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Asymptotic Distribution of Quadratic Forms and Applications

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We consider the quadratic forms

$$Q = \sum_{\substack{1 \leqslant j, k \leqslant N \\ j \neq k}} a_{jk} X_j X_k + \sum_{j=1}^{N} a_{jj} (X_j^2 - \mathbf{E} X_j^2)$$

where X_j are i.i.d. random variables with finite sixth moment. For a large class of matrices (a_{jk}) the distribution of Q can be approximated by the distribution of a second order polynomial in Gaussian random variables. We provide optimal bounds for the Kolmogorov distance between these distributions, extending previous results for matrices with zero diagonals to the general case. Furthermore, applications to quadratic forms of ARMA-processes, goodness-of-fit as well as spacing statistics are included.

KEY WORDS: Independent random variables; quadratic forms; asymptotics of distribution; limit theorems; Berry–Esseen bounds.

1. INTRODUCTION AND RESULTS

Let $X, X_1, X_2, \ldots,$ be independent identically distributed (i.i.d.) random variables (r.v.'s) with $\mathbf{E} X = 0$. Let $A = \{a_{jk}\}_{j,k=1}^{N}$ denote an $N \times N$ matrix with a_{jk} possibly depending on N (and other parameters). Consider the quadratic form

$$Q = \sum_{\substack{1 \leqslant j, k \leqslant N \\ j \neq k}} a_{jk} X_j X_k + \sum_{j=1}^{N} a_{jj} (X_j^2 - \mathbf{E} X_j^2) \tag{1.1}$$

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Here the case $N=\infty$ is included (see Remark 1.8). The aim of this paper is to define an appropriate distribution and to estimate the error of this approximation explicitly in terms of A and the distribution of X_j.

The class of approximating distribution of the statistics Q is the same as for U-statistics or von Mises statistics (see, for example, Sevastjanov[25]). Note that the diagonal elements of the quadratic forms Q are essential for the description of the limit distribution in terms of characteristics of the matrix A.

If $V^2=\sum_{j=1}^N a_{jj}^2=0$, the approximating distributions of Q depend on the mean and variance of X only (see Rotar'[20]). They may be represented as the distribution of a quadratic form in Gaussian r.v.'s. The accuracy of this approximation has been studied in the papers of Rotar' *et al.*[8, 21] The optimal bounds in the i.i.d. case have been obtained in Götze and Tikhomirov.[16]

If $V^2\neq 0$ the approximating distribution for Q depends on $\mathbf{E}X^3$ and $\mathbf{E}X^4$. For example, if $\mathbf{E}X^2=1$, then the statistic

$$Q=N^{-1/2}\sum_{j=1}^{N}(X_j^2-1)+N^{-1}\sum_{\substack{1\leqslant j,k\leqslant N\\ j\neq k}}X_jX_k$$

is approximated by the distribution of the r.v.

$$G=\sqrt{\mu_4-1-\mu_3^2}\;Y+(Y_1+\mu_3/2)^2-1-\mu_3^2/4$$

where $\mu_k:=\mathbf{E}X^k$ and $Y, Y_1,\ldots$ denote i.i.d. Gaussian r.v.'s with $\mathbf{E}Y=0$ and $\mathbf{E}Y^2=1$. This is easily seen by representing Q in the form

$$Q=N^{-1}\left(\sum_{j=1}^{N}X_j\right)^2-N^{-1}\left(\sum_{j=1}^{N}X_j^2\right)+N^{-1/2}\sum_{j=1}^{N}(X_j^2-1)$$

and by introducing the bivariate vectors

$$\zeta_j=(X_j^2-1,\,X_j)$$

Using of the CLT for the i.i.d. random vectors ζ_j, $j=1,\ldots,N$ yields an approximate distribution given by G.

It is easy to see that (1.1) is an orthogonal decomposition of Q into a "quadratic" and a "linear" part as studied by Hoeffding. In the case when the "linear" part dominates the distribution of Q, the normal approximations have been studied intensively in Bentkus *et al.*[6] and Alberink and Bentkus.[1] The class (1.1) may be regarded as a special type of generalized von Mises statistics of the form $\sum_{j,k=1}^{N} h_{jk}(X_j, X_k)$ which has been intensively studied in the "identical" case $h_{jk}(x, y) = h(x, y)$ in Götze,[13, 14] Bentkus and Götze.[3–5]

Let $\mathbf{X} = (X_1, ..., X_N) \in R^N$. Denote by A_0 the matrix with entries $A_0(j, j) = 0$, and $A_0(j, k) = a_{jk}$, if $1 \leqslant j, k \leqslant N$, $j \neq k$. Let $\lambda_1, ..., \lambda_N$ be the eigenvalues A_0.

Let $\mathbf{Y}$ be a standard Gaussian vector in $\mathbb{R}^N$ and let $\bar{\mathbf{Y}}$ be an independent copy of $\mathbf{Y}$. Let

$$G = \mu_2 \langle A_0 \mathbf{Y}, \mathbf{Y} \rangle + \mu_3 \mu_2^{-1/2} \langle \mathbf{d}, \mathbf{Y} \rangle + \sqrt{\bar{\mu}_4} \, \langle \mathbf{d}, \bar{\mathbf{Y}} \rangle \tag{1.2}$$

where the vector $\mathbf{d} \in \mathbb{R}^N$ has coordinates $(a_{11}, ..., a_{NN})$, and $\bar{\mu}_4 = \mu_4 - \mu_2^2 - \mu_3^2 \mu_2^{-1}$. Here and in what follows $\langle \cdot, \cdot \rangle$ denote the scalar product in Euclidean spaces.

Note that the inequality $|\mathbf{E}(X^2 - \mathbf{E}X^2) X|^2 \leqslant \mathbf{E}(X^2 - \mathbf{E}X^2)^2 \, \mathbf{E}X^2$ implies that $\bar{\mu}_4 \geqslant 0$. The equality $\bar{\mu}_4 = 0$ implies that X is a Rademacher r.v. such $\mathbf{P}\{X = \pm 1\} = 1/2$, so that in this case $Q = \sum_{j \neq k} a_{jk} X_j X_k$. This has been studied in [Götze and Tikhomirov[16]].

The aim of this paper is to study the error of the approximation of Q by G explicitly, that is in terms of moments of X_j, norms and a finite number of the largest eigenvalues of A. This allows us to apply the results for instance to infinite dimensional forms ($N = \infty$) in the moving average processes without change in Corollary 2.1. (About the choice of such approximate distribution see Remark 1.8 below.)

Let us denote

$$\mathscr{L}_j^2 = \sum_{k=1}^{N} a_{jk}^2, \qquad \mathscr{L}^2 = \max_{1 \leqslant j \leqslant N} \mathscr{L}_j^2, \qquad \|A\|^2 = \sum_{j,k=1}^{N} a_{jk}^2$$

Let F denote the distribution function (d.f.) of X and $\operatorname{Var}(Q)$ denotes the variance of Q,

$$\operatorname{Var}(Q) = 2(\|A\|^2 - V^2) \, \mu_2^2 + V^2(\mu_4 - \mu_2^2)$$

We will write $\Phi(x)$ for the standard Gaussian distribution function. We introduce the following quantities

$$\delta(A, F) := \sup_x |\mathbf{P}\{Q \leqslant x\} - \mathbf{P}\{G \leqslant x\}|$$

and

$$\Delta(A, F) := \sup_x |\mathbf{P}\{Q/\sqrt{\operatorname{Var}(Q)} \leqslant x\} - \Phi(x)|$$

Throughout this paper C (indexed or not) denotes absolute constants, whereas $C(\cdot,\cdot)$ is used for constants depending on the arguments in parentheses. By Λ_k, $k=1, 2,..., N$, we denote the eigenvalues of A ordered in such a way that $|\Lambda_1| \geqslant \cdots \geqslant |\Lambda_N|$. Note that $\sum_{k=1}^{N} \Lambda_k^2 = \|A\|^2$. We consider the case with non dominating "linear" part. Unlike the case of a matrix A with zero diagonal there is not a single class of distributions which approximates the distribution of Q *uniformly* and with prescribed accuracy on a large classes of coefficients matrices A in the presence of sufficiently large diagonal. Thus we shall assume at first that there exists some absolute positive constant b_1^2 such that

$$1 - V^2/\|A\|^2 \geqslant b_1^2 \tag{1.3}$$

This means that the "non-diagonal" part of $\|A\|^2$ is uniformly in N bounded away from zero and thus has a non negligible influence on the distribution of Q. Introduce in addition $\beta_k = \mathbf{E}\,|X|^k$. Assume that

$$\mu_6 = \beta_6 < \infty \tag{1.4}$$

In the case where the maximal absolute eigenvalue $|\Lambda_1|$ of A is small with respect to $\|A\|$ the distribution of Q is approximately normal, and so we have

Theorem 1.1. Assume that the conditions (1.3) and (1.4) hold. Then there exists an absolute constant C such that

$$\Delta(A, F) \leqslant C b_1^{-2} \alpha_1\, |\Lambda_1|\, \|A\|^{-1}$$

where $\alpha_1 = (\beta_3^2 + V\, \|A\|^{-1}\, \beta_6)\, \mu_2^{-3}$.

If there are $q \geqslant 1$ eigenvalues of A which are not "small," i.e., the inequality

$$|\Lambda_q| \geqslant b_0\, \|A\| \tag{1.5}$$

holds for some $b_0 > 0$, then the distribution of Q may be approximated by a weighted χ^2-distribution. The weights of corresponding χ^2-components are the eigenvalues of A. The optimal estimate (with respect to $\mathscr{L}\, \|A\|^{-1}$) we get, due to our methods when $q = 17$, namely:

Theorem 1.2. Assume that the conditions (1.4) and (1.5) hold with $q = 17$. Then there exists a constant C such that

$$\delta(A, F) \leqslant C \alpha_1 \mathscr{L}\, \|A\|^{-1}\, (b_0)^{-17/4}$$

Without loss of generality we can assume that $\mathscr{L}\, \|A\|^{-1} \leqslant b_0/2$. This assumption and condition (1.5) for $q=1$ together imply (1.3) with $b_1^2 = b_0^2/4$.

Note that the class of quadratic forms (1.1) includes by an obvious choice of a_{jk} for example quadratic forms of sums of d-dimensional random vectors, like

$$Q = \sum_{i,\, j=1}^{d} q_{ij} S_{m,\, j} S_{m,\, i}, \qquad \text{where} \quad N = md, \qquad \text{and}$$

$$S_{m,\, j} = (Z_{1,\, j} + \cdots + Z_{m,\, j})\, m^{-1/2}, \qquad j = 1,..., d, \quad \text{and} \quad Z_{l,\, j} = X_{jd+l}$$

In this case the non zero eigenvalues of A are proportional to the d eigenvalues of the symmetric matrix $Q = (q_{ij})$. Thus the problem of approximating (1.1) is an extension of the old problem of determining the uniform rate of convergence as $m \to \infty$ in the multivariate CLT for e.g. ellipsoids in Euclidian space $\mathbb{R}^d$ or Hilbert space, $d = \infty$, (provided that Q is of trace class). It is well known, see Senatov[24] that a rate of order $m^{-1/2}$ (resp. $\|A\|^{-1/2}$) requires that 6 absolute eigenvalues of Q and hence of A have to be uniformly bounded away from zero. For an Edgeworth type approximation of order m^{-1} (resp. $\|A\|^{-1}$) this number has to be at least 12, see Götze and Ulyanov.[17] As for upper bound in this problem compare the review of Götze at ICM (1998). In the case when the diagonal of A contributes significantly to the approximating distribution, and $\bar{\mu}_4 > 0$ (that is, the r.v. X is not a Rademacher r.v.), the following results hold.

Theorem 1.3. Assume that condition (1.4) is fulfilled. If (1.5) holds with $q = 9$, and for some $b_2^2 > 0$

$$V^2\, \|A\|^{-2} \geqslant b_2^2 \tag{1.6}$$

then there exists a constant C such that

$$\delta(A, F) \leqslant C b_1^{-9/4} b_2^{-2} \alpha_1 \mathscr{L}\, \|A\|^{-1}\, \mu_2^2 \bar{\mu}_4^{-1}$$

The bound in Theorem 1.2 simplifies if the skewness of X_j vanishes. We have

Theorem 1.4. Assume that the condition (1.4) is fulfilled. If (1.5) holds for $q = 13$, and in addition $\mu_3 = 0$, then there exists a constant C such that

$$\delta(A, F) \leqslant C \alpha_1 \mathscr{L}\, \|A\|^{-1}\, b_1^{-13/4}$$

For the further results in the cases $2 \leqslant q \leqslant 16$ we have

Corollary 1.5. Let the condition (1.3) be fulfilled. Assume that condition (1.5) holds for some $2 \leqslant q \leqslant 8$. Then

$$\delta(A, F) \leqslant C b_2^{-2} b_0^{-q/2} \beta_6 \mu_2^{-3} \begin{cases} (\mathscr{L}/\|A\|)^{4/(12-q)}, & \text{if} \quad 2 \leqslant q \leqslant 7 \\ (\mathscr{L}/\|A\|) \log^+(\mathscr{L}/\|A\|), & \text{if} \quad q = 8 \end{cases}$$

Corollary 1.6. Let conditions (1.3) be fulfilled. Assume that condition (1.5) holds for some $2 \leqslant q \leqslant 16$. Then

$$\delta(A, F) \leqslant C \beta_6 \mu_2^{-3} b_0^{-q/2} \begin{cases} (L/\|A\|)^{4/(20-q)}, & \text{if} \quad 2 \leqslant q \leqslant 15 \\ (\mathscr{L}/\|A\|) \log^+(\mathscr{L}/\|A\|), & \text{if} \quad q = 16 \end{cases}$$

Corollary 1.7. Let conditions (1.3) be fulfilled and $\mu_3 = 0$. Assume that condition (1.5) holds for some $2 \leqslant q \leqslant 12$. Then

$$\delta(A, F) \leqslant C \beta_6 \mu_2^{-3} b_0^{-q/2} \begin{cases} (\mathscr{L}/\|A\|)^{4/(16-q)}, & \text{if} \quad 2 \leqslant q \leqslant 11 \\ (\mathscr{L}/\|A\|) \log^+(\mathscr{L}/\|A\|), & \text{if} \quad q = 12 \end{cases}$$

The estimates in Theorems 1.1–1.4 and in the Corollaries 1.5–1.7 a formulated for any fixed matrix A. Since the constants C are some absolute constants, we can consider these results as asymptotic results concerned with a sequence of quadratic forms

$$Q^{(n)} = \sum_{\substack{1 \leqslant j, k \leqslant N \\ j \neq k}} a_{jk}^{(n)} X_j X_k + \sum_{j=1}^{N} a_{jj}^{(n)} (X_j^2 - \mathbf{E} X_j^2)$$

as $n \to \infty$, where the parameter n is not necessarily identical to N. For given constants b_0 and b_1 the results are uniform for all matrices satisfying the conditions (1.3) and (1.5) or (1.3) and (1.6).

Uniform approximations of the same accuracy for cases between conditions (1.5) and (1.6) are difficult to prove since they lie on the border zone of the validity of the two approximations we have to use in these cases.

In the literature only special cases of quadratic forms with non-zero diagonal were considered so far: Rotar'[20] proved that the distribution of Q is close to that of G assuming that V is small with respect to $\|A\|$. Guttorp and Lockhard[19] obtained some modification of Rotar's result in the case of non-zero diagonal. Rotar' and collaborators[8, 21] gave an estimate of the rate of convergence of $\delta(A, F)$. Optimal bounds in the case $V = 0$ were obtained in Götze and Tikhomirov,[16] called GT for short in what follows. Here we extend this result to quadratic forms with non-zero diagonal part.

The rest of this paper is divided into Sections 2–6. In Section 2 we describe some applications of our main results to quadratic forms in moving average processes and to quadratic forms in spacings. Sections 3–6 are devoted to the proofs of Theorems 1.1–1.4. These proofs are based on a symmetrization inequality bounding characteristic functions (c.f.) of quadratic forms at *arbitrary* frequencies due to Götze (13) and the differential inequality method of Tikhomirov[26] for approximating c.f. at *small* frequencies. For quadratic forms this method differs from Stein's differential equation method. Since the computations are rather involved we shall give an outline of this method at the beginning of Section 3. The details of these proofs are provided in Sections 4 and 5. In Section 6 we shall derive the results of Theorem 1.1–1.4 using the bounds of Section 6 which are based on GT.

Remark 1.8. An inspection of the proofs of Theorems 1.1–1.4 shows that they still hold in the case, when $N=\infty$ and $\|A\|<\infty$.

Without loss of generality we shall assume that

$$\|A\|^2 = \sum_{j=1}^{N} \Lambda_k^2 = 1 \tag{1.7}$$

Remark 1.9. To find a suitable approximating distribution for quadratic forms with non-vanishing diagonal we consider a representation of Q as in the case of bivariate U-statistics in Bentkus and Götze (Ref. 5, p. 459). In the general case of (1.1) we may represent Q as a sum of a linear and quadratic form in a $2N$-dimensional random vector.

Let $u_1,\dots, u_N \in \mathbb{R}^N$ be the corresponding to $\lambda_1,\dots, \lambda_N$ eigenvectors A. Introduce the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{2N}$ with coordinates $(a_{11},\dots, a_{NN}, 0,\dots, 0)$ and $(X_1^2-\mathbf{E}X_1^2,\dots, X_N^2-\mathbf{E}X_N^2, \langle u_1, \mathbf{X}\rangle,\dots, \langle u_N, \mathbf{X}\rangle)$ respectively. Let $\widetilde{\mathbf{D}}$ denote the diagonal $2N\times 2N$ matrix with the first N diagonal entries equal to 0 and the last N diagonal entries equal to $\lambda_1,\dots, \lambda_N$ respectively. Then we may write Q as

$$Q = \langle \mathbf{a}, \mathbf{b}\rangle + \langle \widetilde{\mathbf{D}}\mathbf{b}, \mathbf{b}\rangle \tag{1.8}$$

Denote by $\mathbf{B}$ the covariance operator of the random vector $\mathbf{b}$. It is not difficult to see that the variance of Q depends only on $\mathbf{a}$, $\widetilde{\mathbf{D}}$ and $\mathbf{B}$. We may therefore expect that the limit distribution of Q also depends only on $\mathbf{B}$ (assuming that $\mathbf{a}$ and $\widetilde{\mathbf{D}}$ are fixed).

Let $\mathbf{g}$ denote a Gaussian random vector with covariance operator $\mathbf{B}$. Denote by G a statistics resulting from replacing $\mathbf{b}$ in (1.8) by $\mathbf{g}$. We shall now transform G_N to a simpler form. Let $\mathbf{U}$ denote the orthogonal matrix such that $\mathbf{D} := \mathbf{U}^{-1} A_0 \mathbf{U}$ is diagonal. Let $\mathbf{I}$ denote the $N \times N$ identity matrix. In these notations we get, after some trivial calculations,

$$\mathbf{B} = \begin{pmatrix} (\mu_4 - \mu_2^2)\, \mathbf{I} & \mu_3 \mathbf{U}^{-1} \\ \mu_3 \mathbf{U} & \mu_2 \mathbf{I} \end{pmatrix} \tag{1.9}$$

Using (1.9), we can now represent $\mathbf{g}$ as

$$\mathbf{g} = \begin{pmatrix} \sqrt{\bar{\mu}_4}\, \overline{\mathbf{Y}} + \mu_3 \mu_2^{-1/2} \mathbf{U}^{-1} \mathbf{Y} \\ \sqrt{\mu_2}\, \mathbf{Y} \end{pmatrix} \tag{1.10}$$

Substituting in (1.8) the vector $\mathbf{g}$ in the form (1.10) instead of $\mathbf{b}$ and recalling that $A_0 = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$ we get (1.2). We have used here that the vector $\mathbf{U}^{-1}\mathbf{Y}$ is standard Gaussian too.

2. APPLICATIONS

2.1. Moving-Average Processes

We shall assume that the r.v.'s X_j, $j = 0, \pm 1, \ldots$, are centered, i.i.d. with the variance one. Let $Z_1, Z_2, \ldots$ denote a process with

$$Z_k = \sum_{j=-\infty}^{\infty} b_{k-j} X_j, \qquad k = 1, 2, \ldots \tag{2.1}$$

where $\{b_j\}$ denotes a sequence of real numbers such that $\sum_{j=-\infty}^{\infty} b_j^2 < \infty$. This class contains, e.g., the class of autoregressive processes of order q ($AR(q)$) moving average processes of order p ($MA(p)$) and mixed autoregressive/moving average processes of order q, p ($ARMA(q, p)$). According to the rate of decrease of the sequence $\{b_j\}$, $j = 0, \pm 1, \ldots$, such processes are called long-range dependent or short memory processes with innovations $X_1, X_2, \ldots$. See Giraitis and Surgailis,[11] Samorodnitsky and Taqqu,[23] and Beran.[7]

We would like to study the distribution of the quadratic form

$$Q = \sum_{j,k=1}^{N} h_{jk} (Z_j Z_k - \mathbf{E} Z_j Z_k)$$

Without loss of generality we may assume the matrix $H = (h_{jk})_{j,k=1}^{N}$ to be symmetric.

Introduce the matrix $A=(a_{lm})$, $l, m=0, \pm 1,...$ defined by the equality

$$a_{lm}=\sum_{j,k=1}^{N} h_{jk} b_{j-l} b_{k-m}$$

With these notations we have

$$Q=\sum_{l,m=-\infty}^{\infty} a_{lm}(X_l X_m - \mathbf{E} X_l X_m) \tag{2.2}$$

Let $\|A\|_2^2=\sum_{l,m=-\infty}^{\infty} a_{lm}^2$. Under our assumptions $\|A\|_2<\infty$. We may use therefore the results of Theorems 1.1–1.4 to investigate the distribution of the quadratic form Q with $N=\infty$ (see Remark 1.8).

As a first application of Theorem 1.1 we determine the rate of approximation for the result of Giraitis and Surgailis.[12] Let $u(\theta)$ denote the spectral density of the sequence $Z_1, Z_2,...,$ and let h_j denote the jth Fourier coefficient of a function $v(\theta)=\sum_{j=-\infty}^{\infty} h_{|j|} e^{ij\theta}$. The functions $u(\theta)$ and $v(\theta)$ are integrable real symmetric functions on $[-\pi, \pi]$ that are bounded on subintervals that do not contain the origin. Take $h_{ij}=h_{|i-j|}$ and consider the quadratic form Q defined by (2.2). Assuming that $\{Z_k, k=0, \pm 1,...\}$ is a strongly dependent stationary Gaussian sequence, Avram[2] and Fox and Taqqu[9] proved the asymptotic normality of Q. Giraitis and Surgailis[12] extended their results to include the stationary long-range dependent non-Gaussian case.

Assume that

$$u(\theta) \leqslant C\,|\theta|^{-\alpha}, \quad |v(\theta)| \leqslant C\,|\theta|^{-\beta}, \qquad \text{where} \quad \alpha<1, \ \beta<1 \ \text{and} \ \alpha+\beta<\tfrac{1}{2} \tag{2.3}$$

Example of a function satisfying conditions of such type can be found, e.g., in Fox and Taqqu (Ref. 9, p. 214). [For more details about these conditions and about the limit theorems for quadratic forms in long-range dependence r.v.'s see, e.g., Fox and Taqqu,[9] Giraitis and Surgailis,[11] Giraitis and Surgailis[12] and Beran (Ref. 7, Chap. 3)]. Using the results of Fox and Taqqu[9] and Giraitis and Surgailis[12] and Taqqu[10] we derive the following convergence estimates.

Corollary 2.1. Assume that (2.3) and (1.4) are satisfied. For any $\varepsilon>0$ there exists a constant $C=C(\varepsilon, \alpha, \beta)$ such that

$$\sup_x |\mathbf{P}\{Q/\sqrt{\operatorname{Var} Q} \leqslant x\} - \Phi(x)| \leqslant C \mu_6 \mu_2^{-3} N^{-(1/2-(\alpha+\beta))+\varepsilon}$$

Proof. Note that for any symmetric matrix A and for any integer $p \geqslant 1$ the following inequality holds

$$|\Lambda_1| \leqslant (\mathrm{Tr}(A^{2p}))^{1/2p} \tag{2.4}$$

By Theorem 1b in Fox and Taqqu[9] it follows that for $p > 1/(\alpha+\beta)$ and any ε there exists some constant $C = C(\varepsilon, f, g)$ such that

$$\mathrm{Tr}(A^{2p}) \leqslant CN^{2p(\alpha+\beta)+\varepsilon} \tag{2.5}$$

Furthermore, by Theorem 1a in Fox and Taqqu[9] there exist positive constants $C_1 = C_1(\alpha, \beta)$ and $C_2 = C_2(\alpha, \beta)$ such that

$$C_1 N \leqslant \|A\|^2 \leqslant C_2 N \tag{2.6}$$

Now (2.5) and (2.6) together imply

$$|\Lambda_1| \; \|A\|^{-1} \leqslant CN^{-(1/2-(\alpha+\beta))+\varepsilon} \tag{2.7}$$

Repeating the proof of Theorem 1a in Fox and Taqqu,[9] we obtain that for a sufficiently large N and for some constant $\delta = \delta(\alpha, \beta) > 0$

$$1 - V^2 \, \|A\|^{-2} \geqslant \delta \tag{2.8}$$

The inequalities (2.7), (2.8) and Theorem 1.1 together conclude the proof. $\square$

In the case

$$\sum_{k=1}^{\infty} |r_k| = \Gamma_0 < \infty, \qquad r_k = \mathbf{E} Z_1 Z_{k+1} \tag{2.9}$$

and

$$\sum_{k=1}^{\infty} |h_k| = \Gamma_1 < \infty \tag{2.10}$$

we obtain a more precise result.

Corollary 2.2. Assume that (2.9) and (1.4) are satisfied. Assume also that

$$|v(\theta)| \leqslant C \, |\theta|^{-\beta}$$

where $\beta < 1/2$. Then there exists a constant C such that

$$\sup_x |\mathbf{P}\{Q/\sqrt{\operatorname{Var} Q} \leqslant x\} - \Phi(x)| \leqslant C\Gamma_0 \mu_6 \mu_2^{-3} N^{-1/2} \sum_{k=1}^{N} |h_k|$$

If in addition (2.10) holds, then

$$\sup_x |\mathbf{P}\{Q/\sqrt{\operatorname{Var} Q} \leqslant x\} - \Phi(x)| \leqslant C\Gamma_0 \Gamma_1 \mu_6 \mu_2^{-3} N^{-1/2}$$

Proof. For the proof of this result we need to find only a lower bound for $\|A\|$ and an upper bound for $|\Lambda_1|$. The lower bound for $\|A\|$ follows from results of Giraitis and Taqqu.[10] Note that (2.7) implies $|u(\theta)| \leqslant \Gamma_0$. Consequently the conditions (2.12) and (2.13) of Theorem 2.4 in Giraitis and Taqqu[10] are fulfilled with $\alpha = 0$. In particular this theorem implies that there exist constants C_1, C_2 such that

$$C_1 N \leqslant \|A\|^2 = \operatorname{Tr}(HRHR) \leqslant C_2 N \tag{2.11}$$

where $H = (h_{|l-j|})_{l,j=1}^{N}$, $R = (r_{|l-j|})_{l,j=1}^{N}$.

To bound $|\Lambda_1|$ note that

$$|\Lambda_1| = |\Lambda_1(HR)| \leqslant |\Lambda_1(H)|\,|\Lambda_1(R)|$$

where $\Lambda_1(HR)$, $\Lambda_1(H)$, $\Lambda_1(R)$ denote the maximal absolute eigenvalue of the matrix in parenthesis. Since H and R are Toeplitz matrices we have

$$|\Lambda_1(H)| \leqslant 2 \sum_{k=1}^{N} |h_k|, \qquad |\Lambda_1(R)| \leqslant 2\Gamma_0$$

Finally we obtain that

$$|\Lambda_1|\,\|A\|^{-1} \leqslant CN^{-1/2} \sum_{k=1}^{N} |h_k|$$

Combining the last inequality and Theorem 1.1, we finally obtain the stated result. □

2.2. *AR*(1) Processes

Let $Z_1, Z_2, \ldots$ denote a stationary process such that

$$Z_{k+1} = \beta Z_k + X_{k+1}$$

where X_j, $j=0, \pm 1,\ldots$ are i.i.d. r.v.'s and $|\beta|<1$. It is well known that

$$Z_k = \sum_{s=0}^{\infty} \beta^s X_{k-s} \tag{2.12}$$

From (2.12) it follows that

$$r_k = \mathbf{E} Z_0 Z_k = (k+1)\, \beta^k, \qquad \text{for} \quad k \geqslant 0 \tag{2.13}$$

Let $\lambda_1^{(h)},\ldots, \lambda_N^{(h)}$ denote eigenvalues of the matrix $H=(h_{jk})$ ordered in such a way that $|\lambda_1^{(h)}| \geqslant \cdots \geqslant |\lambda_N^{(h)}|$. Note that the spectral density $u(\theta)$ of the sequence $Z_1, Z_2,\ldots$ satisfies the inequalities

$$(1-\beta^4)/(1+|\beta|)^4 \leqslant u(\theta) \leqslant (1-\beta^4)/(1-|\beta|)^4 \tag{2.14}$$

The relations (2.13) and (2.14) imply that all eigenvalues of the matrix $R=(r_{|j-k|})$ are contained in the interval $[(1-\beta^4)/(1+|\beta|)^4,\ (1-\beta^4)/(1-|\beta|)^4]$, see for instance Grenander and Szegő (Ref. 18, p. 65). In particular, the spectral norm of the matrix R^{-1} is bounded as follows,

$$\|R^{-1}\| \leqslant (1+|\beta|)^4/(1-\beta^4) \tag{2.15}$$

Write $\mathscr{L}_h^2 = \max_k \sum_j h_{jk}^2$. Then the following result holds.

Corollary 2.3. Let $Z_1, Z_2,\ldots$ denote an $AR(1)$ process defined in (2.10). Assume that (2.11) holds and that the innovations X_j satisfy (1.4). Furthermore assume that

$$|\lambda_q^{(h)}| \geqslant b_0, \qquad \text{for} \quad q \geqslant 17 \tag{2.16}$$

Then there exists a constant $C=C(b_0)$ such that

$$\sup_x |\mathbf{P}\{Q \leqslant x\} - \mathbf{P}\{G \leqslant x\}| \leqslant C \mu_6 \mu_2^{-3} \mathscr{L}_h \|H\|_2^{-1}$$

Proof. This is an immediate consequence of

$$\|R^{-1}\|^{-1}\, |\lambda_q^h| \leqslant |\Lambda_q| \leqslant \|R\|\, |\lambda_q^h|$$

together with the inequalities (2.15), (2.16) and Theorem 1.2. □

2.3. Goodness-of-Fit Tests

Let $U_{(1)} < \cdots < U_{(N)}$ denote the order statistics of a sample of size n from a distribution on $[0, 1]$. A number of tests of the hypothesis that the distribution is uniform in $[0, 1]$ are based on statistics of the form

$$T = \sum_{l, j=1}^{N} h_{lj} \bar{U}_l \bar{U}_l$$

where $\bar{U}_j = U_{(j)} - j/(N+1)$, $j = 1, \ldots, N$. (See for instance Guttorp and Lockhard[19]). They consider the following statistic

$$T = \frac{1}{(1+\bar{X})^2} \langle A\mathbf{X}, \mathbf{X} \rangle$$

where $\mathbf{X} = (X_1, \ldots, X_{N+1})$ is a vector of i.i.d. centered exponentials (X_1 has density $p(x) = \exp\{-(x+1)\}$ for $x \geqslant -1$) and

$$\bar{X} = \frac{1}{N+1} \sum_{k=1}^{N+1} X_k$$

Here the matrix A is defined by the following equality

$$A = (I - J)\, F^T H F (I - J)$$

where F is the $n \times (N+1)$ matrix with entries $F(i, j) = \mathbf{I}_{\{i \geqslant j\}}/(N+1)$, $\mathbf{I}_{\{B\}}$ denotes the indicator function of an event B, I is the $(N+1) \times (N+1)$ identity matrix, and J denotes the $(N+1) \times (N+1)$ matrix with entries $J_{ij} = 1/(N+1)$. It is easy to see that

$$\mathbf{P}\left\{\left|1 - \frac{1}{(1+\bar{X})^2}\right| \leqslant CN^{-1/2} \sqrt{\log^+ N}\right\} \leqslant CN^{-1/2} \tag{2.17}$$

The last inequality implies

$$\sup_x |\mathbf{P}\{T \leqslant x\} - \mathbf{P}\{\langle A\mathbf{X}, \mathbf{X} \rangle \leqslant x\}| \leqslant CN^{-1/2} \sqrt{\log^+ N} \tag{2.18}$$

Consider now the quadratic form

$$Q = \langle A\mathbf{X}, \mathbf{X} \rangle - \mathbf{E}\langle A\mathbf{X}, \mathbf{X} \rangle$$

Write

$$T_1 = T - \mathbf{E}T$$

Recall that

$$G = \mu_2 \sum_{1 \leqslant i \neq j \leqslant N} a_{ij} Y_i Y_j + \mu_3 \mu_2^{-1/2} \sum_{i=1}^{N} a_{ii} Y_i + \sqrt{\bar{\mu}_4}\, V Y$$

where $Y, Y_1, \dots$ are i.i.d. standard Gaussian r.v.'s and note that in our case X_1 is a centered exponential distributed r.v. with $\mu_2 = 1$, $\mu_3 = 2$, $\mu_4 = 21$, hence

$$G = \sum_{1 \leqslant i \neq j \leqslant N} a_{ij} Y_i Y_j + 2 \sum_{i=1}^{N} a_{ii} Y_i + 4 V Y$$

We can reformulate Theorem 1.1 in this case as follows.

Corollary 2.4. There exists an absolute constant C such that

$$\sup_x |\mathbf{P}\{T_1 / \sqrt{\mathrm{Var}(T_1)} \leqslant x\} - \Phi(x)| \leqslant C(|A_1| \, \|A\|^{-1} + N^{-1/2} \sqrt{\log^+ N})$$

Write

$$\delta_N = \sup_x |\mathbf{P}\{T_1 \leqslant x\} - \mathbf{P}\{G \leqslant x\}|$$

Then Theorem 1.2 can be applied as follows.

Corollary 2.5. Assume that (1.5) is fulfilled for some $q \geqslant 17$. Then there exists a constant $C(b_0)$ such that

$$\delta_N \leqslant C(b_0)(\mathscr{L} \, \|A\|^{-1} + N^{-1/2} \sqrt{\log^+ N})$$

Note that Corollary 2.5 may be extended to the case $q < 17$ using Corollaries 6.1–6.3.

2.4. Spacing Statistics

Consider the m-spacing statistics

$$S_m = \sum_{k=0}^{N} (U_{k+m} - U_k)^2$$

where $U_0 = 0$, $U_{N+1} = 1$, $U_{N+1+k} = 1 + U_k$. Here we obtain the following result.

Corollary 2.6. Let $m \geqslant 1$. Then there exists a constant C such that

$$\sup_x |\mathbf{P}\{(S_m - \mathbf{E}S_m)/\sqrt{\operatorname{Var}(S_m)} \leqslant x\} - \Phi(x)| \leqslant CN^{-1/2}(\sqrt{\log^+ N} + \sqrt{m})$$

Proof. It follows from results of Guttorp and Lockhard (Ref. 19, Section 3) that

$$a_{ij} = \frac{(m - |i-j|_N)_+}{m^2} - \frac{1}{(N+1)} \tag{2.19}$$

where $|i-j|_N = \min\{|i-j|, N+1-|i-j|\}$ and $x_+ = \max\{0, x\}$. We note that $a_{ij} = a_{|i-j|}$, i.e., the matrix A is a Toeplitz matrix. A simple calculation shows that there exist constants C_1 and C_2 such that

$$C_1 N^{1/2} m^{-1/2} \leqslant \|A\| \leqslant C_2 N^{1/2} m^{-1/2} \tag{2.20}$$

From (2.19) we obtain that

$$\sum_{k=1}^{N} |a_k| \leqslant C \tag{2.21}$$

Using (2.20) inequality (2.21) yields

$$|\Lambda_1| \, \|A\|^{-1} \leqslant C \sqrt{m/N}$$

The last inequality and Corollary 2.4 together imply the result. □

Corollary 2.7. Let $m/N \to \gamma > 0$. Then there exist a constant C such that

$$\sup_x |\mathbf{P}\{T_1 \leqslant x\} - \mathbf{P}\{G \leqslant x\}| \leqslant CN^{-1/2} \sqrt{\log^+ N}$$

Proof. From results Guttorp and Lockhard[19] it follows that for any $q \geqslant 1$ there exist constants $\delta_q > 0$ such that for any $N \geqslant 1$

$$|\Lambda_q| \geqslant \delta_q$$

It is easy to show that

$$\mathscr{L}^2 \, \|A\|^{-2} \leqslant \sum_{k=1}^{N} a_k^2 \left(\sum_{k=0}^{N} (N-k) \, a_k^2 \right)^{-1} \leqslant CN^{-1} \tag{2.22}$$

The inequality (2.22) and Corollary 2.5 together imply the result. □

In the sequel we write $\mathrm{e}\{x\} := \exp\{itx\}$. Thus, the c.f. of Q and G can be written as

$$f(t) := \mathbf{E} \exp\{itQ\} = \mathbf{E}\mathrm{e}\{Q\}, \qquad g(t) := \mathbf{E} \exp\{itG\} = \mathbf{E}\mathrm{e}\{G\}$$

3. THE PROOF OF THEOREMS 1.1–1.6

At first we find a representation of both characteristic function as solution some non-homogeneous linear differential equations with identity linear part and different but "small" free terms. To bound these free terms for large frequencies t we shall use the method of symmetrization which was developed for quadratic forms in GT. Let $\mathbf{X}$ and $\mathbf{Y}$ denote the random vectors in $\mathbb{R}^N$ with coordinates $X_1,..., X_N$ and $Y_1,..., Y_N$ respectively. By $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ we denote the Euclidean scalar product and the Euclidean norm in $\mathbb{R}^N$ respectively. Introduce the vector $\mathbf{d}$ with coordinates $d_1,..., d_N$, where $d_i = a_{ii}$ for all i, and the diagonal matrix D with diagonal entries $D(i, i) = d_i$. Hence we may split A as $A = A_0 + D$. In these notation we have

$$\begin{aligned} Q &= \langle A\mathbf{X}, \mathbf{X} \rangle - \mu_2 \operatorname{Tr} A \\ G &= \mu_2 \langle A_0 \mathbf{Y}, \mathbf{Y} \rangle + \mu_3 \mu_2^{-1/2} \langle \mathbf{d}, \mathbf{Y} \rangle + \sqrt{\bar{\mu}_4}\, VY \end{aligned}$$

3.1. Outline of the Differential Equation Method for Quadratic Forms

In order to illustrate our approach we start with the case of one eigenvalue $\lambda_1 \neq 0$ only and $\mu_2 = 1$, where Q is given by $(N^{-1/2} \sum_{j=1}^N X_j + a_1)^2$. Here

$$f'(t) = i \mathbf{E} Q \mathrm{e}\{Q\} = i(q(t) + 2\varphi(t) + a_1^2 f(t))$$

and

$$\begin{aligned} q(t) &= N^{-1} \sum_{j=1}^{N} \mathbf{E} X_j S \mathrm{e}\{Q\} \\ \varphi(t) &= N^{-1/2} \sum_{k=1}^{N} a_1 \mathbf{E} X_k \mathrm{e}\{Q\} \end{aligned}$$

with $S = \sum_{k=1}^N X_k$. For any $j = 1,..., N$ we may write S in the form $S = X_j + S_j$, and Q in the form $Q = Q_j + \Delta_j$, where $Q_j = S_j^2$. Note that $\Delta_j = X_j^2 + 2a_1 X_j + 2X_j S_j$. Expanding the functions $S\mathrm{e}\{Q\}$ and $\mathrm{e}\{Q\}$ in powers of X_j and Δ_j and using the fact that X_j does not depend on S_j and Q_j, we obtain

$$
\begin{aligned}
\varphi(t) &= N^{-1/2} \sum_{k=1}^{N} a_1 \mathbf{E} X_k (\mathrm{e}\{Q - Q_k\} - 1)\, \mathrm{e}\{Q_k\} \\
&= 2itN^{-1} \sum_{k=1}^{N} a_1 \mathbf{E} X_k^2 S_k \mathrm{e}\{Q_k\} + 2itN^{-1} \sum_{k=1}^{N} a_1^2 \mathbf{E} X_k^2 \mathrm{e}\{Q_k\} \\
&\qquad + itN^{-3/2} \sum_{k=1}^{N} a_1 \mathbf{E} X_k^3 \mathrm{e}\{Q_k\} + \mathcal{N}_1(t) \\
&= 2it\varphi(t) + ita_1^2 f(t) + \mathcal{N}_2(t)
\end{aligned}
$$

Similarly we have for $q(t)$

$$
\begin{aligned}
q(t) &= N^{-1} \sum_{j=1}^{N} \mathbf{E} X_j S_j \mathrm{e}\{Q_j\} + N^{-1} \sum_{j=1}^{N} \mathbf{E} X_j^2 \mathrm{e}\{Q_j\} \\
&\qquad + N^{-1} \sum_{j=1}^{N} \mathbf{E} X_j S_j (\mathrm{e}\{\Delta_j\} - 1)\, \mathrm{e}\{Q_j\} \\
&\qquad + N^{-1} \sum_{j=1}^{N} \mathbf{E} X_j^2 (\mathrm{e}\{\Delta_j\} - 1)\, \mathrm{e}\{Q_j\} \\
&= f(t) + 2itN^{-2} \sum_{j=1}^{N} \mathbf{E} X_j^2 S_j^2 \mathrm{e}\{Q_j\} + ita_1 N^{-3/2} \sum_{j=1}^{N} \mathbf{E} X_j^2 S_j \mathrm{e}\{Q_j\} + \mathcal{N}_3(t) \\
&= f(t) + 2itq(t) + 2ita_1 \varphi(t) + \mathcal{N}_1(t)
\end{aligned}
$$

Here and below $\mathcal{N}(t)$ with indices shall denote remainder terms of "smaller" order. Rewriting these representations we immediately get

$$
\varphi(t) = \frac{2ita_1}{1-2it} f(t) + \mathcal{N}_3(t)
$$

and

$$
q(t) = i \frac{1}{1-2it} f(t) + \frac{2ita_1^2}{1-2it} \varphi(t) + \mathcal{N}_4(t)
$$

Substituting $\varphi(t)$ in the last equation, we conclude the equation for $f'(t)$

$$
\begin{aligned}
f'(t) &= i\,(1-2it)^{-1} f(t) + i \left(\frac{2it}{1-2it} \right)^2 a_1^2 f(t) \\
&\qquad + i \frac{4ita_1^2}{1-2it} f(t) + ia_1^2 f(t) + \mathcal{N}_5(t) \\
&= i(1-2it)^{-1} f(t) + i \frac{a_1^2}{(1-2it)^2} f(t) + \mathcal{N}_6(t)
\end{aligned}
$$

In comparison the limiting c.f. $\tilde{g}(t)$ of the r.v. $T=\sum_k \lambda_k(Y_k+a_k)^2$, $\sum_k |\lambda_k| < \infty$, satisfies the equation

$$\tilde{g}'(t) = i\,\mathrm{Tr}((I-2itA)^{-1}A)\,\tilde{g}(t) + i\langle Ua, (I-2itA)^{-2}\,Ua\rangle\,\tilde{g}(t)$$
$$\tilde{g}(0) = 1$$

where A denotes a symmetric operator with eigenvalues λ_k, $k=1,\ldots,N$, and U denote the unitary operator such that UAU^{-1} is diagonal. It is easy to see that in this case $\mathrm{Tr}(I-2itA)=(1-2it)^{-1}$, and $\langle (I-2itA)^{-2}\,Ua, U^Ta\rangle = a_1^2/(1-2it)^2$. The equations for $f'(t)$ and $\tilde{g}'(t)$ imply that $f(t)$ is close to $\tilde{g}(t)$, if $\mathcal{N}_6(t)$ is small. To investigate the c.f. of of general quadratic form $Q=\langle \mathbf{X}, A\mathbf{X}\rangle$ we use a similar approach. The details of these proofs are provided in Sections 4 and 5.

3.2. The General Differential Equations Bounds

Let R_t denote the operator $R_t=(I-2it\mu_2 A_0)^{-1}$, where I is the identity operator in $\mathbb{R}^N$, and $\mathbf{d}^{(t)}$ denotes the vector with coordinates $\mathbf{d}^{(t)}(j) = R_t A(j,j)$, for $j=1,\ldots,N$. Recall that $\mathbf{d}$ denotes the vector with coordinates $\mathbf{d}(j)=A(j,j)$. Note that the spectrum of the operator R_t is contained in the unit disk. This implies for instance that for $m=1,2,\ldots$

$$\begin{aligned}
&\sum_{\substack{1\leqslant l\leqslant N\\ l\neq j}} |R_t^m A(l,j)|^2 \leqslant \sum_{1\leqslant l\leqslant N} |A(l,j)|^2 = \mathcal{L}_j^2\\
&\sum_{j=1}^N |R_t^m A(j,j)|^2 \leqslant \sum_{j=1}^N \mathcal{L}_j^2 = \|A\|^2
\end{aligned} \tag{3.1}$$

In the following θ denotes different complex numbers with $|\theta|\leqslant 1$. For $j=1,\ldots,N$ and for $p,q=0,1,\ldots$ we introduce the functions

$$\mathcal{H}_j(p,q,\eta,\zeta) = |\mathbf{E}[\langle \eta, R_t A_0 \mathbf{X}_j\rangle^p\,\langle \zeta, A_0\mathbf{X}_j\rangle^q\,\mathrm{e}\{Q_j + \langle \zeta, A_0\mathbf{X}_j\rangle\}]|$$

and similarly for Gaussian r.v.'s

$$\tilde{\mathcal{H}}_j(p,q,\eta,\zeta) = |\mathbf{E}[\langle \eta, R_t A_0 \mathbf{Y}_j\rangle^p\,\langle \zeta, A_0\mathbf{Y}_j\rangle^q\,\mathrm{e}\{G_j + \langle \zeta, A_0\mathbf{Y}_j\rangle\}]|$$

where η and ζ denote some nonrandom vectors. We introduce also the functions

$$\mathcal{H}_j(p,q,\eta_j) := \mathcal{H}_j(p,q,\eta_j,\tau\eta_j) \qquad \text{and} \qquad \tilde{\mathcal{H}}_j(p,q,\eta_j) := \mathcal{H}_j(p,q,\eta_j,\tau\eta_j)$$

where η_j denotes a deterministic vector with zero coordinates except for the jth coordinate. Here and in the following let τ denote a uniformly distributed r.v. independent of all other r.v.'s. Set

$$\varkappa(t) = \max_{1 \leqslant j \leqslant N} \{(\max\{\mathscr{H}_j(p, q, \eta_j), \tilde{\mathscr{H}}_j(p, q, \eta_j)\})/(\mathscr{L}_j^{p+q} \|\eta_j\|^{p+q} \beta_{p+q})\}$$

In these notation we have

$$\max\{\mathscr{H}_j(p, q, \eta_j), \tilde{\mathscr{H}}_j(p, q, \eta_j)\} \leqslant \mathscr{L}_j^{p+q} \|\eta_j\|^{p+q} \beta_{p+q} \varkappa(t) \tag{3.2}$$

Remark. Introduce the following *generic* error term

$$\mathscr{N}(t) = \theta C t^2 \mathscr{L} \varkappa(t)$$

This means that for simplicity we shall omit indices of the various remainder terms $\mathscr{N}(t)$ occurring in the following arguments.

The representation for the characteristic function $f(t)$ give us the following Theorem.

Theorem 3.1. The characteristic function $f(t)$ satisfies the following differential equation

$$f'(t) = i\mathscr{J}(t)\, f(t) + (\beta_3^2 + V(|\mu_3|\, \beta_5\, |t| + \beta_6))\, \mathscr{N}(t)$$

where

$$\mathscr{J}(t) = \mu_2 \operatorname{Tr}(R_t A_0) - t^2 \mu_3^2 \langle \mathbf{d}, R_t^2 A_0 \mathbf{d}\rangle - 2t^2 \mu_3^2 \langle \mathbf{d}, R_t A_0 \mathbf{d}\rangle + it(\mu_4 - \mu_2^2)\langle \mathbf{d}, \mathbf{d}\rangle$$

The proof of Theorem 3.1 will be given in Section 4. Analogously to $f(t)$ we have the representation for the characteristic function $g(t)$.

Theorem 3.2. The characteristic function $g(t)$ satisfies the following equality

$$\begin{aligned} g'(t) &= i\mathscr{J}(t)\, g(t) + (\mu_2 + |\mu_3|\, \mu_2^{-1/2} V)^2\, (\mu_2 + |t|\, |\mu_3|\, \mu_2^{1/2} V)\, \mathscr{N}(t) \\ &= i\mathscr{J}(t)\, g(t) + (\beta_3^2 + V(|\mu_3|\, \beta_5\, |t| + \beta_6))\, \mathscr{N}(t) \end{aligned}$$

The bounds for $|f(t)|$, $|g(t)|$ and for the function $\mathscr{N}(t)$ for the arbitrary frequencies t give us the theorems below. Since $|f(t)|$, $|g(t)|$ and $|f(t) - g(t)|$ are odd functions we consider throughout the paper positive frequencies $t \geqslant 0$ only. We write $\bar{\mathscr{L}} = (1 - V^2)^{-1/2}\, \mathscr{L}$ and use the notations (β, μ, b_l, λ_j, Λ_j) introduced in Section 1.

Theorem 3.3. Assume that (1.3) holds. Then for all $t \geqslant 0$ there exists an absolute constant $\delta_0 > 0$ such that

$$|f(t)| \leqslant \exp\{-\delta_0 (t \wedge T_0)^2 (1-V^2)\, \mu_2^2 \overline{\mathscr{L}}^{1/4}\} + \mathbf{I}\{t > \tfrac{1}{8}\, \beta_3^{-2} \mu_2^2 \, |\lambda_1|^{-1}\} + C \overline{\mathscr{L}}^{3/2} \tag{3.3}$$

and

$$\begin{aligned}|g(t)| \leqslant {} & (\exp\{-\tfrac{1}{2}(2(1-V^2)\, \mu_2^2 + V^2 \mu_3^2 \mu_2^{-1})\, t^2\} + \mathbf{I}\{t > \tfrac{1}{8}\, \mu_2^{-1}\, |\lambda_1|^{-1}\}) \\ & \times \exp\{-\tfrac{1}{2}\, \bar{\mu}_4 V^2 t^2\}\end{aligned} \tag{3.4}$$

where $t \wedge T_0 = \min\{t, T_0\}$, $T_0 = \frac{1}{8}\, \beta_3^{-2} \mu_2^2 \, |\lambda_1|^{-1} \, (\log^+ |\lambda_1|)^{-1/2}$.

Under assumption (1.5) there exists an absolute constant $\delta_1 > 0$ such that for $\frac{1}{8}\, \beta_3^{-2} \mu_2^2 \mathscr{L}^{-1} (\log^+ \mathscr{L})^{-1} \leqslant t \leqslant \frac{1}{8}\, \beta_3^{-2} \mu_2^2 \mathscr{L}^{-1}$ we have

$$|f(t)| \leqslant (1 + \delta_0 \mu_2^2 b_0^2 (1-V^2)\, t^2 \overline{\mathscr{L}}^{1/4q})^{-q/16} + C \overline{\mathscr{L}}^{3/2} \tag{3.5}$$

and

$$|g(t)| \leqslant (1 + \mu_2^2 b_0^2 t^2)^{-q/4} \exp\{-\tfrac{1}{2}\, \bar{\mu}_4 V^2 t^2\} \tag{3.6}$$

For $j = 1, \ldots, N$ introduce the functions

$$\mathscr{H}_j(p, q, \eta_j) := \mathscr{H}_j(p, q, \eta_j, \tau \eta_j) \qquad \text{and} \qquad \widetilde{\mathscr{H}}_j(p, q, \eta_j) := \mathscr{H}_j(p, q, \eta_j, \tau \eta_j)$$

where η_j denotes a deterministic vector with zero coordinates except for the jth coordinate.

Theorem 3.4. Assume that the conditions of Theorem 3.3 are fulfilled. Then

$$\max\{\widetilde{\mathscr{H}}_j(p, q, \eta_j), \mathscr{H}_j(p, q, \eta_j)\} \leqslant C \, \|\eta_j\|^{p+q} \, \mathscr{L}_j^{p+q} \beta_{p+q} \varkappa_1(t) \tag{3.7}$$

and

$$|f(t)| \leqslant C \varkappa_1(t) \tag{3.8}$$

where for some positive δ_0

$$\varkappa_1(t) = \exp\{-\delta_0 (1-V^2)\, \mu_2^2 t^2\} + C \overline{\mathscr{L}}^3 + \mathbf{I}\{t \geqslant \mu_2^{-1} \, |\lambda_1|^{-1} \wedge \mathscr{L}^{-1} (\log^+ |\overline{\mathscr{L}}|)^{-1}\}$$

If in addition we assume (1.5), then

$$\max\{\widetilde{\mathscr{H}}_j(p, q, \eta_j), \mathscr{H}_j(p, q, \eta_j)\} \leqslant C \, \|\eta_j\|^{p+q} \, \beta_{p+q} \mathscr{L}_j^{p+q} \varkappa_2(t) \tag{3.9}$$

and

$$|f(t)| \leqslant C\varkappa_2(t) \tag{3.10}$$

with

$$\varkappa_2(t) = (1+\delta_0\mu_2^2 b_0^2(1-V^2)\, t^2)^{-q/8} + C\bar{\mathscr{L}}^6 + \mathbf{I}\,\{t \geqslant \mu_2^{-1}(\mathscr{L}\log^+ \bar{\mathscr{L}})^{-1}\}$$

Corollary 3.5. For the function $\kappa(t)$ defined in Section 3 we have the following bound. If (1.3) holds then

$$\varkappa(t) \leqslant \varkappa_1(t)$$

if in addition we assume (1.5) then

$$\varkappa(t) \leqslant \varkappa_2(t)$$

Put $\beta(t) = f(t) - g(t)$. Theorems 3.1, 3.2 together imply that $\beta(t)$ satisfies the differential equation

$$\beta'(t) = i\mathscr{I}(t)\,\beta(t) + \bar{\delta}(t) \tag{3.11}$$

with $\beta(0) = 0$ and $\bar{\delta}(t) = (\beta_3^2 + V(|\mu_3|\beta_5\,|t| + \beta_6))\,\mathscr{N}(t)$.

The equation (3.11) has the unique solution

$$\beta(t) = \int_0^t \exp\left\{i\int_u^t \mathscr{I}(z)\,dz\right\}\bar{\delta}(u)\,du \tag{3.12}$$

Let us derive Theorem 1.1–1.4 from (3.12). We shall use Esseen's inequality

$$\delta(A, F) \leqslant 2\int_0^T |\beta(t)|\,\frac{dt}{t} + C\mu_2^{-1}T^{-1}$$

for any $T > 0$ and write

$$\delta(A, F) \leqslant 2(I_0 + I_1 + I_2) + C\mu_2^{-1}T^{-1} \tag{3.13}$$

where $I_0 = \int_0^{T_0} |\beta(t)|\,\frac{dt}{t}$, $I_1 = \int_{T_0}^T |f(t)|\,\frac{dt}{t}$ and $I_2 = \int_{T_0}^T |g(t)|\,\frac{dt}{t}$. Here T and T_0 will be chosen later.

Proof of Theorem 1.1. We consider the case $t \geqslant 0$ only. Using that the spectrum of the operator $R_t^m A_0$ for $m = 1, 2$ is contained in the disk of radius $|\lambda_1|$, and that

$$\operatorname{Tr}(R_t A_0) = 2it\mu_2 \operatorname{Tr}(R_t A_0^2) = 2it\mu_2 \operatorname{Tr} A_0^2 - 4t^2\mu_2^2 \operatorname{Tr}(R_t A_0^3)$$

it is not difficult to show that

$$\operatorname{Re}(i\mathscr{J}(t)) \leqslant -t\sigma^2 + 4\,|\lambda_1|\,\mu_2^2 t^2\,\operatorname{Tr}(A_0^2) + 3t^2\mu_3^2\,|\lambda_1|\,\langle \mathbf{d}, \mathbf{d}\rangle \tag{3.14}$$

where

$$\sigma^2 = (\mu_4 - \mu_2^2)\,V^2 + 2\mu_2^2(1 - V^2)$$

Since $\mu_3^2\mu_2^{-1} \leqslant \mu_4 - \mu_2^2$, inequality (3.14) implies that for $u \leqslant t \leqslant \frac{1}{8}\,|\lambda_1|^{-1}\,\mu_2^{-1}$

$$\operatorname{Re}(i\mathscr{J}(t)) \leqslant -\tfrac{5}{8}\,t\sigma^2$$

After integration we obtain

$$\left|\exp\left\{i\int_u^t \mathscr{J}(z)\,dz\right\}\right| \leqslant \exp\{-\tfrac{5}{16}\,\sigma^2(t^2 - u^2)\} \tag{3.15}$$

Put $T_0 = \frac{1}{8}\,\beta_3^{-2}\mu_2^2(|\lambda_1|^{-1} \wedge (\mathscr{L}\log^+\mathscr{L})^{-1})$. By Corollary 3.5 and definition of the function $\varkappa_1(t)$ we get for $t \leqslant T_0$

$$|\bar{\delta}(t)| \leqslant C(\beta_3^2 + V(|\mu_3|\,\beta_5\,|t| + \beta_6))\,\mathscr{L}t^2(\exp\{-\delta_0\mu_2^2(1 - V^2)\,t^2\} + C\bar{\mathscr{L}}^3) \tag{3.16}$$

By Hölder's inequality $\beta_3^{-2}\mu_2^2 \leqslant \mu_2^{-1}$ and $|\mu_3|\,\beta_5\mu_2^{-1} \leqslant \beta_6$. Note that for $k \geqslant 0$

$$\int_0^t u^{k+1}\exp\{\alpha^2u^2\}\,du \leqslant \alpha^{-2}t^k\exp\{\alpha^2t^2\} \tag{3.17}$$

Combining (3.12), and (3.15)–(3.17), we obtain, for $t \leqslant T_0$

$$|\beta(t)| \leqslant C(\beta_3^2 + \beta_6 V\,|t|\,\mu_2)\,\mathscr{L}\sigma^{-2}t(\exp\{-\delta_0\mu_2^2(1 - V^2)\,t^2\} + C\bar{\mathscr{L}}^3) \tag{3.18}$$

By inequality (3.18)

$$I_0 \leqslant C(\beta_3^2 + V\beta_6)\,\mu_2^{-3}\mathscr{L}b_1^{-2} \tag{3.19}$$

Let $T = \frac{1}{8}\,\beta_3^{-2}\mu_2^2\,|\lambda_1|^{-1}$. If $T_0 \geqslant T$, put

$$I_1 = 0, \qquad I_2 = 0 \tag{3.20}$$

Note that $b_1^{-1}\mathscr{L} \geqslant \bar{\mathscr{L}} \geqslant \mathscr{L}$. Using (3.3) for $T_0 \leqslant t \leqslant T$ we have

$$|f(t)| \leqslant \exp\{-\delta_0\mu_2^2 b_1^2 T_0^2\mathscr{L}^{1/4}\} + C\mathscr{L}^{3/2} \tag{3.21}$$

Thus we get for $T \geqslant T_0$, by inequality (3.21) $TT_0^{-1} \leqslant \log^+ \mathscr{L}$

$$I_1 \leqslant (\exp\{-\delta_0 \mu_2^2 b_1^2 T_0^2 \mathscr{L}^{1/4}\} + C\mathscr{L}^{3/2}) \log(\log^+ \mathscr{L}) \tag{3.22}$$

The inequality (3.4) yields

$$I_2 \leqslant \sigma^{-2} T_0^{-2} \exp\{-\sigma^2 T_0^2\} \tag{3.23}$$

The inequalities (3.13) and (3.19)–(3.23) together imply

$$\delta(A, F) \leqslant C\beta_6 \mu_2^{-3}(\mathscr{L} + |\lambda_1|)\, b_1^{-2} \tag{3.24}$$

Write G as a sum of independent r.v. $(Y_k + b_k)^2\, \lambda_k$, $1 \leqslant k \leqslant N$, for some appropriate b_k. By well-known estimates for the normal approximation of $g(t)$ [see Petrov (Ref. 22, Chap. 5, § 2)] we get

$$\begin{aligned}|g(t) - \exp\{-\tfrac{1}{2}\sigma^2 t^2\}| \leqslant Ct^3\, |\lambda_1|\, ((1 - V^2)\, \mu_2^2 t\lambda_1 + \mu_3^2 \mu_2^{-1} V^2) \exp\{-\tfrac{1}{4}\sigma^2 t^2\} \\ + \mathbf{I}\{t \geqslant \beta_3^{-2} \mu_2^2\, |\lambda_1|^{-1}\}\end{aligned} \tag{3.25}$$

Note that $|\lambda_1| \leqslant 2\, |\Lambda_1|$. An application of Esseen's inequality together with the inequalities (3.24) and (3.25) completes the proof of Theorem 1.1 □

Proof of Theorem 1.2. We denote by $\overline{R_t}$ the complex conjugate operator to the operator R_t, i.e., $\overline{R_t} = (I + 2i\mu_2 t A_0)^{-1}$. In this notation we have

$$\begin{aligned}\operatorname{Re}(i \operatorname{Tr}(R_t A_0)) &= -2t\mu_2 \operatorname{Tr}(R_t \overline{R_t}\, A_0^2) \\ \operatorname{Re}(i\langle \mathbf{d}, R_t^m A_0 \mathbf{d}\rangle) &= -2^m t\mu_2 \langle \mathbf{d}, (R_t \overline{R_t})^m\, A_0^2 \mathbf{d}\rangle, \qquad m = 1, 2\end{aligned}$$

Some easy calculation shows that

$$\begin{aligned}\operatorname{Re}(i\mathscr{J}(t)) = {} & -2t\mu_2^2 \operatorname{Tr}(R_t \overline{R_t}\, A_0^2) - t(\mu_4 - \mu_2^2)\langle \mathbf{d}, \mathbf{d}\rangle \\ & + 4t^3 \mu_3^2 \mu_2 \langle \mathbf{d}, (I + R_t \overline{R_t})\, R_t \overline{R_t}\, A_0^2 \mathbf{d}\rangle\end{aligned} \tag{3.26}$$

By definition of $R_t \overline{R_t}$ we have

$$4t^2 \mu_2^2 R_t \overline{R_t}\, A_0^2 = I - R_t \overline{R_t} \tag{3.27}$$

Equalities (3.26) and (3.27) imply

$$\begin{aligned}\operatorname{Re}(i\mathscr{J}(t)) = {} & -2t\mu_2^2 \operatorname{Tr}(R_t \overline{R_t}\, A_0^2) - t(\mu_4 - \mu_2^2)\langle \mathbf{d}, \mathbf{d}\rangle \\ & + t\mu_3^2 \mu_2^{-1} \langle \mathbf{d}, (I + R_t \overline{R_t})(I - R_t \overline{R_t})\, \mathbf{d}\rangle\end{aligned}$$

From the last equality it follows for $t \geqslant 0$

$$\mathrm{Re}(i\mathscr{J}(t)) \leqslant -2t\mu_2^2\,\mathrm{Tr}(R_t\overline{R_t}\,A_0^2) - t\bar{\mu}_4\langle \mathbf{d}, \mathbf{d}\rangle \tag{3.28}$$

where $\bar{\mu}_4 = \mu_4 - \mu_2^2 - \mu_3^2\mu_2^{-1}$. Let $s^2(t) = 1 + 4\lambda_q^2\mu_2^2t^2$. For $t \geqslant 0$ and $q \geqslant 1$

$$-\mathrm{Re}(i\mathscr{J}(t)) \leqslant q\mu_2^2\,\frac{2\lambda_q^2 t}{s^2(t)} - \bar{\mu}_4 V^2 t \tag{3.29}$$

Since $-2\mu_2^2\lambda_q^2\int_u^t zs(z)^{-2}\,dz = \frac{1}{2}\log(s(u)/s(t))$, for $0 \leqslant u \leqslant t$.

$$\left|\exp\left\{\int_u^t i\mathscr{J}(z)\,dz\right\}\right| \leqslant \left(\frac{s(u)}{s(t)}\right)^{q/2} \exp\left\{-\frac{1}{2}\,(t^2-u^2)\,\bar{\mu}_4V^2\right\}$$

Thus we get

$$|\beta(t)| \leqslant \int_0^t |\bar{\delta}(u)| \left(\frac{s(u)}{s(t)}\right)^{q/2} \exp\left\{-\frac{1}{2}\,(t^2-u^2)\,\bar{\mu}_4V^2\right\} du \tag{3.30}$$

Put $T_0 = \frac{1}{8}\,\beta_3^{-2}\mu_2^2\mathscr{L}^{-1}/\log^+\mathscr{L}$ and $T = \frac{1}{8}\,\beta_3^{-2}\mu_2^2\mathscr{L}^{-1}$. By Remark 3.5 and by definition of the function of $\varkappa_2(t)$, in Theorem 3.4, we have for $t \leqslant T_0$,

$$|\bar{\delta}(t)| \leqslant C(\mathscr{L}+\mathscr{M})(\beta_3^2 + V(|\mu_3|\,\beta_5 t + \beta_6))\,t^2((s(\sqrt{\delta_0}\,b_1 t))^{-q/4} + \mathscr{L}^6) \tag{3.31}$$

Note that, for $0 \leqslant u \leqslant t$

$$(s(u)/s(t))^{q/2}\,s(\alpha u)^{-q/4} \leqslant s(t)^{-q/4}\max\{1, \alpha^{-q/4}\} \tag{3.32}$$

The inequalities (3.30)–(3.32) and (3.17) together imply

$$\begin{aligned} |\beta(t)| &\leqslant C(\mathscr{L}+\mathscr{M})(\beta_3^2 + V(|\mu_3|\,\beta_5 t + \beta_6))(b_1^2\delta_0)^{-\frac{q}{8}} \\ &\quad \times (s(t)^{-q/4} + \mathscr{L}^6)\min\{t^3, t(V^2\bar{\mu}_4)^{-1}\} \end{aligned} \tag{3.33}$$

for $t \leqslant T_0$. By (3.5) we have

$$|f(t)| \leqslant (1 + \delta_0\mu_2^2 b_0^2(1-V^2)\,t^2\overline{\mathscr{L}}^{1/4q})^{-q/16} + C\overline{\mathscr{L}}^{3/2} \tag{3.34}$$

for $T_0 \leqslant t \leqslant T$. Note that for any $m \geqslant 0$

$$\int_0^{T_0} \frac{t^m}{1+(\alpha t)^{q/4}}\,dt \leqslant C \begin{cases} \alpha^{-q/4} T_0^{4(m+1)-q/4}, & \text{if} \quad 2 \leqslant q \leqslant 4(m+1)-1 \\ \alpha^{-(m+1)}\log^+ T_0, & \text{if} \quad q = 4(m+1) \\ \alpha^{-(m+1)}, & \text{if} \quad q = 4(m+1)+1 \end{cases} \tag{3.35}$$

The inequalities (3.33) and (3.35) together imply

$$I_0 \leqslant C(\mathscr{L}+\mathscr{M})((\beta_3^2+V\beta_6)(y_1(q)+\mathscr{L}^6 z_1)+V\,|\mu_3|\,\beta_5(y_2(q)+\mathscr{L}^6 z_2)) \tag{3.36}$$

where, for $p=1, 2$,

$$z_p=\min\{T_0^{p+2}, (V^2\bar{\mu}_4)^{-1}\,T_0^p\}, \qquad \text{and} \qquad y_p(q)=\min\{y_{p1}(q), y_{p2}(q)\}$$

$$y_{p,1}(q)=\begin{cases}(|\lambda_q|\,\mu_2)^{-q/4}\,T_0^{4(p+2)-q/4}, & \text{if} \quad 2\leqslant q\leqslant 4(p+2)\\ (|\lambda_q|\,\mu_2)^{-(p+2)}\log^+ T_0, & \text{if} \quad q=4(p+2)\\ (|\lambda_q|\,\mu_2)^{-(p+2)}, & \text{if} \quad q=4(p+2)+1\end{cases}$$

and

$$y_{p,2}(q)=(V^2(\mu_4-\mu_2^2))^{-1}\begin{cases}(|\lambda_q|\,\mu_2)^{-q/4}\,T_0^{4p-q/4}, & \text{if} \quad 2\leqslant q\leqslant 4p-1\\ (|\lambda_q|\,\mu_2)^{-p}\log^+ T_0, & \text{if} \quad q=4p\\ (|\lambda_q|\,\mu_2)^{-p}, & \text{if} \quad q=4p+1\end{cases}$$

By inequality (3.34), for $q>8$,

$$I_1\leqslant C\beta_3^2\mu_2^{-3}(b_0 b_1)^{-q/8}\,\mathscr{L} \tag{3.37}$$

If condition (1.5) holds for $q\geqslant 17$, we obtain from inequality (3.36)

$$I_0\leqslant C(\mathscr{L}+\mathscr{M})(b_1^2\delta_0)^{\frac{q}{8}}\,((\beta_3^2+V\beta_6)(\mu_2 b_0)^{-3}+V\,|\mu_3|\,\beta_5(\mu_2 b_0)^{-4}) \tag{3.38}$$

Inequality (3.6) yields

$$I_2\leqslant C\beta_3^2\mu_2^{-3}\bar{\mu}_4^{-1}\mathscr{L} \tag{3.39}$$

Combining the inequalities (3.13) and (3.36)–(3.39) completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. To prove Theorem 1.3 we use inequality (3.13) with $T=\frac{1}{8}\,\beta_3^{-2}\mu_2^2\mathscr{L}^{-1}$ and $T_0=\frac{1}{8}\,\beta_3^{-2}\mu_2^2\mathscr{L}^{-1}(\log^+\mathscr{L})^{-1}$. By inequality (3.33) we have

$$I_0\leqslant C(\mathscr{L}+\mathscr{M})((\beta_3^2+V\beta_6)\,\mu_2^{-3}b_0^{-1}+V\,|\mu_3|\,\beta_5\mu_2^{-4}b_0^{-2})\,b_1^{-9/4}b_2^{-2}\mu_2^2\,\bar{\mu}_4^{-1} \tag{3.40}$$

The inequalities (3.13), (3.37), (3.39) and (3.40) together imply the result. □

Proof of Theorem 1.4. To prove inequality (1.11) we put $T = \frac{1}{8}\beta_3^{-2}\mu_2^2\mathscr{L}^{-1}$ and $T_0 = \frac{1}{8}\beta_3^{-2}\mu_2^2\mathscr{L}^{-1}(\log^+\mathscr{L})^{-1}$. Inequality (6.23) yields

$$I_0 \leqslant C b_0^{-3} b_1^{-13/4} \alpha_1 \mathscr{L} \tag{3.41}$$

The inequalities (3.13), (3.37), (3.39) and (3.41) together complete the proof. □

An inspection of the arguments between (3.36)–(3.38) shows that for $2 \leqslant q \leqslant 16$ we have the results of Corollaries 1.5–1.7.

4. A DIFFERENTIAL EQUATION FOR $f(t)$. PROOF OF THEOREM 3.1

We reformulate here

Theorem 3.1. The characteristic function $f(t)$ satisfies the following differential equation

$$f'(t) = i\mathscr{J}(t)\, f(t) + (\beta_3^2 + V(|\mu_3|\, \beta_5\, |t| + \beta_6))\, \mathscr{N}(t)$$

where

$$\mathscr{J}(t) = \mu_2 \operatorname{Tr}(R_t A_0) - t^2\mu_3^2\langle \mathbf{d}, R_t^2 A_0 \mathbf{d}\rangle - 2t^2\mu_3^2\langle \mathbf{d}, R_t A_0 \mathbf{d}\rangle + it(\mu_4 - \mu_2^2)\langle \mathbf{d}, \mathbf{d}\rangle$$

Let $\mathbf{x}_j$ denote the vector with zero coordinates except for the jth coordinate, where $\mathbf{x}_j^{(j)} = X_j$ and let $\mathbf{y}_j$ denote a similar vector with $\mathbf{y}_j^{(j)} = Y_j$. We recall that $\mathbf{X} = \sum_{j=1}^N \mathbf{x}_j$, and $\mathbf{Y} = \sum_{j=1}^N \mathbf{y}_j$, and introduce the notation

$$\mathbf{X}_j := \mathbf{X} - \mathbf{x}_j = \sum_{\substack{1 \leqslant l \leqslant N \\ l \neq j}} \mathbf{x}_l, \qquad \mathbf{Y}_j := \mathbf{Y} - \mathbf{y}_j = \sum_{\substack{1 \leqslant l \leqslant N \\ l \neq j}} \mathbf{y}_l$$

To study the behavior of $f(t)$, we need to investigate the auxiliary functions

$$h(t) = \mathbf{E}\langle R_t A\mathbf{X}, \mathbf{X}\rangle\, \mathrm{e}\{\langle A\mathbf{X}, \mathbf{X}\rangle\} \tag{4.1}$$

and

$$\varphi^{(m)}(\eta_l) = \mathbf{E}\langle \eta_l, R_t^m A\mathbf{X}\rangle\, \mathrm{e}\{\langle A\mathbf{X}, \mathbf{X}\rangle\}, \qquad \eta \in \mathbb{R}^N \tag{4.2}$$

Here η_l, $l = 1,\ldots, N$ denote, similar to $\mathbf{x}_j$, N-dimensional vectors with coordinates, say $\eta_l(j)$, $j = 1,\ldots, N$, derived from a N-dimensional vector $\eta = (\eta(1),\ldots, \eta(N))$ such that $\eta_l(j) = 0$ if $j \neq l$ and $\eta_l(l) = \eta(l)$. Recall that $\mathbf{d}$ denotes the vector with coordinates $\mathbf{d}(j) = A(j, j)$ and D denotes the diagonal matrix with elements $D(j, j) = A(j, j)$, R_t denotes the operator $R_t = (I - 2i\mu_2 A_0)^{-1}$.

Lemma 4.1. For any $l = 1,\dots, N$ and for $m = 0, 1$ we have

$$\varphi^{(m)}(\eta_l) = it\mu_3\langle \mathbf{d}, R_t^{m+1} A\eta_l\rangle f(t) + \eta(l)(\beta_3\beta_2 + V\beta_5)\,\mathscr{L}_l\mathscr{N}(t)$$

Represent the function $h(t)$ in the form

$$h(t) = h^{(1)}(t) + h^{(2)}(t)$$

where

$$h^{(1)}(t) = \sum_{j=1}^{N} \mathbf{E}[\langle \mathbf{x}_j, R_t A \mathbf{X}_j\rangle\,\mathrm{e}\{Q\}], \qquad h^{(2)}(t) = \sum_{j=1}^{N} \mathbf{E}[\langle \mathbf{x}_j, R_t A \mathbf{x}_j\rangle\,\mathrm{e}\{Q\}]$$

Lemma 4.2. The function $h^{(1)}(t)$ satisfies the following relation

$$h^{(1)}(t) = 2it\mathbf{E}\langle \mathbf{X}, AR_tA\mathbf{X}\rangle\,\mathrm{e}\{Q\} + \mathscr{I}_1(t)\,f(t) + (\beta_3^2 + V(\beta_5\beta_1 + |t|\,|\mu_3|\,\beta_5))\,\mathscr{N}(t)$$

with

$$\mathscr{I}_1(t) = -t^2\mu_3^2\langle \mathbf{d}, R_t^2 A\mathbf{d}\rangle - 2it\mu_2^2\,\mathrm{Tr}(R_tAD)$$

Lemma 4.3. The function $h^{(2)}(t)$ satisfies

$$h^{(2)}(t) = \mathscr{I}_2(t)\,f(t) + (V(|\mu_3|\,\beta_5\,|t| + \beta_6) + \beta_3^2)\,\mathscr{N}(t)$$

with

$$\mathscr{I}_2(t) = \mu_2\,\mathrm{Tr}(R_t\,A) + it(\mu_4 - \mu_2^2)\,\mathrm{Tr}\{R_tAD\} - 2t^2\mu_3^2\langle \mathbf{d}, R_tA\mathbf{d}^{(t)}\rangle$$

Introduce the function $r(t) = 2it\mu_2\mathbf{E}\langle \mathbf{X}, DR_tA\mathbf{X}\rangle\,\mathrm{e}\{Q\}$.

Lemma 4.4. The function $r(t)$ satisfies the following relation

$$r(t) = \mathscr{I}_3(t)\,f(t) + V\beta_2\beta_4\mathscr{N}(t)$$

with $\mathscr{I}_3(t) = 2it\mu_2^2\,\mathrm{Tr}(R_tAD)$.

About the definition of $\mathscr{N}(t)$ see Remark for the Theorem 3.1. Introduce the function

$$\alpha(t) = \mathscr{I}(t) - \mathscr{I}_1(t) - \mathscr{I}_2(t) - \mathscr{I}_3(t) = \alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t)$$

where

$$\begin{aligned}
\alpha_1(t) &= \mu_2(\mathrm{Tr}((R_t - I))\,A) - \mathrm{Tr}(R_t A_0))\\
\alpha_2(t) &= it(\mu_4 - \mu_2^2)(\mathrm{Tr}\{R_t AD\} - \langle \mathbf{d}, \mathbf{d}\rangle)\\
\alpha_3(t) &= -t^2\mu_3^2\langle \mathbf{d}, R_t^2(A - A_0)\,\mathbf{d}\rangle\\
\alpha_4(t) &= -2t^2\mu_3^2(\langle \mathbf{d}, R_t A(\mathbf{d}^{(t)} - \mathbf{d}))\rangle + \langle \mathbf{d}, R_t D\mathbf{d}\rangle)
\end{aligned}$$

A bound for $|\alpha(t)|$ is given in the following Lemma.

Lemma 4.5. There exist an absolute constant C such that

$$|\alpha(t)| \leqslant CV(\beta_4\beta_2 + \mu_3^2(1 + \beta_2\,|t|))\, t^2\mathscr{L}$$

Below we will use the notation

$$\begin{gathered}
Q_j = \langle A\mathbf{X}_j, \mathbf{X}_j\rangle, \qquad \Delta_j = Q - Q_j = 2\langle \mathbf{x}_j, A\mathbf{X}_j\rangle + \langle \mathbf{x}_j, A\mathbf{x}_j\rangle\\
\mathscr{E}_j = \langle \eta_l, R_t^{m+1}A\mathbf{x}_j\rangle, \qquad \mathscr{E}'_j = \langle \mathbf{x}_j, R_t A\mathbf{x}_j\rangle, \qquad \mathscr{T}_j = \langle \mathbf{x}_j, A\mathbf{x}_j\rangle\\
\mathscr{F}_j = \langle \mathbf{x}_j, A\mathbf{X}_j\rangle, \qquad \text{and} \qquad \mathscr{S}_j = \langle \mathbf{x}_j, R_t A\mathbf{X}_j\rangle
\end{gathered}$$

We recall that τ denote a uniformly distributed r.v. independent of all other r.v.'s. Furthermore, in the proofs of Lemmas 4.1–4.5 we assume without loss of generality that $\mu_2 = \beta_2 = 1$.

Proof of Lemma 4.1. For fixed l with $1 \leqslant l \leqslant N$ consider the function

$$q(\eta, t) = \mathbf{E}\langle \eta_l, R_t^{m+1}A\mathbf{X}\rangle\, \mathrm{e}\{\langle A\mathbf{X}, \mathbf{X}\rangle\}$$

The function q and other functions of η we need to introduce depend on the chosen indices l as well as m. Since $\mathbf{x}_j$ and Q_j are independent, we have $\sum_{j=1}^N \mathbf{E}\mathscr{E}_j \mathrm{e}\{Q_j\} = 0$, and we can represent $q(\eta, t)$ in the form

$$q(\eta, t) = q_1(\eta, t) + q_2(\eta, t) + q_3(\eta, t) \tag{4.3}$$

where

$$\begin{aligned}
q_1(\eta, t) &= \sum_{j=1}^N \mathbf{E}\mathscr{E}_j(\mathrm{e}\{2\mathscr{F}_j\} - 1)\,\mathrm{e}\{Q_j\}\\
q_2(\eta, t) &= \sum_{j=1}^N \mathbf{E}\mathscr{E}_j(\mathrm{e}\{\mathscr{T}_j\} - 1)\,\mathrm{e}\{Q_j\}\\
q_3(\eta, t) &= \sum_{j=1}^N \mathbf{E}\mathscr{E}_j(\mathrm{e}\{2\mathscr{F}_j\} - 1)(\mathrm{e}\{\mathscr{T}_j\} - 1)\,\mathrm{e}\{Q_j\}
\end{aligned}$$

Note that

$$\mathscr{F}_j = X_j \sum_{k \neq j} a_{kj} X_l, \qquad \mathscr{T}_j = d_j X_j^2, \qquad \mathscr{E}_j = R_t^{m+1} A(j, l)\, X_j \eta(l)$$

Since

$$\mathbf{E} \left| \sum_{k : k \neq j} a_{kj} X_k \right| \leqslant \left(\sum_{k : k \neq j} a_{kj}^2 \right)^{1/2} \leqslant \mathscr{L}_j \qquad \text{and} \qquad \sum_{k : k \neq l} |R_t^{m+1} A(k, l)|^2 \leqslant \mathscr{L}_l^2$$

we obtain

$$q_3(\eta, t) = \eta(l)\, \tilde{q}_3(t)$$

where

$$|\tilde{q}_3(t)| \leqslant C t^2 \mathscr{L}_l \max_{1 \leqslant j \leqslant N} \mathbf{E}\, |X_j|^3\, \mathscr{H}_j(0, 1, \mathbf{x}_j)$$

The last inequality, inequalities (3.1) and (3.2) together imply that

$$q_3(\eta, t) = \eta(l)\, \beta_5 \mathscr{L}_l V \mathscr{N}(t) \tag{4.4}$$

Furthermore we can represent the function $q_1(\eta, t)$ in the form

$$q_1(\eta, t) = q_4(\eta, t) + q_5(\eta, t) \tag{4.5}$$

where

$$q_4(\eta, t) = 2it \sum_{j=1}^{N} \mathbf{E} \mathscr{E}_j \mathscr{F}_j \mathrm{e}\{Q_j\}$$

$$q_5(\eta, t) = -4t^2 \sum_{j=1}^{N} \mathbf{E} (1-\tau)^2\, \mathscr{E}_j \mathscr{F}_j^2 \mathrm{e}\{Q_j + 2\tau \mathscr{F}_j\}$$

The function $q_5(\eta, t)$ may be represented as follows

$$q_5(\eta, t) = \eta(l)\, \tilde{q}_5(t) \tag{4.6}$$

where

$$|\tilde{q}_5(t)| \leqslant C t^2 \sum_{j=1}^{N} |R_t^{m+1} A(l, j)|\, \mathbf{E}\, |X_j|\, \mathscr{H}_j(0, 2, \mathbf{x}_j)$$

Using (3.1) and (3.2) from (4.6) it follows that

$$q_5(\eta, t) = \eta(l)\, \beta_3 \beta_2 \mathscr{L}_l \mathscr{N}(t) \tag{4.7}$$

In order to investigate the function $q_4(\eta, t)$, we decompose it as follows

$$q_4(\eta, t) = q_6(\eta, t) + q_7(\eta, t) + q_8(\eta, t)$$

where

$$q_6(\eta, t) = 2it \sum_{j=1}^{N} \mathbf{E}\langle \eta_l, R_t^{m+1} A\bar{\mathbf{x}}_j\rangle\langle \bar{\mathbf{x}}_j, A\mathbf{X}\rangle \,\mathrm{e}\{Q\}$$

$$q_7(\eta, t) = -2it \sum_{j=1}^{N} \mathbf{E}\langle \eta_l, R_t^{m+1} A\bar{\mathbf{x}}_j\rangle\langle \bar{\mathbf{x}}_j, A\mathbf{X}\rangle(\mathrm{e}\{\Delta_j\} - 1)\,\mathrm{e}\{Q_j\}$$

$$q_8(\eta, t) = -2it \sum_{j=1}^{N} \mathbf{E}\langle \eta_l, R_t^{m+1} A\bar{\mathbf{x}}_j\rangle\langle \bar{\mathbf{x}}_j, A\mathbf{x}_j\rangle \,\mathrm{e}\{Q_j\}$$

and $\bar{\mathbf{x}}_j$ is an independent copy of $\mathbf{x}_j$ independent of all other r.v.'s. It is easy to see that

$$q_6(\eta, t) = 2it\,\mathbf{E}\langle \eta_l, R_t^{m+1} A^2\mathbf{X}\rangle \,\mathrm{e}\{Q\} \tag{4.8}$$

Similarly as in (4.4), (4.7), we prove that

$$q_7(\eta, t) = \eta(l)\, \tilde{q}_7(t)$$

where

$$|\tilde{q}_7(t)| \leqslant C\,|t|^2 \sum_{j=1}^{N} (|R_t^{m+1} A(j, l)|\; d_j^2(\mathbf{E}\,|X_j|^3\, \mathscr{H}_j(0, 0, \mathbf{x}_j) + \mathbf{E}\,|X_j|\, \mathscr{H}_j(0, 1, \mathbf{x}_j))$$
$$+ |d_j|\, \mathbf{E}\,\bar{X}_j^2 \mathscr{H}_j(0, 1, 0, \mathbf{x}_j, \bar{\mathbf{x}}_j) + \mathbf{E}\,|\bar{X}_j|\, \mathscr{H}_j(0, 1, 1, \mathbf{x}_j, \bar{\mathbf{x}}_j))$$

Here we used the notation

$$H(p, q, r, \xi, \eta, \zeta) = \mathbf{E}\langle \eta, R_t A\mathbf{X}_j\rangle^p\,\langle \xi, A\mathbf{X}_j\rangle^q\,\langle \zeta, A\mathbf{X}_j\rangle^r\,\mathrm{e}\{\langle \mathbf{X}_j, A\mathbf{X}_j\rangle$$
$$+ \tau\langle \xi, A\mathbf{X}_j\rangle\}$$

The functions $H(p, q, r, \xi, \eta, \zeta)$ are bounded in the same way as $H(p, q+r, \xi, \eta)$. Using (3.1) we get

$$q_7(\eta, t) = \eta_l(\beta_3 + V\beta_5)\,\mathscr{N}(t) \tag{4.9}$$

It is evident that

$$q_8(\eta, t) = 0 \tag{4.10}$$

From (4.5)–(4.10) it follows that

$$q_1(\eta, t) = 2it\mathbf{E}\langle B_l\eta,\, R_t^{m+1}A^2\mathbf{X}\rangle\, \mathrm{e}\{Q\} + \eta(l)(\beta_3 + V\beta_5)\, \mathscr{L}_l\mathscr{N}(t) \tag{4.11}$$

Representing the function $q_2(\eta, t)$ in the form

$$q_2(\eta, t) = q_9(\eta, t) + q_{10}(\eta, t)$$

where

$$q_9(\eta, t) = it \sum_{j=1}^{N} \mathbf{E}\mathscr{E}_j\mathscr{T}_j\mathrm{e}\{Q_j\}$$

$$q_{10}(\eta, t) = -t^2 \sum_{j=1}^{N} \mathbf{E}(1-\tau)^2\, \mathscr{E}_j\mathscr{T}_j^2\mathrm{e}\{Q_j + \tau\mathscr{T}_j\}$$

and proceeding similarly as in (4.4), (4.7) and (4.9), we obtain that

$$q_{10}(\eta, t) = \eta(l)\, \tilde{q}_{10}(t)$$

where

$$|\tilde{q}_{10}(t)| \leqslant Ct^2 \sum_{j=1}^{N} |R_t^{m+1}A(j, l)|\, d_j^2\mathbf{E}\, |X_j|^5\, \mathscr{H}_j(0, 0, \mathbf{x}_j)$$

The last inequality, inequalities (3.1) and (3.2) imply that

$$q_{10}(\eta, t) = \eta(l)\, V\beta_5\mathscr{L}_l\mathscr{N}(t) \tag{4.12}$$

Finally, we shall determine an asymptotic approximation of $q_9(\eta, t)$. Since $\mathbf{x}_j$ and $\mathbf{X}_j$ are independent, we have the following identity

$$q_9(\eta, t) = it\left(\sum_{j=1}^{N} \mathbf{E}\mathscr{E}_j\mathscr{T}_j\right)\mathbf{E}\mathrm{e}\{Q\} + q_{11}(\eta, t) \tag{4.13}$$

Here $q_{11}(\eta, t) = -t^2\sum_{j=1}^{N} \mathbf{E}\mathscr{E}_j\mathscr{T}_j\mathbf{E}\varDelta_j\mathrm{e}\{Q_j + \tau\varDelta_j\}$. Similarly as in (4.4), (4.7), (4.9) we obtain

$$q_{11}(\eta, t) = \eta(l)\, \tilde{q}_{11}(t) \tag{4.14}$$

where

$$|\tilde{q}_{11}(t)| \leqslant C\beta_3t^2 \sum_{j=1}^{N} |R_t^{m+1}A(j, l)|\, \mathscr{L}_j\, |d_j|\, (|d_j|\, \mathbf{E}X_j^2\mathscr{H}_j(0, 0, \mathbf{x}_j) + \mathbf{E}\mathscr{H}_j(0, 1, \mathbf{x}_j))$$

Using inequalities (3.1), (3.2) and (4.14) we show that

$$q_{11}(\eta, t) = \eta(l)\, V\beta_3 \mathcal{L}_l \mathcal{N}(t) \tag{4.15}$$

From (4.12)–(4.15) it follows that

$$q_2(\eta, t) = it\mu_3 \langle \mathbf{d}, R_t^{m+1} AB_l \eta\rangle\, \mathbf{E}\mathrm{e}\{Q\} + \eta(l)(\beta_3 + V\beta_5)\, \mathcal{L}_l \mathcal{N}(t) \tag{4.16}$$

Relations (4.2), (4.9) and (4.16) together imply

$$\begin{aligned} q(\eta, t) &= \mathbf{E}\langle \eta_l, R_t^{m+1} A\mathbf{X}\rangle\, \mathrm{e}\{\langle A\mathbf{X}, \mathbf{X}\rangle\} \\ &= 2it\, \mathbf{E}\langle \eta_l, R_t^{m+1} A^2\mathbf{X}\rangle\, \mathrm{e}\{Q\} + it\mu_3\langle \mathbf{d}, R_t^{m+1}\, A\eta_l\rangle\, \mathbf{E}\mathrm{e}\{Q\} \\ &\qquad + \eta(l)(\beta_3\beta_2 + V\beta_5)\, \mathcal{L}_l \mathcal{N}(t) \end{aligned} \tag{4.17}$$

Since $A^2 = A_0 A + DA$ and $R_t(I - 2itA_0) = I$, by subtracting the first term on the left hand side of (3.19) from both sides and using the definition (4.2) of $\varphi^{(m)}(\eta_l)$ we obtain

$$\varphi^{(m)}(\eta_l) = it\mu_3\langle \mathbf{d}, R_t^{m+1} A\eta_l\rangle\, \mathbf{E}\mathrm{e}\{Q\} + q_{12}(\eta, t) + \eta(l)(\beta_3\beta_2 + V\beta_5)\, \mathcal{L}_l \mathcal{N}(t)$$

where

$$q_{12}(\eta, t) = 2it\mathbf{E}\langle B_l\eta, R_t^{m+1} DA\mathbf{X}\rangle\, \mathrm{e}\{Q\}$$

Since $\sum_{j=1}^N \mathbf{E}\langle B_l\eta, R_t^{m+1} DA\mathbf{x}_j\rangle\, \mathrm{e}\{Q_j\} = 0$, we have

$$q_{12}(\eta, t) = \eta(l)\, \tilde{q}_{12}(t) \tag{4.18}$$

where

$$|\tilde{q}_{12}(t)| \leqslant Ct^2 \sum_{j=1}^N |R_t^{m+1} DA(j, l)|\, (|d_j|^2\, \mathbf{E}\, |X_j|^3\, \mathcal{H}_j(0, 0, \mathbf{x}_j) + \mathbf{E}\, |X_j|\, \mathcal{H}_j(0, 1, \mathbf{x}_j))$$

Similarly as in (4.4), it follows

$$q_{12}(\eta) = \eta(l)\, \beta_3 \mathcal{L}_l \mathcal{N}(t) \tag{4.19}$$

The result of Lemma 4.1 now follows from (4.18) and (4.19). □

Proof of Lemma 4.2. Consider the function $h^{(1)}(t)$. Since $\mathbf{x}_j$ does not depend on $\mathbf{X}_j$ and Q_j, we get $\sum_{j=1}^N \mathbf{E}\mathcal{S}_j \mathrm{e}\{Q_j\} = 0$. Decomposing $h^{(1)}(t)$ we have

$$h^{(1)}(t) = u_1(t) + u_2(t) + u_3(t)$$

where

$$u_1(t)=\sum_{j=1}^{N}\mathbf{E}\mathscr{S}_j(\mathrm{e}\{2\mathscr{F}_j\}-1)\,\mathrm{e}\{Q_j\},\qquad u_2(t)=\sum_{j=1}^{N}\mathbf{E}\mathscr{S}_j(\mathrm{e}\{\mathscr{T}_j\}-1)\,\mathrm{e}\{Q_j\}$$

$$u_3(t)=-2t^2\sum_{j=1}^{N}\mathbf{E}\mathscr{T}_j\mathscr{S}_j\mathscr{F}_j\,\mathrm{e}\{Q_j+\tau_1\mathscr{T}_j+\tau_2\mathscr{F}_j\}$$

Proceeding similarly as in (3.6), we obtain the estimate

$$|u_3(t)|\leqslant C\,|t|^2\sum_{j=1}^{N}|d_j|\,\mathbf{E}X_j^2\mathscr{H}_j(1,1,\mathbf{x}_j)$$

The last inequality and inequalities (3.1) and (3.2) together imply that

$$u_3(t)=V\beta_4\mathscr{N}(t) \tag{4.20}$$

The function $u_1(t)$ can be written as

$$u_1(t)=u_4(t)+u_5(t) \tag{4.21}$$

where

$$u_4(t)=2it\sum_{j=1}^{N}\mathbf{E}\mathscr{S}_j\mathscr{F}_j\mathrm{e}\{Q_j\},\qquad u_5(t)=-4t^2\sum_{j=1}^{N}\mathbf{E}(1-\tau)^2\,\mathscr{S}_j\mathscr{F}_j^2\mathrm{e}\{Q_j+\tau\mathscr{F}_j\}$$

Since is $\mathbf{x}_j$ independent of $\mathbf{X}_j$ and Q_j for $j=1,\dots,N$, we have

$$u_4(t)=2it\sum_{j=1}^{N}\mathbf{E}\langle B_jR_tA\mathbf{X}_j,A\mathbf{X}_j\rangle\,\mathrm{e}\{Q_j\}$$

where B_j denotes the covariance operator of $\mathbf{x}_j$, $B_j(l,k)=\delta_{kj}\delta_{lj}$, $j=1,\dots,N$ and δ_{kj} denotes the Kronecker delta. We represent $u_4(t)$ in the form

$$u_4(t)=u_6(t)+u_7(t)+u_8(t)$$

where

$$u_6(t)=2it\sum_{j=1}^{N}\mathbf{E}\langle B_jR_tA\mathbf{X},A\mathbf{X}\rangle\,\mathrm{e}\{Q\}$$

$$u_7(t)=-2it\sum_{j=1}^{N}\mathbf{E}\langle B_jR_tA\mathbf{X},A\mathbf{X}\rangle(\mathrm{e}\{\varDelta_j\}-1)\,\mathrm{e}\{Q_j\}$$

$$u_8(t)=-2it\sum_{j=1}^{N}\mathbf{E}\langle B_jR_tA\mathbf{x}_j,A\mathbf{x}_j\rangle\,\mathrm{e}\{Q_j\}$$

Since $\sum_{j=1}^{N} B_j = I$, we get

$$u_6(t) = 2it\,\mathbf{E}\langle \mathbf{X}, AR_tA\mathbf{X}\rangle\, \mathrm{e}\{Q\} \tag{4.22}$$

Similarly as in (4.4), we derive

$$\begin{aligned}|u_7(t)| \leqslant C\,|t|^2 \sum_{j=1}^{N} (|R_tA(j,j)|\,(\mathbf{E}\mathscr{H}_j(0,2,\mathbf{x}_j) + \mathbf{E}\mathscr{H}_j(1,1,\mathbf{x}_j)\\ + |d_j|\,\mathbf{E}X_j^2\mathscr{H}_j(0,1,\mathbf{x}_j) + d_j^2\mathbf{E}\mathscr{H}_j(0,0,\mathbf{x}_j)) + \mathbf{E}\mathscr{H}_j(1,1,1,e_j,e_j,\mathbf{x}_j))\end{aligned}$$

Here $e_j = (0,\ldots,0,1,0,\ldots,0)$. An easy computation shows that

$$u_7(t) = V\beta_3^2\mathscr{N}(t) \tag{4.23}$$

Using the independence of $\mathbf{x}_j$ and Q_j, and replacing Q_j by Q, we obtain

$$\begin{aligned}u_8(t) &= -2it\sum_{j=1}^{N}\mathbf{E}\langle B_jR_tA\mathbf{x}_j, A\mathbf{x}_j\rangle\,\mathbf{E}\mathrm{e}\{Q\} + \mathscr{N}(t)\\ &= -2it\,\mathrm{Tr}\{R_tAD\}\,f(t) + \mathscr{N}(t)\end{aligned} \tag{4.24}$$

We used here also that B_j is symmetric and $B_jA\mathbf{x}_j = D\mathbf{x}_j$. The function $u_5(t)$ can be bounded as follows

$$|u_5(t)| \leqslant C\,|t|^2\sum_{j=1}^{N}\mathbf{E}\mathscr{H}_j(1,2,\mathbf{x}_j) \tag{4.25}$$

Combining the inequalities (3.1) and (3.2) with inequality (4.24) implies

$$u_5(t) = \beta_3^2\mathscr{N}(t) \tag{4.26}$$

The relations (4.21)–(4.26) together imply

$$u_1(t) = -2it\,\mathrm{Tr}\{R_tAD\}\,f(t) + 2it\,\mathbf{E}\langle\mathbf{X}, AR_tA\mathbf{X}\rangle\,\mathrm{e}\{Q\} + \beta_3^2\mathscr{N}(t) \tag{4.27}$$

Consider now the function $u_2(t)$. Write

$$u_2(t) = u_9(t) + u_{10}(t)$$

where

$$u_9(t) = it\sum_{j=1}^{N}\mathbf{E}\mathscr{S}_j\mathscr{T}_j\mathrm{e}\{Q_j\}, \qquad u_{10}(t) = \sum_{j=1}^{N}\mathbf{E}(1-\tau)^2\,\mathscr{S}_j\mathscr{T}_j^2\mathrm{e}\{Q_j + \tau\mathscr{T}_j\}$$

It is easy to see that

$$|u_{10}(t)| \leqslant C\,|t|^2 \sum_{j=1}^{N} d_j^2 \mathbf{E} X_j^4 \mathscr{H}_j(1, 0, \mathbf{x}_j) = V\beta_5 \mathscr{N}(t) \tag{4.28}$$

Using that $\mathbf{x}_j$ and Q_j are independent, we get

$$u_9(t) = it \sum_{j=1}^{N} \mathbf{E}\langle \mathbf{x}_j, D\mathbf{x}_j\rangle\, \varphi_j^{(1)}(\mathbf{x}_j) + \theta C t^2 \mathscr{P} \tag{4.29}$$

where

$$\mathscr{P} = \sum_{j=1}^{N} (d_j^2 \mathbf{E}\,|X_j|^4\, \mathscr{H}_j(0, 1, \mathbf{x}_j) + |d_j|\, \mathbf{E}\,|X_j|^2\, \mathscr{H}_j(1, 1, \mathbf{x}_j))$$

Applying Lemma 4.1 and inequalities (3.1) and (3.2) to the right hand side of (4.29), we see that

$$\begin{aligned} u_9(t) &= \mu_3 (it)^2 \sum_{j=1}^{N} \mathbf{E}\langle \mathbf{x}_j, D\mathbf{x}_j\rangle \langle \mathbf{d}, R_t^2 A\mathbf{x}_j\rangle\, f(t) + V(\beta_3 + |\mu_3|\,|t|\,\beta_5)\,\mathscr{N}(t) \\ &= -t^2 \mu_3^2 \langle \mathbf{d}, R_t^2 A\mathbf{d}\rangle\, f(t) + V(\beta_3 + |t|\,|\mu_3|\,\beta_5)\,\mathscr{N}(t) \end{aligned} \tag{4.30}$$

By (4.30)–(4.32) we obtain

$$u_2(t) = -t^2 \mu_3^2 \langle \mathbf{d}, R_t^2 A\mathbf{d}\rangle\, f(t) + V\beta_5 (1 + |t|\,|\mu_3|)\,\mathscr{N}(t) \tag{4.31}$$

Combining (4.20), (4.27) and (4.31), we get

$$\begin{aligned} h^{(1)}(t) = &-t^2 \mu_3^2 \langle \mathbf{d}, R_t^2 A\mathbf{d}\rangle\, f(t) - 2it\,\mathrm{Tr}\{R_t AD\}\, f(t) \\ &+ 2it\,\mathbf{E}\langle \mathbf{X}, AR_t A\mathbf{X}\rangle\, \mathrm{e}\{Q\} + (V(1 + |t|\,|\mu_3|)\,\beta_5) + \beta_3^2)\,\mathscr{N}(t) \end{aligned} \tag{4.32}$$

□

Proof of Lemma 4.3. Represent the function $h^{(2)}(t)$ in the form

$$h^{(2)}(t) = v_1(t) + v_2(t)$$

where

$$v_1(t) = \sum_{j=1}^{N} \mathbf{E}\mathscr{E}_j' \mathrm{e}\{Q_j\}, \qquad v_2(t) = \sum_{j=1}^{N} \mathbf{E}\mathscr{E}_j' (\mathrm{e}\{\Delta_j\} - 1)\, \mathrm{e}\{Q_j\}$$

We shall study these functions separately. Consider first $v_1(t)$. We have

$$v_1(t) = v_3(t) + v_4(t) + v_5(t) \tag{4.33}$$

where

$$v_3(t) = \operatorname{Tr}(R_t A)\, f(t), \qquad v_4(t) = -it \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j \mathbf{E}\mathscr{T}_j \mathbf{E}e\{Q_j\}$$

$$v_5(t) = t^2 \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j \mathbf{E}(1-\tau)^2\, \Delta_j^2\, e\{Q_j + \tau\Delta_j\}$$

Here we used that $\mathbf{E}\mathscr{F}_j e\{Q_j\} = 0$. It is easy to show that

$$\begin{aligned} |v_5(t)| \leqslant C\, |t|^2 \sum_{j=1}^{N} |R_t A(j, j)|\, (|d_j|^2\, \mathbf{E}X_j^4\, \mathscr{H}_j(0, 0, \mathbf{x}_j) \\ + 2\, |d_j|\, \mathbf{E}X_j^2\, \mathscr{H}_j(0, 1, \mathbf{x}_j) + \mathbf{E}\mathscr{H}_j(0, 2, \mathbf{x}_j)) \end{aligned} \tag{4.34}$$

This yields

$$v_5(t) = (V\beta_4 + 1)\, \mathscr{N}(t) \tag{4.35}$$

Using the independence of $\mathbf{x}_j$ and $\mathbf{X}_j$, approximating Q_j by Q, and using (4.33), we conclude

$$v_4(t) = -it\, \operatorname{Tr}\{R_t A D\}\, f(t) + \mathscr{N}(t) \tag{4.36}$$

The relations (4.33)–(4.36) together imply that

$$v_1(t) = \operatorname{Tr}(R_t A)\, f(t) - it\, \operatorname{Tr}\{R_t A D\}\, f(t) + (V\beta_4 + 1)\, \mathscr{N}(t) \tag{4.37}$$

Now write the function $v_2(t)$ in the form

$$v_2(t) = v_6(t) + v_7(t) + v_8(t) \tag{4.38}$$

where

$$v_6(t) = \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j (e\{2\mathscr{F}_j\} - 1)\, e\{Q_j\}, \qquad v_7(t) = \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j (e\{\mathscr{T}_j\} - 1)\, e\{Q_j\}$$

$$v_8(t) = -2t^2 \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j \mathscr{T}_j \mathscr{F}_j\, e\{Q_j + \tau_1 \mathscr{T}_j + \tau_2 \mathscr{F}_j\}$$

It is easy to see that

$$|v_8(t)| \leqslant C\,|t|^2 \sum_{j=1}^{N} |R_t A(j, j)|\, |d_j|\, \mathbf{E}\,|X_j|^4\, \mathscr{H}_j(0, 1, \mathbf{x}_j)$$

and by inequalities (3.1) and (3.2)

$$v_8(t) = V\beta_5 \mathscr{N}(t) \tag{4.39}$$

The functions $v_6(t)$ and $v_7(t)$ may be written in the form

$$v_6(t) = v_9(t) + v_{10}(t)$$

where

$$v_9(t) = 2it \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j \mathscr{F}_j \mathrm{e}\{Q_j\}, \qquad v_{10}(t) = -4t^2 \sum_{j=1}^{N} \mathbf{E}(1-\tau)^2\, \mathscr{E}'_j \mathscr{F}_j^2 \mathrm{e}\{Q_j + \tau\mathscr{F}_j\}$$

and

$$v_7(t) = v_{11}(t) + v_{12}(t) \tag{4.40}$$

where

$$v_{11}(t) = it \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j \mathscr{T}_j \mathrm{e}\,\{Q_j\}, \qquad v_{12}(t) = -t^2 \sum_{j=1}^{N} \mathbf{E}(1-\tau)^2\, \mathscr{E}'_j \mathscr{T}_j^2 \mathrm{e}\{Q_j + \tau\mathscr{T}_j\}$$

The asymptotic approximations of these functions can be determined in the same way as for the function $u_9(t)$ above. We get

$$v_{12}(t) \leqslant \sum_{j=1}^{N} |R_t A(j, j)|\, |d_j|^2\, \mathbf{E} X_j^6 \mathscr{H}(0, 0, \mathbf{x}_j) = \beta_6 \mathscr{N}(t) \tag{4.41}$$

Replacing $\mathbf{x}_j$, $j = 1, \ldots, N$, by independent copies, and approximating Q_j and $\mathscr{X}_j$ by Q and $\mathbf{X}$ respectively, we get

$$v_9(t) = v_{13}(t) + v_{14}(t)$$

where

$$v_{13}(t) = 2it \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j \varphi_j^{(0)}(\mathbf{x}_j)$$

$$v_{14}(t) = -2t^2 \sum_{j=1}^{N} \mathbf{E}\langle \hat{\mathbf{x}}_j, R_t A \hat{\mathbf{x}}_j\rangle \langle \hat{\mathbf{x}}_j, A\mathbf{X}\rangle\, \Delta_j \mathrm{e}\{Q_j + \tau\Delta_j\}$$

and

$$v_{11}(t) = v_{15}(t) + v_{16}(t)$$

where

$$v_{15}(t) = it\mu_4 \operatorname{Tr}\{R_t AD\}\, f(t), \qquad v_{16}(t) = -t^2 \sum_{j=1}^{N} \mathbf{E}\mathscr{E}'_j \mathscr{T}_j \mathbf{E}\varDelta_j \mathrm{e}\{Q_j + \tau\varDelta_j\}$$

Using Lemma 4.1 and inequalities (3.1) and (3.2), we get

$$v_9(t) = -2t^2\mu_3^2\langle \mathbf{d}, R_t A\mathbf{d}^{(t)}\rangle\, f(t) + V\, |\mu_3|\, (|t|\, \beta_5 + \beta_3)\, \mathscr{N}(t) \tag{4.42}$$

From (4.38)–(4.42) it follows that

$$\begin{aligned} v_2(t) = {} & -2t^2\mu_3^2\langle \mathbf{d}, R_t A\mathbf{d}^{(t)}\rangle\, f(t) + it\mu_4 \operatorname{Tr}\{R_t AD\}\, f(t) \\ & + (V(|\mu_3|\, |t|\, \beta_5 + \beta_6) + \beta_3^2)\, \mathscr{N}(t) \end{aligned} \tag{4.43}$$

Finally, by (4.37) and by (4.43), we have

$$\begin{aligned} h^{(2)}(t) = {} & \operatorname{Tr}(R_t A)\, f(t) + it(\mu_4 - 1) \operatorname{Tr}\{R_t AD\}\, f(t) - 2t^2\mu_3^2\langle \mathbf{d}, R_t A\mathbf{d}^{(t)}\rangle\, f(t) \\ & + ((|\mu_3|\, \beta_5\, |t| + \beta_6)\, V + \beta_3^2)\, \mathscr{N}(t) \end{aligned}$$

□

Proof of Lemma 4.5. Represent the function $r(t)$ in the form

$$r(t) = r_1(t) + r_2(t)$$

where

$$r_1(t) = 2it \sum_{j=1}^{N} \mathbf{E}\langle \mathbf{x}_j, DR_t A\mathbf{X}_j\rangle\, \mathrm{e}\{Q\}$$

$$r_2(t) = 2it \sum_{j=1}^{N} \mathbf{E}\langle \mathbf{x}_j, DR_t A\mathbf{x}_j\rangle\, \mathrm{e}\{Q\}$$

Since

$$\mathbf{E}\langle \mathbf{x}_j, DR_t A\mathbf{X}_j\rangle\, \mathrm{e}\{Q_j\} = 0 \tag{4.44}$$

we have

$$r_1(t) = -2t^2 \sum_{j=1}^{N} \mathbf{E}\langle \mathbf{x}_j, DR_t A\mathbf{X}_j\rangle\, \varDelta_j \mathrm{e}\{Q_j + \tau\varDelta_j\}$$

An easy computation shows that

$$|r_1(t)| \leqslant C\,|t|^2 \sum_{j=1}^{N} (|d_j|\,\mathbf{E}\mathscr{H}_j(1, 1, \mathbf{x}_j) + d_j^2\,\mathbf{E}X_j^2\,\mathscr{H}_j(1, 0, \mathbf{x}_j))$$

Applying inequalities (3.1) and (3.2), we obtain

$$r_1(t) = V\beta_3\mathcal{N}(t) \tag{4.45}$$

We decompose the function $r_2(t)$ as follows

$$r_2(t) = r_3(t) + r_4(t)$$

where

$$r_3(t) = 2it \sum_{j=1}^{N} \mathbf{E}\langle \mathbf{x}_j, DR_t A\mathbf{x}_j\rangle\ \mathbf{E}\mathrm{e}\{Q_j\}$$

$$r_4(t) = -2t^2 \sum_{j=1}^{N} \mathbf{E}\langle \mathbf{x}_j, DR_t A\mathbf{x}_j\rangle\ \Delta_j \mathrm{e}\{Q_j + \tau\Delta_j\}$$

The function $r_4(t)$ satisfies

$$\begin{aligned} |r_4(t)| &\leqslant C\,|t|^2 \sum_{j=1}^{N} |d_j|\,|R_t A_{jj}|\,(\mathbf{E}X_j^2\,\mathscr{H}_j(0, 1, \mathbf{x}_j) + |d_j|\,\mathbf{E}X_j^4\,\mathscr{H}_j(0, 0, \mathbf{x}_j)) \\ &= V\beta_4\mathcal{N}(t) \end{aligned} \tag{4.46}$$

Replacing $\mathbf{E}\mathrm{e}\{Q_j\}$ by $f(t)$ and estimating the error as above, we find the asymptotic behavior of $r_3(t)$. We get

$$r_3(t) = 2it\,\mathrm{Tr}(DR_t A)\,f(t) + V\beta_4\mathcal{N}(t) \tag{4.47}$$

The relations (4.46) and (4.47) together complete the proof. □

Proof of Lemma 4.5. We derive separate bounds for the functions $\alpha_j(t)$, $j = 1, 2, 3, 4$. Throughout the proof we shall use the identities

$$\mathrm{Tr}(A_0 D^m) = 0, \qquad m = 0, 1 \tag{4.48}$$

and

$$R_t - I = R_t(I - R_t^{-1}) = 2itR_t A_0 \tag{4.49}$$

which imply

$$\alpha_1 = (\mathrm{Tr}((R_t - I)(A - A_0)) = \mathrm{Tr}((R_t - \mathrm{I})\ D) = 2it\ \mathrm{Tr}(R_t A_0 D)$$

Using (4.48) with $m = 0$, and (4.49) again we obtain

$$\alpha_1(t) = 2it\ \mathrm{Tr}((R_t - I)\ A_0 D) = -4t^2\ \mathrm{Tr}(R_t A_0^2 D) \tag{4.50}$$

It is easy to proof that

$$|\mathrm{Tr}(R_t(A_0)^2\ D| \leqslant \sum_{j=1}^{N} |d_j|\ |R_t A_0^2(j, j)| \leqslant V \mathscr{L}$$

This inequality and the relation (4.49) together imply

$$|\alpha_1(t)| \leqslant 4t^2 V \mathscr{L} \tag{4.51}$$

Using

$$\langle \mathbf{d}, \mathbf{d} \rangle = \mathrm{Tr}(D^2) = \mathrm{Tr}(AD) \tag{4.52}$$

we can write

$$\alpha_2(t) = it(\mu_4 - 1)(\mathrm{Tr}(R_t - I)\ AD) = -2t^2(\mu_4 - 1)(\mathrm{Tr}(R_t A_0 AD)$$

From the last inequality it follows

$$\alpha_2(t) = (\mu_4 - 1)(\tfrac{1}{2}\,\alpha_1(t) - 2t^2\ \mathrm{Tr}(R_t A_0 D^2)) \tag{4.53}$$

Furthermore,

$$|\mathrm{Tr}(R_t A_0 D^2)| \leqslant \sum_{j=1}^{N} d_j^2\ |R_t A_0(j, j)| \leqslant V \mathscr{L} \left(\sum_{j=1}^{N} |R_t A_0(j, j)|^2 \right)^{1/2} \leqslant V \mathscr{L} \tag{4.54}$$

From (4.51)–(4.54) it follows

$$|\alpha_2(t)| \leqslant Ct^2 \beta_4\ V \mathscr{L} \tag{4.55}$$

By inequality $|\langle \mathbf{d}, R_t^2 D \mathbf{d} \rangle| \leqslant \mathscr{L}\ \|\mathbf{d}\|^2$, we have

$$|\alpha_3(t)| \leqslant Ct^2 \mu_3^2 V^2 \mathscr{L} \tag{4.56}$$

Furthermore,

$$|\langle \mathbf{d}, R_t^2 D \mathbf{d}^{(t)} \rangle| \leqslant V\ \|D \mathbf{d}^{(t)}\| \leqslant V^2 \mathscr{L} \tag{4.57}$$

Let $\mathbf{d}^{(1)}$, $\mathbf{d}^{(2)}$ denote the vectors with coordinates $\mathbf{d}^{(1)}(j) = R_t A_0^2(j, j)$ and $\mathbf{d}^{(2)}(j) = R_t A_0 D(j, j)$ for all j. Using (4.50), we obtain with this notation

$$\langle \mathbf{d}, R_t A_0(\mathbf{d} - \mathbf{d}^{(t)}) \rangle = 2it \langle \mathbf{d}, R_t A_0 \mathbf{d}^{(1)} \rangle + 2it \langle \mathbf{d}, R_t A_0 \mathbf{d}^{(2)} \rangle$$

It is easy to see that

$$\mathbf{d}^{(2)} = D \mathbf{d}^{(0)}$$

where $\mathbf{d}^{(0)}$ denote the vector with coordinates $\mathbf{d}^{(0)}(j) = R_t A_0(j, j)$, and

$$\|\mathbf{d}^{(2)}\| \leqslant V \mathscr{L}$$

Hence

$$|\langle \mathbf{d}, R_t A_0 \mathbf{d}^{(2)} \rangle| \leqslant V^2 \mathscr{L} \tag{4.58}$$

Next we note that

$$\begin{aligned} \|\mathbf{d}^{(1)}\|^2 &= \sum_{j=1}^{N} |R_t A_0^2(j, j)|^2 = \sum_{j=1}^{N} |\langle R_t A_0 e_j, A_0 e_j \rangle|^2 \\ &\leqslant \sum_{j=1}^{N} |\langle A_0 e_j, A_0 e_j \rangle|^2 \leqslant \sum_{j=1}^{N} \mathscr{L}_j^4 \leqslant \mathscr{L}^2 \end{aligned}$$

This inequality implies that

$$|\langle \mathbf{d}, R_t A_0 \mathbf{d}^{(t)} \rangle| \leqslant V \, |\mathbf{d}^{(1)}| \leqslant V \mathscr{L} \tag{4.59}$$

The relations (4.57)–(4.59) together imply

$$|\alpha_4(t)| \leqslant \mu_3^2 V \, |t|^2 \, (1 + |t|) \, \mathscr{L} \tag{4.60}$$

The result now follows from inequalities (4.51), (4.55), (4.56), and (4.60). $\square$

Proof of Theorem 3.1. By Lemmas 4.2–4.5 we have

$$\begin{aligned} h(t) &= 2it \mathbf{E} \langle \mathbf{X}, A_0 R_t A \mathbf{X} \rangle \, \mathrm{e}\{Q\} + (\mathscr{I}_1(t) + \mathscr{I}_2(t) + \mathscr{I}_3(t)) \, f(t) \\ &\quad + (\beta_3^2 + V(|\mu_3| \, \beta_5 \, |t| + \beta_6)) \, \mathscr{N}(t) \end{aligned} \tag{4.61}$$

Since $R_t A - 2it A_0 R_t A = (\mathrm{I} - 2it A_0) \, R_t A = A$, the equality (4.61) together with the definitions (4.1) of $h(t)$ and of $f(t)$ implies

$$\begin{aligned} f'(t) &= i \mathbf{E} \langle \mathbf{X}, A\mathbf{X} \rangle \, \mathrm{e}\{Q\} - i \, \mathrm{Tr}(A) \, f(t) \\ &= i \widetilde{\mathscr{I}}(t) \, f(t) + (\beta_3^2 + V(|\mu_3| \, \beta_5 \, |t| + \beta_6) \, \mathscr{N}(t) \end{aligned}$$

where $\tilde{\mathcal{J}}(t) = \mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t)$. Lemma 4.5 yields

$$\tilde{\mathcal{J}}(t) = \mathcal{J}(t) + \alpha(t) = \mathcal{J}(t) + \theta CV(\beta_4 + \mu_3^2(1+|t|))\, t^2 \mathcal{L} \qquad (4.62)$$

From inequalities (3.1) and (3.2), relations (4.63), and (4.64) together now imply the result. □

5. A DIFFERENTIAL EQUATION FOR $g(t)$

We reformulate here

Theorem 3.2. The characteristic function $g(t)$ satisfies the following equality

$$g'(t) = i\mathcal{J}(t)\, g(t) + (\mu_2 + |\mu_3|\, \mu_2^{-1/2} V)^2\, (\mu_2 + |t|\, |\mu_3|\, \mu_2^{1/2} V)\, \mathcal{N}(t)$$

$$= i\mathcal{J}(t)\, g(t) + (\beta_3^2 + V(|\mu_3|\, \beta_5\, |t| + \beta_6))\, \mathcal{N}(t)$$

By Hölders inequality $|\mu_3|^2 \leqslant \beta_6$, $|\mu_3|^3 \leqslant |\mu_3|\, \beta_5 \mu_2^{1/2}$, $\mu_2^3 \leqslant \beta_3^2$. Using these inequalities and the condition $V \leqslant 1$ which follows from (1.7) we obtain that

$$(\mu_2 + |\mu_3|\, \mu_2^{-1/2} V)^2\, (\mu_2 + |t|\, |\mu_3|\, \mu_2^{1/2} V) \leqslant C(\beta_3^2 + V(|\mu_3|\, \beta_5\, |t| + \beta_6))$$

We need to proof the first equality only.

In order to investigate $g'(t)$, we consider the functions

$$\varphi(t) = \mu_2 \mathbf{E}\langle \mathbf{Y}, A_0 \mathbf{Y}\rangle\, \mathrm{e}\{G\}, \qquad \tilde{\varphi}(t) = \mu_2 \mathbf{E}\langle \mathbf{Y}, R_t A_0 \mathbf{Y}\rangle\, \mathrm{e}\{G\}$$

$$\psi^{(m)}(\eta) = \mu_3 \mu_2^{-1/2} \mathbf{E}\langle \eta, (R_t A_0)^m\, \mathbf{Y}\rangle\, \mathrm{e}\{G\}$$

$$\tilde{\psi}^{(m)}(\eta) = \mu_3 \mu_2^{-1/2} \mathbf{E}\langle \eta, R_t\, (R_t A_0)^m\, \mathbf{Y}\rangle\, \mathrm{e}\{G\}$$

where $m = 0, 1$.

The following lemmas are similar to Lemmas 4.2–4.5.

Lemma 5.1. The functions $\psi^{(m)}(\eta)$, $m = 0, 1$, satisfy the relation

$$\psi^{(m)}(\eta) = it\mu_3^2 \mu_2^{-1} \langle \eta, R_t (R_t A_0)^m\, \mathbf{d}\rangle\, g(t)$$

$$+ |\mu_3|\, \mu_2^{-1/2}\, \|\eta\|\, (\mu_2 + |\mu_3|\, \mu_2^{-1/2} V)^2\, \mathcal{N}(t)$$

Lemma 5.2. The function $\varphi(t)$ can be represented as follows

$$\varphi(t)=\mu_2\,\mathrm{Tr}(R_tA_0)\,g(t)-t^2\mu_3^2\langle\mathbf{d},R_t^2\mathbf{d}\rangle\,g(t)$$
$$+(\mu_2+|t|\,V\,|\mu_3|\,\mu_2^{1/2}))(\mu_2+V\,|\mu_3|\,\mu_2^{-1/2})^2\,\mathcal{N}(t)$$

Similarly as in Section 4 we will use the notation

$$\mathbf{Y}_j=\mathbf{Y}-\mathbf{y}_j,\qquad \widetilde{\mathcal{F}}_j=\langle\mathbf{y}_j,A_0\mathbf{Y}_j\rangle,\qquad \widetilde{\mathcal{S}}_j=\langle\mathbf{y}_j,R_tA_0\mathbf{Y}_j\rangle$$
$$\widetilde{\mathcal{E}}_j=\langle\eta,R_t(R_tA_0\,)^m\,\mathbf{y}_j\rangle,\qquad \widetilde{\mathcal{T}}_j=\langle\mathcal{D},\mathbf{y}_j\rangle$$
$$G_j=\mu_2\langle\mathbf{Y}_j,A_0\mathbf{Y}_j\rangle+\mu_3\mu_2^{-1/2}\langle\mathbf{d},\mathbf{Y}_j\rangle+\sqrt{\bar{\mu}_4}\,VY$$
$$\widetilde{\Delta}_j=G-G_j=2\mu_2\langle\mathbf{y}_j,A_0\mathbf{Y}_j\rangle+\mu_3\mu_2^{-1/2}\langle\mathbf{d},\mathbf{y}_j\rangle=2\mu_2\widetilde{\mathcal{F}}_j+\mu_3\mu_2^{-1/2}\widetilde{\mathcal{T}}_j$$

In the proofs of Lemmas 5.1–5.2 we shall assume without loss of generality that $\mu_2=1$.

Proof of Lemma 5.1. Since $\mathbf{E}\widetilde{\mathcal{E}}_j\,\mathrm{e}\{G_j\}=0$, we can represent the function $\tilde{\psi}^{(m)}(\eta)$ in the form

$$\tilde{\psi}^{(m)}(\eta)=\mu_3\sum_{j=1}^{N}\mathbf{E}\widetilde{\mathcal{E}}_j(\mathrm{e}\{\widetilde{\Delta}_j\}-1)\,\mathrm{e}\{G_j\}=\tilde{\psi}_1^{(m)}(\eta)+\tilde{\psi}_2^{(m)}(\eta)+\tilde{\psi}_3^{(m)}(\eta)\tag{5.1}$$

where

$$\tilde{\psi}_1^{(m)}(\eta)=2it\mu_3\sum_{j=1}^{N}\mathbf{E}\widetilde{\mathcal{E}}_j\widetilde{\mathcal{F}}_j\mathrm{e}\{G_j\},\qquad \tilde{\psi}_2^{(m)}(\eta)=it\mu_3^2\sum_{j=1}^{N}\mathbf{E}\widetilde{\mathcal{E}}_j\widetilde{\mathcal{T}}_j\mathrm{e}\{G_j\}$$
$$\tilde{\psi}_3^{(m)}(\eta)=-t^2\mu_3\sum_{j=1}^{N}\mathbf{E}(1-\tau)^2\,\widetilde{\mathcal{E}}_j\widetilde{\Delta}_j^2\,\mathrm{e}\{G_j+\tau\widetilde{\Delta}_j\}$$

Using inequalities (3.1) and (3.2) we obtain that

$$|\tilde{\psi}_3^{(m)}(\eta)|=\|\eta\|\,|\mu_3|\,(1+V^2\,|\mu_3|)^2\,\mathcal{N}(t)\tag{5.2}$$

Note that $\sum_{j=1}^{N}\langle\eta,R_t(R_tA_0)^m\,e_j\rangle\langle e_j,A_0\mathbf{Y}\rangle=\langle(R_tA_0)^m\,\eta,R_tA_0\mathbf{Y}\rangle$ and rewrite the functions $\tilde{\psi}_i^{(m)}(\eta)$, $i=1,2,$ as follows

$$\tilde{\psi}_1^{(m)}(\eta)=2it\mu_3\mathbf{E}\langle\eta,R_t(R_tA_0)^m\,A_0\mathbf{Y}\rangle\,\mathrm{e}\{G\}+\varepsilon_1(t,\eta)\qquad\text{and}$$
$$\tilde{\psi}_2^{(m)}(\eta)=it\mu_3^2\langle\eta,R_t(R_tA_0)^m\,\mathbf{d}\rangle\,g(t)+\varepsilon_2(t,\eta)$$

where

$$\varepsilon_1(t,\eta) = -2t^2\mu_3 \sum_{j=1}^{N} \mathbf{E}\langle B_j(R_t A_0)^m \eta, R_t A_0 \mathbf{Y}_j\rangle \tilde{\Delta}_j \mathrm{e}\{G_j + \tau\tilde{\Delta}_j\}$$

$$\varepsilon_2(t,\eta) = -t^2\mu_3^2 \sum_{j=1}^{N} \langle B_j(A_0 R_t)^m \eta, R_t \mathbf{d}\rangle \mathbf{E}\tilde{\Delta}_j \mathrm{e}\{G_j + \tau\tilde{\Delta}_j\}$$

It is not difficult to prove that

$$|\varepsilon_1(t,\eta)| \leqslant Ct^2 \|\eta\| |\mu_3| \sum_{j=1}^{N} (|\mu_3| |d_j| \mathbf{E} |Y_j| \mathscr{H}_j(1, 0, B_j(R_t A_0)^m \eta, \mathbf{y}_j) + \mathbf{E}\tilde{\mathscr{H}}_j(1, 1, B_j(R_t A_0)^m \eta, \mathbf{y}_j))$$

By the inequality $\sum_{j=1}^{N} \|B_j(A_0 R_t)^m \eta\|^2 \leqslant \|\eta\|^2$ and by inequalities (3.1) and (3.2) we have

$$\varepsilon_1(t,\eta) = \|\eta\| |\mu_3| (1+|\mu_3| V) \mathscr{N}(t) \tag{5.3}$$

For $\varepsilon_2(t,\eta)$ we have

$$|\varepsilon_2(t,\eta)| \leqslant C |\mu_3|^2 |t|^2 \|R_t \mathbf{d}\| \sum_{j=1}^{N} \|B_j(A_0 R_t)^m \eta\| \times (|\mu_3| |d_j| \mathbf{E} Y_j^2 \tilde{\mathscr{H}}_j(0, 0, \mathbf{y}_j) + \mathbf{E}\tilde{\mathscr{H}}_j(1, 0, \mathbf{y}_j)) \tag{5.4}$$

The relations (5.1)–(5.4) together imply that

$$\tilde{\psi}^{(m)}(\eta) = 2it\mu_3 \mathbf{E}\langle \eta, R_t(R_t A_0)^m A_0 \mathbf{Y}\rangle \mathrm{e}\{G\} + it\mu_3^2 \langle \eta, R_t(R_t A_0)^m \mathbf{d}\rangle g(t) + |\mu_3| \|\eta\| (1+|\mu_3| V)^2 \mathscr{N}(t) \tag{5.5}$$

Note that the operators R_t and A_0 commute. Proceeding similarly as in Lemma 4.1, we derive from (5.5) that

$$\psi^{(m)}(\eta) = it\mu_3 \langle \eta, R_t(R_t A_0)^m \mathbf{d}\rangle g(t) + |\mu_3| \|\eta\| (1+|\mu_3| V)^2 \mathscr{N}(t) \qquad \square$$

Introduce the abbreviation $\mathscr{Z}_j = \langle \mathbf{y}_j, R_t A_0 \mathbf{y}_j\rangle$.

Proof of Lemma 5.2. We write $\tilde{\varphi}(t)$ in the form

$$\tilde{\varphi}(t) = \tilde{\varphi}^{(1)}(t) + \tilde{\varphi}^{(2)}(t)$$

where

$$\tilde{\varphi}^{(1)}(t) = \sum_{j=1}^{N} \mathbf{E}\,\mathscr{Z}_j \mathrm{e}\{G\} \qquad \tilde{\varphi}^{(2)}(t) = \sum_{j=1}^{N} \mathbf{E}\,\widetilde{\mathscr{S}}_j \mathrm{e}\{G\}$$

Furthermore, we have

$$\tilde{\varphi}^{(1)}(t) = u_{11}(t) + u_{12}(t)$$

where

$$u_{11}(t) = \sum_{j=1}^{N} \mathbf{E}\,\mathscr{Z}_j \mathrm{e}\{G_j\}, \qquad u_{12}(t) = \sum_{j=1}^{N} \mathbf{E}\,\mathscr{Z}_j (\mathrm{e}\{\widetilde{\Delta}_j\} - 1)\,\mathrm{e}\{G_j\}$$

Since $\mathbf{y}_j$ and G_j are independent, we obtain

$$u_{11}(t) = \mathrm{Tr}(R_t A_0)\, g(t) + \sum_{j=1}^{N} \mathbf{E}\,\mathscr{Z}_j \mathbf{E}(1 - \mathrm{e}\{\widetilde{\Delta}_j\})\,\mathrm{e}\{G_j\} \tag{5.6}$$

Since in the Gaussian case $\mathbf{E}\,\widetilde{\Delta}_j \mathrm{e}\{G_j\} = 0$, we can write

$$u_{11}(t) = \mathrm{Tr}(R_t A_0)\, g(t) + \varepsilon_3(t) \tag{5.7}$$

where

$$\varepsilon_3(t) = -t^2 \sum_{j=1}^{N} \mathbf{E}\,\mathscr{Z}_j \mathbf{E}(1-\tau)^2\, \widetilde{\Delta}_j^2\, \mathrm{e}\{G_j + \tau \widetilde{\Delta}_j\}$$

Note that

$$|\varepsilon_3(t)| \leqslant C t^2 \sum_{j=1}^{N} |R_t A_0(j, j)|\, (\mu_3^2\, |d_j|^2\, \mathbf{E}\, Y_j^4 \widetilde{\mathscr{H}}_j(0, 0, \mathbf{y}_j)$$
$$+ |\mu_3|\, |d_j|\, \mathbf{E}\, |Y_j|^2\, \widetilde{\mathscr{H}}_j(0, 1, \mathbf{y}_j) + \mathbf{E}\, \widetilde{\mathscr{H}}_j(0, 2, \mathbf{y}_j))$$

This implies

$$\varepsilon_3(t) = (1 + \mu_3^2 V)^2\, \mathscr{N}(t) \tag{5.8}$$

Since Y_j, $j = 1, \ldots, N$, are Gaussian, we have

$$\mathbf{E}\,\mathscr{Z}_j \widetilde{\mathscr{F}}_j \mathrm{e}\{G_j\} = 0 \qquad \text{and} \qquad \mathbf{E}\,\mathscr{Z}_j \widetilde{\mathscr{T}}_j \mathrm{e}\{G_j\} = 0$$

Hence,

$$u_{12}(t)=\sum_{j=1}^{N}\mathbf{E}(1-\tau)\,\mathscr{L}_j\widetilde{\Delta}_j^2\,\mathrm{e}\{G_j+\tau\widetilde{\Delta}_j\}$$

Similarly as above we deduce that

$$u_{12}(t)=(1+|\mu_3|\,V)^2\,\mathcal{N}(t) \tag{5.9}$$

The relations (5.6)–(5.9) together imply that

$$\tilde{\varphi}^{(1)}(t)=\mathrm{Tr}(R_tA_0)\,g(t)+(1+|\mu_3|\,V)^2\,\mathcal{N}(t) \tag{5.10}$$

Now we shall study the asymptotic approximation of the function $\tilde{\varphi}^{(2)}(t)$. We write

$$\tilde{\varphi}^{(2)}(t)=u_{13}(t)+u_{14}(t)$$

where

$$u_{13}(t)=\sum_{j=1}^{N}\mathbf{E}\widetilde{\mathscr{S}}_j\mathrm{e}\{G_j\}=0$$

$$u_{14}(t)=\sum_{j=1}^{N}\mathbf{E}\widetilde{\mathscr{S}}_j(\mathrm{e}\{\widetilde{\Delta}_j\}-1)\,\mathrm{e}\{G_j\}$$

Expand the function $v_{13}(t)$ as follows

$$u_{14}(t)=u_{15}(t)+u_{16}(t)+u_{17}(t) \tag{5.11}$$

where

$$u_{15}(t)=2it\sum_{j=1}^{N}\mathbf{E}\widetilde{\mathscr{S}}_j\widetilde{\mathscr{F}}_j\mathrm{e}\{G_j\},\qquad u_{16}(t)=it\mu_3\sum_{j=1}^{N}\mathbf{E}\widetilde{\mathscr{S}}_j\widetilde{\mathscr{T}}_j\mathrm{e}\{G_j\}$$

$$u_{17}(t)=-t^2\sum_{j=1}^{N}\mathbf{E}(1-\tau)^2\,\widetilde{\mathscr{S}}_j\widetilde{\Delta}_j^2\mathrm{e}\{G_j+\tau\widetilde{\Delta}_j\}$$

It is not difficult to prove that

$$|u_{17}(t)|\leqslant Ct^2\sum_{j=1}^{N}(\mu_3^2d_j^2\mathbf{E}Y_j^4\widetilde{\mathscr{H}}_j(1,0,\mathbf{y}_j)$$
$$+2\,|\mu_3|\,|d_j|\,\mathbf{E}Y_j^2\widetilde{\mathscr{H}}_j(1,1,\mathbf{y}_j)+\mathbf{E}\widetilde{\mathscr{H}}_j(1,2,\mathbf{y}_j))$$

This implies

$$u_{17}(t) = (1+|\mu_3|\, V)^2\, \mathcal{N}(t) \tag{5.12}$$

The functions $u_{15}(t)$ and $u_{16}(t)$ may be written in the form

$$u_{15}(t) = 2it\,\mathbf{E}\langle \mathbf{Y},\, A_0 R_t A_0 \mathbf{Y}\rangle\, \mathrm{e}\{G\} + \varepsilon_4(t)$$
$$u_{16}(t) = it\mu_3 \mathbf{E}\langle R_t A_0 \mathbf{d},\, \mathbf{Y}\rangle\, \mathrm{e}\{G\} + \varepsilon_5(t)$$

where

$$\varepsilon_4(t) = -2t^2 \sum_{j=1}^{N} \mathbf{E}\widetilde{\mathscr{S}}_j \widetilde{\mathscr{F}}_j \widetilde{\Delta}_j \mathrm{e}\{G_j + \tau\widetilde{\Delta}_j\}$$
$$\varepsilon_5(t) = -t^2\mu_3 \sum_{j=1}^{N} \mathbf{E}\widetilde{\mathscr{S}}_j \widetilde{\mathscr{T}}_j \widetilde{\Delta}_j \mathrm{e}\{G_j + \tau\widetilde{\Delta}_j\}$$

Similarly as in (5.6), we have

$$|\varepsilon_4(t)| \leqslant Ct^2 \sum_{j=1}^{N} (\mathbf{E}\widetilde{\mathscr{H}}_j(1,2,\mathbf{y}_j) + |\mu_3|\, |d_j|\, \mathbf{E}\, |Y_j|^2\, \widetilde{\mathscr{H}}_j(1,1,\mathbf{y}_j)) \tag{5.13}$$

and

$$|\varepsilon_5(t)| \leqslant Ct^2 \sum_{j=1}^{N} (|\mu_3|^2\, d_j^2 \mathbf{E}\, |Y_j|^2\, \widetilde{\mathscr{H}}_j(1,0,\mathbf{y}_j) + |\mu_3|\, |d_j|\, \mathbf{E} Y_j \widetilde{\mathscr{H}}_j(1,1,\mathbf{y}_j)) \tag{5.14}$$

By Lemma 5.1 and by inequalities (3.1), (3.2) we get

$$\begin{aligned} u_{16}(t) &= it\widetilde{\psi}^{(1)}(\mathbf{d}) + (1+|\mu_3|\, V)^2\, \mathcal{N}(t) \\ &= -t^2\mu_3^2\langle \mathbf{d},\, R_t^2 A_0 \mathbf{d}\rangle\, g(t) + |\mu_3|\, V(1+|\mu_3|\, V)(1+|t|\,(1+|\mu_3|\, V))\, \mathcal{N}(t) \end{aligned} \tag{5.15}$$

Now (5.6)–(5.15) imply

$$\begin{aligned} \tilde{\varphi}(t) = {} & 2it\,\mathbf{E}\langle \mathbf{Y},\, A_0 R_t A_0 \mathbf{Y}\rangle\, \mathrm{e}\{G\} + \mathrm{Tr}(R_t A_0)\, g(t) \\ & - t^2\mu_3^2\langle \mathbf{d},\, R_t^2 A_0 \mathbf{d}\rangle\, g(t) + (1+|\mu_3|\, V)^2\, (1+|t|\, |\mu_3|\, V)\, \mathcal{N}(t) \end{aligned}$$

Since $(I - 2itA_0)\, R_t = I$, the last equality together with the definition of $\tilde{\varphi}$ implies the result. □

Proof of Theorem 3.2. Note that

$$g'(t) = i\varphi(t) + i\psi^{(0)}(\mathbf{d}) + i\sqrt{\bar{\mu}_4}\, V\, \mathbf{E}\, Y\, \mathrm{e}\{G\}$$

Since Y is a Gaussian r.v. independent of $\langle \mathbf{Y}, A_0\mathbf{Y}\rangle + \mu_3\langle \mathbf{d}, \mathbf{Y}\rangle$, it follows by some straightforward computation that

$$i\sqrt{\bar{\mu}_4}\, V\, \mathbf{E}\, Y\, \mathrm{e}\{G\} = -t\bar{\mu}_4\langle \mathbf{d}, \mathbf{d}\rangle\, g(t) \tag{5.16}$$

Now Lemmas 5.1, 5.2 and the equality (5.16) together imply

$$\begin{aligned} g'(t) &= i\,\mathrm{Tr}(R_t A_0)\, g(t) - it^2\mu_3^2\langle \mathbf{d}, R_t^2 A_0 \mathbf{d}\rangle\, g(t) - t\mu_3^2\langle \mathbf{d}, R_t\mathbf{d}\rangle\, g(t) \\ &\quad - t(\mu_4 - 1 - \mu_3^2\langle \mathbf{d}, \mathbf{d}\rangle)\, g(t) + (1 + |\mu_3|\, V)^2(1 + |t|\, |\mu_3|\, V)\, \mathcal{N}(t) \end{aligned}$$

From the equality $R_t - I = 2itR_tA_0$ it follows that

$$t\mu_3^2\langle \mathbf{d}, (R_t - I)\,\mathbf{d}\rangle = 2it^2\mu_3^2\langle \mathbf{d}, R_tA_0\mathbf{d}\rangle$$

This implies the result. □

6. THE METHOD OF SYMMETRIZATION

In this section we consider the bounds for $|f(t)|$ and $|g(t)|$ and for some auxiliary functions introduced in Section 3. Since $|f(t)|$, $|g(t)|$ and $|f(t) - g(t)|$ are odd functions we consider throughout the paper positive frequencies $t \geqslant 0$ only. We recall that

$$\mathbf{X} = \sum_{j=1}^{N} \mathbf{x}_j, \quad \mathbf{Y} = \sum_{j=1}^{N} \mathbf{y}_j, \qquad \mathbf{X}_j = \mathbf{X} - \mathbf{x}_j, \qquad \mathbf{Y}_j = \mathbf{Y} - \mathbf{y}_j$$

$$Q_j = \langle A\mathbf{X}_j, \mathscr{X}_j\rangle, \qquad G_j = \mu_2\langle \mathbf{Y}_j, A_0\mathbf{Y}_j\rangle + \mu_3\mu_2^{-1/2}\langle \mathbf{d}, \mathbf{Y}_j\rangle + \sqrt{\bar{\mu}_4}\,\langle \mathbf{d}, Z\rangle$$

Here $\mathbf{x}_j$ denote the vector with zero coordinates except for the jth coordinate, where $\mathbf{x}_j^{(j)} = X_j$ and let $\mathbf{y}_j$ denote a similar vector with $\mathbf{y}_j^{(j)} = Y_j$. We have defined the operator R_t by the equality $R_t = (I - 2itA)^{-1}$. In Section 3 we introduced the functions

$$\mathscr{H}_j(p, q, \eta, \zeta) = |\mathbf{E}\langle \eta, R_tA_0\mathbf{X}_j\rangle^p\, \langle \zeta, A_0\mathbf{X}_j\rangle^q\, \mathrm{e}\{Q_j + \langle \zeta, A_0\mathscr{X}_j\rangle\}|$$

and similarly for Gaussian r.v.'s

$$\tilde{\mathscr{H}}_j(p, q, \eta, \zeta) = |\mathbf{E}\langle \eta, R_tA_0\mathbf{Y}_j\rangle^p\, \langle \zeta, A_0\mathbf{Y}_j\rangle^q\, \mathrm{e}\{G_j + \langle \zeta, A_0\mathbf{Y}_j\rangle\}|$$

where η and ζ denote some non random vectors. Denote by $\mathbf{I}_{\{B\}}$ the indicator function of an event B. Throughout $\overline{X}$ will be an independent copy of X independent of all other r.v.'s, and $\widetilde{X}$ will denote a symmetrization of X, i.e., $\widetilde{X} = X - \overline{X}$. For convenience we shall state the results of GT we are going to use. We write $\overline{\mathscr{L}} = (1-V^2)^{-1/2} \mathscr{L}$ and use the notations (β, μ, b_l, λ_j, Λ_j) introduced in Section 1. For readers convenience we recall below the theorems, which will be proved.

Theorem 3.3. Assume that (1.3) holds. Then for all $t \geqslant 0$ there exists an absolute constant $\delta_0 > 0$ such that

$$|f(t)| \leqslant \exp\{-\delta_0 (t \wedge T_0)^2 (1-V^2) \mu_2^2 \overline{\mathscr{L}}^{1/4}\} + \mathbf{I}\{t > \tfrac{1}{8} \beta_3^{-2} \mu_2^2 |\lambda_1|^{-1}\} + C\overline{\mathscr{L}}^{3/2} \tag{6.1}$$

and

$$|g(t)| \leqslant (\exp\{-\tfrac{1}{2} (2(1-V^2) \mu_2^2 + V^2 \mu_3^2 \mu_2^{-1}) t^2\} + \mathbf{I}\{t > \tfrac{1}{8} \mu_2^{-1} |\lambda_1|^{-1}\}) \times \exp\{-\tfrac{1}{2} \bar{\mu}_4 V^2 t^2\} \tag{6.2}$$

where $t \wedge T_0 = \min\{t, T_0\}$, $T_0 = \frac{1}{8} \beta_3^{-2} \mu_2^2 |\lambda_1|^{-1} (\log^+ |\lambda_1|)^{-1/2}$.

Under assumption (1.5) there exists an absolute constant $\delta_0 > 0$ such that for $\frac{1}{8} \beta_3^{-2} \mu_2^2 \mathscr{L}^{-1} (\log^+ \mathscr{L})^{-1} \leqslant t \leqslant \frac{1}{8} \beta_3^{-2} \mu_2^2 \mathscr{L}^{-1}$ we have

$$|f(t)| \leqslant (1 + \delta_0 \mu_2^2 b_0^2 (1-V^2) t^2 \overline{\mathscr{L}}^{1/4q})^{-q/16} + C\overline{\mathscr{L}}^{3/2} \tag{6.3}$$

and

$$|g(t)| \leqslant (1 + \mu_2^2 b_0^2 t^2)^{-q/4} \exp\{-\tfrac{1}{2} \bar{\mu}_4 V^2 t^2\} \tag{6.4}$$

For $j = 1,\ldots, N$ introduce the functions

$$\mathscr{H}_j(p, q, \eta_j) := \mathscr{H}_j(p, q, \eta_j, \tau\eta_j) \qquad \text{and} \qquad \widetilde{\mathscr{H}}_j(p, q, \eta_j) := \mathscr{H}_j(p, q, \eta_j, \tau\eta_j)$$

where η_j denotes a deterministic vector with zero coordinates except for the jth coordinate.

Theorem 3.4. Assume that the conditions of Theorem 3.1 are fulfilled. Then

$$\max\{\widetilde{\mathscr{H}}_j(p, q, \eta_j), \mathscr{H}_j(p, q, \eta_j)\} \leqslant C \|\eta_j\|^{p+q} \mathscr{L}_j^{p+q} \beta_{p+q} \varkappa_1(t) \tag{6.5}$$

and

$$|f(t)| \leqslant C\varkappa_1(t) \tag{6.6}$$

where for some positive δ_0

$$\varkappa_1(t)=\exp\{-\delta_0(1-V^2)\,\mu_2^2t^2\}+C\bar{\mathscr{L}}^3+\mathbf{I}\{t \geqslant \mu_2^{-1}\,|\lambda_1|^{-1} \wedge \mathscr{L}^{-1}(\log^+|\bar{\mathscr{L}}|)^{-1}\}$$

If in addition we assume (1.5), then

$$\max\{\tilde{\mathscr{H}}_j(p,q,\eta_j),\mathscr{H}_j(p,q,\eta_j)\} \leqslant C\,\|\eta_j\|^{p+q}\,\beta_{p+q}\mathscr{L}_j^{p+q}\varkappa_2(t) \tag{6.7}$$

and

$$|f(t)| \leqslant C\varkappa_2(t) \tag{6.8}$$

with

$$\varkappa_2(t)=(1+\delta_0\mu_2^2b_0^2(1-V^2)\,t^2)^{-q/8}+C\bar{\mathscr{L}}^6+\mathbf{I}\{t \geqslant \mu_2^{-1}(\mathscr{L}\log^+\bar{\mathscr{L}})^{-1}\}$$

Proof of Theorem 3.4. Let $\varepsilon_1,\ldots,\varepsilon_N$ denote i.i.d. Bernoulli r.v.'s with

$$\mathbf{P}\{\varepsilon_1=1\}=1-\mathbf{P}\{\varepsilon_1=0\}=p$$

Write $\bar{a}_{jk}=(1-V^2)^{-1/2}\,a_{jk}$, and $\bar{A}_0=(1-V^2)^{-1/2}\,A_0$, $\|\bar{A}_0\|=1$. Note that the inequality

$$\begin{aligned}|f(t)|^4 &\leqslant \mathbf{E}\exp\left\{it\sum_{j,k=1}^N a_{jk}(\varepsilon_j-\varepsilon_k)^2\,\tilde{X}_j\tilde{X}_k\right\}\\ &=\mathbf{E}\exp\left\{it(1-V^2)^{1/2}\sum_{j,k=1}^N \bar{a}_{jk}(\varepsilon_j-\varepsilon_k)^2\,\tilde{X}_j\tilde{X}_k\right\}\end{aligned} \tag{6.9}$$

still holds if $a_{jj}\neq 0$. Moreover, the right hand side of (6.9) is independent of the diagonal of the matrix A. Denote by $\tilde{X}^{(H)}$ the truncation of $\tilde{X}$, i.e.,

$$\tilde{X}^{(H)}=\tilde{X}\mathbf{I}\{|\tilde{X}| \leqslant H\}$$

where $\mathbf{I}\{B\}$ denotes the indicator of an event B. Introduce the notation

$$\sigma^2(H)=\mathbf{E}(\tilde{X}^{(H)})^2, \qquad \mu_4(H)=\mathbf{E}(\tilde{X}^{(H)})^4$$

The Lemma 3.2 in GT and inequity (6.9) together imply that for all $t \geqslant 0$, $K>0$,

$$|f(t)|^4 \leqslant \mathbf{E} \exp\left\{\tfrac{1}{4} i(t \wedge T_0)\, \sigma(H) \sum_{j,\, k \in \mathbb{N}} a_{jk} \varepsilon_k (1-\varepsilon_j)\, Y_j Y_k\right\} + \mathbf{I}\{|t| > \gamma \bar{\mathscr{L}}^{-1}\}$$

$$+ \mathbf{P}\left\{\mu_2^{1/2} \max_{k \in \mathbb{N}} \left|\sum_{j \in \mathbb{N}} a_{jk}(1-\varepsilon_j)\, Y_j\right| \geqslant K \bar{\mathscr{L}} \log^+ \bar{\mathscr{L}}\right\}$$

$$+ \mathbf{P}\left\{\max_{j \in \mathbb{N}} \left|\sum_{k \in \mathbb{N}} a_{jk} \varepsilon_k \widetilde{X}_l^{(H)}\right| > K \bar{\mathscr{L}}\right\} \tag{6.10}$$

where $T_0 = \gamma \bar{\mathscr{L}}^{-1} (\log^+ \bar{\mathscr{L}})^{-1}$, $t \wedge T_0 = \min\{t, T_0\}$ and $\gamma = \frac{\sigma(H)}{2K\sqrt{\mu_4(H)}}$. Note that the first function in the right hand side of (6.10) is the conditionally Gaussian characteristic function and therefore it is nonnegative. Without loss of generality we shall assume that $\mathscr{L}_j \neq 0$. Let $\rho_j = \mathscr{L} \mathscr{L}_j^{-1} (\log^+ \mathscr{L}_j)^{-1}$ and $\varkappa = \frac{e}{H^2} \log^+ \mathscr{L}$. By Remark 5.3 in GT for $\bar{\mathscr{L}} \leqslant e^{-\frac{1}{2}}$ we have

$$\mathbf{P}\left\{\max_{1 \leqslant j \leqslant N} \left|\sum_{k \in \mathbb{N}} a_{kj} \mathrm{e}_k \widetilde{X}_k^{(H)}\right| > 2K\mathscr{L}\right\}$$

$$\leqslant \mathscr{L}^{KH^{-1} - 2 - p\varkappa} + p^{K/H} \mathscr{L}^{-2} (\log^+ \mathscr{L})^{-2K/H} \tag{6.11}$$

and by Lemma 5.5 in GT

$$\mathbf{P}\left\{\max_{1 \leqslant j \leqslant N} \left|\sum_{k \in \mathbb{N}} a_{kj} (1-\varepsilon_k)\, Y_k\right| > K\mathscr{L}\, |\log^+ \bar{\mathscr{L}}|\right\} \leqslant \bar{\mathscr{L}}^{\frac{K^2}{2\mu_2} - 2} \tag{6.12}$$

The conclusion of Theorem 3.3 now follow from inequalities (6.10)–(6.12). Choosing $H = 2\beta_3 \mu_2^{-1}$, $K = 32H$, and $p = \bar{\mathscr{L}}^{1/4}$, we obtain (5.1). Choosing $H = 2\beta_3 \mu_2^{-1}$, $K = 32H$, and $p = \bar{\mathscr{L}}^{1/(4q)}$, we obtain (5.3). See also inequalities (3.24) and (3.30) in GT. Inequalities (6.2) and (6.4) now follow from the inequality

$$|g(t)| \leqslant \left|\exp\left\{i\mu_2 \int_0^t \mathrm{Tr}(R_u A_0)\, du\right\}\right| \exp\{-1/2 \bar{\mu}_4 t^2\}$$

and the relation $R_u = I + 2iu\mu_2 R_u A_0$. □

Proof of Theorem 3.4. Let J_0, J_1 a partition of $\{1, 2, \ldots, N\}$ and let $J_{i1}, \ldots, J_{i5}$ denote a partition of J_i, $i = 0, 1$, into five subsets. Let p, q denote integers such that $p + q \leqslant 3$. Repeating the proof of Lemma 3.3 in GT we obtain that

$$\mathscr{H}_j(p, q, \eta, \zeta) \leqslant C \max_{i=0,1} \max_{1 \leqslant l \leqslant 5} (\mathbf{E}\, |\langle \eta, R_t A Z_{i,l} \rangle|^p \, |\langle \zeta, A Z_{i,l} \rangle|^q)$$
$$\times \Big(\max_{1 \leqslant l \leqslant 5} \mathbf{E} \exp\{2it \langle \widetilde{Z}_0, A \widetilde{Z}_{1,l} \rangle\}$$
$$+ \max_{1 \leqslant l \leqslant 5} \max_{1 \leqslant m \leqslant 5} \mathbf{E} \exp\{2it \langle \widetilde{Z}_{1l}, A \widetilde{Z}_{0m} \rangle\}\Big)^{1/2}$$

where $Z_l := \sum_{k \in J_l, k \neq j} \mathbf{x}_k$, $Z_{0l} := \sum_{k \in J_{0l}, k \neq j} \mathbf{x}_k$, $l = 1,..., 5$.

Note that

$$\max_{1 \leqslant l \leqslant 5} \mathbf{E}\, |\langle \eta_j, R_t A Z_{i,l} \rangle|^p \, |\langle \eta_j, A Z_{i,l} \rangle|^q$$
$$\leqslant \|\eta_j\|^{p+q} \, \beta_r^{(p+q)/r} \left(\sum_{l \neq j} |R_t A(l, j)|^2 \right)^{p/2} \left(\sum_{l \neq j} |A(l, j)|^2 \right)^{q/2}$$

where $r = \max\{p+q, 2\}$. Using inequality (3.1) we obtain that

$$\max_{1 \leqslant l \leqslant 5} \mathbf{E}\, |\langle \eta_j, R_t A Z_{i,l} \rangle|^p \, |\langle \eta_j, A Z_{i,l} \rangle|^q \leqslant \|\eta_j\|^{p+q} \, \mathscr{L}_j^{p+q} \beta_r^{(p+q)/r}$$

Now we can repeat the proof of Theorem 1 in GT. The inequality (6.5) follows from Lemmas 3.4, and inequalities (6.8)–(6.13) in GT. If (1.5) is fulfilled, the inequalities (6.16)–(6.19) in GT imply (6.7). The inequality (6.6) follows from Lemmas 3.4, 4.1, 5.4, and 5.5 in GT. Finally, the inequality (6.8) follows as well from Lemmas 3.4, 4.2, 5.4, and 5.5 in GT. □

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