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Time Series Analytics

111-1 Homework #03
Due at 23h59, October 2, 2022; files uploaded to NTU-COOL

1. (10%) y_t is a stationary process with the autocovariance function γ_k. Define $\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t$. Show that
$$\mathrm{V}[\bar{y}] = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\gamma_k = \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1 - \frac{|k|}{n}\right)\gamma_k.$$

2. (10%) Assume x_t is a stationary process and define $y_t = \begin{cases} x_t & \text{for odd } t \\ x_t + 3 & \text{for even } t \end{cases}$
   (a) Show that $\mathrm{COV}[y_t,\ y_{t-k}]$ is independent of t for all lags k.
   (b) Is y_t stationary?

3. (10%) Let y_t be a stationary process with an autocovariance function γ_k.
   (a) Show that $z_t = \nabla y_t = y_t - y_{t-1}$ is stationary by finding the mean and autocovariance function for z_t.
   (b) Show that $w_t = \nabla^2 y_t = z_t - z_{t-1} = y_t - 2y_{t-1} + y_{t-2}$ is stationary.

4. (15%) Let x_t be stationary with $\mathrm{E}[x_t] = 0, \mathrm{V}[x_t] = 1,$ autocorrelation function ρ_k. Define that μ_t is a nonconstant function and σ_t is a positively nonconstant **function** (that is to say: μ_t and σ_t are both deterministic and in function of t). Now we observe a time series formulated as
$$y_t = \mu_t + \sigma_t x_t.$$
   (a) Find the mean and autocovariance function of y_t.
   (b) Show that the autocorrelation of y_t depends only on the lag k. Is y_t stationary?
   (c) Let $\mu_t = \mu_0$ be a constant value. Justify that y_t is still nonstationary.

5. (5%) Let x_t be the series of the "expected" measurements during the production process. Because the measuring tool itself won't be perfect, we observe $y_t = x_t + e_t$, assuming x_t and e_t are independent. In general, we call x_t the **signal** and e_t the **measurement (white) noise**.

   If x_t is stationary with the autocorrelation function ρ_k, show that y_t is also a stationary process with
$$\mathrm{corr}(y_t, y_{t-k}) = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}, \text{for } k \geq 1.$$
   $\frac{\sigma_x^2}{\sigma_e^2}$ is usually referred to as the **signal-to-noise ratio**, or **SNR** for short.
   The larger the SNR, the closer the autocorrelation function of the observed series y_t is to the autocorrelation function of the desired signal x_t.

6. (10%) Suppose $y_t = \alpha_0 + \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)]$, where $\alpha_0, f_1, f_2, \ldots, f_q$ are constants and $\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_q$ are independent random variables with zero means and variances $\mathrm{V}[\alpha_i] = V[\beta_i] = \sigma_i^2$. Show that y_t is stationary and find its autocovariance function.

   (Hint: show $\mathrm{COV}[y_t, y_s]$ depends only on $t - s$.)

1. (10%) y_t is a stationary process with the autocovariance function γ_k. Define $\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t$. Show that

$$V[\bar{y}] = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k = \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)\gamma_k.$$

Given. y_t is a stationary process with autocovariance function γ_k

$$\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t$$

Precision of $\bar{y}$: $V[\bar{y}] = \frac{\gamma_0}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k\right] = Var\left[\frac{1}{n}\sum_{t=1}^{n} y_t\right]$

$$= \frac{1}{n^2} Var\left[\sum_{t=1}^{n} y_t\right]$$

$$= \frac{1}{n^2} Cov\left[\sum_{t=1}^{n} y_t, \sum_{t=1}^{n} y_s\right]$$

$$= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n} \gamma_{t-s}$$

$$\boxed{\text{Theorem: Variance of sum} \quad Var\left(\sum_i X_i\right) = \sum_i Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)}$$

Let $k = t-s$

$$V[\bar{y}] = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{t-k=1}^{n}\gamma_k$$

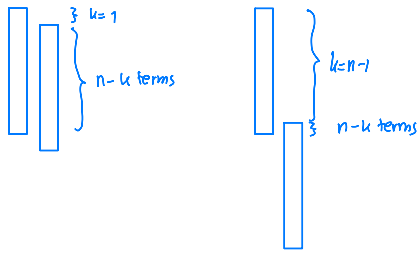
$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{t=k+1}^{n+k}\gamma_k$$

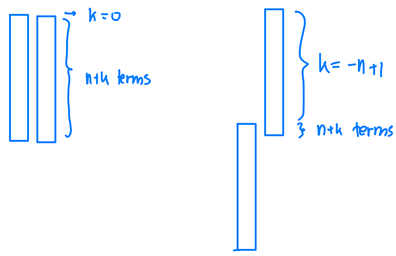
$$= \frac{1}{n^2}\left(\sum_{k=1}^{n-1}(n-k)\gamma_k + \sum_{k=-n+1}^{0}(n+k)\gamma_k\right)$$

$$= \frac{1}{n^2}\sum_{k=-n+1}^{n-1}\left((n-k)\gamma_k + (n+k)\gamma_k\right)$$

$$= \frac{1}{n^2}\sum_{k=-n+1}^{n-1}(n-|k|)\gamma_k$$

$$= \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)\gamma_k$$





2. (10%) Assume x_t is a stationary process and define $y_t = \begin{cases} x_t & \text{for odd } t \\ x_t + 3 & \text{for even } t \end{cases}$
   (a) Show that $\mathrm{COV}[y_t,\ y_{t-k}]$ is independent of t for all lags k.
   (b) Is y_t stationary?

Given: x_t is stationary

$$y_t = \begin{cases} x_t & \text{for odd } t \\ x_t + 3 & \text{for even } t \end{cases}$$

a) Show that $\mathrm{cov}[y_t, y_{t-k}]$ is independent of t for all lags k

$$\begin{aligned} \mathrm{cov}[y_t, y_{t-k}] &= E[(y_t - \mu)(y_{t-k} - \mu)] \\ &= E[y_t y_{t-k} - \mu y_t - \mu y_{t-k} + \mu^2] \\ &= E[y_t]E[y_{t-k}] - \cancel{(E[y_t])^2} - E[y_t]E[y_{t-k}] + \cancel{(E[y_t])^2} \\ &= 0 \end{aligned}$$

Since $\mathrm{cov}[y_t, y_{t-k}] = 0$, and x_t is stationary, the $\mathrm{cov}[y_t, y_{t-k}]$ is independent of t for all lags k. //

b) Is y_t stationary

Consider t and k are both odd, meaning $t-k$ is even

↳ $\mathrm{cov}[y_t, y_{t-k}] = \mathrm{cov}[x_t, x_t + 3] = E[(x_t - E[x_t])(x_t - 3 - E[x_t - 3])]$

Consider t and k are both even, meaning $t-k$ is even

↳ $\mathrm{cov}[y_t, y_{t-k}] = \mathrm{cov}[x_t + 3, x_t + 3] = E[(x_t + 3 - E[x_t + 3])(x_t + 3 - E[x_t + 3])]$

Consider t and k are odd & even, meaning $t-k$ is odd

↳ $\mathrm{cov}[y_t, y_{t-k}] = \mathrm{cov}[x_t, x_t] = E[(x_t - E[x_t])(x_t - E[x_t])]$

Consider t and k are even & odd, meaning $t-k$ is odd

↳ $\mathrm{cov}[y_t, y_{t-k}] = \mathrm{cov}[x_t + 3, x_t] = E[(x_t + 3 - E[x_t + 3])(x_t - E[x_t])]$

For all cases, y_t depends on t

Since y_t is $x_t + 3$ when t is even, and x_t when t is odd, y_t depends on t and thus, y_t is not a stationary time series. //

3. (10%) Let y_t be a stationary process with an autocovariance function γ_k.
(a) Show that $z_t = \nabla y_t = y_t - y_{t-1}$ is stationary by finding the mean and autocovariance function for z_t.
(b) Show that $w_t = \nabla^2 y_t = z_t - z_{t-1} = y_t - 2y_{t-1} + y_{t-2}$ is stationary.

a) $z_t = \nabla y_t = y_t - y_{t-1}$

Since y_t is a stationary process, it must have a constant mean μ over time and autocovariance function γ_k

Mean (Expectation)

$$E[z_t] = E[y_t - y_{t-1}]$$
$$E[z_t] = E[y_t] - E[y_{t-1}]$$
$$= \mu - \mu$$
$$= 0$$

Note: the mean function is a constant and is independent of time

Autocovariance

The autocovariance function of $\{z_t\}$ is $Cov(z_t, z_{t-k})$

$$Cov(z_t, z_{t-k}) = Cov(y_t - y_{t-1}, y_{t-k} - y_{t-k-1})$$ ← Apply the distributive property of covariance

$$= Cov(y_t, y_{t-k}) + Cov(y_t, -y_{t-k-1}) + Cov(-y_{t-1}, y_{t-k}) + Cov(-y_{t-1}, -y_{t-k-1})$$

$$= Cov(y_t, y_{t-k}) - Cov(y_t, y_{t-k-1}) - Cov(y_{t-1}, y_{t-k}) + Cov(y_{t-1}, y_{t-k-1})$$

$$= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k$$

Note: the autocovariance function is independent of time

Since both the mean and autocovariance function are independent of time, z_t is stationary

b) $w_t = \nabla z_t = y_t - 2y_{t-1} + y_{t-2}$

Since it is proven that the first-order difference of y_t, "z_t", is stationary, then the first-order difference of z_t, "w_t" must also be stationary.

4. (15%) Let x_t be stationary with $\mathrm{E}[x_t]=0, \mathrm{V}[x_t]=1$, autocorrelation function ρ_k. Define that μ_t is a nonconstant function and σ_t is a positively nonconstant **function** (that is to say: μ_t and σ_t are both deterministic and in function of t). Now we observe a time series formulated as

$$y_t = \mu_t + \sigma_t x_t.$$

(a) Find the mean and autocovariance function of y_t.
(b) Show that the autocorrelation of y_t depends only on the lag k. Is y_t stationary?
(c) Let $\mu_t = \mu_0$ be a constant value. Justify that y_t is still nonstationary.

a) Mean function of y_t

$$E(y_t) = E(\mu_t + \sigma_t X_t)$$
$$= \mu_t + \sigma_t E[X_t]$$
$$E(y_t) = \mu_t \text{ //}$$

Autocovariance function of y_t

$$Cov(y_t, y_{t-k}) = Cov(\mu_t + \sigma_t X_t, \mu_{t-k} + \sigma_{t-k} X_{t-k})$$
$$= Cov(\sigma_t X_t, \sigma_{t-k} X_{t-k})$$
$$= \sigma_t \sigma_{t-k} Cov(X_t, X_{t-k})$$
$$= \sigma_t \sigma_{t-k} \left[Corr(X_t, X_{t-k}) \sqrt{Var(X_t) Var(X_{t-k})} \right]$$
$$= \sigma_t \sigma_{t-k} (\rho_k \sqrt{1 \cdot 1})$$
$$Cov(y_t, y_{t-k}) = \sigma_t \sigma_{t-k} \rho_k \text{ //}$$

b) Consider the variance

$$Var(y_t) = Var(\mu_t + \sigma_t X_t)$$
$$= Var(\sigma_t X_t)$$
$$= \sigma_t^2 Var(X_t)$$
$$Var(y_t) = \sigma_t^2$$

Consider the autocorrelation function

$$Corr(y_t, y_{t-k}) = \frac{Cov(y_t, y_{t-k})}{\sqrt{Var(y_t) Var(y_{t-k})}}$$
$$= \frac{\cancel{\sigma_t \sigma_{t-k}} \rho_k}{\sqrt{\cancel{\sigma_t^2 \sigma_{t-k}^2}}}$$
$$Corr(y_t, y_{t-k}) = \rho_k$$

Since the autocorrelation of y_t is ρ_k
for all cases, autocorrelation of y_t does not depend on t since X_t is stationary.
Thus, autocorrelation of y_t only depends on the lag k //

However, since $y_t = \mu_t + \sigma_t x_t$, and μ_t and σ_t are deterministic and are functions of t, y_t is NOT stationary

c) By having a mean function that is constant over time t, the mean function becomes stationary.
↳ $y_t = \mu + \sigma_t X_t$

The autocovariance is still $Cov(y_t, y_{t-k}) = \sigma_t \sigma_{t-k} \rho_k$.
While the mean function is constant, the autocovariance function is not.
Thus, y_t is still nonstationary //

5. (5%) Let x_t be the series of the "expected" measurements during the production process. Because the measuring tool itself won't be perfect, we observe $y_t = x_t + e_t$, assuming x_t and e_t are independent. In general, we call x_t the **signal** and e_t the **measurement (white) noise**.

If x_t is stationary with the autocorrelation function ρ_k, show that y_t is also a stationary process with

$$\operatorname{corr}(y_t, y_{t-k}) = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}, \text{ for } k \geq 1.$$

$\frac{\sigma_x^2}{\sigma_e^2}$ is usually referred to as the **signal-to-noise ratio**, or **SNR** for short.

The larger the SNR, the closer the autocorrelation function of the observed series y_t is to the autocorrelation function of the desired signal x_t.

a) Mean function of y_t

$$E[y_t] = E[x_t + e_t]$$
$$= E[x_t] + E[e_t] \quad - \mu_t$$

- Note that x_t is stationary, meaning the mean function $E[x_t]$ is a constant
- Since "white noise" is random, due to it being the measurement of the environment, the mean function $E[e_t]$ doesn't depend on time t
- Thus $E[y_t]$ is stationary.

Variance of y_t

$$Var[y_t] = Var[x_t + e_t]$$
$$= Var[x_t] + Var[e_t]$$
$$= \sigma_x^2 + \sigma_e^2$$

Autocovariance function of y_t

$$Cov[y_t, y_{t-k}] = Cov[x_t + e_t, x_{t-k} + e_{t-k}]$$
$$= Cov[x_t, x_{t-k}]$$
$$= \rho_k$$

Note: the autocovariance function is independent of time

Since both the mean and autocovariance function are independent of time, y_t is stationary //

Autocorrelation function of y_t

$$Corr[y_t, y_{t-k}] = \frac{Cov[y_t, y_{t-k}]}{\sqrt{Var(y_t)\, Var(y_{t-k})}}$$
$$= \frac{\rho_k}{\sqrt{(\sigma_x^2 + \sigma_e^2)(\sigma_x^2 + \sigma_e^2)}}$$
$$= \frac{\rho_k}{\sigma_x^2 + \sigma_e^2}$$

$$Corr[y_t, y_{t-k}] = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}$$

//

6. (10%) Suppose $y_t = \alpha_0 + \sum_{i=1}^{q}[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)]$, where $\alpha_0, f_1, f_2, \dots, f_q$ are constants and $\alpha_1, \alpha_2, \dots, \alpha_q, \beta_1, \beta_2, \dots, \beta_q$ are independent random variables with zero means and variances $V[\alpha_i] = V[\beta_i] = \sigma_i^2$. Show that y_t is stationary and find its autocovariance function.

(Hint: show $\mathrm{COV}[y_t, y_s]$ depends only on $t - s$.)

Mean function of y_t

$$E[y_t] = E\left[a_0 + \sum_{i=1}^{q}(a_i \cos(2\pi f_i t) + b_i \sin(2\pi f_i t))\right]$$

$$= a_0 + \sum_{i=1}^{q} \cancel{E[a_i] \cos(2\pi f_i t)}^{0} + \cancel{E[b_i] \sin(2\pi f_i t)}^{0}$$

$$= a_0$$

Note: the mean function is a constant and is independent of time

Autocovariance Function of y_t

$$\mathrm{Cov}[y_t, s_t] = \mathrm{Cov}\left[\sum_{i=1}^{k} a_i \cos(2\pi f_i t) + b_i \sin(2\pi f_i t),\ \sum_{i=1}^{k} a_i \cos(2\pi f_i s) + b_i \sin(2\pi f_i s)\right]$$

$$= \sum_{i=1}^{k} \mathrm{Cov}(a_i \cos(2\pi f_i t) + a_i \sin(2\pi f_i s)) + \sum_{i=1}^{k} \mathrm{Cov}(b_i \cos(2\pi f_i t) + b_i \sin(2\pi f_i s))$$ ← Theorem: $\mathrm{Var}[X] = \mathrm{Cov}[X, X]$

$$= \sum_{i=1}^{k} \mathrm{Var}[a_i]\,(\cos(2\pi f_i t) + \sin(2\pi f_i s)) + \sum_{i=1}^{k} \mathrm{Var}[b_i]\,(\cos(2\pi f_i t) + \sin(2\pi f_i s))$$

$$= \frac{\sigma^2}{2}\sum_{i=1}^{k}\left[\cos(2\pi f_i (t-s)) + \sin(2\pi f_i (t+s))\right] + \frac{\sigma^2}{2}\sum_{i=1}^{k}\left[\cos(2\pi f_i (t-s)) + \sin(2\pi f_i (t+s))\right]$$

$$\mathrm{Cov}[y_t, s_t] = \sigma^2 \sum_{i=1}^{k} \cos(2\pi f_i (t-s))$$

Note: the autocovariance function is only dependent on $t-s$

Since both the mean and autocovariance function are independent of time, y_t is stationary