National Taiwan University
Institute of Industrial Engineering
Jakey BLUE

Time Series Analytics

111-1 Homework #07

Due at 23h59, November 27, 2022; files uploaded to NTU-COOL

1. (10%) Given the model $y_t = a + bt + c_t + x_t$, where a, b are constants, c_t is deterministic and periodic with period s and x_t is a SARIMA$(p, 0, q) \times (P, 1, Q)_s$. What is the model for $w_t = y_t - y_{t-s}$?

2. (10%) Identify the following as certain multiplicative SARIMA models:
   (a) $y_t = 0.5y_{t-1} + y_{t-4} - 0.5y_{t-5} + a_t - 0.3a_{t-1}$
   (b) $y_t = y_{t-1} + y_{t-12} - y_{t-13} + a_t - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13}$

3. (10%) If the characteristic function of an AR time series model is
$$(1 - 1.6B + 0.7B^2)(1 - 0.8B^{12})$$
   (a) Is the model stationary?
   (b) Identify the model as a certain SARIMA model.

4. (15%) Suppose $y_t = y_{t-4} + a_t$ with $y_t = a_t$, for $t = 1, 2, 3, 4$.
   (a) Find the variance function for y_t.
   (b) Find the autocorrelation function for y_t.
   (c) Identify the model for y_t as a certain SARIMA model.

5. (15%) Consider the famous time series data "co2" (monthly carbon dioxide through 11 years in Alert, Canada).
   (a) Fit a deterministic regression model in terms of months and time. Are the regression coefficients significant? What is the adjusted R-squared? (Note that the month variable should be treated as categorical and transformed into 11 dummy variables.)
   (b) Identify, estimate the SARIMA model for the co2 level.
   (c) Compare the two models above, what do you observe?

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1. (10%) Given the model $y_t = a + bt + c_t + x_t$, where a, b are constants, c_t is deterministic and periodic with period s and x_t is a SARIMA$(p,0,q) \times (P,1,Q)_s$. What is the model for $w_t = y_t - y_{t-s}$?

Given $y_t = a + bt + c_t + x_t$

a, b : constant

c_t : deterministic, period s

x_t : SARIMA $(p,0,q) \times (P,1,Q)_s$

Find the model for $w_t = y_t - y_{t-s}$

$w_t = \cancel{a} + bt + c_t + x_t - [\cancel{a} + b(t-s) + c_{t-s} + x_{t-s}]$

$w_t = -bs + c_t - c_{t-s} + x_t - x_{t-s}$

$w_t = -bs + \nabla_s c_t + \nabla_s x_t$

w_t follows the ARMA$(p,q) \times (P,Q)_s$ with "bs" as an added constant

2. (10%) Identify the following as certain multiplicative SARIMA models:
(a) $y_t = 0.5y_{t-1} + y_{t-4} - 0.5y_{t-5} + a_t - 0.3a_{t-1}$
(b) $y_t = y_{t-1} + y_{t-12} - y_{t-13} + a_t - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13}$

a) $Y_t = 0.5Y_{t-1} + Y_{t-4} - 0.5Y_{t-5} + a_t - 0.3a_{t-1}$

$\underbrace{Y_t - Y_{t-4}}_{I} = 0.5\underbrace{(Y_{t-1} - Y_{t-5})}_{AR} + \underbrace{a_t - 0.3a_{t-1}}_{MA}$

$\because$ note that $s = 4$

Y_t follows ARIMA$(1,0,1) \times (0,1,0)_4$ model, with $\Phi_1 = 0.5$, $\theta_1 = 0.3$, $s = 4$

b) $Y_t = Y_{t-1} + Y_{t-12} - Y_{t-13} - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13}$

$(Y_t - Y_{t-1}) - (Y_{t-12} - Y_{t-13}) = -0.5a_{t-1} - 0.5a_{t-12} + (0.5)(0.5)a_{t-13}$

Y_t follows ARIMA $(0,1,1) \times (0,1,1)_{12}$ model with $\theta_1 = 0.5$, $\Theta_1 = 0.5$, $s = 12$

3. (10%) If the characteristic function of an AR time series model is

$$(1 - 1.6B + 0.7B^2)(1 - 0.8B^{12})$$

(a) Is the model stationary?
(b) Identify the model as a certain SARIMA model.

AR Time Series $(1 - \underset{\phi_1}{1.6}B + \underset{\phi_2}{0.7}B^2)(1 - \underset{\Phi}{0.8}B^{12})$

a) Since $\Phi_1 = 0.8$, the seasonal part of the model is stationary

Check the roots of the nonseasonal AR(2) part.

The stationary conditions for AR(2) are

$\phi_1 + \phi_2 < 1$ $\quad -1.6 + 0.7 < 1$: ✓

$\phi_2 - \phi_1 < 1$ $\quad -1.6 - 0.7 < 1$: ✓

$|\phi_2| < 1$ $\quad |0.7| < 1$: ✓

Thus the nonseasonal part is stationary

$\therefore$ Since both the seasonal and nonseasonal parts are stationary, the complete model is stationary

b) Identify the model as a certain ARIMA model

The model is ARIMA $(2,0,0) \times (0,1,0)_{12}$ with $\phi_1 = -1.6$, $\phi_2 = 0.7$, $\Phi = 0.8$, $s = 12$

$0 = (1 - 1.6B + 0.7B^2)(1 - 0.8B^{12}) Y_t$

$0 = (1 - 0.8B^{12} - 1.6B + (1.6)(0.8)B^{13} + 0.7B^2 - (0.7)(0.8)B^{14}) Y_t$

$Y_t = (0.8B^{12} + 1.6B - (1.6)(0.8)B^{13} - 0.7B^2 + (0.7)(0.8)B^{14}) Y_t$

$Y_t = 1.6Y_{t-1} - 0.7Y_{t-2} + 0.8Y_{t-12} - (1.6)(0.8)Y_{t-13} + (0.7)(0.8)Y_{t-14} + a_t$

4. (15%) Suppose $y_t = y_{t-4} + a_t$ with $y_t = a_t$, for $t = 1, 2, 3, 4$.
   (a) Find the variance function for y_t.
   (b) Find the autocorrelation function for y_t.
   (c) Identify the model for y_t as a certain SARIMA model.

$y_t = y_{t-4} + a_t$, $y_t = a_t$ for $t = 1,2,3,4$

a) Variance function of y_t

Consider $t = 4k + r$

$r = 1,2,3,4$ // indicating quarter

$k = 0,1,2,3 \ldots$ // indicating year

$E[Y] = 0$

$$
\begin{aligned}
Y_t &= Y_{t-4} + a_t \\
&= (Y_{t-8} + a_{t-4}) + a_t \\
&= (Y_{t-12} + a_{t-8}) + a_{t-4} + a_t \\
&\vdots \\
&= a_t + a_{t-4} + a_{t-8} + \ldots + a_{r+8} + a_{r+4} + a_r
\end{aligned}
$$

There are $k+1$ occurences of a_r in Y_t

The variance is $Var[Y_t] = (k+1)\sigma_a^2$

b) Find the autocorrelation function of y

Let s be $> t$

Consider $s = 4j + i$

$i = 1,2,3,4$ // indicating quarter

$j = 0,1,2,3 \ldots$ // indicating year

$Cov(Y_t, Y_s) = Cov(a_t + a_{t-4} + a_{t-8} + \ldots + a_{r+8} + a_{r+4} + a_r)$

In the case of $r \neq i$ there won't be overlaps occuring between the two sets of noises a and thus the covariance will be 0.

In the case of $r = i$, then it's when s and t is referring to the same year, same quarter. Thus the noises a's will be identical from r until t

The $cov(Y_t, Y_s)$ will be $Var(Y_t)$

Thus $Cov(Y_t, Y_s) = \begin{cases} (k+1)\sigma_e^2 & ; \text{ when } r = i \\ 0 & ; \text{ otherwise} \end{cases}$

Because when r is equal to i, it means that $s-t$ is divisible by 4.

The correlation $Corr(Y_t, Y_s) = \begin{cases} \dfrac{\sqrt{k+1}}{\sqrt{j+1}} & \text{if } s = 4j+i \text{ and } t = 4k+r \text{ where } r = 1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$

When t increases, the autocorrelation of the same quarter in consecutive years will have high positive correlation. For different quarters, will be uncorrelated.

c) Identify the model for Y_t as a certain seasonal ARIMA model

Recall that $Y_t - Y_{t-4} = a_t$ is SARIMA $(0,0,0) \times (0,1,0)_4$

So, the yearly seasonal difference is white noise.