)^2}{2} - 3x = 2022$, so $(3+x)^2 - 6x = 4044$. Expanding gives $x^2 + 9 = 4044$, so $x = \sqrt{4035} \approx \boxed{63.52}$.

■

Problem 7:

(pog) A number is called *decent* if it can be written as $2^a \cdot 3^b$, where a and b are positive integers less than 10. What is the probability that the product of two different, randomly chosen decent numbers is a perfect cube? Express your answer as a common fraction.

Answer (1/9):

Let the two numbers be $2^p \cdot 3^q$ and $2^r \cdot 3^s$. There are $9 \cdot 9 = 81$ possibilities for the first number and 80 possibilities for the second number. Since $2^{p+r} \cdot 3^{q+s}$ is a perfect cube, we get that $p + r$ and $q + s$ must both be a multiple of 3.

Case 1: $2^p \cdot 3^q$ is a perfect cube

The probability that $2^p \cdot 3^q$ is a perfect cube is $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. Then, regardless of (p, q), we can choose (r, s) in $3 \cdot 3 = 9$ ways such that $p + r$ and $q + s$ are both a multiple of 3, except when $(p, q) = (r, s)$, as the numbers must be distinct. Hence, the probability for this case is $\frac{1}{9} \cdot \frac{9-1}{80} = \frac{1}{90}$.

Case 2: $2^p \cdot 3^q$ is not a perfect cube

The probability that $2^p \cdot 3^q$ is a perfect cube is $1 - \frac{1}{9} = \frac{8}{9}$. Then, regardless of (p, q), we can choose (r, s) in $3 \cdot 3 = 9$ ways such that $p + r$ and $q + s$ are both a multiple of 3. Hence, the probability for this case is $\frac{8}{9} \cdot \frac{9}{80} = \frac{1}{10}$.

Hence, our answer is $\frac{1}{10} + \frac{1}{90} = \boxed{\frac{1}{9}}$.

OR

Let the two numbers be $2^p \cdot 3^q$ and $2^r \cdot 3^s$. There are 81 possibilities for the first number and $81 - 1$ possibilities for the second number, for a total of $81 \cdot 80 = 6480$ possibilities. Since $2^{p+r} \cdot 3^{q+s}$ is a perfect cube, we get that $p + r$ and $q + s$ must both be a multiple of 3.

Case 1: $2^p \cdot 3^q$ is a perfect cube

There are $3 \cdot 3 = 9$ possibilities for $2^p \cdot 3^q$. Then, regardless of (p, q), we can choose (r, s) in $3 \cdot 3 = 9$ ways such that $p + r$ and $q + s$ are both a multiple of 3, except when $(p, q) = (r, s)$, as the numbers must be distinct. Hence, there are $9 \cdot (9 - 1) = 72$ possibilities for (p, q, r, s) in this case.

Case 2: $2^p \cdot 3^q$ is not a perfect cube

There are $81 - 9 = 72$ possibilities for $2^p \cdot 3^q$. Then, regardless of (p, q), we can choose (r, s) in $3 \cdot 3 = 9$ ways such that $p + r$ and $q + s$ are both a multiple of 3. Hence, there are $72 \cdot 9 = 648$ possibilities for (p, q, r, s) in this case.

Hence, our answer is $\frac{72+648}{6480} = \boxed{\frac{1}{9}}$. ■

Problem 8:

(pog) The first four terms of a geometric sequence are a, b, c, and d. If

$$a + 3b + 3c + d = 128 \quad \text{and} \quad 27a + 27b + 9c + d = 250,$$

then $d = \frac{m}{n}$, where m and n are relatively prime positive integers. What is the value of mn?

Answer (1372):

Let the common ratio of the sequence be r. Then $b = ar$, $c = ar^2$, and $d = ar^3$, so $a + 3ar + 3ar^2 + ar^3 = 128$ and $27a + 27ar + 9ar^2 + ar^3 = 250$. Note that the equations are equal to the expanded form of $a(r + 1)^3$ and $a(r + 3)^3$, so hence

$$\frac{a(r+1)^3}{a(r+3)^3} = \frac{128}{150},$$

so $\frac{(r+1)^3}{(r+3)^3} = \frac{64}{125}$ and $\frac{r+1}{r+3} = \frac{4}{5}$. Cross-multiplying gives $5(r + 1) = 4(r + 3)$, so $5r + 5 = 4r + 12$ and $r = 7$. Hence, $a(7 + 1)^3 = 128$, so $a = \frac{1}{4}$ and $d = \frac{343}{4}$, for an answer of $343 \cdot 4 = \boxed{1372}$. ■