SOLUTION TO PROBLEM 11865

ARSHAK MINASYAN

Problem 11865. Let a_n be a monotone decreasing sequence of non-negative real numbers. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is finite if and only if

$$\lim_{n\to+\infty} a_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}(a_n - a_{n+1})\log n < +\infty$$

Solution.

Proof. Let us first of all note that the following relationship is true for any natural n:

$$\frac{1}{n+1} < \log\frac{n+1}{n} < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{1}$$

(the proof of this fact seems pretty straightforward, so we will skip it.)
Then using this relationship we proceed the solution of the problem.

- $\Leftarrow$
  Note that the second sum can be written in the following way
  $$\sum_{n=1}^{\infty} a_{n+1}\log\frac{n+1}{n} < +\infty$$
  Using (1) and the fact that if $\sum_{n=1}^{\infty}\frac{a_{n+1}}{n+1}$ is finite then adding a number a_1 can not force the sum to diverge, thus
  $$\sum_{n=1}^{\infty}\frac{a_n}{n} = a_1 + \sum_{n=1}^{\infty}\frac{a_{n+1}}{n+1} < a_1 + \sum_{n=1}^{\infty} a_{n+1}\log\frac{n+1}{n} < +\infty, \tag{2}$$
  which leads to the end of the proof of sufficient condition.
- $\Rightarrow$
  Firstly, we will prove that $\sum_{n=1}^{\infty}(a_n - a_{n+1})\log n$ converges. Using the representation from previous part we get
  $$\sum_{n=1}^{\infty}(a_n - a_{n+1})\log n = \sum_{n=1}^{\infty} a_{n+1}\log\frac{n+1}{n} < \sum_{n=1}^{\infty}\frac{a_n}{n} < +\infty, \tag{3}$$
  which leads us to the end of the first part of necessary condition.

Secondly, as $\{a_n\}$ is a monotone decreasing sequence of nonnegative real numbers, then the limit exists and we assume that $\lim_{n\to\infty} a_n = c \neq 0 \implies a_n > c, \ \forall n \in \mathbb{N}$.

$$\sum_{n=1}^{\infty} \frac{a_n}{n} > c \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

which is a contradiction. Hence, $\lim_{n\to\infty} a_n = 0$.

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Arshak Minasyan, National Research University Higher School of Economics, Moscow, Russia
E-mail address: arsh.minasyan@gmail.com