1 Motivation for Ricci Flows in manifold untangling or de-curving

In 1982, Richard Hamilton introduced the idea of the Ricci flow in the hope of evolving the shape of an arbitrary Riemannian manifold into that of a 3-sphere. The idea is to start from an arbitrary Riemannian manifold with a metric g_0 and then "evolve" that metric according to an equation that looks like a "Heat Equation":

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \text{ with } g(0) = g_0 \tag{1}$$

All this work was motivated by the Poincare conjecture that states that a closed simply connected 3-manifold is **homeomorphic** to the 3-sphere.

2 Heat Equation on a circle

Let us consider the circle in 1D, denoted $\mathbb{S}^1$, and $u(\theta)$ a function of the angle describing the position of a point on the circle. θ lives in the interval $[0, 2\pi]$. We consider u to represent a **temperature** field. As a consequence, because of the periodicity of $\mathbb{S}^1$, we require u to be 2π-periodic. We also wish to simplify things a bit by letting u to belong to $\mathcal{C}^k(\mathbb{S}^1)$, which indicates that u is continuous, and has continuous i-th derivatives until order k.

2.1 Fundamental equation

The master equation that will evolve the temperature field u is given by the "Heat Equation" which is a second order partial differential equation. That equation can be recast into the following adimensional form (after proper rescaling of time, angle and the function itself):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \text{ for all } t > 0 \text{ and } \theta \in \mathbb{S}^1 \tag{2}$$

The initial condition is given by a starting temperature field:

$$u(0, \theta) = g(\theta) \tag{3}$$

2.2 General solution

The classic technique of separation of variables can be used to solve that equation. The idea is to make the ansatz that the solution can be written as a product of two functions. The first only depends on time while the second only depends on θ:

$$u(t, \theta) = T(t)\Theta(\theta) \tag{4}$$

which, once injected into the original Heat Equation leads to the following separation:

$$\frac{T'(t)}{T(t)} = -\lambda = \frac{\Theta''(\theta)}{\Theta(\theta)} \tag{5}$$

The only possible solution to this equation is that both functions on each side of the equality are actually constant. That constant is called a *separation constant* and we denote it by the following character : $-\lambda$.

Using the above, we end up with two ordinary differential equations to solve. We don't forget that the solution we seek $u(t, \theta)$ has to satisfy the properties we intended, namely 2π-periodicity and the fact that since we describe a temperature that latter cannot increase indefinitely to $+\infty$. The equations are:

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 \\ T' + \lambda T &= 0 \end{aligned} \tag{6}$$

Solutions of the first equation can be generally written as:

$$\Theta_k(\theta) = A_k \cos(k\theta) + B_k \sin(k\theta), \text{ where } k = \sqrt{\lambda} \tag{7}$$

k is in general a complex number, but since we require the function to be *periodic*, we see that k has to be real and so λ has to be **non-negative**. If $\lambda = 0$ then the solution is a constant. But periodicity puts constraints on the values of k too. Indeed, the function needs to be 2π-periodic. So as a consequence, k can only be restricted to the set of non-negative integers, $\mathbb{N}$.

From the solutions of the first equation, we immediately solve for the second and find the following general solution:

$$T_k(t) = e^{-k^2 t} \tag{8}$$

The general solution is then:

$$u_k(t, \theta) = e^{-k^2 t} \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right) \tag{9}$$

Since the Heat Equation is **linear**, we see that **any** linear combination of the solutions $u_k(t, \theta)$ is also a solution. So all in all, the most general solution is a Fourier series:

$$u(t, \theta) = A_0 + \sum_{k=1}^{+\infty} e^{-k^2 t} \left[A_k \cos(k\theta) + B_k \sin(k\theta)\right] \tag{10}$$

Using the initial condition, $u(0, \theta) = g(\theta)$, we end up with:

$$g(\theta) = A_0 + \sum_{k=1}^{+\infty} \left[A_k \cos(k\theta) + B_k \sin(k\theta)\right] \tag{11}$$

and we only have to decompose $g(\theta)$ in Fourier Series to find out what the values of A_k and B_k are. More specifically, we have the following formulas:

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta \\ A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(k\theta)d\theta \\ B_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(k\theta)d\theta \end{aligned} \tag{12}$$

2.3 Very general solution properties

Among the many fascinating properties that solutions to the Heat Equation have, we will list a few here:

1. if u and v are solutions to the Heat equation, and that at $t = 0$ (initial time), we have $u(0, \theta) < v(0, \theta)$, then at **any later** time, we also have $u(t, \theta) \leq v(t, \theta)$ for all time t and angle θ.

2. if $g(\theta)$ is the initial temperature field, so $g(\theta) = u(0, \theta)$, then at **any later** time, u is **bounded** by the *minimum* and *maximum* of g.

3. u converges to its limit (a uniform temperature, which equals the mean of the temperature field) **exponentially** quickly (in L^2 norm) and **uniformly** in θ.

3 How to straighten up a closed 1D curve ?

This section points to a talk given by Andrejs Treibergs that shows how to use the Ricci flow to solve the "straightening" problem: Is it possible to deform **continuously** a closed planar curve in such a way that:

- The parts that are bent the most are unbent the fastest
- The curve does not cross itself
- The deformation limit is a circle

The answer to these questions are given in the talk by Andrejs Treibergs at the following address : `http://www.math.utah.edu/~treiberg/CurvFlowSlides.pdf`.