Notes on Optimziation

Convex and Nonconvex Optimization

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1 Basic Optimization

1.1 Convex sets

Define a set $\mathcal{A}$ to be *affine* when $\forall x_1, x_2 \in \mathcal{A}$ the point $x = \alpha x_1 + (1-\alpha)x_2$, for any $\alpha \in \mathbb{R}$. An easy example of affine set is the set of solutions of $Ax = b$ for $x \in \mathbb{R}^n$, $A \in \mathbb{M}^{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$, since if $x_1, x_2 \in \{x : Ax = b\}$ then for any α it holds $x = \alpha x_1 + (1-\alpha)x_2 \in \{x : Ax = b\}$.

The definition of a set $\mathcal{A}$ being *convex* is as follows $\forall x_1, x_2 \in \mathcal{A}$ the point $x = \alpha x_1 + (1-\alpha)x_2$, for any $\alpha \in [0,1]$. The geometrical meaning of a convex set is a set of points of a line segment between given two given points.

Remark 1.1. *Note that all affine sets are convex as well.*

Remark 1.2. *example of ball, which loses it's convexity*

We call a set a *convex combination* of points $x_1, \ldots, x_k$ the set of all points x:

$$x = \sum_{i=1}^{k} \alpha_i x_i, \quad \alpha_i \geq 0 \, \forall i \in [1,k], \quad \sum_{i=1}^{k} = 1 \tag{1}$$

Remark 1.3. *example of triangle from 3 points*

Define *convex hull* to be a set of all convex combinations of given points. In case a set is convex then it coincides with it's convex hull. We denote the convex hull of set C as $\mathbf{conv}\, C$, which is also known to be the minimal convex set that contains C.

The *convex cone* is a set of points x:

$$x = \theta_1 x_1 + \theta_2 x_2, \quad \theta_1, \theta_2 \geq 0 \tag{2}$$

Exercise 1.1. *Classify the objects using the above definitions:*

- *Hyperplane* $\{x : (a,x) = b\}$. *affine, convex, not a cone.*
- *Half-space* $\{x : (a,x) \leq b\} \equiv \{x : (a, x - x^\circ) \leq 0\}$. *not affine, convex, not a cone.*
- *Euclidean Balls. The ball with radius r and center $x_\circ$ we denote the following set $B_r(x_\circ)$:*

$$B_r(x_\circ) = \{x : \|x - x_\circ\| \leq r\} = \{x : x_\circ + ru, \|u\|_2 \leq 1\} \tag{3}$$

- *Euclidean Ellipsoid. The ellipsoid with center $x_\circ$ and matrix P we denote the following set $E_P(x_\circ)$:*

$$E_P(x_\circ) = \{x : (x - x_\circ)^T P^{-1} (x - x_\circ) \leq 1\} = \{x : x_\circ + Au, \|u\|_2 \leq 1\}, \tag{4}$$

  where P is a symmetric and positive definite matrix. Note that the second

- *Euclidean norm cone (ice-cream cone, light cone):*

$$\mathcal{K} = \{(x,t) : \|x\|_2 \leq t\}, \tag{5}$$

  where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Remark 1.4. *plot of the set*

We call a set $\mathcal{P}$
$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\} \tag{6}$$
a *polyhedron*. Note that the set $\mathcal{P}$ can be empty.

Now, let's dig deeper into the convex sets. The natural question is how to check whether a given set is convex or not?

The possible ways are:

(a) Apply definition.

(b) Derive your set using a simple (convex) set and operations preserving convexity.

(c) Intersection of two or more convex sets.

(d) The image of a convex set is convex as well using an affine function $f : \mathbb{R}^n :\rightarrow \mathbb{R}^m$, $f(x) = Ax + b$.
If $S \subset \mathbb{R}^n$ is convex then
$$f(s) = \{f(x) : x \in S\} \quad \text{is convex as well} \tag{7}$$
If $C \subset \mathbb{R}^m$ is convex then
$$f^{-1}(c) = \{x : f(x) \in C\} \quad \text{is convex as well} \tag{8}$$

(e) Projection.

(f) Perspective function. A function $P : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$
$$P(x,t) = \frac{x}{t} \tag{9}$$
with $\mathbf{dom}\, P = \{(x,t) : t > 0\}$ is called a perspective function.

Note that applying an affine function and a perspective function along with the property (c) of checking a set to be convex one can derive that the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$
$$f(x) = \frac{Ax + b}{(c,x) + d} \tag{10}$$
is convex as well.

1.2 Convex Functions

We call a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *convex* if
$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \tag{11}$$
holds $\forall\, \alpha \in [0,1]$.

Remark 1.5. *picture of illustration of a convex function in* $\mathbb{R}$

Note that the definition is formally given for $x \in \mathbb{R}^n$, but there are some issues with checking whether the function in $\mathbb{R}^n$ are convex or not. For this purpose one can apply so called *line-search*, which states that if $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as follows
$$g(t) = f(x + d \cdot t) \text{ if } x, d \in \mathbb{R}^n, x + td \in \mathbf{dom} f. \tag{12}$$
is convex then then $f(\cdot)$ is convex as well in argument $x \in \mathbb{R}$.

1.2.1 Operations of preserving convexity

Here we list some operations that preserve the convexity of given function

(a) $\sum_{i=1}^{n} \alpha_i f_i, \ \alpha_i \geq 0$.

(b) Composition with affine function.

(c) Point-wise maximum: $f = \max_i f_i(x)$, where f_i's are convex.

Example 1.1. *An interesting example of convex function is*

$$f(x) = \sum_{i=1}^{r} x_{[i]}, \tag{13}$$

where $x_{[i]}$ is i-th order statistics, i.e. $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. To see this take

$$f(x) = \max_i \{x_{1i} + \cdots + x_{ri}\} \quad i = 1, \ldots, C_n^r, \tag{14}$$

hence by the point-wise maximum property the function in (13) *is convex.*

(d) Minimization. The function $g(y)$ is convex if

$$g(y) = \min_{x \in C(y)} f(x), \tag{15}$$

if the initial function $f(x)$ is convex and the set C parametrized on y is convex as well.

(e) Perspective.