Probability

Schwartz

September 20, 2017

Statistics, Data Science, and other Languages

- *acrolect*: dialect considered better than others
- *basilect*: dialect considered lower status than others
- *creole*: mixture of a European language combining one or more other languages spoken as a first language
- *diglossia*: which a language exists in a formal or literary form and an informal form with each used situationally
- *interlanguage*: mixture of two languages used by people learning a new language that uses features of the first language mixed with those of the new language
- *lexicon*: the total stock of elements that carry meaning
- *lingua franca*: language used to communicate by people with different first languages
- metalanguage: language for discussing language
- *pidgin*: language of two or more languages for communication by people with different first languages
- *prose*: ordinary written language, as opposed to poetry
- *register*: language used in a particular situation or for communicating with a particular group of people
- *sublanguage*: a variety of language with its own terms and expressions used by a particular group or to talk about a particular subject, for example, the language used by doctors to talk to each other about medicine
- vernacular: language spoken in a particular group/area, when it is different from the formal written language

- Machine languages: interpreted by hardware
- Assembly languages: machine language wrapper
- System languages: low-level tasks like memory/process management (e.g. C/++, Java)
- High-level languages: machine independent*
- Scripting languages: very high-level, expressive, and powerful (Python, R, JavaScript, Perl)
- Embedded languages: executes inside text (e.g. server and client side JavaScript)
- Compiled languages: verify, tee up, then run*
- Interpreted languages: run as is (e.g., Python, R)
- Declarative languages: what, not how (e.g., SQL)
- Imperative languages: execution of serial orders*
- Functional languages: functions only (e.g., Lisp)
- Iterative languages: generators (e.g., Python)
- Object oriented languages: classes*
- Procedural languages: functions*$^{\text{plus, e.g., SQL}}$
- Visual languages: non-text based
- Esoteric languages: not intended to be used

* C/++, Java, Python, R

Objectives

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- Counting
- Random Variables
  - Marginal, Joint, and Conditional distributions
- Distributions
  - Representations: pmf/pdf/mgf/characteristic functions
  - Examples: Bernoulli, Binomial, Geometric, Multinomial, Poisson Uniform, Normal, χ^2, Gamma, Exponential, Beta
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- Exposure and comfort with a wide range of sophisticated statistical distribution theory concepts and notations

Why *I* (Scott) think YOU (<you>) should care

If you're

- Pro $\Longrightarrow$ Refresher
- Intermediate $\Longrightarrow$ Solidify
- Noob $\Longrightarrow$ Exposure (and organization for starting points)

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   - street cred that takes you to the next level and looks impressive
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3. the whole inference/prediction methodology cosmology *'thing'*
   - it's part of the general theoretical framework/landscape and all

Counting

a.k.a., *combinatorics* – the discipline of mathematics dedicated to *counting*

- Permutation: How many ways can you *permute* things

- Combination: How many ways can you *combine* things

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  - The number of k-sized subsets of m things ($k < m$) is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

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- If there are m ways event A can happen
- And there are n ways event B can happen... then?

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- *How many ways can* $A \cdot B \cdot C$ *happen?*

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$$\binom{5}{2} \Longrightarrow \Pr(PPSSS) = 1 \Big/ \binom{5}{2} = \frac{2!3!}{5!} = \frac{1}{10}$$

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The world-famous “stars and bars” solution

The way you get good at solving these types of problems is by knowing what the solutions are

Problem 3:

```
'''
The scenario: Donald Trump becomes president and make the python built-in
              'set' class illegal. What do we do?
'''
```

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$$Pr(Hillary_{Wins} | Hillary_{Awesome}) = 0.98$$
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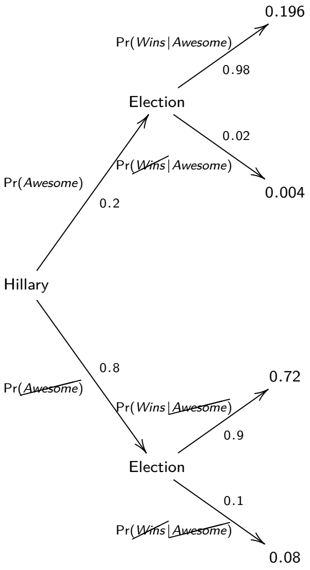

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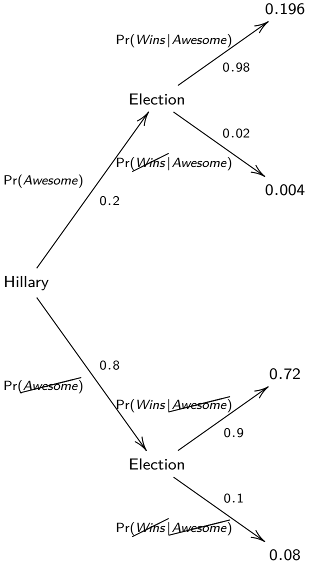

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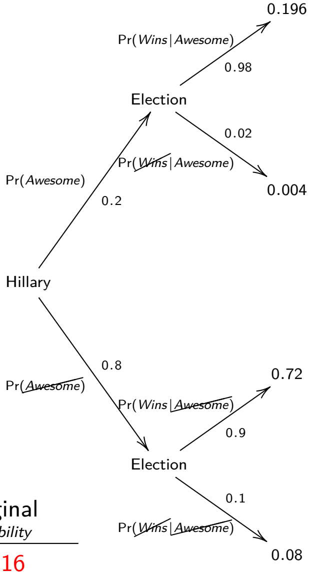

	Awesome Hillary	~~*Awesome*~~ Hillary	Marginal *Probability*
Wins Hillary	0.196	0.720	0.916
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Marginal *Probability*	0.200	0.800	1.000

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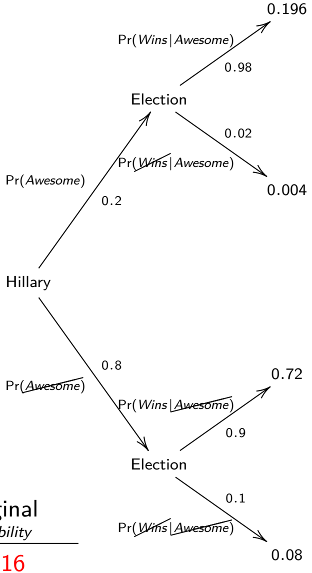

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\begin{aligned}
\mathrm{Pr}(Trump_{Fantastic}) &= 0.20 \\
Pr(DS_{Job} | Trump_{Fantastic}) &= 0.98 \\
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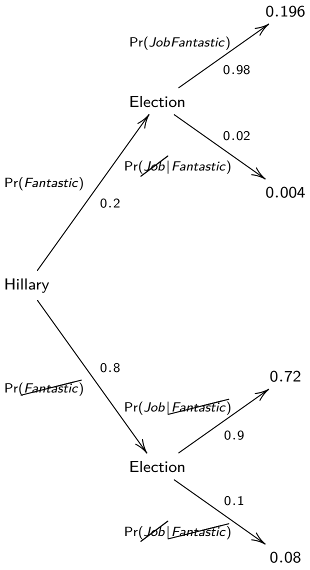

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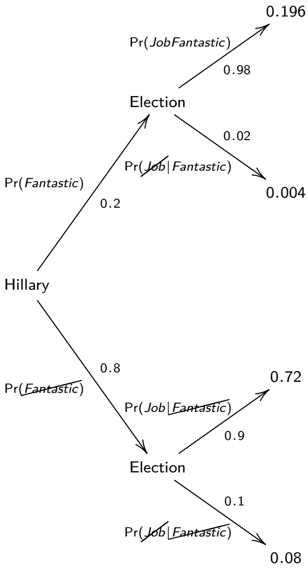

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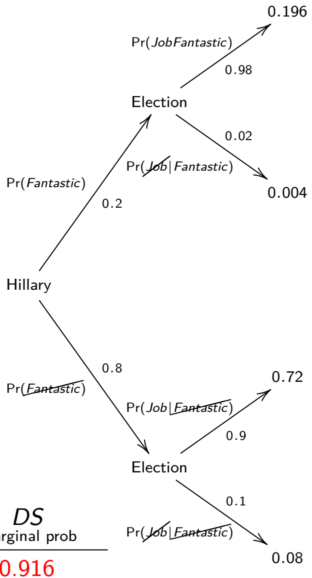

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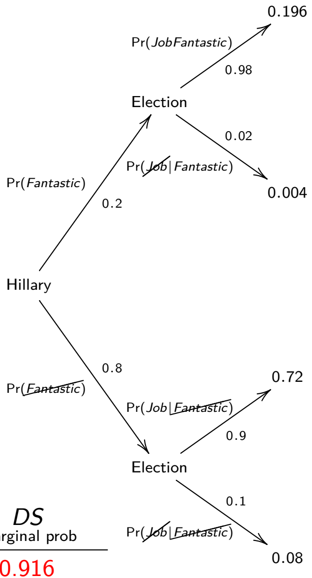

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Problem 4:

Three types of fair coins are in an urn:

HH, HT, and TT

You pull a coin, flip it, and it comes up H

- What is the probability it comes up H if you flip it a second time?

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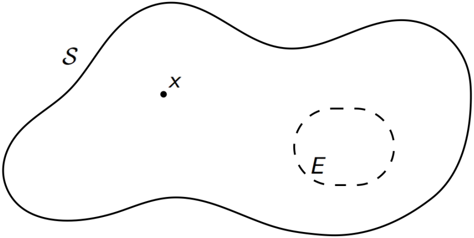

Support space $\mathcal{S}$, event E, and outcome x for random variable X

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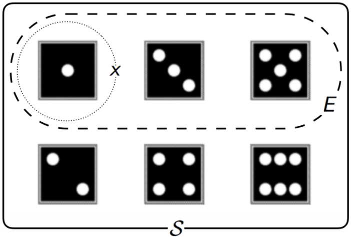
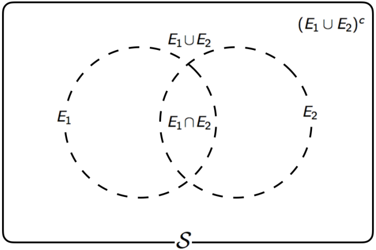
- An event E is a subset of the sample space, i.e., $E \subset \mathcal{S}$



Support space $\mathcal{S}$, event E, and outcome x for random variable X

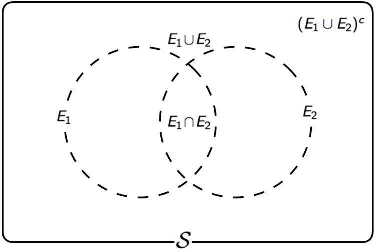
Random Variables: Obvious Rules



Venn Diagram

- $\Pr(E^c) = 1 - \Pr(E)$

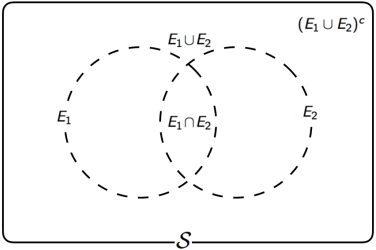
Random Variables: Obvious Rules



Venn Diagram

- $\Pr(E^c) = 1 - \Pr(E)$
- $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$

Random Variables: Obvious Rules



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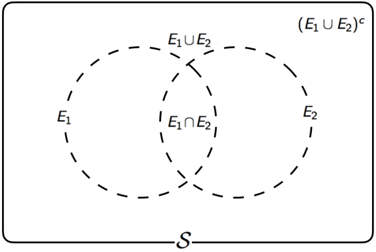

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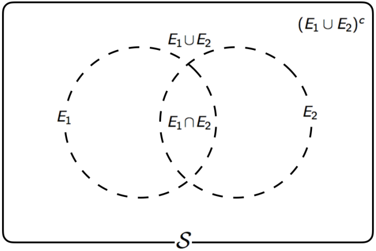
- $\Pr(E \cap E^c) = 0$

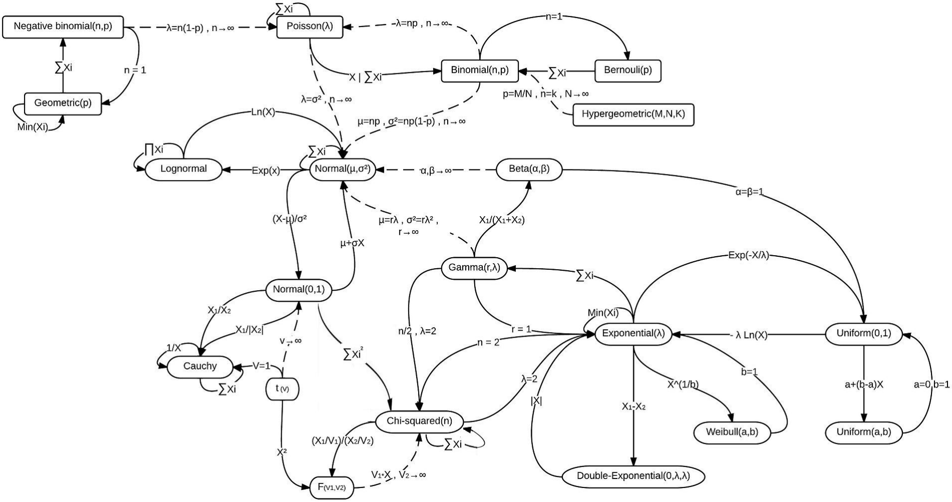
Random Variables: Obvious Rules



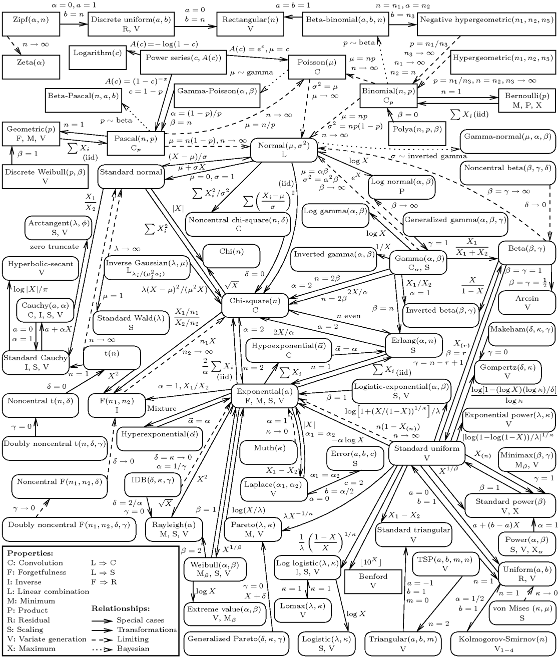
Venn Diagram

- $\Pr(E^c) = 1 - \Pr(E)$
- $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$
- $\Pr(E_1 \cap E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cup E_2)$
- $\Pr(E \cap E^c) = 0$
- DeMorgan's Laws
  - $\Pr\left((E_1 \cup E_2)^c\right) = Pr\left(E_1^c \cap E_2^c\right)$
  - $\Pr\left((E_1 \cap E_2)^c\right) = Pr\left(E_1^c \cup E_2^c\right)$

Distributions



More Distributions



Discrete Distributions: *Bernoulli*

The "coin flip"

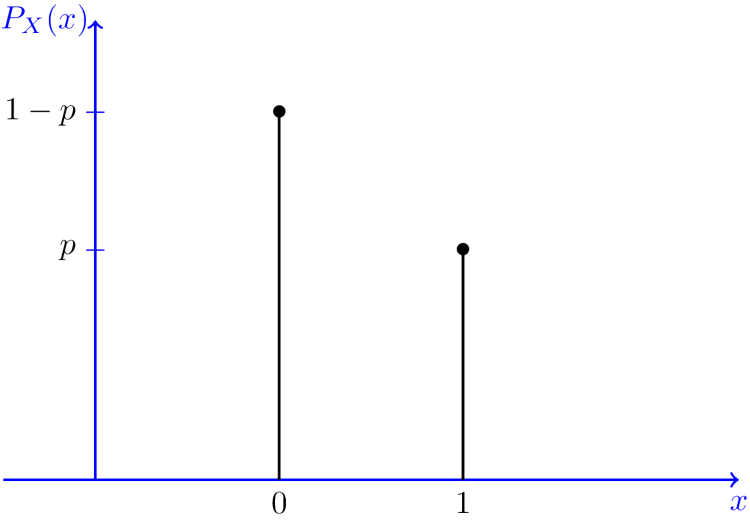
$$x \in \{0, 1\}$$

$$\Pr(X = x|p) = p^x(1-p)^{1-x}$$

$$p \in [0, 1]$$

Discrete Distributions: *Bernoulli*

$$X \sim Bernoulli(p)$$



Discrete Distributions: *Geometric*

The "how many times until"

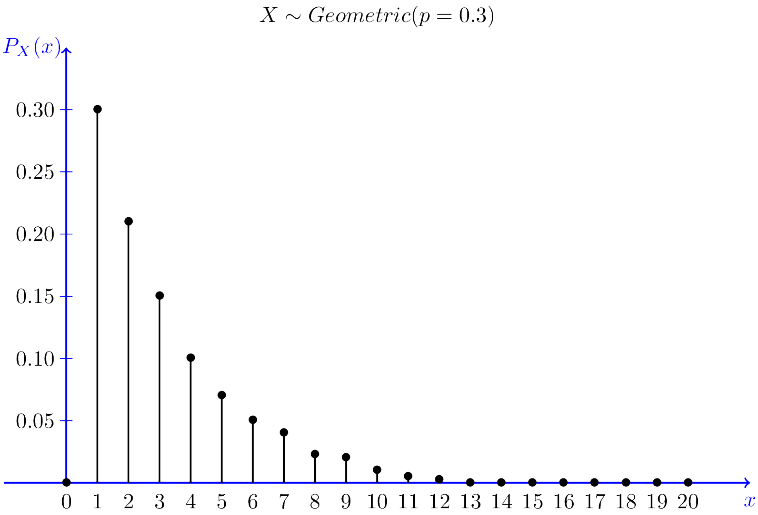
$$x \in \{0, 1, \cdots \infty\}$$

$$\Pr(X = x|p) = (1-p)^{x-1}p$$

$$p \in [0, 1]$$

"If at first you don't succeed, Try, try, try again" – William Edward Hickson

Discrete Distributions: *Geometric*



Discrete Distributions: *Binomial*

The "number of success in *n* trials"

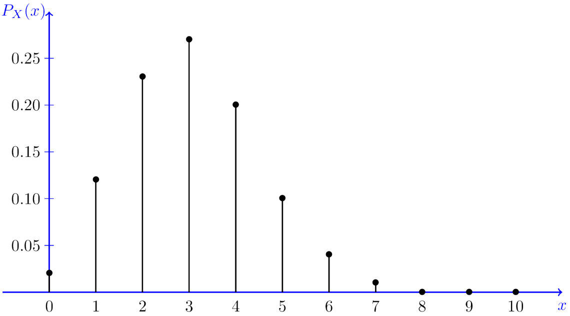
$$x \in \{1, 2, \cdots n\}$$

$$\Pr(X = x|p, n) = \binom{n}{x} p^x (1-p)^{n-x}$$

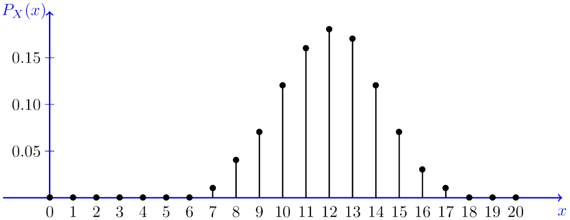
$$p \in [0, 1]$$

Discrete Distributions: *Binomial*

$X \sim Binomial(n = 10, p = 0.3)$



$X \sim Binomial(n = 20, p = 0.6)$



Discrete Distributions: *Multinomial*

The "fancy binomial"

$$\mathbf{x} = (x_1, x_2, \cdots, x_k)$$
$$x_j \in \{0, 1, \cdots m\} \text{ such that } \sum x_j = m$$

$$\Pr(\mathbf{X} = \mathbf{x} | \theta_1, \theta_2, \cdots \theta_k, m) = \frac{m!}{x_1! x_2! \cdots x_k!} \prod_{j=1}^{k} \theta_j^{x_j}$$

$$\theta_j \in [0, 1] \text{ such that } \sum \theta_j = 1$$

Discrete Distributions: *Poisson*

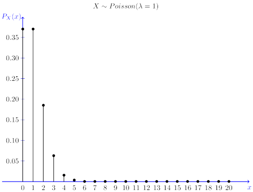
The "number of arrivals"

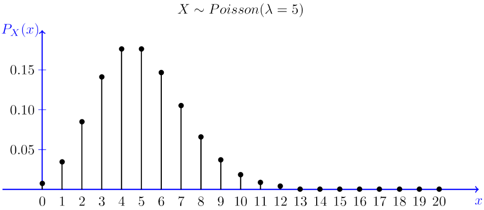
$$k \in \{0, 1, \cdots \infty\}$$

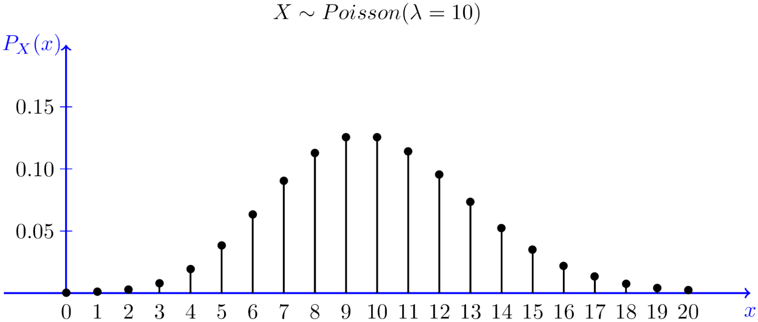
$$\Pr(X = k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\lambda \in \mathbb{R}^+$$

Discrete Distributions: *Poisson*







Discrete Distributions: *Poisson* $\approx$ *Bionomial*

- If $n\theta = \lambda$ then $\theta = \frac{\lambda}{n}$ so that

$$
\begin{aligned}
&= \binom{n}{k} \theta^k (1-\theta)^{n-k} \\
&= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n \cdot (n-1) \cdots (n-k+1)}{n^k k!} \lambda^k \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \\
&\approx \frac{1}{k!} \lambda^k e^{-\lambda} 1 \ (\text{as } n \to \infty) \\
&= \frac{\lambda^k e^{-\lambda}}{k!}
\end{aligned}
$$

Continuous Distributions: *Uniform*

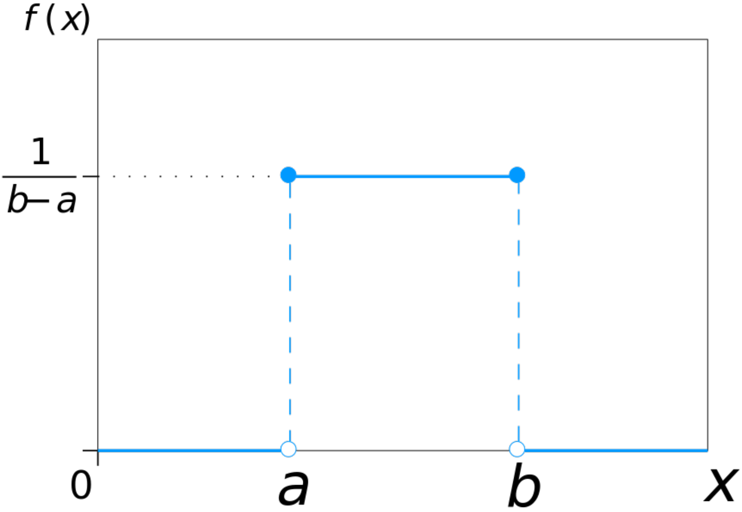
The "random continuous number"

$$u \in \mathbb{R}$$

$$f(X = u|a, b) = \frac{1}{b - a} 1_{[a,b]}(u)$$

$$a, b \in \mathbb{R}, a < b$$

Discrete Distributions: *Uniform*



Continuous Distributions: *Normal*

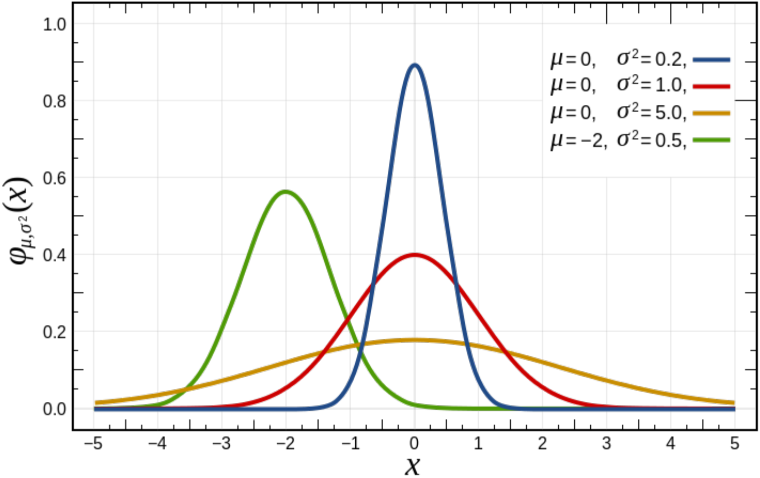
The "bell curve"

$$x \in \mathbb{R}$$

$$f(X = x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}^+$$

Continuous Distributions: *Normal*



Continuous Distributions: *Normal*^{2}

- If $X_j \sim$ *Normal* (μ, σ^2) are normal random variables

Continuous Distributions: *Normal*[2]

- If $X_j \sim \mathit{Normal}\,(\mu, \sigma^2)$ are normal random variables

$$Z_j = \frac{X_i - \mu}{\sigma}$$

Continuous Distributions: *Normal*[2]

- If $X_j \sim \textit{Normal}\left(\mu, \sigma^2\right)$ are normal random variables

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are standard normal random variables

Continuous Distributions: *Normal*[2]

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$$\text{i.e., } Z_j \sim Normal\left(0, 1\right), \text{ and}$$

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  $$\text{i.e., } Z_j \sim Normal\,(0, 1)\,, \text{ and}$$

  $$\sum_{j=1}^{k} Z_i^2 \sim \chi_k^2$$

Continuous Distributions: *Normal*[2]

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  is *chi-squared* random variable with k “degrees of freedom”

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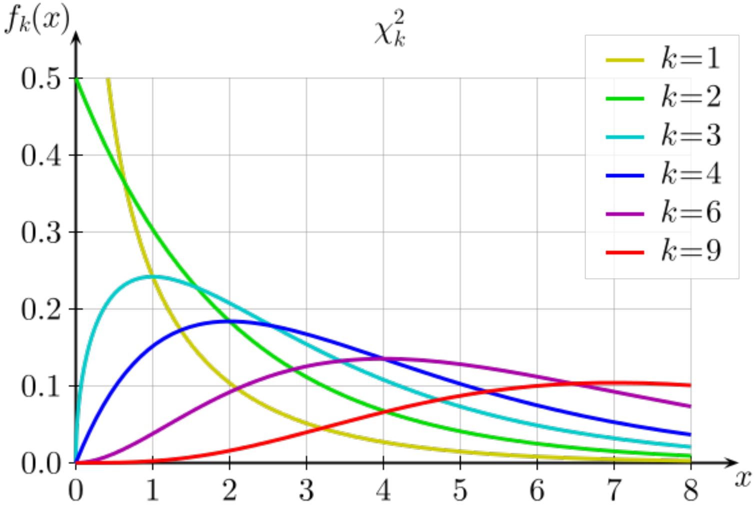
The χ_{df}^2 distribution is a key distribution in hypothesis testing

Continuous Distributions: *Normal*^{2} : χ^2_{df}



Continuous Distributions: *Gamma*

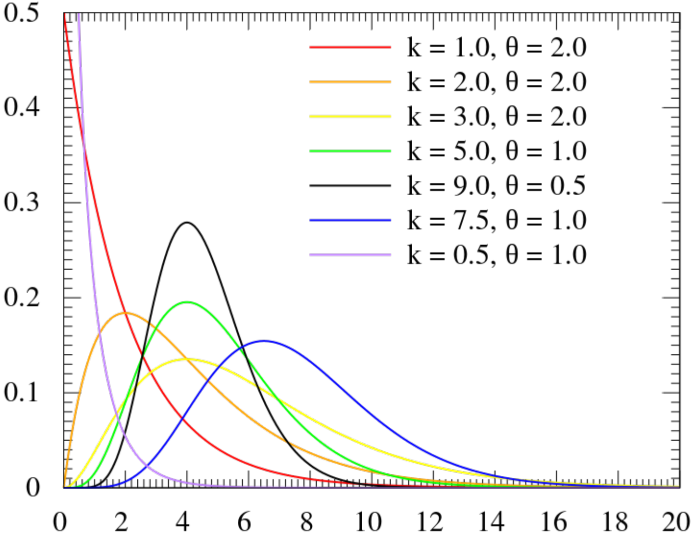
The "Bayesian model for variance"

$$x \in \mathbb{R}^+$$

$$f(X = x|\theta, k) = \frac{\theta^k}{\Gamma(k)} x^{k-1} e^{-x\theta}$$

$$\theta \in \mathbb{R}^+$$

Continuous Distributions: *Gamma*



Continuous Distributions: *Gamma ($\theta = 1/2$, Chi-squared)*

- We previously derived the χ^2_{df} distribution as the "sum of squared standard normal distributions"
- and noted its importance in hypothesis testing

The χ^2_{df} is also a special case of the gamma distribution

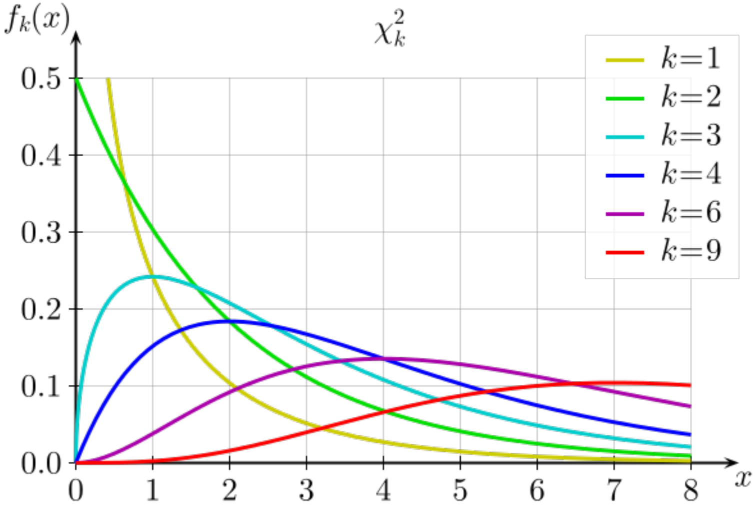
$$x \in \mathbb{R}^+$$

$$f(X = x|k) = \frac{\frac{1}{2}^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$$

$$k \in \mathbb{R}^+$$

Bonus: if $X \sim \chi^2_v$ and $X \sim \chi^2_w$, then $\frac{\frac{1}{v}\chi^2_v}{\frac{1}{w}\chi^2_w} \sim F_{v,w}$

Continuous Distributions: *Gamma* ($\theta = 1/2, \chi^2_{df}$)



Continuous Distributions: *Gamma (k=1, Exponential)*

The *Exponential* is another special case of the gamma

$$x \in \mathbb{R}^{+}$$

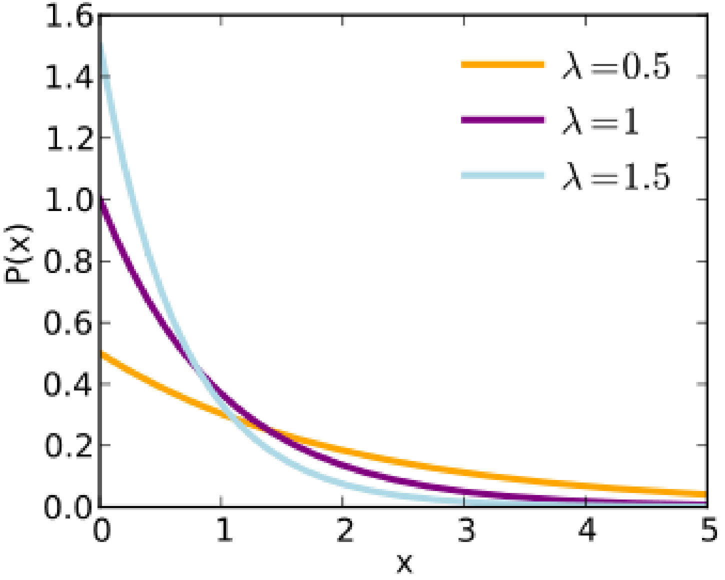
$$f\left(X = x|\theta\right) = \theta e^{-x\theta}$$

$$\theta \in \mathbb{R}^{+}$$

- The Exponential is often used to model time to failure
- It has an interesting "ageless" property, however, in that

  $\Pr(X = x + c|x = 0) = \Pr(X = x + c|x)$ for any value of x

Continuous Distributions: *Gamma (k=1, Exponential)*



Continuous Distributions: *Beta*

The "distribution for modeling random probabilities"

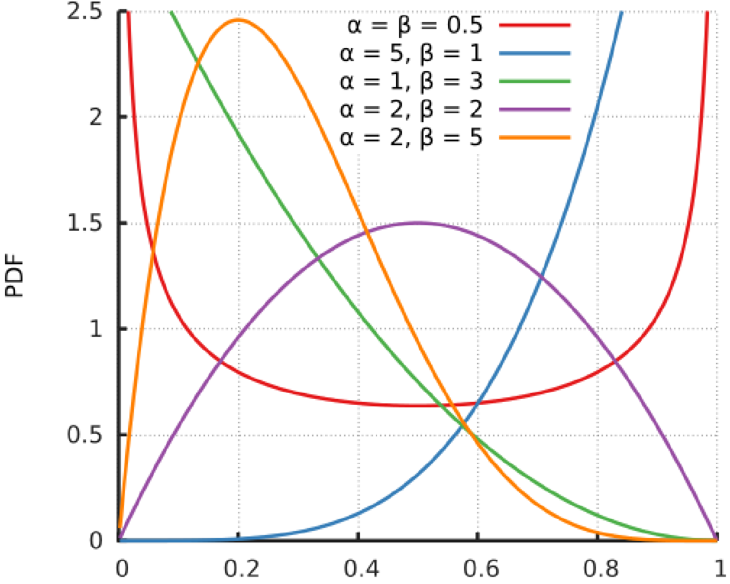
$$p \in [0, 1]$$

$$f(X = p|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1 - p)^{\beta-1}$$

$$\alpha, \beta \in \mathbb{R}^+$$

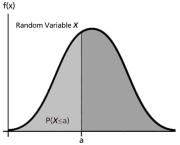
$\alpha = \beta = 1$ results in a *uniform distribution* over the unit interval

Continuous Distributions: *Beta*

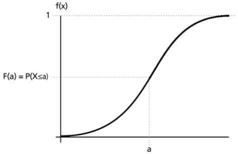


PDF/*PMF*, *CDF*, and characteristic functions

1. We have thus far defined the distribution of a random variable in terms of the *PDF and PMF*



2. We can equivalently alternatively define the distribution of X by it's *cumulative distribution function*



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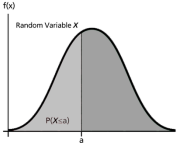

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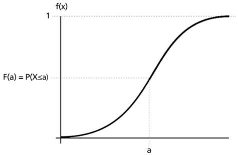

3. Yet another way to define the distribution of X is by its *moment generating function* or its *characteristic function*:

$$\mathrm{E}\left[tX\right], \text{ or } \mathrm{E}\left[itX\right], \text{ respectively}$$

PDF / PMF, *CDF*, and characteristic functions

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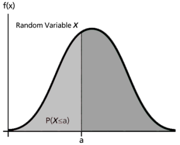

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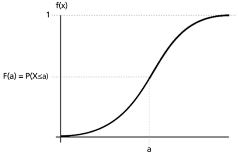

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Interestingly, the characteristic function of $X + Y$ for *independent* random variables X and Y is the product of the characteristic functions of X and Y

Discrete Distributions: *Poisson* + *Poisson*

- The characteristic function of a *Poisson* random variable is

$$e^{\lambda\left(e^{it}-1\right)}$$

Discrete Distributions: *Poisson* + *Poisson*

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- For X and Y Poisson distributed random variables with parameters λ_X and λ_Y, the characteristic function of $X+Y$ is

$$e^{\lambda_X\left(e^{it}-1\right)}e^{\lambda_Y\left(e^{it}-1\right)} = e^{(\lambda_X+\lambda_Y)\left(e^{it}-1\right)}$$

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  So $X+Y$ is Poisson distributed with parameter $\lambda_X+\lambda_Y$

- Quiz: name the distributions of $X+X$ and $Y+Y$ if

$$X \sim \mathit{Bernoulli}(\theta) \text{ and } Y \sim \mathit{Binomial}(\theta, n)$$

  with respective characteristic functions

$$1-\theta+\theta e^{it} \text{ and } \left(1-\theta+\theta e^{it}\right)^n$$

Continuous Distributions: *Normal* + *Normal*

- The characteristic function of a normal random variable is

$$e^{it\mu - \frac{1}{2}t^2\sigma^2}$$

Continuous Distributions: *Normal* + *Normal*

- The characteristic function of a normal random variable is

$$e^{it\mu - \frac{1}{2}t^2\sigma^2}$$

What is the distribution of $X + Y$ if

$$X \sim \mathit{Normal}\left(\mu_X, \sigma_X^2\right) \text{ and } Y \sim \mathit{Normal}\left(\mu_Y, \sigma_Y^2\right)?$$

Continuous Distributions: $X_1 + X_2 + \cdots + X_n \sim Normal$

- The moment generating function (MGF) of a normal random variable is

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The MGF of the sum of n *arbitrary* $i.i.d.$ random variables $\sum_{i=1}^{n} X_i$ is

$$\left(1 + t\mathrm{E}[X] + \frac{t^2\mathrm{E}[X^2]}{2!} + \frac{t^3\mathrm{E}[X^3]}{3!} + \cdots\right)^n$$

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Using $\log\left((1+u)^n\right) = n\left(u - \frac{u^2}{2!} + \frac{u^3}{3!} - \cdots\right)$, we have

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Using $\log\left((1+u)^n\right) = n\left(u - \frac{u^2}{2!} + \frac{u^3}{3!} - \cdots\right)$, we have

$$\exp\left(n\left(t\mathrm{E}[X] + \frac{t^2\mathrm{E}[X^2]}{2} + \cdots - \frac{t^2\mathrm{E}[X]^2}{2} - \cdots + \cdots\right)\right)$$

$$= e^{tn\mathrm{E}[X] + \frac{1}{2}t^2 n(\mathrm{E}[X^2] - \mathrm{E}[X]^2) + n(\cdots)}$$

$$\approx e^{tn\mathrm{E}[X] + \frac{1}{2}t^2 n(\mathrm{E}[X^2] - \mathrm{E}[X]^2)} \text{ as } n \to \infty$$

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$$\exp\left(n\left(t\mathrm{E}[X] + \frac{t^2\mathrm{E}[X^2]}{2} + \cdots - \frac{t^2\mathrm{E}[X]^2}{2} - \cdots + \cdots\right)\right)$$

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$$\approx e^{tn\mathrm{E}[X] + \frac{1}{2}t^2 n(\mathrm{E}[X^2] - \mathrm{E}[X]^2)} \text{ as } n \to \infty \quad \text{so } \sum_{i=1}^{n} X_i \sim N(n\mathrm{E}[X], n(\mathrm{E}[X^2] - \mathrm{E}[X]^2))$$

Continuous Distributions: $X_1 + X_2 + \cdots + X_n \sim Normal$

- The moment generating function (MGF) of a normal random variable is

$$e^{t\mu + \frac{1}{2}t^2\sigma^2}$$

The MGF of the sum of n *arbitrary i.i.d.* random variables $\sum_{i=1}^{n} X_i$ is

$$\left(1 + t\mathrm{E}[X] + \frac{t^2\mathrm{E}[X^2]}{2!} + \frac{t^3\mathrm{E}[X^3]}{3!} + \cdots\right)^n$$

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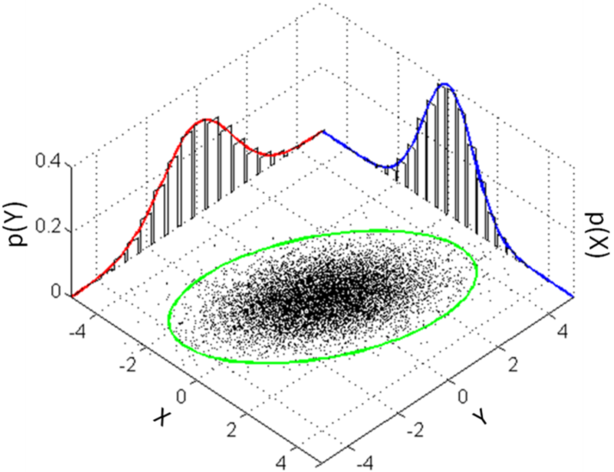
The binomial distribution with large n is approximately normal: why?

Joint Distributions



This multivariate normal joint distribution models the strength of relationship between X and Y with a correlation parameter ρ and it's marginal distributions that are themselves normally distributed

Joint Distributions

- *Errata:* joint distribution **is allowed/encouraged** on campus

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$$\Pr(X_1, X_2, \cdots, X_k)$$

- We actually already saw an example of a *joint distribution*

$$\Pr(\mathbf{X} = \mathbf{x} | \theta_1, \theta_2, \cdots \theta_k, m) = \frac{m!}{x_1! x_2! \dots x_k!} \prod_{j=1}^{k} \theta_j^{x_j}$$

Do these X_j's have some *dependence*?

Joint Distributions: *discrete*

- *Joint distributions* factor as *conditional* $\times$ *marginal* dist.'s

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Joint Distributions: *continuous*

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$$f(\mathbf{X}_1, \mathbf{X}_2) = f(\mathbf{X}_1 | \mathbf{X}_2) f(\mathbf{X}_2)$$

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For Continuous DISTRIBUTIONS

- The *Expected Value* of X $\mathsf{E}_X[X] = \int_{x \in \mathcal{S}_X} x\, f(X = x) dx$
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Statistics are functions of the original random variables *and hence are themselves random variables*

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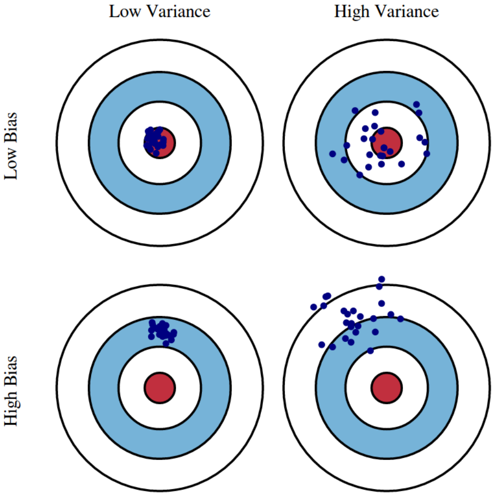
- The *Sample Correlation** $R_{XY} = \frac{S_{XY}}{\sqrt{S_X^2 S_Y^2}} \in [-1, 1]$

*not robust... correlation of ranks?

Why $n-1$?

$$
\begin{aligned}
E\left[\sum_{i=1}^{n}\left(x_i^2-\frac{1}{n}\sum_{j=1}^{n}x_j\right)^2\right] &= E\left[\sum_{i=1}^{n}\left(x_i^2-\frac{2x_i}{n}\sum_{j=1}^{n}x_j+\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)^2\right)\right]\\
&= E\left[\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j+\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j\right]\\
&= E\left[\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{j\neq i}x_ix_j+\frac{1}{n}\sum_{i=1}^{n}x_i^2+\frac{1}{n}\sum_{j\neq i}x_ix_j\right]\\
&= E\left[\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{i=1}^{n}x_i^2+\frac{1}{n}\sum_{i=1}^{n}x_i^2-\frac{2}{n}\sum_{j\neq i}x_ix_j+\frac{1}{n}\sum_{j\neq i}x_ix_j\right]\\
&= E\left[\frac{n-1}{n}\sum_{i=1}^{n}x_i^2-\frac{1}{n}\sum_{j\neq i}x_ix_j\right]=\frac{n-1}{n}\sum_{i=1}^{n}E\left[x_i^2\right]-\frac{1}{n}\sum_{j\neq i}E\left[x_ix_j\right]\\
&= \frac{n-1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2)-\frac{1}{n}\sum_{j\neq i}\mu^2 \quad (why?)\\
&= (n-1)(\sigma^2+\mu^2)-\frac{n^2-n}{n}\mu^2=(n-1)\sigma^2
\end{aligned}
$$

Bias versus Variance of Estimators



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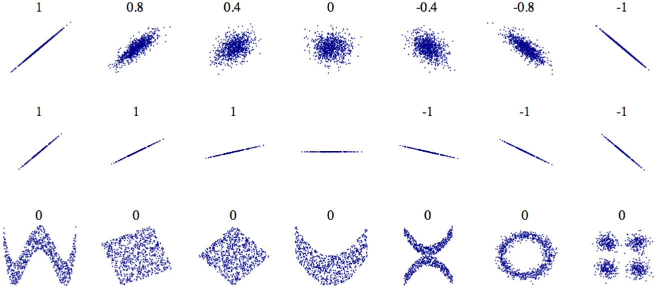

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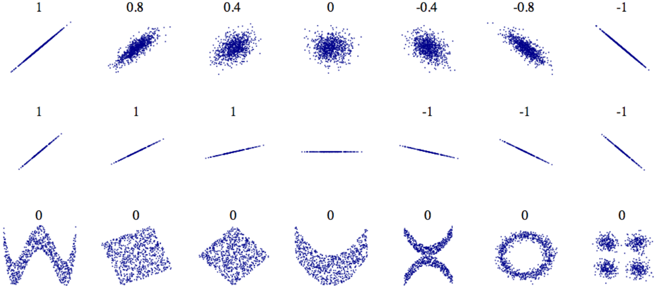

*Mutually exclusive events E_1 and E_2 quite dependent (as opposed to independent)