SPARSE RECOVERY AND THE GEOMETRY OF HIGH-DIMENSIONAL RANDOM MATRICES

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1. Introduction: A brief history of sparse recovery

The uniform uncertainty principle – a precursor to compressed sensing, by Donoho and Stark.

We write ℓ_2^n for the real or complex Hilbert space of functions on a set of size n, equipped with the standard inner product. When the set is chosen as $\{0, 1, \dots, n-1\}$, we write the canonical basis as $\{e_j\}_{j=0}^{n-1}$ and define the discrete Fourier transform of $f \in \ell_2^n$ in the usual way.

Theorem 1.1 (Donoho & Stark (1989))**.** *Let $n \in \mathbb{N}$ be prime, $1 \leq s \leq n$, and T, Ω subsets of $\mathbb{Z}_n$ with $|T| \leq s$ and $|\Omega| \geq s$, then if $f : \mathbb{Z}_n \to \mathbb{C}$, is obtained by $f = \sum_{j \in T} c_j e_j \neq 0$, then restricting the discrete Fourier transform $\hat{f}$ to Ω is non-vanishing, $\hat{f}|_\Omega \not\equiv 0$.*

The relevance for sparse recovery is that Ω can be chosen first, then the kernel of $f \mapsto \hat{f}|_\Omega$ does not contain vectors that are obtained from *any* linear combination of a small number of basis vectors.

Definition 1.2. *A vector $f \in \ell_2^n$ is at most s-sparse, abbreviated $f \in \sum_s$, if $f = \sum_{j \in T} c_j e_j$ with $T \subset \{1, 2, \dots, n\}$, $|T| \leq s$.*

We note that the set Σ_s is a union of subspaces, the s-dimensional coordinate hyperplanes. The kernel of the restricted Fourier transform intersects Σ_s only in the zero vector. This can be used for an injectivity statement for the restricted Fourier transform.

Corollary 1.3. *Let $s \leq n/2$, $\Omega \subset \mathbb{Z}_n$ of size $|\Omega| \geq 2s$, then $f \mapsto \hat{f}|_\Omega$ is injective on Σ_s.*

Proof. This is a consequence of the fact that the difference of any two s-sparse vectors is $2s$-sparse. Under the assumption on the size of Ω, if the difference of any two s-sparse vectors maps to zero, then the vectors are identical. □

Question 1.4. *What about "nearly" sparse f and observing a restriction of a "noisy" $\hat{f}$? Can we find conditions for approximate recovery, assuming s and m and n are appropriately chosen?*

We wish to measure the accuracy of an approximate solution $\hat{f}$ to the input signal f in terms of the Euclidean norm on ℓ_n^2, so if $h = \hat{f} - f$ is the reconstruction error, we wish to control $\|h\| = (\sum_{i=1}^n |h_j|^2)^{1/2}$. In order to derive error bounds, we use a strengthening of the properties of the kernel of the maps used for measuring, see the papers by Candès, Romberg and Tao (2006) as well as by Candès (2008).

Definition 1.5. *Let A be an $m \times n$ matrix. For $s \le n$, the restricted isometry constant δ_s of A is the smallest quantity such that*

$$(1-\delta_s)\|x\|^2 \le \|Ax\|^2 \le (1+\delta_s)\|x\|^2$$

holds for each $x \in \Sigma_s \subset \ell_n^2$.

This statement amounts to saying that a selection of any s column vectors of the matrix $A = [a_1\, a_2\, \cdots\, a_n]$ is a nearly orthonormal system.

Under this assumption, we wish to use a non-linear signal recovery strategy by ℓ^1-norm minimization. Here and in the following, we denote the ℓ^1-norm of $x \in \ell_n^2$ by

$$\|x\|_1 = \sum_{i=1}^n |x_i| \,.$$

Strategy. Let $x^\natural \in \ell_2^n$, $y = Ax^\natural + \eta$, $\|\eta\| \le \epsilon$, then choose

$$\hat{x} \in \arg\min\{\|x\|_1 : \|Ax - y\| \le \epsilon\}.$$

Assuming we know the error tolerance ϵ, the choice of the subset in which the ℓ_1-norm is minimized ensures that $x^\natural$ is included in this set.

The objective is to show that if $x^\natural$ is well approximated by an s-sparse vector x_0, then the recovery error can be bounded by the approximation error $\|x^\natural - x_0\|$ and the measurement error tolerance ϵ. A nice result of this type was derived by Candès.

Definition 1.6. *Given any $x^\natural$ and $s \le n$, then a* best s-sparse approximation *to $x^\natural$ is given by x_0 such that*

$$\|x^\natural - x_0\| \le \min\{\|x^\natural - z\| : z \in \Sigma_s\}\,.$$

In practice, x_0 obtained from $x^\natural$ by choosing a set J_0 of size s containing indices corresponding to entries of $x^\natural$ having the s largest magnitudes and then setting entries x_j with $j \notin J_0$ equal to zero.

Theorem 1.7 (Candès (2008))**.** *If the $m \times n$ matrix A has the restricted isometry constant $\delta_{2s} < \sqrt{2} - 1$ and $x^\natural$, η, ϵ and $\hat{x}$ are as described, and if x_0 is a best s-sparse approximation of $x^\natural$, then*

$$\|x^\natural - \hat{x}\| \le 2\Big(1 - \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}\Big)^{-1}\Big(\frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\epsilon + \big(1 + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}\big)\frac{\|x^\natural - x_0\|_1}{\sqrt{s}}\Big)\,.$$

Question 1.8. *How do we construct a matrix A with small δ_s?*

A randomization approach is surprisingly powerful.

Theorem 1.9 (Foucart and Rauhut, p. 293)**.** *Choosing the matrix A such that its entries are i.i.d. normally distributed with expected value 0 and variance $1/m$, then for any $t > \sqrt{s/m}$, the probability of getting a large constant δ_s is bounded by*

$$\mathbb{P}[\delta_s > 2t + t^2] \le 2e^{s\ln(en/s) - m(t - \sqrt{s/m})^2/2}\,.$$

Hence, if s/m is sufficiently small, we have a negative exponent even for relatively large n/s. If we choose t such that $2t + t^2 = \sqrt{2} - 1$ and hold s/m and m/n fixed, then as n grows the probability of failure (violating the bound for the RIP constant) decays exponentially. This randomized construction then satisfies the conditions for sparse recovery by ℓ^1-minimization with overwhelming probability.

2. Geometric conditions for sparse recovery

We recall that a simple, necessary condition for sparse recovery is that no $2s$-sparse vectors are in the kernel of the (sensing) matrix A, $\mathcal{N}(A) = \{x \in \ell_2^n : Ax = 0\}$. We wish to describe a geometric condition that ensures recovery guarantees for approximately sparse signals $x^\natural$ and noisy measurements. The main idea to achieve stability and robustness is to enlarge the kernel of A and postulate a norm equivalence on this enlarged set.

Definition 2.1. *We say that $A : \ell_2^n \to \ell_2^m$ satisfies the (ρ, α)-robust width property with respect to ℓ_1 if for each $x \in \mathbb{R}^n$ with $\|Ax\| < \alpha\|x\|$, we have the inequality between ℓ_2 and ℓ_1-norms*

$$\|x\|_2 \leq \rho \|x\|_1 \,.$$

Remark 2.2. *The set*

$$\mathcal{N}_\alpha(A) = \{x \in \mathbb{R}^n : \|Ax\| < \alpha\|x\|\} \cup \{0\}$$

satisfies that if A' is close (in operator norm) to A, $\|A - A'\| < \alpha$, then

$$\mathcal{N}(A') \subset \mathcal{N}_\alpha(A)$$

because if $A'x = 0$, then $Ax = (A - A')x$ and so $\|Ax\| \leq \|A - A'\|\|x\|$. Thus, $x = 0$ or $\|Ax\| < \alpha\|x\|$. Consequently, if no non-zero element in Σ_s is in $\mathcal{N}_\alpha(A)$, then the simple necessary condition for sparse recovery also holds for the perturbed measurement matrix A'.

The condition $\Sigma_s \cap \mathcal{N}_\alpha(A) = \{0\}$ is a consequence of choosing sufficiently small ρ with the robust width property: If $x \in \Sigma_s$, then the Cauchy-Schwarz inequality gives

$$\|x\|_1 \leq \sqrt{s}\|x\|_2 \,.$$

If, on the other hand, $x \in \mathcal{N}_\alpha(A)$, then the robust width property implies

$$\|x\|_1 \geq \|x\|_2/\rho \,.$$

Hence, if $\rho < 1/\sqrt{2}$, then only $x = 0$ satisfies both inequalities.

2.1. Robust width implies robust and stable sparse recovery by ℓ^1-minimization.

Lemma 2.3. *Given $a \in \Sigma_s \subset \mathbb{R}^n$, $z \in \mathbb{R}^n$, then we can find z_1, z_2 such that $z = z_1 + z_2$,*

$$\|a + z_1\|_1 = \|a\|_1 + \|z_1\|_1$$

and

$$\|z_2\|_1 \leq \sqrt{s}\|z_2\|_2 \leq \sqrt{s}\|z\|_2 \,.$$

Proof. Let $z_1 = \sum_{j:a_j=0}\langle z, e_j\rangle e_j$, then a and z_1 have disjoint supports, so the ℓ_1-norm is additive. Moreover, $z_2 = z - z_1$ has support of size at most s, so the Cauchy-Schwarz inequality gives the first claimed estimate for the norm of z_2. The orthogonality between z_1 and z_2 then provides the last step, $\|z_2\|_2 \leq \|z\|_2$. □

Theorem 2.4. *If $A : \ell_2^n \to \ell_2^m$ has the (ρ, α)-robust width property w.r.t. ℓ_1 and $2\rho\sqrt{s} \leq 1/2$, then the ℓ_1-minimization strategy for sparse recovery gives that for any $a \in \Sigma_s$,*

$$\|\hat{x} - x^\natural\| \leq 4\rho\|x^\natural - a\|_1 + \frac{2}{\alpha}\epsilon\,.$$

Proof. Let $a \in \Sigma_s$ and $z = \hat{x} - x^\natural$, then split $z = z_1 + z_2$ with z_1 and z_2 according to the lemma, so

$$\|a + z_1\|_1 = \|a\|_1 + \|z_1\|_1$$

and

$$\|z_2\|_1 \leq \sqrt{s}\|\hat{x} - x^\natural\|_2\,.$$

Next, we show that $\|z_1\|_1$ is bounded by $\|x^\natural - a\|_1$ and $\|z_2\|_1$. To this end, we use the sequence of triangle inequalities and the fact that $x^\natural$ is included in the set over which the ℓ^1-norm is minimized, so $\|x^\natural\|_1 \geq \|\hat{x}\|_1$. Combining this yields

$$\begin{aligned}
\|a\|_1 + \|x^\natural - a\|_1 &\geq \|x^\natural\|_1 \\
&\geq \|\hat{x}\|_1 = \|x^\natural + \hat{x} - x^\natural\|_1 \\
&= \|a + (x^\natural - a) + z_1 + z_2\|_1 \\
&\geq \|a + z_1\|_1 - \|x^\natural - a\|_1 - \|z_2\|_1 \\
&= \|a\|_1 + \|z_1\|_1 - \|x^\natural - a\|_1 - \|z_2\|_1\,.
\end{aligned}$$

Rearranging terms gives

$$\|z_1\|_1 \leq 2\|x^\natural - a\|_1 + \|z_2\|_1\,.$$

We conclude that the ℓ_1-norm of the recovery error is bounded by

$$\|\hat{x} - x^\natural\|_1 \leq \|z_1\|_1 + \|z_2\|_1 \leq 2\|x^\natural - a\|_1 + 2\|z_2\|_1\,. \tag{1}$$

To finish the proof, we distinguish two cases. If

$$\|\hat{x} - x^\natural\|_2 \leq \frac{2}{\alpha}\epsilon$$

then there is nothing further to show. Thus, we need to show a bound including the approximation error if $\|\hat{x} - x^\natural\|_2 > \frac{2}{\alpha}\epsilon$. In this case, we obtain with the triangle inequality and the consistency condition $\|A\hat{x} - y\| = \|A\hat{x} - Ax^\natural - \eta\| \leq \epsilon$ that

$$\|A(\hat{x} - x^\natural)\|_2 \leq \|A\hat{x} - Ax^\natural - \eta\|_2 + \|\eta\|_2 \leq 2\epsilon\,.$$

In combination with the assumed lower bound on the recovery error, we then get

$$\|A(\hat{x} - x^\natural)\|_2 < \alpha\|\hat{x} - x^\natural\|_2\,,$$

so $\hat{x} - x^\natural$ is in the approximate null-space $\mathcal{N}_\alpha(A)$. The robust width property then implies

$$\|\hat{x} - x^\natural\|_2 \leq \rho \|\hat{x} - x^\natural\|_1 \,.$$

Next, using the properties from the lemma, this gives

$$\|z_2\|_1 \leq \sqrt{s}\|\hat{x} - x^\natural\|_2 \leq \rho\sqrt{s}\|\hat{x} - x^\natural\|_1 \,.$$

In inequality (1), the second term is then according to the lemma and the robust-width property bounded by $2\|z_2\|_1 \leq 2\sqrt{s}\|\hat{x} - x^\natural\|_2 \leq 2\sqrt{s}\rho\|\hat{x} - x^\natural\|_1$. We can thus bound the recovery error in terms of the approximation error,

$$\|\hat{x} - x^\natural\|_1 \leq \frac{2}{1 - 2\sqrt{s}\rho}\|x^\natural - a\|_1$$

all in terms of the ℓ_1-norm. Using the robust width again, we then have

$$\|\hat{x} - x^\natural\| \leq \rho\|\hat{x} - x^\natural\|_1 \leq \frac{2\rho}{1 - 2\sqrt{s}\rho}\|x^\natural - a\|_1 \,.$$

Now the assumption on ρ allows us to estimate the denominator by $1 - 2\sqrt{s}\rho \geq 1/2$, so combining the estimates for the two cases we considered gives

$$\|\hat{x} - x^\natural\| \leq \max\{4\rho\|x^\natural - a\|_1, \frac{2}{\alpha}\epsilon\} \,.$$

Further estimating the maximum by the sum of the two terms gives the claimed bound □

2.2. **Restricted isometry property implies robust width.** Next, we show that the condition of robust with is weaker than the restricted isometry property.

We prepare this result with two lemmas. The first one is a tail bound for vectors. Given $h \in \ell_n^2$, we iteratively apply the best s-term approximation to it to obtain h_0 such that

$$\|h - h_0\| \leq \min\{\|h - z\| : z \in \Sigma_s\}$$

and inductively select h_{j+1} as optimal s-sparse approximation to the tail $h - \sum_{k=0}^{j} h_k$,

$$\|h - \sum_{k=0}^{j+1} h_k\| \leq \min\{\|h - \sum_{k=0}^{j} h_k - z\| : z \in \Sigma_s\} \,.$$

Lemma 2.5 (Candès (2008)). *Let $h \in \ell_n^2$ and $\{h_j\}$ selected as described above, then*

$$\|h - h_0 - h_1\| \leq s^{-1/2}\|h - h_0\|_1 \,.$$

Proof. Using the equivalence of norms and the fact that the entries in h_j are monotonically decreasing in magnitude, for $j \geq 1$, we have

$$\|h_j\| \leq \sqrt{s}\|h_j\|_\infty \leq \frac{1}{\sqrt{s}}\|h_{j-1}\|_1 \,.$$

Since the supports of h_j are disjoint, we can combine terms because of the additivity of the ℓ^1-norm to obtain

$$\sum_{j\geq 2}\|h_j\| \leq s^{-1/2}\sum_{j\geq 1}\|h_j\|_1 = s^{-1/2}\|h - h_0\|_1 \,.$$

Combined with the triangle inequality, we conclude

$$\|h - h_0 - h_1\| \leq \sum_{j\geq 2} \|h_j\| \leq s^{-1/2}\|h - h_0\|_1 \,.$$

□

The next lemma states that h_j and h_l with $j \neq l$ retain (near)-orthogonality when A is applied to them. This is a consequence of converting between norms and inner products with the parallelogram identity.

Lemma 2.6 (Candès (2008))**.** *If A has the restricted isometry constant δ_{2s}, and $h_j, h_l \in \Sigma_s$ are orthogonal $\langle h_j, h_l\rangle = 0$, then*

$$|\langle Ah_j, Ah_l\rangle| \leq \delta_{2s}\|h_j\|\|h_l\| \,.$$

Proof. By the homogeneity of both sides in h_j and h_l, it is enough to consider the case $\|h_j\| = \|h_l\| = 1$. From the restricted isometry constant and the fact that the sum or difference of two s-sparse vectors is $2s$-sparse, we get

$$(1 - \delta_{2s})\|h_j \pm h_l\|^2 \leq \|A(h_j \pm h_l)\|^2 \leq (1 + \delta_{2s})\|h_j \pm h_l\|^2 \,.$$

Using the parallelogram identity, this yields

$$|\langle Ah_j, Ah_l\rangle| = \frac{1}{4}\left|\|A(h_j + h_l)\|^2 - \|A(h_j - h_l)\|^2\right| \leq \frac{1}{4}|(1+\delta_2 s\|h_j+h_l\|^2-(1-\delta_{2s})\|h_j-h_l\|^2| = \delta_{2s} \,.$$

In the last step, we have used the normalization and orthogonality if h_j and h_l. □

Next we show that the condition of the restricted isometry property implies robust width. This is indirectly a consequence of the results in the paper by Cahill and Mixon, because robust width is equivalent to performance guarantees for sparse recovery by ℓ^1-norm minimization. Hence, the proof of robust and stable recovery based on the RIP implies the robust width property. Here, we wish to give a more direct explanation for this implication.

Theorem 2.7. *If the $m \times n$ matrix A has restricted isometry constant δ_{2s}, then for any $\alpha < \frac{1-\delta_{2s}}{\sqrt{1+\delta_{2s}}}$, there exists $\rho > 0$ such that if $h \in \mathbb{R}^n$ satisfies $\|Ah\| < \alpha\|h\|$, then $\|h\|_2 \leq \rho\|h\|_1$.*

Proof. Given $\alpha > 0$ as assumed, let $h \in \mathcal{N}_\alpha(A)$, $\eta = Ah$, so $\|\eta\| < \alpha\|h\|$.

We split h into its components given by successive s-sparse approximations. Using the restricted isometry property,

$$\begin{aligned}(1 - \delta_{2s})\|h_0 + h_1\|^2 &\leq \|A(h_0 + h_1)\|^2 \\ &= |\langle A(h_0 + h_1), A(h - \sum_{j\geq 2} h_j)\rangle| \\ &\leq |\langle A(h_0 + h_1), Ah\rangle| + |\langle A(h_0 + h_1), A\sum_{j\geq 2} h_j\rangle|.\end{aligned}$$

The first term is bounded by

$$|\langle A(h_0+h_1), Ah\rangle| \le \|A(h_0+h_1)\|\|Ah\| \le \sqrt{1+\delta_{2s}}\|h_0+h_1\|\alpha\|h\|$$

through the restricted isometry constant and the assumption on $h \in \mathcal{N}_\alpha(A)$.

The second term can be bounded by

$$|\langle A(h_0+h_1), A\sum_{j\ge 2} h_j\rangle| \le |\langle Ah_0, A\sum_{j\ge 2}\rangle| + |\langle Ah_1, A\sum_{j\ge 2}\rangle|$$
$$\delta_{2s}(\|h_0\|+\|h_1\|)\sum_{j\ge 2}\|h_j\| \le \sqrt{2}\delta_{2s}\sqrt{\|h_0\|^2+\|h_1\|^2}s^{-1/2}\|h-h_0\|_1$$

where we have used the second lemma and converted between the sum of the norms to the Euclidean norm of $(\|h_0\|, \|h_1\|)$.

We combine the estimates for the two terms and simplify to get

$$(1-\delta_{2s})\|h_0+h_1\|^2 \le \sqrt{1+\delta_{2s}}\|h_0+h_1\|\alpha\|h\| + \sqrt{2}\delta_{2s}\|h_0+h_1\|s^{-1/2}\|h-h_0\|_1\,.$$

Rearranging terms gives

$$\|h_0+h_1\| \le \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\alpha\|h\| + \frac{\sqrt{2}\delta_{2s}}{s^{1/2}(1-\delta_{2s})}\|h-h_0\|_1\,.$$

We conclude by applying the triangle inequality and inserting the estimates

$$\|h\|_2 \le \|h-h_0-h_1\| + \|h_0+h_1\| \le \Big(1+\frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}\Big)s^{-1/2}\|h-h_0\|_1 + \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\alpha\|h\|_2\,.$$

Rearranging terms and estimating $\|h-h_0\|_1 \le \|h\|_1$ gives the desired result $\|h\| \le \rho\|h\|_1$ with

$$\rho = \Big(1-\frac{\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\alpha\Big)^{-1}\Big(1+\frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}\Big)s^{-1/2}\,.$$

□

3. Chernoff bounds for matrices

In this section we review a matrix concentration result by Tropp and examine its consequences.

We recall the classical Chernoff bound: If random variables $X_1, X_2, \ldots, X_m$ are independent with values in $[0, R]$ and the expected value of their sum is $\mu = \sum_{j=1}^m \mathbb{E}[X_j]$, then for $0 \le \delta \le 1$, then

$$\mathbb{P}[\sum_{j=1}^m X_j \ge (1+\delta)\mu] \le e^{-\delta^2\mu/3R}$$

and

$$\mathbb{P}[\sum_{j=1}^m X_j \le (1-\delta)\mu] \le e^{-\delta^2\mu/2R}\,.$$

We compare this with a theorem by Tropp.

Theorem 3.1 (Tropp (2012))**.** *Let $X_1, X_2, \dots X_m$ be independent, random real-valued symmetric $s \times s$ matrices satisfying the operator inequality $0 \le X_j \le RI$ for each j and $\sum_{j=1}^m \mathbb{E}[X_j] = \mu I$, then if $0 \le \delta \le 1$,*

$$\mathbb{P}[\lambda_{\max}(\sum_{j=1}^m X_j) \ge (1+\delta)\mu] \le s e^{-\delta^2 \mu/3R}$$

and

$$\mathbb{P}[\lambda_{\min}(\sum_{j=1}^m X_j) \le (1-\delta)\mu] \le s e^{-\delta^2 \mu/2R} .$$

In the following, we examine the bound for $\lambda_{\max}$. A first step is the use of the Markov inequality.

Lemma 3.2. *Let $\{X_j\}_{j=1}^m$ be as above, $\theta > 0$, then*

$$\mathbb{P}[\lambda_{\max}(\sum_{j=1}^m X_j) \ge t] \le e^{-\theta t}\mathbb{E}[\mathrm{tr}[e^{\theta \sum_{j=1}^m X_j}]] .$$

Proof. We first scale the inequality with θ and use monotonicity of the exponential,

$$\begin{aligned}\mathbb{P}[\lambda_{\max}(\sum_{j=1}^m X_j) \ge t] &= \mathbb{P}[\lambda_{\max}(\theta\sum_{j=1}^m X_j) \ge \theta t] \\ &= \mathbb{P}[e^{\lambda_{\max}(\theta \sum_{j=1}^m X_j)} \ge e^{\theta} t] \\ &= e^{-\theta t}\mathbb{E}[e^{\lambda_{\max}(\theta \sum_{j=1}^m X_j)}].\end{aligned}$$

The last step is the application of Markov's inequality.

Next, we replace $\lambda_{\max}$ with the sum over all eigenvalues,

$$e^{\lambda_{\max}(\theta \sum_{j=1}^m X_j)} = \lambda_{\max}\left(e^{\theta \sum_{j=1}^m X_j}\right) \le \mathrm{tr}[e^{\theta \sum_{j=1}^m X_j}] .$$

□

For the next estimate, we require a concavity result by Lieb and its consequences.

Theorem 3.3. *Let Y be a strictly positive definite $s \times s$ matrix, then the map*

$$X \mapsto \mathrm{tr}[e^{\ln X + \ln Y}]$$

is a concave real-valued function on strictly postive definite $s \times s$ matrices.

Corollary 3.4. *Let Y be a strictly positive definite $s \times s$ matrix and X a random strictly positive definite $s \times s$ matrix, then*

$$\mathbb{E}[\mathrm{tr}[e^{\ln X + \ln Y}]] \le \mathrm{tr}[e^{\ln \mathbb{E}[X] + \ln Y}] .$$

This result can be iterated to treat more terms in the exponent.

Corollary 3.5. *Let $X_1, X_2, \ldots, X_m$ be independent, random strictly positive definite matrices, then*

$$\mathbb{E}[\mathrm{tr}[e^{\sum_{j=1}^m \ln X_j}]] \leq \mathrm{tr}[e^{\sum_{j=1}^m \ln \mathbb{E}[X_j]}]$$

Proof. In the j-th step, fix $X_{j+1}, \ldots X_m$, define

$$Y_j = e^{\sum_{k=1}^{j-1} \ln \mathbb{E}[X_k] + \sum_{k=j+1}^m \ln X_k}$$

and then average over X_j using the preceding corollary. □

In our context, this gives the following estimate.

Corollary 3.6. *With X_j as above,*

$$\mathbb{P}[\lambda_{\max}(\sum_{j=1}^m X_j) \geq t] \leq e^{-\theta t}\,\mathrm{tr}[e^{\sum_{j=1}^m \ln \mathbb{E}[e^{\theta X_j}]}]\,.$$

To estimate further, we require three additional lemmas.

Lemma 3.7. *If X is a random Hermitian matrix with $0 \leq X \leq I$, then*

$$\mathbb{E}[e^{\theta X}] \leq I + (e^\theta - 1)\mathbb{E}[X]\,.$$

Proof. Exercise. □

Lemma 3.8. *The function $f(t) = \ln t$ is operator monotone on the set of strictly positive definite matrices.*

Proof. See Bhatia's book on Positive Definite Matrices. □

Lemma 3.9. *If f is monotonic on $\mathbb{R}$, then $f \mapsto \mathrm{tr}[f(X)]$ is monotonic on Hermitian matrices.*

Proof. Using the spectral representation and Weyl's theorem on the ordering of eigenvalues. □

Now we are ready to prove the matrix version of Chernoff's inequality.

Proof of Tropp's Matrix Chernoff inequality. Consider the special case $R = 1$, otherwise replace X_j by X_j/R and μ by μ/R.

By operator concavity of the logarithm,

$$\sum_{j=1}^m \ln \mathbb{E}[e^{\theta X_j}] \leq m \ln\Big(\frac{1}{m}\sum_{j=1}^m \mathbb{E}[e^{\theta X_j}]\Big)\,.$$

Using this inequality in the exponent together with the monotonicity of the trace exponential function, we have

$$\mathrm{tr}[\exp(\sum_{j=1}^m \ln \mathbb{E}[e^{\theta X_j}]) \leq \mathrm{tr}[\exp(m \ln(\frac{1}{m}\sum_{j=1}^m \mathbb{E}[e^{\theta X_j}]))]\,.$$

Next, we replace the trace by s times the largest eigenvalue and use monotonicity properties,

$$\begin{aligned}\mathrm{tr}[\exp(\sum_{j=1}^m \ln \mathbb{E}[e^{\theta X_j}])] &\le s\lambda_{\max}(\exp(\sum_{j=1}^m \ln \mathbb{E}[e^{\theta X_j}])) \\ &= s\exp(m \ln \lambda_{\max}(\frac{1}{m}\sum_{j=1}^m \ln \mathbb{E}[e^{\theta X_j}])\end{aligned}$$

Now applying the second lemma gives

$$\begin{aligned}\mathrm{tr}[\exp(\sum_{j=1}^m \ln \mathbb{E}[e^{\theta X_j}])] s\exp(m\ln\lambda_{\max}(I+\frac{1}{m}\sum_{j=1}^m (e^\theta-1)\mathbb{E}[X_j])) \\ = s\exp(m\ln(1+\frac{e^\theta-1}{m}\lambda_{\max}(\sum_{j=1}^m \mathbb{E}[X_j]))) \\ \le s\exp((e^\theta-1)\lambda_{\max}(\sum_{j=1}^m \mathbb{E}[X_j])).\end{aligned}$$

We recall that the expected value of $\sum_{j=1}^m X_j$ is μI, so using the bound obtained with Markov's inequality together with the directly preceding result and the choice $\theta = \ln(1+\delta)$ and $t = (1+\delta)\mu$ gives

$$\mathbb{P}[\lambda_{\max}(\sum_{j=1}^m X_j) \ge (1+\delta)\mu] \le e^{-(1+\delta)\mu\ln(1+\delta)} s e^{\delta\mu}$$

Finally, inspecting the Taylor expansion of $\ln(1+\delta)$ shows

$$e^{\delta-(1+\delta)\ln(1+\delta)} \le e^{-\delta^2/3}\,.$$

The general case for which we assume $0 \le X_j \le RI$ can now be obtained by rescaling, so in that case μ is replaced by μ/R. □

4. Applications of matrix concentration

4.1. **A covariance estimate.** Let $\{\eta_j\}_{j=1}^m$ be independent, identically distributed with $\mathbb{E}[\eta_j] = 0$, $\|\eta_j\| \le \sqrt{M}$ and covariance matrix $C = \mathbb{E}[\eta_1\eta_1^*]$. We wish to estimate C with the empirical covariance matrix $\frac{1}{m}\sum_{j=1}^m \eta_j\eta_j^*$ obtained from observing m outcomes.

Lemma 4.1. *The operator norm of C is bounded by $\|C\| \le M$.*

Proof. We have by the convexity of the norm

$$\|C\| = \|\mathbb{E}[\eta_1\eta_1^*]\| \le \mathbb{E}[\|\eta_1\eta_1^*\|]$$

The norm of the rank-one Hermitian $\eta_1\eta_1^*$ equals the squared vector norm, so $\|\eta_1\|^2 \le M$ finishes the proof. □

A natural measure for concentration should scale with $\|C\|$, so we want to estimate the probability of the set

$$\{\eta_j : \|\frac{1}{m}\sum_{j=1}^m \eta_j\eta_j^* - C\| \geq \delta\|C\|\}\,.$$

Assuming C is strictly positive definite, let $\xi_j = C^{-1/2}\eta_j$, then $E[\xi_j\xi_j^*] = I$. We also define $Y_j = \eta_j\eta_j^*$ and $X_j = C^{-1/2}Y_jC^{-1/2}$.

Corollary 4.2. *The bound for the operator norm of each X_j is*

$$\|X_j\| = \|C^{-1/2}Y_jC^{-1/2}\| \leq \|C^{-1}\|\|Y_j\| \leq M\|C^{-1}\|\,.$$

Theorem 4.3. *Let $\{\eta_j\}$ be as described, $Y_j = \eta_j\eta_j^*$, then*

$$\mathbb{P}[\|\frac{1}{m}\sum_{j=1}^m Y_j - C\| \geq \delta\|C\|] \leq 2se^{-m\delta^2/(3M\|C^{-1}\|)}\,.$$

Proof. We note that with our choice of ξ_j, $\mu = m$. Comparing X_j and $Y_j = C^{1/2}X_jC^{1/2}$ shows that

$$-\delta I \leq \sum_{j=1}^m X_j - mI \leq \delta I$$

is equivalent to

$$-\delta C/m \leq \frac{1}{m}\sum_{j=1}^m Y_j - C \leq \delta C/m\,.$$

Further weakening the estimate increases the probability, so if $-\delta I \leq \sum_{j=1}^m X_j - mI \leq \delta I$, then

$$-(\delta\|C\|/m)I \leq \frac{1}{m}\sum_{j=1} Y_j - C \leq (\delta\|C\|/m)I\,.$$

This means the probability of the complement is bounded by

$$\mathbb{P}[\|\frac{1}{m}\sum_{j=1}^m Y_j - C\| \geq \delta\|C\|/m] \leq \mathbb{P}[\|\sum_{j=1}^m X_j - mI\| \geq \delta] \leq 2se^{-\delta^2 m/3R}$$

with $R = M\|C^{-1}\|$. $\square$

A more sophisticated form of sampling in which small vectors are rejected gives better bounds when the smallest eigenvalue of C is close to zero, see Vershynin's book.

4.2. **Randomized construction of sensing matrices.** We review the paper by Rudelson and Vershynin on the construction of sensing matrices with entries that are of equal magnitude. An example of this type of measurement is the random frequency selection in the complex case, or a random subsampling of the rows of Hadamard matrices in the real case. We allow for more randomness and consider matrices whose entries are a constant with independent, uniformly at random chosen signs.

Theorem 4.4 (Rudelson and Vershynin)**.** *There are constants* $C, c > 0$ *such that for any* $t > 1$*,* $n, s > 2$*, if a random matrix* A *is chosen whose number of rows is binomially distributed among the outcomes* $\{1, 2, \ldots, n\}$ *with average value*

$$\overline{m} = \mathbb{E}[m] = Cts \ln n \ln(Cts \ln n) \ln^2 s$$

and the entries are chosen among $\{\pm\frac{1}{\sqrt{\overline{m}}}\}$ *with equal probability, then* A *has restricted isometry constant* $\delta_s \leq \sqrt{2} - 1$ *with probability* $1 - 5e^{-ct}$.