MAGNETIC FIELD EXTRAPOLATION INTO A FULL SPHERICAL VOLUME

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1 Setting up the Problem

Working in spherical polar coordinates (r, θ, ϕ), we wish to extrapolate the magnetic field measured at the photosphere (located at $r = 1$) into the coronal volume above the photosphere out to a spherical source surface at $r = r_s$.

2 Potential Extrapolation via Spherical Harmonic Decomposition

If the magnetic field vector $\boldsymbol{B}$ is assumed potential (current free) within the coronal volume, then $\boldsymbol{\nabla} \times \boldsymbol{B} = 0$, allowing a magnetic potential Φ to be defined such that

$$\boldsymbol{B} = -\boldsymbol{\nabla}\Phi. \tag{1}$$

Since $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ in the solution space (and everywhere else we hope), the potential Φ satisfies a Laplace equation,

$$\nabla^2\Phi = 0. \tag{2}$$

The solution to equation (2) in spherical coordinates is

$$\Phi(r, \theta, \phi) = \sum_{\ell,m} \left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)} \right] Y_\ell^m(\theta, \phi), \tag{3}$$

where the coefficients A_ℓ^m and B_ℓ^m are determined from the imposed radial boundary conditions. The angular variation of Φ in equation (3) has been decomposed into spherical harmonic functions Y_ℓ^m, defined

$$Y_\ell^m(\theta, \phi) = C_\ell^m P_\ell^m(\cos\theta)\, e^{im\phi}, \tag{4}$$

where the functions $P_\ell^m(\cos\theta)$ are the associated Legendre functions, and where the coefficients C_ℓ^m are defined

$$C_\ell^m = (-1)^m \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{\frac{1}{2}}. \tag{5}$$

The photospheric boundary condition on Φ is

$$\left. \frac{\partial\Phi}{\partial r} \right|_{r=1} = -B_r(1, \theta, \phi), \tag{6}$$

where $B_r(1, \theta, \phi)$ denotes the radial magnetic field measured at the photosphere. If we denote the spherical harmonic coefficients of $B_r(1, \theta, \phi)$ as F_ℓ^m, such that

$$B_r(1, \theta, \phi) = \sum_{\ell,m} Y_\ell^m F_\ell^m, \tag{7}$$

equation (6) implies

$$\sum_{\ell,m} Y_\ell^m \left[A_\ell^m \ell - B_\ell^m (\ell+1) \right] = -\sum_{\ell,m} Y_\ell^m F_\ell^m, \tag{8}$$

or

$$A_\ell^m \ell - B_\ell^m (\ell+1) = -F_\ell^m, \tag{9}$$

where equation (3) has been used to calculate the radial derivative of Φ. At the source surface, the magnetic field is assumed purely radial, such that $\Phi = 0$ at $r = r_s$, making

$$A_\ell^m r_s^\ell + B_\ell^m r_s^{-(\ell+1)} = 0 \tag{10}$$

from equation (3). Once the coefficients F_ℓ^m are determined, the system of equations (9) and (10) is then solved to yield the values of A_ℓ^m and B_ℓ^m for each (ℓ, m) pair:

$$A_\ell^m = -r_s^{-(2\ell+1)} B_\ell^m \tag{11}$$

$$B_\ell^m = \frac{F_\ell^m}{1 + \ell + \ell r_s^{-(2\ell+1)}}. \tag{12}$$

Once A_ℓ^m and B_ℓ^m (and thus Φ) are known, the magnetic field $\boldsymbol{B}$ anywhere within the coronal volume can then be calculated[1] using equation (1), giving

$$B_r = -\sum_{\ell,m} Y_\ell^m \left[A_\ell^m \ell r^{\ell-1} - B_\ell^m (\ell+1) r^{-(\ell+2)} \right] \tag{13}$$

$$B_\theta = -\frac{1}{r\sin\theta} \sum_{\ell,m} Y_\ell^m \left\{ R_\ell^m (\ell-1) \left[A_{\ell-1}^m r^{\ell-1} + B_{\ell-1}^m r^{-\ell} \right] - R_{\ell+1}^m (\ell+2) \left[A_{\ell+1}^m r^{\ell+1} + B_{\ell+1}^m r^{-(\ell+2)} \right] \right\} \tag{14}$$

$$B_\phi = -\frac{1}{r\sin\theta} \sum_{\ell,m} im Y_\ell^m \left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)} \right], \tag{15}$$

where the step factor R_ℓ^m is defined

$$R_\ell^m = \left[\frac{\ell^2 - m^2}{4\ell^2 - 1} \right]^{\frac{1}{2}}. \tag{16}$$

3 Derivatives of $\boldsymbol{B}$

For reference, we list various derivatives of $\boldsymbol{B}$ in terms of the harmonic coefficients, organized both by variable and by order. First-order derivatives of B_r are as follows:

$$\frac{\partial B_r}{\partial r} = -\sum_{\ell,m} Y_\ell^m \left[A_\ell^m \ell(\ell-1) r^{\ell-2} + B_\ell^m (\ell+1)(\ell+2) r^{-(\ell+3)} \right] \tag{17}$$

$$\begin{aligned} \frac{\partial B_r}{\partial \theta} = -\sum_{\ell,m} Y_\ell^m \Big\{ & R_\ell^m (\ell-1) \left[A_{\ell-1}^m (\ell-1) r^{\ell-2} - B_{\ell-1}^m \ell r^{-(\ell+1)} \right] \\ & - R_{\ell+1}^m (\ell+2) \left[A_{\ell+1}^m (\ell+1) r^\ell - B_{\ell+1}^m (\ell+2) r^{-(\ell+3)} \right] \Big\} \end{aligned} \tag{18}$$

$$\frac{\partial B_r}{\partial \phi} = -\sum_{\ell,m} im Y_\ell^m \left[A_\ell^m \ell r^{\ell-1} - B_\ell^m (\ell+1) r^{-(\ell+2)} \right] \tag{19}$$

[1] A detailed derivation of equation (14) is presented in Appendix A.

First-order derivatives of B_θ are:

$$\sin\theta\frac{\partial B_\theta}{\partial r} = -\sum_{\ell,m} Y_\ell^m \Big\{ R_\ell^m(\ell-1)\left[A_{\ell-1}^m(\ell-2)r^{\ell-3} - B_{\ell-1}^m(\ell+1)r^{-(\ell+2)}\right] - R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m \ell r^{\ell-1} - B_{\ell+1}^m(\ell+3)r^{-(\ell+4)}\right]\Big\} \tag{20}$$

$$\begin{aligned}\sin\theta\frac{\partial(\sin\theta B_\theta)}{\partial\theta} = -\sum_{\ell,m}^{m} Y_\ell^m \Big\{ & R_\ell^m R_{\ell-1}^m(\ell-1)(\ell-2)\left[A_{\ell-2}^m r^{\ell-3} + B_{\ell-2}^m r^{-\ell}\right] \\ & - \left[(R_\ell^m)^2(\ell+1)(\ell-1) + (R_{\ell+1}^m)^2\ell(\ell+2)\right]\left[A_\ell^m r^{\ell-1} + B_\ell^m r^{-(\ell+2)}\right] \\ & + R_{\ell+1}^m R_{\ell+2}^m(\ell+2)(\ell+3)\left[A_{\ell+2}^m r^{\ell+1} + B_{\ell+2}^m r^{-(\ell+4)}\right]\Big\}\end{aligned} \tag{21}$$

$$\sin\theta\frac{\partial B_\theta}{\partial\phi} = -\sum_{\ell,m} imY_\ell^m \Big\{ R_\ell^m(\ell-1)\left[A_{\ell-1}^m r^{\ell-2} + B_{\ell-1}^m r^{-(\ell+1)}\right] - R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m r^{\ell} + B_{\ell+1}^m r^{-(\ell+3)}\right]\Big\} \tag{22}$$

First-order derivatives of B_ϕ are:

$$\sin\theta\frac{\partial B_\phi}{\partial r} = -\sum_{\ell,m} imY_\ell^m\left[A_\ell^m(\ell-1)r^{\ell-2} - B_\ell^m(\ell+2)r^{-(\ell+3)}\right] \tag{23}$$

$$\sin\theta\frac{\partial(\sin\theta B_\phi)}{\partial\theta} = -\sum_{\ell,m} imY_\ell^m \Big\{ R_\ell^m(\ell-1)\left[A_{\ell-1}^m r^{\ell-2} + B_{\ell-1}^m r^{-(\ell+1)}\right] - R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m r^{\ell} + B_{\ell+1}^m r^{-(\ell+3)}\right]\Big\} \tag{24}$$

$$\sin\theta\frac{\partial B_\phi}{\partial\phi} = \sum_{\ell,m} m^2 Y_\ell^m\left[A_\ell^m r^{\ell-1} + B_\ell^m r^{-(\ell+2)}\right] \tag{25}$$

Second-order derivatives of B_r are:

$$\sin\theta\frac{\partial^2 B_r}{\partial r\partial\theta} = -\sum_{\ell,m} Y_\ell^m \Big\{ R_\ell^m(\ell-1)\left[A_{\ell-1}^m(\ell-1)(\ell-2)r^{\ell-3} + B_{\ell-1}^m\ell(\ell+1)r^{-(\ell+2)}\right] - R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m\ell(\ell+1)r^{\ell-1} + B_{\ell+1}^m(\ell+2)(\ell+3)r^{-(\ell+4)}\right]\Big\} \tag{26}$$

$$\frac{\partial^2 B_r}{\partial r\partial\phi} = -\sum_{\ell,m} imY_\ell^m\left[A_\ell^m\ell(\ell-1)r^{\ell-2} + B_\ell^m(\ell+1)(\ell+2)r^{-(\ell+3)}\right] \tag{27}$$

$$\sin\theta\frac{\partial^2 B_r}{\partial\theta\partial\phi} = -\sum_{\ell,m} imY_\ell^m \Big\{ R_\ell^m(\ell-1)\left[A_{\ell-1}^m(\ell-1)r^{\ell-2} - B_{\ell-1}^m\ell r^{-(\ell+1)}\right] - R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m(\ell+1)r^{\ell} - B_{\ell+1}^m(\ell+2)r^{-(\ell+3)}\right]\Big\} \tag{28}$$

Second-order derivatives of B_θ are:

$$\sin\theta\frac{\partial^2(\sin\theta B_\theta)}{\partial r\partial\theta} = -\sum_{\ell,m} Y_\ell^m\Bigg\{R_\ell^m R_{\ell-1}^m(\ell-1)(\ell-2)\left[A_{\ell-2}^m(\ell-3)r^{\ell-4} - B_{\ell-2}^m \ell r^{-(\ell+1)}\right]$$
$$- \left[(R_\ell^m)^2(\ell+1)(\ell-1) + (R_{\ell+1}^m)^2\ell(\ell+2)\right]\left[A_\ell^m(\ell-1)r^{\ell-2} - B_\ell^m(\ell+2)r^{-(\ell+3)}\right]$$
$$+ R_{\ell+1}^m R_{\ell+2}^m(\ell+2)(\ell+3)\left[A_{\ell+2}^m(\ell+1)r^{\ell} - B_{\ell+2}^m(\ell+4)r^{-(\ell+5)}\right]\Bigg\} \tag{29}$$

$$\sin\theta\frac{\partial^2 B_\theta}{\partial r\partial\phi} = -\sum_{\ell,m} imY_\ell^m\Bigg\{R_\ell^m(\ell-1)\left[A_{\ell-1}^m(\ell-2)r^{\ell-3} - B_{\ell-1}^m(\ell+1)r^{-(\ell+2)}\right]$$
$$- R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m\ell r^{\ell-1} - B_{\ell+1}^m(\ell+3)r^{-(\ell+4)}\right]\Bigg\} \tag{30}$$

$$\sin\theta\frac{\partial^2(\sin\theta B_\theta)}{\partial\theta\partial\phi} = -\sum_{\ell,m} imY_\ell^m\Bigg\{R_\ell^m R_{\ell-1}^m(\ell-1)(\ell-2)\left[A_{\ell-2}^m r^{\ell-3} + B_{\ell-2}^m r^{-\ell}\right]$$
$$- \left[(R_\ell^m)^2(\ell+1)(\ell-1) + (R_{\ell+1}^m)^2\ell(\ell+2)\right]\left[A_\ell^m r^{\ell-1} + B_\ell^m r^{-(\ell+2)}\right]$$
$$+ R_{\ell+1}^m R_{\ell+2}^m(\ell+2)(\ell+3)\left[A_{\ell+2}^m r^{\ell+1} + B_{\ell+2}^m r^{-(\ell+4)}\right]\Bigg\} \tag{31}$$

Second-order derivatives of B_ϕ are:

$$\sin\theta\frac{\partial^2(\sin\theta B_\phi)}{\partial r\partial\theta} = -\sum_{\ell,m} imY_\ell^m\Bigg\{R_\ell^m(\ell-1)\left[A_{\ell-1}^m(\ell-2)r^{\ell-3} - B_{\ell-1}^m(\ell+1)r^{-(\ell+2)}\right]$$
$$- R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m\ell r^{\ell-1} - B_{\ell+1}^m(\ell+3)r^{-(\ell+4)}\right]\Bigg\} \tag{32}$$

$$\sin\theta\frac{\partial^2 B_\phi}{\partial r\partial\phi} = \sum_{\ell,m} m^2Y_\ell^m\left[A_\ell^m(\ell-1)r^{\ell-2} - B_\ell^m(\ell+2)r^{-(\ell+3)}\right] \tag{33}$$

$$\sin\theta\frac{\partial^2(\sin\theta B_\phi)}{\partial\theta\partial\phi} = \sum_{\ell,m} m^2Y_\ell^m\Bigg\{R_\ell^m(\ell-1)\left[A_{\ell-1}^m r^{\ell-2} + B_{\ell-1}^m r^{-(\ell+1)}\right]$$
$$- R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m r^{\ell} + B_{\ell+1}^m r^{-(\ell+3)}\right]\Bigg\} \tag{34}$$

Third-order derivatives are:

$$
\begin{aligned}
\sin\theta\frac{\partial^3 B_r}{\partial r\partial\theta\partial\phi} = -\sum_{\ell,m} imY_\ell^m \Bigg\{ & R_\ell^m(\ell-1)\left[A_{\ell-1}^m(\ell-1)(\ell-2)r^{\ell-3} + B_{\ell-1}^m\ell(\ell+1)r^{-(\ell+2)}\right] \\
& - R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m\ell(\ell+1)r^{\ell-1} + B_{\ell+1}^m(\ell+2)(\ell+3)r^{-(\ell+4)}\right]\Bigg\}
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\sin\theta\frac{\partial^3(\sin\theta B_\theta)}{\partial r\partial\theta\partial\phi} = -\sum_{\ell,m} imY_\ell^m \Bigg\{ & R_\ell^m R_{\ell-1}^m(\ell-1)(\ell-2)\left[A_{\ell-2}^m(\ell-3)r^{\ell-4} - B_{\ell-2}^m\ell r^{-(\ell+1)}\right] \\
& - \left[(R_\ell^m)^2(\ell-1)(\ell+1) + (R_{\ell+1}^m)^2\ell(\ell+2)\right]\left[A_\ell^m(\ell-1)r^{\ell-2} - B_\ell^m(\ell+2)r^{-(\ell+3)}\right] \\
& + R_{\ell+1}^m R_{\ell+2}^m(\ell+2)(\ell+3)\left[A_{\ell+2}^m(\ell+1)r^{\ell} - B_{\ell+2}^m(\ell+4)r^{-(\ell+5)}\right]\Bigg\}
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\sin\theta\frac{\partial^3(\sin\theta B_\phi)}{\partial r\partial\theta\partial\phi} = \sum_{\ell,m} m^2Y_\ell^m \Bigg\{ & R_\ell^m(\ell-1)\left[A_{\ell-1}^m(\ell-2)r^{\ell-3} - B_{\ell-1}^m(\ell+1)r^{-(\ell+2)}\right] \\
& - R_{\ell+1}^m(\ell+2)\left[A_{\ell+1}^m\ell r^{\ell-1} - B_{\ell+1}^m(\ell+3)r^{-(\ell+4)}\right]\Bigg\}
\end{aligned}
\tag{37}
$$

A Derivation of Equation (14)

Since derivatives of θ involving spherical harmonic expansions are somewhat subtle (not to mention messy), we now give a detailed derivation of equation (14). From equations (1)–(5), we have

$$
\begin{aligned}
B_\theta &= -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} && (38)\\
&= -\frac{1}{r}\sum_{\ell,m}\left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)}\right]\frac{\partial Y_\ell^m}{\partial\theta} && (39)\\
&= -\frac{1}{r\sin\theta}\sum_{\ell,m}\left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)}\right]\left[\left(C_\ell^m\sin\theta\,\frac{dP_\ell^m(\cos\theta)}{d\theta}\right)e^{im\phi}\right]. && (40)
\end{aligned}
$$

The associated Legendre functions $P_\ell^m(x)$ satisfy a multitude of recursion relations. The relation applicable here involves the derivative of $P_\ell^m(x)$ and functions of neighboring degrees $\ell \pm 1$, namely,

$$
(x^2-1)\,\frac{dP_\ell^m(x)}{dx} = -(\ell+m)\left(\frac{\ell+1}{2\ell+1}\right)P_{\ell-1}^m + (\ell-m+1)\left(\frac{\ell}{2\ell+1}\right)P_{\ell+1}^m. \tag{41}
$$

Making the substitution $x = \cos\theta$ yields

$$
\sin\theta\,\frac{dP_\ell^m(\cos\theta)}{d\theta} = -(\ell+m)\left(\frac{\ell+1}{2\ell+1}\right)P_{\ell-1}^m + (\ell-m+1)\left(\frac{\ell}{2\ell+1}\right)P_{\ell+1}^m, \tag{42}
$$

which can be used to resolve the derivative of $P_\ell^m(\cos\theta)$ in equation (40), from which we obtain

$$C_\ell^m \sin\theta \frac{dP_\ell^m(\cos\theta)}{d\theta} = C_\ell^m \left[-(\ell+m)\left(\frac{\ell+1}{2\ell+1}\right)P_{\ell-1}^m + (\ell-m+1)\left(\frac{\ell}{2\ell+1}\right)P_{\ell+1}^m\right] \tag{43}$$

$$= -(\ell+m)\left(\frac{\ell+1}{2\ell+1}\right)\left(\frac{C_\ell^m}{C_{\ell-1}^m}\right)C_{\ell-1}^m P_{\ell-1}^m + (\ell-m+1)\left(\frac{\ell}{2\ell+1}\right)\left(\frac{C_\ell^m}{C_{\ell+1}^m}\right)C_{\ell+1}^m P_{\ell+1}^m \tag{44}$$

$$= -(\ell+m)\left(\frac{\ell+1}{2\ell+1}\right)\left[\left(\frac{2\ell+1}{2\ell-1}\right)\left(\frac{\ell-m}{\ell+m}\right)\right]^{\frac{1}{2}} C_{\ell-1}^m P_{\ell-1}^m \tag{45}$$

$$+ (\ell-m+1)\left(\frac{\ell}{2\ell+1}\right)\left[\left(\frac{2\ell+1}{2\ell+3}\right)\left(\frac{\ell+m+1}{\ell-m+1}\right)\right]^{\frac{1}{2}} C_{\ell+1}^m P_{\ell+1}^m \tag{46}$$

$$= -(\ell+1)\left[\frac{\ell^2-m^2}{4\ell^2-1}\right]^{\frac{1}{2}} C_{\ell-1}^m P_{\ell-1}^m + \ell\left[\frac{(\ell+1)^2-m^2}{4(\ell+1)^2-1}\right]^{\frac{1}{2}} C_{\ell+1}^m P_{\ell+1}^m \tag{47}$$

$$= -(\ell+1)R_\ell^m C_{\ell-1}^m P_{\ell-1}^m + \ell R_{\ell+1}^m C_{\ell+1}^m P_{\ell+1}^m, \tag{48}$$

where in the last step we have used the definition of R_ℓ^m from equation (16). Substituting equation (48) into equation (40), we obtain

$$B_\theta = -\frac{1}{r\sin\theta}\sum_{\ell,m}\left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)}\right]\left[\ell\, R_{\ell+1}^m Y_{\ell+1}^m - (\ell+1)R_\ell^m Y_{\ell-1}^m\right], \tag{49}$$

where we have used the definition of Y_ℓ^m in equation (4).

The last part of the derivation involves reindexing the sum over ℓ so that we are summing quantities involving Y_ℓ^m instead of $Y_{\ell-1}^m$ or $Y_{\ell+1}^m$. Doing so yields

$$B_\theta = -\frac{1}{r\sin\theta}\left\{\sum_{\ell,m}\left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)}\right]\left[\ell\, R_{\ell+1}^m Y_{\ell+1}^m\right] - \sum_{\ell,m}\left[A_\ell^m r^\ell + B_\ell^m r^{-(\ell+1)}\right]\left[(\ell+1)R_\ell^m Y_{\ell-1}^m\right]\right\} \tag{50}$$

$$= -\frac{1}{r\sin\theta}\left\{\sum_{\ell,m}\left[A_{\ell-1}^m r^{\ell-1} + B_{\ell-1}^m r^{-\ell}\right](\ell-1)R_\ell^m Y_\ell^m - \sum_{\ell,m}\left[A_{\ell+1}^m r^{\ell+1} + B_{\ell+1}^m r^{-(\ell+2)}\right](\ell+2)R_{\ell+1}^m Y_\ell^m\right\} \tag{51}$$

$$= -\frac{1}{r\sin\theta}\sum_{\ell,m} Y_\ell^m\left\{\left[A_{\ell-1}^m r^{\ell-1} + B_{\ell-1}^m r^{-\ell}\right](\ell-1)R_\ell^m - \left[A_{\ell+1}^m r^{\ell+1} + B_{\ell+1}^m r^{-(\ell+2)}\right](\ell+2)R_{\ell+1}^m\right\}, \tag{52}$$

which after some rearranging is equivalent to equation (14).