## Normal Distribution

When $X$ has a normal distribution with mean $\mu$ and standard deviation $\sigma$, we can standardize $X$ as follows:

$$Z = \frac{X - \mu}{\sigma}.$$

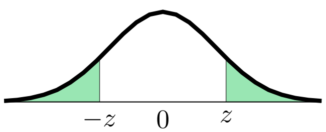
This $Z$, called z-score, has a standard normal distribution (i.e. normal distribution with mean 0 and s.d. 1), of which the values of the distribution function are usually given in a table.

The table of $F(z)$

| $z$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | $\cdots$ | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | $\cdots$ | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | $\cdots$ | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | $\cdots$ | .8997 | .9015 |

For example, $F(1.04) = 0.8508$, which means that $P(Z \leq 1.04) = 0.8508$.

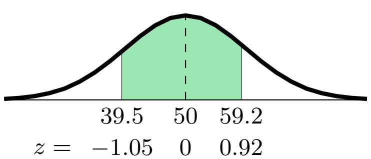
The table of standard normal distribution usually shows the values of $F(z)$ only when $z$ is positive. To get $F(z)$ when $z$ is negative, we can use the symmetry of the normal curve:




$$F(-z) = P(Z \leq -z) = P(Z > z) = 1 - F(z).$$

For example, $F(-1.04) = 1 - F(1.04) = 0.1492$.

**Example 1.** Suppose that $X$ is normally distributed with mean 50 and s.d. 10. Find $P(39.5 < X < 59.2)$.

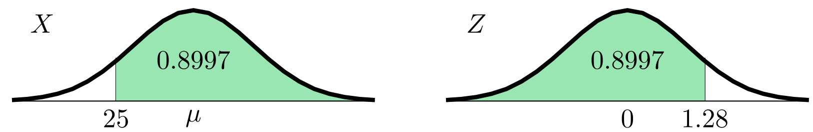



**Solution.**

$$\begin{aligned}
&P(39.5 < X < 59.2)\\
&= P\left(\frac{39.5 - 50}{10} < \frac{X - 50}{10} < \frac{59.2 - 50}{10}\right)\\
&= P(-1.05 < Z < 0.92)\\
&= P(Z < 0.92) - P(Z < -1.05)\\
&= F(0.92) - F(-1.05)\\
&= F(0.92) - (1 - F(1.05))\\
&= 0.8212 - (1 - 0.8531) = 0.6743. \quad \blacksquare
\end{aligned}$$

**Example 2.** Suppose that $X$ is normally distributed with s.d. 10 and $P(X > 25) = 0.8997$. Find $EX$.

**Solution.** Let $\mu$ be the mean of $X$. Note that, since $P(X > 25)$ is greater than $0.5$, $25$ must be less than $\mu$. In particular, the value $25$ has a negative z-score.




Notice from the table that $0.8997 = F(1.28)$, where $F$ is the distribution function for $Z$. So by the symmetry of the normal curve, we have that $25$ corresponds to the z-score $-1.28$. That is, $(25 - \mu)/10 = -1.28$. Thus, $\mu = 37.8$. $\blacksquare$

## Uniform Distribution

$X$ has a uniform distribution over an interval $(a, b)$ if it takes on the values in that interval and its density function is

$$f(x) = \frac{1}{b - a}, \quad a \leq x \leq b.$$

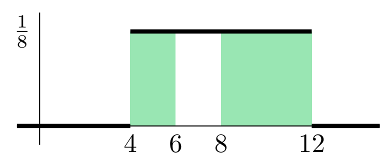
It follows that $EX = \frac{a+b}{2}$ (i.e. the midpoint of the interval) and $\operatorname{var} X = \frac{(b-a)^2}{12}$. Note that the variance naturally depends only on the length of the interval.

The easiest way to compute the probability of $X$ when $X$ is uniform over some interval is by finding the rectangular area under the density curve over that interval.

**Example 3.** $X$ is uniform over the interval $(4, 12)$. Compute $P(|X - 7| > 1)$.

**Solution.** The density function of $X$ is given by $f(x) = \frac{1}{12-4} = \frac{1}{8}$ for $4 \leq x \leq 12$. Then

$$\begin{aligned}
P(|X - 7| > 1) &= P(X < 6 \text{ or } X > 8)\\
&= P(X < 6) + P(X > 8)\\
&= (6 - 4)\tfrac{1}{8} + (12 - 8)\tfrac{1}{8}\\
&= 0.75. \quad \blacksquare
\end{aligned}$$



## Exponential Distribution

$X$ has an exponential distribution if it takes on the non-negative values and its density function has the form

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

The distribution function of $X$ is

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

It follows that

$$P(X > x) = 1 - F(x) = e^{-\lambda x}, \quad x \geq 0.$$

$EX = \frac{1}{\lambda}$ and $\operatorname{var} X = \frac{1}{\lambda^2}$.

**Non-Aging Property.** When $X$ is exponentially distributed, we have that

$$\begin{aligned} P(X > t+h | X > t) &= \frac{P(X > t+h \text{ and } X > t)}{P(X > t)} \\ &= \frac{P(X > t+h)}{P(X > t)} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = P(X > h). \end{aligned}$$

In words, when $X$ models the lifetime of an object, this property states that: given that this object has lasted for at least $t$ units of time, the probability that it will last for at least $h$ more units is equal to the probability that it will last for at least $h$ units of time from the beginning. The fact that it has lasted for $t$ units of time does not have any effect on how longer it will last after that.

**Example 4.** Suppose that the waiting time for the next bus at a bus stop is exponentially distributed with mean 10 minutes. Find the probability that you will have to wait for at least 5 minutes but no longer than 15 minutes until the next bus arrives.

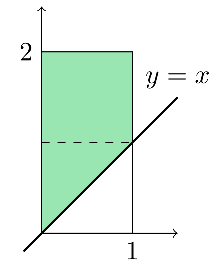
**Solution.** Let $X$ be the waiting time until the next bus. Then $X$ is exponentially distributed with mean $10$. Since $EX = \frac{1}{\lambda}$, we have that $\frac{1}{\lambda} = 10$. That is, $\lambda = 0.1$. Then we want to find $P(5 < X < 15)$.

$$\begin{aligned} P(5 < X < 15) &= P(X > 5) - P(X > 15) \\ &= e^{-(0.1)5} - e^{-(0.1)15} = e^{-0.5} - e^{-1.5}. \quad \blacksquare \end{aligned}$$

## More Examples

**Example 5.** Suppose that $X$ and $Y$ are two independent, uniformly distributed random variables, where $X$ is uniform over the interval $(0, 1)$ and $Y$ is uniform over $(0, 2)$. Find the probability that $Y > X$.

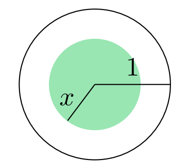
**Solution.** Because $X$ and $Y$ are independent and uniform, the point $(X, Y)$ is distributed uniformly over the 1 x 2 rectangle on the xy-plane as shown. Note that $Y > X$ if the point $(X, Y)$ lies above the line $y = x$. Thus,




$$P(Y > X) = \frac{\text{area of shaded region}}{\text{area of 2 x 1 rectangle}} = \frac{1.5}{2} = 0.75. \ \blacksquare$$

**Example 6.** Suppose that we choose a random point inside a unit circle (i.e. a circle of radius 1). Let $X$ be the distance between the chosen point and the center. Find $EX$ and $\operatorname{var} X$.

**Solution.** Notice first that the range of $X$ is from $0$ to $1$. We start by finding the distribution function. For $0 < x < 1$, we have that $P(X \leq x)$ is the probability that the chosen point is within the circle of radius $x$ with the same center as that of the unit circle. Thus,




$$F(x) = P(X \leq x) = \frac{\text{area of circle of radius } x}{\text{area of unit circle}} = \frac{\pi x^2}{\pi 1^2} = x^2 \quad \text{when } 0 < x < 1.$$

Then we can find the density function by taking the derivative of $F(x)$.

$$f(x) = F'(x) = 2x, \quad 0 < x < 1.$$

Now that we have the density function, we can use it to find $EX$ and $\operatorname{var} X$.

$$\begin{aligned} EX &= \int_0^1 x f(x) dx = \int_0^1 x(2x) dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}. \\ E(X^2) &= \int_0^1 x^2 f(x) dx = \int_0^1 x^2(2x) dx = \left[\frac{2}{4}x^4\right]_0^1 = \frac{1}{2}. \\ \operatorname{var} X &= E(X^2) - (EX)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}. \end{aligned}$$

So the mean and the variance of $X$ are $\frac{2}{3}$ and $\frac{1}{18}$, respectively. $\blacksquare$

**Example 7.** Define $Y = \frac{1}{X+1}$ where $X$ is as defined in Example 6. Find the mean and the variance of $Y$.

**Solution 1.** We can use the density function for $X$ from Example 6.

$$EY = E(\tfrac{1}{X+1}) = \int_0^1 \frac{1}{x+1} f(x)dx = \int_0^1 \frac{2x}{x+1} dx$$
$$= \int_0^2 \left(2 - \frac{2}{x+1}\right) dx = \left[2x - 2\ln(x+1)\right]_0^1 = 2 - 2\ln 2.$$

$$E(Y^2) = E\left(\tfrac{1}{(X+1)^2}\right) = \int_0^1 \frac{1}{(x+1)^2} f(x)dx = \int_0^1 \frac{2x}{(x+1)^2} dx.$$

We substitute $u = x + 1$. (Note the change of the limit of integration.)

$$E(Y^2) = \int_1^2 \frac{2(u-1)}{u^2} du = \int_1^2 \left(\frac{2}{u} - \frac{2}{u^2}\right) du$$
$$= \left[2\ln u + \frac{2}{u}\right]_1^2 = (2\ln 2 + 1) - (2\ln 1 + 2) = 2\ln 2 - 1.$$

Thus, $\operatorname{var} Y = E(Y^2) - (EY)^2 = (2\ln 2 - 1) - (2 - 2\ln 2)^2$. ■

**Solution 2.** We can find the distribution function and the density function for $Y$ first. Note that $Y$ ranges from $\frac{1}{2}$ to $1$, as $X$ ranges from $0$ to $1$. So for $\frac{1}{2} < y < 1$,

$$\begin{aligned} P(Y \le y) &= P(\tfrac{1}{X+1} \le y) \\ &= P(X + 1 \ge \tfrac{1}{y}) \\ &= P(X \ge \tfrac{1}{y} - 1) \\ &= 1 - F_X(\tfrac{1}{y} - 1) \quad \text{where } F_X \text{ denotes the cdf for } X \\ &= 1 - \left(\tfrac{1}{y} - 1\right)^2 \quad \text{by Example 6 (Note that } 0 < \tfrac{1}{y} - 1 < 1.) \\ &= \frac{2}{y} - \frac{1}{y^2}. \end{aligned}$$

Thus the density function for $Y$ is $f_Y(y) = F_Y'(y) = -\dfrac{2}{y^2} + \dfrac{2}{y^3}$ for $\frac{1}{2} < y < 1$. Then

$$EY = \int_{1/2}^1 y f_Y(y)dy = \int_{1/2}^1 y\left(-\frac{2}{y^2} + \frac{2}{y^3}\right) dy = \int_{1/2}^1 \left(-\frac{2}{y} + \frac{2}{y^2}\right) dy$$
$$= \left[-2\ln y - \frac{2}{y}\right]_{1/2}^1 = (-2\ln 1 - 2) - (-2\ln\tfrac{1}{2} - 4) = 2 - 2\ln 2.$$

(Recall that $\ln 1 = 0$ and $\ln\frac{1}{2} = \ln(2^{-1}) = -\ln 2$.)

$$E(Y^2) = \int_{1/2}^1 y^2 f_Y(y)dy = \int_{1/2}^1 y^2\left(-\frac{2}{y^2} + \frac{2}{y^3}\right) dy = \int_{1/2}^1 \left(-2 + \frac{2}{y}\right) dy$$
$$= \left[-2y + 2\ln y\right]_{1/2}^1 = (-2 + 2\ln 1) - (-1 + 2\ln\tfrac{1}{2}) = 2\ln 2 - 1.$$

Therefore, $\operatorname{var} Y = E(Y^2) - (EY)^2 = (2\ln 2 - 1) - (2 - 2\ln 2)^2$. ■

## Exercises

1. Suppose that $X$ has a normal distribution with mean 50 and variance 25. Use the table of standard normal distribution on the first page to find the following:
   (a) $P(X > 45.4)$
   (b) $P(X < 55.7)$
   (c) $P(45.4 < X < 55.7)$
   (d) $P(|X - 50| < 5.4)$
   (e) value $e$ such that $P(X < e) = 0.8888$
   (f) value $f$ such that $P(X > f) = 0.8888$
   (g) value $g$ such that $P(X < g) = 0.1271$
   (h) value $h$ such that $P(X > h) = 0.1271$
2. Suppose that $X$ is uniformly distributed with mean 7 and variance 3. Compute $P(3 < X < 6)$. (*Hint:* Find the interval over which $X$ is distributed first.)
3. The lifetime of a radioactive atom is exponentially distributed with half-life 2 days. This means that the probability that this atom will last longer than 2 days is equal to 0.5.
   (a) Find the average lifetime of this atom.
   (b) Find the probability that this atom will decay within 1.5 day.
4. Let $X$ be defined as in Example 6. Define $Y = e^{2X}$. Compute $EY$ by using the density function for $X$ found in Example 6.
5. Let $X$ and $Y$ be two independent random variables that are both uniformly distributed over the interval $(0, 1)$. Define $Z$ to be the sum of $X$ and $Y$. Then we know that the value of $Z$ is between $0$ and $2$.
   (a) Compute $EZ$ and $\operatorname{var} Z$.
   (b) Compute $P(Z \le 0.4)$, and $P(Z \le z)$ when $0 < z \le 1$ in general. (*Hint:* Draw the line $y = 0.4 - x$.)
   (c) Compute $P(Z \le 1.4)$, and $P(Z \le z)$ when $1 \le z < 2$ in general.