Multi-Armed Bandit

Schwartz

September 24, 2017

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Objectives

1. Bayesian versus Frequentist A/B testing
2. Bayesian Priors
3. Multi-armed bandit deployment
   - Regret: all about it and having none of it
   - ϵ-greedy, softmax, UCB1, and the Bayesian Bandit

A/B Testing reminder

		
Trial 1		
Trial 2		
Trial 3		
Trial 4		
$\vdots$		
Trial $n_A + n_B$		
Total		

A/B Testing reminder

		
Trial 1	$\emptyset$	
Trial 2		
Trial 3		
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Trial 1	$\emptyset$	
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A/B Testing reminder

		
Trial 1	$\emptyset$	
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Trial 3		✓
Trial 4		
$\vdots$		
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Total		

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Trial 1	$\emptyset$	
Trial 2		$\emptyset$
Trial 3		$\checkmark$
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Total		

A/B Testing reminder

		
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Trial $n_A + n_B$		
Total	$\hat{p}_A = \frac{\sum X_A^{(i)}}{n_A}$	$\hat{p}_B = \frac{\sum X_B^{(j)}}{n_B}$

A/B Testing reminder

$X_A^{(i)} \sim \text{Bern}(\theta_A)$ $X_B^{(j)} \sim \text{Bern}(\theta_B)$		
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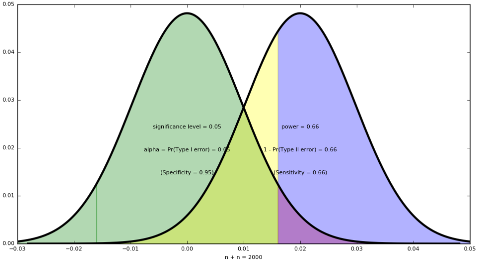
Trial 2		$\emptyset$
Trial 3		$\checkmark$
Trial 4	$\emptyset$	
$\vdots$		
Trial $n_A + n_B$		
Total	$\hat{p}_A = \frac{\sum X_A^{(i)}}{n_A}$	$\hat{p}_B = \frac{\sum X_B^{(j)}}{n_B}$

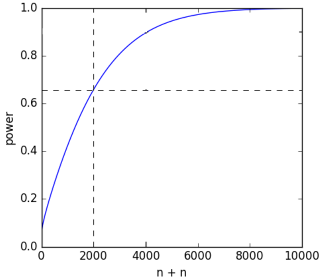
$$\hat{p}_A - \hat{p}_B \sim N\left(\theta_A - \theta_B, \frac{\theta_A(1-\theta_A)}{n_A} + \frac{\theta_B(1-\theta_B)}{n_B}\right)$$

and if $\theta_A = \theta_B = \theta$ then $\hat{p}_A - \hat{p}_B \sim N\left(0, \frac{2\hat{p}(1-\hat{p})}{n_A+n_B}\right)$

A/B **Hypothesis Testing** *power calculations*

- Effect size is $0.06 - 0.04$
- With $2n$ alternating trials
  $\hat{p} = 0.05$
- $se = \sqrt{2\frac{\hat{p}(1-\hat{p})}{n}}$





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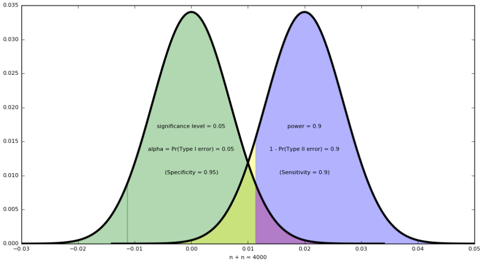

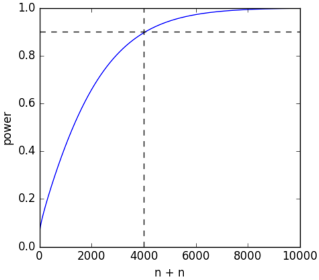

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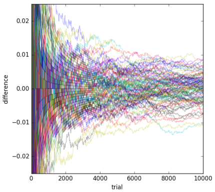

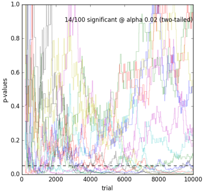

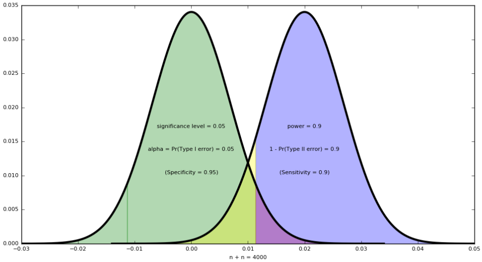

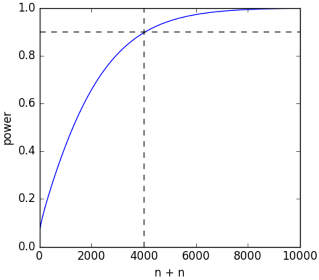

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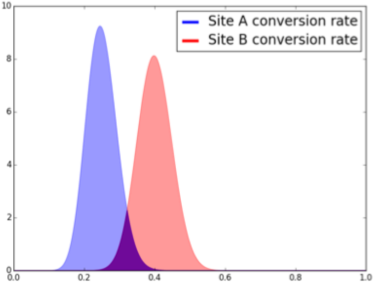

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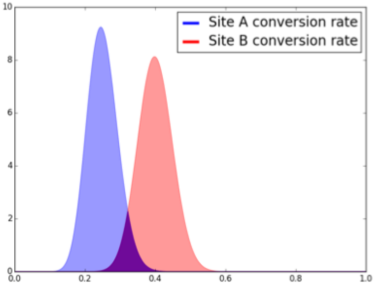

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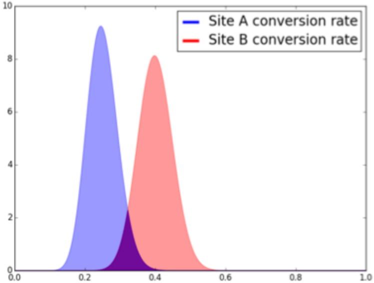

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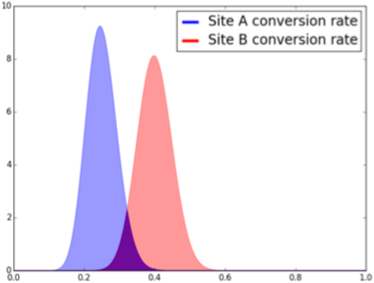

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- Previously, confidence intervals/p-values characterized the (sampling) variability of statistical estimation procedures

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- Data X that informs our knowledge of θ_A and θ_B allows us to "update" our beliefs about θ_A and θ_B using *Bayes' Theorem*

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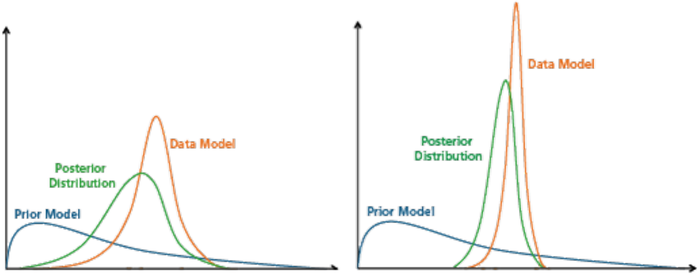
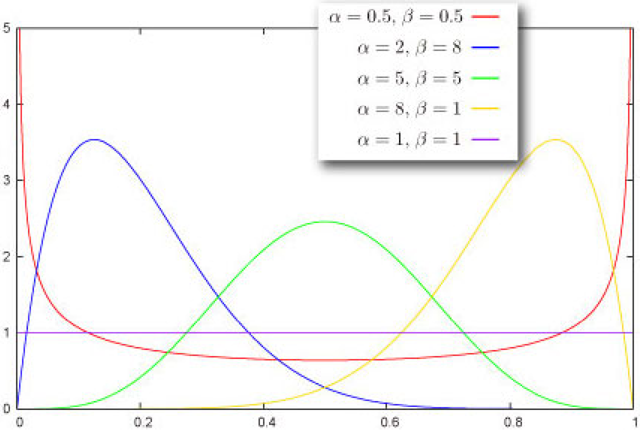


Beta Distribution



$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

Bayesian A/B Testing

- Likelihood

$$
\begin{aligned}
f(X_1, X_2, \cdots, X_n|\theta) &= f(X_1|\theta) f(X_2|\theta) \cdots f(X_n|\theta) \\
&= \prod_{i=1}^{n} \theta^{X_i} (1-\theta)^{1-X_i} \qquad \text{[beta(s)]} \\
&= \theta^{\sum_{i=1}^{n} X_i} (1-\theta)^{n - \sum_{i=1}^{n} X_i}
\end{aligned}
$$

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- Conjugate* Prior

$$f(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)+\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \qquad \text{[beta]}$$

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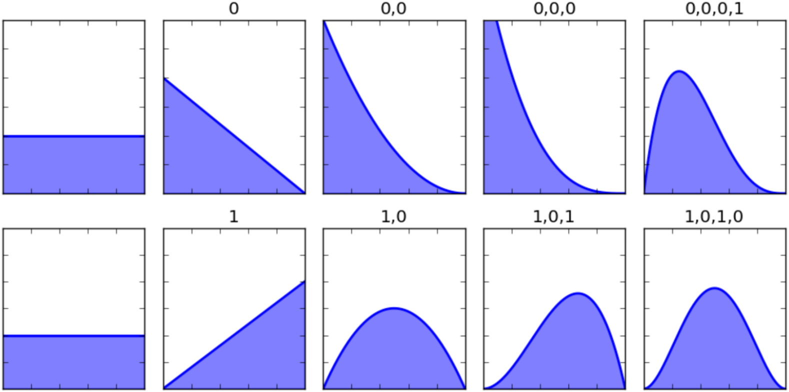
- Posterior

$$f(\theta|X) \propto \theta^{\sum X_i}(1-\theta)^{n-\sum X_i}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$
$$= \theta^{\alpha+\sum X_i-1}(1-\theta)^{\beta+n-\sum X_i-1} \qquad \text{[beta]}$$

Beta Distribution Updates



$$\text{Beta}\left(\alpha + \sum X_i, \beta + n - \sum X_i\right)$$

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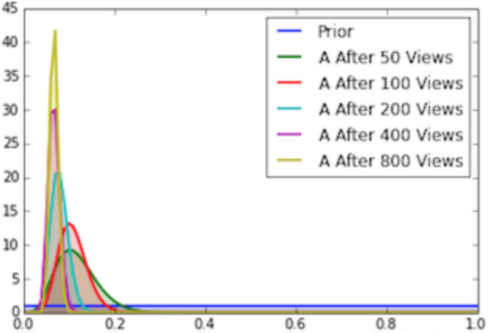

Probability of Success

Rev. Thomas Bayes (1701 – 1761)

$$f(\theta|X) = \frac{f(X|\theta)f(\theta)}{f(X)} = \text{Beta}\left(\alpha + \sum X_i, \beta + n - \sum X_i\right)$$

A brief synopsis of the world of the world to date

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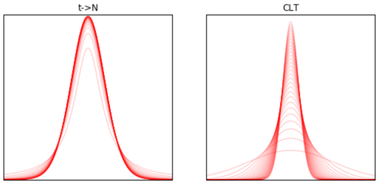

$$n \longrightarrow \infty$$

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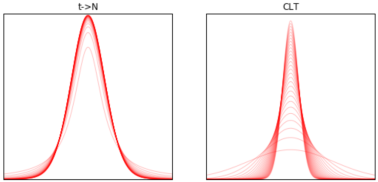

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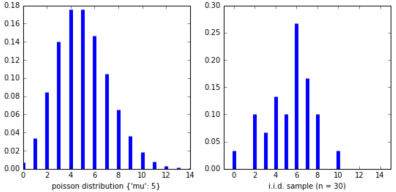

nonparametrics

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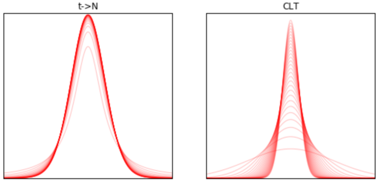

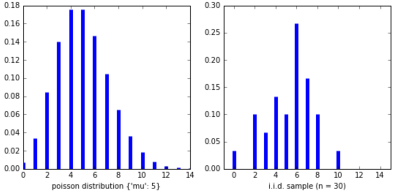

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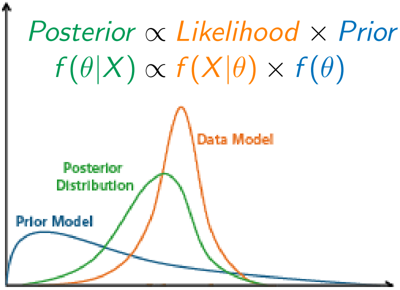
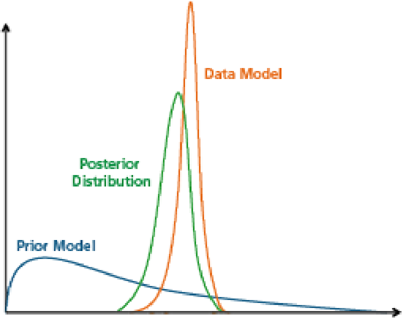
nonparametrics

$$\textit{Posterior} \propto \textit{Likelihood} \times \textit{Prior}$$

$$f(\theta|X) \propto f(X|\theta) \times f(\theta)$$





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   E.g. trees grow in complexity as data becomes richer while a normal distribution is defined by μ and σ^2 regardless of n.

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Truth / Trial	0.01	0.02	0.04	0.03	0.05	0.04	0.02
1							
2							
⋮							

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Exploration & Exploitation

The Multi-Armed Bandit and *Regret*

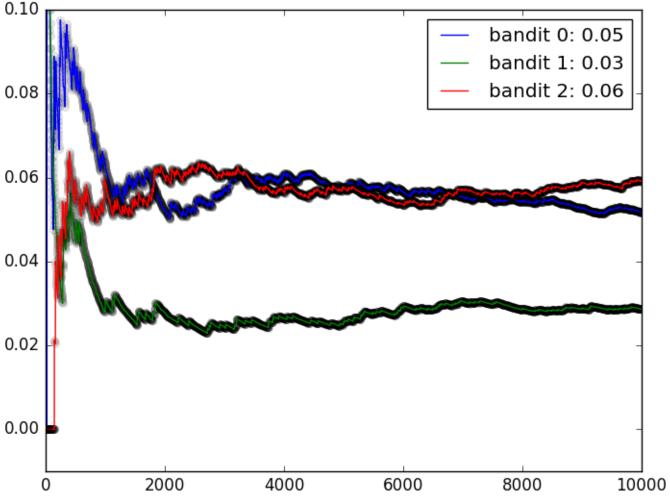
$$\text{regret} = \sum_t^T \max_k \theta_k - \theta^{(t)}$$

$$= T \cdot \max_k \theta_k - \sum_t^T \theta^{(t)}$$

The Multi-Armed *random* Bandit

Random: *A/B[/C/D/E/F/G]-test*

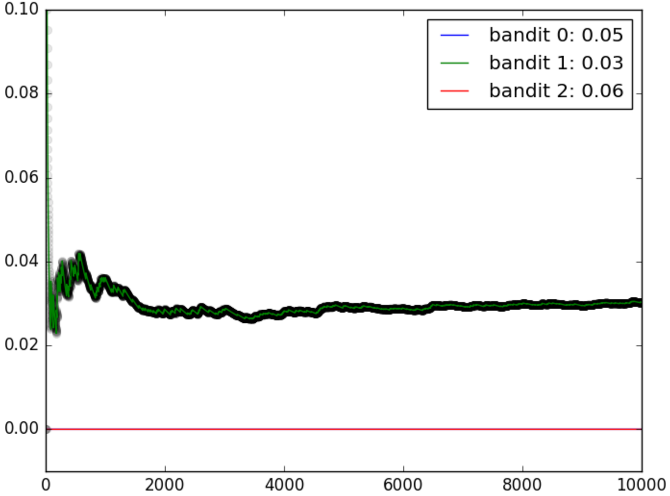
[when should you test?]



The Multi-Armed *max* Bandit

Max: *use the (currently) best performing option*

[perhaps after a burn-in?]

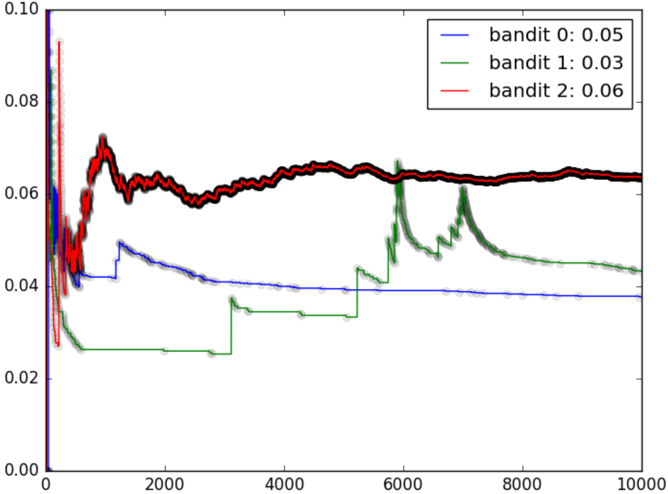


The Multi-Armed *softmax* Bandit

Softmax: *Select proportionally to the currently estimated success rates*

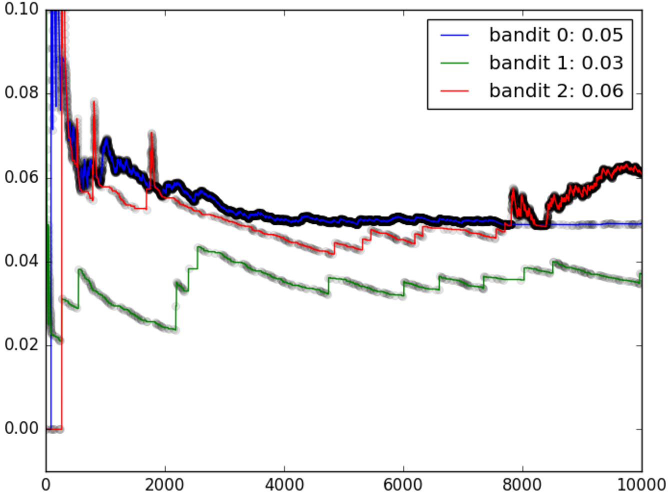
$$\Pr(\text{select } k) \propto \exp(\hat{p}_k/\tau), \text{ i.e.,}$$

$$\Pr(\text{select } k) = \frac{\exp(\hat{p}_k/\tau)}{\sum \exp(\hat{p}_k/\tau)}$$



The Multi-Armed ϵ-*greedy* Bandit

ϵ-greedy: *Select randomly with probability* ϵ
otherwise use the current best option

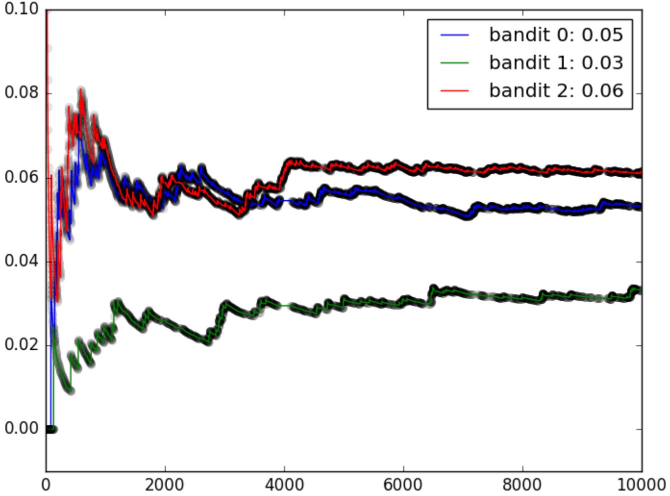


The Multi-Armed *UCB1* Bandit

UCB1: *Select the option which has the highest* **possible** *conversion potential*

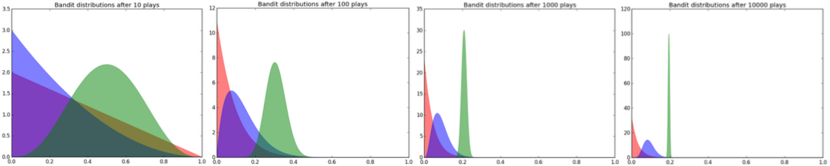
$$\max_k \hat{p}_k + \sqrt{\frac{2 log \sum n_k}{n_k}}$$

with n_k trials for bandit k



The Multi-Armed *Bayesian* Bandit

- *Sample current posteriors $\rightarrow$ Use the best $\rightarrow$ Update that posterior*



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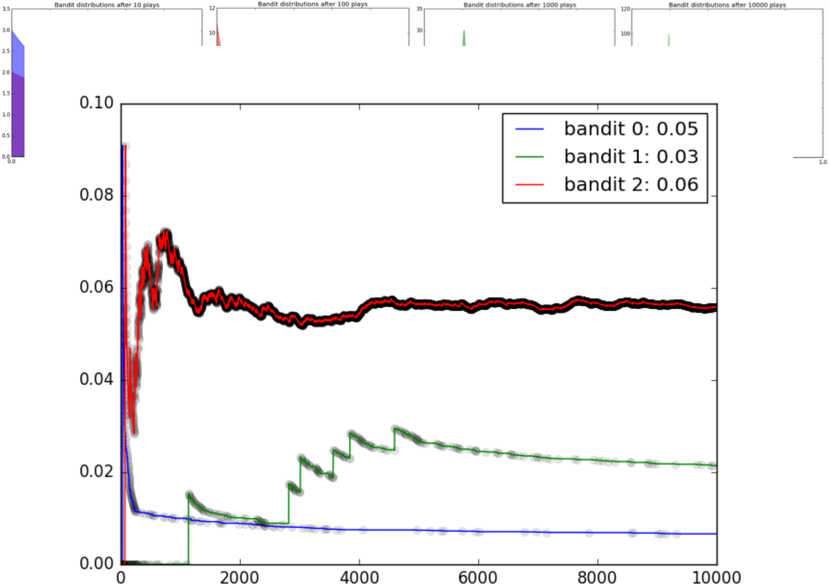

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