FYS4150 - Project 4
Phase Transitions in Magnetic Systems

Adam Niewelgowski[1]
[1]Department of Physics, University of Oslo, N-0316 Oslo, Norway

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Abstract

In this project we wish to investigate phase transition in Ising model using one of Markov chain Monte Carlo method; Metropolis algorithm. Before we move onto larger systems with spin grid size of 100, we benchmark our model with analytical solution for tiny lattice. Then we extract the equalibrium time. This is later used as wait time to avoid noise in data for investigation of phase transition. Nummeric data and code can be found on GitHub page.

Contents

1 Introduction

The Ising model is a very simple and common model to investigate various systems, usually used in statistical mechanics to investigate ferromagnetic materials, but can be used for different studies as well (e.g [Bialek, 2012]) This model is based on discrate variables representing dipole spin, values can either be $+1$ or -1.

In our project, we will be using Metropolis algorithm. We want to investigate how different thermodynamic quantities, such as mean energy, mean magnetisation, heat capacity and magnetic susceptibility behave with time and temperature for different lattice sizes. For very small lattice, we will compare nummerical computations with analytical results to make sure our simulation is behaving as it is supposed to.

We will be studying equalibrium time for an ordered and disordered initial state. Finally we look closer at the critical temperature which represents phase transition in our Ising model.

First of all we will present the theory behind the system and model. Later we will talk about the nummerical technicalities and Metropolis algorithm. Finally we will introduce results and discuss them.

2 Theoretical model

In this section we will have a closer look at the Ising model, thermodynamic quantities and phase transition.Then we present the Metropolis algorithm and periodic boundary conditions.

2.1 Ising model

We have L $\times$ L lattice of spins which can point either up or down. In our case it will be a micro-canonical system, that means the isolated system which cannot exchange energy with environment.

In the simplest Ising model the energy is given by

$$E = -J \sum_{<kl>}^{N} s_k s_l - B \sum_{k}^{N} s_k$$

where $s_k = \pm 1$ is spin, $N = L^2$ is the total number of spins, J is a coupling constant expressing the strength of the interaction between neighboring spins, that is why we sum over $< kl >$, this indicates that we sum only over the neighbour spins. B is an external magnetic field. In our case $B = 0$ and $J = 1$ so we have even simpler expression

$$E = -\sum_{<kl>}^{N} s_k s_l \tag{1}$$

Magnetisation is defined as

$$M = \sum_{i} s_i \tag{2}$$

where we simply sum over all the spins in our system.

These quantities are usefull to calculate heat capacity and magnetic susceptibility (derived in [Malthe-Sorenssen, 2016])

$$C_v = \frac{\langle E^2 \rangle - \langle E \rangle^2}{kT^2} \tag{3}$$

$$\chi = \frac{\langle M^2 \rangle - \langle M \rangle^2}{kT} \tag{4}$$

where T is the temperature of the system and k is Boltzmann's constant and will be equal 1 for the rest of the project.

2.2 Thermodynamic quantities

To be able to check our model, and to have some more insight into the thermodynamics we use Boltzmann's statistics for our system. We have the probability distribution

$$P(E) = \frac{e^{-E/kT}}{Z}$$

where E is the energy of a given state, k is Boltzmann's constant and T is the temperature. Since probability distribution has to be normalized we can find partition function

$$Z = \sum_i e^{-\beta E_i} \tag{5}$$

where we have substituted $\beta = \frac{1}{kT}$, and the sum goes over all the possible microstates.

n'th moment of a given quantity X can be expressed using the probability distribution as follows

$$\langle X^n \rangle = \frac{1}{Z} \sum_i X_i^n e^{-\beta E_i} \tag{6}$$

where again, we sum over all the possible microstates. Moments will be usefull to calculate heat capacity and magnetic susceptibility.

2.3 Phase transition

In Ising model phase transition occurs at a critical temperature, T_C. At this temperature, one can see spikes in thermodynamic quantities. In different words, discontinuity of the derivative of these quantities. This happens since the properties of the system change quickly and significantly, at the transition. We can model these quantities as power laws near this temperature T_C

$$\begin{aligned} \langle M(T) \rangle &\sim (T - T_C)^\beta \\ C_v(T) &\sim |T_C - T|^\alpha \\ \chi(T) &\sim |T_C - T|^\gamma \end{aligned}$$

where $\alpha = 0$, $\beta = 1/8$ and $\gamma = 7/4$. These are so called critical exponents.

In order to T_C we would need to simulate infinitely huge lattice. This is impossible with our computational capacity, so we are required to estimate with a finite and relativly small size

$$T_C(L) - T_C(L = \infty) = aL^{-1/\nu} \tag{7}$$

where $\nu = 1$ and a is a constant that can be found as follows

$$a = \frac{T_C(L_1) - T_C(L_2)}{L_1^{-1} - L_2^{-1}}$$

applying two different lattice sizes make it possible to compute a, what makes it further possible to find $T_C(L = \infty)$ with equation (7).

2.4 Metropolis algorithm

Metropolis algorithm is a Monte Carlo method we used to simulate our system. This algorithm is based on the ratio of probabilities as explained in [Hjorth-Jensen, 2015]. A simple step overview of the algorithm can be

1. From spin grid, randomly choose one spin

2. Calculate the energy (ΔE), as the selected spin was flipped

3. Generate a random number, ζ from an uniform distribution between 0 and 1

4. If $\zeta \leq e^{-\beta \Delta E}$ flip the randomly selected spin and update thermodynamic quantities.

repeat this N times.

2.4.1 Periodic boundary conditions

In reality, spin lattice in materials is so huge that we can safely say it is infinit, so the problem of boundaries is negligible. With our finite lattice size we cannot negligible this aspect and we have to handle the end-points. One way is by saying that the end-points does not have any neighbour. The other way is to imitate the infinity lattice by saying that the end-point on one given side of the lattice is a neighbour of end-point of opposite side. This way corresponds better with reality and gives a good approximation of the infinit grid.

2.5 Steady-state time

We are interested in whether the time of reaching steady-state depends on the initial state of the system and temperature. We investigate two different start configuration of our lattice. Ordered state, with all of the spins pointing in one particular direction or disordered state where spin orientation is based on uniform distribution. We will compare these two initial states, by looking at some signs of equalibrium. This state is characterized by small variations in the system, most of the spins will be in their stable configuration so few flips will occur. We expect it to happen after some time measured in number of Monte-Carlo steps. Mean magnetisation and mean energy will reach the equalibrium as well. These can help to detect the stady-state time.

3 Method

In this section we will only focus on the nummerical method, implementation of the algorithm and all needed features. Analytical solution of a small, 2x2 lattice can be found in appendix (A).

3.1 Energy

3.1.1 Ising model initial energy

Following the description of Ising model from section (2.1) we can calculate energy of the initial system by looping through all the spins and calculating

```
for i in range(N_spins):
    for k in range(N_spins):
        energy -= lattice(i,j)*(lattice(i-1, j) + lattice(i, j-1))
```

3.1.2 Metropolis algorithm and Boltzmann's factor

In Metropolis algorithm, as described in section (2.4), we have to ΔE. Since we flip only one spin at the time, we will calculate this energy differance easly by acting like one spin is already flipped (notice the positive sign in "dE")

```
for i in range(N_spins):
    for k in range(N_spins):
        x = int(random.uniform(0,1)*N_spins)
        y = int(random.uniform(0,1)*N_spins)
```

```
dE = 2*lattice(x,y)*(lattice(x+1,y)+lattice(x-1,y)+lattice(x,y+1)+
    lattice(x,y-1))
if(random.uniform(0,1) <= boltzmann_factor(dE)):
    lattice(x,y) *= -1 //actually flipping this spin
    energy += dE
    magnetisation += 2*lattice(x,y)
```

periodic boundary conditions are not shown here, see (3.2) It is shown here ([Hjorth-Jensen, 2017]), that ΔE can only take five values $\Delta E = \{-8, -4, 0, 4, 8\}$ $[J]$. This allows us to precalculate the Boltzmann's factor and save a lot of time in this time consuming calculation.

3.2 Periodic boundary conditions

To be able to find ΔE, we need to implement periodic boundary conditions. We can make use of modular division (%) to loop from one end to the opposite in our grid. We need a function which returns

$$\begin{aligned} 0 \quad & \text{when } i = N\_spins \\ N \quad & \text{when } i = 0 \end{aligned}$$

this function can be implemented by

$$periodic(i) = (i + N\_spins)\%N\_spins$$

3.3 MC cycles vs. precision

As we know, Markov chain Monte Carlo simulations are characterized by simple rule, the more cycles in simulation the more precise results will be. To find a good balance between precision and computation time, we simply compare output we get using different amount of cycles. We will use graphical approach to judge the precision and simple time calculation to keep a good balance.

3.4 Steady state time investigation

We wish to study how the steady state time (time to reach most likey state) in our system varies with the temperature of the system and the initial spin configuration. Again, straight graphical approach will be employed. We plot mean energy and mean magnetisation as a function of Monte-Carlo steps, to find out the time significant variations dies out. Same idea can be used for time development of amount of accepted flips in the system as discussed in section (2.5).

Using this time as the wait time for sampling the data, we collect data to plot the porbability distribution of energy in the system. We simply count how many times a given energy appears in our calculation.

3.5 Phase transition

Now, when we know the approximate equalibrium time, and amount of MC cycles which give us good precision with decent calculation time, we can run heavier calculation in order to investigate the phase transition in our Ising model. We will use equalibrium time to wait with collecting the data to avoid noise.

From ([Hjorth-Jensen, 2018]) we know the analytical value of $T_C(L = \infty) \approx 2.269$. We can use this fact to simulate our system for different lattice sizes near this temperature. We will be looking for the spikes in heat capacity and magnetic susceptibility to extract the critical temperature for phase transition in our model for a given size.

3.5.1 Parallelization of the code; MPI

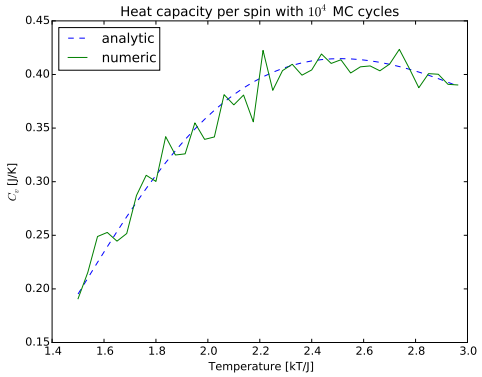
We achieve higher accuracy with more MC cycles, as mentioned in section (3.3). To be able to simulate our model with satisfying resolution and lattice size up to $L = 100$, we can parallelize our code to speed it up. Since the whole simulation is based on randomness and we are calculating statistical quantities like mean/expecting values, we can simply divide work capacity equally on two nodes. After each node ends its calculation, we collect and sum all the data and calculate the rest sequentially with one working node.

4 Results

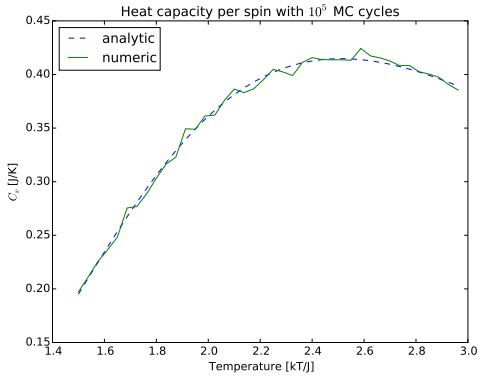
In this section we present the results from the simulation. Discuss of these comes in the following section.

4.1 2x2 lattice

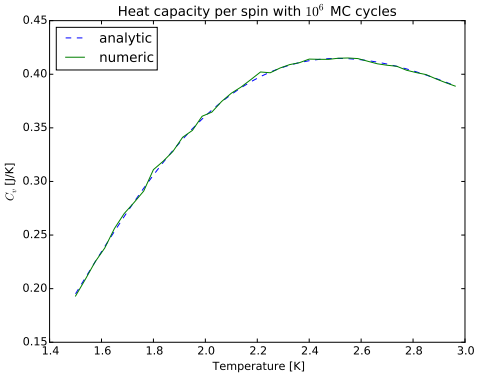
Beggining with small system, to make sure that our model is giving correct output. We pick heat capacity to plot and against the analytical expression since this is the most complex function in our computations.



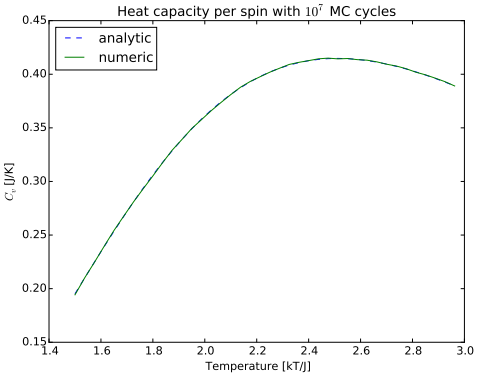
(a) 10^4



(b) 10^5



(c) 10^6



(d) 10^7

Figure 1: Heat capacity as function of temperature for a range $[1.5, 3.0]\ kT/J$. Different number of MC cycles compared with analytical solution

Additionally we have looked at the quantities for 1.0[kT/J] temperature

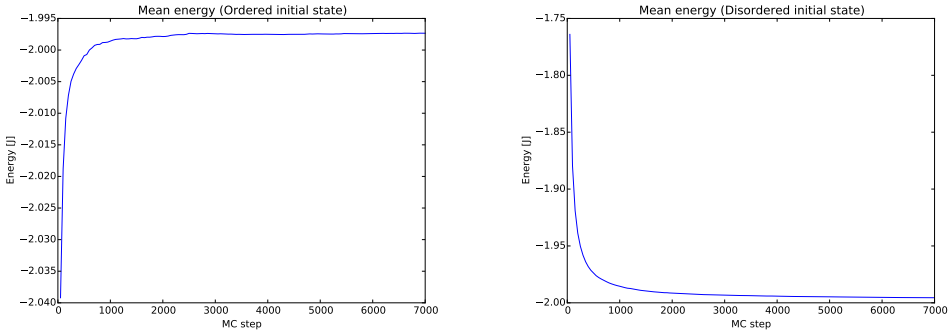
MC cyclses	10^4	10^5	10^6	10^7	exact
$\langle E \rangle$ [J]	-1.9926	-1.9963	-1.9961	-1.9960	-1.9960
$\langle M \rangle$ [J/T]	3.5838	3.7814	3.9935	3.9933	3.9946
C_v [J/K]	0.04785	0.0292	0.03116	0.03187	0.0321
χ [J s^2]	3.5838	3.7814	3.9935	3.9933	3.9933
Calculation Time [s]	0.0982	1.0143	9.7854	96.7403	—

Table 1: Precision and efficiency comparision

Based on precision and computation time from this table, we have chosen to run full simulations with 10^6 cycles.

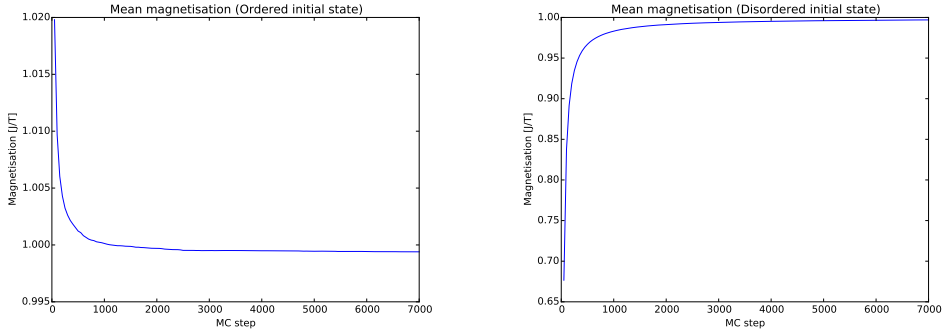
4.2 Most likely state

We used slightly larger system, 20×20 lattice, to investigate the time required to arrive at the most likely state. As mentioned in section (2.5), we study thermodynamic quantities for two different temperatures, 1.0 and 2.4 [kT/J]. Additionally we plot accepted flips, first as function of time, then as function of temperature on the range $[1.0, 2.4]$ [kT/J].



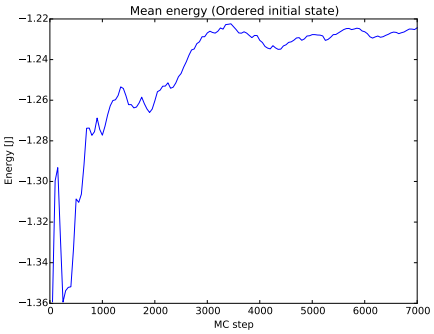
(a) Ordered initial state (b) Disordered initial state

Figure 2: Mean energy, $\langle E \rangle$ in the system as function of MC cycles, temperature = 1 kT/J (ordered and disordered state)

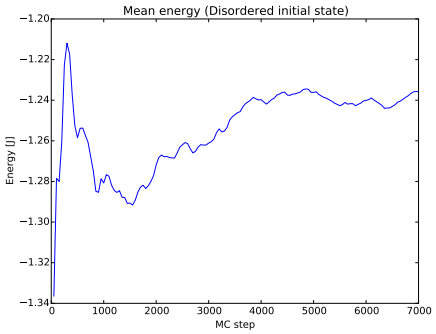


(a) Ordered initial state (b) Disordered initial state

Figure 3: Mean magnetisation, $\langle |M| \rangle$ in the system as function of MC cycles, temperature = 1 kT/J (ordered and disordered state)

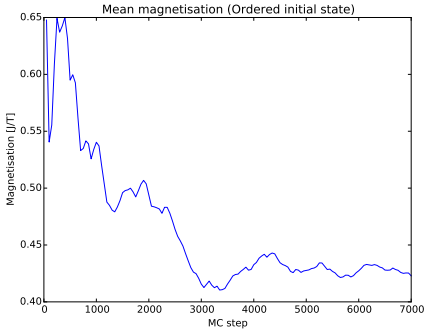


(a) Ordered initial state

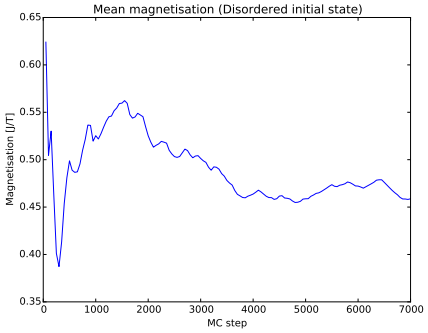


(b) Disordered initial state

Figure 4: Mean energy, $\langle E \rangle$ in the system as function of MC cycles, temperature = 2.4 kT/J (ordered and disordered state)

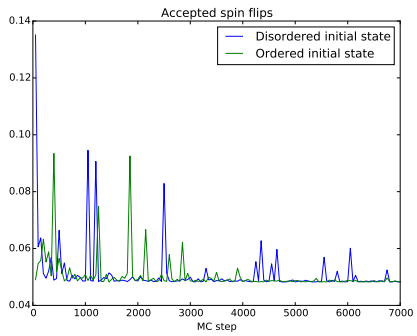


(a) Ordered initial state

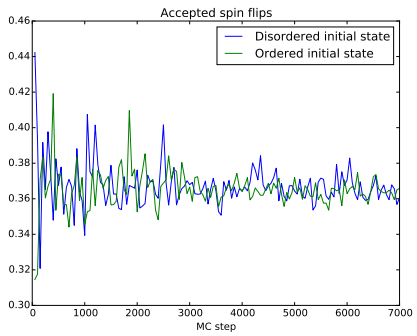


(b) Disordered initial state

Figure 5: Mean magnetisation, $\langle |M| \rangle$ in the system as function of MC cycles, temperature = 2.4 kT/J (ordered and disordered state)



(a) Temperature = 1 kT/J



(b) Temperature = 2.4 kT/J

Figure 6: Accepted number of spin flips, scaled with total number of spins at a given moment, in the system as function of MC cycles. Odreded and disordered initial states.

To be on the safe side, we pick 7000 steps to be our wait limit for collecting the data in further simulations.

Here we plot the number of accepted flips in Metropolis algorithm as function of temperature

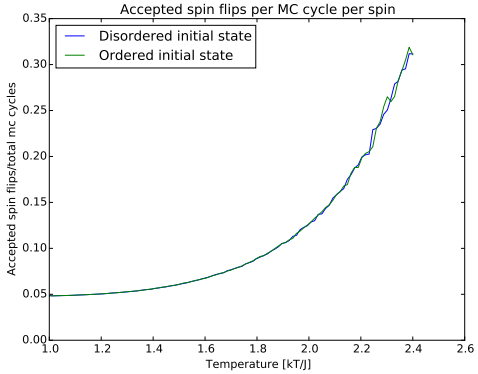
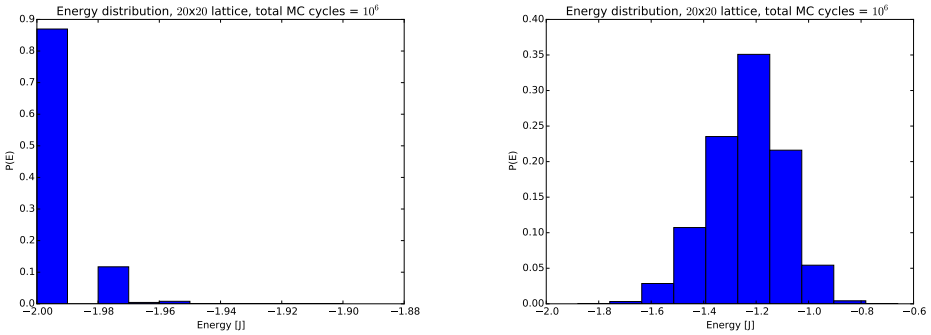


Figure 7: Accepted number of spin slips, scaled with total number of spins, in the system as function of temperature for a range $[1.0, 2.4]$ kT/J

4.3 Probability distribution

Here are the probability distributions plotted for two different temperatures (1.0 and 2.4 [kT/J]) as described in section (3.4)



(a) 1 kT/J temperature

(b) 2.4 kT/J temperature

Figure 8: Probability distribution of energy in 20x20 lattice simulated with 10^6 MC steps (data sampled after the equalibrium time, 7000 steps)

We have computed the variance of energy, by simply setting

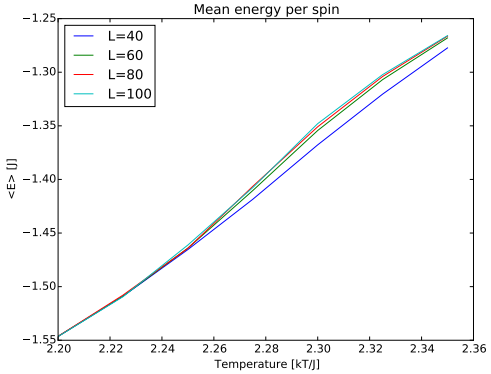
$$\mathrm{Var}[E] = \langle E^2 \rangle - \langle E \rangle^2$$

where the energy, E, is per spin.

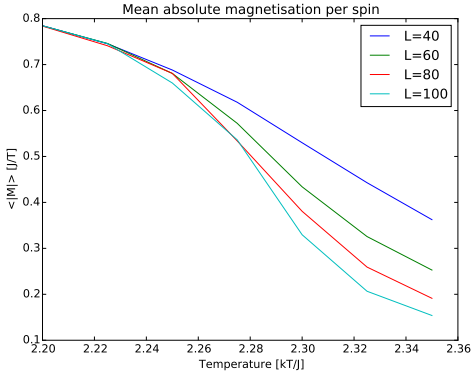
$$\sigma^2_{T=1.0} = 9.2 \times 10^{-5} \tag{8}$$
$$\sigma^2_{T=2.4} = 0.02 \tag{9}$$

4.4 Phase transition

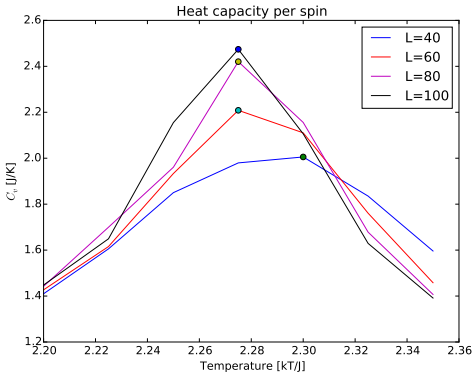
Knowing the equalibrium time and a good amount of MC cyclses which is giving us good tradeoff between computation time and precision allows us to simulate for bigger systems. To investigate phase transition in our Ising model, we pick $L = 40, 60, 80\text{and}100$ and plot the thermodynamic quantities.



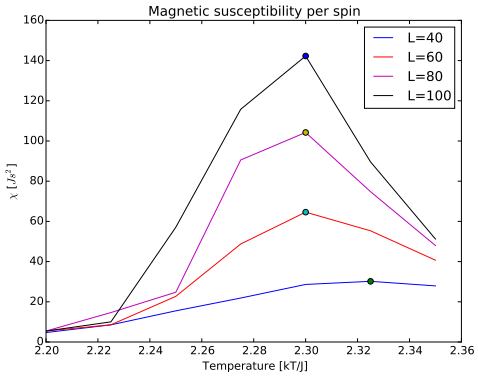
(a) Mean energy per



(b) Mean absolute magnetisation



(c) Heat capacity



(d) Magnetic susceptibility

Figure 9: Thermodynamic quantities per spin for different lattice sizes as function of temperature. Simulated for 10^6 MC-cyclses excluding 7000 equalibrium steps.

Printed critical temperature for different quantities

From heat capacity C_v we have

$$L = 40 \longrightarrow T_C = 2.3 \tag{10}$$
$$L = 60, 80, 100 \longrightarrow T_C = 2.275 \tag{11}$$

From magnetic susceptibility we have

$$L = 40 \longrightarrow T_C = 2.325 \tag{12}$$
$$L = 60, 80, 100 \longrightarrow T_C = 2.3 \tag{13}$$

4.4.1 Paralellizing code with MPI

The speed-up in our code after paralellizing with MPI as described in section (3.5.1). These timings are for small, 2x2 system with 10^4 cycles only, just to see the differance.

	without MPI	with MPI
time [s]	0.087844	0.049845

Table 2: Comparision of computation time with and without MPI

5 Discussion

In this section we will discuss the results presented in the previous section.

5.1 2x2 lattice

As expected, in figure (1) we can clearly see that our nummeric solution is approaching the exact curve with increasing total number of MC cycles. Random fluctiuation is dying out quickly, in figure (1d) we cannot see any significant differance. A zoomed figure

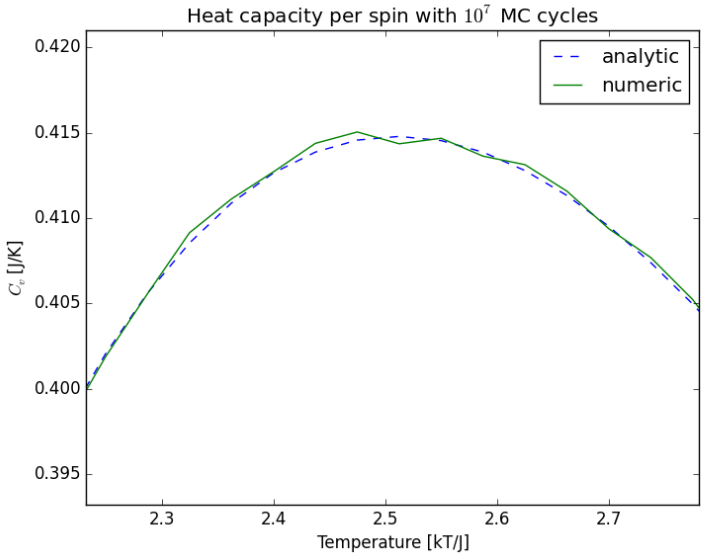


Figure 10: Zoomed version of figure (1d)

shows us that there still are some small deviations. To compare this not only graphically, we composed table (1) where we can see all of the thermodynamic quantities. We can see that we do not have 100% match between exact values, but the accuracy increases significantly. Notice how three of the quantities ($\langle E \rangle$, $\langle M \rangle$ and χ) are changing with only fourth decimal point and heat capacity with third decimal point for 10^6 and 10^7. While the computation time (not paralellized) increases by factor of 10. To keep the good balance between precision and calculation time, we pick 10^6 cycles in the rest of the calculations.

5.2 Most likely state

Figures from (2) to (5) shows how mean energy and mean magnetisation evolves in time. This was done for two different initial states and two different temperatures. We can point out the interesting characteristics of these plots

- First off all we see that for temperature = 1.0 [kT/J] the system is moving very rappidly, without bigger fluctuations toward equalibrium, both for the ordered and disordered initial state. Steady state for lower temperatures is close to an ordered state, so I would expect more fluctuations and slightly later equalibrium for disordered initial state, but this is a random calculation and graphical analysis only so this might be correct as well. As a confirmation we can look at figure (6a) where we see that number of accepted flips in ordered initial state goes to equalibrium much faster than for disordered state.
- For higher temperatures, such as 2.4 [kT/J], system needs more time to reach, more or less, stable state since there is more energy in the system. See figure (6b), this confirms that

higher temperature gives more chaotic system, where more flips are required. Higher energy makes electron's motion faster what gives spins more freedom. Mean magnetisation is lower because of that, since less spins are pointing in one direction.

Additionally, in figure (7) we can see how the accepted number of flips increases exponentially with temperature, this confirms the argument used previously in this section.

5.3 Probability distribution

As discussed in previous section, higher temperature gives more spin freedom, what leads to more accepted flips. This means that for higher temperature we have more varying energy. This can be seen in figure (8b). Energy occurrence takes form of Gaussian distribution confirming Central Limit Theroem, since Metropolis algorithm is accepting flips based on a random, uniform distribution. It looks like mean value is approximately -1.2. We can check if the extracted variance will match the result shown in the plot. Standard deviation is given as

$$\sigma_{T=2.4} = \sqrt{\sigma^2_{T=2.4}} = \sqrt{0.02} \approx 0.141$$

This looks like very accurate value according to the histogram.

For low temperature, (8a), we see that system's most likely energy is the lowest energy -2. This means that all of the spins points up in this state. Standard deviation in this case is close to zero, but it does not tell us much about the energy distribution which is very simple.

5.4 Phase transition

In figure (9) we see thermodynamic quantities for different lattice sizes as function of temperature. We know that T_C is dependend on size of the lattice, however the same T_C occurs for three sizes. This is caused by the temperature resolution which was only $\Delta T = 0.025$ for all of the sizes. With smaller temperature step we would get different temperatures for every lattice size. Because of time limitations, we performed calculations with only this resolution, extraction and analysis of critical temperature is still possible.

Following the description in section (2.3) we will use values from (10) and (12), to find constant "a", and then plug into equation (7). Choosing data from C_v with $L_1 = 40$ and $L_2 = 100$, we get (Note that taking data from χ will give the same results)

$$a = \frac{2.3 - 2.275}{40^{-1} - 100^{-1}} \approx 1.67$$

critical temperature then is

$$T_C(L = \infty) = T_C(L) - aL^{-1} = 2.275 - 1.67 \cdot 100^{-1} \approx 2.2583$$

knowing the exact value from section (3.5) we can calculate relative error

$$\epsilon_{rel} = \frac{|2.269 - 2.2583|}{2.269} = 0.47\%$$

Using same data but $L_2 = 60$ will give less accurate value with relative error $\epsilon = 1.05\%$.

5.4.1 MPI paralellization

As we can see in table (2), using two nodes while running MPI-code gave us twice as fast computation time. Gathering data is not taken into the account while taking the calculation time. This is as expected, splitting work between two working nodes will give around x2 speed-up.

6 Conclusion

We have studied phase transition using Ising model. We employed Monte Carlo method, Metropolis algorithm to plot thermodynamic quantities in various systems. We noticed that our model was giving very precise output for small system with only 2x2 lattice. We found out how much time is required for the system to reach equalibrium, and decided to wait 7000 steps before sampling data to investigate phase transition. Critical temperature we found using larger systems, $L = 100$, turned out to match exact value of $T_C = 2.269$.

A Analytical solution of a 2x2 lattice with periodic boundry conditions

These results will serve as a benchmark for our simulation in the first stage in the project. We will derive analytical solution for thermodynamic quantities, $\langle E \rangle$, $\langle |M| \rangle$, C_v and χ. As described in section (2.1), we have two spins in each dimension which can point either up or down. This i one of the situations with the labels

$$\begin{bmatrix} \uparrow_{s1} & \uparrow_{s2} \\ \uparrow_{s3} & \uparrow_{s4} \end{bmatrix} \tag{14}$$

To stay consisten with other sections we remember that $L = 2$ and $N = L^2 = 4$. First we are interested in energy and magnetisation given by equation (1) and (2). Magnetisation is straight forward, sum over all the spins. For energy we get

$$E = -(s_1 s_2 + s_2 s_1 + s_1 s_3 + s_3 s_1 + s_2 s_4 + s_4 s_2 + s_3 s_4 + s_4 s_3) \tag{15}$$

Let's analyse different scenarios: we have L^N different configurations, in this case it is 16.

- **4 up, 0 down:** Only one way of having all the spins pointing up like in (14)
  $\rightarrow E_{4\uparrow} = -8 \quad \wedge \quad M_{4\uparrow} = 4$
- **3 up, 1 down:** There are 4 different ways for this configuration to occur. One opposite sign in equation (15) will result in half of the expression to have opposite sign so
  $\rightarrow E_{3\uparrow,1\downarrow} = 0 \quad \wedge \quad M_{3\uparrow,1\downarrow} = 2$
- **2 up, 2 down:** This situation needs to be splitted in two. Since energy depends on the neighbours only, it depends which two spins will point down, and which up.
  - **Mixed** when one spin points down, its neighbours need to point up, let's say $s1$ and $s4$ point down while $s2$ and $s3$ point up. We have two ways of this state
    $\rightarrow E_{2\uparrow,mixed} = 8 \quad \wedge \quad M_{2\uparrow,mixed} = 0$
  - **Not-Mixed** all the situations with one neighbour having opposite spin and one the same. This can occur in 4 different ways
    $\rightarrow E_{2\uparrow,not-mixed} = 0 \quad \wedge \quad M_{2\uparrow,not-mixed} = 0$
- **1 up, 3 down:** simillar situation to configuration with 3 spins up and 1 down. Magnetisation in this case is negative.
  $\rightarrow E_{1\uparrow} = 0 \quad \wedge \quad M_{1\uparrow} = -2$
- **0 up, 4 down:** same situation as for opposite configuration, with negative magnetisation
  $\rightarrow E_{0\uparrow} = -8 \quad \wedge \quad M_{0\uparrow} = -4$

table 13.4 in ([Hjorth-Jensen, 2015]) summarizes this in a compact way. We have partition function given by (5), for this lattice we have

$$Z = \sum_{i=1}^{16} e^{-\beta E_i}$$

calculations give

$$Z = 2e^{-8\beta} + 2e^{8\beta} + 12 = 4\cosh 8\beta + 12$$

from equation (6) we have first energy moment

$$\langle E \rangle = \frac{1}{Z} \sum_i \Delta E_i e^{-\beta \Delta E_i}$$

what results in

$$\langle E \rangle = \frac{1}{Z}(16e^{-8\beta} - 16e^{8\beta}) = \frac{-8\sinh(8\beta)}{\cosh(8\beta)+3}$$

With some algebra simplifications underway, second energy moment is given with following expression

$$\langle E^2 \rangle = \frac{1}{Z}\sum_i \Delta E_i^2 e^{-\beta \Delta E_i} = \frac{64\cosh(8\beta)}{\cosh(8\beta)+3}$$

now the heat capacity C_v is easy to find (look equation (3))

$$C_v = \frac{\langle E^2 \rangle - \langle E \rangle^2}{kT^2} = \frac{1}{kT^2}\left[\frac{64\cosh(8\beta)}{\cosh(8\beta)+3} - \left(\frac{-8\sinh(8\beta)}{\cosh(8\beta)+3}\right)^2\right]$$

Magnetic susceptibility is easier to find, because of fact that

$$\langle M \rangle = \frac{1}{Z}\sum_{i=1}^{16} M_i e^{-\beta E_i} = 4e^{8\beta} + 8 - 8 - 4e^{-8\beta} = 0$$

however, second moment gives

$$\langle M^2 \rangle = \frac{1}{Z}\sum_{i=1}^{16} M_i^2 e^{-\beta E_i} = \frac{8(e^{8\beta}+1)}{\cosh(8\beta)+3}$$

Setting up to express χ

$$\chi = \frac{\langle M^2 \rangle - \langle M \rangle^2}{kT} = \frac{1}{kT}\left(\frac{8(e^{8\beta}+1)}{\cosh(8\beta)+3}\right)$$

Additionally we find expression for mean absolute magnetisation

$$\langle |M| \rangle = \frac{1}{Z}\sum_{i=1}^{16} |M_i| e^{-\beta E_i} = \frac{1}{Z}\left(4e^{8\beta} + 2\cdot 4 + 2\cdot 4 + 4e^{8\beta}\right) = \frac{2e^8+4}{\cosh(8\beta)+3}$$

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