Conditional Probability

The probability of event A, given event B as a condition, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 1. Toss a fair coin 4 times. Given that the coin came up tails at least once, what is the probability that the coin came up heads at least twice?

Solution. Note that there are $2^4 = 16$ possible, equally likely outcomes of tossing a fair coin 4 times. Let A be the event that the coin came up heads at least twice, and let B be the event that the coin came up tails at least once. We want to find $P(A|B)$, which by definition is equal to $\frac{P(A\cap B)}{P(B)}$. Let's start by computing $P(B)$:

$$P(B) = 1 - P(B^{\complement}) = 1 - P(\text{no tails}) = 1 - \frac{1}{16} = \frac{15}{16}.$$

For $P(A \cap B)$, notice that $A \cap B$ means that we got at least 2 heads and at least 1 tails. This means either that we got 2 heads and 2 tails or that we got 3 heads and 1 tails. Thus,

$$P(A \cap B) = P(\text{2H,2T}) + P(\text{3H,1T}) = \binom{4}{2}\frac{1}{16} + \binom{4}{3}\frac{1}{16} = \frac{10}{16}.$$

($P(\text{2H,2T}) = \binom{4}{2}\frac{1}{16}$ because there are $\binom{4}{2}$ outcomes that have 2 heads and 2 tails, and the probability of each outcome is $\frac{1}{16}$. $P(\text{3H,1T})$ is computed similarly.)

Therefore, $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{10}{16}}{\frac{15}{16}} = \dfrac{2}{3}$. ■

Independence

Two events A and B are independent if $P(A|B) = P(A)$,
or equivalently, if $P(A \cap B) = P(A)P(B)$.

Example 2. Are events A and B in the solution of Example 1 above independent?

Solution. We must check whether $P(A \cap B)$ is equal to $P(A)P(B)$. We have already computed $P(A \cap B)$ and $P(B)$, so let's compute $P(A)$.

$$P(A) = P(\text{2H,2T}) + P(\text{3H,1T}) + P(\text{4H,0T}) = \binom{4}{2}\frac{1}{16} + \binom{4}{3}\frac{1}{16} + \binom{4}{4}\frac{1}{16} = \frac{11}{16}.$$

So $P(A \cap B) = \frac{10}{16}$ but $P(A)P(B) = \frac{11}{16} \cdot \frac{15}{16} \neq \frac{10}{16}$. Therefore, A and B are not independent. (Alternatively, we can also check that $P(A|B) \neq P(A)$.) ■

Mutual Independence Three events A, B and C are (mutually) independent if they are pairwise independent and $P(A \cap B \cap C) = P(A)P(B)P(C)$.

The notion of mutual independence can be generalized to any number of events.

Example 3. Roll a fair regular six-sided die four times. Assume that the four rolls are independent. **(a)** Find the probability that the die shows number 1 only once on the second roll. **(b)** Find the probability that the die shows 1 only once.

Solution. **(a)** Here we want four events to happen: the first roll is not 1; the second roll is 1; the third roll is not 1; and the fourth roll is not 1. Since the four rolls are independent, the four events are independent, so we can simply multiply together the probabilities of these events. Therefore, the answer is $\frac{5}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{1}{6}\left(\frac{5}{6}\right)^3$.

(b) Here number 1 can appear on any of the four rolls, so there are four possibilities: 1XXX, X1XX (as in part a), XX1X, and XXX1. In each possibility, number 1 appears on one roll, and does not appear on the other three rolls, so the probability that this happens is $\frac{1}{6}\left(\frac{5}{6}\right)^3$. (See the answer for part a.) Now we sum up the probabilities for all four possibilities to get the final answer: $4 \times \frac{1}{6}\left(\frac{5}{6}\right)^3$. ■

Adding vs Multiplying

As a rule of thumb,
we *add* two probabilities to get the probability of A *or* B (i.e. $A \cup B$), and
we *multiply* two probabilities to get the probability of A *and* B (i.e. $A \cap B$).
However, there are some exceptions.

Adding. This comes from the property of probability that $P(A \cup B) = P(A) + P(B)$ when A and B are disjoint. So we have to make sure that A and B are disjoint. If they are not disjoint, we have to subtract the probability that both A and B happen, as in the property of probability that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example 4. If we roll a fair die once. The probability that we get number 1 or number 4 is $\frac{1}{6} + \frac{1}{6} = \frac{2}{6}$. Here the events "getting 1" and "getting 4" are disjoint (as we cannot get 1 and 4 at the same time), so we can simply add the probabilities.

However, if roll a fair die twice. The probability that we get number 1 on the first roll or number 4 on the second roll is $\frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$. Here the two events are not disjoint, as we can possibly get 1 on the first roll and also 4 on the second roll with probability $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. (Notice that we get $\frac{1}{36}$ by multiplying probabilities because we want both 1 on the first roll *and* 4 on the second roll.) ■

One technique in finding probability that is frequently used is to divide the event into *disjoint* cases, find the probability of each case, and then add up the probabilities. See Example 2, where we divided A into three cases to find $P(A)$.

Multiplying. Some people tend to remember that $P(A \cap B) = P(A)P(B)$ but forget that this is true if A and B are independent. If they are not independent, we cannot use this equation. Instead, we have to use another equation, which comes from the definition of $P(A|B)$:

$$P(A \cap B) = P(B)P(A|B)$$
$$P(\text{two events happen}) = P(\text{event 1})P(\text{event 2 given event 1}).$$

We are still multiplying probabilities, but the probability is sometimes conditional.

Example 5. From a standard deck of 52 cards, draw two random cards, one at a time without replacement. Find the probability that both cards are diamonds. (There are 13 cards of diamonds in the deck.)

Solution. Here the two events are "first card is ♦" and "second card is ♦". We can see that the first event affects the second; when we draw the second card, the number of cards of diamonds remaining is either 12 or 13, depending on whether or not the first card we drew was a diamond. So we have to use the general equation:

$$P(\text{both cards are } \blacklozenge) = P(\text{1st card } \blacklozenge)P(\text{2nd card } \blacklozenge \mid \text{1st card } \blacklozenge) = \frac{13}{52} \cdot \frac{12}{51}. \quad \blacksquare$$

Remark. If we replace the card in the deck every time we draw a card, then the probability that both cards are diamonds is $\frac{13}{52} \cdot \frac{13}{52}$, as the two cards are now independent.

Warning. Sometimes the equation $P(A \cap B) = P(B)P(A|B)$ does not help in finding $P(A \cap B)$ because $P(A|B)$ is not straigtforward to find. In this case we have to interpret $A \cap B$ as one event and try to compute the probability in another way. In Example 1, we did not use the equation $P(A \cap B) = P(B)P(A|B)$ to find $P(A \cap B)$, and we can see that $P(A|B)$ was not easy to find. (In fact, we computed $P(A \cap B)$ in order to find $P(A|B)$.)

Other Notes

Maximum Likelihood Estimate. Estimate an unknown by finding the value of that unknown that maximizes the probability of what we observe.

Be Careful. "Independent" and "disjoint" have different meanings. Two events are disjoint when they do not have any outcome in common. Some two events are independent but not disjoint, and some two events are disjoint but not independent.

Exercises

1. Redo Example 1, but instead of using a fair coin, use a biased coin that shows heads 75% of the time.

2. Roll four fair dice.

   (a) Find the probability that exactly two dice show number 1.
   (b) Find the probability that at least one die shows number 1.
   (c) Find the probability that numbers 1, 2, 3 and 4 are shown (in any order).
   (d) Find the probability that the four dice show four different numbers.
   (e) Given that the dice show four different numbers, find the probability that number 1 is one of them.

3. From a standard deck of 52 cards, draw two random cards, one at a time without replacement. What is the probability that the second card is a diamond? (Hint: Consider two cases, depending on the first card.)

4. Let's play the following game. You roll a fair die once. If the die shows 1, 2 or 3, you will toss a fair coin once. If the die shows 4 or 5, you will toss a fair coin twice. And if the die shows 6, you will toss a fair coin three times. You win if a coin shows heads at least once.

   (a) Given that you rolled a 5, find the probability that you win.
   (b) Find the probability that you win. (Hint: Consider all scenarios in which you win.)
   (c) Your friend played this game and you find out later that he won, but you do not know how the game went. Find the probablity that your friend got number 5 in the first step.