## Permutation vs Combination

**Problem.** There are 5 people: A, B, C, D, E.
(a) How many ways can we arrange three of them in a line?
(b) How many ways can we form a group of three people?

**Solution.** Notice that the order is important in (a) but not in (b).

(a) **Task:** Fill three spots in the line with three people chosen from A, B, C, D, E.

**Step 1:** Choose one person to fill in the left spot. There are 5 ways to do this.

**Step 2:** Choose one person to fill in the middle spot. There are 4 ways to do this, as there are four remaining people to choose from after step 1.

**Step 3:** Choose one person to fill in the right spot. There are 3 ways to do this, as there are three people to choose from. *Task completed.*

So there are $5 \times 4 \times 3 = \frac{5!}{2!} = \frac{5!}{(5-3)!} = 60$ ways to arrange three people in a line.

(b) Consider the arrangements in part (a) that consist of A, B, and C. If we choose A, B, and C to be in the line, there are $3! = 3 \times 2 \times 1 = 6$ ways to order them:

ABC, ACB, BAC, BCA, CAB, CBA.

We can see that for each combination of three people, there are 3! ways to arrange them. Since there are $5!/2! = 60$ possible arrangements, there must be $(5!/2!)/3! = \frac{5!}{2!3!} = \binom{5}{3} = 10$ combinations of three people, hence the answer. ■

## MISSISSIPPI Problem

**Problem.** How many ways can we arrange the letters in MISSISSIPPI?

**Solution.** We use the multiplication principle.

**Task:** Fill 11 spots with 4 S's, 4 I's, 2 P's and 1 M.

**Step 1:** Choose 4 spots for 4 S's. $\binom{11}{4}$ ways.

**Step 2:** Choose 4 spots for 4 I's. There are 7 spots remaining, so $\binom{7}{4}$ ways.

**Step 3:** Choose 2 spots for 2 P's. There are 3 spots remaining, so $\binom{3}{2}$ ways.

**Step 4:** Put M in the only remaining spot. There is 1 way to do this. *Task completed.*

So there are
$$\binom{11}{4}\binom{7}{4}\binom{3}{2} = \frac{11!}{4!7!} \cdot \frac{7!}{4!3!} \cdot \frac{3!}{2!1!} = \frac{11!}{4!4!2!1!}$$
ways to arrange these letters. Notice that 11 is the total number of letters, and 4, 4, 2 and 1 are the numbers of S's, I's, P's and M's. ■

Some more examples on the number of ways to shuffle the letters:

CCC: $\frac{3!}{3!} = 1$ MEME: $\frac{4!}{2!2!} = 6$ TROLOLOL: $\frac{8!}{3!3!1!1!} = 1,120$

## Probability - Basics

### Basic properties of probability

1. $0 \le P(A) \le 1$. Probability of an event is a number between 0 and 1.
2. $P(\Omega) = 1$. The probability that something in the sample space will occur is 1. $P(\emptyset) = 0$. The probability that nothing in the sample space will occur is 0.
3. $P(A \cup B) = P(A) + P(B)$ when $A$ and $B$ are *disjoint* events.
4. $P(A^{\complement}) = 1 - P(A)$.
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

**Probability of events with equally likely outcomes.** If there are a finite number of outcomes and all of them are equally likely, then the probability of an event is the number of outcomes in that event divided by the total number of outcomes: $P(A) = \frac{|A|}{|\Omega|}$.

**Example.** Order 5 people A, B, C, D and E in a line randomly. What is the probability that A and B stand next to each other?

**Solution.** The number of possible outcomes is $5!$, as this is the number of ways to arrange 5 people. Now we are interested in the event when A and B stand next to each other, so we have to count the number of arrangements in which A and B stand next to each other.

**Step 1:** Arrange A, C, D and E. There are 4! ways.

**Step 2:** Put B next to A. There are 2 ways: B can be either to the left or to the right of A. *Task completed.*

So there are $4! \times 2$ possible arrangements in which A and B are next to each other. Therefore, the probability that A and B stand next to each other is $\frac{4! \times 2}{5!} = \frac{2}{5}$. ■

## Exercises

1. In how many ways can we arrange 5 people A, B, C, D, and E so that D and E do not stand next to each other?

2. You want to make a weekly exercise schedule. In each week, you plan to have three days of cardio training, three days of weight training, and one rest day. You don't want to do both cardio and weight training on the same day. In how many ways can you make a schedule? (One example of schedule: M/W/F-Cardio, T/Th/Sat-Weight, Sun-Rest.)

3. A six-sided die is biased in such a way that the number 5 is 5 times more likely to show up than any of the other numbers. That is, $P(5) = 5P(1) = 5P(2) = 5P(3) = 5P(4) = 5P(6)$. Let $A$ be the event that the number is even, and let $B$ be the event that the number shown is greater than 3. Describe the following events, list all the outcomes and find the probability.

| events | meaning | elements | probability |
|---|---|---|---|
| $\Omega$ | any number | $\{1,2,3,4,5,6\}$ | 1 |
| $A \cap B$ | even and greater than 3 | $\{4,6\}$ | 0.2 |
| $A \cup B$ | | | |
| $A^{\mathsf{C}}$ | | | |
| $B^{\mathsf{C}}$ | | | |
| $A \cap B^{\mathsf{C}}$ | | | |
| $A^{\mathsf{C}} \cup B$ | | | |
| $(A \cup B)^{\mathsf{C}}$ | | | |
| $A^{\mathsf{C}} \cup B^{\mathsf{C}}$ | | | |

4. Toss a fair coin ten times. What is the probability that it will land on heads exactly five times?

5. In the game of Keno, the player selects 20 distinct numbers from 1 to 80 (inclusive), and then twenty distinct winning numbers are selected, also from 1 to 80. The player wins money according to how many numbers that he or she has chosen match the winning numbers. What is the probability that a player match exactly 8 winning numbers in one game?

6. In how many ways can we arrange 5 people A, B, C, D, and E so that A stands next to B or C (or both)? (Hint: See property 5 of probability.)