2

Characteristics of a stochastic process. Mean and covariance functions. Characteristic functions

Theoretical grounds

In this chapter we consider random functions with the phase space being either real line $\mathbb{R}$ or complex plane $\mathbb{C}$.

Definition 2.1. *Assume that* $\mathsf{E}|X(t)| < +\infty$, $t \in \mathbb{T}$. *Function* $\{a_X(t) = \mathsf{E}X(t), t \in \mathbb{T}\}$ *is called* the mean function *(or simply* the mean*) of the random function* X. *Function* $\tilde{X}(t) = X(t) - a_X(t), t \in \mathbb{T}$ *is called the* centered *(or* compensated*) function, corresponding to function* X.

Recall that *covariance* of two real-valued random variables ξ and η, both having the second moment, is defined as $\operatorname{cov}(\xi,\eta) = \mathsf{E}(\xi - \mathsf{E}\xi)(\eta - \mathsf{E}\eta) = \mathsf{E}\xi\eta - \mathsf{E}\xi\mathsf{E}\eta$. If ξ,η are complex-valued and $\mathsf{E}|\xi|^2 < +\infty, \mathsf{E}|\eta|^2 < +\infty$ then $\operatorname{cov}(\xi,\eta) = \mathsf{E}(\xi - \mathsf{E}\xi)(\overline{\eta - \mathsf{E}\eta}) = \mathsf{E}\xi\overline{\eta} - \mathsf{E}\xi\mathsf{E}\overline{\eta}$ (here "$\overline{}$", the overbar, is a sign of complex conjugation).

Definition 2.2. *Assume that* $\mathsf{E}|X(t)|^2 < +\infty$, $t \in \mathbb{T}$. *Function*

$$R_X(t,s) = \operatorname{cov}(X(t),X(s)), \quad t,s \in \mathbb{T}$$

is called the covariance function *(or simply* the covariance*) of the random function* X. *If* X,Y *are two functions with* $\mathsf{E}|X(t)|^2 < +\infty, \mathsf{E}|Y(t)|^2 < +\infty, t \in \mathbb{T}$*, then* $\{R_{X,Y}(t,s) = \operatorname{cov}(X(t),Y(s)), t,s \in \mathbb{T}\}$ *is called* the mutual covariance function *for the functions* X,Y.

Definition 2.3. *Let* $\mathbb{T}$ *be some set, function* K *be defined on* $\mathbb{T} \times \mathbb{T}$*, and take values in* $\mathbb{C}$. *Function* K *is* nonnegatively defined *if*

$$\sum_{j,k=1}^{m} K(t_j,t_k)c_j\overline{c}_k \geq 0$$

for any $m \in \mathbb{N}$ *and any* $t_1,\ldots,t_m \in \mathbb{T}, c_1,\ldots,c_m \in \mathbb{C}$.

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This definition is equivalent to the following one.

Definition 2.4. *Function $K : \mathbb{T}\times\mathbb{T}\to\mathbb{C}$ is* nonnegatively defined *if for any $m\in\mathbb{N}$ and any $t_1,\dots,t_m\in\mathbb{T}$ the matrix $K_{t_1\dots t_m}=\{K(t_j,t_k)\}_{j,k=1}^m$ is nonnegatively defined.*

Proposition 2.1. *Covariance R_X of an arbitrary stochastic process X is nonnegatively defined. And vice versa, if $a:\mathbb{T}\to\mathbb{C}$ and $K:\mathbb{T}\times\mathbb{T}\to\mathbb{C}$ are some functions and K is nonnegatively defined, then on some probability space there exists random function X such that $a=a_X, K=R_X$.*

Remark 2.1. Recall that the mean vector and covariance matrix for a random vector $\xi=(\xi_1,\dots,\xi_m)$ are $a_\xi=(\mathsf{E}\xi_j)_{j=1}^m$ and $R_\xi=(\mathrm{cov}(\xi_j,\xi_k))_{j,k=1}^m$, respectively. If the conditions of Proposition 2.1 hold, then for any $m\in\mathbb{N}, t_1,\dots,t_m\in\mathbb{T}$ the covariance matrix for the vector $(X(t_1),\dots,X(t_m))$ is equal to $K_{t_1\dots t_m}$ (see Definition 2.4) and the mean vector is equal to $a_{t_1\dots t_m}=(a(t_j))_{j=1}^m$.

Recall that for a random vector $\xi=(\xi_1,\dots,\xi_m)$ with real-valued components, its *characteristic function* (or equivalently, *common characteristic function* of the random variables $\xi_1,\dots,\xi_m$) is defined by

$$\phi_\xi(z)=\mathsf{E}e^{i(\xi,z)_{\mathbb{R}^m}}=\mathsf{E}e^{i\sum_{j=1}^m \xi_j z_j},\quad z=(z_1,\dots,z_m)\in\mathbb{R}^m.$$

Theorem 2.1. *(The Bochner theorem) An arbitrary function $\phi:\mathbb{R}^m\to\mathbb{C}$ is a characteristic function of some random vector if and only if the following three conditions are satisfied.*
(1) $\phi(0)=1$.
(2) ϕ is continuous in the neighborhood of 0.
(3) For any $m\in\mathbb{N}$ and $z_1,\dots,z_m\in\mathbb{R}, c_1,\dots,c_m\in\mathbb{C}$

$$\sum_{j,k=1}^m \phi(z_j-z_k)c_j\bar{c}_k\ge 0.$$

Definition 2.5. *Let X be a real-valued random function. For a fixed $m\ge 1$ and $t_1,\dots,t_m\in\mathbb{T}$, the common characteristic function of $X(t_1),\dots,X(t_m)$ is denoted by $\phi^X_{t_1,\dots,t_m}$ and is called the* (m-dimensional) characteristic function *of the random function X. The set $\{\phi^X_{t_1,\dots,t_m}, t_1,\dots,t_m\in\mathbb{T}, m\ge 1\}$ is called* the set *(or* the family*) of finite-dimensional characteristic functions of the random function X.*

Mean and covariance functions of a random function do not determine the finite-dimensional distributions of this function uniquely (e.g., see Problem 6.7). On the other hand, the family of finite-dimensional characteristic functions of the random function X has unique correspondence to its finite-dimensional characteristics because the characteristic function of a random vector determines the distribution of this vector uniquely. The following theorem is the reformulation of the Kolmogorov theorem (Theorem 1.1) in terms of characteristic functions.

Theorem 2.2. *Consider a family* $\{\phi_{t_1,\dots,t_m} : \mathbb{R}^m \to \mathbb{C}, t_1,\dots,t_m \in \mathbb{T}, m \geq 1\}$ *such that for any* $m \geq 1, t_1,\dots,t_m \in \mathbb{T}$ *the function* $\phi_{t_1,\dots,t_m}$ *satisfies the conditions of the Bochner theorem. The following* consistency conditions *are necessary and sufficient for such a random function X to exist that the family* $\{\phi_{t_1,\dots,t_m} : \mathbb{R}^m \to \mathbb{C}, t_1,\dots,t_m \in \mathbb{T}, m \geq 1\}$ *is the family of its finite-dimensional characteristic functions.*

(1) For any $m \geq 1, t_1,\dots,t_m \in \mathbb{T}, z_1,\dots,z_m \in \mathbb{R}$ *and any permutation* $\pi : \{1,\dots,m\} \to \{1,\dots,m\}$,

$$\phi_{t_1,\dots,t_m}(z_1,\dots,z_m) = \phi_{t_{\pi(1)},\dots,t_{\pi(m)}}\left(z_{\pi(1)},\dots,z_{\pi(m)}\right).$$

(2) For any $m > 1, t_1,\dots,t_m \in \mathbb{T}, z_1,\dots,z_{m-1} \in \mathbb{R}$,

$$\phi_{t_1,\dots,t_m}(z_1,\dots,z_{m-1},0) = \phi_{t_1,\dots,t_{m-1}}(z_1,\dots,z_{m-1}).$$

Bibliography

[9], Chapter II; [24], Volume 1, Chapter IV, §1; [25], Chapter I, §1; [79], Chapter 16.

Problems

2.1. Find the covariance function for (a) the Wiener process; (b) the Poisson process.

2.2. Let W be the Wiener process. Find the mean and covariance functions for the process $X(t) = W^2(t), t \geq 0$.

2.3. Let W be the Wiener process. Find the covariance function for the process X if
(a) $X(t) = W(1/t),\ t > 0$.
(b) $X(t) = W(e^t),\ t \in \mathbb{R}$.
(c) $X(t) = W\left(1-t^2\right),\ t \in [-1,1]$.

2.4. Let W be the Wiener process. Find the characteristic function for $W(2)+2W(1)$.

2.5. Let N be the Poisson process with intensity λ. Find the characteristic function for $N(2)+2N(1)$.

2.6. Let W be the Wiener process. Find:
(a) $\mathsf{E}(W(t))^m, m \in \mathbb{N}$.
(b) $\mathsf{E}\exp(2W(1)+W(2))$.
(c) $\mathsf{E}\cos(2W(1)+W(2))$.

2.7. Let N be the Poisson process with intensity λ. Find:
(a) $\mathsf{P}(N(1)=2, N(2)=3, N(3)=5)$.
(b) $\mathsf{P}(N(1)\leq 2, N(2)=3, N(3)\geq 5)$.
(c) $\mathsf{E}(N(t)+1)^{-1}$.
(d) $\mathsf{E}N(t)(N(t)-1)\cdot\dots\cdot(N(t)-k), k \in \mathbb{Z}^+$.

2.8. Let W be the Wiener process and $f \in C([0,1])$. Find the characteristic function for random variable $\int_0^1 f(s)W(s)\,ds$ (the integral is defined for every ω in the Riemann sense; see Problem 1.25). Prove that this random variable is normally distributed.

2.9. Let W be the Wiener process, $f \in C([0,1])$, $X(t) = \int_0^t f(s)W(s)\,ds$, $t \in [0,1]$. Find $R_{W,X}$.

2.10. Let N be the Poisson process, $f \in C([0,1])$. Find the characteristic functions of random variables: (a) $\int_0^1 f(s)N(s)\,ds$; (b) $\int_0^1 f(s)dN(s) \equiv \sum f(s)$, where summation is taken over all $s \in [0,1]$ such that $N(s) \neq N(s-)$.

2.11. Let N be the Poisson process, $f,g \in C([0,1])$, $X(t) = \int_0^t f(s)N(s)\,ds$, $Y(t) = \int_0^t g(s)dN(s)$, $t \in [0,1]$. Find: (a) $R_{N,X}$; (b) $R_{N,Y}$; (c) $R_{X,Y}$.

2.12. Find all one-dimensional and m-dimensional characteristic functions: (a) for the process introduced in Problem 1.2; (b) for the process introduced in Problem 1.4.

2.13. Find the covariance function of the process $X(t) = \xi_1 f_1(t) + \cdots + \xi_n f_n(t)$, $t \in \mathbb{R}$, where $f_1, \ldots, f_n$ are nonrandom functions, and $\xi_1, \ldots, \xi_n$ are noncorrelated random variables with variances $\sigma_1^2, \ldots, \sigma_n^2$.

2.14. Let $\{\xi_n, n \geq 1\}$ be the sequence of independent square integrable random variables. Denote $a_n = \mathsf{E}\xi_n, \sigma_n^2 = \operatorname{Var}\xi_n$.
(1) Prove that series $\sum_n \xi_n$ converges in the mean square sense if and only if the series $\sum_n a_n$ and $\sum_n \sigma_n^2$ are convergent.
(2) Let $\{f_n(t),\ t \in \mathbb{R}\}_{n\in\mathbb{N}}$ be the sequence of nonrandom functions. Formulate the necessary and sufficient conditions for the series $X(t) = \sum_n \xi_n f_n(t)$ to converge in the mean square for every $t \in \mathbb{R}$. Find the mean and covariance functions of the process X.

2.15. Are the following functions nonnegatively defined: (a) $K(t,s) = \sin t \sin s$; (b) $K(t,s) = \sin(t+s)$; (c) $K(t,s) = t^2 + s^2$ $(t,s \in \mathbb{R})$?

2.16. Prove that for $\alpha > 2$ the function $K(t,s) = \frac{1}{2}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right)$, $t,s \in \mathbb{R}^m$ is not a covariance function.

2.17. (1) Let $\{X(t),\ t \in \mathbb{R}^+\}$ be a stochastic process with independent increments and $\mathsf{E}|X(t)|^2 < +\infty, t \in \mathbb{R}^+$. Prove that its covariance function is equal to $R_X(t,s) = F(t \wedge s)$, $t,s \in \mathbb{R}^+$, where F is some nondecreasing function.
(2) Let $\{X(t), t \in \mathbb{R}^+\}$ be a stochastic process with $R_X(t,s) = F(t \wedge s), t,s \in \mathbb{R}^+$, where F is some nondecreasing function. Does it imply that X is a process with independent increments?

2.18. Let N be the Poisson process with intensity λ. Let $X(t) = 0$ when $N(t)$ is odd and $X(t) = 1$ when $N(t)$ is even.
(1) Find the mean and covariance of the process X.
(2) Find $R_{N,X}$.

2.19. Let W and N be the independent Wiener process and Poisson process with intensity λ, respectively. Find the mean and covariance of the process $X(t) = W(N(t))$. Is X a process with independent increments?

2.20. Find $R_{X,W}$ and $R_{X,N}$ for the process from the previous problem.

2.21. Let N_1, N_2 be two independent Poisson processes with intensities λ_1, λ_2, respectively. Define $X(t) = (N_1(t))^{N_2(t)}$,$t \in \mathbb{R}^+$ if at least one of the values $N_1(t)$, $N_2(t)$ is nonzero and $X(t) = 1$ if $N_1(t) = N_2(t) = 0$. Find:
(a) The mean function of the process X
(b) The covariance function of the process X

2.22. Let X, Y be two independent and centered processes and $c > 0$ be a constant. Prove that $R_{X+Y} = R_X + R_Y, R_{\sqrt{c}X} = cR_X, R_{XY} = R_X R_Y$.

2.23. Let K_1, K_2 be two nonnegatively defined functions and $c > 0$. Prove that the following functions are nonnegatively defined: (a) $R = K_1 + K_2$; (b) $R = cK_1$; (c) $R = K_1 \cdot K_2$.

2.24. Let K be a nonnegatively defined function on $\mathbb{T} \times \mathbb{T}$.
(1) Prove that for every polynomial $P(\cdot)$ with nonnegative coefficients the function $R = P(K)$ is nonnegatively defined.
(2) Prove that the function $R = e^K$ is nonnegatively defined.
(3) When it is additionally assumed that for some $p \in (0,1)$ $K(t,t) < p^{-1}$, $t \in \mathbb{T}$, prove that the function $R = (1 - pK)^{-1}$ is nonnegatively defined.

2.25. Give the probabilistic interpretation of items (1)–(3) of the previous problem; that is, construct the stochastic process for which R is the covariance function.

2.26. Let $K(t,s) = ts, t,s \in \mathbb{R}^+$. Prove that for an arbitrary polynomial P the function $R = P(K)$ is nonnegatively defined if and only if all coefficients of the polynomial P are nonnegative. Compare with item (1) of Problem 2.24.

2.27. Which of the following functions are nonnegatively defined: (a) $K(t,s) = \sin(t-s)$; (b) $K(t,s) = \cos(t-s)$; (c) $K(t,s) = e^{-(t-s)}$; (d) $K(t,s) = e^{-|t-s|}$; (e) $K(t,s) = e^{-(t-s)^2}$; (f) $K(t,s) = e^{-(t-s)^4}$?

2.28. Let $K \in C([a,b] \times [a,b])$. Prove that K is nonnegatively defined if and only if the integral operator $A_K : L_2([a,b]) \to L_2([a,b])$, defined by

$$A_K f(t) = \int_a^b K(t,s) f(s)\, ds, \quad f \in L_2([a,b]),$$

is nonnegative.

2.29. Let A_K be the operator from the previous problem. Check the following statements.
(a) The set of eigenvalues of the operator A_K is at most countable.
(b) The function K is nonnegatively defined if and only if every eigenvalue of the operator A_K is nonnegative.

2.30. Let $K(s,t) = F(t-s)$, $t,s \in \mathbb{R}$, where the function F is periodic with period 2π and $F(x) = \pi - |x|$ for $|x| \le \pi$. Construct the Gaussian process with covariance K of the form $\sum_n \varepsilon_n f_n(t)$, where $\{\varepsilon_n, n \ge 1\}$ is a sequence of the independent normally distributed random variables.

2.31. Solve the previous problem assuming that F has period 2 and $F(x) = (1-x)^2$, $x \in [0,1]$.

2.32. Denote $\{\tau_n, n \ge 1\}$ the jump moments for the Poisson process $N(t), \tau_0 = 0$. Let $\{\varepsilon_n, n \ge 0\}$ be i.i.d. random variables that have expectation a and variance σ^2. Consider the stochastic processes $X(t) = \sum_{k=0}^{n} \varepsilon_k$, $t \in [\tau_n, \tau_{n+1})$, $Y(t) = \varepsilon_n$, $t \in [\tau_n, \tau_{n+1})$, $n \ge 0$. Find the mean and covariance functions of the processes X, Y. Exemplify the models that lead to such processes.

2.33. A radiation measuring instrument accumulates radiation with the rate that equals a Roentgen per hour, right up to the failing moment. Let $X(t)$ be the reading at point of time $t \ge 0$. Find the mean and covariance functions for the process X if $X(0) = 0$, the failing moment has distribution function F, and after the failure the measuring instrument is fixed (a) at zero point; (b) at the last reading.

2.34. The device registers a Poisson flow of particles with intensity $\lambda > 0$. Energies of different particles are independent random variables. Expectation of every particle's energy is equal to a and variance is equal to σ^2. Let $X(t)$ be the readings of the device at point of time $t \ge 0$. Find the mean and covariance functions of the process X if the device shows
(a) Total energy of the particles have arrived during the time interval $[0,t]$.
(b) The energy of the last particle.
(c) The sum of the energies of the last K particles.

2.35. A Poisson flow of claims with intensity $\lambda > 0$ is observed. Let $X(t), t \in \mathbb{R}$ be the time between t and the moment of the last claim coming before t. Find the mean and covariance functions for the process X.

Hints

2.1. See the hint to Problem 2.17.

2.4. Because the variables $(W(1), W(2))$ are jointly Gaussian, the variable $W(2) + 2W(1)$ is normally distributed. Calculate its mean and variance and use the formula for the characteristic function of the Gaussian distribution. Another method is proposed in the following hint.

2.5. $N(2) + 2N(1) = N(2) - N(1) + 3N(1)$. The values $N(2) - N(1)$ and $N(1)$ are Poisson-distributed random variables and thus their characteristic functions are known. These values are independent, that is, the required function can be obtained as a product.

2.6. (a) If $\eta \sim \mathcal{N}(0,1)$, then $\mathsf{E}\eta^{2k-1} = 0, \mathsf{E}\eta^{2k} = (2k-1)!! = (2k-1)(2k-3)\cdots 1$ for $k \in \mathbb{N}$. Prove and use this for the calculations.
(b) Use the explicit formula for the Gaussian density.
(c) Use formula $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ and Problem 2.4.

2.10. (a) Make calculations similar to those of Problem 2.8.
(b) Obtain the characteristic functions of the integrals of piecewise constant functions f and then uniformly approximate the continuous function by piecewise constant ones.

2.17. (1) Let $s \le t$; then values $X(t) - X(s)$ and $X(s)$ are independent which means that they are uncorrelated. Therefore $\operatorname{cov}(X(t),X(s)) = \operatorname{cov}(X(t)-X(s),X(s)) + \operatorname{cov}(X(s),X(s)) = \operatorname{cov}(X(t\wedge s),X(t\wedge s))$. The case $t \le s$ can be treated similarly.

2.23. Items (a) and (b) can be proved using the definition. In item (c) you can use the previous problem.

2.24. Proof of item (1) can be directly obtained from the previous problem. For the proof of items (2) and (3) use item (1), Taylor decomposition of the functions $x \mapsto e^x, x \mapsto (1-px)^{-1}$ and a fact that the pointwise limit of a sequence of nonnegatively defined functions is also a nonnegatively defined function. (Prove this fact!).

Answers and Solutions

2.1. $R_W(t,s) = t\wedge s, R_N(t,s) = \lambda(t\wedge s)$.

2.2. $a_X(t) = t,\ R_X(t,s) = 2(t\wedge s)^2$.

2.3. For arbitrary $f : \mathbb{R}^+ \to \mathbb{R}^+$, the covariance function for the process $X(t) = W(f(t)), t \in \mathbb{R}^+$ is equal to $R_X(t,s) = R_W(f(t), f(s)) = f(t)\wedge f(s)$.

2.8. Let $I_n = n^{-1}\sum_{k=1}^n f(k/n)W(k/n)$. Because the process W a.s. has continuous trajectories and the function f is continuous, the Riemann integral sum I_n converges to $I = \int_0^1 f(t)W(t)\,dt$ a.s. Therefore $\phi_{I_n}(z) \to \phi_I(z), n \to +\infty, z \in \mathbb{R}$. Hence,

$$\mathsf{E}e^{izI_n} = \mathsf{E}e^{izn^{-1}\sum_{k=1}^n f(k/n)W(k/n)} = \mathsf{E}e^{i\sum_{k=1}^n \left[zn^{-1}\sum_{j=k}^n f(j/n)\right](W(k/n)-W((k-1)/n))}$$
$$= \prod_{k=1}^n e^{-(2n)^{-1}\left[zn^{-1}\sum_{j=k}^n f(j/n)\right]^2} \to e^{-\left(z^2/2\right)\int_0^1\left(\int_t^1 f(s)\,ds\right)^2 dt}, \quad n \to \infty.$$

Thus I is a Gaussian random variable with zero mean and variance $\int_0^1\left(\int_t^1 f(s)\,ds\right)^2 dt$.

2.9. $R_{W,X}(t,s) = \int_0^s f(r)(t\wedge r)\,dr$.

2.10. (a) $\phi(z) = \exp\left(\lambda\int_0^1\left[e^{iz\int_t^1 f(s)\,ds} - 1\right]dt\right)$.
(b) $\phi(z) = \exp\left(\lambda\int_0^1\left[e^{izf(t)} - 1\right]dt\right)$.

2.11. $R_{N,X}(t,s) = \lambda^2 \int_0^s f(r)(t\wedge r)\,dr$, $R_{N,Y}(t,s) = \lambda^2 \int_0^{t\wedge s} g(r)\,dr$, $R_{X,Y}(t,s) = \lambda^2 \times \int_0^t f(u)\left[\int_0^{u\wedge s} g(r)\,dr\right] du$.

2.12. (a) Let $0 \le t_1 < \cdots < t_n \le 1$; then $\phi_{t_1,\dots,t_m}(z_1,\dots,z_m) = t_1 e^{iz_1+\cdots+iz_m} + (t_2-t_1)e^{iz_2+\cdots+iz_m} + \cdots + (t_m - t_{m-1})e^{iz_m} + (1-t_m)$.
(b) Let $0 \le t_1 < \cdots < t_n \le 1$, then

$$\phi_{t_1,\dots,t_m}(z_1,\dots,z_m) = \Big[F(t_1)e^{iz_1n^{-1}+\cdots+iz_mn^{-1}} + (F(t_2)-F(t_1))e^{iz_2n^{-1}+\cdots+iz_mn^{-1}} + \cdots + (F(t_m)-F(t_{m-1}))e^{iz_mn^{-1}} + (1-F(t_m))\Big]^n.$$

2.13. $R_X(t,s) = \sum_{k=1}^n \sigma_k^2 f_k(t) f_k(s)$.

2.15. (a) Yes; (b) no; (c) no.

2.17. (2) No, it does not.

2.18. (1) $a_X(t) = \frac{1}{2}\left(1+e^{-2\lambda t}\right), R_X(t,s) = \frac{1}{4}\left(e^{-2\lambda|t-s|} - e^{-2\lambda(t+s)}\right)$.
(2) $R_{N,X}(t,s) = -\lambda(t\wedge s)e^{-2\lambda s}$.

2.19. $a_X \equiv 0, R_X(t,s) = \lambda(t\wedge s)$. X is the process with independent increments.

2.20.

$$R_{X,W}(t,s) = \mathsf{E}[N(t)\wedge s] = e^{-\lambda t}\left[\sum_{k<s} \frac{k(\lambda t)^k}{k!} + s\cdot \sum_{k\ge s}\frac{(\lambda t)^k}{k!}\right], \quad R_{X,N} \equiv 0.$$

2.21. $a_X(t) = \exp\left[\lambda_1 t e^{\lambda_2 t} - (\lambda_1+\lambda_2)t\right]$; function R_X is not defined because $\mathsf{E}X^2(t) = +\infty, t > 0$.

2.25. There exist several interpretations, let us give two of them.

The first one: let $R = f(K)$ and $f(x) = \sum_{m=0}^\infty c_m x^m$ with $c_m \ge 0, m \in \mathbb{Z}^+$. Let the radius of convergence of the series be equal to $r_f > 0$ and $K(t,t) < r_f, t \in \mathbb{R}^+$. Consider a triangular array $\{X_{m,k}, 1 \le k \le m\}$ of independent centered identically distributed processes with the covariance function K. In addition, let random variable ξ be independent of $\{X_{m,k}\}$ and $\mathsf{E}\xi = 0, \mathsf{D}\xi = 1$. Then the series $X(t) = \sqrt{c_0}\xi + \sum_{m=1}^\infty \sqrt{c_m}\prod_{k=1}^m X_{m,k}(t)$ converges in the mean square for any t and the covariance function of the process X is equal to R.

The second one: using the same notations, denote $c = \sum_{k=0}^\infty c_k, p_k = c_k/c$, $k \ge 0$. Let $\{X_m, m \ge 1\}$ be a sequence of independent identically distributed centered processes with the covariance function K, and ξ be as above. Let η be the random variable, independent both on ξ and the processes $\{X_m, m \ge 1\}$, with $P(\eta = k) = p_k, k \in \mathbb{Z}^+$. Consider the process $X(t) = \sqrt{c}\prod_{k=1}^\eta X_k(t)$ assuming that $\prod_{k=1}^0 X_k(t) = \xi$. Then the covariance function of the process X is equal to R. In particular, the random variable η should have a Poisson distribution in item (2) and a geometric distribution in item (3).

2.26. Consider the functions $R_k = (\partial^{2k}/\partial t^k \partial s^k)R, k \geq 0$. These functions are nonnegatively defined (one can obtain this fact by using either Definition 2.3 or Theorem 4.2). Function R_k can be represented in the form $R_k = P_k(K)$, where the absolute term of the polynomial P_k equals the kth coefficient of the polynomial P multiplied by $(k!)^2$. Now, the required statement follows from the fact that $Q(t,t) \geq 0$ for any nonnegatively defined function Q.

2.27. Functions from the items (b), (d), (e) are nonnegatively defined; the others are not.

2.28. Let K be nonnegatively defined. Then for any $f \in C([a,b])$,

$$\begin{aligned}(A_K f, f)_{L_2([a,b])} &= \int_a^b \int_a^b K(t,s) f(t) f(s)\, ds dt \\ &= \lim_{n\to\infty} \sum_{j,k=1}^{n} \left(\frac{b-a}{n}\right)^2 K\left(a + \frac{j(b-a)}{n}, a + \frac{k(b-a)}{n}\right) \geq 0\end{aligned}$$

because every sum under the limit sign is nonnegative. Because $C([a,b])$ is a dense subset in $L_2([a,b])$ the above inequality yields that $(A_K f, f)_{L_2([a,b])} \geq 0$, $f \in L_2([a,b])$. On the other hand, let $(A_K f, f)_{L_2([a,b])} \geq 0$ for every $f \in L_2([a,b])$, and let points $t_1, \dots, t_m$ and constants $z_1, \dots, z_m$ be fixed. Choose m sequences of continuous functions $\{f_n^1, n \geq 1\}, \dots, \{f_n^m, n \geq 1\}$ such that, for arbitrary function $\phi \in C([a,b])$, $\int_a^b \phi(t) f_n^j(t)\, dt \to \phi(t_j), n \to \infty, j = 1, \dots, m$. Putting $f_n = \sum_{j=1}^m z_j f_n^j$, we obtain that $\sum_{j,k=1}^m z_j z_k K(t_j, t_k) = \lim_{n\to\infty} \int_a^b \int_a^b K(t,s) f_n(t) f_n(s)\, ds dt = \lim_{n\to\infty} (A_K f_n, f_n) \geq 0$.

2.29. Statement (a) is a particular case of the theorem on the spectrum of a compact operator. Statement (b) follows from the previous problem and theorem on spectral decomposition of a compact self-adjoint operator.

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