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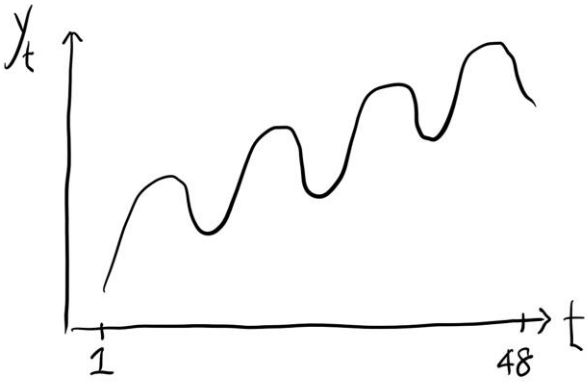
Time Series Analytics

111-1 Homework #01

Due at 23h59, September 11, 2021; files uploaded to NTU-COOL

1. (10%) Write down the scientific procedures to simulate a time series of length 48 similar to the trend below. Use any functions you prefer.



2. (10%) Simulate a time series of length 48 following the settings below
$$Y_t = \cos\left[2\pi\left(\frac{t}{12} + \Phi\right)\right] \text{ for } t = 0, 1, 2, \dots, 47,$$
where Φ is selected from a uniform distribution on the interval $[0, 1]$.

3. (10%) X and Y are two dependent random variables and $V[X] = V[Y]$, find $COV[X + Y, X - Y]$.

4. (15%) Suppose $E[X] = 3, V[X] = 9, E[Y] = 4, V[Y] = 16$, and $Corr(X, Y) = 0.25$. Find:
   a. $V[X + Y]$
   b. $COV[X, X + Y]$
   c. $Corr(X + Y, X - Y)$

3. (10%) X and Y are two dependent random variables and $\mathrm{V}[X] = \mathrm{V}[Y]$, find $\mathrm{COV}[X+Y, X-Y]$.

Given
1) X and Y are two dependent random variables
2) $V[X] = V[Y]$

Required
$COV[X+Y, X-Y]$

Solution
From definition

$\hookrightarrow V[X] = E[X^2] - (E[X])^2$ —(1)

From definition

$\hookrightarrow V[X] = \sigma_X^2 = E[(X-\mu_X)^2]$

From definition, variance of X is a covariance of X with itself, thus :

$$\begin{aligned} \hookrightarrow COV[X,Y] &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y] \\ &= E[XY] - \mu_X\mu_Y - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y} \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

$COV[X,Y] = E[XY] - E[X]E[Y]$ —(2)

From (1) :

$$\begin{aligned} COV[X+Y, X-Y] &= E[X^2 - Y^2] - E[X+Y]\,E[X-Y] \\ &= E[X^2] - E[Y]^2 - (E[X] + E[Y])(E[X] - E[Y]) \\ &= E[X^2] - E[Y^2] - \left[(E[X])^2 - (E[Y])^2\right] \\ &= \left[E[X^2] - (E[X])^2\right] - \left\{E[Y^2] - (E[Y])^2\right] \end{aligned}$$

From (2) : $COV[X+Y, X-Y] = V[X] - V[Y]$

Since $V[X] = V[Y]$, $COV[X+Y, X-Y] = 0$

Answer
$COV[X+Y, X-Y] = 0$

4. (15%) Suppose $E[X] = 3, V[X] = 9, E[Y] = 4, V[Y] = 16$, and $Corr(X, Y) = 0.25$. Find:
   a. $V[X + Y]$
   b. $COV[X, X + Y]$
   c. $Corr(X + Y, X - Y)$

a) $V[X+Y]$

From definition

↳ $cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

From definition

↳ $Corr[X,Y] = \dfrac{COV[X,Y]}{\sqrt{V[X]}\sqrt{V[Y]}}$

$0.25 = \dfrac{COV[X,Y]}{\sqrt{9}\ \sqrt{16}}$

$COV[X,Y] = 3$

From definition

↳ $V[X] = E[(X-\mu_X)^2]$

$V[X+Y] = E[(X+Y-\mu_X-\mu_Y)^2]$

$= E[(X-\mu_X + Y-\mu_Y)^2]$

$= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2]$

$= E[(X-\mu_X)^2] + 2E[(X-\mu_X)(Y-\mu_Y)] + E[(Y-\mu_Y)^2]$

$V[X+Y] = V[X] + V[Y] + 2\,COV[X,Y]$

$= 9 + 16 + 2(3)$

Answer $V[X+Y] = 31$

b) $COV[X, X+Y]$

From 4a), $COV[X,Y] = 3$

↳ $3 = E[XY] - (3)(4)$

$E[XY] = 15$

From definition,

↳ $cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$COV[X,Y] = E[XY] - E[X]E[Y]$

$COV[X,X+Y] = E[X(X+Y)] - E[X]E[X+Y]$

$= E[X^2] - E[X]^2 + E[XY] - E[X]E[Y]$

$= V[X] + E[XY] - E[X]E[Y]$

$= 9 + 15 - (3)(4)$

Answer $COV[X,X+Y] = 12$

c) $Corr[X+Y, X-Y]$

From definition, variance of X is a covariance of X with itself, thus:

↳ $cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$= E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y]$

$= E[XY] - \mu_X\mu_Y - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y}$

$= E[XY] - E[X]E[Y]$

$COV[X,Y] = E[XY] - E[X]E[Y]$ — (1)

From (1): $COV[X+Y, X-Y] = V[X] - V[Y]$

$COV[X+Y, X-Y] = 9 - 16 = -7$

From definition

↳ $V[X] = E[(X-\mu_X)^2]$

$V[X-Y] = E[(X-Y-\mu_X+\mu_Y)^2]$

$= E[(X-\mu_X - Y+\mu_Y)^2]$

$= E[((X-\mu_X) - (Y-\mu_Y))^2]$

$= E[(X-\mu_X)^2 - 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2]$

$= E[(X-\mu_X)^2] - 2E[(X-\mu_X)(Y-\mu_Y)] + E[(Y-\mu_Y)^2]$

$V[X-Y] = V[X] + V[Y] - 2\,COV[X,Y]$

$= 9 + 16 - 2(3)$

$V[X-Y] = 19$

From definition

↳ $Corr[X,Y] = \dfrac{COV[X,Y]}{\sqrt{V[X]}\sqrt{V[Y]}}$

$Corr[X+Y, X-Y] = \dfrac{COV[X+Y, X-Y]}{\sqrt{V[X+Y]}\sqrt{V[X-Y]}}$

$= \dfrac{-7}{\sqrt{31}\ \sqrt{19}}$

Answer $Corr[X+Y, X-Y] = -0.2884$