Problem Set I. Basic Probability

Probability and Beyond
January 23, 2017

Due: January 23, 11:59 pm

Directions: It is strongly recommended to solve the problems on your own. If you end up consulting outside sources, cite your sources. Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

Problem 1.1. Let X is a discrete random variable with the following distribution law (probability mass function)

$$\mathbb{P}(X = n) = \frac{e^{-\mu n}(\mu n)^{n-1}}{n!}, \tag{1}$$

where μ is a parameter and $\mu \in [0, 1]$. Find the expected value of X, $\mathbb{E}X$.

Problem 1.2. Let $U \sim \mathcal{U}[0, 1]$. Let $Y = F^{-1}(U)$ where F is a continuous cdf on the real line. Show that the distribution of Y is F.

Problem 1.3. Find all values of parameters C and m (where appears) such that the following functions are probability density functions (pdf)?

$$a)\ f(x) = \frac{C}{\sqrt{x(1-x)}}, x \in (0, 1),\ \text{this is known as 'arc sin law' pdf}$$

$$b)\ f(x) = C \exp(-x - e^{-x}), x \in \mathbb{R},\ \text{this is known as 'extreme-value distribution' pdf}$$

$$c)\ f(x) = \frac{C}{(1 + x^2)^m}, x \in \mathbb{R}$$

Problem 1.4. Let X and Y be random variables with pdf's $f(x)$ and $g(x)$, respectively. Prove that for all $\alpha \in [0, 1]$ the function $h(x) = \alpha f(x) + (1-\alpha)g(x)$ is a pdf as well. Describe the random variable with pdf of $h(x)$.

Problem 1.5. Find all possible values of α such that $\mathbb{E}|X|^\alpha$ is finite, i.e. $< +\infty$, if the density of random variable X is given by

$$a)\ f(x) = e^{-x}, x \geq 0$$

$$b)\ f(x) = \frac{C}{(1 + x^2)^m}, x \in \mathbb{R}$$

Problem 1.6. Two uncorrelated random variables take only two values each. Can they be dependent? If yes, bring an example, otherwise prove that they are always independent.
Problem 1.7. Let ξ is a random variable. Find the minimum of the following functions of a:

$$a)\,\mathbb{E}(\xi - a)^2 \tag{2}$$

$$b)\,\mathbb{E}|\xi - a| \tag{3}$$

$$c)\,\mathbb{E}\left[q(\xi - a)^+ + (1-q)(\xi - a)^-\right], \tag{4}$$

where $q \in (0, 1)$, $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.
Problem 1.8. A random vector X has a density $p(x)$. Find the pdf and cumulative distribution function (cdf) of $Y = AX$, where A is non-degenerate square matrix.
Problem 1.9. Let $\xi_1, \dots \xi_n$ are non-degenerate independent and identically distributed random variables and $\xi_i \geq 0$ a.s. $\forall i \in [1, n]$. Define

$$\eta_k = \frac{\xi_k}{\xi_1 + \dots \xi_n} \forall\, k \in [1, n] \tag{5}$$

Find $\mathbb{E}\eta_i$ and $\mathrm{corr}(\eta_i, \eta_j), \forall i \neq j$ and $i = 1, \dots, n$.
Problem 1.10. Let $\xi \sim \mathrm{Poiss}(\lambda)$ and $\zeta \sim \mathrm{Poiss}(\mu)$ are independent random variables. Find the distribution of $\xi + \zeta$.
Problem 1.11 Suppose we have the convergence of ξ_n to ξ in p-mean, which means

$$\lim_{n\to\infty} \mathbb{E}|\xi_n - \xi|^p = 0 \tag{6}$$

For which values of q it holds

$$\lim_{n\to\infty} \mathbb{E}|\xi_n - \xi|^q = 0? \tag{7}$$

Note that both $p, q \geq 1$.