Bayesian Importance Sampling (Task 1)

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General idea of Importance Sampling:
Given target distribution p and proposal distribution q, we aim to estimate

$$E_p[f(x)] = E_q[f(x)\frac{p(x)}{q(x)}]$$

Apply Importance Sampling with:
Target distribution: $p(\theta|x)$, which is a posterior
Proposal distribution: $p(\theta)$, which is a prior

Assume x is given (θ conditioned on x), let $E[g(\theta)]$ be the expectation of a function of θ we want to estimate, we have

$$\begin{aligned} E[g(\theta)] &= \int g(\theta)p(\theta|x)\,d\theta \\ &= \int g(\theta)\frac{p(\theta|x)}{p(\theta)}p(\theta)\,d\theta \qquad \text{(Apply IS)} \end{aligned}$$

where $IW(\theta) = \frac{p(\theta|x)}{p(\theta)}$ is the (unnormalized) importance weights.

Apply Bayesian theorem:

$$p(\theta|x) = \frac{p(\theta)f(x|\theta)}{\int_\theta p(\theta)f(x|\theta)} \propto p(\theta)f(x|\theta)$$

Thus, unnormalized importance weights $IW(\theta) = \frac{p(\theta|x)}{p(\theta)} \propto f(x|\theta)$. – QED

a. The normalized importance weights might be better since normalized $Cf(x|\theta)$ is just $f(x|\theta)$, which is the likelihood.

b. According to [1], a good proposal distribution $p(\theta)$ should satisfy:
1. $IW(\theta) = \frac{p(\theta|x)}{p(\theta)} \propto f(x|\theta)$ is bounded and $p(\theta)$ has heavier tails than $p(\theta|x)$.(seems not a condition for $p(\theta)$)
2. For the estimate to have a low variance, $IW(\theta) = \frac{p(\theta|x)}{p(\theta)}$ should be large only when $g(x)$ is very small.

References

[1] Geof H Givens and Jennifer A Hoeting. *Computational statistics*, volume 703. John Wiley & Sons, 2012.