Problem Set II. Probability Inequalities

Probability and Beyond
January 25, 2017

Due: 25 January, 11:59 pm

Directions: It is strongly recommended to solve the problems on your own. If you end up consulting outside sources, cite your sources. Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

Problem 2.1. (Bonferroni's inequality)
Prove that

$$\mathbb{P}\left(\cup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) \tag{1}$$

Problem 2.2. Let ξ and η are random variables with zero mean, unit variance and the correlation coefficient ρ. Prove that

$$\mathbb{E} \max\{\xi^2, \eta^2\} \leq 1 + \sqrt{1 - \rho^2} \tag{2}$$

Problem 2.3. Prove that

$$|\text{med } \xi - \mathbb{E}\xi| \leq \sqrt{\mathbb{V}ar\xi} \tag{3}$$

Problem 2.4. Assume that $\mathbb{E}\xi^2 < \infty$, then prove

$$\mathbb{P}(\xi = 0) \leq \frac{\mathbb{V}ar\xi}{\mathbb{E}\xi^2} \tag{4}$$

Problem 2.5. Let $a \leq X \leq b$ a.s. and $\mathbb{E}X = \mu$. Prove that

$$\mathbb{E}e^{\lambda X} \leq e^{\frac{\lambda^2\sigma^2}{2} + \lambda\mu} \quad \text{for all } \lambda \in \mathbb{R} \tag{5}$$

for some σ. Find the value of σ and prove that it is optimal (i.e. the smallest possible).
Problem 2.6. If $X_1, \ldots, X_n$ are independent with $\mathbb{P}(a \leq X_i \leq b) = 1$ and $\mathbb{E}X_i = \mu$. Then, prove that

$$|\overline{X} - \mu| \leq \sqrt{\frac{(b-a)^2}{2n} \log\left(\frac{2}{\delta}\right)} \tag{6}$$

with probability at least $1-\delta$.

Problem 2.7. Exercise 2.1 from lecture notes.

Problem 2.8. Let $X \sim \mathcal{N}(0,1)$. Prove that

$$\mathbb{P}(|X| > \varepsilon) \leq \frac{2e^{-\varepsilon^2/2}}{\varepsilon}. \tag{7}$$

Problem 2.9. Let random variable X satisfies

$$\mathbb{E}e^{\lambda X} \leq e^{\frac{\sigma^2\lambda^2}{2}+\lambda\mu} \tag{8}$$

for some $\sigma > 0$ and $\mu \in \mathbb{R}$. Show that

- $\mathbb{E}\, X = \mu$.
- $\mathbb{V}ar\, X \leq \sigma^2$.

Problem 2.10. Recall that we call a random variable X with $\mathbb{E}X = \mu$ sub-Gaussian if, there exists $\sigma > 0$ such that

$$\mathbb{E}e^{\lambda(X-\mu)} \leq e^{\frac{\sigma^2\lambda^2}{2}} \quad \text{for all } \lambda \in \mathbb{R} \tag{9}$$

a) Show that X is sub-Gaussian, iff $-X$ is sub-Gaussian.

b) Prove that random variables X with symmetric Bernoulli distribution (Rademacher distribution) is sub-Gaussian. Find σ.

c) If X is sub-Gaussian with $\mathbb{E}X = \mu$ then prove that

$$\mathbb{P}(|X-\mu| > t) \leq 2e^{\frac{-t^2}{2\sigma^2}} \tag{10}$$

d) Let X be a sub-Gaussian random variables with zero mean. Prove that for any $\alpha > 0$

$$\mathbb{E}|X|^\alpha \leq \alpha 2^{\alpha/2}\sigma^\alpha\Gamma(\alpha/2) \tag{11}$$

Problem 2.11. Let centered random variable X be a sub-Exponential with parametesr (σ^2, b), then prove that

$$\mathbb{P}(|X-\mu| > t) \leq \begin{cases} 2e^{-\frac{t^2}{2\sigma^2}}, t \in [0, \frac{\sigma^2}{b}) \\ 2e^{-\frac{t}{2b}}, t \geq \frac{\sigma^2}{b} \end{cases} \tag{12}$$