What Formal Languages can Transformers Learn In-Context?

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Problem Definition

We investigate the capabilities of transformer models in learning formal languages in an in-context setting:

- **Task**: Given a set of in-context examples $\{(x_i, y_i)\}_{i=1}^n$ from a formal language L, generate a new string x_{n+1} that belongs to L.
- **Key Questions**:
  - How does the sample complexity n scale for languages in REG, CFL, CSL, and RE?
  - What are the theoretical bounds on transformer model complexity for each language class?
  - Can transformers achieve efficient in-context learning for context-free and more complex languages?
  - What are the fundamental limitations of in-context learning for recursively enumerable languages?
- **Approach**: Analyze transformers' ability to implicitly learn and apply the rules of formal grammars from examples, without explicit training.

Background

- Transformers act as algorithm approximators, can perform in-context learning at least as well as gradient descent
  - [1] showed decoder-only transformers are few shot learners
  - [2] showed they learn by gradient descent on in-context examples!
  - [3] showed they can actually do better than gradient descent and select better algorithms
  - [4] showed they perform higher order optimization on ICL examples, approximating iterative Newton's method

Methodology

- Analyze in-context learning capabilities for each language class:
  - Regular Languages (REG)
  - Context-Free Languages (CFL)
  - Context-Sensitive Languages (CSL)
  - Recursively Enumerable Languages (REL)
- Derive bounds on:
  - Model complexity (attention heads, hidden dimension, layers)
  - Sample complexity for in-context examples
  - Convergence rates in the in-context setting

Key Results: Regular Languages

- Model Specification for in-context learning:
  - $H = O(\log(|Q| + |\Sigma|))$
  - $D = O(\sqrt{|Q| \log |\Sigma|})$
  - $L = O(\log n)$
- In-context Sample Complexity ($\sim$ Coupon Collector's Problem [5]):

$$n \geq O\left(\frac{|Q| \cdot |\Sigma|}{l} \cdot \log\left(\frac{|Q| \cdot |\Sigma|}{\delta}\right)\right)$$

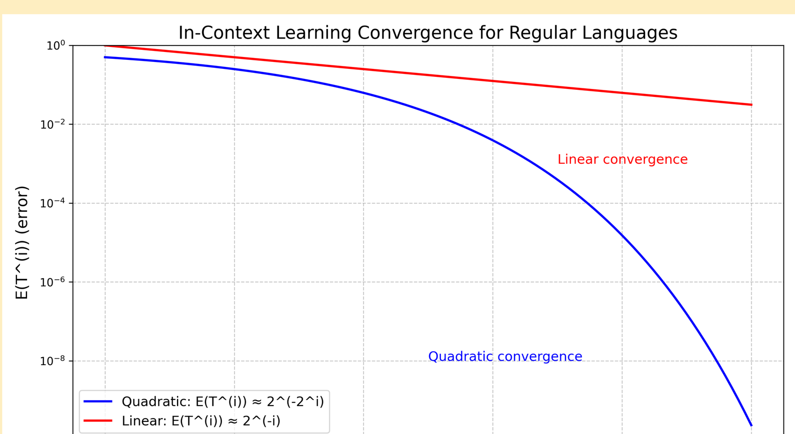
- Potential quadratic convergence in context:

$$E(T^{(i+1)}) \leq c \cdot (E(T^{(i)}))^2$$

- Where:
  - H: Number of attention heads, D: Hidden dimension, L: Number of layers
  - $|Q|$: Number of states in minimal DFA
  - $|\Sigma|$: Size of the alphabet
  - n: Number of in-context examples
  - l: Average length of example strings
  - δ: Confidence parameter
  - $E(T^{(i)})$: Error after i samples

Toy Example Convergence Rate Derivation

1. Model state as probability distribution over DFA states: $p = [p_0, p_1]$
2. Error defined as: $E(T^{(i)}) = 1 - p_{\text{correct}}$
3. Key insight: Each transformer layer can combine multiple transitions
4. For a single transition: $p' \approx 1 - (1 - p)^2$ (error squared)
5. Generalizing: $E(T^{(i+1)}) \leq c \cdot (E(T^{(i)}))^2$, for some constant $c < 1$
6. This quadratic reduction leads to convergence rate of $O(2^{-2^i})$



Key Results: Context-Free Languages

- In-context model with k implicit stack-like representations:
  - $H = O(\log(|R| + k))$
  - $D = O(\sqrt{|R| \log(|V| + |\Sigma|) + k \log(|V| + |\Sigma|)})$
  - $L = O(\log n + \log k)$, $k = O(\log n)$
- Naive Bound on In-context Sample Complexity:

$$n \geq O\left(\frac{|R| \cdot \log(|V| + |\Sigma|) \cdot \log(1/\varepsilon)}{\log(H \cdot D) + \log k}\right) \cdot \log(1/\delta)$$

- **Tighter Lattice-Based Bound:**

$$n \geq O(|R| \cdot \log(|V| + |\Sigma|) \cdot (\log \log |R| + \log(1/\varepsilon)))$$

- Where:
  - $|R|$: Number of production rules
  - $|V|$: Number of non-terminal symbols
  - k: Number of implicit stack-like representations
  - ε: Desired accuracy

Key Results: Context-Sensitive Languages

- In-context model with m tape-like representations in output:
  - $H = O(\log(|P| + m))$
  - $D = O(\sqrt{|P| \log(|V| + |\Sigma|) + m \log(|V| + |\Sigma| + 1)})$
  - $L = O(\log n \cdot \log m)$, $m = O(n)$
- In-context Sample Complexity:

$$n \geq O\left(\frac{|P| \cdot \log(|V| + |\Sigma|) \cdot \log(1/\varepsilon)}{\log(H \cdot D) + \log m}\right) \cdot \log(1/\delta)$$

- Where:
  - $|P|$: Number of production rules in CSG
  - m: Number of implicit tape-like representations

Recursively Enumerable Languages

- Conjecture: No finite transformer model can in-context learn to probabilistically generate from all RELs (the non-probabilistic case is trivial)
- In-context approximation possibilities:
  - Bounded-length input approximations
  - Probabilistic in-context recognition
  - In-context learning of decidable subsets

Conclusions & Future Work

- Clear correspondence between transformer in-context learning capabilities and formal language complexity
- Efficient in-context learning of regular and context-free languages
- Significant increase in sample complexity for in-context learning of context-sensitive languages
- Future questions and directions:
  - Can we find the convergence rate on CFL, CSL, and so on? Empirical testing suggests similar rate but we don't know why.
  - Can we find 'scaling laws' for in-context learning?
  - Optimize transformer architectures for in-context learning
  - Find out how the model represents grammar inference algorithms internally in hidden states and activations. (Significantly, this can help investigate the behavior of "meta-parameters' that seem to emerge at inference time.)

References

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