Logistic Regression and the ROC curve

Schwartz

September 30, 2017

Odd, even at best

In 2015 Leicester City was given 5000 to 1 odds to win the English Premier League. Actually, these are the longest odds ***ever seen*** for ***any*** top tier sporting league... ***ever***. To put this in perspective, the current odds out of Vegas for “the most unlikely team to win the 2016/2017 NFL season” – woefully disastrous Cleveland* Browns – are 200 to 1.

Since the clubs inception in 1890, Leicester City has only managed to appear in the Premier league 10 seasons. They had only been promoted the previous season and just barely escaped relegation in their final match that season. Only five teams – Arsenal, Chelsea, Liverpool, Man. City, and Man. U. – have held the trophy for the past 21 seasons.

Only a few stout souls put money down on Leicester City last year. And when Leicester City (*literally against all odds*) won the premiership last season in absolutely stunning, unbelievable, and unprecedented fashion, those stout souls got paid. Everyone, that is, except for John Micklethwait. John M has made the same bet – 20 pounds ($29) that Leicester will win their division – every August for the past 20 years. Every year, that is, except this one. Last year he moved from London to New York and missed placing his bet. That’s a pity for John M because if he had made his bet he would have won 100,000 pounds, or $145,355.

Overall, $3,000 was bet on Leicester City last season. The *unprecedented* $15,000,000 payout nearly bankrupted the bookmakers. John M got $0.

*Cleveland’s 52-year championship drought ended with the 2015/16 NBA season

Odds

$$\text{Odds} = \frac{p}{1-p} \implies \text{p} = \frac{Odds}{1+Odds} = \frac{1}{1+Odds^{-1}}$$

$$1-p = \frac{1}{1+Odds}$$

Objectives

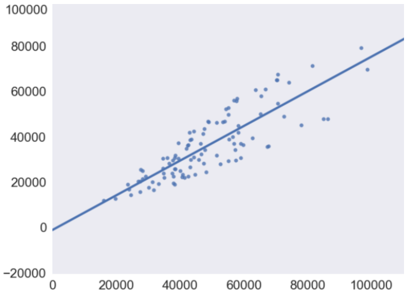
Morning

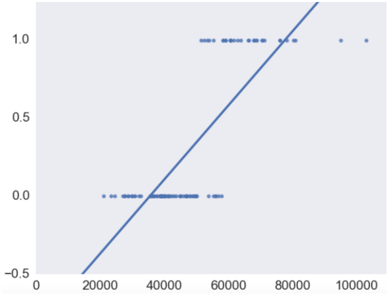
- Know why logistic regression is a thing:
  - Classification vs. Regression
  - Link functions
- Interpreting Logistic Regression
  - Fitted Values (probabilities)
  - Coefficients (log odds ratios)

Afternoon

- T+, T-, F+, F- and other terminology
  - Confusion Matricies
- Thresholding Classification rules
  - ROC curves

Linear Regression





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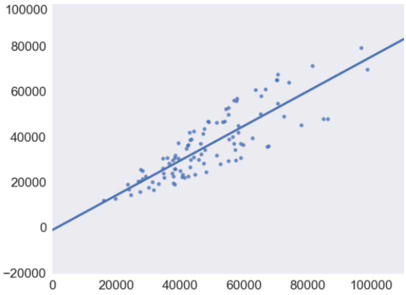

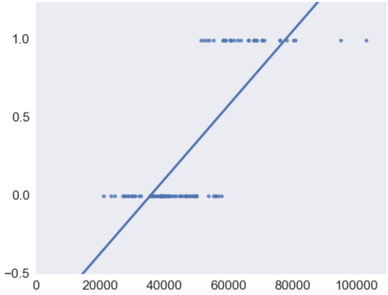

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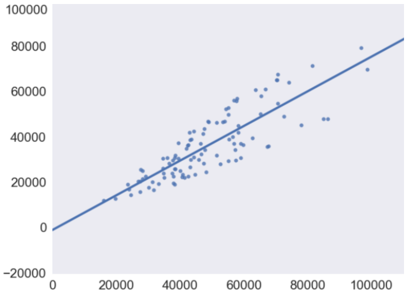

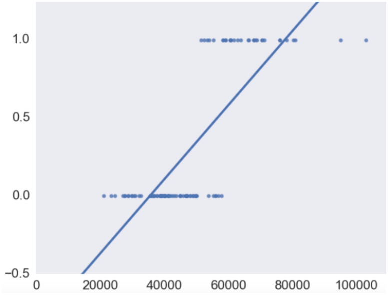

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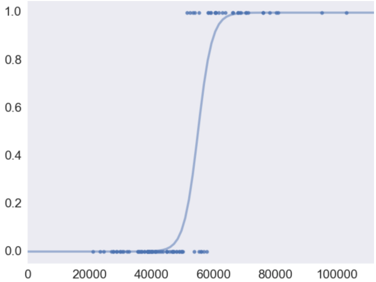
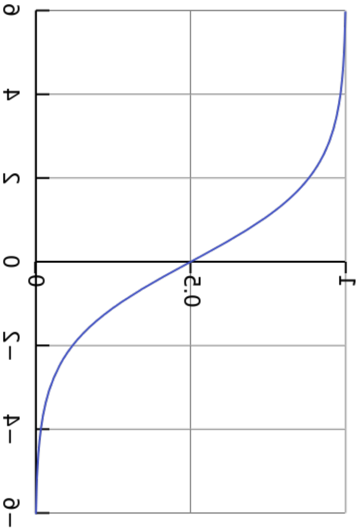
How about this instead $\Longrightarrow$



Link functions

- The "logit"

$$g(p) = \log\left(\frac{p}{1-p}\right)$$



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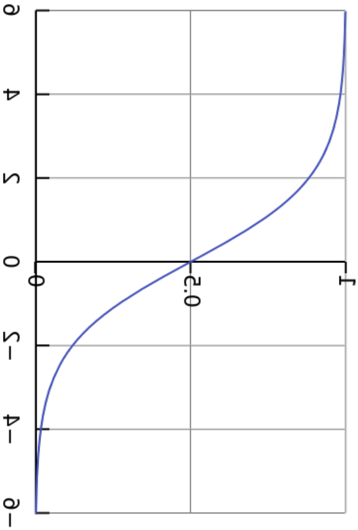
$$p \in [0,1] \mapsto Z \in \mathbb{R}$$



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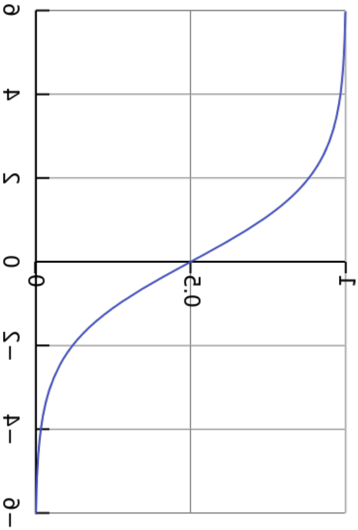

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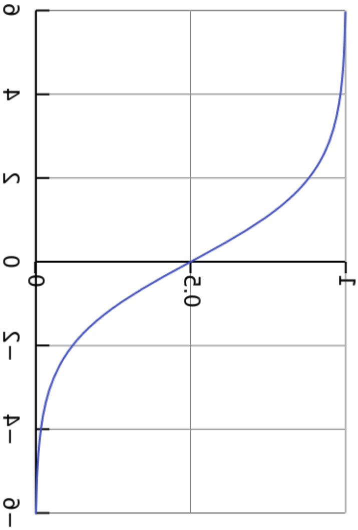
Don't be at odds with odds!



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$$\hat{Y}_i = \Pr(Y = 1) = \mathrm{E}[Y] = g^{-1}(Z)$$

because how else can Z stay between 0 and 1??

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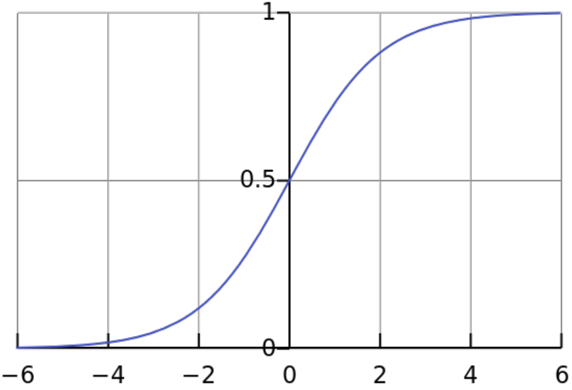
So $g(p) = Z = \log\left(\frac{p}{1-p}\right) \in \mathbb{R}$ (which is called the logit function)

and $\underline{Z = \beta_0 + \beta_1 x_1 + \cdots + \beta_m x_m \in \mathbb{R} \text{ models the log odds}}$

Linear model on log odds $\Longrightarrow$ transformed to probabilities

Standard logistic (sigmoid) function



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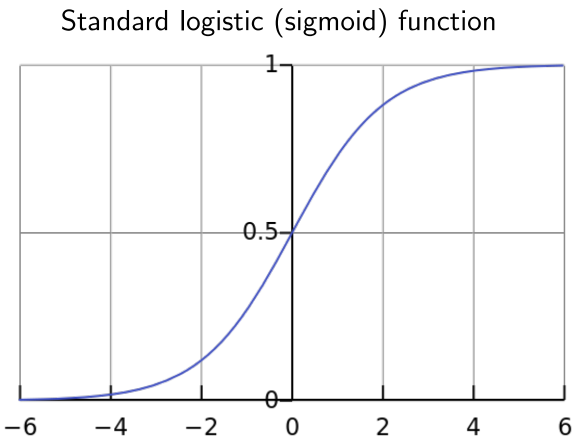

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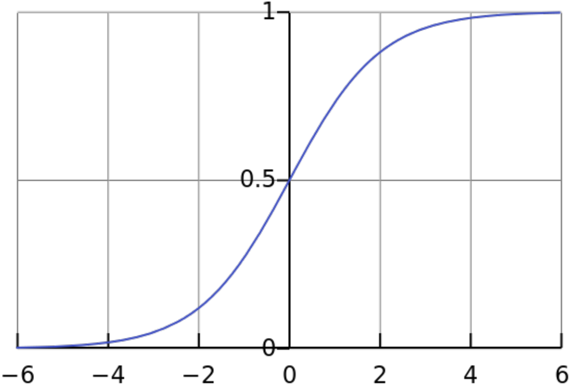

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- I.e., *the odds* are linear in x on a multiplicative, i.e., odds increase with x on a *logorithmic* scale with base $\exp(\beta_j)$
- The *log* odds $log\left(\frac{Pr(Y=1|x)}{Pr(Y=0|x)}\right)$ are on a linear scale
  $(\beta_0 + \beta_1 x_1 + \cdots + \beta_n x_m)$

The Odds Ratio (OR)

- Equivalently, $\exp(\beta_j)$ is the *odds ratio (OR)* between 1-unit differences in x_j (e.g., 0 versus 1) when other x's are constant

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since

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- So β_j is the change in $log(OR)$ for one unit changes in x_j...

Logistic Regression *Likelihood* and *Deviance*

- Likelihood

$$f(\mathbf{Y}|\beta, \mathbf{x}) = \prod \left(\frac{1}{1 + e^{-\mathbf{x}_i^T \boldsymbol{\beta}}} \right)^{Y_i} \left(\frac{1}{1 + e^{\mathbf{x}_i^T \boldsymbol{\beta}}} \right)^{1-Y_i}$$

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- Deviance

$$D_M = \; -2 \left(\log f(\mathbf{Y}|\hat{\beta}, \mathbf{x}) - \log f(\mathbf{Y}|\mathbf{Y}) \right)$$
$$\overset{\text{approx.}}{\sim} \; \chi^2_{n-p-1}$$

$$n = \text{sample size}$$
$$p = \text{number of coefficients in model } M$$
$$f(\mathbf{Y}|\mathbf{Y}) = \text{saturated model } (\mathbf{Y} \text{ perfectly predicted})$$

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[what are *residuals*?] [what are *"residuals"* in logistic regression?]

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- MLE

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- What if, for some λ, we choose $\boldsymbol{\beta}$ to minimize

$$-\prod \left(\frac{1}{1+e^{-\mathbf{x}_i^T\boldsymbol{\beta}}}\right)^{Y_i} \left(\frac{1}{1+e^{\mathbf{x}_i^T\boldsymbol{\beta}}}\right)^{1-Y_i} + \lambda||\boldsymbol{\beta}||^2?$$

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  - predicted probabilities are only $\sim$1 or $\sim$0 (separated classes)
  - There is covariate multicollinearity (as with linear regression)
- What if, for some λ, we choose $\boldsymbol{\beta}$ to minimize

$$-\prod \left(\frac{1}{1+e^{-\mathbf{x}_i^T\boldsymbol{\beta}}}\right)^{Y_i} \left(\frac{1}{1+e^{\mathbf{x}_i^T\boldsymbol{\beta}}}\right)^{1-Y_i} + \lambda|\boldsymbol{\beta}|_1?$$

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- `http://www.ats.ucla.edu/stat/mult_pkg/faq/general/Psuedo_RSquareds.htm`

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How else could you compare nested or non-nested models?

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- Balancing observational comparison groups on *propensity scores* $\Pr(T|x)$ which controls bias from group covariate composition differences

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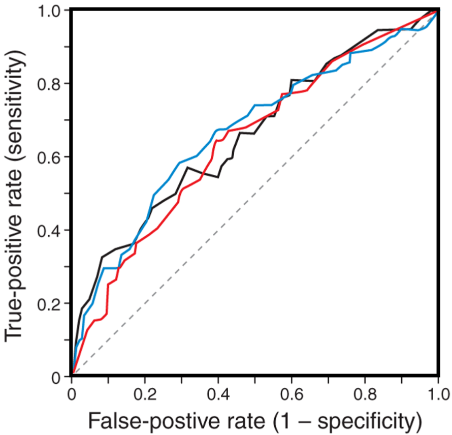
- I.e, Type I & II error rates **are 1-Specificity & 1-Sensitivity**

ROC/AUC



1.0
0.8
0.6
0.4
0.2
0
0
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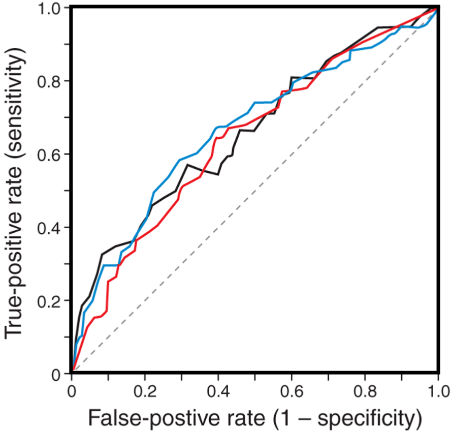
True-positive rate (sensitivity)
False-postive rate (1 – specificity)

ROC/AUC

$Power = 1 - \beta$



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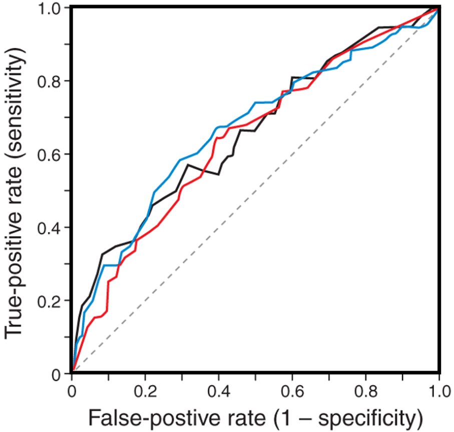

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`https://www.youtube.com/watch?v=JAQC59ArFJw`
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  "Do we call hypotheses **accurately**?"

And just a one or two more...

		Predicted condition			
	Total population	Predicted Condition positive	Predicted Condition negative	Prevalence $= \frac{\Sigma \text{ Condition positive}}{\Sigma \text{ Total population}}$	
True condition	condition positive	**True positive**	**False Negative** (Type II error)	True positive rate (TPR), Sensitivity, Recall $= \frac{\Sigma \text{ True positive}}{\Sigma \text{ Condition positive}}$	False negative rate (FNR), Miss rate $= \frac{\Sigma \text{ False negative}}{\Sigma \text{ Condition positive}}$
	condition negative	**False Positive** (Type I error)	**True negative**	False positive rate (FPR), Fall-out $= \frac{\Sigma \text{ False positive}}{\Sigma \text{ Condition negative}}$	True negative rate (TNR), Specificity (SPC) $= \frac{\Sigma \text{ True negative}}{\Sigma \text{ Condition negative}}$
	Accuracy (ACC) $= \frac{\Sigma \text{ True positive} + \Sigma \text{ True negative}}{\Sigma \text{ Total population}}$	Positive predictive value (PPV), Precision $= \frac{\Sigma \text{ True positive}}{\Sigma \text{ Test outcome positive}}$	False omission rate (FOR) $= \frac{\Sigma \text{ False negative}}{\Sigma \text{ Test outcome negative}}$	Positive likelihood ratio (LR+) $= \frac{\text{TPR}}{\text{FPR}}$	Diagnostic odds ratio (DOR) $= \frac{\text{LR+}}{\text{LR}-}$
		False discovery rate (FDR) $= \frac{\Sigma \text{ False positive}}{\Sigma \text{ Test outcome positive}}$	Negative predictive value (NPV) $= \frac{\Sigma \text{ True negative}}{\Sigma \text{ Test outcome negative}}$	Negative likelihood ratio (LR–) $= \frac{\text{FNR}}{\text{TNR}}$	

Thanks, Wiki!

(You're the besht!)